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On the lattice of varieties of almost-idempotent semirings

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submitted by **Dipl.-Math. Burkhard Michalski**
born on the November 10, 1991 in Münchberg

Assessor: Prof. Dr. Udo Hebisch
Prof. Dr. Bernhard Ganter

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1. Introduction

Counting is one of the first things a child gets taught. Similarly the natural numbers $\mathbb{N}$ were the first algebraic structure in which mankind calculated – starting from ancient Egypt and way before. Without doubt the set of natural numbers with ordinary addition and multiplication is the most famous semiring but nowadays a broad variety of different semirings are used in computer sciences and other applications that – on the first glimpse – seem not to be connected to algebra and semirings at all. This ranges from graph theory and combinatorics over game theory and statistics to automata theory or cryptography and as such gives motivation for further study of semirings and their properties.

To give some structure to the class $\mathbb{SR}$ of semirings many authors have done research on the lattice $\mathcal{L}(\mathbb{SR})$ of semiring varieties. E.g. Pastijn et al. determined the complete lattice $\mathcal{L}(\mathbb{I})$ of all subvarieties of the variety $\mathbb{I}$ of idempotent semirings in [1] and [2] which provides a huge step to the aim of determining the complete lattice of semiring varieties. They proved that the lattice is generated by eleven semirings and consists of 78 varieties. So it seems reasonable to continue this work and have a look at non-idempotent semirings. In [3] Sen and Bhuniya introduced so called almost-idempotent semirings as a generalization of idempotent semirings (cf. Definition 2.19) which seems like a good angle of attack.

The aim of this thesis was to inspect the structure of the lattice $\mathcal{L}(\mathbb{IA})$ of varieties of almost-idempotent semirings. In [4], among others, the lattice $\mathcal{L}(\mathbb{IA}_2)$ generated by almost-idempotent semirings with two elements was determined. It was shown that the lattice $\mathcal{L}(\mathbb{IA}_2)$ is a boolean algebra with 32 elements. Based on those results I started looking at almost-idempotent semirings with three elements in chapter 3. Eleven non-isomorphic proper almost-idempotent semirings were generated by a python program and characterized. In section 3.2 the full context that generates $\mathcal{L}(\mathbb{IA}_3)$ is given and every context implication proved. In the end we receive a lattice consisting of 19901 varieties. So the idea was to use Attribute Exploration (cf. section 2.2.3) to determine the complete lattice $\mathcal{L}(\mathbb{IA})$ starting from the context in chapter 3. But it quickly turned out that the aim was too ambitious since the context grew rapidly. Reducing the research to only commutative semirings and later on introducing the variety $\mathbb{V}$ of commutative semirings that additionally satisfy $xy \approx xy + x$ I found two construction methods for chains of semirings (cf. Example 4.3 and Lemma 4.12). In Lemmas 4.6, 4.15 and 4.16 we finally see that those chains of semirings generate infinite chains of varieties in $\mathcal{L}(\mathbb{V})$. Thus in contrary to the expectation that $\mathcal{L}(\mathbb{IA})$ would be finite and quite small it turns out that even the subvariety $\mathbb{V}_2$ (cf. Remark 4.7) of $\mathbb{COM}$ has an infinite amount of subvarieties.

2. Preliminaries

First off, note that we will denote the set $\{0, 1, 2, \ldots\}$ of natural numbers by $\mathbb{N}$ and the set $\{1, 2, 3, \ldots\}$ of positive natural numbers by $\mathbb{N}_+$ throughout this thesis. We will start with a short recap about varieties. These notations and results are taken from Burris and Sankappanavar in [5]. Afterwards we introduce semirings and almost-idempotent semirings as their specialization.

Definition 2.1 (Cf. [5], Definition 9.3)**.** A *variety* $\mathbb{V}$ is a class of algebras of the same type that is closed under taking subalgebras, homomorphic images and direct products. We denote the *lattice of subvarieties of* $\mathbb{V}$ by $\mathcal{L}(\mathbb{V})$.

Definition 2.2. Let Σ be a set of identities of the same type, and define $M(\Sigma)$ to be the class of algebras satisfying Σ. A class K of algebras is and *equational class* if there is a set of identities Σ such that $K = M(\Sigma)$. In this case we say that K is *defined* by Σ.

Remark 2.3. We use the notation $K = [\Sigma]$ instead of $K = M(\Sigma)$ since it frequently is used in literature.

The following theorem, commonly known as Birkhoff's Theorem, is essential for research on varieties and lattices of varieties.

Theorem 2.4 ([5], Theorem 11.9)**.** *K is an equational class if and only if K is a variety.*

Example 2.5. Clearly, the class $\mathbb{A}$ of all algebras of a given type is a variety. Its equational basis is $x = x$ which obviously is satisfied by every algebra. So $\mathbb{A} = [x = x]$ holds whereas $\mathbb{T} = [y = x]$ is the class of trivial algebras in $\mathbb{A}$ that consist of only one element.

Definition 2.6. For any algebra $A \in \mathbb{A}$ we define $\mathcal{HSP}(A)$ as the smallest variety $\mathbb{V} \in \mathcal{L}(\mathbb{A})$ such that $\mathbb{V}$ still contains A and call $\mathcal{HSP}(A)$ the variety *generated* by A. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ for $n \in \mathbb{N}_+$ be a non-empty set of algebras in $\mathbb{A}$. Then we call

$$\mathcal{HSP}(\mathcal{A}) = \mathcal{HSP}(A_1) \vee \mathcal{HSP}(A_2) \vee \ldots \vee \mathcal{HSP}(A_n)$$

the variety *generated* by $\mathcal{A}$. For brevity we often write $\mathcal{HSP}(A_1, A_2, \ldots, A_n)$ instead of $\mathcal{HSP}(\{A_1, A_2, \ldots, A_n\})$.

Remark 2.7. The notation $\mathcal{HSP}(\cdot)$ originates from closure regarding to **h**omomorphic images, taking **s**ubalgebras and direct **p**roducts.

2.1. Semirings

The following definitions of semirings and their specializations were already given by Hebisch and Weinert in [6]. As this thesis mainly bases upon the work done by Shao and Ren, we will illustrate those concepts by the two-element semirings given in [4].

Definition 2.8 ([6], Definition 2.1)**.** An algebra $(S, +, \cdot)$ is called *semiring* if its additive reduct $(S, +)$ is a commutative semigroup, its multiplicative reduct $(S, \cdot)$ is a semigroup and the distributive laws

$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{1}$$

$$(x + y) \cdot z = x \cdot z + y \cdot z. \tag{2}$$

hold for all $x, y, z \in S$. The variety of semirings will be denoted by $\mathbb{SR}$.

Remark 2.9. From now on we assume that $\cdot$ binds stronger than $+$. Thus we can omit $\cdot$ in most terms and write ab instead of $a \cdot b$.

Definition 2.10 ([6], Definition 2.6 c))**.** Let $(S, +, \cdot)$ be a semiring. When $(S, +)$ is an idempotent semigroup, i.e.

$$x + x = x \tag{3}$$

holds for all $x \in S$, it is called *additively idempotent*. The variety of additively idempotent semirings will be denoted by $\mathbb{SL}^+$ since their additive reduct is a semilattice. Such semirings are often called *semilattice-ordered semigroups*.

Example 2.11. In [4] the two-element semirings L_2, R_2, M_2, D_2, T_2 and N_2 are introduced (these can be found in Appendix A). Those are – apart from isomorphic images – the only two-element semirings in $\mathbb{SL}^+$. Thus they generate $\mathbf{S}_2 = \mathcal{HSP}(L_2, R_2, M_2, D_2, T_2, N_2)$ as a subvariety of $\mathbb{SL}^+$. In [4], Corollary 2.3 Shao and Ren showed that an equational basis of $\mathbf{S}_2$ is given by

$$xyzt = xzyt, \tag{4}$$

$$(xy)^2 = xy, \tag{5}$$

$$x + yz = x + yz + xz + yx \tag{6}$$

and thus

$$\mathbf{S}_2 = [xyzt = xzyt,\ (xy)^2 = xy,\ x + yz = x + yz + xz + yx].$$

Lemma 2.12 (Cf. [7], Chapter 1.6)**.** *Let X be a (finite) set of variables – called* alphabet*. Furthermore*

$$X^+ = \{x_1 x_2 \cdots x_k \mid k \in \mathbb{N}_+,\ x_1, x_2, \ldots, x_k \in X\}$$

is the set of all nonempty words *over X. Whereas ε is the* empty word *consisting of no variables and*

$$X^* = X^+ \cup \{\varepsilon\}$$

is the set of all words *over X. Then a binary operation $\cdot$ is defined on X^+ by juxtaposition/concatenation*

$$a \cdot b = ab$$

for all $a, b \in X^+$ and $(X^+, \cdot)$ is the free semigroup an A.

Example 2.13 ([8], Example 2.2)**.** Let S be a semigroup. We put, for any $Q, R \in \mathfrak{P}(S)$,

$$Q \cdot R = \{qr \mid q \in Q, r \in R\}.$$

Then $(\mathfrak{P}(S), \cdot, \cup)$ and $(F(S), \cdot, \cup)$ are semilattice-ordered semigroups. Here, as usual, $\cup$ denotes the set-theoretical union.

$$F(S) = \{A \subset S : \ |A| \in \mathbb{N}_+\}$$

is the set of all finite subsets of S.

Lemma 2.14 (See [8], Theorem 2.5)**.** *The structure $(F(X^+), \cdot, \cup)$ together with the embedding $\kappa : x \to \{x\}$, $x \in X$, is a free object on the set X in the variety of all semilattice-ordered semigroups.*

Remark 2.15. We will denote $(F(X^+), \cdot, \cup)$ by $(P_{fin}(X^+), +, \cdot)$, writing $\cup$ as addition and following the notation of Shao and Ren in [4]. By former lemma $(P_{fin}(X^+), +, \cdot)$ is the free additively idempotent semiring – the *free ai-semiring.*

Definition 2.16. Let $(P_{fin}(X^+), +, \cdot)$ be the free ai-semiring over an alphabet X. We call the expression $u \approx v$ an *ai-semiring identity* if $u, v \in P_{fin}(X^+)$. Alternatively we sometimes write

$$u_1 + u_2 + \ldots + u_m \approx v_1 + v_2 + \ldots + v_n$$

instead of

$$\{u_i \mid i = 1, \ldots, m\} \approx \{v_j \mid j = 1, \ldots, n\}.$$

Let S be a semiring. Then $u \approx v$ is *satisfied* in S – denoted by $S \models u \approx v$ – if and only if $\varphi(u) = \varphi(v)$ for every homomorphism $\varphi : P_{fin}(X^+) \to S$.

Definition 2.17. Let $(S, +, \cdot) \in \mathbb{SL}^+$ be a semiring. We call $(S, +, \cdot)$ a *commutative* semiring if it satisfies

$$xy \approx yx. \tag{7}$$

The subvariety of all commutative semirings in $\mathbb{SL}^+$ will be denoted by $\mathbb{COM}$.

Example 2.18. It is easy to see that the semirings M_2, D_2, T_2, N_2 are commutative whereas L_2 and R_2 are the only two non-commutative semirings of order two. Hence the variety $\mathbb{COM} \cap \mathbf{S}_2$ is a proper subvariety of $\mathbf{S}_2$.

Definition 2.19. Let $(S, +, \cdot) \in \mathbb{SL}^+$ be a semiring. Then $(S, +, \cdot)$ is called *idempotent* if and only if it satisfies

$$x^2 \approx x. \tag{8}$$

The subvariety of all idempotent semirings in $\mathbb{SL}^+$ is accordingly denoted by $\mathbb{I}$.
The semiring $(S, +, \cdot)$ is called *almost-idempotent* (cf. [9]) if and only if it satisfies

$$x^2 + x \approx x^2. \tag{9}$$

We denote the subvariety of all almost-idempotent semirings in $\mathbb{SL}^+$ by $\mathbb{IA}$.

Example 2.20. Looking at semirings of order two again we see that L_2, R_2, M_2, D_2 are idempotent whereas T_2 and N_2 are not. But since

$$\begin{aligned} x^2 + x &\approx x^2 + x^2 && \text{by (8)} \\ &\approx x^2 && \text{by (3)} \end{aligned}$$

holds for every semiring $S \in \mathbb{I}$ the variety $\mathbb{I}$ is a subvariety of $\mathbb{IA}$. Furthermore (9) is satisfied in T_2 and $\varphi(x) = 0$ contradicts $x^2 \approx x$ in T_2. So we get $T_2 \in \mathbb{IA} \setminus \mathbb{I}$ and thus $\mathbb{I} \subset \mathbb{IA}$.

As already mentioned Pastijn at al. determined the complete lattice $\mathcal{L}(\mathbb{I})$. We are going to investigate $\mathcal{L}(\mathbb{IA})$.

2.2. Formal Concept Analysis

2.2.1. Concept Lattices

The concepts and notations in Section 2.2 follow Ganter and Obiedkov [10]. Results given here without proof can also be found there.

Definition 2.21. Let G and M be two sets and $I \subseteq G \times M$ a binary relation on $G \times M$. We call the triple $\mathbb{K} = (G, M, I)$ a *formal context*. The elements $g \in G$ are called *objects*, the elements $m \in M$ *attributes*. For any object $g \in G$ and any attribute $m \in M$ the relation gIm respectively $(g, m) \in I$ expresses that object g has the attribute m.
A formal context can easily be represented as a table where the rows are objects and columns are attributes. The cell in row g and column m gets marked with an ✓ if gIm holds and remains empty otherwise.

Example 2.22. Earlier in Example 2.11 we introduced the semirings L_2, R_2, M_2, D_2, T_2 and N_2 and will now use them as objects in our context. In Examples 2.18 and 2.20 we already had a glimpse at equations that are or are not satisfied in those semirings. Those equations will be our attributes and the relation gIm can be read as "semiring g *satisfies* equation m". Thus we receive the following context:

	$xy \approx yx$	$x^2 \approx x$	$x^2 + x \approx x^2$
L_2		✓	✓
R_2		✓	✓
M_2	✓	✓	✓
D_2	✓	✓	✓
T_2	✓		✓
N_2	✓		

Definition 2.23. Let (G, M, I) be a context. For any subset $A \subseteq G$ of objects we define

$$\varphi(A) = \{m \in M \mid gIm \ \forall g \in A\}$$

and for any subset $B \subseteq M$ of attributes analogous

$$\psi(B) = \{g \in G \mid gIm \ \forall m \in B\}.$$

Lemma 2.24 ([11], Hilfssatz 10)**.** *Let (G, M, I) be a context. Moreover let $A, A_1, A_2 \subseteq G$ be sets of objects and $B, B_1, B_2 \subseteq M$ be sets of attributes. Then*

1. *$A \subseteq \psi(\varphi(A))$ and $B \subseteq \varphi(\psi(B))$,*
2. *$A_1 \subseteq A_2 \Rightarrow \varphi(A_2) \subseteq \varphi(A_1)$ and $B_1 \subseteq B_2 \Rightarrow \psi(B_2) \subseteq \psi(B_1)$,*
3. *$\varphi(A) = \varphi(\psi(\varphi(A)))$ and $\psi(B) = \psi(\varphi(\psi(B)))$*

hold.

Proof. 1. : Assume $g_0 \in A$ then $g_0 I m$ holds for all $m \in \varphi(A)$ and hence $g_0 \in \psi(\varphi(A)) = \{g \in G \mid \forall m \in \varphi(A) : gIm\}$.
2. : Assume $m \in \varphi(A_2)$ then gIm holds for all $g \in A_2$. Since $A_1 \subseteq A_2$ holds gIm is also true for all $g \in A_1$. In consequence $m \in \varphi(A_1)$ is satisfied.
3. : From 1 and 2 we get $\varphi(A) \supseteq \varphi(\psi(\varphi(A)))$. On the other hand from 1 we also get $B \subseteq \varphi(\psi(B))$, consequently with $B = \varphi(A)$ finally $\varphi(A) = \varphi(\psi(\varphi(A)))$.
Derivation for attributes are analogous. □

Definition 2.25. A *formal concept* of a context (G, M, I) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $\varphi(A) = B$ and $\psi(B) = A$. Then A is called the *extent* and B is called the *intent* of the formal concept (A, B). With $\mathfrak{B}(G, M, I)$ we denote the set of all formal concepts of the context (G, M, I).

Example 2.26. In continuation of Example 2.22 we calculate every formal concept in the given context. For this we calculate the sets $\psi(B)$ and $\varphi(\psi(B))$ for every subset $B \subseteq \{xy \approx yx, x^2 \approx x, x^2 + x \approx x^2\}$:

1. $\psi(\emptyset) = \{L_2, R_2, M_2, D_2, T_2, N_2\}$ and
$\varphi(\psi(\emptyset)) = \emptyset$,
2. $\psi(\{xy \approx yx\}) = \{M_2, D_2, T_2, N_2\}$ and
$\varphi(\psi(\{xy \approx yx\})) = \{xy \approx yx\}$,
3. $\psi(\{x^2 \approx x\}) = \{L_2, R_2, M_2, D_2\}$ and
$\varphi(\psi(\{x^2 \approx x\})) = \{x^2 \approx x, x^2 + x \approx x^2\}$,
4. $\psi(\{x^2 + x \approx x^2\}) = \{L_2, R_2, M_2, D_2, T_2\}$ and
$\varphi(\psi(\{x^2 + x \approx x^2\})) = \{x^2 + x \approx x^2\}$,
5. $\psi(\{xy \approx yx, x^2 \approx x\}) = \{M_2, D_2\}$ and
$\varphi(\psi(\{xy \approx yx, x^2 \approx x\})) = \{xy \approx yx, x^2 \approx x, x^2 + x \approx x^2\}$,
6. $\psi(\{xy \approx yx, x^2 + x \approx x^2\}) = \{M_2, D_2, T_2\}$ and
$\varphi(\psi(\{xy \approx yx, x^2 + x \approx x^2\})) = \{xy \approx yx, x^2 + x \approx x^2\}$,
7. $\psi(\{x^2 \approx x, x^2 + x \approx x^2\}) = \{L_2, R_2, M_2, D_2\}$ and
$\varphi(\psi(\{x^2 \approx x, x^2 + x \approx x^2\})) = \{x^2 \approx x, x^2 + x \approx x^2\}$,
8. $\psi(\{xy \approx yx, x^2 \approx x, x^2 + x \approx x^2\}) = \{M_2, D_2\}$ and
$\varphi(\psi(\{xy \approx yx, x^2 \approx x, x^2 + x \approx x^2\})) = \{xy \approx yx, x^2 \approx x, x^2 + x \approx x^2\}$.

