Generalized Differential and Integral Categories:

Differential Rota-Baxter Algebra, Ribenboim Power Series, and Quantum Calculus

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Abstract

This paper provides two generalizations of differential and integral categories: Leibniz and generalized Rota-Baxter categories, which capture certain algebraic structures, and q-categories, which capture structures of quantum calculus.

In the search for new examples of differential and integral categories, it was observed that many structures were not quite examples but satisfied certain properties and not others. This leads us to the definition of Leibniz, Rota-Baxter and proto-FTC categories.

In generalizing Rota-Baxter categories further to an arbitrary weight, we show that we recapture Ribenboim's generalized power series as a monad on vector spaces with a generalized integral transformation. This also subsumes the renormalization operator on Laurent series, which has applications in the quantum realm.

Finally, we define quantum differential and quantum integral categories, show that they recapture the usual notions of quantum calculus on polynomials, and construct a new example to indicate their potential usefulness outside of that specific setting.

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1 Introduction

Linear logic began with the work of Girard in [21], and captures the structure of classical logic while also adding mechanisms to track resources and make resource-dependent deductions. Its category-theoretic content was later expanded upon by Seely in [39].

Shortly thereafter, Ehrhard began building categorical models of linear logic [15, 16], and observed that in his models there were natural notions of differentiation. This led him, along with Regnier, to introduce a differential logical operator and the differential λ-calculus [17, 18].

To reconstruct Ehrhard and Regnier's abstract notion of differentiation in a purely categorical setting, Blute, Cockett and Seely introduced *differential categories* in [7]. Their main ingredient was a *differential combinator*, an operator that could be applied to maps that were in some abstract sense "smooth" and that produced a smooth assignment of linear maps, inspired by the Jacobian of multivariable calculus. This formulation was then transported across the monoidal-closed adjunction to accommodate not-necessarily-closed categories.

The advent of differential categories launched an effort to categorify the setting of smooth maps, differentiable manifolds and tangent bundles from differential geometry and topology. Via the intermediary of *cartesian differential categories*, also introduced by Blute, Cockett and Seely [6], and *differential restriction categories*, introduced by Cockett, Cruttwell and Gallagher in [11], this culminated in the generalization by Cockett and Cruttwell in [12] of the *tangent categories* first introduced in [37]. Tangent categories axiomatize the general tangent bundle functor, which is a functor $\mathbb{T}\colon \mathcal{C} \to \mathbb{C}$ satisfying certain equations. Current work expanding and unifying these concepts is being done by many of the authors above, as well as others, with particular mention for work in dualizing to the *integral category* setting due to J.S. Lemay among others [3, 13, 31].

The work of this paper began with the observation that several well-known structures satisfy the Leibniz rule of differential categories or the Rota-Baxter rule (i.e. integration by parts) of integral categories, but not other rules like the chain rule or U-substitution. These structures seemed important enough to be worthy of axiomatization as generalizations of differential and integral categories.

This led to exploring Rota-Baxter algebra in more detail, and remarking that one of its applications to quantum renormalization, as studied by Ebrahimi-Fard and Guo in [14], appeared to constitute an algebra modality. It was then natural to further generalize the definition of Rota-Baxter category to include arbitrary weights, and show that this does indeed relate to renormalization. In fact, it turns out to categorically capture *generalized power series*, with the broad class of Rota-Baxter operations on them explored by Ribenboim in works like [35].[2]

We proceed along this path first by following the construction of the free differential Rota-Baxter algebra. We show that the free Rota-Baxter algebra

[2] We refer to these generalized power series as *Ribenboim power series*.

(of weight 0) monad makes the category of vector spaces into a Rota-Baxter category. We similarly show that the free differential algebra monad gives a Leibniz category, before uniting the two structures to show that the free differential Rota-Baxter algebra is an example of both. Next, we generalize to an arbitrary weight, and describe precisely how Ribenboim power series constitute a monad and induce a Rota-Baxter category structure of weight -1. Particular attention is given to the example of Laurent series and the renormalization operator.

The second part of this paper is the definition and description of *quantum differential and integral categories*. Studying the Leibniz and Rota-Baxter rules, along with parallel study of categorical quantum information theory and a cognizance of previous connections between categorical logic and quantum algebra, led to an examination of *quantum calculus*. Quantum calculus, as an oversimplification, might be described as calculus without limits; that is, rather than utilizing the continuous notion of an infinitesimal, quantum calculus instead uses an incompressible discrete quantity in similar contexts. It turns out that a surprising amount of calculus can be rederived in this setting, and leads to formulations of the quantum Leibniz rule, quantum Stokes Theorem, and quantum integration, among others. Much of this work is elucidated by Kac and Cheung in [28].

It seemed natural then to categorify quantum calculus, following the model of differential categories. In so doing we end up making use of the notions of categorical scalar and categorical basis, already current in categorical quantum information theory (see e.g. Abramsky and Coecke [1], Vicary [26]). After showing that the categorical structure does indeed capture the basic notion of quantum differentiation of polynomials, we proceed to the more interesting step of describing a new quantum differential example on the category *Rel* of sets and relations, based on the finitary multiset comonad. It is hoped that these structures can provide further insight into the abstract quantum setting in the future.

What's New: The definitions of Leibniz, Rota-Baxter and proto-FTC categories are new to this paper, as are the proofs that certain known structures (the free differential, Rota-Baxter, and differential Rota-Baxter, algebras) are examples of them. The definition of a module modality associated to an algebra modality is a new one, though examples of it have previously been implied in the literature. The characterization of Ribenboim power series as an algebra modality with Rota-Baxter category structure is new as well, as are the particular distributive laws of monads used in the proofs. Finally, the definition of quantum differential and quantum integral categories in analogy with Blute, Cockett and Seeley's structures is new, as are the demonstrations of the examples of them.

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2 Differential, Integral and FTC Categories

In this section we'll recall the fundamental definitions of differential and integral categories, and we'll define the new Leibniz and (weight-zero) Rota-Baxter categories, the utility of which will be argued in the following section. We also demonstrate some propositions relating these categories with the structures of modules with differentiation and integration in the category $\mathbb{K}$-Vec.

2.1 Differential Categories

Differential categories were first described in [7] extending ideas from linear logic and differential lambda calculus. In categorical logic, the internal hom models classical implication in cartesian closed categories, and in certain categories this can be decomposed into a linear implication and a comonad:

$$A \Rightarrow B = !A \multimap B$$

Note that the right-hand side of this equation has the form of a coKleisli map. Then, with the intuition that maps in a given monoidal category should be thought of as linear, and maps in the coKleisli category should be thought of as smooth, the authors of [7] reasoned about the form a categorical "derivative" should take by analogy with vector calculus as follows.

If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function, then its Jacobian matrix $\mathbf{J}$ takes in a choice of point $x = (x_1, \ldots, x_m)$ to produce a linear map $\mathbf{J}(x)\colon \mathbb{R}^m \to \mathbb{R}^n$. Taking this choice of point is itself a smooth function, and so given a smooth map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ we get a smooth map $D[f]\colon \mathbb{R}^m \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

In the notation of categorical logic, this is $D[f]\colon !A \to A \multimap B$. For increased generality, we would like to work in categories that are not necessarily closed, so this map is shifted back across the monoidal-closed adjunction to obtain a transformation of the following type:

$$f\colon !A \to B \quad \mapsto \quad D(f)\colon A \otimes !A \to B$$

Keeping this intuition in mind, we now review the formal definition.

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category enriched over commutative monoids, where the tensor distributes over the sum of maps, and let $\mathcal{C}$ be equipped with a comonad $(!, \delta, \varepsilon)$, where δ is the comonadic comultiplication and ε is the comonadic counit.

Such a comonad is a *coalgebra modality* if each object of the form $!A$ is equipped with a coalgebra structure $(!A, \Delta, e)$, where $\Delta\colon !A \to !A \otimes !A$ and $e\colon !A \to I$ and such that $(!A, \Delta, e)$ is a comonoid and δ is a morphism of these comonoids.

Definition 2.1.1. *A category $\mathcal{C}$ with such a coalgebra modality $!$ is a* differential category *if it is also equipped with a collection of maps $D_{A,B}\colon Hom(!A, B) \to Hom(A \otimes !A, B)$, a transformation natural in A and B called a* differential combinator*, satisfying the four properties below. For diagrammatic intuition, we shall sometimes write $f; g := g \circ f$.*

(Derivative of Constants rule) $D(e) = 0$

(Leibniz Rule)

$$D(\Delta; (f \otimes g)) = (id \otimes \Delta); (D(f) \otimes g) \; + \; (id \otimes \Delta); (\sigma \otimes id); (f \otimes D(g))$$

where σ is the monoidal symmetry, $f \colon !A \to B$, $g \colon !A \to C$, and we have omitted the monoidal associativity isomorphism.

(Derivative of Linear Maps Rule) $D(\varepsilon; f) = (1 \otimes e); f$ where $f \colon A \to B$ and we have omitted the monoidal unit isomorphism.

(Chain Rule)

$$D(\delta; !f; g) = (id \otimes \Delta); (D(f) \otimes \delta!f); D(g)$$

where $f \colon !A \to B$, $g \colon !B \to C$ and we have again omitted the associativity isomorphism.

The definition of $D_{A,B}$ being natural in A and B is that it carries commuting diagrams to commuting diagrams, like so:

$$\begin{array}{ccc} \begin{tikzcd} !A \arrow[r, "f"] \arrow[d, "!u"] & B \arrow[d, "v"] \\ !C \arrow[r, "g"] & D \end{tikzcd} & \Rightarrow & \begin{tikzcd} A\otimes!A \arrow[r, "D(f)"] \arrow[d, "u\otimes!u"] & B \arrow[d, "v"] \\ C\otimes!C \arrow[r, "D(g)"] & D \end{tikzcd} \end{array}$$

In particular, employing two identity maps gives the following relation:

$$\begin{array}{ccc} \begin{tikzcd} !A \arrow[r, "id"] \arrow[d, "!id"] & !A \arrow[d, "f"] \\ !A \arrow[r, "f"] & B \end{tikzcd} & \Rightarrow & \begin{tikzcd} A\otimes!A \arrow[r, "D(id)"] \arrow[d, "id\otimes!id"] & !A \arrow[d, "f"] \\ A\otimes!A \arrow[r, "D(f)"] & B \end{tikzcd} \end{array}$$

If we write $D(id_{!A}) = d_A$, this shows that $D(f) = f \circ d_A = d_A; f$. Thus d_A can in fact be used to express the entire range of the map D. We call this $d_A \colon A\otimes!A \to !A$ a *deriving transformation*.

It is proved in [7] that an additive symmetric monoidal category with a coalgebra modality is a differential category (i.e. it has a differential combinator) if and only if it has a deriving transformation. Thus we may work exclusively with deriving transformations, and their versions of the rules above, to prove statements about differential categories.

(d1) Derivative of Constants Rule:

$$d; e = 0$$

(d2) Leibniz Rule:

$$d; \delta = (id \otimes \Delta); (d \otimes id) + (id \otimes \Delta); (\sigma \otimes id); (id \otimes d)$$

(d3) Derivative of Linear Maps Rule:

$$d; \varepsilon = (id \otimes e); \rho$$

(d4) Chain Rule:

$$d; \delta = (id \otimes \Delta); (d \otimes \delta); d$$

The dual of a differential category is a codifferential category, built using a monad and the analogous notions of algebra modality and coderiving transformation. We will sometimes be loose with the terminological distinction, sometimes referring to codifferential categories simply as "differential categories". Nearly all of our constructions later on will be codifferential categories.

2.1.1 Examples

1. Let *Rel* be the category of sets and relations, additively enriched via the union and monoidal via the standard cartesian product. Let ! be the *bag functor* on *Rel*; that is, elements of $!A$ are finitary multisets of elements of A, or sets of elements of A where each element may be counted repeatedly, up to finitely many times.

   Define natural transformations $\delta_A \colon !A \to !!A$ and $\varepsilon_A \colon !A \to A$ as follows: δ relates a multiset X to all multisets of multisets whose union is X, and ε relates each singleton multiset in $!A$ to its element. It is easy to see this makes $(!, \delta, \varepsilon)$ into a comonad.

   Next, let $\Delta \colon !A \to !A \times !A$ relate all multisets X to all pairs of multisets whose union is X, and let $e \colon !A \to I$ relate only the empty multiset to the single element of $I = \{*\}$. This defines a coalgebra modality for !.

   We can define a natural transformation $d_A \colon A \times !A \to !A$ by

   $$d_A(a_0, \{\!\{a_1, \ldots, a_n\}\!\}) = \{\!\{a_0, a_1, \ldots, a_n\}\!\}$$

   which is just adding the extra element to the multiset. This is a deriving transformation on *Rel* (see [7] for a sketch of the proof), and thus makes *Rel* into a differential category.

2. The category $\mathbb{K}$-Vec of vector spaces over a field $\mathbb{K}$ can be equipped with the free commutative algebra monad S, and we may then look at the opposite category, $\mathbb{K}$-Vec^{op}. This opposite category's coKleisli subcategory

(that is, the category whose objects and arrows are of the form $S(V)$ and $S(f)\colon S(V) \to W$ for all $\mathbb{K}$-vector spaces V and linear maps $f\colon V \to W$, respectively) is precisely the category of (multivariable) polynomial functions on basis elements, and the opposite of the monad S gives a coalgebra modality. Define a deriving transformation $d_V : S(V) \to V \otimes S(V)$ on basis elements x_i by

$$d(f) = \sum_i x_i \otimes \frac{\partial f}{\partial x_i}.$$

This makes $\mathbb{K}$-Vec^{op} into a differential category, and moreover recaptures the standard notion of polynomial differentiation, further justifying the definition of differential category. For a more detailed treatment and proof of this example, see [7].

2.2 Integral Categories

Next we recall the related notion of an integral category.

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category enriched over commutative monoids, where the tensor distributes over the sum of maps, equipped with a comonad $(!, \delta, \varepsilon)$ which is also a coalgebra modality $(!, \Delta, e)$, as in the differential category case above.

Definition 2.2.1. *The category $\mathcal{C}$ is an* integral category *if it is further equipped with an* integral combinator*, a natural transformation $S_{A,B}\colon Hom(A\otimes !A, B) \to Hom(!A, B)$, satisfying the properties below [13].*

(Additivity) $S(f+g) = S(f) + S(g)$ and $S(0) = 0$.

(Linear Substitution) If the left square commutes, then the right square does also:

$$\begin{array}{ccccccc} A\otimes !A & \xrightarrow{f} & B & & !A & \xrightarrow{S(f)} & B \\ \downarrow{\scriptstyle h\otimes !h} & & \downarrow{\scriptstyle k} & \Rightarrow & \downarrow{\scriptstyle !h} & & \downarrow{\scriptstyle k} \\ C\otimes !C & \xrightarrow{g} & D & & !C & \xrightarrow{S(g)} & D \end{array}$$

(Integral of Constants Rule) $S((id \otimes e); \rho) = \varepsilon$ where $\rho_A\colon A \otimes I \to A$ is the monoidal right unitor transformation.

(Rota-Baxter Rule) $\Delta; (S(f) \otimes S(g)) = S((id \otimes \Delta); S(f) \otimes g)) + S((id \otimes \Delta); (\sigma \otimes id); (f \otimes S(g)))$.

(Interchange Rule) $S(id \otimes S(id \otimes f)) = S((id \otimes S(id \otimes f)); (\sigma \otimes id))$.

Similarly to the differential case, a consequence of the naturality of S is that we can instead work with the map $s := S(id)$, which is known as an *integral transformation*. Once again, an integral category described via an integral combinator and one described via an integral transformation are equivalent [31]. The properties above are often more easily demonstrated in terms of s.

(s1) Integral of Constants Rule:

$$s; (id \otimes e); \rho; = \varepsilon$$

(s2) Rota-Baxter Rule:

$$\Delta; (s \otimes s) = s; (id \otimes \Delta); (id \otimes id \otimes s) + s; (id \otimes \Delta); (\sigma \otimes id); (s \otimes id \otimes id)$$

(s3) Interchange Rule:

$$s; (id \otimes s) = s; (id \otimes s); (\sigma \otimes id)$$

2.2.1 Examples

1. *Rel* is an integral category when equipped with the same finite bag comonad as in the differential category example above. Define a natural transformation $s_A \colon !A \to A \otimes !A$ by

   $$\langle\!| a_0, \ldots, a_n |\!\rangle \, s_A(b_0, \langle\!| b_1, \ldots, b_n |\!\rangle) \iff \langle\!| a_0, \ldots, a_n |\!\rangle = \langle\!| b_0, b_1, \ldots, b_n |\!\rangle$$

   which is the converse relation to adding the extra element to the bag. Then s is an integral transformation.

2. $\mathbb{K}$-Vec^{op} is an integral category when equipped with the same symmetric tensor algebra monad as in the differential category example above. Define a natural transformation $s_V \colon V \otimes SV \to SV$ on basis elements by

   $$s_V(x_0 \otimes (x_1 \otimes \cdots \otimes x_n)) = \frac{1}{n+1}(x_0 \otimes x_1 \otimes \cdots \otimes x_n)$$

   where $\frac{1}{n+1} \in \mathbb{K}$ is notation for the n-fold sum inverse $(1 + \cdots + 1)^{-1}$. This requires that this sum not be zero; to ensure this without added complication, we'll specify that $\mathbb{K}$ must have characteristic 0. Then s is an integral transformation.

2.3 FTC Categories

In calculus, differentiation and integration are famously related by the fundamental theorem of calculus, (the first part of) which states $\frac{d}{dx}\int_0^x f(t)dt = f(x)$. After generalizing the two operations into the categorical setting, a natural next step is to do the same for the fundamental theorem.

Definition 2.3.1. *Let* $(\mathcal{C}, \otimes, I, !, \delta, \varepsilon, \Delta, e, D, S)$ *be a category which is a differential category with respect to* D *and an integral category with respect to* S. *Then* $(\mathcal{C}, D, S)$ *is an* FTC category *if for all* $f\colon !A \to B$ *we have*

$$S(D(f)) = f$$

or equivalently if

$$d; s = id\ .$$

2.3.1 Examples

Both the *Rel* and $\mathbb{K}$-Vec^{op} examples considered in the previous subsections are FTC categories. It is particularly obvious in the *Rel* case, where the differential and integral transformations are defined to be converse to each other. In the Vec case, the differential mimics polynomial differentiation while the integral mimics polynomial integration, so it is not difficult to show the fundamental theorem.

2.4 Modules with Differentiation and Integration

The notion of a module with differentiation has long been important in algebraic geometry and commutative algebra [3, 25, 34], and was first incorporated into the differential categorical setting via Kähler differentials as Kähler categories in [5]. On the other hand, the notion of module with integration seems never to have been defined before it was reverse-engineered from the study of integral categories in [3]. We'll recall both definitions here.

Definition 2.4.1. *Let* A *be a commutative* $\mathbb{K}$*-algebra, and let* M *be a right* A*-module. Then* M *is a* module with differentiation *if it is equipped with a map* $\partial\colon A \to M$ *such that* $\forall a, b, \in A$ *we have*

$$\partial(ab) = \partial(a) \cdot b + \partial(b) \cdot a$$

where juxtaposition is the algebra multiplication and $\cdot$ *is the module action. Note that this is just the Leibniz rule, and the map* ∂ *is thus called a* derivation.

Similarly, M *is a* module with integration *if it is equipped with a map* $\pi\colon M \to A$ *such that* $\forall m, n \in M$ *we have*

$$\pi(m)\pi(n) = \pi(m \cdot \pi(n)) + \pi(n \cdot \pi(m))\ .$$

Note that this is just the Rota-Baxter rule, and the map π *is thus called an* integration. *We formally define the Rota-Baxter rule in the next section.*

We will be interested in these types of modules because in many cases they constitute a preliminary step towards codifferential and cointegral categories.

2.5 Leibniz, Rota-Baxter and Proto-FTC Categories

As mentioned above, there appears to be some utility in describing category-theoretic structures similar to differential and integral categories, but only satisfying a subset of the standard rules. Specifically, we make the following definitions.

Definition 2.5.1. *A monoidal category* $(\mathcal{C}, \otimes, I)$ *equipped with a coalgebra modality* $(!, \delta, \varepsilon, \Delta, e)$ *is a* Leibniz category *if it is further equipped with a natural transformation* $d\colon A \otimes !A \to !A$ *satisfying the constant differentiation rule and the Leibniz rule; that is, satisfying properties* $(d1)$ *and* $(d2)$ *of a deriving transformation as given above. This* d *is then a* Leibniz transformation*.*

Similarly, such a category is a (weight-0) Rota-Baxter category[3] *if it is further equipped with a natural transformation* $s\colon !A \to A \otimes !A$ *satisfying the constant integration rule and the Rota-Baxter rule; that is, satisfying properties* $(s1)$ *and* $(s2)$ *of an integral transformation as given above. This* s *is then a* Rota-Baxter transformation*.*

The structures dual to these will be of importance to us as well, but again we will sometimes be loose with the terminology and refer to coLeibniz and coRota-Baxter categories simply as Leibniz and Rota-Baxter categories, respectively.

Note that we define Rota-Baxter categories for weight 0 only here. Rota-Baxter algebras (and categories) of arbitrary weights are discussed in section 4 below.

Clearly every differential category is a Leibniz category. That the converse is not true is a demonstration of the utility of the Leibniz category definition, and is what we will spend much of the first part of this paper showing. The main Leibniz example for us will be the free differential algebra. Of course, the same statements hold with respect to integral and Rota-Baxter categories, and this will be likewise explored below. The main Rota-Baxter example for us will be the free Rota-Baxter algebra.

Further evidence of this sort of utility would be if some structure existed satisfying the Leibniz and Rota-Baxter category properties as well as the fundamental theorem of calculus, but not the other differential or integral rules.

Definition 2.5.2. *A category* $(\mathcal{C}, d, s)$ *is a* proto-FTC category *if* $(\mathcal{C}, d)$ *is a Leibniz category,* $(\mathcal{C}, s)$ *is a Rota-Baxter category, and* d *and* s *satisfy the dual of the first part of fundamental theorem of calculus:*

$$d; s = id$$

The defined categorical structures work in the direction dual to the usual rules of differentiation and integration, so it is sensible to stipulate that the fundamental theorem hold in this direction also. Again, most of our examples below are co-examples, if you will, and in these cases the fundamental theorem holds in the usual sense.

[3]There is another, entirely different, notion of Rota-Baxter category defined in [10]. We proceed with the name here because it is the most natural, and because the original structure does not seem to have been much made use of.

3 The Free Differential Rota-Baxter Algebra

In this section we'll introduce the Rota-Baxter equation, and present the free Rota-Baxter algebra, largely following [22]. We'll then show that the latter can be captured in the categorical structure of a Rota-Baxter category. We'll proceed similarly with the free differential algebra, and show that it can be captured in the structure of a Leibniz category, before finishing with the composition of the two structures as a proto-FTC category induced by the free differential RB-algebra.

