BEST SIMULTANEOUS APPROXIMATION IN NORMED LINEAR SPACES

A thesis submitted in partial fulfilment of the

requirements for the degree of

Masters of Science

at

RHODES UNIVERSITY

by

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DECEMBER 2013

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Chapter 1

INTRODUCTION

Let $(X, \|\cdot\|)$ be a normed linear space and K be subset of X. For each $x \in X$, we define the distance from x to K by

$$d(x, K) = \inf_{y \in K} \|x - y\|. \tag{1.1}$$

If there exist a $y_0 \in K$ such that (1.1) holds, then y_0 is called a **best approximation** to x in K. Denote by

$$P_K(x) := \{y \in K \mid \|x - y\| = d(x, K) \tag{1.2}$$

the set of all best approximations to x in K.

The map $P_K : X \to 2^K$ defined above is called the metric projection of X onto K or the proximity map.

The problem of best approximation investigates, among others, the following questions:

1. **Existence of best approximations**: What conditions on X and/or K ensure that the set $P_K(x)$ is nonempty for each $x \in X$?

2. **Uniqueness of best approximations**: If there exists an element of best approximation, is it unique?

3. **Characterization of best approximations**: Given an element y_0 of a subset K of X, and $x \in X$ how does one recognize whether or not y_0 is a best approximation to x in K?

In this thesis we consider the problem of simultaneously approximating elements of a set $B \subset X$ by a single element of a set $K \subset X$. This type of a problem arises when the element to be approximated is not known precisely but is known to belong to a set.Thus, best simultaneous approximation is a natural generalization of best approximation which has been studied extensively. The theory of best simultaneous approximation has been studied by many authors, see for example [4], [8], [25], [28], [26] and [12] to name but a few.
Analogous to the theory of best simultaneous approximation is the theory of relative Chebyshev Centres introduced by Garkavi [13], which has also been researched by several Mathematicians e.g., [1], [18] and [12]. In the theory of relative Chebyshev centres, $d(B, K)$ is denoted as $rad_B(K)$ and it is called the **restricted radius** of B in K, the set of best simultaneous approximation $\mathcal{S}_K(B)$ is denoted as $Cent_K(B)$ and it is called the **restricted centre** of B in K. While the notations and definitions of the relative Chebyshev centre and the best simultaneous approximation of elements are different, the ultimate aim and the general concept are the same. In this thesis, we shall study best simultaneous approximation in a **Real** normed space. Hence $(X, \|\cdot\|)$ will always be real.

1.1 Statement of the Problem

Let $(X, \|\cdot\|)$ be a normed linear space and K be subset of X. Denote by $\mathcal{C}(X)$ a collection of subsets of X. Given a $B \in \mathcal{C}(X)$, denote by

$$d(B, K) = \inf_{k \in K} \sup_{s \in B} \|s - k\|. \tag{1.3}$$

the distance from B to a subset K of X.

An element $y_0 \in K$ is called a **best simultaneous approximation** (b.s.a) to a set $B \in \mathcal{C}(X)$ in the set K if

$$\sup_{s\in B} \|s - y_0\| = d(B, K).$$

The set of all best simultaneous approximations to $B \in \mathcal{C}(X)$ in K is denoted by $\mathcal{S}_K(B)$. That is,

$$\mathcal{S}_K(B) = \{y \in K \mid \sup_{s\in B} \|s - y\| = d(B, K)\}. \tag{1.4}$$

Equation (1.4) defines a set-valued map from $\mathcal{C}(X)$ into the set of subsets of K. The map

$$\mathcal{S}_K : \mathcal{C}(X) \to 2^K$$

defined by equation (1.4) is called the **best simultaneous approximation operator**.

Note that if $B = \{x\}$, a singleton, then

$$d(B, K) = d(\{x\}, K) = \inf_{y\in K} \|x - y\| = d(x, K) \quad \text{and}$$

$$\mathcal{S}_K(\{x\}) = P_K(x),$$

the metric projection.

Thus, if the set B consists of a single element $\{x\} \subset X$ then the best simultaneous approximation to the set B from K is the same as the best approximation to x from K and the simultaneous proximity map $\mathcal{S}_K(B)$ is called the metric projection $P_K(x)$.

Definition 1.1. *Let $K \subseteq X$ be nonempty and $\mathcal{C}(X)$ be a collection of subsets of X. Then K is said to be:*

(i) **simultaneous proximinal** *relative to $\mathcal{C}(X)$ if for each $B \in \mathcal{C}(X)$, the set $\mathcal{S}_K(B)$ is nonempty; i.e., each $B \in \mathcal{C}(X)$ has at least one best simultaneous approximation.*

(ii) **simultaneous semi-Chebyshev** *relative to $\mathcal{C}(X)$ if for each $B \in \mathcal{C}(X)$, the set $\mathcal{S}_K(B)$ is at most singleton; i.e., each $B \in \mathcal{C}(X)$ has at most one best simultaneous approximation.*

(iii) **simultaneous Chebyshev** *relative to $\mathcal{C}(X)$ if for each $B \in \mathcal{C}(X)$, the set $\mathcal{S}_K(B)$ is a singleton; i.e., each $B \in \mathcal{C}(X)$ has exactly one best simultaneous approximation.*

It is clear from Definition 1.1 that a simultaneous Chebyshev set is one which is both simultaneous proximinal and simultaneous semi-Chebyshev.

The problem of simultaneous best approximation is concerned with the following questions:

1. **Existence of best simultaneous approximations**: i.e., given any $B \in \mathcal{C}(X)$, what conditions on X and/or K will ensure that $\mathcal{S}_K(B)$ is nonempty?

2. **Uniqueness of best simultaneous approximations**: Given any $B \in \mathcal{C}(X)$, under what conditions on X and/or K is the set $\mathcal{S}_K(B)$ a singleton?

3. **Characterization** of best simultaneous approximations: given an element y_0 of a subset K of X, and a $B \in \mathcal{C}(X)$ how does one recognize whether or not y_0 is a best simultaneous approximation to B in K?

4. **Computation** of best simultaneous approximation: are there algorithms for constructing a best simultaneous approximation to a $B \in \mathcal{C}(X)$?

5. **Degree, or Error** of best simultaneous approximation: can one compute the error $d(B, K)$, or at least get a good upper and/or lower bounds for it?

6. **Continuity** of best simultaneous approximation: how does the set of best simultaneous approximations to B, $\mathcal{S}_K(B)$, depend on $B \in \mathcal{C}(X)$?

In this thesis we will seek to address the questions of **existence, uniqueness**, and **characterization** of best simultaneous approximations.

1.1.1 Some Properties of the distance functional and the simultaneous proximity map

In this subsection we look at some properties of the distance functional defined by equation (1.3) and the simultaneous proximity map given by equation (1.4).

Lemma 1.2. *Let K be a non-empty subset of a normed linear space $(X, \|\cdot\|)$ and $B \in \mathcal{C}(X)$ be non-empty and bounded. Then the functional $\phi : K \to \mathbb{R}$ defined by*

$$\phi(y) = \sup_{s \in B} \|s - y\|$$

is continuous on K. In fact, ϕ is Lipschitz continuous on K.

Proof. For any $s \in B$ and $y, z \in K$, we have, by the triangle inequality, that

$$\|s - y\| \leq \|s - z\| + \|y - z\|.$$

Taking the supremum over all $s \in B$, we have

$$\sup_{s \in B} \|s - y\| \leq \sup_{s \in B} \|s - z\| + \|y - z\|, \quad \text{i.e.,}$$

$$\phi(y) - \phi(z) \leq \|y - z\|. \tag{1.2.1}$$

Interchanging the roles of y and z in equation (1.2.1), we get

$$\phi(z) - \phi(y) \leq \|y - z\|. \tag{1.2.2}$$

From equations (1.2.1) and (1.2.2), we have that

$$|\phi(y) - \phi(z)| \leq \|y - z\|,$$

which proves the result. ∎

Proposition 1.3. *Let K be non-empty subset of a normed linear space X and $B \in \mathcal{C}(X)$ be non-empty. Then for every $y \in X$ and every $\lambda \in \mathbb{R} \setminus \{0\}$,*

1. $d(B+y, K+y) = d(B, K)$.
2. $d(\lambda B, \lambda K) = |\lambda| d(B, K)$.
3. $\mathcal{S}_{K+y}(B+y) = \mathcal{S}_K(B) + y$.
4. $\mathcal{S}_{\lambda K}(\lambda B) = \lambda \mathcal{S}_K(B)$.

Proof.

1. From equation (1.3), we have

$$\begin{aligned} d(B+y, K+y) &= \inf_{k \in K} \sup_{s \in B} \|s + y - (k+y)\| \\ &= \inf_{k \in K} \sup_{s \in B} \|s - k\| \\ &= d(B, K) \end{aligned}$$

2.

$$\begin{aligned} d(\lambda B, \lambda K) &= \inf_{k \in K} \sup_{s \in B} \|\lambda s - \lambda k\| \\ &= \inf_{k \in K} \sup_{s \in B} \|\lambda (s - k)\| \\ &= |\lambda| \inf_{k \in K} \sup_{s \in B} \|s - k\| \\ &= |\lambda| d(B, K) \end{aligned}$$

3.

$$\begin{aligned} k_0 \in \mathcal{S}_{K+y}(B+y) &\iff \sup_{s \in B} \|s + y - k_0\| = \inf_{k \in K} \sup_{s \in B} \|(s+y) - (k+y)\| \\ &\iff \sup_{s \in B} \|s - (k_0 - y)\| = \inf_{k \in K} \sup_{s \in B} \|s - k\| \\ &\iff k_0 - y \in \mathcal{S}_K(B) \\ &\iff k_0 \in \mathcal{S}_K(B) + y \end{aligned}$$

4. Now

$$
\begin{aligned}
k_0 \in \mathcal{S}_{\lambda K}(\lambda B) &\iff \sup_{s\in B}\|\lambda s - k_0\| = \inf_{k\in K}\sup_{s\in B}\|\lambda s - \lambda k\| \\
&\iff |\lambda| \sup_{s\in B}\|s - \frac{1}{\lambda}k_0\| = |\lambda| \inf_{k\in K}\sup_{s\in B}\|s - k\| \\
&\iff \sup_{s\in B}\|s - \frac{1}{\lambda}k_0\| = \inf_{k\in K}\sup_{s\in B}\|s - k\| \\
&\iff \frac{1}{\lambda}k_0 \in \mathcal{S}_K(B) \\
&\iff k_0 \in \lambda \mathcal{S}_K(B). \qquad \blacksquare
\end{aligned}
$$

Proposition 1.4. *Let K be a subset of a normed linear space X. If K is simultaneous proximinal relative to $\mathcal{C}(X)$, then it is closed.*

Proof. Assume that there is an $x \in \overline{K} \setminus K$. Then, for each $y \in K$,

$$d(\{x\}, K) = d(x, K) = 0 < \|x - y\|.$$

Therefore, no element of K is a best simultaneous approximation to the set $\{x\}$, contradicting that K is simultaneous proximinal. ■

In the next Proposition we show that if $\mathcal{C}(X)$ is the collection of bounded subsets of X, denoted by $\mathcal{CB}(X)$, then the set of best simultaneous approximations is closed and bounded.