Thus every tuple $(\psi(B), B)$ with $\varphi(\psi(B)) = B$ is a formal concept in the given context. Those are

- $B_0 = (\{M_2, D_2\}, \{xy \approx yx,^2 \approx x, x^2 + x \approx x^2\})$,
- $B_1 = (\{L_2, R_2, M_2, D_2\}, \{x^2 \approx x, x^2 + x \approx x^2\})$,
- $B_2 = (\{M_2, D_2, T_2\}, \{xy \approx yx, x^2 + x \approx x^2\})$,
- $B_3 = (\{L_2, R_2, M_2, D_2, T_2\}, \{x^2 + x \approx x^2\})$,
- $B_4 = (\{M_2, D_2, T_2, N_2\}, \{xy \approx yx\})$,
- $B_5 = (\{L_2, R_2, M_2, D_2, T_2, N_2\}, \emptyset)$.

Definition 2.27. Let (G, M, I) be a context and $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}(G, M, I)$ two formal concepts with $A_1 \subseteq A_2$. Then we call (A_1, B_1) a *subconcept* of (A_2, B_2) respectively (A_2, B_2) a *superconcept* of (A_1, B_1) and write $(A_1, B_1) \leq (A_2, B_2)$.

Theorem 2.28 ([10], Theorem 1)**.** *The concept lattice of any formal context* (G, M, I) *is a complete lattice. For an arbitrary set*

$$\{(A_t, B_t) \mid t \in T\} \subseteq \mathfrak{B}(G, M, I)$$

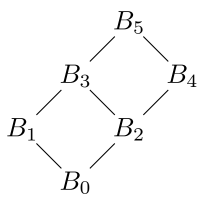
of formal concepts, the supremum is given by

$$\bigvee_{t \in T} (A_t, B_t) = \left(\psi(\varphi(\bigcup_{t \in T} A_t)), \bigcap_{t \in T} B_t \right)$$

and the infimum is given by

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \varphi(\psi(\bigcup_{t \in T} B_t)) \right).$$

Example 2.29. In continuation of Example 2.26 we now can draw the complete lattice of concepts in the context:



2.2.2. Implications

Definition 2.30. Let (G, M, I) be a context and $A, B \subseteq M$. We call the pair (A, B) an *implication* and write $A \to B$. A subset $T \subseteq M$ of attributes *respects* an implication $A \to B$ if and only if $A \not\subseteq T \vee B \subseteq T$ holds. Furthermore T respects a set of implications $\mathcal{I} \subseteq \mathfrak{P}(M) \times \mathfrak{P}(M)$ if it respects every implication in $\mathcal{I}$.

Definition 2.31. Let $\mathbb{K} = (G, M, I)$ be a context and $\mathcal{T}$ a set of subsets of the attributes M – so $\mathcal{T} \subseteq \mathfrak{P}(M)$. Furthermore let $A \to B$ be an implication with $A, B \subseteq M$. Then $\mathcal{T}$ *satisfies* $A \to B$ if and only if

$$\forall T \in \mathcal{T} : A \not\subseteq T \vee B \subseteq T.$$

Hence every element of $\mathcal{T}$ respects $A \to B$.
An implication $A \to B$ is *satisfied* in $\mathbb{K}$ if it is satisfied in the set $\mathcal{T} = \{\varphi(g) \mid g \in G\}$ of object intents. We call an implication that is satisfied in an context $\mathbb{K}$ a *context implication* and denote the set of all context implications of $\mathbb{K}$ by $\mathfrak{L}(\mathbb{K})$.

Lemma 2.32. *Let $\mathbb{K} = (G, M, I)$ be a context and $B \subseteq M$ a set of attributes. Then $B \to \varphi(\psi(B))$ is satisfied in $\mathbb{K}$.*

Proof. Assume there exists $g \in G$ with $B \subseteq \varphi(g) \wedge \varphi(\psi(B)) \not\subseteq \varphi(g)$. By Lemma 2.24 $\psi(\varphi(g)) \subseteq \psi(B)$ and moreover $\varphi(\psi(B)) \subseteq \varphi(\psi(\varphi(g))) = \varphi(g)$ are consequences of $B \subseteq \varphi(g)$ in contradiction to $\varphi(\psi(B)) \not\subseteq \varphi(g)$. □

Example 2.33. With $B = \{x^2 \approx x\}$ we receive $\varphi(\psi(B)) = \{x^2 \approx x, x^2 + x \approx x^2\}$ in the context given in Example 2.22. As we already proved this implication is true in $\mathbb{SL}^+$. Note that this is the only non-trivial implication in the context. So by proving it we showed that e.g. L_2 and R_2 are not distinguishable due to a lack of attributes. We will solve this problem in Example 2.42.

Definition 2.34. Let $\mathbb{K} = (G, M, I)$ be a context and $A, B \subseteq M$ sets of attributes. An implication $A \to B$ is a *consequence* of a set of implications $\mathcal{I} \subseteq \mathfrak{P}(M) \times \mathfrak{P}(M)$ if and only if every set of attributes $T \subseteq M$ that respects $\mathcal{I}$ respects $A \to B$ as well. We will denote the set of all consequences of $\mathcal{I}$ by $\mathfrak{C}(\mathcal{I})$.

Definition 2.35. For a context $\mathbb{K} = (G, M, I)$ we call a set of implications $\mathcal{I} \subseteq \mathfrak{P}(M) \times \mathfrak{P}(M)$ *complete* when every context implication in $\mathbb{K}$ is a consequence of $\mathcal{I}$ – so $\mathfrak{C}(\mathcal{I}) = \mathfrak{L}(\mathbb{K})$. We call $\mathcal{I}$ *irredundant* if no implication $A \to B \in \mathcal{I}$ is already a consequence of $\mathcal{I} \setminus \{A \to B\}$.

Definition 2.36. Let $\mathbb{K} = (G, M, I)$ be a context. A set of implication $\mathcal{I} \subseteq \mathfrak{L}(\mathbb{K})$ is called *base* of $\mathfrak{L}(\mathbb{K})$ if and only if $\mathfrak{C}(\mathcal{I}) = \mathfrak{L}(\mathbb{K})$ holds and no $\mathcal{S} \subset \mathcal{I}$ with $\mathfrak{C}(\mathcal{S}) = \mathfrak{L}(\mathbb{K})$ exists.

Remark 2.37. By former definition every basis $\mathcal{I} \subseteq \mathfrak{L}(\mathbb{K})$ of $\mathfrak{L}(\mathbb{K})$ is complete and irredundant in a given context $\mathbb{K} = (G, M, I)$.

Definition 2.38. Let $\mathbb{K} = (G, M, I)$ be a context. A set $P \subseteq M$ is called *pseudo-closed* if $P \neq \varphi(\psi(P))$ holds and $\varphi(\psi(Q)) \subseteq P$ is satisfied for all proper pseudo-closed subsets $Q \subset P$ of P.

Remark 2.39. Note that former definition is recursive. Hence we need to assume that M is finite from now on.

Theorem 2.40 ([10], Theorem 7)**.** *Let $\mathbb{K} = (G, M, I)$ be a context. Then the set*

$$\mathcal{S} = \{P \to \varphi(\psi(P)) \mid P \textit{ pseudo-closed}\}$$

is complete and irredundant.

The set

$$\mathcal{S}' = \{P \to (\varphi(\psi(P)) \setminus P) \mid P \text{ pseudo-closed}\}$$

is sometimes called the *Duquenne-Guigues-Basis*. Duquenne and Guigues called it the *canonical basis*.

Example 2.41. We calculate the canonical basis for the context given in Example 2.22. So we are interested in the set of pseudo contents. In Example 2.42 we already calculated $\varphi(\psi(P))$ for every $P \subseteq M$. Only

- $P_1 = \{x^2 \approx x\}$ and
- $P_2 = \{xy \approx yx, x^2 \approx x\}$

satisfy the inequality $P_i \neq \varphi(\psi(P_i))$. So P_1 is a pseudo content. It remains to check whether P_2 is a pseudo content or not. Obviously $P_1 \subset P_2$ holds but since $\varphi(\psi(P_1)) = \{x^2 \approx x, x^2 + x \approx x^2\} \nsubseteq \{xy \approx yx, x^2 \approx x\} = P_2$ is true P_2 is no pseudo content. Thus the canonical basis of the context is

$$\mathcal{S} = \{\{x^2 \approx x\} \to \{x^2 + x \approx x^2\}\}.$$

In Example 2.20 we already proved that this implication holds true for any semiring in $\mathbb{SL}^+$.

2.2.3. Attribute Exploration

One often faces the situation that the calculated concept lattice for the given context fulfills implications that are not true in reality. Hence the context is too special and needs to be enlarged. This can be achieved by adding one or more new objects to the context that contradict those implications. As we are dealing with implications of equations the search for counterexamples is rather simple; programs like *Mace4*[1] are designed exactly for this purpose. In Chapter 3 we are facing a similar but different problem: we know that our set of objects (almost-idempotent semirings of order 3) is complete but the set of attributes (equations) is not. To solve this we inspect the dual context (M, G, I^{-1}) with $gIm \Leftrightarrow mI^{-1}g$. We again can compute implications in the new context. Those look like

$$\{S_i \mid i \in J_0\} \to \{S_j \mid j \in J_1\}$$

with index sets J_0 and J_1 and $S_i, S_j \in \mathbb{SL}^+$. To prove those we need to show that every equation that is satisfied in every S_i is also satisfied in any S_j. If the implication is not true we have to add an equation as counterexample that is satisfied in every S_i but not in S_j.

Example 2.42. In the context given in Example 2.22 we know that the set of objects is complete since the objects are up to isomorphic images all semirings of order 2. As pointed out in Example 2.33 the semirings L_2 and R_2 are not distinguishable in this context. By adding the equation $xy \approx x$, which is satisfied in L_2 but not in R_2, we receive an updated context:

[1] See `http://www.cs.unm.edu/~mccune/prover9/`

	$xy \approx yx$	$x^2 \approx x$	$x^2 + x \approx x^2$	$xy \approx x$
L_2		✓	✓	✓
R_2		✓	✓	
M_2	✓	✓	✓	
D_2	✓	✓	✓	
T_2	✓		✓	
N_2	✓			

But again the semirings M_2 and D_2 are not distinguishable in the new context. This can be solved by adding the equation $x + yx \approx x$. We receive the updated context:

	$xy \approx yx$	$x^2 \approx x$	$x^2 + x \approx x^2$	$xy \approx x$	$x + yx \approx x$
L_2		✓	✓	✓	
R_2		✓	✓		✓
M_2	✓	✓	✓		
D_2	✓	✓	✓		✓
T_2	✓		✓		
N_2	✓				

By this procedure we can enlarge the set of attributes. We repeat those steps until we can prove all implications of the contexts canonical basis. In this example the final concept lattice would have no dependencies between objects at all and thus the lattice $\mathcal{L}(\mathbf{S}_2)$ is a boolean algebra. For a complete set of equations see Table 3 in [4].

The example demonstrates a process called *attribute exploration*. In this thesis we use that technique to explore the lattice of varieties of almost-idempotent semirings. We start with almost-idempotent semirings with up to three elements as objects. Afterwards we add equations as attributes to the context until we are able to prove every implication of its canonical basis. Thus we can ensure that the final lattice indeed is the complete lattice of varieties generated by almost-idempotent semirings with up to three elements.

3. The lattice $\mathcal{L}(\mathbb{IA}_3)$

3.1. Generating semirings

In this chapter we will determine the lattice $\mathcal{L}(\mathbb{IA}_3)$ of all subvarieties of $\mathbb{IA}$ generated by proper almost-idempotent semirings of maximal order 3. Note that this is not necessarily the complete lattice of subvarieties of $\mathbb{IA}_3$, the variety generated by all almost-idempotent semirings of maximal order 3. We start with some definitions we will use later on. These are mainly used to characterize the semirings that will generate the lattice. A lot of this definitions are well-known. I.e., one finds definitions for size, content, head and tail of words respectively sets of words already in [1], [2] or [4]. We recap those here and introduce new ones.

Definition 3.1. Let X be an alphabet and $x \in X$. For any finite word $w \in X^+$ we define

I. the *count* $c_x(w)$ *of* x *in* w recursively:

1. $c_x(x) = 1$,
2. $c_x(y) = 0 \quad y \in X, y \neq x$,
3. $c_x(w_1 w_2) = c_x(w_1) + c_x(w_2) \quad w_1, w_2 \in X^+$.

So $c_x(w)$ is the number of occurrences of the variable x in the word w.

II. the *content* $c(w)$ of w via

$$c(w) = \{x \in X \mid c_x(w) \geq 1\}.$$

III. the *size* $|w|$ of w via

$$|w| = \sum_{x \in X} c_x(w).$$

So $|w|$ is the number of variables occurring in w.

IV. the head $h(w)$ respectively tail $t(w)$ of w as the set containing the first respectively last variable occurring in w.

Definition 3.2. Let X be an alphabet, $u = \{u_1, \ldots, u_m : u_i \in X^+\}$ a set of words over the alphabet X and $k \in \mathbb{N}_+$. We define

I. the set $c(u)$ of wordcontents in u via

$$c(u) = \{c(u_i) \mid u_i \in u\}.$$

II. the content of all words with size k via

$$C_k(u) = \bigcup_{\substack{u_i \in u \\ |u_i|=k}} c(u_i),$$

the content of all words with size greater or equal k via

$$C_{k+}(u) = \bigcup_{i=k}^{\infty} C_i(u)$$

and the content $C(u) = C_{1+}(u)$ of u.

III. the set $H(u)$ of all *heads* in u via

$$H(u) = \bigcup_{u_i \in u} h(u_i)$$

and the sets $H_k(u)$ and $H_{k+}(u)$ analogous to $C_k(u)$ respectively $C_{k+}(u)$.

IV. the set $T(u)$ of all *tails* in u via

$$T(u) = \bigcup_{u_i \in u} t(u_i)$$

and the sets $T_k(u)$ and $T_{k+}(u)$ analogous to $C_k(u)$ respectively $C_{k+}(u)$.

V. the *body* in respect to the head[tail] $B^h(u)[B^t(u)]$ as the set of all variables occurring in u after the first[last] variable of every word in u was removed.

VI. the set of variables in u that occur at least twice in at least one word via

$$Q(u) = \{x \in X \mid \exists u_i \in u : c_x(u_i) \geq 2\}.$$

VII. the set

$$\mathbb{Q}(u) = \{\{x, y\} \in \mathfrak{P}(X),\ x, y \notin Q(u) \mid \exists u_i \in u :\ c_x(u_i) = c_y(u_i) = 1\}.$$

VIII. the set

$$\mathfrak{F}(u) = \{P \in \mathfrak{P}(C(u) \setminus Q(u)) \mid \forall u_i \in u\ \exists! x \in c(u_i) : x \in P, c_x(u_i) = 1\}$$

of possible choices of variables in u such that in every word exactly one variable was picked.

IX. the set

$$\mathbb{C}^{\perp}(u) = \{c(u_i) \mid u_i \in u,\ \nexists u_j \in u :\ \ c(u_j) \subset c(u_i)\}$$

of minimal wordcontents in u.

X. the set

$$\mathbb{C}^{\perp}_{k+}(u) = \{c(u_i) \mid u_i \in u,\ |u_i| \geq k,\ \nexists u_j \in u : |u_j| \geq k,\ c(u_j) \subset c(u_i)\}$$

of minimal wordcontents among words with at least size k in u.

Remark 3.3. Obviously the head of a word with size 1 is the same as its content. In consequence

$$H(u) = C_1(u) \cup H_{2+}(u)$$

and

$$H(u) = C_1(u) \setminus H_{2+}(u) \cup H_{2+}(u)$$

hold for any set $u = \{u_1, \dots u_m\}$ of words over a given alphabet. Furthermore

$$C_{2+}(u) = B^t(u) \cup T_{2+}(u)$$

yields since every variable in a word with size at least 2 has to be in its tail or its body in respect to the tail. Dually

$$C_{2+}(u) = B^h(u) \cup H_{2+}(u)$$

and consequently

$$C_{2+}(u) = B^h(u) \cup B^t(u)$$

is satisfied and by former thoughts

$$C(u) = H(u) \cup B^h(u)$$

and dually

$$C(u) = T(u) \cup B^t(u)$$

hold. Finally

$$C(u) = C_1(u) \setminus C_{2+}(u) \cup C_{2+}(u)$$

is satisfied trivially.

We will now turn our attention to proper almost-idempotent semirings of order 3. But first off, let's recapitulate the characterization for additively idempotent semirings with two elements given by Shao and Ren.

Lemma 3.4 ([4], Lemma 1.1)**.** *Let $u \approx v$ be an ai-semiring-identity with*

$$u = \{u_i \mid i = 1, \dots, m\}$$

and

$$v = \{v_j \mid j = 1, \dots, n\}.$$

Then

1. $L_2 \models u \approx v$ *if and only if* $H(u) = H(v)$,
2. $R_2 \models u \approx v$ *if and only if* $T(u) = T(v)$,
3. $M_2 \models u \approx v$ *if and only if* $C(u) = C(v)$,
4. $D_2 \models u \approx v$ *if and only if* $\forall u_i \in u\ \exists v_j \in v : c(v_j) \subseteq c(u_i)$ *and* $\forall v_j \in v\ \exists u_i \in u : c(u_i) \subseteq c(v_j)$,
5. $T_2 \models u \approx v$ *if and only if* $\{u_i \in u \mid |u_i| \geq 2\} \neq \emptyset$ *and* $\{v_j \in v \mid |v_j| \geq 2\} \neq \emptyset$.

