Steady States and Stability of the Bistable Reaction-Diffusion Equation on Bounded Intervals

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Abstract

Reaction-diffusion equations have been used to study various phenomena across different fields. These equations can be posed on the whole real line, or on a subinterval, depending on the situation being studied. For finite intervals, we also impose diverse boundary conditions on the system. In the present thesis, we solely focus on the bistable reaction-diffusion equation while working on a bounded interval of the form $[0, L]$ ($L > 0$). Furthermore, we consider both mixed and no-flux boundary conditions, where we extend the former to Dirichlet boundary conditions once our analysis of that system is complete. We first use phase-plane analysis to set up our initial investigation of both systems. This gives us an integral describing the transit time of orbits within the phase-plane. This allows us to determine the bifurcation diagram of both systems. We then transform the integral to ease numerical calculations. Finally, we determine the stability of the steady states of each system.

Résumé

Les équations à réaction-diffusion sont utilisées depuis longtemps pour étudier plusieurs phénomènes à travers diverses domaines. Selon la situation donnée, l'interval sur lequel nous travaillons peut être borné ou non-borné. De plus, le terme de réaction et les conditions aux limites imposées peuvent varier. Dans cette thèse, nous allons nous concentrer sur l'équation de réaction-diffusion bistable tout en travaillant sur un intervalle de la forme $[0, L]$ avec $L > 0$. On considère les conditions aux limites mixtes et Neumann. En plus, on étend le premier cas aux conditions aux limites de Dirichlet. Pour créer le fondement de notre investigation, nous allons utiliser une analyse de plan de phase pour les deux systèmes. Ceci nous donne une intégrale qui décrit le temps de transit des orbites dans le plan de phase. Cette intégrale nous permet de déterminer les bifurcations qui prennent place. Ensuite, ayant comme but de faciliter les calculs numériques, nous trouvons une différente formule pour décrire cette intégrale. Finalement, nous trouvons la stabilité de chaque système.

Dedications

As challenges in my life became increasingly arduous, my parent's support and encouragement rose. As such, I dedicate this work to them. Without them, this could not have been possible, let alone conceivable.

Acknowledgements

I would like thank my supervisor, Dr. Victor LeBlanc. His guidance and patience throughout this process was paramount to my success. Without his exceptional devotion, this thesis could not have come to fruition. I would also like to thank the University of Ottawa for their support.

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Chapter 1

Introduction

Historically, reaction-diffusion equations have been used in many different fields. In physics, one might use the heat equation to model or measure the distribution of heat over time (see [15] and [8]). In ecology, it is used to model the population of a species in an ecosystem (see [3]). The system, depending on certain parameters, will indicate whether the population will flourish and survive or if the species will eventually meet its doom. Similarly as in the previous model, we use this system to determine whether or not two different but interacting species can live together in a stable manner. The interaction between two species is often either the competition over the same resources or predation of one species by the other (see [3] and [12]). Moreover, in cell biology, the effects of introducing a new chemical in a cell can be modelled using reaction-diffusion equations. When introduced, the new chemical diffuses within the cell and reacts with other existing chemicals (see [21]).

In this thesis, we shall focus on the one-component diffusion equation. As the name suggests, a one-component reaction-diffusion equation models a single population, chemical etc. As it only models one thing in particular, it only contains one equation and it is of the following form

$$u_t(x,t) = Du_{xx}(x,t) + f(u(x,t)), \tag{1.0.1}$$

where $D > 0$ is some constant. Typically, D is much less than 1. The term $Du_{xx}(x,t)$ is called the diffusion term while the term $f(u(x,t))$ is called the reaction term. To bring some context as to what these two terms mean with respect to a solution, $u(x,t)$, of the system, consider the following example. Suppose $u(x,t)$ models the population of a species in an ecosystem, then the diffusion term represents the movement of this species throughout the ecosystem. The larger D is, the faster the species in question spreads. The reaction term, in this case, represents the net growth rate of this species (see [23]).

The reaction-diffusion equation can be considered (in terms of the spatial variable x) on the infinite interval $(-\infty,\infty)$ as well as on a bounded interval depending on the situation being modelled. In the first instance, we look for solutions that are travelling waves, i.e., we seek solutions of the form $u(x,t) = u(x+ct,0)$ for some $c \in \mathbb{R}$. In order for these solutions to be considered travelling waves, they must have horizontal limits and be nearly flat at both extremities of the infinite interval. Thus, we assume that both limits, $\lim_{x\to\pm\infty} u(x,t)$, exist. In addition, we also assume that both limits, $\lim_{x\to\pm\infty} u_x(x,t)$, exist and are equal to zero. When $\lim_{x\to\infty} u(x,t) = \lim_{x\to-\infty} u(x,t)$, we call $u(x,t)$ a *pulse*. If $\lim_{x\to\infty} u(x,t) \neq \lim_{x\to-\infty} u(x,t)$, then we call $u(x,t)$ a *wavefront*.

In contrast, when we are working on a closed interval $[a,b]$, we force conditions at the boundary of the interval. Some examples of boundary conditions include, but are not limited to, the *first-type* or *Dirichlet* boundary conditions, the *mixed* or *Cauchy* boundary conditions as well as the *no-flux* or *Neumann* boundary conditions. Respectively, these boundary conditions are of the form

$$u(a) = u(b) = 0, \quad u(a) = u_x(b) = 0 \quad \text{and} \quad u_x(a) = u_x(b) = 0.$$

Throughout this thesis, we shall use the reaction term $f(u)$ above as

$$f(u) = u(1-u)(u-\alpha) \tag{1.0.2}$$

where α is some constant in the interval $(0, \frac{1}{2})$. This assumption will allow us to simplify our calculations in the bistable case when assuming that the reaction term has three zeroes. In biology, species whose reaction-diffusion model uses this $f(u)$ are said to possess a *strong Allee effect.* A species exhibiting a strong Allee effect signifies that there is an Allee threshold, that is, a population threshold for which the number of individuals will start to decline and eventually the species will die off. In the case of $f(u(x,t))$, the Allee threshold is represented by α. When $u(x,t) > \alpha$, $f(u(x,t))$ is positive and so the population is growing. When $u(x,t) < \alpha$, $f(u(x,t))$ is negative. Therefore the population is decreasing. See [5].

When the spatial variable lies within the infinite interval, we have existence of solutions in the bistable and monostable cases (see [23]). In particular, in the bistable instance with $D = 1$, we have an infinite number of solutions of the form

$$u(x,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right),$$

where $c = \frac{1}{\sqrt{2}}(1-2\alpha)$ and $\beta \in \mathbb{R}$ (see [9]). Indeed, we have

$$u_t(x,t) = \frac{c}{4\sqrt{2}} \operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right) \quad \text{and} \tag{1.0.3}$$

$$u_{xx}(x,t) = -\frac{\tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)}{8}. \tag{1.0.4}$$

Moreover, since we have

$$\frac{1}{2}\left(1 + \tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right)\frac{1}{2}\left(1 - \tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right)$$

$$
\begin{aligned}
&= \frac{1}{4}\left(1 - \tanh^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right) \\
&= \frac{1}{4}\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right),
\end{aligned}
$$

the reaction term in the equation is

$$
f(u(x,t)) = \frac{1}{4}\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\left(\frac{1}{2}\left(1+\tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right) - \alpha\right).
$$

Combining this with (1.0.4) yields

$$
\begin{aligned}
u_{xx}(x,t) + f(u(x,t)) = &-\frac{\tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)}{8} \\
&+ \frac{1}{4}\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\left(\frac{1}{2}\left(1+\tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right) - \alpha\right).
\end{aligned}
$$

Factoring the term $\frac{1}{4}\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)$ and simplifying gives us

$$
\begin{aligned}
u_{xx}(x,t) + f(u(x,t)) =& \frac{1}{4}\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\left(\frac{1}{2} - \alpha\right) \\
=& \frac{c}{4\sqrt{2}}\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right) \\
=& u_t(x,t).
\end{aligned}
$$

as required. In addition, we also have

$$
\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} \frac{1}{2}\left(1+\tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right) = 1
$$

and

$$\lim_{x\to-\infty} u(x,t) = \lim_{x\to-\infty} \frac{1}{2}\left(1+\tanh\left(\frac{x+ct+\beta}{2\sqrt{2}}\right)\right) = 0.$$

Finally, we have

$$\lim_{x\to\pm\infty} u_x(x,t) = \lim_{x\to\pm\infty} \frac{1}{2\sqrt{2}}\operatorname{sech}^2\left(\frac{x+ct+\beta}{2\sqrt{2}}\right) = 0.$$

Consequently, $u(x,t)$ is a wavefront solution.

This leads to an interesting question: Do stable patterned steady states exist on a bounded domain with a one-component reaction-diffusion equation?

When discussing chemicals reacting in a cell, we often discuss the possibility of cell polarisation. Cell polarisation is the process of achieving an equilibrium of the distribution of the chemicals reacting in the cell. In addition, starting from one side of the cell and ending on the other side, the chemical distribution decreases. In [14], the authors demonstrate that cell polarisation is possible in a two-component reaction-diffusion equation with no-flux boundary conditions. Naturally, one might wonder if cell polarisation is possible in a one-component system, i.e., is it possible to obtain a stable amount of a single chemical within a cell long term. As shown in this thesis, cell polarisation is not possible in the one component reaction-diffusion equation with no-flux boundary conditions. However, cell polarisation is possible if the boundary conditions are mixed and if the coefficient D is small enough.

Before embarking on our analysis of the one-component reaction-diffusion equation with mixed and no-flux boundary conditions, let us first do a quick review of some of the literature.

In Chapter 17 of [11], the author discusses the one-component reaction-diffusion equation with Dirichlet boundary conditions in the bistable case. He scales the diffusion coefficient to 1 and works on the interval $[0, L]$ with $L > 0$. He finds an equality involving L and the transit time of steady states that must be satified. This allows him to utilize L as a bifurcation parameter. He goes on to say that below a certain threshold there are no non-constant steady states and once L is large enough, the system has two non-constant steady states. He also mentions, without proof, that one steady state is stable whereas the other is unstable.

In [7], the authors explore the reaction-diffusion equation on the whole real line. They assume that the reaction term has two zeros and that the diffusion coefficient is equal to 1. They then show that stable solutions exist as long as certain conditions on the reaction term are met.

In [10], the author considers a discrete time reaction-diffusion equation that is analagous to our system. He proves that waves cannot propagate (as long as the coupling d is small enough) because there are an infinite amount of stable patterned steady states which block wave propagation. In contrast, the author in [24] shows that for a coupling d sufficiently large, wave propagation is achievable in this same system. Therefore, there must be some critical coupling d^* where for $d < d^*$ propagation fails and for $d > d^*$, propagation is successful. In [2], they look and acquire a first approximation for this critical value d^*. In addition, they determine that near this critical value, the failure (or success) is determined by the slow passage near some limit point.

In [18], the authors consider the bistable reaction-diffusion equation for various boundary conditions. Notably, they consider Dirichlet, Neumann and periodic boundary conditions. They set the diffusion parameter to one and consider the problem on the interval $[-L, L]$ $(L > 0)$. Naturally, they use L as their bifurcation parameter. In the first case, they show that the bifurcation diagram depends strongly on the roots of the reaction term. They group the last two boundary conditions in the same case. They find that there is no bifurcation. There is only one possible patterned steady state at a time. In addition, they also study a time map, a function that describes the transit time of orbits in the phase-plane for Dirichlet and Neumann boundary conditions. In the former they show that this time map has either one or two critical points. In the latter, they show that this map is increasing, i.e., orbits that are further away from the center have a longer time of transit than those that are closer.

In [6], the author examines the reaction-diffusion equation with no-flux boundary conditions. He assumes that the reaction term depends on some parameter λ. In addition, the author presumes that 0 is a steady state of the system. The author uses λ to determine the stability around 0 and finds a pitchfork bifurcation.

Before attempting to show these aforementioned results, we shall first cover some analyses and notions that apply to both the mixed and no flux boundary cases. We show that investigating on a bounded interval of the form $[0, L]$ $(L > 0)$ is the same as investigating on the interval $[0, 1]$ modulo a change in the coefficient D. Then, we move on to a phase-plane analysis of both systems. This allows us to find two integrals that describe the time of transit of different orbits within the phase-plane. Determining that an orbit needs to have a time of transit equal to 1 in order to be a steady state solution, we decide to investigate these integrals. We first take a look at the mixed boundary conditions. We determine that we must have a saddle-node bifurcation. This leads to a quest to find a new form of the integral, which describes the transit time of orbits satisfying the boundary conditions, in order to ease numerical calculations. Afterwards, we determine the stability around the saddle-node steady state and conclude that stable patterned steady states exist under the right conditions. Then, we extend the case to Dirichlet boundary conditions and offer our thoughts on the parameter α. We conclude this chapter with an example. We then move on to the system with no-flux boundary conditions and repeat the same process as when the boundary conditions were mixed. However, this time, we find that stable patterned steady states do not exist under any circumstances.

Chapter 2

Preliminaries

In this chapter we discuss notions such as eigenvalues and eigenfunctions which come from the separation of variables technique for solving certain partial differential equations. In addition, we use phase-plane analysis to find the steady states of our systems. As a result, we encourage the reader to look up textbooks such as [13] and [19] that discuss these concepts.

Consider the reaction-diffusion equation

$$u_t(x,t) = Du_{xx}(x,t) + f(u(x,t)), \quad x \in [0,L], \quad t \geq 0 \tag{2.0.1}$$

where $f(u) = u(1-u)(u-\alpha)$, $D > 0$, $0 < \alpha < 1/2$ and $L \geq 0$. We also assume mixed and no-flux boundary conditions. Throughout this section, we shall introduce concepts that will apply to the reaction-diffusion equation regardless of whether the boundary conditions are mixed or no-flux. In addition, we shall also do some analysis that can be done for both boundary conditions simultaneously without much difficulty.

2.1 Eigenvalues and Eigenfunctions

A steady state solution to (2.0.1) is a solution that does not change with time, i.e., if $u_0(x)$ is a steady state solution to (2.0.1), then $(u_0(x))_t = 0$ and we have

$$0 = D(u_0(x))_{xx} + f(u_0(x)).$$

In addition, $u_0(x)$ must satisfy all boundary conditions imposed on the system. In order to analyse the stability of the steady states of the reaction-diffusion equation, one will often look at the eigenvalues and eigenfunctions of the steady state solutions. To obtain these, we linearize the system around the steady state $u_0(x)$. We set $u(x,t) = u_0(x) + \tilde{u}(x,t)$ and replace in (2.0.1) to get

$$\begin{aligned} &(u_0(x) + \tilde{u}(x,t))_t = D((u_0(x))_{xx} + \tilde{u}_{xx}(x,t)) + f(u_0(x) + \tilde{u}(x,t)) \\ \Longrightarrow\ &\tilde{u}_t(x,t) = D(u_0(x))_{xx} + D\tilde{u}_{xx}(x,t) + f(u_0(x) + \tilde{u}(x,t)). \end{aligned}$$

Note that when $u(x,t)$ is sufficiently close to $u_0(x)$, we have

$$f(u(x,t)) = f(u_0(x) + \tilde{u}(x,t)) \approx f(u_0(x)) + f'(u_0(x))\tilde{u}(x,t).$$

Subsequently, we now have

$$\tilde{u}_t(x,t) = D(u_0(x))_{xx} + D\tilde{u}_{xx}(x,t) + f(u_0(x)) + f'(u_0(x))\tilde{u}(x,t).$$

Considering that $u_0(x)$ is a steady state solution, we then obtain an equation in $\tilde{u}(x,t)$

$$\tilde{u}_t(x,t) = D\tilde{u}_{xx}(x,t) + f'(u_0(x))\tilde{u}(x,t).$$

In order for $u(x,t)$ to be a perturbed solution close to $u_0(x)$, we set $\tilde{u}(x,t) = e^{-\delta t}v(x)$. Applying this to the previous equation leads to

$$-\delta e^{-\delta t}v(x) = De^{-\delta t}v_{xx}(x) + f'(u_0(x))e^{-\delta t}v(x).$$

Dividing both sides by $e^{-\delta t}$ yields the eigenvalue equation associated with the steady state solution $u_0(x)$

$$-\delta v(x) = Dv_{xx}(x) + f'(u_0(x))v(x).$$

Finally, we apply the appropriate boundary conditions to $v(x)$ and we call $v(x)$ an eigenfunction associated with the eigenvalue $-\delta$. Note that, since we chose $e^{-\delta t}$ rather than $e^{\delta t}$, $\delta > 0$ implies that $u(x,t)$ tends towards the steady state solution and $\delta < 0$ means that we are heading away from the steady state solution as $t \to \infty$.

2.2 Equivalence of Bounded Intervals

In this section we want to show that the analysis of a one-component reaction-diffusion system on the bounded interval $[0, L]$ with $L > 0$ is equivalent (modulo a change of variables) to the analysis of the same system on the bounded interval $[0, 1]$. In order to do so, consider the system

$$u_t(x,t) = Du_{xx}(x,t) + f(u(x,t)),$$

with $D > 0$ and $x \in [0, L]$ $(L > 0)$. Let $y = \frac{x}{L}$ so that $y \in [0, 1]$. In addition, let $v(y,t)$ be such that $v(y,t) = u(x,t)$. Then, we have $v_t(y,t) = u_t(x,t)$, and so,

$$v_t(y,t) = u_t(x,t) = Du_{xx}(x,t) + f(u(x,t)) = \frac{D}{L^2}v_{yy}(y,t) + f(v(y,t)).$$

Thus, we end up with

$$v_t(y,t) = D' v_{yy}(y,t) + f(v(y,t)),$$

where $D' = D/L^2$. In addition, we clearly have $\frac{1}{L^n}\frac{d^2v}{dx^2}(y,t) = \frac{d^2u}{dx^2}(x,t)$. Hence, for any homogeneous boundary conditions that $u(x,t)$ satisfies on the interval $[0,L]$, $v(y,t)$ satisfies the same boundary conditions on the interval $[0,1]$. As a result, any analysis that we make on the one-component reaction-diffusion system while considering the interval $[0,1]$ can be easily translated to the interval $[0,L]$ $(L > 0)$.

2.3 Phase-Plane Analysis

Consider the reaction-diffusion system in (2.0.1). In order to find the steady states ($u_t(x,t) = 0$) of this system, we set $v(x) = u_x(x)$ and solve the equation $u_t(x,t) = 0$. We obtain the system

$$\begin{aligned} u_x(x) &= v(x), \\ v_x(x) &= \frac{-f(u(x))}{D}. \end{aligned} \tag{2.3.1}$$

The associated Jacobian matrix associated to the system in (2.3.1) is

$$J(u,v) = \begin{bmatrix} 0 & 1 \\ \frac{-f'(u)}{D} & 0 \end{bmatrix}. \tag{2.3.2}$$

The system in (2.3.1) has three equilibrium points, $(0,0)$, $(\alpha,0)$ and $(1,0)$. Using the Jacobian matrix given in (2.3.2), we can find the eigenvalues associated with each equilibrium point

$$(0,0): \quad J(0,0) = \begin{bmatrix} 0 & 1 \\ \frac{\alpha}{D} & 0 \end{bmatrix} \implies \gamma_{1,2} = \pm\sqrt{\frac{\alpha}{D}},$$

$$(\alpha, 0): \quad J(\alpha, 0) = \begin{bmatrix} 0 & 1 \\ \frac{-\alpha(1-\alpha)}{D} & 0 \end{bmatrix} \implies \gamma_{1,2} = \pm i\sqrt{\frac{\alpha(1-\alpha)}{D}},$$

$$(1, 0): \quad J(1, 0) = \begin{bmatrix} 0 & 1 \\ \frac{1-\alpha}{D} & 0 \end{bmatrix} \implies \gamma_{1,2} = \pm\sqrt{\frac{1-\alpha}{D}}.$$

From this, we can determine that the points $(0, 0)$ and $(1, 0)$ are both saddles, whereas the point $(\alpha, 0)$ is a center; see Figure 2.1.

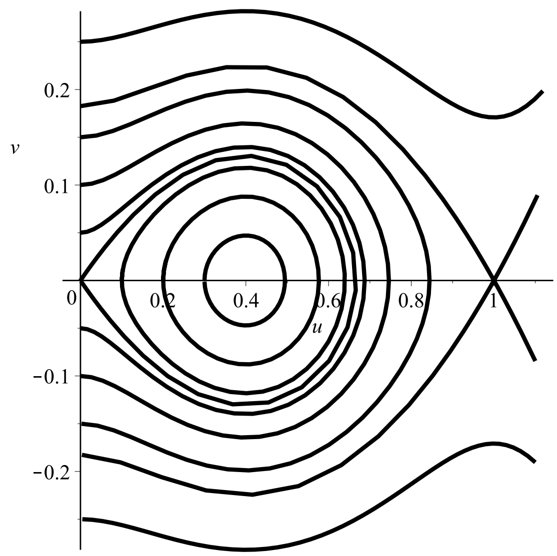

Figure 2.1: This is part of the phase-plane of our system with $\alpha = 0.4$ and $D = 1$.

Using the boundary conditions, we can now trace orbits in the phase-plane for both cases.

Mixed boundary conditions imply that $u(0) = 0$ and $v(1) = 0$. The former implies that orbits start on the v-axis with $v > 0$ while the latter means the orbit stops on the u-axis with $\alpha < u < 1$. See Figure 2.2.

On the other hand, no-flux conditions, along with the fact that $(\alpha, 0)$ is a center, imply that orbits either start and end on some point of the form $(u, 0)$ with $0 < u < \alpha$

or are constant (0, α or 1). See Figure 2.2.

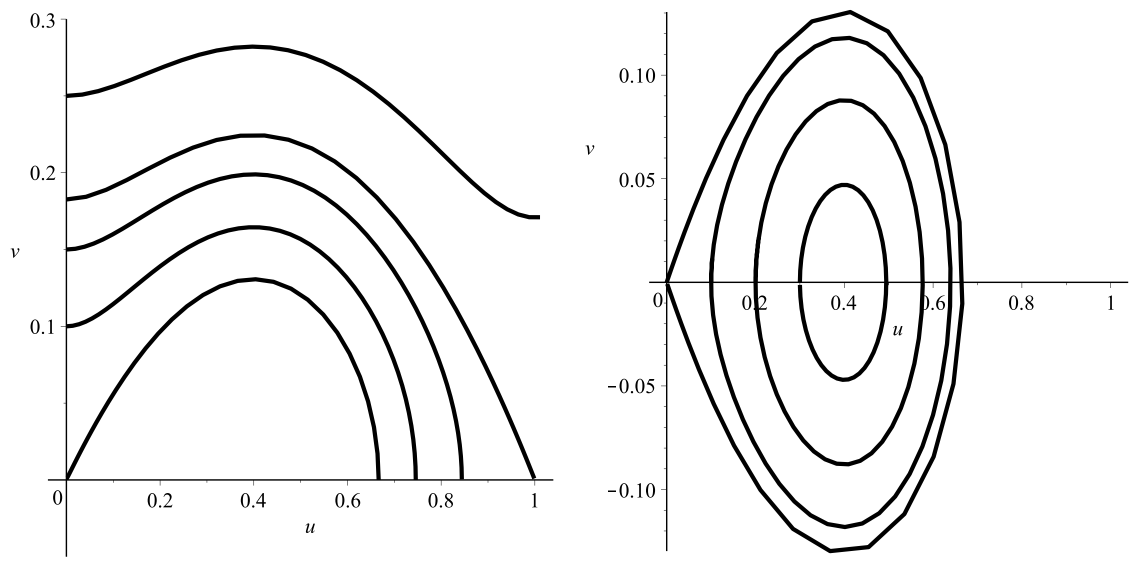

Figure 2.2: On the left, we have some orbits for the mixed boundary conditions. We can see that if we start too high on the v-axis, the saddle at $(1, 0)$ pulls the orbit away so that it never reaches the u-axis. On the right, we have some orbits pertaining to the no-flux boundary conditions. Both sides use $\alpha = 0.4$ and $D = 1$.

Let us try to solve the ordinary differential equation system (2.3.1). We first note that

$$\frac{dv}{du}\frac{du}{dx}(x) = \frac{dv}{dx}(x) = \frac{-f(u(x))}{D}.$$

Thus, as $\frac{du}{dx}(x) = v(x)$, we have

$$vdv = \frac{-f(u)}{D}du.$$

Integrating both sides yields

$$\frac{v^2}{2} = \int \frac{-f(u)}{D}du = \frac{1}{D}\left(\frac{u^4}{4} - \frac{(1+\alpha)}{3}u^3 + \frac{\alpha}{2}u^2 + c\right),$$

where c is the integration constant. Consequently, we must have

$$v = \pm\frac{1}{\sqrt{D}}\sqrt{\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 \pm \lambda}, \tag{2.3.3}$$

where $\pm\lambda = 2c$. This function describes the orbits in the phase-plane. We want to assume that $\lambda > 0$ in both cases. When an orbit starts on the v-axis with $v > 0$, we have $c > 0$ and so we shall use λ for the mixed boundary conditions. When orbits are surrounding $(\alpha, 0)$, c is negative. Thus we shall use $-\lambda$ for the no-flux boundary conditions.

Now, by the definition of $v(x)$, we have $u_x(x) = v(x)$, and so, we have

$$u_x(x) = \pm\frac{1}{\sqrt{D}}\sqrt{\frac{u^4(x)}{2} - \frac{2(1+\alpha)}{3}u^3(x) + \alpha u^2(x) \pm \lambda}.$$

Therefore,

$$\pm\sqrt{D}\frac{du}{\sqrt{\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 \pm \lambda}} = dx.$$

Recall that we are working on the bounded interval $[0, 1]$. In the mixed boundary case, applying the boundary conditions yields

$$\pm\sqrt{D}\int_0^{u^*} \frac{du}{\sqrt{\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 + \lambda}} = 1, \tag{2.3.4}$$

where u^* is the smallest real root of the polynomial $\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 + \lambda$. We show that u^* is real in Remark 2.3.1. In the no-flux boundary instance, we obtain

$$\pm\sqrt{D}\int_{u_1}^{u_2} \frac{du}{\sqrt{\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 - \lambda}} = 1, \tag{2.3.5}$$

where u_1 and u_2 are the second and third smallest real roots (respectively) of the polynomial $\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 - \lambda$. We show that u_2 and u_3 are both real in Remark

2.3.1.

Equations (2.3.4) and (2.3.5) indicate that not every orbit in the phase-plane is a steady state solution to our system. Henceforth, we need to make sure that the time of transit of solutions to the equation in (2.0.1) is equal to 1.

