Multi-Period Portfolio Optimization

Yi, Lan

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Abstract

In this thesis, we focus our study on the multi-period portfolio selection problems with different investment conditions. We first analyze the mean-variance multi-period portfolio selection problem with stochastic investment horizon. It is often the case that some unexpected endogenous and exogenous events may force an investor to terminate her investment and leave the market. We give the assumption that the uncertain investment horizon follows a given stochastic process. By making use of the embedding technique of Li and Ng (2000), the original nonseparable problem can be solved by solving an auxiliary problem. With the given assumption, the auxiliary problem can be translated into one with deterministic exit time and solved by dynamic programming. Furthermore, we consider the mean-variance formulation of multi-period portfolio optimization for asset-liability management with an exogenous uncertain investment horizon. Secondly, we consider the multi-period portfolio selection problem in an incomplete market with no short-selling or transaction cost constraint. We assume that the sample space is finite, and the number of possible security price vector transitions is equal to the number of securities. By introducing a family of auxiliary markets, we connect the primal problem to a set of optimization problems without no short-selling or without transaction costs constraint. In the no short-selling case, the auxiliary problem can be solved by using the martingale method of Pliska (1986), and the optimal terminal wealth of the original constrained problem can be derived. In the transaction cost case, we find that the dual problem, which is to minimize the optimal value for the set of optimization problems, is equivalent to the primal problem, when the primal problem has a solution, and we thus

characterize the optimal solution accordingly.

摘 要

本论文主要研究在不同投资条件下的离散时间多阶段投资模型。论文的第一部分主要分析了具有随机终止时间的均值-方差模型。投资者常常因为一些不可预测的外生因素或者内生因素而终止投资，退出金融市场。我们假设在模型中随机终止时间的概率分布已知。利用Li和Ng (2000)使用的嵌入方法，我们可以通过引入目标函数可分的辅助问题，解决原来不可分的终止时间不确定的均值-方差投资问题。因为我们已知随机终止时间的概率分布，因此辅助问题可以转化为确定时间的动态优化问题，并且可以通过动态优化方法得到该问题的解析解。特别地，我们考虑了终止时间不确定的资产负债优化问题。论文的第二部分主要分析了带约束的多阶段投资模型。我们假设金融市场的样本空间是有限的，而且市场上资产的个数与资产价格的转移向量维数是相等的。这个假设使得无套利市场成为完全市场。通过引入一系列的辅助市场，我们将原来带约束的问题跟一系列不带约束的辅助问题联系起来。最小化这组辅助问题的目标函数即为原问题的对偶问题。我们证明了对偶问题的最优目标函数值就是原问题的最优目标函数值。在没有卖空的市场情况下，这样的辅助问题可以通过Pliska (1986)的鞅方法来解决，而对偶问题可以通过动态优化得到最优解，从而我们可以得到原问题的最优终止财富和最优目标函数值。在有交易成本的市场情况下，我们发现如果原问题存在最有解，则对偶问题的最优解就是原问题的最优解。同时，我们也给出了最优策略的一些性质。

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Notation

$(\Omega, \mathscr{F}, P)$	: Filtered probability space.
$E(X)$	: The expectation of any random variable X.
$E(X\|\mathcal{F})$	: Conditional expectation of X given $\mathcal{F}$.
$Var(X)$	: The variance of any random variable X.
$Cov(X, Y)$	: The covariance between any random variables X and Y.
$\mathscr{F}$	: The filtration of financial market.
$\mathcal{F}_t$	: The σ-algebra produced by assets' prices at time t.
$\mathbb{R}$	: The set of real numbers.
$\mathbb{R}^n$	: The set of n-dimension vectors where all elements are real numbers.
$\mathbb{Z}_+$	: The set of positive integer numbers.
$\mathbb{H}$	: Hilbert Space, an complete vector space in which distances and angles can be measured.
$\mathscr{L}_2$	: The set of $\mathbb{R}$-valued random variables X such that $E\|X\|^2 < \infty$.
$\mathbf{0}$	: A zero column vector.
$\mathbf{e}$	: A column vector with all elements equal to 1.
1_A	: The indicator function of the set A.

Chapter 1

Introduction

Nowadays, leaving all your money in a bank account is no longer a good choice because of the existence of various financial securities in the financial market. How to successfully manage the money has become increasingly important since your investment decision today influences what you will have in the following five months or five years, or even in your life time. To best construct and manage your own portfolio is called portfolio optimization, which actually requires an integration of science and art.

1.1. Literature review

The portfolio selection theory offers the guidance for the mission of how to find an optimal distribution of the wealth among various assets. It has been widely accepted that the portfolio management strategies can be classified into two frameworks, namely the mean-variance approach introduced by Markowitz [38] and the expected utility maximization theory firstly studied by von Neumann and Morgenstern [52]. Markowitz [38] pioneered the modern investment science by developing the mean-variance formulation for single-period portfolio allocation that measures the investment risk by the variance of the final wealth, and identifies the best investment strategy by balancing the trade-off between the expected return and the investment risk. His model mathematically results in a quadratic programming (QP) problem which can be solved numerically by standard QP algorithms. Different values of risk then lead to different wealth allocations to the risky assets. When short-selling is allowed and the stock covariance matrix is positive definite, Merton [42] obtains efficient portfolios and the efficient frontier analytically by employing the Lagrangian multiplier. More specifically, the efficient frontier can be described as the set of portfolios satisfying the constrained minimization problem. By employing Lagrangian multiplier, the constrained minimization problem is solved explicitly.

It is natural to extend portfolio selection model from the static setting to the dynamic ones, including those of multi-period and continuous time portfolio selections. However, these extensions have generally taken a different track to Markowitz's original formulation, e.g., [9], [20], [24], [25], [44], [47] for the multi-period case and [13], [18], [19], [21], [26], [29], [40], [45] for the continuous time case. Dynamic portfolio selection problem was pioneered by Merton [40] [41] in a continuous-time expected utility framework. He solves continuous-time consumption-investment problem by using dynamic programming. This work has been generalized substantially over the past decades. Multi-period portfolio selection was dominated by the results of maximizing expected utility functions of the terminal wealth for decades. Rather than treating the variance and expectation of terminal wealth of a portfolio as separate quantities and finding the relationship between them, the expected utility of terminal wealth is considered instead. The conflicting 'profit seeking yet risk averse' nature of the investor is captured by the utility function. It should be noted that mean-variance analysis and expected utility formulation are two different schools for dealing with portfolio selections. One major difficulty in extending Markowitz's model to the multi-period or continuous time settings is that the variance of terminal wealth involves a square of the expectation of terminal wealth, which is hard to analyze due to its non-separability in the dynamic programming sense. Only up to 2000, Li and Ng [33] overcame the difficulty of nonseparability in the mean-variance formulation for multi-period portfolio selection by using an embedding technique and solved the problem analytically. Shortly after the work of Li and Ng, Zhou and Li [57] investigated the continuous time mean-variance problem with deterministic, time-varying coefficients and formulated it as a stochastic linear-quadratic optimal control problem. The continuous-time mean-variance portfolio selection model is formulated as a bi-criteria optimization problem. By putting weights on the two criteria, a single objective stochastic control problem is formulated, which is 'embedded' into a class of auxiliary stochastic linear-quadratic problems. In their linear-quadratic formulation, the dollar amounts, rather than the proportions of wealth, in individual assets are used to define the trading strategy. This leads to a dynamic system that is linear in both the state (i.e. the level of wealth) and the control variables (i.e. the trading strategies). Together with the quadratic form of the objective function, this formulation falls naturally into the realm of stochastic linear-quadratic control. Moreover, since there is no running cost in the object function, the resulting problem is inherently an indefinite stochastic linear-quadratic control problem. This gives rise to the efficient frontier in a closed form for the original portfolio selection problem.

After extending the portfolio selection problems from single-period financial model to dynamic one, many research directions have been explored. One is to consider other investment risk, such as uncertainty of investment horizon in portfolio selection problem, and another one is to analyze portfolio selection problems in frictional market with no-short-selling constraint or with transaction cost.

Investment risk

It is said that 'Nothing ventured, nothing gained'. In financial market, it is reasonable for you to bear more risk while you expect more profits. Of course, risk comes from uncertainty. The research field under the assumption of uncertainty is one important line to the development of financial theories. All the papers we mentioned above are related to this critical issue. Markowitz (1952) [37] presents the foundational work which concentrates on the optimization of investment trading strategies under uncertainty. Actually, the investment portfolio theory is related to the deduction of risks in financial markets. Markowitz's theory brought the idea of diversification in building an efficient portfolio, which offers us a well-known advice: 'Don't put all your eggs in one basket'.

The risk considered above comes from the randomness of the securities' prices, which is called *market risk*. Another kind of risk, which comes from the randomness of the investment horizon, is called *timing risk*. It was introduced in Blanchet-Scalliet, EL Karoui and Martellini [5].

The research results in the literature on portfolio selection with uncertain investment horizon have been limited, though Merton [41] addresses a dynamic optimal portfolio selection problem for an investor who will retire at an uncertain time. Similar work in discrete case can be traced back to Yaari [55] and Hakansson [23]. More recently, Karatzas and Wang [30] consider an optimal dynamic investment problem with an assumption that markets are complete and the eventual exit is a completely endogenous factor- a stopping time of asset price filtration. Blanchet-Scalliet, EL Karoui and Martellini [5] investigate the pricing problems associated with an uncertain time-horizon. Martellini and Urosevic [39] first propose the concept of exit time risk and show that the mean-variance efficient frontier in such a case, where the exit time is independent of the portfolio performance (exogenous exit), coincides with the traditional mean-variance efficient frontier with fixed exit time. Conversely, when the exit time is dependent on portfolio performance (endogenous exit), the set of mean-variance efficient portfolio may rely on the exit time distribution.

Frictional market

In general, a financial market consists of several tradeable instruments whose prices are known, and a set of contingent claims (random variable which representing a payoff at a given time T). The market is *complete* if every contingent claims can be replicated perfectly by holding a portfolio of only the tradable instruments. Otherwise, the market is *incomplete*.

When market is frictionless (i.e., no transaction cost exists and short-selling is allowed), it is complete if and only if the number of market states equals the number of independent vectors in the set of instruments' prices.

When market is frictional, it will become incomplete even with the above conditions. Portfolio selection with prohibition of short-selling and portfolio selection with transaction costs are two examples of a more general class of problems termed constrained portfolio selection. In the past decade, the constrained portfolio selection problem has been extensively studied, e.g., [14], [15], [43], [50], [53], [54]. However, again the continuous-time expected utility model has been dominating the literature. As we know, linear-quadratic control theory is a powerful tool to derive explicit forms of the optimal state feedback control and the optimal cost value through the Riccati equation. What essentially enables the existence of a closed-form solution is that the control is not constrained. As to the constrained case, it becomes much more complicated. Li, Zhou and Lim [34] solved the mean-variance portfolio selection with no-shorting constraint by making use of the stochastic linear-quadratic control and viscosity verification theorem. In Li, Zhou and Lim's paper, the mean-variance portfolio selection problems in continuous-time under the constraint that short-selling of stocks is prohibited is studied. The problem is formulated as a stochastic optimal linear-quadratic control problem. A continuous function is constructed via two Riccati equations, which is then shown to be a viscosity solution to the Hamilton-Jacobi-Bellman equation. Solving these Riccati equations enables one to explicitly obtain the efficient frontier and efficient investment strategies for the original mean-variance problem, using the viscosity verification theorem established in [58].

1.2. Multi-period financial model

In this thesis, we concentrate on multi-period model of financial market. In the financial market, investor can re-allocate her wealth among different securities every time period, which could be a year, a month, a day or even several minutes, (problem dependent). We consider the multi-period security market model with T trading dates (indexed by $0, 1, \cdots, T-1$), and the time horizon T is finite. There are n risky securities and one bond in the market.

The probability space of the market is $(\Omega, \mathscr{F}, P)$. The filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions, $\mathcal{F}_0 = \sigma\{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$, and $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T$. The rate of return of the riskless security between time periods t and $t+1$ is denoted by r_t^0, and those of the risky assets are denoted by a vector $r_t = (r_t^1, \cdots, r_t^n)'$ for $t = 0, 1, \cdots, T-1$, where r_t^i is the random return for security i between time t and $t+1$. r_t^0 and r_t^i are $\mathcal{F}_{t+1}$-measurable. Denote $R_t := r_t - r_t^0\mathbf{e}$ for $t = 0, 1, \cdots, T-1$, where $\mathbf{e} = (1, 1, \cdots, 1)'$.

A special situation under our investigation is discrete state financial model, termed discrete multi-period model. In this kind of models, the sample space Ω is finite.

Consider that an investor enters in the financial market with an initial wealth v_0. The investor can allocate her wealth among the $n+1$ assets. The wealth can be reallocated among the $(n+1)$ assets at the beginning of each of the following T consecutive time periods. Let V_t be the wealth of the investor at the beginning of the tth period, and let π_t^i be the amount of wealth invested in the ith risky asset at the beginning of the tth period. Let $\{\pi_t\}$ be $\mathscr{F}$-adapted process, with $\pi_t = (\pi_t^1, \cdots, \pi_t^n)'$. So $V_t - \sum_{i=1}^n \pi_t^i$ is the amount of wealths invested in the risk-free asset. The relationship between wealth at periods t and $t+1$ is

$$V_{t+1} = V_t r_t^0 + \pi_t' R_t \quad for \quad t = 0, 1, \cdots, T-1.$$

1.3. Outlet of the thesis

There are two major parts in this thesis. Part one concentrates on the mean-variance multi-period portfolio selection problem with a stochastic investment horizon. Part two studies multi-period portfolio selection problems in a frictional market.

In Chapter 2, we define time uncertainty, and study the mean-variance multi-period portfolio selection problem under a state-dependent exogenous uncertain exit time. We adopt the embedding technique in Li and Ng (2000) to solve the mean-variance portfolio selection problem. Specifically, we can see that state-independent exit time is just a special case of state-dependent one. We compare the case to the case with a certain exit time, and find that adding the uncertain exit time increases the investment risk.

In Chapter 3, we consider the multi-period asset-liability mean-variance portfolio selection problem with an uncertain investment horizon, while the distribution of the investment horizon is assumed to be known. The uncertain investment horizon in this chapter is a special case of the general exogenous stochastic investment horizon investigated in Chapter 2. We adopt the same assumption as in Leippold et al. [31], i.e., liability is exogenous. With the given distribution of the exit time, the problem under investigation can be translated into a problem with a deterministic investment horizon which can be solved analytically by the embedding technique of Li and Ng [33].

In Chapter 4, we study a mean-variance optimal portfolio selection problem with no short-selling constraint. This constrained optimization problem is difficult to solve by using dynamic programming. We thus develop our approach based on the application of convex duality. Convex duality methods establish a connection between the original problem and its dual problem. It is generally the case that the dual problem is easier to solve than the primal problem. Using the solution to the dual problem, it allows us to construct the solution to the primal problem.

In Chapter 5, another kind of friction, transaction cost, is considered in the optimal portfolio selection problem. By introducing a set of auxiliary martingales, we transform the primal problem to a set of optimization problems without transaction costs. We find that optimizing a portfolio in the frictional market is equivalent to minimizing the optimal value among the set of auxiliary optimization problems, when the primal problem has a solution. We further characterize the optimal solution accordingly. Our result is similar to the counterpart in continuous-time, although we consider a discrete-time multiple-risky-assets portfolio.

Finally, we give the conclusion and outline briefly some future areas of investigation related to the work in this thesis.

Part I

Multi-Period Portfolio Optimization under Uncertain Investment Horizon

Chapter 2

Multi-Period Portfolio Selection with Stochastic Investment Horizon

2.1. Introduction

"凡事预则立，不预则废。"

The above adage from *The Book of Rites* - a famous ancient book of China, tells us that the nature is full of uncertainty and we should prepare well if we want to succeed. With respect to our investment activities, we should take into account any possible uncertainty in the economy when we make investment decisions.

An assumption often taken for granted in general portfolio selection models is that the investment horizon is deterministic, which implies that an investor knows with certainty the exit time at the beginning of her investment. However, an investment horizon, in the real world, is always unknown when an investor starts her investment. There are many exogenous and endogenous factors that can drive the exit strategy of an investor. Sudden huge consumption, serious illness, retirement and etc. are market-unrelated exogenous reasons to force an investor to exit the financial market. At the same time, there also exit some market-related exogenous reasons, e.g., an anticipation for long-turn depression of financial market could make some investors to exit market earlier. While the exogenous reasons are independent of the investor's investment policy, endogenous factors are policy-dependent. For example, the investor may decide to exit the market once her wealth hits her investment target, or the investor carefully searches for a stopping time to maximize the expected utility of her terminal wealth. In such situations, the exit time is determined endogenously.

Recognizing a clear gap between theory and practice, it seems sympathetic for us to relax the restrictive assumption that the investment horizon is pre-fixed with certainty. Research on this subject was actually pioneered by Yaari (1965)[55], who deals with the problem of optimal consumption for an individual with uncertain date of death, under a pure deterministic investment environment. Other related works include Hakansson (1969)[23], Merton (1971)[41], Karatzas and Wang (2000)[30], Browne (2000)[8], Guo and Hu (2005)[22] and Martellini (2006)[39]. Karatzas and Wang (2000)[30] address the optimal dynamic investment problem in a complete market with an assumption that the uncertain investment horizon is a stopping time of asset price filtration. A different problem of minimizing the expected time to beat a benchmark is addressed in Browne [8], where the exit time is a random variable related to the portfolio. The uncertain exit time concerned in these two works is endogenous. Martellini [39] analyzes a static mean-variance portfolio selection problem for both the situations where exit time is independent and dependent of asset returns. Exogenous and endogenous exit times are considered, respectively, in these two different cases consider.

Multi-period mean-variance portfolio optimization problem with uncertain exit time is studied in Guo and Hu [22], where the uncertain exit time is exogenous. Although the exogenous exit time has been investigated in the investment literature since Yaari (1965), the only case concerned about is market-independent exit time. That means, the probability of the exit time is independent of the financial market. To our best knowledge, the only papers that consider market-dependent exogenous exit time are Blanchet-Scalliet (2005), which applies the uncertain time horizon into dynamic asset pricing theory, and Blanchet-Scalliet et al (2008), which incorporates an uncertain time horizon into a continuous-time optimal portfolio selection problem.

In this chapter, a market-dependent exogenous exit time is introduced into the multi-period portfolio selection problem. It is the first work to consider a state-dependent exit time in a multi-period mean-variance portfolio selection setting. By introducing the uncertain exit time, there are two kinds of uncertainties in our portfolio model, return risk and exit risk. To solve the mean-variance portfolio selection problem, we adopt the embedding technique in Li and Ng (2000). Both analytical optimal policy and the efficient frontier are derived. Furthermore, we can conclude that the state-independent exit time is just a special case of the state-dependent one. We compare this case with cases with certain exit time, and find that adding the uncertain exit time increases the investment risk.

This chapter is organized as follows. After giving an introduction of the time uncertainty, we describe in Section 2.3 the mean-variance portfolio selection model with uncertain exit time. We derive in Section 2.4 the analytical solution by using dynamic programming, and obtain the efficient frontier. We study a special case where the uncertain exit time is state-independent in Section 2.5, and compare cases with uncertain or certain exit time by means of an example. Finally, we summarize the findings of this chapter in Section 2.6.

2.2. Exit-time uncertainty

Similar to Blanchet-Scalliet et al. [5], we assume that the investor's investment time-horizon is a positive random variable τ.

We denote by $\mathscr{F} = \{\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_t \cdots\}$ the filtration reflecting financial market information, $\mathscr{T} = \{\mathcal{T}_1, \cdots, \mathcal{T}_t \cdots\}$, with $\mathcal{T}_t := \sigma(\tau \wedge t)$ the information about whether the exit has occurred or not. Let the filtration $\mathscr{G} = (\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_t, \cdots)$ represent the total information (not completely available to investor), which is generated by filtrations $\mathscr{F}$ and $\mathscr{T}$. Denote $\mathscr{A} := \{\mathcal{A}_t\}$ as an enlargement filtration of $\mathscr{F}$, and $\mathcal{F}_t \subseteq \mathcal{A}_t \subseteq \mathcal{G}_t$. Filtration $\mathscr{A}$ presents all the available information to investors.

Notice that the assets' prices at time t are $\mathcal{F}_t$-measurable, hence $\mathcal{A}_t$-measurable and $\mathcal{G}_t$-measurable. The event $\{\tau > t\}$ is $\mathcal{G}$-measurable, but may not be $\mathcal{A}$-measurable. If τ is an $\mathscr{F}$-stopping time, we have $\mathcal{G}_t = \mathcal{A}_t = \mathcal{F}_t$. However, in this study, we suppose that $\mathcal{A}_t \subset \mathcal{G}_t$, that is, the random variable τ is not an $\mathscr{A}$-stopping time, so the event $\{\tau > t\}$ is not $\mathcal{A}_t$-measurable, which means we can not imply whether or not the exit has occurred by time t under the σ-algebra $\mathcal{A}_t$.

We suppose that the probability of the event $\{t < \tau\}$ is $\mathcal{A}_t$-measurable, which is pre-given. Denote the conditional probability of $\{\tau \leq t\}$ as $P_t = P(\tau \leq t|\mathcal{A}_t)$. Assume that $P_t = P(\tau \leq t|\mathcal{A}_t)$ is an increasing process with respect to t. A sufficient condition for this assumption is that $P(\tau \leq t|\mathcal{A}_t) = P(\tau \leq t|\mathcal{A}_\infty)$. In this chapter, we make the following assumption.

Assumption 2.1

$$P(\tau > t|\mathcal{A}_t) = P(\tau > t|\mathcal{A}_\infty). \tag{2.1}$$

To understand the above definitions, we consider the following example. While an investor invests her money in the financial market, she is waiting at the same time for a gold mining opportunity. Once the opportunity is available and is more profitable than the market portfolio, she will exit the market and invest all her money on the gold mining project. However, whether the gold mining opportunity will be available at time t is unknown under the information $\mathcal{F}_t$. Assume that the availability of gold mining is described as a poisson process with density $\widehat{\lambda}_t$, which is $\mathcal{A}_t$-measurable random variable. Let $\{M_t\}$ be the return process of market portfolio, which is $\mathcal{F}_t$-measurable. So the probability that $\{\tau \leq t\}$ happens can be determined by

$$P_t = P(\tau \leq t \mid \mathcal{A}_t) = 1 - \exp\{-\sum_{s=1}^{t} \lambda_s\}, \tag{2.2}$$

where $\lambda_t := f(\widehat{\lambda}_t, M_t)$ is determined by $\widehat{\lambda}_t$ and the return of market portfolio at time t. Actually, λ_t can be thought as the average failure rate (exit occurrence) during the interval $(t-1, t]$. Notice that λ_t is $\mathcal{A}_t$-measurable, so is P_t. A more specific example will be given in Example 2.1.

Assumption 2.2 *The random time τ is finite almost surely, i.e., $P(\tau < \infty) = 1$.*

Given a constant T, we define a stochastic process ξ_t as follows,

$$\xi_t := P(\tau = t|\mathcal{A}_t) = \begin{cases} P_1 & t = 1; \\ P_t - P_{t-1} & t = 2, \cdots, T-1; \\ 1 - P_{T-1} & t = T. \end{cases} \tag{2.3}$$

It is easy to check that $\sum_{t=1}^{T} \xi_t = 1$ and ξ_t is $\mathcal{A}_t$-measurable.

Remark 2.1 *In above example, if the density $\widehat{\lambda}_t$ is constant, then*

$$P_t = P(\tau \leq t \mid \mathcal{F}_t) = 1 - \exp\{-\sum_{s=1}^{t} \lambda_s\}.$$

λ_t is $\mathcal{F}_t$-measurable, so is P_t and ξ_t. Specifically, if the investor draw her money out of financial market once the gold mining project is available, no matter it is more profitable than the market portfolio or not, λ_t will be $\mathcal{F}_t$-independent, and $\lambda_t = \widehat{\lambda}_t$.

2.3. Problem formulation

We consider a financial market with T trading dates (indexed by $0, 1, \cdots, T-1$), and a finite time horizon T. Uncertainty of the economy is described through a probability space $(\Omega, \mathscr{A}, P)$. Without lose of generality, we suppose $\mathscr{A} = \mathscr{F}$. The results obtained in this chapter can be easily generalized to situations where $\mathscr{A} \supset \mathscr{F}$. There are $(n+1)$ securities, one risk-free asset S_0 and n risky assets $S_1, \cdots, S_n$. An investor enters the financial market with an initial wealth v_0. The investor can allocate her wealth among the $(n+1)$ assets. The wealth can be reallocated among the $(n+1)$ assets at the beginning of each of the following T consecutive time periods until she exits the market. The investor plans to invest her wealth at most for T periods. However, she will exit the market at some random time τ by some reasons related to the financial market. Hence the exiting time is $T \wedge \tau$.

The rate of return of the riskless security between time periods t and $t+1$ within the planning horizon is denoted by r_t^0, and those of the risky assets are denoted by a vector $r_t = (r_t^1, \cdots, r_t^n)'$, where r_t^i is the random return for security i between time periods t and $t+1$. It is assumed in this chapter that vectors

$\widetilde{r}_t = [r_t^0, r_t']'$, $t = 0, 1, \cdots, T-1$, are statistically independent and return $\widetilde{r}_t$ has a know mean $E(\widetilde{r}_t) = [E(r_t^0), E(r_t^1), \cdots, E(r_t^n)]'$ and a known covariance

$$Cov(\widetilde{r}_t) = \begin{bmatrix} \sigma_{t,00} & \cdots & \sigma_{t,0n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,0n} & \cdots & \sigma_{t,nn} \end{bmatrix}.$$

Denote $R_t := r_t - r_t^0\mathbf{e}$ where $\mathbf{e} = (1, 1, \cdots, 1)'$. It is reasonable to assume that $E(\widetilde{r}_t\widetilde{r}_t')$ is positive definite for all time periods, i.e.,

$$E(\widetilde{r}_t\widetilde{r}_t') = \begin{bmatrix} E((r_t^0)^2) & E((r_t^1 r_t^0)) & \cdots & E((r_t^n r_t^0)) \\ E((r_t^0 r_t^1)) & E((r_t^1)^2) & \cdots & E((r_t^n r_t^1)) \\ \cdots & \cdots & \cdots & \cdots \\ E((r_t^0 r_t^n)) & E((r_t^1 r_t^n)) & \cdots & E((r_t^n)^2) \end{bmatrix} > 0, \quad \forall\, t = 0, 1, \cdots, T-1.$$

Suppose that τ is a discrete random processes defined in Section 2.2. Hence, the exit probability is $\xi_t (t = 1, 2, \cdots, T)$, where ξ_t and R_{t-1} can be dependent.

Let V_t be the wealth of the investor at the beginning of the tth period, and let π_t^i be the amount of wealth invested in the ith risky asset at the beginning of the tth period. Let the vector $\pi_t = (\pi_t^1, \cdots, \pi_t^n)'$. So $V_t - \sum_{i=1}^n \pi_i^t$ is the amount of wealth invested in the risk-free asset. The relationship between wealth of periods t and $t+1$ is

$$V_{t+1} = V_t r_t^0 + \pi_t' R_t, \quad t = 0, 1, \cdots, T-1. \tag{2.4}$$

The investor is seeking a best investment strategy $\pi_t = (\pi_t^1, \cdots, \pi_t^n)'$ for $t = 0, 1, \cdots, T-1$, such that (i) the expected value of the uncertain terminal wealth $V_{T\wedge\tau}$ is maximized while the variance of the terminal wealth is not greater than a preselected risk level,

$$(P1(\sigma)) \begin{cases} \max\limits_{\pi} & E(V_{T\wedge\tau}) \\ s.t. & Var(V_{T\wedge\tau}) \le \sigma \text{ and } (2.4), \end{cases}$$

for $\sigma \geq 0$, or (ii) the variance of the uncertain terminal wealth $V_{T\wedge\tau}$ is minimized while the expected terminal wealth is not smaller than a preselected level,

$$(P2(\varepsilon))\begin{cases} \min_{\pi} & Var(V_{T\wedge\tau}) \\ s.t. & E(V_{T\wedge\tau}) \geq \epsilon \ \text{ and } \ (2.4), \end{cases}$$

for $\epsilon \geq 0$.

Using the Lagrangian approach, either problem $(P1(\sigma))$ or $(P2(\varepsilon))$ can be expressed equivalently as

$$(P3(\omega))\begin{cases} \max_{\pi} & E(V_{T\wedge\tau}) - \omega Var(V_{T\wedge\tau}) \\ s.t. & (2.4), \end{cases}$$

where $\omega \in [0, \infty)$, which represents a trade-off between the expected terminal wealth and the associated risk. In this study, we concentrate on problem $(P3(\omega))$.

2.4. Analytical solution to M-P M-V formulation with exit-time uncertainty

2.4.1. Construction of auxiliary problem

Since the M-V formulation is nonseparable in the sense of dynamic programming, we use the embedding technique of Li and Ng (2000). We will prove in the following that the embedding technique still works when the exit time is uncertain. We first introduce an alternative optimization problem $(P4(\lambda, \omega))$:

$$(P4(\lambda,\omega))\begin{cases} \max_{\pi} & E(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2) \\ s.t. & (2.4), \end{cases}$$

Define $\Phi_A(\lambda, \omega)$ to be the set of optimal solutions of problem $(P4(\lambda, \omega))$ and $\Phi_P(\omega)$ to be the set of optimal solutions of problem $(P3(\omega))$, i.e.,

$\Phi_A(\lambda, \omega) = \{\phi \mid \phi \text{ is an optimal solution to } (P4(\lambda, \omega))\}$.

$\Phi_P(\omega) = \{\phi \mid \phi \text{ is an optimal solution to } (P3(\omega))\}$.

Furthermore, we denote a new variable $d(\phi, \omega)$ as a function of ϕ and ω, i.e.,

$$d(\phi, \omega) = 1 + 2\omega E(V_{T\wedge\tau}) \mid_\phi . \tag{2.5}$$

The following two theorems will show the relationship between the original problem $(P3(\omega))$ and the auxiliary problem $(P4(\lambda, \omega))$.

Theorem 2.1 *For any $\phi^* \in \Phi_P(\omega)$, $\phi^* \in \Phi_A(d(\phi^*, \omega), \omega)$.*

Proof. If ϕ^* is a solution of $(P3(\omega))$, but not a solution to $(P4(d(\phi^*, \lambda), \omega))$, there exists a ϕ such that

$$-\omega E(V_{T\wedge\tau}^2(\phi)) + d(\phi^*, \omega)E(V_{T\wedge\tau}(\phi)) > -\omega E(V_{T\wedge\tau}^2(\phi^*)) + d(\phi^*, \omega)E(V_{T\wedge\tau}(\phi^*)),$$

that is

$$(-\omega, d(\phi^*, \omega)) \begin{pmatrix} E(V_{T\wedge\tau}^2(\phi)) \\ E(V_{T\wedge\tau}(\phi)) \end{pmatrix} > (-\omega, d(\phi^*, \omega)) \begin{pmatrix} E(V_{T\wedge\tau}^2(\phi^*)) \\ E(V_{T\wedge\tau}(\phi^*)) \end{pmatrix}. \tag{2.6}$$

Let

$$\begin{aligned} U &= E(V_{T\wedge\tau}(\phi)) - \omega Var(V_{T\wedge\tau}(\phi)) \\ &= E(V_{T\wedge\tau}(\phi)) - \omega[E(V_{T\wedge\tau}^2(\phi)) - E^2(V_{T\wedge\tau}(\phi))]. \end{aligned} \tag{2.7}$$

As U is convex with respect to $E(V_{T\wedge\tau}(\phi))$ and $E(V_{T\wedge\tau}^2(\phi))$, we have

$$\begin{aligned} & U[E(V_{T\wedge\tau}^2(\phi)), E(V_{T\wedge\tau}(\phi))] - U[E(V_{T\wedge\tau}^2(\phi^*)), E(V_{T\wedge\tau}(\phi^*))] \\ \geq\ & \left(\frac{\partial U}{\partial E(V_{T\wedge\tau}^2(\phi))}, \frac{\partial U}{\partial E(V_{T\wedge\tau}(\phi))}\right)|_{\phi^*} \begin{pmatrix} E(V_{T\wedge\tau}^2(\phi)) - E(V_{T\wedge\tau}^2(\phi^*)) \\ E(V_{T\wedge\tau}(\phi)) - E(V_{T\wedge\tau}(\phi^*)) \end{pmatrix} \\ =\ & (-\omega, d(\phi^*, \omega)) \begin{pmatrix} E(V_{T\wedge\tau}^2(\phi)) - E(V_{T\wedge\tau}^2(\phi^*)) \\ E(V_{T\wedge\tau}(\phi)) - E(V_{T\wedge\tau}(\phi^*)) \end{pmatrix} > 0, \end{aligned}$$

which is a contradiction. □

Theorem 2.2 *Assume $\phi^* \in \Phi_A(\lambda^*, \omega)$. A necessary condition for $\phi^* \in \Phi_P(\omega)$ is $\lambda^* = 1 + 2\omega E(V_{T\wedge\tau}) \mid_{\phi^*}$.*

Proof. For fixed ω, the set of all solutions to $((P4(\lambda, \omega))$ can be parameterized by λ. If ϕ^* is an optimal solution of $(P3(\omega))$, then $\phi^* \in \bigcup_\lambda \Phi_A(\lambda, \omega)$. Hence $(P3(\omega))$ is equivalent to the following problem:

$$\begin{aligned} & \max\nolimits_\lambda U[E(V^2_{T\wedge\tau}(\lambda, \omega)), E^2(V_{T\wedge\tau}(\lambda, \omega)] \\ = \; & \max\nolimits_\lambda \{E(V_{T\wedge\tau}(\lambda, \omega)) - \omega[E(V^2_{T\wedge\tau}(\lambda, \omega)) - E^2(V_{T\wedge\tau}(\lambda, \omega))]\}. \end{aligned} \tag{2.8}$$

The necessary condition for optimal λ^* is $\frac{\partial U}{\partial \lambda} \mid_{\lambda^*} = 0$, that is

$$\frac{\partial E(V_{T\wedge\tau}(\lambda^*, \omega))}{\partial \lambda}[1 + 2\omega E(V_{T\wedge\tau}(\lambda^*, \omega))] - \omega \frac{\partial E(V^2_{T\wedge\tau}(\lambda^*, \omega)}{\partial \lambda} = 0. \tag{2.9}$$

On the other hand, because $\phi^* \in \Phi_A(\lambda, \omega)$, the optimality condition for $((P4(\lambda, \omega))$ gives rise,

$$\lambda^* \frac{\partial E(V_{T\wedge\tau}(\lambda^*, \omega))}{\partial \lambda} - \omega \frac{\partial E(V^2_{T\wedge\tau}(\lambda^*, \omega))}{\partial \lambda} = 0. \tag{2.10}$$

These two conditions, (2.10) and (2.11), yield

$$\lambda^* = 1 + 2\omega E(V_{T\wedge\tau}(\lambda^*, \omega)) = [1 + 2\omega E(V_{T\wedge\tau}(\lambda, \omega))] \mid_{\phi^*} .$$

□

Based on these two theorems, we can get the optimal solution to the original problem by solving the auxiliary problem $((P4(\lambda, \omega))$. The objective function of $((P4(\lambda, \omega))$ can be reformulated by using the definition of exit probability ξ_t.

Proposition 2.1 *The auxiliary problem $(P4(\lambda, \omega))$ is equivalent to*

$$\begin{cases} \max\limits_{\pi} & E[\sum_{t=1}^T (\lambda V_t - \omega V_t^2)\xi_t] \\ s.t. & V_{t+1} = V_t r_t^0 + \pi_t' R_t \quad for \quad t = 0, 1, \cdots, T-1. \end{cases} \tag{2.11}$$

Proof. Using the property of conditional probability, we can derive the following,

$$
\begin{aligned}
& E(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2) \\
= \;& E[E[(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2)\mathbf{1}_{\{\tau=1\}} \mid \mathcal{F}_1]] + E[E[(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2)\mathbf{1}_{\{\tau>1\}} \mid \mathcal{F}_1]] \\
= \;& E[(\lambda V_1 - \omega V_1^2)E[\mathbf{1}_{\{\tau=1\}} \mid \mathcal{F}_1]] + E[(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2)\mathbf{1}_{\{\tau>1\}}] \\
= \;& E[(\lambda V_1 - \omega V_1^2)\xi_1] + E[E[(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2)\mathbf{1}_{\{\tau>1\}} \mid \mathcal{F}_2]] \\
= \;& E[(\lambda V_1 - \omega V_1^2)\xi_1] + E[E[(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2)\mathbf{1}_{\{\tau=2\}} \mid \mathcal{F}_2]] \\
& + E[E[(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2)\mathbf{1}_{\{\tau>2\}} \mid \mathcal{F}_2]] \\
= \;& E[(\lambda V_1 - \omega V_1^2)\xi_1] + E[(\lambda V_2 - \omega V_2^2)\xi_2] + E[(\lambda V_{T\wedge\tau} - \omega V_{T\wedge\tau}^2)\mathbf{1}_{\{\tau>2\}}] \\
& \vdots \\
= \;& \sum_{i=1}^{T} E[(V_t - \lambda V_t^2)\xi_t],
\end{aligned}
$$

which proves the equivalence between (2.11) and ($P4(\lambda, \omega)$). □

2.4.2. Analytical form of the optimal dynamic portfolio policy

The optimal solution to the auxiliary problem can be derived analytically by using dynamic programming. In the following, we denote $E_t(\cdot) := E(\cdot \mid \mathcal{F}_t)$ for our convenience.