Remark 3.5. A complete list of the semirings inspected by Shao and Ren in [4] can be found in appendix A. As the semiring T_2 will play a crucial role throughout this thesis we recap its addition and multiplication table here:

$+$	0	1
0	0	1
1	1	1

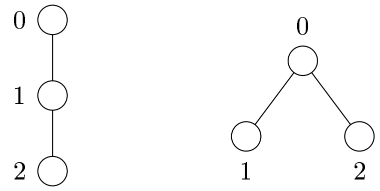
$\cdot$	0	1
0	1	1
1	1	1

Obviously T_2 is a commutative semiring that satisfies $xy \approx xy + x$ and consequently $x^2 \approx x^2 + x$ but not $x^2 \approx x$. These properties will become handy later on. Note that 5 in former lemma is not completely accurate since $x \approx x$ is obviously satisfied in T_2 but does not hold the condition. Hence the correct condition should be

$$(\{u_i \in u \mid |u_i| \geq 2\} \neq \emptyset \wedge \{v_j \in v \mid |v_j| \geq 2\} \neq \emptyset) \quad \vee$$

$$(\{u_i \in u \mid |u_i| \geq 2\} = \emptyset = \{v_j \in v \mid |v_j| \geq 2\} \wedge C(u) = C(v)).$$

The semirings L_2, R_2 ,M_2 and D_2 are multiplicatively idempotent whereas T_2 is the only proper almost-idempotent semiring. As we already have seen they all satisfy $x^2 + x \approx x^2$ and thus generate the variety $\mathbb{IA}_2 \subseteq \mathbf{S}_2$ of almost-idempotent semirings of order 2. Furthermore in [4] it was shown that the lattice generated by those semirings consists of 32 distinct varieties. Moreover an equational base of the variety $\mathbf{S}_2$ generated by all additively idempotent semirings with two elements was given (cf. Example 2.11). Using these results, we can easily determine if a given semiring generates a new variety or already is an element of $\mathbf{S}_2$.
Consequently, it seems reasonable to search for all almost-idempotent non-isomorphic semirings with three elements and filter out those which generate new varieties afterwards. Since for every almost-idempotent semiring $(S, +, \cdot)$ the additive reduct $(S, +)$ forms a semilattice – without loss of generality a $\vee$-semilattice –, we are able to reduce the search space for almost-idempotent semirings with three elements noticeably. The only two $\vee$-semilattices with three elements are the chain and the $\vee$-semi-lattice with two not comparable elements:



We used a Python program that found 44 almost-idempotent semirings $S_0, S_1, \ldots, S_{43}$ (see appendix C) with one of the above additive reducts of which 21 were proper almost-idempotent. Finally, ten of them satisfy the equational basis of $\mathbf{S}_2$ and thus already belong to $\mathcal{HSP}(T_2)$. So only S_3, S_6, S_7, S_{15}, S_{20}, S_{21}, S_{22}, S_{23}, S_{24}, S_{35} and S_{38} are neither already in $\mathbb{I}$ nor in $\mathbb{IA}_2$ and hence generate new varieties in $\mathcal{L}(\mathbb{IA}_3)$.
Similarly to Lemma 3.4 we want to characterize those semirings of order 3 and for this purpose will use a simple trick that was introduced in [4]:
To show that an ai-semiring identity $u \approx v$ with $u = \{u_1, \ldots, u_m\}$ and $v = \{v_1, \ldots, v_n\}$ is derivable from a given set Σ of ai-semiring identities it is sufficient to show that $u \approx u + v_j$ and $v \approx v + u_i$ are derivable from Σ for $u_i \in u$ and $v_j \in v$. Using a symmetry argument it is sufficient to show that $u \approx u + v_j$ is derivable from Σ for

$v_j \in v$. We will use this trick repeatedly in this thesis. The following lemmas will give characterization for every of the eleven three-element proper almost-idempotent semirings generating $\mathcal{L}(\mathbb{IA}_3)$.

Lemma 3.6. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \dots, m\}, \ v = \{v_j \mid j = 1, \dots, n\}.$$

Then the following are equivalent:

1. $S_6 \models u \approx v$,
2. $u \approx v$ *satisfies*

$$B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset \wedge C(u) = C(v) \tag{10}$$

or

$$B^h(u) \cap H(u) = B^h(v) \cap H(v) = \emptyset \wedge H(u) = H(v) \wedge B^h(u) = B^h(v), \tag{11}$$

3. $u \approx v$ *is a consequence of*

$$x + yz \approx xz + yz \tag{12}$$
$$x^2 + u + v \approx x^2 + uv \tag{13}$$

and the identities determining $\mathbb{IA}$.

Proof. $[(3) \Rightarrow (1)]$:
This is trivial since (12) and (13) are satisfied in S_6.
$[(1) \Rightarrow (2)]$:
Without loss of generality let $B^h(u) \cap H(u) \neq \emptyset$ and $B^h(v) \cap H(v) = \emptyset$. Evaluating every $x \in B^h(v)$ at 1 and every $x \in H(v)$ at 2 reduces v to 2 whereas u get reduced to 0 or 1 since there exists a product in u with head 1 or a 2 in it's body. $C(u) = C(v)$ is trivial since M_2 is a subsemiring of S_6.
So let $B^h(u) \cap H(u) = B^h(v) \cap H(v) = \emptyset$. Assume $x \in H(u)$ with $x \notin H(v)$. Evaluating every $y \in H(u)$ at 2 and every other variable at 1 we receive the contradiction $2 = 0$. Finally $B^h(u) = B^h(v)$ is a consequence of $H(u) = H(v)$ since $C(u) = C(v)$ holds.
$[(2) \Rightarrow (3)]$:
First note that

$$xy \approx xy + xy \approx xy + x \tag{14}$$

is a consequence of (12). Furthermore

$$xyz \approx xyz + x \approx xyz + xz \approx xy + xz \approx xy + xzy \approx x + xzy \approx xzy \tag{15}$$

is a consequence of (12) and (14). So assume $u \approx v$ satisfies (10) or (11). Let $p = x_1 \cdots x_k \in v, k \geq 1$. We have to show that $u \approx u + p$ is a consequence of (12), (13), (14), (15) and the identities determining $\mathbb{IA}$. We have two consider the following two cases:

Case 1. $B^h(u) \cap H(u) \neq \emptyset$. So with $t_1, t_3, s_i \in X^*$ and $t_2, r_i \in X^+$ we can write u as

$$\begin{aligned}
u &\approx u + zt_1 + t_2zt_3 + \sum_{i=1}^{k} r_i x_i s_i \\
&\approx u + z + zt_1 + t_2t_3z + \sum_{i=1}^{k} r_i x_i s_i && \text{by (14), (15)} \\
&\approx u + z^2 + zt_1 + t_2t_3z + \sum_{i=1}^{k} r_i x_i s_i && \text{by (12)} \\
&\approx u + z^2 + zt_1 + t_2t_3z + \sum_{i=1}^{k} r_i s_i + \sum_{i=1}^{k} x_i && \text{by (13)} \\
&\approx u + z^2 + zt_1 + t_2t_3z + \sum_{i=1}^{k} r_i s_i + x_1 \cdots x_k && \text{by (13)} \\
&\approx u + p.
\end{aligned}$$

Case 2. $H(u) = H(v) \wedge B^h(u) = B^h(v)$. So with $t, s_i \in X^*$ and $r_i \in X^+$ we can write u as

$$\begin{aligned}
u &\approx u + x_1t + \sum_{i=2}^{k} r_i x_i s_i \\
&\approx u + x_1t + x_1 + \sum_{i=2}^{k} r_i s_i x_i && \text{by (14), (15)} \\
&\approx u + x_1t + x_1 + \sum_{i=2}^{k} r_i s_i x_i + x_1 \cdots x_k && \text{by (12)} \\
&\approx u + p.
\end{aligned}$$

So $u \approx u + p$ can be derived and the statement is shown. □

Lemma 3.7. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \dots, m\}, \ v = \{v_j \mid j = 1, \dots, n\}.$$

Then the following are equivalent:

1. $S_3 \models u \approx v$,
2. $u \approx v$ *satisfies*

$$B^t(u) \cap T(u) \neq \emptyset \wedge B^t(v) \cap T(v) \neq \emptyset \wedge C(u) = C(v) \tag{16}$$

or

$$B^t(u) \cap T(u) = \emptyset \wedge B^t(v) \cap T(v) = \emptyset \wedge T(u) = T(v) \wedge B^t(u) = B^t(v), \tag{17}$$

3. $u \approx v$ is a consequence of

$$x + yz \approx yx + yz \tag{18}$$
$$x^2 + u + v \approx x^2 + uv \tag{13}$$

and the identities determining $\mathbb{IA}$.

Proof. Dually to proof of Lemma 3.6. □

Lemma 3.8. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \ldots, m\}, \; v = \{v_j \mid j = 1, \ldots, n\}.$$

Then the following are equivalent:

1. $S_7 \models u \approx v$

2. $u \approx v$ satisfies

$$\mathfrak{F}(u) = \mathfrak{F}(v) \tag{19}$$

and

$$C(u) = C(v). \tag{20}$$

Proof. $[(1) \Rightarrow (2)]$:
Let $u \approx v$ be satisfied in S_7. Without loss of generality assume $P \in \mathfrak{F}(u)$ with $P \notin \mathfrak{F}(v)$. Evaluating every variable in P at 1 and every other variable at 2 reduces every u_i to 1 and so reduces u to 1. On the other hand, at least one $v_j \in v$ gets reduced to 0 or 2 which consequently reduces v to 0 or 2. The statement $C(u) = C(v)$ is trivial since M_2 is a subsemiring of S_7.
$[(2) \Rightarrow (1)]$:
Assume that (19) and (20) are satisfied in $u \approx v$. Let $\varphi : P_{fin}(X^+) \to S_7$ be any freely chosen but fixed homomorphism from the free ai-semiring to S_7. We have to show that $\varphi(u) = \varphi(v)$ holds. If there exists $x \in C(u) = C(v)$ with $\varphi(x) = 0$ then $\varphi(u) = \varphi(v)$ is trivially satisfied since $0 \cdot x = 0 = x \cdot 0$ and $0 + x = 0 = x + 0$ hold in S_7. So assume there exists no $x \in C(u)$ with $\varphi(x) = 0$. Furthermore if $\varphi(x) = 2$ holds for every $x \in C(u) = C(v)$, then again $\varphi(u) = \varphi(v)$ is satisfied trivially since $2 \cdot 2 = 2$ and $2 + 2 = 2$ hold in S_7. So finally assume there exists a non-empty set $M \subseteq C(u)$ with $\varphi(x) = 1$ for every $x \in M$ and $\varphi(y) = 2$ for every $y \in C(u) \setminus M$. We have to distinguish two cases:

Case 1. $M \in \mathfrak{F}(u) = \mathfrak{F}(v)$. So for every $u_i \in u$ there exists exactly one $x \in u_i$ with $x \in M$ and $c_x(u_i) = 1$. Thus using commutativity of S_7

$$\varphi(u) = \varphi\left(\sum_{i=1}^{n} u_i\right) = \sum_{i=1}^{n} \varphi(u_i) = \sum_{i=1}^{n} 2^{|u_i|-1} \cdot 1 = \sum_{i=1}^{n} 1 = 1$$

holds for u. We get $\varphi(v) = 1$ by symmetry of the arguments.

Case 2. $M \notin \mathfrak{F}(u) = \mathfrak{F}(v)$. So either there exists $u_i \in u$ with $M \cap c(u_i) = \emptyset$ or there exist $t_1, t_2, t_3 \in X^*$ and $u_i = t_1 x t_2 y t_3 \in u$ with $x, y \in M$. In the first case $\varphi(u_i) = 2^{|u_i|} = 2$ holds since there exists at least one $u_j \in u$ with $M \cap c(u_j) \neq \emptyset$. By assumption we get $\varphi(u) = 0$ since $0 + 2 = 0$ and $1 + 2 = 0$ hold in S_7. In the second case notice that

$$\varphi(u_i) = \varphi(t_1) \cdot 1 \cdot \varphi(t_2) \cdot 1 \cdot \varphi(t_3) = 1 \cdot 1 \cdot \varphi(t_1) \cdot \varphi(t_2) \cdot \varphi(t_3) = 0 \cdot \varphi(t_1) \cdot \varphi(t_2) \cdot \varphi(t_3) = 0$$

holds and hence $\varphi(u) = 0$ is satisfied in either case. Since the same arguments hold for v by symmetry we get $\varphi(u) = 0 = \varphi(v)$.

So in either case $\varphi(u) = \varphi(v)$ is satisfied and $u \approx v$ holds in S_7. □

Lemma 3.9. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \dots, m\},\ v = \{v_j \mid j = 1, \dots, n\}.$$

Then the following are equivalent:

1. $S_{15} \models u \approx v$,
2. $u \approx v$ *satisfies*
$$C_{3+}(u) \neq \emptyset,\ C_{3+}(v) \neq \emptyset \tag{21}$$
or
$$C_{3+}(u) = C_{3+}(v) = \emptyset,\ C_2(u) = C_2(v),\ C_1(u) \setminus C_2(u) = C_1(v) \setminus C_2(v), \tag{22}$$
3. $u \approx v$ *is a consequence of*
$$xy \approx yx \tag{7}$$
$$xy + ab \approx xy + ab + ax \tag{23}$$
$$abc \approx abc + x \tag{24}$$
and the identities determining $\mathbb{IA}$.

Proof. $[(3) \Rightarrow (1)]$:
This is trivial since S_{15} is commutative and satisfies (23) and (24).
$[(1) \Rightarrow (2)]$:
Let $u \approx v$ be satisfied in S_{15}. Assume without loss of generality $C_{3+}(u) = \emptyset,\ C_{3+}(v) \neq \emptyset$. Evaluating all variables in u and v at 2 we receive $\varphi(v) = 0$ and $\varphi(u) = 1$ or $\varphi(u) = 2$ which is a contradiction. So let $C_{3+}(u) = \emptyset,\ C_{3+}(v) = \emptyset$ and without loss of generality $x \in C_2(u)$ with $x \notin C_2(v)$. Evaluating x at 1 and every other variable at 2 we receive $\varphi(u) = 0$ and $\varphi(v) = 1$ or $\varphi(v) = 2$. Finally without loss of generality choose $x \in C_1(u) \setminus C_2(u)$ with $x \notin C_1(v) \setminus C_2(v)$. Evaluating x at 0 and every other variable at 2 we receive $\varphi(u) = 0$ and $\varphi(v) = 1$ or $\varphi(v) = 2$.
$[(2) \Rightarrow (3)]$:
Let $u \approx v$ satisfy (21) then $u + p \approx u$ can trivially be derived for any $p \in v$ using only (24). So let $p \in v$ and $u \approx v$ satisfy (22). We will use commutativity without further notice.

Case 1. $p = x$. Then there exists either $u_i \in u$ with $u_i = x$ which is trivial or $u_i = xy$ for some $y \in X$. Thus we can write u as

$$u \approx u + xy$$

$$\begin{aligned}
&\approx u + xy + xy \\
&\approx u + xy + x^2 && \text{by (23)} \\
&\approx u + xy + x^2 + x && \text{by (9)} \\
&\approx u + p.
\end{aligned}$$

Case 2. $p = xy$. Thus there exist $t_1, t_2 \in X$ such that u can be written as

$$\begin{aligned}
u &\approx u + xt_1 + yt_2 \\
&\approx u + xt_1 + yt_2 + xy && \text{by (23)} \\
&\approx u + p.
\end{aligned}$$

Hence in either case $u + p \approx u$ is satisfied in S_{15} and the statement is shown. □

Lemma 3.10. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \dots, m\},\ v = \{v_j \mid j = 1, \dots, n\}.$$

Then the following are equivalent:

1. $S_{20} \models u \approx v$,
2. $u \approx v$ *satisfies*

$$C_{2+}(u) = C_{2+}(v),\ C_1(u) \setminus C_{2+}(u) = C_1(v) \setminus C_{2+}(v), \tag{25}$$

3. $u \approx v$ *is a consequence of*

$$xy \approx yx \tag{7}$$

$$xw + yz \approx xw + xyz \tag{26}$$

   and the identities determining $\mathbb{IA}$.

Proof. $[(3) \Rightarrow (1)]$:
This is trivial since S_{20} is commutative and satisfies (26).
$[(1) \Rightarrow (2)]$:
Let $x \in C_{2+}(u)$ with $x \notin C_{2+}(v)$. Evaluating x at 1 and every other variable at 2 we receive the contradiction $0 = 2$. Thus let $y \in C_1(u) \setminus C_{2+}(u)$ with $y \notin C_1(v) \setminus C_{2+}(v)$. Evaluating y at 0 and every other variable at 2 we receive the contradiction $0 = 2$.
$[(2) \Rightarrow (3)]$:
Let $u \approx v$ satisfy (25). We will distinguish two cases to show that $u + p \approx u$ holds in S_{20} for any $p \in v$.

Case 1. $p = x$. If $x \notin C_{2+}(v)$, then there exists $u_i \in u$ with $u_i = x$ which is trivial. So assume $x \in C_{2+}(v)$. Thus there exists $u_i \in u$ with $u_i = tx$ for some $t \in X^+$. Using (26) and $x^2 + x \approx x^2$ we can derive

$$\begin{aligned}
u_i &\approx tx \approx tx + tx \\
&\approx txx + tx \approx xx + tx && \text{by (26)} \\
&\approx xx + tx + x && \text{by (9)} \\
&\approx tx + x && \text{by (26)} \\
&\approx u_i + x.
\end{aligned}$$

Thus $u \approx u + p$ can be derived.

Case 2. $p = x_1 \cdots x_k$. Since $C_{2+}(u) = C_{2+}(v)$ yields we can without loss of generality write u as

$$u \approx u + x_1t_1 + x_2t_2 + \ldots + x_kt_k$$

with $t_1, t_2, \ldots, t_k \in X^+$. Using (26) repeatedly we derive

$$\begin{aligned} u &\approx u + x_1t_1 + x_2t_2 + \ldots + x_kt_k \\ &\approx u + x_1x_2t_1 + x_1t_1 + x_2t_2 + \ldots + x_kt_k \\ &\approx u + x_1x_2 + x_1t_1 + x_2t_2 + \ldots + x_kt_k \\ &\approx u + x_1x_2x_3 + x_1t_1 + x_2t_2 + \ldots + x_kt_k \\ &\approx \ldots \\ &\approx u + x_1x_2 \cdots x_k + x_1t_1 + x_2t_2 + \ldots + x_kt_k \\ &\approx u + p. \end{aligned}$$

So in either case $u \approx u + p$ can be derived and the statement is shown. □

Lemma 3.11. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \ldots, m\},\ v = \{v_j \mid j = 1, \ldots, n\}.$$

Then the following are equivalent:

1. $S_{21} \models u \approx v$,
2. $u \approx v$ *satisfies*
$$B^h(u) = B^h(v) \tag{27}$$
and
$$C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u) = C_1(v) \setminus C_{2+}(v) \cup H_{2+}(v), \tag{28}$$
3. $u \approx v$ *is a consequence of*
$$yx = y + x^2 \tag{29}$$
and the identities determining $\mathbb{IA}$.