Remark 2.3.1. The orbits in both the mixed and no-flux boundary conditions cases are determined by the function $v(x)$. In addition, for fixed D and α, $v(x)$ is uniquely determined by the value of λ. Thus, studying $v(x)$ and its roots in either instance is equivalent to studying the effects of adding λ to or subtracting λ from the following polynomial (see Figure 2.3 for a visualisation of the polynomial)

$$h(u) = \frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2.$$

This polynomial has a double root $u_{0,1} = 0$ as well as two more real roots, $u_{3,2} = \frac{2}{3}(1+\alpha) \pm \frac{\sqrt{2}}{3}\sqrt{2\alpha^2 - 5\alpha + 2}$. We claim that for all $0 < \alpha \leq 1/2$, we have $u_2 \leq 1 \leq u_3$. Indeed, it is clear that we have $u_2 \leq 1$, and so, we only need to show that $u_3 \geq 1$. We have

$$u_3 \geq 1 \iff \frac{\sqrt{2}}{3}\sqrt{2\alpha^2 - 5\alpha + 2} \geq \frac{1}{3}(1-2\alpha) \iff 2(2\alpha^2 - 5\alpha + 2) \geq 4\alpha^2 - 4\alpha + 1$$
$$\iff 3 \geq 6\alpha \iff \frac{1}{2} \geq \alpha.$$

Now, when the boundary conditions are mixed, we look for orbits starting on the v-axis with $v > 0$. Thus, we are adding λ to $h(u)$. Graphically, this means we are shifting $h(u)$ upwards. Hence, the double root $u_{0,1}$ becomes complex. In addition, since these orbits must end on the u-axis with $\alpha < u \leq 1$, the roots $u_{2,3}$ need to be real or else the orbit will never reach the u-axis. Subsequently, we need $h(1)$ to be negative or 0 (if $h(1)$ is positive, then the roots $u_{2,3}$ become complex and the orbit does not reach the u-axis), and so, we can only shift $h(u)$ upwards as much as $-h(1)$.

Hence, we obtain an upper bound for λ :

$$\lambda \leq -h(1) = \frac{1}{2} - \frac{2}{3}(1+\alpha) + \alpha = \frac{1}{6}(1-2\alpha).$$

Comparatively, in the no-flux boundary conditions instance, we look for orbits starting (and ending on the same point) on the u-axis with $0 < u < \alpha$. Thus, we are subtracting λ from $h(u)$. Graphically, we are shifting $h(u)$ downwards. Therefore, the double root, $u_{0,1}$, splits into two real ones, a negative root u_0 and a positive root u_1. In fact, the orbit starts at $(u_1, 0)$, loops around α, hitting $(u_2, 0)$ in the process, and comes back to u_1. In order for this orbit to be possible, we need u_1 and u_2 to be real and we also need $u_1 \leq \alpha \leq u_2 \leq 1$. The former implies that we need $h(\alpha)$ to be greater than or equal to 0. Thus, we can only move the quartic as much as the value of $h(\alpha)$, and so, we obtain an upper bound for λ

$$\begin{aligned}\lambda \leq h(\alpha) &= \frac{\alpha^4}{2} - \frac{2}{3}(1+\alpha)\alpha^3 + \alpha^3 = \alpha^3\left(\frac{\alpha}{2} - \frac{2}{3}(1+\alpha) + 1\right) = \alpha^3\left(\frac{1}{3} - \frac{1}{6}\alpha\right) \\ &= \frac{\alpha^3}{6}(2-\alpha).\end{aligned}$$

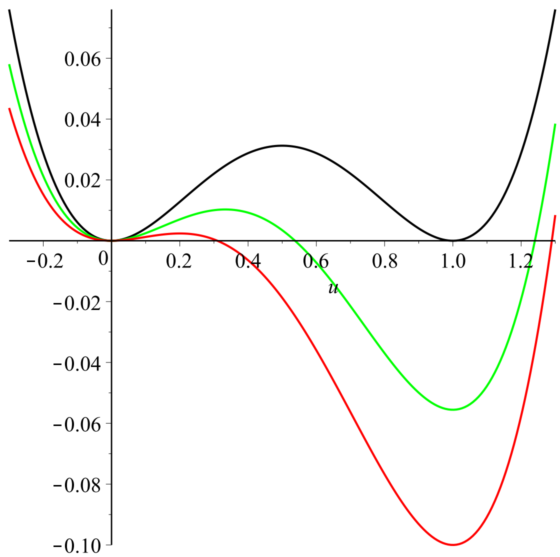

Figure 2.3: Graph of the function $h(u)$ for different values of α. The black colored function represents $\alpha = \frac{1}{2}$, the green colored function is when $\alpha = \frac{1}{3}$, and finally, the red colored function describes $h(u)$ with $\alpha = \frac{1}{5}$.

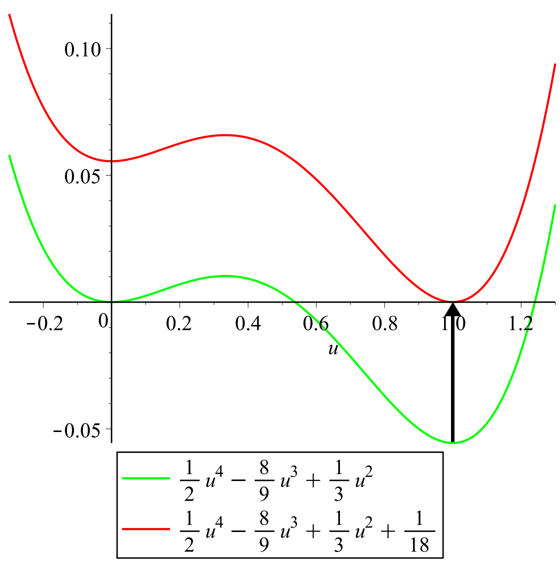

Figure 2.4: Depiction of translating $h(u)$ (when $\alpha = \frac{1}{3}$) upwards by the maximum distance possible, $\frac{1}{6}(1-2\alpha) = \frac{1}{18}$. We can also see that the double root $u_{0,1} = 0$ becomes complex.

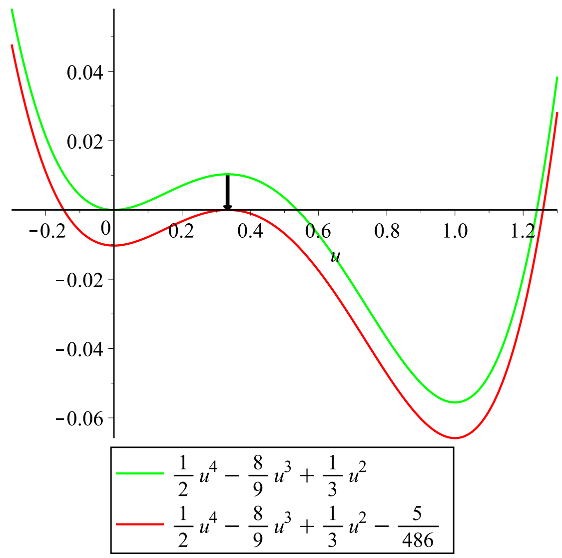

Figure 2.5: Depiction of translating $h(u)$ (when $\alpha = \frac{1}{3}$) downwards by the maximum distance possible, $\frac{\alpha^3}{6}(2-\alpha) = \frac{5}{486}$. In addition, we can see the double root $u_{0,1} = 0$ split into two real roots.

Chapter 3

Mixed Boundary Conditions

In this section, we shall focus on the reaction-diffusion equation with mixed boundary conditions

$$u_t = Du_{xx} + f(u), \quad u(0,t) = u_x(1,t) = 0 \quad (\forall t \geq 0), \quad f(u) = u(1-u)(u-\alpha) \tag{3.0.1}$$

with $D > 0$, $0 < \alpha < 1/2$, $x \in [0,1]$ and $t \geq 0$. In Chapter 2.3, we determined that steady state solutions must satisfy (2.3.4), i.e., the time of transit of steady state solutions must be equal to 1. Thus, in order to figure out how many steady state solutions there are, we must investigate the integral on the left hand side of (2.3.4).

For a given α, the integral given in (2.3.4) depends on two parameters: D and λ. The first parameter simply scales the integral up or down depending on its value, and so, we can omit it for now. Define

$$G(\lambda) = \int_0^{u^*(\lambda)} \frac{du}{\sqrt{\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 + \lambda}}, \tag{3.0.2}$$

where $u^*(\lambda)$ is the first real root of $\frac{u^4}{2} - \frac{2(1+\alpha)}{3}u^3 + \alpha u^2 + \lambda$ and $0 \leq \lambda \leq \frac{1}{6}(1 - 2\alpha)$. So $G(\lambda)$ describes the transit time of steady state solutions when the boundary conditions are mixed and when $D = 1$. We can solve $G(\lambda)$ numerically over the interval $[0, \frac{1}{6}(1 - 2\alpha)]$. We have vertical asymptotes at both ends of the interval. Thus, we must have a global minimum over this interval. In fact, in [18], the authors prove that $G'(\lambda)$ has only one zero and it corresponds to a minimum of $G(\lambda)$. See Figure 3.1 for the graph of $G(\lambda)$ for different values of α. If the minimum is greater than 1, then we do not have any steady states. If the minimum is equal to or less than 1, then we have either 1 or 2 steady state solutions, respectively.

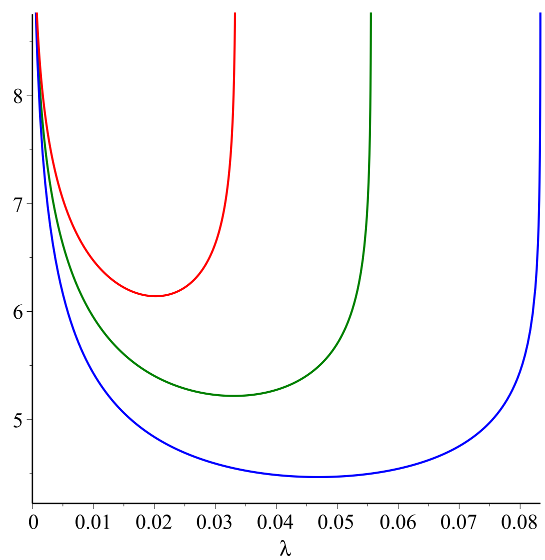

Figure 3.1: Graphs of $G(\lambda)$ for various values of α are shown in this figure. In red we have $\alpha = \frac{2}{5}$. In green we have $\alpha = \frac{1}{3}$ whereas in blue the value of α is $\frac{1}{4}$.

As previously mentioned, D simply scales $G(\lambda)$ downwards. Therefore, if, for a given α, the minimum of $G(\lambda)$ is greater than 1, there must be a certain value of $D = D_{\text{crit}}$ such that the minimum of $\sqrt{D_{\text{crit}}}G(\lambda) = 1$. Consequently, letting D vary leads to a saddle-node bifurcation. Indeed, when $D < D_{\text{crit}}$, we have two steady states. As D approaches D_{crit}, the steady states move toward each other and eventually collide and merge into one steady state when $D = D_{\text{crit}}$. Once D passes D_{crit}, the steady states have annihilated themselves and we are left with nothing. See

Figure 3.2.

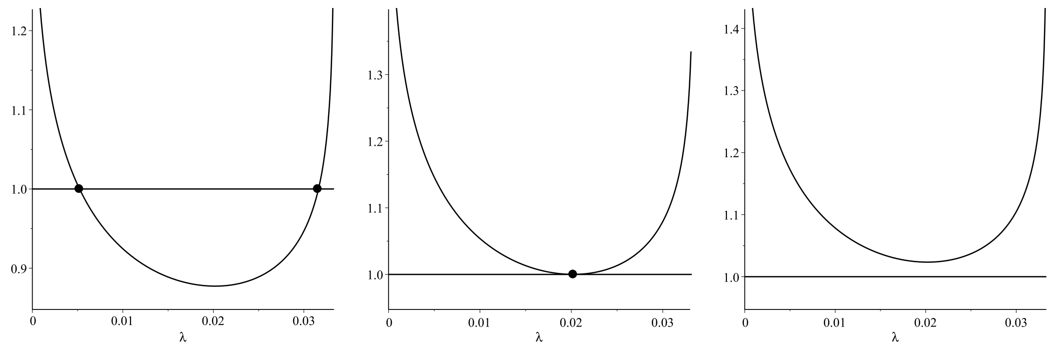

Figure 3.2: We scaled $G(\lambda)$ with $\alpha = \frac{2}{5}$ for different values of D. On the left, we have $D = \frac{1}{49}$ while in the middle we have $D = D_{\text{crit}} \approx \frac{1}{(6.14056)^2} \approx 0.02652$. On the right, we have $D = \frac{1}{36}$. The points indicate when $\sqrt{D}G(\lambda)$ is equal to 1, and so, there is a steady state solution for those values of λ.

It is now clear that in order to determine the existence of stable patterned steady states of the system given in (3.0.1), we must investigate the stability around the steady state solution that appears when $\sqrt{D_{\text{crit}}}G(\lambda) = 1$. However, before conducting this analysis, we shall first find another form for $G(\lambda)$ in order to facilitate the process of numerically solving $G(\lambda)$ and thus, finding its minimum.

3.1 Integral Formula

It is not obvious what the graph of the integral in (2.3.4) is with respect to λ. For this reason, we will dedicate this section to the decryption of the integral given in (3.1.1), i.e., we will transform the integral into a more familiar form. In order to achieve this, we will assume $D = 1$. With this assumption, the integral in (2.3.4) yields

$$\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}}, \tag{3.1.1}$$

with

$$F(u,\lambda) = \frac{u^4}{2} - \frac{2}{3}(1+\alpha)u^3 + \alpha u^2 + \lambda,$$

where $\lambda > 0$ and u_2 is the smallest real root of $F(u,\lambda)$. When $D \neq 1$, we will simply multiply the new form of the integral in (3.1.1) by $\sqrt{D}$.

The first step to changing the integral in (3.1.1) is to transform the function $F(u,\lambda)$. More precisely, we want $F(u,\lambda)$ to be of the following form

$$F(u,\lambda) = \left(A_1(u-a_1)^2 + B_1(u-a_2)^2\right)\left(A_2(u-a_1)^2 + B_2(u-a_2)^2\right),$$

where A_1, A_2, B_1, B_2, a_1 and a_2 are constants. This leads to the following lemma.

Lemma 3.1.1. *Let $F(u) = b_4u^4 + b_3u^3 + b_2u^2 + b_1u + b_0$ with $b_0, \ldots, b_4 \in \mathbb{R}$. Suppose that $F(u)$ has two complex roots, $u_0 = a + bi$ and $u_1 = a - bi$ $(a, b \in \mathbb{R})$ and two real roots u_2 and u_3 with $0 < u_2 < u_3$. Then, there exists constants A_1, A_2, B_1, B_2, a_1 and a_2 such that*

$$F(u) = \left(A_1(u-a_1)^2 + B_1(u-a_2)^2\right)\left(A_2(u-a_1)^2 + B_2(u-a_2)^2\right). \tag{3.1.2}$$

Proof: Set

$$S_1(u) = b_4(u-u_2)(u-u_3) \quad \text{and} \quad S_2(u) = (u-u_0)(u-u_1) = u^2 - 2au + a^2 + b^2. \tag{3.1.3}$$

Note that, by definition of $S_1(u)$ and $S_2(u)$, we have

$$F(u) = S_1(u)S_2(u). \tag{3.1.4}$$

In order to turn (3.1.4) into (3.1.2), we must first find ϵ such that $S_1(u) - \epsilon S_2(u)$ is a perfect square. Expanding $S_1 - \epsilon S_2$ gives us

$$\begin{aligned} S_1 - \epsilon S_2 &= b_4 u^2 - b_4(u_2+u_3)u + b_4 u_2 u_3 - \epsilon u^2 + 2\epsilon a u - \epsilon(a^2+b^2) \\ &= (b_4 - \epsilon)\, u^2 + (2\epsilon a - b_4(u_2+u_3))\, u + b_4 u_2 u_3 - \epsilon(a^2+b^2). \end{aligned}$$

We want $S_1 - \epsilon S_2$ to be a perfect square, thus, we need the discriminant to be zero, i.e.,

$$(2\epsilon a - b_4(u_2+u_3))^2 - 4\,(b_4-\epsilon)\left(b_4 u_2 u_3 - \epsilon(a^2+b^2)\right) = 0.$$

Expanding and then rearranging yields

$$\begin{aligned} 0 &= 4a^2\epsilon^2 - 4ab_4(u_2+u_3)\epsilon + b_4^2(u_2+u_3)^2 - 4\left(b_4^2 u_2 u_3 - b_4(a^2+b^2)\epsilon - b_4 u_2 u_3 \epsilon + (a^2+b^2)\epsilon^2\right) \\ &= \left(4a^2 - 4(a^2+b^2)\right)\epsilon^2 + 4b_4\left((a^2+b^2) + u_2 u_3 - a(u_2+u_3)\right)\epsilon + b_4^2\left((u_2+u_3)^2 - 4u_2 u_3\right) \end{aligned}$$

Simplifying then gives us

$$0 = -4b^2\epsilon^2 + 4b_4\left(a^2+b^2+u_2u_3 - a(u_2+u_3)\right)\epsilon + b_4^2(u_2-u_3)^2. \tag{3.1.5}$$

Therefore, we must have

$$\begin{aligned} \epsilon_{1,2} = &\frac{-4b_4\left(a^2+b^2+u_2u_3-a(u_2+u_3)\right)}{-8b^2} \\ &\pm \frac{\sqrt{16b_4^2\left(a^2+b^2+u_2u_3-a(u_2+u_3)\right)^2 + 16b^2b_4^2(u_2-u_3)^2}}{-8b^2}. \end{aligned} \tag{3.1.6}$$

Given this information, we now know that

$$S_1(u) - \epsilon_1 S_2(u) = (b_4 - \epsilon_1)\,(u - a_1)^2 \tag{3.1.7}$$

and

$$S_1(u) - \epsilon_2 S_2(u) = (b_4 - \epsilon_2)(u - a_2)^2, \tag{3.1.8}$$

where

$$a_1 = \frac{b_4(u_2 + u_3) - 2\epsilon_1 a}{2(b_4 - \epsilon_1)} \quad \text{and} \quad a_2 = \frac{b_4(u_2 + u_3) - 2\epsilon_2 a}{2(b_4 - \epsilon_2)}. \tag{3.1.9}$$

From (3.1.7), we obtain

$$S_1(u) = (b_4 - \epsilon_1)(u - a_1)^2 + \epsilon_1 S_2(u). \tag{3.1.10}$$

Combining this with (3.1.8), we get

$$(b_4 - \epsilon_1)(u - a_1)^2 + (\epsilon_1 - \epsilon_2) S_2(u) = (b_4 - \epsilon_2)(u - a_2)^2,$$

and so,

$$S_2(u) = \left(\frac{\epsilon_1 - b_4}{\epsilon_1 - \epsilon_2}\right)(u - a_1)^2 + \left(\frac{b_4 - \epsilon_2}{\epsilon_1 - \epsilon_2}\right)(u - a_2)^2. \tag{3.1.11}$$

Now, combining (3.1.10) and (3.1.11), we can find $S_1(u)$

$$S_1(u) = (b_4 - \epsilon_1)(u - a_1)^2 + \epsilon_1 \left(\frac{b_4 - \epsilon_2}{\epsilon_1 - \epsilon_2}\right)(u - a_2)^2 - \left(\frac{b_4 - \epsilon_1}{\epsilon_1 - \epsilon_2}\right)(u - a_1)^2,$$

and so,

$$S_1(u) = (\epsilon_1 - b_4)\frac{\epsilon_2}{\epsilon_1 - \epsilon_2}(u - a_1)^2 + (b_4 - \epsilon_2)\frac{\epsilon_1}{\epsilon_1 - \epsilon_2}(u - a_2)^2. \tag{3.1.12}$$

Subsequently, we have

$$S_1(u) = A_1(u - a_1)^2 + B_1(u - a_2)^2 \quad \text{and} \quad S_2(u) = A_2(u - a_1)^2 + B_2(u - a_2)^2, \tag{3.1.13}$$

where

$$A_1 = (\epsilon_1 - b_4)\,\frac{\epsilon_2}{\epsilon_1 - \epsilon_2},\ A_2 = \left(\frac{\epsilon_1 - b_4}{\epsilon_1 - \epsilon_2}\right),\ B_1 = (b_4 - \epsilon_2)\,\frac{\epsilon_1}{\epsilon_1 - \epsilon_2},\ B_2 = \left(\frac{b_4 - \epsilon_2}{\epsilon_1 - \epsilon_2}\right). \tag{3.1.14}$$

Consequently, we finally have (3.1.2). ∎

In the mixed boundary case, $F(u,\lambda)$ has two complex roots, $u_0 = a + bi$ and $u_1 = a - bi$ $(a, b \in \mathbb{R})$, and two real roots, $0 < u_2 < u_3$. Thus the requirements of Lemma 3.1.1 are satisfied, and so, we have

$$\begin{aligned}\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} &= \int_0^{u_2} \frac{du}{\sqrt{S_1(u)S_2(u)}} \\ &= \int_0^{u_2} \frac{du}{\sqrt{\left(A_1(u-a_1)^2 + B_1(u-a_2)^2\right)\left(A_2(u-a_1)^2 + B_2(u-a_2)^2\right)}}.\end{aligned}$$

Note that the constants A_1, A_2, B_1, B_2, a_1 and a_2 are given in the proof of Lemma 3.1.1. We then set $t = \frac{u-a_1}{u-a_2}$ so that $du = \frac{(u-a_2)^2}{a_1-a_2}dt$ and $t^2 = \frac{(u-a_1)^2}{(u-a_2)^2}$. A change of variable then leads to

$$\begin{aligned}&\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} \\ &= \int_0^{u_2} \frac{du}{\sqrt{\left(A_1(u-a_1)^2 + B_1(u-a_2)^2\right)\left(A_2(u-a_1)^2 + B_2(u-a_2)^2\right)}} \\ &= \frac{1}{a_1 - a_2} \int_{\frac{a_1}{a_2}}^{\frac{u_2-a_1}{u_2-a_2}} \frac{(u-a_2)^2 dt}{\sqrt{\left(A_1(u-a_1)^2 + B_1(u-a_2)^2\right)\left(A_2(u-a_1)^2 + B_2(u-a_2)^2\right)}} \\ &= \frac{1}{a_1 - a_2} \int_{\frac{a_1}{a_2}}^{\frac{u_2-a_1}{u_2-a_2}} \frac{(u-a_2)^2 dt}{\sqrt{(u-a_2)^4 \left(A_1 \frac{(u-a_1)^2}{(u-a_2)^2} + B_1\right)\left(A_2 \frac{(u-a_1)^2}{(u-a_2)^2} + B_2\right)}} \\ &= \frac{1}{a_1 - a_2} \int_{\frac{a_1}{a_2}}^{\frac{u_2-a_1}{u_2-a_2}} \frac{dt}{\sqrt{\left(A_1 t^2 + B_1\right)\left(A_2 t^2 + B_2\right)}}.\end{aligned}$$

In order to continue, we need to do another change of variables, notably, we must set $t = \sqrt{\frac{-B_1}{A_1}}\sec(\theta)$. However, we want to make sure that $\frac{-B_1}{A_1} > 0$. Consequently, we first need to investigate the signs of the constants A_1, A_2, B_1 and B_2. In order to explore these constants, we must analyze ϵ_1 and ϵ_2 given in (3.1.6).

Let us start by showing that a (the real part of the complex roots in Lemma 3.1.1) is negative using (3.1.4). This will help us gain more information on both ϵ_1 and ϵ_2.

Lemma 3.1.2. *Let $F(u)$ be as in Lemma 3.1.2. Suppose $b_1 = 0$. Then the real part of the complex roots is negative.*

Proof: From (3.1.4), we have

$$\begin{aligned}
F(u,\lambda) = S_1(u)S_2(u) &= b_4(u-u_2)(u-u_3)(u^2-2au+a^2+b^2) \\
&= b_4(u^2-(u_2+u_3)u+u_2u_3)(u^2-2au+a^2+b^2) \\
&= b_4(u^4-2au^3+(a^2+b^2)u^2-(u_2+u_3)u^3+2a(u_2+u_3)u^2 \\
&\quad -(a^2+b^2)(u_2+u_3)u+u_2u_3u^2-2au_2u_3u+u_2u_3(a^2+b^2)) \\
&= b_4(u^4-(2a+u_2+u_3)u^3+(a^2+b^2+2a(u_2+u_3)+u_2u_3)u^2 \\
&\quad -((a^2+b^2)(u_2+u_3)+2au_2u_3)u+u_2u_3(a^2+b^2))
\end{aligned}$$

The coefficients in front of u must be equal on both sides. Hence,

$$0 = (a^2+b^2)(u_2+u_3)+2au_2u_3.$$

Solving for a yields

$$a = \frac{-(a^2+b^2)(u_2+u_3)}{2u_2u_3}.$$

Since u_2 and u_3 are both positive, we can deduce that a is negative. ∎

We now have enough information to investigate ϵ_1 and ϵ_2.

Lemma 3.1.3. *Let the notation be as in Lemma 3.1.2. Suppose* $b_1 = 0$. *Then,* $\epsilon_1 < 0 < \epsilon_2$ *and* $\epsilon_2 > b_4$.

Proof: Recall that ϵ_1 and ϵ_2 are given in (3.1.6). It is clear that ϵ_1 is negative since

$$\sqrt{16b_4^2\left(a^2+b^2+u_2u_3-a(u_2+u_3)\right)^2+16b^2b_4^2(u_2-u_3)^2} > 4b_4\left(a^2+b^2+u_2u_3-a(u_2+u_3)\right).$$

By Lemma 3.1.2, a is negative. In addition, u_2, u_3 are both positive by hypothesis, and so, the term $(a^2+b^2+u_2u_3-a(u_2+u_3))$ is positive. Thus, we can conclude that ϵ_2 is positive. In addition, from (3.1.8) and the fact that $S_1(u_2) = 0$, we have

$$-\epsilon_2 S_2(u_2) = (b_4-\epsilon_2)(u_2-a_2)^2.$$

Thus,

$$\epsilon_2 = -\frac{(b_4-\epsilon_2)(u_2-a_2)^2}{S_2(u_2)}.$$

Considering that ϵ_2 is positive and that $S_2(u)$ is strictly positive by definition, we must then have $\epsilon_2 > b_4$. ∎

The information we have gathered pertaining to ϵ_1 and ϵ_2 allows us to trivially know the signs of the constants A_1, A_2, B_1 and B_2 using (3.1.14). This gives us the following corollary.

