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Geometry of Deformed Special Relativity

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"Explanations exist: they have existed for all times, for there is always an easy solution to every problem-neat, plausible and wrong."

H.L. Mencken

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Abstract

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Geometry of Deformed Special Relativity

by Vuyile SIXABA

We undertake a study of the classical regime in which Planck's constant and Newton's gravitational constant are negligible, but not their ratio, the Planck mass, in hopes that this could possibly lead to testable quantum gravity (QG) effects in a classical regime. In this quest for QG phenomenology we consider modifications of the standard dispersion relation of a free particle known as deformed special relativity (DSR). We try to geometrize DSR to find the geometric origin of the space-time and momentum space. In particular, we adopt the framework of Hamilton geometry which is set up on phase space, as the cotangent bundle of configuration space in order to derive a purely phase space formulation of DSR. This is necessary when one wants to understand potential links of DSR with modifications of quantum mechanics such as Generalised Uncertainty Principles. It is subsequently observed that space-time and momentum space emerge naturally as curved and intertwined spaces. In conclusion we mention examples and applications of this framework as well as potential future developments.

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Part I

Deformed Special Relativity

Chapter 1

Introduction

The 20th century marked an era of great advancement in theoretical physics with the conception of the classical theory of gravity and the small scale quantum theory. There have been extensive research and observations for these theories respectively and they have revolutionalized our understanding of modern physics. General Relativity(GR) provides a comprehensive way of describing cosmological physics and similarly Quantum Mechanics (QM) gives an accurate description of atomic and subatomic physics. As astonishing as these theories are at describing the physics in their respective domains, they offer strikingly different pictures of physical reality in which the description of reality given by the two theories seems to be quite contradictory. Quantum mechanics and/or quantum field theory, a relativistic version of quantum mechanics, are formulated in a fixed background space, called Minkowski space and posses a probabilistic nature [1], whereas general relativity is completely backroung independent and is a classical theory which interprets gravity as a geometric property of spacetime. When applied to systems where neither one or the other is neglible, these theories turn out to be incompatible, as a result of the properties we have mentioned. Thus one requires, for such a system, a theory which reconciles both GR and QM in order to describe the physics of the system. Such a theory of quantizing gravity is termed *quantum gravity*. The search for a quantum theory of gravity has brought about new fields of research [2-4] such as string theory and loop quantum gravity to name a few. These fields aim to give a mathematical explanation of quantum gravity. However we currently do not posses sensitive enough intrumentation and/or apparatus to experimentally observe effects of quantum gravity. This restriction developed a new area of research called quantum gravity phenomenology in which a "bottom-up" approach is adopted [5].

There are certain complications which arise when one considers a quantum theory of gravity. There are 'factual' and there are conceptual difficulties with such a theory [6]. The factual hurdles include

1. The difficulty to test any proposal of quantum gravity due to the inaccessible regimes at which quantum gravity is expected to manifest. We do not currently have the technology to reach these regimes.

2. There does not seem to be agreement on what sort of data and predictions we might expect from such a theory. We do not know what to expect inside a black hole for example.

There are of course also the conceptual difficulties one may encounter in a theory of quantum gravity, such as

1. One might be faced with current conceptual obstacles that arise from the constituent theories of quantum gravity.

2. Also one will have to face the problem of conceptualising a theory that combines these two theories *viz* QM and GR.

With these problems in mind, we ask ourselves , how can one define a conceptual framework in a mathematical consistent language which could represent an unknown quantum theory of gravity for which we have no tangible experimental evidence [7]. Theories which tackle the problem head on such as string theory or loop quantum gravity are commonly refered to as "top-down" approaches, and are more prone to such difficulties.

A quantum theory of gravity suggests the existence of a minimal length scale in nature, see [8] and has predicted an essential modification of Special Relativity [9], the so called Deformed Special Relativity. The existence of such a minimal length motivates the origin of the general uncertainty principle and non-commutative geometries [10], [11]. One can obtain numerous motivations of the existence of a minimal length, such as black hole physics and or space-time fuzziness amongst others. The correspondence between length and momentum thus enforces a maximal momentum/energy corresponding to the minimal length scale. The energy required to probe a region above this maximal energy is less than the energy required to form a mini black hole in that region of sphere [8], Thus this means that any attempt to probe a phenomena above this maximal energy will lead to the formation of a black hole in that region of space, thus preventing any measurements in that region of space. In string theory, the minimal length is the order of magnitude of the oscillating strings that form elementary particles, such that lengths shorter than this do not make sense. In this thesis, we will be using the term minimal length scale l_p as the second observer invariant scale in Deformed Special Relativity with

$$l_p = 1.61622 \times 10^{-35} m \tag{1.1}$$

Deformed Special Relativity (DSR) is a proposal of how the theory of Special Relativity might experience drawbacks when energies close to the Planck energy E_p are considered. Planck energy is

$$E_p = 1.22 \times 10^{28} eV \tag{1.2}$$

The theory of DSR thus claims to be a theory which is valid in the semi-classical regime, which lies between a full theory of Quantum Gravity and the general relativistic regime. The idea of such a theory has been around for over 15 [96],[97] years and since its inception it has attracted quite a lot of attention as some believe qualitative predictios of DSR could be tested experimentally in the very near future. In DSR, in contrast to Special Relativity, one postulates the existence of two observer-independent scales, the speed of light c and a planck mass $\kappa \sim \frac{1}{l_p}$. The appearance of a minimal length in quantum gravity also results in the modification of the Heisenberg Uncertainty Principle (HUP) of quantum mechanics. The HUP is a principle which asserts to two conjugate observables that they cannot simultaneously be precisely measured. When one is measured with precision, we lose all information of the other. Due to the minimal length, an extra term in the right hand side of the HUP inequality is introduced and the result of this deformation is called Generalized Uncertainty Relation (GUP) which possesses a minimal variance for the measurement of distances [12].

1.1 Quantum Gravity

The problem of marrying Quantum Mechanics (QM) and General Relativity (GR) is one of the most difficult tasks of modern theoretical physics, which to this day, has not found a consistent and satisfactory solution. Our current understanding of gravity is based on Einstein's 1916 General Relativity. GR is a classical mechanics theory which does not take into consideration any quantum properties of particles. There have been numerous experimental tests and observations which enforced our trust and belief in GR, such as the motion of celestial bodies and predictions of black holes etc [6],[13]. In experiments concerning planetary motion, gravitational interaction is usually the dominant force over other forces, this is mainly due to the classical nature of planets. Planets are made up of elementary particles, each having its own energy/mass. Inspite of the fact that these elementary particles carry only a small amount of energy, in which quantum properties would come to play, say atoms, the additive nature of energy in forming celestial bodies results in the planet having a huge amount of energy/mass, in which quantum effects can be safely ignored. In the GR sense, the huge mass increases the curvature of spacetime around the mass, thus gravitation dominates over other interactions. Furthermore, as we have mentioned, a planet satisfies the conditions under which quantum theory is in the classical limit: in the description of the orbits of the planets the quantum properties of the composing particles can be safely neglected [14].

In the process of trying to reconcile these two major theories, QM and GR, there have been undeniable progress despite of the lack of a solution. Research in QG have sprung new theories over the years such as loop quantum gravity (LQG) and string theory to mention the most notable [3][4]. LQG tries to canonically quantize general relativity by starting from a regime in which $\hbar$ is negligible and jumping straight to quantum gravity regime. While on the other hand string theory involves the introduction of a new object, the string, and retains the explicit connection with both quantum theory and the low-energy description of spacetime [15]. String theory starts by [15] neglecting G and aims to arrive at a regime where there is quantum gravity and all interactions unified. Huge mathematical progress has been obtained from both these theories and others, however they are still faced with conceptual and experimental difficulties. No one has been able to use these theories to conduct experiments and observe effects of quantum gravity. So naturally, one may ask whether there is a theory which is conceptually and experimentally accessible to us, that can be somehow taken as a starting point in the search for the quantum gravity theory.

There has been a lot of attention directed at trying to find observable evidence for the quantization of gravity by developing phenomenological models. Such models will be able to quantify possible quantum gravitational effects and can ideally be tested experimentally. This subfield of quantum gravity, which focuses on finding these models is termed *Quantum Gravity Phenomenology*. As an example for such phenomenological models, there has been a growing interest in a regime in which both $\hbar$ and G are negligible, however their ratio $\sqrt{\frac{\hbar}{G}}$ is constant. This ratio is a unit we have encounted already, and we call it the Planck mass. This model is in a relativistic setting i.e we take $c = 1$, with the regime being termed Deformed Special Relativity (DSR), which formaly is the regime in which [16]

$$\hbar \to 0 \ , G \to 0, \tag{1.3}$$

with

$$\frac{h}{G} \tag{1.4}$$

fixed. There is a growing hope that a theory of DSR would become neccesary and sufficient for describing relativistic particles with high momenta travelling in spacetime with negligible gravity. As is, it currently seems like it would be a hard task to derive the complete form of a DSR theory from first principles, thus it is hoped that soon DSR will be able to be derived as an appropriate limit of loop

quantum gravity [16]
This thesis makes an attempt at geometrizing this theory of DSR and hopefully get a clearer understanding of multiple quantum gravity phenomenological models.

1.2 Quantum Gravity Phenomenology

As we have been reiterating, the search for a quantum theory of gravity has been one of the main aims of theoretical physics for many years now. However the efforts in this direction have been often hindered by the lack of experimental or observational tests able to select among, or at least be able to constrain, the numerous quantum gravity models proposed so far. The last decades have witnessed an increasing number of ideas about observable phenomena where QG could play a key role [8],[16],[17]. The following is a partial list of the phenomena of the possible observable quantum gravity and include the loss of quantum coherence or rather quantum decoherence (the wave-like property of quantum objects), deviations from Newton's laws, black holes produced in colliders (large hadron colliders) and violation of spacetime symmetries to name a few.

Quantum gravity phenomenology is a broad field encompassing many possible effects arising from a more fundamental description of spacetime such as the phenomena we listed above including cosmological perturbations. Many but not all of these phenomena arise from the addition of a Planck scale. In order to have a permissible quantum theory of gravity, such a theory must be able to make obvious predictions that in principle can be tested by experiments. This is where quantum gravity phenomenology comes in, as a "bottom-up" approach to a quantum theory of gravity. Contrary to popular belief, loop quantum gravity does make definite predictions such as that of measuring any physical area (e.g. total cross-section of a scattering process) and predicting that it must be given by the discrete spectrum of area [19]. Also in [20], there are proposed possible effects of angle quantization on scattering. The more pressing issue here is that these predictions demand experiments or observations probing the Planck scale, which as we have mentioned earlier is currently out of reach. However, the existence of Planck stars, which are structures considered to replace the singularity in the traditional definition of a black hole, could produce detectable signals and the phenomenological consequences of Planck stars in gamma-ray bursts have been studied in [21]. Planck stars are thought of as allowing general relativity to come back into play in a quantum gravty regime (inside a black hole). Instead of a singularity in a black hole, which "destroys" information, the information is rather stored in a Planck star and as a Planck star expandes to a point where it passes the event horizon, the black hole disintegrates and the information is released back into the universe While Lorentz invariance can be made manifest in LQG, additionally in the spin foam theory [22], many other models suggest the violation of Lorentz invariance in the Planck regime which will modify the standard Lorentzian energy-momentum dispersion relation at lower-energy scales. Observations of Ultra High Energy Cosmic Rays (UHECR), the most energetic particles that have been observed thus far, put strong constraints on deviations from the Lorentzian mass-shell relation. In the future, with improved precision of equipment, further investigation of such particles will either provide stronger constraining evidence or reveal the breakdown of the Lorentz invariance and enable theorists to disqualify different models in quantum gravity phenomenology.

As we have mentioned Quantum gravity phenomenology requires of course a combination of theory and experiments. It does not however adopt any particular preconcieved opinion concerning the structure of spacetime at short distances like the mathematically deserving string theory, LQG or non-commutative geometry. Although it must follow closely the few indications that these theories have and will provide (e.g. minimal length in string theory). We are thus guided by our expectation that quantum gravity research should actually proceed in small incremental steps starting from what we currently know and combining mathematical physics studies with experimental studies to reach deeper layers of understanding of the problem of quantizing gravity.

Experimental Search and Predictions

Neutrino Physics

Neutrinos are one of the fundemental particles which make up the universe. They are similar to the more familiar electron, with one major difference in that neutrinos do not carry charge i.e. they are electrically neutral and thus not affected by the electromagnetic forces that act on electrons. These particles are interesting for the purpose of testing weak quantum gravitational effects. One reason being, they have a very small mass. Another being, their weak interaction which enables them even with high energies to travel long, possibly cosmological distances without being disturbed [23]. Depending on their source, one can categorize neutrino experiments and these include Earth based neutrinos (like those from reactors or colliders), solar neutrinos, atmospheric neutrinos and cosmogenic neutrinos to name a few [24]. Cosmogenic neutrinos [25] are those that can reach the highest energies and longest travel times, with their flux being small at high energies such that with these neutrinos, collecting useful data is difficult. It has however been suggested in [26], to use cosmogenic neutrinos to strengthen bounds on Lorentz Invariance Violations (LIV) with experiments in the near future. LIV is the violation of the standard Special Relativity minkowski Lorentz Invariance. Lorentz transformations include boosts and rotations. One can learn more about LI in any Special Relativity undergraduate physics textbook. In [27], it has been pointed out that one can use cosmogenic neutrinos to possibly test the modification of the disperion relation to high precisions if a baseline of $L \sim \pi m_p^4/E^5$ [24] could be reached, where m_p is the Planck mass. This is where one would find, the flux-ratio of different neutrino species would become sensitive to the distance. It has also been proposed to actually combine neutrino measurements with the detection of gamma-ray bursts to better constrain modified dispersion relations in DSR and LIV [27]

Astrophysical

In astrophysics, there are astrophysical processes which occur at higher energies than is currently accessible to human experiments like particle colliders, moreover the less controlled experimental environments these processes occur in adds more uncertainty in these experiments. Due to this, astrophysical and collider constraints often complement each other [23]. There are some similarities on the constraints coming from ultra high energetic cosmic rays (UHECR) and collider phyics, however additional theoretical and experimental uncertainties come into play in UHECR due to higher energies.

One of the most studied predictions of DSR, which we will see in an experiment in the following chapter, is that of an energy-dependence in the arrival time of higly energetic photons from distant γ-ray bursts (GRB) [23]. One can additional observe such an effect from Lorentz Invariance violation (LIV), however models steming from LIV are constrained already, whereas there are no such constraints in DSR, thus in [28], dispersion relation modifications of first order in the parameter α are considered. This time delay ΔT one expects between two photons with an energy difference ΔE for such a first order modification of the standard dispersion relation is given by

$$\Delta T = L\frac{E_\gamma}{M_p} + \mathcal{O}\left(\frac{E_\gamma^2}{M_p^2}\right) \tag{1.5}$$

where E_γ is the energy of the more energetic photon, M_p the Planck-mass and L is the distance travelled by the photons. In 2, inserting values in this equation we will find that this time delay can be of order of seconds for a certain distance and a specific value α. GRB090510 came with one of the best limits for α to date, which is $\alpha < 1.2$, for the case in which higher energetic photons are slowed down [29]. There has been a few suggestions [30][31], who assert that a modified dispersion relation of LIV and DSR may be tested with weak gravitational lensing. However, we acknowledge that this effect is out of possible precision currently.

1.3 Motivating Deformed Special Relativity

In 1905, Albert Einstein proposed a theory of Special Relativity (SR) [32]. This was and still is a generally accepted and experimentally confirmed theory perfectly describes the relationship between space and time. In special relativity, the concept of absolute time is abondoned and time is treated on equal footing as space. SR has been most successful in explaining all physical phenomena in flat space and high speeds. Einstein postulated a universal constant c, the speed of light in his theory and asserted that no particle can exceed this speed and all observers in a flat space-time would observe this quantity to be the same at all times. Deformed Special Relativity (DSR) is a proposal of how Einstein's theory of Relativity might experience changes when high energies are considered. DSR therefore claims to be a theory which is valid in the semi-classical regime, which lies between a full theory of Quantum Gravity and the general relativistic regime. In physics, we have what are referred to as fundamental constants. These are the physical quanitites that stay invariant and constant with time under any circumstances. Those of importance to us are: the speed of light c, Newton's gravitational constant G and Planck's constant $\hbar$. Consider a physical system which is characterized by the length l, time t and mass m [33]. One can use (l, t, m) to define quantities that asymptotically classify the space of physical regimes, the characteristic velocity $v = \frac{l}{t}$, the action $s = \frac{ml^2}{t}$ and characteristic position $x = \frac{l^3}{mt^2}$. For example, consider the region when the speed of the physical system v is comparable to that of the speed of light c. If in this region we have a flat background and no quantum effects i.e $G \to 0$ and $\hbar \to 0$, then we can expect Special Relativity to fully explain physical phenomena in such a region. Similarly, Quantum field theory (QFT) will describe regions where we have relativistic and quantum effects i.e $v \sim c$ and an arbitrary $\hbar$. Fig 1.1 below was obtained from [33] and attempts to describe the asymptotic physical regimes on an (l, t, m) space. The region described by Deformed Special Relativity is the region of constant Planck mass $\kappa = \sqrt{\frac{\hbar}{G}}$, where $G \to 0$ and $\hbar \to 0$.

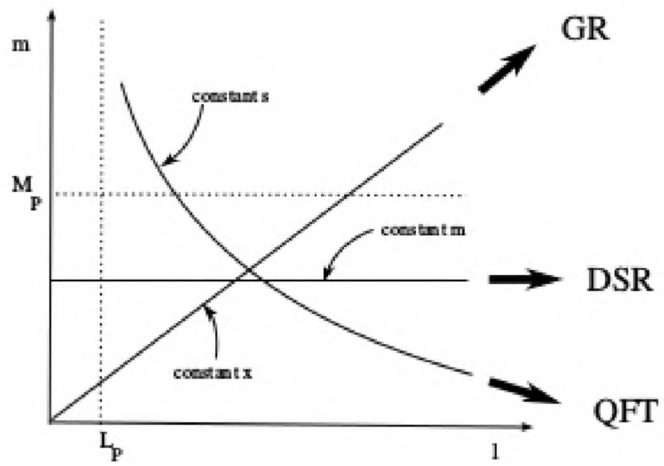

FIGURE 1.1: Asymptotic regions of the space of physical regimes, obtained from [33] (pardon the resolution, as it is at its best)

In the heart of a DSR theory lies the concept of this second invariant constant κ which as we will later see, has far-reaching consequences. Extending Special

Relativity from one invariant constant c, to a theory with two invariant constants c and κ, was an idea originally proposed over 15 years ago in [34], [35]. Since its inception, it has recieved notable attention [9], [36], [37] and it is believed [33] that qualitative predictions of DSR could be tested experimentally in the near future. In the next chapter we will show how the introduction of a second invariant scale affects the general form of the algebra of symmetry generators of Special Relativity i.e. the linear Poincare group and later the effects it has on dispersion relations of particles. One should note that this is a direct consequence of the postulates a DSR theory should possses which are as follows [16]

1. We assume the relativity principle holds, i.e all inertial observers are equivalent just like it is postulated in Special Relativity.
2. Instead of one observer independent scale c, there is an additional scale M_P with dimensions of mass. It should be noted that in the limit when this mass scale goes to infinity, DSR becomes the normal Special Relativity

Since these postulates affect the Poincare group, it follows that it should also be expected that the standard dispersion relation $E^2 = p^2 + m^2$ is to be replaced by some nonlinear mass-shell relation [13], which should involve the second scale in such a way that in the limit that the Planck mass, which we will now write as κ, goes to infinity i.e $\kappa \to \infty$, the dispersion relation simplifies to the standard one. Also including κ in the "new" dispersion relation enforces it to be kept invariant by symmetry transformations [16]

1.3.1 The minimal/fundamental length scale

In 1.3, we motivated the development of Deformed Special Relativity. In this subsection we make sense of the maximal energy, the Planck mass M_p, introduced above. We follow closely the argument presented in [38], by considering a physical system in a region of length scale L, with a mass M enclosed in this region. According to Einstein's General Relativity, one can bound the mass M by the mass of the Swarzschild black hole with a radius of the same length L i.e would be bound by

$$M_L = \frac{L}{2G} \tag{1.6}$$

where M_L is the minimal mass required for a Swarzschild black hole to form. It should be noted that for a particle at rest, a bound in mass is equivalent to a bound in the energy. In light of this, it is apparent that the energy of a phenomenon of length scale L should not exceed M_L, thus having a maximal energy.

Dealing with pheonomena in quantum gravity, one can not leave out quantum theory. Thus taking into consideration quantum mechanics, a system of length scale L will also have a minimal mass δM which one can think of as a notion of quanta of mass linked to the system, which is given by

$$\delta M = \frac{\hbar}{L} \tag{1.7}$$

Another way to understand δM would be to consider it as the uncertainty in the mass of such a system. This quanta of mass for objects e.g. an electron, is usually very small. Since we are working in a regime where we reconcile general relativity and quantum mechanics, we thus get both minimal and maximal bound on masses in regions of length L i.e.

$$\frac{\hbar}{L} = M_{min} \leq M \leq M_{max} = \frac{L}{G} \tag{1.8}$$

One then argues that the Planck scale L_P is the length at which the mass quanta δM become comparable to the maximal length M_L

$$M_{min} = M_{max} \Rightarrow L = L_P = \sqrt{G\hbar} \tag{1.9}$$

Thus when the two masses become comparable, one expects there to be remniscence of both general relativity and quantum theory and the mass corresponding to this regime is the Planck length $M_P = \sqrt{\frac{\hbar}{G}}$. One of the paradoxes of DSR is the high energies of macroscopic objects which exceed the maximal mass M_L, hence such a situation would not make sense to be handled by a theory with a maximal energy. This paradox is commonly referred to as the "soccer-ball" problem. We will give a formal explanation/discussion of this paradox in the next chapter. One should note that in this regime, contrary to Special Relativity, we expect the mass of the object to be relevant in it's kinematics/dynamics , thus reminiscing general relativity.

Again to emphasise, DSR is a theory with a minimal length scale or equivalently maximal energy/mass which is to be made an observer independent quantity. In the following section we see how one can derive a deformation of the Heisenberg Uncertainty Principle (HUP) including how this deformed HUP is equivalent to a theory of DSR.