Proof. $[(3) \Rightarrow (1)]$:
This is trivial since S_{21} satisfies (29).
$[(1) \Rightarrow (2)]$:
Let $u \approx v$ be satisfied in S_{21} and $x \in B^h(u)$ with $x \notin B^h(v)$. Evaluating x at 1 and every other variable at 2 we receive the contradiction $0 = 1$ if $x \in C(v)$ and $0 = 2$ else. So let $x \in C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u)$ with $x \notin C_1(v) \setminus C_{2+}(v) \cup H_{2+}(v)$. Evaluating x at 1 and every other variable at 2 we receive the contradiction $1 = 2$ if $x \notin C(v)$ and $1 = 0$ else.
$[(2) \Rightarrow (3)]$:
Assume that $u \approx v$ satisfies (27) and (28). We need to show that $u + p \approx u$ is derivable from (29) and the identities determining $\mathbb{IA}$ for any $p \in v$. First notice that $xy \approx xy + x$, $xy \approx xy + y$ and $yxz \approx yxz + x$ are all consequences of (29). We have to consider the following two cases:

Case 1. $p = x$.
If $x \in C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u)$, then there exists $u_i \in u$ with $u_i = x$ which is trivial or $u_i \in u$ with $u_i = xy$ and $y \in X^+$. Using (29) we derive

$$u_i \approx xy \approx xy + x \approx u_i + p.$$

Hence $u + p \approx u$ holds in S_{21}. Conversely assume $x \notin C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u)$. So there exists $u_i \in u$ with $u_i = t_1 x t_2$ and $t_1 \in X^+$, $t_2 \in X^*$. Using (29) we derive

$$u_i \approx t_1 x t_2 \approx t_1 x t_2 + x \approx u_i + p.$$

Case 2. $p = x_1 \cdots x_k$.
Since $u + p \approx u$ satisfies (28), there exists $u_i \in u$ with $u_i = x_1 t$ for some $t \in X^*$. Furthermore, as $u + p \approx u$ satisfies (27), there exists $u_{i_j} \in u$ with $u_{i_j} = s_j x_j t_j$ with $s_j \in X^+$ and $t_j \in X^*$ for every $j = 2, \ldots k$. Hence u can be written as

$$\begin{aligned}
u &\approx u + x_1 t + \sum_{i=2}^{k} s_i x_i t_i \\
&\approx u + x_1 t + x_1 + \sum_{i=2}^{k} s_i + \sum_{i=2}^{k} x_i^2 + \sum_{i=2}^{k} t_i^2 && \text{by (29)} \\
&\approx u + x_1 t + x_1 x_2 + \sum_{i=2}^{k} s_i + \sum_{i=3}^{k} x_i^2 + \sum_{i=2}^{k} t_i^2 && \text{by (29)} \\
&\approx u + x_1 t + x_1 x_2 x_3 + \sum_{i=2}^{k} s_i + \sum_{i=4}^{k} x_i^2 + \sum_{i=2}^{k} t_i^2 && \text{by (29)} \\
&\approx \ldots \\
&\approx u + x_1 t + x_1 \cdots x_k + \sum_{i=2}^{k} s_i + \sum_{i=2}^{k} t_i^2 && \text{by (29)} \\
&\approx u + p.
\end{aligned}$$

Thus $u + p \approx u$ can be derived and the statement is shown. □

Lemma 3.12. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \ldots, m\}, \; v = \{v_j \mid j = 1, \ldots, n\}.$$

Then the following are equivalent:

1. $S_{23} \models u \approx v$,
2. $u \approx v$ *satisfies*
$$B^t(u) = B^t(v) \tag{30}$$
and
$$C_1(u) \setminus C_{2+}(u) \cup T_{2+}(u) = C_1(v) \setminus C_{2+}(v) \cup T_{2+}(v), \tag{31}$$

3. $u \approx v$ *is a consequence of*

$$yx \approx y^2 + x \tag{32}$$

and the identities determining $\mathbb{IA}$.

Proof. Dually to proof of Lemma 3.11 □

Lemma 3.13.
$$\sum_{i=2}^{k}\sum_{j=1}^{i-1} x_i x_j \approx \prod_{i=1}^{k} x_i + \sum_{i=2}^{k}\sum_{j=1}^{i-1} x_i x_j \tag{33}$$

is a consequence of

$$x_2x_1 + x_3x_1 + x_3x_2 \approx x_2x_1 + x_3x_1 + x_3x_2 + x_1x_2x_3 \tag{34}$$

Proof. We will show the statement by induction. First note that the case $k = 3$ is given by (34). So propose that (33) it true for some fixed k. We will show that

$$\sum_{i=2}^{k+1}\sum_{j=1}^{i-1} x_i x_j \approx \prod_{i=1}^{k+1} x_i + \sum_{i=2}^{k+1}\sum_{j=1}^{i-1} x_i x_j$$

can be derived from (33) and (34).

$$\begin{aligned}
&\sum_{i=2}^{k+1}\sum_{j=1}^{i-1} x_i x_j \\
&\approx \sum_{i=1}^{k} x_{k+1}x_i + \sum_{i=2}^{k}\sum_{j=1}^{i-1} x_i x_j \\
&\approx \sum_{i=1}^{k} x_{k+1}x_i + \sum_{i=2}^{k}\sum_{j=1}^{i-1} x_i x_j + \prod_{i=1}^{k} x_i && \text{by (33)} \\
&\approx \sum_{i=1}^{k} x_{k+1}x_i + \sum_{i=2}^{k}\sum_{j=1}^{i-1} x_i x_j + x_k\prod_{i=1}^{k-1} x_i + x_{k+1}\prod_{i=1}^{k-1} x_i && \text{by (33)} \\
&\approx \sum_{i=2}^{k+1}\sum_{j=1}^{i-1} x_i x_j + \prod_{i=1}^{k+1} x_i && \text{by(33) and (34)}
\end{aligned}$$

□

Lemma 3.14. *Let* $u \approx v$ *be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \dots, m\},\ v = \{v_j \mid j = 1, \dots, n\}.$$

Then the following are equivalent:

1. $S_{24} \models u \approx v$

2. $u \approx v$ *satisfies*

$$\mathbb{Q}(u) = \mathbb{Q}(v), \tag{35}$$
$$Q(u) = Q(v), \tag{36}$$
$$C(u) = C(v). \tag{37}$$

3. $u \approx v$ is a consequence of

$$xy \approx yx, \tag{7}$$
$$xy \approx xy + x, \tag{14}$$
$$xy + xz + yz \approx xy + xz + yz + xyz, \tag{34}$$
$$x^2 + y \approx x^2 + xy \tag{38}$$

and the identities determining $\mathbb{IA}$.

Proof. $[(3) \Rightarrow (1)]$:
This is trivial since every equation in 3 is satisfied in S_{24}.
$[(1) \Rightarrow (2)]$:
Assume that $u \approx v$ is satisfied in S_{24}. Without loss of generality choose $\{x, y\} \in \mathbb{Q}(u)$ with $\{x, y\} \notin \mathbb{Q}(v)$. Evaluating x and y at 1 and every other variable at 2 we receive the contradiction $1 = 0$. The same point yields when choosing $x \in Q(u)$ with $x \notin Q(v)$. The statement $C(u) = C(v)$ is trivial since M_2 is a subsemiring of S_{24}.
$[(2) \Rightarrow (3)]$:
Assume that $\mathbb{Q}(u) = \mathbb{Q}(v) \wedge Q(u) = Q(v) \wedge C(u) = C(v)$ is satisfied for an ai-semiring identity $u \approx v$. It is sufficient to show that $u + p \approx u$ can be derived from (7), (14), (34), (38) and the identities determining $\mathbb{IA}$ for any $p \in v$. For $p = x_1 \cdots x_k, k \geq 1$ we have to consider three cases:

Case 1. $c(p) \cap (C(v) \setminus Q(v)) = \emptyset$:
So every variable in p is an element in $Q(v)$. Thus for $t_1, \ldots t_k \in X^*$ we can write u using the commutativity without further notice as

$$\begin{aligned}
u &\approx u + \sum_{i=1}^{k} t_i x_i^2 \\
&\approx u + \sum_{i=1}^{k} t_i x_i^2 + \sum_{i=1}^{k} x_i^2 && \text{by (14)} \\
&\approx u + \sum_{i=1}^{k} x_i^2 + x_1 \\
&\approx u + \sum_{i=1}^{k} x_i^2 + x_1 x_2 && \text{by (38)} \\
&\approx \ldots \\
&\approx u + \sum_{i=1}^{k} x_i^2 + x_1 x_2 \cdots x_k && \text{by (38)} \\
&\approx u + x_1 \cdots x_k \\
&\approx u + p
\end{aligned}$$

Case 2. $|c(p) \cap (C(v) \setminus Q(v))| = 1$:
So there exists exactly one variable in p that is not in $Q(v)$. Let without loss of generality be x_1 that variable. Thus for $t_1, \ldots t_k \in X^*$ we can write u using the

commutativity without further notice as

$$\begin{aligned}
u &\approx u + t_1x_1 + \sum_{i=2}^{k} t_ix_i^2 \\
&\approx u + t_1x_1 + x_1 + \sum_{i=2}^{k} t_ix_i^2 + \sum_{i=2}^{k} x_i^2 && \text{by (14)} \\
&\approx u + x_1 + \sum_{i=2}^{k} x_i^2 \\
&\approx u + x_1x_2 + \sum_{i=2}^{k} x_i^2 && \text{by (38)} \\
&\approx \dots \\
&\approx u + x_1x_2\cdots x_k + \sum_{i=2}^{k} x_i^2 && \text{by (38)} \\
&\approx u + p
\end{aligned}$$

Case 3. $|c(p) \cap (C(v) \setminus Q(v))| > 1:$
Let without loss of generality $x_{l+1} \dots x_k \in Q(v)$ be the variables in p that are elements in $Q(v)$. Thus for $t_{i,j}, t_i \in X^*$ we can write u using the commutativity without further notice as

$$\begin{aligned}
u &\approx \sum_{i=2}^{l}\sum_{j=1}^{i-1} t_{i,j}x_ix_j + \sum_{i=l+1}^{k} t_ix_i^2 \\
&\approx \sum_{i=2}^{l}\sum_{j=1}^{i-1} t_{i,j}x_ix_j + \sum_{i=2}^{l}\sum_{j=1}^{i-1} x_ix_j + \sum_{i=l+1}^{k} t_ix_i^2 + \sum_{i=l+1}^{k} x_i^2 && \text{by (14)} \\
&\approx \sum_{i=2}^{l}\sum_{j=1}^{i-1} x_ix_j + \sum_{i=l+1}^{k} x_i^2 \\
&\approx x_1x_2\cdots x_l + \sum_{i=2}^{l}\sum_{j=1}^{i-1} x_ix_j + \sum_{i=l+1}^{k} x_i^2 && \text{by (34) and Lemma 3.13} \\
&\approx x_1x_2\cdots x_l + \sum_{i=l+1}^{k} x_i^2 \\
&\approx x_1x_2\cdots x_lx_{l+1} + \sum_{i=l+1}^{k} x_i^2 && \text{by (38)} \\
&\approx \dots \\
&\approx x_1x_2\cdots x_lx_{l+1}\cdots x_k + \sum_{i=l+1}^{k} x_i^2 && \text{by (38)} \\
&\approx u + p
\end{aligned}$$

Hence in any case $u + p \approx u$ can be derived and the statement is shown. □

Lemma 3.15. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \dots, m\},\ v = \{v_j \mid j = 1, \dots, n\}.$$

Then the following are equivalent:

1. $S_{35} \models u \approx v$
2. $u \approx v$ *satisfies*
$$H_{2+}(u) = H_{2+}(v) \tag{39}$$
and
$$C_1(u) \setminus H_{2+}(u) = C_1(v) \setminus H_{2+}(v) \tag{40}$$
3. $u \approx v$ *is a consequence of*
$$xy \approx xz \tag{41}$$
and the identities determining $\mathbb{IA}$.

Proof. $[(3) \Rightarrow (1)]$:
This is trivial since (41) is satisfied in S_{35}.
$[(1) \Rightarrow (2)]$:
Assume that $u \approx v$ is satisfied in S_{35}. Let $x \in C_1(u) \setminus H_{2+}(u)$ with $x \notin C_1(v) \setminus H_{2+}(v)$. Evaluating x at 1 and every other variable at 2 we receive the contradiction $1 = 0$ if $x \in H_{2+}(v)$ or $1 = 2$ else. So let $x \in H_{2+}(u)$ with $x \notin H_{2+}(v)$. Evaluating x at 1 and every other variable at 2 we receive the contradiction $0 = 2$ if $x \notin C_1(v)$ and $0 = 1$ else.
$[(2) \Rightarrow (3)]$:
Assume that $u \approx v$ satisfies (39) and (40). We need to prove that $u + p \approx u$ can be derived from (41) and the identities determining $\mathbb{IA}$ for every $p \in v$. We distinguish two cases:

Case 1. $p = x$. If $x \notin H_{2+}(u)$ then there exists $u_i \in u$ with $u_i = x$ which is trivial. So assume $x \in H_{2+}(u)$. Thus there exists $u_i \in u$ with $u_i = xt$ for some $t \in X^+$ and u can be written as

$$u \approx u + xt \approx u + x^2 \approx u + x^2 + x \approx u + x \approx u + p.$$

Case 2. $p = x_1 \cdots x_k$. So there exists $u_i \in u$ with $u_i = x_1 t$ for some $t \in X^+$ and u can be written as

$$u \approx u + x_1 t \approx u + x_1 \cdots x_k \approx u + p.$$

So $u + p \approx u$ is derivable in either case and the statement is shown. □

Lemma 3.16. *Let $u \approx v$ be an ai-semiring identity with*

$$u = \{u_i \mid i = 1, \dots, m\},\ v = \{v_j \mid j = 1, \dots, n\}.$$

Then the following are equivalent:

1. $S_{22} \models u \approx v$

2. *$u \approx v$ satisfies*
$$T_{2+}(u) = T_{2+}(v) \tag{42}$$
and
$$C_1(u) \setminus T_{2+}(u) = C_1(v) \setminus T_{2+}(v) \tag{43}$$

3. *$u \approx v$ is a consequence of*
$$xy \approx zy \tag{44}$$
and the identities determining $\mathbb{IA}$.

Proof. Dually to proof of Lemma 3.15. □

Lemma 3.17. *Let $u \approx v$ be an ai-semiring identity with*
$$u = \{u_i \mid i = 1, \dots, m\},\ v = \{v_j \mid j = 1, \dots, n\}.$$
Then the following are equivalent:

1. *$S_{38} \models u \approx v$,*

2. *$u \approx v$ satisfies*
$$\mathbb{C}^{\perp}_{2+}(u) = \mathbb{C}^{\perp}_{2+}(v) \tag{45}$$
and
$$\{u_i \in u :\ |u_i| = 1,\ c(u_i) \notin \mathbb{C}^{\perp}_{2+}(u)\} = \{v_j \in v :\ |v_j| = 1,\ c(v_j) \notin \mathbb{C}^{\perp}_{2+}(v)\}. \tag{46}$$

3. *$u \approx v$ is a consequence of*
$$xy \approx yx \tag{7}$$
$$xy \approx xy + xyz, \tag{47}$$
and the identities determining $\mathbb{IA}$.

Proof. $[(3) \Rightarrow (1)]$:
This is trivial since S_{38} is commutative and (47) is satisfied in S_{38}.
$[(1) \Rightarrow (2)]$:
Let $u \approx v$ be satisfied in S_{38}. Assume without loss of generality that there exists $c(u_i) \in \mathbb{C}^{\perp}_{2+}(u)$ with $c(u_i) \notin \mathbb{C}^{\perp}_{2+}(v)$. So for every $v_j \in v$ there exists some $x \in v_j$ with $x \notin u_i$. Evaluating all variables in $c(u_i)$ at 1 and every other variable at 2 reduces u_i and consequently also u to 0, whereas v gets reduced to 1 or 2. Let without loss of generality $x \in \{u_i \in u :\ |u_i| = 1,\ c(u_i) \notin \mathbb{C}^{\perp}_{2+}(u)\}$ but $x \notin \{v_j \in v :\ |v_j| = 1,\ c(v_j) \notin \mathbb{C}^{\perp}_{2+}(v)\}$. Evaluating x at 1 and every other variable at 2 we receive the contradiction $1 = 2$ if $\{x\} \notin \mathbb{C}^{\perp}_{2+}(v)$ and $1 = 0$ otherwise.
$[(2) \Rightarrow (3)]$:
Notice that $x^2y \approx xy$ can be derived from (47) and (9):
$$x^2y \approx (x^2 + x)y \approx x^2y + xy \approx xy.$$
Assume that $u \approx v$ satisfies (45) and (46). We need to prove that $u + p \approx u$ can be derived from (47) and the identities determining $\mathbb{IA}$ for every $p \in v$. We distinguish two cases:

Case 1. $p = x$. Then there exists $u_i \in u$ with either $u_i = x$ or $u_i = x^l, l \geq 2$. Since $x^l + x \approx x^l$ holds in $\mathbb{IA}$ we can derive $u + p \approx u$ trivially.

Case 2. $p = x_1 \cdots x_k, k \geq 2$. If $c(p) \in \mathbb{C}^{\perp}_{2+}(v)$ then there exists $u_i \in u$ with $c(p) = c(u_i)$. Using $x^2y \approx xy$ we can derive $u + p \approx u$ easily. Otherwise if $c(p) \notin \mathbb{C}^{\perp}_{2+}(v)$ then there exists $u_i \in u$ with $c(u_i) \subset c(p)$ and $|u_i| \geq 2$. Using $xy \approx xy + xyz$ we can derive $u_i = u_i + p$ and hence $u + p \approx u$.

So in either case $u + p \approx u$ is derivable and the statement is shown. □

With the last lemma we characterized every semiring that generates new varieties in $\mathcal{L}(\mathbb{IA}_3)$. As already mentioned, there may be subvarieties of $\mathbb{IA}_3$ that are not generated by semirings of order 3, but those are not part of this research. Using these lemmas we will now introduce a context and finally prove that the context generates a lattice that is isomorph to $\mathcal{L}(\mathbb{IA}_3)$.