Corollary 3.1.4. *Let the notation be as in Lemma 3.1.3. Then* A_1, A_2 *and* B_2 *are positive and* B_1 *is negative.*

Proof: This immediately follows Lemma 3.1.3 and (3.1.14). ∎

Corollary 4.1.3 solves our issue regarding the sign of $\frac{-B_1}{A_1}$. However, the change of variable $t = \sqrt{\frac{-B_1}{A_1}}\sec(\theta)$ leads to another issue. In particular, the problem arises when calculating the upper limit. Indeed, consider $S_1(u)$ in (3.1.3). By definition, we have $S_1(u_2) = 0$, and so, recalling that $S_1(u)$ has another form given in (3.1.13), we have $\frac{(u_2-a_1)^2}{(u_2-a_2)^2} = \frac{-B_1}{A_1}$. Therefore, $\frac{u_2-a_1}{u_2-a_2} = \pm\sqrt{\frac{-B_1}{A_1}}$. Up to this point, it is unclear whether $\frac{u_2-a_1}{u_2-a_2}$ is negative or positive. Thus, we introduce the next lemma to solve this issue.

Lemma 3.1.5. *Let the notation be as in Lemma 3.1.3. Suppose $b_4 > 0$. Then $u_2 < a_1 < u_3$ and $a_2 < 0$.*

Proof: Recall that a_1 and a_2 are given in (3.1.9). Let us first show that $u_2 < a_1 < u_3$. By (3.1.12) we have $S_1(a_1) = B_1(a_1 - a_2)^2$. Hence,

$$b_4(a_1 - u_2)(a_1 - u_3) = B_1(a_1 - a_2)^2.$$

By hypothesis $b_4 > 0$. In addition, by Corollary 4.1.3 $B_1 < 0$. Therefore, (a_1-u_2) and $(a_1 - u_3)$ must have opposite sign. Consequently, we necessarily have $u_2 < a_1 < u_3$.

Let us now prove that $a_2 < 0$. By Lemma 3.1.3, $b_4 - \epsilon_2 < 0$ and $\epsilon_2 > 0$. In addition, by Lemma 3.1.2, a is negative. Thus, $b_4(u_2 + u_3) - 2\epsilon_2 a > 0$. Hence, a_2 is negative. ∎

We are now ready for the next (and last) change of variable. We set $t = \sqrt{\frac{-B_1}{A_1}}\sec(\theta)$. Then, $dt = \sqrt{\frac{-B_1}{A_1}}\sec(\theta)\tan(\theta)d\theta$ and $t^2 = \frac{-B_1}{A_1}\sec^2(\theta)$. The lower limit now becomes $\theta_1 = \arccos\left(\sqrt{\frac{-B_1}{A_1}}\frac{a_2}{a_1}\right)$. In addition, the upper limit becomes

$$\theta_2 = \arccos\left(\sqrt{\frac{-B_1}{A_1}}\frac{u_2 - a_2}{u_2 - a_1}\right) = \arccos(-1) = \pi,$$

where the last equality follows from Lemma 3.1.5. Subsequently, we have

$$\begin{aligned}
\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} &= \frac{1}{a_1 - a_2}\int_{\frac{a_1}{a_2}}^{\frac{u2-a_1}{u2-a_2}} \frac{dt}{\sqrt{(A_1t^2 + B_1)(A_2t^2 + B_2)}} \\
&= \frac{1}{a_1 - a_2}\int_{\theta_1}^{\pi} \frac{\sqrt{\frac{-B_1}{A_1}}\sec(\theta)\tan(\theta)d\theta}{\sqrt{\left(A_1\left(\frac{-B_1}{A_1}\sec^2(\theta)\right) + B_1\right)\left(A_2\left(\frac{-B_1}{A_1}\sec^2(\theta)\right) + B_2\right)}} \\
&= \frac{1}{a_1 - a_2}\sqrt{\frac{-B_1}{A_1}}\int_{\theta_1}^{\pi} \frac{d\theta}{\frac{\cos^2(\theta)}{\sin(\theta)}\sqrt{(-B_1\sec^2(\theta) + B_1)\left(\frac{-B_1A_2}{A_1}\sec^2(\theta) + B_2\right)}} \\
&= \frac{1}{a_1 - a_2}\sqrt{\frac{-B_1}{A_1}}\int_{\theta_1}^{\pi} \frac{d\theta}{\frac{1}{\sin(\theta)}\sqrt{(B_1\cos^2(\theta) - B_1)\left(B_2\cos^2(\theta) - \frac{B_1A_2}{A_1}\right)}}.
\end{aligned}$$

Recall that $\sin^2(\theta) + \cos^2(\theta) = 1$. Consequently, we obtain

$$\begin{aligned}
\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} &= \frac{1}{a_1 - a_2}\sqrt{\frac{-B_1}{A_1}}\int_{\theta_1}^{\pi} \frac{d\theta}{\frac{1}{\sin(\theta)}\sqrt{(B_1\cos^2(\theta) - B_1)\left(B_2\cos^2(\theta) - \frac{B_1A_2}{A_1}\right)}} \\
&= \frac{1}{a_1 - a_2}\sqrt{\frac{-B_1}{A_1}}\int_{\theta_1}^{\pi} \frac{d\theta}{\frac{1}{\sin(\theta)}\sqrt{\left(-B_1\sin^2(\theta)\right)\left(B_2(1 - \sin^2(\theta)) - \frac{B_1A_2}{A_1}\right)}} \\
&= \frac{1}{a_1 - a_2}\sqrt{\frac{-B_1}{A_1}}\int_{\theta_1}^{\pi} \frac{d\theta}{\sqrt{-B_1}\sqrt{\frac{A_1B_2 - B_1A_2}{A_1}}\sqrt{1 - \frac{A_1B_2}{A_1B_2 - B_1A_2}\sin^2(\theta)}} \\
&= \frac{1}{a_1 - a_2}\frac{1}{\sqrt{A_1B_2 - B_1A_2}}\int_{\theta_1}^{\pi} \frac{d\theta}{\sqrt{1 - \frac{A_1B_2}{A_1B_2 - B_1A_2}\sin^2(\theta)}}.
\end{aligned}$$

We thus have so far

$$\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} = \frac{1}{a_1 - a_2} \frac{1}{\sqrt{A_1B_2 - B_1A_2}} \int_{\theta_1}^{\pi} \frac{d\theta}{\sqrt{1 - \frac{A_1B_2}{A_1B_2 - B_1A_2}\sin^2(\theta)}}. \tag{3.1.15}$$

Now, using (3.1.14), let us find the value of $\frac{1}{\sqrt{A_1B_2 - B_1A_2}}$. We have

$$\begin{aligned} A_1B_2 - B_1A_2 &= \left(\epsilon_1 - \frac{1}{2}\right)\frac{\epsilon_2}{\epsilon_1 - \epsilon_2}\left(\frac{\frac{1}{2} - \epsilon_2}{\epsilon_1 - \epsilon_2}\right) - \left(\frac{1}{2} - \epsilon_2\right)\frac{\epsilon_1}{\epsilon_1 - \epsilon_2}\left(\frac{\epsilon_1 - \frac{1}{2}}{\epsilon_1 - \epsilon_2}\right) \\ &= \frac{\left(\epsilon_1 - \frac{1}{2}\right)\left(\frac{1}{2} - \epsilon_2\right)(\epsilon_2 - \epsilon_1)}{(\epsilon_1 - \epsilon_2)^2} = \frac{\left(\frac{1}{2} - \epsilon_1\right)\left(\frac{1}{2} - \epsilon_2\right)}{\epsilon_1 - \epsilon_2}. \end{aligned}$$

Subsequently, we obtain

$$\frac{1}{\sqrt{A_1B_2 - B_1A_2}} = \sqrt{\frac{\epsilon_1 - \epsilon_2}{\left(\frac{1}{2} - \epsilon_1\right)\left(\frac{1}{2} - \epsilon_2\right)}}. \tag{3.1.16}$$

In addition, we can also rewrite the coefficient of $\sin^2(\theta)$

$$\frac{A_1B_2}{A_1B_2 - B_1A_2} = \left(\epsilon_1 - \frac{1}{2}\right)\frac{\epsilon_2}{\epsilon_1 - \epsilon_2}\left(\frac{\frac{1}{2} - \epsilon_2}{\epsilon_1 - \epsilon_2}\right)\frac{\epsilon_1 - \epsilon_2}{\left(\frac{1}{2} - \epsilon_1\right)\left(\frac{1}{2} - \epsilon_2\right)} = \frac{\epsilon_2}{\epsilon_2 - \epsilon_1}. \tag{3.1.17}$$

Let us now calculate $\frac{1}{a_1 - a_2}$. By their definitions given in (3.1.9), we have

$$\begin{aligned} a_1 - a_2 &= -\left(\frac{\frac{1}{2}(u_2 + u_3) - 2\epsilon_2 a}{2\left(\frac{1}{2} - \epsilon_2\right)} - \frac{\frac{1}{2}(u_2 + u_3) - 2\epsilon_1 a}{2\left(\frac{1}{2} - \epsilon_1\right)}\right) \\ &= -\frac{\left(\frac{1}{2} - \epsilon_1\right)\left(\frac{1}{2}(u_2 + u_3) - 2\epsilon_2 a\right) + \left(\frac{1}{2} - \epsilon_2\right)\left(2\epsilon_1 a - \frac{1}{2}(u_2 + u_3)\right)}{2\left(\frac{1}{2} - \epsilon_2\right)\left(\frac{1}{2} - \epsilon_1\right)}. \end{aligned}$$

Simplifying yields

$$a_1 - a_2 = -\frac{-\epsilon_2 a - \epsilon_1 \frac{1}{2}(u_2 + u_3) + \epsilon_1 a + \frac{1}{2}\epsilon_2(u_2 + u_3)}{2\left(\frac{1}{2} - \epsilon_2\right)\left(\frac{1}{2} - \epsilon_1\right)} = \frac{(\epsilon_1 - \epsilon_2)\left(\frac{1}{2}(u_2 + u_3) - a\right)}{2\left(\frac{1}{2} - \epsilon_2\right)\left(\frac{1}{2} - \epsilon_1\right)}. \tag{3.1.18}$$

Before continuing on, let us find $\left(\frac{1}{2}-\epsilon_2\right)\left(\frac{1}{2}-\epsilon_1\right)$. From (3.1.6), we have

$$\epsilon_{1,2}=\frac{-t_2\pm\sqrt{t_2^2-4t_1t_3}}{2t_1},$$

where

$$t_1=-4b^2,\quad t_2=2(a^2+b^2+u_2u_3-a(u_2+u_3))\quad\text{and}\quad t_3=\frac{1}{4}(u_2-u_3)^2.$$

Thus, we have

$$\epsilon_1\epsilon_2=\frac{t_3}{t_1}\quad\text{and}\quad\epsilon_1+\epsilon_2=-\frac{t_2}{t_1}.$$

Subsequently, we get

$$\begin{aligned}\left(\frac{1}{2}-\epsilon_2\right)\left(\frac{1}{2}-\epsilon_1\right)&=\epsilon_1\epsilon_2-\frac{1}{2}(\epsilon_1+\epsilon_2)+\frac{1}{4}=\frac{t_3+\frac{1}{2}t_2+\frac{1}{4}t_1}{t_1}\\&=\frac{\frac{1}{4}(u_2-u_3)^2+4\frac{1}{4}(a^2+b^2+u_2u_3-a(u_2+u_3))+\frac{1}{4}(-4b^2)}{-4b^2}\\&=\frac{1}{4}\frac{(u_2-u_3)^2+4(a^2+b^2+u_2u_3-a(u_2+u_3))-4b^2}{-4b^2}\\&=\frac{1}{4}\frac{(u_2+u_3)^2-4a(u_2+u_3)+4a^2}{-4b^2}.\end{aligned}$$

Consequently, we have

$$\left(\frac{1}{2}-\epsilon_2\right)\left(\frac{1}{2}-\epsilon_1\right)=\frac{-\left(\frac{1}{2}(u_2+u_3)-a\right)^2}{4b^2}. \tag{3.1.19}$$

Combining (3.1.19) with (3.1.18), we get

$$\frac{1}{a_1-a_2}=\frac{\frac{1}{2}(u_2+u_3)-a}{(\epsilon_2-\epsilon_1)2b^2}. \tag{3.1.20}$$

We can also combine (3.1.19) with (3.1.16) in order to obtain

$$\frac{1}{\sqrt{A_1B_2 - B_1A_2}} = \sqrt{\frac{\epsilon_1 - \epsilon_2}{\left(\frac{1}{2} - \epsilon_1\right)\left(\frac{1}{2} - \epsilon_2\right)}} = \frac{2b\sqrt{\epsilon_2 - \epsilon_1}}{\left(\frac{1}{2}(u_2 + u_3) - a\right)}.$$

Combining this result with (3.1.20) and (3.1.17), we can complete the computation of the integral in (3.1.15)

$$\begin{aligned}
\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} &= \frac{1}{a_1 - a_2}\frac{1}{\sqrt{A_1B_2 - B_1A_2}} \int_{\theta_1}^{\pi} \frac{d\theta}{\sqrt{1 - \frac{A_1B_2}{A_1B_2 - B_1A_2}\sin^2(\theta)}} \\
&= \frac{\left(\frac{1}{2}(u_2 + u_3) - a\right)}{(\epsilon_2 - \epsilon_1)2b^2} \frac{2b\sqrt{\epsilon_2 - \epsilon_1}}{\left(\frac{1}{2}(u_2 + u_3) - a\right)} \int_{\theta_1}^{\pi} \frac{d\theta}{\sqrt{1 - \frac{\epsilon_2}{\epsilon_2 - \epsilon_1}\sin^2(\theta)}} \\
&= \frac{1}{b\sqrt{\epsilon_2 - \epsilon_1}} \int_{\theta_1}^{\pi} \frac{d\theta}{\sqrt{1 - \frac{\epsilon_2}{\epsilon_2 - \epsilon_1}\sin^2(\theta)}}.
\end{aligned}$$

Finally, we obtain

$$\int_0^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} = \frac{1}{b\sqrt{\epsilon_2 - \epsilon_1}} \left(\int_0^{\pi} \frac{d\theta}{\sqrt{1 - \frac{\epsilon_2}{\epsilon_2 - \epsilon_1}\sin^2(\theta)}} - \int_0^{\theta_1} \frac{d\theta}{\sqrt{1 - \frac{\epsilon_2}{\epsilon_2 - \epsilon_1}\sin^2(\theta)}} \right), \tag{3.1.21}$$

where ϵ_1 and ϵ_2 are given in (3.1.6), b is the coefficient of the imaginary part of the complex roots, $\theta_1 = \arccos\left(\sqrt{\frac{-B_1}{A_1}\frac{a_2}{a_1}}\right)$, A_1, B_1 are given in (3.1.14) and a_1 along with a_2 are given in (3.1.9).

Recall from Section 2.1 that in order for a function $u(x)$ to be a steady state solution of the system in (3.0.1), $u(x)$ needs to satisfy (2.3.4). Using (3.1.21), this is equivalent to

$$\sqrt{D}\frac{1}{b\sqrt{\epsilon_2-\epsilon_1}}\left(\int_0^{\pi}\frac{d\theta}{\sqrt{1-\frac{\epsilon_2}{\epsilon_2-\epsilon_1}\sin^2(\theta)}}-\int_0^{\theta_1}\frac{d\theta}{\sqrt{1-\frac{\epsilon_2}{\epsilon_2-\epsilon_1}\sin^2(\theta)}}\right)=1.$$

3.2 Stability

Now that we have found a new and easier form to numerically calculate the integral in (2.3.4), we can now focus on determining the stability of the steady states of the system in (3.0.1). As mentioned at the beginning of this chapter, we must investigate the stability around the positive steady state solution, $u_s(x)$, that satisfies

$$D_{\text{crit}}u_s''(x)+f(u_s(x))=0. \tag{3.2.1}$$

The associated eigenvalue problem around the steady state solution is

$$-D_{\text{crit}}\phi''(x)-f'(u_s(x))\phi(x)=\delta\phi(x). \tag{3.2.2}$$

Note that $\phi(x)$ must also satisfy the boundary conditions, i.e., we must have $\phi(0)=\phi'(1)=0$. Our first goal is to show that there is an eigenfunction $\phi_0(x)$ whose associated eigenvalue is $\delta=0$. In order to show this, we first cite the implicit function theorem in Banach spaces.

Theorem 3.2.1 (implicit function theorem)**.** *[1, p. 121] Let $U\subset E$ and $V\subset F$ be open, E,F be Banach spaces and $f:U\times V\to G$ be C^r, $r\geq 1$ and where G is a Banach space. For some $x_0\in U$, $y_0\in V$ assume $\boldsymbol{D}_2f(x_0,y_0):F\to G$ is an isomorphism. Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0,y_0)$ and a unique C^r map $g:U_0\times W_0\to V$ such that for all $(x,w)\in U_0\times W_0$,*

$$f(x,g(x,w))=w.$$

This leads to the following proposition.

Proposition 3.2.2. *There exists an eigenfunction $\phi_0(x)$ of $u_s(x)$ whose associated eigenvalue is zero and such that $\phi_0(0) = \phi_0'(1) = 0$.*

Proof: We set $X = \{u(x) \in C^2[0,1] \mid u(0) = u'(1) = 0\}$ and construct the map

$$\psi \colon X \times \mathbb{R} \to C[0,1], \quad (u(x), D) \mapsto Du''(x) + f(u(x)).$$

We know that $\psi(u_s(x), D_{\text{crit}}) = 0$ by (3.2.1). Therefore, if there does not exist a solution $\phi_0(x) \in X$ of

$$-D_{\text{crit}}\phi''(x) - f'(u_s(x))\phi(x) = 0,$$

then, by the implicit function theorem, we can extend (3.2.1) for values of D that are greater than D_{crit}. This is impossible by the very definition of D_{crit}. As a result, there must be an eigenfunction $\phi_0(x)$ of $u_s(x)$ whose associated eigenvalue is 0. Moreover, since $\phi_0(x) \in X$, $\phi_0(x)$ also satisfies the boundary conditions. ∎

Now that we have existence, we want to know the form of $\phi_0(x)$. This leads to an investigation of (3.2.2) for $\delta = 0$.

Proposition 3.2.3. *Let $u_s(x)$ be the steady state solution of* (3.0.1) *that satisfies* (3.2.1)*. Then the general solution of*

$$-D_{crit}\phi''(x) - f'(u_s(x))\phi(x) = 0 \tag{3.2.3}$$

is

$$\phi(x) = c_1 u_s'(x) + c_2 u_s'(x) \int_0^x \frac{d\tau}{(u_s'(\tau))^2}, \tag{3.2.4}$$

where c_1 and c_2 are constants.

Proof: Consider $\phi(x) = u_s'(x)$. Since $u_s(x)$ is a steady state solution of (3.0.1), we can differentiate both sides of (3.0.1) in order to obtain

$$D_{\text{crit}}(u_s'(x))'' + f'(u_s(x))u_s'(x) = 0. \tag{3.2.5}$$

Thus, $u_s'(x)$ is a solution of (3.2.3). Now, since (3.2.3) is of order 2, we are looking for another solution of (3.2.3) which is linearly independent of $u_s'(x)$, i.e., we want

$$\phi(x) = k_1 v(x) + k_2 u_s'(x),$$

for some constants k_1 and k_2. In order to find $v(x)$, we shall use reduction of order. We set $v(x) = k(x)u_s'(x)$. Differentiating both sides then yields

$$v'(x) = k'(x)u_s'(x) + k(x)u_s''(x).$$

Differentiating again yields

$$\begin{aligned} v''(x) &= k''(x)u_s'(x) + k'(x)u_s''(x) + k'(x)u_s''(x) + k(x)u_s'''(x) \\ &= k''(x)u_s'(x) + 2k'(x)u_s''(x) + k(x)u_s'''(x). \end{aligned}$$

Since $v(x)$ must satisfy (3.2.3), we must have

$$D_{\text{crit}}(k''(x)u_s'(x) + 2k'(x)u_s''(x) + k(x)u_s'''(x)) + f'(u_s(x))k(x)u_s'(x) = 0.$$

Rearranging the terms gives us

$$D_{\text{crit}}(k''(x)u_s'(x) + 2k'(x)u_s''(x)) + k(x)D_{\text{crit}}u_s'''(x) + k(x)f'(u_s(x))u_s'(x) = 0.$$

Combining this with (3.2.5) and dividing by D_{crit} allows us to deduce that

$$k''(x)u_s'(x) + 2k'(x)u_s''(x) = 0.$$

Multiplying by $u_s'(x)$ on both sides yields

$$0 = k''(x)(u_s'(x))^2 + 2k'(x)u_s'(x)u_s''(x) = (k'(x)(u_s'(x))^2)'.$$

Thus, $k'(x)(u_s'(x))^2$ must be some constant A, and so

$$k'(x) = \frac{A}{(u_s'(x))^2}.$$

Integrating then shows

$$k(x) = \int_0^x \frac{A d\tau}{(u_s'(\tau))^2} + c,$$

where c is some constant. Therefore, since $v(x) = k(x)u_s'(x)$, we must have that

$$v(x) = \left(\int_0^x \frac{A d\tau}{(u_s'(\tau))^2} + c \right) u_s'(x) = u_s'(x) \int_0^x \frac{A d\tau}{(u_s'(\tau))^2} + c u_s'(x).$$

Consequently, we obtain

$$\phi(x) = c_1 u_s'(x) + c_2 u_s'(x) \int_0^x \frac{d\tau}{(u_s'(\tau))^2}.$$

∎

From now on, whenever we mention $\phi_0(x)$, we mean the eigenfunction associated to the 0 eigenvalue of $u_s(x)$. Let us now find the form of $\phi_0(x)$ using (3.2.4).

Lemma 3.2.4. *The eigenfunction $\phi_0(x)$ is of the form*

$$\phi_0(x) = cu_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2}, \tag{3.2.6}$$

where c is some constant.

Proof: By (3.2.4), evaluating $\phi_0(x)$ at $x = 0$ gives us

$$0 = \phi_0(0) = c_1 u_s'(0) + c_2 u_s'(0)\int_0^0 \frac{d\tau}{(u_s'(\tau))^2} = c_1 u_s'(0),$$

for some constants c_1 and c_2. Thus, we need $c_1 = 0$. As a result, we must have (3.2.6). ∎

It is not obvious that $u_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2}$ is defined at $x = 1$. Let us show that it is defined at $x = 1$ if the integral diverges. Assume that the integral $\int_0^x \frac{d\tau}{(u_s'(\tau))^2}$ diverges. Then, we have

$$\lim_{x\to 1^-} u_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2} = \lim_{x\to 1^-} \frac{\int_0^x \frac{d\tau}{(u_s'(\tau))^2}}{\frac{1}{u_s'(x)}} = \lim_{x\to 1^-} \frac{\frac{1}{(u_s'(x))^2}}{\frac{-u_s''(x)}{(u_s'(x))^2}} = \lim_{x\to 1^-} \frac{-1}{u_s''(x)}.$$

By (3.2.1), we have $u_s''(x) = \frac{-f(u_s(x))}{D_{\text{crit}}}$. Therefore, we must have

$$\lim_{x\to 1^-} u_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2} = \lim_{x\to 1^-} \frac{D_{\text{crit}}}{f(u_s(x))}.$$

Since $\alpha < u_s(1) < 1$, the limit on the right-hand side exists. Consequently, $\phi_0(x)$ is defined at $x = 1$.

Equation (3.2.6) tells us something quite important pertaining to the oscillation of $\phi_0(x)$ which we sum up in the following corollary.

Corollary 3.2.5. *Let $u_s(x)$ be the steady state solution of* (3.0.1) *satisfying* (3.2.1) *and consider the eigenvalue equation* (3.2.2)*. Then the eigenfunction,* $\phi_0(x)$*, with eigenvalue* $\delta = 0$ *is either strictly positive or strictly negative between* 0 *and* 1*.*

Proof: By (3.2.6), we have $\phi_0(x) = cu_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2}$ where c is some constant. Given that $u_s'(x)$ and $\frac{1}{(u_s'(\tau))^2}$ are both strictly positive on $(0,1)$, ϕ_0 is either strictly positive or strictly negative between 0 and 1 depending on the sign of c. ∎

The purpose behind knowing that $\phi_0(x)$ is strictly positive or strictly negative on the interval $(0,1)$ is so that we can use the Sturm comparison theorem to compare the oscillation occurring in $\phi_0(x)$ with eigenfunctions associated with negative eigenvalues. Then, once we show that $\phi_0(x)$ oscillates more than the eigenfunctions associated with negative eigenvalues, we want to utilize the Sturm-Liouville theorem to get a contradiction.

The next two results are the well known Sturm comparison and Sturm-Liouville theorems.

Theorem 3.2.6 (Sturm Comparison Theorem)**.** *[20, p. 1-2] Consider the equations*

$$\frac{d}{dx}\left(a(x)\frac{du}{dx}(x)\right) + c(x)u(x) = 0, \tag{3.2.7}$$

$$\textit{and} \quad \frac{d}{dx}\left(A(x)\frac{dv}{dx}(x)\right) + C(x)v(x) = 0, \tag{3.2.8}$$

on a bounded open interval $x_1 < x < x_2$*, where* $a(x)$*,* $A(x)$*,* $c(x)$ *and* $C(x)$ *are real-valued continuous functions. Suppose* $c(x) \leq C(x)$ *in the bounded interval* $x_1 < x < x_2$*. If there exists a nontrivial real solution* $u(x)$ *of* (3.2.7) *such that* $u(x_1) = u(x_2) = 0$*, then every real solution* $v(x)$ *of* (3.2.8) *has at least one zero in* (x_1, x_2)*.*

Theorem 3.2.7 (Sturm-Liouville Theorem)**.** *[17, p. 270-272] Consider the equation*

$$-\frac{d}{dx}(p(x)\frac{du}{dx}(x)) + q(x)u(x) - \lambda w(x)u(x) = f(x) \tag{3.2.9}$$

on the interval (a, b). Assume we have the following boundary conditions

$$\cos\alpha u(a) - \sin\alpha u'(a) = 0, \tag{3.2.10}$$

$$\cos\beta u(a) - \sin\beta u'(a) = 0. \tag{3.2.11}$$

In addition, suppose that $p(x)$, $p'(x)$, $q(x)$ and $w(x)$ are real-valued and continuous functions on the in interval (a, b) with $p(x), w(x) > 0$ for $x \in (a, b)$. Moreover, define $\mathcal{L}$ to be the differential operator of the form

$$\mathcal{L}(u(x)) := \frac{1}{w(x)}\left(-\frac{d}{dx}(p(x)\frac{du}{dx}(x)) + q(x)u(x)\right).$$

Finally, we define $D(\mathcal{L})$ to be

$$\mathcal{D}(\mathcal{L}) := \{u \in H^2(a, b) \mid (3.2.10) \text{ and } (3.2.11) \text{ are satisfied}\},$$

where $H^2(a, b)$ is the set of all distributions $u \in l^2(a, b)$ such that $\boldsymbol{D}^\alpha u \in l^2(a, b)$ for $|\alpha| \leq 2$. The following hold :

1. *The eigenvalues of $(\mathcal{D}(\mathcal{L}), \mathcal{L})$ are real.*
2. *The eigenvalues of $(\mathcal{D}(\mathcal{L}), \mathcal{L})$ are bounded below by a constant $\lambda_G \in \mathbb{R}$.*
3. *Eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal in $l^2_w(a, b)$.*
4. *Each eigenvalue has multiplicity one.*

We now have all the tools necessary to show that all of the eigenvalues associated with the saddle node steady state solution $u_s(x)$ are nonnegative which we prove in the following theorem.

