Essays in Behavioral Economics

by

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Abstract

This thesis studies individual choice in both individualistic and interactive decisions, under different situations of risk, uncertainty and time delay.

The first chapter of my dissertation investigates the tendency of human beings to make choices that are biased towards alternatives in the present. I characterize the general class of utilities which are consistent with present-biased behavior. I show that any present-biased preference has a subjective max-min representation, which can be cognitively interpreted as the decision maker considering the most conservative "present equivalents" in the face of subjective uncertainty about future tastes.

The second chapter of my thesis provides desiderata of choice consistency that experimenters should employ while estimating time preferences from choice data. We also show how application of this desiderata can help us learn new insights from previous experimental studies.

The third chapter of my thesis establishes a tight relation between non-standard behaviors in the domains of risk and time by considering a decision maker with non-expected utility preferences who believes that only present consumption is certain while any future consumption is uncertain. We provide the first complete characterization of the two-way relations between i) certainty effect and present bias, and, ii) common ratio effect and the common difference effect. A corollary to our results is that hyperbolic discounting implies the Common Ratio Effect and that quasi-hyperbolic discounting implies the Certainty Effect.

In the fourth chapter of my thesis, I use variation in experimental design (time-discounting) and belief data from subjects to investigate the determinants of behavior in Finitely Repeated Prisoner's Dilemma games.

Lay Summary

My thesis investigates decision making under conditions of risk, uncertainty and temporal delay. In chapters one and three, instead of studying each of these behaviors in isolation, I provide a more comprehensive theory of human behavior by studying the interplay of uncertainty and time as infleuncing factors in different environments. Chapter two uses a meta-study over recent influential experimental papers to inform the design of future experiments investigating temporal-preferences. Chapter four studies the effect of temporal delay (discounting) on human interaction in an environment where there is a tradeoff between individual gain and social surplus.

Preface

A version of Chapter 2 has been published [Chakraborty, A., Calford, E.M., Fenig, G. et al. Exp Econ (2017). doi:10.1007/s10683-016-9506-z]. I was the principal contributor on the project, and hence I am the main author on this project.

Chapter 3 was jointly co-authored with Professor Yoram Halevy, and we equally contributed to this work at every stage of the project.

Chapter 4 involves experimental study and associated methods were approved by the University of British Columbia's Research Ethics Board [Ethics Application #H13-02107].

The author confirms that all the online links provided in the bibliography were functional on July 7th, 2017, prior to submission.

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Dedication

To my mother, who taught me how to be myself. To my father, for sharing with me his passion for academics from very early days. To Dida and Dadu, who never stopped believing in me. To all four of them, once again, for their unconditional love and for always supporting my dreams.

Introduction

The discipline of Economics studies how different incentive structures and economic stimuli shape behavior. As expected of any social science, this requires a two pronged approach grounded in theory and empirics. On one hand, economists provide theories of behavior progressively consistent with the human actions we observe in everyday market and non-market transactions. On the other hand, economists also generate and utilize data obtained from surveys, census, market studies or creative experiments to test and select between competing theories of human behavior.

This thesis combines features of both theoretical and empirical approaches to investigate decision making under conditions of risk, uncertainty, and temporal delay. In its course, it uses existing empirical findings to motivate why particular deviations from classical economic assumptions are necessary for better understanding of human behavior in certain scenarios. Then, precise alternative assumptions[1] are provided to formulate new theories and the empirical implications are discussed subsequently[2]. Below we provide the overarching background and motivation for the questions and results studied in this thesis.

The modal temporal preferences obtained in previous experimental studies show a disproportionate bias for present rewards and consumption (Loewenstein and Prelec [1992], Frederick et al. [2002]), a phenomenon known as Present Bias. Existing theories (Pollak [1968], Harvey [1986], Laibson [1997], Hayashi [2003], Ebert and Prelec [2007]) which try to account for Present Bias, require extraneous assumptions on behavior which are often unrealistic or too strong. In Chapter 1, we ask the question if it is possible to obtain a characterization and model for Present Biased preferences, without having to impose extraneous assumptions on behavior. We further study the consequences of such behavior in practical settings and its welfare implications.

In situations of risk and uncertainty, the modal behavior observed is consistent with conservative behavior under uncertainty, and subsequently a bias for certainty (Allais [1953b], Ellsberg [1961], Gilboa and Schmeidler [1989], Cerreia-Vioglio et al. [2015]). This bears a close resemblance to the temporal behavior of a bias for present rewards (Present Bias), as introduced in the previous paragraph. The

[1] These alternative assumptions are weaker versions of previous classical assumptions.

[2] This is why the following essays are broadly consistent with the topic of Behavioral Economics.

similarity begs the following question: Are preferences under risk and time preferences related, and if so, how could one formally understand this relationship? We answer this question in Chapter 3.

On the empirical side of matters, the importance of developing new experimental methods for studying temporal preferences cannot be overstated. Chapter 2 of this thesis provides benchmarks which should be used to evaluate recently developed experimental methodologies (Andreoni and Sprenger [2012], Augenblick et al. [2015]) in isolation, as well as to compare them to older methodologies (Harrison et al. [2002]).

In light of the preceding questions, it is natural to wonder about how temporal delay (and risk) affects human behavior in interactive environments. One such interactive environment is captured by the Prisoner's Dilemma game (Roth and Murnighan [1983], Bo [2005]), where subjects face a tradeoff between individual gain and social surplus. In Chapter 4 we answer the following question: How could we study human behavior in interactive environments under temporal delay/ discounting, and what could we extrapolate about human motivations from such a study?

Chapter 1

Present Bias

Exponential discounting is extensively used in economics to study the trade-offs between alternatives that are obtained at different points in time. Under exponential discounting, the relative preference for early over later rewards depends only on the temporal distance between the rewards (stationarity). However, recent experimental findings have called the model into question. Specifically, experiments have shown that small rewards in the present are often preferred to larger rewards in the future, but this preference is reversed when the rewards are equally delayed. As an example, consider the following two choices:

Example 1.

A. \$100 today	*vs*	B. \$110 in a week
C. \$100 in 4 weeks	*vs*	D. \$110 in 5 weeks

Many decision makers choose A over B, and D over C. This specific pattern of choice reversal can be attributed to a bias we might have towards alternatives in the present, and hence is aptly called present bias or immediacy effect. This is one of the most well documented time preference anomalies (Thaler 1981; Loewenstein and Prelec 1992; Frederick et al. 2002). If preferences are stable across decision-times and the decision-makers are unable to ward against the behavior of their future selves, the same phenomenon creates dynamic inconsistency in behavior: People consistently fail to follow up on the plans they had made earlier, especially if the plans entail upfront costs but future benefits. Every year many people pledge to exercise more, eat healthier, become financially responsible or quit smoking starting next year but fail to follow through when the occasion arrives, to their own frustration.

There is a big literature on what kind of utility representations could rationalize choices made by a present-biased decision maker (DM), which we succinctly summarize in Table A.1 in Appendix I. Though all of these models capture the behavioral phenomenon of present bias, none of them can be called *the* model of present-biased preferences. Instead they are all models of present bias and *some additional* temporal behavior that is idiosyncratic to the model.[3] Moreover, these

[3] For example, Quasi-Hyperbolic Discounting (called β-δ discounting interchangeably) addition-

additional behavioral features conflict across the models and are often not empirically well-founded. This raises the following natural question: What is the most general model of present-biased preferences? Or alternatively, what general class of utilities is consistent with present-biased behavior? Such a model would be able to represent present-biased preferences without imposing any extraneous behavioral assumption on the decision maker. This paper proposes a behavioral characterization for such general class of utilities. We start by introspecting about what exactly present-biased behavior implies in terms of choices over temporal objects. The following example provides the motivation for our "weak present bias" axiom.

Example 2. Suppose that a DM chooses (B) \$110 in a week over (A) \$100 today. What can we infer about his choice between (B′) \$110 in 5 weeks versus (A′) \$100 in 4 weeks, if we condition on the person being (weakly) present-biased?[4]

Note that B$\succsim$A, implies that a possible present-premium (\$100 is available at the present) and the early factor (\$100 is available 1 week earlier) are not enough to compensate for the size-of-the-prize factor (\$110>\$100). Equally delaying both alternatives preserves the early factor and the size-of-the-prize factor, but, the already inferior \$100 prize further loses its potential present-premium, which should only make the case for the previous preference stronger. Hence, B$\succsim$A must imply B′ $\succsim$A′ to be consistent with a weak notion of present bias.

We use this motivation to define a Weak Present Bias axiom, which relaxes stationarity by allowing for present bias but rules out any choice reversals inconsistent with present bias[5]. We then show that if a decision maker satisfies Weak Present Bias and some basic postulates of rationality, then, his preferences over receiving an alternative (x,t) (that is receiving prize x at time t) can be represented in the following way (henceforth called the *minimum representation*)

$$V(x,t) = \min_{u\in\mathscr{U}} u^{-1}(\delta^t u(x))$$

where $\delta \in (0,1)$ is the discount factor, and $\mathscr{U}$ is a set of continuous and increasing utility functions. The minimum representation can be interpreted as if the DM has

ally assumes Quasi-stationarity: violations of constant discounting happen *only* in the present period and the decision maker (DM)'s discounting between any two future periods separated by a fixed distance is always constant. On the other hand, Hyperbolic Discounting (which subsumes Proportional and Power discounting as special cases) captures the behavior of a decision maker whose discounting between any two periods separated by a fixed distance decreases as both periods are moved into the future.

[4]The choice of monetary reward for this example is without loss of generality. The reader can replace monetary reward with a primary reward in the example, and the main message of this example would still go through undeterred.

[5]Note that we are assuming present-premium$\geq$ 0, thus ruling out the case where it is negative, i.e, something that would be consistent with future bias. This "weak" inequality of present-premium is conceptually equivalent to a "weak" presence of present bias.

not one, but a set $\mathscr{U}$ of potential future tastes or utilities. Each potential future taste (captured by a utility function $u \in \mathscr{U}$) suggests a different *present equivalent*[6] for the alternative (x,t). The DM resolves this multiplicity by considering the *most conservative or minimal present equivalent*. Given that the present equivalent of any prize in the present is the prize itself, the minimum representation has no caution imposed on the present, thus treating present and future in fundamentally different ways. For any prize x received at time $t = 0$, $\min_{u \in \mathscr{U}}(u^{-1}(\delta^0 u(x))) = x$[7], which can be interpreted as if, immediate alternatives are not evaluated through similar standards of conservativeness, as is expected of a DM with present bias. Moreover, the fact that all alternatives are procedurally reduced to present equivalents for evaluation and comparison, underlines the salience of the present to the DM. This is another way in which the psychology of present bias is incorporated in the representation. Our representation nests the classical exponential discounting model as the special case obtained when the set $\mathscr{U}$ is a singleton and hence can be considered a direct generalization of the standard model of stationary temporal preferences.

Our model of decision making nests all the popular models of present-biased discounting as special cases, as those models satisfy all the axioms imposed in our analysis. However, there are several robust empirical phenomena discussed in Sections 1.3 and 1.9 which temporal models like β-δ or hyperbolic discounting cannot account for, but the current model can. For example, Keren and Roelofsma [1995] show that once all prizes under consideration are made risky, they are no longer subject to present-biased preference reversals anymore. In other words, once certainty is lost, present bias is lost too. None of the models of behavior that treat the time and risk components of an alternative separately (for example, any discounted expected or non-expected utility model) can accommodate such behavior. We extend our analysis to a richer domain of preferences over risky timed prospects and provide an extended minimum representation that can account for this puzzling behavioral phenomenon. In Section 1.10 we show how a benevolent social planner can use insights from time-risk behavior to improve the welfare of present-biased individuals. Another choice pattern that most temporal models fail to accommodate is the stake dependence of present bias. For example, a DM might have a bias for the present, but he might also expend considerably more cognitive effort to fight off this bias when the stakes are large. His large stake choices would satisfy stationarity, whereas he would appear to be present-biased in his choices

[6] *Present equivalent* of an alternative (x,t) is the immediate prize that the DM would consider equivalent to (x,t). For a felicity function u defined on the space of all possible prizes x, and a discount factor of δ, the discounted utility from (x,t) is $\delta^t u(x)$. Hence the corresponding present equivalent is $u^{-1}(\delta^t u(x))$.

[7] As, $\delta^0 = 1$, $u^{-1}(\delta^0 u(x)) = u^{-1}(u(x)) = x$ for all $u \in \mathscr{U}$.

over smaller stakes (see Halevy 2015 for supporting evidence). We show how our representation can accommodate such preferences in Section 1.9.

The subjective max-min feature of the functional form has been used previously by Cerreia-Vioglio et al. [2015] in the domain of risk preferences, though they had the minimum replaced by an infimum. In their paper, Cerreia-Vioglio et al. [2015] show that if we weaken the Independence axiom to account for the Certainty Effect (Allais 1953a), we obtain a representation where a decision maker evaluates the certainty equivalent of each lottery with respect to a set of Bernoulli utility functions and then takes the infimum of those values as a measure of prudence. We discuss this connection in greater detail in Section 1.5 and describe how the techniques used in our paper can be used to provide an alternative derivation of their main result in a reduced domain.

The paper is arranged as follows: Section 1.1 defines the novel Weak Present Bias axiom and provides the main representation theorem of the paper. Section 1.2 builds on the main result to provide intuition about the separation of β-δ discounting from Hyperbolic discounting. Section 1.3 extends the main result to a richer domain with risk. Section 1.4 discusses extensions of the representation result to consumption streams. We provide an intuition of the inner workings of the proofs in Section 1.5. Section 1.6 comments on the uniqueness of the results. Section 1.7 surveys the literature closely related to this paper. Sections 1.8 and 1.9 discuss the testability, refutability and empirical content of our model. Sections 1.10-1.11 provide applications, policy implications and extensions of the main results of the paper. The proofs of the main theorems are included in Appendix II.

1.1 Model and the main result

A decision maker has preferences $\succsim$ defined on all timed alternatives $(x,t) \in \mathbb{X} \times \mathbb{T}$ where the first component could be a prize (monetary or non-monetary) and the second component is the time at which the prize is received. Let $\mathbb{T} = \{0,1,2,...\infty\}$ or $T = [0,\infty)$ and $\mathbb{X} = [0,M]$ for $M > 0$. We impose the following conditions on behavior.

A0: $\succsim$ is complete and transitive.

Completeness and transitivity are standard assumptions in the literature, though one can easily argue that they are more normative than descriptive in nature. The few instances of present-biased intransitive preferences studied in the economics literature, notably Read [2001], Rubinstein [2003] and Ok and Masatlioglu [2007] fall outside our domain of consideration due to (A0).

A1: CONTINUITY: $\succsim$ is continuous, that is the strict upper and lower contour sets of each timed alternative is open w.r.t the product topology.

Continuity is a technical assumption that is generally used to derive the continuity of the utility function over the relevant domain. When, $\mathbb{T} = \mathbb{R}_+$, the standard β-δ model does not satisfy continuity at $t = 0$.[8]

A2: DISCOUNTING: For $t,s \in \mathbb{T}$, if $t > s$ then $(x,s) \succ (x,t)$ for $x > 0$ and $(x,s) \sim (x,t)$ for $x = 0$. For $y > x > 0$, there exists $t \in \mathbb{T}$ such that, $(x,0) \succsim (y,t)$.

The Discounting axiom has two components. The first part says that the decision maker always prefers any non-zero reward at an earlier date. The second part states that any reward converges to the zero reward (and hence, continually loses its value), as it is sufficiently delayed.

A3: MONOTONICITY: For all $t \in \mathbb{T}$ $(x,t) \succ (y,t)$ if $x > y$.

The Monotonicity axiom requires that at any point in time, larger rewards are strictly preferred to smaller ones. Finally, in light of Example 1, we formally define Weak Present Bias below.

A4: **WEAK PRESENT BIAS:** If $(y,t) \succsim (x,0)$ then, $(y,t+t_1) \succsim (x,t_1)$ for all $x,y \in X$ and $t,t_1 \in \mathbb{T}$.

To provide context the standard Stationarity axiom is stated below.
Stationarity: $(y,t_1) \succsim (x,t_2)$ if and only if, $(y,t+t_1) \succsim (x,t+t_2)$ for all $x,y \in X$ and $t,t_1,t_2 \in \mathbb{T}$.
Weak present bias as defined in the fourth axiom is the most intuitive weakening of Stationarity in light of the experimental evidence about present bias or immediacy effect. It allows for choice reversals that are consistent with present-bias, something that Stationarity does not allow. On the other hand, having an opposite bias for future consumption is ruled out . [9]Other than all the separable discounting models mentioned in Appendix I, this Weak Present Bias axiom is also satisfied by the non-separable models of present bias proposed by Benhabib et al. [2010]

[8]Pan et al. [2015] axiomatize a model of Two Stage Exponential (TSE) discounting which incorporates the idea of β-δ discounting while maintaining continuity.

[9]Further, $(y,t) \succ (x,0)$ and $(y,t+t_1) \sim (x,t_1)$ is also not consistent with WPB, Continuity and Monotonicity. The reason being that, by Continuity, there would exist $y' < y$, $(y',t) \succ (x,0)$ and $(x,t_1) \succ (y',t+t_1)$. Whereas, $(y,t) \sim (x,0)$ and $(y,t+t_1) \succ (x,t_1)$ is allowed by the postulates A0-4.

[10] and Noor [2011]. This stands testimony to the fact that the Weak Present Bias axiom is able to capture the general behavioral property of present bias in a very succinct way. Now we present our main representation result.

Theorem 3. *The following two statements are equivalent:*

i) The relation $\succsim$ defined on $\mathbb{X} \times \mathbb{T}$ satisfies axioms A0-A4.

ii) For any $\delta \in (0,1)$, there exists a set $\mathscr{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x,t) = \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x)) \tag{1.1}$$

represents the binary relation $\succsim$. The set $\mathscr{U}_\delta$ has the following properties: $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathscr{U}_\delta$. $F(x,t)$ is continuous.

Note that for any timed alternative (x,t), $u^{-1}(\delta^t u(x))$ in (1.1) computes its "present equivalent", the amount in the present which the individual would deem equivalent to (x,t) if u were his utility function. For all present prizes, the present equivalents are trivially equal to the prize itself ($u^{-1}(\delta^0 u(x)) = x \ \forall u$) irrespective of the utility function under consideration, and thus there is no scope or need for prudence. Whereas for timed alternatives in the future, whenever $\mathscr{U}$ is not a singleton, the DM chooses the most conservative present equivalent due to the minimum functional, thus exhibiting prudence. This is the primary intuition of how this functional form treats the present differently from the future and thus incorporates present bias into it. A potential motivation for the minimum representation and differential treatment towards present and future, follows from Loewenstein [1996]'s visceral states argument: "*..immediately experienced visceral factors have a disproportionate effect on behavior and tend to crowd out virtually all goals other than that of mitigating the action, ...but.. people under weigh, or even ignore, visceral factors that they will experience in the future.*" The following example shows an easy application of the theorem to represent present-biased choices.

Example 4. Consider $\mathscr{U} = \{u_1, u_2\}$, where,

$$\begin{aligned} u_1(x) &= x^a \text{ for a>0} \\ u_2(x) &= 1 - \exp(-bx) \text{ for b>0} \end{aligned}$$

[10] Benhabib et al. [2010] introduce the discount factor

$$\Delta(y,t) = \begin{cases} 1 & t = 0 \\ (1-(1-\theta)rt)^{(1-\theta)} - \frac{b}{y} & t > 0 \end{cases}$$

Also consider, $a = .99$, $b = .00021$, $\delta = .91$. One can easily check that a minimum representation with respect to this $\mathscr{U}$ would satisfy Weak Present Bias (also follows from Theorem 3). The minimum representation with respect to this $\mathscr{U}$ would assign the following utilities to the timed alternatives in Example 1.

$$\begin{aligned}
V(100,0) &= \min(100,100) = 100 \\
V(110,1) &= \min(100.056, 99.995) = 99.995 \\
V(100,4) &= \min(68.317, 68.48) = 68.317 \\
V(110,5) &= \min(68.320, 68.344) = 68.320
\end{aligned}$$

Hence,

$$\begin{aligned}
V(100,0) &> V(110,1) \\
V(100,4) &< V(110,5)
\end{aligned}$$

Thus the minimum function with a simple $\mathscr{U}$ can be used to accommodate present biased choice reversals.

1.2 Special cases

This section applies Theorem 3 to a popular model of present bias, the β-δ model (Phelps and Pollak 1968, Laibson 1997). The $\beta - \delta$ model evaluates each alternative (x,t) as $U(x,t) = (\beta + (1-\beta).1_{t=0})\delta^t u(x)$, where u, δ, β have standard interpretation. $1_{t=0}$ is the indicator function that takes value of 1 if $t = 0$ and value 0 otherwise, thus assigning a special role to the present. Given that the $\beta - \delta$ model satisfies Weak Present Bias and all the other axioms included in Theorem 1 (for the discrete case) , any such $\beta - \delta$ representation must have an alternative minimum representation, as shown in Theorem 3.

Below, we consider the simplest possible $\beta - \delta$ representation with linear felicity function $u(x) = x$, $\mathbb{T} = \{0, 1, 2, ..\}$ and construct the corresponding Weak Present Bias representation.

Claim 5. β-δ representation with $u(x) = x$ has an alternative minimum representation.

Proof. Define the functions $u_y : \mathbb{R} \to \mathbb{R}_+$ for all $y \in \mathbb{R}_+$:

$$u_y(x) = \begin{cases} \dfrac{x}{\beta} & \text{for } x \leq \beta\delta y \\ \delta y + (x - \beta\delta y)\dfrac{1-\delta}{1-\beta\delta} & \text{for } \beta\delta y < x \leq y \\ x & \text{for } x > y \end{cases}$$

For any $y \in \mathbb{R}_+$, $x \leq u_y(x) \leq \dfrac{x}{\beta}$ for all $x \in \mathbb{R}_+$. As u_y is an increasing function, it must be that $x \geq u_y^{-1}(x) \geq \beta x$. Since, $x \leq u_y(x)$, we get $\delta^t u_y(x) \geq \delta^t x$, which implies,

$$u_y^{-1}(\delta^t u_y(x)) \geq u_y^{-1}(\delta^t x) \geq \beta\delta^t x$$

Finally, for $x = y$, $\delta^t u_y(x) = \delta^t x < \delta x$ and, hence, $u_y(\delta^t u_y(x)) = \beta\delta^t x$.

Therefore, $V(x,t) = \min_{y\in\mathbb{R}_+} u_y^{-1}(\delta^t u_y(x)) = (\beta + (1-\beta).1_{t=0})\delta^t x$, which finishes our proof.[11] □

This shows that if we start with a rich enough set of piece-wise linear utilities, the minimum representation with respect to that set, is enough to generate behavior consistent with β-δ discounting. In the example above, the set values taken by the set of functions is bounded above and below at each non-zero point x of the domain by $[\dfrac{x}{\beta}, x]$, and this brings us to our next result. Our next theorem characterizes the behavioral axiom necessary and sufficient for the functions in $\mathscr{U}_\delta$ to be similarly bounded.

We start by introducing two more axioms.

A5: EVENTUAL STATIONARITY: For any $x > z > 0 \in \mathbb{X}$, there exists $t_1 \in \mathbb{T}$, such that for $t \geq 0$, $(z,t) \succ (x, t + t_1)$ and $(z,0) \succ (x_t, t_1 + t)$ for any x_t such that $(x,0) \sim (x_t,t)$.

A6: NON-TRIVIALITY: For any $x \in \mathbb{X}$, and $t \in \mathbb{T}$, there exists $z \in \mathbb{X}$, such that $(z,t) \succ (x,0)$.

The last axiom basically means that the space of prizes is rich enough to have exceedingly better outcomes, and it is only needed when $\mathbb{X} = \mathbb{R}_+$, and can be dropped if $\mathbb{X} = [0,M]$. (See Corollary 1)

A5 is the more crucial axiom. That for any $x > z > 0 \in X$, there exists a sufficient delay $\tau_1 \in \mathbb{T}$, such that $(z,0) \succ (x,\tau_1)$ is already implied by Discounting (A2).

[11] This is not necessarily the only possible minimum-representation of the β-δ discounting.

What has been added is the existence of delay t_1 for which we additionally have $(z,t) \succ (x,t+t_1)$ for all $t \geq 0$: This intuitively means once the later larger prize is "sufficiently" delayed, the relative rates at which the attractiveness of the earlier and later rewards fall with further delay (increasing values of t) are consistent with stationarity. This rules out certain preference reversals that were previously allowed under WPB. The last and third part of the axiom, $(z,0) \succ (x_t, t_1+t)$ for any x_t such that $(x,0) \sim (x_t,t)$, also has the same interpretation. The A5 property provides a crucial separation between two popular classes of present-biased discounting functions: β-δ discounting and Hyperbolic discounting , as only the former satisfies it, but the latter does not. We show this more formally in Proposition 25 in Appendix II.

Theorem 6. *Let $\mathbb{T} = \{0,1,2,...\infty\}$ and $\mathbb{X} = \mathbb{R}_+$. The following two statements are equivalent:*

i) The relation $\succsim$ satisfies properties A0-A6.

ii) There exists a set $\mathscr{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x,t) = \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x)) \tag{1.2}$$

represents the binary relation $\succsim$. The set $\mathscr{U}_\delta$ has the following properties: $u(0) = 0$ for all $u \in \mathscr{U}_\delta$, $\sup_u u(x)$ is bounded above, $\inf_u u(x) > 0 \ \forall x > 0$, $\inf_u \dfrac{u(z)}{u(x)}$ is unbounded in z for all $x > 0$. $F(x,t)$ is continuous.

This theorem implies that any "minimum-representation" of hyperbolic discounting must require a set of functions which would take unbounded set values at some point of the domain. The immediate conclusion one can draw from here is that one cannot generate any variant of Hyperbolic discounting (with any felicity function) with a minimum representation over a finite set $\mathscr{U}$ of utilities. This theorem also has a straightforward corollary, where we consider the prize domain $\mathbb{X} = [0,M]$ and drop A6.

Corollary 7. *Let $\mathbb{T} = \{0,1,2,...\infty\}$ and $\mathbb{X} = [0,M]$. The following two statements are equivalent:*

i) The relation $\succsim$ satisfies properties A0-A5.

ii) There exists a set $\mathscr{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x,t) = \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x))$$

represents the binary relation $\succsim$. The set $\mathscr{U}_\delta$ has the following properties: $u(0) = 0$, $u(1) = 1$ for all $u \in \mathscr{U}$, $\inf_u u(x) > 0 \ \forall x$. $F(x,t)$ is continuous.

	Prospect A	Prospect B	% chosing A	% chosing B	N
1	(9,1,0)	(12,.8,0)	58%	42%	142
2	(9,.1,0)	(12,.08,0)	22%	78%	65
3	(9,1,3)	(12,.8,3)	43%	57%	221
4	(100,1,0)	(110,1,4)	82%	18%	60
5	(100,1,26)	(110,1,30)	37%	63%	60
6	(100,.5,0)	(110,.5,4)	39%	61%	100
7	(100,.5,26)	(110,.5,30)	33%	67%	100

Table 1.1: Risk vs No-Risk

1.3 An extension to risky prospects

In this section, we extend the representation derived in Section 1.1 to risk. This extension serves the following three goals. First, it shows that the representation in Section 1 has a natural extension to simple binary lotteries, with zero being one of the lottery outcomes. Second, through the extended representation we are able to accommodate experimental evidence that is inconsistent with most previous temporal models of behavior. Finally, through this extension, we will be able to identify a unique discount factor δ for any DM satisfying certain postulates of behavior.

We start by presenting the experimental evidence from time-risk domain that our model would be able to accommodate, but, the temporal models from Appendix I would not. In the following text, we summarize each alternative by the triplet (x, p, t) where x is a monetary prize, p is the probability with which x is attained at time t. For the first three rows (from Baucells and Heukamp [2010]), x was offered in Euros, and in the next four (taken from Keren and Roelofsma [1995].), x was offered in Dutch Guilder, t was measured in months in Columns 1:3, and measured in weeks in Columns 4:7.

The data can be interpreted in the following way: People have an affinity for both certainty and immediacy. The loss in either certainty or immediacy has a similar disproportionate effect on preferences (compare rows 5 and 6 with row 4, or rows 2-3 with row 1). Most interestingly, there is very little evidence of present-biased reversals over risky prospects (compare rows 6-7, with rows 4-5). It is the latter finding that is at odds with most temporal models of behavior. In fact it rules out *all* discounted expected or non-expected utility functional forms which are separable in the temporal and risk components.[12]

We will consider preferences over triplets $(x,p,t) \in \mathbb{X} \times \mathbb{P} \times \mathbb{T}$, which describe the prospect of receiving a reward $x \in \mathbb{X}$ at time $t \in \mathbb{T}$ with a probability$p \in [0,1]$. $\mathbb{X} = [0,M]$ is a positive reward interval, $\mathbb{P} = [0,1]$ is the unit interval of probability, and $\mathbb{T} = [0,\infty)$ is the time interval. We impose the following conditions on behavior.

B0: $\succsim$ is complete and transitive.

B1: CONTINUITY: $\succsim$ is continuous, that is the strict upper and lower contour sets of each risky timed alternative are open w.r.t the product topology.