Proposition 1.5. *Let K be a closed subset of a normed linear space X and $B \in \mathcal{CB}(X)$ such that $K \cap B = \emptyset$. Then the set $\mathcal{S}_K(B)$ is closed and bounded.*

Proof. Closedness of $\mathcal{S}_K(B)$: Let $y \in \overline{S_K(B)}$. Then there exists a sequence (y_n) in $S_K(B)$ such that $\lim_{n\to\infty} y_n = y$. Since the sequence $(y_n)_n$ is in K and K is closed, we have that $y \in K$. By Lemma 1.2, we have that

$$\sup_{s\in B}\|s - y\| = \lim_{n\to\infty}\sup_{s\in B}\|s - y_n\| = \lim_{n\to\infty} d(B, K) = d(B, K).$$

Therefore $y \in \mathcal{S}_K(B)$ and so $S_K(B)$ is closed.

Boundedness of $\mathcal{S}_K(B)$: Since B is bounded, there is a constant C such that, for each $s \in B$, we have $\|s\| \leq C$. Now, for each $y \in \mathcal{S}_K(B)$ and each $s \in B$, we have

$$\|y\| \leq \|s - y\| + \|s\|.$$

Taking supremum over all $s \in B$, we have

$$\|y\| \leq \sup_{s \in B} \|s - y\| + \sup_{s \in B} \|s\| \leq d(B, K) + C = C'.$$

Hence, $\mathcal{S}_K(B)$ is bounded. ■

Recall that a set $A \subset X$ is said to be convex if $\lambda x + (1 - \lambda)y \in A$ for every $x, y \in A$ and $\lambda \in [0, 1]$.

It is well known, see for example [18], that for a convex subset of K of X, $P_K(x)$, the set of all best approximations to $x \in X$ in K, is convex. In the next proposition we extend this result to best simultaneous approximation and show that the set of best simultaneous approximations to $B \in C(X)$ in K, a convex subset of X, is also convex.

Proposition 1.6. *Let K be a convex subset of a normed linear space $(X, \|\cdot\|)$ and $B \in \mathcal{C}(X)$. Then the set $\mathcal{S}_K(B)$ is convex.*

Proof. If the set $\mathcal{S}_K(B)$ is empty or a singleton, then it is obviously convex. Let $y, z \in \mathcal{S}_K(B)$ and $\lambda \in [0, 1]$. Since $y, z \in K$ and K is convex, we have that $\lambda y + (1 - \lambda)z \in K$. Then, for any $s \in B$,

$$\begin{aligned} \|s - [\lambda y + (1 - \lambda)z]\| &= \|\lambda(s - y) + (1 - \lambda)(s - z)\| \\ &\leq \lambda\|s - y\| + (1 - \lambda)\|s - z\|. \end{aligned}$$

Taking supremum over all $s \in B$, we have

$$\begin{aligned} d(B, K) \leq \sup_{s \in B} \|s - \lambda y + (1 - \lambda)z\| &\leq \lambda \sup_{s \in B} \|s - y\| + (1 - \lambda) \sup_{s \in B} \|s - z\| \\ &= \lambda d(B, K) + (1 - \lambda)d(B, K) = d(B, K). \end{aligned}$$

Therefore

$$\sup_{s\in B}\|s-\lambda y+(1-\lambda)z\|=d(B,K)$$

and so $\lambda y+(1-\lambda)z\in\mathcal{S}_K(B)$. ■

It follows from Proposition 1.6 that if K is convex, then the set of best simultaneous approximations to B in K, if it is nonempty, either contains one element or infinitely many elements.

From the theory of best approximation we know that if we are approximating an element $x\in X$ from a set $K\subset X$ and $x\notin K$, then a best approximation must lie in the boundary, ∂K, of K. The next proposition is a natural extension of this result in the setting of best simultaneous approximation.

Proposition 1.7. *Let K be a closed convex subset of a normed linear space $(X,\|\cdot\|)$ and $B\in\mathcal{C}(X)$ such that $K\cap B=\emptyset$. Then $\mathcal{S}_K(B)\subset\partial K$.*

Proof. Let $y\in\mathcal{S}_K(B)$ be arbitrary. Then $\sup_{s\in B}\|s-y\|=d(B,K)$. Let $r=d(B,K)$. since $K\cap B=\emptyset$, we have $r>0$. Assume if possible that $y\notin\partial K$. So $y\in intK$(interior of K). Thus there exists $\epsilon>0$ such that

$$S=\{x\in X|\|x-y\|<\epsilon\}\subset K.$$

Let $\epsilon_0=\epsilon(r+\epsilon)^{-1}$, $s\in B$ and $y_s=y+\epsilon_0(s-y)\in X$. Note that $0<\epsilon_0<1$. Since

$$\|y_s-y\|=\epsilon_0\|s-y\|\leq\epsilon_0 r<\epsilon,$$

$y_s\in S\subset K$ and for each $s\in B$

$$r=d(B,K)\leq\sup_{t\in B}\|t-y_s\|. \qquad (1.7.1)$$

Since equation (1.7.1) is true for all $s\in B$, it follows that

$$r\leq\inf_{s\in B}\sup_{t\in B}\|t-y_s\|.$$

On the other hand, for each $t, s \in B$ we have

$$\|t - y_s\| = \|(t - y) - \epsilon_0(s - y)\|.$$

This implies that

$$\begin{aligned} r &\leq \inf_{s\in B} \sup_{t\in B} \|t - y_s\| = \inf_{s\in B} \sup_{t\in B} \|(t - y) - \epsilon_0(s - y)\| \\ &\leq \sup_{t\in B} \|(t - y) - \epsilon_0(t - y)\| = (1 - \epsilon_0) \sup_{t\in B} \|t - y\| = (1 - \epsilon_0) \sup_{t\in B} \|t - y\| \\ &= (1 - \epsilon_0) r < r, \end{aligned}$$

which is absurd. This completes the proof. ■

Chapter 2

EXISTENCE OF ELEMENTS OF BEST SIMULTANEOUS APPROXIMATION

In this Chapter we address the question of existence of best simultaneous approximations. That is, given a subset K of X and a non-empty $B \in \mathcal{C}(X)$, does there always exist an element of best simultaneous approximation to B from K? What conditions on K and/or X ensure the existence of an element of best simultaneous approximation to a $B \in \mathcal{C}(X)$?

Definition 2.1. *Let K be a subset of a normed linear space $(X, \|\cdot\|)$. A sequence $(y_n)_n \subseteq K$ is called a* **minimizing sequence** *for a set $B \in \mathcal{C}(X)$ if*

$$\sup_{s \in B} \|s - y_n\| \to d(B, K) \quad as \quad n \to \infty.$$

Note that if $B = \{x\} \subset X$, a singleton set, then Definition 2.1 reduces to the usual one [See, for example [18] Pg 376].

Definition 2.2. *A non empty subset K of a normed linear space $(X, \|\cdot\|)$ is said to be* **approximatively compact** *with respect to $B \in \mathcal{CB}(X)$ if every minimizing sequence for B in K has a subsequence which converges to a point in K.*

In the context of best approximations the concept of approximative compactness was introduced by Efimov and Stechkin [11].

In the paper of Beer and Pai [2], "approximative compactness" is referred to as "*cent-compact*".

Singer [27] showed that an approximative compact subset of a normed linear space is proximinal. The following Theorem shows that this result extends, in a natural way, to the setting of best simultaneous approximations.

Theorem 2.3. *An approximatively compact subset K of a normed linear space $(X, \|\cdot\|)$ is simultaneous proximinal relative to $\mathcal{C}(X)$.*

Proof. Let $B \in \mathcal{C}(X)$ and $(y_n)_n$ a minimizing sequence for B; i.e.,

$$\sup_{s \in B} \|s - y_n\| \to d(B, K) \quad \text{as} \quad n \to \infty.$$

Since K is approximatively compact, there is a subsequence $(y_{n_k})_k$ of $(y_n)_n$ which converges to some $y \in K$. By Lemma 1.2, we have that

$$\sup_{s \in B} \|s - y\| = \lim_{k \to \infty} \sup_{s \in B} \|s - y_{n_k}\| = d(B, K).$$

Therefore y is a simultaneous best approximation of B in K. ■

Proposition 2.4. *If K is an approximatively compact subset of a normed linear space $(X, \|\cdot\|)$ and $B \in \mathcal{CB}(X)$, then the set $\mathcal{S}_K(B)$ is compact.*

Proof. Let $(y_n)_n$ be a sequence in $\mathcal{S}_K(B)$. Then, for each $n \in \mathbb{N}$,

$$\sup_{s \in B} \|s - y_n\| = d(B, K),$$

and so $\sup_{s \in B} \|s - y_n\| \to d(B, K)$ as $n \to \infty$. That is, $(y_n)_n$ is a minimizing sequence for B. Since K is approximatively compact there is a subsequence $(y_{n_k})_k$ of $(y_n)_n$ which converges to some point $y \in K$. Since an approximatively compact subset of a normed linear space is closed and $\mathcal{S}_K(B)$ is closed when K is closed [see Proposition 1.5], we have that $y \in \mathcal{S}_K(B)$. Hence $\mathcal{S}_K(B)$ is compact. ■

Lemma 2.5. *Let K be a non empty subset of a normed linear space $(X, \|\cdot\|)$. Every minimizing sequence for $B \in \mathcal{CB}(X)$ in K is bounded.*

Proof. Let $B \in \mathcal{CB}(X)$ and $(y_n)_n$ a minimizing sequence for B. Then

$$\sup_{s\in B} \|s - y_n\| \to d(B, K) \quad \text{as} \quad n \to \infty.$$

Since the sequence $\left(\sup_{s\in B} \|s - y_n\|\right)_n$ converges, it is bounded; i.e., there is a constant $C > 0$ such that

$$\sup_{s\in B} \|s - y_n\| \leq C \quad \text{for all} \quad n \in \mathbb{N}.$$

Also, since B is bounded, there is a constant $C' > 0$ such that

$$\sup_{s\in B} \|s\| \leq C'.$$

Now, for each $n \in \mathbb{N}$, we have that

$$\|y_n\| \leq \|s - y_n\| + \|s\| \leq \sup_{s\in B} \|s - y_n\| + \sup_{s\in B} \|s\| \leq C + C'.$$

Hence, the sequence $(y_n)_n$ is bounded. ■

Definition 2.6. *A non empty subset K of a normed linear space $(X, \|\cdot\|)$ is said to be* **boundedly compact** *if every bounded sequence in K has a subsequence which converges to a point in K.*

Proposition 2.7. *A non empty subset K of a normed linear space $(X, \|\cdot\|)$ is boundedly compact if and only if for each closed ball S in X, the set $K \cap S$ is compact.*

Proof. Assume that K is boundedly compact, and let S a closed ball in X and $(y_n)_n$ a sequence in $K \cap S$. Then the sequence $(y_n)_n$ is bounded. Since K is boundedly compact, the sequence $(y_n)_n$ has a subsequence $(y_{n_k})_k$ which converges to some point $y \in K$. Since S is closed, we have that $y \in S$. That is, $y \in K \cap S$, and so $K \cap S$ is compact.

Conversely, assume that for each closed ball S in X, the set $K \cap S$ is compact. Let $(y_n)_n$ be a bounded sequence in K. Then, there is a constant $r > 0$ such that $\|y_n\| \leq r$ for each $n \in \mathbb{N}$. That is, for each $n \in \mathbb{N}$,

$$y_n \in B(0, r) = B_r := \{z \in X \mid \|z\| \leq r\}.$$

Since, by our assumption, the set $K \cap B_r$ is compact, the sequence $(y_n)_n$ has a subsequence $(y_{n_k})_k$ which converges to some point $y \in K \cap B_r$. So, starting with a bounded sequence $(y_n)_n$ in K, we have shown that it has a subsequence which converges to a point in K. Hence K is boundedly compact. ■

Theorem 2.8. *Let K be a non empty subset of a normed linear space $(X, \|\cdot\|)$. Then K is boundedly compact if any of the following is true:*

(i) K is compact;

(ii) K is a closed subset of a finite-dimensional subspace of X;

(iii) K is a finite-dimensional subspace of X.