Theorem 2.3 *The optimal solution of the auxiliary problem at each time period t is of the following form,*

$$\pi_t^*(V_t, \gamma) \;=\; \frac{\gamma}{2}u_t(\gamma) - K_t V_t, \tag{2.12}$$

where

$$\gamma = \frac{\lambda}{\omega}, \tag{2.13}$$

$$u_{t-1} = E_{t-1}(R_{t-1}R'_{t-1}(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + A_t)), \tag{2.14}$$

$$K_{t-1} = r_t^0 E_{t-1}(R_{t-1}R'_{t-1}(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + B_t)), \tag{2.15}$$

$$\begin{aligned} A_{t-1} =& r_{t-1}^0[E_{t-1}(\xi_t + A_t) \\ &- E_{t-1}(R_{t-1}(\xi_t + A_t))'E_{t-1}(R_{t-1}R'_{t-1}(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + B_t))], \end{aligned} \tag{2.16}$$

$$\begin{aligned} B_{t-1} =& (r_{t-1}^0)^2[E_{t-1}(\xi_t + B_t) \\ &\times E_{t-1}(R_{t-1}(\xi_t + B_t))'E_{t-1}(R_{t-1}R'_{t-1}(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + B_t))], \end{aligned} \tag{2.17}$$

$$\begin{aligned} C_{t-1} =& C_t + \\ &\frac{\lambda^2}{4\omega}E_{t-1}(R_{t-1}(\xi_t + A_t))'E_{t-1}(R_{t-1}R'_{t-1}(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + A_t)), \end{aligned} \tag{2.18}$$

with the following boundary conditions,

$$\begin{aligned} u_{T-1} =& E_{T-1}(R_{T-1}R'_{T-1}\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T), \\ K_{T-1} =& r_{T-1}^0 E_{T-1}(R_{T-1}R'_{T-1}\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T), \\ A_{T-1} =& r_{T-1}^0[E_{T-1}(\xi_T) - E_{T-1}(R_{T-1}\xi_T)'E_{T-1}(R_{T-1}R'_{T-1}\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T)], \\ B_{T-1} =& (r_{T-1}^0)^2[E_{T-1}(\xi_T) - E_{T-1}(R_{T-1}\xi_T)'E_{T-1}(R_{T-1}R'_{T-1}\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T)], \\ C_{T-1} =& \frac{\lambda^2}{4\omega}E_{T-1}(R_{T-1}\xi_T)'E_{T-1}(R_{T-1}R'_{T-1}\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T). \end{aligned}$$

Proof. Denote the benefit-to-go at stage t by

$$f_t(V_t) = \max_{\pi_{t-1},\cdots,\pi_{T-1}} E[\sum_{s=t}^{T}(\lambda V_s - \omega V_s^2)\xi_s \mid \mathcal{F}_{t-1}],$$

for $t = 1, 2, \cdots, T$. Note that $f_t(V_t)$ can be further expressed as

$$f_t(V_t) = \max_{\pi_{t-1},\cdots,\pi_{T-1}} E\{(\lambda V_t - \omega V_t^2)\xi_t + E[\sum_{s=t+1}^{T}(\lambda V_s - \omega V_s^2)\xi_s \mid \mathcal{F}_t] \mid \mathcal{F}_{t-1}\}$$

$$= \max_{\pi_{t-1}} E[(\lambda V_t - \omega V_t^2)\xi_t + f_{t+1}(V_{t+1}) \mid \mathcal{F}_{t-1}],$$

for $t = 1, 2, \cdots, T-1$, and the boundary condition is

$$f_T(V_T) = \max_{\pi_{T-1}} E[(\lambda V_T - \omega V_T^2)\xi_T \mid \mathcal{F}_{T-1}].$$

The dynamic programming algorithm starts from stage T. For given $\mathcal{F}_{T-1}$, the optimization problem is

$$\begin{aligned}
& f_T(V_T) \\
=& \max_{\pi_{T-1}} E_{T-1}[(\lambda V_T - \omega V_T^2)\xi_T] \\
=& \max_{\pi_{T-1}} E_{T-1}\{[\lambda(V_{T-1}r_{T-1}^0 + R_{T-1}'\pi_{T-1})\xi_T] - \omega(V_{T-1}r_{T-1}^0 + R_{T-1}'\pi_{T-1})^2\xi_T\} \\
=& \max_{\pi_{T-1}} E_{T-1}\{[\lambda V_{T-1}r_{T-1}^0\xi_T - \omega V_{T-1}^2(r_{T-1}^0)^2\xi_T] \\
&+ [\lambda\xi_T R_{T-1}'\pi_{T-1} - 2\omega V_{T-1}r_{T-1}^0\xi_T R_{T-1}'\pi_{T-1} - \omega\xi_T\pi_{T-1}'R_{T-1}R_{T-1}'\pi_{T-1}]\}.
\end{aligned}$$

Maximization of the above function with respect to π_{T-1} yields,

$$\pi_{T-1}^* = E_{T-1}(R_{T-1}R_{T-1}'\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T)(\frac{\lambda}{2\omega} - V_{T-1}r_{T-1}^0).$$

Substituting π_{T-1}^* back to $f_T(V_T)$ yields to optimal benefit-to-go at given $\mathcal{F}_{T-1}$,

$$f_T^*(V_T) = \lambda A_{T-1}V_{T-1} - \omega B_{T-1}V_{T-1}^2 + C_{T-1},$$

where

$$\begin{aligned}
A_{T-1} =& r_{T-1}^0[E_{T-1}(\xi_T) - E_{T-1}(R_{T-1}\xi_T)'E_{T-1}(R_{T-1}R_{T-1}'\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T)], \\
B_{T-1} =& (r_{T-1}^0)^2[E_{T-1}(\xi_T) - E_{T-1}(R_{T-1}\xi_T)'E_{T-1}(R_{T-1}R_{T-1}'\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T)], \\
C_{T-1} =& \frac{\lambda^2}{4\omega}E_{T-1}(R_{T-1}\xi_T)'E_{T-1}(R_{T-1}R_{T-1}'\xi_T)^{-1}E_{T-1}(R_{T-1}\xi_T).
\end{aligned}$$

Therefore, the benefit-to-go at stage $T-1$ is

$$
\begin{aligned}
& f_{T-1}(V_{T-1}) \\
&= \max_{\pi_{T-2}} E_{T-2}[(\lambda V_{T-1} - \omega V_{T-1}^2)\xi_{T-1} + f_T^*(V_T)] \\
&= \max_{\pi_{T-2}} E_{T-2}\{\lambda(\xi_{T-1} + A_{T-1})V_{T-1} - \omega(\xi_{T-1} + B_{T-1})V_{T-1}^2 + C_{T-1}\} \\
&= \max_{\pi_{T-2}} E_{T-2}\{[\lambda V_{T-2} r_{T-2}^0(\xi_{T-1} + A_{T-1}) - \omega V_{T-2}^2 (r_{T-2}^0)^2(\xi_{T-1} + B_{T-1})] \\
&\quad + C_{T-1} + [\lambda(\xi_{T-1} + A_{T-1})R_{T-2}'\pi_{T-2} - 2\omega V_{T-2} r_{T-2}^0(\xi_{T-1} + B_{T-1})R_{T-2}'\pi_{T-2} \\
&\quad - \omega(\xi_{T-1} + B_{T-1})\pi_{T-2}' R_{T-2} R_{T-2}' \pi_{T-2}]\},
\end{aligned}
$$

which has a same structure as the original utility function at stage T.

Assume that the derived utility function has a similar form at stage t, $1 \leq t \leq T-2$, to the original utility function at stage T. The benefit-to-go at stage t is

$$
\begin{aligned}
f_t(V_t) &= \max_{\pi_{t-1}} E_{t-1}[(\lambda V_t - \omega V_t^2)\xi_t + f_{t+1}(V_{t+1})] \\
&= \max_{\pi_{t-1}} E_{t-1}\{\lambda(\xi_t + A_t)V_t - \omega(\xi_t + B_t)V_t^2 + C_t\} \\
&= \max_{\pi_{t-1}} E_{t-1}\{[\lambda V_{t-1} r_{t-1}^0(\xi_t + A_t) - \omega V_{t-1}^2 (r_{t-1}^0)^2(\xi_t + B_t)] \\
&\quad + C_t + [\lambda(\xi_t + A_t)R_{t-1}'\pi_{t-1} - 2\omega V_{t-1} r_{t-1}^0(\xi_t + B_t)R_{t-1}'\pi_{t-1} \\
&\quad - \omega(\xi_t + B_t)\pi_{t-1}' R_{t-1} R_{t-1}' \pi_{t-1}]\}.
\end{aligned}
$$

Maximizing the above function derives the optimal policy at given $\mathcal{F}_{t-1}$,

$$
\begin{aligned}
\pi_{t-1}^* =& E_{t-1}(R_{t-1} R_{t-1}'(\xi_t + B_t))^{-1} \\
& \times [\frac{\lambda}{2\omega} E_{t-1}(R_{t-1}(\xi_t + A_t)) - V_{t-1} r_{t-1}^0 E_{t-1}(R_{t-1}(\xi_t + B_t))],
\end{aligned}
$$

and the cost-to-go $f_{t-1}(V_{t-1})$ is:

$$\begin{aligned}
f_{t-1}(V_{t-1}) &= \max_{\pi_{t-1}} E_{t-1}[(\lambda V_{t-1} - \omega V_{t-1}^2)\xi_{t-1} + \lambda A_{t-1}V_{t-1} - \omega B_{t-1}V_{t-1}^2 + C_{t-1}] \\
&= \max_{\pi_{t-2}} E_{t-2}\{[\lambda V_{t-2}r_{t-2}^0(\xi_{t-1} + A_{t-1}) - \omega V_{t-2}^2(r_{t-2}^0)^2(\xi_{t-1} + B_{t-1})] \\
&\quad + C_{t-1} + [\lambda(\xi_{t-1} + A_{t-1})R_{t-1}'\pi_{t-2} - 2\omega V_{t-2}r_{t-2}^0(\xi_{t-1} + B_{t-1})R_{t-2}'\pi_{t-2} \\
&\quad - \omega(\xi_{t-1} + B_{t-1})\pi_{t-2}'R_{t-2}R_{t-2}'\pi_{t-2}]\},
\end{aligned}$$

where

$$\begin{aligned}
A_{t-1} =& r_{t-1}^0[E_{t-1}(\xi_t + A_t) \\
& - E_{t-1}(R_{t-1}(\xi_t + A_t))'E_{t-1}(R_{t-1}R_{t-1}'(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + B_t))], \\
B_{t-1} =& (r_{t-1}^0)^2[E_{t-1}(\xi_t + B_t) \\
& - E_{t-1}(R_{t-1}(\xi_t + B_t))'E_{t-1}(R_{t-1}R_{t-1}'(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + B_t))], \\
C_{t-1} =& C_t + \\
& \frac{\lambda^2}{4\omega}E_{t-1}(R_{t-1}(\xi_t + A_t))'E_{t-1}(R_{t-1}R_{t-1}'(\xi_t + B_t))^{-1}E_{t-1}(R_{t-1}(\xi_t + A_t)).
\end{aligned}$$

□

We can see that the second term in $\pi_t^*(V_{t-1}, \gamma)$ is linear with respect to the wealth V_{t-1} and is independent of γ, and the first term is a linear function of γ. Substituting (2.12) into the equation of wealth dynamics yields the dynamics of the wealth under policy $\pi_t^*(V_t, \gamma)$,

$$V_{t+1}(\gamma) = (r_t^0 - K_t'R_t)V_t(\gamma) + \frac{\gamma}{2}R_t'u_t. \tag{2.19}$$

We derive V_t for $t = 1, 2, \cdots, T$ by solving the above dynamics as follows

$$V_t = M_tV_0 + \frac{\gamma}{2}N_t, \quad t = 1, 2, \cdots, T, \tag{2.20}$$

where

$$M_t = \prod_{i=0}^{t-1}(r_i^0 - K_i'R_i), \tag{2.21}$$

$$N_t = \sum_{i=0}^{t-1}[\prod_{j=i+1}^{t-1}(r_j^0 - K_j'R_j)]u_i'R_i. \tag{2.22}$$

Squaring both sides of(2.19) yields

$$V_{t+1}^2(\gamma) = (r_t^0 - K_t'R_t)^2V_t^2(\gamma) + \gamma(r_t^0 - K_t'R_t)V_t(\gamma)R_t'u_t + \frac{\gamma^2}{4}u_t'R_tR_t'u_t. \tag{2.23}$$

Similarly, we derive V_t^2 for $t = 1, 2, \cdots, T$ as follows by solving the above dynamics

$$V_t^2 = I_tV_0^2 + \gamma J_tV_0 + \frac{\lambda^2}{4}L_t, \tag{2.24}$$

where

$$I_t = \prod_{i=0}^{t-1}(r_i^0 - K_i'R_i)^2, \tag{2.25}$$

$$J_t = \sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0 - K_j'R_j)^2u_i'R_i\prod_{s=0}^{i}(r_s^0 - K_s'R_s), \tag{2.26}$$

$$L_t = \sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0 - K_j'R_j)^2(u_i'R_i)^2 + 2\sum_{i=1}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0 - K_j'R_j)^2u_i'R_i[\sum_{s=0}^{i-1}\prod_{l=s+1}^{i}t(r_l^0 - K_l'R_l)u_s'R_s]. \tag{2.27}$$

Notice that $E(V_{T\wedge\tau}) = E(\sum_{t=1}^{T}V_t\xi_t)$ and $E(V_{T\wedge\tau}^2) = E(\sum_{t=1}^{T}V_t^2\xi_t)$, the expectation of terminal wealth $V_{T\wedge\tau}$ and $V_{T\wedge\tau}^2$ are

$$E(V_{T\wedge\tau}) = V_0E(\sum_{t=1}^{T}M_t\xi_t) + \frac{\gamma}{2}E(\sum_{t=1}^{T}N_t\xi_t), \tag{2.28}$$

$$E(V_{T\wedge\tau}^2) = V_0^2E(\sum_{t=1}^{T}I_t\xi_t) + \gamma V_0E(\sum_{t=1}^{T}J_t\xi_t) + \frac{\gamma^2}{4}E(\sum_{t=1}^{T}L_t\xi_t). \tag{2.29}$$

The variance of the uncertain terminal wealth under portfolio policy $\pi_t^*(V_t, \gamma)$ can be expressed in the terms of γ by using (2.28) and (2.29) ,

$$\begin{aligned} Var(V_{T\wedge\tau}(\gamma)) &= E(V_{T\wedge\tau}^2) - E^2(V_{T\wedge\tau}) \\ &= V_0^2 I + V_0\gamma J + \frac{\gamma^2}{4} L, \end{aligned} \tag{2.30}$$

where

$$I = [E(\sum_{t=1}^{T} I_t\xi_t) - (E(\sum_{t=1}^{T} M_t\xi_t))^2], \tag{2.31}$$

$$J = [E(\sum_{t=1}^{T} J_t\xi_t) - E(\sum_{t=1}^{T} M_t\xi_t)E(\sum_{t=1}^{T} N_t\xi_t)], \tag{2.32}$$

$$L = [E(\sum_{t=1}^{T} L_t\xi_t) - E^2(\sum_{t=1}^{T} N_t\xi_t)]. \tag{2.33}$$

Rewrite $E(V_{T\wedge\tau})$ as follows

$$E(V_{T\wedge\tau}) = V_0 M + \frac{\gamma}{2} N, \tag{2.34}$$

where

$$M = E(\sum_{t=1}^{T} M_t\xi_t), \tag{2.35}$$

$$N = E(\sum_{t=1}^{T} N_t\xi_t). \tag{2.36}$$

Note that the expected uncertain terminal wealth $E(V_{T\wedge\tau}(\gamma))$ is an increasing linear function of γ while the variance $Var(V_{T\wedge\tau}(\gamma))$ is a quadratic function of γ. We express $U(E(V_{T\wedge\tau}), Var(V_{T\wedge\tau}))$ as a function of γ,

$$\begin{aligned} &U(E(V_{T\wedge\tau}), Var(V_{T\wedge\tau})) \\ = \; & V_0 M + \frac{\gamma}{2} N - \omega[V_0^2 I + V_0\gamma J + \frac{\gamma^2}{4} L]. \end{aligned} \tag{2.37}$$

It can be seen that U is a concave function of γ. Differentiating (2.37) with respect to γ yields

$$\frac{\partial U}{\partial \gamma} = N/2 - \omega(V_0 J + L\gamma/2). \tag{2.38}$$

The optimal γ must satisfy the optimality condition of $\frac{\partial U}{\partial \gamma} = 0$, that is,

$$\gamma^* = -\frac{2J}{L}V_0 + \frac{N}{L\omega}. \tag{2.39}$$

Notice that $\lambda^* = \omega\gamma^*$ satisfies the condition that $\lambda^* = 1 + 2\omega E(V_{T\wedge\tau}) \mid_{\phi^*}$. Actually, the necessary condition for optimal λ^* is $\frac{\partial U}{\partial \lambda}|_{\lambda^*} = 0$. Since $\gamma = \frac{\lambda}{\omega}$ and ω is given, the necessary condition is equivalent to $\frac{\partial U}{\partial \gamma}|_{\gamma^*} = 0$.

Substituting the optimal γ^* in (2.39) into equation (2.12) yields the optimal multi-period portfolio policy for $(P3(\omega))$,

$$\begin{aligned}\pi_{t-1}^* &= -r_{t-1}^0 E_{t-1}(R_{t-1}R_{t-1}'(\xi_t + B_t))^{-1} E_{t-1}(R_{t-1}(\xi_t + B_t))V_{t-1} \\ &\quad +(-\frac{J}{L}V_0 + \frac{N}{2L\omega})E_{t-1}(R_{t-1}R_{t-1}'(\xi_t + B_t))^{-1} E_{t-1}(R_{t-1}(\xi_t + A_t)).\end{aligned}$$

2.5. Special cases of stochastic investment horizon

2.5.1. State-independent uncertain exit time

When the uncertain exit time is state-independent, the stochastic process ξ_t defined in (2.3) satisfies

$$\xi_t := P(\tau = t|\mathcal{A}_t) = P(\tau = t|\mathcal{A}_\infty) = P(\tau = t),$$

which is $\mathcal{A}$-independent. Hence ξ_t and R_{t-1} are independent for any t.

In this situation, the optimal solution of the auxiliary problem at each time period t can be formulated as the following,

$$\pi_t^* = \frac{\gamma}{2} u_t - K_t V_t, \tag{2.40}$$

where

$$\begin{aligned}
\gamma &= \frac{\lambda}{\omega}, \\
u_{t-1} &= \frac{\xi_t + A_t}{\xi_t + B_t} E(R_{t-1} R_{t-1}')^{-1} E(R_{t-1}), \\
K_{t-1} &= r_{t-1}^0 E(R_{t-1} R_{t-1}')^{-1} E(R_{t-1}), \\
A_{t-1} &= r_{t-1}^0 (\xi_t + A_t)[1 - E(R_{t-1})' E(R_{t-1} R_{t-1}')^{-1} E(R_{t-1})], \\
B_{t-1} &= (r_{t-1}^0)^2 (\xi_t + B_t)[1 - E(R_{t-1})' E(R_{t-1} R_{t-1}')^{-1} E(R_{t-1})], \\
C_{t-1} &= C_t + \frac{\lambda^2}{4\omega} \frac{(\xi_t + A_t)^2}{\xi_t + B_t} E(R_{t-1})' E(R_{t-1} R_{t-1}')^{-1} E(R_{t-1}),
\end{aligned}$$

with the following boundary condition

$$\begin{aligned}
u_{T-1} &= E(R_{T-1} R_{T-1}')^{-1} E(R_{T-1}), \\
A_{T-1} &= r_{T-1}^0 \xi_T [1 - E(R_{T-1})' E(R_{T-1} R_{T-1}')^{-1} E(R_{T-1})], \\
B_{T-1} &= (r_{T-1}^0)^2 \xi_T [1 - E(R_{T-1})' E(R_{T-1} R_{T-1}')^{-1} E(R_{T-1})], \\
C_{T-1} &= \frac{\lambda^2}{4\omega} \xi_T E(R_{T-1})' E(R_{T-1} R_{T-1}')^{-1} E(R_{T-1}).
\end{aligned}$$

The expectation and variance of the uncertain terminal wealth under portfolio policy π_t^* can be expressed as follows,

$$E(V_{T\wedge\tau}) = V_0 M + \frac{\gamma}{2} N, \tag{2.41}$$

$$Var(V_{T\wedge\tau}(\gamma)) = V_0^2 I + V_0 \gamma J + \frac{\gamma^2}{4} L, \tag{2.42}$$

where

$$
\begin{aligned}
M_t &= \prod_{i=0}^{t-1}(r_i^0 - K_i' R_i), \\
N_t &= \sum_{i=0}^{t-1}[\prod_{j=i+1}^{t-1}(r_j^0 - K_j' R_j)]u_i' R_i, \\
I_t &= \prod_{i=0}^{t-1}(r_i^0 - K_i' R_i)^2, \\
J_t &= \sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0 - K_j' R_j)^2 u_i' R_i \prod_{s=1}^{i}(r_s^0 - K_s' R_s), \\
L_t &= \sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0 - K_j' R_j)^2 (u_i' R_i)^2 \\
&\quad + 2\sum_{i=1}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0 - K_j' R_j)^2 u_i' R_i [\sum_{s=0}^{i-1}\prod_{l=s+1}^{i}(r_l^0 - K_l' R_l)u_s' R_s],
\end{aligned}
$$

and

$$
\begin{aligned}
M &= E(\sum_{t=1}^{T} M_t \xi_t) = \sum_{t=1}^{T}\xi_t \prod_{i=0}^{t-1}(r_i^0)(1 - Y_i), \\
N &= E(\sum_{t=1}^{T} N_t \xi_t) = \sum_{t=1}^{T}\xi_t \sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0)(1 - Y_j)Y_i \frac{\xi_{i+1} + A_{i+1}}{\xi_{i+1} + B_{i+1}}, \\
I &= [E(\sum_{t=1}^{T} I_t \xi_t) - (E(\sum_{t=1}^{T} M_t \xi_t))^2] \\
&= \sum_{t=1}^{T}\xi_t \prod_{i=0}^{t-1}(r_i^0)^2(1 - Y_i)^2 - [\sum_{t=1}^{T}\xi_t \prod_{i=0}^{t-1}(r_i^0)(1 - Y_i)]^2, \\
J &= [E(\sum_{t=1}^{T} J_t \xi_t) - E(\sum_{t=1}^{T} M_t \xi_t)E(\sum_{t=1}^{T} N_t \xi_t)] \\
&= -\sum_{t=1}^{T}\xi_t \prod_{i=0}^{t-1}(r_i^0)(1 - Y_i) \times \sum_{t=1}^{T}\xi_t \sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0)(1 - Y_j)Y_i \frac{\xi_{i+1} + A_{i+1}}{\xi_{i+1} + B_{i+1}},
\end{aligned}
$$

$$
\begin{aligned}
L =&[E(\sum_{t=1}^{T} L_t\xi_t) - E^2(\sum_{t=1}^{T} N_t\xi_t)] \\
=&\sum_{t=1}^{T}\xi_t\sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0)^2(1-Y_j)Y_i(\frac{\xi_{i+1}+A_{i+1}}{\xi_{i+1}+B_{i+1}})^2 \\
&-[\sum_{t=1}^{T}\xi_t\sum_{i=0}^{t-1}\prod_{j=i+1}^{t-1}(r_j^0)(1-Y_j)Y_i\frac{\xi_{i+1}+A_{i+1}}{\xi_{i+1}+B_{i+1}}]^2,
\end{aligned}
$$

with

$$
Y_t = E(R_t)'E(R_tR_t')^{-1}E(R_t).
$$

The optimal γ in such a case becomes

$$
\gamma^* = -\frac{2J}{L}V_0 + \frac{N}{L\omega}. \tag{2.43}
$$

Substituting the optimal γ^* in (2.43) into equation (2.40) yields the optimal multi-period portfolio policy,

$$
\begin{aligned}
\pi_t^* =& -r_t^0E(R_tR_t')^{-1}E(R_t)V_t \\
&+(-\frac{J}{L}V_0 + \frac{N}{2L\omega})\frac{\xi_{t+1}+A_{t+1}}{\xi_{t+1}+B_{t+1}}E(R_tR_t')^{-1}E(R_t).
\end{aligned}
$$

This result is consistent with the result of Guo and Hu (2005).

2.5.2. Deterministic exit time

If we define the stochastic process ξ_t as

$$
\xi_t := \begin{cases} 0 & 0 \le t \le T-1, \\ 1 & t = T, \end{cases}
$$

then the M-P portfolio selection model with stochastic investment horizon reduces to the case with a deterministic investment horizon. The result we derived in the last section can be the simplified with this specific ξ_t.

When the exit time is certain, the optimal portfolio policy is

$$\pi_t^* = \frac{\gamma}{2}u_t - K_t V_t, \tag{2.44}$$

where

$$\begin{aligned}
\gamma &= \frac{\lambda}{\omega}, \\
u_{t-1} &= \frac{A_t}{B_t}E(R_{t-1}R'_{t-1})^{-1}E(R_{t-1}) = \frac{1}{\prod_{i=t}^{T-1} r_i^0}E(R_{t-1}R'_{t-1})^{-1}E(R_{t-1}), \\
K_{t-1} &= r_{t-1}^0 E(R_{t-1}R'_{t-1})^{-1}E(R_{t-1}), \\
A_{t-1} &= r_{t-1}^0 A_t[1 - E(R_{t-1})'E(R_{t-1}R'_{t-1})^{-1}E(R_{t-1})], \\
B_{t-1} &= (r_{t-1}^0)^2 B_t[1 - E(R_{t-1})'E(R_{t-1}R'_{t-1})^{-1}E(R_{t-1})],
\end{aligned}$$

with the following boundary condition

$$\begin{aligned}
&u_{T-1} = E(R_{T-1}R'_{T-1})^{-1}E(R_{T-1}), \\
&A_{T-1} = r_{T-1}^0[1 - E(R_{T-1})'E(R_{T-1}R'_{T-1})^{-1}E(R_{T-1})], \\
&B_{T-1} = (r_{T-1}^0)^2[1 - E(R_{T-1})'E(R_T R'_{T-1})^{-1}E(R_{T-1})].
\end{aligned}$$

The expectation and variance of the uncertain terminal wealth under portfolio policy π_t^* can be expressed as follows,

$$E(V_T) = V_0 M + \frac{\gamma}{2}N, \tag{2.45}$$

$$Var(V_T) = V_0^2 I + V_0\gamma J + \frac{\gamma^2}{4}L, \tag{2.46}$$

where

$$\begin{aligned}
M =& E(\prod_{i=0}^{T-1}(r_i^0 - K_i'R_i)) = \prod_{i=0}^{T-1} r_i^0(1 - Y_t), \\
N =& E(\sum_{i=0}^{T-1}[\prod_{j=i+1}^{T-1}(r_j^0 - K_j'R_j)]u_i'R_i) = \sum_{i=0}^{T-1}[\prod_{j=i+1}^{T-1} r_j^0(1 - Y_j)]Y_i\frac{1}{\prod_{l=i+1}^{T} r_l^0} \\
=& \sum_{i=0}^{T-1}[\prod_{j=i+1}^{T-1}(1 - Y_j)]Y_i = 1 - \prod_{i=0}^{T-1}(1 - Y_i),
\end{aligned}$$

$$
\begin{aligned}
I =& E(\prod_{i=0}^{T-1}(r_i^0 - K_i'R_i)^2) - [E(\prod_{i=0}^{T-1}(r_i^0 - K_i'R_i))]^2 \\
=& \prod_{i=0}^{T-1}(r_i^0)^2(1-Y_i)(1-\prod_{i=0}^{T-1}(1-Y_i)), \\
J =& E[(\prod_{i=0}^{T-1}(r_i^0 - K_i'R_i))(\sum_{i=0}^{T-1}[\prod_{j=i+1}^{T-1}(r_j^0 - K_j'R_j)]u_i'R_i)] \\
& - E(\prod_{i=0}^{T-1}(r_i^0 - K_i'R_i))E(\sum_{i=0}^{T-1}[\prod_{j=i+1}^{T-1}(r_j^0 - K_j'R_j)]u_i'R_i) \\
=& E[(\prod_{i=0}^{T-1}(r_i^0 - K_i'R_i))(1-\prod_{i=0}^{T-1}(1-K_i/r_i^0R_i))] \\
& - \prod_{i=0}^{T-1} r_i^0(1-Y_t)(1-\prod_{i=0}^{T-1}(1-Y_i)) \\
=& -\prod_{i=0}^{T-1} r_i^0(1-Y_t)(1-\prod_{i=0}^{T-1}(1-Y_i)), \\
L =& E[(\sum_{i=0}^{T-1}[\prod_{j=i+1}^{T-1}(r_j^0 - K_j'R_j)]u_i'R_i)^2] - [E(\sum_{i=0}^{T-1}[\prod_{j=i+1}^{T-1}(r_j^0 - K_j'R_j)]u_i'R_i)]^2 \\
=& \prod_{i=0}^{T-1}(1-Y_i) - \prod_{i=0}^{T-1}(1-Y_i)^2,
\end{aligned}
$$

with $Y_t = E(R_t)'E(R_tR_t')^{-1}E(R_t)$.

The optimal γ is

$$
\gamma^* = 2\prod_{i=0}^{T-1} r_i^0 V_0 + \frac{1}{\prod_{i=0}^{T-1}(1-Y_i)\omega}. \tag{2.47}
$$

Substituting the optimal γ^* in (2.47) into equation (2.44) yields the optimal multi-period portfolio policy,

$$
\begin{aligned}
\pi_t^* =& -r_t^0 E(R_tR_t')^{-1}E(R_t)V_t \\
& +(\prod_{i=0}^{T-1} r_i^0 V_0 + \frac{1}{2\prod_{i=0}^{T-1}(1-Y_i)\omega})\frac{1}{\prod_{i=t+1}^{T-1} r_i^0}E(R_tR_t')^{-1}E(R_t).
\end{aligned}
$$

This result is just consistent with the result of Li and Ng (2000).

2.5.3. Illustrative examples

The following examples illustrate the effect of time risk on the M-V efficient frontier.

Example 2.1 Consider a financial market with investment horizon $T = 3$. There are one risk free asset and two risky assets in the market. Suppose that the economy has a discrete sample space Ω (totally 54 samples). The riskless return is 1.08 and risk returns are listed in Table 2.1. An investor enters the financial market and invests her money among these three assets. At the same time, she is waiting for a gold mining opportunity. If this opportunity is available and is more profitable than market portfolio (a portfolio with half on each asset), she will exit the market. The failure rate of gold mining opportunity $\widehat{\lambda}_t$ is also listed in Table 2.1.

The failure rate λ_t is pre-determined for given $\widehat{\lambda}_t$ (the failure rate increases as the market portfolio return decreases). The corresponding failure rate λ_t are showed in Table 2.2. So the cumulative probability P_t can be calculated by using formulation (2.2). Therefore, the probability of stochastic investment horizon can be determined by using (2.3), and they are also listed in Table 2.2.

Example 2.2 Consider the same economy as in Example 2.1 except that $\widehat{\lambda}_1 \equiv 0.18, \widehat{\lambda}_2 \equiv 0.24$. So that the sample space becomes $\Omega = \{\omega_1, \cdots, \omega_{27}\}$. An investor enters the financial market with one unit of wealth. She is trying to find the best allocation of her wealth among these three assets. At the same time, she is waiting for a gold mining opportunity. If this opportunity is available and is more profitable than market portfolio (a portfolio with half on each asset), she will exit the market. Hence the probability of exit time are listed in Table 2.2. The investor would like to maximize $E(x_{3\wedge\tau}) - 2Var(x_{3\wedge\tau})$, where $x_{3\wedge\tau}$ is the wealth at the exit time.

Using the result derived in Section 2.4.2, we can get the efficient frontiers of wealth at $t = 1, 2, 3$, which are showed in Figure 2.1. It is obvious that the longer the investment horizon is, the higher the efficient frontier is. We also compare the efficient frontier of terminal wealth in the certain exit time case to that of the uncertain investment horizon case in Figure 2.2. We can see that the efficient frontier under the certain-exit-time case is above that of the uncertain case. Uncertain investment horizon actually adds more risk in the investment.

Example 2.3 Consider an investor enters the financial market with one unit of wealth at the very beginning. She plan to stay in the financial market at most $T = 4$ period. However, she will be forced to exit the market for some market-independent exogenous reason. Suppose the uncertain exit time τ has pre-given exit probability $P(\tau = i) = 0.1i$ for $i = 1, 2, 3, 4$. We use the same market data as Example 2 in Li and Ng (2000). The investor is trying to find the best allocation of her wealth among three risky securities, A, B, C and a risk free security D. The expected returns for risky securities, A, B and C are $E(r_t^A) = 1,162$, $E(r_t^B) = 1.246$, and $E(r_t^C) = 1.228$, $t = 1, 2, 3, 4$, and the return for risk free asset D is 1.04. The covariance of $r = [r_t^A, r_t^B, r_t^C]'$ is

$$Cov(r) = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix}, \quad t = 1, 2, 3, 4.$$

The investor would like to maximize$E(x_{4\wedge\tau}) - 2Var(x_{4\wedge\tau})$. We can calculate

$$M = 0, 1088, \quad N = 0, 8917,$$

$$I = 0.1229, \quad J = -0.097, \quad L = 0.0879.$$

The mean-variance efficient frontier is this case is given as the follows:

$$Var(x_{4\wedge\tau}) = 0.1105E^2(x_{4\wedge\tau}) - 0.239E(x_{4\wedge\tau}) + 0.1479.$$

The associated optimal portfolio policy is given as follows:

$$\pi_t^* = x_t - K_t V_t,$$

where

$$K_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}, \quad t = 1, 2, 3, 4.$$

$$x_1 = \begin{bmatrix} 1.3410 \\ 2.1755 \\ 7.7477 \end{bmatrix}, x_2 = \begin{bmatrix} 1.3588 \\ 2.2044 \\ 7.8505 \end{bmatrix}, x_3 = \begin{bmatrix} 1.3713 \\ 2.2247 \\ 7.9227 \end{bmatrix}, x_4 = \begin{bmatrix} 1.4013 \\ 2.2733 \\ 8.0960 \end{bmatrix}.$$

We compare the result with that of the certain exit time case in Example 2 of Li and Ng (2000),

$$\pi_t^* = x_t - K_t V_t,$$

where

$$K_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}, \quad t = 1, 2, 3, 4.$$

$$x_1 = \begin{bmatrix} 3.5440 \\ 5.7494 \\ 20.4751 \end{bmatrix}, x_2 = \begin{bmatrix} 3.6858 \\ 5.9794 \\ 21.2941 \end{bmatrix}, x_3 = \begin{bmatrix} 3.8332 \\ 6.2185 \\ 22.1459 \end{bmatrix}, x_4 = \begin{bmatrix} 3.9865 \\ 6.4673 \\ 23.0317 \end{bmatrix}.$$

We can see that the second part K_t of the optimal policy are the same for the two different cases. When the exit time is uncertain, the investor invests few wealth on risky assets than that of the certain exit time case. The mean-variance efficient frontiers of the two different cases are showed in Figure 2.3.