3.1 Rota-Baxter Algebras (of Weight 0)

The primary new structure we'll look at in this section is the Rota-Baxter algebra. Let R be an algebra and $P\colon R \to R$ a linear map. (R, P) is a *Rota-Baxter* or *RB-algebra (of weight 0)* if P satisfies the *Rota-Baxter equation*:

$$P(x)P(y) \;=\; P(xP(y)) \;+\; P(P(x)y)$$

We specify weight 0 here because there is a more general formula for RB-algebras of arbitrary weight, which we will discuss in the next section.

3.1.1 Examples

1. Any algebra is an RB-algebra in a trivial way if $P = 0$ is the zero map. This is uninteresting, but does demonstrate that Rota-Baxter algebras are not a restricted class of algebras, but rather a generalization of them.

2. The most relevant example for us is standard function integration. Let R be the algebra of continuous functions on $\mathbb{R}$, and let $P\colon R \to R$ be the map defined by $P(f)(x) = \int_0^x f(t)dt$. We observe that the Rota-Baxter equation in this case is just a rearrangement of the integration-by-parts formula; if $F(x) := P(f)(x) := \int_0^x f(t)dt$ and $G(x) := P(g)(x) := \int_0^x g(t)dt$, then we have:

$$\begin{aligned}\int_0^x F'(t)G(t)dt = F(t)G(t)\mid_0^x - \int_0^x F(t)G'(t)dt \\ \Rightarrow P(fP(g))(x) = P(f)(x)P(g)(x) - P(P(f)g)(x) \\ \Rightarrow P(f)P(g) = P(fP(g)) + P(P(f)g)\end{aligned}$$

   This example demonstrates that just as differential algebras satisfying the Leibniz rule are a generalization of differential calculus, so are RB-algebras satisfying the Rota-Baxter equation a generalization of integral calculus.

3.2 Shuffle Product and Free Rota-Baxter Algebra

As a preliminary, we'll recall the definition of the shuffle product. Since the formal statement is rather technical, we refer the interested reader to [22], and proceed instead with the more intuitive version.

Let $T(A)$ be the tensor algebra on a commutative algebra $(A, \cdot)$ over a field $\mathbb{F}$ as defined above. If $w_1 \in A^{\otimes m} \subseteq T(A)$ and $w_2 \in A^{\otimes n} \subseteq T(A)$ are homogeneous (i.e. pure) tensors viewed as words of length m and n respectively, a *shuffle* of w_1 and w_2 is a permutation of the concatenated word $w_1 w_2$ such that the internal order of each of w_1 and w_2 is preserved. The set of all shuffles on a pair of words is denoted $\mathrm{Sh}(w_1, w_2)$. The *shuffle product* of w_1 and w_2 is the sum of all their shuffles:

$$w_1 \sqcup\!\sqcup\, w_2 \;=\; \sum_{w \in \mathrm{Sh}(w_1,w_2)} w$$

Equivalently, the shuffle product can be defined recursively [22], and this definition is often more useful in proofs. If $w = w_1 \otimes w'$ and $u = u_1 \otimes u'$, we can define:

$$w \sqcup\!\sqcup\, u \;=\; w_1 \otimes (w' \sqcup\!\sqcup\, u) + u_1 \otimes (w \sqcup\!\sqcup\, u')$$

Denote by $\sqcup\!\sqcup A$ the underlying space of $T(A)$ equipped with the shuffle product instead of the concatenation product. Note that by convention, if one of the shuffle operands is an element of the field $\mathbb{F}$, the shuffle product reverts to the scalar product. Then $\sqcup\!\sqcup A$ is a commutative algebra with unit $1_{\mathbb{F}}$.

It turns out we need to add a bit more to obtain the free commutative RB-algebra on a commutative algebra. Define

$$\lozenge(A) \;=\; A \otimes \sqcup\!\sqcup A$$

with the multiplication $(a_0 \otimes a') \diamond (b_0 \otimes b') := a_0 \cdot b_0 \otimes (a' \sqcup\!\sqcup\, b')$, i.e. multiply the A factors together and the $\sqcup\!\sqcup A$ factors together separately. We call this the *augmented shuffle product* on A. Define also $P\colon \lozenge A \to \lozenge A$ to be the linear extension of:

$$P(a_0 \otimes a') \;=\; 1_A \otimes (a_0 \otimes a'), \quad a' \in A^{\otimes n}, n \geq 1,$$

$$P(a_0 \otimes 1_{\mathbb{F}}) \;=\; 1_A \otimes a_0$$

This is a Rota-Baxter operator, and indeed $(\lozenge A, \diamond, P)$ is the free commutative Rota-Baxter algebra on A [22].

3.3 Two Rota-Baxter Modules with Integration

We'd like to capture the free RB-algebra as a module with integration, and we encounter two ways to do so. It turns out that each have their own advantages and disadvantages, which we'll explore below.

Since $A \otimes \sqcup\!\sqcup A$ already has the form of a module with integration, we might like to use $\sqcup\!\sqcup S$ as our monad, where S is the free commutative algebra monad on a vector space, which gives for a vector space V the same structure as the tensor algebra $T(V)$ except with symmetrized tensors [20].

The first issue we encounter is that $\sqcup\!\sqcup S$ is not a monad, but a weaker structure called a quasimonad. We'll give the details of this structure and prove that $\sqcup\!\sqcup S$ satisfies it in the next subsection.

Next we must define a map $\Pi_{ш}: V \otimes ШSV \to ШSV$ capturing the structure of P. It is easy to see that $ImP \subseteq 1_{SV} \otimes Ш(SV)$, and that we have an isomorphism $\alpha\colon (ШSV, ш) \cong (1_{\mathbb{F}} \otimes ШSV, \diamond)$; this was proven for a general commutative algebra A in [22], and so holds for SV. Thus we can define

$$\Pi_{ш} = \alpha^{-1} \circ P \circ (i_V \otimes id)$$

where i_V is the inclusion of V into SV.

Proposition 3.3.1. *$(V \otimes ШSV, \Pi_{ш})$ is a module with integration.*

Proof. $\alpha^{-1} \circ P \circ (i_V \otimes id)$ is linear since $(i_V \otimes id)$, P and α^{-1} are. Then for $a = u_0 \otimes (a')$ and $b = v_0 \otimes (b')$:

$$\begin{aligned}\Pi_{ш}(a) \,ш\, \Pi_{ш}(b) &= (u_0 \otimes a') \,ш\, (v_0 \otimes b') \\ &= u_0 \otimes (a' \,ш\, (v_0 \otimes b')) + v_0 \otimes ((u_0 \otimes a') \,ш\, b') \\ &= \Pi_{ш}(a \cdot \Pi_{ш}(b)) + \Pi_{ш}(b \cdot \Pi_{ш}(a))\end{aligned}$$

Above, $\cdot$ is the usual module action on the free $ШSV$-module on V. That is, the action simply applies the multiplication of the $ШSV$ factor, leaving the V factor untouched. $\square$

We move on now to the second approach, which also gives a module with integration over a vector space. Let $\mathcal{B}_V$ be a specified basis of V.

Now let's make $\Diamond \circ S$ our functor (more precisely, $U \circ \Diamond \circ S$ for U the forgetful functor down to the category of vector spaces). Since $\Diamond S$ gives the free Rota-Baxter algebra on a vector space, we hope to obtain a true monad out of it, but the situation is more complicated. We will explain in what sense this is true in the next subsection; for now, we proceed assuming $\Diamond S$ is a monad.

We want to give a module with integration structure using $\Diamond S$; that is, we want a map $\Pi_\diamond\colon V \otimes \Diamond S(V) \to \Diamond S(V)$ satisfying the Rota-Baxter equation. For $w_i \in SV$ and $y_0 \in \mathcal{B}_V$, define

$$\begin{aligned}\Pi_\diamond\colon y_0 \otimes (w_0 \otimes (w_1 \otimes \cdots \otimes w_n)) &\mapsto 1_{\mathbb{F}} \otimes (w_0 \otimes \cdots \otimes w_n) \\ y_0 \otimes 1_{\mathbb{F}} &\mapsto 1_{\mathbb{F}} \otimes y_0\end{aligned}$$

and extend linearly.

Proposition 3.3.2. *$(V \otimes \Diamond SV, \Pi_\diamond)$ is a module with integration.*

Proof. Let $x = x_0 \otimes (u_0 \otimes (u_1 \otimes \cdots \otimes u_m))$, $y = y_0 \otimes (w_0 \otimes (w_1 \otimes \cdots \otimes w_n))$, $u = u_0 \otimes \cdots \otimes u_m$, $u^+ = u_1 \otimes \cdots \otimes u_m$, and similarly for w and w^+. Then:

$$\begin{aligned}\Pi_\diamond(x) \diamond \Pi_\diamond(y) &= (1_{\mathbb{F}} \otimes u) \diamond (1_{\mathbb{F}} \otimes w) \\ &= 1_{\mathbb{F}} \otimes (u \,ш\, w) \\ &= 1_{\mathbb{F}} \otimes \Big[u_0 \otimes (u^+ \,ш\, w) + w_0 \otimes (w^+ \,ш\, u)\Big] \\ &= \Pi_\diamond\Big(x_0 \otimes (u_0 \otimes (u^+ \,ш\, w))\Big) + \Pi_\diamond\Big(y_0 \otimes (w_0 \otimes (w^+ \,ш\, u))\Big) \\ &= \Pi_\diamond\Big(x \cdot \Pi_\diamond(y)\Big) + \Pi_\diamond\Big(y \cdot \Pi_\diamond(x)\Big)\end{aligned}$$

where we've used the recursive definition of the shuffle product. $\square$

3.4 The Rota-Baxter Quasimonad

As mentioned above, the $\Bbbk$ functor is not a true monad, but rather a quasi-monad, as defined in [3]. This structure is the same as what is called an *r-unital monad* in [43]. We recall the definition and some basic properties.

Definition 3.4.1. *Let $\mathcal{C}$ be a category, $F\colon \mathcal{C} \to \mathcal{C}$ a functor, and $\mu\colon FF \to F$ and $\eta\colon id_{\mathcal{C}} \to F$ natural transformations.*

1. *(F, μ) is a* functor with multiplication *if $F\mu; \mu = \mu_F; \mu$.*
2. *(F, μ, η) is a* q-unital monad *if (F, μ) is a functor with multiplication. Note that nothing further is required of η at this stage.*
3. *A q-unital monad is* regular *if $\eta = \eta; F\eta; \mu$.*
4. *A q-unital monad is* compatible *if $\mu = F\eta_F; \mu_F; \mu$, or (equivalently by the functor with multiplication equation) if $\mu = F\eta_F; F\mu; \mu$.*
5. *(F, μ, η) is a* quasimonad *if it is a regular compatible q-unital monad.*

A monad is always induced by an adjunction; similarly, a quasimonad is always induced by a regular pairing, defined as follows.

Definition 3.4.2. *Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be functors.*

1. *A* pairing *between L and R is a pair of natural transformations $\alpha_{A,B}\colon Hom_{\mathcal{D}}(LA, B) \to Hom_{\mathcal{C}}(A, RB)$ and $\beta_{A,B}\colon Hom_{\mathcal{C}}(A, RB) \to Hom_{\mathcal{D}}(LA, B)$.*
2. *A pairing (α, β) is* regular *if $\alpha; \beta; \alpha = \alpha$ and $\beta; \alpha; \beta = \beta$.*

We then have the following theorem, proven in [43].

Theorem 3.4.3. *A functor $F\colon \mathcal{C} \to \mathcal{C}$ is a quasimonad if and only if $\exists L, R, \alpha, \beta$ such that $F = L; R$ and (α, β) is a regular pairing.*

We claim that $\Bbbk S$ is a quasimonad. Before proving this, we must establish some notation for writing elements of $\Bbbk SV$, $\Bbbk S \Bbbk SV$, and $\Bbbk S \Bbbk S \Bbbk SV$.

If a basis element of $\Bbbk SV$ has the form

$$w_{ij}^{k\ell} = (v_{11}^{ijk\ell} \otimes_s \cdots \otimes_s v_{1m_1^{ijk\ell}}^{ijk\ell}) \otimes \cdots \otimes (v_{n_{ijk\ell}1}^{ijk\ell} \otimes_s \cdots \otimes_s v_{n_{ijk\ell}m_{n_{ijk\ell}}^{ijk\ell}}^{ijk\ell})$$

then we can write a basis element of $\Bbbk S \Bbbk SV$ as

$$\omega_{k\ell} = (w_{11}^{k\ell} \otimes_s \cdots \otimes_s w_{1m_1^{k\ell}}^{k\ell}) \otimes \cdots \otimes (w_{n_{k\ell}1}^{k\ell} \otimes_s \cdots \otimes_s w_{n_{k\ell}m_{n_{k\ell}}^{k\ell}}^{k\ell})$$

and a basis element of $\Bbbk S \Bbbk S \Bbbk SV$ as

$$\vartheta = (\omega_{11} \otimes_s \cdots \otimes_s \omega_{1m_1}) \otimes \cdots \otimes (\omega_{n1} \otimes_s \cdots \otimes_s \omega_{nm_n}) \ .$$

Define $\mu_V \colon \sqcup\!\sqcup S \sqcup\!\sqcup SV \to \sqcup\!\sqcup SV$ by

$$\mu(\omega_{k\ell}) = \overline{\omega_{k\ell}} := (w_{11}^{k\ell} \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_{1m_1^{k\ell}}^{k\ell}) \otimes \cdots \otimes (w_{n_{k\ell}1}^{k\ell} \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_{n_{k\ell}m_{n_{k\ell}}^{k\ell}}^{k\ell})$$

and $\eta_V \colon V \to \sqcup\!\sqcup SV$ by

$$\eta(v) = ((v))\ ,$$

which includes v first as $(v) \in SV$ and then as $((v)) \in \sqcup\!\sqcup SV$.

Proposition 3.4.4. $(\sqcup\!\sqcup S, \mu, \eta)$ *is a quasimonad.*

Proof. This proof is similar to a result in [3]. First we show it is a functor with multiplication. Observe:

$$\begin{aligned}
\mu \circ \sqcup\!\sqcup S\mu(\vartheta) &= \mu((\overline{\omega_{11}} \otimes_s \cdots \otimes_s \overline{\omega_{1m_1}}) \otimes \cdots \otimes (\overline{\omega_{n1}} \otimes_s \cdots \otimes_s \overline{\omega_{nm_n}})) \\
&= (\overline{\omega_{11}} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \overline{\omega_{1m_1}}) \otimes \cdots \otimes (\overline{\omega_{n1}} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \overline{\omega_{nm_n}})
\end{aligned}$$

Meanwhile, we have:

$$\begin{aligned}
\mu \circ \mu_{\sqcup\!\sqcup S}(\vartheta) &= \mu((\omega_{11} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \omega_{1m_1}) \otimes \cdots \otimes (\omega_{n1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \omega_{nm_n})) \\
&= (\overline{\omega_{11}} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \overline{\omega_{1m_1}}) \otimes \cdots \otimes (\overline{\omega_{n1}} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \overline{\omega_{nm_n}})
\end{aligned}$$

The latter equality holds because the "outer" shuffle of 'ω's and the "inner" shuffle of 'w's don't interact by definition of μ. Thus the functor with multiplication equation holds.

Regularity is shown as follows:

$$\begin{tikzcd}
v \arrow[r, mapsto, "\eta"] \arrow[rrr, bend right, "\eta"'] & ((v)) \arrow[r, mapsto, "\sqcup\!\sqcup S\eta"] & ((((v)))) \arrow[r, mapsto, "\mu"] & ((v))
\end{tikzcd}$$

Similarly straightforward is compatibility, when we use the equivalent formulation:

$$\begin{tikzcd}
\omega_{k\ell} \arrow[d, mapsto, "\sqcup\!\sqcup S\eta_{\sqcup\!\sqcup S}"] \arrow[ddd, dashed, bend right=40, "\mu"', <->] \\
((w_{11}^{k\ell}))\otimes_s\cdots\otimes_s((w_{1m_1^{k\ell}}^{k\ell})))\otimes\cdots\otimes(((w_{n_{k\ell}1}^{k\ell}))\otimes_s\cdots\otimes_s((w_{n_{k\ell}m_{n_{k\ell}}^{k\ell}}^{k\ell}))) \arrow[d, mapsto, "\sqcup\!\sqcup S\mu"] \\
\omega_{k\ell} \arrow[d, mapsto, "\mu"] \\
\mu(\omega_{k\ell})
\end{tikzcd}$$

Thus $(\sqcup\!\sqcup S, \mu, \eta)$ is a quasimonad. $\square$

3.5 The Rota-Baxter Monad and Algebra Modalities

We are also interested in building an algebra modality with $\lozenge S$. In general, a composition of two monads may not be a monad; we need an extra property known as a distributive law [4].

Definition 3.5.1. *Let (T, μ^T, η^T) and (S, μ^S, η^S) be monads. Then a* distributive law (of T over S) *is a natural transformation $\ell\colon ST \to TS$ such that the following diagrams commute:*

$$\begin{tikzcd} STT \arrow[r, "\ell_T"] \arrow[d, "S\mu^T"] & TST \arrow[r, "T\ell"] & TTS \arrow[d, "\mu^T_S"] \\ ST \arrow[rr, "\ell"] & & TS \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "S\eta^T"'] \arrow[dr, "\eta^T_S"] & \\ ST \arrow[rr, "\ell"] & & TS \end{tikzcd}$$

$$\begin{tikzcd} SST \arrow[r, "S\ell"] \arrow[d, "\mu^S_T"] & STS \arrow[r, "\ell_S"] & TSS \arrow[d, "T\mu^S"] \\ ST \arrow[rr, "\ell"] & & TS \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl, "\eta^S_T"'] \arrow[dr, "T\eta^S"] & \\ ST \arrow[rr, "\ell"] & & TS \end{tikzcd}$$

The following theorem is well-known; for one proof, see [4].

Theorem 3.5.2. *If ℓ is a distributive law of T over S, then TS is a monad with $\mu^{TS} = T\ell_S; TT\mu^S; \mu^T_S$ and $\eta^{TS} = \eta^S; \eta^T_S$.*

To show that $\lozenge S$ is a monad, we will define a distributive law of $\lozenge$ over S below, but first we need to explicitly construct $\mu^\lozenge$ and $\eta^\lozenge$. These are implicit in works like [22] and [23], but never seem to have been made explicit.

Let A be a commutative algebra and (R, P) a Rota-Baxter algebra. It is proven in [22] that the freeness adjunction is given by:

$$Hom_{CAlg}(A, UR) \cong Hom_{RBAlg}(\lozenge A, R)$$

$$\Big(a \mapsto f(a)\Big) \mapsto \Big(a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \mapsto f(a_0) \cdot P(f(a_1) \cdot P(\cdots))\Big)$$

Tracing $id\colon UR \to UR$ through this adjunction gives the counit:

$$\epsilon\colon a_0 \otimes (a_1 \otimes \cdots \otimes a_m) \mapsto a_0 \cdot P\Big(a_1 \cdot P(\cdots)\Big)$$

We claim that the unit is

$$\eta\colon a \mapsto a \otimes 1_{\mathbb{F}} \ ,$$

since tracing this map through the adjunction and recalling that $P\colon \lozenge A \to \lozenge A$ is defined by $P(w) = 1_{\mathbb{F}} \otimes w$ gives the map

$$\begin{aligned} a_0 \otimes (a_1 \otimes \cdots \otimes a_n) &\mapsto (a_0 \otimes 1) \diamond P((a_1 \otimes 1) \diamond P(\cdots)) \\ &= a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \end{aligned}$$

which is $id\colon \lozenge A \to \lozenge A$.

Our monadic unit is then simply $\eta^{\Diamond} := \eta$, and our multiplication $U\epsilon_{\Diamond A}\colon \Diamond\Diamond A \to \Diamond A$ is then given by:

$$\mu^{\Diamond}\colon w_0 \otimes (w_1 \otimes \cdots \otimes w_n) \mapsto w_0 \diamond (1_{\mathbb{F}} \otimes (w_1 \diamond (1_{\mathbb{F}} \otimes (\cdots))))$$

We now define a natural transformation $\ell\colon S\Diamond \to \Diamond S$ by

$$\begin{aligned} \ell_V\colon & (v_{10} \otimes (v_{11} \otimes \cdots \otimes v_{1m_1})) \otimes_s \cdots \otimes_s (v_{n0} \otimes (v_{n1} \otimes \cdots \otimes v_{nm_n})) \\ & \mapsto ((v_{10}) \otimes ((v_{11}) \otimes \cdots \otimes (v_{1m_1}))) \diamond \cdots \diamond ((v_{n0}) \otimes ((v_{n1}) \otimes \cdots \otimes (v_{nm_n}))) \end{aligned}$$

where $(v_{ij}) := \eta^S(v_{ij}) \in SV$.

Proposition 3.5.3. *The natural transformation ℓ is a distributive law of $\Diamond$ over S.*

Proof. For the top-right diagram, we have:

$$\begin{aligned} \ell \circ S\eta^{\Diamond}(v_1 \otimes_s \cdots \otimes_s v_n) &= \ell((v_1 \otimes 1_{\mathbb{F}}) \otimes_s \cdots \otimes_s (v_n \otimes 1_{\mathbb{F}})) \\ &= (v_1 \otimes 1_{\mathbb{F}}) \diamond \cdots \diamond (v_n \otimes 1_{\mathbb{F}}) \\ &= (v_1 \otimes_s \cdots \otimes_s v_n) \otimes 1_{\mathbb{F}} \\ &= \eta^{\Diamond}_S(v_1 \otimes_s \cdots \otimes_s v_n) \end{aligned}$$

For the bottom-right diagram, we have:

$$\begin{aligned} \ell \circ \eta^S_{\Diamond}(v_0 \otimes (v_1 \otimes \cdots \otimes v_n)) &= \ell((v_0 \otimes (v_1 \otimes \cdots \otimes v_n))) \\ &= (v_0) \otimes ((v_1) \otimes \cdots \otimes (v_n)) \\ &= \Diamond\eta^S(v_0 \otimes (v_1 \otimes \cdots \otimes v_n)) \end{aligned}$$

For the bottom-left diagram, we first establish some notation. Write $v^i_j := v^i_{j0} \otimes (v^i_{j1} \otimes \cdots \otimes v^i_{jm_j})$, $w_i := v^i_1 \otimes_s \cdots \otimes_s v^i_{n_i}$, and then write $\overline{v^i_j} := \Diamond\eta^S(v^i_j)$. Then the diagram is verified by:

$$\begin{tikzcd} w_1 \otimes_s \cdots \otimes_s w_k \arrow[r, mapsto, "S\ell"] \arrow[d, mapsto, "\mu^S_{\Diamond}"] & (\overline{v^1_1} \diamond \cdots \diamond \overline{v^1_{n_1}}) \otimes_s \cdots \otimes_s (\overline{v^k_1} \diamond \cdots \diamond \overline{v^k_{n_k}}) \arrow[r, mapsto, "\ell_S"] & (\overline{v^1_1} \diamond \cdots \diamond \overline{v^1_{n_1}}) \diamond \cdots \diamond (\overline{v^k_1} \diamond \cdots \diamond \overline{v^k_{n_k}}) \arrow[d, mapsto, "\Diamond\mu^S"] \\ v^1_1 \otimes_s \cdots \otimes_s v^k_{n_k} \arrow[rr, mapsto, "\ell"] & & \overline{v^1_1} \diamond \cdots \diamond \overline{v^k_{n_k}} \end{tikzcd}$$

The top-left diagram is the most involved. First, we modify some of the notation used above. We again write $v^i_j = v^i_{j0} \otimes (v^i_{j1} \otimes \cdots \otimes v^i_{jm_j})$ and $\overline{v^i_j} := \Diamond\eta^S(v^i_j)$, but we redefine w_i so that $w_i := v^i_0 \otimes (v^i_1 \otimes \cdots \otimes v^i_{n_i})$.