Proof.

(i) Assume K is compact and let $(y_n)_n$ be a bounded sequence in K. Since K is compact, the sequence $(y_n)_n$ has a subsequence $(y_{n_k})_k$ that converges to a point in K. Hence K is boundedly compact.

(ii) Let K be a closed subset of a finite-dimensional subspace of $(X, \|\cdot\|)$. Since the intersection of a norm-closed ball with a closed subset of a finite-dimensional subspace is compact, it follows from Proposition 2.7 that K is boundedly compact.

(iii) Let K be a finite-dimensional subspace of $(X, \|\cdot\|)$. Since every finite-dimensional subspace of $(X, \|\cdot\|)$ is closed, it follows from (ii) that K is boundedly compact. ■

Proposition 2.9. *Every boundedly compact subset K of a normed linear space $(X, \|\cdot\|)$ is approximatively compact.*

Proof Let $B \in \mathcal{CB}(X)$ and $(y_n)_n$ a minimizing sequence for B. Then, by Lemma 2.5, the sequence $(y_n)_n$ is bounded. Since K is boundedly compact, the sequence $(y_n)_n$ has a subsequence $(y_{n_k})_k$ which converges to some element $y \in K$. Hence K is approximatively compact. ■

Corollary 2.10. *Let X be a normed linear space. Then*

1. *Every compact subset of X is simultaneous proximinal relative to $\mathcal{CB}(X)$.*
2. *Every closed subset of a finite-dimensional subspace of X is simultaneous proximinal relative to $\mathcal{CB}(X)$.*
3. *Every finite-dimensional subspace of X is simultaneous proximinal relative to $\mathcal{CB}(X)$.*

Theorem 2.11. *For a Banach space $(X, \|\cdot\|)$, the following statements are equivalent:*

(1) X is reflexive;

(2) Every closed convex subset of X is simultaneous proximinal relative to $\mathcal{CB}(X)$;

(3) Every closed linear subspace of X is simultaneous proximinal relative to $\mathcal{CB}(X)$;

(4) Every closed hyperplane in X is simultaneous proximinal relative to $\mathcal{CB}(X)$.

Proof. (1) $\Rightarrow$ (2): Let K be a closed convex subset of X and $B \in \mathcal{CB}(X)$. For each $n \in \mathbb{N}$, let

$$K_n = \{y \in K \mid \sup_{s \in B} ||s - y\| \leq d(B, K) + \frac{1}{n}\}.$$

Then (K_n) is a decreasing sequence of nonempty, closed, convex and bounded subsets of X. It follows that

$$\mathcal{S}_K(B) = \{y \in K \mid \sup_{s \in B} ||s - y|| = d(B, K)\} = \bigcap_{n \in \mathbb{N}} K_n \neq \emptyset.$$

That is, K is simultaneous proximinal relative to $\mathcal{CB}(X)$.

(2) $\Rightarrow$ (3): This is obvious since every closed linear subspace is a closed convex set.

(3) $\Rightarrow$ (4): This is obvious since every closed hyperplane is a closed linear subspace.

(4) $\Rightarrow$ (1): Let $x^* \in S(X^*)$ and

$$H = \{x \in X \ : \ x^*(x) = 1\}.$$

Then H is a closed hyperplane in X and so is simultaneous proximinal relative to $\mathcal{CB}(X)$. It follows that

$$\mathcal{S}_H(\{0\}) = P_H(0) \neq \emptyset.$$

Let $x_0 \in \mathcal{S}_H(\{0\})$. Then

$$\|x_0\| = d(0, H) = 1$$

and $x^*(x_0) = 1$. That is, x^* attains its norm at x_0. By James' Theorem [16], X is reflexive. ■

Sehgal V.M. and S.P. Singh [26] have proved that the distance functional $\phi : K \to \mathbb{R}$ defined by: $\phi(y) = \sup_{s \in B} \|s - y\|$ is weakly lower-semicontinuous. Recall that a bounded closed and convex subset of a reflexive space is weakly-compact. Using these facts we can prove that a reflexive subspace is simultaneous proximinal relative to $\mathcal{CB}(X)$.

Theorem 2.12. *Let K be a reflexive subspace of a normed linear space $(X, \|\cdot\|)$. Then for any non-empty $B \in \mathcal{CB}(X)$, there exists a best simultaneous approximation $y_0 \in K$ to B.*

Proof: Since B is bounded, there exists an $M \in \mathbb{R}$ such that $\|s\| \leq M$ for each $s \in B$. Define a subset Y of K by

$$Y = \{y \in K : \|y\| \leq 2 \sup_{s\in B} \|s\|\}.$$

We show that $d(B, Y) = d(B, K)$.

Since $0 \in Y$, we have that

$$d(B, Y) = \inf_{y\in Y} \sup_{y\in B} \|s - y\| \leq \sup_{s\in B} \|s - 0\| = \sup_{s\in B} \|s\| \leq M.$$

Now, if $z \in K \setminus Y$, then $\|z\| > 2 \sup_{s\in B} \|s\|$ and

$$\sup_{s\in B} \|s - z\| \geq \|z\| - \sup_{s\in B} \|s\| > \sup_{s\in B} \|s\| \geq d(B, Y). \tag{2.12.1}$$

If $z \in Y$, then

$$d(B, Y) = \inf_{y\in Y} \sup_{s\in B} \|s - y\| \leq \sup_{s\in B} \|s - z\|. \tag{2.12.2}$$

From (2.12.1) and (2.12.2), we have that, for all $z \in K$,

$$d(B, Y) \leq \sup_{s\in B} \|s - z\|. \tag{2.12.3}$$

From (2.12.3), we have that

$$d(B, Y) \leq d(B, K). \tag{2.12.4}$$

But since $Y \subset K$, we must have that

$$d(B, K) \leq d(B, Y). \tag{2.12.5}$$

We conclude from (2.12.4) and (2.12.5) that $d(B, K) = d(B, Y) \leq M$, and this value cannot be assumed by a $y \in K \setminus Y$, because of the strict inequality in (2.12.1). Hence if a best simultaneous approximation to B exists, it must lie in Y. By the reflexivity of K, the set Y is weakly-compact. Therefore, there exists a $y_0 \in Y$ such that

$$\sup_{s\in B} \|s - y_0\| = d(B, Y) = d(B, K).$$

Hence y_0 is a best simultaneous approximation to B from K. ■

Chapter 3

UNIQUENESS OF ELEMENTS OF BEST SIMULTANEOUS APPROXIMATION

In Chapter 2 we found out that if the set $K \subset X$ satisfies certain conditions, then we can be sure to find an element say, $y_0 \in K$ that is a best simultaneous approximation to $B \in \mathcal{C}(X)$. The natural question to ask is, is such an element unique or can we find other elements in K that are also best simultaneous approximations to $B \in \mathcal{C}(X)$? Convexity of $\mathcal{S}_K(B)$ implies that if such an element is not unique then there are infinitely many best simultaneous approximations to the set B. We will try to answer this question in this Chapter.

Definition 3.1. *A normed linear space* $(X, \|\cdot\|)$ *is said to be* **strictly convex** *(or* **rotund***) if for* x, y *in* X, $x \neq y$, $\|x\| = \|y\| = 1$, *and* $0 < \lambda < 1$ *imply that* $\|\lambda x + (1-\lambda)y\| < 1$.

Geometrically, a strictly convex space is one in which the boundary of the unit ball contains no line segments. There are useful alternate formulations of strict convexity as shown in the next Lemma.

Lemma 3.2. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) X *is strictly convex;*

(ii) $x, y \in X \setminus \{0\}$ *and* $\|x+y\| = \|x\| + \|y\|$, *imply that* $x = \lambda y$ *for some* $\lambda > 0$;

(iii) $\|x\| = \|y\| = \|\frac{x+y}{2}\| = 1$ *implies* $x = y$;

(iv) $extU(X) = S(X)$.

Proof of Lemma 3.2 can be found in [18] pg 384.

Definition 3.3. *Let $(X, \|\cdot\|)$ be a normed linear space. A set $B \in \mathcal{CB}(X)$ is said to be* **remotal** *with respect to a subset K of X if for each $y \in K$ there is an element s_0 in B such that*

$$\|s_0 - y\| = \sup_{s\in B} |\|s - y\|.$$

It is obvious that a compact subset of a normed linear space is remotal with respect to any subset of X.

Theorem 3.4. *If K is a closed convex subset of a strictly convex normed linear space $(X, \|\cdot\|)$ and B a subset of X which is remotal with respect to K, then the set $\mathcal{S}_K(B)$ is at most a singleton.*

Proof. If $\mathcal{S}_K(B)$ is empty or a singleton, then there is nothing to prove. Assume that $y_1, y_2 \in \mathcal{S}_K(B)$. Then, since the set $\mathcal{S}_K(B)$ is convex, we have that $y_0 = \frac{1}{2}(y_1 + y_2) \in \mathcal{S}_K(B)$. Since B is remotal with respect to K, there is an $s_0 \in B$ such that

$$\sup_{s\in B} \|s - y_0\| = \|s_0 - y_0\|.$$

Now, since

$$\|s_0 - y_1\| \leq \sup_{s \in B} \|s - y_1\| = \delta, \qquad \|s_0 - y_2\| \leq \sup_{s \in B} \|s - y_2\| = \delta \quad \text{and}$$

$$\|s_0 - y_0\| = \left\| s_0 - \frac{1}{2}(y_1 + y_2) \right\| = \left\| \left(\frac{s_0 - y_1}{2} \right) + \left(\frac{s_0 - y_2}{2} \right) \right\| = \delta,$$

it follows that

$$\|s_0 - y_1\| = \delta \quad \text{and} \quad \|s_0 - y_2\| = \delta.$$

Strict convexity of X implies that $s_0 - y_1 = s_0 - y_2$ and so $y_1 = y_2$. ■

As noted above, a compact subset of a normed linear space X is remotal with respect to *any* subset of X, and so the following result is obvious:

Corollary 3.5. *If K is a closed convex subset of a strictly convex normed linear space $(X, \|\cdot\|)$, then for each $B \in \mathcal{K}(X)$, where $\mathcal{K}(X)$ is the collection of compact subsets of X, the set $\mathcal{S}_K(B)$ is at most a singleton.*

Definition 3.6. [18] *Let $(X, \|\cdot\|)$ be a normed linear space and $B \in \mathcal{CB}(X)$. The set B is said to be* **sup-compact** *with respect to a subset K of X if for each $y \in K$ each maximizing sequence $(s_n)_n \subset B$, i.e., a sequence satisfying $\|y - s_n\| \to \sup_{s \in B} \|s - y\|$, has a subsequence convergent in B.*

Lemma 3.7. *If $B \in \mathcal{CB}(X)$ is sup-compact with respect to $K \subset X$, then it is remotal with respect to K.*

Proof. Let $B \subset X$ be sup-compact with respect to $K \subset X$, $y \in K$ and $(s_n)_n$ be a maximizing sequence in B. Then $(s_n)_n$ has a subsequence $(s_{n_k})_k$ convergent to $s_0 \in B$. By continuity of the norm we have, for each $y \in K$

$$\|y - s_0\| = \lim_{k \to \infty} \|y - s_{n_k}\| = \sup_{s \in B} \|s - y\|$$

Hence B is remotal with respect to K. ■

Combining Theorem 3.4 and Lemma 3.7 we get the following Corollary.