B2: DISCOUNTING: For $t,s \in \mathbb{T}$, if $t > s$ then $(x,p,s) \succ (x,p,t)$ for $x,p > 0$ and $(x,p,s) \sim (x,p,t)$ for $x = 0$ or $p = 0$. For $y > x > 0$, there exists $T \in \mathbb{T}$ such that, $(x,q,0) \succsim (y,1,T)$.

B3: PRIZE AND RISK MONOTONICITY: For all $t \in \mathbb{T}$, $(x,p,t) \succsim (y,q,t)$ if $x \geq y$ and $p \geq q$. The preference is strict if at least one of the two following inequalities is strict.

Note that the first four axioms are just extensions of A0-A3.

B4: WEAK PRESENT BIAS: If $(y,1,t) \succsim (x,1,0)$ then, $(y,1,t+t_1) \succsim (x,1,t_1)$ for all $x,y \in \mathbb{X}$, $\alpha \in [0,1]$ and $t,t_1 \in \mathbb{T}$.

B5: PROBABILITY-TIME TRADEOFF: For all $x,y \in \mathbb{X}$, $p,q,\theta \in (0,1]$, and $t,s,D \in \mathbb{T}$, $(x,p\theta,t) \succsim (x,p,t+D) \implies (y,q\theta,s) \succsim (y,q,s+D)$.

The fifth axiom (used previously in Baucells and Heukamp 2012a) says that passage of time and introduction of risk have similar effects on behavior, and there is a consistent way in which time and risk can be traded off across the domain of behavior. This axiom implies calibration properties as well that we will utilize in the proofs, and it will be crucial to pin down a unique discount factor δ for any DM. Additionally, (B4) when combined with (B5) captures a decision maker's joint bias towards certainty as well as the present, i.e, it embeds Weak Present Bias *as well as Weak Certainty Bias*[13] in itself. This underlines the insight that once

[12]Rows 1 and 3 also imply the same.

[13]Weak Certainty Bias can be defined on $\mathbb{X} \times \mathbb{P}$ in the following fashion: If $(y,p) \succsim (x,1)$ then, $(y,p\alpha) \succsim (x,\alpha)$ for all $x,y \in \mathbb{X}$ and $\alpha \in [0,1]$.

risk and time can be traded-off, Weak Present Bias and Weak Certainty Bias are behaviorally equivalent. Similar relations between time and risk preferences have been elaborated on previously by Halevy [2008], Baucells and Heukamp [2012a], Saito [2015], Fudenberg and Levine [2011], Epper and Fehr-Duda [2012] and Chakraborty and Halevy [2015]. In Section 1.5, we will discuss how the Weak Certainty Bias postulate connects the current work to previous literature on risk preferences.

We are now ready for our next result.

Theorem 8. *The following two statements are equivalent:*

i) The relation $\succsim$ *on* $\mathbb{X} \times \mathbb{P} \times \mathbb{T}$ *satisfies properties B0-B5.*

ii) There exists a unique $\delta \in (0,1)$ *and a set* $\mathscr{U}$ *of monotonically increasing continuous functions such that* $F(x,p,t) = \min_{u \in \mathscr{U}} (u^{-1}(p\delta^t u(x)))$ *represents the relation* $\succsim$*. For all the functions* $u \in \mathscr{U}$*,* $u(M) = 1$ *and* $u(0) = 0$*. Moreover,* $F(x,p,t)$ *is continuous.*

The next example shows a potential application of this representation in light of Keren and Roelofsma [1995]'s experimental results.

Example 9. Consider the set of functions $\mathscr{U}$ and parameters considered in Example 4. When applied to the representation derived in Theorem 8, they predict the following choice pattern.

$$
\begin{aligned}
V(100,1,0) &> V(110,1,1) \\
V(100,1,4) &< V(110,1,5) \\
\\
V(100,.5,0) &< V(110,.5,1) \\
V(100,.5,4) &< V(110,.5,5)
\end{aligned}
$$

Note that this is exactly the choice pattern obtained in the original Keren and Roelofsma [1995] experiment: time and risk affect choices in similar ways, and once certainty is removed present bias disappears.

1.4 Extension to consumption streams

In this section, we extend the representation derived in Section 1.1 to deterministic consumption streams. The DM's preferences $\succsim$ are defined over $[0,\infty)^T$, the set of all consumption streams of finite length $T > 1$. We impose the following conditions on behavior.

D0: $\succsim$ is complete and transitive.

D1: CONTINUITY: $\succsim$ is continuous, that is the strict upper and lower contour sets of each consumption stream are open w.r.t the product topology.

D2: DISCOUNTING: If $0 \leq s < t \leq T-1$, then
$(0,..\underbrace{y}_{\text{in period } s},..,0) \succsim (0,..\underbrace{y}_{\text{in period } t},..,0)$ for $y \geq 0$ with the relation being strict if and only if $y > 0$. Further, for $y_0 > x > 0$, and for any sequences $(y^1, y^2, y^3, ..y^m)$ and $(n^1, n^2, .., n^m)$, where, $(0,..0,\underbrace{y^{i-1}}_{\text{in period } n^i},0..,0) \succsim (y^i, 0, .., 0)\ \forall i \in \{1,2,...,m\}$,

$0 < n^i \leq T-1$ and $\sum_1^m n^i = t$, there exists $t \in \mathbb{N}$ such that, $y_m \leq x$.

D3: MONOTONICITY: For any $(x_0, x_1, ..x_{T-1}),\ (y_0, y_1, ..y_{T-1}) \in [0,\infty)^T$, $(x_0, x_1, ..x_{T-1}) \succsim (y_0, y_1, ..y_{T-1})$ if $x_t \geq y_t$ for all 0≤t $\leq T-1$. The preference is strict if at least one of the inequalities is strict.

D4: WEAK PRESENT BIAS: If $(0,..\underbrace{y}_{\text{in period } t},..,0) \succsim (x,0,..,0)$ then,

$(0,..\underbrace{y}_{\text{in period } t+t_1},..,0) \succsim (0,.\underbrace{x}_{\text{in period } t_1}.,0)$ for all $x, y \in \mathbb{X}$ and $t, t_1 \in \mathbb{T}$.

Note that the first five axioms are alternative restatements of A0-A4 in the current domain, but the Discounting axiom warrants some independent discussion. As before, the second part of the Discounting axiom states that any period-consumption keeps falling arbitrarily in present-equivalent value, as one increases the total discounting it is subjected to. Due to the added restriction that the DM can only consider time delays of upto $T-1$ periods for $T \geq 2$, we have approximated arbitrary delays by a sequence of delays, none greater than $T-1$. But, the restatement is also a stronger version of the former (under WPB) as it also imposes path independence (by stating the axiom for arbitrary sequences $(n^i)_{i=1}^m$ of delays instead of requiring it to hold for a particular sequence of delays, that sum to t) while achieving this total discounting. This is necessary while working with the non-compact prize space of $[0,\infty)$.

D5: STRONG ADDITIVITY: For any pair of orthogonal[14] consumption bundles $(x_0, x_1, ..x_{T-1})$, $(y_0, y_1, ..y_{T-1}) \in [0,\infty)^T$, if, $(x_0, x_1, ..x_{T-1}) \sim (z_0, 0, .., 0)$ and $(y_0, y_1, ..y_{T-1}) \sim (z'_0, 0, .., 0)$, then,
$(x_0 + y_0, x_1 + y_1, ..x_{T-1} + y_{T-1}) \sim (z_0 + z'_0, 0, .., 0)$.

Orthogonality of consumption vectors imply that $x_t > 0$ only if $y_t = 0$, and $y_t > 0$ only if $x_t = 0$ for all t. The fifth axiom implies the standard notion of Additivity used in axiomatizations of additive representation of streams, and is hence named Strong Additivity.

We are now ready for our next result.

Theorem 10. *The following two statements are equivalent:*
i) The relation $\succsim$ on $[0,\infty)^T$ satisfies properties D0-D5.
ii) For any $\delta \in (0,1)$, there exists a set $\mathscr{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x_0, x_1, .., x_{T-1}) = x + \sum_{1}^{T-1} \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x_t))$$

represents the binary relation $\succsim$. The set $\mathscr{U}_\delta$ has the following properties: $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathscr{U}_\delta$. $F(.)$ is continuous.

It is worth noting that this extension to streams required a strong notion of additivity, and hence, the resulting representation on streams is not as general as the one derived in the previous domain. For example, the representation here does not nest the classical exponentially discounted additive utility representation in its most general form.

1.5 An outline of the proofs

This section outlines the proofs of Theorems 3-8 chronologically and places the methodology used in the proofs in the context of recent literature.
We will provide the outline for the case of $\mathbb{T} \in [0,\infty)$, as it is less technical but conveys the main idea behind the proofs nonetheless. For any timed alternative (z,τ), there exists $x \in \mathbb{X}$ such that $(z,\tau) \sim (x,0)$. This follows from monotonicity, continuity, connectedness of the prize-domain and this guarantees that any (timed)

[14] Two vectors are orthogonal if their dot product is zero.

alternative has a well defined present equivalent with respect to $\succsim$. It is easy to see that when $\tau = 0$, one must have $z = x$. Given the present equivalents with respect to $\succsim$ are well defined, one possible utility representation $V : \mathbb{X} \times \mathbb{T} \to \mathbb{R}_+$ is the function that assigns to every alternative (z, τ), the present equivalent according to the relation $(z, \tau) \sim (x, 0)$. The crux of the remaining proof lies in showing that there exists a set of utilities $\mathscr{U}_\delta$ such that the previously defined V function can be rewritten as

$$V(z, \tau) = x \quad = \quad \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^\tau u(z))$$

The proof is constructive. For any point $x^* \in (0, M)$, we construct a function $u_{x^*}(.)$ in the following steps.
i) We assign $u_{x^*}(0) = 0$, $u_{x^*}(x^*) = 1$.
ii) For any $x \in (x^*, M]$, we find $t > 0$ such that $(x, t) \sim (x^*, 0)$. Define, $u_{x^*}(x) = \delta^{-t}$ (for any $\delta \in (0, 1)$ under consideration) and re-label x as x_t.
iii) For $y \in (0, x^*)$, define $u_{x^*}(y) = \min\{\delta^\tau : \ (x_t, t + \tau) \sim (y, 0)\}$ for some t from step (ii).
We show that the minimum is well defined in step (iii), and the constructed $u_{x^*}()$ is strictly increasing, continuous, and has the following crucial property: If $(z, t) \sim (x, 0)$ then, $\delta^t u_{x^*}(z) \geq u_{x^*}(x)$ and subsequently, $u_{x^*}^{-1}(\delta^t u_{x^*}(z)) \geq x$, with the weak inequality replaced by equality if $x = x^*$. The asymmetric construction of $u_{x^*}()$ on the left and right of x^* is crucial for this to hold.
Next we define $\mathscr{U}_\delta = \{u_{x^*}(.) : \ x^* \in (0, M)\}$. It readily follows from the aforementioned property of constructed utility functions that $\min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u_x(z)) = x$ whenever $(z, t) \sim (x, 0)$.

Theorem 6 builds on these methods and insights of Theorem 3. Eventual Stationarity gurantees that the functions in $\mathscr{U}$ can be constrcuted in a way such that for any two points $x < y$ there exists t_1 for which $u(x) > \delta^{t_1} u(y)$ for all $u \in \mathscr{U}$. Now when one normalizes, $u(1) = 1$ for all $u \in \mathscr{U}$, using the condition mentioned in the previous sentence, one additionally obtains that $\sup_u u(x)$ is bounded above and $\inf_u u(x) > 0 \ \forall x > 0$.

Theorem 8 connects time and risk in the following way: Given the Probability-Time Tradeoff axiom, the $\mathbb{X} \times \mathbb{P} \times \mathbb{T}$ domain is isomorphic to either of the reduced domains of $\mathbb{X} \times \mathbb{P}$ or $\mathbb{X} \times \mathbb{T}$. For example, there exists unique $\delta \in (0, 1)$ such that $(x, p, t) \sim (x, p\delta^t, 0)$ and $(x, p, t) \sim (x, 1, t + \log_\delta p)$ for all $x \in \mathbb{X}$ and $p \in \mathbb{P}$. This theorem restricts its domain to $\mathbb{T} = \mathbb{R}_+$, unlike Theorem 3, which holds equally for $\mathbb{T} = \mathbb{N}_0$ as well. The axioms on $\mathbb{X} \times \mathbb{P} \times \mathbb{T}$ domain imply completeness, transitivity, continuity, risk monotonicity (Discounting respectively), Weak Certainty Bias (Weak Present Bias respectively) for a preference defined on the reduced domain of $\mathbb{X} \times \mathbb{P}$ ($\mathbb{X} \times \mathbb{T}$ respectively for $\mathbb{T} = \mathbb{R}_+$). Proving Theorem 8, now reduces to proving that there is a minimum representation on $\mathbb{X}\times\mathbb{P}$ or $\mathbb{X}\times\mathbb{T}$ of the forms $\min_{u\in\mathscr{U}}(u^{-1}(pu(x)))$ or $\min_{u\in\mathscr{U}}(u^{-1}(\delta^t u(x)))$ respectively. Additionally, proving any one of the representations from the implied axioms on the relevant domain is equivalent to proving all of the representations on the respective domains. This flexibility is allowed by the Probability Time Tradeoff axiom. In the Appendix, we prove how the reduction from the richer domain to $\mathbb{X}\times\mathbb{P}$ or $\mathbb{X}\times\mathbb{T}$ works, and then prove that a relation on $\mathbb{X}\times\mathbb{P}$ satisfies completeness, transitivity, continuity, risk monotonicity and Weak Certainty Bias *if and only if* the relation on $\mathbb{X}\times\mathbb{P}$ can be represented by the functional form of $\min_{u\in\mathscr{U}}(u^{-1}(pu(x)))$.

This result on the reduced $\mathbb{X}\times\mathbb{P}$ domain brings us to a very interesting connection that the present work has with Cerreia-Vioglio et al. [2015]. In that paper, the authors consider preferences over lotteries ($\mathscr{L}$) defined over a compact real interval $[w,b]$ of outcomes. To account for violations of the Independence Axiom[15] based on a DM's bias towards certainty or sure prizes[16], they relax it in favor of Negative Certainty Independence (NCI) axiom defined below.

NCI: (Dillenberger 2010) For $p,q\in\mathscr{L}$, $x\in[w,b]$, and $\lambda\in(0,1)$,

$$q\trianglerighteq L_x \implies \lambda p+(1-\lambda)q\trianglerighteq \lambda p+(1-\lambda)L_x$$

Cerreia-Vioglio et al. [2015] show that if $\trianglerighteq$ satisfies NCI and some basic rationality postulates, then there exists a set of continuous and strictly increasing functions $\mathscr{W}$, such that the relation $\trianglerighteq$ can be represented by a continuous function $V(p)=\inf_{u\in\mathscr{W}}c(p,u)$, where $c(p,u)$ is the certainty equivalent of the lottery p with respect to $u\in\mathscr{U}$. The proof of their theorem has the following steps: From $\trianglerighteq$, they construct a partial relation $\trianglerighteq'$ which is the largest sub-relation of the original preference $\trianglerighteq$ that satisfies the Independence axiom. By Cerreia-Vioglio [2009], $\trianglerighteq'$ is reflexive, transitive (but possibly incomplete), continuous and satisfies Independence. Next, following Dubra et al. [2004] [17], there exists a set $\mathscr{W}$ of continuous functions on $[w,b]$ that constitutes an Expected Multi-Utility representation of $\trianglerighteq'$, that is, $p\trianglerighteq' q$ if and only if $E_v(p)\geq E_v(q)$ for all $v\in\mathscr{W}$. Now taking an infimum of the present equivalents with respect to all the functions in $\mathscr{W}$ yields a representation that assigns to each lottery its certainty equivalent implied by the relation $\trianglerighteq$.

This NCI axiom when reduced to the domain of binary lotteries on $\mathbb{X}\times\mathbb{P}$, conveys the same behavior as the Weak Certainty Bias axiom we have discussed

[15] For $p,q,r\in L$, and $\lambda\in(0,1)$, $p\trianglerighteq q$ if and only if $\lambda p+(1-\lambda)r\trianglerighteq\lambda q+(1-\lambda)r$.

[16] We denote the lottery that gives the outcome $x\in[w,b]$ for sure as $L_x\in\mathscr{L}$.

[17] Dubra et al. [2004] define a convex cone in the linear space generated by the lotteries related by $\trianglerighteq'$ and then apply an infinite-dimensional version of the separating hyperplane theorem to establish the existence of $\mathscr{W}$.

above and have used in the proof of our theorem. Our representation over $\mathbb{X} \times \mathbb{P}$ is a minimum representation that is an exact parallel of the infimum representation obtained by Cerreia-Vioglio et al. [2015]. This is no coincidence: we provide an alternative derivation of Cerreia-Vioglio et al. [2015]'s result in a reduced domain of lotteries for similar behavior and show that their infimum representation can be replaced with a minimum representation under the implied axioms in our domain. Our proof is essentially constructive, as illustrated in Claim 5, and it does not use any intermediate results (for example, results from Dubra et al. [2004]).

The similarity in functional forms naturally prompts the question: Could the proof in Cerreia-Vioglio et al. [2015] be applied directly to our representation theorems? The answer to the question is negative for the following two reasons. Firstly, when an NCI-like axiom (Weak Certainty Bias) is imposed on my restricted domain of binary lotteries, the results from Cerreia-Vioglio et al. [2015] no longer follow as corollaries of their main theorem due to the reduced strength of the implied axioms. This follows the usual relation between size of domain and strength of axiom. Secondly, there is no way of starting with an appropriately defined axiom of present bias on consumption streams (instead of timed payments) and reaching a present-biased utility representation on streams by using the route (Present Bias)⇔(NCI)⇔(Multi EU) ⇔ (Present-biased representation), under any equivalence of time and implicit risk necessary for the first and last steps.

1.6 Uniqueness

The uniqueness results discussed here are formulated keeping the main representation theorem of the paper in mind, but they apply equally to the other representation theorems with minor adjustments. We start with a crucial question about the representation: Could we have come across an alternative representation for the same preferences without the exponential discounting part inside the present equivalents? For example, could we have ended up with a representation of the form:

$$V'(x,t) = \min_{u \in \mathscr{U}} u^{-1}(\Delta(t)u(x)) \tag{1.3}$$

where $\Delta(t)$ is some time-decreasing *discount function* other than exponential discounting, for example the hyperbolic one? Note that this is an interesting question, as a positive answer would open the door to representations where the present equivalents are taken with respect to hyperbolic or quasi-hyperbolic discounting. However, the answer is negative. If we start with any $\Delta(t)$ such that $\dfrac{\Delta(t+t_1)}{\Delta(t)} \neq$

$\Delta(t_1)$ for some t, t_1 , there would either 1) exist some binary relation which satisfies all the axioms in this paper, but cannot be represented by the representation in (1.3), or 2) the representation in (1.3) with a permissible set of utilities $\mathscr{U}$ would represent preferences which do not satisfy at least one of the axioms in this paper, thus breaking the two-way relation between the axioms and representation.

Proposition 11. *Given the axioms A0-4, the representation in (1.3) is unique in the discounting function $\Delta(t) = \delta^t$ inside the present equivalent function.*

Proof. See Appendix II. □

One of the limitations of representations over $\mathbb{X} \times \mathbb{T}$ space (the domain used in Sections 1 and 2) is the lack of uniqueness in terms of the discount factor δ. We inherit the non-uniqueness of δ in Theorems 3-6 from Fishburn and Rubinstein [1982]. Fishburn and Rubinstein [1982] impose A0-A3 along with Stationarity on preferences to derive a exponential discounting representation. In their representation, given those conditions on preferences, and given $\delta \in (0,1)$, there exists a continuous increasing function f such that (x,t) is weakly preferred to (y,s) if and only if $\delta^t u(x) \geq \delta^s u(y)$. They have the following result: if (u,δ) is a representation for a preference $\succsim$ then so is (v,β) where $\beta \in (0,1)$ and $v = u^{\frac{\log\beta}{\log\delta}}$. Same holds for our representations in Theorems 3-6: if $(\delta,\mathscr{U})$ is a representation of $\succsim$, then so is $(\alpha,\mathscr{F})$,where $\mathscr{F}$ is constructed by the functions $v = u^{\frac{\log\beta}{\log\delta}}$ for $u \in \mathscr{U}$. Obviously this is a restriction imposed by working on the prize-time domain and we can no longer consider δ as a measure of impatience. To put things in perspective, in a seminal paper Koopmans [1972] instead considers the richer domain of consumption streams, and under the additional assumptions of separability and stationarity, he derives a time-separable additive exponential discounting representation of behavior. In Theorem 8 we provide a representation over a richer domain where the discount factor $\delta \in (0,1)$ is unique.
Next, we show that the set of functions in the representation in (1.1) is unique up to its convex closure. Define

$$\mathscr{F} = \{u : [0,M] \to \mathbb{R}_+ : u(0) = 0,\ u \text{ is strictly increasing and continuous}\}$$

Define the topology of compact convergence on the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Also, let $co(A)$ and $\bar{A}$ define the convex hull and closure of the set A (with respect to the defined topology), and $\bar{co}(A)$ define the convex closure of the set A.

Proposition 12. *If $\mathscr{U}, \mathscr{U}' \subset \mathscr{F}$ are such that $\bar{co}(\mathscr{U}) = \bar{co}(\mathscr{U}')$, and the functional form in (1.1) allows for a continuous minimum representation for both of those sets, then,* $\min_{u\in\mathscr{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathscr{U}'} u^{-1}(\delta^t u(x))$.

Proof. See Appendix II. □

Proposition 13. *i) If there exists a concave function $f \in \mathscr{U}$, and if $\mathscr{U}_1$ is the subset of convex functions in $\mathscr{U}$, then* $\min_{u\in\mathscr{U}}(u^{-1}(\delta^t u(x))) = \min_{u\in\mathscr{U}\setminus\mathscr{U}_1}(u^{-1}(\delta^t u(x)))$.

ii) If $u_1, u_2 \in \mathscr{U}$ and u_1 is concave relative to u_2, then, $\min_{u\in\mathscr{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathscr{U}\setminus\{u_2\}} u^{-1}(\delta^t u(x))$.

Proof. See Appendix II. □

1.7 Related literature

This paper is closely linked to the literature that explores the conditions under which a "rational" person may have present-biased preferences. Sozou [1998], Dasgupta and Maskin [2005] and Halevy [2008] explain particular uncertainty conditions that can give rise to present-biased behavior. While telling an uncertainty story *sufficient* to explain present bias, all these models explicitly assume the particular structure of risk or uncertainty with relevant risk attitude, and these assumptions are central to establishing behavior consistent with present bias in the respective models. In this paper we deviate from this norm: we do not explicitly assume any uncertainty framework or uncertainty attitude. But we still obtain a subjective state space representation that *is necessary and sufficient* for present bias. The set of future tastes $\mathscr{U}$ can be considered to be the subjective state-space, and the decision maker considers the most conservative state dependent utility $\min_{u\in\mathscr{U}} u^{-1}(\delta^t u(x))$ to evaluate each timed alternative.

Our representation looks similar to the max-min expected utility representation of Gilboa and Schmeidler [1989] used in the uncertainty or ambiguity aversion literature, though there is no objective state space or uncertainty defined in our setup. We have already discussed the connection of our paper with Cerreia-Vioglio et al. [2015] in terms of the similarity in representation. There are other variants of the minimum or infimum functional in previous literature, for example, Cerreia-Vioglio [2009] and Maccheroni [2002], used in different contexts.

There is also a sizable literature on the behavioral characterizations of temporal preferences, that the current project adds to. Olea and Strzalecki [2014], Hayashi [2003] and Pan et al. [2015] characterize the behavioral conditions necessary and sufficient for β-δ discounting, Loewenstein and Prelec [1992] characterize Hyperbolic discounting, and, Koopmans [1972], Fishburn and Rubinstein [1982] do the same for exponential discounting. Gul and Pesendorfer [2001] study a two-period model where individuals have preferences over sets of alternatives that represent second-period choices. Their axioms provide a representation that identifies the decision maker's commitment ranking, temptation ranking and cost of self-control.

1.8 Properties of the representation

We propose an alternative notion of "present premium" comparison below. The present premium can be considered as the maximal amount of future consumption one is willing to forego to have the residual moved to the present. For example, if $(y,t) \sim (x,0)$, then the present premium of (y,t) is $(y-x) \geq 0$.

Consider the following partial relation defined on the set of binary relations $\succsim$ over $\mathbb{X} \times \mathbb{T}$.

Definition 14. $\succsim_1$ allows a higher premium to the present than $\succsim_2$ if for all $x, y \in \mathbb{X}$ and $t \in \mathbb{T}$

$$(x,t) \succsim_1 (y,0) \quad \Longrightarrow \quad (x,t) \succsim_2 (y,0)$$

The next result connects this notion of comparative present premia to our representation.

Theorem 15. *Let $\succsim_1$ and $\succsim_2$ be two binary relations which allow for minimum representation w.r.t sets $\mathscr{U}_{\delta,1}$ and $\mathscr{U}_{\delta,2}$ respectively. The following two statements are equivalent:*

i) $\succsim_1$ allows a higher premium to the present than $\succsim_2$.

ii) Both $\mathscr{U}_{\delta,1}$ and $\mathscr{U}_{\delta,1} \cup \mathscr{U}_{\delta,2}$ provide minimum representations of $\succsim_1$.

Proof. See Appendix II. □

One might wonder if there could also be a representation theorem similar to Theorem 3 for an appropriately defined Weak Future Bias axiom. Below we define Weak Future Bias, and provide a corresponding representation.

A4*: **WEAK FUTURE BIAS:** If $(x,0) \succsim (y,t)$ then, $(x,t_1) \succsim (y,t+t_1)$ for all $x,y \in X$ and $t,t_1 \in \mathbb{T}$.
This is an alternative relaxation of Stationarity that is complementary to WPB. Weak Present Bias, when combined with Weak Future Bias yields the Stationarity Axiom. We now present the following result.

Theorem 16. *Let $\mathbb{T} = [0,\infty)$ and $\mathbb{X} = [0,M]$. The following two statements are equivalent:*

i) The relation $\succsim$ satisfies properties A0-A3 and A4.*

ii) There exists a set $\mathscr{U}_\delta$ of monotonically increasing continuous functions such that

$$F(x,t) = \max_{u \in \mathscr{U}} u^{-1}(\delta^t u(x))$$

represents the binary relation $\succsim$. The set $\mathscr{U}_\delta$ has the following properties: $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathscr{U}_\delta$. $F(x,t)$ is continuous.

As expected Weak Future Bias is characterized by a weakly optimistic attitude towards the future. The proof is similar to that of Theorem 3, and is hence omitted.

Testable Implications

The major testable condition in the paper comes from the Weak Present Bias axiom: If $(y,t) \succsim (x,0)$ then, $(y,t+t_1) \succsim (x,t_1)$ for all $x,y \in X$ and $t,t_1 \in \mathbb{T}$. Stated in terms of the contra-positive, If $(x,t_1) \succ (y,t+t_1)$ for some $x,y \in X$ and $t,t_1 \in \mathbb{T}$, $t,t_1 > 0$, then, $(x,0) \succ (y,t)$. Intuitively speaking, this model only allows preference reversals that arise from present bias (as restricted by the Weak Present Bias axiom). So any temporal preference that stems from any other behavioral phenomenon would refute the model.

1.9 Stake dependent Present Bias

Consider a decision maker who makes the following 2 pairs of choices.

Example 17.

\$100 today	$\succ$	\$110 in a week
\$110 in 5 weeks	$\succ$	\$100 in 4 weeks
\$11 in a week	$\sim$	\$10 today
\$11 in 5 weeks	$\sim$	\$10 in 4 weeks

Both pairs of choices are consistent with Weak Present Bias, but there is a classical choice reversal (or a local violation of Stationarity) only in the first pair.[18] This kind of choice is at odds with all the models of present bias that we have mentioned other than the one in this paper, but not necessarily at odds with economic intuition. For example, if a DM's present bias is driven by the psychological fear of future uncertainty, the higher the stake, the higher would be the manifestation of this fear, and the more present-biased he would appear. The opposite phenomenon,

[18] This kind of behavior closely parallels the "magnitude effect": in studies that vary the outcome sizes, subjects appear to exhibit greater patience toward larger rewards. For example, Thaler [1981] finds that respondents were on average indifferent between \$15 now and \$60 in a year, \$250 now and \$350 in a year, and \$3000 now and \$4000 in a year, suggesting a (yearly) discount factor of 0.25, 0.71 and 0.75 respectively.

when a subject appears strictly present-biased for smaller stakes but appears stationary at larger stakes (for the same set of temporal values) can happen, if the subjects get better at temporal decisions at higher stakes due to cognitive optimization. None of the models in Appendix I can account for the behavior in Example 17,[19] whereas, the simple minimum function mentioned in Example 4 can account for such choices. There is scope to run future experiments to test for such stake dependent behavior. The closest precedent for such an experimental design appears in Halevy [2015] where the author finds evidence of stake dependent present bias.

1.10 Application to a timing game

In this section we are going to study dynamic decision-making games for a present-biased DM whose preferences are consistent with the time-risk relations outlined in Keren and Roelofsma [1995]. Present-biased preferences, when extended to a dynamic context[20], create time inconsistent preferences, which in turn results in inefficient decision making and loss in long-term welfare. The goal of this section is to convince the reader about the importance of axiomatization of risk-time relations, by showing that risk-time relations have important welfare implications for such a present-biased individual.