A general term in $S\Diamond\Diamond V$ has the form $w_1 \otimes_s \cdots \otimes_s w_k$. We'll demonstrate the $k = 1$ and $k = 2$ cases; the general k case is analogous to $k = 2$, but using the multi-factor Rota-Baxter rule and the multi-recursive property of $\sqcup\!\sqcup$ instead of the usual two-factor versions.

When $k = 1$, writing $w := w_1$ we have:

$$\begin{tikzcd}
(w) \arrow[r, mapsto, "\ell_\Diamond"] \arrow[d, mapsto, "S\mu^\Diamond"'] & (v_0)\otimes\Big((v_1)\otimes\cdots\otimes(v_n)\Big) \arrow[r, mapsto, "\Diamond\ell"] & \overline{v_0}\otimes(\overline{v_1}\otimes\cdots\otimes\overline{v_n}) \arrow[d, mapsto, "\mu_S^\Diamond"] \\
(v_0\diamond P(v_1\diamond P(\cdots))) \arrow[rr, mapsto, "\ell"] & & \overline{v_0}\diamond P(\overline{v_1}\diamond P(\cdots))
\end{tikzcd}$$

When $k=2$, observe that we have

$$\begin{aligned}
\ell\circ S\mu^\Diamond(w_1\otimes_s w_2) &= \ell\Big((v_0^1\diamond P(v_1^1\diamond P(\cdots)))\otimes_s(v_0^2\diamond P(v_1^2\diamond P(\cdots)))\Big)\\
&= (\overline{v_0^1}\diamond P(\overline{v_1^1}\diamond P(\cdots)))\diamond(\overline{v_0^2}\diamond P(\overline{v_1^2}\diamond P(\cdots)))\\
&= (\overline{v_0^1}\diamond\overline{v_0^2})\diamond(P(\overline{v_1^1}\diamond P(\cdots))\diamond P(\overline{v_1^2}\diamond P(\cdots)))
\end{aligned}$$

as well as

$$\begin{aligned}
\Diamond\ell\circ\ell_\Diamond(w_1\otimes_s w_2) &= \Diamond\ell\Big(\big((v_0^1\otimes\big((v_1^1)\otimes\cdots\otimes(v_{n_1}^1)\big)\big)\diamond\big((v_0^2)\otimes\big((v_1^2)\otimes\cdots\otimes(v_{n_2})\big)\big)\Big)\\
&= \Diamond\ell\Big(\big((v_0^1)\otimes_s(v_0^2)\big)\otimes\big(((v_1^1)\otimes\cdots\otimes(v_{n_1}^1))\sqcup\big((v_1^2)\otimes\cdots\otimes(v_{n_2}^2)\big)\big)\Big)\\
&= (\overline{v_0^1}\diamond\overline{v_0^2})\otimes\big((\overline{v_1^1}\otimes\cdots\otimes\overline{v_{n_1}^1})\sqcup(\overline{v_1^2}\otimes\cdots\otimes\overline{v_{n_2}^2})\big)\\
&=: (*)
\end{aligned}$$

We claim that

$$\mu_S^\Diamond((*)) = (\overline{v_0^1}\diamond\overline{v_0^2})\diamond(P(\overline{v_1^1}\diamond P(\cdots))\diamond P(\overline{v_1^2}\diamond P(\cdots)))$$

which would complete the proof. We prove this by induction on n_1 and n_2.

Suppose first that $n_1=n_2=1$. Then we have

$$\begin{aligned}
\mu_S^\Diamond\Big((\overline{v_0^1}\diamond\overline{v_0^2})\otimes\big((\overline{v_1^1})\sqcup(\overline{v_1^2})\big)\Big) &= \mu_S^\Diamond\Big((\overline{v_0^1}\diamond\overline{v_0^2})\otimes\big((\overline{v_1^1})\otimes(\overline{v_1^2})+(\overline{v_1^2})\otimes(\overline{v_1^1})\big)\Big)\\
&= (\overline{v_0^1}\diamond\overline{v_0^2})\diamond\big(P(\overline{v_1^1}\diamond P(\overline{v_1^2}))+P(\overline{v_1^2}\diamond P(\overline{v_1^1}))\big)\\
&= (\overline{v_0^1}\diamond\overline{v_0^2})\diamond\big(P(\overline{v_1^1})\diamond P(\overline{v_1^2})\big)
\end{aligned}$$

by definition of $\sqcup$ and the Rota-Baxter rule for P.

Now assume the inductive hypothesis for n_1 and n_2 and all shorter lengths, and add one tensor factor to one of them; without loss of generality, suppose we're adding a factor $(\overline{v})$ to the beginning of the first tensor. Then, writing

$$(**) := (\overline{v_0^1}\diamond\overline{v_0^2})\otimes\Big(\big((\overline{v})\otimes(\overline{v_1^1})\otimes\cdots\otimes(\overline{v_{n_1}^1})\big)\sqcup\big((\overline{v_1^2})\otimes\cdots\otimes(\overline{v_{n_2}^2})\big)\Big)$$

we have

$$\begin{aligned}
\mu_S^{\Diamond}((**)) = \mu_S^{\Diamond}\Big((\overline{v_0^1} \diamond \overline{v_0^2}) \otimes \Big[(\overline{v}) \otimes \Big(((\overline{v_1^1}) \otimes \cdots \otimes (\overline{v_{n_1}^1})) \sqcup ((\overline{v_1^2}) \otimes \cdots \otimes (\overline{v_{n_2}^2}))\Big) \\
+ (\overline{v_1^2}) \otimes \Big(((\overline{v} \otimes \overline{v_1^1}) \otimes \cdots \otimes (\overline{v_{n_1}^1})) \sqcup ((\overline{v_2^2}) \otimes \cdots \otimes (\overline{v_{n_2}^2}))\Big)\Big]\Big) \\
= (\overline{v_0^1} \diamond \overline{v_0^2}) \diamond \Big[P\Big(\overline{v} \diamond P(\overline{v_1^1} \diamond P(\cdots)) \diamond P(\overline{v_1^2} \diamond P(\cdots))\Big) \\
+ P\Big(\overline{v_1^2} \diamond P(\overline{v_2^2} \diamond P(\cdots)) \diamond P(\overline{v} \diamond P(\overline{v_1^1} \diamond P(\cdots))\Big)\Big] \\
= (\overline{v_0^1} \diamond \overline{v_0^2}) \diamond \Big[P\Big(\overline{v} \diamond P(\overline{v_1^1} \diamond P(\cdots))\Big) \diamond P\Big(\overline{v_1^2} \diamond P(\overline{v_2^2} \diamond P(\cdots))\Big)\Big]
\end{aligned}$$

by the recursive definition of $\sqcup$, commutativity of $\diamond$, and the Rota-Baxter rule for P. This proves the inductive step, completing the proof of the top-left diagram. Thus ℓ is a distributive law of monads.

□

As for the algebra modality, we can in fact define almost the same structure on both our (quasi)monads. For $\sqcup\!\sqcup$, define $(m_{\sqcup}, e_{\sqcup}) = (\sqcup, i_{\mathbb{F}})$, and for $\Diamond S$, define $(m_{\diamond}, e_{\diamond}) = (\diamond, i_{\mathbb{F}} \otimes i_{\mathbb{F}})$, where i is the inclusion.

Proposition 3.5.4. *$(\sqcup\!\sqcup S, m_{\sqcup}, e_{\sqcup})$ and $(\Diamond S, m_{\diamond}, e_{\diamond})$ are algebra modalities (except that $\sqcup\!\sqcup S$ is a quasimonad instead of a monad).*

Proof. That each acts with an algebra structure is clear. It remains to show that each μ is an algebra map for the respective modality. We'll show this holds for $\sqcup\!\sqcup$; the proof for $\Diamond$ is similar.

Using the same notation as above, the multiplicative and unital diagrams are:

$$\begin{tikzcd}
\omega_{k\ell}^1 \otimes \omega_{k\ell}^2 \arrow[r, mapsto, "\mu\otimes\mu"] \arrow[d, "m"] & \overline{\omega_{k\ell}^1} \otimes \overline{\omega_{k\ell}^2} \arrow[d, "m"] \\
\omega_{k\ell}^1 \sqcup \omega_{k\ell}^2 \arrow[r, mapsto, "\mu"] & \overline{\omega_{k\ell}^1} \sqcup \overline{\omega_{k\ell}^2}
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathbb{F}} \arrow[r, mapsto, "e"] \arrow[dr, mapsto, "e"] & 1_{\mathbb{F}} \arrow[d, "\mu"] \\
& 1_{\mathbb{F}}
\end{tikzcd}$$

As in the proof that $(\sqcup\!\sqcup S, \mu)$ is a functor with multiplication, we have used the fact that the shuffle internal to each ω does not interact with the shuffle of the two 'ω's together.

□

3.6 Rota-Baxter Category Structures

The calculations above showing that $\sqcup\!\sqcup$ and $\Diamond$ are modules with integration are also precisely those demonstrating that they each make $\mathbb{K}$-Vec into a Rota-Baxter category. It remains to examine which other properties of an integral category they satisfy.

It turns out that neither $\sqcup\!\sqcup$ nor $\Diamond$ in general satisfy any of the other integral category structural properties, lending support to the utility of the Rota-Baxter category definition. First we show that the constant rule is satisfied for each. For $\sqcup\!\sqcup$ we have the following diagram:

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "\cong"] & \sqcup\!\sqcup SV \\ V \otimes \mathbb{F} \arrow[r, "id\otimes e"] & V \otimes \sqcup\!\sqcup SV \arrow[u, "\Pi_{\sqcup\!\sqcup}"'] \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r, mapsto] \arrow[d, mapsto] & v \\ v \otimes 1_{\mathbb{F}} \arrow[r, mapsto] & v \otimes 1_{\mathbb{F}} \arrow[u, mapsto] \end{tikzcd}$$

For $\Diamond$, we similarly have:

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "\cong"] & \Diamond SV \\ V \otimes \mathbb{F} \arrow[r, "id\otimes e"] & V \otimes \Diamond SV \arrow[u, "\Pi_{\diamond}"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & 1_{\mathbb{F}} \otimes x \\ x \otimes 1_{\mathbb{F}} \arrow[r, mapsto] & x \otimes 1_{\mathbb{F}} \arrow[u, mapsto] \end{tikzcd}$$

Now recall that the integration of linear maps rule states:

$$2 \cdot \Pi \circ (id \otimes \eta) \;=\; m \circ (\eta \otimes \eta)$$

In the $\sqcup\!\sqcup$ case, we have:

$$\begin{aligned} 2 \cdot \Pi_{\sqcup\!\sqcup} \circ (id \otimes \eta)(a_1 \otimes a_2) &= 2 \cdot \Pi_{\sqcup\!\sqcup}(a_1 \otimes (a_2)) \\ &= 2 \cdot a_1 \otimes a_2 \\ &= a_1 \otimes a_2 + a_1 \otimes a_2 \\ &\neq a_1 \otimes a_2 + a_2 \otimes a_1 \\ &= (a_1) \sqcup\!\sqcup (a_2) \\ &= m \circ (\eta \otimes \eta)(a_1 \otimes a_2) \end{aligned}$$

A similar argument for $\Diamond$ shows neither satisfies the linear rule in general. Next, recall the Fubini rule states, for σ the tensor symmetry:

$$\Pi \circ (id \otimes \Pi) \;=\; \Pi \circ (id \otimes \Pi) \circ (\sigma \otimes id)$$

But observe for $\sqcup\!\sqcup$:

$$\begin{aligned} \Pi_{\sqcup\!\sqcup} \circ (id \otimes \Pi_{\sqcup\!\sqcup})(a_0 \otimes a_0' \otimes (a_1 \otimes \cdots \otimes a_n)) &= \Pi_{\sqcup\!\sqcup}(a_0 \otimes (a_0' \otimes a_1 \otimes \cdots \otimes a_n)) \\ &= a_0 \otimes a_0' \otimes a_1 \otimes \cdots \otimes a_n \\ &\neq a_0' \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n \\ &= \Pi_{\sqcup\!\sqcup} \circ (id \otimes \Pi_{\sqcup\!\sqcup})(a_0' \otimes a_0 \otimes (a_1 \otimes \cdots \otimes a_n)) \\ &= \Pi_{\sqcup\!\sqcup} \circ (id \otimes \Pi_{\sqcup\!\sqcup}) \circ (\sigma \otimes id)(a_0 \otimes a_0' \otimes (a_1 \otimes \cdots \otimes a_n)) \end{aligned}$$

A similar argument for $\Diamond$ shows neither satisfies the Fubini rule in general. Thus both provide examples of Rota-Baxter categories that are not integral categories.

3.7 Free Differential Algebra

Before we combine Leibniz and Rota-Baxter structures using the Ш and ◊ (quasi)monads, we'll define the free differential algebra on an algebra and prove that it provides another example of a Leibniz category structure on Vec. We'll also make use of the free differential algebra monad later on.

Let S be the free symmetric algebra construction as above. For a set X, let $\Delta(X) := X \times \mathbb{N}$, and in particular we can apply Δ to a vector space V over a field $\mathbb{F}$. It is clear that Δ is a functor, and so $\Xi := U \circ S \circ \Delta$ is a functor $Vec \to Vec$.

It is proven in [23] that $(S \circ \Delta(X), d_X)$ is the free (commutative) differential algebra on X, where d_X is defined as the linear and Leibniz extension of:

$$\begin{aligned} d_X((x,n)) &= (x, n+1) \\ d_X(1_{\mathbb{F}}) &= 0 \end{aligned}$$

Thus we'd like to show that Ξ, when equipped with an appropriate deriving transformation capturing d_X, induces a Leibniz category. We'll start by describing the monad structure on Ξ.

Define:

$$\begin{aligned} \eta_V &: V \to \Xi V \\ & \quad x \mapsto (x, 0) \\ \mu_V &: \Xi\Xi V \to \Xi V \\ & \quad (w_1, m_1) \otimes \cdots \otimes (w_\ell, m_\ell) \mapsto d_V^{m_1}(w_1) \otimes \cdots \otimes d_V^{m_\ell}(w_\ell) \\ & \quad 1_{\mathbb{F}} \in \Xi\Xi V \mapsto 1_{\mathbb{F}} \in \Xi V \end{aligned}$$

Proposition 3.7.1. (Ξ, μ, η) *is a monad.*

Proof. We must show commutativity of the following monad diagrams:

$$\begin{tikzcd} \Xi \arrow[r, "\eta_\Xi"] \arrow[d, "\Xi\eta"'] \arrow[dr, equal] & \Xi^2 \arrow[d, "\mu"] \\ \Xi^2 \arrow[r, "\mu"] & \Xi \end{tikzcd} \qquad \begin{tikzcd} \Xi^3 \arrow[r, "\Xi\mu"] \arrow[d, "\mu_\Xi"'] & \Xi^2 \arrow[d, "\mu"] \\ \Xi^2 \arrow[r, "\mu"] & \Xi \end{tikzcd}$$

For the unital equations, we have:

$$\begin{aligned} \mu \circ \Xi\eta_V((v_1, n_1) \otimes \cdots \otimes (v_k, n_k)) &= \mu(((v_1, 0), n_1) \otimes \cdots \otimes ((v_k, 0), n_k)) \\ &= d_V^{n_1}(v_1, 0) \otimes \cdots \otimes d_V^{n_k}(v_k, 0) \\ &= (v_1, n_1) \otimes \cdots \otimes (v_k, n_k) \\ &= d_V^0((v_1, n_1) \otimes \cdots \otimes (v_k, n_k)) \\ &= \mu((v_1, n_1) \otimes \cdots \otimes (v_k, n_k), 0) \\ &= \mu \circ \eta_{\Xi V}((v_1, n_1) \otimes \cdots \otimes (v_k, n_k)) \end{aligned}$$

For the multiplicative equations, we must show that

$$\mu \circ \Xi\mu_V\Big(\big((w_{1_1}, m_{1_1})\otimes\cdots\otimes(w_{\ell_1}, m_{\ell_1}), p_1\big)\otimes\cdots\otimes\big((w_{1_s}, m_{1_s})\otimes\cdots\otimes(w_{\ell_s}, m_{\ell_s}), p_s\big)\Big)$$

is equal to

$$\mu_{\Xi V}\Big(\big((w_{1_1}, m_{1_1})\otimes\cdots\otimes(w_{\ell_1}, m_{\ell_1}), p_1\big)\otimes\cdots\otimes\big((w_{1_s}, m_{1_s})\otimes\cdots\otimes(w_{\ell_s}, m_{\ell_s}), p_s\big)\Big).$$

Call the former $(*)$ and the latter $(**)$. Then we have:

$$\begin{aligned}
(*) &= \mu\left((d_V^{m_{1_1}}(w_{1_1})\otimes\cdots\otimes d_V^{m_{\ell_1}}(w_{\ell_1}), p_1)\otimes\cdots\otimes(d_V^{m_{1_s}}(w_{1_s})\otimes\cdots\otimes d_V^{m_{\ell_s}}(w_{\ell_s}), p_s)\right)\\
&= \bigotimes_{j=1}^{s}\left(\sum_{i=1}^{\ell_j} d_V^{m_{1_j}}(w_{1_j})\otimes\cdots\otimes d_V^{p_j}(d_V^{m_{i_j}}(w_{i_j}))\otimes\cdots\otimes d_V^{m_{\ell_j}}(w_{\ell_j})\right)\\
&= \mu\left(\bigotimes_{j=1}^{s}\left(\sum_{i=1}^{\ell_j}(w_{1_j}, m_{1_j})\otimes\cdots\otimes(w_{i_j}, m_{i_j}+p_j)\otimes\cdots\otimes(w_{\ell_j}, m_{\ell_j})\right)\right)\\
&= \mu\left(\bigotimes_{j=1}^{s} d_V^{p_j}((w_{1_j}, m_{1_j})\otimes\cdots\otimes(w_{\ell_j}, m_{\ell_j}))\right)\\
&= (**)
\end{aligned}$$

Note that we have used the Leibniz property of the differential d_V.

□

Next we'll need the algebra modality. It can be built from the obvious choices, with the unit being inclusion and the multiplication being the concatenation product.

Proposition 3.7.2. *(Ξ, m, e) forms an algebra modality.*

Proof. It is clear that this structure acts as an algebra, and easy to see that it is natural. It remains to prove that μ is an algebra homomorphism. The unital equation is demonstrated by:

$$\begin{tikzcd}
1_{\mathbb{F}} \arrow[r, mapsto, "e"] \arrow[dr, mapsto, "e"'] & 1_{\mathbb{F}} \arrow[d, mapsto, "\mu"] \\
& 1_{\mathbb{F}}
\end{tikzcd}$$

If we denote by (*) the expression

$$m\circ(\mu\otimes\mu)\Big(((w_{1_1}, n_{1_1})\otimes\cdots\otimes(w_{k_1}, n_{k_1}))\otimes((w_{1_2}, n_{1_2})\otimes\cdots\otimes(w_{k_2}, n_{k_2}))\Big)$$

and by (**) the expression

$$\mu \circ m\Big(((w_{1_1}, n_{1_1})\otimes\cdots\otimes(w_{k_1}, n_{k_1})) \otimes ((w_{1_2}, n_{1_2})\otimes\cdots\otimes(w_{k_2}, n_{k_2}))\Big)$$

for an arbitrary element of $\Xi\Xi A \otimes \Xi\Xi A$, then the multiplicative algebra map equation is given by:

$$\begin{aligned}
(*) &= m\Big((d_V^{n_{1_1}}(w_{1_1})\otimes\cdots\otimes d_V^{n_{k_1}}(w_{k_1})) \otimes (d_V^{n_{1_2}}(w_{1_2})\otimes\cdots\otimes d_V^{n_{k_2}}(w_{k_2}))\Big) \\
&= d_V^{n_{1_1}}(w_{1_1})\otimes\cdots\otimes d_V^{n_{k_1}}(w_{k_1}) \otimes d_V^{n_{1_2}}(w_{1_2})\otimes\cdots\otimes d_V^{n_{k_2}}(w_{k_2}) \\
&= \mu\Big((w_{1_1}, n_{1_1})\otimes\cdots\otimes(w_{k_1}, n_{k_1}) \otimes (w_{1_2}, n_{1_2})\otimes\cdots\otimes(w_{k_2}, n_{k_2})\Big) \\
&= (**)
\end{aligned}$$

□

Before we define our deriving transformation, it is worth noting that μ satisfies an additional nice property.

Proposition 3.7.3. *The map $\mu_V : \Xi\Xi V \to \Xi V$ is a map of differential algebras.*

Proof. That μ is an algebra map was shown in 3.7.2, so it remains to show that it respects the differential. We observe:

$$\begin{aligned}
d_V \circ \mu_V\left((w_1, m_1)\otimes\cdots\otimes(w_\ell, m_\ell)\right) &= d_V\left(d_V^{m_1}(w_1)\otimes\cdots\otimes d_V^{m_\ell}(w_\ell)\right) \\
&= \sum_{i=1}^{\ell} d_V^{m_1}(w_1)\otimes\cdots\otimes d_V^{m_i+1}(w_i)\otimes\cdots\otimes d_V^{m_\ell}(w_\ell) \\
&= \mu\left(\sum_{i=1}^{\ell}(w_1, m_1)\otimes\cdots\otimes(w_i, m_i+1)\otimes\cdots\otimes(w_\ell, m_\ell)\right) \\
&= \mu \circ d_{\Xi V}\left((w_1, m_1)\otimes\cdots\otimes(w_\ell, m_\ell)\right)
\end{aligned}$$

□

Finally we must give a deriving transformation $\delta_V : \Xi V \to V \otimes \Xi V$ to complete the Leibniz category structure. For each V, pick some vector $v \in V$. Define:

$$\begin{aligned}
\delta_V : w &\mapsto v \otimes d_V(w) \\
1_{\mathbb{F}} &\mapsto 0
\end{aligned}$$

Here d is the differential of the free differential algebra structure on ΞV, as described above.