1.4 Generalized Uncertainty Principle

According to the Heisenberg uncertainty principle (HUP)[39], which represents one of the fundamental properties of quantum systems, there should be a fundamental limit for the measurement accuracy, with which conjugate pairs of physical observables, like position and momentum can not be precisely measured at the same time. In other words, the more precisely one quantity is measured, the less precise the other one can be measured. In quantum mechanics, when one talks about physical observables, one refers to operators that live in a Hilbert space. Suppose we have an observable A, then the uncertainty in A is defined as the standard deviation of A, i.e. ΔA given by $< (\Delta A)^2 >=< A^2 > - < A >^2$, where $< A >$ is its expectation value. Now when one employs the Schwartz inequality $< \alpha|\alpha >< \beta|\beta > \geq \, | < \alpha|\beta > |^2$, this implies that with, $|\alpha >= \Delta A|\alpha' >$ and $|\beta >= \Delta B|\beta' >$, and after a bit of algebra,we get the HUP

$$\Delta A \Delta B \geq \frac{1}{2}| < [A, B] > | \tag{1.10}$$

As an example of the HUP, we take position and momentum operators $\hat{x}$ and $\hat{p}$ respectively. These operators satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. Consequently, their measurement uncertainties, Δx and Δp respectively have the relations

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1.11}$$

Thus this means when one wants to detect an arbitrarily small length scale, one has to use tools of sufficiently high momentum or equivalently energy. However, test particles of significantly high energy to resolve distances as small as those of Planck length are predicted to gravitationally curve [18] and thus signficantly disturb the structure of spacetime with which they are meant to probe. Therefore, in addition to the expected quantum uncertainty, another uncertainty creeps in, that which arises from spacetime fuzziness fluctuations at the Planck scale.

The canonical commutation relations between momentum operator $\hat{p}$ and position operator $\hat{x}$ are in flat Minkowski space-time with metric $\eta_{\mu\nu}$, thus the relations can be expressed as

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar\eta_{\mu\nu} \tag{1.12}$$

However as we have mentioned, a test particle of sufficient energy, will cause gravitational disturbances, thus spacetime will be curved and have a different metric $f_{\mu\nu}$. This metric should be asymptotically flat, such that it decomposes into $f_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where now $h_{\mu\nu}$ is the perturbation to the flat backrgound. The commutation relations between position and momentum then become

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar f_{\mu\nu} \tag{1.13}$$

The function f should contain the minimal length scale, such that in the limit when we go to normal length scales we obtain the usual HUP. This generalization, which includes scales at the Planck length is termed the General Uncertainty Relation (GUP) [40], [41]. In a GUP, an extra term in the right hand side of Heisenberg inequality forces the existence of a minimal variance for the measurement of distances [12]. We will also see later how the General Uncertainty Principle and Deformed Special Relativity are equivalent and how one can recover the modification of the dispersion relation from them. We will not, however, go in depth of the physics behind this equivalence, we shall only mathematically show this.

There are various ways of studying or deriving GUP, one of which includes non-commutative spacetimes. Firstly, we motivate, how a spacetime can become non-commutative. The minimal length reciprocal can be associated with a maximal mass as we have mentioned, which in turn gives us a maximal energy, the Planck energy mentioned above. The energy required to probe above this maximal energy scale is more than the energy to form a mini black hole, thus this means that any attempt to probe a phenomena above this energy scale could lead to a formation of a black hole in that region of space which in turn prevents any measurements in that region of space. The fuzziness of spacetime in such a scenario is also termed non-commutativity. Next we provide an alternative to showing how one can attain GUP from non-commutative spacetimes. Non-commutative geometry is a field of mathematics in which coordinates or observables do not commute [52]. In quantum gravity problems arise because, unlike other interactions, gravity is dealt with in a background, the spacetime. Other interaction quantization leave this frame invariant however, in GR, gravity interacts with matter and is technically the curving of spacetime due to matter, thus when quantizing gravity one is also faced with the problem of quantizing spacetime. Spacetime becomes an active agent in this quantization process, which poses a problem in such a way that when probing objects at distance of Planck length, measurements become fuzzy; exact locations are substituted by probabilities of finding an object in a given region of space at a given instant of time. This results in the coordinates of spacetime being non-commutative. In non-commutative geometry we have that these spacetime coordinates x^μ, which are associated to a quantum mechanics Hermitian operator $\hat{x}^\mu$, do not commute, i.e these coordinates have a commutation relation of the form.

$$[\hat{x}^\mu, \hat{x}^\nu] = i\lambda^{\mu\nu} \tag{1.14}$$

where here $\lambda^{\mu\nu}$ is a real-valued, antisymmetric two tensor with dimensions of the order of Planck length i.e $[\lambda^{\mu\nu}] \sim l_p^2$. This is the deformation parameter, and in the limit where $\lambda^{\mu\nu} \to 0$, we recover the ordinary commutative spacetime. An important point to note here is that, this tensor is not a dynamical field and it defines a preferred frame, thus breaking Lorentz invariance. A physical interpretation of the deformation parameter is that it represents the smallest observable area in the $\mu\nu$-plane.This is a very good example of a system in noncommuting coordinates as the particle moves in a noncommutative space-time.

To illustrate the idea that a modified commutation relation can lead to a theory of GUP, we consider the variables p_μ and x_μ which we will allow to obey the standard commutation relations i.e.

$$[x^\mu, x^\nu] = 0 \quad [x^\mu, p_\nu] = i\delta^\mu_\nu \quad [p_\mu, p_\nu] = 0 \tag{1.15}$$

Now suppose we define a new quantity $\tilde{p}_\mu = f(p_\mu)$, where the function f must have an inverse i.e. be bijective, so that $f^{-1}(\tilde{p}_\mu) = p_\mu$. Then for the "new" variables x^μ an $\tilde{p}_\mu$, one has the following commutation relations

$$[x^\mu, x^\nu] = 0 \quad [x^\mu, \tilde{p}_\nu] = i\frac{\partial f}{\partial p_\mu} \quad [\tilde{p}_\mu, \tilde{p}_\nu] = 0 \tag{1.16}$$

Now we should recall that for any two operators A and B, the uncertainty principle between the two operators is stated as

$$\Delta A \Delta B \geqslant \frac{1}{2} |< [A, B] >| \tag{1.17}$$

Thus for the operators x^μ and $\tilde{p}_\mu$, we have

$$\Delta x^\nu \Delta \tilde{p_\mu} \geqslant \frac{1}{2} |< \frac{\partial f}{\partial p_\mu} >| \tag{1.18}$$

Now when one inserts the expression for the variance on the r.h.s, i.e use the expression $< p^2 > - < p >^2 = \Delta p^2$ and rewrites the r.h.s in terms of $\Delta \tilde{p}^2$ and $< \tilde{p} >^2$ then one can undoubtedly divide through by Δp^2 and get

$$\Delta x_\mu \Delta \tilde{p}_\mu \geqslant \frac{1}{2} |< \frac{\partial f}{\partial p_\mu} >| \tag{1.19}$$

Which is another form of GUP.

1.5 Summary

In this chapter, we have layed down the problem of quantum gravity and how phenomenology is required instead of theories (string theory and loop quantum gravity) ,which currently do not have experimental proof, to try and solve this problem. In all theories of quantum gravity, there is a recurring idea of a minimal length scale at which the fundemental laws of nature limit our ability to probe aribitrarily short distances. We saw that at the Planck scale i.e. minimal length, notions which rely on a metric structure appear to have an inherent difficulty to investigation. It has been shown in the literature [42][43], that different analysis based on contrasting ways of looking at the quantum gravity problem all give the same prediction in terms of the nature of space-time at the scales of the Planck length. We further provided a motivation of a theory of Deformed Special Relativity (DSR) in which one extends the theory of Special Relativity by introducing as an observer-independent scale the Planck mass. We looked at how this regime is asymptotically attained from well studied theories. Moreover, we introduced the idea of a Generalzied Uncertainty principle in which, classically, the commutation relations recieve a correction due to the fuzziness of spacetime when test particles of high energies are used to proble distances at Planck length. We noted that DSR and GUP are equivalent and this remains to be shown within the thesis.

1.6 Thesis Overview

This thesis is organized as follows. In the next chapter we examine in detail the theory of Deformed Special Relativity by appealing to an algebraic formulation of spacetime. The algebras of Minkowski spacetime in Special Relativity are deformed in such a way as to encapsulate the idea of a second observer independent constant. We then look at how DSR is constructed in different bases, which we show are all mathematically equivalent. We examine the non-linearity of transformation laws in DSR. Lorenz invariance has the potential to be broken in a theory with a second observer-independent scale. One then requires to modify the standard dispersion relation to contain a minimal length scale. In the next chapter, we also give an introduction to modified dispersion relations(MDR). Further we mention a few parodoxes or problems that a theory of DSR may bring about which include non-locality. We study in some detail the principle of Relative locality in a means to understand how to get rid of absolute locality since one cannot with absolute certainty say two observers infer the same space time in a theory of DSR. We end the chapter by looking at an experiment which shows how a theory of DSR is a non-local theory. In chapter 3, we look at the geometrization of a theory of DSR. We however, take a step back and review the underlying theory needed for this geometrization. We study the mechanics of Lagrange and Hamilton, which allows the formulation of mechanics on phase space, the cotangent bundle of a base manifold. In section 3.4, we introduce the framework of Hamilton geometry in hopes of a geometrized theory of DSR. In this framework we look at how phase space can be decomposed into spacetime and momentum space and look at the individual geometry of these spaces respectively. Moreover, we look at the symmetries of phase space in such a framework and then look at a few examples to get a feel of the theory. In chapter four we look at the effect of the entanglement of spacetime and momentum space as subspaces of phase space. We see here that the physical coordinates of spacetime are not those that are canonical to momenta but "new" effective coordinates that have a momentum dependence. This is shown to lead to an already established result of DSR in Snyder's basis.

Chapter 2

Deformed Special Relativity

One of the most unsolved problems in physics today is finding a theory of Quantum Gravity (QG) i.e. a theory that would successfully describe the physics in a regime where both Quantum Mechanics and General Relativity would have to be taken into account. As we have mentioned in the last chapter, such regimes include, for example, the physics inside a black hole and also the physics that would be required to describe what happens close to the initial singularity of the universe. Notable theories that have been considered for addressing this problem include, string theory and loop quantum gravity. Each of these theories is based on radically different ideas of what the fundamental physics will eventually look like. However, a common property of these approaches is, as we reiterate, the lack of testable predictions made by them, largely due to the lack of equipment that can reach energies high enough to probe this regime. It is widely known in the literature [43], that the boundary of the regime at which a theory of quantum gravity will become the dominating theory is given by the Planck scale. As noted above, the fundamental constants of physics viz. G (Newton's constant), $\hbar$ (Planck's constant) and c (speed of light), can be combined in such a way as to give new constants of dimensions of length, time, energy and mass. These so-called "new" four constants are the Planck length $l_p = \sqrt{\hbar G/c^3}$, Planck time $T_p = \sqrt{\hbar G/c^5}$, Planck energy $E_p = \sqrt{\hbar c^5/G}$ and the Planck mass $M_p = \sqrt{\hbar c/G}$ respectively. It is generally assumed, regardless of what the theory of QG might turn out to be, one should find that in the limit where the energies considered are well below E_p, the theory reduces to the well-known theory of Special Relativity (SR) and GR. Thus one should expect a semiclassical regime to lie between the quantum gravity regime and SR/GR. This is precisely the regime where Deformed Special Relativity claims to be the dominating theory. Presently it would be plausable to concentrate on this semiclassical theory and work our way upwards to QG, since we have no clear picture of what quantum gravity looks like fully. One can then hope, with some inspiration from SR/GR and the little we do know about QG, we might actually be able to construct a theory in this semiclassical regime. In this chapter we will be concentrating on this semiclassical regime, for which we assert DSR to be the main candidate for a correct theory here.

Deformed Special Relativity (DSR) is a semiclassical theory that is built by considering Einstein's theory of Special Relativity when high energies, close to the Planck energy are considered. Thus DSR is claimed [44] to be the theory that lies between a full QG theory and a general relativistic regime. The idea of such a theory has been around since the beginning of the 21st century [45], and has since recieved immense attention from the scientific community [37][46][47], due to the possibility of qualitative predictions that could be tested experimentally in the near future [23]. DSR is based on two fundamental assumptions i.e. the principle of relativity, which is also a fundamental principle in Special Relativity, and the postulate of the existence of two scales. In Special Relativity, there exist an observer-independent scale, which is the speed of light c. Moreover, DSR postulates in addition to the speed of light another scale *viz.* the mass scale κ which is inversely related to the Planck length i.e. $l = \frac{1}{\kappa}$. However, having this additional scale instantly faces one with a rudimentary problem, which can be summarized in the question: In which observer's inertial frame does one measure the energy

E_p? Since two observers would be measuring the energy in each of their respective inertial frames, clearly these observers could disagree about a measured E_p. The obvious solution to such a problem is to introduce the scale κ as an invariant scale, such that all observers observe the same value regardless of which rest frame it is measured in. The introduction of such a scale, as we will later see, has consequences on how one writes the transformation laws between frames of distinct observers. Simply stated, we have to re-write the Lorentz transformations of Special Relativity in such a way that the energy E_p will be the same no matter what transformation we apply. As one would expect DSR may somewhat replace Special Relativity as a theory which describes the relativistic kinematics and dynamics of particles when energies close to the Planck energy are considered.

Relative Locality (RL) is a result of DSR and focuses on the nonlocalities and other features of deformed symmetries models, introducing some sort of momentum space curvature that influences the localization process at a characteristic scale that we assume to be of the order of the Planck scale l_p [48]. In RL, one discards the notion that, what we observe around us is space or space-time and replaces this with the idea that we observe momentum space when we look around us. Our most fundamental measurements are all about energy-momentum quanta we absorb and emit. When looking at an object, one sees photons arriving with different momenta and energies at different angles. In the RL framework, one takes phase space as the arena for non-quantum physics rather than space-time. In absolute locality, which was proposed by Einstein, it is postulated that all observers live in the same space-time. Moreover this assumption is equivalent to having a flat momentum space. However in Relative locality, the notion of absolute locality becomes obsolete and different observers see different space-times and these space-times are energy and momentum dependent [44].

2.1 The Mathematical Structure of Deformed Special Relativity

Now that we have been able to motivate the introduction of an additional invariant energy scale E_p, we may want to answer the question: What effects does this new invariant scale have on the mathematical structure of the theory that emerges from imposing such an additional assumption to SR. Particularly, one would be interested in knowing whether or not in this regime the Poincare algebra of the symmetry generators in Special Relativity still hold. In [16], it is shown that this is no longer the case, and that an additional invariant scale is not supported by the Poincare algebra. However, in the same article, a solution to this is proposed by suggesting a deformation of the usual Poincare algebra to get an algebra that allows for an additional invariant scale. This algebra is called the κ-Poincare algebra. κ-Poincare algebra is attained by deforming the Poincare algebra, and the right-hand side of the commutation relations which make up the usual algebra now contain the parameter κ, which is associated with the second invariant scale. κ-Poincare algebras have the crucial property that they are distinguishable from normal Lie groups in that they are quantum deformed algebras (or Hopf algebras) [16]. The interested reader can find a good explanation on Hopf algebras in [49]. For a given Lie algebra, there is a corresponding quantum deformed algebra, essentially the quantum algebra is the generalization of a Lie algebra, thus having the property of a universal enveloping algebra[16]. Consider for instance the quantum deformed group of the Lie group $SO(3,1)$ which will be denoted as $SO_q(3,1)$. At this point it should be mentioned that for the κ-Poincare algebra, one should recover the usual Poincare algebra in the limit where $\kappa \to \infty$. In our example the q is related to the κ scale and thus should bring us back to the undeformed Lie group in the appropriate limit. For illustration, we consider the algebraic part of the Hopf algebra $SO_q(3,1)$ and write it as follows [16]

$$[M_{2,3}, M_{1,3}] = \frac{1}{z}\sinh(zM_{1,2})\cosh(zM_{0,3}) \tag{2.1}$$
$$[M_{2,3}, M_{1,2}] = M_{1,3} \tag{2.2}$$
$$[M_{2,3}, M_{0,3}] = M_{0,2} \tag{2.3}$$
$$[M_{2,3}, M_{0,2}] = \frac{1}{z}\sinh(zM_{0,3})\cosh(zM_{1,2}) \tag{2.4}$$
$$[M_{1,3}, M_{1,2}] = -M_{2,3} \tag{2.5}$$
$$[M_{1,3}, M_{0,3}] = M_{0,1} \tag{2.6}$$
$$[M_{1,3}, M_{0,1}] = \frac{1}{z}\sinh(zM_{0,3})\cosh(zM_{1,2}) \tag{2.7}$$
$$[M_{1,2}, M_{0,2}] = -M_{0,1} \tag{2.8}$$
$$[M_{1,2}, M_{0,1}] = M_{0,2} \tag{2.9}$$
$$[M_{0,3}, M_{0,2}] = M_{2,3} \tag{2.10}$$
$$[M_{0,3}, M_{0,1}] = M_{1,3} \tag{2.11}$$
$$[M_{0,2}, M_{0,1}] = \frac{1}{z}\sinh(zM_{1,2})\cosh(zM_{0,3}) \tag{2.12}$$

where now $z = \ln(q)$ and the generators of the algebra $M_{\mu\nu}$. An interesting and important point to note in the commutation relations above for the generators of the quantum deformed algebra is that on the right-hand side, we do not have linear functions of the generators, as it is in the Lie algebra case, but instead we now have analytic functions of them [16]. An immediate consequence of this algebra having analytic functions on the r.h.s is that one can use any analytic function of the generators to define a new basis for the quantum deformed algebra. This stands in comparison to the Lie algebra case, where only linear combinations of generators are allowed to be bases. This freedom in the choice of basis has led to several different proposals of bases for the κ-Poincare algebra. We will see later that all these bases are equivalent.

One will see that, following this discussion, this freedom in choosing bases has led to several different proposals of bases for the κ-Poincare algebra. Up until this point, the different bases that have been used in the literature include the bi-cross product basis, the Maguejo-Smolin basis and the classical basis. One has to however note, that a common feature of these different bases is that they are chosen in such a way as to leave the Lorentz subalgebra of the κ-Poincare unchanged. For the sake of completion, one should mention that there is also another basis in the literature called the standard basis, in which the Lorentz sector is not of the form of the common Lorentz algebra. The commutators of the κ-Poicare algebra, which involve the boosts and the momentum generators differs from the standard Poincare sector or simply put, the Poincare algeba is deformed in the Poincare sector. It is usually due to this that this framework is dubbed Deformed Special Relativity. There have been articles, that suggest the D in DSR stands for doubly, such that one has Doubly Special Relativity, due to the addition of a second observer-independent scale. The reason for leaving the Lorentz sector of the κ-Poincare algebra unmodified and only modifiying the remaining commutators, is that if the Lorentz sector was modified as well, then upon integration, one would not obtain a group but a quasigroup [50]. It should be said that although mathematically we have the freedom to choose an arbitrary basis for the κ-Poincare algebra, one might expect that there should ultimately be a physical argument which rules in favour of one particular basis and against any other basis.

Note, that in the algebra above, in the limit $z \to 0$ the algebra becomes the standard algebra $SO(3,1)$, and this is the reason for using the notation $SO_q(3,1)$. The $SO(3,1)$ algebra is known to be the $2+1$ dimensional de Sitter algebra it has been well known for quite some time how one can retrieve the $2+1$ dimensional Poincare algebra from it. Firstly, we need the energy and momentum generators which have the correct physical dimension, noting that the generators $M_{0,\mu}$ are dimensionless. Normally, identifying the three-momenta P_μ as the generators $M_{0,\mu}$ when they have been appropriately rescaled and taking the Inomu-Wigner contraction yields our $2+1$ Poincare algebra [16]. However, it becomes more complex in the quantum algebra case as the contraction poses trickery since one still has to convince themselves that after the contraction the structure obtained is still a quantum algebra.

In an attempt to contract the algebra (2.12), we first rescale some of the generators by an appropriate scale, which we get from combining certain dimensionful constants which one get from the definition of z. We do this because momenta are dimensionful whereas the generators M in(2.12) are dimensionless. These combinations are as follows

$$E = \sqrt{\Lambda}\hbar M_{0,3} \tag{2.13}$$

$$P_i = \sqrt{\Lambda}\hbar M_{0,i} \tag{2.14}$$

$$M = M_{1,2} \tag{2.15}$$

$$N_i = M_{i,3} \tag{2.16}$$

When one then takes into consideration the relation $z \approx \sqrt{\Lambda}\hbar/\kappa$, which holds for small Λ, from (2.1) of (2.12) we find that

$$[N_2, N_1] = \frac{\kappa}{\hbar\sqrt{\Lambda}} \sinh(\hbar\sqrt{\Lambda}/\kappa M)\cosh(E/\kappa) \tag{2.17}$$

In a similar fashion, from (2.12) of (2.12), we find

$$[P_2, P_1] = \sqrt{\Lambda}\hbar\kappa \sinh(\sqrt{\Lambda}\hbar/\kappa M)\cosh(E/\kappa) \tag{2.18}$$

One can do similar substitutions for the rest of the commutators in (2.12). Keeping κ constant and taking the contraction limit $\Lambda \to 0$ one obtains the three dimensional κ-Poincare algebra in the standard basis

$$[N_i, N_j] = -M\epsilon_{ij}\cosh(E/\kappa) \tag{2.19}$$

$$[M, N_i] = \epsilon_{ij}N^j \tag{2.20}$$

$$[N_i, E] = P_i \tag{2.21}$$

$$[N_i, P_j] = \delta_{ij}\kappa \sinh(E/\kappa) \tag{2.22}$$

$$[M, P_i] = \epsilon_{ij}P^j \tag{2.23}$$

$$[E, P_i] = 0 \tag{2.24}$$

$$[P_2, P_1] = 0 \tag{2.25}$$

Now, in the latter algebra, one can easily notice that in the limit $\kappa \to \infty$, one recovers the standard Poincare algebra. Also, one should note that in this algebra, both the Lorentz and translation sectors are deformed. However, recall that since quantum algebras posses analytic functions in their commutation relations, one can arbitrarily redefine the generator basis by choosing other forms of the analytical functions. It turns out that one can find a basis where the Lorentz part of the algebra becomes undeformed. This is the bicrossproduct. In [16], the Deformed Special Relativity model based on this basis is called the DSR1. In $2+1$

dimensions, the κ-Poincare algebra has the form

$$[N_i, N_j] = -\epsilon_{ij} M \tag{2.26}$$

$$[M, N_i] = \epsilon_{ij} N^j \tag{2.27}$$

$$[N_i, E] = P_i \tag{2.28}$$

$$[N_i, P_j] = \delta_{ij} \frac{\kappa}{2} \left(1 - e^{-2E/\kappa} + \frac{\vec{P}^2}{\kappa^2} \right) - \frac{1}{\kappa} P_i P_j \tag{2.29}$$

$$[M, P_i] = \epsilon_{ij} P^j \tag{2.30}$$

$$[E, P_i] = 0 \tag{2.31}$$

$$[P_1, P_2] = 0 \tag{2.32}$$

We can then notice that this algebra is just the $2+1$ analogue of (2.12). In conclusion, in the case of $2+1$ dimensional quantum gravity on de Sitter space, in the flat space, i.e. vanishing cosmological constant limit the standard Poinare algebra is replaced by κ-Poincare algebra.