Consider the following game adopted from O'Donoghue and Rabin [1999]. Suppose a DM gets a coupon to watch a free movie, over the next four Saturdays. He has to redeem the coupon an hour before the movie starts. His free ticket is issued subject to availability of tickets, and if there are no available tickets, the coupon is wasted. Hence there is some risk while redeeming the coupon. The movies on offer are of increasing quality- the theater is showing a mediocre movie this week, a good movie next week, a great movie in two weeks and Forrest Gump in three weeks. Our DM perceives the quality of these movies as 30, 40, 60 and 90 on a scale of $0-100$. In our problem, the DM can make a decision maximum 4 times, at $\tau = 1,2,3,4$ (measured in weeks). The DM's utility at calendar time τ from watching a movie of quality x with probability p at calendar time $t+\tau$(in weeks) is given by:

$$U^{\tau}(x,p,\tau+t) = \begin{cases} p^{100}\alpha^{t}x & \text{for } p^{100}\alpha^{t} \geq \alpha^{\frac{1}{2}} \\ \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} p\beta^{t}x & \text{for } p^{100}\alpha^{t} < \alpha^{\frac{1}{2}} \end{cases}$$

[19] For example, if one tries to fit a β-δ model to this data, the second pair of choices immediately suggest $\beta = 1$, which in turn is inconsistent with the first pair of choices.

[20] We are imposing Time Invariance of preferences following Halevy [2015]. We will make precise assumptions about sophitication/ naivete as we go.

Where, $\beta = .99$, $\alpha = (.99)^{100} \approx .36$. This utility function (which is inspired by Pan et al. [2015]'s Two Stage Exponential discounting model) has the following interpretation: The DM has a long run weekly discount factor of .99 that sets in after a delay of half a week for $p = 1$. Before reaching the cut-off, the DM is extremely impatient, with a smaller discount factor of $\alpha = \beta^{100} \approx .36$, and hence is biased towards the present and very short-run outcomes. Similarly, the DM also proportionally undervalues probabilities close to 1. The utility function(s) U^τ define a preference that satisfies all the axioms in Section 1.3, and hence have a minimum representation. The DM is time-inconsistent, as his preferences between future options differ between any two decision periods τ_1 and τ_2 for $\tau_1, \tau_2 \in \{1,2,3,4\}$. Let us assume that the DM is aware of his future preferences, that is she is sophisticated, a notion pioneered by Pollak [1968]. We are going to use the following notion of equilibrium for this game.

Definition 18. (O'Donoghue and Rabin [1999]) A Perception Perfect Strategy for *sophisticates* is a strategy $s^s = (s^s_1, s^s_2, s^s_3, s^s_4)$, such that such that for all $t < 4$, $s^s_t = Y$ if and only if $U^t(t) \geq U^t(\tau')$ where $\tau' = \min_{\tau > t}\{s^s_\tau = Y\}$.

In any period, sophisticates correctly calculate when their future selves would redeem the coupon if they wait now. They then decide on redeeming the coupon if and only if doing it now is preferred to letting their future selves do it. We consider two cases:

Case 1: Suppose, there is not much demand for movie tickets in that city, and the DM knows that he can always book a ticket through his coupon and $p = 1$ for all alternatives under consideration.

In this case, the unique Perception Perfect Strategy is $s^s = (Y,Y,Y,Y)$. The knowledge that the future selves are going to be present biased creates an unwinding effect: The period 2 sophisticate would choose to use the coupon towards the good movie as he knows that the period 3 sophisticate would end up using the coupon for the great movie instead of going for Forrest Gump due to present bias. The period 1 sophisticate in turns correctly understands that waiting now would only result in watching the good movie and hence decides to see the mediocre movie right now instead.

Case 2: Suppose, due to persistent demand for movie tickets in that city, and the DM knows that redeeming a coupon results in a movie ticket in only 99% of cases.

The unique Perception Perfect Strategy is $s^s = (N,N,N,Y)$. The unwinding from the previous case does not happen here due to the risk involved in redeeming the coupon. Once the present is risky (equivalent to having a front end delay due to Probability Time Tradeoff), the bias previously assigned to the present vanishes,

		t				s^s_τ
		1	2	3	4	
τ	4				90	Y
	3			60	54.2	Y
	2		40	36.1	53.6	Y
	1	30	24	35.8	53	Y

		t				s^s_τ
		1	2	3	4	
τ	4				54.2	Y
	3			36.1	53.6	N
	2		24	35.8	53	N
	1	18	24	35.8	52.57	N

Table 1.2: Utilities of different selves under Case 1 (Left) and Case 2 (Right)

stopping the unraveling. The DM waits until the final period to cash in his coupon when the expected returns are the highest to the long run self.

The Left pane of Table 1.2 is for Case 1 ($p = 1$), the right table is for Case 2 ($p = .9$). The entries in the table provide $U^\tau(x, p, t)$, and explain the equilibria. The sophisticated DM compares the quantities in red row-wise for each τ when making a decision.

It would be instructive to compare the two cases in terms of welfare implications. Since present-biased preferences are often used to model self-control problems rooted in the pursuit of immediate gratification, we would compare welfare from the long run perspective. This outcome in Case 1 is consistent with the following general result in O'Donoghue and Rabin [1999]: When benefits are immediate, the sophisticates "preprorate", i.e, they do it earlier than it might be optimal. For example, considering the long term self's interests, given a long term weekly discount factor of .99 for movie quality, the equilibrium outcome of watching the mediocre movie (quality of 30) in the first week, instead of Forrest Gump (quality of 90) definitely results in sub-optimal welfare in Case 1. For example, considering the choices from a $\tau = 0$ self gives $U^0(30, 1, 1) = 18$, and $U^0(90, 1, 4) = 53$. On the other hand, the introduction of a small amount of risk in Case 2, stops the unraveling in terms of "preprorating" (preponing consumption), thus helping the DM attain the most efficient outcome in equilibrium, thus reversing the O'Donoghue and Rabin [1999] result. In fact, not only is the highest level of available welfare achieved in Case 2 after the introduction of risk, *the equilibrium welfare improves from Case 1 to Case 2 in the absolute sense, even though apriori Case 2 seems to be worse than Case 1 for the DM*!

$$U^0(30, 1, 1) = 18 \quad < \quad U^0(90, .99, 4) = 52$$

This is an interesting application of how introducing a dominated menu of choices can result in absolute welfare improvement.

What would happen if the DM had the same preferences $U^\tau()$, but, instead was unaware that his preferences were dynamically inconsistent? Let us consider the extreme case (popularly called “naïveté” in the literature) where the DM thinks that his future selves’ preferences would be identical to his current selves’. We will call such a DM naive, and use the following equilibrium notion to characterize their behavior.

Definition 19. A Perception Perfect Strategy for naifs is a strategy $s^n = (s^n_1, s^n_2, s^n_3, s^n_4)$, such that such that for all $t < 4$, $s^n_t = Y$ if and only if $U^t(t) \geq U^t(\tau)$ for all $\tau > t$.

The naive DM, acting under his false belief of time consistency, redeems the coupon in the current period if and only if it yields him the highest payoff among the remaining periods. Table 1.2 tells us that in Case 1, $s^n = (N,N,Y,Y)$, and in Case 2, $s^n = (N,N,N,Y)$. Thus the introduction of risk in this example also helps a naive DM make the most efficient choice in equilibrium.

1.11 Choice over timed bads

Most of the discussion on Present Bias till now has been centered around timed prizes or consumption, in general objects which are desirable. The central result of this paper is that Present Bias (as defined in A4 in Section 1.1) over such outcomes, can be represented by a minimum representation. This section would provide us the answers to the following two natural follow-up questions: 1) What would Present Bias look like when timed undesireable-goods or bads (for example, effort) are concerned? 2) What would be a utility representation of such preferences?

We would consider the richer domain that includes risk, without loss of generality. The DM has preferences over triplets (x, p, t), which describe the prospect of receiving an undesirable good $x \in \mathbb{X}$ at time $t \in \mathbb{T}$ with a probability$p \in [0, 1]$. We impose the following conditions on behavior.

C0: $\succsim$ is complete and transitive.

C1: CONTINUITY: $\succsim$ is continuous, that is the strict upper and lower contour sets of each timed alternative are open w.r.t the product topology.

The first two axioms are identical to axioms B0 and B1 used in Section 1.3.

C2: DISCOUNTING: For $t, s \in \mathbb{T}$, if $s > t$ then $(x, p, s) \succ (x, p, t)$ for $x, p > 0$ and $(x, p, s) \sim (x, p, t)$ for $x = 0$ or $p = 0$. For $x > y > 0$, there exists $T \in \mathbb{T}$ such that, $(x, q, 0) \succsim (y, 1, T)$.

C3: PRIZE AND RISK MONOTONICITY: For all $t \in \mathbb{T}$, $(x,p,t) \succsim (y,q,t)$ if $y \geq x$ and $q \geq p$. The first binary relation is strict if at least one of the 2 following relations are strict and if $y, q > 0$.

Discounting and Monotonicity have been adapted in the most intuitive way. People want to delay bad outcomes and they prefer when bad outcomes are less likely. Also when bad outcomes are concerned, more is worse.

C4: WEAK PRESENT BIAS: If $(x,1,0) \succsim (y,1,t)$ then, $(x,1,t_1) \succsim (y,1,t+t_1)$ for all $x, y \in X$ and $t, t_1 \in \mathbb{T}$.

The Weak Present Certainty Bias requires that given the present and certainty are special, a DM would try to avoid bad outcomes which are in the present and are certain. Moreover, loss of certainty or immediacy can only make bad outcomes better.

C5: PROBABILITY-TIME TRADEOFF: For all $x, y \in \mathbb{X}$, $p \in (0,1]$, and $t, s \in \mathbb{T}$, $(x, p\theta, t) \succsim (x,p,t+D) \implies (y, q\theta, s) \succsim (y,q,s+D)$.

The Probability-Time tradeoff axiom is unchanged and has the same interpretation as before.

Theorem 20. *The following two statements are equivalent:*

i) The relation $\succsim$ *on* $\mathbb{X} \times \mathbb{P} \times \mathbb{T}$ *satisfies properties C0-C5.*

ii) There exists a unique $\delta \in (0,1)$ *and a set* $\mathscr{U}$ *of monotinically decreasing continuous functions such that*

$$F(x,p,t) = \max_{u \in \mathscr{U}} -u^{-1}(p\delta^t u(x)) = -\min_{u \in \mathscr{U}} u^{-1}(p\delta^t u(x))$$

represents the relation $\succsim$*. For all the functions* $u \in \mathscr{U}$*,* $u(M) = -1$ *and* $u(0) = 0$*. Moreover, F(x,p,t) is continous.*

Conclusion

This paper provides an intuitive behavioral definition of (Weak) Present Bias and characterizes a general class of utility functions consistent with such behavior. Our utility representation can be interpreted as if a DM is unsure about future tastes and present bias arises as an outcome of his cautious behavior in the face of uncertainty about future tastes. Given most of the previous models of present bias have extraneous behavioral assumptions over and above present bias which are often empirically unsupported, we believe that our representation theorem is an important theoretical development in this literature. Having a more general representation for present bias, also helps us accommodate empirical phenomenon (for example, stake dependent present biased behavior) that previous models could not account for. We have extended the model to incorporate time-risk relations in behavior and provided an example where this relation can be utilized for welfare improving policy design. Given the axiomatic nature of our work, we provide simple testable conditions necessary and sufficient for our utility representations. These conditions can be easily taken to the laboratory or field to be empirically tested. We hope that this paper generates further interest in theoretical and applied work directed towards forming a better understanding of intertemporal preferences.

Chapter 2

External and Internal Consistency of Choices made in Convex Time Budgets

2.1 Introduction

Andreoni and Sprenger (2012, henceforth AS) introduce Convex Time Budgets (CTB) to experimentally measure intertemporal substitution. In their design the subject faces linear experimental budgets, which allow her to choose interior allocations between payments at two time periods (henceforth c_t, c_{t+k}). One can rationalize such interior allocations if the subject's preferences between c_t and c_{t+k} are (weakly) convex. It thus provides a way to adjust the measurement of subjective discount rates for intertemporal substitution.

There are some basic properties that allocations in the Andreoni and Sprenger design should satisfy in order to be rationalizable by a very general model of intertemporal choice: allocations should satisfy wealth monotonicity (normality);[21] c_t should be weakly decreasing in interest rate (demand monotonicity);[22] allocations should be consistent with impatience.[23,24]

The AS design includes nine choicesets per subject, where each choiceset is a collection of five CTB tasks between payments at t and at $t+k$ (where $t = 0, 7, 35$ and $k = 35, 70, 98$ measured in days). Eight out of the nine choicesets contain a wealth shift which could be used to test for wealth monotonicity. Demand monotonicity is tested by the other four CTB tasks within a choiceset. Impatience is

[21] c_t and c_{t+k} should be weakly increasing in wealth, holding interest rate constant.

[22] c_t is a weakly decreasing function of the interest rate, holding the dates t and $t+k$ and wealth normalized to the later date constant.

[23] As the later (earlier) date is shifted away from the present, c_t should weakly increase (decrease), holding the earlier (later) date, price ratio and wealth constant.

[24] The various monotonicity criteria for which we evaluate the empirical demand should not be confused with monotonicity of the utility function with respect to (c_t, c_{t+k}). In particular, wealth and demand monotonicity are consequences of the very weak assumption that c_t and c_{t+k} are normal goods.

tested by comparing across choicesets belonging to the same subject.[25]

AS included three choice lists (MPLs) that correspond to three choicesets. Each one of these choice lists included four pairwise choices that corresponded to CTBs. In other words, on these lines of the choice list a subject was asked to make a pairwise choice between the two points in which each CTB intersects the horizontal axis ($c_{t+k} = 0$) and the vertical axis ($c_t = 0$). In the CTB task the menu of allocations the subject was allowed to choose from included these two allocations *and* all interior allocations. We use this set-up to test for violations of the Weak Axiom of Revealed Preference (WARP), which requires that if an alternative is chosen from a menu and is available in a sub-menu then it should be chosen from the sub-menu as well. If in the pairwise choice a subject chooses one corner while in the CTB she chooses the opposite corner this contradicts WARP. The implication is that there exist no complete and transitive preference that can rationalize these choices.

In this study we document the level of adherence of choices (at the individual level) to the above very mild external and internal consistency requirements. We find a very high level of WARP violations among the many subjects who made corner choices. Violations for all three internal measures of monotonicity are concentrated in subjects who make interior choices and thereby take advantage of the novel feature of Andreoni and Sprenger's experimental design. Wealth monotonicity violations are more prevalent and pronounced than either demand or impatience monotonicity violations.

We believe that the findings reported here make it very challenging for one to claim that choices made in CTB experiments reflect on deep and stable preferences. We urge researchers to study the source of the documented problematic behavior in order to decide if it is inherent to CTB or could be attributed to the implementation of CTB in AS (2012).

[25]When evaluating wealth monotonicity we allow for the non-generic possibility of linear preferences with marginal rate of substitution between c_t and c_{t+k} equal to the gross interest rate over k days in which the wealth shift occurs, i.e. $1+r = 1.25$. In this case, the demand is a correspondence and wealth monotonicity as defined above need not hold (we thank Jim Andreoni and Charlie Sprenger for bringing up this possibility). However, to be consistent with this knife edge case, subjects need to satisfy: (1) $c_t^* = 0$ for all $r > 0.25$ and $c_{t+k}^* = 0$ for all $r < 0.25$. (2) In every choiceset (t,k') such that $k' < k$: $c_t^* = 0$ for all $r \geq 0.25$. (3) In every choice set (t,k') such that $k' > k$: $c_{t+k'}^* = 0$ for all $r \leq 0.25$. (1) follows from linearity and (2-3) follow since the daily rate changes as k varies.

2.2 Quantitative evaluation

2.3 Corner choices

Although the CTB design allowed for interior choices, 70% of all choices were made at the corners of the budget set. 36 of the 97 subjects made *only* corner choices. There is little within subject variation and between subject heterogeneity among these subjects. Nineteen of these subjects had the exact same choice sequence for all tasks: they chose the later-larger reward whenever the "gross interest rate" was greater than 1. Four other subjects chose the later-larger reward for all 45 CTB tasks, irrespective of interest rate and time horizon.

2.4 WARP violations

Out of the 36 subjects who made all corner choices in CTB, we found 43 violations of WARP.[26] This is especially impressive if one considers that 17 of them always chose later consumption in the CTB and switched immediately in the choice lists (always chose later consumption). Therefore WARP violations could be detected only among the remaining 19 subjects. The direction of WARP violations is not random: 34 violations are in the direction of exhibiting less impatience in CTB than in choice list, while only 9 are in the opposite direction.

Since these subjects did not exhibit any curvature in their CTB choices, we can directly estimate their discount factor based on the 3 choice lists and the corresponding CTBs. One should not adjust for curvature for these subjects, since their intertemporal decisions did not suggest any concavity of the felicity function. The results are plotted in the attached Figure 2.1.[27] We find that for 11 of them the discount factor estimated from CTB data would be higher than the one estimated from choice list data, while for two subjects the relation between the discount factors would be in the opposite direction. Note that the choices made by the 17 subjects who always chose later consumption can be rationalized with a discount factor of 1, and one cannot form a point estimate of the discount factors of 4 other subjects who chose always immediate consumption in at least one of the three CTBs.[28]

[26] The discussion in this subsection ignores indifferences since we believe that the evidence is systematic and cannot be accounted for by the knife-edge arguments of linear preferences.

[27] AS' Figure 4A is similar, but we restrict to subjects who made only corner choices and therefore there is no need to adjust for concavity

[28] If one estimates a quasi-hyperbolic model based on these three CTBs or CLs, the conclusions do not change. In particular, the present-bias parameter (beta) under both elicitation methods is exactly 1 for 28 out of the 32 subjects.

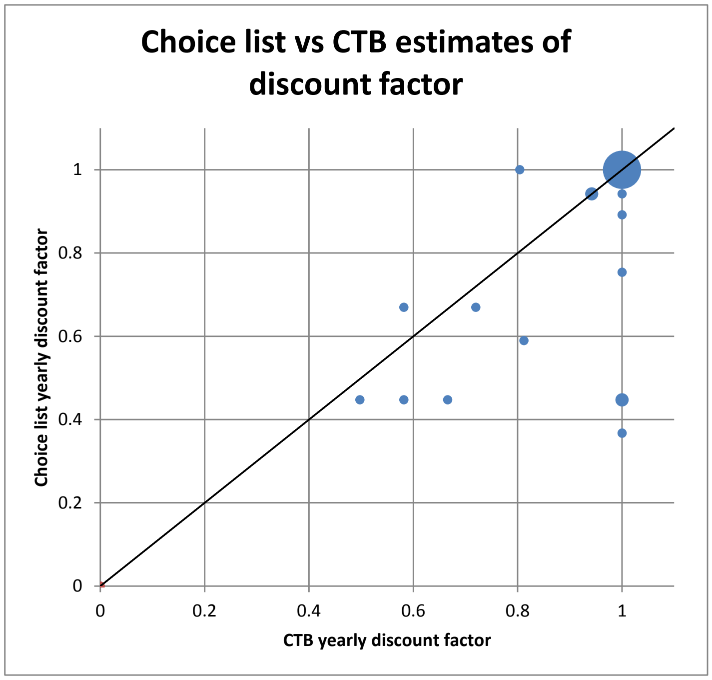


Figure 2.1: Choice list vs CTB estimates of discount factor for the 36 all-corner subjects

Among the other 61 subjects who made at least a single interior choice in the 45 CTBs tasks we find similar directional effect of WARP violations. If one of the three choicesets that has a comparable choice list has all corner choices, we find 23 WARP violations in the direction of exhibiting lower impatience in CTB than in choice list and none in the opposite direction. In choicesets with interior CTB choices (where the potential to observe direct WARP violation is smaller) we found 10 violations in the direction of exhibiting lower impatience in CTB than in choice list and 5 in the opposite direction.

One interpretation to WARP violation (following Ok, Ortoleva and Riella, 2015) is that CTB generates some reference dependence; alternatively it is possible that many subjects become really confused when presented with CTBs. In any case, the fact that a larger menu changes optimal choice systematically cannot be reconciled with a standard model of choice rationalized by a complete and transitive preferences, as the discounted utility model.

2.5 Demand and wealth monotonicity

As the 36 subjects with all corner choices did not take advantage of the convexification offered by the CTB, we believe it would be misleading to include them in evaluating CTB for internal consistency (monotonicity). Hence, the analysis below concentrates on the 61 subjects with at least one interior choice.

2.6 Frequency

Table 2.1 reports the frequency of choicesets that have wealth or demand monotonicity violations as a function of the number of interior choices made in a choiceset.

The frequency of demand monotonicity violations is below 10% for choicesets that contain 4 or fewer interior choices. However, more than 36% of choicesets with all interior choices have demand monotonicity violations. The frequency of wealth monotonicity violations is considerably higher: around half of the choicesets with at least one interior choice have a wealth monotonicity violation.

Table 2.2 reports, for the 61 subjects with at least one interior choiceset, the distribution of subjects satisfying wealth and demand monotonicity as a function of the number of interior choicesets. A choiceset is considered interior if at least a single choice (out of five) is not at the corners of the budget line ($c_t, c_{t+k} > 0$).[29] Table 2.2 reveals that more than half of the 61 subjects violate monotonicity in at least half of their interior choicesets (the shaded entries in the table).

2.7 Magnitude

The two tables above demonstrate the high frequency of non-monotone choices in interior choicesets, especially as a response to wealth changes. We now turn to measure the magnitude of these behaviors. We calculate the magnitude of a wealth monotonicity violation by the number of tokens required to be reallocated to eliminate the violation at the higher wealth level. Our wealth monotonicity measure differs substantially from that reported in footnote 25 by Andreoni and Sprenger for four reasons (presented in decreasing order of importance).

[29] Table 2.2 excludes subjects who made all corner solutions. Among the 36 subjects who made only corner choices, we find only one non-monotonic choiceset.

# of interior choices in a choiceset	# of choice-sets	# of choicesets that exhibit demand monotonicity violations	# of choicesets that exhibit wealth monotonicity violations	# of choicesets that exhibit either wealth or demand monotonicity violations
0	435*	1	9	10
1	101	10	26	34
2	78	5	31	34
3	80	6	47	48
4	63	6	47	47
5	116	42	56	76
Total	873	70	216	249

*324 out of the 435 choicesets with no interior choice (almost 75%) belong to the 36 subjects with only corner solutions.

Table 2.1: Demand and wealth monotonicity violations as a function of number of interior choices

# of interior* choicesets	# of monotone interior* choicesets										
	0	1	2	3	4	5	6	7	8	9	Total
1	0	2									2
2	1	0	0								1
3	2	0	0	2							4
4	0	2	0	0	1						3
5	1	1	0	0	2	1					5
6	0	2	0	0	0	1	1				4
7	0	0	1	1	0	0	0	2			4
8	0	3	2	0	0	2	1	1	0		9
9	1	8	5	4	2	0	2	2	0	5	29
Total	5	18	8	7	5	4	4	5	0	5	61

*A choiceset is considered "interior" if at least a single choice (out of 5) is not at the corners of the budget line.

Table 2.2: Joint frequency of number of interior choicesets (by subjects) and number of interior choicesets that do not violate (demand and wealth) monotonicity (by subject), restricted to subjects who have at least one interior choiceset.

First, when calculating non-monotonicity, AS mistakenly defined non-monotonicity that is expressed as an over-allocation to c_{t+k} (and under-allocation to c_t) as a negative number, while non-monotonicity that is expressed as an under-allocation to c_{t+k} (and over-allocation to c_t) as a positive number. Averaging these two wealth monotonicity violations cancels out at the aggregate level. For example, if choices are generated at random using a uniform distribution over the token allocated to c_t, independently among the two budget lines, the AS measure of wealth monotonicity violation would equal zero in expectation. The expected value of our measure would be approximately 27 tokens (out of 100 tokens).[30]

Second, when calculating the average adjustment required to restore monotonicity, AS use a denominator that includes all choicesets with a wealth shift, rather than just choicesets which have a wealth monotonicity violation. We believe that the AS approach is not advisable chiefly because, by including the 36 subjects who made only corner choices (and had no wealth monotonicity violation), it artificially dilutes the magnitude of monotonicity violations performed by subjects who responded to the convexification offered by the CTB design by making interior choices. Third, we consider the knife edge case of linear preferences with marginal rate of substitution between c_t and c_{t+k} equals the gross interest rate in which the wealth comparative statics is performed as discussed in footnote 25. This correction applies to 8 choicesets. Fourth, when calculating adjustments AS allow for fractional token adjustments. Given that subjects were only able to allocate integer values of tokens we believe it is more appropriate to calculate the adjustment values using whole tokens.[31]

We find that there are 216 violations of wealth monotonicity, with an average size of 23.2 tokens, which is 23.2% of the experimental budget or \$4.64 of c_t at the higher wealth level. That is, conditional on violating wealth monotonicity, the magnitude of the measure is almost as high as the equivalent measure calculated for random choice. Andreoni and Sprenger report an average adjustment of just 1.67 tokens to restore wealth monotonicity. It is important to note that the canceling out occurs mainly at the population level rather than the individual level. Allowing for only individual-level canceling out reduces our measure to only 17.9 tokens.

We calculate the magnitude of demand monotonicity violations by finding the minimal amount of c_t that needs to be reallocated per choiceset to restore monotonicity. There are 70 choicesets with demand monotonicity violations, with an average size of 17.4 tokens and a value (at time t) of \$3.02.[32]

[30] Note that the probability that a pair of choices violate wealth monotonicity in this case is 80%.

[31] Indeed, AS use integer number of tokens when calculating the magnitude of demand monotonicity violations.

[32] AS report 8 demand monotonicity violations for the ($t = 7, k = 70$) choiceset with an average magnitude of 24.6 tokens; in comparison, we find only 7 violations with an average magnitude of

Another measure of the degree of non-monotonicity within a choiceset is to calculate the smallest number of choices that must be removed from a choiceset to restore monotonicity.[33] For the 249 choicesets that exhibit at least one non-monotonicity, the average number of data points that must be removed is 1.2. This figure includes the 179 choicesets that exhibit only wealth non-monotonicity and therefore require the removal of only a single data point; for the 70 choicesets that exhibit demand non-monotonicity the average number of data points that must be removed is 1.8.

2.8 Impatience monotonicity

Turning to impatience, there are 10 pairs of choicesets across which either t is constant and k varies, or $t+k$ is constant and t varies; these are the only pairs of choicesets in which it is possible to test for impatience. In a comparable pair of choicesets (in the sense described above), we test for impatience monotonicity as described in footnote 23 for all pairs of choice tasks (one in each choiceset) with the same prices.

We find that 47 of the 97 subjects satisfy the impatience criterion for all 10 pairs of choicesets; restricting the sample to the 61 subjects with at least one interior choice, we find that only 12 subjects made choices consistent with impatience monotonicity, and that 17 subjects violate impatience monotonicity in at least 5 of the 10 choiceset comparisons.

2.9 Monotonicity index

Finally, we calculate an index that measures the (approximate) minimal number of data points that need to be eliminated from an individual's dataset in order to be consistent with the three monotonicity requirements.[34] Out of the 36 subjects with no interior choice, 35 subjects satisfy all monotonicity measures.[35] Out of the 61

23.4 tokens in this choiceset. AS appear to have erroneously included additional adjustments for c_{t+k}, and correcting for this reduces both the number and magnitude of demand monotonicity violations.

[33]When removing data points to restore monotonicity we also consider joint violations of demand and wealth monotonicity.

[34]This index is close in spirit to the Houtman-Maks (1985) index which is used to calculate the maximal set of observations in a dataset that is consistent with the Generalized Axiom of Revealed Preference (GARP). Because the AS (2012) design has no power to detect violations of GARP, any choices made in a choiceset can be rationalized by a utility function, and by Afriat's theorem the utility function can be chosen to be increasing in (c_t, c_{t+k}). This, however, should not be confused with wealth monotonicity, which is a property of the demand function.

[35]For the other subject, one needs to remove a single choice.

subjects with at least a single interior choice, in 22 datasets we need to remove four or fewer choices,[36] in 21 datasets we need to remove between five to nine choices (more than 10% of choices) and in an additional 18 datasets one needs to remove 10 or more choices (more than 20% of the total number of choices).

2.10 Conclusion

Andreoni and Sprenger's proposal to use CTB in order to measure time preferences represents a potentially important methodological advance. In principle, assuming discounted expected utility, such a method can allow a researcher to calculate a more precise measurement of the discount function by controlling for intertemporal substitution. However, our examination of data gathered by Andreoni and Sprenger (2012) using this method uncovers serious problems.

Subjects who made only corner choices in CTB violate WARP very frequently relative to the pairwise choice benchmark. This fact suggests that corner choices in CTB cannot be interpreted as reflecting reasoned behavior or deep preferences, but are heavily influenced by confusion or some reference introduced by the CTB. As a whole, the bias of WARP violations relative to the pairwise choice benchmark is in the direction of lower impatience (higher discount factor). This explains why AS do not require the concavity adjustment used in other studies in order to estimate similar discount factors.

Subjects with interior choices are broadly consistent with demand monotonicity (except when all choices are interior) and the evidence for impatience monotonicity violations is moderate. However, the high frequency and substantial magnitude of wealth monotonicity violations in this data suggest that interior choices made in CTB (responding to the convexification) may not reflect reasoned behavior and stable preferences as well.