Table 2.1: Risky returns r_1, r_2, r_3 and failure rate $\widehat{\lambda}_1, \widehat{\lambda}_2$ in Example 2.1

ω	$r_1(1)$	$r_2(1)$	$r_3(1)$	$r_1(2)$	$r_2(2)$	$r_3(2)$	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$
ω_1	1.1697	1.2258	0.8997	1.2776	0.9934	1.0292	0.18	0.24
ω_2	1.1697	1.2258	1.1548	1.2776	0.9934	1.5623	0.18	0.24
ω_3	1.1697	1.2258	1.0399	1.2776	0.9934	1.2076	0.18	0.24
ω_4	1.1697	0.8409	1.2362	1.2776	1.6678	1.3599	0.18	0.24
ω_5	1.1697	0.8409	1.2234	1.2776	1.6678	1.2717	0.18	0.24
ω_6	1.1697	0.8409	1.3665	1.2776	1.6678	1.0603	0.18	0.24
ω_7	1.1697	1.1455	1.2334	1.2776	1.4281	1.0825	0.18	0.24
ω_8	1.1697	1.1455	1.0842	1.2776	1.4281	1.3757	0.18	0.24
ω_9	1.1697	1.1455	1.2080	1.2776	1.4281	0.9684	0.18	0.24
ω_{10}	1.2771	1.3005	1.0401	1.2406	1.4795	1.4743	0.18	0.24
ω_{11}	1.2771	1.3005	1.1596	1.2406	1.4795	1.4123	0.18	0.24
ω_{12}	1.2771	1.3005	1.1562	1.2406	1.4795	1.0059	0.18	0.24
ω_{13}	1.2771	1.1058	1.1620	1.2406	1.4110	1.1684	0.18	0.24
ω_{14}	1.2771	1.1058	1.1236	1.2406	1.4110	0.8989	0.18	0.24
ω_{15}	1.2771	1.1058	1.2943	1.2406	1.4110	0.6024	0.18	0.24
ω_{16}	1.2771	1.3692	0.9356	1.2406	1.1561	1.5342	0.18	0.24
ω_{17}	1.2771	1.3692	1.2137	1.2406	1.1561	1.0944	0.18	0.24
ω_{18}	1.2771	1.3692	1.2702	1.2406	1.1561	1.3417	0.18	0.24
ω_{19}	0.9607	0.8504	1.2503	1.3380	1.6880	1.3144	0.18	0.24
ω_{20}	0.9607	0.8504	1.2318	1.3380	1.6880	1.2523	0.18	0.24
ω_{21}	0.9607	0.8504	1.1669	1.3380	1.6880	0.9526	0.18	0.24
ω_{22}	0.9607	1.2514	1.2438	1.3380	1.3195	0.9692	0.18	0.24
ω_{23}	0.9607	1.2514	1.2307	1.3380	1.3195	1.1366	0.18	0.24
ω_{24}	0.9607	1.2514	1.1311	1.3380	1.3195	0.8994	0.18	0.24
ω_{25}	0.9607	1.2176	1.1164	1.3380	1.2615	0.9374	0.18	0.24
ω_{26}	0.9607	1.2176	1.1262	1.3380	1.2615	1.6763	0.18	0.24
ω_{27}	0.9607	1.2176	0.9838	1.3380	1.2615	1.2623	0.18	0.24

ω	$r_1(1)$	$r_2(1)$	$r_3(1)$	$r_1(2)$	$r_2(2)$	$r_3(2)$	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$
ω_{28}	1.1697	1.2258	0.8997	1.2776	0.9934	1.0292	0.21	0.26
ω_{29}	1.1697	1.2258	1.1548	1.2776	0.9934	1.5623	0.21	0.26
ω_{30}	1.1697	1.2258	1.0399	1.2776	0.9934	1.2076	0.21	0.26
ω_{31}	1.1697	0.8409	1.2362	1.2776	1.6678	1.3599	0.21	0.26
ω_{32}	1.1697	0.8409	1.2234	1.2776	1.6678	1.2717	0.21	0.26
ω_{33}	1.1697	0.8409	1.3665	1.2776	1.6678	1.0603	0.21	0.26
ω_{34}	1.1697	1.1455	1.2334	1.2776	1.4281	1.0825	0.21	0.26
ω_{35}	1.1697	1.1455	1.0842	1.2776	1.4281	1.3757	0.21	0.26
ω_{36}	1.1697	1.1455	1.2080	1.2776	1.4281	0.9684	0.21	0.26
ω_{37}	1.2771	1.3005	1.0401	1.2406	1.4795	1.4743	0.21	0.26
ω_{38}	1.2771	1.3005	1.1596	1.2406	1.4795	1.4123	0.21	0.26
ω_{39}	1.2771	1.3005	1.1562	1.2406	1.4795	1.0059	0.21	0.26
ω_{40}	1.2771	1.1058	1.1620	1.2406	1.4110	1.1684	0.21	0.26
ω_{41}	1.2771	1.1058	1.1236	1.2406	1.4110	0.8989	0.21	0.26
ω_{42}	1.2771	1.1058	1.2943	1.2406	1.4110	0.6024	0.21	0.26
ω_{43}	1.2771	1.3692	0.9356	1.2406	1.1561	1.5342	0.21	0.26
ω_{44}	1.2771	1.3692	1.2137	1.2406	1.1561	1.0944	0.21	0.26
ω_{45}	1.2771	1.3692	1.2702	1.2406	1.1561	1.3417	0.21	0.26
ω_{46}	0.9607	0.8504	1.2503	1.3380	1.6880	1.3144	0.21	0.26
ω_{47}	0.9607	0.8504	1.2318	1.3380	1.6880	1.2523	0.21	0.26
ω_{48}	0.9607	0.8504	1.1669	1.3380	1.6880	0.9526	0.21	0.26
ω_{49}	0.9607	1.2514	1.2438	1.3380	1.3195	0.9692	0.21	0.26
ω_{50}	0.9607	1.2514	1.2307	1.3380	1.3195	1.1366	0.21	0.26
ω_{51}	0.9607	1.2514	1.1311	1.3380	1.3195	0.8994	0.21	0.26
ω_{52}	0.9607	1.2176	1.1164	1.3380	1.2615	0.9374	0.21	0.26
ω_{53}	0.9607	1.2176	1.1262	1.3380	1.2615	1.6763	0.21	0.26
ω_{54}	0.9607	1.2176	0.9838	1.3380	1.2615	1.2623	0.21	0.26

Table 2.2: Failure rate λ, cumulative probability P and exit probability ξ in Example 2.2

ω	λ_1	λ_2	P_1	P_2	ξ_1	ξ_2	ξ_3
ω_1	0.20	0.20	0.1813	0.3297	0.1813	0.1484	0.6703
ω_2	0.20	0.20	0.1813	0.3297	0.1813	0.1484	0.6703
ω_3	0.20	0.20	0.1813	0.3297	0.1813	0.1484	0.6703
ω_4	0.20	0.15	0.1813	0.2953	0.1813	0.1140	0.7047
ω_5	0.20	0.15	0.1813	0.2953	0.1813	0.1140	0.7047
ω_6	0.20	0.15	0.1813	0.2953	0.1813	0.1140	0.7047
ω_7	0.20	0.10	0.1813	0.2592	0.1813	0.0779	0.7408
ω_8	0.20	0.10	0.1813	0.2592	0.1813	0.0779	0.7408
ω_9	0.20	0.10	0.1813	0.2592	0.1813	0.0779	0.7408
ω_{10}	0.15	0.05	0.1393	0.1813	0.1393	0.0420	0.8187
ω_{11}	0.15	0.05	0.1393	0.1813	0.1393	0.0420	0.8187
ω_{12}	0.15	0.05	0.1393	0.1813	0.1393	0.0420	0.8187
ω_{13}	0.15	0.15	0.1393	0.2592	0.1393	0.1199	0.7408
ω_{14}	0.15	0.15	0.1393	0.2592	0.1393	0.1199	0.7408
ω_{15}	0.15	0.15	0.1393	0.2592	0.1393	0.1199	0.7408
ω_{16}	0.15	0.12	0.1393	0.2366	0.1393	0.0973	0.7634
ω_{17}	0.15	0.12	0.1393	0.2366	0.1393	0.0973	0.7634
ω_{18}	0.15	0.12	0.1393	0.2366	0.1393	0.0973	0.7634
ω_{19}	0.30	0.10	0.2592	0.3297	0.2592	0.0705	0.6703
ω_{20}	0.30	0.10	0.2592	0.3297	0.2592	0.0705	0.6703
ω_{21}	0.30	0.10	0.2592	0.3297	0.2592	0.0705	0.6703
ω_{22}	0.30	0.20	0.2592	0.3935	0.2592	0.1343	0.6065
ω_{23}	0.30	0.20	0.2592	0.3935	0.2592	0.1343	0.6065
ω_{24}	0.30	0.20	0.2592	0.3935	0.2592	0.1343	0.6065
ω_{25}	0.30	0.16	0.2592	0.3687	0.2592	0.1095	0.6313
ω_{26}	0.30	0.16	0.2592	0.3687	0.2592	0.1095	0.6313
ω_{27}	0.30	0.16	0.2592	0.3687	0.2592	0.1095	0.6313

ω	λ_1	λ_2	P_1	P_2	ξ_1	ξ_2	ξ_3
ω_{28}	0.25	0.25	0.2212	0.3935	0.2212	0.1723	0.6065
ω_{29}	0.25	0.25	0.2212	0.3935	0.2212	0.1723	0.6065
ω_{30}	0.25	0.25	0.2212	0.3935	0.2212	0.1723	0.6065
ω_{31}	0.25	0.18	0.2212	0.3495	0.2212	0.1283	0.6505
ω_{32}	0.25	0.18	0.2212	0.3495	0.2212	0.1283	0.6505
ω_{33}	0.25	0.18	0.2212	0.3495	0.2212	0.1283	0.6505
ω_{34}	0.25	0.14	0.2212	0.3229	0.2212	0.1017	0.6771
ω_{35}	0.25	0.14	0.2212	0.3229	0.2212	0.1017	0.6771
ω_{36}	0.25	0.14	0.2212	0.3229	0.2212	0.1017	0.6771
ω_{37}	0.18	0.1	0.1647	0.2442	0.1647	0.0795	0.7558
ω_{38}	0.18	0.1	0.1647	0.2442	0.1647	0.0795	0.7558
ω_{39}	0.18	0.1	0.1647	0.2442	0.1647	0.0795	0.7558
ω_{40}	0.18	0.18	0.1647	0.3023	0.1647	0.1376	0.6977
ω_{41}	0.18	0.18	0.1647	0.3023	0.1647	0.1376	0.6977
ω_{42}	0.18	0.18	0.1647	0.3023	0.1647	0.1376	0.6977
ω_{43}	0.18	0.16	0.1647	0.2882	0.1647	0.1235	0.7118
ω_{44}	0.18	0.16	0.1647	0.2882	0.1647	0.1235	0.7118
ω_{45}	0.18	0.16	0.1647	0.2882	0.1647	0.1235	0.7118
ω_{46}	0.32	0.14	0.2739	0.3687	0.2739	0.0948	0.6313
ω_{47}	0.32	0.14	0.2739	0.3687	0.2739	0.0948	0.6313
ω_{48}	0.32	0.14	0.2739	0.3687	0.2739	0.0948	0.6313
ω_{49}	0.32	0.22	0.2739	0.4123	0.2739	0.1384	0.5877
ω_{50}	0.32	0.22	0.2739	0.4123	0.2739	0.1384	0.5877
ω_{51}	0.32	0.22	0.2739	0.4123	0.2739	0.1384	0.5877
ω_{52}	0.32	0.2	0.2739	0.4055	0.2739	0.1316	0.5945
ω_{53}	0.32	0.2	0.2739	0.4055	0.2739	0.1316	0.5945
ω_{54}	0.32	0.2	0.2739	0.4055	0.2739	0.1316	0.5945

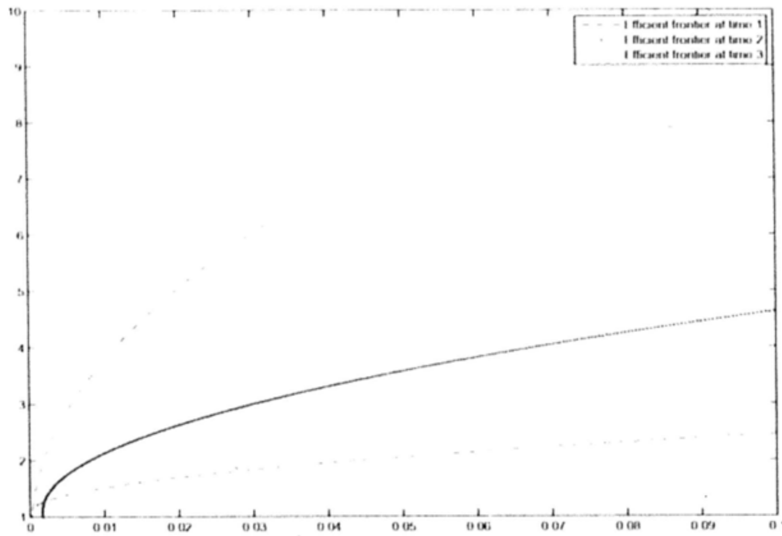

Figure 2.1: Efficient frontiers when exit time is uncertain in Example 2.2

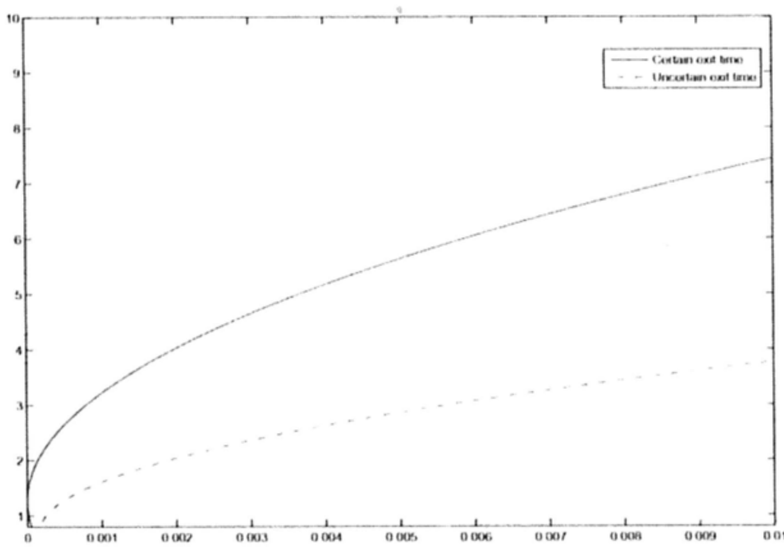

Figure 2.2: Efficient frontiers with and without uncertain investment horizon in Example 2.2

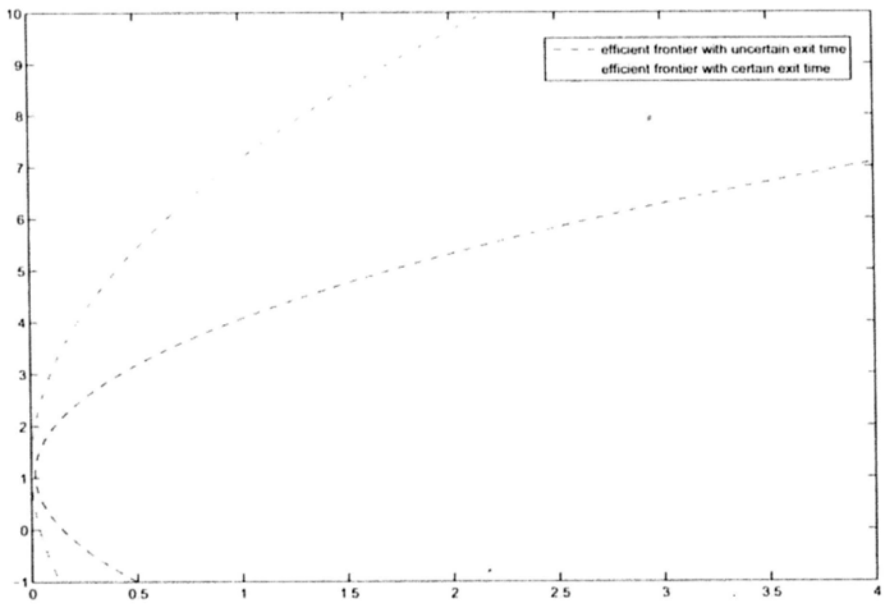

Figure 2.3: Efficient frontiers with and without uncertain exit time in Example 2.3

2.6. Summary

In this chapter, we firstly introduced multi-period mean-variance portfolio selection problem with a state-dependent uncertain exit time. This new formulation is practically meaningful since most investors do not know exactly when they will exit the financial market at the beginning of their investment. Introducing state-dependent uncertain investment horizon is actually presenting a market-dependent exit strategy. However, this market-dependent exit strategy does not relate to investment policy.

In order to deal with such a problem, we used the embedding technique of Li and Ng (2000). The optimal policy of the original inseparable problem has been derived by solving a separable auxiliary problem. Different from Blanchet-Scalliet et al. [39], who used martingale approach to analyze the continuous-time utility maximization problem with time uncertainty, we investigated the mean-variance portfolio selection problem by using dynamic programming. The mean-variance efficient frontier under the optimal policy has also been derived. We have also analyzed the special case where the exit time is state-independent, and have compared the result to the certain-exit-time case. We found that introducing uncertain exit time adds extra risk to the investment.

Chapter 3

Multi-Period Portfolio Selection for Asset-Liability Management with Uncertain Investment Horizon

3.1. Introduction

Liability management is essential for the success of financial institutions, such as pension funds and banks. The past decade has witnessed increasing attention which many investment institutions have paid to take liabilities into account when building up their portfolios.

Asset-liability (AL) management under the mean-variance criteria was first investigated by Sharpe and Tint [49] in a single-period setting. Multi-period asset-liability management under the mean-variance framework was studied in Leippold et al. [31], in which an analytical optimal policy and efficient frontier for the multi-period AL management problem were derived by using the embedding technique of Li and Ng [33]. Chiu and Li [11] further generalized the problem to a continuous-time setting, derived the analytical optimal trading strategy and obtained the optimal initial funding ratio.

In this chapter, we introduce an state-independent uncertain investment horizon which was discussed in Chapter 2 into the multi-period AL mean-variance portfolio selection problem, while the distribution of the investment horizon is assumed to be known. We adopt the same assumptions as in Leippold et al. [31]. With the given distribution of the exit time, the problem under investigation can be translated into a problem with a deterministic investment horizon which can be also solved analytically by the embedding technique of Li and Ng [33]. Different from Guo and Hu [22] where a market is consisted of one risk free asset and n risky assets, we consider i) a more general market model with $(n+1)$ assets which can be all risky, and ii) a liability which significantly affects the optimal trading strategy.

The remaining of the chapter is organized as follows. In Section 3.2, we use the embedding technique of Li and Ng [33] to formulate a tractable problem of the multi-period asset-liability portfolio optimization problem with uncertain investment horizon. We derive in Section 3.3 an analytic solution of the optimal policy for the case with two assets and an exogenous liability. In Section 3.4, we obtain the mean-variance efficient frontier analytically. In Section 3.5, we extend the results to general cases with arbitrary number of multiple assets. Finally, we summarize the chapter in Section 3.6.

3.2. Problem formulation

We assume that an investor, with an initial wealth x_0 and initial liability l_0 at time $t = 0$, plans to invest her wealth in the financial market within a time horizon of T periods. At the beginning of each period, $t = 0, 1, \cdots, T-1$, she can rebalance her portfolio. However, she may be forced to exit the market at period τ (before the final stage T) by some uncontrollable exogenous reasons. Suppose that τ is a discrete random variable with the following probability distribution,

$$P_t = \begin{cases} \widetilde{P}_t, & t = 1, 2, \cdots, T-1, \\ 1 - \sum_{s=0}^{T-1} \widetilde{P}_s, & t = T. \end{cases}$$

To start with, we confine our discussion in this section to situations with only two assets and one liability. Extension to situations with an arbitrary number of assets will be studied in Section 3.5.

Denote the returns of the two assets between time t and $t+1$ by r_t^0 and r_t, respectively, and the return of the liability at time t by q_t. We assume that the returns of financial instruments are statistically independent among different time periods.

Let $\gamma_t = (r_t^0, r_t, q_t)'$. We have

$$\begin{aligned} E(\gamma_t) &= (Er_t^0, Er_t, Eq_t)', \\ Cov(\gamma_t) &= E(\gamma_t\gamma_t') - E(\gamma_t)E(\gamma_t'). \end{aligned}$$

It is reasonable to assume that all the matrices

$$E(\gamma_t\gamma_t') = Cov(\gamma_t) + E(\gamma_t)E(\gamma_t'), \quad t = 0, 1, \cdots, T-1,$$

are positive definite.

Let the aggregated value of assets be x_t at time t. The random aggregated value of assets at time $t+1$ is given as

$$x_{t+1} = r_t^0(x_t - u_t) + r_t u_t = r_t^0 x_t + R_t u_t, \tag{3.1}$$

where $R_t = r_t - r_t^0$, u_t is the amount invested in the asset with return r_t, and $x_t - u_t$ is the amount invested in the asset with return r_t^0.

As the same as in Leippold et al. [31], we assume in this chapter that the liability is exogenous, thus uncontrollable. More specifically, the dynamics of the value of the liability is not affected by the trading strategy of the investor. Let the value of the liability at time t be l_t. Then the random value of the liability at time $t+1$ is given as,

$$l_{t+1} = q_t l_t. \tag{3.2}$$

We denote the exit time by $T \wedge \tau = \min\{T, \tau\}$ and define the final surplus, i.e., the difference between assets and liabilities at the exit time, as $S_{T\wedge\tau} := x_{T\wedge\tau} - l_{T\wedge\tau}$.

The optimization problem of multi-period AL management under a mean-variance framework in this chapter can be now formulated as follows,

$$(P1) \quad \begin{cases} \max\limits_{u} & E(S_{T\wedge\tau}) \\ \text{s.t.} & Var(S_{T\wedge\tau}) \leq \sigma \; and \; (3.1) - (3.2), \end{cases} \tag{3.3}$$

for a given $\sigma \geq 0$. Varying the value of σ from 0 to ∞ yields the set of efficient solutions.

The mean-variance portfolio selection problem in (P1) can be also posed in an alternative form as follows,

$$(P2) \quad \begin{cases} \min\limits_{u} & Var(S_{T\wedge\tau}) \\ \text{s.t.} & E(S_{T\wedge\tau}) \geq \epsilon \; and \; (3.1) - (3.2), \end{cases} \tag{3.4}$$

for a give $\epsilon \geq 0$. Varying the value of ϵ also yields the set of efficient solutions.

Another formulation to generate the set of efficient solutions is given as follows,

$$(P3(\omega)) \quad \begin{cases} \max_{u} & E(S_{T\wedge\tau}) - \omega Var(S_{T\wedge\tau}) \\ \text{s.t.} & (3.1)-(3.2), \end{cases} \tag{3.5}$$

for a given $\omega \geq 0$. Varying the value of ω from 0 to ∞ yields the set of efficient solutions.

As indicated in Li and Ng [33], all formulations (P1)-(P3) are difficult to be solved directly because of their non-separability in the sense of dynamic programming. In Li and Ng [33], the relationship between the multi-period mean-variance portfolio selection problem with a fixed investment horizon and a separable portfolio selection problem with a quadratic utility function is investigated and the analytical solution is derived by using an embedding scheme. Fortunately, Theorems 1 and 2 in Li and Ng [33] can be also applied to the current formulation with an uncertain investment horizon. We now consider the following auxiliary problem:

$$(P4(\lambda,\omega)) \quad \begin{cases} \max_{u} & E(\lambda S_{T\wedge\tau} - \omega S_{T\wedge\tau}^2) \\ \text{s.t.} & (3.1)-(3.2). \end{cases} \tag{3.6}$$

Let $\Phi_A(\lambda,\omega)$ be the set of optimal solutions of problem $(P4(\lambda,\omega))$ and $\Phi_P(\omega)$ be the set of optimal solutions of problem $(P3(\omega))$, respectively,

$$\Phi_A(\lambda,\omega) = \{\pi \mid \pi \text{ is an optimal solution of } (\text{P4}(\lambda,\omega))\},$$

$$\Phi_P(\omega) = \{\pi \mid \pi \text{ is an optimal solution of } (\text{P3}(\omega))\}.$$

Define

$$d(\pi,\omega) = 1 + 2\omega E(S_{T\wedge\tau}) \mid_{\pi}. \tag{3.7}$$

Since the uncertain investment horizon described in this chapter is just a special case of that in chapter 2, so the Theorem 2.1 and Theorem 2.2 still hold in this chapter.

3.3. Portfolio optimization with two assets and an exogenous liability

In this section, we first convert problem $(P4(\lambda,\omega))$ to a portfolio selection problem with a certain exit time and then solve the resulting problem by using dynamic programming. For $t = 0, 1, \cdots, T-1$, we adopt the following notations,

$$z_t = \begin{pmatrix} x_t \\ l_t \end{pmatrix}, e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, h_t = \begin{pmatrix} u_t \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$B_t = \begin{pmatrix} r_t^0 & 0 \\ 0 & q_t \end{pmatrix}, A_t = \begin{pmatrix} R_t & 0 \\ 0 & 0 \end{pmatrix},$$

where matrix B_t can be viewed as a benchmark return matrix and A_t can be regarded as the excess return matrix relative to the benchmark for the asset and liability.

Using the notations above to rewrite the problem $(P4(\lambda,\omega))$ into a matrix form yields the following equivalent form of $(P4(\lambda,\omega))$:

$$(P5(\lambda,\omega)) \quad \begin{cases} \max\limits_{h} & E(\lambda e' z_{T\wedge\tau} - \omega z'_{T\wedge\tau} e e' z_{T\wedge\tau}) \\ s.t. & z_{t+1} = B_t z_t + A_t h_t, h_t = u_t e_1, t = 0, 1, \cdots, T-1. \end{cases} \tag{3.8}$$

As the objective function in $(P5(\lambda,\omega))$ can be rewritten as the following summation according to the distribution of the investment horizon,

$$E(\lambda e' z_{T\wedge\tau} - \omega z'_{T\wedge\tau} e e' z_{T\wedge\tau}) \quad = \quad E[\textstyle\sum_{t=0}^{T} (\lambda e' z_t - \omega z'_t e e' z_t) P_t],$$

Problem $(P5(\lambda,\omega))$ can be translated into the following equivalent formulation with a certain exit time,

$$(P5(\lambda,\omega)) \quad \begin{cases} \max\limits_{h} & E[\sum_{t=0}^{T}(\lambda e' z_t - \omega z_t' e e' z_t) P_t] \\ s.t. & z_{t+1} = B_t z_t + A_t h_t, h_t = e_1 u_t, t = 0, 1, \cdots, T-1. \end{cases}$$

The optimal solution of problem $(P5(\lambda,\omega))$ can be derived analytically by using dynamic programming, which is summarized in the following theorem.

Theorem 3.1 *The optimal policy of problem $(P5(\lambda,\omega))$ is given as follows,*

$$u_t^* = \frac{\lambda}{2\omega}\frac{E(e_1' A_t' F_{t+1})}{E(e_1' A_t' \overline{D}_{t+1}\overline{D}_{t+1}' A_t e_1)} - \frac{E(e_1' A_t' \overline{D}_{t+1}\overline{D}_{t+1}' B_t z_t)}{E(e_1' A_t' \overline{D}_{t+1}\overline{D}_{t+1}' A_t e_1)}, \tag{3.9}$$

for $t = 0, 1, \cdots, T-1$, where

$$\overline{D}_t' = \begin{pmatrix} e' B_t^{T-t-1}(P_T)^{\frac{1}{2}} \\ \vdots \\ e' B_t^{s-t-1}(P_s)^{\frac{1}{2}} \\ \vdots \\ e'(P_t)^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \overline{r}_t^{T-t-1}(P_T)^{\frac{1}{2}} & -\overline{q}_t^{T-t-1}(P_T)^{\frac{1}{2}} \\ \vdots & \vdots \\ \overline{r}_t^{s-t-1}(P_s)^{\frac{1}{2}} & -\overline{q}_t^{s-t-1}(P_s)^{\frac{1}{2}} \\ \vdots & \vdots \\ (P_t)^{\frac{1}{2}} & -(P_t)^{\frac{1}{2}} \end{pmatrix}, \tag{3.10}$$

$$F_t = \sum_{s=t+1}^{T} B_t^{s-t-1} e P_s + e P_t = \begin{pmatrix} \sum_{s=t+1}^{T} \overline{r}_t^{s-t-1} P_s + P_t \\ -\sum_{s=t+1}^{T} \overline{q}_t^{s-t-1} P_s - P_t \end{pmatrix}, \tag{3.11}$$

$$B_t^0 = \begin{pmatrix} \overline{r}_t^0 & 0 \\ 0 & \overline{q}_t^0 \end{pmatrix}, \tag{3.12}$$

$$B_t^i = \begin{pmatrix} \overline{r}_t^i & 0 \\ 0 & \overline{q}_t^i \end{pmatrix}, \tag{3.13}$$

with

$$\begin{aligned}
\overline{r}_t^0 &= r_t^0 - R_t\frac{E(R_t r_t^0)}{E[(R_t)^2]},\\
\overline{q}_t^0 &= q_t - R_t\frac{E(R_t q_t)E[\sum_{s=t+2}^{T}\overline{r}_{t+1}^{s-t-2}\overline{q}_{t+1}^{s-t-2}P_s + P_{t+1}]}{E[(R_t)^2]E[\sum_{s=t+2}^{T}(\overline{r}_{t+1}^{s-t-2})^2P_s + P_{t+1}]},\\
\overline{r}_t^i &= r_t^0\overline{r}_{t+1}^{i-1} - R_t\overline{r}_{t+1}^{i-1}\frac{E(R_t r_t^0)}{E[(R_t)^2]},\\
\overline{q}_t^i &= q_t\overline{q}_{t+1}^{i-1} - R_t\overline{q}_{t+1}^{i-1}\frac{E(R_t q_t)E[\sum_{s=t+2}^{T}\overline{r}_{t+1}^{s-t-2}\overline{q}_{t+1}^{s-t-2}P_s + P_{t+1}]}{E[(R_t)^2]E[\sum_{s=t+2}^{T}(\overline{r}_{t+1}^{s-t-2})^2P_s + P_{t+1}]},
\end{aligned}$$

for $i = 1, \cdots, T-t-1$.

Proof. We use dynamic programming to solve the problem $(P5(\lambda,\omega))$

$$\begin{cases} \max\limits_{h} & E[\sum_{t=0}^{T}(\lambda e'z_t - \omega z_t'ee'z_t)P_t] \\ \text{s.t.} & z_{t+1} = B_t z_t + A_t h_t,\ h_t = e_1 u_t,\ t = 0, 1, \cdots, T-1. \end{cases}$$

Let

$$f_T(z_T) = (\lambda e'z_T - \omega z_T'ee'z_T)P_T,$$

$$f_t(z_t) = \max_h E[\sum_{s=t}^{T}(\lambda e'z_s - \omega z_s'ee'z_s)P_s],$$

for $t = 0, 1, \cdots, T-1$.

The dynamic programming algorithm starts from stage $T-1$. For given z_{T-1}, the optimization problem is given as follows,

$$\begin{aligned}
& f_{T-1}(z_{T-1}) \\
&= \max_h E[(\lambda e'z_{T-1} - \omega z_{T-1}'ee'z_{T-1})P_{T-1} + f_{T,T}(z_T)] \\
&= \max_h E\{(\lambda e'z_{T-1} - \omega z_{T-1}'ee'z_{T-1})P_{T-1} + [\lambda e'(B_{T-1}z_{T-1} + A_{T-1}h_{T-1}) \\
&- \omega(B_{T-1}z_{T-1} + A_{T-1}h_{T-1})'ee'(B_{T-1}z_{T-1} + A_{T-1}h_{T-1})]P_T\} \\
&= \max_u E\{(\lambda e'z_{T-1} - \omega z_{T-1}'ee'z_{T-1})P_{T-1} + [\lambda e'(B_{T-1}z_{T-1} + A_{T-1}e_1u_{T-1}) \\
&- \omega(B_{T-1}z_{T-1} + A_{T-1}e_1u_{T-1})'ee'(B_{T-1}z_{T-1} + A_{T-1}e_1u_{T-1})]P_T\}.
\end{aligned}$$

Maximizing the above expression with respect to u_{T-1} gives:

$$u^*_{T-1} = \frac{\lambda}{2\omega}\frac{E(e'_1 A'_{T-1} F_T)}{E(e'_1 A'_{T-1}\overline{D}_T\overline{D}'_T A_{T-1}e_1)} - \frac{E(e'_1 A'_{T-1}\overline{D}_T\overline{D}'_T B_{T-1}z_{T-1})}{E(e'_1 A'_{T-1}\overline{D}_T\overline{D}'_T A_{T-1}e_1)},$$

where $F_T = eP_T$ and $\overline{D}'_T = e'(P_T)^{\frac{1}{2}}$. Furthermore, we have

$$\begin{aligned}
z_T &= (B_{T-1}z_{T-1} + A_{T-1}h_{T-1}) \\
&= [B_{T-1}z_{T-1} - A_{T-1}e_1\frac{E(e'_1 A'_{T-1}ee'B_{T-1}z_{T-1})}{E(e'_1 A'_{T-1}ee'A_{T-1}e_1)}] \\
&+ \frac{\lambda}{2\omega}A_{T-1}e_1\frac{E(e'_1 A'_{T-1}e)}{E(e'_1 A'_{T-1}ee'A_{T-1}e_1)}, \\
S_T &= e'z_T = e'B^0_{T-1}z_{T-1} + \frac{\lambda}{2\omega}R_{T-1}Y_{T-1},
\end{aligned}$$

where

$$\begin{aligned}
e'B^0_{T-1}z_{T-1} &= e'[B_{T-1}z_{T-1} - A_{T-1}e_1\frac{E(e'_1 A'_{T-1}ee'B_{T-1}z_{T-1})}{E(e'_1 A'_{T-1}ee'A_{T-1}e_1)}], \\
Y_{T-1} &= \frac{E(e'_1 A'_{T-1}e)}{E(e'_1 A'_{T-1}ee'A_{T-1}e_1)} = \frac{E(R_{T-1})}{E[(R_{T-1})^2]},
\end{aligned}$$

with

$$\begin{aligned}
B^0_{T-1} &= \begin{pmatrix} \overline{r}^0_{T-1} & 0 \\ 0 & \overline{q}^0_{T-1} \end{pmatrix}, \\
\overline{r}^0_{T-1} &= r^0_{T-1} - R_{T-1}\frac{E(R_{T-1}r^0_{T-1})}{E[(R_{T-1})^2]}, \\
\overline{q}^0_{T-1} &= q_{T-1} - R_{T-1}\frac{E(R_{T-1}q_{T-1})}{E[(R_{T-1})^2]}.
\end{aligned}$$

Because

$$\begin{aligned}
&E[(e'B^0_{T-1}z_{T-1})(R_{T-1}Y_{T-1})] \\
&= \frac{E(e'_1 A'_{T-1}e)}{E(e'_1 A'_{T-1}ee'A_{T-1}e_1)}\times \\
&E\{(e'A_{T-1}e_1)'e'[B_{T-1}z_{T-1} - A_{T-1}e_1\frac{E(e'_1 A'_{T-1}ee'B_{T-1}z_{T-1})}{E(e'_1 A'_{T-1}ee'A_{T-1}e_1)}]\}
\end{aligned}$$

$$= \frac{E(e_1'A_{T-1}'e)}{E(e_1'A_{T-1}'ee'A_{T-1}e_1)}\times$$

$$\{E(e_1A_{T-1}'ee'B_{T-1}z_{T-1}) - E(e_1A_{T-1}'ee'A_{T-1}e_1)\frac{E(e_1'A_{T-1}'ee'B_{T-1}z_{T-1})}{E(e_1'A_{T-1}'ee'A_{T-1}e_1)}\}$$

$$= 0,$$

$$E[(R_{T-1}Y_{T-1})^2]$$

$$= E[(e'A_{T-1}e_1Y_{T-1})^2] = E[(e'A_{T-1}e_1)^2]\frac{E^2(e_1'A_{T-1}'e)}{E^2(e_1'A_{T-1}'ee'A_{T-1}e_1)}$$

$$= \frac{E^2(e_1'A_{T-1}'e)}{E(e_1'A_{T-1}'ee'A_{T-1}e_1)},$$

$$E(R_{T-1}Y_{T-1})$$

$$= E(e'A_{T-1}e_1Y_{T-1}) = E(e'A_{T-1}e_1)\frac{E(e_1'A_{T-1}'e)}{E(e_1'A_{T-1}'ee'A_{T-1}e_1)}$$

$$= \frac{E^2(e_1'A_{T-1}'e)}{E(e_1'A_{T-1}'ee'A_{T-1}e_1)},$$

we get the following equations,

$$E[(e'B_{T-1}^0z_{T-1})(R_{T-1}Y_{T-1})] = 0,\ \ E[(R_{T-1}Y_{T-1})^2] = E(R_{T-1}Y_{T-1}).$$

Therefore, substituting u_{T-1}^* back to $f_{T-1}(z_{T-1})$ yields the optimal benefit-to-go at given z_{T-1},

$$f_{T-1}(z_{T-1})$$

$$= E\{(\lambda e'z_{T-1} - \omega z_{T-1}'ee'z_{T-1})P_{T-1} + [\lambda(\widetilde{F}_{T-1}'z_{T-1} + \frac{\lambda}{2\omega}\chi_{T-1}^2Y_{T-1})$$

$$- \omega(\widetilde{F}_{T-1}'z_{T-1} + \frac{\lambda}{2\omega}\chi_{T-1}^1Y_{T-1})^2]\}$$

$$= -\omega z_{T-1}'E(D_{T-1})z_{T-1} + \lambda E(F_{T-1})'z_{T-1} + C_{T-1},$$

where

$$\widetilde{F}_{T-1} = B_{T-1}^0{}'eP_T,$$

$$\chi_{T-1}^1 = R_{T-1}(P_T)^{\frac{1}{2}},$$

$$\chi_{T-1}^2 = R_{T-1}P_T,$$

$$F_{T-1} = eP_{T-1} + \widetilde{F}_{T-1} = \begin{pmatrix} P_{T-1} + \overline{r}^0_{T-1}P_T \\ -P_{T-1} - \overline{q}^0_{T-1}P_T \end{pmatrix},$$

$$D_{T-1} = \overline{D}_{T-1}\overline{D}'_{T-1} = ee'P_{T-1} + B^{0\prime}_{T-1}ee'B^0_{T-1}P_T,$$

$$C_{T-1} = \frac{\lambda^2}{4\omega}E(\chi^2_{T-1}Y_{T-1}),$$

with

$$\overline{D}'_{T-1} = \begin{pmatrix} \widetilde{D}'_{T-1} \\ e'(P_{T-1})^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \overline{r}^0_{T-1}(P_T)^{\frac{1}{2}} & -\overline{q}^0_{T-1}(P_T)^{\frac{1}{2}} \\ (P_{T-1})^{\frac{1}{2}} & -(P_{T-1})^{\frac{1}{2}} \end{pmatrix},$$

and $\widetilde{D}'_{T-1} = e'B^0_{T-1}(P_T)^{\frac{1}{2}}$.