Theorem 3.2.8. *Let $u_s(x)$ be the steady state solution of (3.0.1) satisfying (3.2.1). Then all of the eigenvalues associated with $u_s(x)$ are greater than or equal to 0.*

Proof: Assume that there is an eigenfunction, $\phi_1(x)$, of $u_s(x)$ associated with a negative eigenvalue, δ_1, i.e., $\phi_1(x)$ is a solution of (3.2.2) with $\delta_1 < 0$. In addition, let $\phi_0(x)$ be the eigenfunction associated to the 0 eigenvalue of $u_s(x)$. We have the following equations

$$D\phi_1''(x) + (f'(u_s(x)) + \delta_1)\phi_1(x) = 0,$$
$$D\phi_0''(x) + f'(u_s(x))\phi_0(x) = 0.$$

We clearly have $f'(u_s(x))+\delta_1 < f'(u_s(x))$, and so, by the Sturm comparison theorem, $\phi_0(x)$ oscillates more than $\phi_1(x)$. By Corollary 3.2.5, $\phi_0(x)$ is strictly negative or strictly positive on $[0, 1]$. Subsequently, ϕ_1 must either be strictly negative or strictly positive on $(0, 1)$. (Suppose there exists $x_1 \in (0, 1)$ such that $\phi_1(x_1) = 0$. Then, there must be $x_2 \in (0, x_1)$ such that $\phi_0(x_2) = 0$ by the Sturm comparison theorem. However, this is a contraction per Corollary 3.2.5.)

Now, in the context of the SturmLiouville theorem, we have $w(x) = 1$. Thus, by the third result of the Sturm-Liouville theorem (Theorem 3.2.7 (3)), the functions $\phi_0(x)$ and $\phi_1(x)$ are orthogonal in $l^2(0, 1)$, i.e.,

$$\int_0^1 \phi_0(x)\phi_1(x)dx = 0.$$

This is clearly a contradiction since both $\phi_0(x)$ and $\phi_1(x)$ do not change sign over the interval $[0, 1]$. ∎

Theorem 3.2.9. *Let $u_s(x)$ be the steady state solution of* (3.0.1) *satisfying* (3.2.1). *After the bifurcation occurs, the zero eigenvalue splits into two eigenvalues of opposite signs.*

Proof: Let $G(\lambda)$ be as in (3.0.2). The time of transit of steady states for different values of λ is given by $G(\lambda)$ when $D = 1$. When D is different from 1, the transit time is given by

$$T(D, \lambda) := \sqrt{D}G(\lambda).$$

Define λ_{crit} to be the value of λ where G achieves its minimum, i.e.,

$$G_{\min} := G(\lambda_{\text{crit}}) := \min_{0<\lambda<\frac{1}{6}(1-2\alpha)} G(\lambda).$$

As mentioned at the beginning of this chapter, $G(\lambda)$ is concave up with a single critical point. Therefore, the Taylor series approximation of $G(\lambda)$ around λ_{crit} is

$$G(\lambda) = G_{\min} + \gamma(\lambda - \lambda_{\text{crit}})^2 + ... \tag{3.2.12}$$

Now, define D_{crit} to be the solution of

$$T(D_{\text{crit}}, \lambda) = \sqrt{D_{\text{crit}}}G_{\min} = 1.$$

So,

$$D_{\text{crit}} = \frac{1}{G_{\min}^2}. \tag{3.2.13}$$

Now, let $\omega \geq 0$ and set $D = D_{\text{crit}} - \omega$. Let us solve $\sqrt{D}G(\lambda) = 1$ for $\lambda = \lambda(\omega)$. Using the Taylor approximation given in (3.2.12), we obtain

$$\sqrt{D_{\text{crit}} - \omega}\left(G_{\min} + \gamma(\lambda - \lambda_{\text{crit}})^2 + ...\right) = 1.$$

Therefore, we get

$$G_{\min} + \gamma(\lambda - \lambda_{\text{crit}})^2 + ... = \frac{1}{\sqrt{D_{\text{crit}} - \omega}}.$$

Combining this expression with the evaluation of the Taylor series approximation of the function $\frac{1}{\sqrt{x}}$ around D_{crit} at $D_{\text{crit}} - \omega$ then yields

$$G_{\min} + \gamma(\lambda - \lambda_{\text{crit}})^2 + ... = \frac{1}{\sqrt{D_{\text{crit}}}} + \frac{\omega}{2D_{\text{crit}}^{\frac{3}{2}}}.$$

Using (3.2.13) and then solving for λ gives us

$$\lambda = \lambda_{\text{crit}} \pm \frac{\sqrt{\omega}}{\sqrt{2\gamma} D_{\text{crit}}^{\frac{3}{4}}} + ... \tag{3.2.14}$$

Let $\sigma = \frac{1}{\sqrt{2\gamma} D_{\text{crit}}^{\frac{3}{4}}}$ and set

$$v(u, \lambda) = \frac{1}{\sqrt{D}} \sqrt{\frac{u^4}{2} - \frac{2}{3}(1+\alpha)u^3 + \alpha u^2 + \lambda},$$

with $u_x = v$, and $u(0) = u_x(1) = 0$. When $u = 0$, we have that $v(0, \lambda) = \sqrt{\frac{\lambda}{D}}$. So, combining this with (3.2.14), we obtain the following

$$v(0, \lambda) = \sqrt{\frac{\lambda}{D}} = \frac{\sqrt{\lambda_{\text{crit}} \pm \sigma\sqrt{\omega}}}{\sqrt{D_{\text{crit}} - \omega}} + ...$$

Using a Taylor series expansion around $\omega = 0$ yields

$$v(0, \lambda) = \frac{\sqrt{\lambda_{\text{crit}} \pm \sigma\sqrt{\omega}}}{\sqrt{D_{\text{crit}} - \omega}} + ... = \frac{\sqrt{\lambda_{\text{crit}}}}{\sqrt{D_{\text{crit}}}} \pm \frac{\sigma}{2\sqrt{\lambda_{\text{crit}}} D_{\text{crit}}} \sqrt{\omega} + ...$$

Note that since $u_s(x)$ is the saddle node solution, we must have $u_s'(0) = \frac{\sqrt{\lambda_{\text{crit}}}}{\sqrt{D_{\text{crit}}}}$.

Now, set

$$u(x) = u_s(x) + \sqrt{\omega} p(x), \tag{3.2.15}$$

where $p(0) = p'(1) = 0$ and $p'(0) = \frac{\sigma}{2\sqrt{\lambda_{\text{crit}}} D_{\text{crit}}}$. Let $f(u) = u(1-u)(u-\alpha)$ and consider the equation

$$(D_{\text{crit}} - \omega)u'' + f(u) = 0. \tag{3.2.16}$$

Replacing u with (3.2.15) gives us

$$(D_{\text{crit}} - \omega)(u_s'' + \sqrt{\omega}p'') + f(u_s(x) + \sqrt{\omega}p(x)) = 0.$$

Combining the above equation with the Taylor series of $f(u)$ around u_s and evaluated at $u_s(x) + \sqrt{\omega}p(x)$, we obtain

$$D_{\text{crit}}u_s'' + D_{\text{crit}}\sqrt{\omega}p'' - \omega u_s'' - \omega\sqrt{\omega}p'' + f(u_s) + f'(u_s)\sqrt{\omega}p(x) = 0.$$

Considering that u_s satisfies $D_{\text{crit}}u_s'' + f(u_s) = 0$ and only keeping the higher order terms, we get

$$D_{\text{crit}}p'' + f'(u_s)p = 0.$$

By Proposition 3.2.3, we know that p must be of the form

$$p(x) = c_1 u_s'(x) + c_2 u_s'(x) \int_0^x \frac{d\tau}{(u_s'(\tau))^2},$$

where c_1 and c_2 are constants. However, as $p(0) = 0$, we must have that $c_1 = 0$. In addition, we must have $p'(0) = \frac{\sigma}{2\sqrt{\lambda_{\text{crit}}D_{\text{crit}}}}$. Thus, as $p'(x) = c_2\left(u_s''(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2} + \frac{1}{u_s'(x)}\right)$ we get

$$\frac{\sigma}{2\sqrt{\lambda_{\text{crit}}D_{\text{crit}}}} = \frac{c_2}{u_s'(0)} = c_2\frac{\sqrt{D_{\text{crit}}}}{\sqrt{\lambda_{\text{crit}}}}.$$

Therefore, we must have

$$c_2 = \frac{\sigma}{2D_{\text{crit}}},$$

and so,

$$p(x) = \frac{\sigma}{2D_{\text{crit}}}u_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2}.$$

Let us now consider the linearized stability equation associated with (3.2.16)

$$-(D_{\text{crit}} - \omega)\psi'' - f'(u)\psi = \delta\sqrt{\omega}\psi. \tag{3.2.17}$$

We let $\psi = \phi_0 + \sqrt{\omega}\eta$ with $\eta(0) = \eta'(1) = 0$ and replace it in (3.2.17) :

$$-(D_{\text{crit}} - \omega)(\phi_0'' + \sqrt{\omega}\eta'') - f'(u)(\phi_0 + \sqrt{\omega}\eta) = \delta\sqrt{\omega}(\phi_0 + \sqrt{\omega}\eta).$$

Approximating $f'(u)$ with its Taylor series around u_s and evaluating at $u = u_s + \sqrt{\omega}p$ yields

$$-(D_{\text{crit}} - \omega)(\phi_0'' + \sqrt{\omega}\eta'') - (f'(u_s) + f''(u_s)\sqrt{\omega}p)(\phi_0 + \sqrt{\omega}\eta) = \delta\sqrt{\omega}(\phi_0 + \sqrt{\omega}\eta).$$

Only keeping the higher order terms gives us

$$-D_{\text{crit}}(\phi_0'' + \sqrt{\omega}\eta'') - f'(u_s)\phi_0 - f'(u_s)\sqrt{\omega}\eta - f''(u_s)\sqrt{\omega}p\phi_0 = \delta\sqrt{\omega}\phi_0.$$

Noting that $-D_{\text{crit}}\phi_0'' - f'(u_s)\phi_0 = 0$, then dividing by $\sqrt{\omega}$ and rearranging gives us

$$-D_{\text{crit}}\eta'' - f'(u_s)\eta = (f''(u_s)p + \delta)\phi_0. \tag{3.2.18}$$

By Proposition 3.2.3, we know that the solutions to the homogeneous equation of (3.2.18) are of the form $c_1 u_s'(x) + c_2 u_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2}$ for some c_1 and c_2. Using the variation of parameters method, we look for solutions of the form $A(x)u_s'(x) + B(x)u_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau))^2}$ for some $A(x)$ and $B(x)$. Using known formulas, we get that

$$A(x) = -\int_0^x \phi_0^2(\delta + f''(u_s)p)dx + k_1 \quad \text{and} \quad B(x) = \int_0^x u_s'\phi_0(\delta + f''(u_s)p)dx + k_2,$$

where k_1 and k_2 are some constants.

We want to find the value of δ. In order to do so, we shall multiply each side of (3.2.18) by ϕ_0 and then integrate between 0 and 1. Let us first calculate the left hand side :

$$\int_0^1 (-D_{\text{crit}}\eta'' - f'(u_s)\eta)\phi_0 dx = \int_0^1 -D_{\text{crit}}\eta''\phi_0 dx - \int_0^1 f'(u_s)\eta\phi_0 dx$$
$$= \int_0^1 -D_{\text{crit}}\eta''\phi_0 dx + \int_0^1 D_{\text{crit}}\eta\phi_0'' dx,$$

where the last equality follows from $-D_{\text{crit}}\phi_0'' - f'(u_s)\phi_0 = 0$. Using integration by parts then yields

$$\int_0^1 (-D_{\text{crit}}\eta'' - f'(u_s)\eta)\phi_0 dx = D_{\text{crit}} \left((\eta\phi_0')|_0^1 - (\eta'\phi_0)|_0^1 \right) = 0.$$

Hence, we obtain

$$0 = \int_0^1 (f''(u_s)p + \delta)\phi_0^2 dx = \delta \int_0^1 \phi_0^2 dx + \int_0^1 \phi_0^2 f''(u_s) p dx.$$

Consequently, we get that

$$\delta = -\frac{\int_0^1 \phi_0^2 f''(u_s) p dx}{\int_0^1 \phi_0^2 dx}. \tag{3.2.19}$$

Now, set $u(x)$ to be

$$u(x) = u_s(x) - \sqrt{\omega} p(x) = u_s(x) + \sqrt{\omega}(-p(x)), \tag{3.2.20}$$

with $p(0) = p'(1) = 0$ and $p'(0) = \frac{-\sigma}{2\sqrt{\lambda_{\text{crit}} D_{\text{crit}}}}$. Considering the same equation as before, (3.2.16), and replacing u with (3.2.20) within it, we will now obtain

$$-D_{\text{crit}} p'' - f'(u_s) p = 0.$$

This is the same equation as before. Thus, we must have

$$p(x) = \frac{\sigma}{2D_{\text{crit}}} u_s'(x) \int_0^x \frac{d\tau}{(u_s'(\tau))^2}.$$

Let us now consider the linearized stability equation associated with the previous equation :

$$-(D_{\text{crit}} - \omega)\psi'' - f'(u)\psi = -\delta'\sqrt{\omega}\psi. \tag{3.2.21}$$

This time we set $\psi = \phi_0 - \sqrt{\omega}\eta$ and replace it in (3.2.21) :

$$-(D_{\text{crit}} - \omega)(\phi_0'' - \sqrt{\omega}\eta'') - f'(u)(\phi_0 - \sqrt{\omega}\eta) = -\delta'\sqrt{\omega}(\phi_0 - \sqrt{\omega}\eta).$$

Using the same steps as we previously did, we obtain

$$-D_{\text{crit}}\eta'' - f'(u_s)\eta = (f''(u_s)p + \delta')\phi_0.$$

Similarly, we obtain solutions $\eta(x) = A(x)u_s'(x) + B(x)u_s'(x)\int_0^x \frac{d\tau}{(u_s'(\tau)^2}$ with

$$A(x) = -\int_0^x \phi_0^2(\delta' + f''(u_s)p)dx + k_1 \quad \text{and} \quad B(x) = \int_0^x u_s'\phi_0(\delta' + f''(u_s)p)dx + k_2,$$

where k_1 and k_2 are some constants.

Repeating the same procedure as before, we obtain

$$\delta' = -\frac{\int_0^1 \phi_0^2 f''(u_s)pdx}{\int_0^1 \phi_0^2 dx}. \tag{3.2.22}$$

Consequently, we can see that the eigenvalues δ and δ' have the same sign. Therefore, when the steady state $u_s(x)$ splits into two steady states, one of these steady states will have strictly positive eigenvalues whereas the other steady state will have one eigenvalue that is negative and the others will be positive. ∎

The results shown in this chapter indicate that as long as the diffusion parameter is small enough, stable patterned steady states exist. When modelling the population of a species in an ecosystem using mixed boundary conditions, this means that the population will be able to flourish as long as they do not spread towards the hostile environment faster than they reproduce.

3.3 Extension to the Dirichlet Boundary Conditions

During this chapter we analysed the one-component reaction-diffusion equation with mixed boundary conditions. However, we also inadvertently studied the same system with Dirichlet (or first-type) boundary conditions. Indeed, we looked at orbits whose starting point was on the v-axis with $v > 0$ and ending point was on the u-axis with $\alpha < u < 1$. These orbits are described by $v(x)$ in (2.3.3). In comparison, with Dirichlet boundary conditions, the orbits are described by $v^2(x)$. They start on the v-axis with $v > 0$. Then, instead of stopping on the u-axis, the orbit continues and loops around the center point, taking the reflection of the path it just took back to the v-axis. See Figure 3.3 for a visualisation.

If $G^*(\lambda)$ evaluates the time of transit of these orbits for different values of λ, then we have $G^*(\lambda) = 2G(\lambda)$ for all $\lambda \in (0, \frac{1}{2}(1 - 2\alpha))$. Since $G^*(\lambda)$ is simply a multiple of $G(\lambda)$, the same analysis that we have done in this chapter can be done on the system with Dirichlet boundary conditons as well. The results will be the same, modulo a factor of 2 appearing in certain instances. Thus, we also have a saddle-node bifurcation.

3.4 Thoughts on the Parameter α

Throughout this chapter we have discussed the stability of steady states of the system given in (3.0.1) for a fixed value of α. We have seen that there is a saddle-node bifurcation as the coefficient D varies. Naturally, one could wonder about the effects

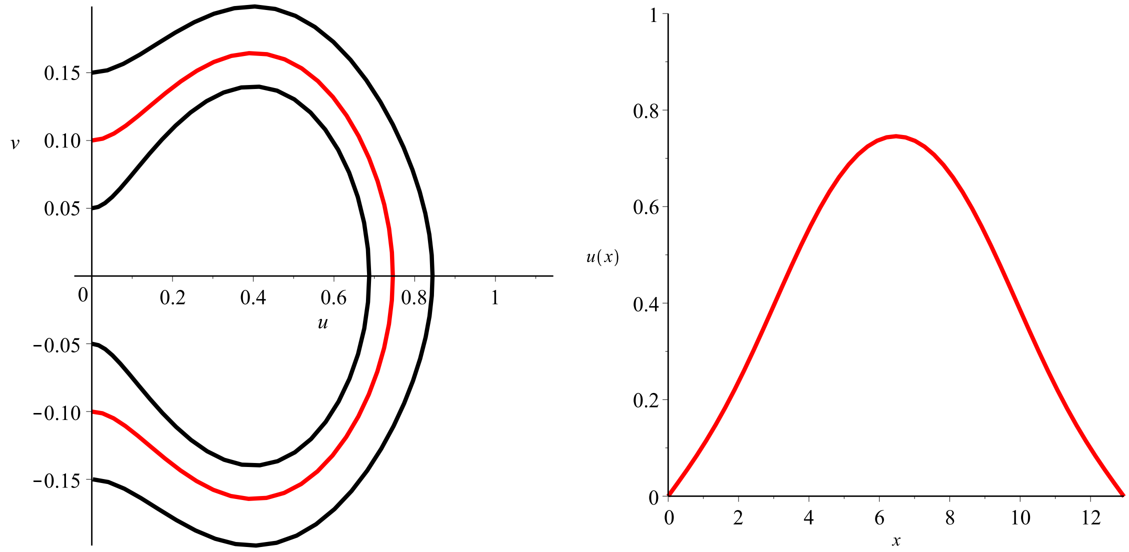

Figure 3.3: On the left we have an example of orbits of the 1-component reaction-diffusion equation with Dirichlet boundary conditions. In this instance, we have $\alpha = \frac{2}{5}$ and $D = 1$. On the right, we show the trajectory of an orbit as a function of x.

of α on the system. In other words, what role does α play on the stability of patterned steady states of our system? How does α affect the existence of stable patterned steady states? We offer our thoughts on the matter.

In order to do some analysis, we consider $G(\lambda)$ (given in (3.0.2)) to be a function of both α and λ: $G(\lambda) = G(\alpha, \lambda)$. As a consequence, we must also consider D_{crit} (given at the beginning of this chapter) to be a function of α: $D_{\text{crit}} = D_{\text{crit}}(\alpha)$. Figure 3.1 shows $G(\alpha, \lambda)$ for different values of α. From this figure, it seems that $G(\alpha, \lambda)$ is an increasing function of α. If this is true for all possible values of α, then for $\alpha_1 \leq \alpha_2$ we have $\min_{\frac{1}{6}(1-2\alpha_1)} G(\alpha_1, \lambda) \leq \min_{\frac{1}{6}(1-2\alpha_2)} G(\alpha_2, \lambda)$. Therefore,

$$D_{\text{crit}}(\alpha_1) = \frac{1}{(\min_{0<\lambda<\frac{1}{6}(1-2\alpha_1)} G(\alpha_1, \lambda))^2} \geq \frac{1}{(\min_{0<\lambda<\frac{1}{6}(1-2\alpha_2)} G(\alpha_2, \lambda))^2} = D_{\text{crit}}(\alpha_2).$$

From this, we can determine that for larger values of α is, smaller values of D are required to have non-trivial stable solutions. Alternatively, the smaller α is, the larger D needs to be. This simply means the following. Suppose we fix D such that $D = D_{\text{crit}}(\alpha_2)$ for some $\alpha_2 \in (0, \frac{1}{2})$. Then, for any $\alpha_1 \leq \alpha_2$, stable patterned steady states exist since $D_{\text{crit}}(\alpha_2) \leq D_{\text{crit}}(\alpha_1)$.

Suppose that we take our initial condition, $u_0(x)$, to be the straight line whose slope is equal to $v(0)$ where $v(x)$ is given in (2.3.3), i.e., $u_0(x) = \sqrt{\frac{\lambda}{D}}x$. Suppose that $D < D_{\text{crit}}(\alpha)$ and let $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ be such that $G(\alpha, \lambda_{1,2}(\alpha)) = 1$ and $\lambda_1(\alpha) < \lambda_2(\alpha)$. If we want our solution to eventually morph into the stable patterned steady state solution, then we need $\lambda > \lambda_1(\alpha)$. We claim that the smaller α is, the smaller $\lambda_1(\alpha)$ becomes, and so, the lower bound on λ decreases. Indeed, suppose $\alpha_1 < \alpha_2$ and assume that $D < D_{\text{crit}}(\alpha_2)$. We need to show that $\lambda_1(\alpha_1) < \lambda_1(\alpha_2)$. We have

$$G(\alpha_1, \lambda_1(\alpha_2)) < G(\alpha_2, \lambda_1(\alpha_2)) = 1,$$

where the first inequality follows from $G(\alpha, \lambda)$ being an increasing function of α. Thus,

$$G(\alpha_1, \lambda_1(\alpha_2)) < 1.$$

Since $G(\alpha, \lambda)$ is concave up, we must have

$$\lambda_1(\alpha_1) < \lambda_1(\alpha_2) < \lambda_2(\alpha_1)$$

as required.

To put what we have done in this section into perspective, suppose our system models the population of a species within an ecosystem. Then, in terms of the possibility of reaching a non-zero stable population, species with a lower extinction threshold have access to a higher possible dispersal speeds. They also require a lower minimum initial population (for initial conditions of the form $u_0(x) = \sqrt{\frac{\lambda}{D}}$) to reach a patterned stable population. In addition, suppose we have two distinct species which have the same movement speed but different extinction thresholds. Furthermore, assume that the species whose extinction threshold is higher can reach a stable population. Then, we know that it is also possible for the other species to reach a stable population for suitable initial conditions.

To add on what was said in the last paragraph, species whose extinction threshold is lower also have access to a spatially smaller ecosystem. Indeed, recall from Chapter 2.2 that studying our system on the interval $[0, L]$ is equivalent to studying the system on the unit interval. The only exception is that our diffusion parameter changes to $\frac{D}{L^2}$. Thus, increasing the diffusion parameter can also be achieved by decreasing the size of the ecosystem.

In conclusion, a smaller α seems to be beneficial in terms of existence of stable patterned steady states of the system. It seems that it allows the use of a wider range of the coefficient D and slope of our initial condition.

3.5 Example

Consider the reaction-diffusion system given in (3.0.1) with $\alpha = \frac{1}{3}$ and define $G(\lambda)$ as in (3.0.2). The graph of $G(\lambda)$ is depicted in Figure 3.1. The minimum of $G(\lambda)$ for $\lambda \in (0, \frac{1}{6}(1-2\alpha)) = (0, \frac{1}{18})$ is approximately 5.2187 and is achieved at $\lambda_{\text{crit}} \approx 0.0329$. Thus, $D_{\text{crit}} \approx \frac{1}{(5.2187)^2} \approx 0.0367$.

If $D = \frac{1}{36} < D_{\text{crit}}$, we have two steady states. See Figure 3.4. They appear when $\lambda \approx 0.0094$ and $\lambda \approx 0.0523$. Recall that orbits are described by $v(x)$ in (2.3.3). Thus, when $\lambda \approx 0.0094$, the orbit begins at the point $(0, \sqrt{\frac{\lambda}{D}}) = (0, 0.5817)$ and ends at the point $(0.6168, 0)$. When $\lambda \approx 0.0523$, the orbit begins at the point $(0, \sqrt{\frac{\lambda}{D}}) = (0, 1.3722)$ and ends at the point $(0.9255, 0)$. The first steady state, the one associated with $\lambda \approx 0.0094$, is unstable while the second steady state is stable. Therefore, for initial conditions of the form $u_0(x) = \sqrt{\frac{\lambda}{D}}x$ where $\lambda \in (0, \frac{1}{18})$, solutions will be pushed back towards 0 if $\lambda < 0.0094$. If $\lambda > 0.0094$, then solutions will tend towards $u(x)$

where $u'(x)$ is the second steady state solution. See Figure 3.5.

If $D = D_{\text{crit}} \approx 0.0367$, we have one steady state and it appears when $\lambda = \lambda_{\text{crit}} \approx 0.0329$. See Figure 3.6. The steady state starts at the point $\left(0, \sqrt{\frac{\lambda}{D_{\text{crit}}}}\right) = (0, 0.9468)$ and ends at the point $(0.7732, 0)$. For initial conditions of the form $u_0(x) = \sqrt{\frac{\lambda}{D_{\text{crit}}}}x$, if $\lambda < 0.0329$ then solutions will be dragged back down to 0. However, if $\lambda > 0.0329$, then solutions will be pulled towards $u(x)$ where $u'(x)$ is the steady state solution of our system. See Figure 3.7.

If $D = \frac{1}{25} > D_{\text{crit}}$, we have no steady states. See Figure 3.8. The coefficient D is not small enough to scale $G(\lambda)$ down to the line $\lambda = 1$. As a result, the steady states are lost. No matter what the slope of initial conditions is (for initial conditions of the form $u_0(x) = \sqrt{\frac{\lambda}{D}}x$), solutions will eventually be pulled down to 0 and become flat. See Figure 3.9.