Unfortunately, the data does not permit us to identify the source of these severe problems. It could be systematic to CTB or a result of AS' experimental interface. We believe that further investigation into the origin of the serious problems documented in the present study is crucial for an informed interpretation of existing and new experiments utilizing CTB.

[36]Only 9 of the 61 subjects made choices fully consistent with monotonicity.

Chapter 3

Allais meets Strotz: Remarks on the relation between Present Bias and the Certainty Effect

3.1 Introduction

Almost all decisions involve outcomes that are uncertain, realized at different points in time, or both. For example, following a strict and often unpleasant diet program requires some motivation about future gains accruing from it, which are quite often uncertain. There has been persistent interest in the fields of Psychology and Economics to understand how behaviors across risky and temporal domains might be related to each other. The standard approach of modeling intertemporal preferences is through the use of geometric (constant, exponential) discounting in which the payoff of a stream of consumption is aggregated through a (delay-geometric) weighting that results in a present discounted value. This is mirrored in the risk domain, as the canonical model for risk behavior is expected utility which aggregates the utility of each possible alternative by weighting it by its probability. But the similarities do not end here as both models contain similar inadequacies as descriptive models. First, preferences are disproportionately sensitive to certainty (certainty effect) and to the present (present bias/immediacy effect/diminishing impatience). Second, proportional changes in probabilities or equal changes in time delays for timed consumption affect preferences disproportionately (common ratio effect and common difference effect respectively).[37] This two-way relation is well accepted in the Psychological literature [Keren and Roelofsma, 1995, Green and Myerson, 2004, Weber and Chapman, 2005, Chapman and Weber, 2006, to name a few] and there is an understanding that the existence of such mirroring behaviors is not a mere coincidence, but points to a common fundamental property of decision making that manifests itself in different domains of behavior [Prelec and Loewenstein, 1991]. There are many ways in which this relation between risk and

[37] Often times certainty effect and present bias are taken as special cases of common ratio effect and common difference effect, respectively.

temporal behavior can be motivated. Delayed rewards or consumption can be inherently risky, as there might be events between the current date and the promised date which interfere in the process of acquiring the reward/consumption. On the other hand, Rachlin et al. [1986, 2000] suggested that the certain value of probabilistic rewards may be expressed not directly by probabilities but by mean waiting time, and the form of the waiting-time discount function is similar to that used in a model of temporal behavior consistent with present bias. This relation has also been analyzed in more recent works in Economics [Halevy, 2008, Saito, 2011, Baucells and Heukamp, 2012b, Epper and Fehr-Duda, 2012, Saito, 2015]. Given this is a two way relation, none of risk or temporal behaviors have primacy over the other, so any formalization of this relation would necessarily involve the two-way feature discussed above. The goal of this paper is to provide a formal characterization of this relation in the most natural setting. We start by showing how previous attempts at this endeavor have failed to achieve this goal. To be more specific, we show that though the formalization in the direction from certainty effect to diminishing impatience has been correctly posited in the literature, it is the converse relation that still lacks formal rigor. We provide a formal characterization of the two-way relations between i) certainty effect and present bias, and, ii) common ratio effect and the common difference effect. A corollary to our results is that hyperbolic discounting implies the Common Ratio Effect and that quasi-hyperbolic discounting implies the Certainty Effect.

The next section provides a brief acknowledgment to the prior unsuccessful attempts made in this literature to establish risk-time equivalence relations. In Section 3 we suggest an intuitive extension to the existing notion of diminishing impatience, which when used in the analytical framework provided by Halevy [2008], re-establishes the severed connection between non-standard behavior over time and under risk.

3.2 Background

The idea that diminishing impatience (hyperbolic discounting, present bias) may be related to the certainty of the present and the risk associated with future rewards, was formalized by Halevy [2008]. In this model, every consumption path $\mathbf{c} = (c_0, c_1, c_2, \ldots)$ is subject to a constant hazard rate of termination (r). The decision maker chooses among consumption paths as if she calculates present discounted utility for every possible length of the path (periods before stopping). The distribution over present discounted utilities is then evaluated using Rank Dependent Utility (RDU) with probability weighting function $g(\cdot)$, which models preferences that are disproportionately sensitive to certainty. The crucial behavioral axiom accommodates dynamic inconsistency between optimal choices at the present and the immediate future ($t = 1$) only if there is uncertainty concerning consumption in the immediate future, drawing an important qualitative distinction between the effect of randomness in the immediate future and stochastic consumption in later periods ($t = 2, 3, \ldots$).[38] Together with other standard axioms on the DM's preferences over stochastic consumption streams, they are then represented by the utility function:

$$U(\mathbf{c}, r) = \sum_{t=0}^{\infty} g\left((1-r)^t\right) \delta^t u(c_t) \tag{3.1}$$

where δ is a constant pure time preference parameter and $u(\cdot)$ is her felicity function. The decision maker's impatience at time t is then the ratio of her discount function at periods t and $t+1$. Halevy [2008] defines diminishing impatience if the impatience is maximized at $t = 0$, and Theorem 1 in his paper claims equivalence between diminishing impatience and increasing elasticity of $g(\cdot)$. To prove his claim, Halevy [2008] proceeds in two steps. First, diminishing impatience holds if and only if the weighting function satisfies a certain functional inequality.[39] Second, he invokes an equivalence result from Segal [1987, Lemma 4.1] between the above functional inequality and increasing elasticity of $g(\cdot)$. Saito [2011] correctly points out that Segal did not prove that increasing elasticity of the weighting function is necessary for the functional inequality, and provides an example of a DM who exhibits diminishing impatience but her weighting function's elasticity is not increasing (and therefore does not exhibit the common ratio effect).[40] Saito [2011] attempts to establish the sought equivalence between present bias and the certainty effect (Claim 3 in his paper) by retaining the first part of Halevy's argument, and noting that the functional inequality is equivalent (by definition) to the certainty effect for RDU.

We show that diminishing impatience as defined by Halevy [2008] and used by Saito [2011] does *not* imply the certainty effect. In light of this new finding, the equivalence results of Halevy [2008] and Saito [2011] reduce to a one-directional implication from the domain of risk to the domain of time. We provide details in Appendix.

[38] Which is impossible to draw in a framework in which consumption occurs only in a single period.

[39] The functional inequality is a special case of Kahneman and Tversky [1979, pg. 282] subproportionality which characterizes common-ratio violations for RDU. Kahneman and Tversky also state the equivalence claimed later by Segal [1987, Lemma 4.1], which is used in the second part of Halevy's argument.

[40] In particular, Saito [2011] shows that Tversky and Kahneman [1992] weighting function for gains with $\gamma = 0.5$ exhibits diminishing impatience but does not possess increasing elasticity around 0 and does not satisfy the common ratio effect.

3.3 Definitions and Results

3.3.1 The Certainty and Common Ratio Effects

Let (x,p) be a lottery that pays x with probability $0 \leq p \leq 1$ and 0 with probability $1-p$. A DM exhibits *Strict Certainty Effect* if for every $x,y \in \mathbb{R}_+$ and $p,q \in (0,1)$: $(x,1) \sim (y,q) \Rightarrow (x,p) \prec (y,pq)$. She exhibits *Certainty Effect* if for every $x,y \in \mathbb{R}_+$ and $p,q \in (0,1)$: $(x,1) \sim (y,q) \Rightarrow (x,p) \preceq (y,pq)$ and there exist p,q for which the preference is strict. If the DM's preferences are represented by RDU then Strict Certainty Effect is equivalent to the following restriction on the weighting function:[41]

$$g(pq) > g(p)\,g(q) \tag{3.2}$$

A DM exhibits *Strict Common Ratio Effect* if for every $x,y \in \mathbb{R}_+$ and $p,q \in (0,1)$, $\ell \in (0,1]$: $(x,\ell) \sim (y,q\ell) \Rightarrow (x,p\ell) \prec (y,pq\ell)$. She exhibits *Common Ratio Effect* if the implied preference is weak and there exist p,q,ℓ for which the preference is strict. If the DM's preferences are represented by RDU then Strict Common Ratio Effect is equivalent to the following restriction on the weighting function:[42]

$$\frac{g(\ell)}{g(p\ell)} > \frac{g(q\ell)}{g(pq\ell)} \tag{3.3}$$

3.3.2 Diminishing Impatience

We assume that the DM's preferences over stochastic consumption paths satisfy the behavioral axioms in Halevy [2008] and are represented by (3.1). The discount function at period t is: $\Delta(t) = \beta^t g\left((1-r)^t\right)$ and her (one period) impatience at t is $\Delta(t)/\Delta(t+1)$. In Halevy [2008] and Saito [2011], the definition of Diminishing Impatience (DI) is restricted to only one-period delay. It implies that for all natural numbers t: $\Delta(0)/\Delta(1) > \Delta(t)/\Delta(t+1)$ which is satisfied if and only if for every $r \in (0,1)$ and $t \in \mathbb{N}$:[43]

$$g\left((1-r)^{t+1}\right) > g(1-r)\,g\left((1-r)^t\right) \tag{3.4}$$

Both Halevy [2008] and Saito [2011] state without proof that (3.2) holds if and only if (3.4) holds. Although the direction (3.2)→(3.4) is immediate,[44] we provide in Appendix A.2.1 a counter-example which shows that the converse in not

[41] *Certainty Effect* implies weak inequality in (3.2) for every $x,y \in \mathbb{R}_+$ and $p,q \in (0,1)$ and existence of p,q for which (3.2) is satisfied with strict inequality.

[42] *Common Ratio Effect* implies weak inequality in (3.3) and existence of p,q,ℓ for which the inequality in (3.3) is strict.

[43] Note that this is equivalent to writing $g\left(r^{t+1}\right) > g(r)\,g\left(r^t\right)\ \forall r \in (0,1)$ and $t \in \mathbb{N}$.

[44] Define $p := 1-r$ and $q := (1-r)^t$

true in general. In other words, DI as defined above does *not* imply the certainty effect for arbitrary weighting functions. Intuitively, the certainty effect implies a bias towards certainty irrespective of how risky the alternative is, the dual to which would be a bias towards the present ($t = 0$) irrespective of the delay in the compared consumption. In evaluating the reason for the severed connection between time and risk preferences, we note that the definition of diminishing impatience used in the literature focuses on a delay of a single period, thus only comparing $\Delta(t)$ to $\Delta(t+1)$ as t increases from 0. This one-period definition fails to generalize to longer delays, and thus fails to account for present bias behaviorally.[45]

Motivated by the behavioral literature in general, and the quasi-hyperbolic discounting model in particular, which focus on the failure of stationarity independently of the delay under consideration,[46] we suggest to compare $\Delta(t)$ to $\Delta(t+k)$ for arbitrary $k \geq 1$.

Definition 21. The decision maker exhibits *Delay Independent Diminishing Impatience (DIDI)* if $\frac{\Delta(0)}{\Delta(k)} > \frac{\Delta(t)}{\Delta(t+k)}$ $\forall k, t \in \mathbb{N}$, where $\Delta(t)$ is the decision maker's time discounting at period t.

DIDI requires impatience to diminish for all possible delays ($k \geq 1$), hence is a strengthening of the standard definition,[47] which is satisfied by the quasi hyperbolic discounting model (see the Proposition below). For preferences represented by (3.1) DIDI holds if and only if for every $r \in (0,1)$ and $t, k \in \mathbb{N}$: $g\left((1-r)^{t+k}\right) > g\left((1-r)^{k}\right) g\left((1-r)^{t}\right)$.

Hyperbolic discounting motivates the definition of Strongly Diminishing Impatience as $\frac{\Delta(t)}{\Delta(t+1)} > \frac{\Delta(t')}{\Delta(t'+1)}$ $\forall t' > t \in \mathbb{N}$. Note that Strongly Diminishing Impatience too, is restricted to only one-period delays, and hence similar to Definition 21, we strengthen this measure to be delay independent:

Definition 22. The decision maker exhibits *Delay Independent Strongly Diminishing Impatience (DISDI)* if $\frac{\Delta(t)}{\Delta(t+k)} > \frac{\Delta(t')}{\Delta(t'+k)}$ $\forall k, t' > t \in \mathbb{N}$, where $\Delta(t)$ is the decision maker's time discounting at period t.

If preferences are represented by (3.1) then DISDI holds if and only if for every $r \in (0,1)$ and $t < t'$, $k \in \mathbb{N} \backslash \{0\}$: $\frac{g\left((1-r)^{t}\right)}{g\left((1-r)^{t+k}\right)} > \frac{g\left((1-r)^{t'}\right)}{g\left((1-r)^{t'+k}\right)}$.

[45] For further discussion and intuition see the introductory discussion in Appendix A.2.1.

[46] Halevy [2015] provides a formal definition and recent experimental evidence for stationarity in a dynamic setting.

[47] DI is the special case of DIDI where delay $k = 1$. An implication of the counter-example provided in Appendix A.2.1 is that DI does not imply DIDI.

Proposition. *Quasi-hyperbolic discounting satisfies DIDI (but not DISDI), Hyperbolic discounting satisfies DISDI (and hence DIDI).*

Proof. In case of quasi-hyperbolic discounting: $U = u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t)$, and for $\beta < 1$:

$$\frac{\Delta(0)}{\Delta(k)} = \frac{1}{\beta\delta^k} > \frac{\beta\delta^t}{\beta\delta^{t+k}} = \frac{1}{\delta^k} = \frac{\Delta(t)}{\Delta(t+k)} = \frac{\Delta(t')}{\Delta(t'+k)}$$

The last equality holds for $t, t' > 0$. Hence, quasi-hyperbolic discounting satisfies DIDI, but not DISDI.

In Hyperbolic Discounting the discount function for period t is given by $\Delta(t) = \frac{1}{1+\rho t}$ for $\rho > 0$. For arbitrary k, and $t' > t$,

$$\frac{\Delta(t)}{\Delta(t+k)} = 1 + \frac{\rho k}{1+\rho t} > 1 + \frac{\rho k}{1+\rho t'} = \frac{\Delta(t')}{\Delta(t'+k)}$$

Hence, hyperbolic discounting satisfies DISDI (and hence DIDI). □

3.3.3 The Relation between Risk and Time Preferences

As noted above, the effect of risk attitude on intertemporal preferences in (3.1) is straightforward. We summarize this relation below.

Claim. Strict Certainty Effect (3.2) implies Delay Independent Diminishing Impatience (DIDI), and the Strict Common Ratio Effect (3.3) implies Delay Independent Strongly Diminishing Impatience (DISDI).

The following Theorem proves the converse direction (though in a weaker form that does not substantiate an equivalence), that is - how the DM's intertemporal preferences determine her risk attitudes.[48] The result is direct and comprehensive in the sense that it does not rely on any intermediate connections through properties (e.g, convexity, increasing elasticity) of the weighting function.

Theorem. *Consider a DM represented by (3.1) with continuous $g(\cdot)$.*

1. *Delay Independent Strongly Diminishing Impatience implies the Common Ratio Effect (and the Certainty Effect).*

2. *Delay Independent Diminishing Impatience implies the Certainty Effect.*

[48]Note that although the Theorem does not imply *Strict* Common Ratio/Certainty Effects, it is *inconsistent* with expected utility since even the weaker forms imply the existence of probabilities for which (3.2) and (3.3) are satisfied with strict inequality.

Proof. (1) Consider a sequence $\{\frac{m_i}{n_i}\}_{i=1}^{\infty}$ of rational numbers that converges to $\frac{\ln p}{\ln q\ell}$, where m_i, n_i are positive integers. Similarly, consider a sequence $\{\frac{a_i}{b_i}\}_{i=1}^{\infty}$ of rational numbers that converges to $\frac{\ln \ell}{\ln q\ell}$, where a_i, b_i are positive integers. Note that $\frac{\ln \ell}{\ln q\ell} < 1$, so we can choose $\{\frac{a_i}{b_i}\}_{i=1}^{\infty}$ such that $a_i < b_i$. Now, given this sequences, define a sequence $\{r_i\}$, such that $q\ell = r_i^{n_i b_i}$, that is $r_i = (q\ell)^{\frac{1}{n_i b_i}} < 1$. Note that as $\frac{a_i}{b_i}$ converges to $\frac{\ln \ell}{\ln q\ell}$, $r_i^{a_i n_i} = (q\ell)^{\frac{a_i}{b_i}}$ converges to $(q\ell)^{\frac{\ln \ell}{\ln q\ell}} = \ell$. Similarly, as $\frac{m_i}{n_i}$ converges to $\frac{\ln p}{\ln q\ell}$, $r_i^{m_i b_i} = (q\ell)^{\frac{m_i}{n_i}}$ converges to $(q\ell)^{\frac{\ln p}{\ln q\ell}} = p$.

Now using DISDI, $\forall i$:

$$\frac{g\left(r_i^{a_i n_i}\right)}{g\left(r_i^{a_i n_i + m_i b_i}\right)} > \frac{g\left(r_i^{n_i b_i}\right)}{g\left(r_i^{n_i b_i + m_i b_i}\right)}$$

Using the continuity of g, as $i \to \infty$, the Common Ratio Effect follows:

$$\frac{g(\ell)}{g(p\ell)} \geq \frac{g(q\ell)}{g(pq\ell)}$$

(2) Let $p, q \in (0,1)$ and assume without loss of generality that $p < q$. Consider a sequence $\{\frac{m_i}{n_i}\}_{i=1}^{\infty}$ of rational numbers that converges to $\frac{\ln p}{\ln q}$, where m_i, n_i are positive integers. Given this sequence, define a sequence $\{r_i\}$, such that $q = r_i^{n_i}$, that is: $r_i = q^{\frac{1}{n_i}} < 1$. Note that as $\frac{m_i}{n_i}$ converges to $\frac{\ln p}{\ln q}$, $r_i^{m_i} = q^{\frac{m_i}{n_i}}$ converges to $q^{\frac{\ln p}{\ln q}} = p$.

Now, $\forall i$:

$$\begin{aligned} g(r_i^{m_i + n_i}) &> g(r_i^{m_i})g(r_i^{n_i}) \\ g(r_i^{m_i} q) &> g(r_i^{m_i})g(q) \end{aligned}$$

Using the continuity of g, Certainty Effect follows: $g(pq) \geqslant g(p)g(q)$. □

Corollary. *Consider a DM represented by (3.1) with continuous $g(\cdot)$.*

1. *Hyperbolic discounting implies the Common Ratio Effect (and the Certainty Effect).*

2. *Quasi-hyperbolic discounting implies Certainty Effect.*

It is important to recall that preferences represented by (3.1) are defined over consumption streams in discrete time (following Koopmans, 1960).[49] It follows that all notions of diminishing impatience (as DI, DIDI, DISDI) are required to hold only for natural numbers, while risk preferences (*Certainty Effect*, *Common Ratio Effect*) are defined over lotteries with probabilities in the simplex. With this insight, it is not surprising that properties of risk preferences manifest themselves in the time domain. The counter-example in the Appendix together with the Theorem demonstrate that the opposite direction can be established as well, but the notion of diminishing impatience must be appropriately defined so it will not be delay dependent. We believe that these new notions (DIDI and DISDI) are very intuitive and reflect the natural meaning of diminishing impatience. Moreover, in light of recent work generalizing hyperbolic discounting to continuous time [Webb, 2016] we conjecture that continuous adaptations of DIDI and DISDI will be required in order to create the link from time to risk, though this remains for future work as the behavioral underpinning of (3.1) are stated in discrete time.

[49]This framework is considerably different from Fishburn and Rubinstein [1982] whose domain includes payments of $\$x$ at time t, which is applicable to more selective environments (as bargaining).

Chapter 4

Drivers of Cooperation in Finitely Repeated Prisoner's Dilemma

Prisoner's dilemma is probably one of the most popular games in economics as it pits two fundamental human actions against one another - Cooperation and Defection. From human-beings to competing firms, from feuding countries to animals in the eco-system- wherever there is a possible interaction of interests, one often faces a choice between cooperation and defection against others, the latter coming at a personal gain and social cost. As a result, many economic and strategic interactions we see around us can be represented simply as finitely repeated Prisoners dilemma (FRPD) games. One standard result from Microeconomic Theory is that under standard assumptions, no cooperation can be supported in any subgame perfect Nash equilibrium of a Finitely Repeated Prisoner's dilemma(FRPD) game.

		Column Player	
		Defect (F)	Cooperate (C)
Row Player	Defect (F)	b,b	c,d
	Cooperate (C)	d,c	a,a

$c>a>b>d>0$

This paper investigates the question of how the gains from potential future periods (shadow of the future) determine behavior in FRPD games. We vary the future gains in a 5-period FRPD game by imposing (and varying) exponential discounting (δ^t) over periods $t = 1-5$ of the 5-period game. The discount factors used are $\delta = 1, 3/4, 3/8, 1/4$. As the discounting increases (and δ decreases), for any fixed first-period payoff matrix, the payments diminish at a higher rate across rounds 1 to 5. In a few of our treatments, we also ask subjects about their beliefs on their partners' actions, to pin down the driving force behind cooperation and defection in the games, and see how subject beliefs react to their exxperiences. The game horizon and payoffs are chosen in such a way that an egoist (a player with is only concerned about maximizing payoffs subject to beliefs) would never cooperate[50]

[50] See Appendix A.3

under the $\delta = 1/4$ treatment. Such an egoist would also never cooperate in the last two periods of the $\delta = 3/8$ treatment, and never cooperate at all if she believes that the her partner plays any variety of threshold strategies [51]. Cooperation under reputation equilibrium Kreps et al. [1982] is possible in the $\delta = 1, 3/4$ treatments, and it should decrease with δ. The predictions from egoistic cooperation driven by reputation contrast sharply with that from theories that assume that subjects' actual utilities are determined by joint strategies and payoffs of them and their partners. For example, any theory that assumes that subjects get a fixed boost in utility from playing kind/ altruistic (cooperative) strategies would suggest that cooperation might increase disproportionately in later rounds of the low δ treatments. Similarly, Rabin [1993] would suggest that as long as reciprocity minded people put a high enough belief on their partner being kind in their actions, they might reciprocate with kindness too, as long as the utility gains from their kind action supersede the utility loss from getting a sub-optimal payoff. So, with the monetary loss from sub-optimal actions diminishing in the low δ treatments (especially in the later rounds), subjects are more likely to be kind, especially if they expect kindness from their partners.

We find the following results: 1) Cooperation in the first period of a FRPD game decreases monotonically the more the future is discounted. First period cooperation is highly correlated with subjects' beliefs on their partners' cooperating in the same period, and, their partners' propensity to cooperate in the future in response to cooperation. 2) Higher the discounting (and lower the final period discounted stakes), higher is the observed final-period cooperation, and this is robust to subjects gaining a considerable amount of game-experience. Final period cooperation is also highly correlated with the subject's belief of their partners cooperating in the final period. 3) Subjects systemcatically over-estimate their partner's propensity of engaging in cooperative behavior. 4) Reported beliefs are consistent with reasonable learning, and move in predictable directions in response to good and bad outcomes. 5) Justifying aggregate subject behavior requires the use of both egoistic and altruistic theories.

In Section 4.1, we provide an overview of the related literature. Section 4.2 discusses the experimental design. Section 4.3 provides the main results and analysis from the experimental data, and Section 4.4 concludes.

4.1 An overview of the literature

There have been many experimental studies about both finitely and infinitely repeated Prisoner's Dilemma games. Below we describe the relevant literature sepa-

[51] Strategies that conditionally cooperate till some period, and revert to always defecting thereafter.

rately for studies on finitely repeated PD games and infinitely repeated PD games.

The effect of the scope of future cooperation on current behavior has been studied in detail in the domain of infinitely repeated PD games. Infinitely repeated PD games are implemented in the lab using a random termination protocol, i.e, after each period, the game ends with some predetermined probability. This probability of continuation is a dual of the δ used in our setting, under Expected Utility, and it determines the shadow of the future in infinitely repeated PD games. Roth and Murnighan [1983] vary the probability of continuation in their experimental setting and find that higher the probability, the greater the number of players who cooperated in the first round of the game. Bo [2005] replicates that higher continuation probabilities result in higher cooperation levels, and additionally shows that it is not just the higher number of expected periods of play, but the higher probability of repeated interaction that drives this behavior.[52]

In the following, we will discuss the experimental literature that studies the determinants of cooperation in FRPD games. Andreoni and Miller [1993] control subjects' beliefs over the value of building a reputation (Kreps et al. [1982]) by varying the probability that subjects interact with a pre-programmed opponent (a computer that plays a Tit-For-Tat strategy). In their study, cooperation falls through the rounds of FRPD and higher beliefs about playing the computer are more conducive to higher cooperation. Cooper et al. [1992] compare behavior in one-shot PDs to that in FRPDs and observe higher cooperation rates in the FRPD. The authors find evidence of both reputation building and altruism and they conclude that neither model can explain all the features of the data on its own. There is some dispersed evidence about how cooperation in FRPD might be affected by the shadow of the future. Bereby-Meyer and Roth [2006] find more cooperation in round one of FRPDs than in the one-shot games, which is equivalent of comparing first round cooperation rates of $\delta = 1$ and $\delta = 0$ in our setting. In the FRPD games conducted by Bo [2005] first round cooperation rates are higher in games with a longer horizon, consistent with the hypothesis that shadow of the future might drive cooperation even in FRPD games. Embrey et al. [2015] identify how the value of cooperation can be captured by the "size of the basin of attraction of Always Defect", and how it is an important determinant of cooperation in FRPD games in the previous literature. Beside their comprehensive meta-study, they also design a new experiment that compares two treatments in which the horizon of the repeated game is varied, but the value of cooperation is kept constant. One can think of our experiment as a dual to theirs, as we keep the horizon of the repeated

[52]The paper compares first period cooperation rates from infinitely repeated games with x number of expected periods to that of finitely repeated games having x periods, and finds that the former is bigger.

game constant, but vary the value of cooperation. Charness et al. [2016] show that higher monetary payoffs from cooperation are associated with substantially higher cooperation rates in one shot PD games.

There is also some work about how beliefs might evolve and affect PD play. For example, Kagel and McGee [2016a,b] have both individual play and team play in their FRPD games and, analysis of team dialogues show signicant discrepancies between subjects' inferred beliefs and those underlying standard models of cooperation in the FRPD. Cox et al. [2015] reveal second-mover histories from an earlier sequential-move FRPD game to the first-mover. They unexpectedly find higher cooperation rates when histories are revealed. They also provide an accompanying theory in which players decide on conditional cooperation based only on naive prior beliefs about what strategy their opponent is playing.

4.2 Experimental design

A total of 132 subjects participated in 5 sessions between November 19, 2015 to December 3, 2015, where subjects played 5-period FRPD-D games, for the values $\delta = 1, 3/4, 3/8, 1/4$. Each subject played under each of the four treatments in these Within sessions. Sessions lasted approximately one hour and a total of 132 subjects participated in these sessions. Within any game, as the rounds progressed, the stage payments diminished according to the particular δ employed in that game. The payoff matrix for the first period for any treatment was fixed at:

		Column Player	
		Defect	Cooperate
Row Player	Defect	1200, 1200	2600, 200
	Cooperate	200, 2600	2000, 2000

Subjects were be divided into two groups. In each match, a subject from the first group was matched with a new subject from the second group using turn-pike matching. Two subjects would never meet in more than a game. Within every game or match, the subjects could see the past actions taken by their partner, but they could not see their partner's actions in previous games. This information protocol coupled with the matching procedure ensured that actions taken within a match or game should not influence their or their partner's behavior in future matches. The fact that no information of previous matches was provided to their new partners and the payoff structures in each treatment was common knowledge for the subjects. The block of treatments were repeated twice in the same order, so, each subject played 2 matches under each treatment, thus playing a total of $(2 \times 4) = 8$ matches/ games. At the end of the experiment, one of the 8 games was randomly chosen, the total points or lab currency earned by the subject in that game was converted into dollars at an exchange rate of 300 points = $1 and paid to the subject. The subjects also received a $5 show-up fee.

The order of treatments in the Within sessions was randomized at the session level: At the beginning of each session, a coin toss by the experimenter decided if the treatments in that session were arranged in the order

$$\delta = 1, 3/4, 3/8, 1/4, \ 1, 3/4, 3/8, 1/4$$

(block of treatments repeated twice) or in the opposite order. Over all, 3 sessions were ran in the former order and 2 in the latter order. A considerable time was spent at the beginning of each session, to make sure that the subjects understood the game, the payment scheme, the matching protocol and the interface thoroughly. The subject instructions and screen shots of the GUI are included in the Appendix.

As a robustness checks for the results, four more sessions were run, with Between design, two each for $\delta = 1/4$ and $\delta = 3/4$. In the four Between sessions (held in April, 2017) the subjects played 8 games under a single δ, thus giving them more time to learn about the particular treatment, and eliminating any possibility of cross-treatment effects. The subjects were also asked to answer the following four prediction questions at the start of each game with a new partner:

- How likely is your partner to play L on the first round of the this game?
- How likely is your partner to play L in the very next round if you played T in the previous round of the game?
- How likely is your partner to play L in the very next round if you played B in the previous round of the game?
- How likely is your partner to play L on the very last (5th) round of the this game?