3.2. The complete lattice $\mathcal{L}(\mathbb{IA}_3)$

As we have seen we got eleven three-element proper almost-idempotent semirings that generated new varieties in $\mathcal{L}(\mathbb{IA}_3)$. Moreover L_2, R_2, M_2, D_2 and T_2 generated $\mathcal{L}(\mathbb{IA}_2) \subseteq \mathcal{L}(\mathbb{IA}_3)$. Finally, SL_2^0, SR_2^0, M_2^0, B and B^* (see Appendix B) are the only five three-element idempotent semirings introduced by Pastijn et al. Hence these 21 semirings generated the complete lattice $\mathcal{L}(\mathbb{IA}_3)$. Using these as attributes and 28 equations as objects we built a context (see Appendix D). Calculating the canonical basis in this context we receive the following 30 implications:

1. $M_2^0 \to M_2, D_2,$
2. $SL_2^0 \to L_2, D_2,$
3. $SR_2^0 \to R_2, D_2,$
4. $B \to L_2, M_2, D_2,$
5. $B^* \to R_2, M_2, D_2,$
6. $S_3 \to M_2, T_2,$
7. $S_6 \to M_2, T_2,$
8. $S_7 \to M_2, T_2,$
9. $S_{15} \to T_2,$
10. $S_{20} \to M_2, T_2,$
11. $S_{21} \to M_2, T_2,$
12. $S_{22} \to R_2, T_2,$
13. $S_{23} \to M_2, T_2,$
14. $S_{24} \to M_2, T_2,$

15. $S_{35} \to L_2, T_2$,
16. $S_{38} \to D_2, T_2$

that are trivially satisfied since the two-element semirings on the right side are subsemirings of the three-element semiring on the left side. Furthermore the implications

17. $S_{21}, S_{23}, M_2, T_2 \to S_{20}$,
18. $S_{24}, M_2, D_2, T_2 \to S_7$,

are self-dual whereas the implications

19. $S_{24,} L_2, M_2, T_2 \to S_6$,
20. $S_{21}, L_2, M_2, T_2 \to S_6$,
21. $S_7, L_2, M_2, T_2 \to S_6$,
22. $S_7, S_{21}, M_2, T_2 \to S_6$,
23. $S_6, S_{21}, S_{35}, L_2, M_2, T_2 \to S_{20}$,
24. $S_{21}, M_2, D_2, T_2 \to S_6$

are pairwise dual to

25. $S_{24}, R_2, M_2, T_2 \to S_3$,
26. $S_{23}, R_2, M_2, T_2 \to S_3$,
27. $S_7, R_2, M_2, T_2 \to S_3$,
28. $S_7, S_{23}, M_2, T_2 \to S_3$,
29. $S_3, S_{22}, S_{23}, R_2, M_2, T_2 \to S_{20}$,
30. $S_{23}, M_2, D_2, T_2 \to S_3$.

So proving implications (17) - (24) is sufficient to ensure that the concept lattice generated by the context would not change if additional equations were added. Hence in fact it is isomorphic to the lattice $\mathcal{L}(\mathbb{IA}_3)$. Note that we are done when proving the following implications

I. $S_{21}, S_{23} \to S_{20}$,

II. $S_{24}, D_2 \to S_7$,

III. $S_{24}, L_2 \to S_6$,

IV. $S_{21}, L_2 \to S_6$

V. $S_7, L_2 \to S_6$,

VI. $S_7, S_{21} \to S_6$,

VII. $S_{21}, S_{35} \to S_{20}$,

VIII. $S_{21}, D_2 \to S_6$,

since (17) - (24) are direct consequences of (I) - (VIII). From now on let $u \approx v$ be an ai-semiring identity with

$$u = \{u_i \mid i = 1, \ldots, m\},\ v = \{v_j \mid j = 1, \ldots, n\}.$$

Then the following lemmas prove exactly these implications.

Lemma 3.18. *$S_{20} \models u \approx v$ if $S_{21} \models u \approx v$ and $S_{23} \models u \approx v$.*

Proof. Let $u \approx v$ be satisfied in S_{21} and S_{23} hence

$$B^h(u) = B^h(v), \tag{27}$$
$$C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u) = C_1(v) \setminus C_{2+}(v) \cup H_{2+}(v), \tag{28}$$
$$B^t(u) = B^t(v), \tag{30}$$
$$C_1(u) \setminus C_{2+}(u) \cup T_{2+}(u) = C_1(v) \setminus C_{2+}(v) \cup T_{2+}(v) \tag{31}$$

are satisfied. We have to show that

$$C_{2+}(u) = C_{2+}(v),\ C_1(u) \setminus C_{2+}(u) = C_1(v) \setminus C_{2+}(v) \tag{25}$$

holds. First note that $C_{2+}(u) = B^h(u) \cup B^t(u) = B^h(v) \cup B^t(v) = C_{2+}(v)$ yields by (27) and (30). So assume $x \in C_1(u) \setminus C_{2+}(u)$. Hence especially $x \notin C_{2+}(u)$ and in consequence $x \notin H_{2+}(u)$ and $x \notin T_{2+}(u)$ hold. Thus $x \in C_1(v) \setminus C_{2+}(v)$ is a consequence of (28) respectively (31). □

Lemma 3.19. *$S_7 \models u \approx v$ if $S_{24} \models u \approx v$ and $D_2 \models u \approx v$.*

Proof. Let $u \approx v$ be an ai-identity which is satisfied in S_{24} and D_2. Then $u \approx v$ satisfies

$$\mathbb{Q}(u) = \mathbb{Q}(v), \tag{35}$$
$$Q(u) = Q(v), \tag{36}$$
$$C(u) = C(v), \tag{37}$$
$$\forall u_i \in u\ \exists v_j \in v : c(v_j) \subseteq c(u_i)\ \wedge\ \forall v_j \in v\ \exists u_i \in u : c(u_i) \subseteq c(v_j). \tag{48}$$

We have to show that $u \approx v$ satisfies

$$\mathfrak{F}(u) = \mathfrak{F}(v) \tag{19}$$

and

$$C(u) = C(v) \tag{20}$$

and thus is satisfied in S_7. Notice that (20) is trivially satisfied since it is equivalent to (37). So assume $P \in \mathfrak{F}(u)$ which implies $\nexists x, y \in P : \{x, y\} \in \mathbb{Q}(u) = \mathbb{Q}(v)$ and $P \cap Q(u) = P \cap Q(v) = \emptyset$ by definition. So for every $v_j \in v$ there exists at most one $x \in P$ with $x \in c(v_j)$. Notice that $\forall v_j \in v \exists c(t) \in c(u) : c(t) \subseteq c(v_j)$ holds since $u \approx v$ is satisfied in D_2. But as $P \in \mathfrak{F}(u)$ holds every $c(t) \in c(u)$ contains at least one $x \in P$. Therefore every $v_j \in v$ contains at least one $x \in P$. By former thoughts every $v_j \in v$ contains exactly one $x \in P$ and in consequence $P \in \mathfrak{F}(u)$ yields. Using a symmetry argument the statement $\mathfrak{F}(u) = \mathfrak{F}(v)$ is shown. Hence $u \approx v$ is satisfied in S_7. □

Lemma 3.20. *$S_6 \models u \approx v$ if $S_{24} \models u \approx v$ and $L_2 \models u \approx v$.*

Proof. Let $\approx v$ be satisfied in S_{24} and L_2. Thus $u \approx v$ satisfies

$$\mathbb{Q}(u) = \mathbb{Q}(v), \tag{35}$$
$$Q(u) = Q(v), \tag{36}$$
$$C(u) = C(v), \tag{37}$$
$$H(u) = H(v) \tag{49}$$

We have to show that

$$B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset \wedge C(u) = C(v) \tag{10}$$

or

$$B^h(u) \cap H(u) = \emptyset \wedge B^h(v) \cap H(v) = \emptyset \wedge H(u) = H(v) \wedge B^h(u) = B^h(v) \tag{11}$$

holds. If $B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset$ holds, then (10) is a direct consequence of (37). So without loss of generality assume $B^h(u) \cap H(u) = \emptyset$ and there exists $x \in B^h(v) \cap H(v) = B^h(v) \cap H(u)$. Thus either there exists $v_j \in v$ with $c_x(v_j) \geq 2$ which means $x \in Q(v)$ holds or $x \notin Q(v)$ yields and there exists $y \in H(v) = H(u)$ with $\{x, y\} \in \mathbb{Q}(v)$. Any case contradicts either (35) or (36). Hence $B^h(v) \cap H(v) = \emptyset$ is satisfied and (11) is a direct consequence. □

Lemma 3.21. *$S_6 \models u \approx v$ if $S_{21} \models u \approx v$ and $L_2 \models u \approx v$.*

Proof. Let $u \approx v$ be satisfied in S_{21} and L_2 hence

$$B^h(u) = B^h(v), \tag{27}$$
$$C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u) = C_1(v) \setminus C_{2+}(v) \cup H_{2+}(v), \tag{28}$$
$$H(u) = H(v) \tag{50}$$

are satisfied. First note that M_2 is a subsemiring of S_{21} hence $C(u) = C(v)$ is satisfied additionally. We have to show that $u \approx v$ satisfies

$$B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset \wedge C(u) = C(v) \tag{10}$$

or

$$B^h(u) \cap H(u) = B^h(v) \cap H(v) = \emptyset \wedge H(u) = H(v) \wedge B^h(u) = B^h(v). \tag{11}$$

This is trivial since $C(u) = H(u) \cup B^h(u) = H(v) \cup B^h(v) = C(v)$ yields. □

Lemma 3.22. *$S_6 \models u \approx v$ if $S_7 \models u \approx v$ and $L_2 \models u \approx v$.*

Proof. Let $u \approx v$ be satisfied in S_7 and L_2 hence

$$\mathfrak{F}(u) = \mathfrak{F}(v), \tag{19}$$
$$C(u) = C(v), \tag{20}$$
$$H(u) = H(v) \tag{51}$$

are satisfied. We will show that

$$B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset \wedge C(u) = C(v) \tag{10}$$

or

$$B^h(u) \cap H(u) = B^h(v) \cap H(v) = \emptyset \wedge H(u) = H(v) \wedge B^h(u) = B^h(v) \tag{11}$$

holds for $u \approx v$. Assume $H(u) \cap B^h(u) \neq \emptyset \wedge H(v) \cap B^h(v) \neq \emptyset$. Then the statement follows trivially by (20). So conversely assume $H(u) \cap B^h(u) = \emptyset$. Since $H(u)$ contains the first variable of every $u_i \in u$, we receive $H(u) \in \mathfrak{F}(u) = \mathfrak{F}(v)$. So $H(v) \in \mathfrak{F}(v)$ yields and by definition of $\mathfrak{F}(v)$ we get $H(v) \cap B^h(v) = \emptyset$. Finally note that $C(u) = H(u) \cup B^h(u)$ yields hence $B^h(u) = B^h(v)$ holds. □

Lemma 3.23. *$S_6 \models u \approx v$ if $S_7 \models u \approx v$ and $S_{21} \models u \approx v$.*

Proof. Let $u \approx v$ be satisfied in S_7 and S_{21}, hence

$$\mathfrak{F}(u) = \mathfrak{F}(v), \tag{19}$$
$$C(u) = C(v), \tag{20}$$
$$B^h(u) = B^h(v), \tag{27}$$
$$C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u) = C_1(v) \setminus C_{2+}(v) \cup H_{2+}(v) \tag{28}$$

are satisfied in $u \approx v$. We have to show that $u \approx v$ satisfies

$$B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset \wedge C(u) = C(v) \tag{10}$$

or

$$B^h(u) \cap H(u) = B^h(v) \cap H(v) = \emptyset \wedge H(u) = H(v) \wedge B^h(u) = B^h(v). \tag{11}$$

Assume $H(u) \cap B^h(u) \neq \emptyset$ and $H(v) \cap B^h(v) \neq \emptyset$. Then the statement follows trivially from (20).
Conversely, assume $H(u) \cap B^h(u) = \emptyset$. Since $H(u) \cup B^h(u) = C(u)$ yields for every semiring and there exists exactly one header in every $u_i \in u$, we receive $H(u) \in \mathfrak{F}(u) = \mathfrak{F}(v)$. So there exists exactly one $x_j \in H(u)$ with $c_{x_j}(v_j) = 1$ for every $v_j \in v$ by definition of $\mathfrak{F}(v)$. But since $x_j \notin B^h(u) = B^h(v)$ holds by (27), we receive $x_j \in H(v)$, hence $H(v) \supseteq H(u)$. But if $H(v) \supset H(u)$ was satisfied, then there exists $v_j \in v$ with $H(u) \cap c(v_j) = \emptyset$. This contradicts $H(u) \in \mathfrak{F}(v)$. Hence $H(u) = H(v)$ and finally $B^h(v) \cap H(v) = \emptyset$ hold. □

Lemma 3.24. *$S_{20} \models u \approx v$ if $S_{21} \models u \approx v$ and $S_{35} \models u \approx v$.*

Proof. Let $u \approx v$ be satisfied in S_{21} and S_{35}. Thus

$$B^h(u) = B^h(v), \tag{30}$$
$$C_1(u) \setminus C_2(u) \cup H_{2+}(u) = C_1(v) \setminus C_2(v) \cup H_{2+}(v), \tag{31}$$
$$H_{2+}(u) = H_{2+}(v), \tag{42}$$
$$C_1(u) \setminus H_{2+}(u) = C_1(v) \setminus H_{2+}(v) \tag{43}$$

hold. We have to show that

$$C_{2+}(u) = C_{2+}(v),\ C_1(u) \setminus C_{2+}(u) = C_1(v) \setminus C_{2+}(v) \tag{25}$$

is satisfied. Obviously

$$C_{2+}(u) = B^h(u) \cup H_{2+}(u) = B^h(v) \cup H_{2+}(v) = C_{2+}(v)$$

holds and with (43) we get

$$C_1(u) \setminus C_{2+}(u) = C_1(v) \setminus C_{2+}(v).$$

Hence $u \approx v$ is satisfied in S_{20}. □

Lemma 3.25. *$S_6 \models u \approx v$ if $S_{21} \models u \approx v$ and $D_2 \models u \approx v$.*

Proof. Let $u \approx v$ be satisfied in S_{21} and D_2 hence

$$B^h(u) = B^h(v), \tag{27}$$
$$C_1(u) \setminus C_{2+}(u) \cup H_{2+}(u) = C_1(v) \setminus C_{2+}(v) \cup H_{2+}(v), \tag{28}$$
$$\forall u_i \in u \; \exists v_j \in v : c(v_j) \subseteq c(u_i) \;\wedge\; \forall v_j \in v \; \exists u_i \in u : c(u_i) \subseteq c(v_j) \tag{52}$$

are satisfied. First note that M_2 is a subsemiring of S_{21} hence $C(u) = C(v)$ is satisfied additionally. We have to show that $u \approx v$ satisfies

$$B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset \wedge C(u) = C(v) \tag{10}$$

or

$$B^h(u) \cap H(u) = B^h(v) \cap H(v) = \emptyset \wedge H(u) = H(v) \wedge B^h(u) = B^h(v). \tag{11}$$

If $B^h(u) \cap H(u) \neq \emptyset \wedge B^h(v) \cap H(v) \neq \emptyset$ holds, then (10) follows trivially. So assume $B^h(u) \cap H(u) = \emptyset$. Obviously $B^h(u) = B^h(v)$ is satisfied and with $C(u) = C(v)$ we get $H(u) \subseteq H(v)$. So let $x \in H(v) \cap B^h(v)$ with $x \notin H(u)$. We have to distinguish two cases. If $x \in H_{2+}(v)$, then $x \in C_1(v) \setminus C_{2+}(v) \cup H_{2+}(v)$ holds which contradicts (28). So assume $x \in C_1(v)$. But since $u \approx v$ is satisfied in D_2, there exists $u_i \in u$ with $c(u_i) \subseteq \{x\}$. Hence $x \in C_1(u)$ which contradicts $B^h(u) \cap H(u) = \emptyset$. Thus $H(u) = H(v)$ and finally $B^h(v) \cap H(v) = \emptyset$ yield. □

Using former eight lemmas we ultimately showed that the lattice generated by the given context in fact is isomorphic to $\mathcal{L}(\mathbb{IA}_3)$. The lattice consequently consists of 19901 varieties[2]. Due to the size we relinquish including an image of the lattice.

Remark 3.26. Even though the focus of this chapter was to generate the lattice $\mathcal{L}(\mathbb{IA}_3)$, we found equational bases for some subvarieties of $\mathbb{IA}_3$ during this research. We will list them here without further proof. Let $\alpha, \beta \in X \cup \{\varepsilon\}$ then

1. $\mathcal{HSP}(S_{24}, S_{38}) = [x^2 + y \approx x^2 + xy, x^2y + ab \approx x^2y + ab + abx, xy + \alpha yz + \beta xz \approx xy + \alpha yz + \beta xz + xyz]$,

2. $\mathcal{HSP}(S_{21}, S_{23}) = [xy \approx xy + x + y, xy + yz \approx xy + yz + xyz, xyz + ab \approx xyz + ab + ayb, xy + ab \approx xy + ab + xb]$,

3. $\mathcal{HSP}(S_{21}, S_{38}) = [xyz \approx xzy, xy + ab \approx xy + ab + ayb, x\alpha + a^2b = x\alpha + ab + xab]$,

[2]The complete lattice and the amount of varieties can be calculated using programs like ConExp (See `http://conexp.sourceforge.net/` or `http://www.upriss.org.uk/fca/fcasoftware.html`).

4. $\mathcal{HSP}(S_{23}, S_{38}) = [xyz \approx yxz, xy + ab \approx xy + ab + axb, \alpha x + ab^2 = \alpha x + ab + abx]$,
5. $\mathcal{HSP}(S_{22}, S_{23}) = [xy \approx xy + y, xy + ab \approx xy + ab + axb]$,
6. $\mathcal{HSP}(S_{21}, S_{35}) = [xy \approx xy + x, xy + ab \approx xy + ab + ayb]$.

So in total we gave an equational basis for 16 of 19901 varieties. Finding bases for the remaining varieties is an open task for further research. Especially, the equational basis for $\mathcal{HSP}(S_7)$ turned out to be a tough nut. Maybe one even needs an infinite set of equations.

4. Structure of the lattice $\mathcal{L}(\mathbb{IA})$

In the former chapter we determined the lattice of varieties of almost-idempotent semirings with three elements. It is natural to use Formal Context Analysis to extend that lattice until the complete lattice $\mathcal{L}(\mathbb{IA})$ of varieties of almost-idempotent semirings is reached. The hope was that adding a few more semirings of higher order is sufficient to generate the complete lattice. But it turns out that the lattice is far bigger than expected.

Let $\mathbb{V}$ be the subvariety of $\mathbb{COM}$ characterized through the additional equation

$$xy \approx xy + x. \tag{14}$$

We will prove that the varieties $\mathbb{V}_2 = [x^3 \approx x^2], \mathbb{V}_3 = [x^4 \approx x^3], \mathbb{V}_4 = [x^5 \approx x^4], \ldots$ are all distinct subvarieties of $\mathbb{V}$ and thus form an infinite chain in $\mathcal{L}(\mathbb{V})$. Consequently, it seems reasonable to look at those subvarieties and to try to determine their complete lattice. But again we construct an infinite chain of subvarieties of $\mathbb{V}_n,\ n \geq 2$.