Corollary 3.8. *Let K be a closed convex subset of a strictly convex normed linear space $(X, \|\cdot\|)$. If $B \in \mathcal{CB}(X)$ is sup-compact with respect to K, then the set $\mathcal{S}_K(B)$ is at most a singleton.*

Definition 3.9. [1] *A normed linear space $(X, \|\cdot\|)$ is said to be* **strictly convex with respect to its linear subspace** K *if its sphere contains no line segment parallel to a line segment in K, i.e., $x, y \in X$,*

$$\left\{ \|x\| = \|y\| = \|\frac{x+y}{2}\| = 1, x - y \in K \right\} \Rightarrow x = y.$$

It is clear that X is strictly convex if and only if it is strictly convex with respect to itself and if X is strictly convex with respect to K, then it is strictly convex with respect to every $G \subset K$. Furthermore, each subspace $X_0, K \subset X_0 \subset X$, is strictly convex with respect K, and in particular K itself is a strictly convex subspace.

Theorem 3.10. [1] *For a subspace K of a normed linear space $(X, \|\cdot\|)$, the following statements are equivalent.*

(i) X is strictly convex with respect to K;

(ii) $\mathcal{S}_K(B)$ is at most a singleton for every subset B of X that is remotal with respect to K;

(iii) $\mathcal{S}_K(B)$ is at most a singleton for every compact subset B of X;

(iv) For every set $\{x_1, x_2\} \subset X$ of two elements, $\mathcal{S}_K(\{x_1, x_2\})$ is at most a singleton;

(v) Every line segment in K is a Chebyshev set.

Proof. (i) $\Rightarrow$ (ii). If $\mathcal{S}_K(B)$ is empty, then there is nothing to prove. Otherwise, we may assume that $d(B, K) = 1$ and let $y_1, y_2 \in \mathcal{S}_K(B)$. Then since

$\mathcal{S}_K(B)$ is convex, we have that $\frac{y_1+y_2}{2} \in \mathcal{S}_K(B)$. Using the fact that B is remotal, it follows that there exists an $s_0 \in B$ such that

$$\left\| s_0 - \frac{y_1 + y_2}{2} \right\| = \sup_{s \in B} \left\| s - \frac{y_1 + y_2}{2} \right\| = 1.$$

Now since $\|s_0 - y_1\| \leq d(B, K) = 1, \quad \|s_0 - y_2\| \leq d(B, K) = 1,$ and $\left\| s_0 - \frac{y_1+y_2}{2} \right\| = \frac{1}{2}\|s_0 - y_1\| + \frac{1}{2}\|s_0 - y_2\| \leq 1$, it follows that

$$\|s_0 - y_1\| = \|s_0 - y_2\| = 1.$$

Since $(s_0 - y_1) - (s_0 - y_2) = y_2 - y_1 \in K$, and X is strictly convex with respect to K, it follows that $y_1 = y_2$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv): This is obvious.

(iv) $\Rightarrow$ (i): Assume (i) is not satisfied. then there exist $x, z \in X$ such that

$$\|x\| = \|z\| = \left\| \frac{x + z}{2} \right\| = 1, \text{and}\, x - z \in K, \text{ but}\, x \neq z. \tag{3.10.1}$$

By homogeneity of the norm, we assume $d(\{\frac{x+z}{2}, -(\frac{x+z}{2}\}, K) = 1$.
Let $B = \left\{ \frac{x+z}{2}, -\left(\frac{x+z}{2}\right) \right\}$. We show that $\frac{x-z}{2} \in \mathcal{S}_K(B)$ and $\frac{z-x}{2} \in \mathcal{S}_K(B)$.
Note that since K is a linear subspace and $x - z \in K$, it follows that both $\frac{x-z}{2}$ and $\frac{z-x}{2}$ are elements of K. Now, since

$$\left\| \frac{x + z}{2} - \left(\frac{x - z}{2} \right) \right\| = \|z\| = 1,$$

$$\left\| -\left(\frac{x + z}{2} \right) - \left(\frac{x - z}{2} \right) \right\| = \|x\| = 1,$$

$$\left\| \frac{x + z}{2} - \left(\frac{z - x}{2} \right) \right\| = \|x\| = 1,$$

$$\left\| -\left(\frac{x+z}{2}\right) - \left(\frac{z-x}{2}\right)\right\| = \|z\| = 1,$$

we have that

$$\left\{\frac{x-z}{2}, \frac{z-x}{2}\right\} \subset \mathcal{S}_K\left(\left\{\frac{x+z}{2}, \frac{-(x+z)}{2}\right\}\right),$$

which contradicts (iv). Thus the statements (i)-(iv) are equivalent.

(i) $\Rightarrow$ (v): If there is a line segment in K which is not Chebyshev, then there are points $y_1, y_2 \in K, y_1 \neq y_2$ and a point $x \notin [y_1, y_2]$ such that $[y_1, y_2] = P_{[y_1,y_2]}(x)$. Then $\|x - y_1\| = \|x - y_2\| = \|x - \frac{y_1+y_2}{2}\|$ and since $(x - y_1) - (x - y_2) = y_2 - y_1 \in K$, (i) is contradicted.

(v) $\Rightarrow$ (i): Assume (i) is not satisfied. Then there are distinct points $u, v \in X$ satisfying (3.10.1). By the Hahn-Banach theorem, there is an $x^* \in S(X^*)$ such that

$$x^*\left(\frac{u+v}{2}\right) = \left\|\frac{u+v}{2}\right\| = 1.$$

Then

$$\begin{aligned} 1 = x^*\left(\frac{u+v}{2}\right) &= \frac{1}{2}x^*(u) + \frac{1}{2}x^*(v) \\ &= \left|\frac{1}{2}x^*(u) + \frac{1}{2}x^*(v)\right| \leq \frac{1}{2}|x^*(u)| + \frac{1}{2}|x^*(v)| \leq 1. \end{aligned}$$

This shows that

$$\frac{1}{2}|x^*(u)| + \frac{1}{2}|x^*(v)| = 1$$

Claim: $x^*(u) = 1 = x^*(v)$. Note first that

$$x^*(u) \leq |x^*(u)| \leq \|x^*\|\|u\| = 1.$$

Similarly, $x^*(v) \leq 1$. If $x^*(u) < 1$, then, since $\frac{1}{2}x^*(u) + \frac{1}{2}x^*(v) = 1$, we must have that $x^*(v) > 1$, which is impossible. Therefore, $x^*(u) = 1$. Similarly, $x^*(v) = 1$.

Each y in the line segment $[0, u-v]$ can be written as
$y = \lambda.0 + (1-\lambda)(u-v) = (1-\lambda)(u-v)$ for some $\lambda \in [0,1]$.
Therefore

$$x^*(y) = x^*((1-\lambda)(u-v)) = (1-\lambda)(x^*(u) - x^*(v) = 0.$$

We have shown that there is an $x^* \in S(X^*)$ such that

$$\left.\begin{aligned} x^*(y) = 0 \ \text{ for all } \ y \in [0, u-v], \ \text{ and} \\ x^*(u-0) = x^*(u) = 1 = \|u\| = \|u-0\|, \\ x^*(u-(u-v)) = x^*(v) = 1 = \|v\| = \|u-(u-v)\|. \end{aligned}\right\}$$

This shows that 0 and $u - v$ are both best approximations to u from the line segment $[0, u-v]$. This shows that the line segment $[0, u-v]$ is not Chebyshev, which shows that (v) fails. ■

Amir and Ziegler [1] have shown that if $dimK = 1$, then X is strictly convex if and only if K is a Chebyshev set. They also gave some examples of spaces which are not strictly convex with respect to any subspace of dimension greater or equal to two:

1. $C_0(T)$, the space of continuous functions which "vanish at infinity" on a topological space T;
2. $L_1(\mu)$ for μ any measure;
3. the space $C_1[a,b]$ of continuous real-valued functions with the L_1-norm.

Definition 3.11. *A normed linear space $(X, \|\cdot\|)$ is said to be* **uniformly convex** *if, given any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that whenever $x, y \in X$, $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \epsilon$, then $\left\|\frac{x+y}{2}\right\| < 1 - \delta(\epsilon)$.*

The concept of uniformly convex spaces was introduced by Clarkson[5] in 1936. It is easy to show that a normed linear space $(X, \|\cdot\|)$ is a uniformly

convex if and only if whenever (x_n) and (y_n) are sequences in X such that $\|x_n\| \stackrel{n\to\infty}{\longrightarrow} d$, $\|y_n\| \stackrel{n\to\infty}{\longrightarrow} d$ and $\left\|\frac{1}{2}(x_n + y_n)\right\| \stackrel{n\to\infty}{\longrightarrow} d$, then $\|x_n - y_n\| \stackrel{n\to\infty}{\longrightarrow} 0$.
The following are examples of uniformly convex spaces:

1. The L_p spaces for $1 < p < \infty$. The proof follows easily from the Clarkson inequality.

2. The ℓ_p spaces for $1 < p < \infty$.

3. Every inner product space.

Note that L_1 and L_∞ are not uniformly convex.

It was shown by Milman[19] and Pettis [24] that every uniformly convex space is reflexive.

It is well-known [See for example [15] pg.66] that a closed convex subset of a uniformly convex Banach space is Chebyshev. We state this result formally:

Theorem 3.12. *Let K be a closed convex subset of a uniformly convex Banach space $(X, \|\cdot\|)$. Then K is Chebyshev.*

The following result is a natural extension of Theorem 3.12 to the setting of simultaneous best approximations.

Theorem 3.13. *Let K be a closed convex subset of a uniformly convex Banach space $(X, \|\cdot\|)$. Then for each $B \in \mathcal{CB}(X)$, we have that the set $\mathcal{S}_K(B)$ is a singleton.*

Proof. Let $B \in \mathcal{CB}(X)$. Then, by $1 \Rightarrow 2$ of Theorem 2.11, we have that $\mathcal{S}_K(B)$ is nonempty.

Let $y_1,\ y_2 \in \mathcal{S}_K(B)$. Since the set $\mathcal{S}_K(B)$ is convex [see Proposition 1.6], we have that $y_0 = \frac{1}{2}(y_1 + y_2) \in \mathcal{S}_K(B)$. That is,

$$\sup_{s\in B} \|s - y_0\| = \sup_{s\in B} \|s - y_1\| = \sup_{s\in B} \|s - y_2\| = d(B, K) := d.$$

By the characterisation of a supremum, there is a sequence (s_n) in B such that $\|s_n - y_0\| \to d$ as $n \to \infty$. Now, since $y_0 - s_n = \frac{1}{2}(y_1 - s_n) + \frac{1}{2}(y_2 - s_n)$ and for $j = 1, 2$ and $n \in \mathbb{N}$,

$$\|y_j - s_n\| \leq \sup_{s \in B} \|y_j - s\| \leq d,$$

we must have that, for each $j = 1, 2$

$$\lim_{n \to \infty} \|y_j - s_n\| = d.$$

That is, for each $j = 1, 2$,

$$\|y_1 - s_n\| \overset{n\to\infty}{\longrightarrow} d, \quad \|y_2 - s_n\| \overset{n\to\infty}{\longrightarrow} d, \quad \text{and} \quad \|\frac{1}{2}(y_1 - s_n) + \frac{1}{2}(y_2 - s_n)\| = \|y_0 - s_n\| \overset{n\to\infty}{\longrightarrow} d.$$

By uniform convexity of X, we have that

$$\|y_1 - y_2\| = \|\frac{1}{2}(y_1 - s_n) - \frac{1}{2}(y_2 - s_n)\| \overset{n\to\infty}{\longrightarrow} 0.$$

It follows that $y_1 = y_2$. ■

If we wish to have uniqueness of the set of best approximations for an arbitrary bounded subset B of X, it will be necessary to strengthen the convexity hypothesis. More precisely, we need the following generalisation of uniform convexity: the concept of **uniform convexity in every direction** whose geometrical significance is that the collection of all chords of the unit ball that are parallel to a fixed direction and whose lengths are bounded below by a positive number has the property that the midpoints of the chords lie uniformly deep inside the unit ball. This notion of convexity was introduced by A.L Garkavi [14] to characterize normed linear spaces for which every bounded subset has at most one Chebyshev centre.

