The fattening phenomenon for level set solutions of the mean curvature flow

by

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Abstract

Level set solutions are an important class of weak solutions to the mean curvature flow which allow the flow to be extended past singularities. Unfortunately, when singularities do develop it is possible for the Hausdorff dimension of the level set solution to increase. This behaviour is referred to as the fattening phenomenon. The purpose of this thesis is to discuss this phenomenon and to provide concrete examples, focusing especially on its relation to the uniqueness of smooth solutions. We first discuss the definition of level set solutions in arbitrary codimension, due to Ambrosio and Soner. We then prove some technical results about distance solutions, a type of set-theoretic subsolution to level set solutions. These include a new method of gluing together distance solutions. Next, we present several known results on the fattening phenomenon in the context of distance solutions. Finally, we provide a new example by proving that fattening occurs when immersed curves in $\mathbb{R}^3$ develop self-intersections.

Preface

The topic of this thesis was chosen in collaboration with the author's supervisor, Dr. Jingyi Chen. A large portion of this thesis surveys existing results. The organization and presentation of these results is unique to this work. Portions of Sections 3.3 and 4.2 present original results obtained independently by the author.

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Acknowledgments

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Chapter 1

Introduction

The mean curvature flow is a well studied geometric evolution equation for immersed submanifolds of a Riemannian manifold. Briefly, it is a system of parabolic PDE which moves an immersion in the direction of its mean curvature vector H (see Section 2.1 for a full definition). This process is of interest because it can be used to simplify the geometry of a submanifold, and also to find special submanifolds such as minimal surfaces (where $H \equiv 0$). Unfortunately, such applications are often impeded by the development of singularities which prevent flows from being extended for long times.

In order to apply the mean curvature flow in cases when such singularities develop, there have been several attempts to define classes of weak solutions. Broadly, there have been two major types of weak solutions.[1] In 1978, Bakke defined a generalized mean curvature flow in the language of geometric measure theory, which has become known as the Brakke flow [7]. This flow has the advantage of being defined for a very broad class of initial data (a large class of rectifiable varifolds). Furthermore, the well-developed regularity theory of geometric measure theory can be applied in this setting. However, Brakke's definition allows for a great deal of non-uniqueness, and there were certain gaps between the existence and regularity results that

[1] There have also been several less well known formulations, such as the representation of mean curvature flow as a singular limit of Ginzberg-Landau type equations [8] or the method of "ramps" applied in [1].

have proven difficult to address [15].

The next wave of development in weak solutions of the mean curvature flow occurred in the early 1990s. Two groups independently defined notions of a weak mean curvature flow for codimension-1 surfaces represented by level sets of a scalar function, using the theory of viscosity solutions to analyze the singular parabolic equation that defined the motion of these level sets [9, 12]. Critically, such level set flows can be defined for *any* closed initial set and are unique. Subsequently, Ambrosio and Soner extended this definition to higher codimension, and proved that Brakke flows are always contained in the corresponding level set flow [2]. At nearly the same time, di Giorgi developed a purely geometric definition of weak solutions, based on the use of classical solutions as barriers [4]. It was later shown that this definition was, in fact, equivalent to the level set flows [5].

In this thesis, we focus on the level set flow of Ambrosio and Soner in arbitrary codimension. In contrast to Brakke flows, the uniqueness of level set flow is desirable, but also leads to some difficulties. In particular, when there is non-uniqueness among smooth mean curvature flows or Brakke flows, the level set flow tends to develop a particular type of singularity called "fattening" in which the Hausdorff dimension of the solution can increase. For geometric applications, this can be quite undesirable unless fattening is well understood. Our goal is to contribute to the understanding of this phenomenon by summarizing some of the known results in the context of Ambrosio and Soner's work, and also providing a new example of fattening for immersed curves in $\mathbb{R}^3$.

We begin by introducing the precise definition of the mean curvature flow in Section 2.1 and summarizing the basics of viscosity solution theory necessary to understand Ambrosio and Soner's work in Section 2.2. In Chapter 3, we introduce level set flows and prove that they are well-defined and share some basic properties of smooth solutions of the mean curvature flow. We then discuss equivalence of the level set flow with smooth solutions when the later exist, and present a recent result of Hershkovitz which is stronger than that obtained in [2]. Next, we focus our attention on distance solutions, a kind of "set-theoretic subsolution" to level set flows. A highlight of this section is Theorem 3.16, which establishes a new technique to produce distance solutions by a gluing procedure. In Chapter 4, we introduce the fattening phenomenon and explain some existing results using the framework of distance solutions. Finally, we prove Theorem 4.17, which establishes the occurrence of fattening when curves evolving in $\mathbb{R}^3$ develop transverse self-intersections. The proof of this theorem makes crucial use of the gluing result for distance solutions, as well as a construction of such solutions satisfying a degenerate Dirichlet problem.

Chapter 2

Preliminaries

2.1 Mean Curvature Flow

In this thesis, we consider the evolution of immersed submanifolds $M^k \hookrightarrow \mathbb{R}^{n+k}$ by the mean curvature flow. To define this process, first recall that if $F : M \to \mathbb{R}^{n+k}$ is an immersion, the Levi-Civita connection on M with respect to the pullback g of the Euclidean metric is given by

$$\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^\top$$

where $\bar{\nabla}$ is the Euclidean connection on $\mathbb{R}^{n+k}$ and $(\cdot)^\top$ denotes projection onto the tangent space of M. Then the difference between the Euclidean connection and the induced connection

$$A(X, Y) = \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y$$

is a symmetric bilinear form with values in $N_x M$ called the second fundamental form. We can then define the shape operator $A_\nu : T_x M \to T_x M$ for a unit normal vector $\nu \in N_x M$ by

$$g(A_\nu(X), Y) = \langle A(X, Y), \nu \rangle.$$

For an orthonormal basis $\{e_i\}$ of T_xM we define

$$H = \operatorname{tr} A = \sum_i A(e_i, e_i)$$

to be the mean curvature vector of M at x. It can be verified that H is independent of the choice of basis.

On the other hand, we can define the area of the immersion F by

$$A(F) = \int_M \mathrm{dvol}_g$$

where, as above, g is the pullback metric by F. Then if X is a compactly supported normal vector field on M, the first variation of A with respect to X is given by

$$\delta A(F)X = -\int_M \langle H, X\rangle \, \mathrm{dvol}_g \, .$$

From this formula, it is natural to consider the negative gradient flow of A for immersions. This leads to the definition of the mean curvature flow for immersions.

Definition 2.1. *A family of smooth immersions $F : M \times I \to \mathbb{R}^{n+k}$ on a time interval $I \subset \mathbb{R}$ is a smooth mean curvature flow if*

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = H(x,t) \quad \textit{for} \quad x \in M, t \in I \tag{2.1}$$

where $H(x,t)$ is the mean curvature vector of the immersion F at x.

Note that in the following, we will often refer a family of immersed manifolds M_t which are the images of some smooth mean curvature flow as simply a smooth flow. Also, when M is a 1-manifold, the mean curvature flow is traditionally known as the curve shortening flow.

Classical solutions of (2.1) have been studied extensively in both the cases where the codimension $k = 1$ and for higher codimension. Short time existence and uniqueness of smooth flows is well known. Note that by composing F with a diffeomorphism of M it is possible to produce non-identical solutions of (2.1) which have the same image $M_t = F(M,t)$ ([21], Proposition 3.1). Therefore, when discussing uniqueness of smooth solutions to Equation (2.1), we will always consider whether the images M_t are unique, rather than the immersions F themselves.

Proposition 2.2 ([21], Propositions 3.2 and 3.11)**.** *Suppose that $F_0 : M \to \mathbb{R}^{n+k}$ is a smooth immersion of a compact manifold M. Then there exists a smooth flow $F : M \times [0,T) \to \mathbb{R}^{n+k}$ satisfying (2.1) such that $F(\cdot, 0) = F_0$. The images $M_t = F(M,t)$ are uniquely defined by $F_0(M)$. Furthermore, for the maximal such T, we have*

$$\limsup_{t \to T} \max_{M_t} |A|^2 = \infty$$

where $|A|^2 = \sum |A(e_i, e_i)|^2$ for an orthonormal basis $\{e_i\}$ of $T_x M$.

It is also well known that if M is compact the maximal time of existence T to solutions of (2.1) will be finite. Therefore, a major goal of research on the mean curvature flow has been to understand the formation of singularities. In this area, codimension-1 flows are somewhat better understood, in part due to the availability of stronger maximum principles in this setting. For example, the following avoidance property applies.

Proposition 2.3 ([18], Theorem 2.2.1)**.** *Suppose that M_t and N_t are smooth mean curvature flows of n-dimensional submanifolds of $\mathbb{R}^{n+1}$ on $[0,T)$ such that M_0 and N_0 are embedded and disjoint. Then M_t and N_t will each remain embedded and the pair will remain disjoint for all $t < T$.*

When $k > 1$, Proposition 2.3 does not hold. This will become an important issue when we discuss weak solutions to (2.1) in higher codimension.

2.2 Viscosity Methods

The class of weak solutions to the mean curvature flow that we consider are based on the theory of viscosity solutions to second order nonlinear elliptic and parabolic equations introduced by Crandall and Lions. In this section we introduce this theory and prove some basic results about such solutions. The main reference here is the "User's Guide to Viscosity Solutions" [10].

Recall that a function $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is said to be upper (resp. lower) semicontinuous if $\limsup_{x\to x_0} u(x) \leq u(x_0)$ (resp. $\liminf_{x\to x_0} u(x) \geq u(x_0)$). Importantly, upper and lower semicontinuous function achieve their suprema and infima on compact sets. If u is locally bounded we define the upper semicontinuous envelope of u by

$$u^*(x_0) = \limsup_{x\to x_0} u(x),$$

and define the lower semicontinuous envelope u_* analogously. Clearly, the upper semicontinuous envelope is an upper semicontinuous function satisfying $u^*(x) \geq u(x)$, and likewise for the lower semicontinuous envelope.

Our goal is to define a weak notion of sub- and supersolutions to equations of the form

$$u_t + F(u, \nabla u, \nabla^2 u) = 0 \tag{2.2}$$

which applies to functions u which are only semicontinuous. This will be accomplished by requiring a local version of the maximum principle to hold for such solutions. To this end, we make the following definition.

Definition 2.4. *A function $F : \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ (where $\mathcal{S}^d$ is the set of symmetric $d \times d$ matrices) is called degenerate elliptic if*

$$F(r, p, X) \leq F(r, p, Y) \quad \text{for} \quad Y \leq X$$

where $Y \leq X$ if and only if $X - Y$ is a positive semidefinite matrix. Furthermore, we say that F is proper if it is increasing in its first argument.

If F is proper and degenerate elliptic, the problem (2.2) will be called degenerate parabolic. Now we can define our notion of sub- and supersolutions to problems involving such operators.

Definition 2.5. *Suppose that $F : \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \to \mathbb{R}$ is proper, degenerate elliptic, and locally bounded. Let $\Omega \subset \mathbb{R}^d$ be a locally compact, open set. We* *say that an upper semicontinuous function* $u : \Omega \times (0, T) \to \mathbb{R}$ *is a viscosity subsolution of* (2.2) *if we have*

$$\phi_t(x_0, t_0) + F^*(u(x_0, t_0), \nabla\phi(x_0, t_0), \nabla^2\phi(x_0, t_0)) \leq 0 \tag{2.3}$$

for all $\phi \in C^2(\Omega \times (0, T))$ *and* (x_0, t_0) *which are local maxima of* $u - \phi$.

Likewise, we say a lower semicontinuous function u is a viscosity supersolution of (2.2) *if the reverse inequality holds for* F_* *at local minima of* $u - \phi$.

We will call u a viscosity solution of (2.2) if it is both a viscosity subsolution and a viscosity supersolution, and when it is clear, the term "viscosity" will be omitted. Note that this definition is justified in that if u is a C^2 solution of $u_t = F(u, \nabla u, \nabla^2 u)$, it is easy to check by the weak maximum principle that u is a viscosity solution.

We say that $\phi \in C^2$ touches u from above at (x_0, t_0) if $\phi(x_0, t_0) = u(x_0, t_0)$ and $\phi \geq u$ on an open neighborhood of (x_0, t_0). Correspondingly, ϕ touches u from below at (x_0, t_0) if the opposite inequality holds. The following lemma gives a characterization of viscosity sub- and supersolutions using these conditions, which is often easier to use in practice than the original definition.

Lemma 2.6. *A function u is a viscosity subsolution of* (2.2) *if and only if for every* $(x_0, t_0) \in \Omega \times (0, T)$ *and* $\phi \in C^2$ *touching u from above at* (x_0, t_0), (2.3) *holds. Likewise, u is a viscosity supersolution if and only if the corresponding condition holds for* ϕ *touching u from below.*

Proof. Suppose that (2.3) holds for test functions touching u above at (x_0, t_0). Let $\phi \in C^2$ be such that $u - \phi$ has a local maximum at (x_0, t_0). Then $\widetilde{\phi} = \phi - \phi(x_0, t_0) + u(x_0, t_0)$ touches u from above at (x_0, t_0). Thus we have

$$\widetilde{\phi}_t(x_0, t_0) + F^*(u(x_0, t_0), \nabla\widetilde{\phi}(x_0, t_0), \nabla^2\widetilde{\phi}(x_0, t_0)) \leq 0.$$

Since the derivatives of $\widetilde{\phi}$ are equal to those of ϕ, this implies that u is a viscosity subsolution of (2.2).

On the other hand, suppose that u is a viscosity subsolution of (2.2). If that ϕ touches u from above at (u_0, t_0), then $u - \phi$ has a local maximum at (x_0, t_0), so (2.3) holds. □

We now prove two important lemmas which allow us to construct new viscosity solutions from existing ones. First, we define the following weak limit operations.

Definition 2.7. *Suppose that* $(u_n)_{n=1}^{\infty} : \Omega \subset \mathbb{R}^d \to \mathbb{R}$*, we define*

$$\limsup{}^* u(x) = \sup \left\{ \limsup_{n\to\infty} u_n(x_n) \;\middle|\; (x_n)_{n=1}^{\infty} \to x \right\}$$

and

$$\liminf{}_* u(x) = \inf \left\{ \liminf_{n\to\infty} u_n(x_n) \;\middle|\; (x_n)_{n=1}^{\infty} \to x \right\}.$$

These limits are called, respectively, the upper and lower half-relaxed limits of u_n*.*

Note that it is easy to check that the upper (resp. lower) half-relaxed limit of upper (resp.) semicontinuous functions is upper (resp. lower) semicontinuous. It turns out that these operations are the correct limit under which viscosity sub- and supersolutions are preserved.