Assume that the optimal policy and the value function at time $t < T-1$ have the following forms:

$$u^*_t = \frac{\lambda}{2\omega}\frac{E(e'_1A'_tF_{t+1})}{E(e'_1A'_t\overline{D}_{t+1}\overline{D}'_{t+1}A_te_1)} - \frac{E(e'_1A'_t\overline{D}_{t+1}\overline{D}'_{t+1}B_tz_t)}{E(e'_1A'_t\overline{D}_{t+1}\overline{D}'_{t+1}A_te_1)},$$

$$f_t(z_t) = -\omega z'_tE(\overline{D}_t\overline{D}'_t)z_t + \lambda E(F_t)'z_t + \sum_{i=t}^{T-1}C_t,$$

where

$$\widetilde{D}'_k = \begin{pmatrix} e'B^{T-k-1}_k(P_T)^{\frac{1}{2}} \\ \vdots \\ e'B^{s-k-1}_k(P_s)^{\frac{1}{2}} \\ \vdots \\ e'B^0_k(P_{k+1})^{\frac{1}{2}} \end{pmatrix},$$

$$\widetilde{F}_k = \sum_{s=k+1}^{T} B^{s-k-1}_kP_s,$$

$$F_k = \widetilde{F}_k + eP_k = \begin{pmatrix} \sum_{s=k+1}^{T}\overline{r}^{s-k-1}_kP_s + P_k \\ -\sum_{s=k+1}^{T}\overline{q}^{s-k-1}_tP_s - P_k \end{pmatrix},$$

$$\overline{D}'_k = \begin{pmatrix} \widetilde{D}'_k \\ c'(P_k)^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \overline{r}_k^{T-k-1}(P_T)^{\frac{1}{2}} & -\overline{q}_k^{T-k-1}(P_T)^{\frac{1}{2}} \\ \vdots & \vdots \\ \overline{r}_k^{s-k-1}(P_s)^{\frac{1}{2}} & -\overline{q}_k^{s-k-1}(P_s)^{\frac{1}{2}} \\ \vdots & \vdots \\ (P_k)^{\frac{1}{2}} & -(P_k)^{\frac{1}{2}} \end{pmatrix},$$

$$C_k = \frac{\lambda^2}{4\omega} E(\chi_k^2 Y_k),$$

$$Y_k = \frac{E(R_k)}{E[(R_k)^2]} \frac{E[\sum_{s=k+2}^{T} \overline{r}_{k+1}^{s-k-2} P_s + P_{k+1}]}{E[\sum_{s=k+2}^{T} (\overline{r}_{t+1}^{s-k-2})^2 P_s + P_{k+1}]},$$

$$\chi_k^1 = \begin{pmatrix} R_k \overline{r}_{k+1}^{T-k-2}(P_T)^{\frac{1}{2}} \\ \vdots \\ R_k \overline{r}_{k+1}^{s-k-2}(P_s)^{\frac{1}{2}} \\ \vdots \\ R_k (P_{k+1})^{\frac{1}{2}} \end{pmatrix}_{(T-k)\times 1},$$

$$\chi_k^2 = R_k[\sum_{s=k+2}^{T} \overline{r}_{k+1}^{s-k-2} P_s + P_{k+1}],$$

$$B_k^0 = \begin{pmatrix} \overline{r}_k^0 & 0 \\ 0 & \overline{q}_k^0 \end{pmatrix}, B_k^i = \begin{pmatrix} \overline{r}_k^i & 0 \\ 0 & \overline{q}_k^i \end{pmatrix},$$

with

$$\overline{r}_k^0 = r_k^0 - R_k \frac{E(R_k r_k^0)}{E[(R_k)^2]},$$

$$\overline{r}_k^i = r_k^0 \overline{r}_{k+1}^{i-1} - R_k \overline{r}_{k+1}^{i-1} \frac{E(R_k r_k^0)}{E[(R_k)^2]},$$

$$\overline{q}_k^0 = q_k - R_k \frac{E(R_k q_k) E[\sum_{s=k+2}^{T} \overline{r}_{k+1}^{s-k-2} \overline{q}_{k+1}^{s-k-2} P_s + P_{k+1}]}{E[(R_k)^2] E[\sum_{s=k+2}^{T} (\overline{r}_{k+1}^{s-k-2})^2 P_s + P_{k+1}]},$$

$$\overline{q}_k^i = q_k \overline{q}_{k+1}^{i-1} - R_k \overline{q}_{k+1}^{i-1} \frac{E(R_k q_k) E[\sum_{s=k+2}^{T} \overline{r}_{k+1}^{s-k-2} \overline{q}_{k+1}^{s-k-2} P_s + P_{k+1}]}{E[(R_k)^2] E[\sum_{s=k+2}^{T} (\overline{r}_{k+1}^{s-k-2})^2 P_s + P_{k+1}]},$$

for $k = t, t+1, \cdots, T-1$ and $i = 1, \cdots, T-k-1$. Then, the optimization problem at time $t-1$ for a given z_{t-1} is,

$$\begin{aligned} f_{t-1}(z_{t-1}) &= \max_{h} E\{(\lambda e' z_{t-1} - \omega z'_{t-1} e e' z_{t-1}) P_{t-1} + f_t(z_t)\} \\ &= \max_{u} E\{(\lambda e' z_{t-1} - \omega z'_{t-1} e e' z_{t-1}) P_{t-1} \\ &+ [-\omega(B_{t-1} z_{t-1} + A_{t-1} e_1 u_{t-1})' \overline{D}_t \overline{D}'_t (B_{t-1} z_{t-1} + A_{t-1} e_1 u_{t-1}) \\ &+ \lambda F'_t (B_{t-1} z_{t-1} + A_{t-1} e_1 u_{t-1}) + \sum_{i=t}^{T-1} C_i]\}, \end{aligned}$$

Maximizing $f_{t-1}(z_{t-1})$ with respect to u_{t-1} yields

$$u^*_{t-1} = \frac{\lambda}{2\omega} \frac{E(e'_1 A'_{t-1} F_t)}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)} - \frac{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t B_{t-1} z_{t-1})}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)}.$$

Hence,

$$\begin{aligned} z_t &= B_{t-1} z_{t-1} + A_{t-1} h_{t-1} \\ &= [B_{t-1} z_{t-1} - A_{t-1} e_1 \frac{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t B_{t-1} z_{t-1})}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)}] \\ &+ \frac{\lambda}{2\omega} A_{t-1} e_1 \frac{E(e'_1 A'_{t-1} F_t)}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)}, \\ S_t &= e' z_t \\ &= e'[B_{t-1} z_{t-1} - A_{t-1} e_1 \frac{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t B_{t-1} z_{t-1})}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)}] \\ &+ \frac{\lambda}{2\omega} e' A_{t-1} e_1 \frac{E(e'_1 A'_{t-1} F_t)}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)} \\ &= e' B^0_{t-1} z_{t-1} + \frac{\lambda}{2\omega} R_{t-1} Y_{t-1}, \\ \overline{D}'_t z_t &= \overline{D}'_t [B_{t-1} z_{t-1} - A_{t-1} e_1 \frac{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t B_{t-1} z_{t-1})}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)}] \\ &+ \frac{\lambda}{2\omega} \overline{D}'_t A_{t-1} e_1 \frac{E(e'_1 A'_{t-1} F_t)}{E(e'_1 A'_{t-1} \overline{D}_t \overline{D}'_t A_{t-1} e_1)} \\ &= \widetilde{D}'_{t-1} z_{t-1} + \frac{\lambda}{2\omega} \chi^1_{t-1} Y_{t-1}, \end{aligned}$$

$$\begin{aligned}
F_t' z_t &= F_t'[B_{t-1} z_{t-1} - A_{t-1} e_1 \frac{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' B_{t-1} z_{t-1})}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)}] \\
&+ \frac{\lambda}{2\omega} F_t' A_{t-1} e_1 \frac{E(e_1' A_{t-1}' F_t)}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)} \\
&= \widetilde{F}_{t-1}' z_{t-1} + \frac{\lambda}{2\omega} \chi_{t-1}^2 Y_{t-1},
\end{aligned}$$

where

$$\widetilde{D}_{t-1}' = \begin{pmatrix} e' B_{t-1}^{T-t} (P_T)^{\frac{1}{2}} \\ \vdots \\ e' B_{t-1}^{s-t} (P_s)^{\frac{1}{2}} \\ \vdots \\ e' B_{t-1}^{0} (P_t)^{\frac{1}{2}} \end{pmatrix},$$

$$\widetilde{F}_{t-1} = \sum_{i=t+1}^{T} e' B_{t-1}^{i-t} P_i + e' B_{t-1}^{0} P_t = \sum_{i=t}^{T} e' B_{t-1}^{i-t} P_i,$$

$$\chi_{t-1}^1 = \begin{pmatrix} R_{t-1} \overline{r}_t^{T-t-1} (P_T)^{\frac{1}{2}} \\ \vdots \\ R_{t-1} \overline{r}_t^{s-t-1} (P_s)^{\frac{1}{2}} \\ \vdots \\ R_{t-1} (P_t)^{\frac{1}{2}} \end{pmatrix},$$

$$\chi_{t-1}^2 = R_{t-1}[\sum_{s=t+1}^{T} (\overline{r}_t^{s-t-1}) P_s + P_t],$$

$$\begin{aligned}
Y_{t-1} &= \frac{E(e_1' A_{t-1}' F_t)}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)} \\
&= \frac{E(R_{t-1})}{E[(R_{t-1})^2]} \frac{E[\sum_{s=t+1}^{T} \overline{r}_t^{s-t-1} P_s + P_t]}{E[\sum_{s=t+1}^{T} \overline{r}_t^{s-t-1})^2 P_s + P_t]},
\end{aligned}$$

$$B_{t-1}^0 = \begin{pmatrix} \overline{r}_{t-1}^0 & 0 \\ 0 & \overline{q}_{t-1}^0 \end{pmatrix}, \quad B_{t-1}^i = \begin{pmatrix} \overline{r}_{t-1}^i & 0 \\ 0 & \overline{q}_{t-1}^i \end{pmatrix},$$

with

$$\overline{r}_{t-1}^0 = r_{t-1}^0 - R_{t-1} \frac{E(R_{t-1} r_t^0)}{E[(R_{t-1})^2]},$$

$$\overline{q}_{t-1}^{0} = q_{t-1}^{0} - R_{t-1}\frac{E(R_{t-1}q_t^0)E[\sum_{s=t+1}^{T}(\overline{r}_t^{s-t-1}\overline{q}_t^{s-t-1})P_s + P_t]}{E[(R_{t-1})^2]E[\sum_{s=t+1}^{T}(\overline{r}_t^{s-t-1})^2P_s + P_t]},$$

$$\overline{r}_{t-1}^{i} = r_{t-1}^{0}\overline{r}_t^{i-1} - R_{t-1}\overline{r}_t^{i-1}\frac{E(R_{t-1}r_{t-1}^0)}{E[(R_{t-1})^2]},$$

$$\overline{q}_{t-1}^{i} = q_{t-1}^{0}\overline{q}_t^{i-1} - R_{t-1}\overline{q}_t^{i-1}\frac{E(R_{t-1}q_{t-1}^0)E[\sum_{s=t+1}^{T}\overline{r}_t^{s-t-1}\overline{q}_t^{s-t-1}P_s + P_t]}{E[(R_{t-1})^2]E\{\sum_{s=t+1}^{T}(\overline{r}_t^{s-t-1})^2P_s + P_t\}},$$

for $i = 1, 2, \cdots, T-t$.

Furthermore,

$$\begin{aligned}
E[(\chi_{t-1}^1 Y_{t-1})'(\widetilde{D}_{t-1}' z_{t-1})] &= \frac{E(e_1' A_{T-2}' F_{T-1})}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)}\{E[e_1' A_{t-1}' \overline{D}_t \overline{D}_t' B_{t-1} z_{t-1}] \\
&\quad - E[e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1]\frac{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' B_{t-1} z_{t-1})}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)}\} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
E[(\chi_{t-1}^1 Y_{t-1})'(\chi_{t-1}^1 Y_{t-1})] &= E[e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1]\frac{E^2(e_1' A_{t-1}' F_t)}{E^2(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)} \\
&= \frac{E^2(e_1' A_{t-1}' F_t)}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)}, \\
E[\chi_{t-1}^2 Y_{t-1}] &= E(e_1' A_{t-1}' F_t)\frac{E(e_1' A_{t-1}' F_t)}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)} \\
&= \frac{E^2(e_1' A_{t-1}' F_t)}{E(e_1' A_{t-1}' \overline{D}_t \overline{D}_t' A_{t-1} e_1)},
\end{aligned}$$

We can further get the following equations:

$$E[(\chi_{t-1}^1 Y_{t-1})'(\widetilde{D}_{t-1}' z_{t-1})] = 0,$$

$$E[(\chi_{t-1}^1 Y_{t-1})'(\chi_{t-1}^1 Y_{t-1})] = E[\chi_{t-1}^2 Y_{t-1}].$$

Therefore,

$$\begin{aligned}
f_{t-1}(z_{t-1}) = E\{&(\lambda e' z_{t-1} - \omega z_{t-1}' e e' z_{t-1})P_{t-1} \\
&+ [-\omega(B_{t-1} z_{t-1} + A_{t-1} h_{t-1})' \overline{D}_t \overline{D}_t' (B_{t-1} z_{t-1} + A_{t-1} h_{t-1}) \\
&+ \lambda F_t' (B_{t-1} z_{t-1} + A_{t-1} h_{t-1}) + \sum_{i=t}^{T-1} C_i]\}
\end{aligned}$$

—

$$= E\{(\lambda e' z_{t-1} - \omega z'_{t-1} e e' z_{t-1}) P_{t-1} + [\lambda(\widetilde{F}'_{t-1} z_{t-1} + \frac{\lambda}{2\omega} Y_{t-1} \chi^2_{t-1})$$
$$- \omega(\widetilde{D}'_{t-1} z_{t-1} + \frac{\lambda}{2\omega} Y_{t-1} \chi^1_{t-1})'(\widetilde{D}'_{t-1} z_{t-1} + \frac{\lambda}{2\omega} Y_{t-1} \chi^1_{t-1})] + \sum_{i=t}^{T-1} C_i\}$$
$$= E\{(\lambda e' z_{t-1} - \omega z'_{t-1} e e' z_{t-1}) P_{t-1}$$
$$+ [\lambda \widetilde{F}'_{t-1} z_{t-1} - \frac{\lambda^2}{4\omega} Y_{t-1} \chi^2_{t-1} - \omega z_{t-1} \widetilde{D}_{t-1} \widetilde{D}'_{t-1} z_{t-1}] + \sum_{i=t}^{T-1} C_i$$
$$= -\omega z'_{t-1} E(D_{t-1}) z_{t-1} + \lambda E(F_{t-1})' z_{t-1} + \sum_{i=t-1}^{T-1} C_i,$$

where,

$$F_{t-1} = e P_{t-1} + \widetilde{F}_{t-1} = \begin{pmatrix} \sum_{s=t}^{T} \overline{r}_{t-1}^{s-t} P_s + P_{t-1} \\ -(\sum_{s=t}^{T} \overline{q}_{t-1}^{s-t} P_s + P_{t-1}) \end{pmatrix},$$
$$D_{t-1} = e e' P_{t-1} + \widetilde{D}_{t-1} \widetilde{D}'_{t-1} = \overline{D}_{t-1} \overline{D}'_{t-1},$$
$$C_{t-1} = \frac{\lambda^2}{4\omega} E(\chi^2_{t-1} Y_{t-1}),$$

with

$$\overline{D}'_{t-1} = \begin{pmatrix} \widetilde{D}'_{t-1} \\ e'(P_{t-1})^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \overline{r}_{t-1}^{T-t} (P_T)^{\frac{1}{2}} & -\overline{q}_{t-1}^{T-t} (P_T)^{\frac{1}{2}} \\ \vdots & \vdots \\ \overline{r}_{t-1}^{s-t} (P_s)^{\frac{1}{2}} & -\overline{q}_{t-1}^{s-t} (P_s)^{\frac{1}{2}} \\ \vdots & \vdots \\ (P_{t-1})^{\frac{1}{2}} & -(P_{t-1})^{\frac{1}{2}} \end{pmatrix}_{(T-t+2)\times 2}.$$

The theorem is thus proved by mathematical induction. □

3.4. Mean-variance efficient frontier

The efficient mean-variance frontier presents the most important piece of information of a portfolio selection problem. More specifically, the efficient frontier in a mean-variance space shows the relationship between the expected value and the investment risk, characterized by the variance, under the optimal policy. In this section, we derive the mean-variance efficient frontier using the optimal investment policy derived in Section 3.3.

Let

$$Y_t = \frac{E(R_t)}{E[(R_t)^2]} \frac{E[\sum_{s=t+2}^{T} \overline{r}_{t+1}^{s-t-2} P_s + P_{t+1}]}{E[\sum_{s=t+2}^{T} (\overline{r}_{t+1}^{s-t-2})^2 P_s + P_{t+1}]}. \tag{3.14}$$

Under the optimal policy given in Theorem 3.1, $e'z_t$, $e'B_{t-1}^{0} z_{t-1}$, $\cdots$, $e'B_0^{t-1} z_0$ can be expressed as

$$\begin{aligned}
e'z_t &= e'B_{t-1}^{0} z_{t-1} + \tfrac{\lambda}{2\omega} Y_{t-1} R_{t-1}, \\
e'B_{t-1}^{0} z_{t-1} &= e'B_{t-2}^{1} z_{t-2} + \tfrac{\lambda}{2\omega} Y_{t-2} R_{t-2} \overline{r}_{t-1}^{0}, \\
&\vdots \\
e'B_j^{t-1-j} z_j &= e'B_{j-1}^{t-j} z_{j-1} + \tfrac{\lambda}{2\omega} Y_{j-1} R_{j-1} \overline{r}_j^{t-1-j}, \\
&\vdots \\
e'B_1^{t-2} z_1 &= e'B_0^{t-1} z_0 + \tfrac{\lambda}{2\omega} Y_0 R_0 \overline{r}_1^{t-2}, \\
e'B_0^{t-1} z_0 &= \overline{r}_0^{t-2} x_0 - \overline{q}_0^{t-2} l_0,
\end{aligned}$$

for $t = 1, 2, \cdots, T$.

Recall that $S_t = x_t - l_t$. After some manipulations, S_t and $e'B_j^{t-1-j} z_j$ can be expressed as functions of Y_i, R_i and $\overline{r}_i^j$:

$$S_t = e'z_t = e'B_0^{t-1} z_0 + \frac{\lambda}{2\omega}[Y_{t-1}R_{t-1} + \Sigma_{i=0}^{t-2}(Y_i R_i \overline{r}_{i+1}^{t-2-i})], \tag{3.15}$$

$$e'B_j^{t-1-j} z_j = e'B_0^{t-1} z_0 + \frac{\lambda}{2\omega}[\Sigma_{i=0}^{j-1}(Y_i R_i \overline{r}_{i+1}^{t-2-i})]. \tag{3.16}$$

Furthermore, the expectation of S_t and $S_{T\wedge\tau}$ can be derived as follows,

$$\begin{aligned}
E(S_t) &= E(e'B_0^{t-1} z_0) + \tfrac{\lambda}{2\omega} E[Y_{t-1}E_{t-1} + \Sigma_{i=0}^{t-2}(Y_i R_i \overline{r}_{i+1}^{t-2-i})] \\
&= \tfrac{\lambda}{2\omega} G_t + J_t,
\end{aligned}$$

for $t = 1, \cdots, T$, and

$$\begin{aligned} E(S_{T\wedge\tau}) &= E(\Sigma_{t=0}^{T} S_t P_t) \\ &= \Sigma_{t=2}^{T} E\{e' B_0^{t-1} z_0 + \tfrac{\lambda}{2\omega}[Y_{t-1} R_{t-1} + \Sigma_{i=0}^{t-2}(Y_i R_i \overline{r}_{i+1}^{t-2-i})]\} P_t \\ &\quad + E(\tfrac{\lambda}{2\omega} Y_0 R_0 + e' B_0^0 z_0) P_1 + e' z_0 P_0 \\ &= \tfrac{\lambda}{2\omega} G + J, \end{aligned}$$

where

$$G_t = E[Y_{t-1} R_{t-1} + \Sigma_{i=0}^{t-2}(Y_i R_i \overline{r}_{i+1}^{t-2-i})], \tag{3.17}$$

$$J_t = E(e' B_0^{t-1} z_0), \tag{3.18}$$

$$G = \Sigma_{t=2}^{T} E[Y_{t-1} R_{t-1} + \Sigma_{i=0}^{t-2}(Y_i R_i \overline{r}_{i+1}^{t-2-i})] P_t + E(Y_0 R_0) P_1, \tag{3.19}$$

$$J = \Sigma_{t=1}^{T} E(e' B_0^{t-1} z_0) P_t + e' z_0 P_0. \tag{3.20}$$

According to Theorem 2.2, a necessary condition for the solution of $(P4(\lambda^*, \omega))$ to attain the optimality of $(P3(\omega))$ at the same time is

$$\lambda^* = 1 + 2\omega E(S_{T\wedge\tau}) \mid_{\lambda^*} = 1 + 2\omega(\frac{\lambda^*}{2\omega} G + J).$$

Therefore, we have $\lambda^* = \frac{1+2\omega J}{1-G}$.

Now we derive $E(S_{T\wedge\tau}^2)$ in a way similar to the way we derive $E(S_{T\wedge\tau})$. Because

$$\begin{aligned} z_t' e e' z_t &= z_{t-1}' B_{t-1}^{0\,\prime} e e' B_{t-1}^0 z_{t-1} \\ &\quad + \tfrac{\lambda^2}{4\omega^2}(Y_{t-1} R_{t-1})^2 + \tfrac{\lambda}{\omega} Y_{t-1} R_{t-1} e' B_{t-1}^0 z_{t-1}, \\ &\quad \vdots \\ z_j B_j^{t-1-j\prime} e e' B_j^{t-1-j} z_j &= z_{j-1} B_j^{t-j\prime} e e' B_j^{t-j} z_{j-1} \\ &\quad + \tfrac{\lambda^2}{4\omega^2}(Y_{j-1} R_{j-1} \overline{r}_j^{t-1-j})^2 + \tfrac{\lambda}{\omega} Y_{j-1} R_{j-1} \overline{r}_j^{t-1-j} e' B_{j-1}^{t-j} z_{j-1}, \\ &\quad \vdots \\ z_1 B_1^{t-2\prime} e e' B_1^{t-2} z_1 &= z_0 B_0^{t-1\prime} e e' B_0^{t-1} z_0 \\ &\quad + \tfrac{\lambda^2}{4\omega^2}(Y_0 R_0 \overline{r}_1^{t-2})^2 + \tfrac{\lambda}{\omega} Y_0 R_0 \overline{r}_1^{t-2} e' B_0^{t-1} z_1, \\ z_0 B_0^{t-1\prime} e e' B_0^{t-1} z_0 &= (\overline{r}_0^{t-2} x_0 - \overline{q}_0^{t-2} l_0)^2, \end{aligned}$$

the expectation of $(S_t)^2$ can be thus derived as follows by using (3.16) and the above set of equations:

$$\begin{aligned}
E[(S_t)^2] &= E(z_t'ee'z_t) \\
&= \tfrac{\lambda^2}{4\omega}\{E(Y_{t-1}R_{t-1})^2 + \Sigma_{i=0}^{t-2}E[(Y_iR_i\overline{r}_{i+1}^{t-i-2})^2]\} + E(z_0B_0^{t-1'}ee'B_0^{t-1}z_0) \\
&\quad + \tfrac{\lambda}{\omega}E[\Sigma_{i=0}^{t-2}(Y_iR_i\overline{r}_{i+1}^{t-i-2})e'B_i^{t-1-i}z_i + Y_{t-1}R_{t-1}e'B_{t-1}^{0}z_{t-1}] \\
&= \tfrac{\lambda^2}{4\omega^2}M_t + N_t + \tfrac{\lambda}{\omega}Q_t.
\end{aligned}$$

where

$$M_t = E(Y_{t-1}R_{t-1})^2 + \Sigma_{i=0}^{t-2}E[(Y_iR_i\overline{r}_{i+1}^{t-i-2})^2]$$
$$+ 2E[\sum_{i=1}^{t-2} Y_iR_i\overline{r}_{i+1}^{t-i-2}(\sum_{j=0}^{i-1} Y_jR_j\overline{r}_{j+1}^{t-2-j}) + Y_{t-1}R_{t-1}(\sum_{j=0}^{t-2} Y_jR_j\overline{r}_{j+1}^{t-2-j})], \quad (3.21)$$

$$N_t = E(z_0'B_0^{t-1'}ee'B_0^{t-1}z_0), \quad (3.22)$$

$$Q_t = E[\Sigma_{i=0}^{t-2}(Y_iR_i\overline{r}_{i+1}^{t-i-2})e'B_0^{t-1}z_0 + Y_{t-1}R_{t-1}e'B_0^{t-1}z_0]. \quad (3.23)$$

Furthermore, the expectation of $(S_{T\wedge\tau})^2$ can be obtained from the expectation of $(S_t)^2$:

$$\begin{aligned}
E[(S_{T\wedge\tau})^2] &= E[\Sigma_{t=0}^{T}(S_t)^2P_t] \\
&= E[\Sigma_{t=2}^{T}(S_t)^2P_t] + E(S_1)P_1 + E(S_0)P_0 \\
&= \tfrac{\lambda^2}{4\omega^2}M + N + \tfrac{\lambda}{\omega}Q,
\end{aligned}$$

where

$$\begin{aligned}
M &= \Sigma_{t=2}^{T}\{E(Y_{t-1}R_{t-1})^2 + \Sigma_{i=0}^{t-2}E[(Y_iR_i\overline{r}_{i+1}^{t-i-2})^2] \\
&\quad +2E[\sum_{i=1}^{t-2} Y_iR_i\overline{r}_{i+1}^{t-i-2}(\sum_{j=0}^{i-1} Y_jR_j\overline{r}_{j+1}^{t-2-j}) \\
&\quad +Y_{t-1}R_{t-1}(\sum_{j=0}^{t-2} Y_jR_j\overline{r}_{j+1}^{t-2-j})]\}P_t + (Y_0R_0)^2P_1, \quad (3.24) \\
N &= \Sigma_{t=1}^{T}E(z_0B_0^{t-1'}ee'B_0^{t-1}z_0)P_t + z_0'ee'z_0P_0, \quad (3.25) \\
Q &= \Sigma_{t=2}^{T}E[\Sigma_{i=0}^{t-2}(Y_iR_i\overline{r}_{i+1}^{t-i-2})e'B_0^{t-1}z_0 + Y_{t-1}R_{t-1}e'B_0^{t-1}z_0]P_t \\
&\quad +E(Y_0r_0^1e'B_0^0z_0)P_1. \quad (3.26)
\end{aligned}$$

With the above results, we can calculate the variance of S_t and $S_{T\wedge\tau}$ as follows:

$$Var(S_t) = \frac{M_t - G_t^2}{G_t^2}(ES_t - J_t)^2 + 2\frac{Q_t - G_tJ_t}{G_t}(ES_t - J_t) + (N_t - J_t^2),$$

for $t = 1, 2, \cdots, T$, and

$$\begin{aligned} Var(S_{T\wedge\tau}) &= E[(S_{T\wedge\tau})^2] - [E(S_{T\wedge\tau})]^2 \\ &= (\tfrac{\lambda^2}{4\omega^2}M + N + \tfrac{\lambda}{\omega}Q) - (\tfrac{\lambda}{2\omega}G + J)^2 \\ &= \tfrac{\lambda^2}{4\omega^2}(M - G^2) + \tfrac{\lambda}{\omega}(Q - GJ) + (N - J^2). \end{aligned}$$

Finally, we get the expression of the mean-variance efficient frontier of the asset-liability portfolio selection problem with an uncertain exit time.

Proposition 3.1 *The efficient frontier of* $(P3(\omega))$ *is given as,*

$$Var(S_{T\wedge\tau}) = \frac{M - G^2}{G^2}(ES_{T\wedge\tau} - J)^2 + 2\frac{Q - GJ}{G}(ES_{T\wedge\tau} - J) + (N - J^2),$$

where G,J,M,N,Q *are given, respectively, in* (3.19), (3.20), (3.24), (3.25) *and* (3.26).

3.5. Extension to situations with more than two assets

The results in the previous sections can be readily extended to general situations with $n+1$ assets.

Suppose that the returns of the $n+1$ assets' returns at time t are given by r_t^0 and $r_t^i (i = 1, 2, \cdots, n)$, respectively. Define $r_t = (r_t^1, r_t^2, \cdots, r_t^n)$. Note that the difference between the current situation and the previous case with two-assets is that $\tilde{r}_t$ is now a vector. We have the following equations for dynamic evolvement of the wealth and the liability:

$$x_{t+1} = r_t^0 x_t + R_t u_t,\ l_{t+1} = q_t l_t,$$

where $R_t = r_t - (r_t^0, r_t^0, \cdots, r_t^0)_{1\times n}$ and $u_t = (u_t^1, \cdots, u_t^n)'$.

Modifying A_t, h_t and e_1 to the following forms:

$$A_t = \begin{pmatrix} R_t & 0 \\ 0_{1\times n} & 0 \end{pmatrix}, h_t = \begin{pmatrix} u_t \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} e_{n\times n} \\ 0_{1\times n} \end{pmatrix},$$

where $e_{n\times n}$ is the $n \times n$ identity matrix, we have the following dynamic equation for the surplus,

$$z_{t+1} = B_t z_t + A_t h_t,$$

with $h_t = e_1 u_t$.

It becomes clear now that formulation $(P5)$ for situations with more than two assets takes the same form as in the situation with two assets:

$$\begin{cases} \max\limits_{u,v} & E(\lambda S_{T\wedge\tau} - \omega S^2_{T\wedge\tau}) \\ \text{s.t.} & z_{t+1} = B_t z_t + A_t h_t,\ h_t = e_1 u_t, t = 0, 1, 2, \cdots, T-1, \end{cases}$$

except that the control variable u_t is now a vector. It can be verified that the optimal policy for the problem with multiple assets is given as follows:

$$\begin{aligned} u_t^* &= \tfrac{\lambda}{2\omega} E^{-1}(e_1' A_t' \overline{D}_{t+1} \overline{D}_{t+1}' A_t e_1) E(e_1' A_t' F_{t+1}) \\ &\quad - E^{-1}(e_1' A_t' \overline{D}_{t+1} \overline{D}_{t+1}' A_t e_1) E(e_1' A_t' \overline{D}_{t+1} \overline{D}_{t+1}' B_t z_t), \end{aligned}$$

for $t = 0, 1, 2, \cdots, T-1$.

Furthermore, the efficient frontier can be derived as follows,

$$Var(S_{T\wedge\tau}) = \frac{M - G^2}{G^2}(ES_{T\wedge\tau} - J)^2 + 2\frac{Q - GJ}{G}(ES_{T\wedge\tau} - J) + (N - J^2).$$

Note that our result reduces to the result of Guo and Hu [22] when there is no liability and there is one risk free asset among $(n + 1)$ assets in the financial market.

Example 3.1: An investor has 1 unit wealth at the very beginning of the planning horizon, which is resulted from 2 units of assets and 1 unit of liability.

She plans to invest in a time horizon of 4 time periods ($T = 4$). However, the uncertain investment environment may force him out of the market at period τ, where τ is a discrete random variable with a probability distribution of $P_0 = \widetilde{P}_0 = 0$, $P_1 = \widetilde{P}_1 = 0$, $P_2 = \widetilde{P}_2 = 0.5$, $P_3 = 1 - \sum_{i=0}^{2} \widetilde{P}_i = 0.5$.

The investor is trying to find out the best allocation of her wealth among two assets A, B and exogenous liability C to maximize $E(S_{T\wedge\tau})$ while keeping his risk not to exceed certain value. Note that liability C is uncontrollable.

The expected returns for assets A, B and liability C are $E(r_t^A) = 1.159$, $E(r_t^B) = 1.243$ and $E(Q_t^C) = 1.224, t = 0, 1, 2, 3$, while the covariance of $R_t = (r_t^A, r_t^B, q_t^C)'$ is given as

$$Cov(R_t) = \begin{pmatrix} 0.0148 & 0.0185 & 0.0146 \\ 0.0185 & 0.0855 & 0.0105 \\ 0.0146 & 0.0105 & 0.0288 \end{pmatrix}, \quad t = 0, 1, 2, 3.$$

Taking asset A as the benchmark gives rise to

$$\begin{aligned} E(r_t^0) &= E(r_t^A) = 1.159, \\ E(q_t) &= E(q_t^C) = 1.224, \\ E(r_t^1) &= E(r_t^B - r_t^A) = 0.084. \end{aligned}$$

Hence, we get

$$Y_2 = 1.1939, Y_1 = 1.1221, Y_0 = 0.9968,$$

$$G_1 = 0.0837, J_1 = 0.9903, M_1 = 0.0699, N_1 = 0.9020,$$

$$G_2 = 0.1812, J_2 = 0.9618, M_2 = 0.1703, N_2 = 0.7274,$$

$$G_3 = 0.2696, J_3 = 0.9177, M_3 = 0.2993, N_3 = 0.1893,$$

$$G = 0.2348, J = 0.9398, M = 0.2348, N = 0.4583.$$

Finally, the mean-variance efficient frontier of this example problem is derived as,

$$\begin{aligned} Var(S_1) &= 8.9711(ES_1 - 1.1007)^2 - 0.1880, \\ Var(S_2) &= 4.1867(ES_2 - 1.1916)^2 - 0.4187, \\ Var(S_3) &= 2.5980(ES_3 - 1.2710)^2 - 0.9771, \\ Var(S_{T\wedge\tau}) &= 3.2586(ES_{T\wedge\tau} - 1.2282)^2 - 0.6959. \end{aligned}$$

To examine further the impact of the uncertain exit time on the efficient frontier, we compare the efficient frontiers of the above example problem under four different settings of the uncertain exit time:

$$P1 : (P_0, P_1, P_2, P_3) = (0, 0.9, 0.09, 0.01),$$

$$P2 : (P_0, P_1, P_2, P_3) = (0, 0.1, 0.89, 0.01),$$

$$P3 : (P_0, P_1, P_2, P_3) = (0, 0, 0.5, 0.5),$$

$$P4 : (P_0, P_1, P_2, P_3) = (0, 0, 0, 1).$$

Figure 3.1 presents the efficient frontiers of the uncertain terminal wealth $S_{T\wedge\tau}$ under these four different settings of the uncertain exit time. It is evident from Figure 3.1 that the larger the probability for a later exit time, the more dominant the efficient frontier becomes.