Remark 3.7.4. *The element v in the definition above is merely a placeholder element; if we worked over algebras, we would have the canonical choice of 1_A for each algebra A, but the differential map δ is, by its nature, not a map of* *algebras. Thus we must work over Vec and content ourselves with this less natural structure.*

On a more fundamental level, this inelegance is forced upon us by the nature of the codomain of the deriving transformation. While sensible in contexts tightly analogous to function differentiation, the codomain $V\otimes !V$ seems to be too restrictive in more general situations. This indicates that a looser, further modified definition of deriving transformation might be useful in the future. For the purposes of this paper, however, we keep the definition previously given.

Theorem 3.7.5. *(Vec, Ξ, δ) constitute a Leibniz category, but not a differential category.*

Proof. By the previous propositions, the only remaining datum to be proved is that δ is a deriving transformation satisfying the Leibniz and constant differentiation rules. The constant rule is satisfied by definition. As for Leibniz, let $w_1 = (v_1, n_1)\otimes\cdots\otimes(v_k, n_k)$ and $w_2 = (v_{k+1}, n_{k+1})\otimes\cdots\otimes(v_\ell, n_\ell)$ in ΞV. Then we observe:

$$\begin{aligned}
\delta_V(w_1\otimes w_2) &= v\otimes d_V(w_1\otimes w_2) \\
&= v\otimes\left(\sum_{i=1}^{\ell}(v_1,n_1)\otimes\cdots\otimes(v_i,n_i+1)\otimes\cdots\otimes(v_\ell,n_\ell)\right) \\
&= v\otimes\left(\sum_{i=1}^{k}(v_1,n_1)\otimes\cdots\otimes(v_i,n_i+1)\otimes\cdots\otimes(v_k,n_k)\right)\otimes((v_{k+1},n_{k+1})\otimes\cdots\otimes(v_\ell,n_\ell)) \\
&+ v\otimes\left(\sum_{i=k+1}^{\ell}(v_{k+1},n_{k+1})\otimes\cdots\otimes(v_i,n_i+1)\otimes\cdots\otimes(v_\ell,n_\ell)\right)\otimes((v_1,n_1)\otimes\cdots\otimes(v_k,n_k)) \\
&= v\otimes d_V(w_1)\otimes w_2 + v\otimes d_V(w_2)\otimes w_1 \\
&= \delta_V(w_1)\otimes w_2 + \delta_V(w_2)\otimes w_1
\end{aligned}$$

Here we used the symmetrized nature of the tensor product of elements of ΞV. Thus the Leibniz rule is satisfied.

The monad Ξ does not induce a differential category structure, however. The linear derivative rule would imply:

$$\begin{tikzcd}
u\otimes 1_{\mathbb{F}} \cong u \arrow[r, mapsto, "\eta"] \arrow[dr, mapsto, "id\otimes e"'] & (u,0) \arrow[d, mapsto, "\delta_V"] \\
& u\otimes 1_{\mathbb{F}} \neq v\otimes(u,1)
\end{tikzcd}$$

Similarly, the chain rule would imply that

$$v\otimes d(d^{m_1}(w_1)\otimes\cdots\otimes d^{m_k}(w_k)) \;=\; 0$$

which is false in general (for instance, when all $w_i \neq 1$). $\square$

3.8 Free Differential RB-Algebra

We first describe the free differential Rota-Baxter algebra on a differential algebra, and then we explain how it is generalized by each of the $Ш S$ quasimonad and the $\Diamond S$ monad. We also discuss the advantages and disadvantages of each.

Recall that a *differential Rota-Baxter algebra* is an algebra structure (A, d, P) such that (A, d) is a differential algebra, (A, P) is a Rota-Baxter algebra, and the interaction of the two structures satisfies the first fundamental theorem of calculus, namely $d \circ P = id$.

Let $(A, \cdot, d_0)$ be a differential algebra, and $\Diamond A$ the free Rota-Baxter algebra on A. Then define a differential on $\Diamond A$ by

$$\begin{aligned} d(a_0 \otimes (a_1 \otimes \cdots \otimes a_n)) &= d_0(a_0) \otimes (a_1 \otimes \cdots \otimes a_n) \\ &\quad + a_0 \cdot a_1 \otimes (a_2 \otimes \cdots \otimes a_n) \end{aligned}$$

where by convention if $n = 0$ then $d(a_0) = d_0(a_0)$. It is shown in [23] that this structure is the free differential RB-algebra on the differential algebra A.

Let's first look at generalizing this via Ш. To define a quasimonad structure on ШΞ, we must again establish some notation, this time for writing elements of $Ш\Xi V$, $Ш\Xi Ш\Xi V$, and $Ш\Xi Ш\Xi Ш\Xi V$.

If a basis element of $Ш\Xi V$ has the form

$$w^{k\ell}_{ij} = ((v^{ijk\ell}_{11}, r^{ijk\ell}_{11}) \otimes_s \cdots \otimes_s (v^{ijk\ell}_{1m^{ijk\ell}_1}, r^{ijk\ell}_{1m^{ijk\ell}_1})) \otimes \cdots \otimes ((v^{ijk\ell}_{n_{ijk\ell}1}, r^{ijk\ell}_{n_{ijk\ell}1}) \otimes_s \cdots \otimes_s (v^{ijk\ell}_{n_{ijk\ell} m^{ijk\ell}_{n_{ijk\ell}}}, r^{ijk\ell}_{n_{ijk\ell} m^{ijk\ell}_{n_{ijk\ell}}}))$$

then we can write a basis element of $Ш\Xi Ш\Xi V$ as

$$\omega_{k\ell} = ((w^{k\ell}_{11}, r^{k\ell}_{11}) \otimes_s \cdots \otimes_s (w^{k\ell}_{1m^{k\ell}_1}, r^{k\ell}_{1m^{k\ell}_1})) \otimes \cdots \otimes ((w^{k\ell}_{n_{k\ell}1}, r^{k\ell}_{n_{k\ell}1}) \otimes_s \cdots \otimes_s (w^{k\ell}_{n_{k\ell} m^{k\ell}_{n_{k\ell}}}, r^{k\ell}_{n_{k\ell} m^{k\ell}_{n_{k\ell}}}))$$

and a basis element of $Ш\Xi Ш\Xi Ш\Xi V$ as

$$\vartheta = ((\omega_{11}, r_{11}) \otimes_s \cdots \otimes_s (\omega_{1m_1}, r_{1m_1})) \otimes \cdots \otimes ((\omega_{n1}, r_{n1}) \otimes_s \cdots \otimes_s (\omega_{nm_n}, r_{nm_n})) \ .$$

Define $\mu_V \colon Ш\Xi Ш\Xi V \to Ш\Xi V$ by

$$\mu(\omega_{k\ell}) = \overline{\omega_{k\ell}} := (d^{r^{k\ell}_{11}}(w^{k\ell}_{11}) Ш \cdots Ш d^{r^{k\ell}_{1m^{k\ell}_1}}(w^{k\ell}_{1m^{k\ell}_1})) \otimes \cdots \otimes (d^{r^{k\ell}_{n_{k\ell}1}}(w^{k\ell}_{n_{k\ell}1}) Ш \cdots Ш d^{r^{k\ell}_{n_{k\ell} m^{k\ell}_{n_{k\ell}}}}(w^{k\ell}_{n_{k\ell} m^{k\ell}_{n_{k\ell}}}))$$

and $\eta_V \colon V \to Ш\Xi V$ by

$$\eta(v) = ((v, 0)) \ ,$$

which includes v first as $(v, 0) \in \Xi V$ and then as $((v, 0)) \in Ш\Xi V$. This makes ШΞ into a quasimonad; the proof is similar to the proof for $Ш S$ above.

Recall that for the free Rota-Baxter algebra $(V \otimes Ш SV, P_{SV})$ we had $\Pi_Ш = \alpha^{-1} \circ P_{SV} \circ (i \otimes i)$ where $\alpha \colon Ш SV \cong 1_F \otimes Ш SV$. Thus if $(\Xi V, d_0)$ is the free differential algebra, the obvious choice to make $V \otimes Ш\Xi V$ into an FTC module is to define $\delta_Ш \colon Ш\Xi V \to V \otimes Ш\Xi V$ by

$$\delta_Ш \ = \ (p \otimes id) \circ d_{\Diamond \Xi V} \circ \alpha$$

where p is the modified projection map $p\colon \Xi V \to V$ defined by

$$p\colon (u,0) \mapsto u \text{ for } (u,0) \in \Delta V$$

$$p\colon a \mapsto 0 \text{ for all other elements } a \in \Xi V$$

Then we have the following proposition.

Proposition 3.8.1. $(Ш\Xi V, \delta_Ш)$ *is a module with differentiation.*

Proof. Let $a = a_1 \otimes a'$, $b = b_1 \otimes b' \ \in Ш\Xi V$. Then the Leibniz rule is shown as follows:

$$\begin{aligned}
\delta_Ш(a \sqcup b) &= (p \otimes id) \circ d_{\Diamond\Xi V} \circ \alpha(a \sqcup b) \\
&= (p \otimes id) \circ d_{\Diamond\Xi V}(1_{\mathbb{F}} \otimes a \diamond 1_{\mathbb{F}} \otimes b) \\
&= (p \otimes id)\Big(d_{\Diamond\Xi V}(1_{\mathbb{F}} \otimes a) \diamond 1_{\mathbb{F}} \otimes b + d_{\Diamond\Xi A}(1_{\mathbb{F}} \otimes b) \diamond 1_{\mathbb{F}} \otimes a\Big) \\
&= (p \otimes id)\Big(a_1 \otimes a' \diamond 1_{\mathbb{F}} \otimes b + b_1 \otimes b' \diamond 1_{\mathbb{F}} \otimes a\Big) \\
&= (p \otimes id)\Big(a_1 \otimes (a' \sqcup b) + b_1 \otimes (b' \sqcup a)\Big) \\
&= \delta_Ш(a) \cdot b + \delta_Ш(b) \cdot a
\end{aligned}$$

□

Corollary 3.8.2. $(Vec, Ш\Xi, \delta_Ш)$ *has the structure of a Leibniz category, except that* $Ш\Xi$ *is a quasimonad.*

Now, modify the definition of $\Pi_Ш\colon V \otimes ШSV \to V \otimes ШSV$ to be $V \otimes Ш\Xi V \to Ш\Xi V$ in the obvious way. Then in addition:

$$\delta_Ш \circ \Pi_Ш = (p \otimes id) \circ d_{\Xi V} \circ \alpha \circ \alpha^{-1} \circ P_{\Xi V} \circ (i \circ id) = (p \otimes id) \circ d_A \circ P_A \circ (i \otimes id) = (p \otimes id) \circ (i \otimes id) = id$$

Thus the FTC property is captured.

A potential disadvantage of this approach is that if we look closely, it might be argued that we are only superficially "capturing" the differential structure $d_{\Diamond\Xi V}$ of the free differential RB-algebra. As we already exploited above, we have:

$$\begin{aligned}
\delta_Ш(a_1 \otimes a') &= (p \otimes id) \circ d_V(1_{\mathbb{F}} \otimes (a_1 \otimes a')) \\
&= d_\Xi(1_{\mathbb{F}}) \otimes a_1 \otimes a' + 1_{\mathbb{F}} \cdot a_1 \otimes a' \\
&= 0 + a_1 \otimes a' \\
&= a_1 \otimes a'
\end{aligned}$$

So while one term does involve the application of d_V as in the free differential Rota-Baxter algebra, its effect in this construction is always to nullify that term.

Nevertheless, it is an example of the FTC property being satisfied in the Leibniz and Rota-Baxter category setting. Thus we have another example of a Leibniz category and of a Rota-Baxter category, and our first example of a proto-FTC category.

We now turn to $\Diamond$. We must first make $\Diamond\Xi$ into a monad, and again we start with notation.

If a basis element of $\Diamond\Xi V$ has the form

$$\begin{aligned} w_{ij}^{k\ell} = &\Big((v_{01}^{ijk\ell}, r_{01}^{ijk\ell}) \otimes_s \cdots \otimes_s (v_{0m_0^{ijk\ell}}^{ijk\ell}, r_{0m_0^{ijk\ell}}^{ijk\ell})\Big) \\ &\otimes \Big(((v_{11}^{ijk\ell}, r_{11}^{ijk\ell}) \otimes_s \cdots \otimes_s (v_{1m_1^{ijk\ell}}^{ijk\ell}, r_{1m_1^{ijk\ell}}^{ijk\ell})) \otimes \cdots \\ &\cdots \otimes ((v_{n_{ijk\ell}1}^{ijk\ell}, r_{n_{ijk\ell}1}^{ijk\ell}) \otimes_s \cdots \otimes_s (v_{n_{ijk\ell}m_{n_{ijk\ell}}^{ijk\ell}}^{ijk\ell}, r_{n_{ijk\ell}m_{n_{ijk\ell}}^{ijk\ell}}^{ijk\ell}))\Big) \end{aligned}$$

then we can write a basis element of $\Diamond\Xi\Diamond\Xi V$ as

$$\begin{aligned} \omega_{k\ell} = &\Big((w_{01}^{k\ell}, r_{01}^{k\ell}) \otimes_s \cdots \otimes_s (w_{0m_0^{k\ell}}^{k\ell}, r_{0m_0^{k\ell}}^{k\ell})\Big) \\ &\otimes \Big(((w_{11}^{k\ell}, r_{11}^{k\ell}) \otimes_s \cdots \otimes_s (w_{1m_1^{k\ell}}^{k\ell}, r_{1m_1^{k\ell}}^{k\ell})) \otimes \cdots \otimes ((w_{n_{k\ell}1}^{k\ell}, r_{n_{k\ell}1}^{k\ell}) \otimes_s \cdots \otimes_s (w_{n_{k\ell}m_{n_{k\ell}}^{k\ell}}^{k\ell}, r_{n_{k\ell}m_{n_{k\ell}}^{k\ell}}^{k\ell}))\Big) \end{aligned}$$

and a basis element of $\Diamond\Xi\Diamond\Xi\Diamond\Xi V$ as

$$\begin{aligned} \vartheta = &((\omega_{01}, r_{01}) \otimes_s \cdots \otimes_s (\omega_{0m_0}, r_{0m_0})) \\ &\otimes \Big(((\omega_{11}, r_{11}) \otimes_s \cdots \otimes_s (\omega_{1m_1}, r_{1m_1})) \otimes \cdots \otimes ((\omega_{n1}, r_{n1}) \otimes_s \cdots \otimes_s (\omega_{nm_n}, r_{nm_n}))\Big) \ . \end{aligned}$$

Define $\mu_V \colon \Diamond\Xi\Diamond\Xi V \to \Diamond\Xi V$ by

$$\begin{aligned} \mu(\omega_{k\ell}) &= \overline{\omega_{k\ell}} \\ &:= \Big((d^{r_{01}^{k\ell}}(w_{01}^{k\ell}) \diamond \cdots \diamond d^{r_{0m_0^{k\ell}}^{k\ell}}(w_{0m_0^{k\ell}}^{k\ell}))\Big) \\ &\quad \otimes \Big((d^{r_{11}^{k\ell}}(w_{11}^{k\ell}) \diamond \cdots \diamond d^{r_{1m_1^{k\ell}}^{k\ell}}(w_{1m_1^{k\ell}}^{k\ell})) \otimes \cdots \otimes (d^{r_{n_{k\ell}1}^{k\ell}}(w_{n_{k\ell}1}^{k\ell}) \diamond \cdots \diamond d^{r_{n_{k\ell}m_{n_{k\ell}}^{k\ell}}^{k\ell}}(w_{n_{k\ell}m_{n_{k\ell}}^{k\ell}}^{k\ell}))\Big) \end{aligned}$$

and $\eta_V \colon V \to \Diamond\Xi V$ by

$$\eta(v) = ((v, 0)) \otimes 1_{\mathbb{F}} \ ,$$

This makes $\Diamond\Xi$ into a monad; the proof is similar to the proof for $\Diamond S$ above.

Define $\Pi \colon V \otimes \Diamond\Xi V \to \Diamond\Xi V$ by $\Pi(v \otimes a) = 1_{\mathbb{F}} \otimes (a)$. The proofs in the previous subsections are adapted essentially without modification to show that this makes Vec into a Rota-Baxter category.

Let $a = a_0 \otimes (a_1 \otimes \cdots \otimes a_n) \in \Diamond\Xi V$, and write $a^+ = a_1 \otimes \cdots \otimes a_n$, $a^{++} = a_2 \otimes \cdots \otimes a_n$. For each V, choose some vector $v \in V$. Define $\delta_\diamond \colon \Diamond\Xi V \to V \otimes \Diamond\Xi V$ by:

$$\delta(a) = v \otimes (d_{\Xi V}(a_0) \otimes a^+ + a_0 \cdot a_1 \otimes a^{++})$$

We stipulate the sensible convention that if $a^+ = 1_{\mathbb{F}}$, then $a^{++} = 0$.

We can now prove the following.

Proposition 3.8.3. $(Vec, \Diamond\Xi, \delta_\Diamond)$ *is a Leibniz category that is not a differential category.*

Proof. This proof uses techniques similar to the proof that $\lozenge A$ is a differential algebra for any differential algebra A from [23].

Let V be a vector space with chosen vector v. Recall that the Rota-Baxter operator P on $\lozenge\Xi V$ (in fact, on $\lozenge A$ for any algebra A) is defined by $P(a) = 1_A \otimes (a)$. Then we can write $a = (a_0 \otimes 1_{\mathbb{F}}) \diamond P(a^+)$, and with similar notation for another element $b \in \lozenge\Xi V$, we have

$$\begin{aligned} a \diamond b &= (a_0 b_0 \otimes 1_{\mathbb{F}}) \diamond (P(a^+) \diamond P(b^+)) \\ &= (a_0 b_0 \otimes 1_{\mathbb{F}}) \diamond (P(a^+ \diamond P(b^+)) + P(b^+ \diamond P(a^+))) \\ &= (a_0 b_0 \otimes 1_{\mathbb{F}}) \diamond P(a^+ \diamond P(b^+) + b^+ \diamond P(a^+)) \end{aligned}$$

by the Rota-Baxter property and linearity of P. Then writing $d := d_{\Xi V}$ we thus have:

$$\begin{aligned} \delta \circ m(a \otimes b) &= \delta(a \diamond b) \\ &= \delta((a_0 b_0 \otimes 1_{\mathbb{F}}) \diamond P(a^+ \diamond P(b^+) + b^+ \diamond P(a^+))) \\ &= v \otimes \Big((d(a_0 b_0) \otimes 1_{\mathbb{F}}) \diamond P(a^+ \diamond P(b^+) + b^+ \diamond P(a^+)) \\ &\quad + (a_0 b_0 \otimes 1_{\mathbb{F}}) \diamond (a^+ \diamond P(b^+) + b^+ \diamond P(a^+)) \Big) \\ &= v \otimes \Big(((d_0(a_0) b_0 + a_0 d_0(b_0)) \otimes 1_{\mathbb{F}}) \diamond (P(a^+) \diamond P(b^+)) \Big) \\ &\quad + v \otimes \Big((a_0 b_0 \otimes 1_{\mathbb{F}}) \diamond (a^+ \diamond P(b^+) + b^+ \diamond P(a^+)) \Big) \\ &= v \otimes \Big((d_0(a_0) b_0 \otimes 1_{\mathbb{F}}) \diamond (P(a^+) \diamond P(b^+)) \Big) \\ &\quad + v \otimes \Big((a_0 d_0(b_0) \otimes 1_{\mathbb{F}}) \diamond (P(a^+) \diamond P(b^+)) \Big) \\ &\quad + v \otimes \Big((a_0 b_0 \otimes 1_{\mathbb{F}}) \diamond (a^+ \diamond P(b^+) + b^+ \diamond P(a^+)) \Big) \end{aligned}$$

On the other hand, writing

$$(*) = \big((id \otimes m) \circ (\delta \otimes id) + (id \otimes m) \circ (\sigma \otimes id) \circ (id \otimes \delta) \big)(a \otimes b) \ ,$$

we have:

$$\begin{aligned}
(*) &= v \otimes \Big(((d_0(a_0) \otimes 1_{\mathbb{F}}) \diamond P(a^+) + (a_0 \otimes 1_{\mathbb{F}}) \diamond a^+) \diamond b \Big) \\
&\quad + v \otimes \Big(((d_0(b_0) \otimes 1_{\mathbb{F}}) \diamond P(b^+) + (b_0 \otimes 1_{\mathbb{F}}) \diamond b^+) \diamond a \Big) \\
&= v \otimes \Big(((d_0(a_0) \otimes 1_{\mathbb{F}}) \diamond P(a^+) + (a_0 \otimes 1_{\mathbb{F}}) \diamond a^+) \diamond ((b_0 \otimes 1_{\mathbb{F}}) \diamond P(b^+)) \Big) \\
&\quad + v \otimes \Big(((d_0(b_0) \otimes 1_{\mathbb{F}}) \diamond P(b^+) + (b_0 \otimes 1_{\mathbb{F}}) \diamond b^+) \diamond ((a_0 \otimes 1_{\mathbb{F}}) \diamond P(a^+)) \Big) \\
&= v \otimes \Big(((d_0(a_0) \otimes 1_{\mathbb{F}}) \diamond P(a^+)) \diamond ((b_0 \otimes 1_{\mathbb{F}}) \diamond P(b^+)) \Big) \\
&\quad + v \otimes \Big(((a_0 \otimes 1_{\mathbb{F}}) \diamond a^+) \diamond ((b_0 \otimes 1_{\mathbb{F}}) \diamond P(b^+)) \Big) \\
&\quad + v \otimes \Big(((d_0(b_0) \otimes 1_{\mathbb{F}}) \diamond P(b^+)) \diamond ((a_0 \otimes 1_{\mathbb{F}}) \diamond P(a^+)) \Big) \\
&\quad + v \otimes \Big(((b_0 \otimes 1_{\mathbb{F}}) \diamond b^+) \diamond ((a_0 \otimes 1_{\mathbb{F}}) \diamond P(a^+)) \Big) \\
&= v \otimes \Big((d_0(a_0)b_0 \otimes 1_{\mathbb{F}}) \diamond (P(a^+) \diamond P(b^+)) \Big) \\
&\quad + v \otimes \Big((a_0b_0 \otimes 1_{\mathbb{F}}) \diamond (a^+ \diamond P(b^+)) \Big) \\
&\quad + v \otimes \Big((d_0(b_0)a_0) \otimes 1_{\mathbb{F}}) \diamond (P(b^+) \diamond P(a^+)) \Big) \\
&\quad + v \otimes \Big((b_0a_0 \otimes 1_{\mathbb{F}}) \diamond (b^+ \diamond P(a^+)) \Big) \\
&= v \otimes \Big((d_0(a_0)b_0 \otimes 1_{\mathbb{F}}) \diamond (P(a^+) \diamond P(b^+)) \Big) \\
&\quad + v \otimes \Big((d_0(b_0)a_0 \otimes 1_{\mathbb{F}}) \diamond (P(b^+) \diamond P(a^+)) \Big) \\
&\quad + v \otimes \Big((a_0b_0 \otimes 1_{\mathbb{F}}) \diamond (a^+ \diamond P(b^+) + b^+ \diamond P(a^+)) \Big)
\end{aligned}$$

Since these are equal by commutativity of $\diamond$, we've proven the Leibniz rule.