In summary, in $2+1$ gravity, the scale κ, arises naturally. One can also show that instead of the Poincare symmetry we have to now deal with the deformed algebra, that possesses the deformation scale κ.

A crucial point that we will also come across in the later chapters is the consequence of the emergence of the κ-Poincare algebra. As in the standard case this algebra can be interpreted as the algebra of spacetime symmetries [16]. One can easily observe that this algebra can be interpreted as an algebra of Lorentz symmetries of momenta if the momentum space is de Sitter space of curvature κ [16]. It can be shown that one can extend this algebra to the full phase space algebra of a point particle, by adding four non-commutative coordinates. The resulting spacetime of the particle becomes what has been dubbed κ-Minkowski spacetime with the non-commutative structure

$$[x_0, x_i] = -\frac{1}{\kappa} x_i \tag{2.33}$$

The usefullness of this κ-Minkowski spacetime is that one can build field theory on it, which in turn could be used to discuss phenomenological issues that we have mentioned in the introduction.

2.1.1 Deformed Special Relativity in various bases and space-time non-community

In the most part of this chapter, we are using the most frequently used framework in studying the relativity theory with two observed independent kinematical scales. The DSR construction here has been based on the quantum (Hopf) κ-Poincare algebra as previously shown in the latter section. We will later study the geometrization of this theory in 3. In Special Relativity, we have the standard algebra i.e the Poincare algebra, and the κ-Poincare algebra is just a deformation of this standard algebra. However, there have been a variety of DSR theories proposed in the literature, which are different from the one we have discussed already i.e the κ-Poincare theory in the bicrossproduct basis. We will briefly mention this DSR again for completeness here. Moreover, this variety of theories raises questions such as, how many theories of this kind may exist. Are these different theories physically equivalent? Can Quantum Gravity experiments deduce that there should be one unique theory? Here we will see that algebraically, these theories are completely equivalent and are simply expressed in different bases. To each of these different bases, one constructs a "different" DSR theory in appearance.

A very important assumption in the construction of a DSR theory is to not deform the Lorentz algebra i.e the subalgebra of the κ-Poincare algebra. However one should note that this assumption is not satisfied by the so-called standard basis of the κ-Poincare algebra. Thus, we therefore have to assume that the boost generators N_i and the rotation generators M_i satisfy the algebra.

$$[M_i, M_j] = i\epsilon_{ijk}M_k \quad [M_i, N_j] = i\epsilon_{ijk}N_k$$
$$[N_i, N_j] = -i\epsilon_{ijk}M_k$$

Furthermore, another assumption to be put in place is that the action of rotations is not deformed and that generators of momenta commute. We then take these postulates as starting points in defining the action of the Lorentz algebra on the energy-momentum sector. We should recap that a quantum algebra or Hopf algebra is in addtion to the algebra of commutators an algebra that possesses additional structures like the co-product and the antipode. Next we will present three bases of the quantum deformation Poincare algebra *viz* the bicrossproduct, the Maguejo-Smoling and the classical basis. One should note that the classical basis algebraic sectors are identical to the standard Poincare algebra [46].

The bicrossproduct basis

Since we have assumed that the action of rotations is not deformed i.e it is standard, it will suffice to write down the commutators of deformed boost generators with momenta [46]. We get

$$[N_i, p_j] = i\delta_{ij}\left(\frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa}\right) + \frac{1}{2\kappa}\vec{p}^2\right) \tag{2.34}$$
$$= i\frac{1}{\kappa}p_i p_j \tag{2.35}$$

and

$$[N_i, p_0] = ip_i \tag{2.36}$$

One can easily check, see appendix for calculation, that a Casimir operator of the algebra , which is important to note that it lives in the universal enveloping algebra, reads

$$m^2 = \left(2\kappa \sinh\left(\frac{p_0}{2\kappa}\right)\right)^2 - \vec{p}^2 e^{p_0/\kappa} \tag{2.37}$$

Moreover the three-momentum is bounded above $\vec{p}^2 \leq \kappa^2$ for positive κ, and p_0 corresponds to zero energy.

The quantum algebra structure in this basis is provided by the following coproducts Δ and antipodes S, whose only the explicit form of the momenta copoducts

$$\Delta(p_i) = p_i \otimes \mathbb{1} + e^{-p_0/\kappa} \otimes p_i \tag{2.38}$$
$$\Delta(p_0) = p_0 \otimes \mathbb{1} + \mathbb{1} \otimes p_0 \tag{2.39}$$

will be relevant to what follows. Taking these formulas as a starting point, we can now turn to analysis of other bases.

Magueijo-Smolin basis

In a 2002 paper Magueijo and Smolin [51] proposed another DSR theory. A DSR theory that has the boost generators as linear combinations of the usual standard Lorentz generators and the generator of dilation. However it has to be in a way such that the algebra 2.34 holds. In this basis the commutators of four-momenta P_μ and boosts have the following form [46], (note the difference between p_μ and P_μ notation)

$$[N_i, P_j] = i\left(\delta_{ij}P_0 - \frac{1}{\kappa}P_iP_j\right) \tag{2.40}$$

and

$$[N_i, P_0] = i\left(1 - \frac{P_0}{\kappa}\right)P_i \tag{2.41}$$

Calculating the Casimir, one obtains the following

$$M^2 = \frac{P_0^2 - \vec{P}^2}{(1 - P_0/\kappa)^2} \tag{2.42}$$

Now we would like to see if this basis is equivalent to the bicrossproduct one. As mentioned before, these bases are algebraically equivalent since one can easily check that the relation between the variable P_μ and p_μ is given by

$$p_i = P_i, \tag{2.43}$$

$$p_0 = -\frac{\kappa}{2}\log\left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2}\right) \tag{2.44}$$

$$P_0 = \frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa} + \frac{\vec{p}^2}{\kappa^2}\right) \tag{2.45}$$

One should note from the above relations that the maximal momentum in the bicrossproduct basis corresponds to the maximal energy $P_0 = \kappa$ in the M-S basis.

Further we can easily develop the algebra here to a quantum algebra using the formulas above. We define the new co-products using the relation 2.43 and these read as follows

$$\Delta(P_i) = P_i \otimes \mathbb{1} + \left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2}\right)^{1/2} \otimes P_i \tag{2.46}$$

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 - \frac{2}{\kappa}P_0 \otimes P_0 + \frac{1}{\kappa^2}\vec{P}^2 \otimes P_0 + \frac{1}{\kappa}\left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2}\right)^{1/2} \sum P_i \otimes P_i \tag{2.47}$$

The classical basis

For comparison, we present yet another basis which was first described in the early 90's [133]. We call this basis the classical basis in which the classical Poincare algebra is formed by the boosts-momenta commutators with the Lorentz sector. We have

$$[N_i, \mathcal{P}_j] = i\delta_{ij}\mathcal{P}_0 \quad [N_i, \mathcal{P}_0] = i\mathcal{P}_i \tag{2.48}$$

The Casimir for this basis equals, of course the one of special relativity, to wit

$$\mathcal{M}^2 = \mathcal{P}_0^2 - \vec{\mathcal{P}}^2 \tag{2.49}$$

The classical generators $\mathcal{P}_\mu$ are related to the bicrossproduct basis generators by the formulas

$$\mathcal{P}_0 = \kappa \sinh \frac{p_0}{\kappa} + e^{p_0/\kappa} \frac{\vec{p}^2}{2\kappa} \tag{2.50}$$

$$\mathcal{P}_i = e^{p_0/\kappa} p_i \tag{2.51}$$

and one can easily compute the expression for co-product

$$\Delta(\mathcal{P}_0) = \frac{\kappa}{2}\left(K \otimes K - K^{-1} \otimes K^{-1}\right) + \frac{1}{2\kappa}\left(K^{-1}\vec{\mathcal{P}}^2 \otimes K + 2K^{-1}\mathcal{P}_i \otimes \mathcal{P}_i + K^{-1} \otimes K^{-1}\vec{\mathcal{P}}^2\right) \tag{2.52}$$

$$\Delta(\mathcal{P}_i) = \mathcal{P}_i \otimes K + \mathbb{1} \otimes \mathcal{P}_j \tag{2.53}$$

Where

$$K = e^{p_0/\kappa} = \frac{1}{\kappa}\left[\mathcal{P}_0 + \left(\mathcal{P}_0^2 - \vec{\mathcal{P}}^2 + \kappa^2\right)^{1/2}\right]$$

2.1.2 Noncommutatitive space-time

The setting within which spacetime noncommutativity could be formulated and lead to an observable phenomenon, well at least in principle, is of concern. Evidently this setting cannot be based on classical mechanics, where the formalism provides no room for noncommutativity of coordinates. This in itself is not so shocking, since classical mechanics should only emerge as an approximate limit of a quantum theory of mechanics, and the limiting procedure from quantum mechanics to classical mechanics may be such that also the non-commutativity of spacetime coordinates is removed in the classical limit. One of the problems here is that even giving a formulation of κ-Minkowski spacetime non-commutativity in a quantum-mechanics setup is not straightforward. The reader can see in the appendix **??**, this is quite tricky as we tried quantizing the κ-Poincare Hamiltonian and was faced with the problem of making it relativistic. This might be due to the fact that in κ-Minkowski the time coordinate is a noncommutative observable, whereas in the standard formulation of quantum mechanics the time coordinate is just an evolution parameter. Time, according to κ-Minkowski should be an operator that does not commute with the spatial coordinate operators, but in the standard setup of quantum mechanics we are not in the situation of time being described by an operator that commutes with the spatial-coordinate operators: in the standard setup of quantum mechanics time is not an observable at all, it just plays the role of an evolution parameter.

In this section we try and closely follow the (general) step by step procedure on how to construct the space-time commutator algebras given in [46] . We will see that regardless of which choice of basis we make, the space-time commutation relation of κ-Minkowski remains the same.

We show this easily as an example in the *bicrossproduct* basis. It follows that if one is to follow the steps depicted in [46], then

$$< p_i, x^0 x^j >= -\frac{1}{\kappa}\delta_{ij}, \quad < p_i, x_j x_0 >= 0 \tag{2.54}$$

from which we get

$$[x^0, x_i] = -\frac{1}{\kappa} x_i \tag{2.55}$$

The standard relations

$$[p_0, x^0] = i, \quad [p_j, x^j] = -i\delta_i^j \tag{2.56}$$

are then found by using step 3 of [46]. However, we are not done, as it happens that there is another non-vanishing commutator which is give by

$$[p_i, x_0] = -\frac{i}{\kappa} p_i \tag{2.57}$$

and as required of a "nice" algebra, the above algebra satisfies the Jacobi identities.

Now, an interesting question is, do we obtain a similar algebra of the space-time in other bases depicted above. We employ the *Magueijo-Smolin* basis and try to find the non-commutative structure of space-time. Step 1 in [46], suggests we start with

$$[P_\mu, X_\nu] = -\eta_{\mu\nu} \tag{2.58}$$

Now the next step is to observe that the only terms in (2.46) and (2.47) which come into play for computations are the bilinear ones. So we write

$$\Delta(P_i) = \mathbb{1} \otimes P_i + P_i \otimes \mathbb{1} - \frac{1}{\kappa} P_0 \otimes P_i + \ldots \tag{2.59}$$

$$\Delta(P_0) = \mathbb{1} \otimes P_0 + P_i \otimes \mathbb{1} - \frac{2}{\kappa} P_0 \otimes P_0 + \frac{1}{\kappa} \Sigma P_i \otimes P_i + \ldots \tag{2.60}$$

and from this it follows that in the spatial sector an immediate non-vanishing commutator is given by

$$[X_0, X_i] = -\frac{i}{\kappa} X_i \tag{2.61}$$

and from [46] we find remaining commutators to be

$$[P_0, X_i] = -\frac{i}{\kappa} P_i \quad [P_0, X_0] = i\left(1 - \frac{2}{\kappa} P_0\right) \tag{2.62}$$

$$[P_i, X_j] = -i\delta_{ij} \quad [P_i, X_0] = -\frac{i}{\kappa} P_i \tag{2.63}$$

of course the algebra above satisfies the Jacobi identity.

Furthermore, when one employs the *classical basis* $\{\mathcal{X}_\mu, \mathcal{P}_\mu\}$, which is merely a different basis from the previously mentioned, again we start at step one and obtain the relation

$$[\mathcal{P}_\mu, \mathcal{X}_\nu] = -i\eta_{\mu\nu} \tag{2.64}$$

and to get the commutators in the spatial sector as above we similarly take the part of the co-product up to the bilinear terms and thus have

$$\Delta(\mathcal{P}_i) = \mathbb{1} \otimes \mathcal{P}_i + \mathcal{P}_i \otimes \mathbb{1} + \frac{2}{\kappa} \mathcal{P}_0 \otimes \mathcal{P}_i + \frac{1}{\kappa} \mathcal{P}_i \otimes \mathcal{P}_0 + \ldots \tag{2.65}$$

$$\Delta(\mathcal{P}_0) = \mathbb{1} \otimes \mathcal{P}_0 + \mathcal{P}_0 \otimes \mathbb{1} + \frac{1}{\kappa} \Sigma \mathcal{P}_i \otimes \mathcal{P}_i + \ldots \tag{2.66}$$

which leads again to

$$[\mathcal{X}_0, \mathcal{X}_i] = -\frac{1}{\kappa} \mathcal{X}_i \tag{2.67}$$

The rest of the commutation relations can be found from [46] however the point we would like to stress is that regardless of the basis we use, one recovers the spatial sector from energy momentum co-algebra as the κ-Minkowski spacetime.

2.2 Non-linearity of transformation laws in DSR

If one is to derive the transformation rules for a DSR theory, he necessarily has to re-write the principles on which Special Relativity is bases and possibly add additional principles. The first two principles we will state should be familiar to the reader from above and in SR. They are as follows

- *Relativity of inertial frames*: Observers in free, inertial motion are equivalent
- *Equivalence Principle*: Under the influence of gravity, observers in free-fall are inertial observers and are all equivalent to each other.

One then needs to add more principles for a theory of DSR to accomodate for "new" ideas like the additional observer independent scale. These principles are as follows

- *Observer independence of the Planck scale/energy*: In addition to the speed of light, all observers have to agree on the Planck scale as an invariant scale.
- *Principle of correspondence*: In the low energy limit, one has to recover the standard SR and/or GR.

In our next discussion, we follow closely the presentation in [51]. Starting with the Lorentz sector, we derive the symmetry generators for DSR. Noting from the first principle, we must have a transformation group that relates measurements made by different observers. However, the last principle, constrains this transformation group to one that reduces to the usual Lorentz group of SR at the appropriate limit. Our expectation, similar to SR, the transformation group also has to have six parameters i.e. 3 rotations and 3 boosts. However, we find that the only group with these properties is the Lorentz group itself. At this point one faces a problem, in that the Lorentz group does not actually satisfy a theory with an additional Planck scale or Planck energy E_p. One way to get around this is to assume that our symmetry group is actually the Lorentz group, however it should act non-linearly on momentum space. Thus we are then required to write down an explicit form for the action on momentum space, which should be energy dependent and leave the Planck energy invariant. Adding a dilatation term to each boost generator turns out to allow us to achieve this. We remind the reader that this discussion is specific to a DSR theory in the M-S basis.

Here, one should note that we are working in momentum space rather than the configuration space., where by momentum space we are referring to the space P of momentum four-vectors p_α where $\alpha \in \{0, 1, 2, 3\}$ using the signature $(+, -, -, -)$. The standard Lorentz generators are then given by

$$L_{\alpha\beta} = p_\alpha \frac{\partial}{\partial p_\beta} - p_\beta \frac{\partial}{\partial p_\alpha} \tag{2.68}$$

Our strategy as we have stated above, is to leave the rotation generators J_i where $i \in \{1, 2, 3\}$, unchanged but add an additional term to the boost generators K_i. This term we are adding now contains the dilatation relation generator $D = p_\alpha \frac{\partial}{\partial p_\alpha}$, with the obvious action on M, $D \circ p_\alpha = p_\alpha$. The modified generators of boosts are then given by

$$K^i := L^i_0 + l_p p^i D \tag{2.69}$$

where l_p is the Planck length, which is the inverse of E_p.

Recall that we leave the rotation generators unchanged, thus they retain their usual definition

$$J^i = \epsilon^{ijk} L_{jk} \tag{2.70}$$

One crucial point we would like to highlight is that even though, we have made modifications on the generators, they still satisfy the standard Lie algebra for the Lorentz group *viz*

$$\left\{J^i, K^j\right\} = i\epsilon^{ijk} K_k, \ \left\{K^i, K^j\right\} = -i\epsilon^{ijk} J_k, \ \left\{J^i, J^j\right\} = i\epsilon^{ijk} J_k \tag{2.71}$$

Having set up the algebra of the generators, we tackle the next task, which is to exponentiate the boost generators so that we can find the transformation laws of the boosts. One hopes here, at the least the transformation laws we attain resemble those of SR. Firstly, we note that one can generate the modified, non-linear boost generators K^i, by using an energy dependent tranformation $U(p_0) = e^{l_p p_0 D}$

$$K^i = U^{-1}(p_0) L_0^i U(p_0) \tag{2.72}$$

Exponentiating, one then recovers the representation of the Lorentz group we have been looking for, which has the form

$$R[\omega_{\alpha\beta}] = U^{-1}(p_0) e^{\omega_{\alpha\beta} L_{\alpha\beta}} U(p_0) \tag{2.73}$$

The boost transformations are then, with γ being the usual Lorentz factor, given by,

$$\begin{aligned} p_0' &= \frac{\gamma(p_0 - vp_z)}{1 + l_p(\gamma - 1)p_0 - l_p \gamma v p_z} \\ p_z' &= \frac{\gamma(p_z - vp_0)}{1 + l_p(\gamma - 1)p_0 - l_p \gamma v p_z} \\ p_x' &= \frac{p_x}{1 + l_p(\gamma - 1)p_0 - l_p \gamma v p_z} \\ p_y' &= \frac{p_y}{1 + l_p(\gamma - 1)p_0 - l_p \gamma v p_z} \end{aligned}$$

The reader should note that in the limit of small $|p_\alpha|$, these transformations reduce to the usual SR momentum space transformations. Also note that this set of energy-dependent transformations look similar to the standard SR transformations.

In writing down transformation laws which leave the Planck energy invariant, we had to pay the price of having a momentum space P with a constant curvature, thus having a de Sitter or anti-de Sitter geometry depending on which sign the curvature takes [31]. As a consequence of this, now translations on momentum space no longer commute.

2.3 The physics of Deformed Special Relativity

Having gone through the general mathematical structure of DSR, the next step one might want to consider is the kind of physics which might possibly emerge from such a theory of DSR. It should be noted however that since physicists still cannot agree on a single formulation of DSR theory, one can only speculate about the physics as well. In this section, we will have a look at two different aspects of physics, which include the modification of dispersion relations and the possibiliy of having an energy-dependent metric. The modification of the standard dispersion relation is a mathematical consequence of the particular form of our symmetry algebra, thus an integral part of the theory. Whereas the metric energy dependence should be regarded more as an intriguing, bold guess rather than a mathematical result.

2.3.1 Modified Dispersion Relations

Throughout this chapter, we have managed to motivate and introduce the mathematical formulation of an observer-independent energy theory of DSR. Furthermore we looked at the fundamental principles and how they change the transformation laws of SR to accomodate an additional observer-independent scale. Now we move on to consider modifications of the standard relativistic dispersion relation, which is

$$E^2 = p^2 + m^2 \tag{2.74}$$

It should not come as a surprise that we might need to modify this dispersion relation in a theory of DSR, since we need to include somehow the observer-independent energy scale. One kind of modification that is sometimes suggested is of the following form

$$E^2 = p^2 + m^2 + f\left(\frac{E}{E_p}\right) \tag{2.75}$$

where the function f is in powers of E/E_p of order greater than 2. Modified dispersion relations (MDR) of this fashion were first suggested in [34]. Dispersion relations are merely Casimirs of algberas, thus if given the κ-Poincare algebra in a specific basis, one can straight forwardly derive a precise form of the modified dispersion relation. We should put emphasis on the fact that any modified dispersion relation in DSR is observer-independent i.e. under the non-linear momentum space transformations the MDR is left invariant. Recall there were many possible bases of the κ-Poincare algebra, similarly, there are many forms of the function f with the hope that should DSR turn out to be correct, f will be fixed by experimental data. To make things a little more concrete, let us give the explicit form of the modified dispersion relation used in [34]

$$E^2 \approx p^2 + m^2 \pm \frac{2}{n+1}E^2\left(\frac{E}{E_p}\right)^n \tag{2.76}$$

Here we have introduced a modification which is only given up to leading order, with the parameter n being the power of the leading correction. It is exciting to note that possible modifications of the dispersion relation for highly energetic photons can be tested in experiments which are already running or will become feasible in the near future. Thus DSR should be more prioritized in studies of quantum gravity since it provides one of first testable predictions made by any theory of the semi-classical regime, not mentioning the pure quantum gravity regime.

A consequence of modifying the dispersion relation this way is the possibility of an energy-dependent speed of light $c(E)$. We make the assumption that the equation $v = \frac{\partial E}{\partial p}$ still holds in the semi-classical regime, we then find [34] that for the modified dispersion relation (2.76), the speed of light $c(E)$ with an energy dependence is given by

$$c(E) = 1 \pm \left(\frac{E}{E_p}\right)^n \tag{2.77}$$

What should be clear from this relation, is that in the limit $E \to 0$, the speed $c(E)$ tends to the speed of light c. As one should expect an energy-dependent speed of light might lead to conceptual problems of a theory. We will discuss an experiment with an energy dependent speed of light towards the end of this chapter. A crucial point that has to be noted is that what we have done is to take the modified dispersion relation which is in terms of momentum space variables and apply the relation $v = \partial E/\partial p$ which is an equation for calculating the position space quantity v. Thus the conceptual problems one faces because of an energy-dependent speed of light might be attributed to the transition between momentum and configuration space not being done correctly. It could be that the relation $v = \partial E/\partial p$ is not the appropriate relation to use in a semi-classical theory with an observer-independent Planck energy scale. Possibly, using the right relation here might or might not yield a speed of light which is energy-independent.

2.3.2 The Rainbow metric and Spacetime geometry in DSR

In the previous sections we have tried to outline the DSR formalism and make some of the concepts commonly used in the literature more precise. In [46] however, we have seen that there are multiple formulations of the DSR theory based on which basis is being used for the theory. An example is the DSR1 theory [16], which is a formulation in the bicrossproduct basis. In these different formulations, one of the key issues is the derivation and interpretation of the geometry of spacetime which as we have seen above derives as non-commutative geometry. In the next chapter, we will encounter a more geometric approach for recovering the geometry of spacetime. In particular we shall discuss a proposal which stems from considering the modification of the dispersion relation as level sets of a Hamiltonian on phase space. This is a different angle and provides a fresh view on the subject.