3.6. Summary

Multi-period mean-variance portfolio optimization formulation has been extended in this chapter to incorporate exogenous liabilities and uncertain investment horizon into the consideration. Although the model is much more complicated than the case with asset-only and/or with certain investment horizon, the embedding technique of Li and Ng(2000) is still applicable. Both the optimal policy and the mean-variance efficient frontier have been derived analytically.

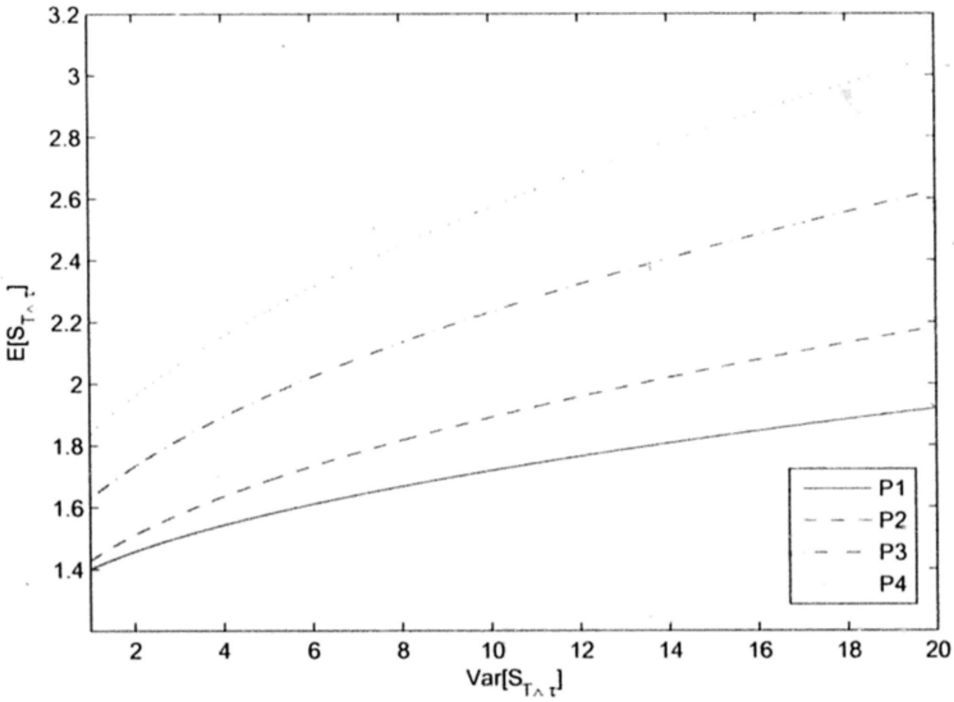

Figure 3.1: Efficient frontiers of $S_{T\wedge\tau}$ under different settings of the uncertain exit time in Example 3.1

Part II

Multi-Period Portfolio Optimization in Frictional Market

Chapter 4

Multi-Period Portfolio Selection without Shorting

4.1. Introduction

In the financial market, most investors purchase financial instruments with expected increasing value. However, if they believe that a financial instrument is overpriced, and want to take advantage of an expected decline in the price, they may sell the instrument short. *Short-selling* or *shorting* is the practice of selling a financial instrument that the seller does not own at the time of the sale. This kind of practice is of a high risk. If the price of the instrument declines, not as the short-seller expected, the short-seller will suffer a big loss, and the loss can be boundless theoretically. So short-selling is not suggested.

When short-selling is not allowed, the multi-period portfolio selection problem is much more difficult to deal with. The portfolio selection problem involving no-shorting-selling constraint has been studied in Xu and Shreve [53][54], in which utility maximization problems without short-selling are investigated by using convex duality analysis. Instead of using optimization method, they used a martingale approach, and showed that an unique equivalent martingale measure exist in the no-arbitrage complete market model.

Convex duality theory was firstly applied to portfolio selection problems in 1975 by Bismut [4]. By applying convex duality theory from Bismut [3], Bismut [4] solved a problem similar to the one solved by Merton [41]. Martingale methodology, which was developed in the papers of Harrison and Kreps [25] and Harrison and Pliska [26], allows the application of convex duality theory to a more general market model. Pliska [45] firstly studied how to use this methodology to maximize the expected utility of terminal wealth. It is obvious that convex duality can be applied to deal with the problems in which there are no portfolio constraints. Xu and Shreve [53][54] showed that convex duality analysis also succeeds in solving problems with no-short-selling constraint.

Li, Zhou and Lim [34] handled the portfolio selection problem under a short-selling prohibition in a different way. They use stochastic linear quadratic control theory to study the constrained problem. After conjecturing a continuous solution to the HJB equation via two Riccati equations, they showed that it is indeed the viscosity solution to the equation. Hence, the efficient investment strategies were derived.

Despite the fact that much work has been done on continuous-time portfolio selection problems involving no-short-selling constraints so far, no progress has been achieved concerning the discrete-time portfolio selection problem with this kind of constraint. This chapter studies an optimal portfolio selection problem with no-short-selling constraint in a discrete-time multi-period financial model by using convex duality analysis. First, we build up the security market model in Section 4.2. We then consider the optimal portfolio selection problem with no-short-selling constraint in Section 4.3. By transforming the original market to some auxiliary markets, the optimal value of the original constrained problem can be derived by the optimal value of the unconstrained problem in the auxiliary markets. In Section 4.4, we use martingale approach to solve the auxiliary unconstrained problems in the auxiliary markets. In Section 4.5, we introduce the dual problem of the original constrained problem, and prove its strong duality. Also, we derive the optimal solution to the dual problem by using dynamic programming. In the last section, we derive the optimal terminal wealth of the original problem, and illustrate our analysis by a numerical example.

4.2. Financial market model

We consider a multi-period model of security market with $T+1$ trading dates (indexed by $0, 1, \cdots, T$). There are n risky securities and one bond in the market. Let $(\Omega, \mathscr{F}, P)$ be the probability space, where $\Omega := \{\omega_1, \cdots, \omega_K\}$ is the sample space with K finite samples. The filtration is denoted by $\mathscr{F} = \{\mathcal{F}_t, t = 0, 1, \cdots, T\}$, where $\mathcal{F}_t$ reveals the information on the economy at time t. Specifically, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We claim that for any $A \in \mathcal{F}\backslash\emptyset$, $P(A) > 0$. We give the following definition according to Pliska [46],

Definition 4.1 *A set* $\mathbf{f}_t = \{A_t^1, \cdots, A_t^u; A_t^i \in \mathcal{F}_t \;\; for \;\; i = 1, \cdots, u\}$ *of subsets of the sample space* Ω *is called basic partition at time* t *of* Ω *if and only if:*

1. *For* $t = 0$, $\mathbf{f}_0 = \{\Omega\}$*;*

2. *For* $t = T$, $\mathbf{f}_T = \{\{\omega_1\}, \cdots, \{\omega_K\}\}$*;*

3. *For every* $i \neq j$, *we have* $A_t^i \cap A_t^j = \emptyset$;

4. *The union* $A_t^1 \cup \cdots \cup A_t^u = \Omega$;

5. *For any element in* $\mathbf{f}_t$, *i.e.* A_t, *there exist some element* $A_{t+1}^1, \cdots, A_{t+1}^l$ *in* $\mathbf{f}_{t+1}$, *such that* $A_t = A_{t+1}^1 \cup \cdots \cup A_{t+1}^l$.

The sequence of basic partitions $\{\mathbf{f}_t\}$ forms the information structure of the financial market. Notice that if $\mathbf{f}_t = \{A_t^1, \cdots, A_t^u\}$, then $\mathcal{F}_t = \sigma(A_t^1, \cdots, A_t^u)$.

Assumption 4.1 (1) *For each element* A_t *in partition* $\mathbf{f}_t$, *there always exist* $n+1$ *element* $A_{t+1}^1, \cdots, A_{t+1}^{n+1}$ *in* $\mathbf{f}_{t+1}$, *such that* $A_t = A_{t+1}^1 \cup \cdots \cup A_{t+1}^{n+1}$; (2) *The probabilities of all elements in sample space* Ω *are equivalent, that is* $P(\omega) = \frac{1}{K}$.

Remark 4.1 *Under Assumption 4.1, there are* $(n+1)^t$ *set in partition* $\mathbf{f}_t$ *for* $t = 0, 1, \cdots, T$. *Therefore,* $K = (n+1)^T$.

To help understand Assumption 4.1, we consider a two period security market model with two risky assets. See Figure 4.1.

In this model, $T = 2$ and $\Omega = \{\omega_1, \cdots, \omega_9\}$. The partitions for $t = 0, 1, 2$ are

$$
\begin{aligned}
\mathbf{f}_0 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\}\}, \\
\mathbf{f}_1 &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\}, \\
\mathbf{f}_2 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}, \{\omega_9\}\}.
\end{aligned}
$$

For $A_1 = \{\omega_4, \omega_5, \omega_6\} \in \mathbf{f}_1$, there exist $\{\omega_4\}, \{\omega_5\}, \{\omega_6\} \in \mathbf{f}_2$, such that $A_1 = \{\omega_4\} \cup \{\omega_5\} \cup \{\omega_6\}$.

Remark 4.2 *With* (1) *in Assumption 4.1, there will be* $(n+1)^t$ *elements in partition* $\mathbf{f}_t$, *denoted by* A_t^j, *with* $j \in I_t := \{1, 2, \cdots, (n+1)^t\}$.

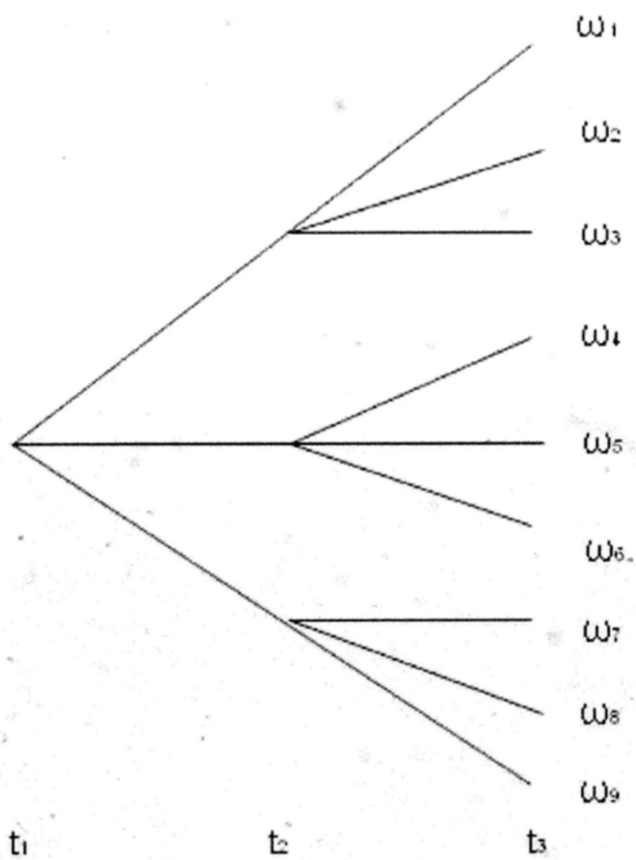

Figure 4.1: Two-period model of security market with two risky assets

Remark 4.3 *For any $A_t^j \in \mathbf{f}_t$, we know that there exist $n+1$ elements in $\mathbf{f}_{t+1}$ such that the union of them equals A_t^j. We denote them as A_{t+1}^l with $l \in I_{t+1}^j := \{(n+1)(j-1)+1, (n+1)(j-1)+2, \cdots, (n+1)j\} \subset I_{t+1}$. Notice that $\cup_{j\in I_t} I_{t+1}^j = I_{t+1}$.*

Remark 4.4 *For any $\omega_i \in \Omega$, there exist $A_t^{i_t} \in \mathbf{f}_t$ for $t = 0, 1, \cdots, T$, such that $\omega_i \in A_T^{i_T} \subset A_{T-1}^{i_{T-1}} \subset \cdots \subset A_1^{i_1} \subset A_0^{i_0} = \Omega$. Denote $\mathbb{I}_0 := \{(i_0, \cdots, i_T) : i_0 \in I_0, i_1 \in I_1^{i_0}, \cdots, i_T \in I_T^{i_{T-1}}\}$, and $\mathbb{I}_\tau^l := \{(i_\tau, \cdots, i_T) : i_\tau = l \in I_\tau, i_{\tau+1} \in I_1^{i_\tau}, \cdots, i_T \in I_T^{i_{T-1}}\}$. We can see that sets $\mathbb{I}_0$ and Ω are of a one-to-one mapping.*

There are $n+1$ securities traded in the market without transaction cost. Denote the stochastic process of the security price as $\mathbf{S} = \{S_t; t = 0, \cdots, T\}$, where $S_t = (S_t(1), \cdots, S_t(n))'$ is a random vector, and the bond price process as $\mathbf{B} = \{B_t; t = 0, \cdots, T\}$, where B_t is a scalar. Let $\mathbf{r}^0 = \{r_t^0; t = 0, \cdots, T-1\}$ be the bond return process where $r_t^0 = B_{t+1}/B_t \geq 0$ for $t = 0, 1, \cdots, T-1$; $\mathbf{r} = \{r_t; t = 0, \cdots, T-1\}$ be the risky securities return process with $r_t = (r_t(1), \cdots, r_t(n))'$ and $r_t(i) = S_{t+1}(i)/S_t(i) \geq 0$. Denote $\mathbf{R} = \{R_t; t = 0, \cdots, T-1\}$ as the premium return process with $R_t = (R_t(1), \cdots, R_t(n))'$ and $R_t(i) = r_t(i) - r_t^0$.

Assumption 4.2 *For arbitrary $A_{t-1}^j \in \mathbf{f}_{t-1}$, the matrix of the securities' prices is given as follows,*

$$\widehat{D}_t(A_{t-1}^j) = \begin{pmatrix} B_t & S_t(1, A_t^{(n+1)(j-1)+1}) & \cdots & S_t(n, A_t^{(n+1)(j-1)+1}) \\ \vdots & \vdots & \vdots & \vdots \\ B_t & S_t(1, A_t^{(n+1)j}) & \cdots & S_t(n, A_t^{(n+1)j}) \end{pmatrix},$$

with $rank(\widehat{D}_t(A_{t-1}^j)) = n+1$.

Assumption 4.2 actually shows that the number of market states equals the number of independent vectors in the set of instruments' prices.

Definition 4.2 *A market which satisfies both Assumption 4.1 and Assumption 4.2 is called a pseudo-complete market.*

We consider an investor in the financial market with initial wealth v_0. She follows a self-financing trading strategies $\pi = \{\pi_t; t = 0, 1, \cdots, T-1\}$, where $\pi_t = (\pi_t(1), \cdots, \pi_t(n))'$ and $\pi_t(i)$ is the amount of money invested in ith risky security at the beginning time t. Let V_t be the value of portfolio at the beginning of time t. The amount of money invested in the bond at the beginning of time t is $V_t - \sum_{i+1}^{n} \pi_t(i)$. So the wealth process satisfies

$$V_{t+1} = V_t r_t^0 + \pi_t' R_t. \tag{4.1}$$

Definition 4.3 *A financial market is said to admit no arbitrage, if for all $t \in \{1, 2, \cdots, T\}$ and all $\mathcal{F}_t$-measurable portfolios π_t, $V_t(r_t^0 - 1) + \pi_t' R_t \geq 0$ P-a.s. implies $V_t(r_t^0 - 1) + \pi_t' R_t = 0$ P-a.s.*

Assumption 4.3 *The financial market is arbitrage-free in the sense of Definition 4.3.*

Assumption 4.1, Assumption 4.2 and Assumption 4.3 make the security market a complete one. In this financial market model, short-selling is prohibited, so we further give the following assumption.

Assumption 4.4 *The investor invests her wealth in the complete market with no short-selling constraint, that is*

$$\pi_t \geq \mathbf{0} \quad for \quad t = 0, 1, \cdots, T-1. \tag{4.2}$$

For our convenience, we introduce the discounted price process $S^* = \{S_t^*; 0 \leq t \leq T\}$ with $S_t^* = (S_t^*(1), \cdots, S_t^*(n))'$ and $S_t^*(i) = S_t(i)/B_t$. The change of the discounted prices of risky security is defined as $\delta_t(i) = (\delta_t(1), \cdots, \delta_t(n))'$ with $\delta_t(i) = S_t^*(i) - S_{t-1}^*(i)$.

4.3. Problem formulation

The multi-period portfolio optimization problem under the mean-variance framework in this chapter can be formulated as follows:

$$\begin{cases} \max & E(V_T) - \omega Var(V_T) \\ s.t. & (4.1) - (4.2), \end{cases}$$

for $\omega \geq 0$. Varying the value of ω yields the set of efficient solutions.

As indicated in Li and Ng (2000), the above problem is difficult to be solved directly because of the non-separability in the sense of dynamic programming. In Li and Ng(2000), the relation between the multi-period mean-variance portfolio selection problem with a fixed investment horizon and a separable portfolio selection problem with a quadratic utility function is investigated, and the analytical solution is derived by using an embedding scheme. Theorems 1 and 2 in Li and Ng (2000) can be also applied to the current model with no-short-selling constraint. We now consider the following auxiliary problem:

$$\begin{cases} \max & E(\lambda V_T - \omega V_T^2) \\ s.t. & (4.1) - (4.2). \end{cases}$$

The objective function of the auxiliary problem is equivalent to the quadratic utility function $U(x) = \beta x - \frac{1}{2}x^2$ with $\beta = \frac{\lambda}{2\omega} > 0$, which is concave and twice continuously differentiable. Note that (1) The first order derivative of $U(x)$ is $U'(x) = \beta - x$; (2) The inverse function of $U'(x)$ is $I(y) := \beta - y$.

The optimal portfolio problem is to maximize the expectation of $U(V_T)$ under the no short selling constraint $\pi_t \in \mathbf{K} = \{\pi \in \mathbb{R}^n; \pi \geq \mathbf{0}\}$. So the constrained optimal portfolio problem is:

$$(P)\begin{cases} \max & EU(V_T) = E[\beta V_T - \frac{1}{2}(V_T)^2] \\ s.t & \pi \in \mathcal{A},\ V_0 = v_0, \end{cases}$$

where $\mathcal{A}$ denotes the set of all admissible trading strategies that belong in $\mathbf{K}$.

Since there is a no short-selling constraint in the optimal portfolio selection problem, Problem (P) is difficult to be solved directly by dynamic programming. We will invoke a solution scheme by introducing a family of unconstrained auxiliary problems.

Denote the support function $\sigma(x)$ of $\mathbf{K}$ by

$$\sigma(x) \equiv \sup_{\pi \in \mathbf{K}} (-\pi' x).$$

The effective domain of σ is the convex cone $\widetilde{\mathbf{K}} = \{x \in \mathbb{R}^n : \sigma(x) < \infty\} = \{x \in \mathbb{R}^n; x \geq \mathbf{0}\}$, and $\sigma(x) \equiv 0$ for $x \in \widetilde{\mathbf{K}}$. We introduce the predictable process $\kappa = \{\kappa_t; t = 0, 1, \cdots, T-1\}$ with $\kappa_t = (\kappa_t(1), \cdots, \kappa_t(n))' \in \widetilde{\mathbf{K}}$ for all $t \geq 0$. Let $\mathcal{N}$ denote the set of all such processes of κ. Define an auxiliary market $\mathbf{M}_\kappa$ for each $\kappa \in \mathcal{N}$ by modifying the return processes for the bond and the risky securities as:

$$\begin{aligned} r_t^0 &\rightarrow r_t^0, & t \geq 0; \\ r_t &\rightarrow r_t + \kappa_t, & t \geq 0. \end{aligned}$$

Specially, the market $\mathbf{M}_0$ with $\kappa_t = \mathbf{0}$, $\forall\, t \geq 0$ is the original market. We consider the following unconstrained optimal portfolio problem in the market $\mathbf{M}_\kappa$:

$$(P^\kappa) \begin{cases} \max & EU(V_T^\kappa) \\ s.t & V_{t+1}^\kappa = V_t^\kappa r_t^0 + \pi_t'(R_t + \kappa_t), \\ & \pi_t \in \mathbb{R}^n. \end{cases}$$

Let $J_\kappa(v_0)$ denote the corresponding optimal objective value in the market $\mathbf{M}_\kappa$.

4.4. Auxiliary markets and auxiliary problems

4.4.1. Signed martingale measure in auxiliary market

In the auxiliary market $\mathbf{M}_\kappa$, we have similar proportions as in Assumption 4.2.

Proposition 4.1 *In auxiliary market $\mathbf{M}_\kappa$, for arbitrary $A_{t-1}^j \in \mathbf{f}_{t-1}$, the matrix D^κ of the securities' prices*

$$D_t^\kappa(A_{t-1}^j) = \begin{pmatrix} B_t^\kappa & S_t^\kappa(1, A_t^{(n+1)(j-1)+1}) & \cdots & S_t^\kappa(n, A_t^{(n+1)(j-1)+1}) \\ \vdots & \vdots & \vdots & \vdots \\ B_t^\kappa & S_t^\kappa(1, A_t^{(n+1)j}) & \cdots & S_t^\kappa(n, A_t^{(n+1)j}) \end{pmatrix}$$

satisfies that $rank(D_t^\kappa(A_{t-1}^j)) = n+1$ for $t = 1, 2, \cdots, T$.

Proof. In the auxiliary market $\mathbf{M}_\kappa$, if the economy at time $t-1$ is in set A_{t-1}^j, then the prices of securities at time t satisfy

$$\begin{aligned} B_t^\kappa &= B_t, \\ S_t^\kappa(l, A_t^{i+(n+1)(j-1)}) &= \frac{S_{t-1}^\kappa(l, A_{t-1}^j)}{S_{t-1}(l, A_{t-1}^j)} S_t(l, A_t^{i+(n+1)(j-1)}) + S_{t-1}^\kappa(l, A_{t-1}^j)\kappa_{t-1}(l, A_{t-1}^j), \end{aligned}$$

for $l = 1, \cdots, n$ and $i = 1, \cdots, n+1$. Since $D_t(A_{t-1}^j)$ is of a full rank, therefore $D_t^\kappa(A_{t-1}^j)$ is still of a full rank. □

As we know, if the market is arbitrage free, then the market which satisfies Proposition 4.1 is complete. However, we can not guarantee that there does not exist arbitrage opportunity in the auxiliary market. So the auxiliary market M_κ for $\kappa \in \mathcal{N}$ is pseudo-complete. Denote the signed martingale measure Q^κ in the market $\mathbf{M}_\kappa$ as follows,

Definition 4.4 *A signed martingale measure is a measure Q such that*

1. $\forall A_i \in \mathcal{F}, i = 1, 2\cdots$, *if* $A_i \cap A_j = \emptyset$, $i \neq j$, *then* $Q(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Q(A_i)$;

2. $\sum_{\omega \in \Omega} Q(\omega) = 1$;

3. *The discounted price satisfies* $E[L \cdot S_{t+1}^*(i)|\mathcal{F}_t] = S_t^*(i)$, $\forall\, t = 0, 1, \cdots, T-1$,

where $L(\omega) = Q(\omega)/P(\omega)$ *denotes the state price density.*

Remark 4.5 *Signed martingale measure could be negative. When* $Q(\omega) > 0$ *for all* $\omega \in \Omega$, *the market will have no arbitrage opportunity, and the signed martingale measure becomes an equivalent risk neutral probability.*

Theorem 4.1 *In pseudo-complete market* M_κ, *there exits a unique signed martingale measure.*

Proof. For arbitrary $\omega_i \in \Omega$, there exist some $A_t^i \in \mathbf{f}_t$ for $t = 0, 1, \cdots, T$, such that $\{\omega_i\} = A_T^{i_T} \subset A_{T-1}^{i_{T-1}} \subset \cdots \subset A_1^{i_1} \subset A_0^{i_0} = \Omega$. Suppose that $Q(\omega_i)$ is the signed martingale measure for ω_i. We define the conditional signed martingale measure as $Q_t(A_t^{i_t}) := Q(A_t^{i_t}|A_{t-1}^{i_{t-1}}) = \frac{Q(A_t^{i_t})}{Q(A_{t-1}^{i_{t-1}})}$. Notice that $Q(\omega_i) = Q(A_T^{i_T})$ and $Q(A_0^{i_0}) = 1$. Therefore,

$$Q(\omega_i) = \Pi_{t=1}^{T} Q_t(A_t^{i_t}).$$

If we can prove that $Q_t(A_t^{i_t})$ is unique, the signed martingale measure $Q(\omega_i)$ is unique too.

For arbitrary $A_t^j \in \mathbf{f}_t$, there exist $n+1$ basic elements A_{t+1}^l $(l \in I_t^j)$. So

$$\sum_{l \in I_t^j} Q_{t+1}(A_{t+1}^l) = \frac{Q(A_{t+1}^{(n+1)(j-1)+1}) + \cdots + Q(A_{t+1}^{(n+1)j})}{Q(A_t^j)} = \frac{Q(A_t^j)}{Q(A_t^j)} = 1. \quad (4.3)$$

From the definition of the signed martingale measure, we know that

$$\sum_{l \in I_t^j} S_{t+1}^{\kappa *}(i, A_{t+1}^l) Q_{t+1}(A_{t+1}^l) = S_t^{\kappa *}(i, A_t^j),\ \forall\, i = 1, 2, \cdots, n. \quad (4.4)$$

With Proposition 4.4 that $D^{\kappa}(A_t^j)$ is of a full rank, we can derive unique $Q_{t+1}(A_{t+1}^l)$ for $l \in I_t^j$ from (4.3) and (4.4). Hence the signed martingale measure is unique. □

Definition 4.5 *For arbitrary $A_{t-1}^\iota \in \mathbf{f}_{t-1}$, we defined the matrix of the price change at time t at the original market as*

$$D_t(A_{t-1}^\iota) = \begin{pmatrix} \delta_t(1, A_t^{(n+1)(\iota-1)+1}) & \delta_t(1, A_t^{(n+1)(\iota-1)+2}) & \cdots & \delta_t(1, A_t^{(n+1)\iota}) \\ \vdots & \vdots & \vdots & \vdots \\ \delta_t(n, A_t^{(n+1)(\iota-1)+1}) & \delta_t(n, A_t^{(n+1)(\iota-1)+2}) & \cdots & \delta_t(n, A_t^{(n+1)\iota}) \end{pmatrix},$$

for $t = 1, 2, \cdots T$, where $\delta_t(j, A_t^i)$ is the change of discounted price of jth security at time t when samples are in set A_t^i.

Remark 4.6 *The matrix $D_t(A_{t-1}^\iota)$ is still of a full rank under Assumption 4.2.*

For our convenience, we define the following three matrices which are modifications of $D_t(A_{t-1}^\iota)$.

Definition 4.6 *Denote $\widetilde{D}^j(A_{t-1}^\iota)$ as an $n \times (n+1)$ matrix, which is resulted from $D_t(A_{t-1}^\iota)$ by replacing the row j with $\mathbf{1}_{1\times(n+1)}$, that is,*

$$\widetilde{D}_t^j(A_{t-1}^\iota) = \begin{pmatrix} \delta_t(1, A_t^{(n+1)(\iota-1)+1}) & \cdots & \delta_t(1, A_t^{(n+1)\iota}) \\ \vdots & \vdots & \vdots \\ \delta_t(j-1, A_t^{(n+1)(\iota-1)+1}) & \cdots & \delta_t(j-1, A_t^{(n+1)\iota}) \\ 1 & \cdots & 1 \\ \delta_t(j+1, A_t^{(n+1)(\iota-1)+1}) & \cdots & \delta_t(j+1, A_t^{(n+1)\iota}) \\ \vdots & \vdots & \vdots \\ \delta_t(n, A_t^{(n+1)(\iota-1)+1}) & \cdots & \delta_t(n, A_t^{(n+1)\iota}) \end{pmatrix}.$$

Definition 4.7 *Denote $D^j_t(A^i_{t-1})$ as an $n \times n$ matrix, which is resulted from $D_t(A^i_{t-1})$ by deleting column j, that is,*

$$D^j_t(A^i_{t-1}) = \begin{pmatrix} \cdots & \delta_t(1, A_t^{(n+1)(i-1)+j-1}) & \delta_t(1, A_t^{(n+1)(i-1)+j+1}) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \delta_t(n, A_t^{(n+1)(i-1)+j-1}) & \delta_t(n, A_t^{(n+1)(i-1)+j+1}) & \cdots \end{pmatrix}.$$

Definition 4.8 *Denote $\widetilde{D}^{l,j}_t(A^i_{t-1})$ as an $n \times n$ matrix, which is resulted from $D^l_t(A^i_{t-1})$ by replacing the row j with $\mathbf{1}_{1\times n}$, that is,*

$$\widetilde{D}^{l,j}_t(A^i_{t-1}) = \begin{pmatrix} \cdots & \delta_t(1, A_t^{(n+1)(i+1)+l-1}) & \delta_t(1, A_t^{(n+1)(i+1)+l+1}) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \delta_t(j-1, A_t^{(n+1)(i+1)+l-1}) & \delta_t(j-1, A_t^{(n+1)(i+1)+l+1}) & \cdots \\ \cdots & 1 & 1 & \cdots \\ \cdots & \delta_t(j+1, A_t^{(n+1)(i+1)+l-1}) & \delta_t(j+1, A_t^{(n+1)(i+1)+l+1}) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \delta_t(n, A_t^{(n+1)(i+1)+l-1}) & \delta_t(n, A_t^{(n+1)(i+1)+l+1}) & \cdots \end{pmatrix}.$$

The following theorem is one of the main results in this chapter. It is useful for us to derive the optimal terminal wealth to the auxiliary problems in Section 4.4.2.

Theorem 4.2 *In the pseudo-complete market $\mathbf{M}_\kappa$, there exists a unique signed martingale measure*

$$Q^\kappa(\omega_i) = \prod_{t=0}^{T-1} [a^{i_t}_{t+1}(i_{t+1}) + b^{i_t}_{t+1}(i_{t+1})'\kappa_t(i_t)], \tag{4.5}$$

with

$$a^{i_t}_{t+1}(i_{t+1}) = \frac{(-1)^m \mid D^m_{t+1}(A^{i_t}_t) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D^j_{t+1}(A^{i_t}_t) \mid}, \tag{4.6}$$

$$b^{i_t}_{t+1}(i_{t+1}) = (b^{i_t}_{t+1}(1, i_{t+1}), \cdots, b^{i_t}_{t+1}(n, i_{t+1}))', \tag{4.7}$$

$$b^{i_t}_{t+1}(j, i_{t+1}) = \frac{(-1)^m \sum_{j=1}^{n} \mid \widetilde{D}^{m,j}_{t+1}(A^{i_t}_t) \mid \frac{B_t}{B_{t+1}} S^*_t(j, A^{i_t}_t)}{\sum_{j=1}^{n+1}(-1)^j \mid D^j_{t+1}(A^{i_t}_t) \mid}, \tag{4.8}$$

where $m = i_{t+1} - i_t + 1$, $S^*_t(j, A^{i_t}_t)$ is the discount price of the jth risky security at time t when samples are in set $A^{i_t}_t$, and $\kappa_t(i_t)$ is a compact notation of $\kappa_t(A^{i_t}_t)$.

Proof. Firstly, we try to prove that the sum of $Q^{\kappa}(\omega_i)$ is equal to one. Actually, we just need to prove that the conditional signed martingale measure $Q_t^{\kappa}(A_t^{i_t}) = Q^{\kappa}(A_t^{i_t} \mid A_{t-1}^{i_{t-1}})$, for any given basic element $A_{t-1}^{i_{t-1}} \in \mathbf{f}_{t-1}$, satisfies

$$\sum_{i_t \in I_t^{i_{t-1}}} Q_t^{\kappa}(A_t^{i_t}) = 1.$$

For $A_t^{i_t}$, the conditional signed martingale measure is

$$Q_{t+1}^{\kappa}(A_{t+1}^{i_{t+1}}) = a_{t+1}^{i_t}(i_{t+1}) + b_{t+1}^{i_t}(i_{t+1})'\kappa_t(i_t),$$

with

$$\begin{aligned} a_{t+1}^{i_t}(i_{t+1}) &= \frac{(-1)^m \mid D_{t+1}^m(A_t^{i_t}) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid}, \\ b_{t+1}^{i_t}(i_{t+1}) &= (b_{t+1}^{i_t}(1, i_{t+1}), \cdots, b_{t+1}^{i_t}(n, i_{t+1}))', \end{aligned}$$

and

$$b_{t+1}^{i_t}(j, i_{t+1}) = \frac{(-1)^m \sum_{j=1}^n \mid \widetilde{D}_{t+1}^{m,j}(A_t^{i_t}) \mid \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid},$$

where $m = i_{t+1} - i_t + 1 \in \{1, 2, \cdots, n+1\}$. Hence,

$$\begin{aligned} \sum_{i_{t+1} \in I_{t+1}^{i_t}} Q_{t+1}^{\kappa}(A_{t+1}^{i_{t+1}}) &= \frac{\sum_{m=1}^{n+1}(-1)^m \mid D_{t+1}^m(A_t^{i_t}) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid} \\ &+ \frac{\sum_{m=1}^{n+1}(-1)^m \sum_{j=1}^n \mid \widetilde{D}_{t+1}^{m,j}(A_t^{i_t}) \mid \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})\kappa_t(j, A_t^{i_t})}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid}. \end{aligned}$$

Because

$$\begin{aligned} &\sum_{m=1}^{n+1}(-1)^m \sum_{j=1}^n \mid \widetilde{D}_{t+1}^{m,j}(A_t^{i_t}) \mid \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})\kappa_t(j, A_t^{i_t}) \\ = &\sum_{j=1}^n [\sum_{m=1}^{n+1}(-1)^m \mid \widetilde{D}_{t+1}^{m,j}(A_t^{i_t}) \mid \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})\kappa_t(j, A_t^{i_t})] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}[(-1) \mid \tilde{D}_{t+1}^{1,j}(A_t^{i_t}) \mid + \mid \tilde{D}_{t+1}^{2,j}(A_t^{i_t}) \mid + \cdots + (-1)^{n+1} \mid \tilde{D}_{t+1}^{n+1,j}(A_t^{i_t}) \mid] \\
&\quad \times \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})\kappa_t(j, A_t^{i_t}) \\
&= -\sum_{j=1}^{n} \mid M_t^j(A_t^{i_t}) \mid \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})\kappa_t(j, A_t^{i_t}) = 0,
\end{aligned}
$$

where

$$
M_t^j(A_t^{i_t}) = \begin{pmatrix}
1 & \cdots & 1 \\
\delta_t(1, A_{t+1}^{(n+1)(i_t-1)+1}) & \cdots & \delta_t(1, A_{t+1}^{(n+1)i_t}) \\
\vdots & \vdots & \vdots \\
\delta_t(j-1, A_{t+1}^{(n+1)(i_t-1)+1}) & \cdots & \delta_t(j-1, A_{t+1}^{(n+1)i_t}) \\
1 & \cdots & 1 \\
\delta_t(j+1, A_{t+1}^{(n+1)(i_t-1)+1}) & \cdots & \delta_t(j+1, A_{t+1}^{(n+1)i_t}) \\
\vdots & \vdots & \vdots \\
\delta_t(n, A_{t+1}^{(n+1)(i_t-1)+1}) & \cdots & \delta_t(n, A_{t+1}^{(n+1)i_t})
\end{pmatrix}. \tag{4.9}
$$

Therefore,

$$
\sum_{i_{t+1} \in I_{t+1}^{i_t}} Q_{t+1}^{\kappa}(A_{t+1}^{i_{t+1}}) = \frac{\sum_{m=1}^{n+1}(-1)^m \mid D_{t+1}^m(A_t^{i_t}) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid} = 1.
$$

We thus conclude that the sum of $Q^{\kappa}(\omega_i)$ is one.