Lemma 2.8 ([10], Lemma 6.1)**.** *If* u_n *is a sequence of viscosity subsolutions of* (2.2)*, and if the upper half-relaxed limit* $\bar{u} = \limsup^* u_n$ *is bounded above, it is also a viscosity subsolution. The same holds for sequences of supersolutions and their lower half-relaxed limit.*

Next, we consider a convolution-type operation which also preserves viscosity solutions.

Definition 2.9. *Let* $f, g : \mathbb{R}^d \to \mathbb{R}$ *and define the supremal convolution of* f *and* g *by*

$$f *^{\sup} g = \sup \left\{ f(y) + g(x - y) \;\middle|\; y \in \mathbb{R}^n \right\}.$$

Likewise, define the infimal convolution of f *and* g *by*

$$f *^{\inf} g = \inf \left\{ f(y) + g(x - y) \;\middle|\; y \in \mathbb{R}^n \right\}.$$

As with the half-relaxed limits, these operations preserve upper and lower semicontinuity respectively.

Lemma 2.10. *Suppose that* $u : \Omega \times (0,T) \to \mathbb{R}$ *is a viscosity subsolution of* $u_t + G(\nabla u, \nabla^2 u) = 0$*, and* $g : \Omega \to \mathbb{R}$ *is an upper semicontinuous function satisfying* $g \leq c$*. Let*

$$\widetilde{u}(\cdot, t) = u(\cdot, t) *^{\sup} g.$$

Then $\widetilde{u}$ *is also a viscosity subsolution. The same holds for viscosity supersolutions and infimal convolution with* $g \geq c$*.*

Proof. (Following [20], Lemma 4.2.) Suppose that ϕ touches $\widetilde{u}$ from above at (x_0, t_0). Because g is upper-semicontinuous and bounded above, there exists $y_0 \in \Omega$ such that $\widetilde{u}(x_0, t_0) = u(y_0, t_0) + g(x_0 - y_0)$. Let $\widetilde{\phi}(x,t) = \phi(x + x_0 - y_0, t) - g(x_0, y_0)$. Then

$$\widetilde{\phi}(y_0, t_0) = \phi(x_0, t_0) - g(x_0, y_0) = u(y_0, t_0).$$

Furthermore, for x, t sufficiently close to (y_0, t_0) we have

$$\widetilde{\phi}(x,t) \geq \widetilde{u}(x + x_0 - y_0, t) - g(x_0, y_0) \geq u(x + x_0 - y_0, t)$$

by definition of $\widetilde{u}$. Hence $\widetilde{\phi}$ touches u from above at (y_0, t_0) and so we have

$$\widetilde{\phi}_t(y_0, t_0) + G(\nabla\widetilde{\phi}(y_0, t_0), \nabla^2\widetilde{\phi}(y_0, t_0)) \leq 0.$$

This implies that the same equation holds for ϕ at (x_0, t_0), and thus $\widetilde{u}$ is a subsolution. The same argument with inequalities reversed applies for supersolutions. □

Note that the procedure of shifting test functions used in the proof of Lemma 2.10 is characteristic of the arguments which will be used later when working with viscosity solutions.

Chapter 3

Level Set Flows and Distance Solutions

3.1 Definition and Fundamentals

In order to define a notion of weak solutions to mean curvature flow using the machinery of viscosity solutions, we must represent an embedded manifold $M^n \hookrightarrow \mathbb{R}^{n+k}$ via a single function u. We then aim to write an equation of the form (2.2) for u which gives an equivalent evolution of the embedding. If $k = 1$, it is natural to represent M_t as a regular level set of a smooth function $u : \mathbb{R}^{n+1} \times (0, T) \to \mathbb{R}$. We begin this section by showing how the corresponding evolution equation ought to be written. The situation is somewhat more complicated for $k > 1$, as M_t must be represented as a level set of u at a singular value.

First, consider a function $u : \mathbb{R}^{n+1} \times (0, T) \to \mathbb{R}$ and a local parameterization of the zero-level set by $\phi : \Omega \subset \mathbb{R}^n \times (0, T) \to \mathbb{R}^{n+1}$ such that $u(\phi(x, t), t) = 0$. Differentiating gives

$$u_t(\phi, t) = -\langle \nabla u(\phi, t), \phi_t \rangle.$$

If the zero-level set is to move normally with speed v (to be determined later), then we take $\phi_t = v\frac{\nabla u}{|\nabla u|}$, which gives

$$u_t(\phi, t) = -|\nabla u(\phi, t)|v.$$

We generalize this to hold on all points to obtain

$$u_t = -|\nabla u|v. \tag{3.1}$$

In the case when $k = 1$, the mean curvature of a level set of u is simply given by

$$H = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Therefore, using $v = H$ in (3.1) the MCF equation for M_t becomes

$$u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{3.2}$$

This equation was considered in context of viscosity solutions by Evans and Spruck [12] and Chen, Giga, and Goto [9]. It was shown that (3.2) corresponds to a degenerate parabolic problem which admits a unique viscosity solution for uniformly continuous initial data u_0. Furthermore, the evolution of the zero level set depends only on its initial geometry, not the choice of u_0, and this evolution agrees with a smooth mean curvature flow if it exists.

Proceeding from this work, Ambrosio and Soner [2] generalized this approach to the case when $k > 1$. This generalization will be the main class of weak solution considered here, so we will explain it in detail. As above, the surface M_t will be represented by the zero level set of $u : \mathbb{R}^{n+k} \times (0, T) \to \mathbb{R}$. However, as noted above, the difficulty is that M_t must be represented by a singular level set of u, and therefore geometric quantities are difficult to compute for this level set. Ambrosio and Soner instead consider regular ε-level sets for ε small. Such level sets are smooth hypersurfaces which "wrap tightly" around M_t so we expect them to have $k-1$ principle curvatures which are very large and n principle curvatures which closely approximate those of M_t nearby. With this intuition, they design a flow which moves regular level sets by the sum of their smallest n principle curvatures. To do so, we first represent the shape operator in terms of the level set function u.

Lemma 3.1. *Suppose that M is the zero level set of $u : \mathbb{R}^{n+k} \to \mathbb{R}$ and $\nabla u \neq 0$ on M. Then M is a smooth manifold with normal vector field $\nu = \frac{\nabla u}{|\nabla u|}$. At $x \in M$, let $B = \frac{1}{|\nabla u|} P_{\nabla u} \nabla^2 u P_{\nabla u}$ where $P_{\nabla u} = I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}$ is the orthogonal projection onto $T_x M$. Then*

$$B = A_\nu \oplus 0$$

where A_ν is the shape operator on $T_x M$ and 0 acts on $N_x M$.

Proof. Note that $B\eta = 0$ for $\eta \in N_x M$, and $BX \in T_x M$ for $X \in T_x M$ by definition of $P_{\nabla u}$ so B splits into these subspaces. We can compute for $X \in T_x M$

$$BX = \frac{1}{|\nabla u|} P_{\nabla u} (\nabla^2 u) X = \frac{\nabla^2 u}{|\nabla u|} X - \frac{1}{|\nabla u|^3} \langle \nabla u, (\nabla^2 u) X \rangle \nabla u.$$

On the other hand, we have

$$\begin{aligned} A_\nu X = \nabla_X \frac{\nabla u}{|\nabla u|} &= \langle X, \nabla |\nabla u|^{-1} \rangle \nabla u + \frac{\nabla^2 u}{|\nabla u|} X \\ &= \frac{\nabla^2 u}{|\nabla u|} X - \frac{1}{|\nabla u|^3} \langle (\nabla^2 u) X, \nabla u \rangle \nabla u. \end{aligned}$$

Thus we have $BX = A_\nu X$. $\square$

By Lemma 3.1, we see that the principle curvatures of a regular level set of u are the eigenvalues of B with eigenvectors orthogonal to ∇u. This motivates us to define $F : \mathbb{R}^{n+k} \setminus \{0\} \times \mathcal{S}^{n+k} \to \mathbb{R}$ by

$$F(p, X) = -\sum_{i=1}^{n} \lambda_i(p, X) \tag{3.3}$$

where $\lambda_i(p, X)$ are the ordered eigenvalues of $P_p X P_p$ with eigenvector orthogonal to p and $P_p = I - \frac{p \otimes p}{|p|^2}$ as in (3.1). Then the equation

$$u_t + F(\nabla u, \nabla^2 u) = 0 \tag{3.4}$$

corresponds to the level set evolution equation (3.1) taking v to be the sum of the smallest n principle curvatures on regular level sets. The zero level sets of viscosity solutions of this equation will be our weak solutions to mean curvature flow.

Definition 3.2. *Suppose that $\Gamma^* \subset \mathbb{R}^d$ is closed and let u_0 be any uniformly continuous function on $\mathbb{R}^d$ such that $\Gamma^* = \{x \mid u_0(x) = 0\}$. Suppose $u \in C(\mathbb{R}^{n+k} \times [0,\infty))$ is a viscosity solution of the problem*

$$\begin{cases} u_t + F(\nabla u, \nabla^2 u) = 0 & \text{on} \quad \mathbb{R}^{n+k} \times (0,\infty) \\ u(x,0) = u_0(x,0) & \text{for} \quad x \in \mathbb{R}^{n+k}. \end{cases} \tag{3.5}$$

We let

$$\Gamma_t = \{x \mid u(x,t) = 0\} \tag{3.6}$$

and call $\Gamma = \bigcup_{t\in[0,\infty)} \Gamma_t \times \{t\}$ the n-dimensional level set flow of Γ^*.

Note that F is in fact a continuous degenerate elliptic operator away from $p = 0$, so (3.4) can be considered in the viscosity sense. Furthermore, we record

$$F_*(0,A) = \min_{|p|=1} F(p,A) \quad \text{and} \quad F^*(0,A) = \max_{|p|=1} F(p,A) \tag{3.7}$$

which will be used later. In [2] the following fundamental facts about solutions to (3.5) were proven.

Proposition 3.3 ([2], Theorems 2.2-2.4)**.** *(a) (Comparison) Suppose that u and v are sub- and supersolutions of (3.4) such that at least one of u or v is uniformly continuous and there exists $K > 0$ such that*

$$|u(x,t)| + |v(x,t)| \leq K(1 + |x|)$$

then $u - v \leq \sup\{u(x,0) - v(x,0) \mid x \in \mathbb{R}^{n+k}\}$.

(b) (Existence) If u_0 is uniformly continuous, there exists a unique uniformly continuous solution u to (3.5) defined on $\mathbb{R}^{n+k} \times [0,\infty)$.

(c) *(Relabeling) If u is a subsolution (resp. supersolution) of* (3.4)*, and $\theta : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and nondecreasing, then $\theta \circ u$ is also a subsolution (resp. supersolution).*

As a demonstration of the methods used in the viscosity solution theory, we use Proposition 3.3 to prove that the n-dimensional level set flow of a set is well-defined.

Lemma 3.4. *Suppose that $\Gamma^* \subset \mathbb{R}^{n+k}$ is closed and u_0 and $\widetilde{u_0}$ are two uniformly continuous function which both have Γ^* as their zero-level set. Let u and $\widetilde{u}$ be the corresponding solutions of* (3.5)*. Then the zero-level set of u is equal to that of $\widetilde{u}$.*

Proof. (Following [2], Theorem 2.5) In particular, we will show that the lemma holds if $u_0(x) = \operatorname{dist}(x, \Gamma)$, from which the full result follows by transitivity. Let Γ_t and $\widetilde{\Gamma}_t$ be the zero-level sets of u and $\widetilde{u}$ respectively.

($\Gamma_t \subset \widetilde{\Gamma}_t$): Let $\omega(s) = \sup \left\{ \widetilde{u_0}(y) \mid \operatorname{dist}(y, \Gamma^*) \le s \right\}$. Then since $\widetilde{u_0}$ is uniformly continuous, $\omega(s)$ is a non-decreasing uniformly continuous function, and thus by Proposition 3.3(c), $\omega \circ u$ is a supersolution of (3.4). Now note that

$$\omega(u_0(x)) = \omega(\operatorname{dist}(x, \Gamma)) \ge \widetilde{u_0}(x).$$

Hence by Proposition 3.3(a), $\widetilde{u} \le \omega \circ u$. Therefore, if $x \in \Gamma_t$, $\omega(u(x,t)) = 0$ so $\widetilde{u}(x,t) = 0$, and so $x \in \widetilde{\Gamma}_t$.

($\widetilde{\Gamma}_t \subset \Gamma_t$): Let $h^\varepsilon(s)$ be a sequence of non-decreasing cutoff functions with $h^\varepsilon \equiv 0$ on $(-\infty, 0]$ and $h^\varepsilon \equiv 1$ on $[\varepsilon, \infty)$. By Proposition 3.3(c), $h^\varepsilon \circ \widetilde{u}$ is a supersolution of (3.4) for each ε. By Lemma 2.8, it follows that $\liminf_* h^\varepsilon \circ \widetilde{u} = 1 - \chi_{\widetilde{\Gamma}_t}$ is also supersolution. Finally, let $v = \min(u, 1)$, so that $v(x, 0) \le 1 - \chi_\Gamma$ and by comparison $v \le 1 - \chi_{\widetilde{\Gamma}_t}$ for all t. This inequality implies that if $x \in \widetilde{\Gamma}_t$, then $v(x,t) = 0$, and so $x \in \Gamma_t$. □

It is an easy computation to check that F satisfies

$$F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X) \tag{3.8}$$

for $\lambda > 0$ and $\sigma \in \mathbb{R}$. This identity allows us to extend the usual scaling of solutions to mean curvature flow to level set flows.

Lemma 3.5. *Suppose* Γ *is the level set flow of* Γ^*. *Then for* $\lambda > 0$, *the level set flow* $\widetilde{\Gamma}$ *of* $\lambda\Gamma^*$ *is given by* $\widetilde{\Gamma}_t = \lambda\Gamma_{\lambda^{-1}t}$.