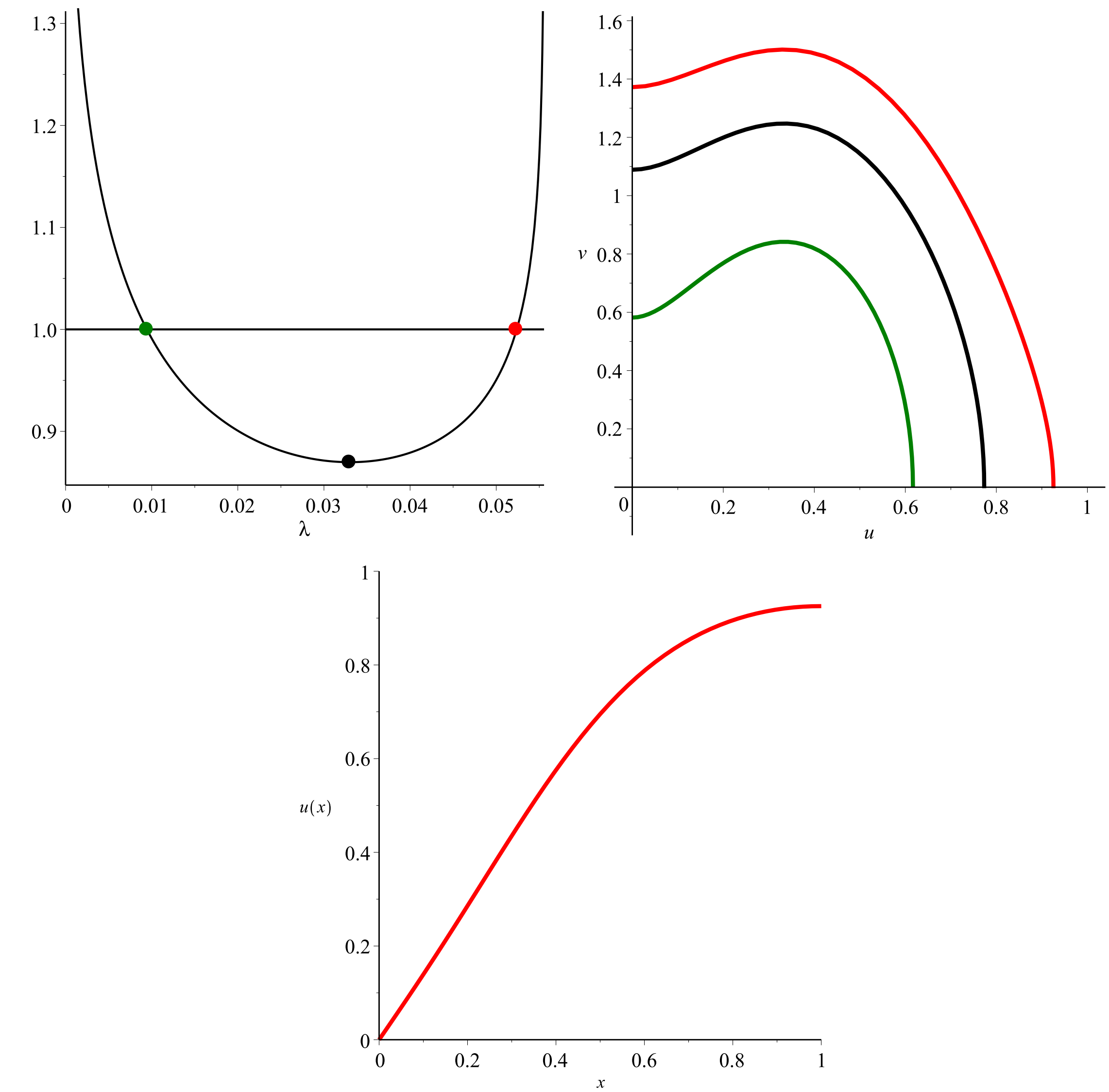

Figure 3.4: At the top left, we have the graph of $\frac{1}{6}G(\lambda)$. It intersects the line $\frac{1}{6}G(\lambda) = 1$ twice, and so we have two steady states. At the top right, we have some orbits in the phase-plane corresponding to our system with $D = \frac{1}{36}$. The red and green orbits correspond to the derivatives of the steady states as a function of the steady states. In other words, the orbits of the steady states in the phase-plane. The red orbit is associated with the stable steady state whereas the green orbit is associated to the unstable steady state. On the bottom, we have the trajectory of the stable steady state as a function of x. This is what we expect solutions of our system to eventually look like, given the right initial condition.

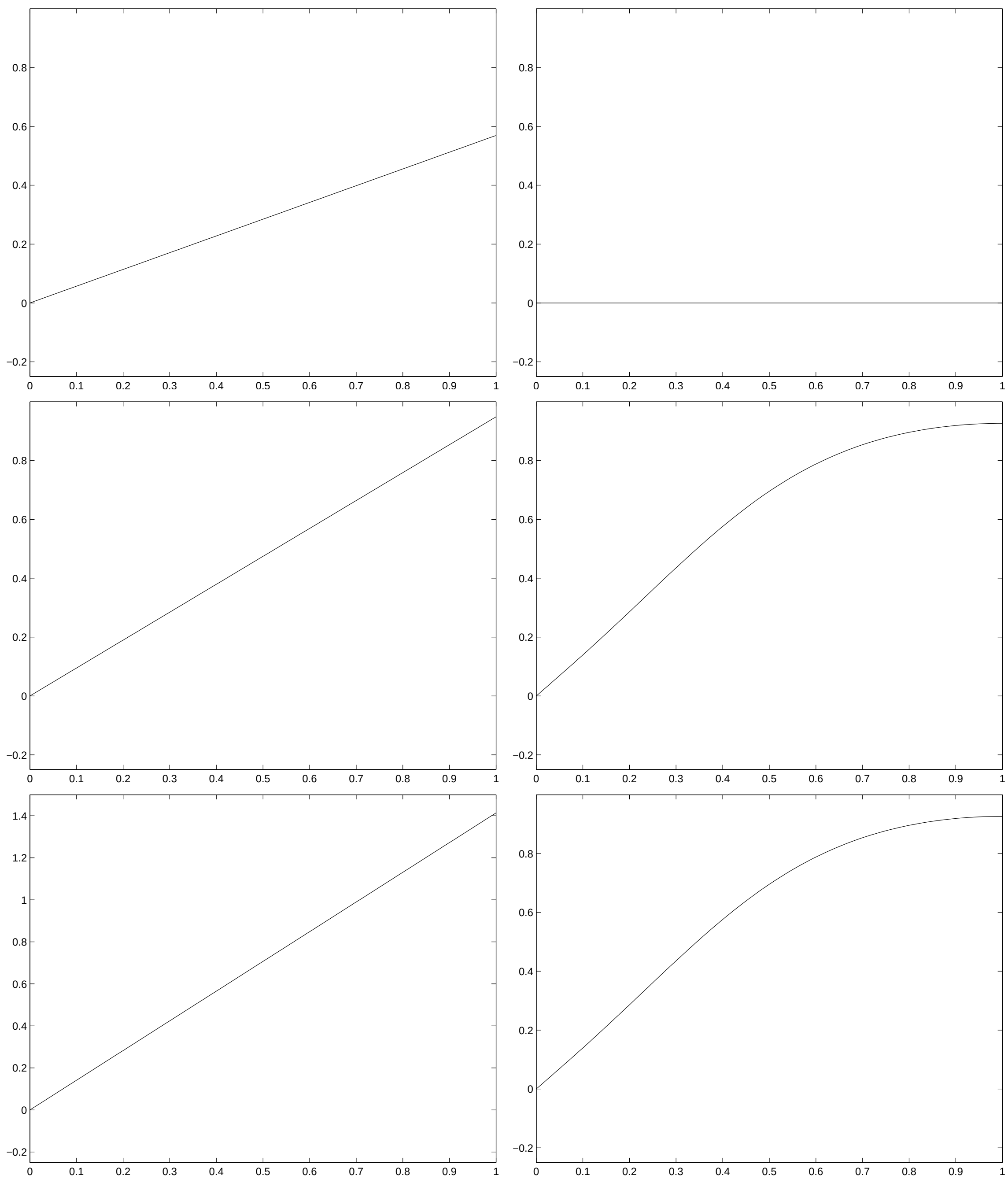

Figure 3.5: Each row has the graph of an initial condition on the left and the graph of the solution of (3.0.1) at $t = 1000$ on the right. Every initial condition is of the form $u_0(x) = \sqrt{\frac{\lambda}{D}}x$. From top to bottom, the values of λ are 0.009, 0.025 and 0.0555. In all cases we have $D = \frac{1}{36}$. We used the function "pdepe" in MATLAB to calculate these solutions.

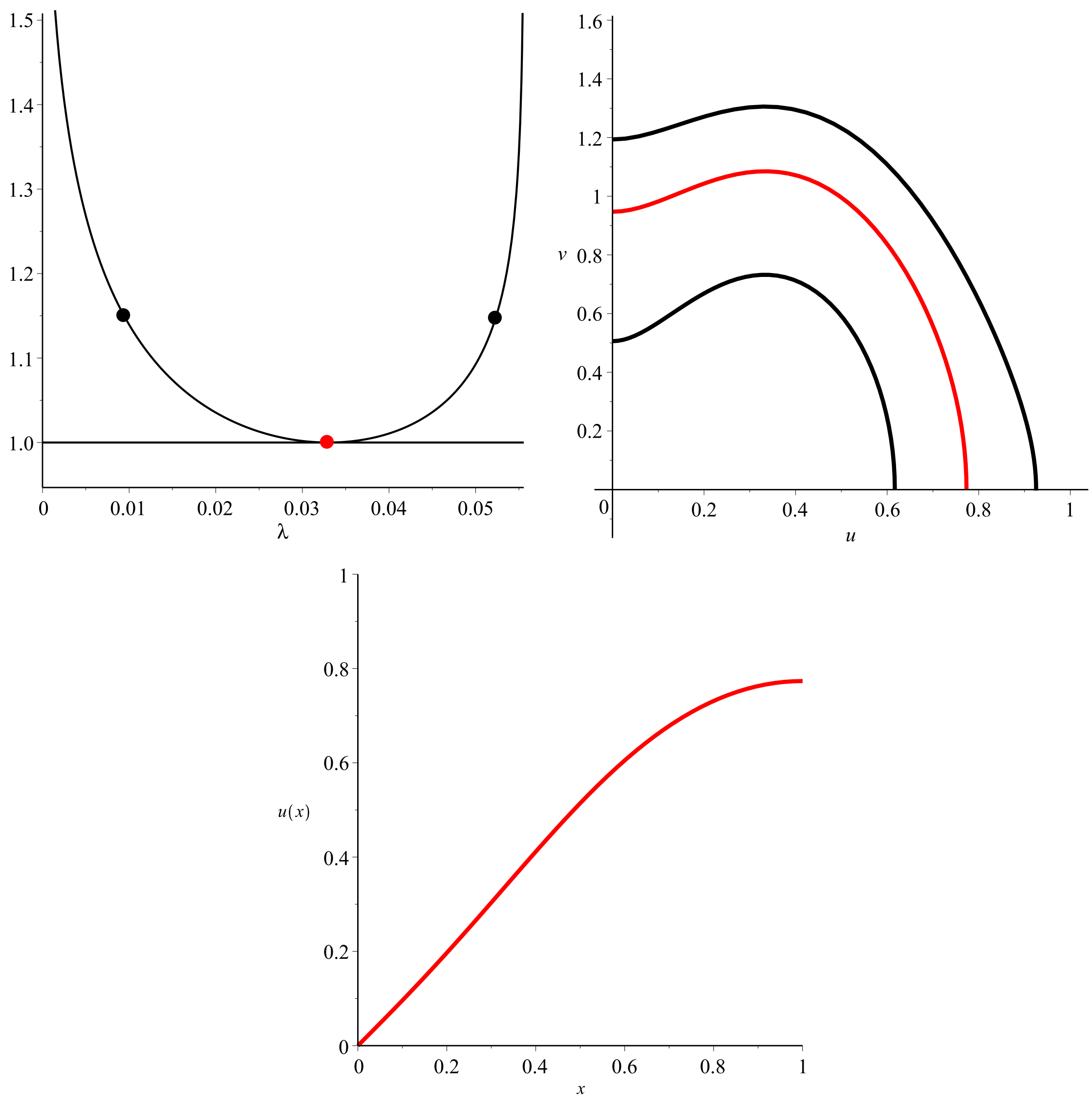

Figure 3.6: At the top left, we have the graph of $\sqrt{0.0367}G(\lambda)$. It intersects the line $\sqrt{0.0367}G(\lambda) = 1$ once, and so we only have one steady state. At the top right, we have some orbits in the phase-plane corresponding to our system with $D = 0.0367$ (D_{crit}). The red orbit corresponds to the steady state of our system. On the bottom, we have the steady state of our system. Given a good choice of an initial condition, this is what solutions of our system should eventually morph into.

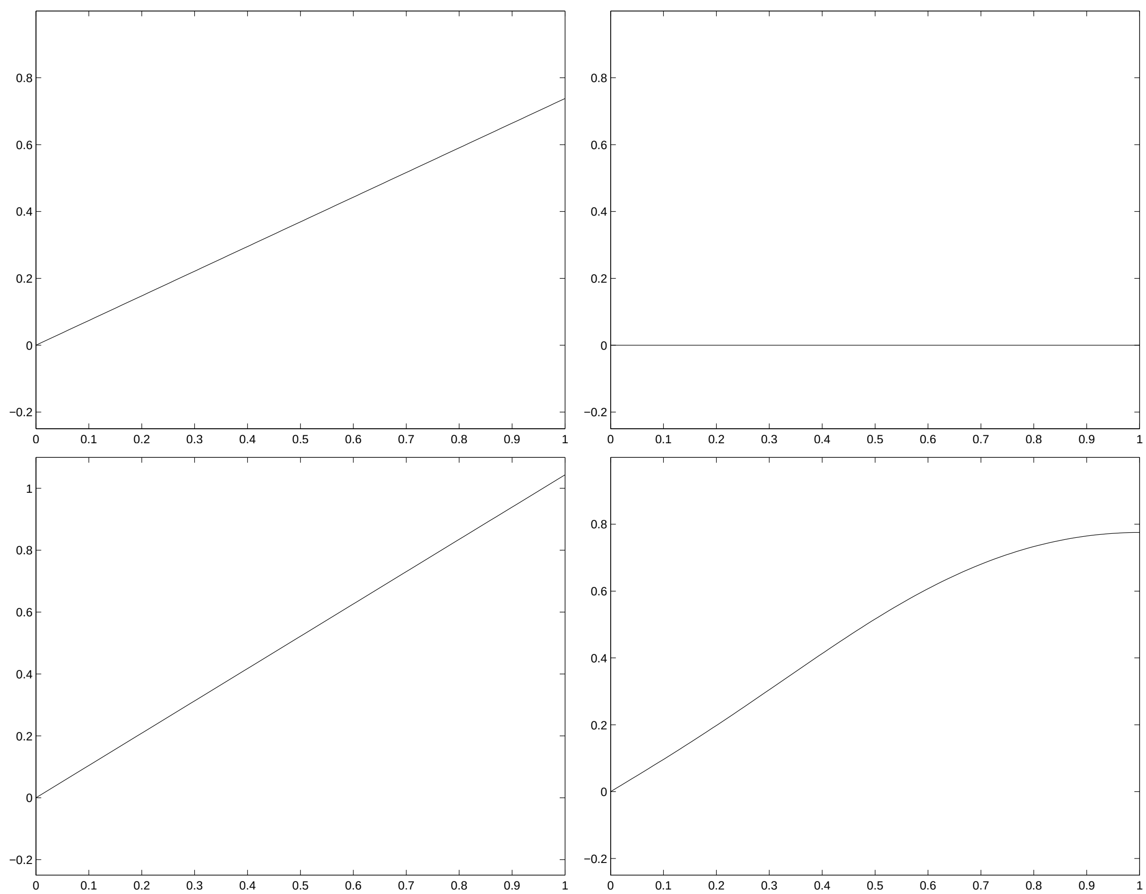

Figure 3.7: Each row has the graph of an initial condition on the left and the graph of the solution of (3.0.1) at $t = 1000$ on the right. Every initial condition is of the form $u_0(x) = \sqrt{\frac{\lambda}{D_{\mathrm{crit}}}}x$. From top to bottom, the values of λ are 0.02 and 0.04. We used the function "pdepe" in MATLAB to calculate these solutions.

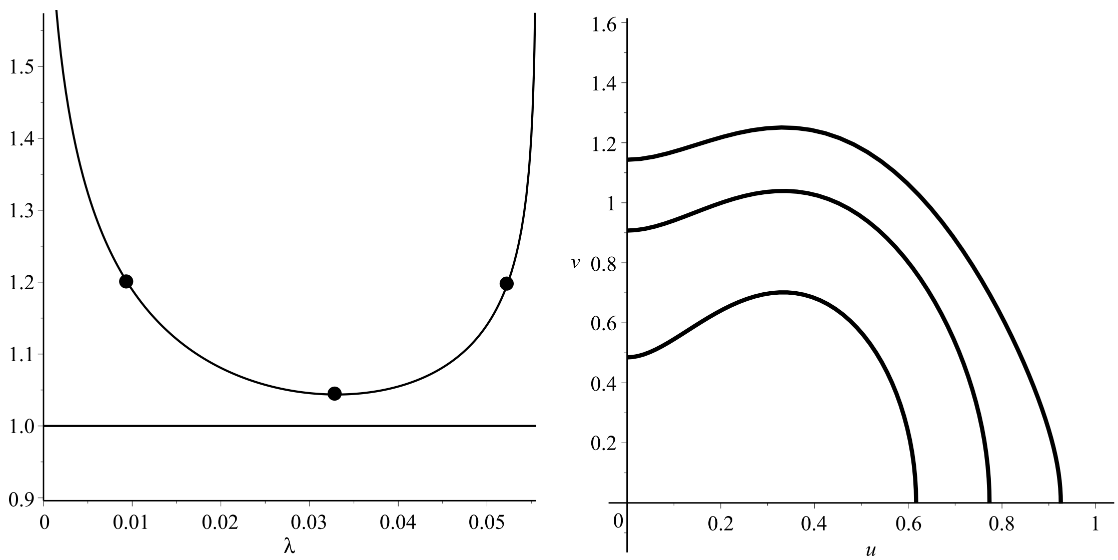

Figure 3.8: On the left, we have the graph of $\frac{1}{5}G(\lambda)$. It does not intersect the line $\frac{1}{5}G(\lambda) = 1$. Thus, there are no steady states. On the right, we have some orbits in the phase-plane corresponding to our system with $D = \frac{1}{25}$.

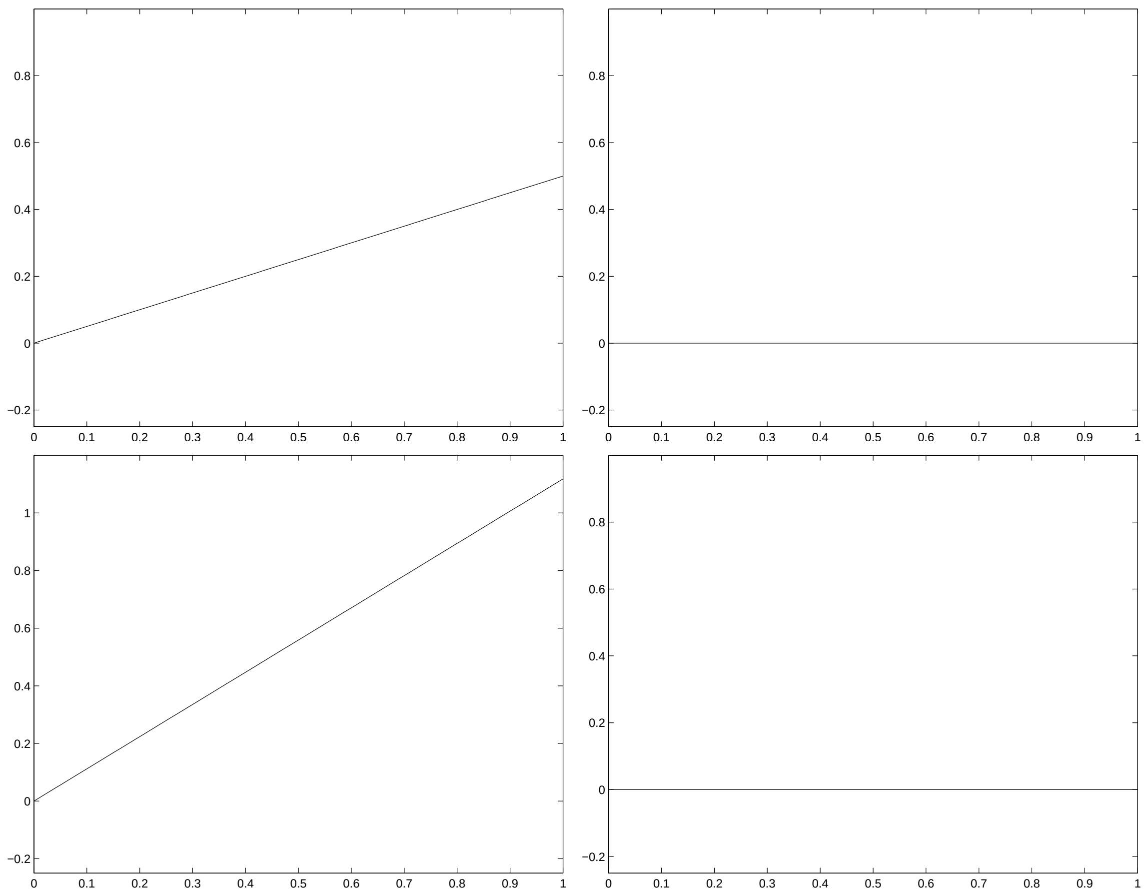

Figure 3.9: Each row has the graph of an initial condition on the left and the graph of the solution of (3.0.1) at $t = 1000$ on the right. Every initial condition is of the form $u_0(x) = \sqrt{\frac{\lambda}{D}}x$. From top to bottom, the values of λ are 0.01 and 0.05. In all cases we have $D = \frac{1}{25}$. We used the function "pdepe" in MATLAB to calculate these solutions.

Chapter 4

No-flux Boundary Conditions

Consider the reaction-diffusion equation

$$u_t(x,t) = Du_{xx}(x,t) + f(u(x,t)), \tag{4.0.1}$$

with no-flux boundary conditions $u_x(0,t) = u_x(1,t) = 0$ ($\forall t \geq 0$) where $f(u) = u(1-u)(u-\alpha)$, $D > 0$, $0 < \alpha < 1/2$, $x \in [0,1]$ and $t \geq 0$. In Chapter 2.3, we discovered that steady states within the phase-plane must have a transit time equal to 1. That is, (2.3.5) must be satisfied. In Chapter 4.1, we find a new form of the integral on the left hand side of (2.3.5) when $D = 1$. In Chapter 4.2, we show that this integral is decreasing as a function of λ. Figure 4.1 shows the graphs of the transit time as a function of λ for three different values of α. Since we know that this integral is decreasing on the interval $(0, \frac{\alpha^3}{6}(2-\alpha))$, we know that there is a minimum. In fact, this minimum is easily calculated (we shall compute this in Chapter 4.1). It is located at $\lambda = \frac{\alpha^3}{6}(2-\alpha)$ and is equal to $\frac{\pi}{\sqrt{\alpha(1-\alpha)}}$. When $D \neq 1$, we simply multiply this minimum by $\sqrt{D}$. Thus, in order for the minimum to be equal to 1, we must solve

$$\frac{\sqrt{D}\pi}{\sqrt{\alpha(1-\alpha)}} = 1.$$

In terms of D, this means

$$D = \frac{\alpha(1-\alpha)}{\pi^2}.$$

So, in order to have steady states, we require D to be less than or equal to $\frac{\alpha(1-\alpha)}{\pi^2}$.

Now, the integral on the left hand side of (2.3.5) tracks the transit time of half orbits. Thus, once the diffusion coefficient D becomes small enough that the minimum of the integral is less than 1, two steady states that are half orbits appear. The first steady state, $v(u(x))$, starts at a point $(u_1, 0)$ with $0 < u_1 < \alpha$, goes up and it starts to go back down once u hits α where it goes back down and ends on a point $(u_2, 0)$ with $\alpha < u_2 < 1$. The other steady state, $-v(u(1-x))$, is simply the reflection (across the u-axis) of the first steady state starting at $(u_2, 0)$ and ending on $(u_1, 0)$. As the parameter D continues to decrease, the minimum will eventually shrink more and more, passing through $\frac{1}{2}$, $\frac{1}{3}$,..., $\frac{1}{n}$ where $n \in \mathbb{N} \setminus \{0\}$. Every time the minimum crosses these thresholds, two new steady states appear. These new steady states are formed by combining n half orbits. For example, once the minimum is smaller than $\frac{1}{3}$ (this means that $D < \frac{\alpha(1-\alpha)}{\pi^2 3^2} = \frac{\alpha(1-\alpha)}{9\pi^2}$), there is a half orbit, v_1 whose transit time is equal to $\frac{1}{3}$. Thus, the two steady states correspond to combining that orbit and its reflective orbit (across the u-axis), v_2, 3 times. The first steady state starts with v_1, then uses v_2 and reuses v_1. The second steady state on the other hand starts with v_2, then uses v_1 and finishes with v_2 once again.

As a result, we have critical values of D constantly appearing. These values appear when the minimum of the transit times hits $\frac{1}{n}$ for some $n \in \mathbb{N} \setminus \{0\}$. Thus, the critical values are the solutions of

$$\frac{\pi\sqrt{D}}{\sqrt{\alpha(1-\alpha)}} = \frac{1}{n},$$

where $n \in \mathbb{N} \setminus \{0\}$. Subsequently, these critical values are

$$D = \frac{\alpha(1-\alpha)}{\pi^2 n^2},$$

where $n \in \mathbb{N} \setminus \{0\}$. As the diffusion parameter D increases, one of these values is approached. Hence, the minimum transit time of orbits approaches a value of $\frac{1}{n}$ ($n \in \mathbb{N} \setminus \{0\}$). Therefore, two steady states shrink closer and closer to α. Once these critical values of D are attained, the two steady states have now shrunk down to α and disappear while α continues to be a steady state. From this analysis, we can see that steady states continuously collide and are absorbed by α. Consequently, studying the system around α is of paramount importance.