For the constants rule, observe that $e(1_{\mathbb{F}}) = 1_{\mathbb{F}} \otimes 1_{\mathbb{F}}$ and so $\delta \circ e(1_{\mathbb{F}}) = 1_{\mathbb{F}} \otimes d_{\Xi A}(1_{\mathbb{F}}) \otimes 1_{\mathbb{F}}$ (remembering our convention that $1_{\mathbb{F}}^+ = 0$). But $d_{\Xi A}(1_{\mathbb{F}}) = 0$, so the preceding tensor product is 0.

The linear maps rule is not satisfied: it is easily verified that

$$\delta \circ \eta(a) = 1_A \otimes (a \otimes 1_{\mathbb{F}}) \neq a \otimes (1_A \otimes 1_{\mathbb{F}}) = (id \otimes e) \circ \rho(a) \ .$$

The chain rule is also not satisfied: observe that for any $a \in A$,

$$(\delta; (\delta \otimes \mu))(a) = (\delta \otimes \mu)((1_A \otimes 1_{\mathbb{F}}) \otimes b) = 0$$

since $\delta(1_A \otimes 1_{\mathbb{F}}) = 0$, where b signifies all the factors irrelevant to this calculation. On the other hand, $\mu; \delta$ in general will not be zero.

□

We've shown that (Vec, δ, Π) is both a Leibniz category and a Rota-Baxter category, but neither a differential nor integral one. Unfortunately, a disadvantage to this approach is that it is not quite a proto-FTC category as defined above. It does, however, satisfy a slightly weaker version of the fundamental theorem of calculus. Let us explain in what sense this is true.

Recall that the differential operator d and RB operator P on the free Rota-Baxter algebra on a differential algebra, as described above, satisfy the fundamental theorem of calculus without modifications [23]. We have

$$d \circ P(a) = d(1_A \otimes a) = a$$

since $d_0(1_A) = 0$.

The categorification given here does not translate the FTC directly, and any that does so within the established differential category setting would have to be truly unusual. The route $A \otimes A \otimes Ш A \to A \otimes Ш A \to A \otimes A \otimes Ш A$ is too bumpy, in the sense that we would have to preserve the leftmost A factor throughout the mappings and pull it back out unscathed at the end.

What does hold true in our case, however, is the following.

Proposition 3.8.4. *For each vector space, there is a nontrivial subspace B of $V \otimes \Diamond \Xi V$ such that the following diagram commutes:*

$$\begin{tikzcd}
B \arrow[r, hook, "\iota"] \arrow[dd, hook, "\iota"'] & V \otimes \Diamond SV \arrow[d, "\Pi"] \\
 & \Diamond SV \arrow[d, "\delta"] \\
 & V \otimes \Diamond SV
\end{tikzcd}$$

Proof. Let $B = v \otimes \Diamond \Xi V$, the subspace of elements of the form $v \otimes a$ for $a \in \Diamond \Xi V$ and v the chosen vector of V used in $\delta_\Diamond$. Then the diagram becomes:

$$\begin{tikzcd}
v \otimes a \arrow[r, mapsto, "\iota"] & v \otimes a \arrow[r, mapsto, "\Pi_\Diamond"] \arrow[rr, mapsto, bend right, "id"'] & v \otimes 1_{\mathbb{F}} \otimes a \arrow[r, "\delta_\Diamond"] & v \otimes (d(1_{\mathbb{F}}) \otimes a + a)
\end{tikzcd}$$

□

To summarize, we've constructed Rota-Baxter categories (which are not integral categories) using $Ш S$, $\Diamond S$, $Ш\Xi$, and $\Diamond\Xi$, and we've constructed Leibniz categories (which are not differential categories) using Ξ, $Ш\Xi$, and $\Diamond\Xi$. The constructions using $Ш\Xi$ and $\Diamond\Xi$ also capture the fundamental theorem of calculus, although each has advantages and disadvantages: $Ш\Xi$ is not a monad but a quasimonad and its capturing of the underlying differential of Ξ is imperfect, but its satisfaction of the FTC is elegant and direct; while $\Diamond\Xi$ involves the choosing of specific vectors in each vector space V and only a restricted version of the FTC is satisfied, but it is a true monad directly capturing the free differential Rota-Baxter algebra adjunction.

4 Weight-λ Rota-Baxter Categories

In this section we generalize our Rota-Baxter categories to accommodate an arbitrary weight, and demonstrate the one of the applications of this is to capture renormalization of Laurent series as used in quantum field theory, and more generally a certain broad class of operators on what Ribenboim called generalized power series, which we will call Ribenboim power series.

4.1 An Arbitrary Weight λ

A Rota-Baxter algebra of weight λ is defined to be an algebra A equipped with a linear map $P\colon A \to A$ satisfying

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy) \ .$$

Of course, taking $\lambda = 0$ gives us back the structure we have up to now been calling a Rota-Baxter algebra. But many applications demand Rota-Baxter algebras of non-zero weight; one of the most interesting is the Laurent series, which plays a key role in renormalization in perturbative quantum field theories [22].

We approach our generalization seeking to replicate the format of Kähler categories. In brief, Kähler categories were designed to capture modules of differential forms, and utilize a derivation $\partial\colon TA \to H(A)$ for some second algebra modality functor H which is a compatible TA-module. It was once believed that all codifferential categories were Kähler, but this has recently been cast into doubt. The case with a minor additional property assumed is proven in [5].

The H above is a structure that merits its own definition.

Definition 4.1.1. *Let (T, m_T, e_T) be an algebra modality. Then an algebra modality (H, m_H, e_H) is a* module algebra modality associated to T *if $H(A)$ is a TA-module for every object A; that is, if there is a map $\bullet\colon H(A)\otimes T(A) \to H(A)$ such that the following diagrams commute:*

$$\begin{tikzcd}
H(A)\otimes T(A)\otimes T(A) \arrow[r, "id\otimes m_T"] \arrow[dr, "\bullet\otimes id"'] & H(A)\otimes T(A) \arrow[r, "\bullet"] & H(A) \\
& H(A)\otimes T(A) \arrow[ur, "\bullet"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
H(A) \arrow[r, "\rho^{-1}"] \arrow[rrr, bend right, "id"'] & H(A)\otimes I \arrow[r, "id\otimes e_T"] & H(A)\otimes T(A) \arrow[r, "\bullet"] & H(A)
\end{tikzcd}$$

Proposition 4.1.2. *Every algebra modality is a module algebra modality associated to itself.*

Proof. Straightforward. $\square$

We'll also need a categorical notion of scalar multiplication, to describe the effect of λ. We employ the same structure used in quantum category theory, as described in [1].

Suppose we are in an additive symmetric monoidal closed category. Then it is straightforward to see that $R = Hom(I, I)$ is a commutative ring, which we call the *ring of scalars*. There is an action of R on arbitrary hom-sets: if $f \in Hom(M, N)$ and $q \in R$, we define $q \cdot f \colon M \to N$ by the formula

$$q \cdot f = M \xrightarrow{\cong} I \otimes M \xrightarrow{q \otimes f} I \otimes M \xrightarrow{\cong} M \ .$$

This action satisfies all of the evident properties of a commutative ring on an abelian group. Furthermore, given $f \in Hom(M, N), g \in Hom(M', N')$, we have

$$q \cdot f \otimes g = f \otimes q \cdot g$$

The action of R also respects composition:

$$(q \cdot f) \circ g = f \circ (q \cdot g) = q \cdot (f \circ g)$$

Write λf for the scalar multiple of λ on $f \colon A \to B$. We are now ready to define our new categories.

Definition 4.1.3. *Let $(\mathcal{C}, \otimes, I, \sigma)$ be a symmetric monoidal closed category equipped with a monad (T, μ, η) that is also an algebra modality (T, m_T, e_T), and let $(H, m_H, e_H, \cdot)$ be a module algebra modality associated to T, where $\bullet \colon H(A) \otimes TA \to H(A)$ is the module action. A* Rota-Baxter transformation of weight λ *for a given scalar λ is a natural transformation $\Pi \colon H \to T$ such that the following equation holds:*

$$(\Pi \otimes \Pi); m^T = (id \otimes \Pi); \bullet; \Pi + (\Pi \otimes id); \sigma; \bullet; \Pi + m^H; \lambda \Pi$$

Equipped with such a Π, $\mathcal{C}$ is called a Rota-Baxter category of weight λ*.*

We begin with a couple of basic examples.

Example 4.1.4. Let $\mathcal{C} = \mathbb{K}\text{-}Vec$ be the category of $\mathbb{K}$-vector spaces, where for simplicity we specify $char\mathbb{K} = 0$, and let S be the usual symmetric tensor algebra monad, so that SV can be viewed as polynomials in basis vectors of V. View this as a module algebra modality over itself. Then for each basis vector $x_i \in V$, define a transformation $\Pi_V^{x_i} \colon SV \to SV$ by:

$$\Pi^{x_i}(x_1^{m_1} \cdots x_n^{m_n}) = \frac{1}{m_i + 1} x_1^{m_1} \cdots x_i^{m_i+1} \cdots x_n^{m_n}$$

This acts as polynomial integration with respect to the variable x_i, and so makes $\mathbb{K}\text{-}Vec$ into a Rota-Baxter category of weight $\lambda = 0$.

Example 4.1.5. Let $T = H$ be any algebra modality on any category $\mathcal{C}$ with $m = \bullet \colon TA \otimes TA \to TA$. For each scalar λ define:

$$\Pi_A^\lambda = -\lambda id_{TA}$$

That is, Π^λ is just scalar multiplication by $-\lambda$. Then, since scalar multiplication is an action of the commutative ring $Hom(I,I)$ on the monoid $Hom(TA,TA)$ respecting composition and tensors, we have:

$$\begin{aligned}
(\Pi \otimes \Pi); m &= \Big((-\lambda id) \otimes (-\lambda id)\Big); m \\
&= \lambda^2 (id \otimes id); m \\
&= \lambda^2 m \\
&= \lambda^2 m + \lambda^2 m - \lambda^2 m \\
&= (-\lambda id \otimes id); \bullet; (-\lambda id) + (-\lambda id); \bullet; (id \otimes -\lambda id) + m_H; \lambda(-\lambda id) \\
&= (\Pi \otimes id); \sigma; \bullet; \Pi + (id \otimes \Pi); \cdot; \Pi + m; \lambda \Pi
\end{aligned}$$

Thus $(\mathcal{C}, T, \Pi)$ is a Rota-Baxter category of weight λ.

4.2 Laurent Series and the Monad G(M,-)

A more noteworthy example is that of Laurent series. These admit a Rota-Baxter operator of degree -1 that is of interest in perturbative quantum field theory [22, 40]. We show that they, along with a much broader class of examples, are captured by our weight-λ Rota-Baxter category structure.

Recall that a Laurent series is a series

$$f(x) = \sum_{n=k}^{\infty} a_n x^n$$

where $a_n \in \mathbb{C}$ and $k \in \mathbb{Z}$. That is, Laurent series are a generalization of power series where there may be a finite singular part of negative degree. These can be equipped with a linear operator

$$P(f(x)) = \sum_{n=k}^{-1} a_n x^n$$

which keeps only the singular part of the series. It is shown in [22] that this is a Rota-Baxter algebra of weight -1.

There is a structure known to capture Laurent series, among other constructions, which we will show is an algebra modality. This is the structure that Ribenboim called generalized power series, which we will call Ribenboim power series. The presentation is based on those in [24] and [35].

Let $(M, +, \leq)$ be a partially ordered monoid. M is *strictly ordered* if

$$s < s' \Rightarrow s + t < s' + t \ \forall s, s', t \in M \ .$$

We will henceforth assume that all the monoids we work with are strictly ordered.

An ordered monoid is *artinian* if all strictly descending chains are finite; that is, if any list $(m_1 > m_2 > \cdots)$ must be finite. It is *narrow* if all discrete subsets are finite; that is, if all subsets of elements mutually unrelated by $\leq$ must be finite.

Definition 4.2.1. *Let V be a vector space, and recall that the* support *of a function $f\colon M \to V$ is defined by $supp(f) = \{m \in M | f(m) \neq 0\}$. Define the* Ribenboim power series from M with coefficients in V *$G(M, V)$ to be the set of functions $f\colon M \to V$ whose support is artinian and narrow.*

If V is also a commutative $\mathbb{K}$-algebra, then $G(M, V)$ is a commutative $\mathbb{K}$-algebra with

$$(f \cdot g)(m) = \sum_{(u,v) \in X_m(f,g)} f(u) \cdot g(v)$$

where

$$X_m(f, g) := \{(u, v) \in M \times M | u + v = m \text{ and } f(u) \neq 0, g(v) \neq 0\}$$

and $M \times M$ has the product ordering. To prove that this sum necessarily exists, we need a couple of well-known results on artinian and narrow sets [36, 27, 35].

Lemma 4.2.2. *Recall that an ordered set is* noetherian *if all strictly ascending chains are finite, and let $(M, \leq)$ be artinian and noetherian. Then M is finite if and only if M is narrow.*

Lemma 4.2.3. *If $(M, \leq_M)$ and $(N, \leq_N)$ are artinian and narrow, then $M \times N$ is artinian and narrow under the product ordering.*

Proposition 4.2.4. *The set $X := X_m(f, g)$ is finite for f, $g \in G(M, V)$.*

Proof. We follow the proof in [35].

Observe that $X \subseteq supp(f) \times supp(g)$, so X is artinian and narrow. Suppose towards a contradiction that X is infinite. Then by the first lemma above, X must not be noetherian. Thus there exists in X an infinite strictly increasing sequence

$$(u_1, v_1) < (u_2, v_2) < \cdots .$$

Note that the strict inequalities under the product ordering in this sequence imply that at least one of the sets $\{u_i\}$, $\{v_i\}$ must be infinite; suppose without loss of generality that it is $\{u_i\}$. There must then exist a strictly increasing subsequence

$$u_{j_1} < u_{j_2} < \cdots .$$

In particular $(u_{j_1}, v_{j_1}) < (u_{j_2}, v_{j_2})$, and recall that by definition of X we have $u_i + v_i = m \ \forall i$. But then by strictness of the ordering, we have

$$m = u_{j_1} + v_{j_1} < u_{j_2} + v_{j_1} \leq u_{j_2} + v_{j_2} = m$$

which is a contradiction. Thus X is finite.

$\square$

A similar argument shows that for a map $h\colon M \to G(M, V)$, the set $X_m(h) = \{(u, v) \in M \times M | u + v = m \text{ and } h(u)(v) \neq 0\}$ is also finite.

Before proceeding with a series of lemmas and propositions to show $G(M, V)$ is induced as an algebra modality, we note that setting $M = \mathbb{Z}$ with the usual ordering and $V = \mathbb{C}$ gives Laurent series; the image of each integer $n \in \mathbb{Z}$ gives the coefficient of x^n, and the artinian property of the function's support ensures a finite singular part. Similarly, $M = \mathbb{N}$ gives the usual power series, and there are additional examples which will be discussed below.

Lemma 4.2.5. *$G(M,V)$ is a vector space.*

Proof. Define $(f+g)(m) := f(m) + g(m)$, $(cf)(m) = cf(m)$, $0_{G(M,V)} = 0$ (the zero map), and $(-f)(m) = -f(m)$. The result then follows immediately from V being a vector space.

$\square$

Proposition 4.2.6. *$G(M,-)$ is a functor $Vec \to Vec$.*

Proof. Let $f\colon V \to W$. Define $G(M,f)\colon G(M,V) \to G(M,W)$ by

$$G(M,f)(g)(m) = f \circ g(m)$$

We must prove that $f \circ g \in G(M,W)$, i.e. that $supp(f \circ g)$ is artinian and narrow. But $supp(f \circ g) \subseteq supp(g)$ since f is linear, and $supp(g)$ is artinian and narrow since $g \in G(M,V)$. Thus $supp(f \circ g)$ is also. It is clear that $G(M,id) = id$ and $G(M, f \circ g) = G(M,f) \circ G(M,g)$. Thus $G(M,-)$ is a functor.

$\square$

Proposition 4.2.7. *$G(M, G(N,V)) \cong G(M \times N, V)$ for ordered monoids M, N and vector space V, where $M \times N$ is equipped with the product ordering $(a,b) \leq (c,d) \iff a \leq c$ and $b \leq d$.*

Proof. Let $f \in G(M \times N, V)$, and define a mapping $(\tilde{-})\colon G(M \times N, V) \to G(M, G(N,V))$ by

$$f \mapsto [\tilde{f}\colon m \mapsto (n \mapsto f(m,n))]$$

We show first that $\tilde{f} \in G(M, G(N,V))$, and then that $(\tilde{-})$ is an isomorphism of vector spaces. Observe that we have:

$$\begin{aligned} m \in supp(\tilde{f}) &\iff \tilde{f}(m) \neq 0 \\ &\iff [n \mapsto f(m,n)] \neq 0 \\ &\iff f(m,n_0) \neq 0 \text{ for some } n_0 \in N \\ &\iff (m,n_0) \in supp(f) \end{aligned}$$

Let $(m_1 > m_2 > \cdots)$ for $m_i \in supp(\tilde{f})$ be a strictly decreasing chain. Then $((m_1,n_0) > (m_2,n_0) > \cdots)$ in $M \times N$ by the product ordering, and this chain must be finite since $(m_i,n_0) \in supp(f)$ $\forall i$ by the above. Thus the chain (m_i) must be finite, and so $supp(\tilde{f})$ is artinian.

Similarly, suppose $M' \subseteq supp(\tilde{f})$ is a discrete subset. Then $M' \times \{n\}$ is a discrete subset of $supp(f)$, and thus is finite. Thus M' is finite as well, and so $supp(\tilde{f})$ is narrow. We've now shown $\tilde{f} \in G(M, G(N,V))$.

Next, we show $\tilde{(-)}$ is linear and bijective. For all $m \in M$, $n \in N$ we have

$$\begin{aligned}\widetilde{(af+bg)}(m)(n) &= (af+bg)(m,n)\\ &af(m,n)+bg(m,n)\\ &a\tilde{f}(m)(n)+b\tilde{g}(m)(n)\end{aligned}$$

which shows linearity. For injectivity, we have:

$$\begin{aligned}\tilde{f}=\tilde{g} &\Rightarrow \tilde{f}(m)=\tilde{g}(m)\ \forall m\in M\\ &\Rightarrow \tilde{f}(m)(n)=\tilde{g}(m)(n)\ \forall m\in M,\ n\in N\\ &\Rightarrow f(m,n)=g(m,n)\ \forall (m,n)\in M\times N\\ &\Rightarrow f=g\end{aligned}$$

For surjectivity, let $h \in G(M, G(N, V))$, and define a function $f\colon M \times N \to V$ by $f(m, n) = h(m)(n)$. It is clear that $\tilde{f} = h$. We claim $f \in G(M \times N, V)$.

Following the same reasoning as above, we have $(m, n) \in supp(f) \iff m \in supp(h)$ and $n \in supp(h(m))$, both of which are artinian and narrow. Let $((m_1, n_1) > (m_2, n_2) > \cdots)$ consist of elements of $supp(f)$. Then for all i either $m_i > m_{i+1}$ or $n_i > n_{i+1}$; for each i choose m_i if the former holds and n_i if the latter holds (if both hold choose either one), and denote the new chain by (p_i). This has subchains (p^m_j) and (p^n_k) consisting of the elements of (p_i) belonging to M and N respectively. Since both these subchains are strictly descending and consist of elements of $supp(h)$ and $supp(h(m_k))$ respectively, they are both finite. Thus (p_i) is finite, so in turn (m_i, n_i) is finite, and $supp(f)$ is artinian.

Similarly, suppose $M' \times N'$ is a discrete subset of $supp(f)$. Then M' is a discrete subset of $supp(h)$, and so is finite. N' is then a union of discrete subsets of $supp(h(m_i))$ for $m_i \in supp(h)$, of which there are only finitely many, and since each of these subsets is finite, the union is finite also. Thus $M' \times N'$ is finite, and $supp(f)$ is narrow. Therefore $f \in G(M \times N, V)$, completing the proof.

□

Proposition 4.2.8. *$G(M, -)$ is a monad, with monadic unit $\eta_V : V \to G(M, V)$ given by*

$$\eta\colon v \mapsto \left[m \mapsto \begin{cases} v & \text{if } m = 0_M \\ 0_V & \text{if } m \neq 0_M \end{cases}\right]$$

and monadic multiplication $\mu_V : G(M, G(M, V)) \to G(M, V)$ given by

$$\mu\colon h \mapsto \left[m \mapsto \sum_{(u,v)\in X_m(h)} h(u)(v)\right] .$$

Proof. We'll show it is a Kleisli monad; that is, for all $f\colon V \to G(M, W)$ and $g\colon W \to G(M, Z)$ it satisfies:

1. $\mu \circ G(M, \eta) = id_{G(M,V)}$
2. $\mu \circ G(M, f) \circ \eta = f$
3. $\mu \circ G\Big(M, (\mu \circ G(M, g) \circ f)\Big) = \mu \circ G(M, g) \circ \mu \circ G(M, f)$

We will write X_m in place of $X_m(f)$ when the function f is clear. Then for the first property, we have:

$$\begin{aligned} f &\overset{G(M,\eta)}{\mapsto} [G(M, \eta)(f) \colon m \mapsto \eta \circ f(m)] \\ &= \left[m \mapsto \begin{cases} f(m) & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \right] \\ &\overset{\mu}{\mapsto} \left[\mu \circ G(M, \eta)(f) \colon m \mapsto \sum_{(u,v) \in X_m} G(M, \eta)(f)(u)(v) \right] \end{aligned}$$

But:

$$\begin{aligned} (u, v) \in X_m(G(M, \eta)(f)) &\iff u + v = m \text{ and } G(M, \eta)(f)(u)(v) \neq 0 \\ &\iff u + v = m \text{ and } v = 0 \text{ (and } f(u) \neq 0) \\ &\iff u = m \text{ and } v = 0 \end{aligned}$$

Thus $\mu \circ G(M, \eta)(f) \colon m \mapsto G(M, \eta)(f)(m)(0) = f(m)$ and so $\mu \circ G(M, \eta)(f) = f$ as required.

For the second property, we have:

$$\begin{aligned} v &\overset{\eta}{\mapsto} \eta(v) \\ &\overset{G(M,f)}{\longmapsto} \left[G(M, f)(\eta(v)) \colon m \mapsto f \circ \eta(v)(m) = \begin{cases} f(v) & \text{if } m = 0 \\ f(0) = 0 & \text{if } m \neq 0 \end{cases} \right] \\ &= \left[m \mapsto \left[n \mapsto \begin{cases} f(v)(n) & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \right] \right] \\ &\overset{\mu}{\mapsto} \left[\mu \circ G(M, f)(\eta(v)) \colon m \mapsto \sum_{(u_1,u_2) \in X_m} G(M, f)(\eta(v))(u_1)(u_2) \right] \end{aligned}$$

Similarly to in the first property, we have:

$$\begin{aligned} (u_1, u_2) \in X_m &\iff u_1 + u_2 = m \text{ and } G(M, f)(\eta(v))(u_1)(u_2) \neq 0 \\ &\iff u_1 + u_2 = m \text{ and } u_1 = 0 \\ &\iff u_1 = 0 \text{ and } u_2 = m \end{aligned}$$

Thus $\mu \circ G(M, f)(\eta(v)) \colon m \mapsto G(M, f)(\eta(v))(0)(m) = f(v)(m)$, and so $\mu \circ G(M, f) \circ \eta = f$ as required.

