Robust Stabilization and Output Regulation of Nonlinear Feedforward Systems and Their Applications

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Contents

Abstract

This thesis contains two parts. The first part studies the global robust stabilization problem of feedforward systems and the second part further addresses the global robust output regulation problem of the same class of nonlinear systems.

The stabilization problem of feedforward systems has absorbed a lot of attention during the past fifteen years. More recently, the stabilization problem of feedforward systems subject to input unmodeled dynamics is studied. Nevertheless, the more realistic case where the system is subject to both time-varying static and dynamic uncertainties has not been adequately investigated. The first part of this thesis focuses on the global robust stabilization problem for various classes of feedforward systems containing both time-varying static and dynamic uncertainties. The major results are summarized as follows.

(i) A pure small gain approach is proposed to handle a disturbance attenuation problem for a class of feedforward systems subject to both dynamic uncertainty and disturbance. Two versions of small gain theorem with restrictions are employed to establish the global attractiveness and local stability of the closed-loop system at the origin, respectively. Unlike Lyapunov's linearization method and asymptotic small gain theorem combined approach, the proposed approach does not require the stabilizability assumption of the Jacobian linearization of the system at the origin.

(ii) A small gain based bottom-up recursive design is developed to solve a global robust stabilization problem for a class of feedforward systems subject to both time-varying static and dynamic uncertainties. Unlike most existing results, our design does not require the bottom dynamics at each recursion be locally exponentially stable.

(iii) The small gain based bottom-up recursive design is further extended to deal with a global robust stabilization problem for a class of feedforward systems which are approximated at the origin by a nonlinear chain of integrators and perturbed by some type of input unmodeled dynamics. Even in the special case where the input unmodeled dynamics is not present, our result is new in the sense that our approach can handle some cases that cannot be handled by any existing approaches.

It is now well known from the general framework for tackling the output regulation problem that the robust output regulation problem can be approached in two steps. In the first step, the problem is converted into a robust stabilization problem of a so-called augmented system which consists of the original plant and a suitably defined dynamic system called an internal model candidate, and in the second step, the robust stabilization

problem of the augmented system is further pursued. The success of the first step depends on whether or not an internal model candidate exists. Even though the first step succeeds, the success of the second step is by no means guaranteed due to at least two obstacles. First, the stabilizability of the augmented system is dictated not only by the given plant but also by the particular internal model candidate employed. Second, the stabilization problem of the augmented system is much more challenging than that of the original plant with the exogenous signal set to 0, because the structure of the augmented system may be much more complex than that of the original plant. Perhaps, it is because of these difficulties, so far almost all papers on semi-global or global robust output regulation problem are focused on the lower triangular systems, feedback linearizable systems and output feedback systems. The second part of this thesis aims to study the global robust output regulation problem of feedforward systems. The major results are summarized as follows.

(i) We first identify structural properties of the plant so that an internal model candidate exists. Then, by looking for a suitable internal model and performing appropriate transformations on the augmented system, we succeed in converting the global robust output regulation problem for a class of feedforward systems into a global robust stabilization problem for a class of feedforward systems subject to both time-varying static and dynamic uncertainties. As a result, the global robust stabilization result obtained in the first part of this thesis is used to solve the global robust output regulation problem for a class of feedforward systems.

(ii) We apply the result of the global robust output regulation problem to solve two trajectory tracking problems for a chain of integrators with uncertain parameters and the Vertical Take-Off and Landing (VTOL) aircraft, respectively. In contrast with the existing designs, for the chain of integrators, our design is low gain and does not need to know the reference trajectory exactly, and for the VTOL aircraft, our design is a complete low gain design and thus is more cost effective.

(iii) We propose a Lyapunov approach to a special case of the output regulation problem, the input disturbance suppression problem for a class of feedforward systems. When the exosystem is known, we solve the problem via dynamic output feedback control. When the exosystem is unknown, we solve the problem via adaptive dynamic state feedback control and we also give the conditions under which an estimated parameter vector can converge to the true parameter vector.