In this section however, we will be discussing the possibility of an energy-dependent spacetime metric which is dubbed the *rainbow* metric. We closely follow the work presented in [31],[52]. We have seen in this chapter the energy-dependence of the deformed transformation laws and the dispersion relation by having introduced the observer-independent energy scale E_p. With this in mind it is natural to ask, is the geometry of spacetime also energy-dependent in itself? This was the question asked in [31] and a detailed discussion of the energy-dependence of the geometry of spacetime can be found in that paper and references therein. As a starting point, [31] assumes an energy dependent metric written as $g(E)$, where the meaning of the quanitity E needs more clarity. E is then defined as the energy of a particle in spacetime as seen by someone observing the particle. However, by this definition, E will have kinetic energy contributions, leading to different observers attributing different energies to one and the same particle. Intuitively one can think of E as the energy with which a particle would probe spacetime, according to some distant observer. Recall, we saw in section 2.2 that introducing the invariant scale E_p resulted in deformations of momentum space, which can be achieved via the action of a map U of momentum space P into itself, $U : P \to P$. Thus having

$$U \circ (E, p_i) = (U_0, U_i) = \left(f\left(\frac{E}{E_p}\right) E, h\left(\frac{E}{E_p}\right) p_i \right) \tag{2.78}$$

In order to obtain a frame in which one can describe a spacetime geometry at very low energies, we make use of the functions f and h to write down energy-dependent orthornormal form fields

$$e_0 = f^{-1}(E/E_p)\tilde{e}_0, \quad e_i = h^{-1}(E/E_p)\tilde{e}_i \tag{2.79}$$

Then if one is to write the metric in terms of these fields we recover

$$g(E) = \eta^{ab} e_a \otimes e_b \tag{2.80}$$

$\tilde{e}_\mu$ are the frame fields, which describes the spacetime geometry at very low energies. A crucial requirement is that in the appropriate limit i.e. $E \to 0$, one recovers the classical spacetime which we usually work with in GR and is energy-independent

$$\lim_{E \to 0} g_{\mu\nu}(E) = g_{\mu\nu}^{classical} \tag{2.81}$$

We have ended up with a one-parameter family of metrics. As we have mentioned a metric with such an energy-dependence is called a rainbow metric and it should be noted that the introduction of an energy-dependent metric is merely a mathematical construct and is not physical. The reader might be asking himself, does this mean that with an increase in the kinetic energy the gravitational field around a particle moving gets stronger? The answer to this is no. The only thing that happens is that an observer sees the particle moving in a metric which depends on the energy of that particular particle. So far this discussion of the rainbow metric has been dealing with a single particle. However, the situation becomes a bit more complicated once one consideres different particles at the same point in spacetime but travelling at different speeds, thus having different energies according to an observer. In a situation like this, the observer will then associate different metrics with each of the particles, thus posing the question of what overall metric the observer will see. In [31], it is suggested that the result should be a superposition of all the different metric, however it is not yet known how exactly the metrics superimpose. We only mention the rainbow metric here for the sake of completeness.

2.4 Some limitations in Deformed Special Relativity

Our discussion so far has concentrated on the formal development of a theory of DSR and we have not mentioned physically observable consequences that the theory of DSR might have. However we cannot ignore this if we ever wish to bring closer theory and experiment. In this section and subsections herein, we will shift attention to a few limitations Deformed Special Relativity might have or paradoxes pertaining DSR. Some of these issues with DSR have cast heavy doubt on some aspects of DSR in the scientific community. We will now continue to outline these arguments. The list of limitations given here is however not exhaustive.

2.4.1 Distance or spacetime fuzziness

The fact that the structure of the Quantum Gravity problem suggests that the classical description of spacetime should give way to a nonclassical one at scales of order of the Planck scale has been extensively investigated in the literature. However in the nonclassical description of spacetime, a key characteristic of quantum theory is the emergence of uncertainties, and one would expect that the position observable would also be affected by uncertainties. In ordinary quantum theory, we can still measure sharply any given observable, at the cost of losing all information on a conjugate observable (HUP), however it appears as though in a theory of quantum gravity the observable distance would be affected by irreducible uncertainties given the minimal length L_P of quantum gravity. This essentially means that the uncertainty in the measurement of distances could not be reduced below the Planck-length level, however there have been measurability bounds of other forms considered [53], where the uncertainty δD is bounded by some function $f(D, L_p)$, which has a dependence on the minimal length. Regardless, the problem of a "fuzzy distance" is still pervasive. This problem of an irreducible measurement uncertainty could be a significant hindrance in some context, one of which would be the noise levels in the readout of a laser interferometer receiving an irreducible contribution from effects of quantum gravity. In principle, classically interferometric noise can be reduced to zero, such as done by the SKA, however already the analysis of ordinary quantum properties of matter includes an extra noise contribution. Taking into account quantum gravity would mean a fundamental uncertainty induced by the Planck scale. This would have an effect on the arms of the interferometer, hence introduce a source of noise.

2.4.2 Planck-scale departures from the equivalence principle

There have been quite a lot of views on the problem of quantizing gravity, however a few [53] seem to suggest the departures of one or the other forms of the equivalence principle i.e the strong or the weak. Here by the weak equivalence principle we mean the principle that states that all laws of motion for freely falling particles are the same as in an unaccelerated reference frame, whereas the strong equivalence principle builds upon the weak, however includes astronomic bodies with gravitational binding energies. We briefly mention here the argument which is based on the observation that locality is a key ingredient of the present formulation of the equivalence principle. In fact, what the equivalent principle advocates is that, given the same intitial conditions, two point particles would go on the same geodesic independently of their mass. But this including extended bodies and delocalised point particles in the equivalence principle, one reaches the conclusion that it does not apply for these objects. Now if the structure of spacetime were to induce an irreducible limit on the localization of particles, one could expect some departures from the equivalence principle.

2.4.3 The *Soccer-ball* Problem

In short, the "soccer-ball" problem is the problem of high energies that large bodies (with large energies exceeding the Planck energy) have in theories with maximal energies. This problem is termed as such due to a soccer ball being an example of a macroscopic body compared to the Planck length. These high energies exceed the Planck energy E_p and thus in DSR poses a paradox. More formally, in an approach to sum momenta when Lorentz symmetry in momentum space is modified, one chooses to maintain observer-independence [130]. In short, the problem is that we find a nonlinear modified Lorentz transformation $\tilde{\Lambda}$ acting on momenta thus one finds that when transforming the addition of momenta, the result differs to when one adds the individually transformed momenta, i.e.

$$\tilde{\Lambda}(\vec{p}_1 + \vec{p}_2) \neq \tilde{\Lambda}(\vec{p}_1) + \tilde{\Lambda}(\vec{p}_2) \tag{2.82}$$

Having this, we ruin one of the motivations to introduce deformed Lorentz-symmetry This then ruins observer-independence which was one of the main motivations to introduce deformed Lorentz-symmetry: If $\tilde{\Lambda}$ is the unit element of the group, then the equation is fulfilled, and thus the restframe in which it is fullfilled singles out a preferred frame

One thus concludes that it is necessary to define a new, non-linear, addition law for momentum that has the property that it remains invariant under Lorentz-transformations and that can be rightfully interpreted as a conserved quantity.

It is presently not known how to define the sum of momenta in approaches that modify Lorentz symmetry in momentum space and mantain observers independence. In a nutshell, the problem is that the modified Lorentz-transformation, $\tilde{\Lambda}$, that acts on momenta is nonlinear, and thus the transformation of the sum of momenta is not the same as sum of the transformation of the momenta

Luckily, it is possible to construct a Lorentz-invariant new addition law without too much trouble. To see how this works, we refer the reader to the elaborate publication [54].

2.4.4 The Box Problem

In DSR literature, the issue of non-locality has been around for some time now. We will try and give a qualitative description of the problem following the paper [55]. In the next section, we study an experiment which leads to effects of non-locality of macroscopic magnitude, as such one would believe they should have been observed a long time ago. However, for the reader not familiar with locality, we wil first try and explained what we mean by locality and non-locality in the context of DSR. By locality/non-locality we mean, when different observers agree or disagree on whether two events in spacetime are happenning at exactly the same point, independent of the observer. This is what is usually meant by a local theory. If a theory is however non-local, then events happening at a specific point in spacetime will only appear to be happening at that point for only one of the observers but not the others.

We have been arguing throughout this thesis that the energy associated to the Planck mass, should have an observer-independent meaning. However, the requirement of assigning an observer independent meaning to the Planck mass requires a modification of Special Relativity (hence DSR) and a new sort of Lorentz-transformations. In DSR as we have stated before, we postulate the Planck mass as an observer-independent invariant. The Lorentz transformations that leave the Planck mass invariant under a boost generically result in a modification of the standard dispersion relation and an energy dependent speed of light [47]. This energy dependent speed of light can, in the low energy be constant, increase or decrease.

In other Deformed Special Relativity considerations [34], we saw that the speed of light is a function of energy $c(E)$, such that this function is the same for all observers. Now suppose we transform the energy E to $\hat{E}$, then we should have that $c(E) = c(\hat{E})$ in contrast to SR where only one speed is invariant under the Lorentz-transformations i.e. $c = 1$. A confusion arises when one thinks about how an energy-dependent speed of light that can take different values can also be observer-independent.

To see how this confusion can be comprehended consider the following scenario. Suppose we have a monotonically decreasing speed of light which reaches zero as the energy approaches the Planck energy. Then when we consider a photon of energy $E = m_P$, this photon should be at rest. Now we put this photon inside a box and let an observer in another frame with speed v observe this photon inside a box. The box moves, relatively to the observer with speed v in the opposite direction of the observer. The box here is taken as a classical, macroscopic, low-energy object. At the instant the photon is put inside the box, the observer observes the photon inside the box, however as time passes, the observer sees the box at a distance from the photon considering the photon should be stationary irrespective of an observer. In contrast to the observer-dependence of 'the same' moment in time that one also has in Special Relativity, this concerns the observer dependence of what happens at the same time and the same place [55] and this gives problems if one wants to accomodate such a scenerario in a local theory. This problem is termed the "box-problem", and further and elaborate discussions of this problem can be found in [55]. The next section discusses the issue of locality or rather non-locality in the DSR context and an experiment depicting this paradox.

2.5 Relative Locality

In Special Relativity, when we infer the coordinates of a distant event we analyze light signals sent between the observer and the event. However, in doing so we throw away information about the energy of the photons. In this section, we ask ourselves questions like, how do we know we live in spacetime? And if so, how do we know we all share the same spacetime? We closely follow [44] in analyzing what happens when one considers the energy of light signals as well. Suppose we want to deduce spacetimes of events by Einstein's localization procedure, however we use two photons, one with the Planck energy and a red photon. Can we be sure that the spacetimes we infer in these cases is the same? Also how can we be sure that when two events are concluded to be at the same spacetime position by one observer, the same holds for another observer at a distance?

In Einstein's two theories of relativity SR and GR, the answer to the latter questions is yes. The simultaneity of objects is relative however locality is absolute. This clearly follows from assuming spacetime is where all of physics occurs. However in the approaches that exist of quantum gravity, the weakening of locality is certainly suggested and that the concept of spacetime is only emergent and should be replaced by something more fundemental (see 3 on how spacetime emerges from DSR). In the search for a full theory of quantum gravity, one would be interested in finding out if it was possible to relax the universal locality assumption in a controlled manner, such that we have a progression in the solution of the problem at hand. In [44] a theory is proposed which make an attempt at answering questions like the ones above, which relaxes the assumption of universal locality and termed it Relative Locality.

In the Relative Locality regime, we again explore a classical non-gravitation regime of quantum gravity as we have done with DSR and try and capture some of the key delocalising features of quantum gravity. In this regime, $\hbar$ and G are both neglected, while their ratio is held fixed:

$$\hbar \Rightarrow 0, \;\; G \Rightarrow 0, \;\; \text{but with fixed} \sqrt{\frac{\hbar}{G}} = M_p \tag{2.83}$$

Here we switch off both quantum mechanics and gravity, but keep the effects due to the presence of the Planck mass. Surprisingly, as one will see [56], this regime includes effects on very large scales which can be explored in astrophysical experiments. Furthermore, since $\hbar$ and G are both zero it can be investigated in simple phenomenological models. *Fig* 2.1 below shows the Relative Locality regime as a limit of the fundemental constants.

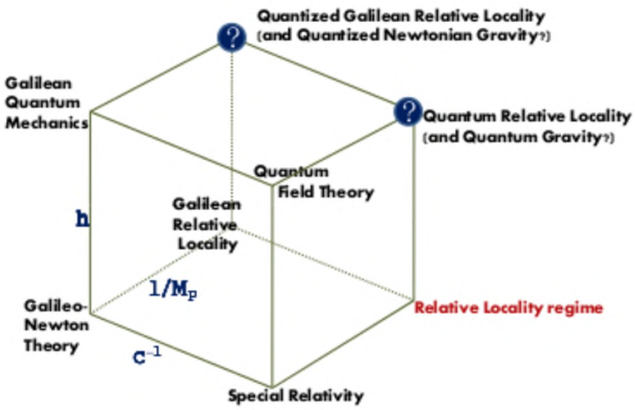

FIGURE 2.1: The limit at which Relative Locality is observed. This diagram is obtained from [131]

When we talk about universal locality of spacetime, this is usually taken to be equivalent to saying that momentum space is a flat linear space. However when we propose [44] a relative principle of locality, we obtain the mass scale M_P as parametrizing the non linearities of momentum space. Remarkably, these non linearities can be understood as introducing on momentum space a non-trivial geometry (an example will be seen in chapter 3). In [44], a precise formulation of the geometry of momentum space from which the consequences for the earlier questions can be exactly derived.

Max Born has been noted as the first person to propose [45] the idea that momentum space should have a non-trivial geometry when quantum gravity effects are taken into consideration as early as the 20th century. In his argument [45], he insists that the validity of quantum mechanics implies that there is in physics an equivalence between space and momentum space. Over the years, this equivalence has been coined the Born reciprocity. When we introduce gravity, this symmetry gets broken since in GR, spacetime is now curved while momentum space is a linear space. Thus for the symmetry to be maintained, allowing momentum space to be curved is a natural way of reconciling gravity with quantum mechanics from this view [44].

In describing the geometry of momentum space, we adopt the operational point of view in which a local observer is equipped with devices to measure the energy and momenta of particles around his vicinity [44]. In order for the observer to be able to determine the proper time we equip him with a clock. Now in this setup, one can determine the geometry of momentum space by also equipping the observer with aparatus which can measure energies like the calorimeter. When one constructs momentum space, similarly to spacetime, one requires a measure of distance i.e. a metric and for a notion of curvature or torsion one requires a connection. In this construction, one assumes that to each choice of the apparatus carried by the observer, there is a preferred coordinate system on momentum space, k_a, equivalently x^a for spacetime. Just as in spacetime, we would like the dynamics not to depend on the choice of the apparatus' coordinates i.e. to be expressed covariantly on our momentum space. We take the energy ground state $k_a = 0$ to be the origin, thus any other value k_a is a measure of momenta or energy above the ground state. Below we will derive a metric on the momentum space and the operational route to such a derivation is requiring the observer to measure only a single particle and as we will see later, be able to define a metric on the space. When one measures multiple particles, this determines the geometry of the space, whether it is curved at one point or straight at another, thus this measurement enables us to define a connection on our space. In this discussion, we employ the reader to study [44] for an extensive derivation of the connection.

In determining the metric geometry, we employ our observer to measure the rest energy or relativistic mass of a particle. This measurement is a function of the four momenta, as we are in momentum space, the energy and 3-momenta of a particle is required. Locally the observer can also measure the kinetic energy K of a particle moving with respect to him. Our interpretation of these measurements will allow us to determine the metric geometry of the space. The relativistic mass, with respect to the coordinate system k_a is taken to be the geodesic distance from the ground state or origin, thus this give

$$D^2(p) = D^2(p,0) = m^2 \tag{2.84}$$

which is the dispersion relation, with m the measured mass and D the geodesic distance of the particle.

Measuring the kineric energy then allows us to define the geodesic distance between a p particle at rest and a particle $p_{\prime}$ of the same mass and with kinetic energy K. That is $D(p) = D(p') = m$ and [44]

$$D^2(p,p') = -2mK \tag{2.85}$$

where we have the minus sign reflecting the Lorentzian of the momentum space. When one uses all these measurements, one can define a metric on momentum space as

$$dk^2 = h_{ab}(k)dk^a dk^b \tag{2.86}$$

As we have mentioned before, for the curious reader an extensive mathematical and experimental derivation of the connection in momentum space, can be found in [44].

Now one notices that within this framework of relative locality, we find that the momentum space P can actually be curved. Thus there cannot be a single spacetime M for every point in momentum space. Therfore, we conclude that for each point $k \in P$ there should be a seperate spacetime M_k compared to a different point $k' \in P$ which would have a spacetime $M_{k'}$. This implies now that the phase space would be the cotangent bundle over momentum space i.e. $\Gamma = T^*P$.

It would be interesting to try and compare the worldlines of particles at different points in momentum space. In this scenerio if two particles' A and B worldlines were to meet, we cannot simply say $x^\mu_A = x^\mu_B$, because this would be intuitively wrong. One cannot assert $x^\mu_A = x^\mu_B$, because they live in different spaces just as in spacetime different momenta p^A_μ and p^B_μ of particles A and B would. Thus in a situation like this, one would ask how would you parallel transport one worldlne of particle A on momentum to that the spacetime of particle B. In the following chapter we see how this works with Deformed Special Relativity.

2.5.1 Deformed Special Relativity, a non-local theory?

In this section, we will construct an experiment which will lead to the conclusion that DSR theory is a non-local theory. More precisely, since we have seen that there is no single formulation of DSR theory that everyone agrees on, one should specify that we are talking about DSR theories, which have an observer-dependent speed of light. This arguement follows closely [28]. Let us introduce a law which gives the phase velocity of a photon in terms of its energy E

$$c(E) \approx \left(1 + \alpha \frac{E}{M_p}\right) + \mathcal{O}(\frac{E^2}{M_p^2} \tag{2.87}$$

where we only consider first order corrections in $\frac{E}{M_p}$ and α is a constant, which is chosen to be negative in order for the velocity to decrease with increasing energy. The crucial point about this relation is that its functional form does not depend on the observer. Thus meaning that, even though the energy E is not invariant under transformations, the relation is. We now continue to describe the experiment.

Consider a gamma-ray burst, which emits two different types of photons. One is a photon with very high energy $E_\gamma \approx 10GeV \approx 10^{-18}M_p$ and the other, a very low energy photon which we use as a reference. We should note that, both photons are detected here using an earth-bound detector and that the distance between the gamma-ray burst and the earth is set to be $L \approx 4Gpc \approx 10^{26}m$. If we are to assume that the low and high energy photons get emitted simultaneously, then due to the energy dependence in the expression for the phase velocity, we find that there is a delay ΔT in the arrival times of the two photons in the detector. This delay is given by

$$\Delta T = L\frac{E_\gamma}{M_p} + \mathcal{O}\left(\frac{E_\gamma^2}{M_p^2}\right) \tag{2.88}$$

When we use the values given above, one finds that this delay is of the order of 1 second. Lets now add a second detail to the experiment.

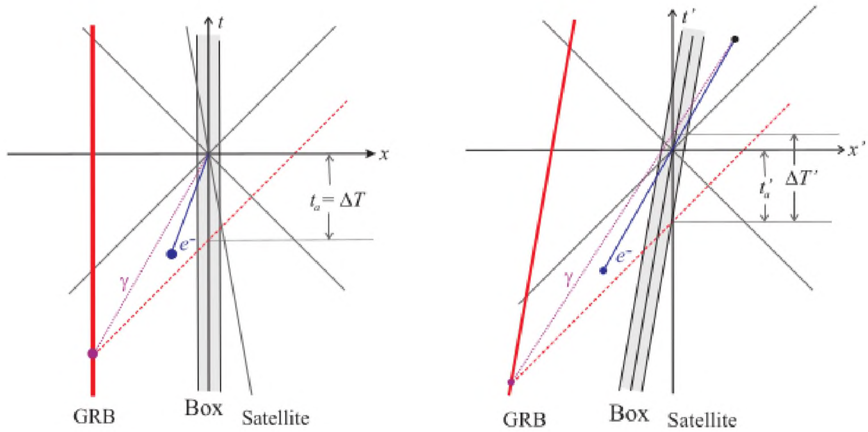

FIGURE 2.2: The setup of the non-locality. Both figures are taken from [28] (resolution is at best)

Consider also an electron with an energy $E_{e^-} \approx 10MeV$, which gets emitted by a source which is directly in the vicinity of the detector on the earth. The source emits the electron in such a way that it arrives at the detector simultaneously with the high energy photon, thus also a time difference $-t_{e^-} = \Delta T$ earlier than the low energy photon. To add, also imagine, the high energy photon and the electron scatter off each other and creat some kind of macroscopic irreversible change. The precise details of this event are not crucial in this discussion.

Now we ask ourselves, what will a second observer see when observing the above setup, in particular to the arrival times of the different photons and the electron in the detector on Earth. Imagine, this second observer Neo, is in a satellite and moving towards the gamma-ray burst. Assuming Neo's satellite is moving at a speed of $v_s = -10km.s^{-1}$ with respect to the detector on the earth, then this leaves a Lorentz factor of $\gamma_s \approx 1 + 10^{-9}$. From Neo's rest frame, the high energy photon and the electron will appear to be blue-shiffted, however the very low energy photon will still be seen as having a very low energy. Thus calling the low energy photon the reference photon. Now, the high energy photon's energy is

$$E_\gamma^{'} = \sqrt{\frac{1-v_s}{1+v_s}}E_\gamma + \mathcal{O}\left(\frac{E_\gamma^2}{M_p^2}\right) \tag{2.89}$$

where E'_γ now indicates the energy in Neo's rest frame as opposed to the earth-bound detector's rest frame energy E_γ. Also making a transformation in the arrival difference time of the low energy photon and the electron t_{e^-} into Neo's frame, we find

$$t'_{e^-} = \frac{L}{\gamma_s} \frac{1/c(E_\gamma) - 1}{1 - v_s} \tag{2.90}$$

Since the high energy photon now has a different energy in Neo's frame, this will affect the speed (2.87), thus we need to calculate the new speed $c(E'_\gamma)$. Here the functional invariance of the expression $c(E)$ comes into play and we have

$$c(E'_\gamma) = 1 - \frac{E'_\gamma}{M_p} = 1 - \sqrt{\frac{1 - v_s}{1 + v_s}} E_\gamma + \mathcal{O}\left(\frac{E^2_\gamma}{M^2_p}\right) \tag{2.91}$$

Now also the distance that has been travelled by the photons becomes $L' = \gamma_s \left(v_s / c(E_\gamma) - 1\right) L$. Putting everything together, we then find that the ΔT in Neo's frame is

$$\Delta T' = \frac{E'_\gamma}{M_p} L' = \frac{1 - v_s}{1 + v_s} \Delta T + \mathcal{O}\left(\frac{E^2_\gamma}{M^2_p}\right) \tag{2.92}$$

Recalling in the earth-bound frame we had $-t_{e^-} = \Delta T$, we now find that in the primes frame we have

$$\Delta T' - t'_{e^-} = \left(\frac{1 - v_s}{1 + v_s} - \frac{1}{\gamma_s(1 - v_s)}\right) + \mathcal{O}\left(\frac{E^2_\gamma}{M^2_p}\right) \tag{2.93}$$

Now if we insert our values for γ_s and v_s, we are left with $\Delta T' - t'_{e^-} \approx 10^{-5} \Delta T$, or in other words, the photon is lagging roughly one kilometer behind the electron. Thus we find that in Neo's frame, the high energy photon and the electron do not arrive in the detector at the same time, as they did in the detector's rest frame. As a result they will not scatter off each other and will not cause the macroscopic event. This is how non-locality in DSR theory arises, if we assume an observer-independent, energy-dependent phase velocity. Given the fact that we have not yet observed any such non-local effects allows us to make the conclusion that we can actually rule out an energy dependence of the form (2.87) up to a high precision. More presicely, since the centre of mass of the high energy photon and the electron is roughly $15 MeV$, the scattering process probes a spacetime distance of the order of $10 fm$. If the distance difference between the arrival of the electron and the high-energy photon in the detector is less than this distance probed by the scattering process, then non-locality is not an issue. Thus if $|\Delta T' - t'_{e^-}| < 10 fm$ is fulfilled then the theory is still local. For boosts of up to about $\gamma_s = 30$, this allows us to find a bound on ΔT, namely $\Delta T < 10^{-23}$s or alternatively we can also find $|\alpha| < 10^{23}$. We see that by the latter inequality, we can rule out modifications in the phase velocity to first order in E/M_p.