Definition 3.14. *Let K be a convex subset of a normed linear space $(X, \|\cdot\|)$. The space X is said to be* **uniformly convex with respect to**

every direction *in K (Uced-K) if for every $k \in K \setminus \{0\}$ and every $\epsilon > 0$, there exists a $\delta = \delta(k, \epsilon) > 0$ such that*

$$\|x\| = \|y\| = 1, \ x - y = \lambda k, \ \left\| \frac{x+y}{2} \right\| > 1 - \delta \Rightarrow |\lambda| < \epsilon.$$

Day, James and Swaminathan [6] have given the following equivalent definition for uniform convexity in every direction:
A normed linear space X is said to be uniformly convex with respect to every direction in $K \subset X$ if there are sequences $(x_n)_n$ and $(y_n)_n$ in X and a nonzero member z of K for which

(a) $\|x_n\| = \|y_n\| = 1$, for every n,

(b) $x_n - y_n = \lambda_n z$, for every n,

(c) $\|x_n + y_n\| \to 2$,

then $\lambda_n \to 0$.

Amir and Ziegler [1] have established that if K is a subspace of X, then uniform convexity of X with respect to every direction in K is both necessary as well as sufficient in order that $\mathcal{S}_K(B)$ be at most a singleton for every bounded subset B of X.

Theorem 3.15. [1] *The space X is Uced-K if and only if $\mathcal{S}_K(B)$ is at most a singleton for every bounded $B \subset X$.*

Proof. $\Rightarrow$: Assume $\mathcal{S}_K(B)$ is not a singleton, and let y_1, y_2 be two distinct elements of $\mathcal{S}_K(B)$. Then by convexity of $\mathcal{S}_K(B)$ we have that $y_0 = \frac{y_1+y_2}{2}$ is also in $\mathcal{S}_K(B)$. That is,

$$\sup_{s \in B} \|s - y_0\| = \sup_{s \in B} \|s - y_1\| = \sup_{s \in B} \|s - y_2\| = d(B, K) := d.$$

By the characterization of the supremum, we can choose a sequence $(x_n) \subset B$ such that $\|y_0 - x_n\| \to d$. Since

$$\begin{aligned} \|y_0 - x_n\| &\leq \frac{1}{2}\|y_1 - x_n\| + \frac{1}{2}\|y_2 - x_n\| \\ &\leq \frac{1}{2}\sup_{s\in B}\|s - y_1\| + \frac{1}{2}\sup_{s\in B}\|s - y_2\| \leq d, \end{aligned}$$

it follows that

$$\frac{1}{2}\lim_{n\to\infty}\|y_1 - x_n\| + \frac{1}{2}\lim_{n\to\infty}\|y_2 - x_n\| = d.$$

Hence $\lim_{n\to\infty}\|y_i - x_n\| = d$, for each $i = 1, 2$. We may assume that $\|y_1 - x_n\| \geq \|y_2 - x_n\|$ and take $z_n = y_1 + t_n(y_2 - y_1)$ with $t_n \geq 1$ chosen so that $\|z_n - x_n\| = \|y_1 - x_n\|$. Let $u_n = (y_1 - x_n)/\|y_1 - x_n\|$ and $v_n = (z_n - x_n)/\|z_n - x_n\|$.

Then

$$\begin{aligned} 2 = \|u_n\| + \|v_n\| \geq \|u_n + v_n\| &\geq \frac{1}{\|y_1 - x_n\|}\|y_1 - x_n + z_n - x_n\| \\ &= \frac{1}{\|y_1 - x_n\|}\|y_1 + y_1 + t_n(y_2 - y_1) - 2x_n\| \\ \text{since } z_n = y_1 + t_n(y_2 - y_1) \quad & \\ &= \frac{1}{\|y_1 - x_n\|}\|2y_1 + t_n(y_2 - y_1) - 2x_n\| \\ &\geq \frac{1}{\|y_1 - x_n\|}\|2y_1 + y_2 - y_1 - 2x_n\| \quad \text{since } t_n \geq 1 \\ &= \frac{1}{\|y_1 - x_n\|}\left\|2\left(\frac{y_1 + y_2}{2} - x_n\right)\right\| \\ &= \frac{1}{\|y_1 - x_n\|}\|2(y_0 - x_n)\| \end{aligned}$$

Then $2 \geq \lim_{n\to\infty}\|u_n + v_n\| \geq \lim_{n\to\infty}\frac{1}{\|y_1 - x_n\|}\|2(y_0 - x_n)\| = 2$
$\therefore$ $\|u_n + v_n\| \to 2$ while $u_n - v_n \in K$ and it does not tend to 0, so that the Uced-K condition is not satisfied.

$\Leftarrow$: Assume that X is not Uced-K. Then there exists $z \in K \setminus \{0\}$ and two sequences $(x_n)_n, (y_n)_n$ in K satisfying $\|x_n\| = \|y_n\| = 1$, $x_n - y_n = \lambda_n z$, $|\lambda_n| \geq \lambda > 0$ and $\left\|\frac{x_n+y_n}{2}\right\| \to 1$. Let $u_n = \frac{x_n+y_n}{2}$, $B = \{\pm u_n; n = 1, 2, ...\}$. Since $\|u_n\| \to 1$, it follows that $d(B, K) = 1$ and $0 \in \mathcal{S}_K(B)$. However, we also have $\pm\frac{\lambda z}{2} \in \mathcal{S}_K(B)$ since

$$\begin{aligned} \left\|u_n \pm \frac{\lambda z}{2}\right\| &= \left\|\frac{x_n + y_n}{2} \pm \frac{\lambda}{2\lambda_n}(x_n - y_n)\right\| \\ &= \left\|(\frac{1}{2} \pm \frac{\lambda}{2\lambda_n})x_n + (\frac{1}{2} \mp \frac{\lambda}{2\lambda_n})y_n\right\| \end{aligned}$$

and since $\left|\frac{\lambda}{2\lambda_n}\right| \leq \frac{1}{2}$ we have

$$\left\|u_n \pm \frac{\lambda z}{2}\right\| \leq \left(\frac{1}{2} \pm \frac{\lambda}{2\lambda_n}\right) + \left(\frac{1}{2} \mp \frac{\lambda}{2\lambda_n}\right) = 1$$

and so $\frac{\lambda z}{2}$ is also in $\mathcal{S}_K(B)$ which is therefore not a singleton. ■

By looking at the previous Chapter and what we have done in this Chapter, it is evident that for a set K subset of X to be simultaneous Chebyshev with respect to $\mathcal{CB}(X)$, we need to impose certain conditions on $B \in \mathcal{C}(X)$, K and the unit ball of the underlying space X. It is important also to note the balance that exists among these conditions; by strengthening some of them, we are allowed to weaken others.

Chapter 4

CHARACTERIZATION OF BEST SIMULTANEOUS APPROXIMATIONS

In this Chapter we consider the problem of characterization of elements of best simultaneous approximation, i.e., the problem of finding necessary and sufficient conditions in order that an element $y_0 \in K$, K a subset of a normed linear space $(X, \|\cdot\|)$, be an element of best simultaneous approximation to a bounded subset B of X, and some consequences of these characterizations. Most of the work in the sequel is adapted from [20], [21] and [22] where similar characterization results are shown for elements of best simultaneous approximation from a closed convex set to a bounded subset of a conditionally complete Lattice Banach space. Our results are shown for best simultaneous approximations from a closed convex set to a remotal subset of a normed linear space X.
We start by stating the following theorems which are the main tools that will be used in the Chapter.

Theorem 4.1. *(Basic Separation Theorem). Let A and B be disjoint convex subsets of X. Assume A has an interior point. Then there exists non-zero* *linear functional $\phi \in X^*$ and $c \in \mathbb{R}$ such that $\phi(x) \geq c$ for all $x \in A$ and $\phi(y) \leq c$ for all $y \in B$.*

Theorem 4.2. *(Krein-Milman Theorem). Let X be a locally convex Hausdorff space, and $K \subset X$ a convex, compact subset. Then*

1. *The extremal set of K is nonempty.*

2. *$K = \overline{co(ext(K))}$ (i.e., K is the closure of the convex hull of the extremal points of K).*

Recall that a closed bounded subset B of a normed linear space $(X, \|\cdot\|)$ is said to be **remotal** if for each $x \in X$ there is an element $s_0 \in B$ such that

$$\|s_0 - x\| = \sup_{s \in B} \|s - x\|.$$

Note that every compact subset of a normed linear space $(X, \|\cdot\|)$ is remotal.

Deutsch and Maserick [7], gave the following main characterization Theorem for elements of best approximation.

Theorem 4.3. *Let K be a convex subset of a normed linear space $(X, \|\cdot\|)$ and $x \in X \setminus K$. An element $y_0 \in K$ is a best approximation to x if and only if there exists an $x^* \in X^*$ such that*

(i) $\|x^*\| = 1$;

(ii) $x^*(y_0 - y) \geq 0$ *for each* $y \in K$;

(iii) $x^*(x - y_0) = \|x - y_0\|$.

The next Theorem is a natural extension of Theorem 4.3 to the setting of best simultaneous approximation.

Theorem 4.4. *Let B be a remotal subset of a normed linear space, K a closed convex subset of X such that $B \cap K = \emptyset$ and $y_0 \in K$.Then $y_0 \in \mathcal{S}_K(B)$ if and only if there is an $x^* \in X^*$ and an $s_0 \in B$ such that* *(i)* $\|x^*\| = 1$;

(ii) $x^*(s_0 - y_0) = \sup_{s\in B} \|s - y_0\|$ *and*

(iii) $x^*(y_0 - y) \geq 0$ *for* $y \in K$.

Proof. $\Rightarrow$: Assume that $y_0 \in \mathcal{S}_K(B)$. Then

$$\sup_{s\in B} \|s - y_0\| = \inf_{y\in K} \sup_{s\in B} \|s - y\| = d(B, K).$$

Since B is remotal, there is an element $s_0 \in B$ such that

$$\|s_0 - y_0\| = \sup_{s\in B} \|s - y_0\| := \delta.$$

Since $B \cap K = \emptyset$ and K is closed, it follows that $\delta > 0$. Let

$$S = \{x \in X \mid \|s_0 - x\| < \delta\}.$$

Then the sets S and K satisfy the conditions of the Separation Theorem (Theorem 4.1). Therefore there is a $\phi \in X^* \setminus \{0\}$ and a $k \in \mathbb{R}$ such that

$$\phi(x) \geq k \text{ for all } x \in S \text{ and } \phi(y) \leq k \text{ for all } y \in K.$$

By continuity of ϕ, we have that $\phi(x) \geq k$ for all $x \in \overline{S}$. It follows that $\phi(y_0) = k$.