Now we try to prove that $E[L \cdot (S_{t+1}^{\kappa *}(i) - S_t^{\kappa *}(i)) | \mathcal{F}_t] = 0, \ \forall\, t = 0, 1, \cdots, T-1.$ Consider an arbitrary time t, and arbitrary event $A_{t-1}^{\zeta} \in \mathbf{f}_{t-1}$ ($\zeta \in I_{t-1}$). For the ith ($i = 1, \cdots, n$) risky security we have

$$
\begin{aligned}
E_{Q^{\kappa}}[(S_t^{\kappa *}(i) - S_{t-1}^{\kappa *}(i)) \mid \mathcal{F}_{t-1}] &= E_{Q^{\kappa}}[\frac{S_{t-1}^{\kappa *}(i)}{S_{t-1}^*(i)}(\delta_t(i) + \frac{B_{t-1}}{B_t} S_{t-1}^*(i)\kappa_{t-1}(i)) \mid \mathcal{F}_{t-1}] \\
&= \frac{S_{t-1}^{\kappa *}(i)}{S_{t-1}^*(i)} E_{Q^{\kappa}}[\delta_t(i) + \frac{B_{t-1}}{B_t} S_{t-1}^*(i)\kappa_{t-1}(i) \mid \mathcal{F}_{t-1}].
\end{aligned}
$$

Hence for any basic element $A_{t-1}^{\varsigma} \in \mathcal{F}_{t-1}$, we have

$$E_{Q^*}[\delta_t(i) + \frac{B_{t-1}}{B_t} S_{t-1}^*(i)\kappa_{t-1}(i) \mid A_{t-1}^{\varsigma}]$$

$$= \sum_{l=(n+1)(\varsigma-1)+1}^{(n+1)\varsigma} Q_t(A_t^l)[\delta_t(i, A_t^l) + \frac{B_{t-1}}{B_t} S_{t-1}^*(i, A_{t-1}^{\varsigma})\kappa_{t-1}(i, A_{t-1}^{\varsigma})]$$

$$= \sum_{m=1}^{n+1} \frac{(-1)^m[\mid D_t^m(A_{t-1}^{\varsigma}) \mid + \sum_{j=1}^n \mid \widetilde{D}_t^{m,j}(A_{t-1}^{\varsigma}) \mid \frac{B_{t-1}}{B_t} S_{t-1}^*(j, A_{t-1}^{\varsigma})\kappa_{t-1}(j, A_{t-1}^{\varsigma})]}{\sum_{j=1}^{n+1}(-1)^j \mid D_t^j(A_{t-1}^{\varsigma}) \mid}$$

$$\times [\delta_t(i, A_t^{(n+1)(\varsigma-1)+m}) + \frac{B_{t-1}}{B_t} S_{t-1}^*(i, A_{t-1}^{\varsigma})\kappa_{t-1}(i, A_{t-1}^{\varsigma})]$$

$$= \frac{\sum_{l=1}^{n+1}(-1)^l \mid D_t^l(A_{t-1}^{\varsigma}) \mid \delta_t(i, A_t^{(n+1)(\varsigma-1)+l})}{\sum_{j=1}^{n+1}(-1)^j \mid D_t^j(A_{t-1}^{\varsigma}) \mid} \tag{4.10}$$

$$+ \frac{\sum_{l=1}^{n+1}(-1)^l \sum_{j\neq i} \mid \widetilde{D}_t^{l,j}(A_{t-1}^{\varsigma}) \mid \frac{B_{t-1}}{B_t} S_{t-1}^*(j,\zeta)\kappa_{t-1}(j,\zeta)\delta_t(i, A_t^{(n+1)(\varsigma-1)+l})}{\sum_{j=1}^{n+1}(-1)^j \mid D_t^j(A_{t-1}^{\varsigma}) \mid} \tag{4.11}$$

$$+ \frac{\sum_{l=1}^{n+1}(-1)^l \mid \widetilde{D}_t^{l,i}(A_{t-1}^{\varsigma}) \mid \delta_t(i, A_t^{(n+1)(\varsigma-1)+l}) + \sum_{l=1}^{n+1}(-1)^l \mid D_t^l(A_{t-1}^{\varsigma}) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D_t^j(A_{t-1}^{\varsigma}) \mid}$$

$$\times \frac{B_{t-1}}{B_t} S_{t-1}^*(i, A_{t-1}^{\varsigma})\kappa_{t-1}(i, A_{t-1}^{\varsigma}) \tag{4.12}$$

$$+ \frac{\sum_{l=1}^{n+1}(-1)^l \sum_{j=1}^{n} \mid \widetilde{D}_t^{l,j}(A_{t-1}^{\varsigma}) \mid \frac{B_{t-1}}{B_t} S_{t-1}^*(j, A_{t-1}^{\varsigma})\kappa_{t-1}(j, A_{t-1}^{\varsigma})}{\sum_{j=1}^{n+1}(-1)^j \mid D_t^j(A_{t-1}^{\varsigma}) \mid}$$

$$\times \frac{B_{t-1}}{B_t} S_{t-1}^*(i, A_{t-1}^{\varsigma})\kappa_{t-1}(i, A_{t-1}^{\varsigma}). \tag{4.13}$$

There are four parts of the above formulation. We will calculate them separately.

For part (4.10), we have

$$\sum_{l=1}^{n+1}(-1)^l \mid D_t^l(A_{t-1}^{\varsigma}) \mid \delta_t(i, A_t^{(n+1)(\varsigma-1)+l}) = -\left| \begin{pmatrix} C_t^{\varsigma} \\ D_t(A_{t-1}^{\varsigma}) \end{pmatrix} \right| = 0,$$

where C_t^{ς} is the ith row of $D_t(A_{t-1}^{\varsigma})$.

For part (4.11), because for $j \neq i$,

$$\sum_{l=1}^{n+1}(-1)^l \mid \widetilde{D}_t^{l,j}(A_{t-1}^\varsigma) \mid \frac{B_{t-1}}{B_t} S_{t-1}^*(j, A_{t-1}^\varsigma)\kappa_{t-1}(j, A_{t-1}^\varsigma)\delta_t(i, A_t^{(n+1)(\varsigma-1)+l})$$

$$= -\left| \begin{pmatrix} C_t^\varsigma \\ \widetilde{D}_t^j(A_{t-1}^\varsigma) \end{pmatrix} \right| \frac{B_{t-1}}{B_t} S_{t-1}^*(j, A_{t-1}^\varsigma)\kappa_{t-1}(j, A_{t-1}^\varsigma) = 0,$$

we have

$$\sum_{l=1}^{n+1}\sum_{j\neq i}(-1)^l \mid \widetilde{D}_t^{l,j}(A_{t-1}^\varsigma) \mid \frac{B_{t-1}}{B_t} S_{t-1}^*(j, A_{t-1}^\varsigma)\kappa_{t-1}(j, A_{t-1}^\varsigma)\delta_t(i, A_t^{(n+1)(\varsigma-1)+l}) = 0.$$

For part (4.12), we have

$$\sum_{l=1}^{n+1}(-1)^l \mid \widetilde{D}_t^{l,i}(A_{t-1}^\varsigma) \mid \delta_t(i, A_t^{(n+1)(\varsigma-1)+l}) + \sum_{l=1}^{n+1}(-1)^l \mid D_t^l(A_{t-1}^\varsigma) \mid$$

$$= (-1)\left| \begin{pmatrix} C_t^\varsigma \\ \widetilde{D}_t^i(A_{t-1}^\varsigma) \end{pmatrix} \right| + (-1)\left| \begin{pmatrix} \mathbf{1}_{1\times(n+1)} \\ D_t(A_{t-1}^\varsigma) \end{pmatrix} \right|$$

$$= (-1)\left| \begin{pmatrix} C_t^\varsigma \\ \widetilde{D}_t^i(A_{t-1}^\varsigma) \end{pmatrix} \right| + (-1)^2\left| \begin{pmatrix} C_t^\varsigma \\ \widetilde{D}_t^i(A_{t-1}^\varsigma) \end{pmatrix} \right| = 0.$$

For part (4.13), we have

$$\sum_{l=1}^{n+1}(-1)^l \sum_{j=1}^{n} \mid \widetilde{D}_t^{l,j}(A_{t-1}^\varsigma) \mid \frac{B_{t-1}}{B_t} S_{t-1}^*(j, A_{t-1}^\varsigma)\kappa_{t-1}(j, A_{t-1}^\varsigma)$$

$$= -\sum_{j=1}^{n} \mid M_t^j(A_{t-1}^\varsigma) \mid \frac{B_{t-1}}{B_t} S_{t-1}^*(j, A_{t-1}^\varsigma)\kappa_{t-1}(j, A_{t-1}^\varsigma) = 0,$$

where $M_t^j(A_{t-1}^\varsigma)$ is defined in (4.9).

Hence,

$$E_Q[\delta_t(i) + \frac{B_{t-1}}{B_t} S_{t-1}^*(i)\kappa_{t-1}(i) \mid \mathcal{F}_{t-1}] = 0, \quad \forall i,\ t.$$

So Q^κ is a signed martingale measure of the auxiliary market. □

Proposition 4.2 *If there is no arbitrage opportunity in the market* $\mathbf{M}_\kappa$, *the pseudo-complete market* $\mathbf{M}_\kappa$ *becomes complete. Furthermore, the unique signed martingale measure satisfies* $Q^\kappa > 0$, *and becomes the risk neutral probability.*

4.4.2. Optimal terminal wealth to auxiliary problems

Now we try to solve the auxiliary problems

$$(P^\kappa)\begin{cases} \max & EU(V_T^\kappa) \\ s.t & V_0 = v_0. \end{cases}$$

Notice that the expected discounted terminal wealth based on the risk neutral probability is equal to the initial wealth, i.e.

$$E_{Q^\kappa}(V_T^\kappa/B_T) = v_0.$$

So (P^κ) is equivalent to the following problem,

$$\begin{cases} \max & EU(V_T^\kappa) \\ s.t & E_{Q^\kappa}(V_T^\kappa/B_T) = v_0 \end{cases}$$

In the previous subsection, we have already formulated the risk neutral probability Q^κ. Therefore, we can derive the optimal terminal wealth to the auxiliary problems by using martingale approach.

Theorem 4.3 *For problem* (P^κ) *with quadratic utility function, the optimal attainable wealth is:*

$$V_T^{\kappa*} = \beta + [\frac{v_0 - \beta E[L^\kappa/B_T]}{E[(L^\kappa)^2/(B_T)^2]}]L^\kappa/B_T,$$

and the optimal objective value is

$$EU(V_T^{\kappa*}) = \frac{1}{2}\beta^2 - \frac{(v_0 - \beta E[L^\kappa/B_T])^2}{2E[(L^\kappa)^2/(B_T)^2]}.$$

Proof. The Lagrangian dual problem can be written as

$$\begin{aligned}
& EU(V_T^\kappa) - \lambda E_{Q^\kappa}(V_T^\kappa) \\
= \; & EU(V_T^\kappa) - \lambda E(V_T^\kappa L^\kappa / B_T) \\
= \; & \sum_{i=1}^{(n+1)^T} P(\omega_i)[U(V_T^\kappa(\omega_i)) - \lambda V_T^\kappa(\omega_i) L^\kappa(\omega_i)/B_T].
\end{aligned}$$

If V_T maximizes (P^κ), then the following necessary condition must be satisfied according to Pliska (1997):

$$U'(V_T^\kappa(\omega)) = \lambda L^\kappa(\omega)/B_T, \quad for \;\; all \;\; \omega \in \Omega,$$

which is equivalent to

$$V_T^\kappa(\omega) = I(\lambda L^\kappa(\omega)/B_T) = \beta - \lambda L^\kappa(\omega)/B_T, \quad for \;\; all \;\; \omega \in \Omega.$$

The value of the parameter λ is the one that makes V_T^κ satisfies $E_{Q^\kappa}(V_T^{\kappa *}) = v_0$. Hence

$$\begin{aligned}
E_{Q^\kappa}((\beta - \lambda L^\kappa/B_T)/B_T) &= E((\beta - \lambda L^\kappa/B_T)L^\kappa/B_T) \\
&= \beta E(L^\kappa/B_T) - \lambda E((L^\kappa)^2/B_T^2) \\
&= v_0.
\end{aligned}$$

Therefore, $\lambda = [\beta E(L^\kappa/B_T) - v_0]/E((L^\kappa)^2/B_T^2)$. Hence, we have

$$V_T^{\kappa *} = \beta + [\frac{v_0 - \beta E[L^\kappa/B_T]}{E[(L^\kappa)^2/B_T^2]}]L^\kappa/B_T, \quad all \;\; \omega \in \Omega,$$

and the optimal objective value is

$$J_\kappa(v_0) = \frac{1}{2}\beta^2 - \frac{(v_0 - \beta E[L^\kappa/B_T])^2}{2E[(L^\kappa)^2/B_T^2]}.$$

□

4.5. Dual problem and its optimal solution

4.5.1. Dual problem and strong duality theory

The following theorem from Pliska [46] shows the relationship between the original constrained problem and its dual problem.

Theorem 4.4 *Suppose that J_0^* is the optimal solution of the primal constrained problem, and J^* is the optimal solution of the dual problem*

$$(D)\ \min_{\kappa \in \mathcal{N}} J_\kappa(v_0),$$

where $J_\kappa(v_0)$ is the optimal objective value in the unconstrained market $\mathbf{M}_\kappa$, associated with the optimal solution κ^*. If the optimal trading strategy π for the unconstrained market $\mathbf{M}_{\kappa^*}$ satisfies

(a) $\pi_t \in \mathbf{K}$,

(b) $\pi_t' \kappa_t^* = 0$, *for all* $t \geq 0$.

Then π is the optimal strategy for the original constrained market, and $J_0 = J^*$.

Actually, for the market $\mathbf{M}_{\kappa^*}$ and the optimal trading strategy π which satisfies (a) and (b), the value of portfolio at time T is

$$\begin{aligned} V_T^{\kappa^*}(\pi) &= v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t'(R_t + \kappa_t^*) \prod_{i=t+1}^{T-1} r_i^0 \\ &= v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t' R_t \prod_{i=t+1}^{T-1} r_i^0 + \sum_{t=0}^{T-1} \pi_t' \kappa_t^* \prod_{i=t+1}^{T-1} r_i^0 \\ &= v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t' R_t \prod_{i=t+1}^{T-1} r_i^0 = V_T^0(\pi). \end{aligned}$$

As π is a feasible solution of the original constrained problem, the expected utility of $V_T^0(\pi)$ is smaller than or equal to the optimal value of the original constrained problem. So we have $J^* = EU(V_T^{\kappa^*}(\pi)) = EU(V_T^0(\pi)) \leq J_0^*$.

On the overhand, for an arbitrary market $\mathbf{M}_\kappa$ and the optimal trading strategy π^* of the original constrained portfolio problem, we have,

$$\begin{aligned} V_T^\kappa(\pi^*) &= v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t^{*'}(R_t + \kappa_t) \prod_{i=t+1}^{T-1} r_i^0 \\ &= v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t^{*'} R_t \prod_{i=t+1}^{T-1} r_i^0 + \sum_{t=0}^{T-1} \pi_t^{*'} \kappa_t \prod_{i=t+1}^{T-1} r_i^0 \\ &\geq \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t^{*'} R_t \prod_{i=t+1}^{T-1} r_i^0 = V_T^0(\pi^*). \end{aligned}$$

Therefore, $J_0^* = EU(V_T^0(\pi^*)) \leq EU(V_T^\kappa(\pi^*)) \leq J_\kappa(v_0)$ for any κ. Hence, $J_0^* \leq J_{\kappa^*}(v_0) = J^*$.

Combining the above two inequalities gives rise $J^* = J_0^*$.

Theorem 4.4 tells us that the optimal value of the dual problem could be equal to the optimal value of primal constrained problem when conditions (a) and (b) are satisfied. So we would like to know whether these conditions are always satisfied or not. The following theorem will answer this question.

Theorem 4.5 *If problem (P) is solvable, then there exists κ^* and π^* such that*

(a) $\pi_t^* \in \mathbf{K}$, $\kappa_t^* \in \widetilde{\mathbf{K}}$ and π^* solves (P^{κ^*});

(b) $\kappa_t^{*'} \pi_t^* = 0$.

Proof. Under Assumption 4.1, we know that there exist $m := (n+1)^t$ elements $A_t^1, \cdots, A_t^m$, such that $A_t^1 \bigcup \cdots \bigcup A_t^m = \Omega$, $A_t^i \cap A_t^j \; \forall i \neq j$, and $\mathcal{F}_t = \sigma(A_t^1, \cdots, A_t^m)$. Let us consider the Lagrangian dual of problem (P),

$$(D_L) \quad \left\{ \begin{array}{ll} \min_{\lambda \geq 0} & \max_{\pi \in \mathbb{R}^n} EU(V_T) + \sum_{t=0}^{T-1} \sum_i^{(n+1)^t} E[\lambda_t' \pi_t | A_t^i] \\ s.t. & V_T = v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} R_t' \pi_t \prod_{i=t+1}^{T-1} r_i^0, \end{array} \right. \tag{4.14}$$

where π_t and λ_t are $\mathcal{F}_t$-measurable (hence $E(\lambda_t'\pi_t|A_t^i)$ is constant vector for $A_t^i \in \mathcal{F}_t$), $E(\lambda_t|A_t^i) \in \mathbb{R}_+^n$ for $t = 0, 1, \cdots, T-1$ and $A_t^i \in \mathcal{F}_t$.

Simply speaking, the above objective function is a concave utility function of the following variables,

$$\pi_0(A_0), \cdots, \pi_t(A_t^1), \cdots, \pi_t(A_t^{(n+1)^t}), \cdots, \pi_{T-1}(A_{T-1}^1), \cdots, \pi_{T-1}(A_{T-1}^{(n+1)^{T-1}}).$$

Correspondingly, the Lagrangian parameters are

$$\lambda_0(A_0), \cdots, \lambda_t(A_t^1), \cdots, \lambda_t(A_t^{(n+1)^t}), \cdots, \lambda_{T-1}(A_{T-1}^1), \cdots, \lambda_{T-1}(A_{T-1}^{(n+1)^{T-1}}).$$

As problem (P) is strictly concave, there is no duality gap between problems (P) and (D_L) from strong duality. Furthermore, a pair $\{(\pi_t^*, \lambda_t^*)\}$ satisfying

$$\frac{1}{(n+1)^t} E[U'(v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t^{*\prime} R_t \prod_{i=t+1}^{T-1} r_i^0) R_t \prod_{i=t+1}^{T-1} r_i^0 | \mathcal{F}_t] + \lambda_t^* = 0, \tag{4.15}$$

$$\lambda_t^{*\prime} \pi_t^* = 0, \ for\ t = 0, 1, \cdots, T-1, \tag{4.16}$$

for each realization $(\pi_t^*(A_t^i), \lambda_t^*(A_t^i))$, solves both the primal and the Lagrangian dual problems.

For the pair (π^*, λ^*) determined by (4.15) and (4.16), let

$$\kappa_t^* = \frac{(n+1)^t \lambda_t^*}{E[U'(v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t^{*\prime} R_t \prod_{i=t+1}^{T-1} r_i^0) \prod_{i=t+1}^{T-1} r_i^0 | \mathcal{F}_t]}. \tag{4.17}$$

The fact of $\pi^* \in \mathbf{K}$ is from the strong duality between the primal problem (P) and the Lagrangian dual (D_L). As

$$\begin{aligned}
\delta(\kappa_t^*) &= \sup_{\pi_t \in \mathbf{K}} -\pi_t' \kappa_t^* \\
&= \sup_{\pi_t \in \mathbf{K}} -\frac{(n+1)^t \pi_t' \lambda_t^*}{E[U'(v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t^{*\prime} R_t \prod_{i=t+1}^{T-1} r_i^0) \prod_{i=t+1}^{T-1} r_i^0 | \mathcal{F}_t]} \\
&\le -\frac{0}{E[U'(v_0 \prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1} \pi_t^{*\prime} R_t \prod_{i=t+1}^{T-1} r_i^0) \prod_{i=t+1}^{T-1} r_i^0 | \mathcal{F}_t]} \\
&< \infty,
\end{aligned}$$

we have $\kappa_t^* \in \widetilde{\mathbf{K}}$.

The following is clear from (4.16),

$$\begin{aligned}\kappa_t^{*'}\pi_t^* &= \frac{(n+1)^t\pi_t^{*'}\lambda_t^*}{E[U'(v_0\prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1}\pi_t^{*'}R_t\prod_{i=t+1}^{T-1} r_i^0)\prod_{i=t+1}^{T-1} r_i^0|\mathcal{F}_t]}\\ &= \frac{0}{E[U'(v_0\prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1}\pi_t^{*'}R_t\prod_{i=t+1}^{T-1} r_i^0)\prod_{i=t+1}^{T-1} r_i^0|\mathcal{F}_t]}\\ &= 0.\end{aligned}$$

Substituting π^* into the optimality condition of problem (P^{κ^*}),

$$E[U'(v_0\prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1}\pi_t^{*'}(R_t+\kappa_t^*)\prod_{i=t+1}^{T-1} r_i^0)(R_t+\kappa_t^*)\prod_{i=t+1}^{T-1} r_i^0|\mathcal{F}_t] = 0$$

yields the following by using (4.15) and (4.17),

$$\begin{aligned}& E[U'(v_0\prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1}\pi_t^{*'}(R_t+\kappa_t^*)\prod_{i=t+1}^{T-1} r_i^0)(R_t+\kappa_t^*)\prod_{i=t+1}^{T-1} r_i^0|\mathcal{F}_t]\\ =\ & E[U'(v_0\prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1}\pi_t^{*'}R_t\prod_{i=t+1}^{T-1} r_i^0)(R_t+\kappa_t^*)\prod_{i=t+1}^{T-1} r_i^0|\mathcal{F}_t]\\ =\ & E[U'(v_0\prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1}\pi_t^{*'}R_t\prod_{i=t+1}^{T-1} r_i^0)R_t\prod_{i=t+1}^{T-1} r_i^0|\mathcal{F}_t]\\ & +E[U'(v_0\prod_{t=0}^{T-1} r_t^0 + \sum_{t=0}^{T-1}\pi_t^{*'}R_t\prod_{i=t+1}^{T-1} r_i^0)\kappa_t^*\prod_{i=t+1}^{T-1} r_i^0|\mathcal{F}_t]\\ =\ & -(n+1)^t\lambda_t^* + (n+1)^t\lambda_t^*\\ =\ & 0.\end{aligned}$$

Therefore, π^* solves (P^{κ^*}). □

Based on Theorem 4.5, we can get the optimal solution to the primal problem (P) by solving the dual problem (D_L). Actually, dual problem (D_L) is much easier to be solved than the primal problem. In the following subsection, we will derive an analytical solution to the dual problem.

4.5.2. Optimal solution to dual problem

Now we come back to the dual problem (D_L). Since we have already derived the optimal terminal wealth to the auxiliary problems, i.e. $J_\kappa(v_0)$, we can reformulate the dual problem as,

$$\min_{\kappa\in\mathcal{N}} J_\kappa(v_0) = \frac{1}{2}\beta^2 - \frac{(v_0 - \beta E[L^\kappa/B_T])^2}{2E[(L^\kappa)^2/B_T^2]}.$$

Since $E[L^\kappa/B_T] = E[L^\kappa]/B_T = 1/B_T$ and $E[(L^\kappa)^2/B_T^2] = E[(L^\kappa)^2]/B_T^2$, the dual problem is equivalent to

$$\min_{\kappa\in\mathcal{N}} \sum_{i=1}^{(n+1)^T} Q^2(\omega_i),$$

i.e.,

$$\min_{\kappa\in\mathcal{N}} \sum_{(i_0,\cdots,i_T)\in\mathbb{I}_0} \prod_{t=0}^{T-1} [a_{t+1}^{i_t}(i_{t+1}) + (b_{t+1}^{i_t}(i_{t+1}))'\kappa_t(i_t)]^2,$$

with

$$\begin{aligned} a_{t+1}^{i_t}(i_{t+1}) &= \frac{(-1)^m \mid D_{t+1}^m(A_t^{i_t}) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid}, \\ b_{t+1}^{i_t}(i_{t+1}) &= (b_{t+1}^{i_t}(1, i_{t+1}), \cdots, b_{t+1}^{i_t}(n, i_{t+1}))', \end{aligned}$$

and

$$b_{t+1}^{i_t}(j, i_{t+1}) = \frac{(-1)^m \sum_{j=1}^n \mid \widetilde{D}_{t+1}^{m,j}(A_t^{i_t}) \mid \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid},$$

where $m = i_{t+1} - i_t + 1$.

The following special properties in the objective function facilitate the solution scheme for the dual problem. Denote

$$\varphi_\tau(i) := \sum_{(i_\tau, i_{\tau+1},\cdots,i_T)\in\mathbb{I}_\tau^i} \prod_{t=\tau}^{T-1} [a_{t+1}^{i_t}(i_{t+1}) + (b_{t+1}^{i_t}(i_{t+1}))'\kappa_t(i_t)]^2.$$

Proposition 4.3 *Under Assumption 4.1, $\varphi_\tau(i)$ satisfies the following recursion,*

$$\begin{aligned}
\varphi_{\tau-1}(l) &= \sum_{(i_{\tau-1},i_{\tau+1},\cdots,i_T)\in \mathbb{I}^l_{\tau-1}} \prod_{t=\tau-1}^{T-1} [a^{i_t}_{t+1}(i_{t+1}) + (b^{i_t}_{t+1}(i_{t+1}))'\kappa_t(i_t)]^2 \\
&= \sum_{i=(n+1)(l-1)+1}^{(n+1)l} \sum_{\substack{i_{\tau-1}=l,\\ (i_\tau,i_{\tau+1},\cdots,i_T)\in \mathbb{I}^i_\tau}} \prod_{t=\tau-1}^{T-1} [a^{i_t}_{t+1}(i_{t+1}) + (b^{i_t}_{t+1}(i_{t+1}))'\kappa_t(i_t)]^2 \\
&= \sum_{i=(n+1)(l-1)+1}^{(n+1)l} [a^l_\tau(i) + (b^l_\tau(i))'\kappa_t(l)]^2 \varphi_\tau(i),
\end{aligned}$$

and the boundary condition is,

$$\varphi_0 : = \sum_{(i_0,\cdots,i_T)\in \mathbb{I}_0} \prod_{t=0}^{T-1} [a^{i_t}_{t+1}(i_{t+1}) + (b^{i_t}_{t+1}(i_{t+1}))'\kappa_t(i_t)]^2.$$

So we have

$$\begin{aligned}
\varphi_0 : &= \sum_{(i_0,\cdots,i_T)\in \mathbb{I}_0} \prod_{t=0}^{T-1} [a^{i_t}_{t+1}(i_{t+1}) + b^{i_t}_{t+1}(i_{t+1})'\kappa_t(i_t)]^2 \\
&= \sum_{l_1\in I^1_1} [a^1_1(l_1) + b^1_1(l_1)'\kappa_0(1)]^2 \{ \sum_{l_2\in I^{l_1}_2} [a^{l_1}_2(l_2) + b^{l_1}_2(l_2)'\kappa_1(l_1)]^2 \{ \sum_{l_3\in I^{l_2}_3} \cdots \\
&\quad \{ \sum_{l_T\in I^{l_{T-1}}_T} [a^{l_{T-1}}_T(l_T) + b^{l_{T-1}}_T(l_T)'\kappa_{T-1}(l_{T-1})]^2 \}\}\}.
\end{aligned}$$

The dual problem is separable and can be solved by dynamic programming.

Theorem 4.6 *If $[\sum_{i=(n+1)(m-1)+1}^{(n+1)m} \varphi_{t+1}(i)\cdot b^m_{t+1}(i) b^m_{t+1}(i)']$ is of a full rank for any t and $m \in I_t$, then the optimal solution of the dual problem is*

$$\kappa^*_t(m) = (\kappa^*_t(1,m),\cdots,\kappa^*_t(n,m))'$$

for $t = 0,1,2,\cdots,T-1$ and $m = 1,2,\cdots,(n+1)^t \in I_t$, with

$$\kappa^*_t(j,m) = \begin{cases} \widetilde{\kappa}_t(j,m) & if \;\; \widetilde{\kappa}_t(j,m) \geq 0 \\ 0 & if \;\; \widetilde{\kappa}_t(j,m) < 0 \end{cases}$$

where $\widetilde{\kappa}_t(m) = (\widetilde{\kappa}_t(1,m), \cdots, \widetilde{\kappa}_t(n,m))'$ *is given by*

$$\begin{aligned}
\widetilde{\kappa}_t(m) &= -\Big[\sum_{i=(n+1)(m-1)+1}^{(n+1)m} \varphi^*_{t+1}(i) \cdot b^m_{t+1}(i) b^m_{t+1}(i)'\Big]^{-1} \\
&\quad \times \Big[\sum_{i=(n+1)(m-1)+1}^{(n+1)m} \varphi^*_{t+1}(i) \cdot b^m_{t+1}(i) \cdot a^m_{t+1}(i)\Big], \\
\varphi^*_{t+1}(m) &= \sum_{i=(n+1)(m-1)+1}^{(n+1)m} [a^m_{t+2}(i) + b^m_{t+2}(i)'\kappa^*_{t+1}(m)]^2 \varphi^*_{t+2}(i).
\end{aligned}$$

Proof. Denote the cost-to-go at stage t by

$$\begin{aligned}
f_t(l_t) &= \min_{\kappa_t,\cdots,\kappa_{T-1}} \sum_{l_{t+1}\in I_t^{l_t}} [a^{l_t}_{t+1}(l_{t+1}) + b^{l_t}_{t+1}(l_{t+1})'\kappa_t(l_t)]^2 \Big\{ \sum_{l_{t+2}\in I_{t+2}^{l_{t+1}}} \cdots \\
&\quad \Big\{ \sum_{l_T \in I_T^{l_{T-1}}} [a_T^{l_{T-1}}(l_T) + b_T^{l_{T-1}}(l_T)'\kappa_{T-1}(l_{T-1})]^2 \Big\},
\end{aligned}$$

for $t = 0, 1, \cdots, T-1$. Note that $f_t(l_t)$ can be further expressed as

$$f_t(l_t) = \min_{\kappa_t} \sum_{l_{t+1}\in I_t^{l_t}} [a^{l_t}_{t+1}(l_{t+1}) + b^{l_t}_{t+1}(l_{t+1})'\kappa_t(l_t)]^2 f_{t+1}(l_{t+1}).$$

At stage $T-1$, for $l_{T-1} \in I_{T-1} = \{1, 2, \cdots, (n+1)^{T-1}\}$, we have

$$\begin{aligned}
f_{T-1}(l_{T-1}) &= \min_{\kappa_{T-1}} \varphi_{T-1}(l_{T-1}) \\
&= \min_{\kappa_{T-1}} \sum_{l_T \in I^{l_{T-1}}(T)} [a_T^{l_{T-1}}(l_T) + b_T^{l_{T-1}}(l_T)'\kappa_{T-1}(l_{T-1})]^2.
\end{aligned}$$

We solve the problem $f_{T-1}(l_{T-1})$ separately for each $l_{T-1} = m$:

$$\begin{cases} \min_{\kappa_{T-1}} & \sum_{i=(n+1)(m-1)+1}^{(n+1)m} [a^m_T(i) + b^m_T(i)'\kappa_{T-1}(m)]^2 \\ s.t & \kappa_{T-1}(m) \geq \mathbf{0}. \end{cases}$$

Consider the corresponding unconstrained problem as follows,

$$\begin{cases} \min_{\kappa_{T-1}} & \sum_{i=(n+1)(m-1)+1}^{(n+1)m} [a^m_T(i) + b^m_T(i)'\kappa_{T-1}(m)]^2 \\ s.t & \kappa_{T-1}(m) \in \mathbb{R}^n. \end{cases}$$

Solving the unconstrained problem, we obtain the optimal solution $\widetilde{\kappa}_{T-1}(m) = (\widetilde{\kappa}_{T-1}(1,m), \cdots, \widetilde{\kappa}_{T-1}(n,m))'$ which is specified as follows,

$$\widetilde{\kappa}_{T-1}(m) = -[\sum_{i=(n+1)(m-1)+1}^{(n+1)m} b_T^m(i) \cdot b_T^m(i)']^{-1} \cdot [\sum_{i=(n+1)(m-1)+1}^{(n+1)m} b_T^m(i) \cdot a_T^m(i)].$$

Therefore, the optimal solution to the constrained problem is

$$\kappa_{T-1}^*(m) = (\kappa_{T-1}^*(1,m), \cdots, \kappa_{T-1}^*(n,m))',$$

where

$$\kappa_{T-1}^*(j,m) = \begin{cases} \widetilde{\kappa}_{T-1}(j,m) & if \quad \widetilde{\kappa}_{T-1}(j,m) \geq 0, \\ 0 & if \quad \widetilde{\kappa}_{T-1}(j,m) < 0. \end{cases}$$

Hence, $\varphi_{T-1}^*(m) = \sum_{i=(n+1)(m-1)+1}^{(n+1)m} [a_T^m(i) + b_T^m(i)'\kappa_{T-1}^*(m)]^2$, and

$$\begin{aligned} f_{T-2}(l_{T-2}) &= \min_{\kappa_{T-2}} \varphi_{T-2}(l_{T-2}) \\ &= \min_{\kappa_{T-2}} \sum_{l_{T-1} \in I_{T-1}^{l_{T-2}}} [a_{T-1}^{l_{T-2}}(l_{T-1}) + b_{T-1}^{l_{T-2}}(l_{T-1})'\kappa_{T-2}(l_{T-2})]^2 \varphi_{T-1}^*(l_{T-1}). \end{aligned}$$

Assume that the utility function has a similar form at stage τ, $0 \leq \tau \leq T-2$. The cost-to-go at stage τ is

$$\begin{aligned} f_\tau(l_\tau) &= \min_{\kappa_\tau} \varphi_\tau(l_\tau) \\ &= \min_{\kappa_\tau} \sum_{l_{\tau+1} \in I_{\tau+1}^{l_\tau}} [a_{\tau+1}^{l_\tau}(l_{\tau+1}) + b_{\tau+1}^{l_\tau}(l_{\tau+1})'\kappa_\tau(l_\tau)]^2 \varphi_{\tau+1}^*(l_{\tau+1}). \end{aligned}$$

We solve the following problem separably for each $l_\tau = m \in I_\tau$:

$$\begin{cases} \min_{\kappa_\tau} & \sum_{i=(n+1)(m-1)+1}^{(n+1)m} [a_{\tau+1}^m(i) + b_{\tau+1}^m(i)'\kappa_\tau(m)]^2 \varphi_{\tau+1}^*(i) \\ s.t & \kappa_\tau(j,m) \geq 0. \end{cases}$$

The optimal solution of the corresponding unconstrained problem is denoted by

$\widetilde{\kappa}_T(m) = (\widetilde{\kappa}_T(1,m), \cdots, \widetilde{\kappa}_T(n,m))'$, where

$$\widetilde{\kappa}_T(m) = -\left[\sum_{i=(n+1)(m-1)+1}^{(n+1)m} \varphi_{T+1}^*(i) \cdot b_{T+1}^m(i) b_{T+1}^m(i)'\right]^{-1}$$
$$\times \left[\sum_{i=(n+1)(m-1)+1}^{(n+1)m} \varphi_{T+1}^*(i) \cdot b_{T+1}^m(i) \cdot a_{T+1}^m(i)\right].$$

Therefore the optimal solution of the constrained problem is $\kappa_T^*(m) = (\kappa_T^*(1,m), \cdots, \kappa_T^*(n,m))'$, where

$$\kappa_T^*(j,m) = \begin{cases} \widetilde{\kappa}_T(j,m) & if \quad \widetilde{\kappa}_T(j,m) \geq 0, \\ 0 & if \quad \widetilde{\kappa}_T(j,m) < 0, \end{cases}$$

and the cost-to-go at stage $T-1$ is

$$\begin{aligned} f_T(l_{T-1}) &= \min_{\kappa_{T-1}} \varphi_{T-1}(l_{T-1}) \\ &= \min_{\kappa_{T-1}} \sum_{l_T \in I_T^{l_T}} [a_T^{l_{T-1}}(l_T) + b_T^{l_{T-1}}(l_T)' \kappa_{T-1}(l_{T-1})]^2 \varphi_T^*(l_T), \end{aligned}$$

with

$$\varphi_T^*(m) = \sum_{i=(n+1)(m-1)+1}^{(n+1)m} [a_{T+1}^m(i) + b_{T+1}^m(i)' \kappa_T^*(m)]^2 \varphi_{T+1}^*(i).$$

□

4.6. Optimal terminal wealth of the primal problem

We have derived the optimal solution κ^* to the dual problem (D_L). The corresponding risk neutral probability is

$$Q^{\kappa^*}(\omega_i) = \prod_{t=0}^{T-1} [a_{t+1}^{i_t}(i_{t+1}) + b_{t+1}^{i_t}(i_{t+1})' \kappa_t^*(i_t)]$$

with κ^* being the optimal one given in Theorem 4.6 and

$$\begin{aligned}
a_{t+1}^{i_t}(i_{t+1}) &= \frac{(-1)^m \mid D_{t+1}^m(A_t^{i_t}) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid}, \\
b_{t+1}^{i_t}(i_{t+1}) &= (b_{t+1}^{i_t}(1, i_{t+1}), \cdots, b_{t+1}^{i_t}(n, i_{t+1}))', \\
b_{t+1}^{i_t}(j, i_{t+1}) &= \frac{(-1)^m \sum_{j=1}^{n} \mid \widetilde{D}_{t+1}^{m,j}(A_t^{i_t}) \mid \frac{B_t}{B_{t+1}} S_t^*(j, A_t^{i_t})}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid},
\end{aligned}$$

where $m = i_{t+1} - i_t + 1$.