Proof. Let u be a solution to (3.5) with initial data u_0 zero on Γ^*. Define $v(x,t) = \lambda^{-1}u(\lambda^{-1}x, \lambda^{-1}t)$. Note that the zero level set of $v(\cdot, 0)$ is $\lambda\Gamma^*$. We claim that v solves (3.4). Suppose that ϕ touches v from below at (x_0, t_0). Define $\widetilde{\phi}(x,t) = \lambda\phi(\lambda x, \lambda t)$. Then we have

$$\widetilde{\phi}(\lambda^{-1}x_0, \lambda^{-1}t_0) = \lambda\phi(x_0, t_0) = \lambda v(x_0, t_0) = u(\lambda^{-1}x_0, \lambda^{-1}t_0)$$

and for x and t sufficiently close to $\lambda^{-1}x_0$ and $\lambda^{-1}t_0$

$$\widetilde{\phi}(x,t) = \lambda\phi(\lambda x, \lambda t) \leq \lambda v(\lambda x, \lambda t) = u(x,t).$$

Hence $\widetilde{\phi}$ touches u from below at $(\lambda^{-1}x_0, \lambda^{-1}t_0)$, and so at this point we have

$$\widetilde{\phi}_t + F(\nabla\widetilde{\phi}, \nabla^2\widetilde{\phi}) \geq 0.$$

Applying the definition of $\widetilde{\phi}$ and (3.8) we have

$$\lambda^2\phi_t(x_0, t_0) + \lambda^2 F(\nabla\phi(x_0, t_0), \nabla^2\phi(x_0, t_0)) \geq 0.$$

Therefore v is a viscosity supersolution of (3.4). The same argument with inequalities reversed shows that v is also a subsolution. Since v is a solution of (3.4), $\widetilde{\Gamma}_t$ is the zero-level set of $v(\cdot, t)$, which is exactly $\lambda\Gamma_{\lambda^{-1}t}$ by the definition of v. □

3.2 Equivalence with Smooth Flows

An important property of any formulation of weak solutions to the mean curvature flow problem is agreement with smooth flows when they exist. In the codimension-1 case, this equivalence was proven in [12] and [9]. In their original paper on level set flows in arbitrary codimension, Ambrosio and Soner proved the following result.

Proposition 3.6. *If $M_t \subset \mathbb{R}^{n+k}$ is a smooth flow of embedded n-dimensional submanifolds on $[0,T)$, then the n-dimensional level set flow Γ of M_0 satisfies $\Gamma_t = M_t$ for $0 \leq t < T$.*

Ambrosio and Soner prove this result by considering the evolution of the distance function $\delta(x,t) = \operatorname{dist}(x, M_t)$. Recall that if, for all $0 < t < T$, M_t has a tubular neighborhood of radius ρ, the distance function $\delta(\cdot,t)$ is smooth on $U = \{(x,t) \mid 0 \leq t < T, 0 < \delta(x,t) < \rho\}$. Ambrosio and Soner show that on U we have

$$F(\nabla\delta, \nabla^2\delta) \leq \delta_t \leq F(\nabla\delta, \nabla^2\delta) + C\delta \tag{3.9}$$

for some constant C independent of x and t. Using these bounds, they modify δ (while maintaining the zero level set) to construct suitable sub- and supersolutions $\underline{u}$ and $\bar{u}$ of (3.5), which are bounded above and below by $\delta(\cdot,0)$ at $t=0$. By the comparison principle of Proposition 3.3(a), if u is the solution of (3.5) with initial data $\delta(\cdot,0)$, we have

$$\underline{u} \leq u \quad \text{and} \quad \bar{u} \geq u.$$

As in the proof of Lemma 3.4, these bounds show that $M_t = \Gamma_t$.

While this is one of Ambrosio and Soner's fundamental results and is important in justifying the definition of the level set flow, it is not entirely satisfying because it says nothing about the case when M_t is a smooth flow on $(0,T)$ but does not extend smoothly to $t=0$. This case is important in applications of the weak solution theory to the question of solvability of the mean curvature flow equation for rough initial data. In particular, establishing equivalence of the level set flow and a smooth flow can be used to prove uniqueness of the smooth flow.

Hershkovits takes this approach in [14] when he considers the short time existence of smooth solutions of the mean curvature flow when the initial data is an ε-Reifenberg set. Briefly, such sets are embedded topological manifolds which have approximate tangent spaces at each point which are allowed to vary according to a scale parameter (the details are not relevant here; see [19] for a complete definition). To apply the level set flow theory of Ambrosio and Soner in this setting, Hershkovits proves a stronger version of Proposition 3.6.

Proposition 3.7. *Suppose that M_t is a smooth flow of embedded n-dimensional submanifolds of $\mathbb{R}^{n+k}$ on $(0,T)$ and M_0 is a connected compact set. Suppose further that for constants $c_1^2 \leq \frac{1}{8}$ and $\frac{1}{4c_1} - c_2 > \sqrt{2n}$, M_t satisfies*

(i) $\sup_{M_t} |A(t)| \leq \frac{c_1}{\sqrt{t}}$

(ii) $d_{\mathcal{H}}(M_t, M_0) \leq c_2\sqrt{t}$

(iii) M_t has a tubular neighborhood of radius at least $\frac{\sqrt{t}}{4c_1}$

where $A(t)$ is the second fundamental form of M_t and $d_{\mathcal{H}}$ is the Hausdorff distance. Then the level set flow of M_0 is equal to M_t on $(0,T)$.

Note that while the constants in the statement of this result may seem somewhat arbitrary, they cannot be easily scaled away (i.e. we cannot trade a worse bound on c_1 for a better bound on c_2) because (i-iii) are all scale independent with respect to the spacetime scaling of mean curvature flow.

To prove this result, Hershkovits establishes a more precise version of (3.9). In particular for (x,t) in the same neighborhood U as above we have

$$\delta_t = F(\nabla\delta, \nabla^2\delta) + \delta \sum_{i=1}^{n} \frac{\langle A(v_i, v_i), \nabla\delta\rangle^2}{1 - \delta\langle A(v_i, v_i), \nabla\delta\rangle} \tag{3.10}$$

where $\{v_i\}$ are principle directions for the shape operator $A_{-\nabla\delta}$ at the closest point on M_t to x, and A is evaluated at the same point.

We will also need the following lemma, which says that n-dimensional level set flows do not cross spheres evolving by the sum of their first n principle curvatures. Note that this is actually a very special property of spheres, since in general the avoidance properties that hold in codimension-1 (c.f. Proposition 2.3) do not apply in higher codimension. We will use this result to bound the rate at which the level set flow can move away from M_0.

Lemma 3.8. *Suppose that* Γ^* *is a closed set and* $v : \mathbb{R}^{n+k} \times (0,T) \to \mathbb{R}$ *is a supersolution of* (3.5)*, with* $v(x,0) = \operatorname{dist}(x, \Gamma^*)$*. Then*

$$v(x_0, t) \geq v(x_0, 0) - \sqrt{2nt} \tag{3.11}$$

for all $x_0 \in \mathbb{R}^{n+k}$*.*

Proof. We fix $x_0 \in \mathbb{R}^{n+k}$ and define

$$u(x,t) = v(x_0, 0) - \sqrt{|x - x_0|^2 + 2kt}.$$

Note that $u(x, 0) = \operatorname{dist}(x_0, \Gamma^*) - |x - x_0|$. Using the fact that the distance function to Γ^* is 1-Lipschitz, we have $u(x, 0) \leq v(x, 0)$. Furthermore, it is easy to check that u is in fact a solution to (3.4). (One can either compute directly, or use the fact that the level sets of u are spheres moving by their first n principle curvatures.) Therefore by Proposition 3.3(a), we have

$$u(x_0, t) \leq v(x_0, t) \implies v(x_0, 0) - \sqrt{2nt} \leq v(x_0, t)$$

which proves the claim. □

Proof of Proposition 3.7. Following Herskovitz, Theorem 1.7 [14], let Γ be the level set flow of M_0 with level set function u solving (3.5) with initial data $\operatorname{dist}(M_0, \cdot)$. The argument of [2] that was discussed above still suffices to show that $M_t \subset \Gamma_t$. We will only consider the more difficult problem of showing that $\Gamma_t \subset M_t$.

To do this, we consider $v(x,t) = \frac{\delta(x,t)}{\sqrt{t}}$ defined on

$$N = \left\{ (x,t) \bigm| 0 \leq t \leq T, \delta(x,t) < \frac{\sqrt{t}}{4c_1} \right\}.$$

Note that $N \cap (\mathbb{R}^{n+k} \times \{0\})$ is empty, and so by (iii) v is smooth on N. Then, a computation using (3.10) and assumptions (i) and (ii) show that v is a classical subsolution of (3.4) on N. (Here is where the condition $c_1^2 < \frac{1}{8}$ is used.)

Now, if we can use the comparison principle to conclude that $u \geq v$, we will be done, since then $x \in \Gamma_t$ would imply $u(x,t) = 0$ and so $v(x,t) = 0$ which implies $x \in M_t$. A (slightly modified version) of the comparison principle will apply if we can show that $u \geq v$ on the parabolic boundary of N which consists of (x,t) such that $t > 0$ and $\delta(x,t) = \frac{\sqrt{t}}{4c_1}$. Using this characterization along with (ii) we have

$$\frac{\sqrt{t}}{4c_1} = \delta(x,t) \leq \delta(x,0) + d_{\mathcal{H}}(M_t, M_0) \leq \delta(x,0) + c_2\sqrt{t}.$$

This implies that $\delta(x,0) \geq \left(\frac{1}{4c_1} - c_2\right)\sqrt{t}$. Applying Lemma 3.8, we have

$$u(x,t) \geq \delta(x,0) - \sqrt{2nt} \geq \left(\frac{1}{4c_1} - c_2 - \sqrt{2n}\right)\sqrt{t} = \alpha v(x,t)$$

where α is a positive constant by the constraints on c_1 and c_2. By the relabeling result Proposition 3.3(c) for solutions of (3.4), αv is also a subsolution on N, and by comparison we have $u \geq \alpha v$ on N. □

3.3 Distance Solutions and Gluing

In the previous section, (3.9) shows that the distance function to a family of smooth manifolds evolving by mean curvature is a classical supersolution of (3.4) within a tubular neighborhood. In fact, we will see below that the distance function is actually a viscosity supersolution everywhere. From this conclusion, Ambrosio and Soner extract the following definition, which can be understood as an intrinsic characterization of "set-theoretic subsolutions" of the level set flow.

Definition 3.9. *Suppose that $\Gamma \subset \mathbb{R}^{n+k} \times (0,T)$ and for each $t \in (0,T)$ supposed $\Gamma_t = \Gamma \cap \mathbb{R}^{n+k} \times \{t\}$ is closed. For any such set write*

$$\delta_\Gamma(x,t) = \operatorname{dist}(x, \Gamma_t)$$

for the spatial distance function. We call Γ an n-dimensional distance solu-

tion if δ_Γ *is a supersolution of* (3.5). *Let*

$$\liminf_{t\to 0} \Gamma_t = \bigcap_{t\in(0,T)} \overline{\bigcup_{s\in(0,t)} \Gamma_s}.$$

Given a closed set $\Gamma^* \subset \mathbb{R}^{n+k}$*, a distance solution is said to satisfy the initial inclusion* $\Gamma_0 \subset \Gamma^*$ *if* $\liminf_{t\to 0} \Gamma_t \subset \Gamma^*$*.*

The following lemma will be used extensively when considering distance solutions in the remainder of this thesis. It greatly simplifies the process of checking whether a set Γ is a distance solution by restricting the class of test functions that must be considered.

Lemma 3.10. *Suppose that* $\Gamma \subset \mathbb{R}^{n+k} \times (0,T)$*. Then* Γ *is a distance solution if for all* C^2 *test function* ϕ *touching* δ_Γ *from below at a point* (y_0, t_0) *on* Γ *we have*

$$\phi_t(y_0, t_0) + F_*(\nabla\phi(y_0, t_0), \nabla^2\phi(y_0, t_0)) \geq 0.$$

Furthermore, if $t \mapsto \Gamma_t$ *is continuous with respect to the Hausdorff distance* $d_{\mathcal{H}}$*, we need only consider* ϕ *and* (y_0, t_0) *such that* $\nabla\phi(y_0, t_0) \neq 0$ *and* $y_0 \in \Gamma_{t_0}$ *is the minimizer in* Γ_{t_0} *of distance to some point outside* Γ*.*

Proof. Let ϕ be an arbitrary C^2 test function which touches δ_Γ from below at $(x_0, t_0) \in \mathbb{R}^{n+k} \times (0,T)$. Let $y_0 \in \Gamma_{t_0}$ be such that $\delta_\Gamma(x_0, t_0) = |x_0 - y_0|$, and define

$$\widetilde{\phi}(x,t) = \phi(x + x_0 - y_0, t) - |x_0 - y_0|.$$

Note that $\widetilde{\phi}(y_0, t_0) = 0$. Let ε be such that $\phi(z,t) \leq \delta_\Gamma(z,t)$ for $(z,t) \in B_{2\varepsilon}((x_0, t_0))$. Let $y \in B_\varepsilon(y_0)$ and $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Then $(y + x_0 - y_0, t) \in B_\varepsilon((x_0, t_0))$ and so

$$\widetilde{\phi}(y,t) = \phi(y + x_0 - y_0) - |x_0 - y_0| \leq \delta_\Gamma(y + x_0 - y_0, t) - |x_0 - y_0|.$$

By the triangle inequality, we have $\delta_\Gamma(y + x_0 - y_0, t) \leq |x_0 - y_0| + \delta_\Gamma(y,t)$, so we obtain

$$\widetilde{\phi}(y,t) \leq \delta_\Gamma(y,t).$$

Thus $\widetilde{\phi}$ is a C^2 function touching δ_Γ from below at (y_0, t_0). By our assumption, we have

$$\widetilde{\phi}_t(y_0, t_0) + F_*(\nabla\widetilde{\phi}(y_0, t_0), \nabla^2\widetilde{\phi}(y_0, t_0)) \geq 0.$$

But the derivatives of ϕ at (x_0, t_0) are equal to those of $\widetilde{\phi}$ at (y_0, t_0), so we have

$$\phi_t(x_0, t_0) + F_*(\nabla\phi(x_0, t_0), \nabla^2\phi(x_0, t_0)) \geq 0.$$

From this it follows that Γ is a distance solution.