At each question the subjects could respond on a scale of 0 to 10, and they were advised to enter a higher number the more likely they thought the event was. They were also provided the following the following reference points:

- A response of 0 (lowest point of the scale) would mean "never".
- 5 (midway point of the scale) would mean "as likely as getting Heads on a fair coin toss/ 50-50 odds",

- 10 (right extreme of the scale) would mean "surely".
- Events more likely than “never” and less likely than heads on a fair coin toss, should be rated between 0 and 5, and so on.

There was a separate paragraph in the instructions which the subjects were advised to read only if they were more comfortable in thinking of likelihoods in terms of probabilities. This para linked how their probability assesments would map to responses on the $0-10$ scale. The prediction tasks were not incentivized, to make sure that prediction incentives could not influence FRPD-D play in any way. A total of 90 subjects participated in the Between sessions. The payment scheme and exchange rate in the Between treatment was identical to those in the Within treatment, and the subjects received $6 as show-up fee.

There is one more crucial aspect of the design that deserves separate mention. The norm in the experimental literature on repeated games is to model the shadow of the future by varying the continuation probabilities, and tie the experimental findings with theoretical predictions by assuming that subjects calculate expected returns in accordance with Expected Utility Theory. Under Expected Utility Theory, the continuation probability is equivalent to a discount factor imposed on future rounds of a repeated game. This norm runs almost in denial of the literature (Kahneman and Tversky [1979], Tversky and Kahneman [1992]) that rejects Expected Utility Theory (EUT) based on lab evidence. Our decision to impose discounting instead of random termination was taken in acknowledgement of the rich set of findings that indicate that subjects frequently violate EUT in their behavior.

4.3 Results

Our analysis will be presented in three parts. We will start by analysis of the data from the Within sessions, then show how the main results about cooperation were replicated in the Between session, and then finally describe the rich beliefs data from the Between sessions.

Twenty seven of the total 132 subjects in the Within sessions always defect, whereas only one subject always cooperates. Hence, there is very little evidence of subjects who play the game as if cooperation was a dominant strategy in the sub-game. Fifty four of the subjects cooperate only thrice among their forty decisions. Average cooperation across the 5 periods of the Within sessions is highest in the $\delta = 1$ treatment, and is lower for all lower values of δ (Table 4.1, Column 1). There is very little evidence of any change in total cooperation for treatments with $\delta \in (0,1)$. One could interpret the data as if the discount factor of $\delta = 1$ is saliently different from all discount factors in terms of being a driver of total cooperation. This mirrors Kahneman and Tversky [1979]'s finding that there is a discrete change in risk behavior when moving from certainty (probability $p = 1$) to risky options, but preferences are less sensitive to moving between two different risky options. The average percentage cooperation in the $\delta = 1$ treatment is close to what was obtained (23.78%) by Bo [2005] over 10 matches in his 4-period FRPD experiment.

Table 4.1: Total Cooperation by treatments

	(1) All data	(2) Block 2
d1	22.20***	20.91***
	(2.301)	(2.704)
d2	15.15***	12.88***
	(1.789)	(2.009)
d3	15.61***	13.48***
	(1.830)	(2.231)
d4	15.08***	13.64***
	(1.720)	(2.198)
N	5280	2640
d1_d2	0.0000570	0.000911
d2_d3	0.775	0.769
d3_d4	0.742	0.946

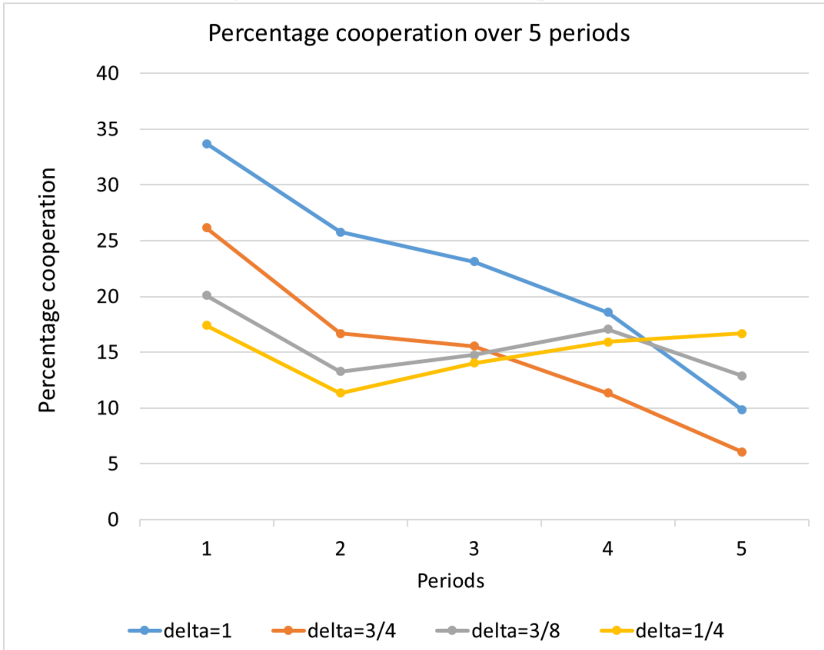
Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

The same results holds when we we allow for learning and only consider the second block of the treatments (Column 2 in Table 4.1) in our analysis. The average cooperation in the second block is lower than the average over first and second blocks.

Though the treatments of $\delta = 3/4, 3/8, 1/2$ have very similar aggregate cooperation levels, the strategy profiles by periods look very different across all the three treatments. Figure 4.1 describes the evolution of cooperation throughout the periods

Figure 4.1: Evolution of cooperation



for the different peroids. The cooperation levels aggregated across all sessions are indeed ranked period by period till the first two periods. The difference in cooperation among the treatments vanishes around the third period and thereafter flips signs! This is why the difference between cooperation levels at $\delta = 3/4, 3/8, 1/2$ vanishes when all five periods are aggregated.

The cooperation rates decline progressively and sharply across the periods in the $\delta = 1,\ 3/4$ treatments, whereas, they are stable in the lower δ treatments. For example, very few people start by cooperating in $\delta = 1/4$ treatment, but the level of cooperation remains stable throughout the later stages of a game. This is in sharp contrast to the cooperation profile in the $\delta = 1$ treatment, where the percentage cooperation is in a steady decline throughout. This results in a cross-over of cooperation rates between the two treatments at period 5, which explains why the $\delta < 1$ treatments result in similar aggregate levels of cooperation. The percentage cooperation in the first period is decreasing in δ. The differences between the first and terminal period cooperation levels across the treatments is presented in Table 4.2.

The first round cooperation rates are decreasing in δ, and the differences are significant for all pairs of discount factors, other than the smallest two, which are

Table 4.2: Comparison of first and terminal periods

	Period 1	Period 5
$\delta_1 = 1$	33.71***	9.85***
	(3.55)	(1.97)
$\delta_2 = \frac{3}{4}$	26.14***	6.06***
	(3.12)	(1.70)
$\delta_3 = \frac{3}{8}$	20.08***	12.88***
	(2.84)	(2.32)
$\delta_4 = \frac{1}{4}$	17.42***	16.67***
	(2.64)	(2.45)
N	1056	1056
$\delta_1 = \delta_2$	.009	.11
$\delta_2 = \delta_3$	.03	.006
$\delta_3 = \delta_4$		.14

Standard errors (clustered at subject level)
are reported in parentheses below
Lower panel reports p-values from F test for $H_0 : \delta_i = \delta_j$
$* \ p < 0.05$, $** \ p < 0.01$, $*** \ p < 0.001$

underpowered. This is consistent with the meta-analysis in Embrey et al. [2015] which finds that across experimental studies, the cooperation in the first period is increasing in the length of the horizon, another metric for shadow of the future. On the other hand, the percentage cooperation in the last period of the $\delta = 1/4$ treatment is significantly higher than those of the $\delta = 1$ and $\delta = 3/4$ treatments.

Table 4.3: Cooperation by treatment and period

	Period1	Period2	Period3	Period4	Period5	Average
1	33.71	25.76	23.11	18.56	9.85	22.20
2	26.14	16.67	15.53	11.36	6.06	15.15
3	20.08	13.26	14.77	17.05	12.88	15.61
4	17.42	11.36	14.02	15.91	16.67	15.08
Total	24.34	16.76	16.86	15.72	11.36	17.01

Part of this cross-over can be attributed higher propensity of subjects cooperating in the terminal period of the game in the $\delta = 1/4,\ 3/8$ treatments. As consistent with previous experiments, a big proportion of the subject pool (62 out of the 132 total subjects) make such non-egoistic choices at least once, but only 17 subjects do it more than once. But this cannot be the whole story behind the cross-over pattern, given the cooperation rates start converging across treatments well before that. Below we study some of the other behavioral channels that could be driving the cross-over in cooperation rates.

We call a choice *Forgiving*, if a subject cooperates while her opponent played Defect against her cooperative move in the last period. *Recooperating* implies that a subject switched back to cooperation in the current period after Defecting in the last period. Both Forgiving and Recooperating play are behaviors which would increase cooperation.

Similarly, we define *Unfolding* as responding to Cooperate-Cooperate in the previous period by Defaulting in the current period. This behavior results in unfolding the Default-Default equilibrium one eventually expects under reputation theories, and hence the name. In the table below we report the relative frequencies and total possible instances of these three kinds of behavior by the Treatments.

As one would expect from the previous results, subjects are significantly more forgiving in the lower δ treatments. For example, they are twice as likely to forgive in the $\delta = 1/4$ treatment than the $\delta = 3/4$ treatment. 82 out of the 132 subjects indulge in Forgiving behavior at least once. There are insignificant minor differences in

Recooperating and Unfolding behavior. Forgiving seems to be the major driving force behind the cooperation rates flipping in later periods. Here is a rough way to understand this: Compared to $\delta = 1/4$, there are respectively 43 and 23 more instances of cooperation in first period in treatments $\delta = 1$ and $\delta = 3/4$. These differences are almost single handedly overcome by 33 and 41 instances of additional instances of Forgiveness observed in the $\delta = 1/4$ treatment compared to the the other two aforementioned treatments. [53]In the four treatments, 82 of the total 132 subjects commit Forgiving behavior at least once.
The more frequent forgiving behavior at the lower δ treatments implies that smaller the payment horizon gets, the more forgiving people are. The fact that the smaller δ treatments have smaller cooperation rates in the earlier periods and have similar rates of Reciprocal play means that it is most likely the lower costs of Forgiving that is driving the crossing over of cooperation rates for the higher periods.

Table 4.4: Relative Frequency of Forgiving , Reciprocal, and Unfolding play

Treatments	Forgiving	Recooperating	Unfolding
1	0.07	0.21	0.17
	(789)	(117)	(150)
2	0.05	0.23	0.16
	(872)	(114)	(70)
3	0.07	0.28	0.14
	(884)	(98)	(74)
4	0.10	0.22	0.14
	(901)	(105)	(50)
Total	0.07	0.23	0.15
	(3446)	(434)	(344)

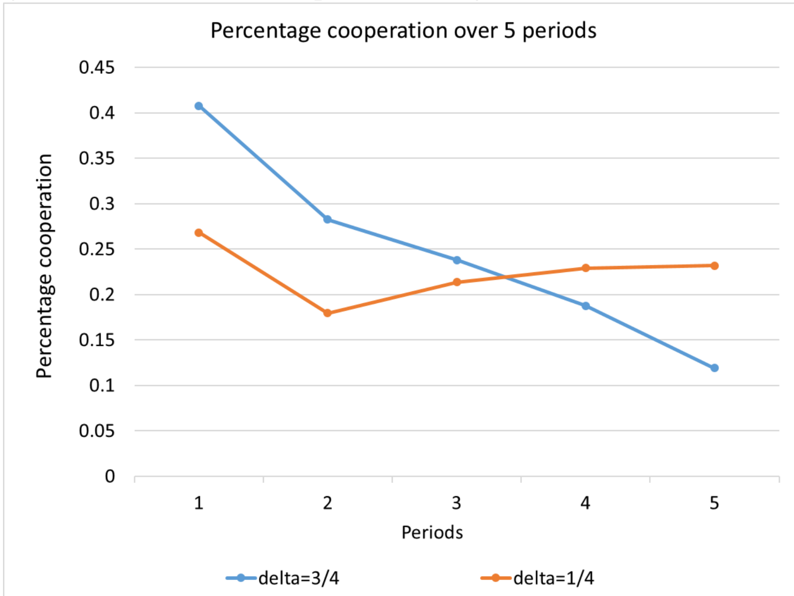
The top entry reports the relative frequency of behaviors by Treatment. The total number of possible observations is reported in brackets below. Treatments 1-4 are in decreasing order from 1 to $1/4$.

Results on cooperation from Between study:

The between study replicates the central finding of cross-over of cooperation trends from the Within study, as shown in Fig 4.2. The average cooperation in the $\delta = 3/4$ treatment is relatively stable, whereas cooperation falls steadily in the $\delta = 1/4$ treatment. In Table 4.5, we compare the first and last period cooperations

[53]Fudenberg et al show that subjects are “Slow to anger and fast to forgive” in PD games where actions are noisy. We see a similar trend here.

Figure 4.2: Evolution of cooperation (All games from Between Session)



of the two treatments for games while allowing for learning in the initial periods. The cross-over pattern is still highly significant.

Beliefs in the Between study:

In the following, we analyze the responses to the four likelihood questions. For each subject, let the quadruple $(g_1,\ g_2,\ g_3,\ g_4)$ contain responses to the following four questions on the $0-10$ scale:

- How likely is your partner to play L in the very next round if you played T in the previous round of the game?
- How likely is your partner to play L in the very next round if you played B in the previous round of the game?
- How likely is your partner to play L on the very last (5th) round of the this game?

Firstly, the reported likelihoods/ beliefs seem consistent with learning. For example, subjects weakly decrease their reported g_1 ($\Delta g_1 \leq 0$) after a bad experience (their partner defecting in the first period of the last game) in 89% of all possible occasions and weakly increase g_1 after a good experience (their partner cooperating in the first period of the last game) on 86% of all occasions. These percentages are

Table 4.5: Comparison of first and terminal periods

	Games 2-8		Games 4-8	
	Period 1	Period 5	Period 1	Period 5
$\delta_2 = \frac{3}{4}$	40.48***	11.90***	40***	9.52***
	(6.12)	(2.78)	(6.25)	(2.73)
$\delta_4 = \frac{1}{4}$	23.80***	21.73***	18.75***	18.33***
	(4.72)	(4.06)	(4.80)	(4.19)
$\delta_2 = \delta_4$	.03	.049	.008	.08

Standard errors (clustered at subject level) are reported in parentheses below

Lower panel reports p-values from F test for $H_0 : \delta_i = \delta_j$

$* \ p < 0.05$, $** \ p < 0.01$, $*** \ p < 0.001$

Table 4.6: Change in beliefs and Forgiving across games (1-8)

	Quarter					Three Quarters				
	g_1	g_2	g_3	g_4	Forgiving	g_1	g_2	g_3	g_4	Forgiving
(game-1)	-.41	-.26	-.10	-.36	-.007	-.19	-.26	-.13	-.23	-.013
	(.07)	(.07)	(.06)	(.07)	(.005)	(.07)	(.07)	(.06)	(.06)	(.004)
constant	5.3	6.5	2.4	5.2	.14	4.9	5.8	2.8	3.8	.12
	(.39)	(.39)	(.26)	(.38)	(.03)	(.37)	(.36)	(.29)	(.37)	(.02)
N	384	384	384	384	1194	336	336	336	336	969

90% and 91% respectively in case of g_4. Note that $\Delta g_i = 0$ is consistent with learning, as we only observe subject responses on a discrete grid, and for small changes in g_i we might not see any changes in their reported beliefs. We can summarize the evolution of beliefs across the games by running a regression of the beliefs against the variable (game-1). The coefficient of regression can be read as the average change in beliefs after every passing game, whereas the coefficient on "constant" provides us average beliefs at the start of the session. As seen in Table 4.6, on average, subjects get more pessimistic about their partners as the session goes on and more games are played. Our finding are in contrast to Cox et al (2015) who find that subjects might have unsophisticated priors. Further, the learning of threshold strategies (that defection should be followed by defection) is slower in the Quarter treatment. There only seems to be significant learning away from Forgiving in the $\delta = 3/4$ treatment only.

In Figure 4.3, we plot the evolution of Period 1 belief g_1 (and and Period 5 belief g_4) against the average cooperation in that period, across the two treatments,

Figure 4.3: Average Beliefs (g_1, g_4) vs Average cooperation across Games 1-8

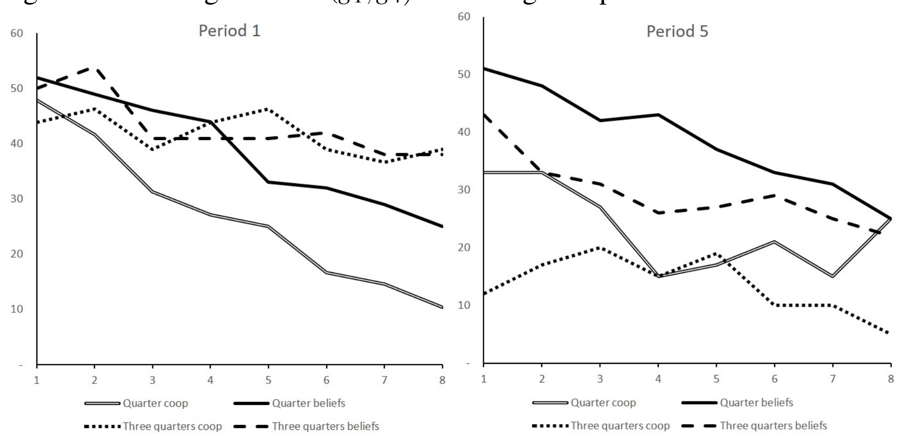


Table 4.7: Behavior and beliefs

	Reported belief	N	% Coop
	0-4	358	16%
g_1	5	198	48%
	6-10	164	53%
	0-4	429	13%
g_4	5	155	21%
	6-10	136	30%

throughout games $1-8$. Under the assumption that the subjects report their probabilistic assessments about the population in their response, we can meaningfully compare it to the actual average response of the population. Other than in the case of first period cooperation in the Three quarters treatment, subjects systematically overestimate the how often cooperation takes place.

Optimistic beliefs about partner's actions are also highly associated with cooperative behavior by the players themselves, as we show in Table 4.7. Given that subjects were provided reference points for 0, 5, 10 on their response scale, we use the most natural way to tabulate the belief data, and look at subject responses. Fisher's exact test and the chi-squared test result in a rejection of the null of equal relative proportions of cooperation with p-values of zero for the tabulations, and suggest that more optimistic a subject was about her partner's responses, the more likely they were to cooperate.

To take the analysis a step further, in Table 4.3 we run a logit regression of first and last period cooperation on the self-reported beliefs, game and treatment dummies. The standard errors are clustered at the subject level, as in the rest of this study. Even after controlling for the treatments and the games, beliefs over partner cooperating in the first period and partner's propensity to reciprocate to cooperation are significant drivers of first period cooperation. Also, as expected higher the odds of partner cooperating after facing a defection, lower is the cooperation in the first period. On the other hand, it seems that among the belief variables, belief about partner cooperating in the final period is the sole determinant of last period cooperation. This result is highly intuitive. We have focussed on forgiving behavior from only the first 4 periods to make sure we do not double account Period 5 behavior. Forgiving behavior (in the first four periods) decreases with increasing beliefs about partner cooperating in the first period, and increases with higher beliefs about the partner cooperating in the last period. The former is consistent with possible disenchantment driven aversion to forgiving, and the latter is consistent with forgiving being driven by optimism about partner's future play.

4.4 Conclusion

This paper povides a systematic analysis of how the the shadow of the future might affect cooperation in FRPD games. The discount factor, which is a measure of the shadow of the future is a significant determinant of first-round cooperation in FRPD games. First period cooperation decreases as the discount factor decreases, consistent with reputation play. But, latter period cooperation is instead driven by behavior consistent with theories of altruism, fairness and reciprocal kindness (Rabin [1993]), and cooperation increases with decreasing discount factor. We find that cooperation in both the first and the last period is driven by beliefs about the partner reciprocating. We also find that subjects are generally over-optimistic in their beliefs about their partners, but their beliefs move in reasonable directions when confronted with good or bad news. Finally, the subject population contains both subjects who look like they are playing egoistic reputation equilibrium, and players whose utility respond to altruistic and reciprocity motives.

Table 4.8: Logit regressions on belief variables and game dummies

	(1) Coop in Period 1	(2) Coop in Period 5	(3) Forgiving in Period<5
main			
1.game	0	0	0
	(.)	(.)	(.)
2.game	-0.0615	0.206*	-0.261*
	(0.254)	(0.309)	(0.150)
3.game	-0.299*	0.103	-0.337
	(0.282)	(0.322)	(0.195)
4.game	-0.252*	-0.508*	-0.660****
	(0.291)	(0.402)	(0.179)
5.game	-0.0443*	-0.566*	-0.533**
	(0.286)	(0.376)	(0.214)
6.game	-0.408*	-0.343	-0.297*
	(0.316)	(0.384)	(0.187)
7.game	-0.368*	-0.613*	-0.489***
	(0.317)	(0.387)	(0.188)
8.game	-0.389*	-0.271*	-0.601***
	(0.314)	(0.346)	(0.200)
T	0.902**	-0.661**	-0.139
	(0.386)	(0.319)	(0.178)
g1	0.216****	-0.0437	-0.0555*
	(0.0610)	(0.0479)	(0.0324)
g2	0.159**	0.0241	0.0166
	(0.0636)	(0.0533)	(0.0303)
g3	-0.0892*	-0.0210	0.0255
	(0.0807)	(0.0676)	(0.0411)
g4	0.0956	0.128**	0.0658*
	(0.0705)	(0.0572)	(0.0360)
_cons	-2.915****	-1.431***	-1.081****
	(0.593)	(0.461)	(0.260)
N	720	720	1594

Standard errors in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$, **** $p < 0.001$

Conclusion

The first chapter provides a novel, testable weakening of classical assumptions to derive a new theory that would best fit Present Biased behavior. We show that any present-biased preference has a max-min representation, which can be cognitively interpreted as if the decision maker considers the most conservative present equivalents in the face of uncertainty about future tastes. We compare our theory with existing theories of temporal behavior (Koopmans [1972], Fishburn and Rubinstein [1982], Harvey [1986], Laibson [1997], Ebert and Prelec [2007]) to discuss the dimensions along which it is an improvement over the latter. We also discuss how previously unsupported behavioral anomalies from the domain of time and risk can now be addressed through this new theory, and how knowledge of such anomalous behavior can in turn be used for better policy design. The third chapter of the thesis shows how time-delay and risk are behavioral duals of one another, and a formal study of any one also conveys fundamental insights about the other. Though this duality had been hypothesized previously in the literature (Green and Myerson [2004], Baucells and Heukamp [2012b], Halevy [2008], Saito [2015]), we provide the first formal derivation of the duality relationship. We show how under the assumptions of Non-Expected Utility and constant hazard rate, one can derive a bias from the present or a bias for certainty from one another, thus completing the characterization result in Halevy [2008]. In chapters one and three, instead of studying behaviors under risk or time-delay in isolation, we provide a more holistic theory of human behavior by studying the joint interplay of uncertainty and time as influencing factors.

Chapters two and four contribute to the literature on the empirical study of preferences. Chapter two uses a meta-study over recent influential experimental papers to inform the design of future experiments investigating temporal-preferences. We provide desiderata of choice consistency that experimenters should employ while estimating time preferences from choice data. We also show how the application of our desiderata can help us learn new insights from recent experimental studies. Chapter four introduces a novel experimental design to study the effect of temporal delay (discounting) on human interaction in an environment where there is a tradeoff between individual gain and social surplus. We find that subject behavior is driven by a combination of altruistic and selfish motives, and selfish motives sometimes drive cooperative play, especially when the future gains from cooperation is large enough. Altruistic play is common only when returns from selfish play are very low. We also see that subjects become more pessimistic about altruistic play from their partners as they gain more experience.

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Appendix A

Appendix

A.1 Appendix to to Chapter 1

Appendix I

MODELS OF PRESENT BIAS

Consider the general separable discounted utility mode defined over timed prospects (x,t)

$$V(x,t) = \Delta(t)u(x)$$

Here, $\Delta(t)$ is the discount factor, and $u(x)$ is the felicity function. Below[54], we give a brief summary of the literature on different discounting models which accommodate present bias, in terms of the discount functions they propose. We also include the exponential discounting model as a point of reference.

Appendix II

Theorem 3: *Let* $\mathbb{T} = \{0, 1, 2, ...\infty\}$ *or* $\mathbb{T} = [0, \infty)$ *and* $\mathbb{X} = [0, M]$ *for* $M > 0$. *The following two statements are equivalent:*

[54] We take the idea of tabular presentation from Abdellaoui et al (2010).

	Model	Author(s)	$\Delta(t)$
0	Exponential discounting	Samuelson (1937)	$(1+g)^{-t}, g > 0$
1	Quasi-hyperbolic discounting	Phelps and Pollak (1968)	$(\beta + (1-\beta)_{t=0})(1+g)^{-t}, \beta < 1, g > 0$
2	Proportional discounting	Herrnstein (1981)	$(1+gt)^{-1}, g > 0$
3	Power discounting	Harvey (1986)	$(1+t)^{-\alpha}, \alpha > 0$
4	Hyperbolic discounting	Loewenstein and Prelec (1992)	$(1+gt)^{-\alpha/\gamma}, \alpha > 0, g > 0$
5	Constant sensitivity	Ebert and Prelec (2007)	$\exp[-(at)^b], a > 0, 1 > b > 0$

Table A.1: Models of temporal behavior

i) The relation $\succsim$ defined on $\mathbb{X} \times \mathbb{T}$ satisfies properties A0-A4.

ii) For any $\delta \in (0,1)$ there exists a set $\mathscr{U}_\delta$ of monotinically increasing continuous functions such that

$$F(x,t) = \min_{u \in \mathscr{U}_\delta} (u^{-1}(\delta^t u(x)))$$

represents the binary relation $\succsim$. Moreover, $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathscr{U}_\delta$. $F(x,t)$ is continuous.

Proof: We start by showing (ii) implies (i). To show Weak Present Bias, we follow the following steps

$$\begin{aligned}
& (y,t) \succsim (x,0) \\
\Longrightarrow \quad & \min_{u \in \mathscr{U}_\delta}(u^{-1}(\delta^t u(y))) \geq \min_{u \in \mathscr{U}}(u^{-1}(u(x))) \\
\Longrightarrow \quad & \min_{u \in \mathscr{U}_\delta}(u^{-1}(\delta^t u(y))) \geq x \\
\Longrightarrow \quad & u^{-1}(\delta^t u(y)) \geq x \qquad \forall u \in \mathscr{U}_\delta
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad & \delta^t u(y) \geq u(x) && \forall u \in \mathscr{U}_\delta \\
\Longrightarrow \quad & \delta^{t+t_1} u(y) \geq \delta^{t_1} u(x) && \forall u \in \mathscr{U}_\delta \\
\Longrightarrow \quad & u^{-1}(\delta^{t+t_1} u(y)) \geq u^{-1}(\delta^{t_1} u(x)) && \forall u \in \mathscr{U}_\delta \\
\Longrightarrow \quad & \min_{u \in \mathscr{U}_\delta}(u^{-1}(\delta^{t+t_1} u(y))) \geq \min_{u \in \mathscr{U}_\delta}(u^{-1}(\delta^{t_1} u(x))) \\
\Longrightarrow \quad & (y, t+t_1) \succsim (x, t_1)
\end{aligned}$$

To show Monotonicity and Discounting, let us show $(x,t) \succ (y,s)$, when, either $x > y$ and $t = s$, or, $x = y$ and $t < s$. As all the functions $u \in \mathscr{U}_\delta$ are strictly increasing, and $\delta \in (0,1)$,

$$\begin{aligned}
\delta^t u(x) &> \delta^s u(y) \ \forall u \in \mathscr{U}_\delta \\
\Longleftrightarrow u^{-1}(\delta^t u(x)) &> u^{-1}(\delta^s u(y)) \ \forall u \in \mathscr{U}_\delta \\
\Longleftrightarrow \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x)) &> \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^s u(y)) \\
\Longleftrightarrow (x,t) &\succ (y,s)
\end{aligned}$$

For proving the second statement under Discounting, start with any $u_1 \in \mathscr{U}_\delta$. For $z > x > 0$, and $\delta \in (0,1)$ there must exist t big enough such that

$$\begin{aligned} u_1(x) &> \delta^t u_1(z) \\ \iff u_1^{-1}(u_1(x)) &> u_1^{-1}(\delta^t u_1(z)) \\ \iff x &> \min_{u \in \mathscr{U}_\delta} u^{-1}((\delta^t u(z)) \end{aligned}$$

Hence, there exists t big enough such that $(x,0) \succ (z,t)$.
That $\succsim$ satisfies continuity follows directly from the definition of continuity on the utility function.
Now, we will prove the other direction of the representation theorem. We will first deal with the case of $\mathbb{T} = [0,\infty)$. A similar proof technique would be used in the proof of Theorem 8.

Proof for the case when $\mathbb{T} = [0,\infty)$.