Thus, the optimal terminal wealth is

$$\begin{aligned}
V_T^{\kappa^*}(\omega_i) &= \beta + \frac{v_0 - \beta E[L^{\kappa^*}/B_T]}{E[(L^{\kappa^*})^2/(B_T)^2]} L^{\kappa^*}(\omega_i)/B_T \\
&= \beta + \frac{v_0 B_T - \beta}{E[(L^{\kappa^*})^2]} L^{\kappa^*}(\omega_i) \\
&= \beta + \frac{1}{(n+1)^T} \frac{v_0 B_T - \beta}{E[(Q^{\kappa^*}(\omega_i)^2]} Q^{\kappa^*}(\omega_i). \qquad (4.18)
\end{aligned}$$

Furthermore, the expectation and variance of the terminal wealth are:

$$E(V_T^*) = a\beta + bv_0, \qquad (4.19)$$

$$Var(V_T^*) = c(v_0 B_T - \beta)^2, \qquad (4.20)$$

and the analytical expression of the mean-variance efficient frontier can be expressed in the following form,

$$Var(V_T^*) = c\left(\frac{aB_T + b}{a} v_0 + \frac{1}{a} E(V_T^*)\right)^2, \qquad (4.21)$$

where

$$\begin{aligned}
a &= 1 - \frac{1}{(n+1)^T} \frac{E[Q^{\kappa^*}(\omega)]}{E[(Q^{\kappa^*}(\omega)^2]}, \\
b &= \frac{1}{(n+1)^T} \frac{E[Q^{\kappa^*}(\omega)]}{E[(Q^{\kappa^*}(\omega))^2]} B_T, \\
c &= \frac{1}{(n+1)^{2T}} \frac{E[(Q^{\kappa^*}(\omega))^2] - (E[Q^{\kappa^*}(\omega)])^2}{(E[(Q^{\kappa^*}(\omega))^2])^2}.
\end{aligned}$$

It can be seen that the expected terminal wealth $E(V_T^*)$ is an increasing linear function of β whereas the variance of the terminal wealth, $Var(V_T^*)$, is a quadratic function of β. From (4.19) and (4.20), we can express the original objective function as a function of β,

$$\begin{aligned} & \widehat{U}(E((V_T^*)^2), E(V_T^*)) \\ = \; & E(V_T^*) - \omega Var(V_T^*) \\ = \; & a\beta + bv_0 - \omega[c(v_0 B_T - \beta)^2]. \end{aligned}$$

Note that $\widehat{U}$ is a concave function of β. The optimal β must satisfy the optimality condition of $\frac{d\widehat{U}}{d\beta} = 0$, i.e.,

$$\beta^* = \frac{a + 2\omega c v_0 B_T}{2\omega c}. \tag{4.22}$$

Substituting the optimal β^* in (4.22) into equation (4.18) yields the optimal terminal wealth of primal problem (P),

$$V_T^{\kappa^*}(\omega_i) = \frac{a}{2\omega c}[1 + \frac{1}{(n+1)^T}\frac{Q^{\kappa^*}(\omega_i)}{E[(Q^{\kappa^*}(\omega_i)^2]}] + v_0 B_T.$$

Example 4.1: We consider a market with one risky security and one bond, and the investment horizon of $T = 3$. The sample space Ω has $K = 8$ elements. Suppose that the bond price is constant, and the prices of the risky security are listed in Table 4.1.

The partitions are

$$\mathbf{f}_1 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\},$$

$$\mathbf{f}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}\}.$$

An investor enters this financial market with initial wealth $v_0 = 1$. She would like to maximize a mean-variance objective of the terminal wealth, $\{E(V_3) - 0.768Var(V_3)\}$, with no-short-selling constraint.

Table 4.1: Prices of risky asset in Example 4.1

ω	S_0	S_1	S_2	S_3
ω_1	5	8	9	12
ω_2	5	8	9	8
ω_3	5	8	6	8
ω_4	5	8	6	5
ω_5	5	4	6	8
ω_6	5	4	6	5
ω_7	5	4	3	5
ω_8	5	4	3	2

Firstly, the signed martingale measure in market $\mathbf{M}_\kappa$ can be calculated by using Theorem 4.2, which are showed in Table 4.2.

Table 4.2: Signed martingale measure of auxiliary market $\mathbf{M}_\kappa$ in Example 4.1

ω	$Q_\kappa(\omega)$
ω_1	$\frac{(1-5\kappa_0(1))(2-8\kappa_1(1))(1-9\kappa_2(1))}{48}$
ω_2	$\frac{(1-5\kappa_0(1))(2-8\kappa_1(1))(3+9\kappa_2(1))}{48}$
ω_3	$\frac{(1-5\kappa_0(1))(1+8\kappa_1(1))(1-6\kappa_2(2))}{36}$
ω_4	$\frac{(1-5\kappa_0(1))(1+8\kappa_1(1))(2+6\kappa_2(2))}{36}$
ω_5	$\frac{(3+5\kappa_0(1))(1-4\kappa_1(2))(1-6\kappa_2(3))}{36}$
ω_6	$\frac{(3+5\kappa_0(1))(1-4\kappa_1(2))(2+6\kappa_2(3))}{36}$
ω_7	$\frac{(3+5\kappa_0(1))(2+4\kappa_1(2))(1-3\kappa_2(4))}{36}$
ω_8	$\frac{(3+5\kappa_0(1))(2+4\kappa_1(2))(2+3\kappa_2(4))}{36}$

Let us solve the following dual problem:

$$\begin{cases} \max & \sum_{i=1}^{8} Q_\kappa(\omega_i)^2 \\ s.t & \kappa_t \geq 0 \ \ for \ \ t = 1, 2, 3. \end{cases}$$

The optimal solution of this dual problem is found to be:

$$\kappa_2(1) = \kappa_2(2) = \kappa_2(3) = \kappa_2(4) = 0,$$

$$\kappa_1(1) = \frac{5}{68}, \quad \kappa_1(2) = 0,$$

$$\kappa_0(1) = 0.$$

We thus derive the optimal parameter β by using (4.22),

$$\beta = 2, \quad a = 0.349, \quad b = 0.651, \quad c = 0.227.$$

The optimal terminal wealth can be derived by using (4.18), and they are listed in Table 4.3. The efficient frontier can be expressed as a quadratic form,

$$Var(V_3) = 1.864(1 - E(V_3))^2.$$

Finally, we can derive the optimal trading strategy of the original problem by using the wealth transition equation $V_{t+1} = V_t r_t^0 + \pi_t' R_t$. The optimal strategies are listed in Table 4.3.

Table 4.3: Optimal terminal wealth and trading strategies in Example 4.1

ω	$V_3(\omega)$	π_1	π_2	π_3
ω_1	1.846853464	1.0287	0	0.4262
ω_2	1.540560391	1.0287	0	0.4262
ω_3	1.770280196	1.0287	0	0.2841
ω_4	1.540560391	1.0287	0	0.2841
ω_5	1.566084814	1.0287	1.2144	0.6807
ω_6	1.131648721	1.0287	1.2144	0.6807
ω_7	1.131648721	1.0287	1.2144	1.569
ω_8	0.263818349	1.0287	1.2144	1.569

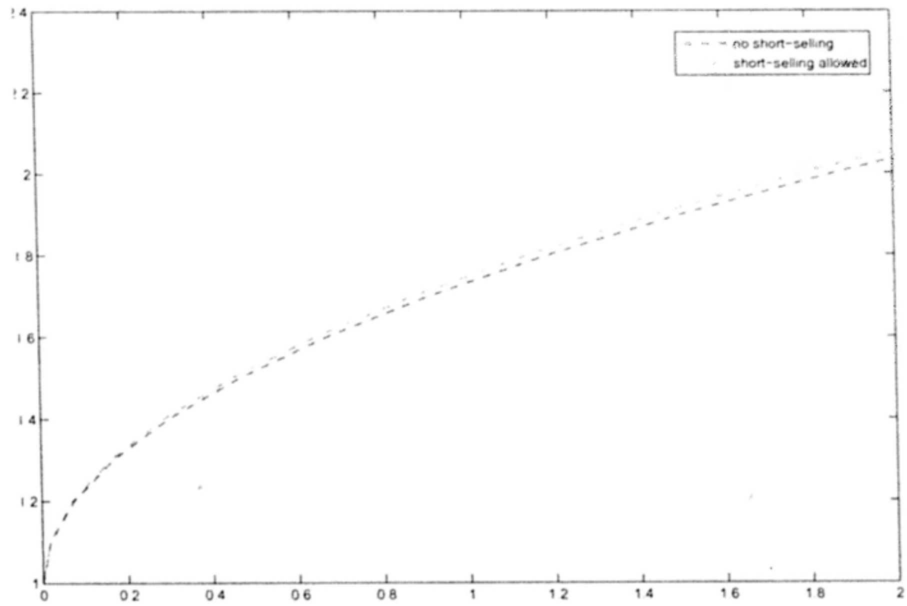

Figure 4.2: Efficient frontiers with and without shorting in Example 4.1

When short-selling is allowed, the efficient frontier can be formulated as

$$Var(V_3) = 1.79(1 - E(V_3))^2.$$

We compare the efficient frontiers with and without short-selling in Figure 4.2. It is obvious that the efficient frontier with shorting dominates that without shorting.

4.7. Summary

We study the optimal mean-variance portfolio selection problem without shorting is studied in the chapter. We connect the original mean-variance problem to an auxiliary problem by using an embedding technique. Since the auxiliary problem is difficult to solve directly, we use duality theory and martingale approach to carry out the analysis.

Since the price processes in discrete-time model can not be easily expressed by differential equations as in continuous-time model, it is difficult for us to derive the equivalent risk neutral probability. Fortunately, we can formulate the risk neutral probability after we define a discrete-time complete market under three assumptions. With this risk neutral probability, the auxiliary problem can be solved by using martingale approach. Finally, a closed form of the optimal terminal wealth is derived.

Chapter 5

Multi-Period Portfolio Selection with Transaction Costs

5.1. Introduction

Let's consider to buy an apple from a fruit market. To purchase the apple, your costs will not only include the price of the apple itself, but also include your efforts to find out your preferred store, and the cost of traveling from your house to the store and back; the costs mentioned above, which are beyond the cost of the apple itself, are called the transaction costs. In financial market, a *transaction cost* is a cost incurred in making a security exchange. For example, investors, when buying or selling a stock, must pay a commission to their broker; that commission is a transaction cost of doing a stock deal.

In the models mentioned in the previous chapters, we assumed that investors trade costless. However, transaction cost can not be ignored in practice. When investors face transaction costs in financial market, their trading strategies may become different. Magill and Constantinides [36], to our best knowledge, first introduced proportional transaction costs to Merton's continuous model. Their paper studied investors' investment behavior by maximizing total discounted utility of consumption, and analyzed the impact of transaction costs on investment. Following this paper, Davis and Norman's work [16] provided a more precise formulation and analysis for a similar problem, and suggested an algorithm of finding the optimal strategy. There are a lot of other works on continuous time portfolio selection with the transaction costs, such as [1], [19], [35], [50], [43]. After Jouini and Kallal [28] analyzed securities markets with transaction costs by using martingale and duality methods, a new methodology emerged to study portfolio selection with transaction costs. Cvitanic and Karatzas [15] used this martingale approach to analyze hedging and portfolio optimization under transaction costs.

Discrete time investment behavior with convex transaction costs was discussed by Constantinides [12]. However, not many noticeable works have been seen in this direction, while some other works studied option pricing model under transaction costs, such as [2], [7]. In this chapter, we try to analyze discrete time portfolio selection model under transaction costs. Our interest in this problem was stimulated by the paper of Cvitanic and Karatzas [15] on the same subject. Their paper studied the portfolio optimization problem under transaction costs in a continuous-time financial model with one-risky-asset, while this chapter tries to extend their work to a discrete-time model with multiple-risky-asset. Different from their paper, this chapter constructs a set of auxiliary markets so as to investigate the relationship between the primal constrained problem and dual problem.

We establish discrete time security market under transaction costs and formulates the portfolio optimization problems with transaction costs in Section 5.2. We formulate in Section 5.3 the corresponding portfolio optimization problems without transaction costs, and derive the optimal solution of this problem. To analyze the optimization problem with transaction costs, we introduce some auxiliary martingales in Section 5.4. After that, we introduce and analyze the auxiliary optimization problems and dual problem. Also, we characterize the optimal admissible solution. In Section 5.5, we reveal the relationship between the dual problem and primal problem. We summarize the main results of this chapter in Section 5.6.

5.2. Problem formulation

We consider the same financial market as in Section 4.2 except that $B_t \equiv 1$. An investor enters in the financial market with initial wealth v_0. She follows a trading strategy featured by $L = \{L_t; t = 0, 1, 2, \cdots, T\}$ and $M = \{M_t; t = 0, 1, 2, \cdots, T\}$, where $L_t = (L_t(1), \cdots, L_t(n))'$ with $L_t(i)$ being the number of units of the ith risky security bought at the beginning of time t, and $M_t = (M_t(1), \cdots, M_t(n))'$ with $M_t(i)$ being the number of units of the ith risky security sold at the beginning of time t. Let us introduce the following set,

$$\mathcal{K} := \{(L_t, M_t) : L_t \geq 0,\ M_t \geq 0,\ \text{for } t = 0, 1, \cdots, T\}.$$

Assumption 5.1 *The investor pays proportional transaction fees when she transfers her money between bank account and stock holdings, while the proportion rate is λ (or μ) for buying (or selling) risky assets.*

Based on Assumption 5.1, the total transaction costs at time t is $\sum_{i=1}^{n}[\lambda L_t(i)S_t(i) + \mu M_t(i)S_t(i)]$.

Let $(\Pi_t(0), \Pi_t(1), \cdots, \Pi_t(n))$ be the processes of holdings in bond and in stocks at the beginning of time t, which are governed by the following equations:

$$\Pi_t(0) = \Pi_{t-1}(0) - \sum_{i=1}^{n}(1+\lambda)L_t(i)S_t(i) + \sum_{i=1}^{n}(1-\mu)M_t(i)S_t(i), \quad (5.1)$$

$$\Pi_t(i) = \Pi_{t-1}(i)\frac{S_t(i)}{S_{t-1}(i)} + [L_t(i) - M_t(i)]S_t(i), \quad (5.2)$$

for $i = 1, 2, \cdots, n$ and $t = 0, 1, \cdots, T$.

Definition 5.1 *A portfolio processes $(L_t, M_t) \in \mathcal{K}$ is called admissible for the initial holding of bond x and stocks $\boldsymbol{y} \in \mathbb{R}^n$, if*

$$\Pi_t(0) + \sum_{i\in I}(1-\mu)\Pi_t(i) + \sum_{i\in \overline{I}}(1+\lambda)\Pi_t(i) \geq 0, \quad (5.3)$$

$\forall I \subseteq \{1, \cdots, n\}$ and $t = 0, 1, \cdots, T$. The set of admissible (L_t, M_t) is denoted by $\mathscr{A}(x, \boldsymbol{y})$.

Remark 5.1 *An admissible portfolio process guarantees no-bankruptcy in the investment process. Consider the situation when $n = 1$, inequality (5.3) reduces to $\Pi_t(0)+(1-\mu)\Pi_t(1) \geq 0$ and $\Pi_t(0)+(1+\lambda)\Pi_t(1) \geq 0$. These inequalities guarantee that the portfolio always be nonnegative.*

We consider a quadratic utility function $U(x) = \beta x - \frac{1}{2}x^2$, $\beta > 0$, which is concave and twice continuously differentiable.

Our optimization problem is to optimize the expected utility of the terminal wealth:

$$V_T = \Pi_T(0) + \sum_{i=1}^{n} g(\Pi_T(i)), \quad (5.4)$$

where

$$g(x) = \begin{cases} (1-\mu)x, & if \quad x \geq 0, \\ (1+\lambda)x, & if \quad x \leq 0. \end{cases}$$

When there is no transaction cost, i.e., $\mu = \lambda = 0$, the above notations reduce to

$$V_T = \sum_{i=0}^{n} \Pi_T(i),$$

with

$$\Pi_t(0) = \Pi_{t-1}(0) - \sum_{i=1}^{n} [L_t(i) - M_t(i)] S_t(i), \tag{5.5}$$

$$\Pi_t(i) = \Pi_{t-1}(i) \frac{S_t(i)}{S_{t-1}(i)} + [L_t(i) - M_t(i)] S_t(i), \tag{5.6}$$

for $i = 1, 2, \cdots, n$. In this case, the wealth process is the same as those in the previous chapters.

The portfolio optimization problem with transaction costs can be formulated as

$$(P) \begin{cases} \max_{(L_t, M_t)} & EU(V_T) = EU(\Pi_T(0) + \sum_{i=1}^{n} g(\Pi_T(i))) \\ s.t. & (5.1) - (5.2), \\ & (L_t, M_t) \in \mathscr{A}(v_0, \mathbf{0}). \end{cases}$$

5.3. Optimal solution to the problem without transaction costs

In this section, we consider first the optimization problem without transaction costs:

$$\begin{cases} \max & E(U(V_T)) = EU(\sum_{i=0}^{n} \Pi_T(i)) \\ \text{s.t.} & (5.5) - (5.6). \end{cases}$$

As the financial market is complete, there exists a unique risk neutral probability.

Theorem 5.1 *Under Assumption 4.1, Assumption 4.2 and Assumption 4.3, the following unique risk neutral probability exists in the market*

$$Q(\omega_i) = \prod_{t=0}^{T-1} \frac{(-1)^m \mid D_{t+1}^m(A_t^{i_t}) \mid}{\sum_{j=1}^{n+1}(-1)^j \mid D_{t+1}^j(A_t^{i_t}) \mid}, \tag{5.7}$$

where $m = i_{t+1} - i_t + 1 \in \{1, 2, \cdots, n+1\}$ and $D_{t+1}^m(A_t^{i_t})$ is defined in Definition 4.7.

As the proof is the same as that of Theorem 4.2, it is omitted here. The optimization problem without transaction costs is thus equivalent to

$$(P_0)\begin{cases} \max & EU(V_T) \\ s.t & E_Q(V_T) = v_0. \end{cases}$$

Theorem 5.2 *For problem (P_0) with quadratic utility function $U(x) = \beta x - \frac{1}{2}x^2$, the optimal attainable wealth is:*

$$V_T^* = \beta + \frac{v_0 - \beta}{E[(L)^2]} L,$$

and the optimal objective value is

$$EU(V_T^*) = \frac{1}{2}\beta^2 - \frac{(v_0 - \beta)^2}{2E[(L)^2]}.$$

where $L = Q/P$ is the state price density.

As the proof is similar with that of Theorem 4.3, it is omitted here.

5.4. Auxiliary optimization problems and dual problem

5.4.1. Auxiliary martingales

We observe in the previous section that there exists a unique risk neutral probability in the financial market without transaction costs. In this section, we introduce first some auxiliary martingales, so that we can similarly define the risk neutral probability in the corresponding auxiliary markets in the next subsection.

Suppose that now $(H_t(0), H_t(1), \cdots, H_t(n))$ is a strictly positive martingale with $H_0(0) = 1$, and $H_0(i) \in [S_0(i)(1-\mu), S_0(i)(1+\lambda)]$ for $i = 1, 2, \cdots, n$. Denote $Z_t(i) := \frac{H_t(i)}{H_t(0)S_t(i)}$. Let

$$\mathcal{L} := \{(H_t(0), H_t(1), \cdots, H_t(n)) : 1 - \mu \leq Z_t(i) \leq 1 + \lambda, \ \forall\, i = 1, 2, \cdots, n\}.$$

Let us consider $H_t^*(0) := E[(n+1)^{-T}Q|\mathcal{F}_t]$, where Q is the variable derived in Theorem 5.1. Obviously, $H_0^*(0) = 1$. We also consider $H_t^*(i) := H_t^*(0)S_t(i)$ $(i = 1, 2, \cdots, n)$ which satisfy $H_0^*(i) \in [S_0(i)(1-\mu), S_0(i)(1+\lambda)]$ and $Z_t^*(i) := \frac{H_t^*(i)}{H_t^*(0)S_t(i)} \equiv 1$ for $t = 0, 1, \cdots, T$ and $i = 1, 2, \cdots, n$. We can verify that $(H^*(0), \cdots, H^*(n)) \in \mathcal{L}$.

We introduce the following equivalent probability measures

$$P_H(A) = E[H_T(0)1_A], \quad A \in \mathcal{F}_T.$$

Remark 5.2 *Specifically, we can denote $P^*(A) = E[H_T^*(0)1_A]$ for $A \in \mathcal{F}_T$ as the probability measure corresponding to $(H^*(0), \cdots, H^*(n))$. Actually, $P^*(A) = Q(A)$, where Q is the risk neutral probability derived in Theorem 5.1.*

For given $(H(0), \cdots, H(n)) \in \mathcal{L}$ and $(\Pi_t(0), \Pi_t(1), \cdots, \Pi_t(n))$ satisfying (5.1) and (5.2), let us denote that

$$\begin{aligned} W_t^{L,M} &= \Pi_t(0) + \sum_{i=1}^{n} Z_t(i)\Pi_t(i) \\ &+ \sum_{\tau=0}^{t}\{-\sum_{i=1}^{n}[Z_\tau(i) - (1+\lambda)]L_\tau(i)S_\tau(i) + \sum_{i=1}^{n}[Z_\tau(i) - (1-\mu)]M_\tau(i)S_\tau(i)\}, \end{aligned} \tag{5.8}$$

for $t = 0, 1, \cdots, T$.

When $(L_t, M_t) \in \mathcal{K}$, we have

$$W_t^{L,M} \geq \Pi_t(0) + \sum_{i=1}^{n} Z_t(i)\Pi_t(i). \tag{5.9}$$

Let us observe the following expectation,

$$\begin{aligned}
&E[H_t(0)W_t^{L,M} \mid \mathcal{F}_{t-1}] \\
&= E[H_t(0)\Pi_t(0) + \sum_{i=1}^{n} H_t(i)\Pi_t(i)/S_t(i) \mid \mathcal{F}_{t-1}] \\
&+ E[H_t(0)\sum_{\tau=0}^{t}\{-\sum_{i=1}^{n}[Z_\tau(i) - (1+\lambda)]L_\tau(i)S_\tau(i)\} \mid \mathcal{F}_{t-1}] \\
&+ E[H_t(0)\sum_{\tau=0}^{t}\{\sum_{i=1}^{n}[Z_\tau(i) - (1-\mu)]M_\tau(i)S_\tau(i)\} \mid \mathcal{F}_{t-1}] \\
&= H_{t-1}(0)\{\Pi_{t-1}(0) + \sum_{i=1}^{n} Z_{t-1}(i)\Pi_{t-1}(i)) \\
&- \sum_{\tau=0}^{t-1}\sum_{i=1}^{n}[Z_\tau(i) - (1+\lambda)]L_\tau(i)S_\tau(i) \\
&+ \sum_{\tau=0}^{t-1}\sum_{i=1}^{n}[Z_\tau(i) - (1-\mu)]M_\tau(i)S_\tau(i)\}.
\end{aligned}$$

Therefore,

$$E[H_t(0)W_t^{L,M} \mid \mathcal{F}_{t-1}] = H_{t-1}W_{t-1}^{L,M}, \ \forall \ 0 \leq t \leq T. \tag{5.10}$$

Hence, $H_t(0)W_t^{L,M}$ is P-martingale, or $W_t^{L,M}$ is P_H-martingale.

5.4.2. Auxiliary unconstrained problems and dual problem

Recall that we have approached a multi-period portfolio selection problem with no-shorting by introducing a set of auxiliary markets in Chapter 4. Similarly, we would like to introduce a set of auxiliary markets corresponding to elements in set $\mathcal{L}$ in this subsection in order for us to tackle (P), the portfolio selection problem with transaction costs. For given $((H_t(0), H_t(1), \cdots, H_t(n)) \in \mathcal{L}$, we define an auxiliary market $\mathscr{M}_H$ by modifying the price processes for the risky securities as:

$$S_t(i) \to Z_t(i)S_t(i), \quad t \geq 0, \tag{5.11}$$

for $i = 1, \cdots, n$. Note that $H_T(0)$ is just the state price density, and the probability, defined as $P_H(A) := E[H_T(0)1_A]$, is the equivalent risk-neutral probability in this auxiliary market. So the terminal wealth with trading strategy (L_t, M_t) is

$$\begin{aligned} W_T &= \Pi_T^{L,M}(0) + \sum_{i=1}^{n} Z_T(i)\Pi_T^{L,M}(i) \\ &+ \sum_{\tau=0}^{T}\{-\sum_{i=1}^{n}[Z_\tau(i) - (1+\lambda)]L_\tau(i)S_\tau(i) + \sum_{i=1}^{n}[Z_\tau(i) - (1-\mu)]M_\tau(i)S_\tau(i)\}, \end{aligned}$$

where $\Pi_T^{L,M}(0)$ and $\Pi_T^{L,M}(i)$ satisfy (5.1) and (5.2). We have already shown that W_t is P_H-martingale from (5.10).

Let us consider the following auxiliary unconstrained portfolio selection problem in the auxiliary market $\mathscr{M}_H$,

$$(A(H)) \begin{cases} \max_{W_T} & EU(W_T) \\ s.t & E[H_T(0)W_T] = v_0. \end{cases}$$

Let $J(v_0)$ be the optimal value of the original problem (P) and $J_H(v_0)$ be the optimal value of problem $(A(H))$. Suppose that $\widehat{W}_T$ is the optimal terminal wealth of problem $(A(H))$. We denote the set of the optimal trading strategies of problem $(A(H))$ as $\Phi_H^0 := \{(L_t, M_t) : W_T^{L,M} = \widehat{W}_T\}$, and denote the set of the optimal admissible trading strategies as $\Phi_H := \{(L_t, M_t) \in \mathscr{A}(v_0, \mathbf{0}) : W_T^{L,M} = \widehat{W}_T\}$. Notice that $\Phi_H \subseteq \Phi_H^0$ and Φ_H may be empty.

We consider the following dual problem:

$$(D) \quad \inf_{H\in\mathcal{L}} \quad J_H(v_0).$$

Similar to Cvitanic et. al. (1996)[15], we assume the following assumption.

Assumption 5.2 *There exists some* $(H_t(0), \cdots, H_t(n)) \in \mathcal{L}$ *that attains the infimum of dual problem* (D).

Remark 5.3 *Unlike the dual problem* $\min_\kappa J_\kappa(v_0)$ *in Chapter 4, which can be solved by dynamic programming, the dual problem* (D) *in this chapter is difficult to analyze because of the ambiguity of* $\mathcal{L}$. *Actually, we have not yet been able to obtain a general existence result of the dual problem.*

For the optimal solution of dual problem (D), i.e. $\widehat{H}$, our question is whether there exists $(\widehat{L}, \widehat{M}) \in \mathscr{A}(v_0, \mathbf{0})$ such that $W_T^{\widehat{L},\widehat{M}}$ defined in (5.8) is equal to the optimal value of the auxiliary problem $(A(\widehat{H}))$. Or is it possible $\Phi_{\widehat{H}} \neq \emptyset$? In the following subsection, we will answer these questions.

5.4.3. Attainability of optimal terminal wealth

Actually, a question whether there exists $(\widehat{L}_t, \widehat{M}_t) \in \mathscr{A}(v_0, \mathbf{0})$ such that $W_T^{\widehat{L},\widehat{M}}$ is equal to the optimal value of the auxiliary problem $(A(\widehat{H}))$ is equivalent to the question whether the optimal value of the auxiliary problem $(A(\widehat{H}))$, i.e. $\widehat{W}_T$, can be hedged with initial value $(v_0, \mathbf{0})$, where the definition of 'hedge' is given as follows:

Definition 5.2 *We call that a trading strategy* (L_t, M_t) *hedges a time* T *contingent claim* $(X(0), \cdots, X(n))$ *with* $X(i)$ *being the amount of asset* i, *which is* $\mathcal{F}_T$*-measurable, starting with initial holdings* $(x, \mathbf{y})$, *if* $\Pi_t^{L,M}(i)$ $(i = 0, 1, \cdots, n)$ *defined in (5.1) and (5.2) satisfy*

$$\begin{aligned}
&\Pi_T^{L,M}(0) + \sum_{i\in I}(1+\lambda)\Pi_T^{L,M}(i) + \sum_{i\in \overline{I}}(1-\mu)\Pi_T^{L,M}(i) \\
&\geq X(0) + \sum_{i\in I}(1+\lambda)X(i) + \sum_{i\in \overline{I}}(1-\mu)X(i),
\end{aligned}$$

for all $I \subseteq \{1, 2, \cdots, n\}$.

Actually, we consider a situation with an initial holding $\mathbf{0} \in \mathbb{R}^n$ in stocks, and want to hedge a given contingent claims $\mathbf{a} := (\widehat{W}_T, 0, \cdots, 0) \in \mathbb{R}^{n+1}$ with admissible strategies. We give the following notation,

$$f(\mathbf{a}; \mathbf{0}) := \inf\{x \in \mathbb{R} / \exists (L_t, M_t) \in \mathscr{A}(x, \mathbf{0}) \text{ and } (L_t, M_t) \text{ hedges } \mathbf{a}\}. \tag{5.12}$$

Therefore, to prove that $\widehat{W}_T$ is attainable is equivalent to prove that $f(\mathbf{a}; \mathbf{0}) \leq v_0$.

Let us take arbitrary $f_0 < f(\mathbf{a}; \mathbf{0})$ and consider the following sets,

$$\begin{aligned}\mathbf{A}_0 = \{&(X(0), X(1), \cdots, X(n)) \in (\mathscr{L}_2^*)^{n+1} : \\ &\exists (L_t, M_t) \in \mathscr{A}(0, \mathbf{0}) \text{ that hedges } (X(0), \cdots, X(n)) \text{ starting with } (0, \mathbf{0})\},\end{aligned} \tag{5.13}$$

$$\mathbf{A}_1 = \{(\widehat{W}_T - f_0, \mathbf{0})\}, \tag{5.14}$$

where $\mathscr{L}_2^* = \mathcal{L}_2(\Omega, \mathscr{F}, P^*)$.

In the following, we investigate properties of these two sets $\mathbf{A}_0$ and $\mathbf{A}_1$, which are parallelto the results in the continuous-time case studied in Cvitanic and Karatzas [15].

Proposition 5.1 $\mathbf{A}_0$ *is convex, and contains the origin* $(0, \mathbf{0})$ *in* $(\mathscr{L}_2^*)^{n+1}$.

Proof. Suppose that $(X^i(0), X^i(1), \cdots, X^i(n)) \in (\mathscr{L}_2^*)^{n+1}$ is hedged correspondingly by $(L_t^i, M_t^i) \in \mathscr{A}(0, \mathbf{0})$ for $i = 1, 2$. Let $(Y^i(0), \cdots, Y^i(n))$ be the value processes. Then

$$\begin{aligned} Y_t^i(0) &= Y_{t-1}^i(0) - \sum_{j=1}^n L_t^i(j)(1+\lambda)S_t(j) + \sum_{j=1}^n M_t^i(j)(1-\mu)S_t(j), \\ Y_t^i(j) &= Y_{t-1}^i(j)\frac{S_t(j)}{S_{t-1}(j)} + \sum_{j=1}^n [L_t^i(j) - M_t^i(j)]S_t(j), \end{aligned}$$

for $j = 1, 2, \cdots, n$. According to the hedging property, we have

$$\begin{aligned} & Y_T^i(0) + \sum_{j\in I}(1-\mu)Y_T^i(j) + \sum_{j\in \overline{I}}(1+\lambda)Y_T^i(j) \\ & \geq X_T^i(0) + \sum_{j\in I}(1-\mu)X_T^i(j) + \sum_{j\in \overline{I}}(1+\lambda)X_T^i(j), \end{aligned}$$

for arbitrary $I \in \{1, 2, \cdots, n\}$ and $i = 1, 2$.

For arbitrary $0 < \eta^i < 1$ for $i = 1, 2$, define

$$(X(0), \cdots, X(n)) := (\sum_i \eta^i X^i(0), \cdots, \sum_i \eta^i X^i(n)) \in (\mathscr{L}_2^*)^{n+1}.$$

It is easy to verify that $(X(0), \cdots, X(n))$ can be hedged by the policy $(L_t, M_t) := (\sum_i \eta^i L_t^i, \sum_i \eta^i M_t^i) \in \mathscr{A}(0, \mathbf{0})$. □

Proposition 5.2 $\mathbf{A}_0$ *is closed in* $(\mathscr{L}_2^*)^{n+1}$.

Proof. Suppose that $\{(X^i(0), X^i(1), \cdots, X^i(n))\}_{i\in\mathbb{Z}_+} \in \mathbf{A}_0$ is a sequence converging to $(X(0), X(1), \cdots, X(n))$ in $(\mathscr{L}_2^*)^{n+1}$. We denote $\{(L_t^i, M_t^i)\}_{i\in\mathbb{N}} \in \mathscr{A}(0, \mathbf{0})$ as the corresponding hedging strategies. For our convenience, we define $(\overline{L}_t^i, \overline{M}_t^i)$ as the cumulative amounts of buying and selling respectively, i.e., $\overline{L}_t^i(j) = \sum_{\tau=1}^t L_\tau^i(j)S_\tau(j)$ and $\overline{M}_t^i(j) = \sum_{\tau=1}^t M_\tau^i(j)S_\tau(j)$. Also, we define the corresponding value processes, respectively, as follows,

$$\begin{aligned} Y_t^i(0) &= -\sum_{j=1}^n(1+\lambda)\overline{L}_t^i(j) + \sum_{j=1}^n(1-\mu)\overline{M}_t^i(j), \\ Y_t^i(j) &= \overline{L}_t^i(j) - \overline{M}_t^i(j) + \sum_{s=0}^{t-1} Y_s^i(j)\frac{S_{s+1}(j) - S_s(j)}{S_s(j)}, \end{aligned}$$

for $j = 1, \cdots, n$. According to the hedging properties, we have

$$\begin{aligned} & Y_T^i(0) + \sum_{j\in I}(1-\mu)Y_T^i(j) + \sum_{j\in \overline{I}}(1+\lambda)Y_T^i(j) \\ & \geq X^i(0) + \sum_{j\in I}(1-\mu)X^i(j) + \sum_{j\in \overline{I}}(1+\lambda)X^i(j), \end{aligned} \tag{5.15}$$

for any $I \subseteq \{1, 2, \cdots, n\}$.

Now we need to prove that sequences $\{\overline{L}_t^i\}$, $\{\overline{M}_t^i\}$ and $\{Y_t^i(j)\}$ $(j = 1, 2, \cdots, n)$ are bounded in the Hilbert Space $\mathbb{H}$.