Finally, if we only consider $(x_0, t_0) \notin \Gamma$ in the proof above, we obtain that δ_Γ is a supersolution of (3.4) on $(\mathbb{R}^{n+k} \times (0, T)) \setminus \Gamma$. For such points ϕ in the above has $\nabla\widetilde{\phi}(x_0, t_0) \neq 0$, since ϕ touches δ_Γ from below at a point away from its zero level set. Furthermore, by definition y_0 is a minimizer in Γ_{t_0} of the distance to x_0. To show that δ_Γ is a supersolution on all of $\mathbb{R}^{n+k} \times (0, T)$ we apply Lemma 3.11 below to δ_Γ. We note that by the assumption of $d_{\mathcal{H}}$-continuity, assumption (i) of the lemma is satisfied. □

Lemma 3.11 ([2], Lemma 3.11)**.** *Suppose that $u : \mathbb{R}^{n+k} \times (0, T)$ is lower semicontinuous and*

(i) for (x, t) such that $u(x, t) = 0$ there exists a sequence $(x_n, t_n) \to (x, t)$ with $t_n < t$ and $u(x_n, t_n) = 0$;

(ii) u is a viscosity supersolution of (2.2) on $\{(x, t) \mid u(x, t) > 0\}$;

(iii) and $u(\cdot, t)$ is K-Lipschitz continuous with K independent of t.

Then u is also a viscosity supersolution of (2.2) on $\mathbb{R}^{n+k} \times (0, T)$.

Note that Lemma 3.10 shows that the distance function to a smooth flow is actually a viscosity supersolution globally since we only need to consider test functions touching δ from below at points on M_t, and δ is a supersolution near M_t. Thus, smooth flows are distance solutions. (In fact, the main idea of Lemma 3.10 is extracted from the proof of this fact in [2].)

A distance solution Γ may be quite poorly behaved. For example, entire connected components Γ_t may disappear instantaneously, and δ need only be lower semicontinuous in time. However, it turns out that *maximal* distance solutions (with respect to containment as sets) are exactly level set flows, which explains how distance solutions may be viewed as subsolutions.

Proposition 3.12. *Suppose that $\Gamma^* \subset \mathbb{R}^{n+k}$ is closed and Γ is the n-dimensional level set flow of Γ^*. Then Γ is the maximal distance solution satisfying the initial inclusion $\Gamma_0 \subset \Gamma^*$.*

Proof. (Following [2], Theorem 4.4) Let u be any solution of (3.5) with zero level set Γ. Let h^ε be as in the proof of Lemma 3.4. Let $\underline{u} = \liminf_* h^\varepsilon \circ u$. As before $\underline{u} = 1 - \chi_\Gamma$ is a supersolution by Lemma 2.8. Now let $v_K(\cdot, t) = K(1 - \chi_\Gamma(\cdot, t)) *^{\inf} g$ where $g(x) = |x|$ and $K > 0$. By Lemma 2.10, v_K is a supersolution. We claim that

$$v_K(x,t) = \min(\delta_\Gamma(x,t), \inf\left\{K + |x-y| \;\middle|\; y \notin \Gamma_t\right\}).$$

The infimum in the definition of the infimal convolution must be attained since g becomes unbounded as $x \to \infty$. Hence there exists y such that

$$v_K(x,t) = K(1 - \chi_\Gamma(y,t)) + |x-y|.$$

If $y \in \Gamma_t$, then $v_K(x,t) = |x-y|$. Otherwise, $v_K(x,t) = K + |x-y|$. Hence

$$v_K(x,t) = \min(\inf\left\{|x-y| \;\middle|\; y \in \Gamma_t\right\}, \inf\left\{K + |x-y| \;\middle|\; y \notin \Gamma_t\right\})$$

which is exactly the claim above. Finally, by Lemma 2.8 $\delta_\Gamma = \liminf_* v_K$ must be a supersolution, and so Γ is a distance solution satisfying the initial inclusion $\Gamma_0 \subset \Gamma^*$.

On the other hand, if $\widetilde{\Gamma}$ is any other distance solution satisfying $\widetilde{\Gamma}_0 \subset \Gamma^*$, we note that $\delta_{\widetilde{\Gamma}} \le u$ by Proposition 3.3(a). Hence $\widetilde{\Gamma} \subset \Gamma$, and so Γ is in fact the maximal distance solution satisfying $\Gamma_0 \subset \Gamma^*$. □

As an example of the utility of this result, we consider the situation in which Γ^* sits in an affine subspace Σ of $\mathbb{R}^{n+k}$. If Γ^* is a submanifold, its smooth mean curvature flow clearly remains in Σ and is equivalent to that obtained when Γ^* is viewed as a subspace of Σ. The following proposition generalizes this fact to level set flows.

Proposition 3.13. *Let $\Sigma \subset \mathbb{R}^{n+k}$ be a d-dimensional affine subspace. Suppose that $\Gamma^* \subset \Sigma$. Let Γ be the n-dimensional level set flow of Γ^* in $\mathbb{R}^{n+k}$ and $\widetilde{\Gamma}$ be the level set flow of Γ^* in Σ. Then $\Gamma = \widetilde{\Gamma}$.*

Proof. Without loss of generality, we identify Σ with the plane $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{n+k}$. We will write $x = (y, z) \in \mathbb{R}^d \times \mathbb{R}^{n+k-d}$ for coordinates in this decomposition.

($\widetilde{\Gamma} \subset \Gamma$): We will show that if $\widetilde{\Gamma}$ is any distance solution as a subset of Σ, then it is also a distance solution in $\mathbb{R}^{n+k}$. By the characterization of level set flows as maximal distance solutions, this will prove $\widetilde{\Gamma} \subset \Gamma$. Let F_d and F_{n+k} refer to the operator F defined in (3.3) for ambient dimensions d and $n+k$ respectively. By Lemma 3.10, we consider a test function $\phi : \mathbb{R}^{n+k} \times (0,T) \to \mathbb{R}$ touching $\delta_{\widetilde{\Gamma}}$ from below at $((y_0, 0), t_0) \in \widetilde{\Gamma}$. We then need to show that

$$\phi_t(y_0, 0, t_0) + (F_{n+k})_*(\nabla\phi(y_0, 0, t_0), \nabla^2\phi(y_0, 0, t_0)) \geq 0.$$

By restricting ϕ to $\widetilde{\phi}(y,t) = \phi(y, 0, t)$, and using the fact that $\widetilde{\Gamma}$ is a distance solution in Σ, we have

$$\phi_t(y_0, 0, t_0) + (F_d)_*(\nabla\widetilde{\phi}(y_0, t_0), \nabla^2\widetilde{\phi}(y_0, t_0)) \geq 0.$$

Write $A = \nabla^2\phi(y_0, 0, t_0)$ and $p = \nabla\phi(y_0, 0, t_0)$, and correspondingly $\widetilde{A} = \nabla^2\widetilde{\phi}(y_0, t_0)$ and $\widetilde{p} = \nabla\widetilde{\phi}(y_0, t_0)$. We will show that

$$(F_d)_*(\widetilde{A}, \widetilde{p}) \geq (F_{n+k})_*(A, p) \tag{3.12}$$

which suffices to establish the necessary inequality. First we consider the case when $p \neq 0$. Recall that the variational characterization of eigenvalues gives

$$\lambda_j(P_p A P_p) = \max_{\substack{S \subset \mathbb{R}^{n+k} \\ \dim(S) \geq n+k-j+1}} \min_{\substack{x \in S \\ |x|=1 \\ \langle x,p \rangle = 0}} \langle P_p A P_p x, x \rangle, \tag{3.13}$$

where the condition $\langle x, p\rangle = 0$ is due to the fact that we consider only eigenvalues with eigenvectors orthogonal to p. On the other hand, we have

$$\lambda_j(P_{\widetilde{p}}\widetilde{A}P_{\widetilde{p}}) = \max_{\substack{S\subset\Sigma \\ \dim(S)\geq d-j+1}} \min_{\substack{x\in S \\ |x|=1 \\ \langle x,\widetilde{p}\rangle=0}} \langle P_{\widetilde{p}}\widetilde{A}P_{\widetilde{p}}x, x\rangle. \tag{3.14}$$

Letting $q = (\widetilde{p}, 0) \in \mathbb{R}^{n+k}$, we can write (3.14) in terms of vectors $x \in \mathbb{R}^{n+k}$ as

$$\lambda_j(P_{\widetilde{p}}\widetilde{A}P_{\widetilde{p}}) = \max_{\substack{S\subset\mathbb{R}^{n+k} \\ \dim(S)\geq d-j+1}} \min_{\substack{x\in S\cap\Sigma \\ |x|=1 \\ \langle x,q\rangle=0}} \langle P_qAP_qx, x\rangle. \tag{3.15}$$

In (3.13) and (3.14) using the symmetry of P_p and P_q and the constraints on x, we have

$$\langle P_pAP_px, x\rangle = \langle Ax, x\rangle \quad \text{and} \quad \langle P_qAP_qx, x\rangle = \langle Ax, x\rangle.$$

Finally, since $d < n + k$, if S is considered in the maximum in (3.13), then it is considered in (3.15) and the corresponding minimum is over a larger set. Hence $\lambda_j(P_{\widetilde{p}}\widetilde{A}P_{\widetilde{p}}) \geq \lambda_j(P_pAP_p)$. From the definition of F_d and F_{n+k}, (3.12) holds. Finally, note that having removed the normalization by $|p|$ and $|\widetilde{p}|$, Equations (3.13) and (3.14) are valid for all p, so the above argument applies in the case when $p = 0$ or $\widetilde{p} = 0$ as well.

($\Gamma \subset \widetilde{\Gamma}$): Using the avoidance of spheres proven in Lemma 3.8, it is easy to see that Γ remains within the subspace Σ. (Choose arbitrarily large spheres tangent to each point on Σ.) We will apply Lemma 3.10 again to show that Γ is a distance solution viewed as a subset of Σ. Therefore, we consider a C^2 test function ϕ on $\Sigma \times (0, T)$ which touches $\delta_\Gamma|_\Sigma$ from below at $(x_0, t_0) \in \Gamma$. Let λ be larger than all of the eigenvalues of $P_{\nabla\phi}\nabla^2\phi P_{\nabla\phi}$ at (x_0, t_0), and define

$$\widetilde{\phi}(y, z, t) = \phi(y, t) + \frac{1}{2}\lambda|z|^2.$$

Using the fact that $\delta_\Gamma(y, z, t) = \sqrt{\delta_\Gamma(y, 0, t)^2 + |z|^2}$, it is easy to check that $\widetilde{\phi}$ touches δ_Γ from below at (x_0, t_0). Then, using the definition of $\widetilde{\phi}$, we can compute that

$$\begin{aligned}\phi_t(x_0,t_0) &= \widetilde{\phi}_t(x_0,t_0)\\ &\geq F_{n+k}(\nabla\widetilde{\phi}(x_0,t_0),\nabla^2\widetilde{\phi}(x_0,t_0))\\ &= F_d(\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)).\end{aligned}$$

where the last equality is because our choice of λ ensures that any new eigenvalues of $P_{\nabla\widetilde{\phi}}\nabla^2\widetilde{\phi}P_{\nabla\widetilde{\phi}}$ are large. □

From Proposition 3.12, we also see that distance solutions provide a simple method of proving "set theoretic lower bounds" on level set flows. In particular, if we can prove that Γ is a distance solution satisfying $\Gamma_0 \subset \Gamma^*$, then the level set flow of Γ^* must include Γ. We will take advantage of this fact in our discussion of the fattening phenomenon in Section 4.2. As a preliminary, we now develop a method to glue together several distance solutions into a new distance solution. Clearly by the properties of viscosity supersolutions, the union of two distance solutions is a distance solution, so instead we consider sets which are distance solutions apart from some "boundary."

Definition 3.14. *Let $\Gamma \subset \mathbb{R}^{n+k} \times (0,T)$ and $\Sigma \subset \Gamma$ be such that Γ_t and Σ_t are closed. The pair (Γ,Σ) will be called an interior distance solution if every point $(x,t) \in \mathbb{R}^{n+k} \times (0,T)$ such that $x \in \Gamma_t \setminus \Sigma_t$ has a neighborhood U on which δ_Γ is a viscosity supersolution of* (3.4)*. The set Σ will be called the boundary of (Γ,Σ).*

The following proposition allows us to construct interior distance solutions by cutting subsets out of distance solutions. (This is also the justification for the terms *interior distance solution* and *boundary.*)

Proposition 3.15. *Let Γ be a distance solution and $\Omega \subset \Gamma$ be such that Ω_t is compact and $t \mapsto \Omega_t$ is continuous with respect to the Hausdorff distance. Let $\partial_\Gamma\Omega$ be the boundary of Ω relative to Γ. Then $(\Omega,\partial_\Gamma\Omega)$ is an interior distance solution.*

Proof. Let (x,t) be such that $x \in \Omega_t \setminus \partial_\Gamma \Omega_t$. Note that $\delta_\Omega \geq \delta_\Gamma$ because $\Omega_t \subset \Gamma_t$. Therefore, we will prove that there exists a neighborhood $V \ni (x,t)$ on which $\delta_\Omega \leq \delta_\Gamma$. Then $\delta_\Omega|_V \equiv \delta_\Gamma|_V$, and so δ_Ω is a supersolution on V.

Since $(x,t) \notin \partial_\Gamma \Omega$, the definition of the relative boundary implies the existence of a neighborhood $U \ni (x,t)$ such that $U \cap \Omega = U \cap \Gamma$. Let $\varepsilon > 0$ be small enough that $K = \{x\} \times [t-\varepsilon, t+\varepsilon] \subset U$. Define $\eta = \inf \left\{ |k - x| \;\middle|\; k \in K \quad \text{and} \quad x \in U^c \right\}$. Note that by the $d_\mathcal{H}$-continuity of $t \mapsto \Omega_t$, $\Omega \setminus U$ is closed, so $\eta > 0$. Furthermore, we can choose $\delta > 0$ such that for all $|t' - t| < \delta$ we have $d_\mathcal{H}(\Omega_t, \Omega_{t'}) < \eta/4$. Now let $V = B_{\eta/4}(x) \times B_{\min(\varepsilon,\delta)}(t)$.

Suppose to the contrary that there exists $(y,s) \in V$ such that $\delta_\Omega(y,s) > \delta_\Gamma(y,s)$. Then we have $d_\mathcal{H}(\Omega_t, \Omega_s) < \eta/4$, so there exists $x' \in \Omega_s$ such that $|x - x'| < \eta/4$. On the other hand, there exists $z \in \Gamma_s \setminus \Omega_s$ such that $\delta_\Gamma(y,s) = |z - y|$. Furthermore, we must have $z \in U^c$, since if $z \in U$ then $z \in U \cap \Gamma_s = U \cap \Omega_s$, but $z \notin \Omega_s$. Therefore $|z - x| \geq \eta$. Also, by definition of z, $|z - y| < |x' - y|$. Then we can compute

$$\eta \leq |z - x| \leq |z - y| + |y - x| < \left|x' - y\right| + |y - x| \leq \left|x' - x\right| + 2|x - y| \leq 3\eta/4.$$

This is a contradiction, so we must have $\delta_\Omega|_V \leq \delta_\Gamma|_V$. □

Now, our main result in this section describes how interior distance solutions may be glued to form distance solutions. The idea is that if the solutions are joined so that each point which minimizes the distance to an external point is in the interior of one of the solutions, then the distance function will not be able to detect the boundaries.