Example 4.1. The semirings S_7 and S_{38} are commutative but do not satisfy (14) whereas S_{15}, S_{20} and S_{24} are commutative and satisfy (14) and thus generate $\mathbb{IA}_3 \cap \mathbb{V}$. So $\mathbb{V}$ is a proper subvariety of $\mathbb{COM}$.
The so called max-plus algebra $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with

$$a \oplus b = \max(a, b)$$

and

$$a \odot b = a + b \text{ if } a, b \in \mathbb{R} \text{ and } -\infty \text{ otherwise}$$

is used for finding critical paths in graphs, in control or automata theory, for analysis of the behaviour of industrial processes, and in many other applications (cf. [12], Example 1.22). It contains the subsemiring $(\mathbb{N}, \oplus, \odot)$ which satisfies (14) and is commutative thus is a semiring in $\mathbb{V}$. Note that 0 is a multiplicatively and additively neutral element in that semiring. We relinquish 0 since we do not want a neutral element and receive the following semiring:

Lemma 4.2. *Let + be the ordinary addition. Then the algebra* $\mathcal{T} = (\mathbb{N}_+, \oplus, \odot)$ *with*

$$a \oplus b = max(a, b)$$

and

$$a \odot b = a + b$$

for every $a, b \in \mathbb{N}$ *is a semiring in* $\mathbb{V}$.

Proof. Obviously the maximum function is an idempotent, commutative and associative mapping from $\mathbb{N}_+ \times \mathbb{N}_+$ to $\mathbb{N}_+$. On the other hand $\odot : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{N}_+$ is commutative and associative since the ordinary addition is commutative and associative. It remains to show that one distributive law and (14) hold. We get

$$a \odot (b \oplus c) = a + \max(b, c)$$

$$
\begin{aligned}
&= \max(a+b, a+c) \\
&= \max(a \odot b, a \odot c) \\
&= a \odot b \oplus a \odot c
\end{aligned}
$$

and

$$a \odot b = a + b = \max(a+b, b) = \max(a \odot b, b) = a \odot b \oplus b$$

for every $a, b, c \in \mathbb{N}$, thus $\mathcal{T} \in \mathbb{V}$. □

Example 4.3. Let $n_1, n_2, n_3 \in \mathbb{N}$ with $n_1 < n_2 < n_3$ and $\rho \subseteq \mathbb{N}_+ \times \mathbb{N}_+$ a congruence on $\mathcal{T}$ with $(n_1, n_3) \in \rho$. Then $(n_1 \oplus n_2, n_3 \oplus n_2) = (\max(n_1, n_2), \max(n_3, n_2)) = (n_2, n_3)$ holds. Thus $(n_2, n_3) \in \rho$ and consequently $(n_1, n_2) \in \rho$ by transitivity, especially $(n_1, n_1{+}1) \in \rho$. Now assume $(n_1, n_1{+}1) \in \rho$ then $(1 \odot n_1, 1 \odot (n_1{+}1)) = (n_1{+}1, n_1{+}2) \in \rho$ and by transitivity $(n_1, n_1+2) \in \rho$. Repeating this step we finally get $(n_1, n_1+k) \in \rho$ for any $k \in \mathbb{N}$.

So for any fixed $n \in \mathbb{N}_+$ the least congruence relation ρ_n on $\mathcal{T}$ with $(n, n+1) \in \rho_n$ is the trivial relation on $\{n, n+1, n+2, \ldots\}$ and the identity relation on $\{1, 2, \ldots, n-1\}$ by former thoughts. We denote the subsemiring of $\mathcal{T}$ generated by ρ_n as $T_n = \mathcal{T}/\rho_n$. Thus T_n is a semiring of order n. T_1 is the trivial semiring whereas T_2 is the semiring mentioned in Example 2.11 and T_3 is isomorph to S_{15}. Since T_n is the factor algebra of $\mathcal{T}$ regarding to ρ_n it in particular is a semiring in $\mathbb{V}$.

Definition 4.4. Let $(S, +, \cdot)$ be a semiring and $\infty \in S$. We call ∞ an *absorbing element* or *absorbing* in S if $x \cdot \infty = \infty \cdot x = \infty$ and $x + \infty = \infty + x = \infty$ hold for any $x \in S$.

Lemma 4.5. *Let $n \in \mathbb{N}_+$ and T_n be the semiring introduced in former Example 4.3 then T_n satisfies*

$$\bigoplus_{i=1}^{n+1} \bigodot_{\substack{j=1\\ i\neq j}}^{n+1} x_j \approx \bigoplus_{i=1}^{n+1} \bigodot_{\substack{j=1\\ i\neq j}}^{n+1} x_j \oplus \bigodot_{j=1}^{n+1} x_j \tag{53}$$

but not

$$\bigoplus_{i=1}^{n} \bigodot_{\substack{j=1\\ i\neq j}}^{n} x_j \approx \bigoplus_{i=1}^{n} \bigodot_{\substack{j=1\\ i\neq j}}^{n} x_j \oplus \bigodot_{j=1}^{n} x_j \tag{54}$$

Proof. Assume $x_1, x_2, \ldots, x_k \in \{1, \ldots, n\}$ for some $k \in \mathbb{N}_+$. In $\mathcal{T}$ we calculate

$$\bigodot_{j=1}^{k} x_j = \bigodot_{j=1}^{k-1} x_j + x_k = \bigodot_{j=1}^{k-2} x_j + x_k + x_{k-1} = \ldots = \sum_{j=1}^{k} x_j.$$

Hence for $k = n-1$ respectively $k = n$ we receive

$$\bigodot_{j=1}^{n} x_j = \sum_{j=1}^{n} x_j \geq n$$

and

$$\bigodot_{j=1}^{n-1} x_j = \sum_{j=1}^{n-1} x_j \geq n-1.$$

Since ρ_n is the trivial congruence relation on $\{n, n+1, n+2, \ldots\}$ we get $\bigodot_{j=1}^{n} x_j = n$ in T_n regardless of the choices of x_j. But as n is absorbing in T_n the equation (53) holds in T_n since a product of at least n factors exists on both sides. On the other hand choose $x_j = 1$ for $j = 1, \ldots, n$. Then

$$\bigodot_{\substack{j=1 \\ i \neq j}}^{n} x_j = \sum_{j=1}^{n-1} x_j = n - 1$$

yields for any $i \in \{1, \ldots, n\}$ and we receive the inequality $n - 1 \neq n$ in (54). □

Lemma 4.6. *Let $n \in \mathbb{N}$ and T_n be the semiring introduced in Example 4.3 then T_n satisfies $x^{n+1} \approx x^n$ but not $x^n \approx x^{n-1}$.*

Proof. Substituting $x \in T_n$ for every variable x_j, $j = 1, \ldots, n+1$, in Lemma 4.5 we get this statement trivially. □

Remark 4.7. *Burnside semigroups* are semigroups that additionally satisfy $x^n \approx x^m$ with $m < n$ (cf. [13]). A semiring with a semilattice as additive and a Burnside semigroup as multiplicative reduct is called *semilattice-ordered Burnside semigroup.* The variety of semilattice-ordered Burnside semigroups satisfying $x^n \approx x^m$ is accordingly denoted by $\mathbf{Sr}(n, m)$. Equations of the form $x^{n+1} \approx x^n$ play a crucial role as shown. Thus introducing the subvarieties $\mathbb{V}_n = [x^{n+1} \approx x^n] = \mathbf{Sr}(n+1, n) \wedge \mathbb{V}$ of $\mathbb{V}$ we see that T_n is in $\mathbb{V}_n$ but not in $\mathbb{V}_{n-1}$. Hence the varieties $\{\mathbb{V}_n\}_{n \in \mathbb{N}_+}$ are all distinct and form the chain

$$\mathbb{I} \cap \mathbb{V} = \mathbb{V}_1 \subset \mathbb{V}_2 \subset \ldots \subset \mathbb{V}_n \subset \mathbb{V}$$

for any $n \in \mathbb{N}_+$. The variety $\mathbb{V}_n$ is a proper subvariety of $\mathbb{V}$ for any $n \in \mathbb{N}$ since $\mathcal{T}$ is in $\mathbb{V}$ but not in $\mathbb{V}_n$. This means that the variety $\mathbb{V}$ has countable infinitely many subvarieties. So determining the complete lattice $\mathcal{L}(\mathbb{V})$ through Formal Concept Analysis is a pointless undertaking.

The following lemmas helps us to inspect some structural properties of the chain $\{\mathbb{V}_n\}_{n \in \mathbb{N}_+}$:

Lemma 4.8. *Let S be an almost-idempotent semiring. Then $x^m + x^n \approx x^{max(n,m)}$ holds in S for $m, n \in \mathbb{N}$. Hence $x^2 \approx x$ can already be derived from $x^n \approx x$ with $n \geq 2$.*

Proof. Obviously the equation holds for $m = n$. So assume without loss of generality $m < n$ thus $n = max(m, n)$. We will show the statement via induction. For $m = 1$ and $n = 2$ the equation follows by definition of an almost-idempotent semiring. So assume that $x + x^n \approx x^n$ yields for a fixed $n \geq 2$. Hence

$$\begin{aligned}
x + x^{n+1} &\approx x + x x^n \\
&\approx x + x\left(x + x^n\right) \\
&\approx x + x^2 + x^{n+1} \\
&\approx x^2 + x^{n+1} \\
&\approx x\left(x + x^n\right) \\
&\approx x\left(x^n\right) \\
&\approx x^{n+1}
\end{aligned}$$

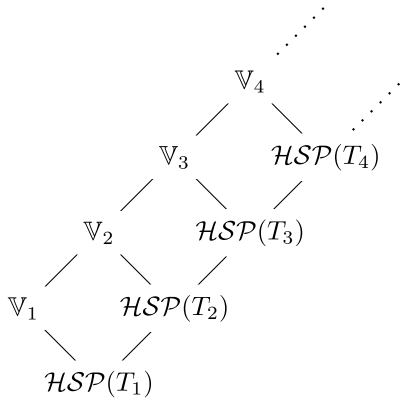


Fig. 1.: Chain $\{\mathbb{V}_n\}_{n\in\mathbb{N}_+}$

and the statement holds for $m = 1$ and any $n \in \mathbb{N}$. For $m > 1$ therefore

$$\begin{aligned} x^m + x^n &\approx x^{m-1}\left(x + x^{n-(m-1)}\right) \\ &\approx x^{m-1}x^{n-(m-1)} \\ &\approx x^n \end{aligned}$$

yields in S. Finally we derive $x^2 \approx x$ from $x^n \approx x$ with $n \geq 2$:

$$x \approx x^n \approx x^2 + x^n \approx x^2 + x \approx x^2.$$

□

To prove the following statement we adapt a technique that was used in [2] to prove Theorem 3.3.

Lemma 4.9. *Let $n \in \mathbb{N}_+$. Furthermore let $\mathbb{W}$ be a subvariety of $\mathbb{IA}$ that does not contain T_n. Then every semiring in $\mathbb{W}$ satisfies $x^{n-1} \approx x^n$, hence $\mathbb{W} \subseteq \mathbb{IA}\wedge[x^{n-1} \approx x^n]$.*

Proof. Let S be any semiring in $\mathbb{W}$ and $x \in S$. By Lemma 4.8

$$T = \{x, x^2, x^3, \dots\}$$

is a subsemiring of S. Since $x^k \neq x^m$ for all $k \neq m$ in $\mathbb{N}_+$ would imply $\mathcal{T} \in \mathbb{W}$ and hence $T_n = \mathcal{T}/\rho_n \in \mathbb{W}$ for all $n \in \mathbb{N}_+$, there is a smallest $k \in \mathbb{N}_+$ such that $x^k = x^m$ for some $m > k$. Now, Lemma 4.8 implies $x^k = x^{k+1}$, and thus $T_k = T \in \mathbb{W}$. Since $k \geq n$ would imply $T_n \in \mathbb{W}$, because T_n is a homomorphic image of T_k in that case, we have $x^k = x^{k+1}$ for some $k < n$, which immediately implies $x^{n-1} = x^n$. This shows that $x^{n-1} \approx x^n$ is satisfied in S. □

Remark 4.10. In consequence of former lemma the intervals $(\mathbb{V}_k, \mathbb{V}_{k+1})$ in Figure 1 are empty for any $k \in \mathbb{N}_+$. Note that this can be generalized to the intervals

$$(\mathbb{IA} \wedge [x^k \approx x^{k+1}], \mathbb{IA} \wedge [x^{k+1} \approx x^{k+2}])$$

but we used this notation in view of the upcoming results.

It seems reasonable to inspect the varieties $\mathbb{V}_n,\ n \geq 2$. Thus we will introduce a construction method for an infinite chain of semirings in any variety $\mathbb{V}_n,\ n \geq 2$. We will use this method to show that in contrast to the finite lattice $\mathcal{L}(\mathbb{V}_1)$ the lattice $\mathcal{L}(\mathbb{V}_n)$ has an infinite amount of varieties.

Lemma 4.11. *Let $(S, +, \cdot)$ be a semiring in $\mathbb{V}$ with $S = \{a_i \mid i \in I\}$ for any set of indices $I \neq \emptyset$. Furthermore let $\infty = a_{i_0} \in S$ be absorbing and $S' = \{a_i' \mid a_i \in S\}$ be a copy of S with $S' \cap S = \emptyset$. We continue the addition $+$ of S on $V(S) = S \cup S'$ through*

$$a_i \oplus a_j = a_i + a_j \tag{55}$$
$$a_i' \oplus a_j' = (a_i + a_j)' \tag{56}$$
$$a_i \oplus a_j' = a_j' \oplus a_i = a_i \tag{57}$$

and the multiplication of S on $V(S)$ through

$$a_i \odot a_j = \infty \tag{58}$$
$$a_i' \odot a_j' = (a_i \cdot a_j)' \tag{59}$$
$$a_i \odot a_j' = a_j' \odot a_i = a_i \cdot a_j \tag{60}$$

for every $i, j \in I$. Then $(V(S), \oplus, \odot)$ is a semiring in $\mathbb{V}$ with ∞ as absorbing element. If $x^{n+1} = x^n$ is satisfied for an $n \geq 2$ and every $x \in S$ then it is also satisfied for every $x \in V(S)$.

Proof. Since $(S, +)$ is idempotent, we receive idempotence of $(V(S), \oplus)$ by (55) and (56). Furthermore, commutativity of $(V(S), \oplus)$ is a consequence of commutativity of $(S, +)$ and (55) - (57).

Commutativity holds in $(V(S), \odot)$ since $(S, \cdot)$ is commutative and by (58) - (60). Consequently, we get $a_i \odot a_j' = a_i \cdot a_j = a_i' \odot a_j$ by (60). Finally, ∞ is absorbing in $(V(S), \odot)$ since it is absorbing in $(S, +, \cdot)$ and by (57), (58) and (60). Furthermore $(V(S), \odot)$ is commutative by (58) - (60).

By former thoughts the only remaining case to show that $x \odot y = x \odot y \oplus x$ holds for every $x, y \in V(S)$ is $x = a_i'$ and $y = a_j$. But by

$$x \odot y = a_i' \odot a_j = a_i \cdot a_j = a_i \cdot a_j + a_i = a_i \cdot a_j \oplus a_i = a_i' \odot a_j \oplus a_i = x \odot y \oplus x$$

we also proved that case.

By (56) and (59) $(S', \oplus, \odot)$ is a subsemiring of $(V(S), \oplus, \odot)$ that is isomorphic to $(S, +, \cdot)$. Furthermore $(S, \oplus)$ is isomorphic to the semigroup $(S, +)$ by (55). Moreover $(S, \oplus, \odot)$ is a zerosemiring with absorbing element ∞ by (58) and idempotence of ∞. Hence to prove that $\oplus$ is associative the following six cases remain:

$$x \oplus (y \oplus z') = x \oplus y = (x \oplus y) \oplus z'$$

$$x \oplus (y' \oplus z) = x \oplus z = (x \oplus y') \oplus z$$
$$x' \oplus (y \oplus z) = y \oplus z = (x' \oplus y) \oplus z$$
$$x \oplus (y' \oplus z') = x \oplus (y + z)' = x = x \oplus z' = (x \oplus y') \oplus z'$$
$$x' \oplus (y \oplus z') = x' \oplus y = y = y \oplus z' = (x' \oplus y) \oplus z'$$

$$x' \oplus (y' \oplus z) = x' \oplus z = z = (x+y)' \oplus z = (x' \oplus y') \oplus z$$

Similar the following six cases remain to show associativity of $\odot$:

$$x \odot (y \odot z') = x \odot (y \cdot z) = \infty = \infty \odot z' = (x \odot y) \odot z'$$

$$x \odot (y' \odot z) = x \odot (y \cdot z) = \infty = (x \cdot y) \odot z = (x \odot y') \odot z$$
$$x' \odot (y \odot z) = x' \odot \infty = \infty = (x \cdot y) \odot z = (x' \odot y) \odot z$$
$$x \odot (y' \odot z') = x \odot (y \cdot z)' = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (x \cdot y) \odot z' = (x \odot y') \odot z'$$
$$x' \odot (y \odot z') = x' \odot (y \cdot z) = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (x \cdot y) \odot z' = (x' \odot y) \odot z'$$
$$x' \odot (y' \odot z) = x' \odot (y \cdot z) = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (x \cdot y)' \odot z = (x' \odot y') \odot z$$

Finally, the following four cases prove distributivity and thus show that $(V(S), \oplus, \odot)$ in fact is a semiring in $\mathbb{V}$:

$$x \odot (y \oplus z') = x \odot y = \infty = \infty \oplus x \odot z' = x \odot y \oplus x \odot z'$$

$$x \odot (y' \oplus z') = x \odot (y+z)' = x \cdot (y+z) = x \cdot y + x \cdot z = x \cdot y \oplus x \cdot z = x \odot y' \oplus x \odot z'$$
$$x' \odot (y \oplus z) = x' \odot (y+z) = x \cdot (y+z) = x \cdot y + x \cdot z = x \cdot y \oplus x \cdot z = x' \odot y \oplus x' \odot z$$
$$x' \odot (y \oplus z') = x' \odot y = x \cdot y = x \cdot y \oplus (x \cdot z)' = x' \odot y \oplus x' \odot z'$$

Assume $x^{n+1} = x^n$ is satisfied for every $x \in S$ and a fixed $n \in \mathbb{N},\ n \geq 2$. By (59) we get

$$(a_i')^{n+1} = (a_i^{n+1})' = (a_i^n)' = (a_i')^n$$

for every $a_i' \in S' \subset V(S)$ and by (58) we get

$$(a_i)^{n+1} = \infty = a_i^n$$

for every $a_i \in S \subset V(S)$. Hence $x^{n+1} = x^n$ holds for every $x \in V(S)$. □

Lemma 4.12. *Let $S \in \mathbb{V}$ be a semiring and the semiring $V(S)$ constructed from S as described in Lemma 4.11. Furthermore let $\eta \notin V(S)$. We continue both operations of $V(S)$ on $S(\eta) = V(S) \cup \{\eta\}$ through*

$$a_i \oplus \eta = \eta \oplus a_i = a_i \tag{61}$$
$$a_i' \oplus \eta = \eta \oplus a_i' = \eta \tag{62}$$
$$\eta \oplus \eta = \eta \tag{63}$$

and

$$a_i \odot \eta = \eta \odot a_i = \infty \tag{64}$$
$$a_i' \odot \eta = \eta \odot a_i' = a_i \tag{65}$$
$$\eta \odot \eta = \infty \tag{66}$$

for every $a_i \in S$ and $a_i' \in S'$. Then $(S(\eta), \oplus, \odot)$ is a semiring in $\mathbb{V}$ with ∞ as absorbing element.
If $x^{n+1} = x^n$ is satisfied for an $n \geq 2$ and every $x \in S$ then it is also satisfied for every $x \in S(\eta)$.