Before ending this section it is worth noting the assumptions that we have made in the derivation of non-local effects. A very crucial assumption is that of an energy-dependent but observer-independent speed of light in position space. And the only reason why one might actually believe that the speed of light does, in fact depend on the energy is because of the modified dispersion relation, which as we have seen before is an inherent feature of the theory of DSR.

2.6 Summary

In this chapter we have introduced Deformed Special Relativity in depth, going through the algebraic construction of the theory. We further showed how the theory of DSR can be expressed in different basis and how these basis are all equivalent. We did not exhaust all the different bases which DSR can currently be constructed in, however we felt the three mentioned here made the point we were trying to make. Furthermore, we mentioned some of the limitations or paradoxes that a theory of DSR could or is facing, and directed the reader to how these are solved in the literature. Moreover we introduced the Relative locality regime in hopes of showing the equivalence between DSR and RL. In our discussion of RL, we mentioned why a theory of RL is needed, furthermore we described how and why momentum space becomes curved in RL. We then showed as an example how one can obtain the metric geometry of momentum space through energy measurements and concluded the chapter by constructing an experiment which led to the conclusion that DSR theory is a non-local theory.

Part II

Hamilton Geometry of Deformed Special Relativity

Chapter 3

Geometrization of Deformed Special Relativity

FIGURE 3.1: Sir William Rowan Hamilton

3.1 Lagrangian and Hamiltonian mechanics: A review

In this section we review the Lagrangian and Hamiltonian mechanics based on the Principle of least action. Further we discuss the necessary structures to geometrize the space of Hamiltonian functions. Moreover we provide useful and coherent ways to analyse Hamiltonian mechanics.

Principle of Least Action

Consider a configuration of a mechanical system evolving in an n-dimensional space, with coordinates $\vec{x} = (x^1, x^2, ..., x^n)$. One may decide to describe this system in generalized coordinates $\vec{q} = (q^1, q^2, ..., q^n)$. The usefullness of these generalized coordinates will soon be apparent. Hamilton's principle otherwise the principle of least action can be simply stated as: Out of all possible paths in configuration space, between two points along which a dynamical system may move from one point to another within a given time interval $\left(t_i, t_f\right)$, the actual path followed by the system is the one which extremizes the line integral of the Lagrangian.

The function $L = L(q, \dot{q}, t)$, which appears in the action integral below is called the Lagrangian. The integral

$$S = \int_{t_i}^{t_f} L(q, \dot{q}, t)dt \tag{3.1}$$

is called the action of the system.

More precisely Hamilton's Principle can be stated according to [57] as follows: *Hamilton's Pricincple: The dynamical behavior of a classical system is completely determined by the Lagrangian $L : TM \to \mathbb{R}$. The dynamical trajectories are the solutions of the variational equation $\delta S = 0$ where S is the functional 3.1, where the variation is over all curves $\gamma : [t_i, t_f] \to M$ with fixed endpoints $\gamma(t_i)$, $\gamma(t_f)$.*

The variation of the above action according to the principle of least action yields a set of equations, called *Euler-Lagrange* equations. These equations specify the dynamics of a conserved mechanical system on the configuration space and are thus the equations of motion. These equations are of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{3.2}$$

Usually, one can obtain Hamilton equations by deriving them from the Euler-Lagrange equations. However, since we did not dwell on the variational principle with the Lagrangian, we will use it with the Hamitlonian. The *Hamiltonian* equations of motion can be directly calculated from the least-action principle. We start by defining the Hamiltonian function H, as the function given by

$$H = \Sigma_i p_i \dot{q}_i - L \tag{3.3}$$

Then using the action equation (3.1) together with this definition, one obtains

$$S = \int L(q, \dot{q}, t)dt = \int (\Sigma_i p_i dq_i - H dt) \tag{3.4}$$

Here in contrast to Lagrange formalism, one considers both p_i and q_i on equal footing and as independent dynamical variables and make the infinitesmal variation

$$q_i \to q_i + \delta q_i \quad p_i \to p_i + \delta p_i \tag{3.5}$$

The corresponding action variation then has the form

$$\delta S = \Sigma \int \left(\delta p_i dq_i + p_i d\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i dt - \frac{\partial H}{\partial p_i}\delta p_i dt\right) \tag{3.6}$$

Integrating the second term by parts and using that $\delta q_i|_{boundary} = 0$ (the variation of q is assumed to vanish at the boundaries) , one can then rewrite this equation as

$$\delta S = \Sigma \int \left[\delta p_i \left(dq_i - \frac{\partial H}{\partial p_i}dt\right) + \delta q_i \left(-dp_i - \frac{\partial H}{\partial q_i}dt\right)\right] \tag{3.7}$$

Since for the actual trajectories, the action variation is trivial i.e $\delta S = 0$, taking δq_i and δp_i to be arbitrary and independent, one concludes that both expressions in round brackets are zero, therefore

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \tag{3.8}$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{3.9}$$

These two equations of motion are dubbed *Hamiltonian equations of motion*.

Hamilton's equations form a system of $2n$ ordinary differential equations of the first order for the $2n$ unknown functions $q^i(t), p_j(t), i, j = 1, 2, ..., n$. Suppose we are given the initial values of these $2n$ functions at some instant in time t_0, we are guaranteed by the standard existence and uniqueness theorems of the theory of differential equations that these equations of motion possess a unique solution on some time interval containing t_0. That is to say, given the initial values of the spacetime coordinates and momenta, Hamilton's equations completely specify the position and momentum coordinates at all other times $t \in (t_0, t_0 + c)$ for some constant $c > 0$.

Unification of coordinates

As we have seen above, in the Hamiltonian formalism, one treats the q and p variables or coordinates on an equal footing. In this section, we explore the unification of these coordinates into a set of $2n$ variables that will be called the "unified coordinates" written ζ^i, with the index $i \in \{1, 2, ..., 2n\}$. These coordinates are divided into two smaller sets, with the first n of the ζ^i representing the configuration space coordinates q^β, and the second n, the momenta p_β. More explicitly,

$$\zeta^i = q^i, \quad i \in \{1, ..., n\} \tag{3.10}$$

$$\zeta^i = p_{i-n}, \quad i \in \{n+1, ..., 2n\} \tag{3.11}$$

The next obvious investigation would be writing Hamilton equations of motion i.e. equations (3.8) and (3.9) in these coordinates. Here one has to be careful reading the indices as they must have different sets in which they run. We want to write $\dot{\zeta}^i = f^i(\zeta)$, where the f^i functions are to be determined using the equations of motion in canonical coordinates. One then finds that the first n of the f^j are given by $\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial \zeta^{i+n}}$, and the last n are $-\frac{\partial H}{\partial q^i} = -\frac{\partial H}{\partial \zeta^i}$, such that the equations of motion are given as [57]

$$\dot{\zeta}^k = \frac{\partial H}{\partial \zeta^{k+n}} \quad k = 1, ...n \tag{3.12}$$

$$\dot{\zeta}^k = -\frac{\partial H}{\partial \zeta^{k-n}} \quad k = n+1, ...2n \tag{3.13}$$

One may then want to compactify these equations into one equation where the indices run in a single set. In order for one to accomplish this, we introduce a $2n \times 2n$ matrix, given by

$$\Theta = \begin{pmatrix} 0_n & -\mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix} \tag{3.14}$$

where $\mathbb{I}_n$ is the identity matrix and 0_n, the null matrix. This matrix satisfies the following properties

$$\Theta^2 = -\mathbb{I}, \;\; \Theta^T = -\Theta \tag{3.15}$$

That is, it satisfies $\Theta^{-1} = -\Theta$ and is antisymmetric. To complete the introduction of this matrix, we note that we will denote by θ^{ij} the elements of the matrix Θ. We can then proceed by denoting $\partial_k = \frac{\partial}{\partial \zeta^k}$, that the canonical equations of motion can be written compactly as

$$\dot{\zeta}^i = \theta^{ij} \partial_j H \tag{3.16}$$

or alternatively

$$\theta_{ki} \dot{\zeta}^i = \partial_k H \tag{3.17}$$

here repeated indices indicate summation over the index.

3.1.1 The Poisson bracket and the symplectic structure

Poisson bracket

Another important idea in Hamiltonian mechanics is that of the Poisson bracket. Generally if $f, g \in \mathcal{F}(T^*M)$, where $\mathcal{F}(T^*M)$ is the functional space of T^*M, and where M is a base manifold with T^*M its cotangent manifold, then the Poisson bracket of f with g is defined as [57]

$$\{f, g\} = (\partial_i f)\theta^{ik}(\partial_k g) = \frac{\partial f}{\partial \zeta^i}\theta^{ik}\frac{\partial g}{\partial \zeta^k} = \frac{\partial f}{\partial q^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q^\alpha} \tag{3.18}$$

where the rightmost expression gives the definition in coordinates which we are yet to define as canonical, which is usually the most useful form. The Poisson bracket in turn satisfies the usual properties of brackets which include bilinearity, antisymmetry and the satisfaction of the Jacobi identity, hence it is a derivative operator if you fix one of its entries. In addition it satisfies a Leibnitz type of product rule [57]:

$$\{f, gh\} = g\{f, h\} + \{f, g\} h \tag{3.19}$$

hence by fixing one of its entries and not the other it becomes a derivative operator due the nature of its definition (defined usig space-time and momentum derivatives).x The aforementioned properties define a specific kind of algebraic structure called a Lie algebra, thus we note that the functional space $\mathcal{F}(T^*M)$ is a Lie algebra under the Poisson bracket. We will see in the forthcoming sections that one can study Hamilton dynamics from the perspective of Lie algebras.

Suppose we consider a dynamical variable $f(q, p, t)$, it is possible to find how this variable varies along the motion using the Lie derivative along the dynamical vector field. Here by dynamical vector field we mean the field representing the flow. With the use of the bracket introduced here, we can find the change with respect to time of the variable f as follows

$$L_\Delta f = \frac{df}{dt} = \{f, H\} + \partial_t f \tag{3.20}$$

where Δ is a covariant derivative. We will discuss what covariant derivative is in the later sections. We can apply this equation to every dynamical variable and by taking f to be the canonical coordinates, one can recover the equations of motion i.e.

$$\dot{\zeta}^i = \left\{\zeta^i, H\right\} \tag{3.21}$$

We can also use (3.20) to indicate the conservation of energy for time-independent Hamiltonians. Due to the antisymmetry of the bracket one should note that $\{f, f\} = 0$ for any $f \in T^*M$. Thus applying this property and using (3.20) we recover

$$\dot{H} = \{H, H\} + \partial_t H = \partial_t H \tag{3.22}$$

that is, the total time derivative of the Hamiltonian is equal to its partial time derivative. However, if the Hamiltonian does not depend on time then $\partial_t H = 0$, thus $\dot{H} = 0$. One recovers the conservation of energy.

Symplectic form

This chapter focuses on the geometrization of Deformed Special Relativity, therefore the next logical step would be looking at how one can geometrize the equations of motion (3.8) and (3.9). In geometrizing these equations, we briefly introduce some of the notation to be used which include the interior product or contraction $i_X : \Omega^p(M) \to \Omega^{p-1}(M)$ where X is a vector field on $\Omega^p(M)$ and where $\Omega^p(M)$ is the p-dimensional space of vector fields on M. We will dwell more on this notion in the upcoming sections of the chapter. We now proceed to see how this relates to the canonical equations. The left and right hand sides of equation (3.16) are respectively given by the one-forms $i_\Delta \omega$ and dH. Where Δ is a dynamical vector field on T^*M with components given by the canonical equations of motion. By canonical we mean equations in terms of the coordinates (p_i, q^i), which satisfy the equation

$$\left\{p_j, q^i\right\} = -\delta^i_j \tag{3.23}$$

Thus in these local coordinates this vector field is of the form

$$\Delta_H = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \dot{p}_\alpha \frac{\partial}{\partial p_\alpha} \tag{3.24}$$

where H indicates a Hamiltonian vector field. Equation (3.16) is then geometrically given by

$$i_{\Delta}\theta = dH \tag{3.25}$$

with the θ_{ij} of (3.16) being components of a two-form θ. To see that θ is here a two-form, note that i_{Δ} is a contraction thus, as stated above $i_{\Delta}\theta$ is a one-form. Using local coordinates on phase space, a two-form is generally of the form [57]

$$\theta = \theta_{\alpha} dq^{\alpha} \wedge dp_{\alpha} + \frac{1}{2}\left(a_{\alpha\beta} dq^{\alpha} \wedge dq^{\beta} + b^{\alpha\beta} dp_{\alpha} \wedge dp_{\beta}\right) \tag{3.26}$$

with constants $a_{\alpha\beta} = -a_{\beta\alpha}$ and $b^{\alpha\beta} = -b^{\beta\alpha}$. Contracting the latter two-form θ with the vector field Δ one recovers

$$i_{\Delta}\theta = \dot{q}^{\mu}\left(\theta^{\beta}_{\mu} dp_{\beta} + a_{\mu\beta} dq^{\beta}\right) - \dot{p}_{\mu}\left(\theta^{\mu}_{\alpha} dq^{\alpha} - b^{\alpha\beta} dp_{\alpha}\right) \tag{3.27}$$

On the other hand, the right-hand side of equation (3.24) is of the form

$$dH = -\dot{p}_{\mu} dq^{\mu} + \dot{q}^{\mu} dp_{\mu} \tag{3.28}$$

thus equating (3.25) and (3.26) shows that $a_{\mu\beta} = b^{\mu\beta} = 0$ and $\theta^{\mu}_{\alpha} = \delta^{\mu}_{\alpha}$ and consequently recovering

$$\theta = dq^{\alpha} \wedge dp_{\alpha} \tag{3.29}$$

We call this two-form the *symplectic form*, thus in order for one to geometrize the equations of motion, the required two-form is given by (3.29). We call a two-form closed if $d\theta = 0$, and one should notice that θ is closed and nondegenerate. Nondegeneracy explicitly states that $i_X\theta = 0$ iff X is the null vector field, with physical consequences given in [57].

Canonical Transformations

A transformation in general is a mapping of variables to "new" variables. Saying a transformation is canonical in classical mechanics limits us to a certain set of transformations. In the next discussion, we consider a system with n degrees of freedom. Suppose that one is faced with a classical system that is described by the Hamiltonian $H(q,p)$ in phase space $\mathbb{P} = T^*M$, where M is the configuration space and T^*M is its cotangent bundle, such that for any point $\eta \in \mathbb{P}, \eta = (q,p)$. The coordinates (q^i, p_i) undergo a time evolution which is explicitly described by the equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \tag{3.30}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \tag{3.31}$$

We say one has a canonical transformation $(q^i, p_i) \to (\tilde{q}^i, \tilde{p}_i)$, if the new variables $\tilde{q}, \tilde{p}$ preserve the form of equations (3.30) and (3.31) respectively. That is we want the coordinate transformations $\tilde{q}^i = \tilde{q}^i(q,p)$ and $\tilde{p}_i = \tilde{p}_i(q,p)$ to satisfy the equations

$$\frac{d\tilde{q}}{dt} = \frac{\partial \tilde{H}}{\partial \tilde{p}} \tag{3.32}$$

$$\frac{d\tilde{p}}{dt} = -\frac{\partial \tilde{H}}{\partial \tilde{q}} \tag{3.33}$$

where $\tilde{H}$ is the "old" Hamiltoninan expressed in the "new" coordinates or alternatively, one could think about it in a more geometric sense in that the following must be satisfied. If we start with canonical coordinates

$$\left\{q^i, p_j\right\} = \delta^i_j \tag{3.34}$$

then the “new" coordinates should also be canonically conjugate, i.e. they should also satisfy

$$\left\{\tilde{q}^i, \tilde{p}_j\right\} = \delta^i_j \tag{3.35}$$

Important conditions on $\tilde{q}$ and $\tilde{p}$ include being differentiable and invertible. If these are satisfied then one is able to write the ”old” Hamiltonian $H(q,p)$ in terms of the the transformed coordinates i.e $\tilde{H}(\tilde{q},\tilde{p})$ is possible. However suppose we are given the Hamiltonian H and a particular canonical transformation, a missing ingredient in order to determine the ”new” Hamiltonian $\tilde{H}$ is the generating function F of canonical transformation. The method to determine $\tilde{H}$ is neatly shown in [58]. Using equations (3.30) - (3.33), and the chain rule one can calculate the L.H.S of (3.32) and (3.33) respectively and find that they reduce to

$$\frac{\partial \tilde{H}}{\partial \tilde{p}} = \frac{\partial H}{\partial \tilde{p}} \{\tilde{q}, \tilde{p}\} - \frac{\partial \tilde{p}}{\partial t} \tag{3.36}$$

$$\frac{\partial \tilde{H}}{\partial \tilde{q}} = \frac{\partial H}{\partial \tilde{q}} \{\tilde{q}, \tilde{p}\} - \frac{\partial \tilde{p}}{\partial t} \tag{3.37}$$

Since $H \in \mathcal{F}(\mathbb{P})$, where $\mathcal{F}(\mathbb{P})$ is the functional space of $\mathbb{P}$, an important point to note is that in this work, the Hamiltonian $\tilde{H}$ must essentially be the original Hamiltonian H re-expressed in terms of the the new variable $(\tilde{q}^i, \tilde{p}_i)$. In that case the equations (3.36) and (3.37) hold, regardless of the Hamiltonian if and only if

$$\left\{\tilde{q}^i, \tilde{p}_j\right\} = \delta^i_j \tag{3.38}$$

and $\tilde{q}^i = \tilde{q}^i(q,p,t) = \tilde{q}^i(q,p)$ and $\tilde{p}_i = \tilde{p}_i(q,p,t) = \tilde{q}_i(q,p)$ i.e new variables must be time independent. After a bit of algebra, it can be shown that (3.38) is equivalent to the equation

$$\frac{\partial}{\partial q^i}\left(\tilde{p}_j \frac{\partial \tilde{q}^j}{\partial p_k}\right) = \frac{\partial}{\partial p_i}\left(\tilde{p}_j \frac{\partial \tilde{q}^j}{\partial q^k} - \tilde{p}_i\right) \tag{3.39}$$

It will be shown in subsequent sections that this condition encodes the preservation of the symplectic form θ. It can be found in [58] that (3.39) is a necessary and sufficient condition for the local existence of the function F , which we’ve called the generating function, such that

$$\tilde{p}_i d\tilde{q}^j - p_j dq^i = dF_{ij} \tag{3.40}$$

One main advantage of using canonical transformations is that these transformations leave invariant the Poisson Bracket defined by$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$, where f and g are functions on the manifold. It is worth noting that there are more general coordinate transformations that leave the Hamiltonian invariant in contrast to those that also satisfy (3.39)

3.2 Hamilton Geometry: The Geometry of Phase space

In general relativity, spacetime is entirely determined by the metric. In a similar fashion, in Hamilton geometry we allow the Hamilton function H of the phase space of free particles to determine the geometry of phase space [59]. This framework allows for a description where spacetime and momentum space become curved and intertwined. Equipped with a Hamilltonian describing the propagation of a free particle on phase space, one then fixes a symplectic structure. We know that in Special Relativity, at each point, a freely falling particle has the dispersion relation

$$E^2 = p^2 + m^2 \tag{3.41}$$

where E is the energy, $\vec{p}$ the three momenta and m the particles's mass. In this framework, we find that the dispersion relations are simply level sets of Hamilton functions on phase space. Given a spacetime metric g and its inverse g^{-1} and the four momentum p of a particle, we can covariantly rewrite the dispersion relation in terms of the Hamiltonian H_g as such

$$H_g(x,p) = g^{ab}(x)p_a p_b = m^2 \tag{3.42}$$

Now the covariant dispersion relation demonstrates that the geometry of spacetimes is derived from second derivatives of H w.r.t the momenta p of particles. Also that particle worldlines are basically determined by Hamilton equations of motion.

In this chapter, we will demonstrate that the Hamiltonian phase space is constructed as a cotangent bundle of a base manifold, furthermore explore the Hamilton geometry framework in detail including a detailed example where we apply the objects of the Hamilton geometry. Moreover, we look at Planck scale modified dispersion relations of free point particles and determine what underlying spacetime geometry one gets.

3.2.1 Phase space as a cotangent bundle

Phase space is the natural setting for studying the dynamics of an N-particle system. Suppose we have a real n dimensional smooth manifold M. Then around any $p \in M$ there exists a local chart $\{U, \phi\}$ where U is an open set in M, and ϕ takes points $p \in U$, and maps them to a local coordinate frame $\left(x^1(t), ..., x^n(t)\right) \in \mathbb{R}^n$ where p exists independent of a coordinate system. We define, the cotangent bundle of M as the manifold $T^*M = \cup_{p\in M} T_p^*M$ then the local induced coordinates on a subset of $\pi^{-1}(U) \in T^*M$ are $\left(x^i, p_i\right)$, where π is the projection defined by $\pi : T^*M \to M, \pi(\rho) = p \;\forall \rho \in T^*M$, *fig.* 3.2.