Theorem 3.16. *Suppose that (Γ^i, Σ^i) for $i = 1, \ldots, N$ are interior distance solutions. Let $\Gamma = \Gamma^1 \cup \cdots \cup \Gamma^N$, and suppose that*

(i) if $x \in \mathbb{R}^{n+k} \setminus \Gamma_t$ and $y \in \Gamma_t$ are such that $\delta_\Gamma(x,t) = |x - y|$, then $y \in \Gamma^j_t \setminus \Sigma^j_t$ for some $j \in 1, \ldots, N$;

(ii) the mapping $t \mapsto \Gamma_t$ is continuous with respect to $d_\mathcal{H}$.

Then Γ is a distance solution.

Proof. In order to appeal to Lemma 3.10, we consider a test functions ϕ touching δ_Γ from below at a point $(y_0, t_0) \in \Gamma$. By assumption (ii), the second part of the lemma allows us to assume further that there exists $x_0 \in \mathbb{R}^{n+k} \setminus \Omega_t$ such that $\delta_\Gamma(x_0, t_0) = |x_0 - y_0|$. By assumption (i) this implies that $y_0 \in \Gamma_t^j \setminus \Sigma_t^j$ for some $j \in 1, \ldots, N$. By the definition of an interior distance solution, there exists a neighborhood $U \ni (y_0, t_0)$ on which δ_{Γ^j} is a supersolution. But since $\Gamma^j \subset \Gamma$, we have $\delta_\Gamma \leq \delta_{\Gamma^j}$. Hence ϕ also touches δ_{Γ^j} from below at (y_0, t_0). Since δ_{Γ^j} is a supersolution, this implies that

$$\phi_t(y_0, t_0) + F(\nabla\phi(y_0, t_0), \nabla^2\phi(y_0, t_0)) \geq 0.$$

Thus the condition of Lemma 3.10 is satisfied, and so Γ is a distance solution. $\square$

Note that, in fact, the finiteness of the collection of interior distance solutions to be glued was not used in the proof of Theorem 3.16. Therefore, any collection of interior distance solutions satisfying assumptions (i) and (ii) may be glued in this fashion. Additionally, the following corollary to Theorem 3.16 will be useful in understanding level set flows in the case when $k = 1$.

Corollary 3.17. *Suppose that Γ is a distance solution with $t \mapsto \Gamma_t$ continuous with respect to $d_{\mathcal{H}}$ and K is the closure of a connected component of $(\mathbb{R}^{n+k} \times (0, T)) \setminus \Gamma$. Then $\Gamma \cup K$ is also a distance solution.*

Proof. Note that the entire space $\mathbb{R}^{n+k} \times (0, T)$ is a distance solution, so the subset K satisfies the assumptions of Proposition 3.15. Hence, $(K, \partial K)$ is an interior distance solution. Since $\partial K \subset \Gamma$, the pair of interior distance solutions $(\Gamma, \emptyset)$ and $(K, \partial K)$ satisfies the assumptions of Theorem 3.16. Thus $\Gamma \cup K$ is a distance solution. $\square$

As an example of Theorem 3.16, consider two round circles C_0^1 and C_0^2 in $\mathbb{R}^2$ incident at a point (see Figure 3.1). Each evolves by mean curvature flow by shrinking about its center, giving two interior distance solutions $(C^1, \emptyset)$ and $(C^2, \emptyset)$. Let L be the line through the centers of C_0^1 and C_0^2

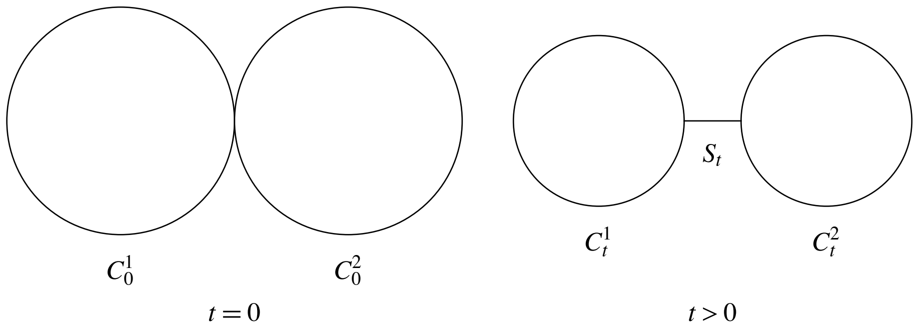

Figure 3.1: Gluing a segment between two circles

and $S_t \subset L$ be the segment between the intersections of L with C_t^1 and C_t^2. Let $p(t)$ and $q(t)$ be the endpoints of S_t. Then by Proposition 3.15, $(S, \{p(t), q(t)\})$ is an interior distance solution. Theorem 3.16 implies that $\Gamma_t = C_t^1 \cup C_t^2 \cup S_t$ is a distance solution. Note that $\Gamma_0 = C_0^1 \cup C_0^2$, so by Proposition 3.12, Γ_t is contained in the level set flow of $C_0^1 \cup C_0^2$. This shows that intersecting smooth manifolds may produce level set flows which are distinct from the union of their smooth flows. This idea is explored in greater detail in Chapter 4.

Chapter 4

The Fattening Phenomenon

4.1 Definition and Previous Results

In this chapter, we consider a particular type of singularity which can arise in the level set flows considered in Section 3.1. As was noted in that section, the level set flow from an arbitrary closed initial set is well-defined and unique. Furthermore, as was shown in Section 3.3, the level set flow must contain all distance solutions. In particular, if there are multiple smooth flows whose images approach the initial set at $t = 0$ (in e.g. Hausdorff distance), they must all be contained in the level set flow. If such non-uniqueness occurs, we expect the level set flow to become large in some sense. This is manifested in the fattening phenomenon.

Definition 4.1. *Let $\Gamma^* \subset \mathbb{R}^{n+k}$ be a closed set, and let Γ be its n-dimensional level set flow on a time interval $I \subset \mathbb{R}$. Following [6], we will say that Γ^* develops α-dimensional fattening at time $t^* \in I$ if*

$$\mathcal{H}^\alpha(\Gamma_t) = 0 \quad \textit{for} \quad t \leq t^* \quad \textit{and} \quad \mathcal{H}^\alpha(\Gamma_t) > 0 \quad \textit{for} \quad t \in (t^*, t^* + \varepsilon)$$

for some $\varepsilon > 0$ and $\alpha \in (n, n+k]$.

The occurrence of fattening for curves in $\mathbb{R}^2$ (with $n = k = 1$) is fully understood and provides a prototype for understanding the relationship between this phenomenon and uniqueness of classical solutions. First, recall the following theorem of Lauer [17].

Proposition 4.2 ([17], Theorem 1.2 and Corollary 9.3)**.** *Suppose that* γ^* *is the continuous image of* S^1 *in* $\mathbb{R}^2$ *and* $\mathcal{H}^1(\gamma^*) < \infty$*. Let* γ *be the* 1*-dimensional level set flow of* γ^**. There exists* $T > 0$ *such that at each time* $0 < t < T$*, the topological boundary* $\partial\gamma_t$ *(viewed as a subset of* $\mathbb{R}^2$*) is the disjoint union of* $N > 0$ *smooth closed curves, each of which evolve by mean curvature flow. Furthermore,* γ^* *is a Jordan curve, then* $N = 1$*.*

Also recall the fact that for a simple smooth closed curve in $\mathbb{R}^2$ evolving by curvature, the enclosed area $A(t)$ satisfies

$$\frac{\partial A}{\partial t} = -2\pi. \tag{4.1}$$

From these facts we summarize the characterization of fattening obtained by Lauer.

Proposition 4.3. *If* γ^* *is as in Proposition 4.2 and* $\partial\gamma_t$ *has* N *components for* $0 < t < T$*, then either*

(i) $N = 1$*,* γ^* *never develops* α*-dimensional fattening for any* $\alpha > 1$*, and there is a unique smooth curve shortening flow of* γ^* *on* $(0, T)$*;*

(ii) or $N > 1$*,* γ^* *develops* 2*-dimensional fattening at* $t = 0$*, and there are at least two smooth curve shortening flows of* γ^* *on* $(0, T)$*.*

Proof. First note that if $N = 1$, then for $0 < t < T$, γ_t is a simple smooth closed curve so no fattening occurs. By the results of [13], the unique classical evolution of this curve will exist up until it shrinks to a round point.

Now, we assume that $N > 1$. Then for any time $0 < t < T$, $\partial\gamma_t$ consists of non-intersecting curves $\gamma_t^1, \ldots, \gamma_t^N$. Without loss of generality, assume that $\gamma_t^1, \ldots, \gamma_t^{N-1}$ are contained in the region bounded by γ_t^N. Let K be the closure of the connected component of $\mathbb{R}^2 \times (0, T) \setminus \partial\gamma$ which is bounded between γ^N and $\gamma^0 = \gamma^1 \cup \cdots \cup \gamma^{N-1}$ (see Figure 4.1). Then by Corollary 3.17, $\Omega = \partial\gamma \cup K$ is a distance solution. In fact, it is easy to see

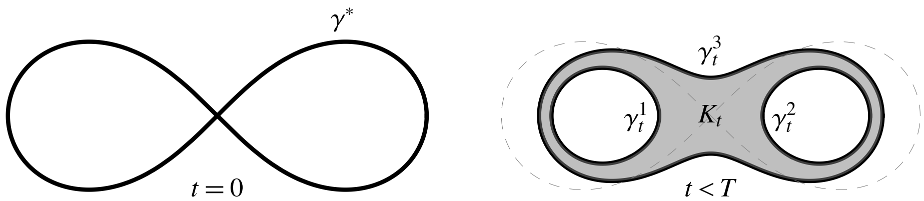

Figure 4.1: Fattening of a figure-eight curve

that Ω must be the level set flow of γ^* as Ω cannot be enlarged without enlarging its boundary. Note that the area of Ω is given by

$$\mathcal{H}^2(\Omega_t) = 2\pi(N-1)t$$

by evolution of enclosed area (4.1) for each of the boundary components of Ω. Finally, further results of Lauer imply that γ^N and γ^0 are both smooth curve shortening flows of γ^* on $(0, T)$. $\square$

Proposition 4.3 shows that in the case of continuous curves of zero Lebesgue measure in $\mathbb{R}^2$, fattening of γ at time $t = 0$ is equivalent to the non-uniqueness of the smooth curve shortening flow originating at γ^*. Such non-uniqueness can be attributed to the possibility of parameterizing the initial data in at least two non-equivalent ways. For example, in Figure 4.1, the initial curve can be parameterized in at least three ways: smoothly by traversing the self-intersection transversely, as a Lipschitz curve with two corners at the self-intersection, or as a Lipschitz image of $S^1 \amalg S^1$. The later parameterizations produce the outer and inner solutions γ_t^3 and $\gamma_t^1 \cup \gamma_t^2$, while the first parameterization produces a self-intersecting distance solution contained in K_t.

For codimension-1 surfaces $\Gamma^* \subset \mathbb{R}^{n+1}$, Ilmanen has proven that a result similar to Proposition 4.3 holds. In particular, Theorems 11.4 and 12.9 of [15] imply that if the n-dimensional level set flow of Γ^* does not develop

n-dimensional fattening, there is a unique "boundary motion" of Γ^* which can be thought of as a kind of maximal Brakke flow. While the details of this result are outside the scope of this thesis, the tools that we have developed allow us to prove that non-fattening implies uniqueness of *smooth* codimension-1 flows. (Note that the while the statement of the this result is not taken directly from any of the references, the principle is well-known and not original to this work.)

Proposition 4.4. *Suppose $\Gamma^* \subset \mathbb{R}^{n+k}$ has an n-dimensional level set flow Γ which does not develop n-dimensional fattening at $t = 0$. Then there is at most one embedded smooth flow M of Γ^* satisfying $\lim_{t\to 0} d_{\mathcal{H}}(M_t, \Gamma^*) = 0$.*

Proof. Suppose that there exist two different smooth flows M and N of Γ^* which approach Γ^* in Hausdorff distance. Note that M and N are distance solutions by the results of Section 3.2. Furthermore, it is easy to see that if $\liminf_{t\to 0} M_t \not\subset \Gamma^*$, for some there would be points in M_t at least ε away from Γ^* for arbitrarily small t. Hence we could not have $\lim_{t\to 0} d_{\mathcal{H}}(M_t, \Gamma^*) = 0$. Therefore, the initial inclusions $M_0 \subset \Gamma^*$ and $N_0 \subset \Gamma^*$ hold.

As in the proof of Proposition 4.3, our approach is to glue in the region between M_t and N_t. Let δ_M and δ_N be the signed distance functions to M_t and N_t, defined so that $\left\{x \mid \delta_M(x,t)\right\}$ is compact, and likewise for δ_N. Define

$$K_t = \left\{x \mid \delta_M(x,t)\delta_N(x,t) \le 0\right\}.$$

It is easy to check that K_t satisfies the assumptions of Corollary 3.17, and so $\Omega = K \cup M \cup N$ is a distance solution. Since M and N are distinct smooth solutions, by continuity K_t (and thus Ω_t) has non-empty interior for $t > 0$. Finally, one can see that

$$d_{\mathcal{H}}(K_t, \Gamma^*) \le d_{\mathcal{H}}(M_t \cup N_t, \Gamma^*) \le \max\left\{d_{\mathcal{H}}(M_t, \Gamma^*), d_{\mathcal{H}}(N_t, \Gamma^*)\right\} \to 0$$

so K satisfies the initial inclusion $K_0 \subset \Gamma^*$. Hence $\Omega \subset \Gamma$ and Γ has n-dimensional fattening at $t = 0$. ☐

As an application of Proposition 4.4, we consider the case when Γ^* is star-shaped about a point $x_0 \in \mathbb{R}^{n+1}$. That is, each ray originating at

x_0 intersects Γ^* exactly once. Under this hypothesis, we will show that the level set flow of Γ^* does not develop n-dimensional fattening for small $t > 0$, and therefore Γ^* admits at most one embedded smooth flow M_t with $\lim_{t\to 0} d_{\mathcal{H}}(M_t, \Gamma^*) = 0$. Note that Soner has proven a similar result using rather different methods [22].