Proof. For every $x^* \in (0,M)$, we are going to provide an increasing utility function u_{x^*} on $[0,M]$ which would have $\delta^\tau u_{x^*}(x) \geq u_{x^*}(y)$ if $(x,\tau) \succsim (y,0)$. Additionally it would also have $\delta^t u_{x^*}(x_t) = u_{x^*}(x^*)$ for all $(x^*,0) \sim (x_t,t)$.
Fix $u_{x^*}(x^*) = 1$, $u_{x^*}(0) = 0$.
For any $x \in (x^*,M]$, by Discounting there exists a delay T large enough, such that $(x^*,0) \succ (x,T)$. Hence, it must be true that $(x,0) \succ (x^*,0) \succ (x,T)$. By Continuity there must exist $t(x) \in \mathbb{T}$ such that, $(x,t(x)) \sim (x^*,0)$. Define the utility at x as

$$u_{x^*}(x) = \delta^{-t(x)} \tag{A.1}$$

It would be helpful to introduce the following additional notation to move seamlessly between prizes and time in terms of indifference of time-prize pairs w.r.t $(x^*,0)$. For $t > 0$, define x_t as the value in $(x^*,M]$ such that $(x_t,t) \sim (x^*,0)$. Using continuity, we can say that all points in the interval $(x^*,M]$ can be written as x_t for some $t > 0$. This notation essentially implements the inverse of the $t(x)$ function defined in the previous paragraph.
Now, for $x \in (0,x^*)$, define

$$u_{x^*}(x) = \inf\{\delta^\tau : \text{ There exists } t \text{ such that } (x_t, t+\tau) \sim (x,0)\} \tag{A.2}$$

Firstly, we will show that the infimum in (A.2) can be replaced by minimum. Let the infimum be obtained at a value $I = \delta^{\tau^*}$. Consider a sequence of delays $\{\tau_n\}$ that converge above to τ^*, and $(x_{t_n}, t_n + \tau_n) \sim (x,0)$. Clearly, $\{t_n\}$ is the corresponding sequence of t's in (A.2). Note that $t_n \in [0,t_{max}]$ where $(x^*,0) \sim (M,t_{max})$. Hence, $\{t_n\}$ must lie in this compact interval, and must have a convergent subsequence

$\{t_{n_k}\}$. If t^* is the corresponding limit of $\{t_{n_k}\}$, we know that $t^* \in [0, t_{max}]$. Similarly, x_t can be considered a continuous function in t (this also follows from the continuity of $\succsim$). Therefore, $x_{t_{n_k}} \to x_{t^*}$ when $t_{n_k} \to t^*$. Thus, we have $(x_{t_{n_k}}, t_{n_k} + \tau_{n_k}) \sim (x, 1)$ for all elements of $\{n_k\}$. As, $n_k \to \infty$, $x_{t_{n_k}} \to x_{t^*}$, $t_{n_k} + \tau_{n_k} \to t^* + \tau^*$. Then, using the continuity of $\succsim$, $(x_{t^*}, t^* + \tau^*) \sim (x, 1)$. Hence, the infimum can be replaced by a minimum.

Now we will show that the utility defined in (A.1) and (A.2) has the following properties : 1) It is increasing. 2) $\delta^t u_{x^*}(x_t) = u_{x^*}(x^*)$ for all $(x^*, 0) \sim (x_t, t)$. 3) $(x, \tau) \succsim (y, 0)$ implies $\delta^\tau u_{x^*}(x) \geq u_{x^*}(y)$, 4) u is continuous. The first two properties are true by definition of u. We will show the third and fourth in some detail.

Consider $(x, \tau) \succsim (y, 0)$. In the case of interest, $\tau > 0$ and hence, $x > y$.

Now let $x > y > x^*$. Let, $u(y) = \delta^{-t_1}$, which means, $(y, t_1) \sim (x^*, 0)$. Given $(x, \tau) \succsim (y, 0)$, we must have

$$(x, \tau + t_1) \succsim (y, t_1) \sim (x^*, 0)$$

Hence, if $(x, t_2) \sim (x^*, 0)$, then,

$$\begin{aligned} t_2 &\geq \tau + t_1 \\ \iff u_{x^*}(x) = \delta^{-t_2} &\geq \delta^{-(\tau + t_1)} \\ \iff \delta^\tau u_{x^*}(x) &\geq \delta^{-t_1} = u_{x^*}(y) \end{aligned}$$

If, $x > x^* > y$, the proof follows from the way the utility has been defined.

Let $y < x < x^*$. Let, $u_{x^*}(x) = \delta^{t_1}$, which means, $(x_t, t + t_1) \sim (x, 0)$ for some $x_t \in [x^*, M]$. Given $(x, \tau) \succsim (y, 0)$, we must have

$$(x_t, t + t_1 + \tau) \succsim (x, \tau) \succsim (y, 0)$$

Hence, $u_{x^*}(y) \leq \delta^{\tau + t_1} = \delta^\tau u_{x^*}(x)$.

Now we turn to proving the continuity of u_{x^*}. The continuity at x^* from the right, or on $(x^*, M]$ is easy to see.

Next, for any $r = \delta^s \in (0, 1)$, define

$$f(r) = \sup\{y : \ (x_t, t + s) \sim (y, 0)\} = \hat{y} \tag{A.3}$$

The supremum can be replaced by a maximum, and the proof is similar to the one before. Suppose there is a sequence of $\{y_n\}$ that converges up to a value $\hat{y}$, and, $(x_{t_n}, t_n + s) \sim (y_n, 1)$. Note that t_n lies in a compact interval $[0, t_{max}]$, and hence has a convergent subsequence t_{n_k} that converges to a point in that interval $\hat{t} \in [0, t_{max}]$. Now, x_t is continuous in t (in the usual sense), and hence, x_{t_n} also converges to $x_{\hat{t}}$. Further, $y_{n_k} \to \hat{y}$ as $n_k \to \infty$. Therefore, using, $(x_{t_{n_k}}, t_{n_k} + s) \sim (y_{n_k}, 0)$, as, $n_k \to \infty$,

it must be that $(x_{\hat{t}}, \hat{t}+s) \sim (\hat{y}, 0)$. Hence, the supremum in (A.3) must have been attained from $x_{\hat{t}}$, and hence the supremum can be replaced by a maximum. Further given this is a maximum, we can say that $\hat{y} \in (0, x^*)$. The f function is well defined, strictly increasing and is the inverse function of u_{x^*} over $r \in (0,1)$ to $(0, x^*)$, in the sense that, $u(f(r)) = r$. This function can be used to show the continuity of u at the point x^*.
Finally, the function u can be easily normalized to have $u_{x^*}(M) = 1$. (By dividing the function from before by $u_{x^*}(M)$.)
Now, consider $\mathscr{U}_\delta = \{u_{x^*}() : \; x^* \in (0, M]\}$. By construction of the functions, it must be that

$$\begin{aligned} (x,t) \succsim (y,0) &\iff \delta^t u(x) \geq u(y) \; \forall u \in \mathscr{U}_\delta \\ (x,t) \sim (y,0) &\iff \delta^t u(x) \geq u(y) \; \forall u \in \mathscr{U}_\delta \\ &\qquad \text{and } \delta^t u_y(x) = u_y(y) \text{ for some } u_y \in \mathscr{U}_\delta \end{aligned}$$

For any (z,τ), consider the sets $\{(y,0) \in \mathbb{X} \times \mathbb{T} : (y,0) \succsim (z,\tau)\}$ and $\{(y,0) \in \mathbb{X} \times \mathbb{T} : (z,\tau) \succsim (y,0)\}$. Both are non-empty, as $(z,0)$ belongs to the first one and $(0,0)$ in the second one. Both sets are closed in the product topology. Their union is connected, and hence there exists an element in their intersection, i.e, there exists a $y_1 \in \mathbb{X}$ such that $(y_1, 0) \sim (x,t)$. By monotonicity this y_1 must be unique. Therefore there must exist a continuous present equivalent utility representation for $\succsim$. We show this formally in the next two paragraphs.
Given $\succsim$ is complete, transitive and satisfies continuity, there exists a continuous function $\bar{F} : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ such that $\bar{F}(a) \geq \bar{F}(b)$ if and only if $a \succsim b$ for $a, b \in \mathbb{X} \times \mathbb{T}$. (Following Theorem 1, Fishburn and Rubinstein [1982]).
We define $G : \mathbb{X} \to \mathbb{R}$ as $G(x) = \bar{F}(x,0)$. The function G would be strictly monotonic and continuous. Also define $F : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ as $F(x,t) = G^{-1}(\bar{F}(x,t))$. As any alternative has a unique present equivalent, F is well defined, is a monotonic continuous transformation of $\bar{F}$ (hence represents $\succsim$) and $F(x,0) = x$ for all $x \in \mathbb{X}$. By definition the function F assigns to every alternative its present equivalent as the corresponding utility. Therefore, the present equivalent utility representation is continuous.
We will show that the function W defined below also assigns to every alternative (z,τ) an utility exactly equal to its present equivalent.

$$W(x,t) \;=\; \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x)) = F(x,t)$$

Consider any $(z,\tau) \sim (y_1, 0)$. By definition of $\mathscr{U}_\delta$ and by construction of its constituent functions, it must be that for all $u \in \mathscr{U}_\delta$, $\delta^\tau u(z) \geq u(y_1)$ and there exists a

function u_{y_1} such that $\delta^\tau u_{y_1}(z) = u(y_1)$. This is equivalent to the following statement: For all $u \in \mathscr{U}_\delta$, $u^{-1}(\delta^\tau u(z)) \geq y_1$ and there exists a function u_{y_1} such that $u_{y_1}^{-1}(\delta^\tau u_{y_1}(z)) = y_1$.
Therefore, $W(z,\tau) = \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^\tau u(z))$ is continuous utility representation of the relation $\succsim$.

□

Proof for the case of $\mathbb{T}$ =$\{0,1,2,...\}$.
This proof would be more technical and we will break down the proof of this case into the following Lemmas.

Lemma 23. *Under Axioms A0-A4, for a fixed x_0, and any x_t and t such that $(x_t,t) \sim (x_0,0)$, there exists a continuous strictly increasing function u such that $\delta^t u(x_t) = u(x_0)$ and $\delta^t u(z_1) \geq u(z_0)$ for all $(z_1,t) \succsim (z_0,0)$. Further, $u(0) = 0$, $u(M) = 1$.*

Proof. By the Discounting axiom, we know that there exists a smallest integer $n \geq 1$ such that $(x_0,0) \succsim (M,n)$. Choose $x_0^* = x_0$. For $0 < t < n$, find x_t^* such that $(x_0^*,0) \sim (x_t^*,t)$. If $(x_0,0) \succ (M,n)$, choose $x_n = M$.
We define x_{-1}^* in the following way

$$x_{-1}^* = \min\{x \in \mathbb{X} : (x,0) \succsim (x_j^*, j+1),\ j = 0,1,2,...n\}$$

The idea is to look at the present equivalents of $(x_j^*, j+1)$ and take the maximum of those present equivalents. The alternative way to express the same is to look at the intersection of the weak upper counter sets of $(x_j^*, j+1)$ on $\mathbb{X} \times \{0\}$, and then take the minimal value from that set.
Next we will use this to define x_{-2}^*, then use x_{-1}^* and x_{-2}^* to define x_{-3}^*. In general, for $i \in \{-1,-2,-3...\}$ define x_i^* recursively as the minimum of the set

$$\{x \in \mathbb{X} : (x,0) \succsim (x_j^*, j-i), j = i+1, i+2, ...n\}$$

The definition uses the same idea as before. We consider the intersection of the weak upper counter sets of $(x_j^*, j-i)$ on $\mathbb{X} \times \{0\}$ and take its minimum. The set is non-empty (x_0^* belongs to it, for example), closed and the minimum exists due to the continuity, monotonicity and discounting properties.
Next we show that for every x_i^* with $i \leq -1$, there exists $j \in \{0,1,..n\}$ such that $(x_j^*, j-i) \succsim (x_i^*,0)$**.** The proof is by induction. For $i = -1$, it is immediate from the definition. Suppose, it holds for all $i \geq -m$. Consider x_{-i-1}^*. By construction, there must exist $k \in \{-m,-m+1,..n\}$ such that $(x_{-i-1}^*,0) \sim (x_k^*, k+i+1)$.

If $k \in \{0,1,..n\}$ we are done already. If not, by the induction hypothesis, there exists $j \in \{0,1,..n\}$ such that $(x_j^*, j-k) \succsim (x_k^*, 0)$, which gives, $(x_j^*, j+i+1) \succsim (x_k^*, k+i+1)$, and hence, $(x_j^*, j+i+1) \succsim (x_{-i-1}^*, 0)$, completing the proof.
Next we will show that the sequence $\{...x_{-2}^*, x_{-1}^*, x_0^*, x_1^*, x_2^*, ..\}$ is converges below to 0. Suppose not (we are going for a proof by contradiction), that is there exists $w > 0$ such that $x_i \geq w$ for all $i \in \mathbb{Z}$. As, $M > z > 0$, there must exist t_1 big enough such that $(z,0) \succ (M,t_1)$. Consider the element $x_{-t_1}^*$ from the sequence in consideration. Using the result from the previous paragraph, it must be true that there exists $j \in \{0,1,..,n\}$, such that $(x_j^*, j+t_1) \succsim (x_{-t_1}^*, 0)$. Now, as $M \geq x_j^*$, we must have, $(M,t_1) \succsim (x_{-t_1}^*, 0) \succ (z,0)$, which provides a contradiction.
Consider any $y_0 \in (x_0^*, x_1^*)$.
We are going to find a $y_1, y_2, ..y_{n-1}$ recursively.
Finding y_1: For each point $y \in (x_1, x_2]$, take reflections of length 1, i.e, find x_y such that $(y,1) \sim (x_y, 0)$. Note that, $(x_1^*, 0) \succ (y,1) \succ (x_0^*, 0)$. Hence, $x_y \in (x_0^*, x_1^*)$. Let, x_{x_2} be the reflection for the point x_2. For any $y \in (x_1^*, x_2^*]$, $f(y) = x_0^* + (x_y - x_0^*)\dfrac{(x_1^* - x_0^*)}{(x_{x_2} - x_0^*)}$. Now, for $y_0 \in (x_0^*, x_1^*)$, define y_1 as $f^{-1}(y_0)$.
We can check that this method satisfies the 2 following conditions:

1) Consider two such sequences, one starting from y_0^1, and another from y_0^2, with $y_0^1 > y_0^2$. We will have $y_1^1 > y_1^2$.

2) All points in intervals (x_1^*, x_2^*) are included by some y_1 from the sequence. This follows from monotonicity and discounting too.

Now, the recursive step:
For each point $y \in (x_i^*, x_{i+1}^*]$, take reflections of length $j \in \{i, i-1.., 1\}$ conditional on those reflections being in the corresponding $(x_{i-j}^*, x_{i+1-j}^*]$ intervals. For any y, at least one of these reflections must exist, and in particular the one with length i always exists, as $(x_1^*, 0) \succ (x_{i+1}^*, i) \succsim (y,i)$ and $(y,i) \succ (x_i^*, i) \sim (x_0^*, 0)$.
Now, for each such reflection, find the corresponding sequence of $\{y_0, y_1, ..y_{i-1}\}$ it belongs to, and denote the smallest y_0 from that collection of sequences as $x_y \in [x_0^*, x_1^*]$. Note that $x_{x_{i+1}} \leq x_1^*$. Define the $1:1$ strictly increasing function f from $(x_{n-1}, x_n]$ to $(x_i^*, x_{i+1}^*]$ in the following way: For any $y \in (x_i^*, x_{i+1}^*]$, $f(y) = x_0^* + (x_y - x_0^*)\dfrac{(x_1^* - x_0^*)}{(x_{x_{i+1}} - x_0^*)}$. Now, define y_i as $f^{-1}(y_0)$. The conditions mentioned above are still satisfied for the extended sequence.
For $i \leq -1$, define y_i recursively in the following way. Start by finding y_i' as the minimum of the set

$$\{y \in \mathbb{X} : (y,0) \succsim (y_j, j-i), j = i+1, i+2, ...n\}$$

Define x_{-i}' as the minimum of the set

$$\{y \in \mathbb{X} : (y,0) \succsim (y_j, j-i), j = i+1, i+2, ...n-1\}$$

Finally, define

$$y_i = x^*_{i+1} - (x^*_{i+1} - y'_i)\frac{(x^*_{i+1} - x^*_i)}{(x^*_{i+1} - x'_i)} \tag{A.4}$$

Given $y^1_0 > y^2_0$ determines the order of $y^1_t > y^2_t$, for $t \in \{1, 2, ..n-2\}$, our inductive procedure make sure this holds true for all $t \leq -1$ too.

One can check for covering properties of the sequences by induction. Suppose all points in the intervals (x^*_i, x^*_{i+1}) are covered by y_i for some sequence, for $i \geq j$ for some integer j. We are going to show that all points in (x^*_{j-1}, x^*_j) are also covered by y_{j-1} for some sequence. Take any point $y \in (x^*_{j-1}, x^*_j)$, and consider its corresponding y' as defined in Equation A.4. Consider the reflections from point y' of sizes $1, ..n-j+1$, i.e, the points at those temporal distances which are indifferent to it, conditional on being in the corresponding intervals. By the induction hypothesis, each of those reflection end points must be coming from some $y_0 \in (x^*_0, x^*_1)$. Take the sequence with smallest y_0, and that sequence would result in having $y \in (x^*_{j-1}, x^*_j)$ as its next element.

Now, define u on $\mathbb{X}$ as follows: Set $u(x^*_n) = u(x^*_n) = 1$. For the sequence $..., x^*_{-2}, x^*_{-1}, x^*_0, x^*_1, ..$, let $u(x^*_i) = \delta^{i-n}$ for all positive and negative integers i. Next, let us define u on (x^*_{n-1}, x^*_n) as any continuous and increasing function with $\inf_{(x^*_{n-1}, x^*_n)} u(x) = \delta = u(x^*_{n-1})$ and $\sup_{(x^*_{n-1}, x^*_n)} u(x) = 1 = u(M)$. We can extend each dual sequence with some as $u(y_i) = \delta^{i-n} u(y_0)$. This finishes the construction of a u that satisfies the conditions mentioned in the Lemma. □

Lemma 24. *Under Axioms A0-A4, there exists a continuous present equivalent utility function $F : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ that represents $\succsim$. Moreover, for $\delta \in (0,1)$, $F(z, \tau) = \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^\tau u(z))$ for some set $\mathscr{U}_\delta$ of strictly monotonic, continuous functions, $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathscr{U}_\delta$.*

Proof. Consider the set $\mathscr{U}_\delta$ of all strictly monotonic, continuous functions u such that $\delta^t u(z_1) \geq u(z_0)$ for all $(z_1, t) \succsim (z_0, 0)$, $u(0) = 0$ and $u(M) = 1$. By the previous Lemma, this set is non-empty, and for any $(z_1, t) \sim (z_0, 0)$ includes a function u, such that $\delta^t u(z_1) = u(z_0)$. By construction of the functions, it must be that

$$\begin{aligned}
(x,t) \succsim (y,0) &\iff \delta^t u(x) \geq u(y)\ \forall u \in \mathscr{U}_\delta \\
(x,t) \sim (y,0) &\iff \delta^t u(x) \geq u(y)\ \forall u \in \mathscr{U}_\delta \\
&\qquad \text{and } \delta^t u_y(x) = u_y(y) \text{ for some } u_y \in \mathscr{U}_\delta
\end{aligned}$$

For any (z,τ), consider the sets $\{(y,0) \in \mathbb{X} \times \mathbb{T} : (y,0) \succsim (z,\tau)\}$ and $\{(y,0) \in \mathbb{X} \times \mathbb{T} : (z,\tau) \succsim (y,0)\}$. Both are non-empty, as $(z,0)$ belongs to the first one and $(0,0)$ in the second one. Both sets are closed in the product topology. Their union is connected, and hence there exists an element in their intersection, i.e, there exists a $y_1 \in \mathbb{X}$ such that $(y_1,0) \sim (x,t)$. By monotonicity this y_1 must be unique. Therefore there must exist a continuous present equivalent utility representation for $\succsim$. We show this formally in the next two paragraphs.
Given $\succsim$ is complete, transitive and satisfies continuity, there exists a continuous function $\bar{F} : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ such that $\bar{F}(a) \geq \bar{F}(b)$ if and only if $a \succsim b$ for $a,b \in \mathbb{X} \times \mathbb{T}$. (Following Theorem 1, Fishburn and Rubinstein [1982]).
We define $G : \mathbb{X} \to \mathbb{R}$ as $G(x) = \bar{F}(x,0)$. The function G would be strictly monotonic and continuous. Also define $F : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ as $F(x,t) = G^{-1}(\bar{F}(x,t))$. As any alternative has a unique present equivalent, F is well defined, is a monotonic continuous transformation of $\bar{F}$ (hence represents $\succsim$) and $F(x,0) = x$ for all $x \in \mathbb{X}$. By definition the function F assigns to every alternative its present equivalent as the corresponding utility. Therefore, the present equivalent utility representation is continuous.
We will show that the function W defined below also assigns to every alternative (z,τ) an utility exactly equal to its present equivalent.

$$W(x,t) \quad = \quad \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x)) = F(x,t)$$

Consider any $(z,\tau) \sim (y_1,0)$. By definition of $\mathscr{U}_\delta$ and by construction of its constituent functions, it must be that for all $u \in \mathscr{U}_\delta$, $\delta^\tau u(z) \geq u(y_1)$ and there exists a function u_{y_1} such that $\delta^\tau u_{y_1}(z) = u(y_1)$. This is equivalent to the following statement: For all $u \in \mathscr{U}_\delta$, $u^{-1}(\delta^\tau u(z)) \geq y_1$ and there exists a function u_{y_1} such that $u_{y_1}^{-1}(\delta^\tau u_{y_1}(z)) = y_1$.
Therefore, $W(z,\tau) = \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^\tau u(z)) = F(z,\tau)$ is a continuous utility representation of the relation $\succsim$. □

Proposition 1: *Given the axioms A0-4, the representation form in (1.3) is unique in the discounting function* $\Delta(t) = \delta^t$ *inside the present equivalent function.*

Proof. We start with the case where $\Delta(t)$ is such that $\dfrac{\Delta(t+t_1)}{\Delta(t)} < \Delta(t_1)$ for some

t, t_1. Consider any singleton $\mathscr{U} = \{u\}$.

$$
\begin{aligned}
(y,t) &\sim (x,0) \\
\implies u^{-1}(\Delta(t)u(y)) &= x \\
\implies \Delta(t)u(y) &= u(x) \\
\implies \Delta(t+t_1)u(y) &= \frac{\Delta(t+t_1)}{\Delta(t)}u(x) < \Delta(t_1)u(x) \\
\implies u^{-1}(\Delta(t+t_1)u(y)) &< u^{-1}(\Delta(t_1)u(x)) \\
\implies (x,t_1) &\succ (y,t+t_1)
\end{aligned}
$$

Hence, the relation implied by the representation contradicts Weak Present Bias.

Now assume the opposite, let there exists some $t, t_1 > 0$ such that $\frac{\Delta(t+t_1)}{\Delta(t)} > \Delta(t_1)$. Now suppose we started with a relation $\succsim$ which has $(y,t) \sim (x,0)$ as well as $(y,t+t_1) \sim (x,t_1)$ for all t, t_1 and some x, y. (This does not necessarily mean that the person's preferences satisfy stationarity in the broader sense as we do not ask this from all x, y.) We will show below that such preferences cannot be represented by the functional form we started with for any set of functions $\mathscr{U}$.

$$
\begin{aligned}
(y,t) &\sim (x,0) \\
\implies \min_{u\in\mathscr{U}}(u^{-1}(\Delta(t)u(y))) &\geq \min_{u\in\mathscr{U}}(u^{-1}(u(x))) = x \\
\implies \Delta(t)u(y) &\geq u(x) \;\; \forall u \in \mathscr{U} \\
\implies \Delta(t+t_1)u(y) &\geq \frac{\Delta(t+t_1)}{\Delta(t)}u(x) > \Delta(t_1)u(x) \;\; \forall u \in \mathscr{U} \\
\implies u^{-1}(\Delta(t+t_1)u(y)) &> u^{-1}(\Delta(t_1)u(x)) \;\; \forall u \in \mathscr{U} \\
\implies \min_{u\in\mathscr{U}}(u^{-1}(\Delta(t+t_1)u(y))) &> \min_{u\in\mathscr{U}}(u^{-1}(\Delta(t_1)u(x))) \\
\implies (y,t+t_1) &\succ (x,t_1)
\end{aligned}
$$

This completes our proof. □

Proposition 2: *If $\mathscr{U}, \mathscr{U}' \subset \mathscr{F}$ are such that $\bar{co}(\mathscr{U}) = \bar{co}(\mathscr{U}')$, and the functional form in (1.1) allows for a continuous minimum representation for both of those sets, then,* $\min_{u\in\mathscr{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathscr{U}'} u^{-1}(\delta^t u(x))$.

Proof. We will prove this in 2 steps.
First we will show that for any set A, $\min_{u\in A} u^{-1}(\delta^t u(x)) = \min_{u\in\bar{A}} u^{-1}(\delta^t u(x))$, where $\bar{A}$ is the closure of the set A.
It is easy to see the direction that $\min_{u\in A} u^{-1}(\delta^t u(x)) \geq \min_{u\in\bar{A}} u^{-1}(\delta^t u(x))$.
We will prove the other direction by contradiction. Suppose, $\min_{u\in A} u^{-1}(\delta^t u(x)) >$

$\min_{u\in\bar{A}} u^{-1}(\delta^t u(x))$. This would imply that there exists $v \in \bar{A}\backslash A$ and some $\varepsilon > 0$, such that $v^{-1}(\delta^t v(x)) + \varepsilon < u^{-1}(\delta^t u(x))$ for all $u \in A$. By definition of the topology of compact convergence and given that v belongs to the set of limit points of A, there must exist a sequence of functions $\{v_n\} \subset A$ which converges to v in the topology of compact convergence , i.e, for any compact set $K \subset \mathbb{R}_+$, $\lim_{n\to\infty} \sup_{x\in K} |v_n(x) - v(x)| = 0$. It can be shown that under this condition, $v_n^{-1}(\delta^t v_n(x))$ would also converge to $v^{-1}(\delta^t v(x))$ where $v_n \in \mathscr{U}$.[55] This constitutes a violation of $v^{-1}(\delta^t v(x)) + \varepsilon < u^{-1}(\delta^t u(x))$ for all $u \in A$. Hence, it must bemin$_{u\in A} u^{-1}(\delta^t u(x)) = \min_{u\in\bar{A}} u^{-1}(\delta^t u(x))$.
As a second part of this proof, we will show that for any set A, $\min_{u\in A}(u^{-1}(\delta^t u(x))) = \min_{u\in co(A)}(u^{-1}(\delta^t u(x)))$.
It is easy to see that $\min_{u\in A}(u^{-1}(\delta^t u(x))) \geq \min_{u\in co(A)} u^{-1}(\delta^t u(x))$, as $A \subset co(A)$.
We will again use proof by contradiction to show the opposite direction. We assume that there exists a $\bar{u} \in \bar{co}(A)$ and $(x,t) \in \mathbb{X} \times \mathbb{T}$, such that $\bar{u} = \sum_{i=1}^n \lambda_i u_i$, $\sum_{i=1}^n \lambda_i = 1$ and $\bar{u}^{-1}(\delta^s \bar{u}(y)) < \min_i u_i^{-1}(\delta^s u_i(y))$. This would imply that $u_i(\bar{u}^{-1}(\delta^s \bar{u}(y))) < \delta^s u_i(y)$ for all i.
Now,

$$\begin{aligned}
\delta^s \bar{u}(y) &= \delta^s \sum_i \lambda_i u_i(y) \\
&= \sum_i \lambda_i \delta^s u_i(y) \\
&> \sum_i \lambda_i u_i(\bar{u}^{-1}(\delta^s \bar{u}(y))) \\
&= \bar{u}(\bar{u}^{-1}(\delta^s \bar{u}(y))) \\
&= \delta^s \bar{u}(y)
\end{aligned}$$

This gives us a contradiction. Note that the equality right after the inequality comes from the definition of $\bar{u}$.
Hence, we have, $\min_{u\in A} u^{-1}(\delta^t u(x)) = \min_{u\in co(A)} u^{-1}(\delta^t u(x))$. □

Proposition 3: *i) If there exists a concave function $f \in \mathscr{U}$, and if $\mathscr{U}_1$ is the subset of convex functions in $\mathscr{U}$, then* $\min_{u\in\mathscr{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathscr{U}\backslash\mathscr{U}_1} u^{-1}(\delta^t u(x))$.
ii) If $u_1, u_2 \in \mathscr{U}$ and u_1 is concave relative to u_2, then, $\min_{u\in\mathscr{U}} u^{-1}(\delta^t u(x)) = \min_{u\in\mathscr{U}\backslash\{u_2\}} u^{-1}(\delta^t u(x))$.

[55] As, $v_n \to v$ in the topology of compact convergence, $v_n \to v$ point wise, hence, $\delta^t v_n(x) \to \delta^t v(x)$. Now, as $v_n^{-1} \to v^{-1}$ compact convergence (proof later in the appendix), $v_n^{-1}(\delta^t v_n(x)) \to v^{-1}(\delta^t v(x))$.