摘要

本文分爲兩個部分。本文的第一部分研究了前饋系統的鎮定問題，而第二部分則研究了前饋系統的輸出調節問題。

在過去的十五年裏，前饋系統的鎮定問題吸引了許多研究者的注意。最近，許多研究者對受到輸入未建模動態影響的前饋系統的鎮定問題進行了研究。然而，現在還沒有人研究同時受到動態和靜態不確定性影響的前饋系統的鎮定問題。本文的第一部分研究了受到動態和靜態不確定性影響的各種類型的前饋系統的鎮定問題。所取得主要結果如下所示：

（1） 我們提出了一種僅基於小增益定理的方法來處理同時受到動態不確定性和外加干擾影響的一類前饋系統的干擾抑制問題。我們用兩種類型的受限小增益定理分別確定閉環系統在平衡點處的局部穩定性和全局收斂性。與 Lyapunov 線性化和小增益定理相結合的方法不同，我們的方法並不需要所研究的系統在平衡點處的線性化可鎮定。

（2） 爲解決同時受到動態和時變靜態不確定性影響的一類更一般的前饋系統的鎮定問題， 我們提出了一種以小增益定理爲基礎的遞迴設計方法。 與大多數已有方法不同，我們的方法並不要求每次遞推過程中位於串聯系統下面的動態子系統局部指數穩定。

（3） 通過對上面提出的以小增益定理爲基礎的遞迴設計方法的推廣，我們研究了在平衡點處可近似爲非線性積分器並且受到輸入未建模動態影響的一類前饋系統的鎮定問題。在不受到輸入未建模動態影響的特殊情況下，我們的鎮定結果也是新的，這是因爲我們的方法仍然可以處理一些不能用已有方法來處理的前饋系統。

根據處理輸出調節問題的一般框架，輸出調節問題可以分爲兩個步驟解決。第一步，將所研究系統的輸出調節問題轉化爲一個增廣系統的鎮定問題，這裏的增廣系統由所研究系統和一個候選內模所組成。第二步，解決增廣系統的鎮定問題。第一步的成功取決於候選內模是否存在。即使第一步成功，因爲兩個障礙，第二步不一定會成功。第一，增廣系統的可鎮定性不僅取決於所研究的系統，還取決於所選取的特殊內模。第二，因爲增廣系統的結構要比所研究系統更加複雜，所以增廣系統的鎮定問題要比在外加信號設置爲零時所研究系統的鎮定問題更加困難。可能由於如上所述的諸多困難，到目前爲止幾乎所有的半全局或全局魯棒輸出調節問題都是針對下三角系統，可反饋線性化系統和輸出反饋系統。本文的第二部分將以前饋系統的全局魯棒輸出調節問題爲主要研究內容。所取得的主要結果如下所示：

（1）爲確保候選內模的存在性，我們首先確定了所研究系統需要滿足的結構條件。然後，通過設計合適的內模和尋找合適的座標和輸出變換，我們成功地將原系統的魯棒輸出調節問題轉化爲同時受到動態和靜態不確定性影響的一類一般的前饋系統

的鎮定問題。最終，我們利用本文第一部分中所得的鎮定結果解決了一類前饋系統的全局魯棒輸出調節問題。

（2） 我們利用所得的全局魯棒輸出調節問題的結果研究了帶有不確定參數的積分器和垂直起降飛行器的軌跡跟蹤問題。與已有設計相比，對於帶有不確定參數的積分器，我們的設計並不需要確切的參考軌跡；對於垂直起降飛行器，我們的設計是低增益的。

（3） 針對一類特殊的前饋系統，我們用 Lyapunov 方法研究了一類特殊的輸出調節問題，即輸入干擾抵消問題。當外部系統已知時，我們設計了動態輸出反饋控制器；當外部系統未知時，我們設計了動態狀態反饋控制器，而且我們還給出了估計參數收斂到其真值的充分條件。

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Chapter 1

Introduction

1.1 Introduction and Literature Survey

The control of nonlinear systems has been one of the major subjects in control theory. Due to the complexity of nonlinear systems, there is no general approach for the control of nonlinear systems. Instead, there are a lot of approaches, each of which is best applicable to a particular class of nonlinear systems. Since the 1990s, some recursive design approaches have been developed for two important classes of nonlinear systems, namely, systems in feedback form and systems in feedforward form. While the research on the first class of systems has reached a certain degree of maturity, the research on the second class of systems is still in its infancy.

Since Teel's seminal work [82], feedforward systems have gradually attracted the attention of the nonlinear control community. It is now known that, feedforward systems occur naturally in the model of many physical systems, such as, the ball and beam system [84], the vertical take-off and landing aircraft [56, 85], the inverted pendulum on a cart [59, 85], the translation and rotational actuator system [33] and the spherical inverted pendulum [1, 52]. Moreover, under certain conditions, a given nonlinear system can be (locally or globally) transformed, via a coordinate or a feedback transformation, into a feedforward system [8, 81, 82]. In particular, a food-chain system [19] is shown to be locally feedback equivalent to a feedforward system in [8]. As a result, it would be meaningful to study the control of feedforward systems.