Proposition 4.5 ([22], Theorem 9.3)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a compact domain and assumed $\Gamma^* = \partial\Omega$ is star-shaped about x_0, then its level set flow Γ does not develop n-dimensional fattening at $t = 0$.*

Proof. Without loss of generality, suppose that $x_0 = 0$. For $x \in \mathbb{R}^{n+1} \setminus \{0\}$ define $p(x)$ to be the unique intersection the ray from 0 through x with Γ^*. Let $\rho(x) = \frac{|p(x)|}{|x|}$. Now define

$$u_0(x) = \begin{cases} 1/2 & \text{if} \quad \rho(x) \le 1/2 \\ 3/2 & \text{if} \quad \rho(x) \ge 3/2 \\ \rho(x) & \text{otherwise.} \end{cases}$$

Note that Γ^* is the 1-level set of u, and u_0 is uniformly continuous. Note that in codimension-1, Lemma 3.4 holds for every level set see e.g. [12]. That is, the level set flow of the λ-level set of u_0 is given by the λ-level sets of the solution u to (3.5) with initial data u_0. In particular Γ is the 1-level set of u.

Now suppose to the contrary that Γ develops n-dimensional fattening at $t = 0$. Let Γ^λ be the level set flow of $\lambda\Gamma^*$. By Lemma 3.5 we have $\Gamma^\lambda_t = \lambda\Gamma_{\lambda^{-1}t}$. On the other hand, for $\frac{1}{2} \le \lambda \le \frac{3}{4}$, $\lambda\Gamma^*$ is exactly the λ-level set of u_0. Therefore for $\frac{1}{2} < \lambda < \frac{3}{2}$, the λ-level sets of u are given by $\frac{1}{\lambda}\Gamma_{\lambda t}$. Thus, if $\mathcal{H}^n(\Gamma_t) > 0$ for $0 < t < t_0$, an uncountable number of level sets of u will have positive $\mathcal{H}^n$-measure at some small positive time, which is impossible. □

In the case when $k > 1$, much less is known about the fattening phenomenon. On one hand, the results of Herskovitz discussed in Section 3.2 show that if the initial data Γ^* is a ε-Reifenberg set with ε sufficiently small, then Γ will not develop α-dimensional fattening at $t = 0$ for any $\alpha > n$. (In fact even if $\dim_{\mathcal{H}} \Gamma^* > n$, then Γ will "thin" down to dimension n for some positive time.) On the other hand, it is possible for sets which are initially smooth to develop self-intersections from which fattening occurs after finite time. One known example of this phenomenon was produced by Bellettini, Novaga, and Paolini. Using methods based on the geometric barrier formulation of di Giorgi, they showed in some special cases that disjoint curves in $\mathbb{R}^3$ develop 3-dimensional fattening at the time of their first transverse intersection [6]. This suggests that, even in the case of curves, there may not be a characterization of fattening as simple as that in Proposition 4.3.

4.2 Fattening of Immersed Curves

In this section we extend the example of Bellettini *et al.* of fattening of curves in $\mathbb{R}^3$ mentioned above. To introduce our results, we first give a more detailed description of this example. Bellettini *et al.* consider a pair of embedded closed curves in $\mathbb{R}^3$ which lie in distinct planes, and which are initially linked. Up until the time at which they intersect, these curves evolve smoothly by curve shortening flow. It is shown that from the time of the intersection onward, there is (in the language of this thesis) a distance solution which remains connected. There is also a disconnected distance solution consisting of the smooth evolutions of the original curves. The authors then use a similar method to that in Corollary 3.17 to prove fattening by joining together these distance solutions. The proof of the existence of the connected distance solution relies heavily on the fact that the initial curves are planar. Hence, the method is not applicable to single curves which develop a self-intersection, such the one shown in Figure 4.2.

Our aim is generalize the example of linked planar curves by proving that if any smooth curve in $\mathbb{R}^3$ (possibly non-compact or with multiple components) evolving by curve shortening flow has a transverse self-intersection, then the corresponding level set flow develops fattening at the time of the self intersection. This result was expected by Bellettini *et al.* and broadens the cases in which fattening is known to occur. As with previous examples

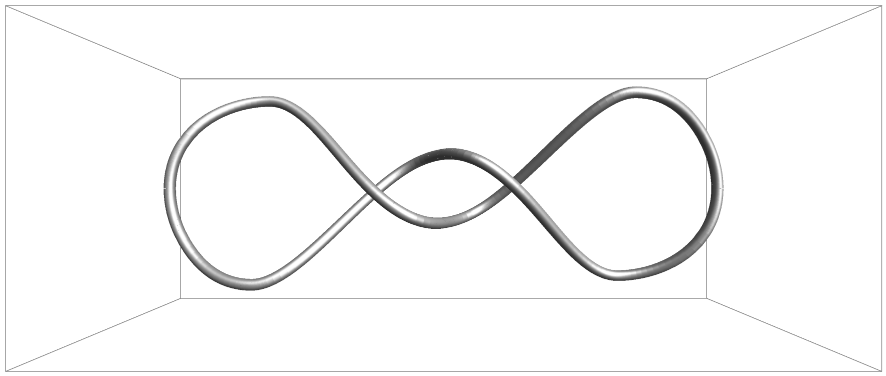

Figure 4.2: A twisted curve in $\mathbb{R}^3$

of fattening, the method will be to apply Theorem 3.16 to prove that the level set flow contains a large set after the time of the intersection. In particular, we consider suitably chosen planes Σ through the intersection point. We show that Σ contains two distinct intersections with the evolving curve after the time of the self-intersection. We then construct a distance solution contained in Σ which connects these two intersections. Finally, we show that all of these distance solutions must be contained in the level set flow.

In this section, M will be a 1-manifold (possibly disconnected) and $\gamma : M \times I \to \mathbb{R}^3$ will be a smooth immersed curve evolving by the curve shortening flow on some time interval I. That is

$$\frac{\partial}{\partial t}\gamma = \kappa N \tag{4.2}$$

where κ is the curvature of γ and N is the Frenet unit normal vector. Note that the combination κN is always well-defined even though N may not be. The unit tangent vector of γ will be denoted by T.

To set up our construction, we first define the type of self-intersections which we consider. The following definition ensures that a self-intersection will immediately break apart and that the curve does not lie in a single plane.

Definition 4.6. *A self-intersection $\gamma(p_1, t) = \gamma(p_2, t)$ for $p_1 \neq p_2$ will be* *called strongly transverse if*

$$\langle \kappa(p_1,t)N(p_1,t), \kappa(p_2,t)N(p_2,t)\rangle < 0 \tag{4.3}$$

and

$$\dim \operatorname{span}\{N(p_1,t), T(p_1,t), N(p_2,t), T(p_2,t)\} = 3. \tag{4.4}$$

Note that the normal vectors in (4.4) *are well-defined because* (4.3) *ensures that* $\kappa(p_1,t)$ *and* $\kappa(p_2,t)$ *are non-zero.*

From now on, we will assume without loss of generality that γ has a strongly transverse self-intersection at $t=0$ such that $\gamma(p_1,0)=\gamma(p_2,0)=0$. We proceed by constructing a large family of planes which have well-controlled intersections with $\gamma(\cdot,t)$ for $t>0$.

Lemma 4.7. *With the above assumptions, for all* $\varepsilon > 0$ *there exists* $\nu \in S^2$ *such that*

$$\left|\frac{\langle \kappa(p_i,0)N(p_i,0), \nu\rangle}{\langle T(p_i,0),\nu\rangle}\right| < \varepsilon \quad \textit{for} \quad i=1,2.$$

and $\langle T(p_i,0), n\rangle \neq 0$ *for* $i=1,2$.

Proof. For brevity, we let $N_i = N(p_i,0)$, $T_i = T(p_i,0)$ and $\kappa_i = \kappa(p_i,0)$ for $i=1,2$.

There are two cases to consider. First, if N_1 and N_2 are linearly dependent, then we can write

$$\left|\frac{\langle \kappa_2 N_2,\nu\rangle}{\langle T_2,\nu\rangle}\right| = \left|\frac{\langle \lambda\kappa_2 N_1,\nu\rangle}{\langle T_2,\nu\rangle}\right|$$

for some λ, and choose ν such that $\langle N_1,\nu\rangle = 0$ while $\langle T_i,\nu\rangle \neq 0$ for $i=1,2$.

On the other hand, suppose that N_1 and N_2 are linearly independent. Then $N_1 \times N_2 \neq 0$. We claim that one of $\langle N_1\times N_2, T_1\rangle$ or $\langle N_1 \times N_2, T_2\rangle$ is non-zero, for if both were zero, then the sets

$$A = \{N_1, N_2, T_1\}$$
$$B = \{N_1, N_2, T_2\}$$

would both be linearly dependent. Thus the nullspace of the matrix X with $A \cup B$ as columns would have dimension at least 2, and X would have rank at most 2, contradicting (4.4). Now, without loss of generality, suppose that $|\langle N_1 \times N_2, T_1 \rangle| > 0$. Then we can choose ν such that $\langle \kappa_2 N_2, \nu \rangle = 0$ and ν is close enough to $N_1 \times N_2$ that

$$\left| \frac{\langle \kappa_1 N_1, \nu \rangle}{\langle T_1, \nu \rangle} \right| < \varepsilon$$

because the norm of the denominator is bounded below as $\nu \to N_1 \times N_2$. $\square$

Lemma 4.7 now allows us to apply the implicit function theorem to choose planes Σ through the origin for which the intersections of Σ and γ move away from each other after $t = 0$ and the curves traced by the intersection points have bounded gradient (in a sense made precise below).

Proposition 4.8. *Suppose that γ is as above. For $\nu \in S^2$, let $\Sigma(\nu)$ be the plane through $0 \in \mathbb{R}^3$ with normal vector ν. There exists a closed ball $N \subset S^2$, a time $T > 0$, and $\delta_0 > 0$ such for any $\nu \in N$, there exist smooth $\alpha_i : [0, T] \to \Sigma(\nu)$ $(i = 1, 2)$ such that*

(i) $\alpha_i(0) = 0$ *and* $\alpha_i(t) \in \gamma(M, t)$*;*

(ii) $\langle \alpha_1'(t), \alpha_2'(t) \rangle < 0$*;*

(iii) and $\left| \left\langle \frac{\alpha_i'(t)}{|\alpha_i'(t)|}, \hat{\alpha} \right\rangle \right| \geq \delta_0$ *for* $t \in [0, T]$

where $\hat{\alpha}$ is a unit vector in the direction $\alpha_1'(0) - \alpha_2'(0)$.

Proof. Let $\varepsilon > 0$ and choose ν_0 according to Lemma 4.7. Choose a neighborhood $U \ni p_1$ on which we can parameterize γ by arc length s such that at $t = 0$, $s = 0$ corresponds to p_1. Consider the function $F : S^2 \times U \times I \to \mathbb{R}$ given by

$$F(n, s, t) = \langle \gamma(s, t), n \rangle.$$

Note that we have $F(\nu_0, 0, 0) = 0$ and $F_s(\nu_0, 0, 0) = \langle T(p_1, 0), \nu_0 \rangle \neq 0$ by 4.7. Hence, by the implicit function theorem, there exists a neighborhood $V \subset$

$S^2 \times I$ containing $(\nu_0, 0)$ and a function $s_1 : V \to M$ such that $s_1(n_0, 0) = 0$ and $F(\nu, s_1(\nu, t), t) = 0$. Let $\alpha_1(\nu, t) = \gamma(s_1(\nu, t), t)$. Define $\alpha_2(\nu, t)$ in the same way, using p_2 in place of p_1.

By definition of F, we have $\alpha_i(\nu, t) \in \Sigma(\nu) \cap \gamma(M, t)$ hence condition (i) is satisfied. The implicit function theorem and the evolution of γ also give

$$\alpha_i'(\nu, 0) = \kappa(p_i, 0)N(p_i, 0) - \frac{\langle \kappa(p_i, 0)N(p_i, 0), \nu \rangle}{\langle T(p_i, 0), \nu \rangle} T(p_i, 0).$$

Using the fact that $\langle \kappa(p_1, 0)N(p_1, 0), \kappa(p_2, 0)N(p_2, 0) \rangle < 0$, we can ensure that the choice of ε above is small enough that $\langle \alpha_1'(\nu_0, 0), \alpha_2'(\nu_0, 0) \rangle$ is negative. (Note also that choosing such ε depends only $|\kappa(p_i, 0)|$.) Furthermore, a simple computation shows that $\langle \alpha_1(\nu_0, 0), \alpha_2(\nu_0, 0) \rangle < 0$ implies that $\hat{\alpha}$ exists and

$$\left| \left\langle \frac{\alpha_i'(\nu_0, 0)}{|\alpha_i'(\nu_0, 0)|}, \hat{\alpha} \right\rangle \right| > 0.$$

Using the fact that conditions (ii) and (iii) hold for $\nu = \nu_0$ and $t = 0$, by continuity of α_1 and α_2 we can choose a cylinder $\overline{B_\rho(\nu_0)} \times [0, T] \subset V$ on which they hold uniformly. □

Note that because the smooth solution γ exists past the time T given in Proposition 4.8, all of the time derivatives of the α_i are uniformly bounded on $[0, T]$. In particular $|\alpha_i'| \leq \kappa_0$ where κ_0 is the upper bound on the curvature of γ on $[0, T]$.

Our goal is now to find, for each $\nu \in N \subset S^2$, a connected set $\eta_\nu \subset \Sigma(\nu) \times [0, T]$ such that

$$\begin{cases} (\eta_\nu, \{\alpha_i(\nu, t) \mid i = 1, 2\}) & \text{is an interior distance solution} \\ (\eta_\nu)_0 = \{0\} & \end{cases} \tag{4.5}$$

Ideally, such a solution would simply be a curve evolving by curve shortening flow with endpoints $\alpha_1(\nu, t)$ and $\alpha_2(\nu, t)$; however, the problem of finding such a curve is ill-posed since the nominal initial curve is not regular. To avoid this problem, we will find such a curve on the time interval $[t_0, T]$ and show that we can obtain a solution to (4.5) by taking a weak limit as $t_0 \searrow 0$. It turns out that this limit may not be a curve, but this is acceptable for our purposes. For the time being, we fix a particular $\nu \in N$ and let $\alpha_i(t) = \alpha_i(\nu, t)$. The gradient bounds above let us represent α_i as a graph over $\hat{\alpha}$.