Proof. Since $(V(S), \oplus, \odot)$ is commutative, we receive commutativity of $(S(\eta), \oplus)$ and $(S(\eta), \odot)$ by (61) - (66). Moreover, $(S(\eta), \oplus)$ is idempotent since $(V(S), \oplus)$ is idempotent and by (63).
The element ∞ is absorbing in $S(\eta)$ since $\infty \in S$ and (61) and (64).
To show that $x \odot y = x \odot y \oplus x$ is satisfied for every $x, y \in S(\eta)$ the following for cases are sufficient:

$$\begin{aligned}
&\eta \odot y = \infty = \infty \oplus \eta = \eta \odot y \oplus \eta && \text{for } y \in S \cup \{\eta\} \\
&\eta \odot a_i' = a_i = a_i \oplus \eta = \eta \odot a_i' \oplus \eta && \\
&x \odot \eta = \infty = \infty \oplus x = x \odot \eta \oplus x && \text{for } x \in S \cup \{\eta\} \\
&a_i' \odot \eta = a_i = a_i \oplus a_i' = a_i' \odot \eta \oplus a_i'. &&
\end{aligned}$$

By (61) and (63) we receive $(S \cup \{\eta\}, \oplus)$ through adjunction of η as neutral element to the semigroup $(S, \oplus)$, hence $(S \cup \{\eta\}, \oplus)$ again is a semigroup. Similar, by (62) and (63) we get $(S' \cup \{\eta\}, \oplus)$ through adjunction of an absorbing element η to the semigroup $(S', \oplus)$. In consequence $(S' \cup \{\eta\}, \oplus)$ again is a semigroup. By this and commutativity the following three cases remain to prove associativity of $(S(\eta), \oplus)$:

$$\begin{aligned}
&\eta \oplus (a_i \oplus a_j') = \eta \oplus a_i = a_i = a_i \oplus a_j' = (\eta \oplus a_i) \oplus a_j' \\
&\eta \oplus (a_i' \oplus a_j) = \eta \oplus a_j = (\eta \oplus a_i') \oplus a_j \\
&a_i \oplus (\eta \oplus a_j') = a_i \oplus \eta = a_i = a_i \oplus a_j' = (a_i \oplus \eta) \oplus a_j'.
\end{aligned}$$

To prove associativity of $(S(\eta), \odot)$ we have to distinguish the following cases:

$$\begin{aligned}
&\eta \odot (x \odot y) = \eta \odot \infty = \infty = \infty \odot y = (\eta \odot x) \odot y && \text{for } x, y \in S \cup \{\eta\} \\
&\eta \odot (\eta \odot a_j') = \eta \odot a_j = \infty = \infty \odot a_j' = (\eta \odot \eta) \odot a_j' && \\
&\eta \odot (a_i' \odot \eta) = \eta \odot a_i = a_i \odot \eta = (\eta \odot a_i') \odot \eta && \\
&\eta \odot (a_i \odot a_j') = \eta \odot (a_i \cdot a_j) = \infty = \infty \odot a_j' = (\eta \odot a_i) \odot a_j' && \\
&\eta \odot (a_i' \odot a_j) = \eta \odot (a_i \cdot a_j) = \infty = a_i \odot a_j = (\eta \odot a_i') \odot a_j && \\
&\eta \odot (a_i' \odot a_j') = \eta \odot (a_i \cdot a_j)' = a_i \cdot a_j = a_i \odot a_j' = (\eta \odot a_i') \odot a_j' && \\
&a_i \odot (\eta \odot a_j') = a_i \odot a_j = \infty = \infty \odot a_j' = (a_i \odot \eta) \odot a_j'. &&
\end{aligned}$$

Since $(S(\eta), \oplus)$ is idempotent – thus $x \odot (y \oplus y) = x \odot y = x \odot y \oplus x \odot y$ holds for every $x, y \in S(\eta)$ – and by commutativity, the following cases remain to show distributivity:

$$\begin{aligned}
&\eta \odot (x \oplus y) = \infty = \infty \oplus \infty = \eta \odot x \oplus \eta \odot y \qquad \text{for } x, y \in S \cup \{\eta\} \\
&\eta \odot (\eta \oplus a_j') = \eta \odot \eta = \infty\infty \oplus \eta = \eta \odot \eta \oplus \eta \odot a_j' \\
&\eta \odot (a_i \oplus a_j') = \eta \odot a_i = \infty = \infty \oplus a_j = \eta \odot a_i \oplus \eta \odot a_j' \\
&\eta \odot (a_i' \oplus a_j') = \eta \odot (a_i + a_j)' = a_i + a_j = a_i \oplus a_j = \eta \odot a_i' \oplus \eta \odot a_j' \\
&a_i \odot (\eta \oplus a_j) = a_i \odot a_j = \infty = a_i \odot \eta \oplus \infty = a_i \odot \eta \oplus a_i \odot a_j \\
&a_i \odot (\eta \oplus a_j') = a_i \odot \eta = \infty = \infty \oplus a_i \odot a_j' = a_i \odot \eta \oplus a_i \odot a_j' \\
&a_i' \odot (\eta \oplus a_j) = a_i' \odot a_j = a_i \cdot a_j = a_i + a_i \cdot a_j = a_i' \odot \eta \oplus a_i \cdot a_j = a_i' \odot \eta \oplus a_i' \odot a_j \\
&a_i' \odot (\eta \oplus a_j') = a_i' \odot \eta = a_i = a_i \oplus (a_i \cdot a_j)' = a_i' \odot \eta \oplus a_i' \odot a_j'.
\end{aligned}$$

Finally assume $x^{n+1} = x^n$ holds for a $n \in \mathbb{N}, n \geq 2$, and for every $x \in S$. Then the equation holds for every $x \in V(S)$ by Lemma 4.11 and for every $x \in S(\eta)$ by (66). $\square$

Example 4.13. Let $T_1 = \{0\}$ be the trivial semiring with $\infty = a_0 = 0$. With $\infty' = a_0' = 2$ we get the two-element mono-semiring $V(T_1) = M_2 = \{0, 2\}$ and with $\eta = 1$ the three-element semiring $T_1(\eta) = S_{20}$ characterized in Lemma 3.10.
If $T_2 = \{a_0 = 0 = \infty, a_1 = 1\}$ is the semiring introduced in [4]. With $I = \{0, 1\}$ and $T_2 = S = \{a_0 = 4, a_1 = 5 = \infty\}$ together with $S' = \{a_0' = 1, a_1' = 2\}$ as notation we get the semiring $V(T_2) = \{1, 2, 4, 5\}$ with following operation tables:

$+$	1	2	4	5
1	1	2	4	5
2	2	2	4	5
4	4	4	4	5
5	5	5	5	5

$\cdot$	1	2	4	5
1	2	2	5	5
2	2	2	5	5
4	5	5	5	5
5	5	5	5	5

With $\eta = 3$ we receive the semiring $T_2(\eta) = \{1, 2, 3, 4, 5\}$ with operation tables

$+$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	4	5
3	3	3	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

$\cdot$	1	2	3	4	5
1	2	2	4	5	5
2	2	2	5	5	5
3	4	5	5	5	5
4	5	5	5	5	5
5	5	5	5	5	5

Remark 4.14. Since $(S', \oplus, \odot) \cong (S, +, \cdot)$ is a subsemiring of $(V(S), \oplus, \odot)$ by construction and furthermore $(V(S), \oplus, \odot)$ is a subsemiring of $(S(\eta), \oplus, \odot)$ we get

$$\mathcal{HSP}(S) \subseteq \mathcal{HSP}(V(S)) \subseteq \mathcal{HSP}(S(\eta)).$$

For any subsemiring T of S obviously $V(T)$ is a subsemiring of $V(S)$ and $T(\eta)$ a subsemiring of $S(\eta)$.
If we start with $S \in V_k$ and iterate the former construction through

1. $S(\eta_1) = S(\eta)$ with $\eta_1 = \eta \notin V(S)$ and
2. $S(\eta_1, \ldots, \eta_{n-1}, \eta_n) = S(\eta_1, \ldots, \eta_{n-1})(\eta_n)$ with $\eta_n \notin V(S(\eta_1, \ldots, \eta_{n-1}))$

we receive

$$\mathcal{HSP}(S(\eta_1)) \subseteq \ldots \subseteq \mathcal{HSP}(S(\eta_1, \ldots, \eta_{n-1})) \subseteq \mathcal{HSP}(S(\eta_1, \ldots, \eta_n)).$$

Note that this is a chain of not necessarily distinct varieties in $\mathbb{V}_k$ for some fixed $k \in \mathbb{N}_+$.

We will now have a deeper look at the chain of semirings building up over T_n and the varieties generated by those. The following two lemmas will give us some information about validity of (67) in that chain.

Lemma 4.15. *Let $k \in \mathbb{N}$, $k \geq 2$ and $n \in \mathbb{N}$. Furthermore let $(T_k, +, \cdot)$ be the semiring introduced in Example 4.3 with absorbing element ∞. Then the subsemiring $(T_k(\eta_1, \ldots, \eta_{n-1}), \oplus, \odot)$ of $(T_k(\eta_1, \ldots, \eta_n), \oplus, \odot)$ is a zero-semiring with absorbing element ∞ and the equation*

$$\bigoplus_{i=1}^{k+n+1} \bigodot_{\substack{j=1 \\ i \neq j}}^{k+n+1} x_j \approx \bigoplus_{i=1}^{k+n+1} \bigodot_{\substack{j=1 \\ i \neq j}}^{k+n+1} x_j \oplus \bigodot_{j=1}^{k+n+1} x_j \tag{67}$$

is satisfied for any $x_j \in T_k(\eta_1, \ldots, \eta_n)$.

Proof. We write $S = T_k(\eta_1, \ldots, \eta_{n-1})$ and introduce a new notation to distinguish the operations of $(T_k(\eta_1, \ldots, \eta_{n-1}), \oplus, \odot)$ and $(T_k(\eta_1, \ldots, \eta_n), \oplus, \odot)$:

$$(S, \boxplus, \boxdot) = (S, \oplus, \odot).$$

We will show this statement via induction. The case $n = 0$ is given by Lemma 4.5. So assume that for some fixed $n-1$ instead of n equation (67) is satisfied in $(S, \boxplus, \boxdot)$. We have to show that (67) is satisfied in $(T_k(\eta_1, \ldots, \eta_n), \oplus, \odot)$.
Note that the subsemiring $(S', \oplus, \odot)$ of $(T_k(\eta_1, \ldots, \eta_n), \oplus, \odot)$ is isomorph to $(S, \boxplus, \boxdot)$. Hence by assumption (67) is satisfied for every evaluation φ with $\varphi(x_j) \in S'$ for every j. So assume there exist at least two variables x_{j_1}, x_{j_2} with $\varphi(x_{j_1}) \in S \cup \{\eta_n\}$ and $\varphi(x_{j_2}) \in S \cup \{\eta_n\}$. In this case there exists a product in (67) containing both variables thus by commutativity, (58), (64), (66) and the absorbing property of ∞ we receive that (67) is satisfied. So finally assume there exists exactly one variable that gets evaluated in S. Without loss of generality, we evaluate x_1 at $a_1 \in S$ and x_j at $a'_j \in S'$ for $j = 2, \ldots k+n+1$. Using the induction hypothesis we derive

$$\begin{aligned}
\varphi\left(\bigoplus_{i=1}^{k+n+1} \bigodot_{\substack{j=1\\ i\neq j}}^{k+n+1} x_j\right) &= a'_2 \odot \cdots \odot a'_{k+n+1} \oplus \bigoplus_{i=2}^{k+n+1} a_1 \bigodot_{\substack{j=2\\ i\neq j}}^{k+n+1} a'_j \\
&= a'_2 \odot \cdots \odot a'_{k+n+1} \oplus a_1 \left(\bigoplus_{i=2}^{k+n+1} \left(\bigodot_{\substack{j=2\\ i\neq j}}^{k+n+1} a'_j\right)\right) \\
&= a'_2 \odot \cdots \odot a'_{k+n+1} \oplus a_1 \left(\bigoplus_{i=2}^{k+n+1} \left(\boxdot_{\substack{j=2\\ i\neq j}}^{k+n+1} a_j\right)'\right) \\
&= a'_2 \odot \cdots \odot a'_{k+n+1} \oplus a_1 \left(\boxplus_{i=2}^{k+n+1} \boxdot_{\substack{j=2\\ i\neq j}}^{k+n+1} a_j\right)' \\
&= a'_2 \odot \cdots \odot a'_{k+n+1} \oplus a_1 \left(\boxplus_{i=2}^{k+n+1} \boxdot_{\substack{j=2\\ i\neq j}}^{k+n+1} a_j \boxplus \boxdot_{j=2}^{k+n+1} a_j\right)' \\
&= a'_2 \odot \cdots \odot a'_{k+n+1} \oplus a_1 \left(\bigoplus_{i=2}^{k+n+1} \left(\boxdot_{\substack{j=2\\ i\neq j}}^{k+n+1} a_j\right)' \oplus \left(\boxdot_{j=2}^{k+n+1} a_j\right)'\right) \\
&= a'_2 \odot \cdots \odot a'_{k+n+1} \oplus a_1 \left(\bigoplus_{i=2}^{k+n+1} \left(\bigodot_{\substack{j=2\\ i\neq j}}^{k+n+1} a'_j\right) \oplus \left(\bigodot_{j=2}^{k+n+1} a'_j\right)\right)
\end{aligned}$$

$$= a_2' \odot \cdots \odot a_{k+n+1}' \oplus \bigoplus_{i=2}^{k+n+1} a_1 \bigodot_{\substack{j=2\\i\neq j}}^{k+n+1} a_j' + a_1 \odot \left(\bigodot_{j=2}^{k+n+1} a_j' \right)$$

$$= \varphi \left(\bigoplus_{i=1}^{k+n+1} \bigodot_{\substack{j=1\\i\neq j}}^{k+n+1} x_j \oplus \bigodot_{j=1}^{k+n+1} x_j \right).$$

Note that the last case yields analogous for $\varphi(x_1) = \eta_n$. □

Lemma 4.16. *Let $k \in \mathbb{N}$, $k \geq 2$ and $n \in \mathbb{N}$. Furthermore let $(T_k, +, \cdot)$ be the semiring introduced in Example 4.3 with absorbing element $k = \infty$. Then the equation*

$$\bigoplus_{i=1}^{k+n} \bigodot_{\substack{j=1\\i\neq j}}^{k+n} x_j \approx \bigoplus_{i=1}^{k+n} \bigodot_{\substack{j=1\\i\neq j}}^{k+n} x_j \oplus \bigodot_{j=1}^{k+n} x_j \tag{68}$$

is not satisfied in $T_k(\eta_1, \ldots, \eta_n)$.