Denote $R_t^i(I) = E^*[X^i(0) + \sum_{j\in I} X^i(j)(1+\lambda) + \sum_{j\in \overline{I}} X^i(j)(1-\mu)|\mathcal{F}_t]$. Since $Y_T^i(0) + \sum_{j\in I}(1-\mu)Y_T^i(j) + \sum_{j\in\overline{I}}(1+\lambda)Y_T^i(j)$ is P^*-supermartingale, we can easily derive that

$$R_t^i(I) \le Y_t^i(0) + \sum_{j\in I} Y_t^i(j)(1+\lambda) + \sum_{j\in\overline{I}} Y_t^i(j)(1-\mu). \tag{5.16}$$

We also denote $C_t^i(j) = \sum_{s=0}^{t-1} Y_s^i(j)\frac{S_{s+1}(j)-S_s(j)}{S_s(j)}$. With this notation, we can rewrite (5.16) as

$$(\lambda+\mu)(\sum_{j\in I}\overline{M}_t^i(j) + \sum_{j\in\overline{I}}\overline{L}_t^i(j)) \le -R_t^i(I) + \sum_{j\in I}(1+\lambda)C_t^i(j) + \sum_{j\in\overline{I}}(1-\mu)C_t^i(j). \tag{5.17}$$

Since

$$\begin{aligned} Y_t^i(j) &\le \overline{L}_t^i(j) + C_t^i(j), \\ Y_t^i(j) &\ge -\overline{M}_j^t(j) + C_t^i(j), \end{aligned}$$

together with (5.17), we can derive that

$$\begin{aligned} \sum_{j=1}^n Y_t^i(j) &\le \sum_{j=1}^n \overline{L}_t^i(j) + \sum_{j=1}^n C_t^i(j) \le -\frac{R_t(I_0)}{\lambda+\mu} + \sum_{j=1}^n \frac{1+\lambda}{\lambda+\mu}C_t^i(j), \\ \sum_{j=1}^n Y_t^i(j) &\ge -\sum_{j=1}^n \overline{M}_t^i(j) + \sum_{j=1}^n C_t^i(j) \ge \frac{R_t(\overline{I}_0)}{\lambda+\mu} - \sum_{j=1}^n \frac{1-\mu}{\lambda+\mu}C_t^i(j), \end{aligned}$$

where $I_0 = \emptyset$ and $\overline{I}_0 = \{1, 2, \cdots, n\}$. Hence,

$$\begin{aligned} |\sum_{j=1}^n Y_t^i(j)| &\le \frac{|R_t(I_0)| + |R_t(\overline{I}_0)|}{\lambda+\mu} + \sum_{j=1}^n \frac{1+\lambda}{\lambda+\mu}|C_t^i(j)| \\ &\le \frac{|R_t(I_0)| + |R_t(\overline{I}_0)|}{\lambda+\mu} + \frac{1+\lambda}{\lambda+\mu}\max_j |\frac{S_{s+1}(j)-S_s(j)}{S_s(j)}| \cdot \sum_{s=0}^{t-1} |\sum_{j=1}^n Y_s^i(j)|. \end{aligned}$$

Based on discrete Gronwall's inequality[1], $\sum_{j=1}^{n} Y_t^i(j)$ is bounded. Therefore, $C_t^i(j)$ is bounded.

Furthermore, R_I^i is bounded. Actually,

$$\sup_{t,i\in\mathbb{Z}} E^*[\sum_I (R_t^i(I))^2] \leq m \sup_{i\in\mathbb{Z}} E^*[(X(0))^2 + \sum_{i=0}^{n}(1+\lambda)(X(i))^2] < \infty,$$

where m is a bounded constant. Hence, both $\{\overline{L}_t^i\}$ and $\{\overline{M}_t^i\}$ are bounded.

Then, there exist converging subsequences of $\{\overline{L}_t^i\}$, $\{\overline{M}_t^i\}$ and $\{Y_t^i\} \in \mathbb{H}$, such that

$$\overline{L}_t^i \to \overline{L}_t, \quad \overline{M}_t^i \to \overline{M}_t, \quad Y_t^i \to Y_t \text{ weekly in } \mathbb{H}, \quad \text{as } i \to \infty.$$

We can then define

$$\begin{aligned} Y_t(0) &= -\sum_{j=1}^{n}(1+\lambda)\overline{L}_t(j) + \sum_{j=1}^{n}(1-\mu)\overline{M}_t(j), \\ Y_t(j) &= \overline{L}_t(j) - \overline{M}_t(j) + \sum_{s=0}^{t-1} Y_s(j)\frac{S_{s+1}(j) - S_s(j)}{S_s(j)}, \end{aligned}$$

for $j = 1, \cdots, n$. Notice that Y_t^i converges to Y_t weekly in $\mathbb{H}$. Therefore $C_t^i(j)$ converges to $C_t(j) := \sum_{s=0}^{t-1} Y_s(j)\frac{S_{s+1}(j)-S_s(j)}{S_s(j)}$.

Since $\overline{L}_t^i$ and $\overline{M}_t^i$ are nondecreasing paths, we can assume that their limits $\overline{L}_t$ and $\overline{M}_t$ are also nondecreasing. So,

$$\begin{aligned} &\lim_{i\to\infty} E^*[\overline{L}_T^i(j)1_A] = E^*[\overline{L}_T(j)1_A], \\ &\lim_{i\to\infty} E^*[\overline{M}_T^i(j)1_A] = E^*[\overline{M}_T(j)1_A], \ \forall A \in \mathcal{F}_T. \end{aligned}$$

As $R_T^i(I) = X^i(0) + \sum_{j\in I} X^i(j)(1+\lambda) + \sum_{j\in\overline{I}} X^i(j)(1-\mu)$, it is obvious that $R_T^i(I)$ converges to $R_T(I) := X(0) + \sum_{j\in I} X(j)(1+\lambda) + \sum_{j\in\overline{I}} X(j)(1-\mu)$ as

[1]Discrete Gronwall's inequality states that if a nonnegative sequence $\{y_n, \ n = 0, \cdots, n\}$ satisfies $y_0 = 0$, $y_n \leq A + Bh\sum_{j=0}^{n-1} y_j$, $1 \leq n \leq N, h = 1/N$, then $\max_{1\leq i\leq N} y_i \leq Ae^B$, where A and B are positive constants independent of h.

$i \to \infty$. Hence,

$$\begin{aligned}(\lambda+\mu)(\sum_{j\in I}\overline{M}_T(j)+\sum_{j\in\overline{I}}\overline{L}_T(j)) &= \lim_{i\to\infty}(\lambda+\mu)(\sum_{j\in I}\overline{M}^i_T(j)+\sum_{j\in\overline{I}}\overline{L}^i_T(j))\\ &\le \lim_{i\to\infty}[-R^i_T(I)+\sum_{j\in I}(1+\lambda)C^i_T(j)\\ &\quad +\sum_{j\in\overline{I}}(1-\mu)C^i_T(j)]\\ &= -R_T(I)+\sum_{j\in I}(1+\lambda)C_T(j)+\sum_{j\in\overline{I}}(1-\mu)C_T(j),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&Y_T(0)+\sum_{j\in I}(1-\mu)Y_T(j)+\sum_{j\in\overline{I}}(1+\lambda)Y_T(j)\\ &\ge X(0)+\sum_{j\in I}(1-\mu)X(j)+\sum_{j\in\overline{I}}(1+\lambda)X(j).\end{aligned}$$

Therefore, $(\overline{L}_t,\overline{M}_t)$ can hedge $(X(0),X(1),\cdots,X(n))$ with initial holding $(0,\mathbf{0})$.

Now we need to verify that $(\overline{L}_t,\overline{M}_t)\in\mathscr{A}(0,\mathbf{0})$. Since $(\overline{L}^i_t,\overline{M}^i_t)\in\mathscr{A}(0,\mathbf{0})$, we have

$$\overline{L}^i(t)\ge 0,\quad \overline{M}^i(t)\ge 0,$$

$$Y^i_t(0)+\sum_{j\in I}(1-\mu)Y^i_t(j)+\sum_{j\in\overline{I}}(1+\lambda)Y^i_t(j)\ge 0,$$

for any $t=0,1,2,\cdots,T$ and $I\subseteq\{1,2,\cdots,n\}$. Also we have already known that

$$\overline{L}^i_t\to\overline{L}_t,\quad \overline{M}^i_t\to\overline{M}_t,\quad Y^i_t\to Y_t,\quad \text{weekly in }\ \mathbb{H},\quad \text{as }\ i\to\infty.$$

Therefore, we can derive that

$$\overline{L}_t\ge 0,\quad \overline{M}_t\ge 0,$$

$$Y_t(0)+\sum_{j\in I}(1-\mu)Y_t(j)+\sum_{j\in\overline{I}}(1+\lambda)Y_t(j)\ge 0,$$

for any $t = 0, 1, 2, \cdots, T$ and $I \subseteq \{1, 2, \cdots, n\}$. According to Definition 5.1, $(\overline{L}_t, \overline{M}_t) \in \mathscr{A}(0, \mathbf{0})$.

□

Proposition 5.3 $\mathbf{A}_1 \cap \mathbf{A}_0 = \emptyset$.

Proof. Suppose that $\mathbf{A}_1 \cap \mathbf{A}_0 \neq \emptyset$. So there exists some $(L_t, M_t) \in \mathscr{A}(0, \mathbf{0})$ such that $X_0 = X^{\mathbf{0}, L, M}$ and $X_i = X_i^{\mathbf{0}, L, M}$ $(i = 1, 2, \cdots, n)$ satisfy

$$X_T(0) + \sum_{j \in I}(1 - \mu)X_T(j) + \sum_{j \in \overline{I}}(1 + \lambda)X_T(j) \ \leq \ \widehat{W}_T - f_0,$$

for arbitrary $I \subseteq \{1, 2, \cdots, n\}$.

But if we define

$$\begin{aligned}\widetilde{X}_t(0) &= X_t^{f_0, L, M}(0) = f_0 + X_t(0), \\ \widetilde{X}_t(i) &= X_t^{f_0, L, M}(i) = X_t(i),\end{aligned}$$

for $i = 1, 2, \cdots, n$, we can derive that

$$\widetilde{X}_T(0) + \sum_{j \in I}(1 - \mu)\widetilde{X}_T(j) + \sum_{j \in \overline{I}}(1 + \lambda)\widetilde{X}_T(j) \ \leq \ \widehat{W}_T,$$

for arbitrary $I \subseteq \{1, 2, \cdots, n\}$, which implies that (L_t, M_t) belongs to $\mathscr{A}(f_0, \mathbf{0})$ and hedges $(\widehat{W}_T, \mathbf{0})$ with initial wealth $(f_0, \mathbf{0})$. This is a contradiction to the fact that $f_0 < f(\mathbf{a}; \mathbf{0})$. □

Suppose that $(\widehat{H}(0), \cdots, \widehat{H}(n)) \in \mathcal{L}$ is the optimal solution of problem (D). We then have the following property about $\widehat{W}_T$.

Theorem 5.3 *For an arbitrary $H \in \mathcal{L}$, and optimal $\widehat{H} \in \mathcal{L}$ for the dual problem (D), we have*

$$E[H_T(0)\widehat{W}_T] \leq E[\widehat{H}_T(0)\widehat{W}_T] = v_0.$$

Proof. Denote $\widetilde{U}(y) = \max U(x) - xy$, and $I(\cdot) = (U'(\cdot))^{-1}$.

For the auxiliary problem $(A(H))$, we introduce the Lagrangian multiplier ξ, and consider the following optimization problem:

$$\max EU(W_T) - \xi E[H_T(0)W_T].$$

The optimal condition dictates the first derivative to be equal to zero, $E[U'(W_T) - \xi H_T(0)] = 0$. According to Pliska (1997), the following necessary conditions must be satisfied,

$$U'(W_T) - \xi H_T(0) = 0.$$

Hence the optimal value satisfies $W_T = I(\xi H_T(0))$ for some ξ. Then the optimal value of the objective function is given by $J_H(v_0) = \min_\xi EU(I(\xi H_T(0))) - \xi E[H_T(0)I(\xi H_T(0))] = \min_\xi E[\widetilde{U}(\xi H_T(0))]$. Specially, for optimal $\widehat{H} \in \mathcal{L}$ of problem (D), $\widehat{W}_T = I(\widehat{\xi}\widehat{H}_T(0))$, and $J_{\widehat{H}}(v_0) = E[\widetilde{U}(\widehat{\xi}\widehat{H}_T(0))]$ for some $\widehat{\xi}$.

For arbitrary $H \in \mathcal{L}$ and fixed arbitrary $0 < \varepsilon < 1$, we introduce the following perturbation,

$$\widetilde{H}^\varepsilon(i) = (1-\varepsilon)\widehat{H}(i) + \varepsilon H(i).$$

Note that $\widetilde{H}^\varepsilon \in \mathcal{L}$, and $U'(\cdot) = -I(\cdot)$. We now consider

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} U_\varepsilon &:= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}[\widetilde{U}(\widehat{\xi}\widehat{H}_T(0)) - \widetilde{U}(\widehat{\xi}\widetilde{H}^\varepsilon_T(0))] \\
&= -\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\widehat{\xi} I(\widehat{\xi}h_\varepsilon)[\widehat{H}_T(0) - \widetilde{H}^\varepsilon_T(0)] \\
&= -\widehat{\xi} I(\widehat{\xi}\widehat{H}_T(0))[\widehat{H}_T(0) - H_T(0)],
\end{aligned}$$

with h_ε being some variable between $\widehat{H}_T(0)$ and $\widetilde{H}^\varepsilon_T(0)$, and $\lim_{\varepsilon \downarrow 0} h_\varepsilon = \widehat{H}_T(0)$.

Hence, using Fatou's Lemma, gives rise the following,

$$\begin{aligned} & E[\widehat{\xi} H_T(0) I(\widehat{\xi}\widehat{H}_T(0)) - \widehat{\xi}\widehat{H}_T(0) I(\widehat{\xi}\widehat{H}_T(0))] \\ = \; & E[\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(\widetilde{U}(\widehat{\xi}\widehat{H}_T(0)) - \widetilde{U}(\widehat{\xi}\widetilde{H}_T^{\varepsilon}(0)))] \\ \leq \; & \lim_{\varepsilon \downarrow 0} E[U_\varepsilon] \leq 0. \end{aligned}$$

The above inequality is satisfied because $\widehat{H} \in \mathcal{L}$ attains the minimum of the dual problem (D), i.e.

$$\inf_{H \in \mathcal{L}} \min_{\xi} E[\widetilde{U}(\xi H_T(0))] = \min_{\xi} \inf_{H \in \mathcal{L}} E[\widetilde{U}(\xi H_T(0))].$$

Therefore $E[\widetilde{U}(\widehat{\xi}\widehat{H}_T(0))] \leq E[\widetilde{U}(\widehat{\xi}\widetilde{H}_T^{\varepsilon}(0)))]$, and hence $E(U_\varepsilon) \leq 0$. □

The following theorem is the main result of this section. It gives a positive answer to the question we asked at the beginning of this section.

Theorem 5.4 $f(\mathbf{a}; 0) \leq \sup_{\mathcal{L}} E[H_T(0)\widehat{W}_T] = v_0$.

Proof. According to Propositions (5.1),(5.2) and (5.3), and the Hahn-Banach separation theorem, there exists a set of non-zero random vectors $(\zeta_0^*, \cdots, \zeta_n^*) \in (\mathscr{L}_2^*)^{n+1}$, such that for any $(X(0), \cdots, X(n)) \in \mathbf{A}_0$,

$$E^*[\zeta_0^* X(0) + \zeta_1^* X(1) + \cdots + \zeta_n^* X(n)] = E[\zeta_0 X(0) + \zeta_1 X(1) + \cdots + \zeta_n X(n)] \leq \; 0, \tag{5.18}$$

$$E^*[\zeta_0^*(\widehat{W}_T - f_0)] = E[\zeta_0(\widehat{W}_T - f_0)] \geq \; 0, \tag{5.19}$$

where $\zeta_i = \zeta_0^* H_T^*(i)$ for $i = 0, 1, \cdots, n$. It is easy to verify that

$$(1-\mu)E[\zeta_0|\mathcal{F}_t] \leq \frac{E[\zeta_i S_T(i)|\mathcal{F}_t]}{S_t(i)} \leq (1+\lambda)E[\zeta_0|\mathcal{F}_t], \tag{5.20}$$

$$\zeta_0 \geq 0 \;\; \zeta_i \geq 0, \;\; E(\zeta_0) > 0, E[\zeta_i S_T(i)] > 0. \tag{5.21}$$

Actually, for fixed $\tau \in \{0, 1, \cdots, T\}$ and an arbitrary bounded nonnegative $\mathcal{F}_t$-measurable random variable θ, we consider a trading strategy starting with $(0, \mathbf{0})$ that buys θ shares of arbitrary stock i at time $t = \tau$, and does nothing otherwise. This is a buy-and-hold strategy. So, the trading strategies are given as follows,

$$M_t^\theta(i) \equiv 0, \ \ L_t^\theta(i) = \theta \mathbf{1}_{t=\tau},$$
$$M_t^\theta(j) \equiv 0, \ \ L_t^\theta(j) \equiv 0, for \ \ j \neq i,$$

and the wealth processes are

$$\begin{aligned} X_t^\theta(0) &= -\theta(1+\lambda)S_\tau(i)\mathbf{1}_{t\geq\tau}, \\ X_t^\theta(i) &= \theta S_t(i)\mathbf{1}_{t\geq\tau}, \\ X_t^\theta(j) &= 0 \ for \ j \neq i, \end{aligned}$$

for $0 \leq t \leq T$. The following process,

$$H_t(0)[X_t^\theta(0) + Z_t(i)X_t^\theta(i)] := \theta[H_t(i) - (1+\lambda)H_t(0)S_t(i)]\mathbf{1}_{t\geq\tau},$$

is a P-supermartingale for every $H \in \mathcal{L}$. Therefore,

$$\begin{aligned} E[H_t(0)(X_t^\theta(0) + Z_t(i)X_t^\theta(i))|\mathcal{F}_\tau] &= \theta\{E[H_t(i)|\mathcal{F}_\tau] - (1+\lambda)S_\tau(i)E[H_t(0)|\mathcal{F}_\tau]\} \\ &= \theta[H_\tau(i) - (1+\lambda)S_\tau(i)H_\tau(0)] \\ &= \theta S_\tau(i)H_\tau(0)[[Z_\tau(i) - (1+\lambda)] \\ &\leq 0 = H_{\tau-1}(0)[X_{\tau-1}^\theta(0) + Z_{\tau-1}(i)X_{\tau-1}^\theta(i)]. \end{aligned}$$

From (5.18), we have

$$\begin{aligned} 0 &\geq E[\zeta_0 X(0) + \zeta_1 X(1) + \cdots + \zeta_n X(n)] \\ &= E\{\theta[\zeta_i S_T(i) - (1+\lambda)\zeta_0 S_\tau(i)]\} \\ &= E\{\theta[E(\zeta_i S_T(i)|\mathcal{F}_\tau) - (1+\lambda)S_\tau(i)E(\zeta_0|\mathcal{F}_\tau)]\}. \end{aligned}$$

From arbitrariness of $\theta \geq 0$, τ and $i = 1, \cdots, n$, we can deduce that

$$\frac{E[\zeta_i S_T(i)|\mathcal{F}_t]}{S_t(i)} \leq (1+\lambda)E[\zeta_0|\mathcal{F}_t].$$

Similarly, if we consider a trading strategy of selling θ shares of arbitrary stock i at time $t = \tau$, we can deduce that

$$(1-\mu)E[\zeta_0|\mathcal{F}_t] \leq \frac{E[\zeta_i S_T(i)|\mathcal{F}_t]}{S_t(i)}.$$

Specifically, for $t = T$ and $t = 0$, we can easily derive

$$\zeta_0 \geq 0 \quad \zeta_i \geq 0, \quad E(\zeta_0) > 0, E[\zeta_i S_T(i)] > 0.$$

If we take $E(\zeta_0) = 1$, (5.19) gives rise,

$$f_0 \leq E[\zeta_0 \widehat{W}_T].$$

Consider arbitrary $0 < \varepsilon < 1$, $(H_0, \cdots, H_n) \in \mathcal{L}$ and define

$$\begin{aligned} \widetilde{H}_t(0) &= \varepsilon H_t(0) + (1-\varepsilon)E[\zeta_0|\mathcal{F}_t] \\ \widetilde{H}_t(i) &= \varepsilon H_t(i) + (1-\varepsilon)E[\zeta_i S_T(i)|\mathcal{F}_t], \end{aligned}$$

for $i = 1, 2, \cdots, n$ and $t = 0, 1, \cdots, T$.

Clearly, these are positive martingales and $\widetilde{H}_0(0) = 1$. Multiplying (5.20) by $(1-\varepsilon)$, we have

$$(1-\varepsilon)(1-\mu)E[\zeta_0|\mathcal{F}_t] \leq (1-\varepsilon)\frac{E[\zeta_i S_T(i)|\mathcal{F}_t]}{S_t(i)} \leq (1-\varepsilon)(1+\lambda)E[\zeta_0|\mathcal{F}_t].$$

Combining the above inequality with

$$\varepsilon(1-\mu)H_t(0) \leq \varepsilon\frac{H_t(i)}{S_t(i)} \leq \varepsilon H_t(0)(1+\lambda)$$

yields

$$(1-\mu)\widetilde{H}_t(0) \leq \frac{\widetilde{H}_t(i)}{S_t(i)} \leq \widetilde{H}_t(0)(1+\lambda),$$

for $t = 0, 1, \cdots, T$. Therefore, $(\widetilde{\Pi}_t(0), \cdots, (\widetilde{\Pi}_t(n)) \in \mathcal{L}$. Hence

$$\begin{aligned}\sup_{\mathcal{L}} E[\Pi_T(0)\widehat{W}_T] &\geq E[\widetilde{\Pi}_T(0)\widehat{W}_T] \\ &= (1-\varepsilon)E[\zeta_0\widehat{W}_T] + \varepsilon E[\Pi_T(0)\widehat{W}_T] \\ &\geq f_0(1-\varepsilon) + \varepsilon E[\Pi_T(0)\widehat{W}_T].\end{aligned}$$

Let $\varepsilon \downarrow 0$ and $f_0 \uparrow f(\mathbf{a}; \mathbf{0})$, from the property in Theorem 5.3, we have

$$f(\mathbf{a}; \mathbf{0}) \leq \sup_{\mathcal{L}} E[\Pi_T(0)\widehat{W}_T] = v_0.$$

□

Remark 5.4 *Theorem 5.4 shows that if there exists an optimal solution $\widehat{\Pi}$ to the dual problem (D), then the corresponding optimal solution $\widehat{W}_T$ to the problem $(A(\widehat{\Pi}))$ can be hedged by some admissible policy $(\widehat{L}_t, \widehat{M}_t) \in \mathscr{A}(v_0, \boldsymbol{0})$, which means that $\widehat{W}_T$ is attainable.*

Furthermore, we give some properties of the optimal admissible trading strategies, i.e., $(\widehat{L}_t, \widehat{M}_t) \in \phi_{\widehat{\Pi}}$, in the following proposition.

Proposition 5.4 *The optimal trading strategy $(\widehat{L}_t, \widehat{M}_t) \in \phi_{\widehat{\Pi}}$ and the optimal terminal wealth of the problem $(A(\widehat{\Pi}))$, i.e. $\widehat{W}_T$, satisfy:*

$$\begin{aligned}((1+\lambda) - \widehat{Z}_t(i))\widehat{L}_t(i) &= 0, \\ ((1-\mu) - \widehat{Z}_t(i))\widehat{M}_t(i) &= 0,\end{aligned}$$

and

$$\widehat{W}_T = \widehat{\Pi}_t(0) + \sum_{i=1}^{n} \widehat{Z}_t(i)\widehat{\Pi}_t(i).$$

Proof. Since $(\widehat{L}_t, \widehat{M}_t)$ can hedge $\widehat{W}_T$, we have

$$\widehat{\Pi}_T(0) + \sum_{i\in I}(1-\mu)\widehat{\Pi}_T(i) + \sum_{i\in \overline{I}}(1+\lambda)\widehat{\Pi}_T(i) \geq \widehat{W}_T,$$

$\forall I \subseteq \{1, \cdots, n\}$. As $1-\mu \leq \widehat{Z}_T(i) \leq 1+\lambda$, we can get the following inequality,

$$\widehat{\Pi}_T(0) + \sum_{i=1}^{n} \widehat{Z}_T(i)\widehat{\Pi}_T(i) \geq \widehat{W}_T.$$

Recall (5.9) that

$$\widehat{\Pi}_T(0) + \sum_{i=1}^{n} \widehat{Z}_T(i)\widehat{\Pi}_T(i) \leq \widehat{W}_T.$$

Therefore, we can derive that $\widehat{\Pi}_T(0) + \sum_{i=1}^{n} \widehat{Z}_T(i)\widehat{\Pi}_T(i) = \widehat{W}_T$, i.e.,

$$\sum_{\tau=0}^{T}\{\sum_{i=1}^{n}[-\widehat{Z}_\tau(i) + (1+\lambda)]\widehat{L}_\tau(i)S_\tau(i) + \sum_{i=1}^{n}[\widehat{Z}_\tau(i) - (1-\mu)]\widehat{M}_\tau(i)S_\tau(i) = 0.$$

Since $[-\widehat{Z}_\tau(i) + (1+\lambda)]\widehat{L}_\tau(i)S_\tau(i) \geq 0$ and $[\widehat{Z}_\tau(i) - (1-\mu)]\widehat{M}_\tau(i)S_\tau(i) \geq 0$ for any τ and i, hence we can derive that

$$\begin{aligned} ((1+\lambda) - \widehat{Z}_t(i))\widehat{L}_t(i) &= 0, \\ ((1-\mu) - \widehat{Z}_t(i))\widehat{M}_t(i) &= 0. \end{aligned}$$

□

5.5. Duality for primal problem

In Section 5.4, we shows that if infimum of the dual problem (D) , i.e. $\widehat{H}$, is attainable, then the optimal terminal wealth of auxiliary problem $(A(\widehat{H}))$, $\widehat{W}_T$, is an attainable contingent claim. The following theorem shows the relationship between the optimal solution of the original problem and the auxiliary problem $(A(\widehat{H}))$.

Theorem 5.5 *Suppose that* $(\widehat{\Pi}(0), \widehat{\Pi}(1), \cdots, \widehat{\Pi}(n)) \in \mathcal{L}$ *is the optimal solution of the dual problem* (D), *the corresponding optimal objective value of* $(A(\Pi))$ *is* $\widehat{W}_T$, *and assume that there exists optimal trading strategy* $(\widehat{L}_t, \widehat{M}_t) \in \Phi_{\widehat{\Pi}}$. *Then* $(\widehat{L}_t, \widehat{M}_t)$ *is the optimal trading strategy to the original problem* (P), *and*

$$J(v_0) = J_{\widehat{\Pi}}(v_0) \le J_\Pi(v_0), \quad \forall \ \Pi \in \mathcal{L},$$

where $J(v_0)$, $J_{\widehat{\Pi}}(v_0)$ *and* $J_\Pi(v_0)$ *are the optimal object values of* (P), $(A(\widehat{\Pi}))$ *and* $(A(\Pi))$, *respectively.*

Proof. It is obvious that $(\widehat{L}_t, \widehat{M}_t)$ is a feasible solution of the original problem, and can hedge $\widehat{W}_T$. Thus

$$\widehat{\Pi}_T(0) + \sum_{i\in I}(1+\mu)\widehat{\Pi}_T(i) + \sum_{i\in \overline{I}}(1-\lambda)\widehat{\Pi}_T(i) \ge \widehat{W}_T,$$

$\forall I \subseteq \{1, \cdots, n\}$. Note that we have

$$\widehat{\Pi}_T(0) + \sum_{i=1}^{n} \widehat{Z}_T(i)\widehat{\Pi}_T(i) = \widehat{W}_T. \tag{5.22}$$

For arbitrary i, if $\widehat{\Pi}_T(i) > 0$, we have the following inequality from the definition of hedge,

$$\widehat{\Pi}_T(0) + \sum_{j\neq i, \widehat{\Pi}_T(j)\ge 0}(1-\mu)\widehat{\Pi}_T(j) + \sum_{j\neq i, \widehat{\Pi}_T(j)\le 0}(1+\lambda)\widehat{\Pi}_T(j) + (1-\mu)\widehat{\Pi}_T(i) \ge \widehat{W}_T. \tag{5.23}$$

With (5.22) and (5.23), we can deduce that $\widehat{Z}_T(i) \le 1-\mu$. Hence, $\widehat{Z}_T(i) = 1-\mu$ on $\{\widehat{\Pi}_T(i) > 0\}$.

Similarly, for arbitrary i, if $\widehat{\Pi}_T(i) < 0$, we have the following inequality,

$$\widehat{\Pi}_T(0) + \sum_{j\neq i, \widehat{\Pi}_T(j)\ge 0}(1-\mu)\widehat{\Pi}_T(j) + \sum_{j\neq i, \widehat{\Pi}_T(j)\le 0}(1+\lambda)\widehat{\Pi}_T(j) + (1+\lambda)\widehat{\Pi}_T(i) \ge \widehat{W}_T. \tag{5.24}$$

With (5.22) and (5.24), we can deduce that $\widehat{Z}_T(i) \geq 1+\lambda$. Hence, $\widehat{Z}_T(i) = 1+\lambda$ on $\{\widehat{\Pi}_T(i) < 0\}$. Thus,

$$\begin{aligned}
&\widehat{\Pi}_T(0) + \sum_{i=1}^{n} \widehat{Z}_T(i)\widehat{\Pi}_T(i) \\
= \;& \widehat{\Pi}_T(0) + \sum_{i=1}^{n} \widehat{\Pi}_T(i)[(1+\lambda)\mathbf{1}_{\{\widehat{\Pi}_T(i)<0\}} + (1-\mu)\mathbf{1}_{\{\widehat{\Pi}_T(i)>0\}}] \\
= \;& \widehat{\Pi}_T(0) + \sum_{i=1}^{n} g_i(\widehat{\Pi}_T(i)).
\end{aligned}$$

Therefore, the corresponding terminal wealth, i.e. $\widehat{V}_T = \widehat{W}_T$, is an attainable wealth of (P). Hence $J(v_0) \geq J_{\widehat{H}}$.

On the other hand, for optimal trading strategy of the original problem (L_t^*, M_t^*), and arbitrary $H \in \mathcal{L}$, we have

$$E[H_T(0)V_T^*] \leq E[H_T(0)W_T^{L^*,M^*}] = v_0.$$

So (L_t^*, M_t^*) is a feasible solution of $(A(H))$. Therefore,

$$J(v_0) = J_{H^*}(v_0) \leq J_H(v_0) \;\; \forall \;\; H \in \mathcal{L},$$

where H^* is the corresponding process that makes $V_T^* = W_T^*$.

Hence, we have

$$J(v_0) = J_{\widehat{H}}(v_0) \leq J_H(v_0) \;\; \forall \;\; H \in \mathcal{L}.$$

□

Based on Theorem 5.5, we can derive the optimal solution of the primal constrained problem (P) by solving an unconstrained problem $(A(\widehat{H}))$ in the auxiliary market $\mathscr{M}_{\widehat{H}}$, if we can find the optimal solution of the dual problem (D). This kind of unconstrained problem has already been analyzed in Section 5.3.

The following example illustrates the relationship between the primal problem and an unconstrained auxiliary problem.

Example 5.1 We consider a market with one risky security and one bond, and the investment horizon is $T = 2$. The sample space Ω has $K = 4$ elements. Suppose that the bond price is constant, and the prices of the risky security are listed in Table 5.1.

Table 5.1: Prices of risky asset at original market in Example 5.1

ω	S_0	S_1	S_2
ω_1	5	8	9
ω_2	5	8	6
ω_3	5	4	6
ω_4	5	4	3

An investor enters this financial market with initial wealth $v_0 = 1$. She is trying to find out the best allocation of her wealth to both the bond and the risky asset to maximize the utility function of the terminal wealth $E(2V_2 - 0.5V_2^2)$. Suppose that she will pay a proportional transaction cost when she transfers her money between bank account and stock holdings. The proportion rate is 0.008 for buying and 0.005 for selling.

By using an enumeration method, we can get the optimal trading strategy by comparing all selling and buying cases. The optimal policy is to buy 0.2 unit of risky asset at initial time $t = 0$, sell 0.3 unit of risky asset when the price goes up to 8 and do nothing when the price goes down to 4 at time $t = 1$. Furthermore, the optimal objective value is 1.62725.

Now we illustrate that this constrained investment problem is equivalent to an unconstrained problem in an auxiliary market $\mathscr{M}_H$ corresponding to some martingale processes $\{(H_t(0), H_t(1))\} \in \mathcal{L}$, which is specified as in Table 5.2. Actually, this specified martingale process is constructed as the one to make the optimal trading strategy to satisfy Proposition 5.4.

Table 5.2: Optimal martingale processes in $\mathcal{L}$ in Example 5.1

ω	$H_0(0)$	$H_0(1)$	$H_0(2)$	$H_1(0)$	$H_1(1)$	$H_1(2)$
ω_1	1	0.5252	0.664	5.04	4.1806	6.0238
ω_2	1	0.5252	0.3864	5.04	4.1806	2.3369
ω_3	1	1.4748	1.0028	5.04	5.8992	5.9867
ω_4	1	1.4748	1.9468	5.04	5.8992	5.8112

The price process of risky asset at the auxiliary market, according to (5.11), is modified as in Table 5.3.

Table 5.3: Prices of risky asset at auxiliary market $\mathcal{M}_H$ in Example 5.1

ω	$Z_0(1)S_0$	$Z_1(1)S_1$	$Z_2(1)S_2$
ω_1	5.04	7.96	9.072
ω_2	5.04	7.96	6.048
ω_3	5.04	4	5.97
ω_4	5.04	4	2.985

At the auxiliary market, $H_0(2)$ is the state price density, and $E[(H_2(0))^2] = 1.3465$. According to Theorem 5.2, the optimal objective value to the unconstrained portfolio selection problem is

$$\frac{1}{2}\beta^2 - \frac{(v_0 - \beta)^2}{2E[(H_2(0))^2]} = \frac{1}{2} \times 2^2 - \frac{(1-2)^2}{2 \times 1.3465} = 1.62725,$$

which is consistent with the result derived by using enumeration method.

5.6. Summary

We have discussed in this chapter a multi-period portfolio selection problem under transaction costs. Our work is an extension of Cvitanic and Karatzas [15]. By introducing a set of auxiliary martingales, we transform the primal problem to a set of optimization problems without transaction costs constraint. We find that optimizing a portfolio in the frictional market is equivalent to minimizing the optimal value for the set of auxiliary unconstrained optimization problems, if it exists. Also, we characterize the optimal solution by investigating its properties.

Our result is similar to the continuous-time case, although we consider a multiple-risky-asset portfolio. Different from continuous-time model, we build up an equivalent relationship between the portfolio selection problem with transaction costs in the original market and the problem without transaction costs in an auxiliary market.

Chapter 6

Conclusions

We have studied in this thesis multi-period portfolio selection problems with different market settings. More specifically, we have considered investment situations with stochastic investment horizon, which introduces an additional line of risk into the discrete-time investment optimization. We also have investigated frictional financial markets by considering no-shorting or considering transaction costs.

We have revealed in the first part of this thesis the impact of the stochastic investment horizon on the efficient frontier of the optimal portfolio. This thesis is the first to analyze mean-variance multi-period portfolio selection with state-dependent stochastic investment horizon. We build up a system of uncertain exit time, which requires much more information in addition to prices of securities. By using dynamic programming, we derived the optimal investment policy and efficient frontier of the portfolio selection problem, which has a similar structure to the case with deterministic investment horizon. However, introducing this new line of risk, uncertainty of investment horizon, into the multi-period portfolio selection increases the investment risk.

There are several possible extensions on this research subject in the future. The first one is to consider the portfolio selection problem with a stochastic investment horizon under general utility criterion or other mean-risk criteria. As we have known, mean-variance criteria has some limitations. For example, extreme events may be ignored when using mean-variance criterion and investor could face a risk with a significant magnitude as a consequence. Adopting some suitable criteria, such as the mean-CVaR criterion, will avoid this kind of problem. The second one is to consider the portfolio selection problems with an uncertain exit time due to endogenous reasons, in which situation the exit time is a stopping time and is dependent on the investor's wealth.

We have considered in the second part of this thesis the multi-period portfolio selection problem in a frictional market with no-shorting or with transaction cost. Actually, when the market is frictional, the portfolio optimization problem will become very complex. We overcome the difficulty in this thesis by a dual method. By constructing dual problems, the optimal solution can be determined by the optimal solutions of the corresponding dual problem. Explicitly, if we can get the optimal solution of the dual problem, we will define a corresponding frictionless auxiliary market. Therefore, the original portfolio selection problem in a frictional market is equivalent to a portfolio selection problem in the frictionless auxiliary market, which is much easier to handle. Our result, which is based on a quadratic utility function, can be extended to general utility function.

There remain some outstanding open problems for us. Although we can derive the optimal solution of the dual problem by using dynamic programming under the situation with no-shorting in Chapter 4, it is difficult for us to solve the dual problem under the situation with transaction costs in Chapter 5, because of the ambiguity of the feasible set $\mathcal{L}$. Another challenging problem is whether we can investigate portfolio optimization problems with other kinds of constraints, such as constraints of upper and lower bounds, by using the duality analysis.

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