Lemma 4.9. *Let $\{e_1, e_2\}$ be orthonormal vectors in $\Sigma(\nu)$ with $e_1 = \hat{\alpha}$, and let $\{x, y\}$ be the corresponding coordinates of $\Sigma(\nu)$. For $i = 1, 2$, there exist functions $\beta_i : [0, B_i] \to \mathbb{R}$ and $x_i : [0, T] \to [0, B_i]$ such that $x_i(0) = 0$ and $\alpha_i(t) = x_i(t)e_1 + \beta_i(x_i(t))e_2$. Furthermore, $|\beta_i'(x)| < \delta_0^{-1}$, the functions x_i are monotonic, and $B_1 < 0 < B_2$.*

Proof. Let $x_i(t) = \langle \alpha_i(t), e_1 \rangle$. Then using (ii) and (iii) from Proposition 4.8, we see that $x_i(t)$ is monotonic and therefore invertible. Thus we can define $\beta_i(x) = \langle \alpha_i(x_i^{-1}(x)), e_2 \rangle$. Clearly this gives $\alpha_i(t) = x_i(t)e_1 + \beta_i(x_i(t))e_2$. We can compute

$$\beta_i'(x) = \frac{\langle \alpha_i'(x_i^{-1}(x)), e_2 \rangle}{\langle \alpha_i'(x_i^{-1}(x)), e_1 \rangle}$$

and using condition (iii) again we have

$$\left|\beta_i'(x)\right| < \frac{1}{\delta_0} \left| \left\langle \frac{\alpha_i'(t)}{|\alpha_i'(t)|}, e_2 \right\rangle \right| \leq \delta_0^{-1}.$$

Finally, we have $B_i = x_i(T)$. Using the monotonicity of the x_i along with the fact that $\langle \alpha_1'(0), \alpha_2'(0) \rangle < 0$ the B_i must have opposite sign. By relabeling if necessary we have $B_1 < 0 < B_2$. □

Using the notation from Lemma 4.9, for a time interval I, let

$$\Omega_I = \left\{ (x, t) \;\middle|\; t \in I \quad \text{and} \quad x_1(t) < x < x_2(t) \right\} \tag{4.6}$$

as usual $\overline{\Omega}_I$ will be the closure of Ω_I and $\partial_P \Omega_I$ will be the parabolic boundary. We may now consider the restricted graphical problem

$$\begin{cases} u_t = \frac{u_{xx}}{1+u_x^2} & \text{in} \quad \Omega_{[t_0,T]} \\ u(x_i(t), t) = \beta_i(x_i(t)) & \text{for} \quad i = 1, 2 \\ u(x, t_0) = u_0(x) & \text{for} \quad x \in [x_1(t_0), x_2(t_0)]. \end{cases} \tag{4.7}$$

By the usual graphical formulation of the curve shortening flow problem, it is clear that the graphs of solutions to (4.7) are solutions to (4.5) on the time interval $[t_0, T]$.

Proposition 4.10. *Let $x_i^0 = x_i(t_0)$ for $i = 1, 2$. Suppose that $u_0 : [x_1^0, x_2^0] \to \mathbb{R}$ is a smooth function which satisfies the compatibility conditions*

$$u_0(x_i^0) = \beta_i(x_i^0) \tag{4.8}$$

$$(u_0)_x(x_i^0) = (\beta_i)_x(x_i^0) \tag{4.9}$$

$$(u_0)_{xx}(x_i^0) = 0. \tag{4.10}$$

Then there exists a smooth solution of (4.7) on $\bar{\Omega}_{[t_0,t_1]}$ for some $t_1 > t_0$.

Proof. We first need to construct a smooth function $G : \overline{\Omega}_{[t_0,T]} \to \mathbb{R}$ such that $G(x_i(t), t) = \beta_i(x_i(t))$ for $i = 1, 2$ and $G(x, t_0) = u_0(x)$, and

$$G_t(x, t_0) = \frac{G_{xx}(x, t_0)}{1 + G_x(x, t_0)^2}. \tag{4.11}$$

The only thing that needs to be checked is that (4.11) holds at $x = x_i^0$, as G can be always be chosen so (4.11) holds for $x \in (x_1^0, x_2^0)$. To do this, we can compute

$$G_x(x_i(t), t)x_i'(t) + G_t(x_i(t), t) = (\beta_i)_x(x_i(t))x_i'(t)$$

which implies that

$$G_t(x_i^0, t_0) = ((\beta_i)_x(x_i^0) - G_x(x_i^0, t_0))x_i'(t_0) = ((\beta_i)_x(x_i^0) - (u_0)_x(x_i^0))x_i'(t_0) = 0$$

using the compatibility condition on u_0. On the other hand,

$$G_{xx}(x_i^0, t_0) = (u_0)_{xx}(x_i^0) = 0$$

so (4.11) holds at $x = x_i^0$. Thus, (4.7) reduces to

$$\begin{cases} u_t = \frac{u_{xx}}{1+u_x^2} \quad \text{in} \quad \Omega_{[t_0,T]} \\ u|_{\partial_P \Omega_{[t_0,T]}} = G|_{\partial_P \Omega_{[t_0,T]}}. \end{cases} \tag{4.12}$$

Standard theory on quasilinear parabolic equations ([16], Theorem VI.4.1) now implies the existence of a smooth solution on $\overline{\Omega}_{[t_0,t_1]}$ for some $t_1 > t_0$. □

In order to complete the construction of solutions to (4.5), we must show that the short time solutions given by (4.10) can be extended to $[t_0, T]$, and that we can obtain a solution in the limit as $t_0 \searrow 0$. The following uniform gradient bound will serve both of these needs.

Proposition 4.11. *Suppose that u is a smooth solution to* (4.7) *on* $\Omega_{[t_0,t_1)}$ *with* $|(u_0)_x| < \delta_0^{-1}$. *Then* $|u_x| \le \delta_0^{-1}$ *on* $\bar{\Omega}_{[t_0,t_1)}$.

Proof. First we show that $|u_x(x_i(t), t)| \le \delta_0^{-1}$ for $t \in [t_0, t_1)$. To simplify matters, we will prove $u_x(x_2(t), t) \le \delta_0^{-1}$. The same proof will work for the other cases. Suppose toward a contradiction that at $t' \in [t_0, t_1)$ we have $u_x(x_2(t'), t') > \delta_0^{-1}$. Let $\ell(x) = \delta_0^{-1}(x - x_2(t')) + \beta_2(x_2(t'))$ describe the line through $\alpha_2(t')$ with slope δ_0^{-1}. We claim that there exists $x' \in (x_1(t'), x_2(t'))$ such that $\ell(x') = u(x', t')$. In particular, note that for some $\varepsilon > 0$ we have $\ell(x_2(t') - \varepsilon) - u(x_2(t') - \varepsilon, t') > 0$ by our assumption on u_x. A simple computation using the bound $(\beta_i)_x < \delta_0^{-1}$ and the fact that $x_1(t') < 0$ shows that $\ell(x_1(t')) - u(x_1(t'), t') < 0$ (see Figure 4.3). Hence by the intermediate value theorem there exists x' as claimed above. Note that the graphs of u and ℓ must cross transversely at $(x', u(x', t'))$, and thus by continuity this crossing must have existed on some time interval before t'.

Let $G(u)$ and $G(\ell)$ denote the graphs of u and ℓ. Both $G(u)$ and $G(\ell)$ move by curve shortening flow on their interiors. Thus the intersection principle of Angenent ([3], Section 5), implies that the number of crossings between these curves is non-increasing as long as the endpoints of $G(u)$ remain disjoint from $G(\ell)$. It is easy to see that t' is the first time that an endpoint of $G(u)$ intersects $G(\ell)$. Furthermore, $G(\ell)$ is disjoint from $G(u_0)$

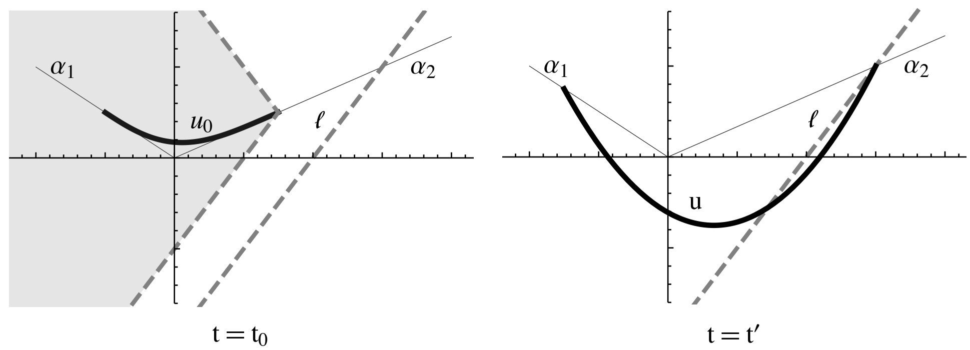

Figure 4.3: Schematic of the proof of the gradient bound in Proposition 4.11. The shaded region on the left shows the area that u_0 may lie in. Note that it must be disjoint from ℓ.

by the assumption that $|u_x| \le \delta_0^{-1}$. Hence we have obtained a contradiction as the number of interior intersections of $G(u)$ and $G(\ell)$ must be zero up to time t'. This completes the proof that $|u_x(x_i(t),t)| \le \delta_0^{-1}$ for $t \in [t_0, t_1)$.

Now, we apply the maximum principle to prove the result on the entire domain. Denote by $\frac{\partial}{\partial s}$ the operator which gives the derivative with respect to arc length. It is well known (see, e.g. [1]) that for a curve moving by curve shortening flow with tangent vector T and V a fixed vector, we have

$$\frac{\partial}{\partial t}\langle T, V\rangle = \frac{\partial^2}{\partial s^2}\langle T, V\rangle + 2\kappa^2\langle T, V\rangle. \tag{4.13}$$

In our case, $\frac{\partial}{\partial s} = \frac{1}{\sqrt{1+u_x^2}}\frac{\partial}{\partial x}$ and we take $V = e_2$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{u_x}{1+u_x^2}\right) = \frac{\partial^2}{\partial s^2}\left(\frac{u_x}{1+u_x^2}\right) + 2\kappa^2\left(\frac{u_x}{1+u_x^2}\right). \tag{4.14}$$

Thus $\langle T, e_2\rangle = \frac{u_x}{1+u_x^2}$ satisfies the parabolic maximum principle. Combining the assumptions and the first part of this proof give bounds on $\partial_P\Omega_{[t_0,t_1)}$, which then extend to the interior. Thus $|\langle T, e_2\rangle| < \frac{\delta_0^{-1}}{1+\delta_0^{-2}}$, from which the conclusion follows. □

It is well known that it is possible to obtain strong curvature bounds for graphical solutions of mean curvature flow satisfying a uniform gradient bound.

Proposition 4.12. *Suppose that* u *is a solution to* (4.7) *with* $|u_x| \leq \delta_0^{-1}$*. Let* $\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$*. Recall that* κ_0 *is the uniform bound on the curvature of the original evolving curve* γ*. We have*

(a) $\left|\frac{\partial^l \kappa}{\partial s^l}\right| \leq C_l(u_0, \delta_0, \|\alpha_1\|_{C^l}, \|\alpha_2\|_{C^l})$ *for* $l = 0, 1, 2, \ldots$*;*

(b) and $(t - t_0)\kappa(x,t)^2 \leq C(\delta_0, \kappa_0)$ *for* $(x,t) \in \Omega_{(t_0,t_1)}$*.*

Proof. These bounds follow from well-known results of Ecker and Huisken [11]. Proposition 4.12(a) is proven via an inductive application of the maximum principle to the evolution equations satisfied by the derivatives of κ ([11], Proposition 4.3). For Proposition 4.12(b), see [11], Proposition 4.4. The only modification necessary is the incorporation of the contribution of the derivatives of the boundary data α_i into the constants. □

These bounds allow us to extend the solution up to time T using the usual long-time existence procedure (see, e.g. [11], Theorem 4.6).

Lemma 4.13. *With the same notation as Proposition 4.10, if* $|(u_0)_x| \leq \delta_0^{-1}$*, there exists a smooth solution of* (4.7) *on* $\Omega_{[t_0,T]}$ *satisfying the bounds of Proposition 4.12.*

Proof. Let $A \subset [t_0, T]$ be the set of times up to which the maximal solution of (4.7) is defined. By Proposition 4.10, A is open and non-empty. Since u_0 satisfies the hypothesis of Proposition 4.11, the maximal solution has bounded gradient for all time, and therefore Proposition 4.12(a) and the Arzela-Ascoli theorem imply that A is closed. Therefore $A = [t_0, T]$. □

In order to make use of the long-time existence theory above, we need to construct appropriate initial data for each time $t_0 > 0$. Note that we do not assert any curvature bounds on our choice of initial data, and in fact, none are possible if $\alpha_1'(0) \neq -\alpha_2'(0)$. Therefore, the curvature independent bound of Proposition 4.12(b) is crucial in obtaining convergence below.

Lemma 4.14. *For each $t_0 > 0$, there exists an initial function $u_0 : [x_1^0, x_2^0] \to \mathbb{R}$, satisfying the compatibility condition of Proposition 4.10 and with $|(u_0)_x| < \delta_0^{-1}$.*

Proof. Let $\varepsilon > 0$ and define

$$\widetilde{u_0}(x) = \begin{cases} \beta_1(x_1^0) + (\beta_1)_x(x_1^0)(x - x_1^0) & \text{if} \quad x_1^0 \le x \le x_1^0 + \varepsilon \\ \beta_2(x_2^0) + (\beta_2)_x(x_2^0)(x - x_2^0) & \text{if} \quad x_2^0 - \varepsilon \le x \le x_2^0 \\ \xi(x) & \text{otherwise} \end{cases}$$

where $\xi(x)$ is the linear function which makes $\widetilde{u_0}$ continuous. By taking ε small enough, we can see that $\widetilde{u_0}$ is Lipschitz with constant less than δ_0^{-1} by using the bounds on the β_i and their derivatives. Thus, we can obtain u_0 by smoothing $\widetilde{u_0}$ slightly near the two corners at $x_1^0 + \varepsilon$ and $x_2^0 - \varepsilon$. □

To obtain weak limits of the solutions to (4.7) on $[t_0, T]$ from which we can construct the desired interior distance solution, we need to ensure that these limits will have sufficient regularity. In particular, we show that half relaxed limits preserve local Lipschitz constants.