However, before conducting our analysis of the system around α, we will first find a new form of the integral on the left hand side of (2.3.5). This will allow us to numerically solve the transit time of orbits much more easily. In addition, it will grant us easy access to the minimum of these transit times for a fixed value of α.

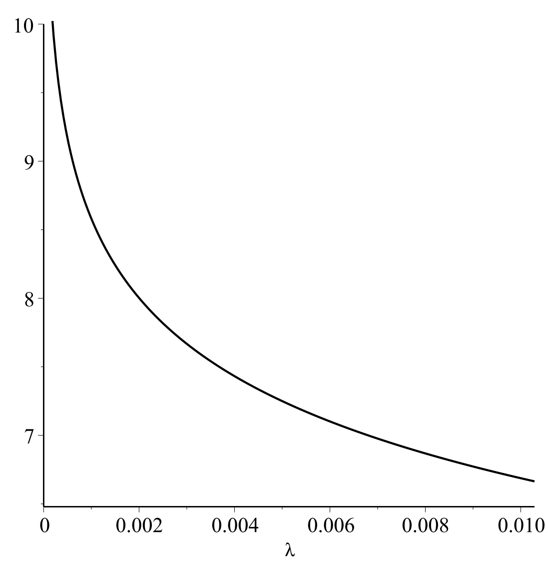

Figure 4.1: We have a graph showing the time of transit of orbits of (2.3.1) for no-flux boundary conditions for $\alpha = \frac{1}{3}$ and with $D = 1$.

4.1 Integral formula

In this section we will transform the following integral

$$\int_{u_1}^{u_2} \frac{du}{\sqrt{F(u,\lambda)}}, \tag{4.1.1}$$

where

$$F(u,\lambda) = \frac{u^4}{2} - \frac{2}{3}(1+\alpha)u^3 + \alpha u^2 - \lambda,$$

where $0 < \alpha < 1/2$ and $\lambda > 0$. In addition, we have four real distinct roots u_0, u_1, u_2 and u_3 such that $u_0 < 0 < u_1 < \alpha < u_2 < 1 < u_3$. We proceed as in Chapter 3.1. We want

$$F(u,\lambda) = \left(A_1(u-a_1)^2 + B_1(u-a_2)^2\right)\left(A_2(u-a_1)^2 + B_2(u-a_2)^2\right),$$

where A_1, A_2, B_1, B_2, a_1 and a_2 are constants.

Lemma 4.1.1. *Let $F(u) = b_4u^4 + b_3u^3 + b_2u^2 + b_1u + b_0$ with $b_0 > 0$ and $b_0, \ldots, b_4 \in \mathbb{R}$. Suppose that $F(u)$ has four real roots, u_0, u_1, u_2 and u_3 with $u_0 < 0 < u_1 < u_2 < 1 < u_3$. Then, there exists constant A_1, A_2, B_1, B_2, a_1 and a_2 such that*

$$F(u) = \left(A_1(u-a_1)^2 + B_1(u-a_2)^2\right)\left(A_2(u-a_1)^2 + B_2(u-a_2)^2\right). \tag{4.1.2}$$

Proof: The proof is similar to the proof of Lemma 3.1.1, thus, we shall only give the values of the important constants and leave the details to the reader. We set

$$S_1(u) = \frac{1}{2}(u_3-u)(u-u_0) \quad \text{and} \quad S_2(u) = (u-u_1)(u_2-u) = -u^2+(u_1+u_2)u-u_1u_2.$$

Thus, $F(u) = S_1(u)S_2(u)$. We want to find ϵ such that $S_1(u) - \epsilon S_2(u)$ is a perfect square. Since this is just a subtraction of quadratics, we only need the discriminant to be zero. Expanding $S_1(u) - \epsilon S_2(u)$ and then setting the discriminant to zero gives us

$$\epsilon_{2,1} = \frac{(u_0+u_3)(u_1+u_2)-2(u_0u_3+u_1u_2)}{2(u_1-u_2)^2} \pm \frac{\sqrt{(2(u_0u_3+u_1u_2)-(u_0+u_3)(u_1+u_2))^2-(u_1-u_2)^2(u_0-u_3)^2}}{2(u_1-u_2)^2}. \quad (4.1.3)$$

Thus,

$$S_1(u) - \epsilon_1 S_2(u) = \left(\epsilon_1 - \frac{1}{2}\right)(u-a_1)^2 \quad (4.1.4)$$

and

$$S_1(u) - \epsilon_2 S_2(u) = \left(\epsilon_2 - \frac{1}{2}\right)(u-a_2)^2, \quad (4.1.5)$$

where

$$a_1 = \frac{\epsilon_1(u_1+u_2)-\frac{1}{2}(u_0+u_3)}{2\left(\epsilon_1-\frac{1}{2}\right)} \quad \text{and} \quad a_2 = \frac{\epsilon_2(u_1+u_2)-\frac{1}{2}(u_0+u_3)}{2\left(\epsilon_2-\frac{1}{2}\right)}. \quad (4.1.6)$$

Using the same procedure as before, we find that

$$S_1(u) = A_1(u-a_1)^2 + B_1(u-a_2)^2 \quad \text{and} \quad S_2(u) = A_2(u-a_1)^2 + B_2(u-a_2)^2,$$

where

$$A_1 = \left(\frac{1}{2}-\epsilon_1\right)\frac{\epsilon_2}{\epsilon_1-\epsilon_2},\ A_2 = \left(\frac{\frac{1}{2}-\epsilon_1}{\epsilon_1-\epsilon_2}\right),\ B_1 = \left(\epsilon_2-\frac{1}{2}\right)\frac{\epsilon_1}{\epsilon_1-\epsilon_2},\ B_2 = \left(\frac{\epsilon_2-\frac{1}{2}}{\epsilon_1-\epsilon_2}\right). \quad (4.1.7)$$

Consequently, we finally have (4.1.2). ∎

The signs of ϵ_1 and ϵ_2 are given by the following lemma.

Lemma 4.1.2. *Let the notation be as in Lemma 4.1.2. Suppose $b_1 = 0$. Then, $\epsilon_2 > \epsilon_1 > b_4$.*

Proof: The proof is similar to the proof of Lemma 3.1.3 ∎

Since $b_4 = \frac{1}{2}$, we immediately get the following corollary regarding the signs of the coefficients given in (4.1.7).

Corollary 4.1.3. *Let the notation be as in Lemma 4.1.2. Then A_1 and A_2 are positive while B_1 and B_2 are negative.*

Proof: This immediately follows Lemma 4.1.2 and (4.1.7). ∎

The following lemma will allow us to determine the integration boundaries.

Lemma 4.1.4. *Let the notation be as in Lemma 4.1.2. Suppose that $b_4 > 0$ and $-u_0 < u_1$. Then $u_1 < a_2 < u_2$ and $a_1 < u_0$.*

Proof: We can use similar techniques used in the proof of Lemma 3.1.5 in order to show that $u_1 < a_2 < u_2$ and that we either have $a_1 < u_0$ or $a_1 > u_3$. In order to prove that we have $a_1 < u_0$, we proceed as follows. By (4.1.9), we have

$$a_2 - a_1 = \frac{(\epsilon_2 - \epsilon_1)((u_0 + u_3) - (u_1 + u_2))}{4(\epsilon_1 - \frac{1}{2})(\epsilon_2 - \frac{1}{2})}.$$

By Lemma 4.1.2, the terms $\epsilon_2 - \epsilon_1$, $\epsilon_1 - \frac{1}{2}$ and $\epsilon_2 - \frac{1}{2}$ are positive. By Corollary 4.2.6, we have $(u_0 + u_3) - (u_1 + u_2) > 0$. Hence, $a_2 > a_1$. Subsequently, we must have $a_1 < u_0$ since $a_2 < u_3$. ∎

Using the previous lemma allows us to determine that

$$\frac{u_1 - a_2}{u_1 - a_1} = -\sqrt{\frac{-A_2}{B_2}} \qquad \text{and} \qquad \frac{u_2 - a_2}{u_2 - a_1} = \sqrt{\frac{-A_2}{B_2}}.$$

Let us now start to calculate (4.1.1). Firstly, we do a change of variable. We set $\sqrt{\frac{-A_2}{B_2}}x = \frac{u-a_2}{u-a_1}$. Then, $du = \sqrt{\frac{-A_2}{B_2}}\frac{1}{a_2-a_1}dx$ and $\frac{-A_2}{B_2}x^2 = (\frac{u-a_2}{u-a_1})^2$. Moreover, when $u = u_1$, $x = -1$ and when $u = u_2$, $x = 1$. Thus, combining this with (4.1.2) yields

$$\int_{u_1}^{u_2} \frac{du}{\sqrt{F(u,\lambda)}}$$

$$= \sqrt{\frac{-A_2}{B_2}} \frac{1}{a_2 - a_1} \int_{-1}^{1} \frac{(u-a_1)^2 dx}{\sqrt{(A_1(u-a_1)^2 + B_1(u-a_2)^2)\,(A_2(u-a_1)^2 + B_2(u-a_2)^2)}}$$

$$= \sqrt{\frac{-A_2}{B_2}} \frac{2}{a_2 - a_1} \int_{0}^{1} \frac{dx}{\sqrt{\left(A_1 - \frac{B_1 A_2}{B_2} x^2\right)(A_2 - A_2 x^2)}}$$

$$= \sqrt{\frac{-1}{B_2 A_1}} \frac{2}{a_2 - a_1} \int_{0}^{1} \frac{dx}{\sqrt{\left(1 - \frac{B_1 A_2}{A_1 B_2} x^2\right)(1 - x^2)}}.$$

One can easily show that

$$\frac{B_1 A_2}{A_1 B_2} = \frac{\epsilon_1}{\epsilon_2}.$$

Hence,

$$\int_{u_1}^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} = \sqrt{\frac{-1}{B_2 A_1}} \frac{2}{a_2 - a_1} \int_{0}^{1} \frac{dx}{\sqrt{\left(1 - \frac{\epsilon_1}{\epsilon_2} x^2\right)(1 - x^2)}}. \tag{4.1.8}$$

In addition, we can also show that

$$\frac{1}{a_2 - a_1} = \frac{4(\epsilon_1 - \frac{1}{2})(\epsilon_2 - \frac{1}{2})}{(\epsilon_2 - \epsilon_1)((u_0 + u_3) - (u_1 + u_2))}, \tag{4.1.9}$$

and

$$\sqrt{\frac{-1}{B_2 A_1}} = \frac{\epsilon_2 - \epsilon_1}{\sqrt{\epsilon_1(\epsilon_1 - \frac{1}{2})(\epsilon_2 - \frac{1}{2})}}.$$

Therefore,

$$\sqrt{\frac{-1}{B_2 A_1}} \frac{1}{a_2 - a_1} = \frac{4\sqrt{(\epsilon_1 - \frac{1}{2})(\epsilon_2 - \frac{1}{2})}}{\sqrt{\epsilon_2((u_0 + u_3) - (u_1 + u_2))}}. \tag{4.1.10}$$

Furthermore, we can show that

$$\sqrt{\left(\epsilon_1 - \frac{1}{2}\right)\left(\epsilon_2 - \frac{1}{2}\right)} = \frac{(u_0 + u_3) - (u_1 + u_2)}{2(u_2 - u_1)}.$$

Combining this with (4.1.10) yields

$$\sqrt{\frac{-1}{B_2 A_1}}\frac{1}{a_2 - a_1} = \frac{2}{(u_2 - u_1)\sqrt{\epsilon_2}}.$$

Finally, this implies, along with (4.1.8), that we have

$$\int_{u_1}^{u_2} \frac{du}{\sqrt{F(u,\lambda)}} = \frac{4}{(u_2 - u_1)\sqrt{\epsilon_2}} \int_0^1 \frac{dx}{\sqrt{\left(1 - \frac{\epsilon_1}{\epsilon_2}x^2\right)(1 - x^2)}}. \tag{4.1.11}$$

Keep in mind that $\frac{\epsilon_1}{\epsilon_2} < 1$. In the next section we prove that the integral on the left hand side of (4.1.11) is decreasing as a function of λ.

4.2 Investigating the Integral Formula

The goal of this section is to shed some light on the rate of change of the left hand side of (4.1.11) with respect to λ. In order to accomplish this, we will study the real roots of fourth degree polynomials. However, before attempting this we shall first establish some terminology. Throughout this section, we will be speaking about the derivatives of the roots of a polynomial with respect to the constant within the polynomial. More concretely, suppose that

$$g(u) = au^4 + bu^3 + cu^2 + du - \lambda$$

is a fourth degree polynomial with four real roots u_0, u_1, u_2, u_3 and real coefficients. We can view $g(u)$ as a function of both u and λ if we let λ vary within an interval, i.e., we would have $g(u) = g(u, \lambda)$. This is turn allows us to view its roots as functions of λ, $u_i = u_i(\lambda)$ for $i = 0, 1, 2, 3$, because increasing or decreasing λ simply moves the polynomial $g(u)$ up or down, if we assume that the other coefficients in $g(u, \lambda)$ are constants with regards to λ. We can then consider the derivatives of the polynomial's roots, $\frac{d}{d\lambda}u_i(\lambda) = u_i'(\lambda)$ $(i = 0, 1, 2, 3)$. In order to avoid clutter and to increase legibility, we will often use u_i and u_i' $(i = 0, 1, 2, 3)$ when referring to $u_i(\lambda)$ and $u_i'(\lambda)$ $(i = 0, 1, 2, 3)$ (respectively) when the context is clear. For example, we might use $u_i(\lambda)$ $(i \in \{0, 1, 2, 3\})$ in the statement of a result but simply use u_i $(i \in \{0, 1, 2, 3\})$ in the proof of said result.

The reason why we must investigate the roots of fourth degree polynomials is because the right hand side of (4.1.11) is constructed using the roots of the polynomial

$$f(u, \lambda) := \frac{u^4}{2} - \frac{2}{3}(1 + \alpha)u^3 + \alpha u^2 - \lambda. \tag{4.2.1}$$

The objective is to obtain enough knowledge about the roots of $f(u, \lambda)$ to prove two things. We want to demonstrate that the term in front of the elliptic integral in (4.1.11) is decreasing. In addition, we want to show that the elliptic integral in (4.1.11) is decreasing. In order to establish that this elliptic integral is decreasing, we will use Proposition 4.2.1. Since $\sqrt{\frac{\epsilon_1}{\epsilon_2}} < 1$ by Lemma 4.1.2, we simply need to verify that $\sqrt{\frac{\epsilon_1}{\epsilon_2}}$ is decreasing where ϵ_1 and ϵ_2 are defined in (4.1.3). These results then entail that the left hand side of (4.1.11) is a product of two positive and decreasing functions, and thus is decreasing itself.

Proposition 4.2.1. *The complete elliptic integral of the first kind*

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - k^2x^2)(1 - x^2)}}$$

is an increasing function of k for $0 < k < 1$.

Proof: We have

$$K'(k) = \frac{d}{dk}\left(\int_0^1 \frac{dx}{\sqrt{(1 - k^2x^2)(1 - x^2)}}\right) = \int_0^1 \frac{d}{dk}\left(\frac{1}{\sqrt{(1 - k^2x^2)(1 - x^2)}}\right)dx$$
$$= \int_0^1 \frac{\frac{-1}{2}(1 - x^2)(-2kx^2)}{((1 - k^2x^2)(1 - x^2))^{\frac{3}{2}}}dx = \int_0^1 \frac{(1 - x^2)kx^2}{((1 - k^2x^2)(1 - x^2))^{\frac{3}{2}}}dx.$$

Since k and x are both in between 0 and 1, the numerator is positive. In addition, the denominator is clearly positive. Thus, the integral obtained in the last equality is positive. Consequently, $K'(k)$ is positive. ∎

From here on, we shall use $u_i(\lambda)$ $(i = 0, 1, 2, 3)$ to denote the four real roots of $f(u, \lambda)$ and $u_i(\lambda) < u_j(\lambda)$ when $i < j$ $(i, j \in \{0, 1, 2, 3\})$ unless specified otherwise. Now, consider ϵ_1 and ϵ_2 given in (4.1.3). Let us transform their numerators. We have

$$\begin{aligned}(u_0 + u_3)(u_1 + u_2) - 2(u_0u_3 + u_1u_2) &= u_0u_1 + u_0u_2 + u_3u_1 + u_3u_2 - 2u_0u_3 - 2u_1u_2 \\ &= u_1(u_0 - u_2) + u_0(u_2 - u_3) + u_1(u_3 - u_2) \\ &+ u_3(u_2 - u_0).\end{aligned}$$

Hence, we have

$$(u_0 + u_3)(u_1 + u_2) - 2(u_0u_3 + u_1u_2) = (u_3 - u_1)(u_2 - u_0) + (u_1 - u_0)(u_3 - u_2). \quad (4.2.2)$$

In addition, we have

$$\begin{aligned}&(u_3 - u_1)(u_2 - u_0) + (u_1 - u_0)(u_3 - u_2) + (u_2 - u_1)(u_3 - u_0) \\ &= (u_3 - u_1)(u_2 - u_0) + (u_1 - u_0)(u_3 - u_2) + (u_2 - u_0)(u_3 - u_0) - (u_1 - u_0)(u_3 - u_0) \\ &= (u_1 - u_0)(u_0 - u_2) + (u_3 - u_1)(u_2 - u_0) + (u_2 - u_0)(u_3 - u_0).\end{aligned}$$

Thus,

$$(u_3-u_1)(u_2-u_0)+(u_1-u_0)(u_3-u_2)+(u_2-u_1)(u_3-u_0) = 2(u_3-u_1)(u_2-u_0). \quad (4.2.3)$$

Similarly, we have

$$(u_3-u_1)(u_2-u_0)+(u_1-u_0)(u_3-u_2)-(u_2-u_1)(u_3-u_0) = 2(u_1-u_0)(u_3-u_2). \quad (4.2.4)$$

Subsequently, combining (4.2.2), (4.2.3) and (4.2.4) allows us to transform the square root term in ϵ_1 and ϵ_2.

$$\begin{aligned}
&(2(u_0u_3+u_1u_2)-(u_0+u_3)(u_1+u_2))^2-(u_1-u_2)^2(u_0-u_3)^2\\
&=((u_3-u_1)(u_2-u_0)+(u_1-u_0)(u_3-u_2))-(u_1-u_2)^2(u_0-u_3)^2\\
&=4(u_3-u_1)(u_2-u_0)(u_1-u_0)(u_3-u_2)
\end{aligned}$$

Therefore,

$$\epsilon_2 = \frac{\left(\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}\right)^2}{2(u_1-u_2)^2}, \quad (4.2.5)$$

and

$$\epsilon_1 = \frac{\left(\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}\right)^2}{2(u_1-u_2)^2} \quad (4.2.6)$$

It is now clear that in order to understand the derivatives (with respect to λ) of $\sqrt{\frac{\epsilon_1}{\epsilon_2}}$ and of the term $\frac{1}{(u_2-u_1)\sqrt{\epsilon_2}}$, we must study the derivatives (with respect to λ) of both $\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}$ and $\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}$.

A good starting point are the roots of $f(u,\lambda)$. Let us look at their derivatives.

Lemma 4.2.2. *We have*

$$u_i'(\lambda) = \frac{-1}{u_i(\lambda)(1-u_i(\lambda))(u_i(\lambda)-\alpha)}.$$

Proof: Since u_i is a root of $f(u,\lambda)$ for $i=0,1,2,3$, we have $f(u_i,\lambda)=0$. for

$i = 0, 1, 2, 3$. Differentiating both sides with respect to λ yields (for $i \in \{0, 1, 2, 3\}$)

$$f_u(u_i, \lambda)u_i' + f_\lambda(u_i, \lambda) = 0.$$

Since $f_u(u, \lambda) = -u(1-u)(u-\alpha)$ and $f_\lambda(u, \lambda) = -1$, we obtain

$$-u_i'(1-u_i')(u_i'-\alpha)u_i' - 1 = 0$$

for $i = 0, 1, 2, 3$. Solving for u_i' ($i = 0, 1, 2, 3$) gives us

$$u_i' = \frac{-1}{u_i'(1-u_i')(u_i'-\alpha)}$$

as required. ∎

From Remark 2.3.1 we know that $u_0(\lambda) \le 0$ and $0 \le u_1(\lambda) \le \alpha \le u_2(\lambda) \le 1$. In addition, since $f(u, \lambda)$ has a minimum at $u = 1$ for all $\lambda \in [0, \frac{\alpha^3}{6}(2-\alpha)]$, we must have $u_3(\lambda) \ge 1$. Thus, along with the previous lemma, we can determine the sign of the derivative of each root. This gives us the following corollary.

Corollary 4.2.3. *The roots $u_1(\lambda)$ and $u_3(\lambda)$ are increasing functions of λ whereas the roots $u_0(\lambda)$ and $u_2(\lambda)$ are decreasing functions of λ.*

Proof: This follows from Lemma 4.2.2 and the fact that we consider $u_0 \le 0 \le u_1 \le \alpha \le u_2 \le 1 \le u_3$. ∎

In both terms, $\sqrt{(u_3-u_1)(u_2-u_0)} \pm \sqrt{(u_1-u_0)(u_3-u_2)}$, there are occurences of the distance between certain roots. Thus, we will most likely need some information on certain distances. For this reason, we introduce the following corollary.

Corollary 4.2.4. *We have $(u_1(\lambda) - u_0(\lambda))' > 0$ and $(u_3(\lambda) - u_2(\lambda))' > 0$.*

Proof: This follows immediately from Corollary 4.2.3. ∎

Before going any further, let us note that if we can show that $\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}$ is increasing and that $\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}$ is decreasing, then it follows that $\sqrt{\frac{\epsilon_1}{\epsilon_2}}$ and $\frac{1}{(u_2-u_1)\sqrt{\epsilon_2}}$ are both decreasing. Let us take a look at the derivatives of $\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}$ and $\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}$. We have

$$\begin{aligned}\left(\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}\right)'\\ =\frac{((u_3-u_1)(u_2-u_0))'}{2\sqrt{(u_3-u_1)(u_2-u_0)}}+\frac{((u_1-u_0)(u_3-u_2))'}{2\sqrt{(u_1-u_0)(u_3-u_2)}}.\end{aligned}$$

Thus, $\left(\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}\right)'\geq 0$ if and only if we have

$$\frac{((u_3-u_1)(u_2-u_0))'}{2\sqrt{(u_3-u_1)(u_2-u_0)}}\geq-\frac{((u_1-u_0)(u_3-u_2))'}{2\sqrt{(u_1-u_0)(u_3-u_2)}}.$$

Consequently, by Corollary 4.2.4 we must have

$$\begin{aligned}\left(\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}\right)'\geq 0\\ \iff\frac{((u_3-u_1)(u_2-u_0))'}{((u_1-u_0)(u_3-u_2))'}\geq-\frac{\sqrt{(u_3-u_1)(u_2-u_0)}}{\sqrt{(u_1-u_0)(u_3-u_2)}}.\end{aligned}\tag{4.2.7}$$

Similarly, we have

$$\begin{aligned}\left(\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}\right)'\\ =\frac{((u_3-u_1)(u_2-u_0))'}{2\sqrt{(u_3-u_1)(u_2-u_0)}}-\frac{((u_1-u_0)(u_3-u_2))'}{2\sqrt{(u_1-u_0)(u_3-u_2)}}.\end{aligned}$$

Therefore, $\left(\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}\right)' \leq 0$ if and only if we have

$$\frac{((u_3-u_1)(u_2-u_0))'}{2\sqrt{(u_3-u_1)(u_2-u_0)}} \leq \frac{((u_1-u_0)(u_3-u_2))'}{2\sqrt{(u_1-u_0)(u_3-u_2)}}.$$

Therefore, applying Corollary 4.2.4 yields

$$\begin{aligned}&\left(\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}\right)' \leq 0\\ &\qquad\qquad\iff \frac{((u_3-u_1)(u_2-u_0))'}{((u_1-u_0)(u_3-u_2))'} \leq \frac{\sqrt{(u_3-u_1)(u_2-u_0)}}{\sqrt{(u_1-u_0)(u_3-u_2)}}. \quad (4.2.8)\end{aligned}$$

Now, recall that we have $u_0 \leq 0 \leq u_1 \leq \alpha \leq u_2 \leq 1 \leq u_3$. Hence, the distance between u_3 and u_1 is greater than the distance between u_3 and u_2. In addition, the distance between u_2 and u_0 is greater than the distance between u_1 and u_0. As a result, we can conclude that we have

$$\frac{\sqrt{(u_3-u_1)(u_2-u_0)}}{\sqrt{(u_1-u_0)(u_3-u_2)}} \geq 1. \quad (4.2.9)$$

Consequently, if we can show that we have

$$-1 \leq \frac{((u_3-u_1)(u_2-u_0))'}{((u_1-u_0)(u_3-u_2))'} \leq 1, \quad (4.2.10)$$

then we know that $\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}$ is increasing and $\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}$ is decreasing. It is now more apparent than ever that we need to look into the rate of change between the distances of roots.

Our new goal is to prove (4.2.10). Thus, we cite the equations used to calculate the roots of fourth degree polynomials in order to shed some light on the inequalities between the sum of some roots.

Lemma 4.2.5. *[16] Let $g(u) = au^4+bu^3+cu^2+du+e$ be a fourth degree polynomial with real coefficients. Let u_0, u_1, u_2 and u_3 be the four real roots of $g(u)$. Then, we* *have*

1. $u_0 = \frac{-b}{4a} - \frac{1}{2}H - \frac{1}{2}\sqrt{2A - \frac{B}{E} - \frac{C}{F} - \frac{G}{4H}}$,
2. $u_1 = \frac{-b}{4a} - \frac{1}{2}H + \frac{1}{2}\sqrt{2A - \frac{B}{E} - \frac{C}{F} - \frac{G}{4H}}$,
3. $u_2 = \frac{-b}{4a} + \frac{1}{2}H - \frac{1}{2}\sqrt{2A - \frac{B}{E} - \frac{C}{F} + \frac{G}{4H}}$,
4. $u_3 = \frac{-b}{4a} + \frac{1}{2}H + \frac{1}{2}\sqrt{2A - \frac{B}{E} - \frac{C}{F} + \frac{G}{4H}}$,

where

1. $A = \frac{b^2}{4a^2} - \frac{2c}{3a}$,
2. $B = 2^{\frac{1}{3}}(12ae - 3bd + c^2)$,
3. $$C = \Big(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4(12ae - 3bd + c^2)^3 + (-72ace + 27ad^2 + 27b^2e - 9bcd + 2c^3)^2}\Big)^{1/3},$$
4. $E = 3aC$,
5. $F = 32^{\frac{1}{3}}a$,
6. $G = -\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}$,
7. $H = \sqrt{A + \frac{B}{E} + \frac{C}{F}}$.