The third property is the most involved. We'll compute the effect of each side of the equation separately. Before that, define the following notation:

$$\begin{aligned}
X_m^1 &:= X_m(G(M,f)(h)) \\
X_m^2 &:= X_m(G(M,g) \circ \mu \circ G(M,f)(h)) \\
X_{m,n}^3 &:= X_n(G(M,g)(f(h(m)))) \\
X_m^4 &:= X_m(G(M,(\mu \circ G(M,g) \circ f))(h))
\end{aligned}$$

For the right side of the third equation, we have:

$$\begin{aligned}
h &\overset{G(M,f)}{\longmapsto} [G(M,f)(h) \colon m \mapsto f \circ h(m)] \\
&\overset{\mu}{\mapsto} \left[\mu \circ G(M,f)(h) \colon m \mapsto \sum_{(u,v) \in X_m^1} f \circ h(u)(v)\right] \\
&= \left[m \mapsto \sum_{(u,v) \in X_m^1} f(h(u))(v)\right] \\
&\overset{G(M,g)}{\longmapsto} \left[m \mapsto \sum_{(u,v) \in X_m^1} g(f(h(u))(v))\right] \\
&= \left[m \mapsto \left[n \mapsto \sum_{(u,v) \in X_m^1} g(f(h(u))(v))(n)\right]\right] \\
&\overset{\mu}{\mapsto} \left[m \mapsto \sum_{(s,t) \in X_m^2} \sum_{(u,v) \in X_s^1} g(f(h(u))(v))(t)\right]
\end{aligned}$$

For the left side, we have:

$$\begin{aligned}
h &\overset{G(M,(\mu \circ G(M,g) \circ f))}{\longmapsto} \left[m \mapsto \left[n \mapsto \sum_{(u,v) \in X_{m,n}^3} g(f(h(m))(u))(v)\right]\right] \\
&\overset{\mu}{\mapsto} \left[m \mapsto \sum_{(s,t) \in X_m^4} \sum_{(u,v) \in X_{s,t}^3} g(f(h(s))(u))(v)\right]
\end{aligned}$$

To prove that these maps are equal, we'll show the sums in the images are equal. To facilitate this, we first give some equivalent conditions to elements being in the various "X" sets.

$$\begin{aligned}
(u,v) \in X^1_s &\iff u+v=s \text{ and } G(M,f)(h)(u)(v) \neq 0 \\
&\iff u+v=s \text{ and } f(h(u))(v) \neq 0
\end{aligned}$$

$$\begin{aligned}
(s,t) \in X^2_m &\iff s+t=m \text{ and } G(M,g) \circ \mu \circ G(M,f)(h)(s)(t) \neq 0 \\
&\iff s+t=m \text{ and } \sum_{(p,q)\in X^1_s} g(f(h(p))(q))(t) \neq 0
\end{aligned}$$

$$\begin{aligned}
(u,v) \in X^3_{s,t} &\iff u+v=t \text{ and } G(M,g)(f(h(s)))(u)(v) \neq 0 \\
&\iff u+v=t \text{ and } g(f(h(s))(u))(v) \neq 0
\end{aligned}$$

$$\begin{aligned}
(s,t) \in X^4_m &\iff s+t=m \text{ and } G(M,(\mu \circ G(M,g) \circ f))(h)(s)(t) \neq 0 \\
&\iff s+t=m \text{ and } \sum_{(p,q)\in X^3_{s,t}} g(f(h(s))(p))(q) \neq 0
\end{aligned}$$

Fix m. Let $(s,t) \in X^2_m$ and $(u,v) \in X^1_s$. We'll show that the term $g(f(h(u))(v))(t) \neq 0$ from the right-hand sum also appears as a term in the left-hand sum; that is, we'll show it is of the form $g(f(h(s'))(u'))(v')$ for some $(s',t') \in X^4_m$ and $(u',v') \in X^3_{s',t'}$.

Set $s' := u$, $u' := v$, $v' := t$, and $t' := v+t$. Then $u'+v' = v+t = t'$, and $g(f(h(s'))(u'))(v') = g(f(h(u))(v))(t) \neq 0$, so $(u',v') \in X^3_{s',t'}$. We also have $s'+t' = u+v+t = s+t = m$, and $\sum_{(p,q)\in X^3_{s',t'}} g(f(h(s'))(p))(q) \neq 0$ since for $(p,q) = (u',v')$, $g(f(h(s'))(u'))(v') \neq 0$. Thus $(s',t') \in X^4_m$, and so this term does indeed appear in the left-hand sum.

Now let $(s,t) \in X^4_m$ and $(u,v) \in X^3_{s,t}$. We want to show that the term $g(f(h(s))(u))(v) \neq 0$ from the left-hand sum also appears as a term in the right-hand sum; that is, that it is of the form $g(f(h(u'))(v'))(t')$ for some $(s',t') \in X^2_m$ and $(u',v') \in X^1_{s'}$. Setting $u' := s$, $v' := u$, $t' := v$, and $s' := s+u$ and reasoning analogously to above shows that it is indeed.

Thus the sums share all the same terms, and so are equal. This completes the proof of the third Kleisli monad property, and thus $G(M,-)$ is a monad.

□

Lemma 4.2.9. *There is a distributive law of monads of $G(M,-)$ over S, the symmetric algebra monad.*

Proof. For this proof, write

$$X_{m,k} := \{(u_1,\ldots,u_k) | \sum_{i=1}^{k} u_i = m\} .$$

Define $\ell_V : SG(M,V) \to G(M,SV)$ by

$$\ell \colon f_1 \otimes \cdots \otimes f_k \mapsto \left[m \mapsto \sum_{(u_1,\ldots,u_k)\in X_{m,k}} f_1(u_1) \otimes \cdots \otimes f_k(u_k) \right]$$

In the top-right diagram, for one side we have:

$$v_1 \otimes \cdots \otimes v_k \overset{\eta^{G(M,-)}_{SV}}{\longmapsto} \left[m \mapsto \begin{cases} v_1 \otimes \cdots \otimes v_k & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \right]$$

For the other side, by definition of η we have:

$$\begin{aligned} v_1 \otimes \cdots \otimes v_k &\overset{S\eta^{G(M,-)}_V}{\longmapsto} \eta^{G(M,-)}(v_1) \otimes \cdots \otimes \eta^{G(M,-)}(v_k) \\ &\overset{\ell}{\mapsto} \left[m \mapsto \sum_{(u_1,\ldots,u_k)\in X_{m,k}} \eta(v_1)(u_1) \otimes \cdots \otimes \eta(v_k)(u_k) \right] \end{aligned}$$

Note that if $m \neq 0$, then no terms survive in this last sum, since at least one u_i must be nonzero, whereas if $m = 0$, then only the term $u_1 = \cdots = u_k = 0$ survives the application of the η maps, producing the single term $v_1 \otimes \cdots \otimes v_k$. Thus the two maps are equal and the diagram is proven.

In the bottom-right diagram, for one side we have:

$$[f \colon m \mapsto f(m)] \overset{G(M,\eta^S_V)}{\longmapsto} [m \mapsto (f(m))]$$

For the other side, we have:

$$\begin{aligned} [f \colon m \mapsto f(m)] &\overset{\eta^S_{G(M,V)}}{\longmapsto} ([f \colon m \mapsto f(m)]) \\ &\overset{\ell}{\mapsto} \left[m \mapsto \sum_{u_1 \in X_{m,1}} (f(u_1)) \right] \\ &= [m \mapsto (f(m))] \end{aligned}$$

Thus the diagram commutes.

Now define:

$$f := (f_{11} \otimes \cdots \otimes f_{nm_1}) \otimes \cdots \otimes (f_{n1} \otimes \cdots \otimes f_{nm_n})$$

In the bottom-left diagram, for one side we have:

$$\begin{aligned} f &\overset{\mu^S_{G(M,V)}}{\longmapsto} f_{11} \otimes \cdots \otimes f_{nm_n} \\ &\overset{\ell_V}{\mapsto} \left[m \mapsto \sum_{u_{11},\ldots,u_{nm_n} \in X_{m,nm_n}} f_{11}(u_{11}) \otimes \cdots \otimes f_{nm_n}(u_{nm_n}) \right] \end{aligned}$$

For the other side, we have:

$$
\begin{aligned}
f \overset{S\ell_V}{\mapsto} & \left[m \mapsto \left(\sum_{(u_{11},\ldots,u_{1m_1})\in X_{m,m_1}} f_{11}(u_{11}) \otimes \cdots \otimes f_{1m_1}(u_{1m_1}) \right) \right] \\
& \otimes \cdots \otimes \left[m \mapsto \left(\sum_{(u_{n1},\ldots,u_{nm_n})\in X_{m,m_n}} f_{n1}(u_{n1}) \otimes \cdots \otimes f_{nm_n}(u_{nm_n}) \right) \right] \\
\overset{\ell_{SV}}{\mapsto} & \left[m \mapsto \sum_{(v_1,\ldots,v_n)\in X_{m,n}} \left(\sum_{(u_{11},\ldots,u_{1m_1}\in X_{v_1,m_1}} f_{11}(u_{11}) \otimes \cdots \otimes f_{1m_1}(u_{1m_1}) \right) \right. \\
& \left. \otimes \cdots \otimes \left(\sum_{(u_{n1},\ldots,u_{nm_n}\in X_{v_n,m_n}} f_{n1}(u_{n1}) \otimes \cdots \otimes f_{nm_n}(u_{nm_n}) \right) \right] \\
\overset{G(M,\mu_V^S)}{\longmapsto} & \left[m \mapsto \sum_{(v_1,\ldots,v_n)\in X_{m,n}} \left(\sum_{(u_{11},\ldots,u_{1m_1}\in X_{v_1,m_1}} f_{11}(u_{11}) \otimes \cdots \otimes f_{1m_1}(u_{1m_1}) \right) \right. \\
& \left. \otimes \cdots \otimes \left(\sum_{(u_{n1},\ldots,u_{nm_n}\in X_{v_n,m_n}} f_{n1}(u_{n1}) \otimes \cdots \otimes f_{nm_n}(u_{nm_n}) \right) \right]
\end{aligned}
$$

Take an arbitrary term in the second image's sum. Then its arguments sum to m, and so it also appears in the first image's sum. Conversely, take an arbitrary term in the first image's sum. Setting $v_i = \sum_{j=1}^{m_i} u_{ij}$ for each i shows that it also appears in the second image's sum. Thus the images are equal, and the diagram commutes.

Finally, for the top-left diagram, let $f_i \in G(M, G(M,V))$ and write

$$f = f_1 \otimes \cdots \otimes f_k \ .$$

Then for one side, we have:

$$
\begin{aligned}
f \overset{S\mu_V^{G(M,-)}}{\longmapsto} & \left[m \mapsto \sum_{(u_1,v_1)\in X_m} f_1(u_1)(v_1) \right] \otimes \cdots \otimes \left[m \mapsto \sum_{(u_k,v_k)\in X_m} f_k(u_k)(v_k) \right] \\
\overset{\ell_V}{\mapsto} & \left[m \mapsto \sum_{(w_1,\ldots,w_k)\in X_m} \left(\sum_{(u_1,v_1)\in X_{w_1}} f_1(u_1)(v_1) \right) \otimes \cdots \otimes \left(\sum_{(u_k,v_k)\in X_{w_k}} f_k(u_k)(v_k) \right) \right]
\end{aligned}
$$

For the other side, we have:

$$
\begin{aligned}
f &\overset{\ell_{G(M,V)}}{\longmapsto} \left[m \mapsto \sum_{(w_1,\ldots,w_k)\in X_m} (n \mapsto f_1(w_1)(n)) \otimes \cdots \otimes (n \mapsto f_k(w_k)(n))\right] \\
&\overset{G(M,\ell_V)}{\longmapsto} \left[m \mapsto \sum_{(w_1,\ldots,w_k)\in X_m} \left(n \mapsto \sum_{(z_1,\ldots,z_k)\in X_n} f_1(w_1)(z_1) \otimes \cdots \otimes f_k(w_k)(z_k)\right)\right] \\
&\overset{\mu_{SV}^{G(M,\ell_V)}}{\longmapsto} \left[m \mapsto \sum_{(u,v)\in X_m} \sum_{(w_1,\ldots,w_k)\in X_u} \sum_{(z_1,\ldots,z_k)\in X_v} f_1(w_1)(z_1) \otimes \cdots \otimes f_k(w_k)(z_k)\right]
\end{aligned}
$$

Take an arbitrary term in this latter sum, $f_1(w_1)(z_1) \otimes \cdots \otimes f_k(w_k)(z_k)$. We'll show that it's also in the former one; that is, we'll show it's of the form $f_1(u_1')(v_1') \otimes \cdots \otimes f_k(u_k')(v_k')$ for $(u_i, v_i) \in X_{w_i'}$ and $(w_1', \ldots, w_k') \in X_m$.

Set $u_i' := w_i$, $v_i' := z_i$, and $w_i' := w_i + z_i$ for all i. Then $u_i' + v_i' = w_i + z_i = w_i'$ so that $(u_i', v_i') \in X_{w_i'}$, and

$$\sum w_i' = \sum (w_i + z_i) = \sum w_i + \sum z_i = u + v = m$$

since $(w_1, \ldots, w_k) \in X_u$, $(z_1, \ldots, z_k) \in X_v$ and $(u, v) \in X_m$. Thus this term is indeed in the former sum.

Now take an arbitrary term in the former sum, $f_1(u_1)(v_1) \otimes \cdots \otimes f_k(u_k)(v_k)$. We'll show that it's also in the latter one; that is, we'll show it's of the form $f_1(w_1')(z_1') \otimes \cdots \otimes f_k(w_k')(z_k')$ for $(w_1', \ldots, w_k') \in X_{u'}$, $(z_1', \ldots, z_k') \in X_{v'}$ and $(u', v') \in X_m$.

Set $w_i' := u_i$, $z_i := v_i$, $u' := \sum u_i$, and $v' := \sum v_i$. Then $\sum w_i' = \sum u_i = u'$, $\sum z_i' = \sum v_i = v'$, and

$$u' + v' = \sum u_i + \sum v_i = \sum (u_i + v_i) = \sum w_i = m$$

since $(u_i, v_i) \in X_{w_i}$ and $(w_1, \ldots, w_k) \in X_m$. Thus this term is indeed in the latter sum.

This shows the sums must be equal, and so the maps are equal. This demonstrates that the top-left diagram commutes, and thus ℓ is a distributive law.

$\square$

A corollary then follows immediately from [4].

Corollary 4.2.10. *The composite functor $G(M, S-) := G(M, -) \circ S$ is a monad on $\mathbb{K}$-Vec.*

In our next proposition, we'll need a version of a lemma taken from [8].

Lemma 4.2.11. *Let S be the symmetric algebra monad. Then the commutative algebra modalities on a category are in bijective correspondence with pairs (T, ψ) where T is a monad and $\psi\colon S \to T$ is morphism of monads.*

Proposition 4.2.12. *The monad $H := G(M, S-)$ is an algebra modality with multiplication and unit defined by*

$$m_H(f \otimes g)(m) = \sum_{(u,v) \in X_m(f,g)} f(u) \otimes_s g(v)$$

$$e_H(1_{\mathbb{K}})(m) = \begin{cases} 1_{\mathbb{K}} & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

We write $f(u) \otimes_s g(v)$ for what is more correctly $m_s(f(u) \otimes g(v))$; in particular, if one of these, say $g(v)$, is equal to $1_{\mathbb{K}}$, then this product is "$f(u) \otimes_s 1$", which of course is just $f(u)$.

Proof. We make use of the previous lemma by defining a transformation $\psi \colon S \to H$ and showing that it is a morphism of monads. We then follow along the bijection and show that it induces the given algebra modality structure. Define $\psi_V \colon SV \to G(M, SV)$ by:

$$\psi(v_1 \otimes \cdots \otimes v_n) = \left[m \mapsto \begin{cases} v_1 \otimes \cdots \otimes v_n & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right]$$

This is clearly natural. We must show it satisfies the commutative diagrams of a monad morphism:

$$\begin{tikzcd} SSV \arrow[r, "S\psi_V"] \arrow[d, "\mu^{SV}"] & SG(M,SV) \arrow[r, "\psi_{G(M,SV)}"] & G(M, SG(M,SV)) \arrow[d, "\mu^{G(M,SV)}"] \\ SV \arrow[rr, "\psi_V"] & & G(M,SV) \end{tikzcd}$$

$$\begin{tikzcd} & V \arrow[dl, "\eta^{SV}"'] \arrow[dr, "\eta^{G(M,SV)}"] & \\ SV \arrow[rr, "\psi_V"] & & G(M,SV) \end{tikzcd}$$

The unit diagram is straightforward. For the multiplication diagram, write

$$v := (v_{11} \otimes \cdots \otimes v_{1n_1}) \otimes \cdots \otimes (v_{m1} \otimes \cdots \otimes v_{mn_m})$$

and

$$[i] := \left[m \mapsto \begin{cases} v_{i1} \otimes \cdots \otimes v_{in_i} & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right] .$$

Then it is easy to see that the path along the bottom of the diagram yields:

$$\psi_V \circ \mu_V^S(v) = \left[m \mapsto \begin{cases} \mu_V^S(v) & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right]$$

For the path along the top of the diagram, first recall that according to the distributive law [4],

$$\mu_V^{G(M,S-)} = \mu_{SV}^{G(M,-)} \circ G(M, G(M, \mu_V^S)) \circ G(M, \ell_{SV}) \ .$$

Using this, the top diagram path becomes

$$\begin{aligned}
v &\overset{S\psi_V}{\longmapsto} [1] \otimes \cdots \otimes [n] \\
&\overset{\psi_{G(M,SV)}}{\longmapsto} \left[m \mapsto \begin{cases} [1] \otimes \cdots \otimes [n] & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right] \\
&\overset{G(M,\ell_{SV})}{\longmapsto} \left[m \mapsto \left[k \mapsto \begin{cases} \sum_{(u_1,\ldots,u_n)\in X_{k,n}} [1](u_1) \otimes \cdots \otimes [n](u_n) & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right] \right] \\
&\overset{G(M,G(M,\mu_V^S))}{\longmapsto} \left[m \mapsto \left[k \mapsto \begin{cases} \sum_{(u_1,\ldots,u_n)\in X_{k,n}} \mu_V^S\left([1](u_1) \otimes \cdots \otimes [n](u_n)\right) & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right] \right] \\
&\overset{\mu_{SV}^{G(M,-)}}{\longmapsto} \left[m \mapsto \sum_{(s,t)\in X_m} \begin{cases} \sum_{(u_1,\ldots,u_n)\in X_{t,n}} \mu_V^S\left([1](u_1) \otimes \cdots \otimes [n](u_n)\right) & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases} \right] \\
&= \left[m \mapsto \begin{cases} \mu_V^S(v) & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right]
\end{aligned}$$

where the final equality is because all other terms in the sums are zero. Thus the diagram commutes. This being the case, the lemma implies that an algebra modality on $G(M, S-)$ is given by:

$$\begin{tikzcd}
m_V^H : G(M,SV) \otimes G(M,SV) \arrow[rr] \arrow[d, "\eta^S_{G(M,SV)} \otimes \eta^S_{G(M,SV)}"] & & G(M,SV) \\
SG(M,SV) \otimes SG(M,SV) \arrow[r, "m^S_{HV}"] & SG(M,SV) \arrow[r, "\psi_{HV}"] & G(M, SG(M,SV)) \arrow[u, "\mu_V^H"']
\end{tikzcd}$$

$$e_V^H : I \xrightarrow{e_V^S} SV \xrightarrow{\psi_V} G(M, SV)$$

Using the same formula for μ^H as before, we compute:

$$\begin{aligned}
f \otimes g &\overset{\eta^S_{HV} \otimes \eta^S_{HV}}{\longmapsto} (f) \otimes (g) \\
&\overset{m^S_{HV}}{\mapsto} f \otimes_s g \\
&\overset{\psi_{HV}}{\mapsto} \left[m \mapsto \begin{cases} f \otimes_s g & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right] \\
&\overset{G(M, ell_{SV})}{\longmapsto} \left[m \mapsto \left[k \mapsto \begin{cases} \sum_{(u,v) \in X_{k,2}} (f(u)) \otimes_s (g(v)) & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right] \right] \\
&\overset{G(M, G(M, \mu^S_V))}{\longmapsto} \left[m \mapsto \left[k \mapsto \begin{cases} \sum_{(u,v) \in X_{k,2}} f(u) \otimes_s g(v) & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right] \right] \\
&\overset{\mu^{G(M,-)}_{SV}}{\longmapsto} \left[m \mapsto \sum_{(s,t) \in X_m} \begin{cases} \sum_{(u,v) \in X_{t,2}} f(u) \otimes_s g(v) & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases} \right] \\
&= \left[m \mapsto \sum_{(u,v) \in X_m} f(u) \otimes_s g(v) \right]
\end{aligned}$$

Thus the multiplication is as we supposed. Similarly, for the unit we have:

$$1_{\mathbb{K}} \overset{e^S_V}{\mapsto} 1_{\mathbb{K}} \overset{\psi_V}{\mapsto} \left[m \mapsto \begin{cases} 1_{\mathbb{K}} & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \right]$$

□

We can now show that Laurent series are recaptured by our algebra modality, along with a wealth of other examples.

Theorem 4.2.13. *Let $M = \mathbb{Z}$, let $\mathcal{C} = \mathbb{C}\text{-}Vec$, and let $\{0\}$ be the zero vector space. Then $G(M, S\{0\})$ and the algebra of Laurent series are isomorphic as algebras.*

Proof. Observe that we have

$$S\{0\} = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} (\{0\}_1 \otimes \cdots \otimes \{0\}_i) \cong \mathbb{C}$$

since $\{0\} \otimes \{0\} \cong \{0\}$ and $V \oplus \{0\} \cong V$. We therefore have:

$$G(\mathbb{Z}, S\{0\}) \cong \{f : \mathbb{Z} \to \mathbb{C} | supp(f) \text{ is artinian and narrow}\}$$

Since the ordering on $\mathbb{Z}$ is total, narrowness imposes no restriction. Artinianness means that any such function f has a lowest $k \in \mathbb{Z}$ mapping to something nonzero. Thus $G(\mathbb{Z}, S\{0\})$ is in bijective correspondence with Laurent series.

Adding functions is equivalent to adding coefficients of each power of x, and the multiplication

$$(f \cdot g)(k) = \sum_{(u,v) \in X_k} f(u)g(v)$$

is equivalent to multiplication of Laurent series. Thus they are isomorphic as algebras.