Lemma 4.15. *Let U be a connected domain in $\mathbb{R}^d$. Suppose that $u_n : U \to \mathbb{R}$ is a sequence such that for each $K \subset\subset U$, $u_n|_K$ is L-Lipschitz for some constant L depending only on K. The upper and lower half relaxed limits $\bar{u} = \limsup^* u_n$ and $\underline{u} = \liminf_* u_n$ also satisfy this property with the same Lipschitz constants.*

Proof. We prove the result for the upper half relaxed limit. Let $x, y \in U$ and $K \subset U$ be a compact set containing x and y. Then $u_n|_K$ is Lipschitz with constant L. Suppose $x_n \to x$ and $y_n \to y$. Without loss of generality we can assume that $x_n \subset K$ and $y_n \subset K$. Then we have

$$|u_n(x_n) - u_n(y_n)| \le L|x_n - y_n| \Rightarrow u_n(x_n) \le L|x_n - y_n| + u_n(y_n).$$

Taking the lim sup of both sides gives

$$\limsup u_n(x_n) \le L|x - y| + \limsup u_n(y_n).$$

Taking the supremum over such sequences x_n and y_n gives

$$\bar{u}(x) - \bar{u}(y) \leq L|x - y|.$$

To obtain the lower bound, the argument can be repeated, switching the roles of x and y. For the lower half-relaxed limit, the same proof applies. □

Finally, we can put together the results above to obtain existence of solutions to (4.5).

Proposition 4.16. *There exists a solution η to* (4.5) *such that η_t consists of the region in $\Sigma(\nu)$ bounded by two Lipschitz curves joining $\alpha_1(t)$ and $\alpha_2(t)$.*

Proof. By Lemma 4.13, for each $n > 1$, we obtain a solution u^n of (4.7) on $\Omega_{[T/n,T]}$ with initial data u_0^n given by Lemma 4.14. If $K \subset\subset \Omega_{(0,T]}$, then for some N and all $n > N$, u^n will be defined on K. The uniform gradient bound of Proposition 4.11 combined with the interior time-derivative bound of Proposition 4.12(b) shows that $u^n|_K$ is L-Lipschitz for some constant depending L on δ_0, κ_0, and K only. Thus, by Lemma 4.15,

$$\bar{u} = \limsup{}^* u^n \quad \text{and} \quad \underline{u} = \liminf{}_* u^n$$

are locally Lipschitz with $\bar{u} \geq \underline{u}$. Furthermore, by Lemma 2.8, $\bar{u}$ (resp. $\underline{u}$) is a subsolution (resp. supersolution) of (4.7) on $\Omega_{(0,T]}$. We define

$$\eta_t = \left\{ xe_1 + ye_2 \;\middle|\; x_1(t) \leq x \leq x_2(t) \quad \text{and} \quad \underline{u}(x) \leq y \leq \bar{u}(x) \right\}$$

for $t > 0$ and $\eta_0 = \{0\}$ and

$$\partial\eta = \left\{ (x,t) \;\middle|\; t \in [0,T], x = \alpha_1(\nu,t) \quad \text{or} \quad x = \alpha_2(\nu,t) \right\}.$$

It remains to be shown that $(\eta, \partial\eta)$ is an interior distance solution. Suppose that $(x,t) \in \eta \setminus \partial\eta$. By a similar argument to that in the proof of Proposition 3.15, there exists a neighborhood $(x,t) \in U \subset \Sigma(\nu) \times I$ such that for $(x_0,t_0) \in U$ and $y_0 \in \eta_t$ with $\delta_\eta(x_0,t_0) = |x_0 - y_0|$, we have $y_0 \in \eta_t \setminus \partial\eta_t \cap U$. We will show that $\delta_\eta|_U$ is a supersolution of (3.4) with $n = k = 1$. The proof of Proposition 3.13 will then imply that $\delta_\eta|_{U\times\mathbb{R}}$ is a supersolution of (3.4) with $k = 2$.

The strategy to do this is to convert test functions for η into test functions for $\underline{u}$ and $\bar{u}$, and then apply the equations obtained using the fact that $\underline{u}$ and $\bar{u}$ are supersolutions and subsolutions of (4.7) (see Figure 4.4). We first consider a point $(p_0, t_0) \in U$ such that p_0 is given in the coordinates used above as (x_0, y_0) and satisfies

$$\underline{u}(x_0, t_0) > y_0.$$

Let $\phi : \Sigma(\nu) \times I \to \mathbb{R}$ be a C^2 function touching δ from below at (p_0, t_0). Define

$$\widetilde{\phi}(q_0, t) = \phi(x + p_0 - q_0, t) - |p_0 - q_0|$$

where $q_0 \in \eta \setminus \partial\eta$ is such that $\delta_\eta(p_0, t_0) = |p_0 - q_0|$. Let

$$Z_t = \left\{ \widetilde{\phi}(x, t) = 0 \;\middle|\; x \in \Sigma(\nu) \right\}.$$

We claim

(i) there exists a neighborhood $V \ni (q_0, t_0)$ such that $Z_t \cap V_t$ lies below the graph of $\underline{u}(\cdot, t)$;

(ii) and $\widetilde{\phi}_y(q_0, t_0) < 0$.

To see (i), let $W \ni (p_0, t_0)$ be such that $\phi \leq \delta_\eta$ on W, and define $V = \{(x - p_0 + q_0, t) \mid (x, t) \in W\}$. Let $(\widetilde{z}, t) \in V \cap Z$. Then there exists $(z, t) \in W$ such that $\widetilde{z} = z - p_0 + q_0$. We then have

$$0 = \widetilde{\phi}(\widetilde{z}, t) = \phi(\widetilde{z} + p_0 - q_0, t) = \phi(z, t) - |p_0 - q_0|.$$

By definition of W, this implies

$$\delta(z, t) \geq \phi(z, t) = |p_0 - q_0|.$$

Since $|\widetilde{z} - z| = |p_0 - q_0|$ either $\widetilde{z}$ lies outside η_t or on the boundary of η_t. Therefore $\widetilde{z}$ must lie below the graph of $\underline{u}$.

To see (ii), note that $\nabla\widetilde{\phi}(q_0, t_0) = \nabla\phi(p_0, t_0) \neq 0$, since ϕ touches a distance function from below away from its zero set. Thus, if $\widetilde{\phi}_y(q_0, t_0) = 0$, the implicit function theorem would give w such that $\widetilde{\phi}((w(y,t), y), t) = 0$ in a neighborhood of (q_0, t_0). Unless the graph of $\underline{u}$ had a vertical tangent at q_0, this would contradict (i). Since $\underline{u}$ is locally Lipschitz, this is impossible. Furthermore, note that for ε small, we have

$$\phi((x_0, y_0 + \varepsilon), t_0) \leq \delta_\eta((x_0, y_0 + \varepsilon), t_0) \leq \delta_\eta((x_0, y_0), t_0) = \phi(p_0, t_0)$$

since p_0 lies below the graph of $\underline{u}$. Hence $\phi_y(p_0, t_0) \leq 0$. Combining these facts gives $\widetilde{\phi}_y(q_0, t_0) < 0$.

From (i) and (ii), the implicit function theorem gives v such that

$$\widetilde{\phi}((x, v(x,t)), t) = 0 \tag{4.15}$$

$$\text{and} \quad v(x,t) \leq \underline{u}(x,t) \tag{4.16}$$

for (x, t) in a neighborhood of $((q_0)_x, t_0)$. Note also that $v((q_0)_x, t_0) = (q_0)_y$, so v touches $\underline{u}$ from below at $((q_0)_x, t_0)$. Therefore, by the fact that $\underline{u}$ is a supersolution of (4.7), we have

$$v_t \geq \frac{v_{xx}}{1 + v_x^2}. \tag{4.17}$$

at $((q_0)_x, t_0)$. Differentiating (4.15) and substituting into (4.17) gives

$$-\frac{\widetilde{\phi}_t}{\widetilde{\phi}_y} - \frac{1}{\widetilde{\phi}_y} F(\nabla\widetilde{\phi}, \nabla^2\widetilde{\phi}) \geq 0$$

at (q_0, t_0). Using (ii) and the definition of $\widetilde{\phi}$, this gives

$$\phi_t + F(\nabla\phi, \nabla^2\phi) \geq 0$$

at (p_0, t_0), which shows that δ_η is a supersolution of (3.4) on the portion of U lying below the graph of $\underline{u}$. The same argument, with inequalities reversed, and using the fact the $\bar{u}$ is a supersolution, implies that δ_η is likewise a

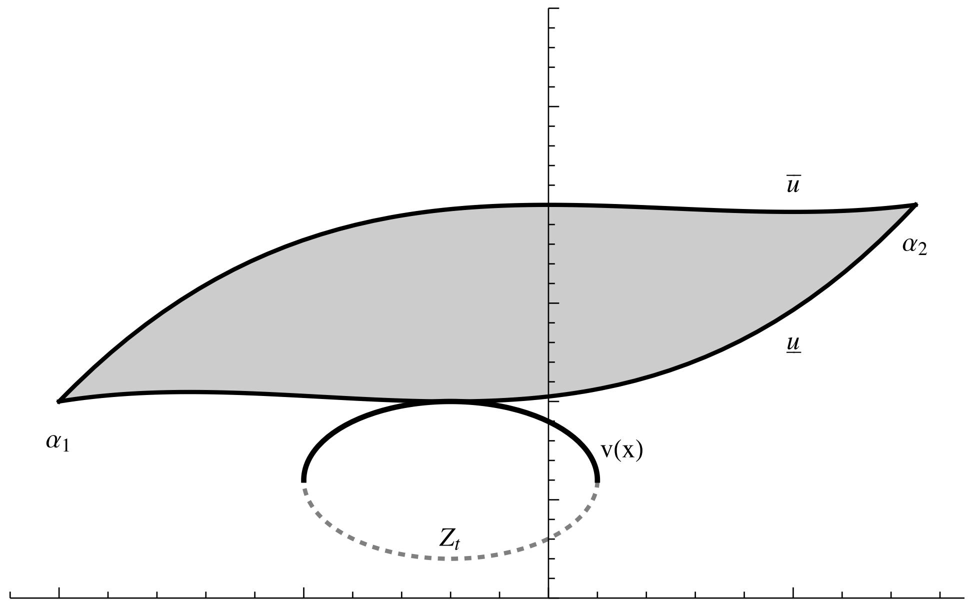

Figure 4.4: Schematic of the proof of Proposition 4.16. The shaded region is the interior of $(\eta_\nu)_t$.

supersolution on the portion of U lying above the graph of $\bar{u}$. Finally, we apply Lemma 3.11 to conclude that δ_η is a supersolution on all of U, and therefore $(\eta, \partial\eta)$ is an interior distance solution. □

Theorem 4.17. *Suppose that* $\gamma : M \times [-\varepsilon, \varepsilon) \to \mathbb{R}^3$ *is a smooth immersed curve evolving by curve shortening flow which has a strongly transverse self-intersection at time* 0. *Then the level set flow* Γ *of* $\gamma_{-\varepsilon}$ *develops* 2-*dimensional fattening at time* 0.

Proof. Let $N \subset S^2$ and $0 < T < \varepsilon$ be as in Proposition 4.8. Then by Proposition 4.16, for each $\nu \in N$, there exists a solution η_ν of (4.5) defined on $[0, T]$. By the construction of these solution, Theorem 3.16 shows that the set

$$\Omega = \gamma(M, [-\varepsilon, \varepsilon)) \cup \bigcup_{\nu} \eta_\nu$$

is a distance solution on $[-\varepsilon, T)$. (Note that $\gamma(M, [-\varepsilon, \varepsilon])$ is a distance solution because it is locally a union of embedded smooth flows.) Therefore $\Omega_t \subset \Gamma_t$ for $-\varepsilon \leq t < T$.

At time $0 < t < T$, there must be at least one curve in N on which the maps $\nu \mapsto \alpha_i(\nu, t)$ are injective and the segment ℓ_ν joining $\alpha_1(\nu, t)$ to $\alpha_2(\nu, t)$ does not pass through the origin. Then each η_ν for ν along this curve contains at least one Lipschitz curve (say the graph of $\underline{u}$). Furthermore, by the condition on ℓ_ν, this curve has a segment with length bounded below which lies only in $\Sigma(\nu)$. This is enough to show that $\mathcal{H}^2(\Gamma_t) > 0$. $\square$

As an example of the type of case which Theorem 4.17 addresses which is not covered by [6], consider the twisted curve depicted in Figure 4.2. By symmetry, it is easy to see that the small central twist must develop a strongly transverse self intersection after a finite amount of time. At this point, we conclude that the level set flow will develop (at least) 2-dimensional fattening.

Chapter 5

Conclusion

In this thesis, we have studied generalized evolutions of submanifolds by mean curvature flow, focusing on the level set solutions of Ambrosio and Soner, and the fattening phenomenon which occurs with such solutions. Apart from presenting some of the existing results on this phenomenon, our main contributions have been to prove Theorem 3.16, a new gluing result for distance solutions, and to use this result to prove Theorem 4.17 which demonstrates the occurrence of fattening when immersed curves develop self-intersections.

Theorem 4.17 may help to understand the fattening phenomenon in higher codimension in several ways. First, it verifies a case of fattening which was suspected, but not proven to occur except in some very special cases [6]. Second, it provides an interesting piece of information when considering the relationship between fattening and non-uniqueness. In particular, the existence results for smooth flows with rough initial data in higher codimension generally require a smallness assumption (on e.g. the Lipschitz or Reifenberg constants of the initial data [14]). On the other hand, such an assumption may not be satisfied by a parameterization of a self-intersecting curve. Thus, it seems plausible that the fattening proven in Theorem 4.17 is not directly due to the existence of multiple smooth solutions, in contrast to the case of curves in $\mathbb{R}^2$ (Proposition 4.3). Finally, we note that while some of results in Section 4.2 were specific to curves, the basic method of constructing multiple codimension-1 distance solutions confined to affine subspaces in order to prove fattening may be applicable in other situations. For example, the same method could feasibly be used to demonstrate fattening when a 2-dimensional surface in $\mathbb{R}^4$ develops a self-intersection along a curve. Constructing such examples of fattening in higher codimension may further illuminate our understanding of this phenomenon, and subsequently allow for more applications of the mean curvature flow.

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