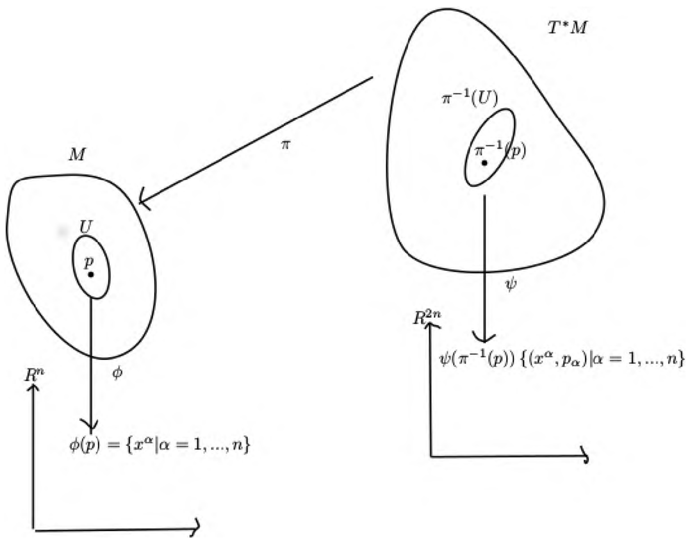

FIGURE 3.2: Construction of Phase space $\mathbb{P} = T^*M$, as cotangent space of configuration space M.

Remark 3.2.0.1. The map $\pi : T^*M \mapsto M$ is called the projection map of the fibre bundle $\left(T^*M, \pi, M, T_p^*M, G\right)$ where these are respectively the total space, base manifold, fibre and Lie group repectively. A lie group is simply a smooth manifold.

In section 3.1.1, we mentioned one-forms without defining what a one-form is. Given a vector $X \in T_xM$ at a point $x \in M$, we call the cotangent vector of X at $x \in M$ a one-form. Here we introduce the one-form in the canonical coordinates and write it as $\Omega_p \in T_p^*M$, $\Omega_p = p_i dx^i$. The exterior derivative of this one-form gives the aforementioned symplectic two-form θ. We call the variable p_i in this definition the momentum variable and it is canonically conjugate to the configuration space variable x^i.

Suppose one wants to perform a coordinate transformation in configuration space M i.e. $x \to \tilde{x}$. Intrinsically this will have an impact on the cotangent bundle of M, thus inducing a change in the local coordinates on T^*M as follows

$$\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n) \quad \tilde{p} = \frac{\partial x^j}{\partial \tilde{x}^i} p_j \tag{3.43}$$

with the tilde indicating transformed coordinates. Now equipped with the local coordinates and coordinate changes on T^*M, we now simply ignore the base manifold for a moment and consider the cotangent bundle as a manifold in its own right. Since T^*M is a differentiable manifold, one can have a look at its tangent and cotangent spaces i.e the tangent respectively cotangent spaces of the cotangent bundle of the base manifold.

As stated before the exterior derivative of the one form Ω is called the symplectic two-form θ and is given by

$$\theta = d\Omega = dp_i \wedge dx^i \tag{3.44}$$

However, what we did not mention was the importance of this structure in our geometry. This is the symplectic structure on phase space T^*M and as a consequence, the manifold induced coordinates (x, p) have the property that the Poisson bracket between them is the canonical one i.e

$$\left\{x^i, p_j\right\} = \frac{\partial x^j}{\partial p_i}\frac{\partial p_j}{\partial x^i} - \frac{\partial p_i}{\partial p_j}\frac{\partial x^j}{\partial x^i} = \delta^i_j \tag{3.45}$$

Having introduced the basic geometrical structures required for phase space analysis, we now embark on our quest on geometrizing DSR. We will introduce any neccessary geometrical entities as we proceed through out the chapter.

3.2.2 Hamilton Geometry

As suggested by [59], here we propose the use of the framework of Hamiltonian geometry of phase space as the natural arena for geometrizing manifestations of quantum gravity in dispersion relations. We have previously stated that in the quantization of gravity one expects to recover a non-commutative space and by DSR, this space is accompanied by a curved momentum manifold of curvature of the order of the Planck scale. In this section and subsections herein, we attempt at showing geometrically how these results are found by closely following the framework layed down in [59]. Quantum gravity phenomenlogy has received a lot of attention recently, as it should since some astrophysical observations are reaching sensitivity levels that allow one to test the consequences of modified dispersion relations on the time of propagation of particles i.e. experiments like Gamma Ray Bursts (GRB) detection[25]. It is however always possible to interpret any dispersion relation as the level sets of a Hamilton function H on phase space

$$H(x, p) = m^2 \tag{3.46}$$

where m is the particle's mass.

First we introduce the Hamiltonian vector field which we only mentioned in passing, this is an essential geometrical structure of phase space and phase space symmetries. The Hamiltonian vector field X_H is

Definition 3.2.1. : The vector field X_H on T^*M defined by

$$dH = -X_H \lrcorner \theta \tag{3.47}$$

is the globally defined Hamiltonian vector field determined by H

here the "left hook product" is defined by

$$(X \lrcorner \theta)(Y) = 2\theta(X, Y) \tag{3.48}$$

The following theorem lists important properties of a Hamilton system

Theorem 3.2.1. *: For a Hamiltonian system (T^*M, θ, H), where H is the Hamiltonian funcion and θ, the symplectic structure the following hold*

1. *There is a unique vector field $X_H \in \mathcal{X}(T^*M)$ for which*

$$i_{X_H}\theta = -dH \tag{3.49}$$

2. *The integral curves of the vector field X_H are given by the Hamilton-Jacobi equations which are as follows*

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \tag{3.50}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \tag{3.51}$$

The curves which solve the equations given in theorem 3.2.1 are called the autoparallels of the Hamilton geometry of the cotangent bundle when the Hamiltonian is homogenous with respect to the momenta as we will see later. Autoparallels of the connection are curves $\gamma : R \rightarrow T^*M$ with purely horizontal tangent. Thus γ is an autoparallel if it satisfies [59]

$$\dot{p}_a + N_{ab}(x.p)\dot{x}^b = 0 \tag{3.52}$$

Another useful proposition is the relationship between θ and the induced Poisson brackets $\{., .\}$ which is as follows

Lemma 3.2.1.1. $\{f, g\} = \theta(X_f, X_g)$ $, \forall f, g \in \mathcal{F}(T^*M)$ $, \forall\ X \in \mathcal{X}(T^*M)$

Proof. $\{f, g\} = X_f g = -X_g f = -df(X_g) = (i_{X_f}\theta)(X_f) = \theta(X_f, X_g)$ □

Thus consequently our equations of motion can be expressed as follows

$$\frac{dx^i}{dt} = \left\{H, x^i\right\} \tag{3.53}$$

$$\frac{dp_i}{dt} = \{H, p_i\} \tag{3.54}$$

This lemma indicates the elegance of using Poisson brackets on phase space.

In the next section we discuss the notion of connections. We already know, a linear connection is a structure on a vector bundle that defines a notion of parallel transport on the bundle. By parallel transport here, we mean a way to "connect" or identify fibers over nearby points. A linear connection can equivalently be specified by a covariant derivative, which is an operator that can differentiate sections of that bundle along tangent directions in the base manifold [60].

Essentially a connection determines the geometry of the manifold it is defined on. One also requires a connection to make sense of the notions of curvature and torsion on manifolds. Next we will define and analyze the connection on phase space T^*M and how this connection allows us to decompose the phase space into horizontal and vertical distributions.

The Non-linear Connection

Consider an n-dimensional, smooth manifold M and T^*M, the cotangent bundle over M. As depicted in *fig* 3.2, let the projection map be such that $\pi : T^*M \to M$. If we choose x^i to be local coordinates in a local chart U on M, then we call (x^i, p_i),$i = 1, ...n$ the manifold induced coordinates and are taken as local coordinates in the local chart $\pi^{-1}(U)$ on T^*M. The momenta p_i is provided by the one-form $\Omega_p = p_i dx^i$ where $\Omega_p \in T^*_x M$, $x \in M$ and $\{dx^i\}$ is the natural basis of $T^*_x M$. In these coordinates the projection map takes the form $\pi(x, p) = x$. The way we have defined coordinates here is such that, it should be clear that a coordinate change $x \to \tilde{x}$ on the base manifold M induces the following coordinate change on the cotangent bundle

$$(x, p) \to (\tilde{x}(x), \tilde{p}(x, p)) \quad , \tilde{p}_a(x, p) = \frac{\partial \tilde{x}^a}{\partial x^b}(x^b) \tag{3.55}$$

Furthermore we consider $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\right\}$, with $i \in \{1, ..., n\}$ the natural basis in $T_{(x,p)}T^*M$ and $\{dx^i, dp_i\}$ the dual basis of $T_{(x,p)}T^*M$. Under the same manifold coordinate transformation as above, these bases transform as [59]

$$\left\{\partial_a = \frac{\partial}{\partial x^a}, \bar{\partial}^a = \frac{\partial}{\partial p_a}\right\} = \left\{\tilde{\partial}_a x^b \partial_b + \tilde{\partial}_a p_b \bar{\partial}^b, \partial_b \tilde{x}^a \bar{\partial}^b\right\} \tag{3.56}$$

$$\{dx^a, dp_a\} \to \left\{\partial_b \tilde{x}^a dx^b, \partial_b \tilde{p}_a dx^b + \bar{\partial}^b \tilde{p}_a dp_b\right\} \tag{3.57}$$

Suppose now, we have a vector $Z \in T_{(x,p)}T^*M$

$$Z = Z^a \frac{\partial}{\partial x^a} + \bar{Z}^a \frac{\partial}{\partial p_a} \tag{3.58}$$

we note that, the differential of the pushforward of π, $d\pi^* : T_{(x,p)}T^*M \to T_x M$, annihilates the $\bar{\partial}$ part of the vector Z i.e.

$$d\pi^*_{(x,p)}(Z^a \partial_a + \bar{Z}^a \bar{\partial}^a) = Z^a \partial_a \tag{3.59}$$

one should keep in mind that a vector field such as that right most of the latter equation is a vector field on a manifold with local coordinates given by $\{x^a\}$.

The kernel of $d\pi^*_{(x,p)}$, that is, the vector space of the part of all the vectors $Z \in T_{(x,p))}T^*M$ that goes to zero is called the *vertical* tangent space to T^*M and is denoted $V_{(x,p)}T^*M$. The existence of this canonical subspace of the tangent space of the cotangent bundle leads to the notion of a connection $\omega_{(x,p)} : T_{(x,p)} \to V_{(x,p)}T^*M$ which is a projection of the tangent space of T^*M to the vertical subspace $V_{(x,p)}T^*M$. In the general theory, connections are associated with mappings, called bundle mappings, that project larger spaces onto smaller ones [61]. This integrable vector space is locally spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$. The kernel of the connection ω, is then the vector subspace of $T_{(x,p)}T^*M$ that houses all the vectors that do not have a vertical part. We call this vector space the *horizontal* tangent space of the cotangent bundle and is denoted by $H_{(x,p)}T^*M$. The general definition of a connection is a specification of a set of directions, called horizontal directions, that are complimentary at each point to the space of vertical directions. Thus the vertical and horizontal vector spaces divide the tangent space into

$$T_{(x,p)}T^*M = V_{(x,p)}T^*M \otimes H_{(x,p)}T^*M \tag{3.60}$$

In manifold induced coordinates (x, p) of the cotangent bundle, the projection ω has the following form

$$\omega_{(x,p)} = \left(dp_a + N_{ab}(x, p)dx^b\right) \otimes \bar{\partial}^a \tag{3.61}$$

Such a projection is called a connection and is defined through its connection coefficients $N_{ab}(x,p)$

By requiring the basis of the tangent space of phase space to transform like tensor components on the base manifold under manifold induced coordinate changes, we define a new basis for $T_{(x,p)}T^*M$, $\{\delta_a, \bar{\partial}^a\}$, called the adapted basis where

$$\delta_a = \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} + N_{ab}\frac{\partial}{\partial p_a} \tag{3.62}$$

where its phase space cotangent space counterpart is $\delta p_b = dp_b + N_{ab}(x,p)dx^a$.

The diagram below is an attempt at visualising this split of phase space tangent space, where $\mathbb{P} = T^*M$. The boxes represent the tangent and cotangent spaces of the cotangent space, hence one can see these boxes being divided into complimentary sections which are the horizotal and the vertical spaces. In this diagram π_* is the pushforward from the horizontal space to the tangent space of the base manifold, hence we can say that vectors live here.

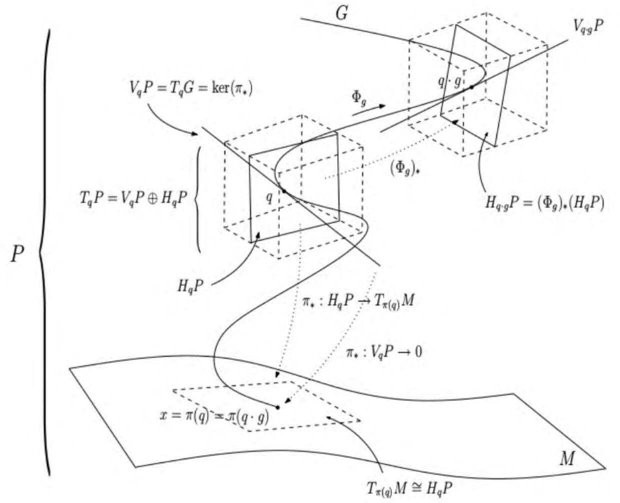

FIGURE 3.3: Visual illustration of the vertical and horizontal distribution as per [62]

If we have a Hamilton system (M, T^*M, H, θ), then the nonlinear connection coefficients $N_{ab}(x, p)$ of ω are defined as

$$N_{\alpha\beta} = \frac{1}{4}\left(\{g_{\alpha\beta}, H\} + g_{ai}\partial_b\bar{\partial}^i H + g_{bi}\partial_a\hat{\partial}^i H\right) \tag{3.63}$$

and can nonlinearly depend on momenta, hence the name nonlinear connection coefficients.

Following this definition we state the following theorem

Theorem 3.2.2. *The Hamiltonian nonlinear connection coefficients are the unique coefficients which satisfy*

$$N_{ab} = N_{ba}, \quad \Delta g^H_{ab} = 0 \tag{3.64}$$

where $g^H_{ab} = \frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}H$

A proof of this theorem is provided in appendix A.2 (see also A.2 for definition of the covariant derivative Δ).

The non-linear connection provides us with means of splitting the tangent space $T_{(x,p)}T^*M$ of phase space i.e we can split it into vertical and horizontal distributions. The vertical distribution is integrable and has dimensions $\dim(V_{(x,p)}T^*M) = \dim(M) = n$. It then naturally follows that the decomposition requires $\dim(H_{(x,p)}T^*M) = n$ since $\dim(T_{(x,p)}T^*M) = 2n$.

Now suppose our connection coefficient is symmetric i.e $N_{ab} = N_{ba}$, then we can rewrite our previous geometric structures in an invariant form i.e we can write the symplectic form θ and Poisson bracket as

$$\theta = \delta p_i \wedge dx^i \tag{3.65}$$

$$\begin{aligned}
\{f, g\} &= \frac{\partial f}{\partial p_i}\frac{\delta g}{\delta x^i} - \frac{\partial g}{\partial p_i}\frac{\delta f}{\delta x^i} \\
&= \frac{\partial f}{\partial p_i}\left(\frac{\partial}{\partial x^i} - N_{ij}\frac{\partial}{\partial p_j}\right)g - \frac{\partial g}{\partial p_i}\left(\frac{\partial}{\partial x^i} - N_{ij}\frac{\partial}{\partial p_j}\right)f \\
&= \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - N_{ij}\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_j} - \frac{\partial g}{p_i}\frac{\partial f}{\partial x^i} + N_{ij}\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_i}
\end{aligned}$$

and these are both equivalent to the previously defined symplectic form and Poisson distribution. This can be seen in appendix A.2 For a metric Hamiltonian

$$g^H_{ab} = \frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}H \tag{3.66}$$

the connection coefficients are basically the Christoffel Symbols of the Levi-Civita connection of the metric with components g_{ab} and is

$$N_{ab}(x, p) = -p_q\Gamma^q_{ab} \tag{3.67}$$

whereby Christoffel Symbols we mean the array of numbers that describe a metric connection, in this case the Levi-Civita connection. They describe how the local coordinate bases change from point to point.

Parallel transport and Autoparallels

In this section we describe the geometric structure of phase space, and establish in a general way the relation between the mathematical, geometrical quantity "connection" prevously defined and see how this geometrical structure will be identified with the physical quantity "force".

Suppose we construct a phase space T^*M, such that each point in T^*M is described by (x^a, p_a). Now consider two points in spacetime, x and x', an infinitesmall distance dx apart. Define $T^*M|_x$ as the subspace of T^*M representing all possible momenta the particle may take at the point x. If, at a time τ , the particle is at point x, we may associate with it some momentum, say $p \in T^*M$. Similarly, if at a time $\tau + d\tau$,the particle is at x' , we may associate it with some $p' \in T^*M$. Now in Newtonian mechanics, the change in momentum over this interval of time $d\tau$ represents the forces acting on the particle i.e $F = \frac{dp}{d\tau}$. However, viewing this same situation geometrically, we see that there is a subtle point here that has been overlooked in this view i.e. the momenta being compared here do not live on the same space.

Note that there is no way *a priori* to associate some element $p \in T^*M|_x$ with some $p' \in T^*M|_{x'}$, since x and x' are infinitesimally displaced and thus the momenta live in different spaces.

We must first parallel transport p' back to $T^*M|_x$ and then we can compute the difference between it and $p \in T^*M|_x$. That is when comparing the momenta of the particle at points x and x', i.e. p and p', one has to pushforward p to the space of p' in order to compare them and or be able to calculate the force F. Thus we need a connection before we can construct the rate of change of momentum representing the Newtonian force. As we have previously defined, we have the non-coordinate basis which is a one form and given by

$$\delta p_a = dp_a + N_{ab}(x,p)dx^b \tag{3.68}$$

where as stated before the functions $N_{ab}(x,p)$ are the connection cooeffictients. Consider first the linear homogenous connection i.e the metric connection $N_{ab}(x,p) = -p_q\Gamma^q_{ab}$. Defining the phase space connection 1-form

$$\omega_{ab} = -\Gamma_{abc}(x)dx^c \tag{3.69}$$

then (3.68) becomes

$$\delta p_a = dp_a - \omega_{ab}p^b \tag{3.70}$$

for this homogenous-linear connection. We can now calculate explicitly the equations for the curves defined by (3.68). Denote by X the unit tangent vector to these curves, we know that $X\lrcorner\delta p_a = 0$ for all a. The equation (3.70) then becomes

$$X\lrcorner\delta p_a = 0 = X\lrcorner dp_a - X\lrcorner(\omega_{ab}p_b) \tag{3.71}$$

$$0 = \frac{dp_a}{d\tau} + \Gamma_{abc}p^b x^c \tag{3.72}$$

where τ is a parametrization of the curve, and where $\frac{dp_a}{dt}$ denotes the derivative along the curve, and where we have taken

$$\frac{dp_a}{d\tau} = X\lrcorner dp_\mu = x^b\frac{\partial p_a}{\partial x^b} \tag{3.73}$$

Assuming a free point particle Lagrangian, with mass m, for which

$$p_a = mx_a \tag{3.74}$$

equation (3.72) becomes

$$\frac{d^2x_a}{d\tau^2} + \Gamma_{abc}x^b x^c = 0 \tag{3.75}$$

where this is the geodesic equation for a Riemannian space. Einstein demonstrated that this equation describes the forces of gravitation, and that the curvature of space is related to stress energy $T_{\mu\nu}$.

Now we relax the assumption of homogenous linearity. Consider connections of the form

$$N_{ab}(p,x) = h_{ab}(x) + g_{abc}(x)p^c \tag{3.76}$$

where h and g are simply functions of x at this point, physical meanings of these quantities will be apparent soon. The reason to assume a connection of this form is that, in the Riemannian limit we want the function g to be the Levi-Civita connection coefficients such that as we have seen in the last chapter, for metric geometry we recover $N_{ab}(x,p) = \Gamma^c_{ab}p_c$. The function h, functions as a "perturbation". Defining the set of forms

$$\omega^0_a = h_{ab}dx^b$$
$$\omega^1_{ac} = g_{abc}dx^b$$

equation 3.68 becomes

$$\delta p_a = dp_a - \omega^0_a - \omega^1_{ab}p^b \tag{3.77}$$

As before, we explicitly calculate the equations for the curves which nullify this one form, by contracting the unit tangent vector X with all of the 1-forms δp_a:

$$X \lrcorner \delta_a = 0 \tag{3.78}$$

$$0 = X \lrcorner dp_a - X \lrcorner \omega^0_a - X \lrcorner \omega^1_{ab}p^b \tag{3.79}$$

Assuming again a point particle Lagrangian, $p_a = mx_a$, this becomes

$$0 = m\frac{dx_a}{d\tau} - x^b h_{ab} - mx^b x^c g_{abc} \tag{3.80}$$

in a form where the mass-dependence of the different terms is explicit,

$$0 = \frac{dx_\mu}{d\tau} - \frac{1}{m}h_{\mu\gamma}x^\gamma - g_{\gamma\alpha\mu}x^\gamma x^\alpha \tag{3.81}$$

Notice that, expressed in this form, the equivalence principle is manifest. Only one term must be without a mass term, the first order term. Mathematically , this must describe paths which particles will follow independent of their mass, which physically corresponds to gravitation, as Einstein's General theory shows. Other geometrically derived forces, the zeroth order term, and the second order terms and higher, correspond to forces whose effects will be dependent on the mass of the particle. This identification of force with connection allows one to identify stress-energy with curvature, where mathematical identities on the curvature automatically produce the necessary conservation theorems on stress-energy. However, the approach described here generalizes Einstein's approach, by going beyond the affine connections derived from a metric in Riemannian geometry, to the most general expression for a non-linear connection on phase space. Whereas in Riemannian geomety, the metric is the fundamental quantity, from which quantities such as the connection and curvature is derived, these connections we consider, are the most fundemental quantities from which other geometric quantities such as curvature is derived. Such connections may not in general be derived from a metric. By adopting this more general framework, we can "geometrize" a wider variety of forces, notably the electromagnetic force.