1.1.1 Stabilization of Feedforward Systems

Stabilization and output regulation are two fundamental problems in the control of nonlinear systems. During the last fifteen years, most of the researchers' attention was devoted to the stabilization problem of feedforward systems. By utilizing the feedforward structure, a general way of handling the stabilization problem of feedforward systems is by setting up a bottom-up recursive design procedure: design a control for the bottom dynamics first and then iterate the design upwards by $n-1$ times. There are two major approaches to the stabilization problem of feedforward systems: the Lyapunov approach and the small gain approach.

Lyapunov Approach

The Lyapunov approach to the state feedback stabilization problem was first introduced by Mazenc and Praly [59] where the feedforward system should satisfy some restrict structural properties. Jankovic, Sepulchre and Kokotovic [29] developed a different Lyapunov approach for an enlarged class of feedforward systems. However, their approach requires, in general, complicate computations to derive the exact cross term in the Lyapunov function. The same authors in [72] refined their results [29] for a special class of feedforward systems as studied in [29, 59] and introduced formally the "forwarding" approach. Nonetheless, to apply the forwarding approach, one still has to solve a series of nonlinear systems and compute a series of integrals. More recently, Krstic provided in [45] closed-form solutions to these nonlinear systems and integrals for some more special classes of feedforward systems than the one studied in [72]. In both [45] and [72], the designed control, by inverse optimality [46, 47, 71], can guarantee the global asymptotic stability of the equilibrium of the feedforward system in the presence of a class of input unmodeled dynamics satisfying certain passivity property.

The Lyapunov approach [29, 59] has been further developed [16, 50, 58, 60, 69, 72, 86, 88]. Mazenc [58] extended the Lyapunov approach [59] to handle a class of feedforward systems which are approximated at the origin by a nonlinear chain of integrators and satisfy some growth conditions. Lin and Qian [50, 69] generalized the work of Mazenc [58] by relaxing both the restriction on the nonlinear chain of integrators and those growth conditions. In the spirit of [50, 59], Ye and Unbehauen [88] designed a global adaptive stabilizer for the feedforward system, as studied in [50, 59, 69], but perturbed by some unknown constant parameter whose bound is not known a priori. A common assumption of

[50, 59, 69, 88] is that the order sequence of the chain of integrators has to be nondecreasing. The case where the assumption does not hold was also studied by Tsinias and Tzamtzi [86], and Frye, Trevino and Qian [16]. The price to be paid for removing the assumption is that the feedforward system has to satisfy some more restrict growth conditions.

The Lyapunov approach to the output feedback stabilization problem has also been widely addressed in the recent literature [9, 14, 16, 40, 44, 61, 66, 68]. Maznec [61] studied the global asymptotic stabilization problem for a class of cascaded systems. For a class of feedforward systems satisfying linear growth condition, a linear output feedback controller is constructed in [14]. Qian and Li [68] proposed a homogeneous domination approach for a class of feedforward systems satisfying some nonlinear growth conditions. Later, the approach of [68] was further extended in [16] to handle feedforward systems which are approximated at the origin by a nonlinear chain of integrators and satisfy some restrict growth conditions. Polendo and Schrader [66] presented an approach for a very special class of feedforward systems. The advantage of their approach is that the maximum amplitude of the nested saturation controller can be set prior to designing the control. More recently, the dynamic high gain scaling technique proposed by Krishnamurthy and Khorrami [39, 40] has been applied to feedforward systems [41, 43, 44]. In particular, [44] studied an output feedback stabilization and disturbance attenuation problem for a class of feedforward systems subject to some dynamic uncertainty satisfying certain ISS property. However, the feedforward systems studied in [41, 43, 44] have to satisfy some bounds which are linearly bounded in the unmeasured states and polynomially bounded in the output.

Small Gain Approach

In [85], Teel generalized his stabilization result of [82] by utilizing the nonlinear small gain technique. In particular, Teel proposed an asymptotic small gain theorem and Lyapunov's linearization method combined approach for constructing a nested saturation control such that the closed-loop system is input-to-state stable (ISS) with restriction on the external disturbance. Following the lines of [85], Angeli, Chitour and Marconi extended the result of [85] in [4] where the feedforward system is allowed to contain some unknown bounded constant parameters. However, their result only holds for the feedforward system where the order of each subsystem is not greater than 3.