□

The same structure but with $M = \mathbb{N}$ is isomorphic to the usual power series over $\mathbb{K}$. There are a number of other interesting examples we can capture as well. The following are taken from [35] and [22].

Example 4.2.14. Let $M = \mathbb{N} \backslash \{0\}$ with the operation of multiplication, equipped with the usual ordering. Then $G(M, SV)$ is the ring of arithmetic functions with values in SV, and m_H is Dirichlet's convolution [35].

Example 4.2.15. Let $M = \mathbb{N} \backslash \{0\}$ with the operation of multiplication as above, but now equipped with the divisibility ordering; that is, $m_1 \leq m_2 \iff m_1 | m_2$. Then $G(M, SV)$ is the subring of arithmetic functions whose support is either finite or contains an infinite divisibility chain of natural numbers. For example, a function whose support is all primes would not be contained in this example, though it would be in the previous example [35].

Many of these are also Rota-Baxter algebras, and make their categories into Rota-Baxter categories of some weight. The most general case is the following one.

Example 4.2.16. Let $\mathcal{C} = \mathbb{K}$-Vec, let M be any strictly ordered monoid, and let M_1, M_2 be such that the disjoint union $M_1 \dot{\cup} M_2 = M$. Define a map $\Pi_V : G(M, SV) \to G(M, SV)$ by

$$\Pi(f)(m) = \begin{cases} f(m) & \text{if } m \in M_1 \\ 0 & \text{if } m \in M_2 \end{cases}$$

It is the major theorem of [24] that, in the context where "SV" is an arbitrary commutative ring, the map Π satisfies the Rota-Baxter equation of weight -1 if and only if M_1 and M_2 are subsemisimple groups of M. The proof of that theorem carries over into this setting, since SV is in particular a commutative ring. Thus we have a large class of examples of Rota-Baxter categories of weight -1.

Example 4.2.17. Let $\mathcal{C} = \mathbb{C}$-Vec with algebra modality H as above. Then, as a particular case of the previous example, the transformation $\Pi \colon HV \to HV$ defined by

$$\Pi(f)(k) = \begin{cases} f(k) & \text{if } k < 0 \\ 0 & \text{if } k \geq 0 \end{cases}$$

makes the category into a Rota-Baxter category of weight $\lambda = -1$. This is equivalent to the Rota-Baxter operator

$$P\left(\sum_{n=k}^{\infty} a_n x^n\right) = \sum_{n=k}^{-1} a_n x^n$$

on Laurent series when $V = 0$. Thus we've recaptured renormalization.

Example 4.2.18. We can also reproduce "perhaps the most important combinatorial example" [38, 22] of a Rota-Baxter algebra. Define an $\mathbb{R}$-algebra by

$$R = \{f : \mathbb{R} \to \mathbb{Q} || supp(f)| < \infty\}$$

with product as in the Laurent series example; that is,

$$(fg)(x) := \sum_{(y,z)\in\{\mathbb{R}\times\mathbb{R}|y+z=x\}} f(y)g(z)$$

Define an operator $P : R \to R$ by

$$P(f)(x) := \sum_{max(0,y)=x} f(y)$$

It is proven in [22] that this is a Rota-Baxter operator of weight -1 on R. Our monad $G(M, S-)$ induces it as follows. Let M be any trivially ordered monoid; that is, let $m_1 \leq m_2\ \forall m_1, m_2 \in M$. Then the artinian and narrow subsets are simply the finite ones, and $G(M, SV)$ is the monoid ring of M with coefficients in SV [35]. In particular, if we set $M = \mathbb{R}$ and work over $\mathbb{Q}$-vector spaces, then $G(\mathbb{R}, S\{0\}) \cong G(\mathbb{R}, \mathbb{Q})$ recovers R. The transformation

$$\Pi(f)(x) = \sum_{max(0,y)=x} f(y)$$

(where the max is taken according to the usual ordering on $\mathbb{R}$, of course) then makes $\mathbb{Q}$-Vec into a Rota-Baxter category of weight -1.

5 Quantum Differential and Integral Categories

In this section, we wish to provide a variation of the standard differential and integral categories that captures quantum calculus. We begin by introducing some background concepts. Then we review the quantum calculus, and introduce our new structure encapsulating it. Finally, we show that the categories as we introduced them can be equivalently characterized in terms of certain equalizers.

5.1 Scalars, Grading and Bases

In the previous section, we described the notion of categorical scalars and scalar multiplication. We will again need these here. We will also need the notion of a general *graded commutative algebra* in an additive symmetric monoidal closed category.

Definition 5.1.1. A *graded commutative algebra* in an additive symmetric monoidal closed category $\mathcal{C}$ consists of an object of the form:

$$A = \bigoplus_{i=0}^{\infty} A_i \text{ together with multiplications } A_i \otimes A_j \to A_{i+j}$$

satisfying evident unital, associativity and commutativity[4] constraints. Note in particular that $A_0 = I$. The object A_i is called the *homogeneous object of degree i*. In categories where the objects have elements, an element of A_i would be called a *homogeneous element*.

We note that in both Rel, the category of sets and relations, and Vec, the category of vector spaces and linear maps, the usual models of the linear logic modalities are in fact graded commutative algebras.

To work in the q-setting, we will need to couple the notion of graded algebra with that of algebra modality.

Definition 5.1.2. An additive symmetric monoidal category has a *graded algebra modality* if it is equipped with a monad (T, μ, η) such that for every object M in $\mathcal{C}$, the object, $T(M)$, has the structure of a graded commutative algebra

$$m : T(M) \otimes T(M) \to T(M), \qquad e : I \to T(M)$$

and this family of graded commutative algebra structures satisfies evident naturality conditions. The only additional requirement from the usual definition of algebra modality is that the map $\mu\colon T^2(M) \to T(M)$ be a graded algebra map.

The linear logic modalities in both Rel and Vec are graded algebra modalities.

Finally, let M be an object in a monoidal category $\mathcal{C}$. A *basis* for M is an isomorphism $\mathcal{B}\colon \bigoplus_{j\in J} I \to M$ for some set J. An *indeterminate* from M is one of the maps $x_k := \mathcal{B} \circ i_k\colon I \to \bigoplus_{j\in J} I \to M$, where i is the biproduct injection.

5.2 The Differential Setting

Variations of the quantum calculus have arisen in a number of settings. For the moment, we follow [28] closely. One defines the q-derivative of a function of 1 variable via the following formula:

$$D_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x}$$

[4]We assume ordinary commutativity as opposed to a graded commutative equation.

Evidently if we are in a setting where one can take limits, then taking the limit of this expression as $q \to 1$ gives the usual $f'(x)$.

So for example, one can see:

$$D_q(x^n) = \frac{q^n - 1}{q - 1} x^{n-1}$$

The following is a helpful bit of notation:

$$[n] = \frac{q^n - 1}{q - 1}$$

In which case, one can write:

$$D_q(x^n) = [n] x^{n-1}$$

In this framework, the Leibniz rule takes on the following form:

$$D_q(f(x)g(x)) = f(qx)D_q(g(x)) + g(x)D_q(f(x)) = f(x)D_q(g(x)) + g(qx)D_q(f(x))$$

As noted in [28], the chain rule of the differential calculus doesn't work in any evident way. In trying to q-differentiate $f(u(x))$, one quickly discovers there is no way to write $D_q(f(x))$ as any kind of multiple of $D_q(u(x))$. However, one can in the case when $u(x) = x^n$. Note that no restrictions on f are necessary. In this case, we get the following equation.

$$D_q(f(x^n)) = D_{q^n}(f)(x^n)D_q(u(x))$$

This indicates that to state a more abstract q-chain rule, we will need to restrict to homogeneous elements.

5.3 q-Partial Derivatives and the q-Differential Rules

In this subsection, our goal is to build the requisite structure to define a q-codifferential category.

For each indeterminate x_k, scalar q, and $n \in \mathbb{N}$, define a *quantum partial deriving transformation* or *q-deriving transformation* as a map

$$\partial_n = \partial_{x_k,q,n} \colon T(M)_n \to T(M)_{n-1} \otimes M$$

satisfying four conditions. The first two are fairly easy to state, but the latter two require some elaboration.

(qd1) (The q-Constants Rule) For all x_k, q:

$$\begin{tikzcd} I \arrow[r, "e"] \arrow[dr] & T(M)_0 \arrow[r, "\partial_0"] & T(M)_{-1} \otimes M \\ & 0 \arrow[ur] & \end{tikzcd}$$

Note that it follows easily from the graded algebra definition of (m, e) that e maps into the degree-zero part of $T(M)$.

(qd3) (q-Linear Maps Rule) For all x_k, q, and where p is the biproduct projection map:

$$\begin{tikzcd} I \arrow[r, "x_k"] \arrow[dr, "x_k"'] & M \arrow[r, "\eta"] & T(M) \arrow[r, "p_1"] & T(M)_1 \arrow[r, "\partial_1"] & T(M)_0 \otimes M \\ & M \arrow[r, "\lambda^{-1}"] & I \otimes M \arrow[urr, "e \otimes id"] & & \end{tikzcd}$$

To adequately capture the quantum Leibniz and chain rules, we'll need some more structure. Both rules rely on knowing the degree of the factor x_k within the greater $x_1^{n_1} \cdots x_m^{n_m}$; the "subdegree" of x_k, if you will. We'll develop a way to describe this idea categorically using bases.

Let $n_k \in \mathbb{N}$ and x_k be an indeterminate from M via basis $\mathcal{B}_1$. Define the map $x_k^{n_k} \colon I \to T(M)_{n_k}$ as follows:

$$I \xrightarrow{\cong} \bigotimes_{r=1}^{n_k} I \xrightarrow{\bigotimes x_k} \bigotimes_{r=1}^{n_k} M \xrightarrow{\bigotimes \eta} \bigotimes_{r=1}^{n_k} T(M) \xrightarrow{\bigotimes m} T(M) \xrightarrow{p} T(M)_{n_k}$$

Additionally let $y_{x_k}^{\ell}$ denote any map of the following form and satisfying the following condition:

$$I \xrightarrow{\cong} \bigotimes_{r=1}^{\ell} I \xrightarrow{\bigotimes_{r=1}^{\ell} x_r} \bigotimes_{r=1}^{\ell} M \xrightarrow{\bigotimes \eta} \bigotimes_{r=1}^{\ell} T(M) \xrightarrow{\bigotimes m} T(M) \xrightarrow{p} T(M)_{\ell}$$

such that $\forall r,\ x_r \neq x_k$. The intuition behind these should be that in a monomial $x_1^{n_1} \cdots x_m^{n_m}$, $x_k^{n_k}$ in our definition corresponds to exactly what it seems, and y^{ℓ} corresponds to the product of all the other factors, where ℓ is the remaining degree. In the intuition, we would thus have $\ell = (\sum_{r=1}^{m} n_r) - n_k$.

To state the rules, we must also make the following definition.

Definition 5.3.1. *We call* $(\mathcal{C}, T, \partial)$ Colbert *if for each* $j \in \mathbb{N}$ *and each indeterminate* x_k *from our chosen basis* $\mathcal{B}_1$ *of* M*, there exists*

1. *some* $n_k \in \mathbb{N}$*,*
2. *some basis* $\mathcal{B}_2$ *of* $T(M)_{n_k}$ *which has* $x_k^{n_k}$ *as an indeterminate,*
3. *some basis* $\mathcal{B}_3$ *of* $T(T(M)_{n_k})_j$ *with some indeterminate* $z \colon I \to T(T(M)_{n_k})_j$*,*
4. *and some* $y^{n_k(j-1)} \colon I \to T(M)_{n_k(j-1)}$ *in the form described above OR such that* $y^{n_k(j-1)} = e$*,*

such that the following diagram commutes:

$$\begin{tikzcd}
I \arrow[r, "\cong"] \arrow[d, "z"] & I \otimes I \arrow[r, "x_k^{n_k} \otimes y^{n_k(j-1)}"] & T(M)_{n_k} \otimes T(M)_{n_k(j-1)} \arrow[d, "m"] \\
T(T(M)_{n_k})_j \arrow[rr, "\mu"] & & T(M)_{n_k j}
\end{tikzcd}$$

The highest n_k such that this holds is called the subdegree of x_k in z*, denoted $subdeg(x_k, z)$. If $z; \mu = y_{x_k}^{n_k j}$ for some $y_{x_k}^{n_k j}$, then we interpret this as $n_k = 0$ and define $subdeg(x_k, z) = 0$.*

We can now describe the q-Leibniz rule. For reference, in the q-calculus, the single-variable Leibniz rule is:

$$\begin{aligned} D_q(f(x)g(x)) &= f(qx)D_q(g(x)) + g(x)D_q(f(x)) \\ &= f(x)D_q(g(x)) + g(qx)D_q(f(x)) \end{aligned}$$

The multivariable version of the rule, using partial derivatives, can be written as follows, where $\vec{x} = x_1^{n_1}, \ldots, x_m^{n_m}$:

$$\begin{aligned} D_{x_k,q}(f(\vec{x})g(\vec{x})) &= f(x_1, \ldots, qx_k, \ldots, x_n)D_{x_k,q}(g(\vec{x})) + D_{x_k,q}(f(\vec{x}))g(\vec{x}) \\ &= f(\vec{x})D_{x_k,q}(g(\vec{x})) + g(x_1, \ldots, qx_k, \ldots, x_n)D_{x_k,q}(f(\vec{x})) \end{aligned}$$

We'd like to generalize this categorically.

Let $z_1 \colon I \to T(M)_i)_j, z_2 \colon I \to T(T(M)_\ell)_r$ be basis maps having the Colbert property with respect to an indeterminate $x_k \colon I \to M$. Then $subdeg(x_k, z_1) = i$ and $subdeg(x_k, z_2) = \ell$. Write $z_1 = z_1; \mu$, $z_2 = z_2; \mu$ for notational convenience. The map ∂ satisfies the q-Leibniz rule if for all such $i, j, \ell, r \in \mathbb{N}$ and all $y_1 := y_{x_k}^{i(j-1)}, y_2 := y_{x_k}^{\ell(r-1)}$ defined as above, the following holds:

(qd2) (q-Leibniz Rule)

$$\begin{aligned} z_1 \otimes z_2; m; \partial_{x_k,q,ij+\ell r} &= z_1 \otimes z_2; ((q^i \cdot id) \otimes \partial_{x_k,q,\ell r}; m \otimes id \\ &\quad + \partial_{x_k,q,ij} \otimes id; id \otimes \sigma; m \otimes id) \\ &= z_1 \otimes z_2; (id \otimes \partial_{x_k,q,\ell r}; m \otimes id \\ &\quad + \partial_{x_k,q,ij} \otimes (q^\ell \cdot id); id \otimes \sigma; m \otimes id) \end{aligned}$$

This has diagrammatic representation:

$$\begin{tikzcd}
I \otimes I \arrow[r, "z_1 \otimes z_2"] \arrow[rrr, bend left, "z_1 \otimes z_2; (q^i \cdot id) \otimes \partial_{x_k,q,\ell r}; m \otimes id + \partial_{x_k,q,ij} \otimes id; id \otimes \sigma; m \otimes id"] \arrow[rrr, bend right, "z_1 \otimes z_2; id \otimes \partial_{x_k,q,\ell r}; m \otimes id + \partial_{x_k,q,ij} \otimes (q^\ell \cdot id); id \otimes \sigma; m \otimes id"'] & T(M)_{ij} \otimes T(M)_{\ell r} \arrow[r, "m"] & T(M)_{i+j} \arrow[r, "\partial_{x_k,q,ij+\ell r}"] & T(M)_{ij+\ell r-1} \otimes M
\end{tikzcd}$$

Next we generalize the q-chain rule. For reference, in the q-calculus the chain rule is:

$$D_q(f(x^n)) \;=\; D_{q^n}(f)(x^n)D_q(x^n)$$

The multivarible version with partial derivatives is as follows, where $\vec{x} = x_1^{n_1}, \ldots, x_m^{n_m}$:

$$D_{x_k,q}(f(\vec{x})) = D_{x_k^{n_k},q^{n_k}}(f)(\vec{x})D_{x_k,q}(\vec{x})$$

We would like μ to be a graded algebra map. Define a grading on $T(T(M))$ by:

$$T(T(M))_k = \bigoplus_{ij=k} T(T(M)_i)_j$$

Then to be a graded algebra map, our μ must satisfy:

$$\mu(T(T(M)_i)_j) \subseteq T(M)_{ij}$$

Let $\mathcal{C}$ be Colbert, and let $subdeg(x_k, z) = i$. We then posit the following as the categorical q-chain rule, for all x_k coupled with all of their respective z related by the Colbert property:

(qd4) (q-Chain Rule)

$$\begin{tikzcd}
I \arrow[rr, "z"] \arrow[d, "z"] & & T(T(M)_i)_j \arrow[d, "\mu"] & \\
T(T(M)_i)_j \arrow[d, "\partial_{x_k^i,q^i,j}"] & & T(M)_{ij} \arrow[r, "\partial_{x_k,q,ij}"] & T(M)_{ij-1} \otimes M \\
T(T(M)_i)_{j-1} \otimes T(M)_i \arrow[rrr, "\mu\otimes\partial_{x_k,q,i}"] & & & T(M)_{i(j-1)} \otimes T(M)_{i-1} \otimes M \arrow[u, "m\otimes id"']
\end{tikzcd}$$

5.4 q-Codifferential Categories

Definition 5.4.1. *A* quantum codifferential *or* q-codifferential category *is a Colbert category $\mathcal{C}$ equipped with a graded algebra modality T and a q-partial deriving transformation ∂.*

The definition of a *q-differential category* is, of course, just the dual.

5.4.1 Examples

Let $\mathcal{C} = Vec$ be the category of vector spaces V with basis X over a field $\mathbb{F}$ and linear maps, and equip it with the symmetric tensor algebra monad S.

Since $S(V) \cong \mathbb{F}[X]$, we can view $S(V)$ as a polynomial algebra in chosen basis vectors of V, and this can be graded via polynomial degree. The monoidal unit is $I = \mathbb{F}$, so categorical scalars $q\colon \mathbb{F} \to \mathbb{F}$ correspond to field scalars $q = q(1_{\mathbb{F}})$, and the notions of scalar multiplication coincide. Similarly, the notions of basis correspond, and indeterminates $x_k\colon \mathbb{F} \to \bigoplus_{j\in J} \mathbb{F} \to V$ correspond to basis vectors. As usual, m and e denote polynomial multiplication and inclusion, respectively.

Let x_k denote basis vectors and $\vec{x} = x_1^{n_1} \otimes \cdots \otimes x_m^{n_m}$. Define $\partial_{x_k} = \partial_{x_k,q,i} \colon S(V)_i \to S(V)_{i-1} \otimes V$ by

$$\partial_{x_k}(\vec{x}) = \frac{\partial_q}{\partial_q x_k}(\vec{x}) \otimes x_k$$

where $\frac{\partial_q}{\partial_q x_k}$ is the quantum partial derivative with respect to x_k, and where $\partial_{x_k}(1_{\mathbb{F}}) := 0$.

Proposition 5.4.2. *(Vec, S, ∂_q) is a q-codifferential category, and its structure coincides with multivariable quantum differentiation of polynomials. The single-variable polynomial q-derivative is captured as a particular case.*

Proof. Write ∂ for $\partial_{x_k,q,i}$ where confusion will not result.

The q-constant rule translates to $\partial(1_{\mathbb{F}}) = 0$, and using the q-calculus we have:

$$\partial(1) := \frac{\partial_q}{\partial_q x_k}(1) \otimes x_k := \frac{1-1}{(q-1)x_k} \otimes x_k = 0$$

The q-linear maps rule translates to $\partial(x_k) = 1 \otimes x_k$ for basis vectors x_k. We have:

$$\partial(x_k) := \frac{\partial_q}{\partial_q x_k}(x_k) \otimes x_k := \frac{q x_k - x_k}{(q-1)x_k} \otimes x_k = 1 \otimes x_k$$

Next, it is clear that Vec is Colbert; if x is a basis vector for V, then $z = x^i$ is a basis vector for $T(V)_i$ and $z' = (x^i)^j$ is a basis vector for $T(T(V)_i)_j$ demonstrating Colbertness for $y = e$, with $subdeg(x, z') = ij$. Thus the notion of subdegree corresponds with our intuition in this case.

For the q-Leibniz rule, let $i = subdeg(x_k, z_1)$, $j = subdeg(x_k, z_2)$. First assume $i, j > 0$. Observe that $w := (z_1 \otimes z_2)(1 \otimes 1) = (x_k^i \cdot y_1) \otimes (x_k^j \cdot y_2)$, where for partial differentiation purposes y_1 and y_2 act as constants. Then we have:

$$\begin{aligned}
m; \partial(w) &= \frac{\partial_q}{\partial_q x_k}(x_k^{i+j} y_1 y_2) \otimes x_k \\
&= \left(\frac{q^{i+j} - 1}{q-1} x_k^{i+j-1} y_1 y_2 \right) \otimes x_k \\
&= \left(\left(q^i \frac{q^j - 1}{q-1} + \frac{q^i - 1}{q-1} \right) x_k^{i+j-1} y_1 y_2 \right) \otimes x_k \\
&= \left(q^i x_k^i y_1 \cdot \frac{\partial_q}{\partial_q x_k}(x_k^j y_2) + \frac{\partial_q}{\partial_q x_k}(x_k^i y_1) \cdot x_k^j y_2 \right) \otimes x_k \\
&= \big((q^i \cdot id) \otimes \partial; m \otimes id + \partial \otimes id; id \otimes \sigma; m \otimes id \big)(w)
\end{aligned}$$

The other q-Leibniz equation is similar.

Finally, for the q-chain rule, let $x = x(1)$ be a basis vector of V, and let $z = z(1)$ be any basis vector of $T(T(V)_i)_j$ satisfying the Colbert property with respect to x. Then $z = \bigotimes_{\ell=1}^m x_\ell^{n_\ell}$ with $x_\ell = x$ for some ℓ, and $i = subdeg(x, z)$.

Write y for the product of all factors of z other than x. Then we have:

$$\begin{aligned}
\mu; \partial_{x,q}(z) &= \frac{\partial_q}{\partial_q x}(z) \\
&= \frac{q^i - 1}{q - 1} x^{i-1} y \\
&= \left(\frac{q^{i^2} - 1}{q^{i^2} - 1}\right) \frac{x^i y}{x^i} \cdot \left(\frac{q^i - 1}{q - 1}\right) x^{i-1} \\
&= \frac{(q^i x)^i y - x^i y}{(q^i x)^i - x^i} \cdot \frac{q^i - 1}{q - 1} x^{i-1} \\
&= \frac{\partial_{q^i}}{\partial_{q^i} x^i}(x^i y) \cdot \frac{\partial_q}{\partial_q x}(x^i) \\
&= \partial_{x_k^i, q^i}; \mu \otimes \partial_{x_k, q}; m \otimes id(z)
\end{aligned}$$

Thus the chain rule is satisfied, and so Vec is a q-codifferential category.