Now, there is a $c \in \mathbb{R}$ (*viz.* $c = \phi(s_0) - k$) such that

$$\phi(s_0 - x) \leq c \text{ for all } x \in \overline{S} \text{ and } \phi(s_0 - y) \geq c \text{ for all } y \in K. \tag{4.4.1}$$

It follows from the above that $0 \neq \phi(s_0 - y_0) = c$. Since S is a ball of positive radius, we have that $c > 0$. Set $x^* = \frac{\delta\phi}{c}$. Then $x^* \in X^*$ and, for any $x \in \overline{S}$,

$$x^*(s_0 - x) = \frac{\delta\phi(s_0 - x)}{c} \leq \delta \tag{4.4.2}$$

and, for all $y \in K$,

$$x^*(s_0 - y) = \frac{\delta\phi(s_0 - y)}{c} \geq \delta. \tag{4.4.3}$$

From equations (4.4.2) and (4.4.3), we have that

$$x^*(s_0 - y_0) = \delta = \sup_{s \in B} \|s - y_0\|, \tag{4.4.4}$$

which establishes (ii).

From equation (4.4.4), we have that

$$\|s_0 - y_0\| \le \sup_{s \in B} \|s - y_0\| = x^*(s_0 - y_0) = |x^*(s_0 - y_0)| \le \|x^*\| \|s_0 - y_0\|.$$

It follows that $\|x^*\| \ge 1$.

Claim: $\|x^*\| = 1$. If not, then there is a $z \in X$ with $\|z\| = 1$ such that $x^*(z) > 1$. Let $w_0 = s_0 - \delta z \in X$. Then $\|s_0 - w_0\| = \delta$ and so $w_0 \in \overline{S}$. But then

$$x^*(s_0 - w_0) = x^*(\delta z) = \delta x^*(z) > \delta,$$

which contradicts equation (4.4.2). It now follows that $\|x^*\| = 1$, which establishes (i).

Noting that $\phi(y_0 - s_0) = -c$ and that $\phi(s_0 - y) \ge c$ for all $y \in K$, it follows that, for all $y \in K$,

$$\phi(y_0 - y) = \phi(y_0 - s_0 + s_0 - y) = \phi(y_0 - s_0) + \phi(s_0 - y) \ge -c + c = 0.$$

Since $\delta > 0$ and $c > 0$, we have that, for all $y \in K$,

$$x^*(y_0 - y) = \frac{\delta \phi(y_0 - y)}{c} \ge 0,$$

which establishes (iii).

$\Leftarrow$: Assume that there is an $x^* \in X^*$ and an $s_0 \in B$ satisfying (i), (ii), and (iii) and let $y \in K$. Then

$$\begin{aligned}
\sup_{s \in B} \|s - y_0\| &= x^*(s_0 - y_0) = x^*(s_0 - y + y - y_0) \\
&= x^*(s_0 - y) + x^*(y - y_0) \\
&\le x^*(s_0 - y) \\
&\le \|s_0 - y\| \\
&\le \sup_{s \in B} \|s - y\|.
\end{aligned}$$

Hence $y_0 \in \mathcal{S}_K(B)$. ■

Remark 4.5. Note that if K is a closed *linear subspace* of X, then condition (iii) of Theorem 4.4 reduces to "$x^*(y) = 0$" for each $y \in K$.

Corollary 4.6. *Let B be a compact subset of a normed linear space, K a closed convex subset of X such that $B \cap K = \emptyset$ and $y_0 \in K$. Then $y_0 \in \mathcal{S}_K(B)$ if and only if there is an $x^* \in X^*$ and an $s_0 \in B$ such that*

(i) $\|x^*\| = 1$;

(ii) $x^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\|$ *and*

(iii) $x^*(y_0 - y) \geq 0$ *for* $y \in K$.

A natural generalization of the problem of characterization of elements of best simultaneous approximation is the problem of *simultaneous characterization* of a set of elements of best simultaneous approximation: given X, K a closed convex subset of X and $B \in \mathcal{CB}(X)$ and a subset M of K, what are the necessary and sufficient conditions in order that *every* element $m_0 \in M$ be an element of best simultaneous approximation of B by the elements of K? We get the following as a Corollary of Theorem 4.4.

Corollary 4.7. *Let B be a remotal subset of a normed linear space X, K a closed convex subset of X such that $B \cap K = \emptyset$ and $M \subset K$. Then $M \subset \mathcal{S}_K(B)$ if and only if there exists $x^* \in X^*$ and $s_0 \in B$ such that*

(i) $\|x^*\| = 1$;

(ii) $x^*(s_0 - m) = \sup_{s \in B} \|s - m\|$ *for all* $m \in M$; *and*

(iii) $x^*(m - y) \geq 0$ *for* $y \in K$, *and all* $m \in M$.

Proof. $\Rightarrow$: Assume that $M \subset \mathcal{S}_K(B)$. Choose and fix $m_0 \in M$. Then, by Theorem 4.4 there exists $x^* \in X^*$ and $s_0 \in B$ such that

$$\|x^*\| = 1;$$

$$x^*(s_0 - m_0) = \sup_{s\in B} \|s - m_0\| = d(B,K); \tag{4.7.1}$$

and

$$x^*(m_0 - y) \geq 0 \text{ for } y \in K. \tag{4.7.2}$$

Let $m \in M$ be arbitrary. Then from (4.7.1)

$$x^*(s_0 - m_0) = \sup_{s\in B} \|s - m_0\| = d(B,K) = \sup_{s\in B} \|s - m\|. \tag{4.7.3}$$

Now,

$$\begin{aligned} x^*(s_0 - m) &\leq \|x^*\|\|s_0 - m\| \leq \sup_{s\in B} \|s - m\| = \sup_{s\in B} \|s - m_0\| \\ &= x^*(s_0 - m_0) = x^*(s_0 - m) + x^*(m - m_0) \\ &\leq x^*(s_0 - m) \text{ since, by (4.7.2), } x^*(m - m_0) \leq 0. \end{aligned}$$

Thus, for each $m \in M$,

$$x^*(s_0 - m) = \sup_{s\in B} \|s - m\| = d(B,K), \quad \text{which proves (ii).}$$

Finally, for each $y \in K$ and each $m \in M$, we have by (4.7.3),

$$\begin{aligned} x^*(m - y) &= x^*(m - s_0) + x^*(s_0 - m_0) + x^*(m_0 - y) \\ &= -d(B,K) + d(B,K) + x^*(m_0 - y) = x^*(m_0 - y) \geq 0. \end{aligned}$$

That is, for all $y \in K$ and all $m \in M$,

$$x^*(m - y) \geq 0.$$

$\Leftarrow$: This is an immediate consequence of Theorem 4.4. ■

4.1 Characterization of elements of best simultaneous approximations in terms of the extremal points of $U(X^*)$

In this section we introduce the concept of extreme points of a set and characterize elements of best simultaneous approximation in terms of extremal subsets of a normed linear space.

Definition 4.8. *Let A be a closed convex subset of a topological linear space X. A non-empty subset $M \subseteq A$ is said to be an* **extremal subset** *of A, if a proper convex combination $\lambda x + (1-\lambda)y$, $0 < \lambda < 1$, of two points x and y of A lies in M only if both x and y are in M.*

An extremal subset of A consisting of just one point is called an extremal point of A. We shall denote by $ext(A)$ the set of all extremal points of A. Using Theorem 4.4, we shall prove the following property of best simultaneous approximations.

Theorem 4.9. *Let B be a remotal subset of a normed linear space, K a closed convex subset of X such that $B \cap K = \emptyset$ and $y_0 \in K$. Then $y_0 \in \mathcal{S}_K(B)$ if and only if there an $s_0 \in B$ and for each $y \in K$ there is an $x^*(= x^*_y) \in X^*$ (which depends on y) such that*

(i) $x^* \in \text{ext}(U(X^*))$;

(ii) $x^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\|$ *and*

(iii) $x^*(y_0 - y) \geq 0$.

Proof. $\Rightarrow$: Assume that $y_0 \in \mathcal{S}_K(B)$. Since B is remotal, there is an element $s_0 \in B$ such that $\|s_0 - y_0\| = \sup_{s \in B} \|s - y_0\|$. Set

$$\mathcal{E} = \{x^* \in U(X^*) \mid \quad x^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\|\},$$

and for each $y \in K$, set

$$\mathcal{E}_y = \{x^* \in \mathcal{E} \mid x^*(y_0 - y) = \sup_{\psi \in \mathcal{E}} \psi(y_0 - y)\}.$$

By Theorem 4.4, the set $\mathcal{E}$ is nonempty. The set $\mathcal{E}$ is a weak*-compact subset of $U(X^*)$. We show that $\mathcal{E}$ is an extremal subset of $U(X^*)$. To that end, let

$\phi \in \mathcal{E}$, $\lambda \in (0,1)$ and $\phi = \lambda x_1^* + (1-\lambda)x_2^*$, where $x_1^*,\ x_2^* \in U(X^*)$. Then, since $\phi \in \mathcal{E}$, we have that

$$\|s_0 - y_0\| = \sup_{s\in B}\|s - y_0\| = \phi(s_0 - y_0) = \lambda x_1^*(s_0 - y_0) + (1-\lambda)x_2^*(s_0 - y_0).$$

Note that $|x_1^*(s_0-y_0)| \le \|s_0-y_0\| (= \sup_{s\in B}\|s-y_0\|)$ and $|x_2^*(s_0-y_0)| \le \|s_0-y_0\|$.

Claim: $x_1^*(s_0 - y_0) = \sup_{s\in B}\|s - y_0\|$ and $x_2^*(s_0 - y_0) = \sup_{s\in B}\|s - y_0\|$.

If $x_1^*(s_0 - y_0) < \sup_{s\in B}\|s - y_0\|$, then, since $0 < \lambda < 1$, we have that

$$\begin{aligned}\sup_{s\in B}\|s - y_0\| &= \phi(s_0 - y_0) = \lambda x_1^*(s_0 - y_0) + (1-\lambda)x_2^*(s_0 - y_0)\\ &< \lambda \sup_{s\in B}\|s - y_0\| + (1-\lambda)\sup_{s\in B}\|s - y_0\| = \sup_{s\in B}\|s - y_0\|,\end{aligned}$$

which is absurd.

A similar conclusion results if one assumes that $x_2^*(s_0 - y_0) < \sup_{s\in B}\|s - y_0\|$.

It is obvious that $\|x_1^*\| = 1 = \|x_2^*\|$. It now follows that $x_1^*,\ x_2^* \in \mathcal{E}$ and so $\mathcal{E}$ is an extremal subset of $U(X^*)$.

For each $y \in K$, the linear functional $\widehat{y_0 - y}$ on X^* given by

$$\widehat{y_0 - y}(x^*) = x^*(y_0 - y), \quad x^* \in X^*$$

is weak*-continuous and therefore must attain its supremum on $\mathcal{E}$. It thus follows that for each $y \in K$, the set $\mathcal{E}_y$ is nonempty. Furthermore, $\mathcal{E}_y$ is a weak*-compact extremal subset of $\mathcal{E}$. By Theorem 4.2, $\mathcal{E}_y$ has an extremal point x^* which must also be an extremal point of $\mathcal{E}$ and also of $U(X^*)$. That is, $x^* \in \text{ext}(U(X^*))$. By Theorem 4.4, there is a $\phi \in \mathcal{E}$ such that $\phi(y_0 - y) \ge 0$. Since $x^* \in \mathcal{E}_y$, we have that

$$x^*(y_0 - y) = \sup_{\psi\in\mathcal{E}}\psi(y_0 - y) \ge \phi(y_0 - y) \ge 0,$$

which establishes (iii).