Proof. We will show this statement via induction. The case $n = 0$ is given by Lemma 4.5.
So by induction hypothesis there exits an evaluation φ that contradicts (68) in the subsemiring $(T_k(\eta_1, \ldots, \eta_{n-1})', \boxplus, \boxdot) = (T_k(\eta_1, \ldots, \eta_{n-1})', \oplus, \odot)$ of $(T_k(\eta_1, \ldots, \eta_n), \oplus, \odot)$ for a fixed $n-1$ instead of n. We will show that

$$\overline{\varphi}(x_i) = \begin{cases} \varphi(x_i) = a_i' \in T(\eta_1, \ldots, \eta_{n-1})', & i = 1, \ldots, k+n-1 \\ \eta_n, & i = k+n \end{cases}$$

contradicts (68) in $(T_k(\eta_1, \ldots, \eta_n), \oplus, \odot)$. First note that

$$\psi : a_i' \to \eta_n \odot a_i' = a_i$$

is a bijective mapping between $(T_k(\eta_1, \ldots, \eta_{n-1})', \oplus, \odot)$ and $(T_k(\eta_1, \ldots, \eta_{n-1}), \oplus, \odot)$. Thus

$$a_i = \psi(a_i') = \eta_n \odot a_i' \neq \eta_n \odot a_i' = \psi(a_j') = a_j \quad \Leftrightarrow \quad a_i' \neq a_j'$$

holds for any $a_i, a_j \in T_k(\eta_1, \ldots, \eta_{n-1})$. So we derive

$$\overline{\varphi} \left(\bigoplus_{i=1}^{k+n} \bigodot_{\substack{j=1\\i\neq j}}^{k+n} x_j \right) = \bigoplus_{i=1}^{k+n-1} \left(\overline{\varphi}(x_{k+n}) \bigodot_{\substack{j=1\\i\neq j}}^{k+n-1} \overline{\varphi}(x_j) \right) \oplus \bigodot_{i=1}^{k+n-1} \overline{\varphi}(x_i)$$

$$= \bigoplus_{i=1}^{k+n-1} \left(\eta_n \bigodot_{\substack{j=1\\i\neq j}}^{k+n-1} a_j' \right) \oplus \bigodot_{i=1}^{k+n-1} a_i'$$

$$= \eta_n \left(\bigoplus_{i=1}^{k+n-1} \bigodot_{\substack{j=1\\i\neq j}}^{k+n-1} a_j' \right) \oplus \left(\boxdot_{i=1}^{k+n-1} a_i \right)'$$

$$= \eta_n \left(\boxplus_{i=1}^{k+n-1} \boxdot_{\substack{j=1\\ i\neq j}}^{k+n-1} a_j \right)' \qquad \text{by (57)}$$

$$\neq \eta_n \left(\boxplus_{i=1}^{k+n-1} \boxdot_{\substack{j=1\\ i\neq j}}^{k+n-1} a_j \boxplus \boxdot_{i=1}^{k+n-1} x_i \right)' \qquad \text{by (68)}$$

$$= \eta_n \left(\boxplus_{i=1}^{k+n-1} \boxdot_{\substack{j=1\\ i\neq j}}^{k+n-1} a_j \boxplus \boxdot_{i=1}^{k+n-1} a_i \right)' \oplus \left(\boxdot_{i=1}^{k+n-1} a_i \right)' \qquad \text{by (57)}$$

$$= \eta_n \left(\bigoplus_{i=1}^{k+n-1} \bigodot_{\substack{j=1\\ i\neq j}}^{k+n-1} a'_j \oplus \bigodot_{i=1}^{k+n-1} a'_i \right) \oplus \bigodot_{i=1}^{k+n-1} a'_i$$

$$= \overline{\varphi}(x_{k+n}) \left(\bigoplus_{i=1}^{k+n-1} \bigodot_{\substack{j=1\\ i\neq j}}^{k+n-1} \overline{\varphi}(x_j) \oplus \bigodot_{i=1}^{k+n-1} \overline{\varphi}(x_i) \right) \oplus \bigodot_{i=1}^{k+n-1} \overline{\varphi}(x_i)$$

$$= \bigoplus_{i=1}^{k+n-1} \bigodot_{\substack{j=1\\ i\neq j}}^{k+n} \overline{\varphi}(x_j) \oplus \bigodot_{i=1}^{k+n} \overline{\varphi}(x_i) \oplus \bigodot_{i=1}^{k+n-1} \overline{\varphi}(x_i)$$

$$= \bigoplus_{i=1}^{k+n} \bigodot_{\substack{j=1\\ i\neq j}}^{k+n} \overline{\varphi}(x_j) \oplus \bigodot_{i=1}^{k+n} \overline{\varphi}(x_i)$$

$$= \overline{\varphi} \left(\bigoplus_{i=1}^{k+n} \bigodot_{\substack{j=1\\ i\neq j}}^{k+n} x_j \oplus \bigodot_{i=1}^{k+n} x_i \right).$$

Thus (68) is not satisfied in $T_k(\eta_1, \ldots, \eta_n)$. □

Remark 4.17. Using the notation $T_k^{(n)} = T_k(\eta_1, \eta_2, \ldots, \eta_n)$ we get $T_k^{(n)} \in \mathbb{V}_k$ for every $k, n \in \mathbb{N}_+$, $k > n$. Moreover, those semirings generate a countable infinite chain

$$\mathcal{HSP}(T_k) \subset \mathcal{HSP}(T_k^{(1)}) \subset \ldots \subset \mathcal{HSP}(T_k^{(n-1)}) \subset \mathcal{HSP}(T_k^{(n)}) \subset \mathbb{V}_k$$

of proper subvarieties in $\mathcal{L}(\mathbb{V}_k)$ for any k. This arises the question if

$$\mathcal{HSP}(T_k^{(n)}) = \mathcal{HSP}(T_2^{(n)}) \vee \mathcal{HSP}(T_k)$$

yields for any $k, n \in \mathbb{N}_+$ which is still open for further research.

5. Conclusion

In this thesis almost-idempotent semirings and their varieties were studied. After some Preliminaries about semirings and Formal Concept Analysis in chapter 2 we discussed almost-idempotent semirings with three elements. Those were generated by a python program and checked against each other for possible isomorphic images. Each of the remaining eleven non-isomorphic semirings was characterized in Lemma 3.6 to 3.17. In section 3.2 we built up the context of almost-idempotent semirings with three elements using 28 equations as attributes and proved its canonical basis. Thus the concept lattice – consisting of 19901 concepts – indeed is the complete lattice $\mathcal{L}(\mathbb{IA}_3)$ of varieties generated by almost-idempotent semirings of order three, but not necessarily the lattice of subvarieties of $\mathbb{IA}_3$.

Calculating all non-isomorphic almost-idempotent semirings with four elements failed due to the sheer amount of such semirings. So we turned our attention to commutative almost-idempotent semirings that additionally satisfy $xy \approx xy + x$ in chapter 4. In Example 4.3 we introduced a construction method for a chain of such and saw in Lemma 4.6 that the subvarieties $\mathbb{V}_k = [x^k \approx x^{k+1}]$ form an infinite chain in $\mathcal{L}(\mathbb{V})$. Consequently, we inspected the subvarieties $\mathbb{V}_k$ and introduced a second construction method for semirings. In Lemmas 4.15 and 4.16 we proved that the subvarieties of $\mathbb{V}_k$ generated by those semirings are all distinct, hence form an infinite chain of subvarieties in $\mathcal{L}(\mathbb{V}_2)$. In contrast, the variety $\mathbb{V}_1$ clearly is a subvariety of $\mathbb{I}$ and only has the variety of bisemilattices $\mathbf{M}$ and the trivial variety $\mathbf{T}$ as subvarieties (cf. [2]). In the end it turns out that the lattice $\mathcal{L}(\mathbb{IA})$ is far bigger than expected and leaves a lot of open questions for further research.

A. Additively idempotent semirings of order 2

L_2

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	1	1

R_2

+	0	1
0	0	1
1	1	1

·	0	1
0	0	1
1	0	1

M_2

+	0	1
0	0	1
1	1	1

·	0	1
0	0	1
1	1	1

D_2

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

N_2

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	0

T_2

+	0	1
0	0	1
1	1	1

·	0	1
0	1	1
1	1	1

B. Idempotent semirings of order 3

These are the five three-element idempotent semirings introduced by Pastijn et al. Obviously each of them is isomorphic to one semiring in Appendix C.

M_2^0

+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

is isomorphic to S_{40}. Note that M_2^0 is denoted by Bi_3 from time to time.

SL_2^0

+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

·	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

is isomorphic to S_{42}.

SR_2^0

+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

·	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

is isomorphic to S_{41}.

B

+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

·	0	1	2
0	0	0	0
1	1	1	1
2	0	1	2

is isomorphic to S_{39}.

B^*

+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

·	0	1	2
0	0	1	0
1	0	1	1
2	0	1	2

is isomorphic to S_{34}.

C. Almost-idempotent semirings of order 3

S_0

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

is commutative and satisfies (4), (5) and (6), hence $S_0 \in \mathbb{IA}_2$.

S_1

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

is commutative and satisfies (4), (5) and (6), hence $S_1 \in \mathbb{IA}_2$.

S_2

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	1	1	1
2	0	0	0

is dual to semiring S_4 and satisfies (4), (5) and (6), hence $S_2 \in \mathbb{IA}_2$.

S_3

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	0	1	2
2	0	0	0

is dual to S_6 and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_4

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	1	0
1	0	1	0
2	0	1	0

is dual to semiring S_2 and satisfies (4), (5) and (6), hence $S_4 \in \mathbb{IA}_2$.

S_5

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	1	0
1	1	1	1
2	0	1	0

is commutative and satisfies (4), (5) and (6), hence $S_5 \in \mathbb{IA}_2$.

S_6

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	0	1	0
2	0	2	0

is dual to semiring S_3 and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_7

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

is commutative and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_8

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

is idempotent and commutative, hence $S_8 \in \mathbb{I}$.

S_9

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	0
1	1	1	1
2	0	0	2

is dual to semiring S_{10} and idempotent, hence $S_9 \in \mathbb{I}$.

S_{10}

+	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

·	0	1	2
0	0	0	2
1	0	1	2
2	0	0	2

is dual to semiring S_9 and idempotent, hence $S_{10} \in \mathbb{I}$.

S_{11}

$+$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$\cdot$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

is commutative and idempotent, hence $S_{11} \in \mathbb{I}$.

S_{12}

$+$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$\cdot$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

is dual to semiring S_{13} and idempotent, hence $S_{12} \in \mathbb{I}$.

S_{13}

$+$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$\cdot$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

is dual to semiring S_{12} and idempotent, hence $S_{13} \in \mathbb{I}$.

S_{14}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

is commutative and satisfies (4), (5) and (6), hence $S_{14} \in \mathbb{IA}_2$.

S_{15}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

is commutative and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{16}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	1

is commutative and satisfies (4), (5) and (6), hence $S_{16} \in \mathbb{IA}_2$.

S_{17}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	1
1	0	1	1
2	0	1	1

is dual to semiring S_{18} and satisfies (4), (5) and (6), hence $S_{17} \in \mathbb{IA}_2$.

S_{18}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	1	1	1
2	1	1	1

is dual to semiring S_{17} and satisfies (4), (5) and (6), hence $S_{18} \in \mathbb{IA}_2$.

S_{19}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	1
1	1	1	1
2	1	1	1

is commutative and satisfies (4), (5) and (6), hence $S_{19} \in \mathbb{IA}_2$.

S_{20}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	0	0
2	0	0	2

is commutative and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{21}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	0	1
2	0	0	2

is dual to semiring S_{23} and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{22}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	2
1	0	0	2
2	0	0	2

is dual to semiring S_{35} and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{23}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	0	0
2	0	1	2

is dual to semiring S_{21} and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{24}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

is commutative and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{25}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

is commutative and idempotent, hence $S_{25} \in \mathbb{I}$.

S_{26}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	1
1	0	1	1
2	0	1	2

is dual to semiring S_{29} and idempotent, hence $S_{26} \in \mathbb{I}$.

S_{27}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

is dual to semiring S_{32} and idempotent, hence $S_{27} \in \mathbb{I}$.

S_{28}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

is dual to semiring S_{36} and idempotent, hence $S_{28} \in \mathbb{I}$.

S_{29}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	1	1	1
2	1	1	2

is dual to semiring S_{26} and idempotent, hence $S_{29} \in \mathbb{I}$.

S_{30}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	1
1	1	1	1
2	1	1	2

is commutative and idempotent, hence $S_{30} \in \mathbb{I}$.

S_{31}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	2
1	1	1	2
2	1	1	2

is dual to semiring S_{37} and idempotent, hence $S_{31} \in \mathbb{I}$.

S_{32}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

is dual to semiring S_{27} and idempotent, hence $S_{32} \in \mathbb{I}$.

S_{33}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

is commutative and idempotent, hence $S_{33} \in \mathbb{I}$.

S_{34}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	2
1	0	1	2
2	0	2	2

is dual to semiring S_{39} and idempotent, hence $S_{34} \in \mathbb{I}$.

S_{35}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	0
1	0	0	0
2	2	2	2

is dual to semiring S_{22} and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{36}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

is dual to semiring S_{28} and idempotent, hence $S_{36} \in \mathbb{I}$.

S_{37}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	1	1
1	1	1	1
2	2	2	2

is dual to semiring S_{31} and idempotent, hence $S_{37} \in \mathbb{I}$.

S_{38}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	2

is commutative and generates a new variety since it does not satisfy the equational basis of $\mathbb{IA}_2$.

S_{39}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	2	2	2

is dual to semiring S_{34} and idempotent, hence $S_{39} \in \mathbb{I}$.

S_{40}

$+$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\cdot$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

is commutative and idempotent, hence $S_{40} \in \mathbb{I}$.

S_{41}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	2
1	0	1	2
2	2	2	2

is dual to semiring S_{42} and idempotent, hence $S_{41} \in \mathbb{I}$.

S_{42}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	0	2
1	1	1	2
2	2	2	2

is dual to semiring S_{41} and idempotent, hence $S_{42} \in \mathbb{I}$.

S_{43}

+	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

·	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

is commutative and idempotent, hence $S_{43} \in \mathbb{I}$.

D. Context generating $\mathcal{L}(\mathbb{IA}_3)$

With equations

$$xy^2z \approx xyz \tag{69}$$
$$xy \approx xy + y \tag{70}$$
$$xy \approx xy + x \tag{71}$$
$$xyz + ab \approx xyz + a^2b \tag{72}$$
$$xyz + ab \approx xyz + ab^2 \tag{73}$$
$$xy + yx \approx xy + yx + x \tag{74}$$
$$x^3 + y \approx x^3 + y + xy \tag{75}$$
$$x^3 + y \approx x^3 + y + yx \tag{76}$$
$$y + x^3 \approx y + x^2 + xyx \tag{77}$$
$$a^2 + xyx + xzx \approx a^2 + xyx + xzx + xyzx \tag{78}$$
$$xy + xz + yz \approx xy + xz + yz + xyz \tag{79}$$
$$xy + xyzxy + yxzyx \approx yxy + xyzxy + yxzyx \tag{80}$$
$$yx + yxzyx + yxzxy \approx yxy + yxzyx + yxzxy \tag{81}$$
$$x + xyzx + xyzyx \approx x + xyzyx \tag{82}$$
$$x + xzyx + xyzyx \approx x + xyzyx \tag{83}$$
$$xyx + y \approx xyx + yx + y \tag{84}$$
$$xyx + y \approx xyx + xy + y \tag{85}$$
$$xa + yx^2z + x \approx xa + yx^2z + xyxz + x \tag{86}$$
$$ax + zx^2y + x \approx ax + zx^2y + zxyx + x \tag{87}$$
$$xyxzxyx + x^2 \approx xyxzxyx + xyx + x^2 \tag{88}$$
$$xyxzxyx + x \approx xyxzxyx + xyzyx + x^2 + x \tag{89}$$
$$x^2y \approx xyx \tag{90}$$
$$yx^2 \approx xyx \tag{91}$$
$$x + x^2 \approx x \tag{92}$$
$$xy^2x \approx y^2x + xyx \tag{93}$$
$$xy^2x \approx xy^2 + xyx \tag{94}$$
$$xy^2x \approx xy^2x + x \tag{95}$$
$$x^3 \approx x^3 + xyx \tag{96}$$

as objects and the semirings generating $\mathcal{L}(\mathbb{IA}_3)$ as attributes we receive the following context:

	S_3	S_6	S_7	S_{15}	S_{20}	S_{21}	S_{22}	S_{23}	S_{24}	S_{35}	S_{38}	M_2^0	SL_2^0	SR_2^0	B	B^*	L_2	R_2	M_2	D_2	T_2
(69)	✓	✓		✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(70)	✓			✓	✓	✓	✓	✓	✓									✓	✓		✓
(71)		✓		✓	✓	✓		✓	✓	✓							✓		✓		✓
(72)	✓			✓	✓		✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(73)		✓		✓	✓	✓	✓			✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(74)	✓	✓		✓	✓	✓	✓	✓	✓	✓							✓	✓	✓		✓
(75)	✓	✓	✓	✓				✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(76)	✓	✓	✓	✓		✓	✓		✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(77)	✓	✓	✓	✓			✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(78)	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(79)	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(80)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓
(81)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓
(82)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓
(83)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓
(84)	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(85)	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(86)	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(87)	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(88)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓
(89)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(90)		✓	✓	✓	✓	✓			✓	✓	✓	✓	✓		✓		✓		✓	✓	✓
(91)	✓		✓	✓	✓		✓	✓	✓		✓	✓		✓		✓		✓	✓	✓	✓
(92)												✓	✓	✓	✓	✓	✓	✓	✓	✓	
(93)	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓	✓		✓		✓		✓	✓	✓	✓
(94)	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓		✓		✓		✓	✓	✓
(95)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓							✓	✓	✓		✓
(96)				✓			✓			✓	✓		✓	✓			✓	✓		✓	✓

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

Die Hilfe eines Promotionsberaters habe ich nicht in Anspruch genommen. Weitere Personen haben von mir keine geldwerten Leistungen für Arbeiten erhalten, die nicht als solche kenntlich gemacht worden sind. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

1. Dezember 2017 — Dipl.-Math. Burkhard Michalski

Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such.

I did not seek the help of a professional doctorate-consultant. Only those persons identified as having done so received any financial payment from me for any work done for me. This thesis has not previously been published in the same or a similar form in Germany or abroad.

1st December 2017 Dipl.-Math. Burkhard Michalski

Glossary

Symbol	Description	Page
B^h	set of all variables occurring among the given set of words after the first variable of every word was removed	20
B^t	set of all variables occurring among the given set of words after the last variable of every word was removed	20
$\mathbb{C}^{\perp}$	set of all minimal contents over the words in the given term	20
$\mathbb{COM}$	variety of all commutative semirings in $\mathbb{SL}^+$	11
C_{k+}	set of all variables that occur in any word with length at least k in the given term	20
C_k	set of all variables that occur in any word with length k in the given term	20
C	set of all variables that occur in the given term	20
c	set of all variables that occur in the given word	19
c_x	number of occurrences of x in a given word	19
$\mathfrak{F}$	set of all possible sets of variables such that exactly one variable in each word of the given term is chosen	20
H	set of all variables occurring first from the left in any word of the given term	20
h	set containing the first variable from the left occurring in the given word	19
$\mathcal{HSP}$	smallest variety that still contains the given algebras	9
$\mathbb{IA}$	variety of all almost-idempotent semirings in $\mathbb{SL}^+$	11
$\mathbb{IA}_2$	variety generated by almost-idempotent semirings with two elements	22
$\mathbb{IA}_3$	variety generated by almost-idempotent semirings with three elements	19
$\mathbb{I}$	variety of all idempotent semirings in $\mathbb{SL}^+$	11
$\mathcal{L}$	complete lattice of subvarieties of the given variety	9
$\mathfrak{L}(\mathbb{K})$	set of all context implications in the context $\mathbb{K}$	15
$\mathbb{N}$	set of natural numbers – $0, 1, 2, 3, \ldots$	9
$\mathbb{N}_+$	set of all positive natural numbers – $1, 2, 3, \ldots$	9
$\mathbb{Q}$	set of pairs of variables that are not in Q but together in any word of the given term	20
Q	set of all variables occurring at least two times in a word of the given term	20
$\mathbf{S}_2$	variety generated by additively idempotent semirings with two elements	10
$\mathbb{SL}^+$	variety of all additively commutative and idempotent semirings	10
$\mathbb{SR}$	variety of all semirings	10

Symbol	**Description**	**Page**
T	set of all variables occurring last from the left in any word of the given term	20
t	set containing the last variable from the left occurring in the given word	19
$\mathbb{V}_n$	variety of all commutative semirings in $\mathbb{IA}$ that additionally satisfy $xy \approx xy + x$ and $x^n \approx x^{n+1}$	45
$\mathbb{V}$	variety of all commutative semirings in $\mathbb{IA}$ that additionally satisfy $xy \approx xy + x$	43

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