Of course, we recover the single-variable case by taking $y = id$, leaving only powers of x in the polynomials. □

We can also equip a q-differential structure onto the finite bag (or multiset) comonad ! on *Rel*. Note that in this case all arrows will be dual to the previous example.

For this category, recall a few facts: the additive enrichment is given by the set-theoretic union, the monoidal structure is given by the cartesian product $\times$, and $I = \{*\}$ is the one-element set. Scalars are then relations $q\colon I \to I$, i.e. either $q = \emptyset$ or $q = id$; denote the former case by $q = 0$ and the latter by $q = 1$.

Every set X can be given a cobasis $\mathcal{B}\colon X \to \bigoplus_{j \in J} I$, where $|J| = |X|$, defined by relating each element of X to a different copy of $*$. This map's inverse is its converse, so it is an isomorphism. Coindeterminates are then the relations $x = p \circ \mathcal{B}\colon X \to \bigoplus_{j \in J} I \to I$, each relating a single element of X to $*$.

The coalgebras $!X$ can be graded by multiset cardinality; formally, let double brackets $\{\!\!\{\}\!\!\}$ denote a multiset, and define $(!X)_n = \{\{\!\!\{x_1, \ldots, x_n\}\!\!\}\}$.

It is easy to see that the comonadic comultiplication $\delta\colon !X \to !!X$ is a graded coalgebra map given the grading on $!!X$ as described in the definition of the chain rule, above. Recall that this comultiplication relates each bag to all bags of bags whose (multiset) union is the original bag, and also that the comonadic counit $\varepsilon\colon !X \to X$ is the partial function from each singleton bag to its single element. Recall also that the coalgebra modality is given by $\Delta\colon !X \to !X \times !X$ relating each bag to all pairs of bags whose union is the bag, and $e\colon !X \to I$ relating only the empty bag to $*$.

To define a q-partial deriving transformation, we must define

$$\partial_{x,q,n}\colon !X_n \times X \to !X_{n+1}$$

for every coindeterminate x, scalar q, and natural number n. But we have only the two scalar cases $q = 0$ or $q = 1$, and $q = 1$ reduces to equations very similar to the standard differential category rules.

The case $q = 0$ first needs some consideration. Observe the following case of the lower branch of the q-linear map rule diagram, for cobasis vector (i.e. element of X) x:

$$(\emptyset, x) \xmapsto{e \times id} (*, x) \xmapsto{\lambda^{-1}} x \xmapsto{x} *$$

Thus to satisfy the rule, we must have $\partial_x((\emptyset, x)) = ⦃x⦄$, even when $q = 0$. However, $q = 0$ ends up killing one of the terms on the sum side of the q-Leibniz rule (since the product of any relation with the empty relation is the empty relation). These properties seem only reconcilable if we set

$$(\emptyset, x)\partial_{x,0,0}\, ⦃x⦄$$

to be the only relation of ∂_0. Then, formally setting

$$\begin{aligned}(⦃x_1, \ldots, x_n⦄\,, x_0)\partial_{x,1,n}\, ⦃y_1, \ldots, y_{n+1}⦄ \iff{}& ⦃y_1, \ldots, y_{n+1}⦄ = ⦃x_0, x_1, \ldots, x_n⦄ \\ & \text{and } x_0 = x\end{aligned}$$

leads to the following proposition.

Proposition 5.4.3. *$(Rel, !, \partial_q)$, where $!$ is the finite bag comonad and ∂_q is as defined above, is a q-differential category.*

Proof. Rel is a Colbert category; the maps $x^n \times y$ simply divide a multiset M containing n copies of x into an x bag and non-x bag, and we can choose z to be the relation "picking out M", i.e. mapping $⦃M⦄ \mapsto *$. The subdegree of x in $!X$ is then the number of copies of x.

Now let $q = 0$. The q-constants rule is trivial, since e only relates the empty set, and ∂ never produces it. The q-linear maps rule was demonstrated above.

For the q-Leibniz rule, refer back to the diagram in the rule's section above, noting however that now the arrows are reversed. The only elements that reach $!X_i \times !X_j$ unkilled are $(⦃x⦄\,, \emptyset)$ along the upper path, $(\emptyset, ⦃x⦄)$ along the lower path, and both of these along the middle path. In all cases one element is the empty set, which no cobasis element z relates to $*$; thus the full composite is the empty relation in all cases, demonstrating that the rule is (trivially) satisfied.

For the q-chain rule, again note that the only element with non-empty relations stemming from it is $(\emptyset, x)$. Then for cobasis map z such that $⦃⦃x⦄⦄\, z*$ (i.e. $⦃⦃x⦄⦄$ is related to $*$ by z), we have:

$$\partial_x; \mu; z((\emptyset, x)) = *$$

Now observe that $n := subdeg(x, z) = 1$. Thus the element $(\emptyset, ⦃x⦄)$ of $!(!X) \times !X$ is related to only $⦃⦃x⦄⦄$ by $\partial_{x^n} = \partial_x$. It is easy to see that $(\emptyset, x)$ is related to $(\emptyset, ⦃x⦄)$ (among other elements) by $(\Delta \times id); (\mu \times \partial_x)$; thus the diagram commutes, and the q-chain rule is satisfied, completing the $q = 0$ case.

The $q = 1$ case essentially reduces to the standard differential category proof, and the proof is analogous.

$\square$

Intuitively, we see that the quantum structure on Rel adds a kind of "on-off switch" to the standard deriving transformation, turning it on when $q = 1$ and off when $q = 0$. When "off", only trivial differential maps remain.

5.5 The Integral Setting

Below, we will not include the quantum version of the U-substitution rule in our definition, due in part to its difficulty to state categorically, but we do mention that the integration of the quantum calculus satisfies it. In describing the integral quantum calculus, we again follow [28].

The q-antiderivative of a function f is defined much as in standard calculus, to be the family $\int f(x)d_qx$ of functions whose q-derivative is f. An added subtlety in the quantum case is that the antiderivative is no longer unique simply up to the addition of a constant, but up to addition of any function g such that $g(x) = g(qx)$.

If we restrict our attention to formal power series, we regain uniqueness up to a constant in the antiderivative. The condition above becomes $g(x) = g(qx)$, which implies that, if $g(x) = \sum_{n=0}^{\infty} c_n x^n$, then $c_n = q^n c_n$ for all n, meaning that $c_n = 0$ $\forall n \geq 1$ and so g is constant.

In this context, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ we also get the following attractively familiar formula for the q-antiderivative:

$$\int f(x)d_qx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{[n+1]} + C$$

The uniqueness for general functions can be similarly enhanced under some additional assumptions; for details see [28].

Our major focus here will be the related notion of the Jackson integral. Suppose $F(x)$ is an antiderivative of $f(x)$, and define the operator M_q by $M_q(F(x)) = F(qx)$. Then by definition:

$$\begin{aligned} f(x) &= D_q F(x) \\ &= \frac{F(qx) - F(x)}{(q-1)x} \\ &= \frac{(M_q - 1)F(x)}{(q-1)x} \end{aligned}$$

Rearranging and formally employing the geometric series expansion gives:

$$\begin{aligned} F(x) &= (1-q)\frac{xf(x)}{1-M_q} \\ &= (1-q)\sum_{j=0}^{\infty} M_q^j(xf(x)) \\ &= (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x) \end{aligned}$$

This formal series is the *Jackson integral*, for which we'll simply write $\int f(x)d_qx$. The conditions under which this converges to a true q-antiderivative are explored in [28].

5.6 The Integral Rules

We'll now examine the Jackson integral in the context of the usual integral category rules, namely integration of constants, integration of linear maps, the Rota-Baxter rule, the U-substitution rule, and the Fubini rule. For our purposes, we will assume all the integrals converge.

If $f(x) = a$ is a constant function, we have:

$$\begin{aligned}\int a\, d_qx &= (1-q)x\sum_{j=0}^{\infty} aq^j \\ &= (1-q)x\frac{a}{1-q} \\ &= ax\end{aligned}$$

Thus the integration of constants rule works exactly as usual.

If $f(x) = x$, we have:

$$\begin{aligned}\int x\, d_qx &= (1-q)x\sum_{j=0}^{\infty} q^j(q^jx) \\ &= (1-q)x^2\sum_{j=0}^{\infty}(q^2)^j \\ &= (1-q)x^2\frac{1}{1-q^2} \\ &= \frac{x^2}{1+q}\end{aligned}$$

This may be rewritten as:

$$(1+q)\int x\, d_qx \;=\; x^2$$

This is similiar, but not identical, to the usual integration of linear maps rule. In fact it becomes the usual rule if we can take the limit as $q \to 1$.

The q-integration by parts formula for the q-calculus is derived in [28] from the q-Leibniz rule, giving the following q-Rota-Baxter rule:

$$\begin{aligned}f(x)g(x) &= \int f(x)d_qg(x) + \int g(qx)d_qf(x) \\ &= \int f(qx)d_qg(x) + \int g(x)d_qf(x)\end{aligned}$$

It should now be easy to see how this is generalized to multivariable functions.

The q-calculus version of the substitution rule is derived in [28], in the limited case where $u(x) = \alpha x^\beta$. For the same reasons as the q-differential chain rule, this is the best we can do in general. The rule is as follows:

$$\int f(u)\, d_q u \;=\; \int f(u(x))\, d_{q^{1/\beta}} u(x)$$

If $f(x,y)$ is a function of two variables, then we have:

$$\begin{aligned}
\int\int f(x,y) d_q x d_q y &= \int (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x, y) d_q y \\
&= (1-q)y \sum_{k=0}^{\infty} q^k \left((1-q)x \sum_{j=0}^{\infty} q^j f(q^j x, q^k y) \right) \\
&= (1-q)x \sum_{k=0}^{\infty} q^k \left((1-q)y \sum_{j=0}^{\infty} q^j f(q^j x, q^k y) \right) \\
&= \int (1-q)y \sum_{k=0}^{\infty} q^k f(x, q^k y) d_q x \\
&= \int\int f(x,y) d_q y d_q x
\end{aligned}$$

Thus the usual Fubini rule applies in the quantum calculus.

5.7 q-Cointegral Categories

We'll make the definition similar to the q-codifferential category definition above.

Definition 5.7.1. *A* q-cointegral category *consists of an additive symmetric monoidal Colbert category equipped with a graded algebra modality and a* q-integral transformation, *i.e. a family of natural transformations indexed by indeterminates $x_k \colon I \to M$, $q \colon I \to I$ and $i \in \mathbb{N}$ of the form $s_{x_k,q,i} \colon T(M)_i \otimes M \to T(M)_{i+1}$ satisfying:*

(qs1) *(q-Integration of Constants)*

$$\lambda^{-1}; (e \otimes id); s = \eta$$

(qs2) *(q-Integration of Linear Maps)*

$$(x_k \otimes x_k); \Big((\eta \otimes id); s_{x_k,q,i} + (\eta \otimes id); q \cdot s_{x_k,q,i} \Big) \;=\; (x_k \otimes x_k); (\eta \otimes \eta); m$$

(qs3) *(The q-Rota-Baxter Rule) For all $w, z\colon I \to TT(M)$ satisfying the Colbert property with respect to x_k, where $n_w := subdeg(x, w)$ and $n_z := subdeg(x, z)$, and writing $\varphi = w \otimes x_k \otimes z \otimes x_k$, the following holds:*

$$\begin{aligned}\varphi; (s_{x_k,q} \otimes s_{x_k,q}); m = \varphi\Big(&(s_{x_k,q} \otimes q^{n_z} \cdot id \otimes id); (m \otimes id); s_{x_k,q}\\ &+ (id \otimes id \otimes s_{x_k,q}); (id \otimes \sigma); (m \otimes id); s_{x_k,q}\Big)\\ = \varphi\Big(&(s_{x_k,q} \otimes id \otimes id); (m \otimes id); s_{x_k,q}\\ &+ (q^{n_w} \cdot id \otimes id \otimes s_{x_k,q}); (id \otimes \sigma); (m \otimes id); s_{x_k,q}\Big)\end{aligned}$$

(qs4) *(The q-Fubini Rule) For x_k, x_ℓ in M:*

$$(s_{x_\ell,q,i} \otimes id); s_{x_k,q,i+1} \ = (id \otimes \sigma); (s_{x_k,q,i} \otimes id); \ s_{x_\ell,q,i+1}$$

5.7.1 Examples

Let $\mathcal{C} = Vec$, S be the symmetric tensor algebra modality, and $\vec{x} = x_1^{n_1} \otimes \cdot \otimes x_m^{n_m}$ for basis vectors x_k in V, as in the q-codifferential category Vec example above. Define

$$\begin{aligned} s_{x_k,q,i}\colon S(V) \otimes V &\to S(V)\\ \vec{x} \otimes x_k &\mapsto \int \vec{x} d_q x_k\end{aligned}$$

where $\int d_q x_k$ is the Jackson integral with respect to x_k.

Proposition 5.7.2. (Vec, S, s) *is a q-cointegral category, and its structure coincides with multivariable quantum Jackson integration of polynomials. The single-variable polynomial q-integral is captured as a particular case.*

Proof. For the q-constant rule, observe that $s \circ (e \otimes id) \circ \lambda^{-1}(x_k) = \int 1 d_q x_k$, which was shown to be equal to x above.

The q-linear maps rule simply translates to $x_k^2 = \int x_k d_q x_k + q \cdot \int x_k d_q x_k$, and this was demonstrated above.

We demonstrated above that this category is Colbert. Any basis vectors w, z satisfying the Colbert property with respect to $x := x_k$ are of the form $w = x^m y$, $z = x^n y'$, where $m = subdeg(x, w)$ and similarly for n and z. Thus the first of the required Rota-Baxter equations becomes:

$$\int x^m y d_q x \int x^n y' d_q x = \int \left(\int x^m y d_q x\right) q^n x^n y' d_q x + \int x^m y \left(\int x^n y' d_q x\right) d_q x$$

This follows easily from the Rota-Baxter property of the Jackson integral of a general function, stated above.

Finally, the Fubini rule is straightforward and follows from the proof above. $\square$

The finite bag comonad once again provides another example. The comonad induces an integral category whose integral transformation is just the converse of the deriving transformation described above.

Proposition 5.7.3. *$(Rel, !, \partial_q)$, where ! is the finite bag comonad and s_q is the converse relation to the deriving transformation d_q defined above, is a q-integral category.*

Proof. The calculations here are straightforward and similar enough to the differential case for both $q = 0$ and $q = 1$; most of the work is done for us by the fact that the additive enrichment in *Rel* is the set-theoretic union.

□

5.8 q-Calculus Categories

A brief mention of the fundamental theorem of calculus is in order here. In elementary calculus, the theorem describes an inverse relationship between the derivative and the integral. The first part of this theorem has been generalized into the notion of an FTC category, for "Fundamental Theorem of Calculus", or more simply a calculus category, as described above. This naturally leads to an equivalent notion in our quantum setting, whose most obvious name is quite fortuitous.

Definition 5.8.1. *A category $\mathcal{C}$ equipped with q-deriving transformation d_q and q-integral transformation s_q is a* quantum calculus category *if it is both a q-(co)differential and q-(co)integral category and satisfies*

$$d \circ s = id.$$

Proposition 5.8.2. *Both our Vec and our Rel quantum structures are quantum calculus categories.*

Proof. Let cx^n be a polynomial in Vec, with c a constant. Then observe:

$$\begin{aligned} \frac{d_q}{d_q x} \int cx^n d_q x &= c \frac{d_q}{d_q x} \frac{x^{n+1}}{[n+1]} \\ &= c \frac{[n+1]}{[n+1]} x^n \\ &= cx^n \end{aligned}$$

The general assertion for multiple variables follows immediately.

The *Rel* example is trivial, since d_q and s_q are defined to be converses.

□

In summary, we've formulated definitions of quantum differential and quantum integral categories in a manner akin to their standard counterparts. However, the machinery required to define q-differential and q-integral categories is more complex than anticipated. In particular, the intricacy of the definition of the Colbert property means the final product lacks somewhat in elegance. It seems necessary, however, given how reliant the quantum calculus operations are on the idea of "subdegree"; the correct exponent of the scalar q must be the subdegree of the variable with respect to which we are taking the derivative or integral, otherwise the rules simply do not hold. A possible direction for future work to mitigate this unseemliness is to refine the notion of grading in the graded algebra modality.

5.9 Characterization in Terms of Limits and Colimits

The definitions of q-differential and q-integral categories above, while phrased in the language of category theory, lack a certain categorical feel. The introduction of the categorical basis maps seems unavoidable, and in any case is not uncommon in quantum applications, but the indeterminate maps employed in the various rules are more reminiscent of elements than of abstract maps. Fortunately, these rules can equivalently be characterized in terms of equalizers with a universal property, a more categorical notion. The resulting diagrams more closely resemble the rules of the non-quantum structures.

Theorem 5.9.1. *A Colbert category $\mathcal{C}$ equipped with maps*

$$\partial_{x_k,q,n}\colon T(M)_n \to T(M)_{n-1} \otimes M$$

for each indeterminate x_k, scalar q, and $n \in \mathbb{N}$ is a q-codifferential category if and only if these maps satisfy the following conditions:

(qd1) *(q-Constants Rule) For all x_k, q:*

$$\begin{tikzcd} I \arrow[r, "e"] \arrow[dr] & T(M)_0 \arrow[r, "\partial_0"] & T(M)_{-1} \otimes M \\ & 0 \arrow[ur] & \end{tikzcd}$$

(qd2') *(q-Leibniz Rule) Let $z_1\colon I \to T(M)_i)_j, z_2\colon I \to T(T(M)_\ell)_r$ be basis maps having the Colbert property with respect to an indeterminate $x_k\colon I \to M$. Then $subdeg(x_k, z_1) = i$ and $subdeg(x_k, z_2) = \ell$. Write $z_1 = z_1;\mu$, $z_2 = z_2;\mu$ for notational convenience. Then for all such $i, j, \ell, r \in \mathbb{N}$ and all $y_1 := y_{x_k}^{i(j-1)}, y_2 := y_{x_k}^{\ell(r-1)}$ defined as above, the indeterminate tensor map $z_1 \otimes z_2$ is an equalizer of the following diagram:*

$$\begin{tikzcd} T(M)_{ij} \otimes T(M)_{\ell r} \arrow[r, "m"] \arrow[rr, bend left, "(q^i\cdot id)\otimes\partial_{x_k,q,\ell r};m\otimes id+\partial_{x_k,q,ij}\otimes id;id\otimes\sigma;m\otimes id"] \arrow[rr, bend right, "id\otimes\partial_{x_k,q,\ell r};m\otimes id+\partial_{x_k,q,ij}\otimes(q^\ell\cdot id);id\otimes\sigma;m\otimes id"'] & T(M)_{i+j} \arrow[r, "\partial_{x_k,q,ij+\ell r}"] & T(M)_{ij+\ell r-1} \otimes M \end{tikzcd}$$

(qd3') *(q-Linear Maps Rule) For all x_k, q, and where p is the biproduct projection map, the indeterminate x_k is an equalizer of the following diagram:*

$$\begin{tikzcd}
M \arrow[r, "\eta"] \arrow[dr, "\lambda^{-1}"] & T(M) \arrow[r, "p_1"] & T(M)_1 \arrow[r, "\partial_1"] & T(M)_0 \otimes M \\
 & I \otimes M \arrow[urr, "e\otimes id"] & &
\end{tikzcd}$$

(qd4') *(q-Chain Rule) Let subdeg$(x_k, z) = i$. Then for all x_k coupled with all of their respective z related by the Colbert property, the indeterminate z is an equalizer of the following diagram:*

$$\begin{tikzcd}
T(T(M)_i)_j \arrow[r, "\mu"] \arrow[d, "\partial_{x_k^i, q^i, j}"] & T(M)_{ij} \arrow[r, "\partial_{x_k, q, ij}"] & T(M)_{ij-1} \otimes M \\
T(T(M)_i)_{j-1} \otimes T(M)_i \arrow[rr, "\mu\otimes\partial_{x_k, q, i}"] & & T(M)_{i(j-1)} \otimes T(M)_{i-1} \otimes M \arrow[u, "m\otimes id"']
\end{tikzcd}$$

Proof. The q-constants rule $(qd1)$ is unchanged. It is clear that if the equalizer versions $(qd2')$, $(qd3')$ and $(qd4')$ of the other rules hold, then the original versions $(qd2)$, $(qd3)$ and $(qd4)$ hold as well. For the converse, we must show that the original versions imply the universal property of the equalizer.

Recall that the basis for M is an isomorphism $\mathcal{B}\colon \bigoplus_{j\in J} I \to M$, and $x_k = \mathcal{B} \circ i_k \colon I \to \bigoplus_{j\in J} I \to M$, where i is the biproduct injection. We'll use these to prove the universal property, starting with the q-linear maps rule.

Suppose we have some object A in $\mathcal{C}$ and map $f\colon A \to T(M) \otimes T(M)$ equalizing the q-linear maps rule diagram. We need to construct a unique map $\xi\colon A \to I$ such that $x_k \circ \xi = f$. Define $\xi = p_k \circ \mathcal{B}^{-1} \circ f$. Then we have

$$\begin{aligned} x_k \circ \xi &= \mathcal{B} \circ i_k \circ p_k \circ \mathcal{B}^{-1} \circ f \\ &= f \end{aligned}$$

as required, and if $g\colon A \to I$ is any map such that $x_k \circ g = f$, then

$$\begin{aligned} x_k \circ g = f &\Rightarrow \mathcal{B} \circ i_k \circ g = f \\ &\Rightarrow i_k \circ g = \mathcal{B}^{-1} \circ f \\ &\Rightarrow g = p_k \circ \mathcal{B}^{-1} \circ f \end{aligned}$$

since $p_k \circ i_k = id$ by definition of a biproduct, proving uniqueness.

The same reasoning with $\xi := (p_1 \otimes p_2) \circ (\mathcal{B}^{-1}_{T(M)_{ij}} \otimes \mathcal{B}^{-1}_{T(M)_{\ell r}}) \circ f$ in the q-Leibniz rule and $\xi := p \circ \mathcal{B}^{-1}_{T(M)_{ij}} \circ f$ in the q-chain rule completes the proof. □

6 Concluding Remarks

The goal of this paper was twofold: first, to define Leibniz and Rota-Baxter categories and demonstrate their usefulness in analyzing structures relating algebra, combinatorics and quantum theory; and second, to create a new kind of differential and integral category capturing quantum calculus parallel to the standard notions and hopefully indicate their potential usefulness beyond that setting.

Beyond the scope of this paper is examining the deeper relationships between these two ideas. Quantum calculus itself is known to have applications to perturbation theory [9], so it would be interesting to connect this to the renormalization example of our Rota-Baxter categories, and examine what relationship exists between this context and our quantum differential and integral categories, if any.

Quantum calculus also has applications to non-commutative geometry [41]. Another interesting path then might be to explore what we can say categorically in this context, perhaps via structures "q-analogous" to the tangent categories of [12]. Hopefully future work can shed some light in these regards.

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