Given a curve $\Gamma : I \in \mathbb{R} \mapsto \Gamma(t) \in T^*M$, the tangent vector $\frac{d\Gamma}{dt}$ can be written in the adapted basis as follows

$$\frac{d\Gamma}{dt} = \frac{dx^a}{dt}\frac{\delta}{\delta x^a} + \frac{\delta p_a}{dt}\frac{\partial}{\partial p_a} = \frac{dx^a}{dt}\delta_a + \frac{\delta p_a}{dt}\bar{\partial}^a \tag{3.82}$$

This curve is horizontal if and only if it lies in the horizontal space [59] i.e. it must be expressed in the basis of the horizontal space as follows

$$\frac{d\Gamma}{dt} = \frac{dx^a}{dt}\delta_a \tag{3.83}$$

thus

$$\frac{\delta p_a}{dt} = 0 \tag{3.84}$$

$$\Rightarrow \frac{dp_a}{dt} - N_{ab}(x,p)\frac{dx^a}{dt} = 0 \tag{3.85}$$

recall the Hamiltonian equations of motion are expressed as

$$\dot{x}^a = \frac{\partial H}{\partial p_a} \tag{3.86}$$

$$\dot{p}_a = -\frac{\partial H}{\partial x^a} \tag{3.87}$$

and using these in (3.85) one obtains the following equation

$$\frac{\partial H}{\partial x^a} + N_{ab}\frac{\partial H}{\partial p_b} = 0 \tag{3.88}$$

$$\left(\frac{\partial}{\partial x^a} + N_{ab}\frac{\partial}{\partial p_b}\right) H = 0 \tag{3.89}$$

$$\delta_a H = 0 \tag{3.90}$$

And one can see that, being horizontal actually means parallel transport or curves are autoparallels in the tangent bundle of T^*M since the force term (3.90) is zero. Recall we have shown that the general Hamiltonian equations of motion were given by

$$\dot{p}_a + N_{ab}\bar{\partial}^b H + \delta_a H = 0 \tag{3.91}$$

Curvature of Spacetime and Momentum space

In $\mathbb{R}^n$, when performing partial derivatives, one easily transports a given vector field $X = X^i e_i$ at x to a nearby tangent space with vector field $\tilde{X} = \tilde{X}^i \tilde{e}_i$. In flat vector spaces this is trivial, however the notion of parallel transport becomes non-trivial for curved manifolds. One requires a structure called a connection, which will allow one to specify how generic tensorial objects are transported along curves on manifolds. We have seen above that with the introduction of the non-linear connection and by identifying its transformation behaviour we were able to split directions on phase space into horizontal and vertical directions. We have seen that the non-linear connection coefficients define the geometry on phase space since they are the coefficients of the dynamical covariant derivative otherwise connection Δ defined in [59]. In order for us to specify the geometry of the spacetime , respectively momentum space we need to find connection coefficients of their respective connections. Thus we introduce linear covariant derivatives of the directional vector spaces.

$$\Delta_{\delta_i}(\delta_j) = F^k_{ij}\delta_k \;\; \Delta_{\delta_a}(\partial^a) = -F^a_{ij}\partial_a \tag{3.92}$$

$$\Delta_{\partial^a}(\delta_i) = C^{ka}_i\delta_k \;\; \Delta_{\partial^b}(\partial^b) = -C^{ba}_i\partial_b \tag{3.93}$$

Now the geometry of spacetime and momentum space will be described by the connection ceofficients given F and C respectively. There are numerous choices for covariant derivatives which satisfy the conditions given above, however on imposing extra conditions which include invariance of the Hamilton metric vertically and horizontally and in addition torsion free, one recovers a unique covariant derivative. By torsion here we mean the ability of a manifold to twist. This covariant derivative is called the Cartan-linear covariant derivative usually denoted Δ^{CL}. The first condition can be explicitly expressed as

$$\Delta^{CL}_{\delta_a} g^H = 0 \quad \Delta^{CL}_{\bar{\partial}_a} g^H = 0 \tag{3.94}$$

along the horizontal and vertical directions respectively. Here H simply denotes that this metric is that of the Hamilton Geometry space. The coefficients of the horizontal and vertical covariant derivatives can then be give by

$$F^a_{bc} = \frac{1}{2} g^{Haq} \left(\delta_b g^H_{cq} + \delta_c g^H_{bq} - \delta_q g^H_{cb} \right) := \Gamma^{\delta a}_{bc} \tag{3.95}$$

and

$$C^{ab}_c = -\frac{1}{2} g^H_{rc} \bar{\partial}^a g^{Hrb} \tag{3.96}$$

With the connection coefficients of the respective vector spaces, one would be interested in finding out the curvature tensors of these spaces, and as it turns out they are give by [59]

$$R^{Hq}_{abc}(x,p)\delta_q = \left(\delta_b \Gamma^{\delta q}_{ac} - \delta_c \Gamma^{\delta q}_{ab} + \Gamma^{\delta q}_{bi} \Gamma^{\delta i}_{ac} - \Gamma^{\delta q}_{ci} \Gamma^{\delta i}_{ab} - R_{ibc} C^{qi}_a \right) \delta_q \tag{3.97}$$

for the horizontal distribution and

$$Q^{abc}_q(x,p)\bar{\partial}^q = \left(\bar{\partial}^b C^{ac}_q - \bar{\partial}^c C^{ab}_q + C^{bi}_q C^{ac}_i - C^{ci}_q C^{ab}_i \right) \bar{\partial}^q \tag{3.98}$$

One can thus deduce from these expressions that our theory is consistent in that, when we have a free particle, the momentum space connection coefficients vanish and the spacetime connection coefficients become the Riemannian Levi-civita connection coefficients.

Furthermore, one should mention that the curvature of phase space should be associated with the nonlinear connection N_{ab}. In the same logic, considering phase space as a manifold with a covariant derivative, the non-linear connection coefficients of Δ should determine the geometry of phase space. Thus if we have a trivial nonlinear connection coefficient then phase space is flat and is isomorphic to $\mathbb{R}^n$ otherwise curved. It would be interesting to see what effect this curvature of phase space has on spacetime and momentum space.

3.3 Spacetime and Phase space: Local theory

In this section we discuss the symmetries and symmetry generators in phase space.

Symmetries on Phase space

Let us suppose we restrict our attention to a case in which specifying an initial data set, possibly an instantaneous dynamical state, for the equations of the theory determines a unique solution defined at all times. Then, roughly speaking a Hamiltonian treatment amounts to the following. The phase space of the theory is the space, T^*M, of all initial data sets. Recall, the Hamiltonian, H, of the theory is the real- valued function on the phase space that assigns to each point of the phase space the energy of the corresponding physical state. We have shown in the preceding sections that a phase space can be equipped with a geometric structure, θ, with the following feature: together H and θ determine a family of curves in T^*M, exactly one passing through each point; each of these curves corresponds to a solution of the theory's equation of motion, in the sense that the two objects pick out the same sequence of instantaneous dynamical states; and for any such solution there is a corresponding curve of this kind. So the structure (T^*M, θ, H) in effect encodes the differential equation of the theory. It is natural to investigate the

Hamiltonian symmetries of the theory: the one-to-one and onto maps from T^*M to itself that preserve both θ and H

We closely follow the symmetry analysis of phase space given in [59]. Suppose that we have a diffeomorphism $\Phi \in T^*M$, we say it is a symmetry of phase space if it leaves the function $H : T^*M \to \mathbb{R}$ i.e the Hamiltonian invariant. Mathematically this statement can be represented as follows

$$H(\Phi(x,p)) = H(x,p) \tag{3.99}$$

A taylor expansion of the diffeomorphism, truncated at first order in ϵ i.e

$$\Phi(x,p) = (x^a + \epsilon\xi^a(x,p), p_a + \epsilon\bar{\xi}_a(x,p)) + \mathcal{O}(\epsilon^2) \tag{3.100}$$

allows us to find the vector field, that generates the symmetry transformation by imposing (3.99), thus this leads to

$$\begin{aligned} H(\phi(x,p)) &= H(x^a + \xi^a(x,p), p_a + \bar{\xi}_a(x,p) \\ = H(x,p) + \epsilon\left(\xi^a(x,p)\partial_a H(x,p) + \bar{\xi}_a(x,p)\bar{\partial}H(x,p)\right) &+ \mathcal{O}(\epsilon^2) \\ &= H(x,p) + \epsilon Z(H)(x,p) + \mathcal{O}(\epsilon^2) \\ &= H(x,p) \end{aligned}$$

From the latter calculation, one can deduce that the vector field $Z = \xi^a(x,p)\partial_a + \bar{\xi}_a(x,p)\bar{\partial}^a$ on T^*M generates Φ. However for the diffeomorphism Φ to be a symmetry, we must have, according to (3.99) that

$$Z(H) = 0 \tag{3.101}$$

We can then state that a symmetry generator in T^*M is a vector field $Z \in T^*M$ that satisifies condition (3.101)

In phase space geometry, an important class of symmetries is that which is associated to a constant of motion of the Hamitlon dynamics. A constant of motion is a quantity $\mathcal{C}$, which typically lives in phase space such that when one takes the poisson backets between the Hamiltonian H and $\mathcal{C}$, the result is zero. This constant of motion must be conserved along solutions of the Hamilton-Jacobi equations, as such any change in this quantity along the solutions $(x(\lambda), p(\lambda))$ should be null. Here λ is an affine parameter. This statement can be summarized mathematically as

$$\frac{d}{d\lambda}\mathcal{C}(x(\lambda), p(\lambda)) = 0 \tag{3.102}$$

Also from the above equation, with the use of poisson brackets we note that

$$\frac{d}{d\lambda}\mathcal{C}(x(\lambda), p(\lambda)) = \dot{x}^a\partial_a\mathcal{C} + \dot{p}_a\bar{\partial}^a\mathcal{C} = \{\mathcal{C}, H\} \tag{3.103}$$

thus for any constant of motion $\mathcal{C}$, we have

$$\{\mathcal{C}, H\} = 0 \tag{3.104}$$

Spacetime to Phase space symmetries

In Hamilton geometry, we came across the notion of a lift from a base manifold M to a the cotangent manifold T^*M, through the projection π defined section 3.2. It is important for one to consider how symmetries on the manifold M translate to symmetries on T^*M. One can generate an infinitesmal diffeomorphism of M by vector fields $X = \eta^a(x)\partial_a \in M$. Such a diffeomorphism on M acts as a change of local coordinates $(x^a) \to (x^a + \eta^a)$. However, as we have seen in previous sections, the way we have defined phase space is such that a coordinate transformation on the manifold M induces a coordinate transformation on T^*M. Keeping this in mind, we see that this diffeomorphism induces on T^*M a coordinate change of the form $(x^a, p_a) \to (x^a + \eta^a, p_a - p_q\partial_a\eta^q)$. Thus a diffeomorphism on M generated by the vector field X induces a diffeomorphism on T^*M generated by the vector field $X^{comp.lift}$. This diffeomorphism is called the complete lift of X from M to T^*M and we expicitely write it as

$$X_q^{comp.lift} = \eta^a\partial_a - p_q\partial_a\bar{\partial}^a \tag{3.105}$$

We then say, a manifold symmetry of the Hamilton geometry is a diffeomorphism ϕ of M whose generating vector field X satisfies

$$X^{comp.lift}(H) = 0 \tag{3.106}$$

3.4 Applications of the theory

In this section we take a look at a few examples illustrating the theory we have developed in the chapter.

Example 3.1: The Spherical Pendulum

We here discuss the spherical pendulum of classical mechanics with modern mathematical techniques, by using spherical coordinates and the traditional frameworks of Lagrange and Hamilton. Further analysing the geometrical symmetries of the system. We will see that this motion is equivalent to a particle of unit mass moving on the surface of the unit sphere S^2 under the influence of constant gravitational force of unit strength. *fig* 3.4 below displays a 3 dimensional spherical pendulum in cartesian coordinates.

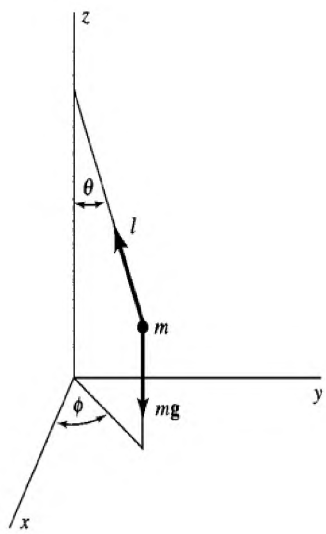

FIGURE 3.4: Spherical Pendulum in 3-D cartesian coordinates

A transformation from Cartesian coordinates to Spherical coordinates allows for an easy treatment of the system's mechanics. This transformation can be seen to be of the form

$$x = \sin\theta\cos\theta \tag{3.107}$$
$$y = \sin\theta\sin\phi \tag{3.108}$$
$$z = -\cos\theta \tag{3.109}$$

where $0 < \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$. Let the generalised coordinates be such that $\vec{q} = (\theta, \phi)$. In general the Lagrangian function L is given by

$$L = T - U \tag{3.110}$$

where T and U are the kinetic and potential energies of the system respectively. In this example, we take $l = 1$ and $m = 1$, where these are respectively the length of the string and the mass of the particle. The string is assumed to have neglible mass. Then the Lagrangian L of the spherical pendulum is

$$L = \frac{1}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + g\cos\theta \tag{3.111}$$

where the second term is the potential energy $U = -g\cos\theta$ and g the gravitational acceleration.

We can then use the legendre transform to define the Hamiltonian function H, defined in phase space and generally given by

$$H = p_\alpha \dot{q}^\alpha - L \tag{3.112}$$

where the p_α are the conjugate canonical momenta to the configuration space coordinates. One can calculate the generalised momentum $p_\alpha \in T_x^*S^2$ at a point x of S^2 by using the simple relation

$$p_\alpha = \frac{\partial L}{\partial \dot{q}} \tag{3.113}$$

following this we get the momentum coordinates are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\phi}} = \sin^2\theta\dot{\phi}$$

which are canonically conjugate to the (θ, ϕ) coordinates. From which, the Hamiltonian function then becomes

$$H = \frac{1}{2}p_\theta^2 + \frac{1}{2\sin^2\theta} - g\cos\theta \tag{3.114}$$

Hamilton's canonical equations are, as previously stated, the equations of motion in the Hamilton formulation. With the aid of (3.114) one recovers these equations to be of the form

$$\frac{dx^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha} \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha} \tag{3.115}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = p_\theta \tag{3.116}$$

$$\dot{\phi} = \frac{1}{\sin^2\theta}p_\phi \tag{3.117}$$

and

$$\dot{p}_\alpha = \frac{\cos\theta}{\sin^3\theta}p_\phi^2 - \sin\theta \tag{3.118}$$

$$\dot{p}_\phi = 0 \tag{3.119}$$

the dynamics of the spherical pendulum system are given by solutions to these Hamiltonian equation. Now for the application of Hamilton geomerty, before calculating any major geometrical structures one has to first obtain the metric g_{ab}^H. With the help of the inverse metric equation

$$g^{H\alpha\beta}(x,p) = \frac{1}{2}\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p_b}H \tag{3.120}$$

simple algebra yields

$$g^{H\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix} \tag{3.121}$$

Thus by inverting the latter equation, one recovers the metric to be

$$g^{H}_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{3.122}$$

this metric allows one to write for the length ds^2

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{3.123}$$

which gives the geometry of the 2-sphere with unit radius. Fig 3.5 above gives a graphical representation of this scenerio. Now using this metric and the Hamiltonian, one finds after long and tedious calculations that the non-linear connection coefficients are

$$N_{H\alpha\beta}(x,p) = \begin{pmatrix} 0 & -\frac{\cos\theta}{\sin\theta}p_\phi \\ -\frac{\cos\theta}{\sin\theta}p_\phi & 0 \end{pmatrix} \tag{3.124}$$

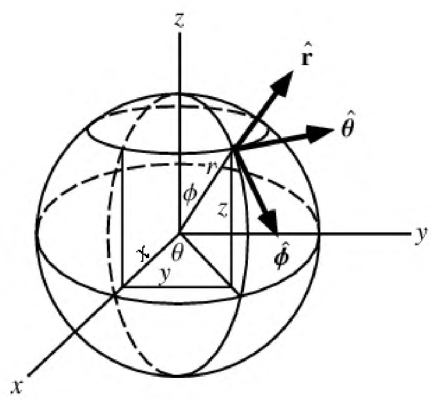

FIGURE 3.5: Geometry of spherical pendulum is isomorphic to S^2

It would be interesting to determine the autoparallels of the spherical pendulum. The general equation for these is

$$\dot{p} + N_{ab}\bar{\partial}^b H = -\delta_a H \tag{3.125}$$

which work out to be

$$\dot{p}_\theta - \frac{p_\phi^2\cos^2\theta}{\sin^3\theta} = -\sin\theta \quad , \dot{p}_\phi - \frac{\cos\theta}{\sin\theta}p_\theta p_\phi = -\frac{\cos\theta}{\sin\theta}p_\theta p_\phi \tag{3.126}$$

In these equations, there seem to be a drag term pulling particles away from the geodesic paths. However these equations of motion simplify to

$$\dot{p}_a + N_{ab}\dot{x}^b = 0 \tag{3.127}$$

$$\dot{p}_\theta - \frac{\cos^2\theta}{\sin^3\theta}p_\phi^2 = 0 \tag{3.128}$$

$$\dot{p}_\phi = 0 \tag{3.129}$$

Thus we can conclude a spherical pendulum follows autoparallels on S^2.

Example 3.2: kappa-Poincare geometry

Suppose for simplicity we are working in $1+1$ dimensions. Consider a particle moving on Minkowskian spacetime and characterized by a curved momentum space. Furthermore suppose the radius of this momentum space is given by a quantum deformation parameter l, in this case. The algebra of symmetries is the κ-Poincare Hopf algebra, which herein is in the bicrossproduct basis. Describing the momentum space as a maximally symmetric manifold guarantees that the algebra has three generators of global symmetry transformations $\mathcal{P}_0, \mathcal{P}_1, \mathcal{N}$. Respectively these are the time translation, space translation and boost in Minkowski spacetime. In first order in l, the algebra of these generators is

$$\{\mathcal{P}_0, \mathcal{P}_1\} = 0 \quad \{\mathcal{P}_0, \mathcal{N}\} = \mathcal{P}_1 \tag{3.130}$$

$$\{\mathcal{P}_1, \mathcal{N}\} = \mathcal{P}_0 - l\left(\mathcal{P}_0^2 + \frac{1}{2}\mathcal{P}_1^2\right) \tag{3.131}$$

with the Casimir reading as follows

$$\mathcal{C} = \mathcal{P}_0^2 - \mathcal{P}_1^2 - l\mathcal{P}_0\mathcal{P}_1^2 \tag{3.132}$$

Note again that for the sake of simplicity we have chosen to work in $1+1$ dimensions, using the canonical phase space representation i.e phase space with coordinates x^μ and p^μ satisfying the standard symplectic structure (3.44). Equipped with these canonical coordinates we can represent the generators as

$$\mathcal{P}_0 = p_0, \tag{3.133}$$

$$\mathcal{P}_1 = p_1, \tag{3.134}$$

$$\mathcal{N} = p_1 x^0 + p_0 x^1 - l\left(x^1 p_0^2 + \frac{1}{2}x^1 p_1^2\right) \tag{3.135}$$

thus having the Casimir

$$\mathcal{C} = p_0^1 - p_1^2 - l p_0 p_1^2 \tag{3.136}$$

The Casimir is a constant of motion on phase space, thus one is able to take it as a Hamitlonian for the purposes of calculations. The Hamiltonian is then

$$H_{\kappa P} = H = p_0^2 - p_1^2 - l p_0 p_1^2 \tag{3.137}$$

Equipped with the Hamiltonian H, the natural step is to determine the phase space geometry associated with it. Finding the metric g^H_{ab} requires one to obtain its inverse. As such the inverse metric is

$$g^{H\alpha\beta} = \frac{1}{2}\bar{\partial}^a\bar{\partial}^b H = \begin{pmatrix} 1 & -l p_1 \\ -l p_1 & -(1 + l p_0) \end{pmatrix} \tag{3.138}$$

thus yielding the metric

$$g^H_{ab} = \begin{pmatrix} \frac{1+lp_0}{lp_1^2-(1+lp_0)} & \frac{lp_1}{(1+lp_0)-lp_1^2} \\ \frac{lp_1}{(1+lp_0)-lp_1^2} & \frac{1}{lp_1^2-(1+lp_0)} \end{pmatrix} \tag{3.139}$$

With the use of the metric, one can further calculate the non-linear connection coefficients $N_{ab}(x,p)$ defined in (3.63), finding

$$N_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.140}$$

This quantity allows one to calculate the curvatures of spacetime and momentum space submanifolds. These are the structures which make parallel transport non-trivial along both spaces. Curvature in spacetime is obtained via the coefficients $\Gamma^{\delta a}_{bc}$ and respectively momentum space curvature via C^{ab}_{c}. We find the coefficients $\Gamma^{\delta a}_{bc}$ to all be identical zero i.e

$$\Gamma^{\delta a}_{bc} = \mathbf{0} \tag{3.141}$$

This result informs us that the spacetime is flat, thus Minkowskian. However, one finds that connection coefficients required for determing the momentum space curvature do not vanish. Most of the coefficients are identical to zero, except for C^{01}_{1} and C^{11}_{0}, i.e.

$$C^{00}_{0} = 0,\ C^{00}_{1} = 0,\ C^{01}_{0} = 0,\ C^{11}_{1} = 0 \tag{3.142}$$

$$C^{01}_{1} = -\frac{1}{2},\ C^{11}_{0} = -\frac{1}{2} \tag{3.143}$$

Thus we can conclude that the momentum space is curved, however finding its curvature requires one to calculate the quantities given by

$$Q^{abc}_{q} = C^{bi}_{q} C^{ac}_{i} - C^{ci}_{q} C^{ab}_{i} \tag{3.144}$$

These result in all other components being zero except for Q^{001}_{1}, which is

$$Q^{001}_{1} = C^{01}_{1} C^{01}_{1} - C^{11}_{0} C^{01}_{1} \tag{3.145}$$

$$= -\frac{l}{2} \times -\frac{l}{2} - \left(-\frac{l}{2}\right)\frac{l}{2} \tag{3.146}$$

$$= \frac{l^2}{2} \tag{3.147}$$

This curvature is that of the de Sitter momentum space with curvature of the oder of the planck mass (see *fig* **??** in co-moving coordinates η^{α}, this figure shows the parts of the de Sitter space that have constant energy and momentum, include infinite momenta,as is the case in Minkowsi space-time). Moreover, a momentum space with the curvature of $\frac{l^2}{2}$ corresponds to a specific non-commutative spacetime. This spacetime is the κ-Minkowski spacetime and is characterized by the commutation relation

$$\left[x^0, x^1\right] = \frac{i}{\kappa} x^1 \tag{3.148}$$

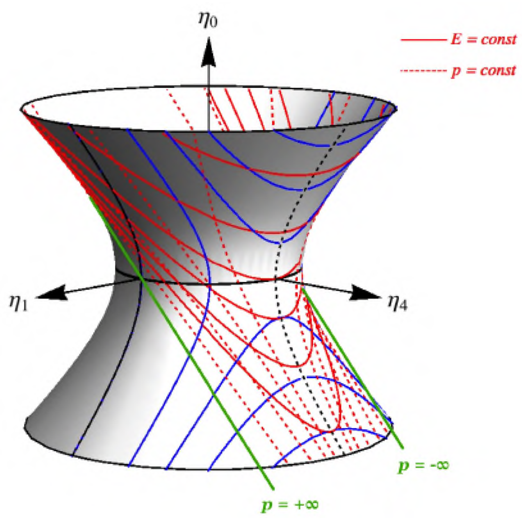

FIGURE 3.6: Diagram depicting a de Sitter space in co-moving coordinates

3.5 Summary

In this chapter we have reviewed Lagrange and Hamilton Mechanics, with emphasis on the latter. In the review of the Hamilton Mechanics, we introduced important topics such as Poisson Brackets, symplectic structure etc. These objects allowed us to make a smooth transition to the Hamilton geometry framework, where we took the notion of cotangent bundle and used it as a phase space. This allowed us to be able to look at deformations of the mass-shell relation as level sets in phase space. In doing this we looked at the autoparallels of phase space and noticed these attain a dragging force whenever we don't have a homogenous Hamiltonian. Furthermore we looked at the non-linear connection-coefficient which allowed us to determine the curvature of phase space and simultaneously allowed us to isolate spacetime and momentum space in phase space. However, we saw that this isolation intertwines spacetime and momentum space whenever the Hamiltonian of a system is not a function of momenta only. We further looked at the symmetries of phase space, which included the constant of motion and also took a look at how one defines a manifold symmetry. Moreover, we took a look at a few examples for an understanding of this framework.