Proof. If a function u is convex,

$$\begin{aligned} u^{-1}(\delta^t u(x)) &= u^{-1}(\delta^t u(x) + (1-\delta^t)u(0)) \\ &\geq u^{-1}(u(\delta^t x + (1-\delta^t)0)) \\ &= \delta^t x \end{aligned}$$

Similarly for concave f , we would have, $f^{-1}(\delta^t f(x)) \leq \delta^t x$ which completes the proof of part (i). Note that this result is expected given concave functions give rise to more conservative present equivalents.
For part (ii), note that

$$\begin{aligned} u_1^{-1}(\delta^t u_1(x)) &= u_1^{-1}(\delta^t u_1(u_2^{-1}(u_2(x)))) \\ &\leq u_1^{-1}(u_1(u_2^{-1}(\delta^t u_2(x)))) \\ &= u_2^{-1}(\delta^t u_2(x)) \end{aligned}$$

Where the inequality arises from the fact that u_1 is concave relative to u_2. □

Proposition 25. *Eventual stationarity is satisfied by β-δ discounting, but not hyperbolic discounting.*

Now for any $x > z > 0 \in X$, choose $t_1 > \log_{\frac{1}{\delta}} \left(\frac{u(x)}{u(z)} \right)$.

$$\begin{aligned} t_1 &> \log_{\frac{1}{\delta}} \left(\frac{u(x)}{u(z)} \right) \\ \iff \left(\frac{1}{\delta} \right)^{t_1} &> \frac{u(x)}{u(z)} \\ \implies u(z) &> \delta^{t_1} u(x) > \beta \delta^{t_1} u(x) \\ \implies \beta \delta^t u(z) &> \beta \delta^{t+t_1} u(x) \\ (z,t) &\succ (x, t+t_1) \end{aligned}$$

Also, $(x,0) \sim (x_t, t)$ implies, $u(x) = \beta \delta^t u(x_t)$, which implies,

$$\begin{aligned} u(z) &> \delta^{t_1} u(x) = \beta \delta^{t+t_1} u(x_t) \\ (z,0) &\succ (x_t, t+t_1) \end{aligned}$$

This shows that $\beta - \delta$ does indeed satisfy A5.
Now consider the simple variant of Hyperbolic discounting model when $\alpha = \gamma = 1$. Fix any felicity function u and $x > z > 0 \in X$. We will show that there does not exist t_1,such that $(z,t) \succ (x, t+t_1)$ for all $t \geq 0$.

$$\begin{aligned}
(z,t) &\succ (x,t+t_1) \text{ for all } t \geq 0 \\
\iff \frac{u(z)}{1+t} &> \frac{u(x)}{1+t+t_1} \text{ for all } t \geq 0 \\
\iff \frac{1+t+t_1}{1+t} &> \frac{u(x)}{u(z)} \text{ for all } t \geq 0 \\
\iff 1+\frac{t_1}{1+t} &> \frac{u(x)}{u(z)} \text{ for all } t \geq 0
\end{aligned}$$

Note that the last statement is not possible, as for fixed t_1 the LHS$\downarrow$ 1 as $t \uparrow \infty$, whereas, the RHS is always a fixed number, that is strictly greater than one. Hence, hyperbolic discounting does not satisfy A5.

Theorem 6: *The following two statements are equivalent:*
i) The relation $\succsim$ satisfies properties A0-A6.
ii) There exists a set $\mathscr{U}_\delta$ of monotinically increasing continuous functions such that

$$F(x,t) = \min_{u \in \mathscr{U}} u^{-1}(\delta^t u(x))$$

represents the binary relation $\succsim$. The set $\mathscr{U}$ has the following properties: $u(0) = 0$ for all $u \in \mathscr{U}$, $\sup_u u(x)$ is bounded above, $\inf_u u(x) > 0 \; \forall x$, $\inf_u \dfrac{u(z)}{u(x)}$ is unbounded in z for all $x > 0$.

Proof : Going from (ii) to (i) :
That (ii) implies Monotonicity, Discounting, Weak Present Bias and Continuity has already been shown in the proof of Theorem 3.
Showing Eventual Stationarity: Given $\sup_u u(x)$ is bounded above and $\inf_u u(x) > 0$, for any choice of $x,z > 0$ and $\delta \in (0,1)$ there exists $t_1 > 0$ big enough such that $\inf_u u(z) > \delta^{t_1} \inf_u u(x)$. This would imply that, *for all $u \in \mathscr{U}$*,

$$u(z) \quad > \quad \delta^{t_1} u(x)$$

and, hence, $(z,0) \succ (x,t_1)$.
Now, for $t > 0$ consider x_t such that $(x_t,t) \sim (x,0)$. By the representation, this implies that there exists $u_1 \in \mathscr{U}$ such that

$$\begin{aligned}
\delta^t u_1(x_t) &= u_1(x) \\
\implies \delta^{t+t_1} u_1(x_t) &= \delta^{t_1} u_1(x) < u_1(z) \\
\implies \min_u u^{-1}(\delta^{t+t_1} u_1(x_t)) &\leq u_1^{-1}(\delta^{t+t_1} u_1(x_t)) < u_1^{-1}(u_1(z)) = \min_u u^{-1}(u(z))
\end{aligned}$$

Hence, $(z,0) \succ (x_t, t+t_1)$.
Similarly, *for all* $u \in \mathscr{U}$,

$$\begin{aligned} \delta^t u(z) &> \delta^{t+t_1} u(x) \\ \implies \min_u u^{-1}(\delta^t u(z)) &> \min_u u^{-1}(\delta^{t+t_1} u(x)) \end{aligned}$$

Hence, $(z,t) \succ (x, t+t_1)$.
Showing Non-triviality: We have that $\inf_u \dfrac{u(z)}{u(x)}$ is unbounded in z for all $x > 0$.
Therefore, for any x, and $t \in \mathbb{T}$, there exists z, such that

$$\begin{aligned} \inf_u \frac{u(z)}{u(x)} &> \delta^{-t} \\ \implies \frac{u(z)}{u(x)} &> \delta^{-t} \ \forall u \in \mathscr{U} \\ \implies \delta^t u(z) &> u(x) \ \forall u \in \mathscr{U} \\ \implies u^{-1}(\delta^t u(z)) &> u^{-1}(u(x)) \ \forall u \in \mathscr{U} \\ \implies \min_u u^{-1}(\delta^t u(z)) &> \min_u u^{-1}(u(x)) \\ (z,t) &\succ (x,0) \end{aligned}$$

To go from the direction (i) to (ii) of Theorem 6, one needs to follow Lemma 26-28.

Lemma 26. *Under Axioms A1-A6, for any* $(x_0,t),(x_t,0)$ *such that* $(x_0,t) \sim (x_t,0)$ *in the original relation, there exists* $u \in \mathscr{U}$ *such that* $\delta^t u(x_t) = u(x_0)$ *and* $\delta^t u(z_1) \geq u(z_0)$ *for all* $(z_1,t) \geq (z_0,0)$*. Moreover, u is strictly monotonic, continuous, and* $u(0) = 0$*,* $u(1) = 1$*.*

Proof. We will prove it for $t = 1$, $x_0, x_t > 0$ and then show the general guideline for a general t.
We define the following procedure: Choose $x_0^* = 1$. Find x_1^* such that $(x_0^*,0) \sim (x_1^*,1)$. We can do it because of the Non-Triviality assumption. Clearly, $x_1^* = x_1$. Next find $x_{-1}^* = \max\{x_{-1}, x'_{-1}\}$ where $(x_0^*,1) \sim_v (x_{-1},0)$ and $(x_1^*,2) \sim_v (x'_{-1},0)$. The value $x_{-1} > 0$ exists because, $(x_0^*,0) \succ (x_0^*,1) \succ (0,1)$, coupled with the fact that $\succsim$ is continuous. Same with x'_{-1}.
Note that $x_0^* > x_{-1}^*$ by discounting. Next going in the opposite direction, we find $x_2^* = \min\{x_2, x'_2, x''_2\}$, where, $(x_1^*,0) \sim_v (x_2,1)$, $(x_0^*,0) \sim_v (x'_2,2)$ and $(x_{-1}^*,0) \sim_v (x''_2,3)$. Next we find $x_{-2}^*, x_3^*, x_{-3}^*, x_4^*, \ldots$ sequentially. Thus one can find a sequence $\ldots x_{-3}^* < x_{-2}^* < x_{-1}^* < x_0^* < x_1^* < x_2^* \ldots$

We will show that this sequence is unbounded above and converges below to 0 . Consider any $z < x_0^*$. By A5, there must exist t_1 such that $(z,0) \succ (x_0^*, t_1)$. and given for any $t > 0$, $(x_0, 0) \succsim (x_t^*, t)$, by monotonicity, it must hold that $(z,0) \succ (x_t^*, t+t_1)$. By definition of x_{-1}^*, either $(x_{-1}^*, 0) \sim (x_0^*, 1)$ or $(x_{-1}^*, 0) \sim (x_1^*, 2)$, if not both. So, by WPB, either $(x_0^*, t_1) \succsim (x_{-1}^*, t_1 - 1)$ or $(x_1^*, t_1 + 1) \succsim (x_{-1}^*, t_1 - 1)$, and hence, either $(z,0) \succ (x_{-1}^*, t_1 - 1)$. One can use the construction of the sequence, and induction, here on, to show that, for any general $0 < i < t_1$, $(z,0) \succ (x_{-i}^*, t_1 - i)$. Hence, it must be that $x_{-t_1}^* \leq z$, which proves that the sequence converges below to zero. To show that the sequence is unbounded above, one uses a similar trick. Consider $z > x_0^*$. There must exist t_2 such that $(x_0^*, t) \succ (z, t+t_2)$ for all $t \geq 0$, and given for any $t > 0$, $(x_{-t}^*, 0) \succsim (x_0^*, t)$, by monotonicity, it must hold that $(x_{-t}^*, 0) \succsim (x_0^*, t) \succ (z, t+t_2)$. By definition of x_1^*, $(x_1^*, 1) \sim (x_0^*, 0) \succ (z, t_2)$. So, by WPB, it must be that $(x_1^*, 0) \succ (z, t_2 - 1)$. ($z < x_1^*$ is trivial and hence neglected). One can use the construction of the sequence, and induction, here on, to show that, for any general $0 < i < t_2$, $(x_i^*, 0) \succ (z, t_2 - i)$. Hence, it must be that $x_{t_2}^* \geq z$, which proves that the sequence diverges to infinity.

Consider any $y_0 \in (x_0^*, x_1^*)$. We find y'_{-1} such that $(y'_{-1}, 0) \sim (y_0^*, 1)$. Finally,

$$y_{-1}^* = x_0^* - (x_0^* - y'_{-1}) \frac{(x_0^* - x_{-1}^*)}{(x_0^* - x_{-1})} \in (x_{-1}^*, x_0^*).$$

The upper bound on y_{-1}^* comes from the fact that $(x_0^* > y'_{-1})$ and the lower bound comes from the fact that y'_{-1} is bounded below by x_{-1}. Note that for $y_0^*, \hat{y}_0 \in (x_0^*, x_1^*)$, $y_0^* > \hat{y}_0$ if and only if $y_{-1}^* > \hat{y}_{-1}$. And finally, for any $y_{-1}^* \in (x_{-1}^*, x_0^*)$ there exists a $y_0^* \in (x_0^*, x_1^*)$ corresponding to it.

Next we will define an inductive procedure to find the other points in such sequences. Let $\mathscr{S}$ be the set of all such sequences. The induction hypothesis is that for every $y_0^* \in (x_0^*, x_1^*)$ we have already defined a corresponding chain[56] $\mathscr{S}_i = y_{-i}^* < \ldots y_{-3}^* < y_{-2}^* < y_{-1}^* < y_0^* < y_1^* < y_2^* .. < y_{i-1}^*$, $i \geq 2$ such that i) $y_n^* \in (x_n^*, x_{n+1}^*)$ for all the elements of all the chains. ii) If we compare the n^{th} elements of 2 chains they are always similarly ranked, regardless of the value of n. iii) If the last element constructed is y_i^* for $i \in \mathbb{N}$ then, any point in (x_n, x_{n+1}) for $n \in \{-i, ..i-1\}$ is part of exactly one chain in $\mathscr{S}_i$.

Finding y_i^* where $i \geq 1$: Note that we can write $x_i^* = \min\{x_i^1, x_i^2, x_i^3 ... x_i^{2i}\}$[57], where $(x_i^1, 1) \sim (x_{i-1}^*, 0), (x_i^2, 2) \sim (x_{i-2}^*, 0).., (x_i^{2i-1}, 2i-1) \sim (x_{-i+1}^*, 0)$. Similarly, $x_{i+1}^* = \min\{x_{i+1}^1, x_{i+1}^2, x_{i+1}^3 ... x_{i+1}^{2i+1}\}$. Define, $x'_{i+1} = \min\{x_{i+1}^1, x_{i+1}^2, x_{i+1}^3 ... x_{i+1}^{2i}\} \geq x_{i+1}^*$. De-

[56] A set paired with a total order.

[57] We are using one extra comparison than that existed in the original construction of the sequence, and this is to make sure that x_i^* has $2i$ comparisons in its construction, just like y_i^*. Given the structure of the sequence we can always add more comparisons than the original, but never have fewer comparisons.

fine $y'_i = \max\{y_i^1, y_i^2, y_i^3, ... y_i^{2i}\}$ where $(y_i^1, 1) \sim (y^*_{i-1}, 0), .., (y_i^{2i}, 2i) \sim (y^*_{-i}, 0)$. Finally, $y^*_i = x^*_i + (y'_i - x^*_i)\frac{(x^*_{i+1} - x^*_i)}{(x'_{i+1} - x^*_i)} \in (x^*_i, x^*_{i+1})$. By monotonicity, $y_i^n \in (x_i^n, x_{i+1}^n)$ for all $n \in \{1, 2, .., 2i\}$. Therefore, $y_i^{'} \in (x^*_i, x_{i+1}^{'})$. Therefore, $y^*_i \in (x^*_i, x^*_{i+1})$, the upper bound comes from the fact that $x'_{i+1} > y'_i$ and the lower bound comes from the fact that y'_i is bounded below by x^*_i. Note that for $y^*_0, \hat{y}^*_0 \in (x^*_0, x^*_1)$, $y^*_0 > \hat{y}^*_0$ if and only if $y^*_i > \hat{y}^*_i$. And finally, for any $\hat{y}^*_i \in (x^*_i, x^*_{i+1})$ there exists a $\hat{y}^*_0 \in (x^*_0, x^*_1)$ corresponding to it. The last part can be shown constructively.

Finding y^*_{-i-1} where $i \geq 1$: Note that $x^*_{-i} = \max\{x^1_{-i}, x^2_{-i}, x^3_{-i} ... x^{2i+1}_{-i}\}$[58], where $(x^1_{-i}, 0) \sim_v (x^*_{-i+1}, 1), (x^2_{-i}, 0) \sim_v (x^*_{-i+2}, 2) .., (x^{2i}_{-i}, 0) \sim_v (x^*_i, 2i)$. Similarly, $x^*_{-i-1} = \max\{x^1_{-i-1}, x^2_{-i-1}, .. x^{2i+1}_{-i-1}, x^{2i+2}_{-i-1}\}$.

Define, $x'_{-i-1} = \max\{x^1_{-i+1}, x^2_{-i+1}, x^3_{-i+1}, ... x^{2i+1}_{-i+1}\} \leq x^*_{-i-1}$.

Define $y'_{-i-1} = \max\{y^1_{-i+1}, y^2_{-i+1}, y^3_{-i+1}, ... y^{2i+1}_{-i+1}\}$ where $(y^1_{-i+1}, 0) \sim_v (y^*_{-i+2}, 1), ..$ $.., (y_{-i+1}, 0) \sim_v (y^*_i, 2i+1)$. Finally, $y^*_{-i-1} = x^*_{-i} - (x^*_{-i} - y'_{-i-1})\frac{(x^*_{-i} - x^*_{-i-1})}{(x^*_{-i} - x'_{-i-1})} \in (x^*_{-i-1}, x^*_{-i})$. By monotonicity, $y^n_{-i-1} \in (x^n_{-i-1}, x^n_{-i})$ for all $n \in \{1, 2, .., 2i+1\}$. Therefore, $y^{'}_{-i-1} \in (x^{'}_{-i-1}, x^*_{-i})$. Therefore, $y^*_{-i-1} \in (x^*_{-i-1}, x^*_{-i})$, the upper bound comes from the fact that $x^*_{-i} > y'_{-i-1}$ and the lower bound comes from the fact that y'_{-i-1} is bounded below by x'_{-i-1}. Note that for $y^*_0, \hat{y}_0 \in (x^*_0, x^*_1)$, $y^*_0 > \hat{y}_0$ if and only if $y^*_{-i-1} > \hat{y}_{-i-1}$. And finally, for any $\hat{y}_{-i-1} \in (x^*_{-i-1}, x^*_{-i})$ there exists a $\hat{y}_0 \in (x^*_0, x^*_1)$ corresponding to it. The last part can be shown inductively. Fix $\hat{y}'_{-i-1}$. Find the points (whenever possible) $z_n \in (x^*_n, x^*_{n+1})$ for $n \in \{-i, -i+1, -i+2, ..i\}$ such that $(\hat{y}'_{-i-1}, 0) \sim_v (z_n, n+i+1)$. Note that we can always find atleast one such z_n.[59] Next, using the induction hypothesis we can map all the z_n's to a $y^*_0 \in (x^*_0, x^*_1)$. We take the maximum of all such y^*_0s and define it as $\hat{y}^*_0$. One can check that starting from this $(\hat{y}_{-i+1}, ..\hat{y}_0, \hat{y}_1..\hat{y}_i)$ would indeed result in ending with the $\hat{y}'_{-i-1}$ we started with. [60]

Now, define u on $\mathbb{X}$ as follows: Set $u_1(x^*_0) = 1$. For the sequence $.., x^*_{-1}, x^*_0, x^*_1, ..$, let $u(x^*_i) = \delta^i$ for all positive and negative integers i. Next, let us define $u_!$ on $(x^*_{-1}, 1)$ as any continuous and increasing function with $\inf_{(x^*_{-1}, 1)} u_1(x) = \delta = u(x^*_{-1})$ and

[58] As before, we are using one extra comparison than that existed in the original construction of the sequence.

[59] There exists k such that $(x^*_{-i}, 0) \sim_v (x^*_{-i+k}, k)$. In general, $(x^*_{-i}, 0) \succsim_v (x^*_{-i+k}, k)$.This implies $(\hat{y}'_{-i}, 0) \succ_v (x^*_{-i+k}, k)$ and $(x^*_{-i+1+k}, k) \succ_v (\hat{y}'_{-i}, 0)$. Hence, there exists $z_{-i+k} \in (x^*_{-i+k}, x^*_{-i+k+1})$ such that $(\hat{y}'_{-i}, 0) \sim_v (z_n, n+i)$.

[60] Suppose not. Given our definition of $\hat{y}_0$, starting from this $(\hat{y}_{-i+1}, ..\hat{y}_0, \hat{y}_1..\hat{y}_i)$ would give us $\hat{y}''_{-i} \geq \hat{y}'_{-i}$. Let, $\hat{y}''_{-i} > \hat{y}'$ and $(\hat{y}''_{-i}, 0) \sim (\hat{y}_{-i+k}, k)$ for $\hat{y}_{-i+k} \in (x_{-i+k}, x_{-i+k+1})$, this being the relation that binds while defining $\hat{y}''_{-i}$. Given, $(\hat{y}_{i+k}, k) \succ_v (\hat{y}'_{-i}, 0)$ and $(\hat{y}'_{-i}, 0) \succ (x'_{-i}, 0) \succsim_v (x^*_{-i+k}, k)$, there would exist $(\hat{y}''_{-i}, 0) \sim (\hat{y}^*_{-i+k}, k)$ for $\hat{y}_{-i+k} \in (x^*_{-i+k}, x^*_{-i+k+1})$ and $\hat{y}_{-i+k} < \hat{y}'_{-i+k}$ which would be a contradiction.

$\sup_{(x^*_{-1},1)} u_1(x) = 1 = u(1)$. We can extend each dual sequence with some $y_0 \in (x^*_{-1}, 1)$ as $u(y_i) = \delta^i u(y_0)$. Finally, define $U(x,t) = \delta^t \dfrac{u_1(x)}{u_1(1)}$ to ensure $u_1(1) = 1$ (note that $u_1(1) > 0$).

It is important to note here that the utility defined retains all the monotonicity, discounting and present bias properties. Consider any $(y,t) \succsim (x,0)$ in the original relation. The element x must belong to one of the sequences defined above. If x_t is the corresponding element to the right in that sequence separated by a distance of t, then, by construction we must have $u(x) = \delta^t u(x_t)$ and $(x,0) \succsim (x_t,t)$. By monotonicity, it would be true that $y > x_t$ and hence, $u(x) < \delta^t u(y)$.

Now we will extend the logic above to a more general case of $(x_0,t),(x_t,0)$ such that $(x_0,t) \sim (x_t,0)$ for $t > 1$.

For $i \in \{1,..t\}$, let x_i be such that $(x_0,0) \sim (x_i,i)$. We define the following procedure: Choose $x^*_0 = x_0$, the same x_0 that was provided in the statement of this Lemma. Find x^*_1 such that $(x^*_0,0) \sim (x^*_1,1)$. Of course, $x^*_1 = x_1$. Next use the iterative format used in Lemma 2 to find $x^*_2, x^*_3, ..x^*_t$.

At each of these steps, by WPB, one would get, $x^*_i = x_i$, ending with $x^*_t = x_t$. We provide a brief outline for this, the proof requires induction.

Let, $x^*_2 = \min\{x_2, x'_2\}$, where, $(x_2,2) \sim_v (x^*_0,0)$ and $(x'_2,1) \sim_v (x^*_1,0)$. By WPB, the latter implies, $(x'_2,2) \succsim_v (x^*_1,1)$. By definition of x^*_1, $(x^*_0,0) \sim_v (x^*_1,1)$. Putting it all together,

$$(x'_2,2) \succsim_v (x^*_1,1) \sim_v (x^*_0,0) \sim_v (x_2,2)$$

Hence, $x'_2 \geq x_2$, and $x^*_2 = x_2$.

Similarly, let $x^*_3 = \min\{x_3, x'_3, x''_3\}$, where, $(x_3,3) \sim_v (x^*_0,0)$, $(x'_3,2) \sim_v (x^*_1,0)$ and $(x''_3,1) \sim_v (x^*_2,0)$.

$$(x'_3,3) \succsim_v (x^*_1,1) \sim_v (x^*_0,0) \sim_v (x_3,3)$$

Also,

$$(x''_3,3) \succsim_v (x^*_2,2) \sim_v (x^*_0,0) \sim_v (x_3,3)$$

And so on. Note that the sequence in which the elements are being found till now has been different that that in Lemma 2. Here on, find the sequence elements in the following order $x^*_{-1}, x^*_{t+1}, x^*_{-2}, x^*_{t+2}, ...$ using the iterative procedure as Lemma 2.

For any $y_0 \in (x^*_0, x^*_1)$, find similar sequences in the same order as we derived the sequence x^*.

Now, define u on $\mathbb{X}$ as before to finish the proof. Note that any u such constructed is strictly monotonic, continuous, and $u(0) = 0$, $u(1) = 1$. □

Lemma 27. *Under Axioms A1-A6, there exists a set of functions $\mathscr{U}$ such that, for all $u \in \mathscr{U}$, u is strictly monotonic, continuous, and $u(0) = 0$, $u(1) = 1$, and $\delta^t u(z_1) \geq u(z_0)$ for all $(z_1,t) \geq (z_0,0)$. Moreover, i) for any $(x,t) \sim (y,0)$, there existsu $\in \mathscr{U}$ such that $\delta^t u(x_t) = u(x_0)$. ii) For $x > 0$, $\inf_{u \in \mathscr{U}} u(x) > 0$, $\sup_{u \in \mathscr{U}} u(x) < \infty$*

Proof. Consider the set $\mathscr{U}$ consisting of all functions u constructed from all the indifference relations $\sim$ in (26). It would suffice to show that $\inf_{u \in \mathscr{U}} u(x) > 0$, $\sup_{u \in \mathscr{U}} u(x) < \infty$.
First we will show that $\inf_{u \in \mathscr{U}} u(x) > 0$. This is trivial for points above $x = 1$. Consider $0 < x < 1$. Suppose we are constructing a function that would respect the relation $(x_0,0) \sim (x_t,t)$.
By A5, there exists t_1 such that $(x,t) \succ (1,t+t_1)$ for all $t \geq 0$ and for any y_i such that such that $(1,0) \sim (y_i,i)$ for $i \geq 0$, $(x,0) \succ (y_i,t_1+i)$. Consider the following cases:

CASE 1: Consider $x_0 < x < 1$. By A5, there exists $t \geq 1$, such that in the sequence constructed, $x_{t-1} < x \leq x_t$. Note that given the construction of the sequence for $(x,0) \sim (x_i,i)$, it must be that for any (x_p,x_q), $p < q$, $(x_p,0) \succsim (x_q,q-p)$). By monotonicity, using $x_{t-1} < x \leq x_t$, for any point x_i in the sequence , $|i| \leq t$, one has $(x_i,0) \succsim (x_t,t-i) \succsim (x,t-i)$. Hence, for any element x_i of the sequence with $i \leq 0$, $(x_i,0) \succsim (x,t-i) \succ (1,t_1+t-i)$, with the last inequality coming from A5.[61] Hence, the $x_{(t+t_1)}$th element of the sequence must be weakly to the right of 1. Thus, $u(x) \geq \dfrac{1}{\delta^{t_1+1}}$.

CASE 2: Consider $x < x_0 < 1$. By construction of the dual sequence $\{..x_{-1},x_0,x_1,..\}$, it must be that $x_{-t_1} \leq x$ and $x_{t_1} \geq 1$. Thus, $u(x) \geq \dfrac{1}{\delta^{2t_1}}$.[62]

Hence, $u(x) \geq \dfrac{1}{\delta^{2t_1}}$ for all $u \in \mathscr{U}$.
Now, showing that $\sup_{u \in \mathscr{U}} u(x) < \infty$. This is trivial for points $x \leq 1$. Consider $x > 1$. By A4, there exists t_1 such that $(1,0) \succ (x,t_1)$ and for any y such that $(x,0) \sim (y,i)$, $i \geq 0$,$(1,0) \succ (y,t_1+i)$. Suppose we are constructing a function that would respect the relation $(x_0,0) \sim (x_t,t)$, and in the process construction a dual sequence $\{..x_{-1},x_0,x_1,..\}$. There are two cases as before.

CASE 1: Consider $x_0 > x > 1$. By A4, there exists $t \geq 1$, such that in the sequence constructed, one has $x_{-t} \leq x < x_{-t+1}$. As before, given the construction of the sequence for $(x,0) \sim (x_i,i)$, it must be that for any (x_p,x_q), $p < q$, $(x_p,0) \succsim$

[61] The property we are using implicitly without proving is the following: In our constructed sequences, x_i always is a direct reflection from $\{x_0,x_{-1},x_{-2},..\}$ when i is positive, and a direct reflection of $\{x_0,x_1,x_2..\}$when i is negative. This follows from WPB.

[62] One can make the bound tighter.

$(x_q, q-p)$.) By monotonicity, using $x_{-t} \leq x$, for any point x_i in the sequence , $|i| \leq t$, $(x,0) \succsim (x_{-t},0) \succsim (x_i, i+t)$. Hence, for any element x_i of the sequence with $i \geq 0$, $(1,0) \succ (x_i, t_1+i+t)$. Thus, $u(x) \leq \dfrac{1}{\delta^{t_1+1}}$.

CASE 2: Consider $x > x_0 > 1$. By construction of the dual sequence $\{..x_{-1}, x_0, x_1, ..\}$, it must be that $x_{-t_1} \leq 1$ and $x_{t_1} \geq x$. Thus, $u(x) \leq \dfrac{1}{\delta^{2t_1}}$.

Hence, $u(x) \leq \dfrac{1}{\delta^{2t_1}}$ for all $u \in \mathscr{U}$. □

Lemma 28. *Under Axioms A1-A6, there exists a continuous present equivalent utility function $F : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ that represents $\succsim$. F is monotonically increasing in x and monotonically decreasing in t.*

Proof. The first part of this proof is very similar to Lemma 24, and we will omit it here. By construction of the set $\mathscr{U}$, $V(x,t) = \min_{v \in \mathscr{U}} v^{-1}(\delta^t v(x))$. Moreover, for all $u \in \mathscr{U}$, $u(0) = 0$, $u(1) = 1$, $\inf_{u \in \mathscr{U}} u(x) > 0$, $\sup_{u \in \mathscr{U}} u(x) < \infty$ for $x > 0$.
Finally, from A6, for any $x > 0$, and $t \in \mathbb{T}$, there exists z such that $(z,t) \succ (x,0)$.

$$\begin{aligned} \delta^t u(z) &> u(x) \;\; \forall u \in \mathscr{U} \\ \implies \frac{u(z)}{u(x)} &> \delta^{-t} \;\; \forall u \in \mathscr{U} \\ \implies \inf_u \frac{u(z)}{u(x)} &\geq \delta^{-t} \;\; \forall u \in \mathscr{U} \end{aligned}$$

But we had started with arbitrary t. Hence, $\inf_u \dfrac{u(z)}{u(x)}$ is unbounded above for any $x > 0$.