$\Leftarrow$: Let $y \in K$. Then there is an $x_y^* \in X^*$ such that (i), (ii) and (iii) hold.

Therefore

$$\begin{aligned}\sup_{s\in B}\|s-y_0\| &= x_y^*(s_0-y_0)=x_y^*(s_0-y)+x_y^*(y-y_0)\\ &\le x_y^*(s_0-y)\le \|x_y^*\|\|s_0-y\|\\ &\qquad\qquad = \|s_0-y\|\le \sup_{s\in B}\|s-y\|.\end{aligned}$$

Hence, $y_0\in\mathcal{S}_K(B)$. ■

Corollary 4.10. *Let B be a remotal subset of a normed linear space $(X,\|\cdot\|)$, K a closed convex subset of X with $B\cap K=\emptyset$ and $y_0\in K$. Then the following statements are equivalent:*

1. *$y_0\in\mathcal{S}_K(B)$.*

2. *There exists $x^*\in ext(U(X^*))$ and $s_0\in B$ such that*

$$|x^*(s_0-y_0)|=\sup_{s\in B}\|s-y_0\| \tag{4.10.1}$$

and

$$|x^*(s_0-y_0)|\le|x^*(s_0-y)| \quad \text{for each} \quad y\in K. \tag{4.10.2}$$

3. *There exists $x^*\in ext(U(X^*))$ and $s_0\in B$, satisfying (4.10.2) and*

$$x^*(y-y_0)x^*(s_0-y_0)\le 0 \quad \text{for each} \quad y\in K.$$

Proof. 1 ⇒ 2: Assume that statement 1 holds. Then conditions (i), (ii) and (iii) of Theorem 4.9 hold. Hence (4.10.1) holds. On the other hand, by conditions (ii) and (iii) of Theorem 4.9 we get

$$\begin{aligned}|x^*(s_0-y_0)| &= x^*(s_0-y_0)=x^*(s_0-y)+x^*(y-y_0)\\ &\le x^*(s_0-y)\le|x^*(s_0-y)| \quad \text{for each} \quad y\in K.\end{aligned}$$

Hence 1⇒2.

$2 \Rightarrow 1$: If we have statement 2, then for each $y \in K$, we have

$$\begin{aligned}\sup_{s\in B}\|s-y_0\| &= |x^*(s_0-y_0)| \leq |x^*(s_0-y)| \\ &\leq \|x^*\|\|s_0-y\| \leq \sup_{s\in B}\|s-y\|.\end{aligned}$$

It follows that

$$\sup_{s\in B}\|s-y_0\| = d(B,K).$$

i.e., $y_0 \in \mathcal{S}_K(B)$. Thus, $2 \Rightarrow 1$.

$1 \Rightarrow 3$: Assume now that statement 1 holds. Then, by Theorem 4.9, we conclude that

$$x^*(y_0-y) \geq 0 \ \text{ for each } \ y \in K, \ \text{ and } \ x^*(s_0-y_0) = \sup_{s\in B}\|s-y_0\| \geq 0.$$

Thus we have

$$x^*(y-y_0)x^*(s_0-y_0) \leq 0 \ \text{ for each } \ y \in K.$$

Therefore $1 \Rightarrow 3$.

$3 \Rightarrow 1$: If statement 3 holds, then there exists $x^* \in ext(U(X^*))$ such that $\|x^*\| = 1$ and satisfying (4.10.2) and

$$x^*(y-y_0)x^*(s_0-y_0) \leq 0 \ \text{ for each } y \in K.$$

Put

$$\psi = \text{sgn}[x^*(s_0-y_0)]x^*.$$

Then, by hypothesis we have $\psi \in ext(U(X^*))$,

$$\psi(y-y_0) = \frac{x^*(s_0-y_0)}{|x^*(s_0-y_0)|}x^*(y-y_0) \leq 0 \ \text{ for each } \ y \in K,$$

and

$$\begin{aligned}\psi(s_0-y_0) &= \text{sgn}[x^*(s_0-y_0)]x^*(s_0-y_0) \\ &= \frac{x^*(s_0-y_0)}{|x^*(s_0-y_0)|}x^*(s_0-y_0) \\ &= [x^*(s_0-y_0)| = \sup_{s\in B}\|s-y_0\|.\end{aligned}$$

Hence, the functional ψ satisfies conditions (i), (ii) and (iii) of Theorem 4.9. Therefore, $y_0 \in \mathcal{S}_K(B)$. Thus, $3 \Rightarrow 1$, which completes the proof. ■

4.2 Characterizations of best simultaneous approximation by elements of Hyperplanes

In this section we shall give various characterization results for elements of best simultaneous approximations from hyperplanes.

Definition 4.11. *A* **hyperplane** *in a normed linear space $(X, \|\cdot\|)$ is any set of the form*

$$H = [x^*; c] = \{x \in X | x^*(x) = c\}$$

where x^* is a non-zero functional on X and c is a scalar.

Definition 4.12. *Let B be a remotal set in a normed linear space X and $r > 0$. The set $A \subset X$ is said to* **support** *the ball*
$S(B, r) = \{y \in X | \sup_{s \in B} \|s - y\| \leq r\}$, or that A is a support set for the ball $S(B, r)$, if

$$D(A, S(B, r)) = \inf_{y \in A} \inf_{s \in S(B,r)} \|y - s\| = 0 \quad \text{and} \quad A \cap IntS(B, r) = \emptyset.$$

Proposition 4.13. *Let X be a normed linear space, A a subset of X, B be a remotal subset of X and $r > 0$. Then A supports the ball $S(B, r)$, if and only if $d(B, A) = r$.*

Proof. Assume that $d(B, A) \neq r$. Assume that $d(B, A) = \alpha$ and $\alpha < r$. Let $\epsilon > 0$ be given such that $\alpha + \epsilon < r$. Then by characterization of the infimum, there exists $y \in A$ such that

$$\sup_{s \in B} \|s - y\| \leq d(B, A) + \epsilon = \alpha + \epsilon < r.$$

This implies that $y \in A \cap IntS(B, r)$, and consequently A does not support the ball $S(B, r)$. If $\alpha > r$, let $\epsilon > 0$ be such that $\alpha > r + \epsilon$. Then, by characterization of the infimum, there exists $x \in A$ such that

$$D(x, S(B, r)) \leq D(A, S(B, r)) + \epsilon \tag{4.13.1}$$

where $D(x, S(B,r)) = \inf_{y \in S(B,r)} \|x - y\|$. Now, we show that

$$\sup_{s \in B} \|s - x\| - r \leq D(x, S(B,r)). \tag{4.13.2}$$

To see this, let $s \in B$ be arbitrary. Then, for every $y \in S(B,r)$ we get

$$\|s - x\| \leq \|s - y\| + \|y - x\|.$$

This implies that for all $y \in S(B,r)$

$$\sup_{s \in B} \|s - x\| \leq \sup_{s \in B} \|s - y\| + \|y - x\| \leq r + \|y - x\|.$$

Thus, we conclude that, for each $y \in S(B,r)$,

$$\sup_{s \in B} \|s - x\| - r \leq \|y - x\|.$$

It follows that

$$\sup_{s \in B} \|s - x\| - r \leq D(x, S(B,r)),$$

and hence we obtain (4.13.2).

From inequalities (4.13.1) and (4.13.2), we have that

$$\begin{aligned} D(A, S(B,r)) &\geq D(x, S(B,r)) - \epsilon \geq \sup_{s \in B} \|s - x\| - r - \epsilon \\ &\geq d(B, A) - r - \epsilon = \alpha - r - \epsilon > 0. \end{aligned}$$

Consequently, A does not support the ball $S(B,r)$.

Conversely, assume that $d(B,A) = r$ and let $\epsilon > 0$ be arbitrary. Then, by characterization of the infimum, there exists $x \in A$ such that $\sup_{s \in B} \|s - x\| < r + \epsilon$. Put

$$z = \frac{\epsilon}{r + \epsilon} s + \frac{r}{r + \epsilon} x.$$

Then we have

$$\sup_{s \in B} \|s - z\| = \frac{r}{r + \epsilon} \sup_{s \in B} \|s - x\| < r.$$

Hence $z \in S(B,r)$ and

$$D(A, S(B,r)) \leq \|x - z\| = \frac{\epsilon}{r + \epsilon} \|s - x\| \leq \frac{\epsilon}{r + \epsilon} \sup_{s \in B} \|s - x\| < \epsilon.$$

Since $\epsilon > 0$, was arbitrary, it follows that $D(A, S(B, r)) = 0$.

On the other hand, if there exists a $y \in A \cap IntS(B, r)$, then

$$d(B, A) \leq \sup_{s \in B} \|s - y\| < r$$

This contradicts the fact that $d(B, A) = r$. Hence $A \cap IntS(B, r) = \emptyset$. ■

Lemma 4.14. *Let K be a closed convex subset of X, B be a remotal set in X with $K \cap B = \emptyset$, $y_0 \in K$ and $r = \sup_{s \in B} \|s - y_0\|$. Then there exists $s_0 \in B$ such that the following assertions are equivalent:*

(1) There exists $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(y_0 - y) = 0$ for all $y \in K$ and $x^*(s_0 - y_0) = r$;

(2) There exists $x^* \in X^*$ such that $\|x^*\| = 1$ and the hyperplane $H = \{y \in X : x^*(s_0 - y) = r\}$ passes through K (that is $K \subset H$) and $d(B, H) = r$.

Proof. (1)⇒ (2) Assume that (1) holds. Then there exists $s_0 \in B$ and $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(s_0 - y) = 0$ $(y \in K)$ and $x^*(s_0 - y_0) = r$. Then

$$H = \{y \in X : x^*(s_0 - y) = r\}$$

is a hyperplane and $y_0 \in H$. Thus, we conclude that

$$\begin{aligned} d(B, H) &\leq \sup_{s \in B} \|s - y_0\| = r = x^*(s_0 - y) = |x^*(s_0 - y)| \\ &\leq \|x^*\| \|s_0 - y\| \leq \sup_{s \in B} \|s - y\| \quad \text{for all } y \in H. \end{aligned}$$

This means that $d(B, H) = r$.

Assume now that $y \in K$ is arbitrary. Because $x^*(y_0 - y) = 0$ for all $y \in K$, it follows that

$$\begin{aligned} x^*(s_0 - y) &= x^*(s_0 - y_0) + x^*(y_0 - y) \\ &= x^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\| = r. \end{aligned}$$

this implies that $K \subset H$. That is H passes through K.

(2) $\Rightarrow$ (1) If (2) holds, then there exists $x^* \in X^*$ and $s_0 \in B$ such that $\|x^*\| = 1$, $K \subset H$ and, in particular we have $y_0 \in H$. Thus we get
$x^*(s_0 - y_0) = r$, and
$x^*(y_0 - y) = x^*(s_0 - y) - x^*(s_0 - y_0) = r - r = 0$ for all $y \in K$, which completes the proof. ■

Lemma 4.15. *Let K be a closed convex subset of X, B be a remotal set in X with $K \cap B = \emptyset$, $y_0 \in K$, and $r = \sup_{s \in B} \|s - y_0\|$. Then there exists $s_0 \in B$ such that the following assertions are equivalent:*

(1) There exists $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(y) = 0$ for all $y \in K$ and $x^*(s_0 - y_0) = r$.

(2) There exists $x^* \in X^*$ such that $\|x^*\| = 1$ and the hyperplane $H = \{y \in X : x^*(y) = 0\}$ passes through K and $d(B, H) = r$.