Chapter 4

DSR as an effective theory of quantum gravity

In the last chapter (3), we found an important result that asserts that in the Hamilton Geometry framework, one generally finds that, particles do not follow autoparallels but rather attain a dragging force-like term

$$\dot{p}_a + N_{ab}\dot{x}^b = -\delta_a H \tag{4.1}$$

where the RHS is the source term.

The physical interpretation of this source term as we have seen in the last chapter, is that for a general Hamiltonian, the motion of particles cannot be understood as free fall motion in a geometry but rather there is a force-like term $\delta_a H$ present which drags the particles away from free fall [59]. This leads one to asking questions like: Are the space-time coordinates we are using in HG really the physical ones? We get a hint at the answer to this question towards the end of chapter 3 when we realise in *example 3.2* that the x^a coordinates, which are canonically conjugate to the p_a coordinates, are not the physical coordinates but rather we get non-commutative coordinates X^a as the physical spacetime coordinates. The non-commutativity of these coordinates is actually equivalent to the curvature of momentum space [16], hence in *e.g. 3.2* the momentum space had a curvature of l^2. This alone leads one to believe there is a relationship between momenta and the physical spacetime coordinates.

If we can back-track for a second, we find further motivation for the existence of these non-commutative coordinates in the search for a quantum theory of gravity. Quantum gravity encompasses both quantum field theory and general relativity. A feature one can expect to obtain in this theory is that of the existence of a minimal length scale. However the issue one usually encounters is how to reconcile such a structure (Planck length) with the requirement of Lorentz invariance.

From the point of view of a static observer at infinity, in GR, a particle of mass m creates a Schwarzschild metric with an event horizon at the distance $r = l_S = 2Gm/c^2$, in which this event horizon is a Lorentz invariant boundary. Under a Lorentz transformation (say boosts), the distance r between the particle and a test particle will get contracted, however this distance will never be smaller than l_S, hence the minimal length l_S in GR. Similarly in QFT, in the presece of a massive field mass m. the Compton length scale $l_C = \hbar/mc$ establishes a minimal length scale. As we have seen in chapter 2, DSR becomes the framework to describe this same physical phenomenon which is due to two different causes in two different theories. The non-commutative geometry of DSR becomes important here as it allows for such a minimal length scale.

In this chapter, we briefly try to tie up the claim that the "new", physical coordinates of spacetime are not those which are conjugate to the momenta with Snyder's theory of DSR [63]. We closely follow [64], and propose to recover a framework with non-commutative space-time coordinates, assuming that the space-time coordinates that we measure are effectively not the usual x_μ but objects which also depend on the momentum p_μ. This introduces a class of momentum-dependent spacetime coordinates X_a. Recall, in the Hamilton Geometry framework we also found that the geometry of spacetime and that of momentum space is intertwined for a general Hamiltonian. Thus as a starting point it would make sense to have momentum-dependent spacetime coordinates. These coordinates are also suitable in that the worldline expressions in X_a coordinates is momentum independent [48], thus one does not encounter problems in fixing a reflexive, symmetric and transitive defintion for a time interval. We show that these effective coordinates naturally lead to a stable $so(4,1)$ (or $so(3,2)$) structure similar to deformed special relativity which represents a minimal length scale.

In considering coordinates which are to be physical spacetime coordinates, it is essential one requires Lorentz covariance of the coordinates. In [64], an analysis of how different cases of momentum dependence affect spacetime coordinates. There is a coordinate shift in p_a and also a p^2-dependent rescaling. In this chapter we are more concerned in the former. The shift is interpreted as a time-lapse or dragging in measurements, whereas the p^2-rescaling has been interpreted as having an impact on the measurement of the object's mass, deforming the surrounding spacetime. We do not have much interest in large masses that can influence the spacetime, thus as we have stated, we do not dwell much on the p^2-rescaled coordinates.

The motivation for effective coordinates of momentum dependence, is that the mass of a particle is fundementally relatable to both the Swarzschild radius and Compton length (see chapter 1). On one hand, black holes are created by objects with huge 4-momenta on small regions, thus resulting in a deformed spacetime in that particular region and affecting the measured spacetime coordinates. As a result it would not be outrageous to think that in such a region, one would not have momentum dependent spacetime coordinates if he is to probe this region. On the other hand, we have also seen in our introduction of GUP (1) that the uncertainty principle relates the measurability of momentum and position, thus noting the position is affected by the momentum.

In our next discussion, we insist that we neither break nor deform Lorentz invariance and that the work done here is with Lorentz covariant objects. The metric signature used is $(+---)$.

4.1 Non-commutativity of effective coordinates

Our starting point here is analyzing the drag effect of the particle motion which is essentially measuring the position a bit later or earlier, with this depending on the drag term whether it is positive or negative. Our class of phase space functions is given by

$$X_\mu = x_\mu - \frac{\psi}{\kappa^2} p_\mu \tag{4.2}$$

on the right hand side, we have put κ for dimension purposes at the moment and we assume it to be an arbitrary mass scale with ψ being a dimensionless Lorentz invariant function on phase space. A constraint on the function ψ is that it has to be a scalar function and however, it can be a function of x^2, p^2 or the dilation $D = x_\mu p^\mu$. Since one would like to focus on the momentum dependence of the effective coordinates, we do not inquire much about the possibility of a x^2 dependence. We must note that a function of p^2 does not change the Poisson brackets, thus not having much physical interest. An elimination of these possibilities then leaves one with the obvious case of a function $\psi(D)$ i.e with a dilation dependence. The Poisson brackets with such a dependence when computed result in

$$\{X_\mu, X_\nu\} = -\frac{\psi'(D)}{\kappa^2} j_{\mu\nu} \tag{4.3}$$

$$\{X_\mu, p_\nu\} = \eta_{\mu\nu} - \psi'(D)\frac{p_\mu p_\nu}{\kappa^2} \tag{4.4}$$

where we have the Lorentz generators $j_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$.

This algebra of position-momentum is very similar to the algebra underlying the Deformed Special Relativity in Snyder's basis, (see [63] for a mathematical construct of Snyder's DSR). Now, if one is to put the constraint that the algebra (4.4) is to close, then this means that the function ψ' is constant[64]. Neglecting this term due to a shift $\pm T p_\mu$, where T is a constant, amounting to a simple time shift on the particle's trajectory. We then have two possibilities for $\alpha = \pm$, up to a renormalisation of the mass scale [64]:

$$X_\mu = x_\mu + \alpha \frac{D}{\kappa^2} p_\mu \tag{4.5}$$

Having the effective coordinate X_μ this way then results on a commutation relation of the form

$$\{X_\mu, X_\nu\} = -\frac{\alpha}{\kappa^2} j_{\mu\nu} \tag{4.6}$$

And the X, j's form a closd Lie algebra, $\mathfrak{so}(4,1)$ for $\alpha = +$ and $\mathfrak{so}(3,2)$for $\alpha = -$.

Notice how this is exactly the structure behind a theory of DSR, see [65]. Now if we are to assume that when we do measurements on spacetime, we measure the coordinates X^a, which are then the more physically relevant coordinates than the $x's$, then one ends up with a non-commutative spacetime of the DSR type. Also this is indicative of the natural appearance of $\mathfrak{so}(4,1)$ and $\mathfrak{so}(3,2)$ structure in special relativity.

It has been shown in [66] that any DSR theory can be understood as a particular coordinate system on four dimensional de Sitter space of momenta imbedded in five dimensional Minkowski space. Here we then see the Lorentz transformations are identified with the $SO(3,1)$ subgroup of $SO(4,1)$ group of the symmetries of de Sitter space while the (non-commutative) positions are taken to be the remaining four translation generators in momentum space wich satisfy the relation

$$[X_\mu, X_\nu] = -\frac{i\hbar^2}{\kappa^2} J_{\mu\nu} = -i l_p^2 J_{\mu\nu} \tag{4.7}$$

where the deformation parameter is $\kappa = \hbar / l_p$. Mathematically, the momentum space is identified as the hyperboloid $SO(4,1)/SO(3,1)$ and the coordinate operators are the de Sitter translation generators. And we recover the mathematical set-up of deformed special relativity (DSR). Here we see that for a DSR theory, the canonical spacetime coordinates do not quite give the physical spacetime we obtain from the Hamilton Geometry framework. Instead what we notice is that there is a class of momentum-dependent spacetime coordinates that are more physically relevant than the configuration space coordinates given by

$$X_\mu = x_\mu - \frac{\psi}{\kappa^2} p_\mu \tag{4.8}$$

Next we show the choice of ψ here which gives us the results of example 3.4 for the κ-Poincare geometry.

We start by assuming *a priori*, a non-trivial coordinate system given by p_α, for the momentum coordinate and likewise by X^α for the spacetime coordinates adding that these coordinates are not canonically conjugate to each other. Where the latter coordinates satisfies the relation

$$\left\{X^\alpha, X^0\right\} = \frac{1}{\kappa} X^\alpha \tag{4.9}$$

The mass-shell relation is defined as a square of the distance from the origin to a point p, with coordinates $p_\alpha(p)$ i.e

$$C(p) = D^2(p) - m^2 \tag{4.10}$$

Thus we need a metric to define the mass-shell relation. In [48], the deSitter momentum space in $1+1$ dimensiona has a metric given by

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -(1+2lp_0) \end{pmatrix} \tag{4.11}$$

where l is the deformation parameter.

Defining the invariant line-element as

$$D^2(p) = \int_0^p \eta^{\alpha\beta} \dot{p}_\alpha \dot{p}_\beta ds \tag{4.12}$$

where s is the affine parameter parametrizing a geodesic $\mu(s)$ connecting a point p in which a particle lies to the origin allows us to then recover a dispersion relation of the form

$$\mathcal{C}(p) = p_0^2 - p_1^2 - lp_0p_1^2 \tag{4.13}$$

We have already pointed out that p_α and X^α are not canonically conjugate, thus this causes a deformation of the symplectic structure between momenta and the spacetime coordinates. This induces a non-trivial poisson bracket between these coordinates given by [48]

$$\begin{aligned} \left\{p_0, X^0\right\} &= 1 \quad \{p_0, X^\alpha\} = 1 \\ \left\{p_\alpha, X^0\right\} &= -lp_0 \quad \left\{p_i, X^i\right\} = \delta^i_j \\ &\qquad \left\{p_\alpha, p^\beta\right\} = 0 \end{aligned}$$

and these accompanied by the relation $\{X^i, X^0\} = lX^i$ satisfy the Jacobi identities. In a symplectic setting we know that for conjugate coordinates x^α and p_α, we have the following poisson brackets

$$\begin{aligned} \left\{x^\alpha, x^\beta\right\} = \{p_\alpha, p_\beta\} &= 0 \\ \{x^\alpha, p_\beta\} &= \delta^\alpha_\beta \end{aligned}$$

Using the Jacobi identities, one can obtain a relationship between the coordinates X^α and x^α, which happens to be known in the literature [48]. This relationship is given by

$$X^\alpha = \left(\delta^\alpha_\beta - l\delta^\alpha_0\delta^j_\beta p_j\right) x^\beta \tag{4.14}$$

with the inverse relation being

$$x^\alpha = \left(\delta^\alpha_\beta + l\delta^\alpha_0\delta^j_\beta p_j\right) X^\beta \tag{4.15}$$

This leads us to conclude that from the class of momentum dependent spacetime coordinates (4.8), the Hamilton Geometry framework picks out one particular physical spacetime given by the coordinates (4.14) where ψ has a linear dependence on the canonical coordinate x^a.

Chapter 5

Conclusion and Final remarks

Now that we have outlined the general features of DSR and discussed some of its aspects in more detail, we want to take a step back and make a few final remarks which will help us understand the bigger picture. What DSR does almost resembles Einstein's theory of relativity in that Einstein took the Euclidean group of the dynamics in a Euclidean space and extended it to the Poincare group, simply by adding boost generators which allowed for more general dynamics on a Riemannian manifold. The Euclidean group is a subgroup of the Poincaré group. Now the idea behind DSR is to take the Poincaré group and yet again realise that it is part of a larger structure, namely a quantum group whose algebra is the κ-Poincaré algebra. In going from the Poincaré algebra to the κ-Poincaré algebra, we are not adding any additional generators and hence we do not extend the dynamics as is done in Einstein's relativity. The only thing we have done is to modify the generators and thereby obtain a more general algebra. It is claimed, see [59], that the Riemannian geometry used in GR is then replaced by the more general Hamiltonian geometry. Then the action of the kinematical group preserves not a Minkowski line element, but a Hamiltonian metric which depends on the cotangent bundle.

It is also interesting to note that in the regime of DSR, spacetime seems to reveal different physical phenomena, which are not significant when we are considering energies much lower than the Planck scale. These physical phenomena include the change in the dispersion relation to a relation whose precise form depends on the energy. Adding to that the possibility of having an energy dependent speed of light, as we discussed in an experiment in 2. Although the analogy might not be one-to-one, however one can think of spacetime as being analogous to a solid body that undergoes changes as it is heated up. In that light, the properties of spacetime also undergo changes as the energy used to probe it increase to the Planck energy. Deformed Special Relativity seems to be a theory which provides a new and exciting way of thinking about the physics at energies close to the Planck scale. The elegance of a DSR theory essentially lies in the fact that it is based on a single and physically reasonable requirement, namely that all observers agree upon the exact value of the length scale of which below this scale Quantum Gravity becomes the dominating theory. This requirement merely imitates the elegance of Einstein's Principle of Relativity where he introduces the speed of light as an invariant scale.

In hopes of trying to find a framework in which spacetime and momentum space were emergent from a theory that is more general, we interpreted a dispersion relation as level sets of a Hamilton function on phase space , which we had derived directly from a modified dispersion relation. These spaces were identified as subspaces of phase space and their geometry was consistently described in the Hamilton Geometry framework. What one finds is that for a general dispersion relation, geometric structures of spacetime and momentum space are dependent on positions and momenta. These structures could include covariant derivatives and curvature as examples. One then, in this framework, generally finds that the geometry of spacetime and that of momentum space are entangled as subspaces of the larger geometry of phase space. A special case where one can have these spaces seperate from each other is for those Hamiltonians who with respect to momenta have third or higher derivatives vanish. In this case, we recover a flat spacetime and likewise, a flat momentum space. In cases where we have a position independent Hamiltonian, we recovered a flat spacetime with a possibly curved momentum space.

Hamilton geometry has been observed to be an optimal framework in describing the level set geometry of Planck scale dispersion relations' modification [59]. A more crucial case for phenomenological purposes is when one introduces particles moving in a curved spacetime, which until now has been a very difficult assignment to describe in a coherent framework.

In standard Hamiltonian mechanics, particle trajectories are determined by the Hamilton-Jacobi equations of motion, however in the Hamilton Geometry framework, these equations become the autoparallel equation of the phase space geometry. Generally the autoparallel equation has a source term, such that the Planck scale effect on phase space is to drag particles away from purely geometric freefall [59]. However special case is when the Hamiltonian is homogenous with respect to momenta, then the source term vanishes and the particle trajectory is solely describe by the autoparallel equation.

As examples of the Hamilton Geometry framework, we analysed the phase space geometry of the κ-Poincare quantum algebra dispersion relation. We saw that this phase space geometry yielded, as subspaces of phase space, a flat spacetime and a curved momentum space of curvature l^2, where l is the Planck-scale deformation parameter. The κ-Poincare algebra dispersion relation when recognized as level sets on phase space yielded a Hamiltonian with no spacetime coordinate dependence, thus from our statement above the curvatures of these spaces is as expected. We concluded based on the results of chapter 2, that the momentum space we recover is the de Sitter momentum space and the spacetime is the κ-Minkowski flat space with the commutation relation $[x^0, x^i] = i1/\kappa x^i$. Furthermore since the κ-Poincare Hamiltonian was homogenous in momenta, the equations of motion were the autoparallels with no force-like term. Moreover, the symmetries in the Hamilton Geometry framework are the symmetries of the Hamiltonian.

For further research in this framework, it will be interesting to devlope a coherent description of particle interactions. However, this is expected to be a non-trivial task since relativistic compatibility would require to modify energy-momentum conservation. As one would see in [59], when spacetime and momentum space are both non-trivially curved, one has to consider tensor fields on phase space to describe the motion of particles. Now identifying these tensors on phase space for the description of the interaction of particles would pose a great challenge. Which would be a very interesting avenue to explore. One suggested path to attacking such a problem would be to look for an appropriate particle's momentum representation. We know on a spacetime manifold M, the momentum is taken as the one form on M, however now one would need to generalize this in terms of tensor fields on phase space. With such an identification, one would proceed to look into particle interactions on phase space formulated in terms of these tensors. With this tensor identification on phase space $\mathcal{P}$, the addition of momentum on $\mathcal{P}$ could be realised by parallel transport of the momenta along autoparallels in momentum space which would resemble what is done in chapter 2 with relative locality where we have taken momentum space to be a curved base manifold and spacetime as a flat space at each point on the base manifold.

Appendix A

Appendix

A.1 Preliminaries on Fibre bundles

In this section we will introduce the reader to the basic definitions and properties of fibre bundles. In chapter 3 we applied these concepts to the study of Hamilton geometry. A formal definition of a fibre bundle can be found in any differential geometry text. We will give the definition according to the book [132]. We define a fibre as follows

Definition A.1.1. Fibre Bundle *Definition*: A differential Fibre Bundle (E, π, M, F, G) consists of the following

1. A differentiable manifold E, called the total space
2. A differentiable manifold M, called the base manifold
3. A differentiable manifold F, called the fibre
4. An onto mapping $\pi : E \mapsto M$ called the projection, which is such that $\pi^{-1}(p) \approx F \forall p \in M$
5. A Lie group, G, known as the structure group. which acts on the fibre
6. And an open covering $\{U_1\}$ of M with diffeomorphisms $\Phi_i : U_i \times F \mapsto \pi^{-1}(U_i)$, called the local trivialisation.

Remark A.1.0.1. One should note that essentially a fibre bundle is the projection π, since all other structures can be defined in terms of the projection map. For example the total space T^*M and the base manifold M are just the range and domain of the projection map respectively.

In fibre bundle language, we would like the total space P to be the cotangent bundle T^*M and the base manifold M to be the spacetime manifold M. First, to each point ρ of P, we can associate a point p on the base manifold, and this is usuallly accomplished by the use of the projection map $\pi : P \mapsto M$ such that $\pi(\rho) = p$. Note that this can be done globally. One should also notice that the projection map is not injective, since we would want it to map entire fibre F_p to points p. This basically captures the fact that to each point p, we are attaching a copy of the fibre bundle F which is enforced by the requirment

$$\pi^{-1}(p) \approx F \tag{A.1}$$

to each point $P \in M$.For us it should be obvious at this point that to each point $p \in M$, we associate a cotangent space $T^*_pM = F_p$ which we will call the fibre to a point on the base manifold. Thus cotagent space is the total space.

Now one takes an open covering of M, $\{U_i\}$, and a set of smooth homeomorphisms, $\{\phi_i\}$ and associate an open set of P given by the preimage $\pi^{-1}(U_i)$ with a product space

$$\phi_i : U_i \times F \mapsto \pi^{-1}(U_i) \tag{A.2}$$

and since this map is between $\phi^{-1}(U_i)$ and $U_i \times F$, we can locally express points in T^*M using points in $U_i \times F$,i.e $\rho = (x^1, p_1) \in T^*M$.

A.2 Proofs

In subsection 3.1.3, it is stated that the transformed definition of the symplectic form and the Poisson bracket are equivalent to the previously defined symplectic form and Poisson bracket. Following is an attempt at a proof of this statement

$$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - N_{ij}\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_i} - \frac{g}{p_i}\frac{\partial f}{\partial x^i} + N_{ij}\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_i} \tag{A.3}$$

$$\Rightarrow \{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial x^i} \tag{A.4}$$

$$\theta = \delta p_i \wedge dx^i = \left(dp_i - N_{ij}dx^j\right) \wedge dx^i = dp_i \wedge dx^i - N_{ij}dx^i \wedge dx^j = dp_i \wedge dx^i \tag{A.5}$$

In section 3.1.3, we defined the nonlinear connection to be of the form

$$N_{ab} = \frac{1}{4}\left(\left\{g^H_{ab}, H\right\} - \partial_a\bar{\partial}^m H g^H_{mb} - \partial_b\bar{\partial}^m H g^H_{am}\right) \tag{A.6}$$

we here show how this is obtained from theorem 1 in [59]. The proof is as follows

Proof.

$$\begin{aligned}
\Delta g^H_{ab} &= 0 \\
\Delta g^H_{ab} &= \left\{g^H_{ab}, H\right\} - Q^m_a g^H_{mb} - Q^m_b g^H_{am} = 0 \\
\left\{g^H_{ab}, H\right\} - 2N_{aq}g^{Hqm}g^H_{mb} - \partial_a\bar{\partial}^m H g^H_{mb} - 2N_{bq}g^{Hqm}g^H_{am} - \partial_b\bar{\partial}^m H g^H_{am} &= 0 \\
\left\{g^H_{ab}, H\right\} - 2N_{ab} - \partial_a\bar{\partial}^m H g^H_{mb} - 2N_{ba} - \partial_b\bar{\partial}^m H g^H_{am} &= 0 \\
4N_{ab} &= \left\{g^H_{ab}, H\right\} - \partial_a\bar{\partial}^m H g^H_{mb} - \partial_b\bar{\partial}^m H g^H_{am} \\
N_{ab} &= \frac{1}{4}\left(\left\{g^H_{ab}, H\right\} - \partial_a\bar{\partial}^m H g^H_{mb} - \partial_b\bar{\partial}^m H g^H_{am}\right)
\end{aligned}$$

□

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