□

Theorem 8: The following two statements are equivalent:

i) The relation $\succsim$ satisfies properties B0-B5.

ii) There exists a continuous function $F : \mathbb{X} \times \mathbb{P} \times \mathbb{T} \to \mathbb{R}$ such that $(x,p,t) \succsim (y,q,s)$ if and only if $F(x,p,t) \geq F(y,q,s)$. The function F is continuous, increasing in x, p and decreasing in $t \in \mathbb{T}$. There exists a unique $\delta \in (0,1)$ and a set $\mathscr{U}$ of monotinically increasing continuous functions such that $F(x,p,t) = \min_{u \in \mathscr{U}} u^{-1}(p\delta^t u(x))$ and $u(0) = 0$ for all $u \in \mathscr{U}$.

Proof. Showing that (ii) implies (i) :
Continuity and monotonicity of $\succsim$ follow from the continuity and monotonicity of

F. Weak Present Bias follows as before.
B5 can be shown in the following way:

$$\begin{aligned}(x,p\theta,t) &\succsim (x,p,t+D)\\ \implies \min_u u^{-1}(p\theta\delta^t u(x)) &\geq \min_u u^{-1}(p\delta^{t+D}u(x))\\ \implies \theta &\geq \delta^D\\ \implies \min_u u^{-1}(q\theta\delta^s u(y)) &\geq \min_u u^{-1}(q\delta^{s+D}u(y))\\ \implies (y,q\theta,s) &\succsim (y,q,s+D)\end{aligned}$$

We will prove the direction (i) to (ii) in the following three steps.

Step 1: Recall the Probability Time Tradeoff axiom: for all $x,y \in \mathbb{X}$, $p,q \in (0,1]$,and $t,s \in \mathbb{T}$, $(x,p\theta,t) \succsim (x,p,t+\Delta) \implies (y,q\theta,s) \succsim (y,q,s+\Delta)$.
This axiom has calibration properties that we will use. Given Monotinicity, $(x,1,0) \succ (x,1,1) \succ (x,0,0)$ for any $x \in \mathbb{X}$. By continuity, there must exist $\delta \in (0,1)$ such that $(x,\delta,0) \sim (x,1,1)$. Note that Probability-Time Tradeoff Axiom helps us write $(x,\delta,\tau+1) \sim (x,1,\tau)$ for all $x \in \mathbb{X}$ and $\tau \in \mathbb{T}$, and further extend it to $(x,q,t) \sim (x,q\delta^t,0)$. For integer t's this follows by induction.
For any integer $b > 0$, there exists $\Delta(\frac{1}{b}) = \delta_1 \in \mathbb{P}$ such that $(x,\delta_1,0) \sim (x,1,\dfrac{1}{b})$. Now applying Probability Time Tradeoff (PTT) repeatedly b times, $(x,1,1) \sim (x,\delta_1^b,0)$, which implies, $\delta_1 = \delta^{\frac{1}{b}}$. For any ratio of positive integers (rational number) $t = \dfrac{a}{b}$, $\Delta(\frac{a}{b}) = \delta^{\frac{a}{b}}$. This argument can be extended to all real $t > 0$. This crucially helps us pin down δ as the discount factor.
Henceforth, we are going to concentrate on finding a representation of the reduced domain of $\mathbb{X} \times [0,1]$. Note that this reduced domain can also be conceptually seen as the set of all binary lotteries that have zero as one of the outcomes.

Step 2: The rest of the proof will have a similar flavor to the ones the reader has already encountered. For every $x^* \in \mathbb{X}$, we are going to provide an increasing utility function u on $[0,M]$ which would respect all the relations of the form $(x,p) \succsim (y,1)$, i.e, have $pu(x) \geq u(y)$ and also have $pu(y) = u(x^*)$ for all $(x^*,1) \sim (y,p)$.
Fix x^*, $u(0) = 0$ and $u(x^*) = 1$. For $x \in (x^*,M]$, define

$$u(x) = \{\frac{1}{p} : (x,p) \sim (x^*,1)\} \tag{A.5}$$

and,

$$x_q = \{x : (x,q) \sim (x^*,1)\} \tag{A.6}$$

The expressions in (A.5) and (A.6) exist due to the continuity of $\succsim$.
Now, for $x \in (0,x^*)$, define

$$u(x) = \inf\{p(q) : (x_q, qp(q)) \sim (x,1), q \leq 1\} \tag{A.7}$$

First, we will show that the infimum in (A.7) can be replaced by minimum. Consider a sequence of probabilities $\{p_n\}$ that converge below to p^*, and $(x_{q_n}, p_n q_n) \sim (x,1)$. Note that $q_n \in [q_{max}, 1]$ where $(x^*, 1) \sim (M, q_{max})$. Hence, $\{q_n\}$ must be bounded by this closed interval, and must have a convergent subsequence $\{q_{n_k}\}$. Let q^* be the corresponding limit, and we know that $q^* \geq q_{max}$. Similarly, x_q can be considered continuous in q (this also follows from the continuity of $\succsim$). Therefore, $x_{q_{n_k}} \to x_{q^*}$ as $q_{n_k} \to q^*$. Also, it must be that $p_{n_k} \to p^*$ as $q_{n_k} \to q^*$. Thus, we have $(x_{q_{n_k}}, p_{n_k} q_{n_k}) \sim (x,1)$ for all elements of $\{n_k\}$. Then, using the continuity of $\succsim$, $(x_{q^*}, p^* q^*) \sim (x,1)$.

$$u(x) = \inf\{p : (x_q, pq) \sim (x,1)\} = \min\{p : (x_q, pq) \sim (x,1)\} = p^*$$

Now we will show that the utility defined in (A.5) and (A.7) has the following properties : 1) It is increasing. 2) $p_1 u(x_1) = u(y_1)$ 3) $(x,p) \succsim (y,1)$, implies $pu(x) \geq u(y)$ 4) u is continuous. The first two properties are true by definition of u. We will show the third in some detail. Consider $(x,p) \succsim (y,1)$. In the case of interest, $p < 1$ and hence, $x > y$. Now let $x > y > x^*$. Let, $u(y) = 1/p_1$, which means, $(y, p_1) \sim (x^*, 1)$. Given $(x,p) \succsim (y,1)$, we must have □

$$(x, pp_1) \succsim (y, p_1) \sim (x^*, 1)$$

Hence,

$$\begin{aligned} u(x) &\geq \frac{1}{pp_1} \\ \iff pu(x) &\geq \frac{1}{p_1} = u(y) \end{aligned}$$

If, $x > x^* > y$, the proof follows from the way the utility has been defined.
Let $y < x < x^*$. Let, $u(x) = p_1$, which means, $(x_q, p_1 q) \sim (x,1)$ for some x_q. Given $(x,p) \succsim (y,1)$, we must have

$$(x_q, p_1 qp) \succsim (x,p) \succsim (y,1)$$

Hence, $u(y) \leq pp_1$.
Now we turn to proving the continuity of u. The continuity at x^* from the right is easy to see.
Next, for any $r \in (0,1)$, define

$$f(r) = \sup\{x \in [0,x^*) : (x_q, qr) \sim (x,1)\} \tag{A.8}$$

The supremum can be replaced by a maximum, and the proof is similar to the one before. Suppose there is a sequence of $\{x_n\}$ that converges up to a value $\hat{x}$, and, $(x_{q(n)}, q(n)r) \sim (x_n, 1)$. Note that $q(n)$ lies in a closed interval, and hence has a convergent subsequence that converges to a point in that interval. Let us call this point $\hat{q}$. Now, x is continuous in q (in the usual sense), and hence, x_n and $x_{q(n)}$ also has a convergent subsequence. The convergent subsequence $\{x_{q(n)}\}$ and $\{x_n\}$ must have the same limit point, let us call it $x_{\hat{q}}$, a point in $[x^*, M]$. Hence, the supremum in (A.8) must have been attained from $x_{\hat{q}}$, and hence the supremum can be replaced by a maximum. The f function is well defined, strictly increasing and is the inverse function of u over $r \in (0,1)$. This function can be used to show the continuity of u at the point x^*.
Finally, the function u can be easily normalized to have $u(M) = 1$.
Step 3: In this step, we construct the $\mathscr{U}$ set as in Theorem Theorem 3, to complete the proof.

Theorem 10: *The following two statements are equivalent:*
i) The relation $\succsim$ on $[0,\infty)^T$ satisfies properties D0-D5.
ii) For any $\delta \in (0,1)$, there exists a set $\mathscr{U}_\delta$ of monotinically increasing continuous functions such that

$$F(x_0, x_1, .., x_{T-1}) = x + \sum_{1}^{T-1} \min_{u \in \mathscr{U}_\delta} u^{-1}(\delta^t u(x_t))$$

represents the binary relation $\succsim$. The set $\mathscr{U}_\delta$ has the following properties: $u(0) = 0$ and $u(M) = 1$ for all $u \in \mathscr{U}_\delta$. $F(.)$ is continuous.

Proof: Going from (ii) to (i), we will show how the representation satisfies D5 and the second property in D2.
Suppose, $(x_0, x_1, .., x_{T-1})$ and $(y_0, y_1, .., y_{T-1})$ are orthogonal. Therefore,

$$\begin{aligned} F(x_0 + y_0, x_1 + y_1, .., x_{T-1} + y_{T-1}) &= F(x_0, x_1, .., x_{T-1}) + F(y_0, y_1, .., y_{T-1}) \\ &= F(z_0, 0, .., 0) + F(z_0', 0, .., 0) \\ &= z_0 + z_0' \\ &= F(z_0 + z_0', 0, .., 0) \end{aligned}$$

To see how Discounting can be derived, start by assuming $y^0 > x > 0$, and choose a function $u_1 \in \mathscr{U}$. As $\delta \in (0,1)$, there must exist t such that $\delta^t u_1(y^0) < u_1(x)$ and hence, $u_1^{-1}(\delta^t u_1(y^0)) < x$. For any sequences $(y^1, y^2, y^3, ..y^m)$ and $(n^1, n^2, .., n^m)$, where, $(0, ..0, \underbrace{y^{i-1}}_{\text{in period } n_i}, 0.., 0) \succsim (y^i, 0, .., 0)\ \forall i \in \{1, 2, ..., m\}$, one must have $\delta^{n_i} u_1(y^i) \geq$

$u_1(y^{i-1})\ \forall i \in \{1,2,...,m\}$.
Multiplying all these inequalities gives us,

$$\begin{aligned}
\delta^{\Sigma n_i} u_1(y^0) &\geq u_1(y^m) \\
\iff y_m &\leq u_1^{-1}(\delta^{\Sigma n_i} u_1(y^0)) \\
&= u_1^{-1}(\delta^t u_1(y^0)) \\
&< u_1^{-1}(u_1(x)) \\
&= x \\
\implies y_m &\leq x
\end{aligned}$$

Now to show the proof for the direction (i) to (ii), we start by following the same steps we used in the proof of Theorem 3 to derive the set $\mathscr{U}_\delta$. There are two points to be noted during the construction of functions in $\mathscr{U}_\delta$. First, only comparisons upto lengths of $T-1$ periods need to be considered. Secondly, in the construction of each function $u \in \mathscr{U}_\delta$, the fact that the interative construction spans over $\mathbb{R}_{\geq 0}$ is guaranteed by the second part of the Discounting axiom. The additive representation across periods follows from using induction and the D5 axiom.

Theorem 5: *Let $\succsim_1$ and $\succsim_2$ be two binary relations which allow for minimum representation w.r.t sets $\mathscr{U}_{\delta,1}$ and $\mathscr{U}_{\delta,2}$ respectively. The following two statements are equivalent:*
i) $\succsim_1$ allows a higher premium to the present than $\succsim_2$.
ii) Both $\mathscr{U}_{\delta,1}$ and $\mathscr{U}_{\delta,1} \cup \mathscr{U}_{\delta,2}$ provide minimum representations for $\succsim_1$.

Proof. The direction from (i) to (ii): Consider any $(x,t) \in \mathbb{X} \times \mathbb{T}$ such that $(x,t) \sim_1 (y,0)$. Using (i), we must have, $(x,t) \succsim_2 (y,0)$.
Hence,

$$\begin{aligned}
\min_{u \in \mathscr{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &\geq y \\
\implies \min_{u \in \mathscr{U}_{\delta,1} \cup \mathscr{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &= y
\end{aligned}$$

Hence,

$$\min_{u \in \mathscr{U}_{\delta,1} \cup \mathscr{U}_{\delta,2}} u^{-1}(\delta^t u(x)) = \min_{u \in \mathscr{U}_{\delta,1}} u^{-1}(\delta^t u(x)) \tag{A.9}$$

To go in the opposite direction, let us assume, $(x,t) \succsim_1 (y,0)$.

Given, (A.9), it must be that

$$\begin{aligned}
\min_{u\in\mathscr{U}_{\delta,1}\cup\mathscr{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &= \min_{u\in\mathscr{U}_{\delta,1}} u^{-1}(\delta^t u(x)) \geq y \\
\implies u^{-1}(\delta^t u(x)) &\geq y \quad \forall u \in \mathscr{U}_{\delta,1}\cup\mathscr{U}_{\delta,2} \\
\implies u^{-1}(\delta^t u(x)) &\geq y \quad \forall u \in \mathscr{U}_{\delta,2} \\
\implies \min_{u\in\mathscr{U}_{\delta,2}} u^{-1}(\delta^t u(x)) &\geq y
\end{aligned}$$

Hence, $(x,t) \succsim_2 (y,0)$, which completes the proof. □

Proposition 29. *Let f_n be a set of bijective, increasing, continuous functions. Let $f_n \to f$ "locally uniformly"/ compactly (equivalent notions in $\mathbb{R}^n$.), where f is bijective, increasing, continuous. Then, $f_n^{-1} \to f^{-1}$ compactly.*

Proof. Consider the composite function $g_n = f_n o f^{-1}$. Note that g_n is also bijective, increasing, continuous. As f_n converges locally uniformly to f, g_n converges locally uniformly to the identity function $g(x)$.
To see this, note that for any $\varepsilon_1 > 0$

$$\begin{aligned}
\sup_{x\in[c,d]} |g_n(x) - g(x)| &= \sup_{x\in[c,d]} |f_n(f^{-1}(x)) - f(f^{-1}(x))| \\
&= \sup_{y\in[f^{-1}(c),f^{-1}(d)]} |f_n(y) - f(y)| \\
&\leq \varepsilon_1
\end{aligned}$$

for $n \geq N_0$ for some N_0.
Choose an interval $[a,b]$. Now, there would exist n_1, n_2 such that $g_n(a-1) \leq a$ and $g_n(b+1) > b$ for $n \geq n_1$ and $n \geq n_2$ respectively. Similarly, there exists n_3 such that $\sup_{x\in[a-1,b+1]} |g_n(x) - g(x)| < \varepsilon$ for $n \geq n_3$.
Finally, for $N \geq max\{n_1, n_2, n_3\}$

$$\begin{aligned}
\sup_{x\in[a,b]} |g_n^{-1}(x) - g(x)| &\leq \sup_{x\in[g_n(a-1),g_n(b+1)]} |g_n^{-1}(x) - x| \\
&= \sup_{t\in[a-1,b+1]} |g_n^{-1}(g_n(t)) - g(t)| \\
&= \sup_{t\in[a-1,b+1]} |t - g(t)| \\
&< \varepsilon
\end{aligned}$$

Therefore, $g_n^{-1} = f o f_n^{-1}$ converges locally uniformly to the identity function. Therefore, f_n^{-1} converges locally uniformly to f^{-1}. □

A.2 Appendix to Chapter 3

A.2.1 Diminishing Impatience does not imply the Certainty Effect

We start by providing a basic intuition of why DI as characterized by (3.4) fails to imply the certainty effect. To complete a proof in the direction from DI to certainty effect one is required to approximate arbitrary probabilities used in lotteries by the total hazard rate of termination over one or multiple periods. More specifically, one needs to approximate the ratio of probabilities $\frac{g(p)}{g(pq)}, \frac{g(1)}{g(q)}$ in the relation (3.2) by the relative hazard rates between two consecutive time-periods in (3.4), $\frac{g((1-r)^t)}{g((1-r)^{t+1})}, \frac{g(1)}{g(1-r)}$ respectively, for some hazard rate r. Given (3.4), we are restricted to establishing the certainty effect relation for p, q combinations which can be approximated as integer exponents of each other, hence the result does not generalize to the certainty effect. Under DISDI we are approximating $\frac{g(p)}{g(pq)}, \frac{g(1)}{g(q)}$ by the relative hazard rates between arbitrarily spaced time-periods in (3.4), $\frac{g((1-r)^t)}{g((1-r)^{t+k})}, \frac{g(1)}{g((1-r)^k)}$. Hence, we are allowed to approximate p, q by different integer exponents of the hazard rate and hence rational exponents of each other (for example, when, $p = r^k, q = r^t$, then $p = q^{\frac{k}{t}}$). Given the rationals are dense in reals (and the integers are not!), a sequence of $\frac{k}{t}$'s can approximate $ln_q p$ and this allows the relation from time to risk be established for general p, q and continuous g. The following counter-example provides the vital step that DI does not imply DISDI.

If (3.4) implied (3.2), then (3.4) would also imply that $\forall r \in (0,1)$ and $\forall m, n \in \mathbb{N}$

$$g(r^{m+n}) > g(r^m)g(r^n) \tag{A.10}$$

We rewrite this problem in an additive form by defining $f(x) = -log(g(e^{-x})) \iff g(x) = e^{-f(-logx)}$. Then $f : (0,\infty) \to (0,\infty)$ is differentiable and increasing, just like the function g. The inequalities under consideration are now:

$$\begin{aligned}
\forall t \in \mathbb{N} \text{ and } \forall r \in (0,1),\ g(r^{t+1}) &> g(r)g(r^t) \\
\iff e^{-f\left(-log(r^{t+1})\right)} &> e^{-f(-log(r^t))}e^{-f(-log(r))} \\
\iff f(-(t+1)log(r)) &< f(-tlog(r)) + f(-log(r))
\end{aligned}$$

Now, defining $x := -log(r) \in (0,\infty)$ for $r \in (0,1)$.

$$f((t+1)x) < f(tx) + f(x) \tag{A.11}$$

Further, the boundary conditions $g(0)=0$ and $g(1)=1$ translate to $f(0)=0$ and $f(\infty)=\infty$.[63]
Similarly, (A.10) converts to

$$f(mx+nx) < f(mx)+f(nx) \ \forall x \in (0,\infty) \, and \, \forall m,n \in \mathbb{N} \tag{A.12}$$

Summing it up, (3.4) implies (A.10), if and only if (A.11) implies (A.12). The next step is to propose a function f which would satisfy (A.11) on all points of its domain, but violate (A.12) for some $x \in \mathbb{R}$ and some $m,n \in \mathbb{N}$.

Instead of providing the function f, we propose it's derivative h, so f can be calculated as $f(x)=\int_0^x h(x)dx$.[64] Let, $k=\frac{20}{1+\sin(\pi/2-.0001)}$ and $\delta = 50k\pi\cos(\pi/2-.0001) \approx .157$.

Let,

$$h(x) = \begin{cases} 11+(1-x)\delta & For\ x<1 \\ 1+\frac{k}{2}+\frac{k}{2}\sin 100\pi(1+\frac{\pi/2-.0001}{100\pi}-x) & For\ 1 \leq x \leq 1.005+\frac{\pi/2-.0001}{100\pi} \\ 1 & For\ 1.005+\frac{\pi/2-.0001}{100\pi} < x < 2-.005 \\ 4+3\sin 100\pi(x-2) & For\ 2-.005 \leq x \leq 2+.005 \\ 7 & For\ 2+.005 < x < 2.5-.005 \\ 4+3\sin 100\pi(2.5-x) & For\ 2.5-.005 \leq x \leq 2.5+.005 \\ 1 & For\ 2.5+.005 < x < 3-.005 \\ 4+3\sin 100\pi(x-3) & For\ 3-.005 \leq x \leq 3+.005 \\ 7 & For\ 3+.005 < x < 5-.005 \\ 4+3\sin 100\pi(5-x) & For\ 5-.005 \leq x \leq 5+.005 \\ 1 & For\ x > 5+.005 \end{cases}$$

f is increasing, twice differentiable and $f(\infty)=\infty$. $h(x)$ is plotted in Figure A.1.

We next show that (A.11) holds.

Lemma 30. *$\forall t \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $\int_0^x h(x)dx > \int_{tx}^{(t+1)x} h(x)dx$.*

Proof. The most intuitive way to check the claim would be to notice that the sinusoids introduced hardly perturb the area under the curve. Figure A.2 illustrates the point in a more clear fashion by considering the function h for a small part of the real line. For all practical purposes, one could go about checking the inequalities by replacing the sinusoid (in black) in Figure 1 by a corresponding discontinuous

[63] Using the extended real line $(\mathbb{R} \cup \infty)$
[64] Recall that $f(0)=0$.

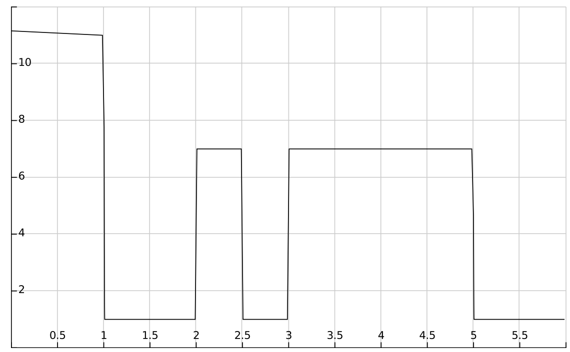

Figure A.1: The function h.

function($\bar{h}(x) = 7$ for $x \leq 2.5$, $\bar{h}(x) = 1$ for $x > 2.5$ as drawn in red). The area between the two curves in $[2.495, 2.5]$ is only $(.005 * 3 - \frac{3}{100\pi}) \approx .005$. Therefore, as long as the inequalities hold with a large enough margin, this intuitive method of approximating sinusoids with flat lines works fine. The area between the two curves in $[2.5, 2.505]$ is also $(.005 * 3 - \frac{3}{100\pi})$. Thus, the two approximations in $[2.495, 2.505]$ are equal and opposite in direction, and the areas under the red and black curves in this region are equal. During our analysis, in some cases there will be multiple approximations in opposite directions which would partially or completely cancel each other out.

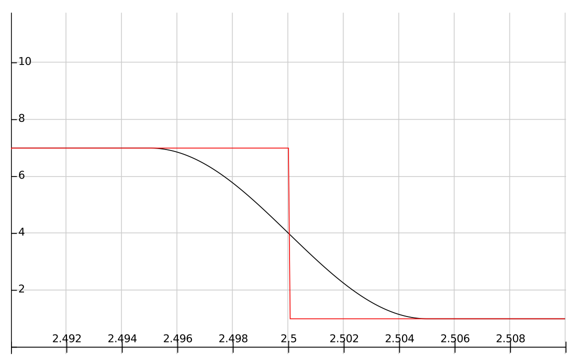

Figure A.2: Function h approximated in a sinusoidal region

Utilizing this intuition more rigorously, one can create upper bounds and lower bounds on $\int_{tx}^{(t+1)x} h(x)dx$ and $\int_0^x h(x)dx$ respectively to complete the proof.
For $0 < x \leq 1$, $\int_0^x h(x)dx > \int_{tx}^{(t+1)x} h(x)dx$ is obvious, as $[0, x]$ contains the highest values obtained by $h(x)$ on the real line.
For, $1 < x \leq \frac{5}{3}$, $\int_0^x h(x)dx = \int_0^1 h(x)dx + \int_1^x h(x)dx > \frac{1}{2}(11 + 11 + \delta) + (x - 1) =$

$10+\frac{\delta}{2}+x$.[65] The inequality holds because $h(x)\geq 1$ with strict inequality for $1\leq x<1.005+\frac{\pi/2-.0001}{100\pi}$, and hence $\int_1^x h(x)dx>x-1$. In the interval $[tx,(t+1)x]$, $h(x)\leq 7$ and after mutual canceling out there are no more than 3 sinusoidal perturbations which could increase the area under the curve. Hence, $\int_{tx}^{(t+1)x}h(x)dx<7x+3(.015-\frac{3}{100\pi})=6x+x+3(.015-\frac{3}{100\pi})\leq 6(\frac{5}{3})+x+3(.015-\frac{3}{100\pi})=10+x+3(.015-\frac{3}{100\pi})$.
For $\frac{5}{3}\leq x\leq 2$, $\int_0^x h(x)dx>10+\frac{\delta}{2}+x$ as before. On the other hand, using the same line of logic as before, $\int_x^{2x}g(x)dx<1.x+6[(4-3)+(2.5-2)]+3.(.015-\frac{3}{100\pi})=9+x+3.(.015-\frac{3}{100\pi})$. Similarly, $\int_{2x}^{3x}h(x)dx\leq 1.x+6[5-2.\frac{5}{3}]+3.(.015-\frac{3}{100\pi})=10+x+3.(.015-\frac{3}{100\pi})$.
Similarly for larger values of x, it can be shown that $\int_0^x h(x)dx>\int_{tx}^{(t+1)x}h(x)dx$. (follows trivially for $x\geq 5$.) □

Now complete the counter-example:

$$\int_0^2 h(x)dx<12+\frac{\delta}{2}+\{.01*10+(.015-\frac{3}{100\pi})\}<14-2(.015-\frac{3}{100\pi})=\int_3^5 h(x)dx$$

The first inequality follows from setting an upper bound on the sinusoidal perturbation introduced around 1.[66] Therefore, $f(5)>f(2)+f(3)$, which provides us with the counter-example to equation (A.12) and hence, to equation (A.10). In other words, as (A.11) does not imply (A.12), (3.4) does not imply (A.10), and hence, (3.4) does not imply (3.2).
That is, even if for all $t\in\mathbb{N}$ and for all $r\in(0,1)$: $g((1-r)^{t+1})>g((1-r)^t)g((1-r))$ it does not imply that $\forall p,q\in(0,1)$: $g(pq)\geqslant g(p)g(q)$.

A.3 Appendix to Chapter 4

Cooperation in low δ treatments

The high and low discount factors would have different predictions under reputation equilibrium. First consider, $\delta\leq 3/8$. Any egoistic player who believes that the other player conditionally cooperates till first defection with probability ρ_0, contemplates the following before making a choice in the first period of the game:
Suppose the subject is in Period 1. The lowest possible payoff from Defecting right away in this game is

[65] $\delta=50k\pi\cos(\pi/2-.0001)=.157$ (approximately)

[66] This particular sinusoid dies down after $1.005+\frac{\pi/2-.0001}{100\pi}<1.01$ and never rises above the $h(x)=1$ line by more than 6 units.

Table A.2: Cooperation by treatment and period, split by order of treatments

	Period1	Period2	Period3	Period4	Period5	Average
1	39.19	29.05	27.03	23.65	12.16	26.22
2	34.46	18.24	15.54	10.14	4.73	16.62
3	22.30	13.51	15.54	15.54	10.81	15.54
4	14.86	7.43	12.16	14.86	14.19	12.70
1	26.72	21.55	18.10	12.07	6.90	17.07
2	15.52	14.66	15.52	12.93	7.76	13.28
3	17.24	12.93	13.79	18.97	15.52	15.69
4	20.69	16.38	16.38	17.24	19.83	18.10

The first 4 rows are from sessions run in decreasing order of δ.
The last 4 rows are from sessions run in increasing order of δ

$$\Pi_D = \rho_0.2600 + (1-\rho_0)1200 + 1200(\delta + \delta^2 + \delta^3 + \delta^4)$$

The highest possible payoff from Cooperating is bounded above by [67]

$$\Pi_C = \rho_0.2000 + (1-\rho_0)200 + \underbrace{2000(\delta + \delta^2 + \delta^3) + 2600\delta^4}_{\text{payoff from cooperating all the way and defecting on Period 5}}$$

Now,

$$\begin{aligned} \Pi_D &> \Pi_C \\ \iff \rho_0.2600 + (1-\rho_0)1200 + 1200(\delta + \delta^2 + \delta^3 + \delta^4) &> \rho_0.2000 + (1-\rho_0)200 \\ &\quad +2000(\delta + \delta^2 + \delta^3) + 2600\delta^4 \\ \iff \rho_0.600 + (1-\rho_0)1000 &> 800(\delta + \delta^2 + \delta^3) \\ &\quad +1400\delta^4 \end{aligned}$$

The LHS is atleast 600. The RHS is increasing in δ and the maximum possible value of RHS is 482 for $\delta \leq 3/8$. Hence, any rational player should Defect right away under $\delta \leq 3/8$. The same argument would hold if the subject was in any other period of the game.
Now, instead assume $\delta \geq 3/4$. If Π_C is the highest possible return from Cooperting

[67] Under the considered δ range, this does strictly better that defecting earlier.

in the present, then,

$$\Pi_C \geq \rho_0.2000 + (1-\rho_0)200 + \underbrace{2600\delta\rho_0 + 1200\delta(1-\rho_0)}_{\text{minimum second round payoff given beliefs}}$$
$$+ \underbrace{1200(\delta^2+\delta^3+\delta^4)}_{\text{minimum continuation payoff from last 3 periods}}$$

For reasonably large ρ_0, $\Pi_c > \Pi_D$. A similar analysis holds for later periods also. In general, when beliefs about the other player being a cooperative behavioral type is high, Cooperation is always justifiable for $\delta \geq 3/7$.
A similar analysis would show that there exist no possible beliefs on the partner that should induce an egoist to cooperate in any period of the $\delta = 1/4$ treatment. The easiest wat to see it is to note that the lowest possible one time gain from defection vs cooperation in any period, overpowers any possible future gains, irrespective of the beleifs held over the other player's action.