This then leads to the following corollary.

Corollary 4.2.6. *We have* $u_1 + u_2 < u_0 + u_3$.

Proof: By Lemma (4.2.5), we have

$$\begin{aligned}
&u_1+u_2\\
&=\frac{1+\alpha}{3}-\frac{1}{2}H+\frac{1}{2}\sqrt{2A-\frac{B}{E}-\frac{C}{F}-\frac{G}{4H}}-\frac{1+\alpha}{3}+\frac{1}{2}H-\frac{1}{2}\sqrt{2A-\frac{B}{E}-\frac{C}{F}+\frac{G}{4H}}\\
&=\frac{2(1+\alpha)}{3}+\frac{1}{2}\left(\sqrt{2A-\frac{B}{E}-\frac{C}{F}-\frac{G}{4H}}-\sqrt{2A-\frac{B}{E}-\frac{C}{F}+\frac{G}{4H}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&u_0+u_3\\
&=\frac{1+\alpha}{3}-\frac{1}{2}H-\frac{1}{2}\sqrt{2A-\frac{B}{E}-\frac{C}{F}-\frac{G}{4H}}-\frac{1+\alpha}{3}+\frac{1}{2}H+\frac{1}{2}\sqrt{2A-\frac{B}{E}-\frac{C}{F}+\frac{G}{4H}}\\
&=\frac{2(1+\alpha)}{3}+\frac{1}{2}\left(\sqrt{2A-\frac{B}{E}-\frac{C}{F}+\frac{G}{4H}}-\sqrt{2A-\frac{B}{E}-\frac{C}{F}-\frac{G}{4H}}\right).
\end{aligned}$$

Therefore, we have $u_1+u_2\leq u_0+u_3$ if and only if

$$\begin{aligned}
&\frac{2(1+\alpha)}{3}+\frac{1}{2}\left(\sqrt{2A-\frac{B}{E}-\frac{C}{F}-\frac{G}{4H}}-\sqrt{2A-\frac{B}{E}-\frac{C}{F}+\frac{G}{4H}}\right)\leq\\
&\qquad\frac{2(1+\alpha)}{3}+\frac{1}{2}\left(\sqrt{2A-\frac{B}{E}-\frac{C}{F}+\frac{G}{4H}}-\sqrt{2A-\frac{B}{E}-\frac{C}{F}-\frac{G}{4H}}\right).
\end{aligned}$$

This holds true if and only if

$$\sqrt{2A-\frac{B}{E}-\frac{C}{F}-\frac{G}{4H}}\leq\sqrt{2A-\frac{B}{E}-\frac{C}{F}+\frac{G}{4H}}.$$

This last inequality hold since $G=\frac{32}{27}(1+\alpha)(2\alpha-1)(\alpha-2)\geq 0$. Therefore, we are done. ∎

The previous result will useful a little later on. Let us continue with our analysis.

Increasing λ in $f(u,\lambda)$ simply moves the curve down, the geometry of the curve stays the same. From this, we obtain the following lemma which will be the foundation of subsequent proofs.

Lemma 4.2.7. *We have*

$$0 = u_0'(\lambda) + u_1'(\lambda) + u_2'(\lambda) + u_3'(\lambda), \tag{4.2.11}$$

$$0 = (u_0(\lambda)u_1(\lambda))' + (u_0(\lambda)u_2(\lambda))' + (u_0(\lambda)u_3(\lambda))' \tag{4.2.12}$$

$$+ (u_1(\lambda)u_2(\lambda))' + (u_1(\lambda)u_3(\lambda))' + (u_2(\lambda)u_3(\lambda))', \tag{4.2.13}$$

$$0 = (u_0(\lambda)u_1(\lambda)u_2(\lambda))' + (u_0(\lambda)u_1(\lambda)u_3(\lambda))' \tag{4.2.14}$$

$$+ (u_0(\lambda)u_2(\lambda)u_3(\lambda))' + (u_1(\lambda)u_2(\lambda)u_3(\lambda))', \tag{4.2.15}$$

$$-2 = (u_0(\lambda)u_1(\lambda)u_2(\lambda)u_3(\lambda))'. \tag{4.2.16}$$

Proof: We have

$$f(u,\lambda) = \frac{1}{2}(u-u_0)(u-u_1)(u-u_2)(u-u_3) =$$

$$\frac{1}{2}u^4 - \frac{1}{2}\sum_{i=0}^{3} u_i u^3 + \frac{1}{2}\sum_{\substack{i,j,i<j\\0<i<2\\1<j<3}} u_i u_j u^2 - \frac{1}{2}\sum_{\substack{i,j,k\\i<j<k\\0<i<1\\1<j<2\\2<k<3}} u_i u_j u_k u + \frac{1}{2}u_0 u_1 u_2 u_3.$$

Subsequently, we have

$$-\frac{2}{3}(1+\alpha) = -\frac{1}{2}\sum_{i=0}^{3} u_i, \quad \alpha = \frac{1}{2}\sum_{\substack{i,j,i<j\\0<i<2\\1<j<3}} u_i u_j,$$

$$0 = -\frac{1}{2}\sum_{\substack{i,j,k\\i<j<k\\0<i<1\\1<j<2\\2<k<3}} u_i u_j u_k, \quad -\lambda = \frac{1}{2}u_0 u_1 u_2 u_3. \tag{4.2.17}$$

Differentiating each side of every equation in (4.2.17) with respect to λ yields (4.2.11)-(4.2.16). ∎

The equations given in Lemma 4.2.7 are fundamental but not quite good enough in their current state. Hence, we need to combine equations given in Lemma 4.2.7 to obtain new ones. Combining (4.2.11) with (4.2.13) and (4.2.11) with (4.2.15) gives light to the following two corollaries.

Corollary 4.2.8. *We have*

$$0 = u_i'(\lambda)(u_n(\lambda) - u_i(\lambda)) + u_j'(\lambda)(u_n(\lambda) - u_j(\lambda)) + u_k'(\lambda)(u_n(\lambda) - u_k(\lambda)), \quad (4.2.18)$$

for $0 \le i < j < k \le 3$*,* $0 \le n \le 3$ *and* $n \ne i, j, k$*.*

Proof: We shall prove it for the case $n = 0, i = 1, j = 2, k = 3$. The other cases are done in a similar fashion. By Lemma 4.2.7 (4.2.11), we have

$$u_0' = -u_1' - u_2' - u_3'. \quad (4.2.19)$$

In addition, by Lemma 4.2.7 (4.2.13), we have

$$\begin{aligned} 0 = &u_0'u_1 + u_0u_1' + u_0'u_2 + u_0u_2' + u_0'u_3 + u_0u_3' + u_1'u_2 + u_1u_2' + u_1'u_3 + u_1u_3' + u_2'u_3 \\ &+ u_2u_3'. \end{aligned}$$

Combining this with (4.2.19), this leads to

$$\begin{aligned} 0 = &- u_1'u_1 - u_2'u_1 - u_3'u_1 + u_0u_1' - u_1'u_2 - u_2'u_2 - u_3'u_2 + u_0u_2' - u_1'u_3 \\ &- u_2'u_3 - u_3'u_3 + u_0u_3' + u_1'u_2 + u_1u_2' + u_1'u_3 + u_1u_3' + u_2'u_3 + u_2u_3' \\ = &-u_1'u_1 - u_2'u_2 - u_3'u_3 + u_0u_1' + u_0u_2' + u_0u_3'. \end{aligned}$$

Therefore,

$$0 = u_1'(u_0 - u_1) + u_2'(u_0 - u_2) + u_3'(u_0 - u_3).$$

∎

Corollary 4.2.9. *We have*

$$0 = u_i'(\lambda)(u_n(\lambda) - u_i(\lambda))(u_j(\lambda) + u_k(\lambda)) + u_j'(\lambda)(u_n(\lambda) - u_j(\lambda))(u_i(\lambda) + u_k(\lambda)) + u_k'(\lambda)(u_n(\lambda) - u_k(\lambda))(u_i(\lambda) + u_j(\lambda)), \quad (4.2.20)$$

for $0 \le i < j < k \le 3$, $0 \le n \le 3$ *and* $n \ne i, j, k$.

Proof: We shall prove this for the case $n = 3, i = 0, j = 1, k = 2$, i.e.,

$$0 = u_0'(u_3 - u_0)(u_1 + u_2) + u_1'(u_3 - u_1)(u_0 + u_2) + u_2'(u_3 - u_2)(u_0 + u_1).$$

The other cases are proved in a similar fashion. By Lemma 4.2.7 (4.2.15), we have

$$\begin{aligned} 0 =&(u_0u_1u_2)' + (u_0u_1u_3)' + (u_0u_2u_3)' + (u_1u_2u_3)' \\ =&u_0'u_1u_2 + u_0u_1'u_2 + u_0u_1u_2' + u_0'u_1u_3 + u_0u_1'u_3 + u_0u_1u_3' \\ &+ u_0'u_2u_3 + u_0u_2'u_3 + u_0u_2u_3' + u_1'u_2u_3 + u_1u_2'u_3 + u_1u_2u_3'. \end{aligned}$$

By Lemma 4.2.7, we know that $u_3' = -u_0' - u_1' - u_2'$. Hence, we then get

$$0 =u_0'u_1u_2 + u_0u_1'u_2 + u_0u_1u_2' + u_0'u_1u_3 + u_0u_1'u_3 - u_0u_1u_0'$$

$$
\begin{aligned}
&- u_0u_1u_1' - u_0u_1u_2' + u_0'u_2u_3 + u_0u_2'u_3 - u_0u_2u_0' - u_0u_2u_1' \\
&- u_0u_2u_2' + u_1'u_2u_3 + u_1u_2'u_3 - u_1u_2u_0' - u_1u_2u_1' - u_1u_2u_2' \\
=&u_0'(u_1u_3 - u_0u_1 + u_2u_3 - u_0u_2) + u_1'(u_0u_3 - u_0u_1 + u_2u_3 - u_1u_2) \\
&+ u_2'(u_0u_3 - u_0u_2 + u_1u_3 - u_1u_2) \\
=&u_0'(u_1(u_3 - u_0) + u_2(u_3 - u_0)) + u_1'(u_0(u_3 - u_1) + u_2(u_3 - u_1)) \\
&+ u_2'(u_0(u_3 - u_2) + u_1(u_3 - u_2))
\end{aligned}
$$

Consequently, we then obtain

$$0 = u_0'(u_3 - u_0)(u_1 + u_2) + u_1'(u_3 - u_1)(u_0 + u_2) + u_2'(u_3 - u_2)(u_0 + u_1).$$

∎

With these new equations on hand, we can now look into the ratio of the derivatives of some roots, which turns out to be a quotient of two products of distances between certain roots.

Lemma 4.2.10. *We have*

$$-\frac{u_2'(\lambda)}{u_1'(\lambda)} = \frac{(u_3(\lambda) - u_1(\lambda))(u_1(\lambda) - u_0(\lambda))}{(u_3(\lambda) - u_2(\lambda))(u_2(\lambda) - u_0(\lambda))}. \tag{4.2.21}$$

Proof: By Corollary 4.2.9, we have

$$0 = u_0'(u_3 - u_0)(u_1 + u_2) + u_1'(u_3 - u_1)(u_0 + u_2) + u_2'(u_3 - u_2)(u_0 + u_1). \tag{4.2.22}$$

In addition, Corollary 4.2.8 gives us

$$u_0'(u_3 - u_0) = -u_1'(u_3 - u_1) - u_2'(u_3 - u_2). \tag{4.2.23}$$

Combining (4.2.22) and (4.2.23) together gives us

$$\begin{aligned}0 =&(-u_1'(u_3 - u_1) - u_2'(u_3 - u_2))(u_1 + u_2) + u_1'(u_3 - u_1)(u_0 + u_2)\\ &+ u_2'(u_3 - u_2)(u_0 + u_1)\\ =&u_1'(u_3 - u_1)(u_0 + u_2 - u_1 - u_2) + u_2'(u_3 - u_2)(u_0 + u_1 - u_1 - u_2)\\ =&u_1'(u_3 - u_1)(u_0 - u_1) + u_2'(u_3 - u_2)(u_0 - u_2).\end{aligned}$$

Thus, we have

$$-u_2'(u_3 - u_2)(u_2 - u_0) = u_1'(u_3 - u_1)(u_1 - u_0).$$

Consequently, we then obtain

$$-\frac{u_2'}{u_1'} = \frac{(u_3 - u_1)(u_1 - u_0)}{(u_3 - u_2)(u_2 - u_0)}.$$

∎

The following lemma gives us an upper bound for the ratios presented in (4.2.21).

Lemma 4.2.11. *We have*

$$\frac{(u_3(\lambda) - u_1(\lambda))(u_1(\lambda) - u_0(\lambda))}{(u_3(\lambda) - u_2(\lambda))(u_2(\lambda) - u_0(\lambda))} \leq 1. \tag{4.2.24}$$

Proof: We have

$$\begin{aligned}
&\frac{(u_3-u_1)(u_1-u_0)}{(u_3-u_2)(u_2-u_0)} \le 1 \\
\Longleftrightarrow\ & (u_3-u_1)(u_1-u_0) \le (u_3-u_2)(u_2-u_0) \\
\Longleftrightarrow\ & u_3u_1 - u_3u_0 - u_1u_1 + u_1u_0 \le u_3u_2 - u_3u_0 - u_2u_2 + u_2u_0 \\
\Longleftrightarrow\ & u_3u_2 - u_2u_2 + u_2u_0 - u_3u_1 + u_1u_1 - u_1u_0 \ge 0 \\
\Longleftrightarrow\ & u_3(u_2-u_1) + u_1(u_1-u_0) - u_2(u_2-u_0) \ge 0.
\end{aligned}$$

Using the equality $u_2(u_2-u_0) = u_2(u_2-u_1+u_1-u_0)$, we then obtain

$$\begin{aligned}
&\frac{(u_3-u_1)(u_1-u_0)}{(u_3-u_2)(u_2-u_0)} \le 1 \\
\Longleftrightarrow\ & u_3(u_2-u_1) + u_1(u_1-u_0) - u_2(u_2-u_1) - u_2(u_1-u_0) \ge 0 \\
\Longleftrightarrow\ & (u_3-u_2)(u_2-u_1) + (u_1-u_2)(u_1-u_0) \ge 0 \\
\Longleftrightarrow\ & (u_3-u_2) - (u_1-u_0) \ge 0 \\
& .
\end{aligned}$$

Consequently, we then obtain

$$\frac{(u_3-u_1)(u_1-u_0)}{(u_3-u_2)(u_2-u_0)} \le 1 \iff u_3+u_0 \ge u_1+u_2.$$

By Corollary 4.2.6, the last statement is true and so, we are done. ∎

This leads to an interesting result involving the derivatives of the roots u_1 and u_2.

Lemma 4.2.12. *We have*

$$-u_2'(\lambda) \leq u_1'(\lambda).$$

Proof: This result follows from Lemma 4.2.10 and Lemma 4.2.11. ∎

Analysing an equation given in Corollary 4.2.8 leads to a result that is similar to Lemma 4.2.12.

Lemma 4.2.13. *We have* $u_1'(\lambda) \geq -u_0'(\lambda)$.

Proof: By Corollary 4.2.8, we have

$$u_0'(u_3 - u_0) + u_1'(u_3 - u_1) + u_2'(u_3 - u_2) = 0.$$

Since $u_2' < 0$ and $u_3 - u_2 > 0$, we must have $u_0'(u_3 - u_0) + u_1'(u_3 - u_1) > 0$. In addition, we know that $u_3 - u_1 < u_3 - u_0$. Consequently, we must then have $u_1' \geq -u_0'$. ∎

This leads to the following lemma.

Lemma 4.2.14. *We have* $((u_2(\lambda) - u_1(\lambda))(u_3(\lambda) - u_0(\lambda)))' \leq 0$.

Proof: Let us first begin with the following

$$\begin{aligned} & ((u_2 - u_1)(u_3 - u_0))' \leq 0 \\ \Longleftrightarrow\ & (u_2 - u_1)'(u_3 - u_0) + (u_2 - u_1)(u_3 - u_0)' \leq 0 \\ \Longleftrightarrow\ & (u_2 - u_1)'(u_3 - u_0) \leq -(u_2 - u_1)(u_3 - u_0)' \\ \Longleftrightarrow\ & \frac{(u_2 - u_1)}{(u_3 - u_0)} \leq -\frac{(u_2 - u_1)'}{(u_3 - u_0)'}. \end{aligned}$$

Now, it is clear that $\frac{(u_2-u_1)}{(u_3-u_0)} \leq 1$. Thus, if we can show that $u_1' \geq -u_0'$ if and only if

$-\frac{(u_2-u_1)'}{(u_3-u_0)'} \geq 1$, then we are done. Indeed, we have

$$-\frac{(u_2-u_1)'}{(u_3-u_0)'} \geq 1 \iff -u_2'+u_1' \geq u_3'-u_0' \iff u_1'+u_0' \geq u_3'+u_2'.$$

By Lemma 4.2.7 (4.2.11), we know that $u_3'+u_2' = -u_1'-u_0'$. Thus, we then obtain

$$-\frac{(u_2-u_1)'}{(u_3-u_0)'} \geq 1 \iff u_1'+u_0' \geq -u_1'-u_0' \iff u_1'+u_0' \geq 0 \iff u_1' \geq -u_0'.$$

∎

We now have enough information to confirm (4.2.10). We start with the lower bound.

Lemma 4.2.15. *We have*

$$\frac{((u_3(\lambda)-u_1(\lambda))(u_2(\lambda)-u_0(\lambda)))'}{((u_1(\lambda)-u_0(\lambda))(u_3(\lambda)-u_2(\lambda)))'} \geq -1. \tag{4.2.25}$$

Proof: Let us start with assuming that

$$\frac{((u_3-u_1)(u_2-u_0))'}{((u_1-u_0)(u_3-u_2))'} \geq -1. \tag{4.2.26}$$

Expanding the left hand side gives us

$$\begin{aligned}
&\frac{(u_2u_3)'+(u_0u_1)'-(u_0u_3)'-(u_1u_2)'}{(u_1u_3)'+(u_0u_2)'-(u_1u_2)'-(u_0u_3)'} \geq -1 \\
\iff &(u_2u_3)'+(u_0u_1)'-(u_0u_3)'-(u_1u_2)' \geq (u_1u_2)'+(u_0u_3)'-(u_1u_3)'-(u_0u_2)'
\end{aligned}$$

since $((u_1-u_0)(u_3-u_2))' > 0$ by Corollary 4.2.4. Thus, (4.2.26) is true if and only if we have

$$(u_2u_3)' + (u_0u_1)' + (u_1u_3)' + (u_0u_2)' \geq 2(u_1u_2)' + 2(u_0u_3)'. \tag{4.2.27}$$

By Lemma 4.2.7 (4.2.13),

$$(u_2u_3)' + (u_0u_1)' + (u_1u_3)' + (u_0u_2)' = -(u_1u_2)' - (u_0u_3)'.$$

Consequently, (4.2.26) is equivalent to

$$-(u_1u_2)' - (u_0u_3)' \geq 2(u_1u_2)' + 2(u_0u_3)',$$

and this is equivalent to

$$0 \geq (u_1u_2)' + (u_0u_3)'.$$

Expanding $(u_1u_2)' + (u_0u_3)'$, using the substitution $u_0' = -u_1' - u_2' - u_3'$ given by Lemma 4.2.7 (4.2.11) and rearranging yields

$$\begin{aligned}(u_1u_2)' + (u_0u_3)' &= u_1'u_2 + u_1u_2' + u_0'u_3 + u_0u_3' \\ &= u_1'u_2 + u_1u_2' - u_1'u_3 - u_2'u_3 - u_3'u_3 + u_0u_3' \\ &= u_1'(u_2 - u_3) + u_2'(u_1 - u_3) + u_3'(u_0 - u_3).\end{aligned}$$

Subsequently, (4.2.26) is true if and only if

$$0 \geq u_1'(u_2 - u_3) + u_2'(u_1 - u_3) + u_3'(u_0 - u_3) \tag{4.2.28}$$

By Corollary 4.2.8, we have

$$u_3'(u_0 - u_3) = u_1'(u_1 - u_0) + u_2'(u_2 - u_0).$$

Therefore the inequality given in (4.2.28) is equivalent to

$$\begin{aligned} 0 \geq & u_1'(u_2 - u_3) + u_2'(u_1 - u_3) + u_1'(u_1 - u_0) + u_2'(u_2 - u_0) \\ = & u_1'(u_1 + u_2 - u_0 - u_3) + u_2'(u_1 + u_2 - u_0 - u_3). \end{aligned}$$

By Corollary 4.2.6, $u_1+u_2-u_0-u_3 < 0$. Thus, (4.2.26) is true if and only if $u_1' \geq -u_2'$. By Lemma 4.2.12, we are done. ∎

We now prove the upper bound of (4.2.10).

Lemma 4.2.16. *We have*

$$\frac{((u_3(\lambda) - u_1(\lambda))(u_2(\lambda) - u_0(\lambda)))'}{((u_1(\lambda) - u_0(\lambda))(u_3(\lambda) - u_2(\lambda)))'} \leq 1. \tag{4.2.29}$$

Proof: We have

$$\begin{aligned} & \frac{((u_3 - u_1)(u_2 - u_0))'}{((u_1 - u_0)(u_3 - u_2))'} \leq 1 \\ \iff & ((u_3 - u_1)(u_2 - u_0))' \leq ((u_1 - u_0)(u_3 - u_2))' \\ \iff & ((u_3 - u_1)(u_2 - u_0) - (u_1 - u_0)(u_3 - u_2))' \leq 0 \\ \iff & (u_3u_2 + u_1u_0 - u_1u_3 - u_0u_2)' \leq 0 \\ \iff & (u_3(u_2 - u_1) + u_0(u_1 - u_2))' \leq 0 \\ \iff & ((u_2 - u_1)(u_3 - u_0))' \leq 0, \end{aligned}$$

where the first statement follows from Corollary 4.2.4. By Lemma 4.2.14, the last if and only if statement is true, and so, we are done. ∎

Our goal to confirm (4.2.10) is done. We only need to sum up our path to get up to this point into two lemmas.

Lemma 4.2.17. *The term* $\sqrt{(u_3 - u_1)(u_2 - u_0)} + \sqrt{(u_1 - u_0)(u_3 - u_2)}$ *is increasing as a function of* λ.

Proof: By Lemma 4.2.15, the lower bound of equation (4.2.10) holds. Thus, combining this with equations (4.2.7) and (4.2.9), we are done. ∎

Lemma 4.2.18. *The term* $\sqrt{(u_3 - u_1)(u_2 - u_0)} - \sqrt{(u_1 - u_0)(u_3 - u_2)}$ *is decreasing as a function of* λ.

Proof: By Lemma 4.2.16, the upper bound of equation (4.2.10) holds. Thus, combining this with equations (4.2.8) and (4.2.9), we are done. ∎

We have finished showing that the numerators of $\sqrt{\epsilon_1}$ and $\sqrt{\epsilon_2}$ are decreasing and increasing respectively. Hence, we are now ready to show that the term $\sqrt{\frac{\epsilon_1}{\epsilon_2}}$ is decreasing.

Lemma 4.2.19. *Let* ϵ_1 *and* ϵ_2 *be as in* (4.1.3)*. Then,* $\sqrt{\frac{\epsilon_1}{\epsilon_2}}$ *is decreasing.*

Proof: By (4.2.5) and (4.2.6), we have

$$\sqrt{\frac{\epsilon_1}{\epsilon_2}} = \frac{\sqrt{(u_3 - u_1)(u_2 - u_0)} - \sqrt{(u_1 - u_0)(u_3 - u_2)}}{\sqrt{(u_3 - u_1)(u_2 - u_0)} + \sqrt{(u_1 - u_0)(u_3 - u_2)}}. \tag{4.2.30}$$

By Lemma 4.2.18, the numerator on the right hand side of (4.2.30) is decreasing whereas by Lemma 4.2.17, the denominator of the right hand side of (4.2.30) is increasing. Consequently, it is clear that $\sqrt{\frac{\epsilon_1}{\epsilon_2}}$ is decreasing. ∎

We are now finally ready to show our ultimate goal of demonstrating that the integral on the left hand side of (4.1.11) is decreasing.

Proposition 4.2.20. *The left hand side of* (4.1.11) *is decreasing.*

Proof: The term in front of the elliptic integral can be rewritten as follows

$$\frac{4}{(u_2-u_1)\sqrt{\epsilon_2}} = \frac{4\sqrt{2}}{\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}}.$$

By Lemma 4.2.17 $\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}$ is increasing, and so, $\frac{4}{(u_2-u_1)\sqrt{\epsilon_2}}$ is decreasing.

By Lemma 4.2.19, we know that $\sqrt{\frac{\epsilon_1}{\epsilon_2}}$ is decreasing. In addition, we know that $0<\sqrt{\frac{\epsilon_1}{\epsilon_2}}<1$. Subsequently, $\int_0^1 \frac{dx}{\sqrt{\left(1-\frac{\epsilon_1}{\epsilon_2}x^2\right)(1-x^2)}}$ is decreasing.