Proof. (1)$\Rightarrow$(2) Assume (1) holds. Then there exists $s_0 \in B$ and $x^* \in X$ such that $\|x^*\| = 1$, $x^*(y) = 0$ $(y \in K)$ and $x^*(s_0 - y_0) = r$. Then

$$H = \{y \in X : x^*(y) = 0\}$$

is a hyperplane and $y_0 \in H$. Thus, for all $y \in H$

$$\begin{aligned} d(B, H) &\leq \sup_{s \in B} \|s - y_0\| = r = x^*(s_0 - y_0) = x^*(s_0 - y) \\ &= |x^*(s_0 - y_0)| \leq \|x^*\| \|s_0 - y\| \leq \sup_{s \in B} \|s - y\|. \end{aligned}$$

This means that $d(B, H) = r$.

(2)$\Rightarrow$(1) If (2) holds, then there exists $s_0 \in B$ and $x^* \in X^*$ such that $\|x^*\| = 1$ and the hyperplane $H = \{y \in X : x^*(y) = 0\}$ passes through K and $d(B, H) = r = \sup_{s \in B} \|s - y_0\|$. This means that $y_0 \in \mathcal{S}_K(B)$. Hence, by

Theorem 4.4, there exists $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(y) = 0$ for all $y \in H$ and $x^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\| = r$.

Therefore, the functional $x^* \in X^*$ satisfies the condition (i), which completes the proof. ■

4.2.1 Separating Hyperplanes

Definition 4.16. *Let S and T be non-empty sets in a normed linear space X, $x^* \in X^*$, and $c \in \mathbb{R}$. The hyperplane $H = \{x \in X : x^*(x) = c\}$ is said to separate S and T if:*

1. *$x^*(x) \leq c$ for every $x \in S$*

2. *$x^*(y) \geq c$ for every $y \in T$.*

If both inequalities are strict we say that H strictly separates the sets, S and T.

The following results extends similar results in the theory of best approximation to the setting of best simultaneous approximation.

Theorem 4.17. *Let K be a closed convex subset of X, B be a remotal set in X with $K \cap B = \emptyset$, $y_0 \in K$, and $r = \sup_{s \in B} \|s - y_0\|$. Then $y_0 \in \mathcal{S}_K(B)$ if and only if there exists $s_0 \in B$ and $x^* \in X^*$ such that $\|x^*\| = 1$ and the hyperplane $H = \{y \in X : x^*(y) = x^*(s_0) - r\}$ passes through y_0, and separates K and the ball $S(B, r) = \{y \in X | \sup_{s \in B} \|s - y\| \leq r\}$.*

Proof. $\Rightarrow$: Assume $y_0 \in \mathcal{S}_K(B)$. Then by Theorem 4.4, there exists $s_0 \in B$ and $x^* \in X^*$ such that

$$\|x^*\| = 1, \tag{4.17.1}$$

$$x^*(y_0 - y) \geq 0, \text{for all } y \in K \tag{4.17.2}$$

and

$$x^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\| = r. \tag{4.17.3}$$

Let $H = \{y \in X : x^*(s_0 - y) = r\}$. Then H is a hyperplane, and by (4.17.3) we have $y_0 \in H$, that is, H passes through y_0.
Also, by (4.17.1) we have that for all $z \in S(B, r)$,

$$\begin{aligned} x^*(s_0 - z) &\leq |x^*(s_0 - z)| \leq \|x^*\| \|s_0 - z\| \\ &\leq \sup_{s \in B} \|s - z\| \leq r \ . \end{aligned}$$

This means that for all $y \in S(B, r)$,

$$x^*(y) \geq x^*(s_0) - r. \tag{4.17.4}$$

Similarly, using equations (4.17.2) and (4.17.3), we have that, for all $y \in K$,

$$x^*(s_0 - y) = x^*(s_0 - y_0) + x^*(y_0 - y) \geq x^*(s_0 - y_0) = r.$$

This means that, for all $z \in K$,

$$x^*(z) \leq x^*(s_0) - r. \tag{4.17.5}$$

In view of (4.17.4) and (4.17.5), it follows that H separates K and the ball $S(B, r)$.

$\Leftarrow$: Let $s_0 \in B$ and $x^* \in X^*$ such that $\|x^*\| = 1$ and the hyperplane $H = \{y \in X : x^*(s_0 - y) = r\}$ passes through y_0 and separates K and the ball $S(B, r)$. Then

$$x^*(s_0 - y_0) = r = \sup_{s \in B} \|s - y_0\|.$$

Since, for each $z \in S(B, r)$, we have that

$$x^*(s_0 - z) \leq |x^*(s_0 - z)| \leq \|x^*\| \|s_0 - z\| \leq \sup_{s \in B} \|s - z\| \leq r,$$

i.e., $x^*(s_0) - r \leq x^*(z)$ and H separates K and $S(B, r)$, it follows that, for each $y \in K$, $x^*(y) \leq x^*(s_0) - r$, i.e. $x^*(s_0 - y) \geq r$. Therefore, for each $y \in K$,

$$x^*(y_0 - y) = x^*(s_0 - y) - x^*(s_0 - y_0) \geq r - r = 0.$$

Hence, by Theorem 4.4, we conclude that $y_0 \in \mathcal{S}_K(B)$. ■

The following Theorem is an immediate consequence of Theorems 4.9 and 4.17.

Theorem 4.18. *Let K be a closed convex subset of X, B be a remotal set in X with $K \cap B = \emptyset$, $y_0 \in K$, and $r = \sup_{s\in B} \|s - y_0\|$. Then $y_0 \in \mathcal{S}_K(B)$ if and only if there exists $s_0 \in B$ and $x^* \in X^*$ such that $x^* \in ext(U(X^*))$ and the hyperplane $H = \{y \in X : x^*(y) = x^*(s_0) - r\}$ passes through y_0 and separates K from the ball $S(B, r)$.*

Proof. $\Rightarrow$: Assume $y_0 \in \mathcal{S}_K(B)$. Then by Theorem 4.9 , there exists $s_0 \in B$ and $x^* \in X^*$ such that

(i) $x^* \in \text{ext}(U(X^*))$;

(ii) $x^*(s_0 - y_0) = \sup_{s\in B} \|s - y_0\| = r$ and

(iii) $x^*(y_0 - y) \geq 0$.

Then $H = \{y \in X : x^*(s_0 - y) = r\}$ is a hyperplane and $y_0 \in H$. That is H passes through y_0.

The rest of the proof follows from the proof of Theorem 4.17. ■

4.3 Characterization of best simultaneous approximations in finite-dimensional subspaces

In this section we give a proof of a version of Theorem 4.4 for the case when both K and B are subsets of finite-dimensional subspaces of the normed linear space $(X, \|\cdot\|)$. To achieve this we shall need the following Lemma.

Lemma 4.19. *Let Y be an n-dimensional subspace of the normed linear space X and let $f \in Y^* \setminus \{0\}$. Then there exists $m \leq n$ linearly independent* *functionals* $x_i^* \in ext(U(X^*))$ *and* m *scalars* $\lambda_i > 0$ *such that* $\sum_{i=1}^{m} \lambda_i = \|f\|$ *and* $f = \sum_{i=1}^{m} \lambda_i x_i^*|_Y$, *i.e.,* $f(y) = \sum_{i=1}^{m} \lambda_i x_i^*(y), \ \ y \in Y$.

Proof of Lemma 4.19 can be found in H. N. Mhasker and D. V. Pai [18] page 418.

Theorem 4.20. *Let* K *be a closed convex subset of an* m*-dimensional subspace of a normed linear space* $(X, \|\cdot\|)$, B *a bounded remotal subset of an* n*-dimensional subspace of* X *such that* $B \cap K = \emptyset$, *and* $y_0 \in K$. *Then* $y_0 \in \mathcal{S}_K(B)$ *if and only if there exist* ℓ *linearly independent functionals* $x_1^*, \ x_2^*, \ \ldots, \ x_\ell^*$ *in* $ext(U(X^*))$, $s_0 \in B$ *and* ℓ *positive scalars* $\lambda_1, \ \lambda_2, \ \ldots, \ \lambda_\ell$ *with* $1 \leq \ell \leq m+n$ *such that*

(i) $x_i^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\|$ *for all* $i = 1, \ 2, \ \ldots, \ \ell$;

(ii) $\sum_{i=1}^{\ell} \lambda_i x_i^*(y_0 - y) \geq 0$ *for all* $y \in K$.

Proof. $\Rightarrow$: Let Y denote an m-dimensional subspace of X containing K, Z an n-dimensional subspace of X containing B and let $X_0 := \operatorname{span}(Y \cup Z)$. Then $\dim(X_0) \leq m+n$. By Theorem 4.4 applied to X_0, $y_0 \in \mathcal{S}_K(B)$ if and only if there exists $x^* \in X_0^*$ and $s_0 \in B$ such that

$\|x^*\| = 1$,

$x^*(y_0 - y) \geq 0$ for all $y \in K$, and

$x^*(s_0 - y_0) = \sup_{s \in B} \|s - y_0\|$.

Now by applying Lemma 4.19, there exist $\ell \leq m+n$ linearly independent functionals $x_i^* \in ext(U(X^*))$ and ℓ scalars $\lambda_i > 0$ such that $\sum_{i=1}^{\ell} \lambda_i = \|x^*\| = 1$

and $x^* = \sum_{i=1}^{\ell} \lambda_i x_i^*|_{X_0}$. Then $\sum_{i=1}^{\ell} \lambda_i x_i^*(y_0 - y) = x^*(y_0 - y) \geq 0$ for each $y \in K$. Then (ii) holds. Also

$$\begin{aligned}
\sup_{s\in B} \|s - y_0\| &= x^*(s_0 - y_0) = \sum_{i=1}^{\ell} \lambda_i x_i^*(s_0 - y_0) \\
&= \left| \sum_{i=1}^{\ell} \lambda_i x_i^*(s_0 - y_0) \right| \leq \sum_{i=1}^{\ell} \lambda_i |x_i^*(s_0 - y_0)| \\
&\leq \sum_{i=1}^{\ell} \lambda_i \sup_{s\in B} \|s - y_0\| = \sup_{s\in B} \|s - y_0\|.
\end{aligned}$$

Hence, equality holds throughout, which gives (i)

$\Leftarrow$: Without loss of generality we may assume that $\sum_{i=1}^{l} \lambda_i = 1$. Let $y \in K$. Now,

$$\begin{aligned}
\sup_{s\in B} \|s - y_0\| &= \sum_{i=1}^{l} \lambda_i \sup_{s\in B} \|s - y_0\| \\
&= \sum_{i=1}^{l} \lambda_i x_i^*(s_0 - y_0) \\
&= \sum_{i=1}^{l} \lambda_i x_i^*(s_0 - y) + \sum_{i=1}^{l} \lambda_i x_i^*(y - y_0) \\
&\leq \sum_{i=1}^{l} \lambda_i x_i^*(s_0 - y) \\
&\leq \left| \sum_{i=1}^{l} \lambda_i x_i^*(s_0 - y) \right| \leq \sum_{i=1}^{l} \lambda_i |x_i^*(s_0 - y)| \\
&\leq \sum_{i=1}^{l} \lambda_i \sup_{s\in B} \|(s - y)\| = \sup_{s\in B} \|(s - y)\|.
\end{aligned}$$

It follows that $y_0 \in \mathcal{S}_K(B)$. ■

Note that in the case where B is a singleton that the above result reduces to the characterization of best approximation from finite dimensional subspaces.

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