Finally, both $\frac{4}{(u_2-u_1)\sqrt{\epsilon_2}}$ and $\int_0^1 \frac{dx}{\sqrt{\left(1-\frac{\epsilon_1}{\epsilon_2}x^2\right)(1-x^2)}}$ are positive. Consequently, since they are also both decreasing, their product is decreasing. ∎

4.3 Stability

We now know that transit times decrease as λ increases. Thus, there must be a minimum on the interval. In fact, the minimum must be reached at the end of the interval, at $\lambda=\frac{\alpha^3}{6}(2-\alpha)$. Let us use (4.2.5) and (4.2.6) along with (4.1.11) in order to compute this minimum for any value of α. When $\lambda=\frac{\alpha^3}{6}(2-\alpha)$, the polynomial in (4.2.1) has a double root at $u_{1,2}=\alpha$. The other two roots are $u_{0,3}=\frac{1}{3}(2-\alpha\pm\sqrt{2}\sqrt{-\alpha^2+\alpha+2})$. As a result, (4.2.5) and (4.2.6) yield

$$\begin{aligned}\frac{\epsilon_1}{\epsilon_2} &= \frac{\left(\sqrt{(u_3-u_1)(u_2-u_0)}-\sqrt{(u_1-u_0)(u_3-u_2)}\right)^2}{\left(\sqrt{(u_3-u_1)(u_2-u_0)}+\sqrt{(u_1-u_0)(u_3-u_2)}\right)^2}\\ &= \frac{\left(\sqrt{(u_3-u_1)(u_1-u_0)}-\sqrt{(u_1-u_0)(u_3-u_1)}\right)^2}{\left(\sqrt{(u_3-u_1)(u_1-u_0)}+\sqrt{(u_1-u_0)(u_3-u_1)}\right)^2}\end{aligned}$$

$$=0.$$

In addition, we also obtain

$$\begin{aligned}\frac{4}{(u_2-u_1)\sqrt{\epsilon_2}} =&\frac{2\sqrt{2}}{\sqrt{(u_3-u_1)(u_1-u_0)}+\sqrt{(u_1-u_0)(u_3-u_1)}}\\ =&\frac{2\sqrt{2}}{\sqrt{(u_3-u_1)(u_1-u_0)}}.\end{aligned}$$

Let us now calculate $(u_3-u_1)(u_1-u_0)$. We have

$$\begin{aligned}&(u_3-u_1)(u_1-u_0)\\ &\quad=\left(\frac{1}{3}(2-\alpha+\sqrt{2}\sqrt{-\alpha^2+\alpha+2})-\alpha\right)\left(\alpha-\frac{1}{3}(2-\alpha-\sqrt{2}\sqrt{-\alpha^2+\alpha+2})\right)\\ &\quad=\left(\frac{2}{3}-\frac{4}{3}\alpha+\frac{\sqrt{2}}{3}\sqrt{-\alpha^2+\alpha+2})\right)\left(\frac{4}{3}\alpha-\frac{2}{3}+\frac{\sqrt{2}}{3}\sqrt{-\alpha^2+\alpha+2})\right)\\ &\quad=-\left(\frac{4}{3}\alpha-\frac{2}{3}\right)^2+\frac{2}{9}(-\alpha^2+\alpha+2)\\ &\quad=-\frac{16}{9}\alpha^2+\frac{16}{9}\alpha-\frac{4}{9}-\frac{2}{9}\alpha^2+\frac{2}{9}\alpha+\frac{4}{9}\\ &\quad=-2\alpha^2+2\alpha\\ &\quad=2\alpha(1-\alpha).\end{aligned}$$

Consequently, we have

$$\frac{4}{(u_2-u_1)\sqrt{\epsilon_2}}=\frac{2\sqrt{2}}{\sqrt{(u_3-u_1)(u_1-u_0)}}=\frac{2}{\sqrt{\alpha(1-\alpha)}}.$$

As a result, the integral on the right hand side of (4.1.11) yields

$$\frac{4}{(u_2-u_1)\sqrt{\epsilon_2}}\int_0^1\frac{dx}{\sqrt{\left(1-\frac{\epsilon_1}{\epsilon_2}x^2\right)(1-x^2)}}=\frac{2}{\sqrt{\alpha(1-\alpha)}}\int_0^1\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{\sqrt{\alpha(1-\alpha)}}.$$

Subsequently, the minimum transit time of half orbits (when $D = 1$) is $\frac{\pi}{\sqrt{\alpha(1-\alpha)}}$.

Now, as discussed at the beginning of this chapter, steady states are absorbed by α as D increases. Hence, we must look at the dynamics of the system around the steady state α. This leads to the first step, linearizing our system around α. We need to find the eigenvalues associated to α in order to figure out if α is stable or unstable. This leads to the following result.

Proposition 4.3.1. *The eigenvalues of α are of the form*

$$\delta = D\pi^2 n^2 - \alpha(1-\alpha),$$

where $n \in \mathbb{N}$. As a result, regardless of the value of D, there is a negative eigenvalue associated with α and so, α is unstable.

Proof: The eigenvalue equation associated with α is

$$-\delta v(x) = Dv_{xx}(x) + \alpha(1-\alpha)v(x). \tag{4.3.1}$$

Rearranging gives us

$$v_{xx} = -\frac{\delta + \alpha(1-\alpha)}{D} v(x). \tag{4.3.2}$$

Suppose $-\delta - \alpha(1-\alpha) > 0$, i.e., $\delta < -\alpha(1-\alpha)$. Then, solutions are of the form

$$v(x) = c_1 \exp\left(\sqrt{-\frac{\delta + \alpha(1-\alpha)}{D}} x\right) + c_2 \exp\left(-\sqrt{-\frac{\delta + \alpha(1-\alpha)}{D}} x\right),$$

where c_1 and c_2 are some constants. In order for $v(x)$ to be an eigenfunction, it needs to satisfy the boundary conditions. Differentiating yields

$$v'(x) = c_1 \sqrt{-\frac{\delta + \alpha(1-\alpha)}{D}} \exp\left(\sqrt{-\frac{\delta + \alpha(1-\alpha)}{D}} x\right)$$

$$- c_2\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}\exp\left(-\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}x\right).$$

Applying the left boundary condition ($v'(0) = 0$) gives us

$$0 = v'(0) = c_1\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}} - c_2\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}} = \sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}(c_1 - c_2).$$

As $\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}} \neq 0$, we must then have $c_1 = c_2$. So $v(x)$ is now of the form

$$v(x) = c_1\exp\left(\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}x\right) + c_1\exp\left(-\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}x\right).$$

The right boundary condition ($v'(1) = 0$) implies

$$0 = v'(1) = c_1\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}\exp\left(\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}\right) - c_1\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}\exp\left(\sqrt{-\frac{\delta+\alpha(1-\alpha)}{D}}\right)$$

Since $-\frac{\delta+\alpha(1-\alpha)}{D}$ is positive, the only way that $v'(1) = 0$ is if $c_1 = 0$ which then implies that $v(x) = 0$.

If $-\delta - \alpha(1-\alpha) = 0$, i.e., if $\delta = -\alpha(1-\alpha)$, then (4.3.2) becomes

$$v''(x) = 0.$$

So, $v(x) = c_1x + c_2$ for some constants c_1 and c_2. The left boundary conditions yields

$$0 = v'(0) = c_1.$$

So $v(x) = c_2$. As $v(x)$ is just a constant, it automatically satisfies the right boundary condition.

If $-\delta - \alpha(1-\alpha) < 0$, i.e., if $\delta > -\alpha(1-\alpha)$, then solutions of (4.3.2) are of the form

$$v(x) = c_1 \sin\left(\sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}x\right) + c_2 \cos\left(\sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}x\right),$$

where c_1 and c_2 are constants. Since $v'(0) = 0$, we obtain

$$0 = \sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}c_1 \cos(0) - \sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}c_2 \sin(0) = \sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}c_1 \cos(0).$$

Therefore, $c_1 = 0$. Thus,

$$v(x) = c_2 \cos\left(\sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}x\right).$$

Using the other condition, $v'(1) = 0$, we then get

$$0 = -\sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}c_2 \sin\left(\sqrt{\frac{\delta + \alpha(1-\alpha)}{D}}\right).$$

Hence, we need $\sqrt{\frac{\delta+\alpha(1-\alpha)}{D}} = \pi n$ for $n \in \mathbb{Z}$. Subsequently, we must have

$$\frac{\delta + \alpha(1-\alpha)}{D} = \pi^2 n^2.$$

Consequently,

$$\delta = D\pi^2 n^2 - \alpha(1-\alpha), \tag{4.3.3}$$

where $n \in \mathbb{N}$. When $n = 0$, we have $\delta = -\alpha(1-\alpha)$. Since $0 < \alpha < \frac{1}{2}$, we must have $\delta < 0$. Therefore α always has a negative eigenvalue. This then implies that α is unstable. ∎

From (4.3.3), we can see that a zero eigenvalue appears when D reaches $\frac{\alpha(1-\alpha)}{\pi^2 n^2}$

$(n \in \mathbb{N} \setminus \{0\})$. When $D < \frac{\alpha(1-\alpha)}{\pi^2 n^2}$, the eigenvalue is negative which corresponds to instability. When $D = \frac{\alpha(1-\alpha)}{\pi^2 n^2}$, two steady states are consumed by α and the eigenvalue is now 0. Once, $D > \frac{\alpha(1-\alpha)}{\pi^2 n^2}$, the eigenvalue is now positive which corresponds to stability. However, we must be careful because α always remains unstable because of its negative eigenvalue that is constantly there.

Let us cite the center manifold theorem that appears in [4].

Theorem 4.3.2. *Consider the following ordinary differential equations in* $\mathbb{R}^{n+m}$*:*

$$\begin{aligned} x' &= Ax + f(x, y), \\ y' &= By + g(x, y), \end{aligned} \tag{4.3.4}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A *and* B *are* $n \times n$ *and* $m \times m$ *constant matrices, respectively, and* f *and* g *are nonlinear maps. Assume that* $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $g\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfy* $f(x, y) = \mathcal{O}(|x+y|^2)$ *and* $g(x, y) = \mathcal{O}(|x+y|^2)$ *as* $(x, y) \to (0, 0)$*. Assume that the spectra* $\sigma(A)$ *and* $\sigma(B)$ *satisfy the following condition:*

$$Re(\sigma(A)) = 0 \quad \textit{and} \quad Re(\sigma(B)) < 0. \tag{4.3.5}$$

Then for every integer $k \geq 1$ *there exists a* $\delta_k > 0$ *such that if* $f \in C^k(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $g \in C^k(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$ *and* $|f|_k + |g|_k \leq \delta_k$*, then there exists a unique* C^k *(global) center manifold of* (4.3.4).

We want to use the center manifold theorem in order to determine the dyamics around α. This will allows us to confirm that a pitchfork bifurcation occurs around α. Then, as α is always unstable, we will be able to deduce that the partterned steady states are unstable. However, the previous theorem is only valid for finite-dimensional systems. Therefore, we refer the reader to [22, Theorem 1] since much more lead up is required for the infinite dimensional instance. Nevertheless, the essence of the center manifold theorem in infinite dimensions is still captured in the finite-dimensional case.

Now, in order to do this analysis, we must split our space into two orthogonal subspaces. One will contain the eigenfunctions, while the other will be perpendicular to this space. Thus, when solutions of our system are close to α, we may write them as follows

$$u(x,t) = \alpha + r(t)\cos(\pi x) + v(x,t), \tag{4.3.6}$$

where $v(x,t)$ and $\cos(\pi x)$ are perpendicular, i.e., we have

$$\int_0^1 \cos(\pi x) v(x,t) dx = 0. \tag{4.3.7}$$

In addition, $v(x,t)$ must satisfy the no-flux boundary conditions. Note that this space splitting is used when the diffusion parameter D is equal to $\frac{\alpha(1-\alpha)}{\pi^2 n^2}$. In particular, this one is for $D = \frac{\alpha(1-\alpha)}{\pi^2}$. Therefore, this will allow us to check the bifurcation diagram when D approaches and then exceeds $\frac{\alpha(1-\alpha)}{\pi^2}$. Verifying the bifurcation diagram around the other critical values of D can be done in an analgous way. We believe that it will give us the same end result, a pitchfork bifurcation.

Before continuing on and substituting (4.3.6) into our initial system given in (4.0.1), let us investigate the repercussions of (4.3.7) in terms of calculating other similar inner products.

Lemma 4.3.3. *Consider a function $v(x,t)$ such that $v_x(0,t) = v_x(1,t) = 0$ and*

$$\int_0^1 \cos(\pi x) v(x,t) dx = 0. \tag{4.3.8}$$

Then, we have

$$\int_0^1 v_t(x,t)\cos(\pi x) dx = 0, \tag{4.3.9}$$

and

$$\int_0^1 v_{xx}(x,t)\cos(\pi x) dx = 0. \tag{4.3.10}$$

Proof: Let us first prove (4.3.9). We have

$$\int_0^1 v_t(x,t)\cos(\pi x)dx = \int_0^1 \frac{d(v(x,t)\cos(\pi x))}{dt}dx = \frac{d}{dt}\left(\int_0^1 (v(x,t)\cos(\pi x))dx\right) = 0.$$

Let us now prove (4.3.10). Integrating by parts twice leads to

$$\begin{aligned}\int_0^1 v_{xx}(x,t)\cos(\pi x)dx &= (v_x(x,t)\cos(\pi x))_0^1 + \pi\int_0^1 v_x(x,t)\sin(\pi x)dx \\ &= \pi\left((v(x,t)\sin(\pi x))_0^1 - \pi\int_0^1 \cos(\pi x)v(x,t)dx\right) \\ &= 0.\end{aligned}$$

∎

We are now ready to check the bifurcation occuring around the steady state α.

Theorem 4.3.4. *Up to leading order, there is a pitchfork birfucation occuring around α when D approaches $\frac{\alpha(1-\alpha)}{\pi^2}$.*

Proof: Set $u(x,t) = \alpha + r(t)\cos(\pi x) + v(x,t)$ where $v(x,t)$ satisfies (4.3.8). By (4.0.1), we obtain

$$r'(t)\cos(\pi x) + v_t(x,t) = -D\pi^2 r(t)\cos(\pi x) + Dv_{xx}(x,t)$$

$$+ (\alpha + r(t)\cos(\pi x) + v(x,t))\left((1-\alpha - r(t)\cos(\pi x) - v(x,t)\right)\left((r(t)\cos(\pi x) + v(x,t)\right),$$

and so,

$$\begin{aligned} r'(t)\cos(\pi x) + v_t(x,t) =& Dv_{xx}(x,t) + (-D\pi^2 - \alpha^2 + \alpha)r(t)\cos(\pi x) \\ &+ (\alpha - \alpha^2 + (2-4\alpha)r(t)\cos(\pi x) - 3r^2(t)\cos^2(\pi x))v(x,t) \\ &+ (1 - 2\alpha - 3r(t)\cos(\pi x))v^2(x,t) - v^3(x,t) \\ &+ (1-2\alpha)r^2(t)\cos^2(\pi x) - r^3(t)\cos^3(\pi x). \end{aligned} \tag{4.3.11}$$

Since both sides are equal, the inner product of both sides with $\cos(\pi x)$ is equal. Thus, combining this with Lemma (4.3.3) yields

$$\begin{aligned} r'(t) =& (-D\pi^2 + \alpha - \alpha^2)r(t) - \frac{3}{4}r^3(t) \\ &+ 2r(t)\left((2-4\alpha)\int_0^1 v(x,t)\cos^2(\pi x)dx - 3\int_0^1 v^2(x,t)\cos^2(\pi x)dx\right) \\ -6r^2(t)&\int_0^1 v(x,t)\cos^3(\pi x)dx + (2-4\alpha)\int_0^1 v^2(x,t)\cos(\pi x)dx - 2\int_0^1 v^3(x,t)\cos(\pi x)dx. \end{aligned} \tag{4.3.12}$$

From (4.3.11), we can also obtain

$$\begin{aligned} v_t(x,t) =& -r'(t)\cos(\pi x) + Dv_{xx}(x,t) + (-D\pi^2 - \alpha^2 + \alpha)r(t)\cos(\pi x) \\ &+ (\alpha - \alpha^2 + (2-4\alpha)r(t)\cos(\pi x) - 3r^2(t)\cos^2(\pi x))v(x,t) \\ &+ (1 - 2\alpha - 3r(t)\cos(\pi x))v^2(x,t) - v^3(x,t) + (1-2\alpha)r^2(t)\cos^2(\pi x) - r^3(t)\cos^3(\pi x). \end{aligned} \tag{4.3.13}$$

We know that $v(x,t) = (r^2(t)q(x) + \text{h.o.t.})$ since we are working on the center manifold. Thus,

$$v_t(x,t) = 2r(t)r'(t)q(x) + \text{h.o.t.} \quad \text{and} \quad v_{xx}(x,t) = r^2(t)q''(x) + \text{h.o.t.}. \tag{4.3.14}$$

Thus, combining this with (4.3.13) gives us

$$
\begin{aligned}
2r(t)r'(t)q(x) = -r'(t)\cos(\pi x) + D(r^2(t)q''(x) + \text{h.o.t.}) + (-D\pi^2 - \alpha^2 + \alpha)r(t)\cos(\pi x) \\
+ (\alpha - \alpha^2 + (2 - 4\alpha)r(t)\cos(\pi x) - 3r^2(t)\cos^2(\pi x))(r^2(t)q(x) + \text{h.o.t.}) \\
+ (1 - 2\alpha - 3r(t)\cos(\pi x))(r^4(t)q^2(x) + \text{h.o.t.}) - (r^6(t)q^3(x) + \text{h.o.t.}) \\
+ (1 - 2\alpha)r^2(t)\cos^2(\pi x) - r^3(t)\cos^3(\pi x).
\end{aligned}
$$

When $v(x,t) = (r^2(t)q(x) + \text{h.o.t.})$, the $r^2(t)$ terms on the left hand side are

$$2(-D\pi^2 - \alpha^2 + \alpha)r^2(t)q(x),$$

whereas the $r^2(t)$ terms on the right hand side are

$$
\begin{aligned}
&(Dq''(x) + (\alpha - \alpha^2)q(x) + (1 - 2\alpha)\cos^2(\pi x))r^2(t) \\
&\qquad = (Dq''(x) + (\alpha - \alpha^2)q(x) + \frac{(1 - 2\alpha)}{2}(\cos(2\pi x) + 1))r^2(t).
\end{aligned}
$$

Therefore, we obtain a second order nonhomogeneous ODE for $q(x)$

$$-Dq''(x) + (-2D\pi^2 - \alpha^2 + \alpha)q(x) = (1 - 2\alpha)\cos^2(\pi x) = \frac{(1 - 2\alpha)}{2}(\cos(2\pi x) + 1). \tag{4.3.15}$$

Note that, since $v_x(0,t) = v_x(1,t) = 0$ ($\forall t \geq 0$), we must have $q'(0) = q'(1) = 0$. Now, the solution to the homogeneous form of (4.3.15) is

$$q(x) = c_1 \cos\left(\sqrt{\frac{-2D\pi^2 + \alpha - \alpha^2}{D}}x\right) + c_2 \sin\left(\sqrt{\frac{-2D\pi^2 + \alpha - \alpha^2}{D}}x\right), \tag{4.3.16}$$

where c_1 and c_2 are constants. In addition, the functions

$$q_1(x) = \frac{1-2\alpha}{2(-2D\pi^2+\alpha-\alpha^2)} \qquad \text{and} \qquad q_2(x) = \frac{1-2\alpha}{2(2D\pi^2+\alpha-\alpha^2)}\cos(2\pi x) \tag{4.3.17}$$

are particular solutions to the second order ordinary differential equations

$$-Dq''(x) + (-2D\pi^2 - \alpha^2 + \alpha)q(x) = \frac{(1-2\alpha)}{2}$$

and

$$-Dq''(x) + (-2D\pi^2 - \alpha^2 + \alpha)q(x) = \frac{(1-2\alpha)}{2}\cos(2\pi x)$$

respectively. Thus, by (4.3.16) and (4.3.17), the general solution to (4.3.15) is

$$\begin{aligned} q(x) = c_1 \cos\left(\sqrt{\frac{-2D\pi^2+\alpha-\alpha^2}{D}}x\right) + c_2 \sin\left(\sqrt{\frac{-2D\pi^2+\alpha-\alpha^2}{D}}x\right) \\ + \frac{1-2\alpha}{2(-2D\pi^2+\alpha-\alpha^2)} + \frac{1-2\alpha}{2(2D\pi^2+\alpha-\alpha^2)}\cos(2\pi x). \end{aligned} \tag{4.3.18}$$

As stated earlier, we need $q'(0) = q'(1) = 0$. For $q'(0) = 0$, this means

$$\begin{aligned} 0 = q'(0) &= -\sqrt{\frac{-2D\pi^2+\alpha-\alpha^2}{D}}c_1 \sin(0) \\ &\quad + \sqrt{\frac{-2D\pi^2+\alpha-\alpha^2}{D}}c_2 \cos(0) - 2\pi\frac{1-2\alpha}{2(2D\pi^2+\alpha-\alpha^2)}\sin(0) \\ &= \sqrt{\frac{-2D\pi^2+\alpha-\alpha^2}{D}}c_2. \end{aligned}$$

Hence, we must have $c_2 = 0$. As for the condition $q'(1) = 0$, we get

$$\begin{aligned} 0 = q'(1) = -\sqrt{\frac{-2D\pi^2+\alpha-\alpha^2}{D}}c_1 \sin\left(\sqrt{\frac{-2D\pi^2+\alpha-\alpha^2}{D}}\right) \\ - 2\pi\frac{1-2\alpha}{2(2D\pi^2+\alpha-\alpha^2)}\sin(2\pi) \end{aligned}$$

$$= -\sqrt{\frac{-2D\pi^2 + \alpha - \alpha^2}{D}} c_1 \sin\left(\sqrt{\frac{-2D\pi^2 + \alpha - \alpha^2}{D}}\right).$$

Therefore, we can deduce that $c_1 = 0$. Consequently, $q(x)$ is given by

$$q(x) = \frac{(1-2\alpha)}{2(2D\pi^2 + \alpha - \alpha^2)} \cos(2\pi x) + \frac{1-2\alpha}{2(-2D\pi^2 + \alpha - \alpha^2)}.$$

Now, we need $D = \frac{\alpha(1-\alpha)}{\pi^2}$. Thus, $q(x)$ becomes

$$q(x) = \frac{(1-2\alpha)}{6\alpha(1-\alpha)} \cos(2\pi x) + \frac{1-2\alpha}{-2\alpha(1-\alpha)}. \tag{4.3.19}$$

Before continuing on, let us compute $\int_0^1 \cos^2(\pi x) q(x) dx$ as we will be needing it shortly. We have

$$\int_0^1 q(x) \cos^2(\pi x) dx = \frac{(1-2\alpha)}{6\alpha(1-\alpha)} \int_0^1 \cos^2(\pi x) \cos(2\pi x) dx - \frac{1-2\alpha}{2\alpha(1-\alpha)} \int_0^1 \cos^2(\pi x) dx$$
$$= \frac{1-2\alpha}{24\alpha(1-\alpha)} - \frac{1-2\alpha}{4\alpha(1-\alpha)} = -\frac{5(1-2\alpha)}{24\alpha(1-\alpha)}.$$

Subsequently, we get

$$\int_0^1 q(x) \cos^2(\pi x) dx = -\frac{5(1-2\alpha)}{24\alpha(1-\alpha)}. \tag{4.3.20}$$

Now, keeping the most significant terms in (4.3.12), i.e., the $r(t)$ and $r^3(t)$ terms (and keeping in mind that $v(x,t) = r^2(t)q(x) + \text{h.o.t.}$), we get

$$r'(t) = (-D\pi^2 + \alpha - \alpha^2) r(t) + \left((4 - 8\alpha) \int_0^1 q(x) \cos^2(\pi x) dx - \frac{3}{4}\right) r^3(t).$$

By (4.3.20), we then get

$$r'(t) = (-D\pi^2 + \alpha - \alpha^2) r(t) + \left(-(4 - 8\alpha) \frac{5(1-2\alpha)}{24\alpha(1-\alpha)} - \frac{3}{4}\right) r^3(t),$$

and so,

$$r'(t) = (-D\pi^2 + \alpha - \alpha^2)r(t) + \left(\frac{-31\alpha^2 + 31\alpha - 10}{12\alpha(1-\alpha)}\right) r^3(t).$$

Therefore, $r'(t) = 0$ if and only if

$$r(t) = 0 \quad \text{or} \quad r(t) = \pm\sqrt{\frac{12(D\pi^2 - \alpha(1-\alpha))\alpha(1-\alpha)}{-31\alpha^2 + 31\alpha - 10}}, \tag{4.3.21}$$

when D is less than $\frac{\alpha(1-\alpha)}{\pi^2}$. Consequently, we do in fact have a pitchfork bifurcation diagram. ∎

We believe that this same method can be used to achieve the same results for the other critical points of D. Thus, a consequence of Theorem 4.3.4 is that the patterned steady states are unstable since α is unstable.

Chapter 5

Conclusion

In this thesis, we have investigated the bistable one-component reaction-diffusion equation. We treated both the mixed and no-flux boundary conditions. We first noticed that studying these systems on an interval of the form $[0, L]$ ($L > 0$) is equivalent to studying the system on the unit interval $[0, 1]$ by dividing the diffusion parameter by the squared length of the initial interval, L^2.

By way of phase-plane analysis, we determined that steady states of both systems must have a transit time equal to 1, which became the foundation of our bifurcation analysis. This condition became essential in finding the number of steady states of both systems.

When the boundary conditions are mixed, our investigation led to a saddle-node bifurcation when the diffusion parameter varied. This allowed us to conclude, after examining the eigenvalues of the steady states, that a non-trivial stable solution can be achieved granted the diffusion parameter is small enough. Our analysis then led us to the Dirichlet boundary conditions. We looked at orbits in the phase-plane corresponding to these boundary conditions. We noticed that these orbits have twice the transit time of orbits corresponding to mixed boundary conditions. This permitted us to extend our analysis of the mixed boundary conditions to the Dirichlet boundary conditions and conclude the same results. We then concluded the analysis with conjectures about the parameter α, which appears in the reaction term. We discussed the possible benefits that a smaller α could have on the existence of stable patterned steady states of the system.

We then moved on to the no-flux boundary conditions, which was the main motivation of this thesis. Our motivation originated from the positive results achieved in the two-component reaction-diffusion equation (see [14]). The phase-plane analysis led to us investigating the steady state α. We determined that the key to understanding the stability of steady states of our system was to comprehend the dynamics around α. This led to the use of the center manifold theorem around α. Then, combining the instability of α with a pitchfork bifurcation occuring around α, we determined that the patterned steady states of our system are unstable. Thus, contrary to the two-component system, it is not possible to attain patterned steady states in the one-component system.

In both instances, we used an integral to calculate the time of transit of orbits in the phase-plane. In order to facilitate calculating this integral for various values of alpha, we found new forms of both integrals. In particular, in the no-flux boundary conditions case, we were able to use this new form to show that the time of transit of orbits in the phase-plane is a decreasing function of λ (the parameter that decides where the the orbits starts).

Although real life problems occur in three dimensions, we studied the reaction-diffusion equation on a one dimensional spatial variable. We do this in order to simplify overly complex situations so that deciphering the problem becomes easier. Doing this means that the solutions to our system represent the density of (among other various possibilities) a population at a certain area of an ecosystem or of a chemical along the length of a cell. As a result, in the context of the population of a species in an ecosystem, we do not account for every direction individuals within the ecosystem can spread. For a chemical in a cell, we do not take into consideration whether the cell diffuses in any direction other than right or left. It would be interesting to see if the results seen in this thesis can extend to a three-dimensional space. However, problems would arise when considering a three-dimensional space. For example, we used a phase-plane analysis to jump start our investigation of our systems. This approach works because the spatial variable lies in one dimension. Hence, we would need to find another way to even begin an analysis of the system in three dimensions. In addition, our boundary conditions would now lie on the boundary of a cube rather than on both ends of a line. It is safe to say that the analysis on a three-dimensional space would be much more complex.

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