The recursive design of [85] lies in how to determine a suitable saturation level λ and a good saturated linear controller F at each recursion. F can be determined in advance and two straightforward methods are given in [85]. It can be seen from the proof of Theorem 3 of [85] that, given any F, the small gain condition can always be satisfied by adjusting only the saturation level λ. Therefore, the main difficulty of the solution of Theorem 5 of [85] is how to determine a suitable saturation level λ at each recursion. Since the designed control at each recursion generally depends on all the states of the cascaded system considered at the recursion, as evident from Theorem 5 of [85], such recursive design usually necessitates that all the states of the feedforward system should be available for feedback. If part of the states of the lower subsystem of the cascade system considered at the recursion is not available for feedback, then as shown in [7], adjusting only the saturation level λ is not enough to guarantee the satisfaction of the small gain condition. As a result, F cannot be determined in advance anymore and should be chosen suitably to render the satisfaction of the small gain condition. In this case, how to determine F becomes a difficult and challenging task. It is because of this difficulty that Arcak, Teel and Kokotovic in [7] provided a different recursive design for a subclass of feedforward systems as studied in [85] in the presence of some type of input unmodeled dynamics. In the recursive design of [7], the designed control at each recursion only depends on the state of the upper subsystem of the cascade system considered at the recursion. Nonetheless, the recursive design of [7] will fail if the feedforward system contains certain time-varying static uncertainty. There are two reasons causing the failure of the recursive design of [7]: first, it relies upon the existence of a time invariant coordinate transformation; second, the argument to show the exponential stability of the Jacobian linearization does not hold when the Jacobian linearization contains certain time-varying parameters.

Inspired by [7] and [17], Marconi and Isidori presented in [53] a new design to handle the case where the feedforward system contains time-varying static uncertainty. The problem is solved by first showing an auxiliary system is ISS with restriction on the exogenous input with arbitrarily small linear gains and then utilizing an adaptation of the asymptotic small gain theorem [28] to conclude the global asymptotic stability. Later, the design of [53] was further extended in [54] to handle a global robust stabilization problem for a class of feedforward systems subject to input unmodeled dynamics.

More recently, a partial state feedback stabilization problem of an affine control cascade system is solved by Kaliora and Astolfi [33] using the linear bounded real lemma and a generalized version of the small gain theorem [70]. In particular, by assuming the controllability of the linearized system, [33] converted the stabilization problem for the original feedforward system into a stabilization problem for its linearized system. The approach of [33] was further extended to handle an output feedback stabilization problem for a class of block feedforward systems in [9].

1.1.2 Output Regulation of Feedforward Systems

Output regulation problem of nonlinear systems has been one of the central control problems for nearly two decades [10, 12, 15, 20, 21, 22, 23, 24, 25, 27, 34, 35, 63, 64, 73, 74]. The research was first focused on the local version of the problem where all the initial conditions and uncertain parameters are assumed to be sufficiently small [10, 20, 23, 24, 27, 63]. The research on the nonlocal version of the problem started in the late 1990s [12, 15, 22, 25, 34, 35, 64, 73, 74]. It is now well known (see e.g. [22]) that the robust output regulation problem can be approached in two steps. In the first step, the problem is converted into a robust stabilization problem of a so-called augmented system which consists of the original plant and a suitably defined dynamic system called an internal model candidate, and in the second step, the robust stabilization problem of the augmented system is further pursued. The success of the first step depends on whether or not an internal model candidate exists which can usually be ascertained by the property of the solution of the regulator equations. Even though the first step succeeds, the success of the second step is by no means guaranteed due to at least two obstacles. First, the stabilizability of the augmented system is dictated not only by the given plant but also by the particular internal model candidate employed. An internal model candidate can be chosen from an infinite set of dynamic systems and a suitable internal model candidate is usually obtained from the past experience and some trial and error. Second, the structure of the augmented system may be much more complex than that of the original plant. Therefore, even though the stabilization of the original plant with the exogenous signal set to 0 is solvable, the stabilization of the augmented system may still be untractable.

Perhaps, it is because of these difficulties, so far almost all papers on semi-global or global robust output regulation problem are focused on the lower triangular systems [12, 22, 25, 74], feedback linearizable systems [34, 35], and output feedback systems [15, 73]. To our knowledge, the only papers that are relevant to the output regulation problem of feedforward systems are [9] and [56]. An approximate and restricted tracking problem for a class of block feedforward systems is studied in [9] via dynamic output feedback control. In [56], the authors deal with an input disturbance suppression problem via dynamic state feedback control. The problem is addressed by converting it into a global robust stabilization problem for a class of feedforward systems subject to input unmodeled dynamics. Several results about this robust stabilization problem have been reported, see e.g., [7, 45, 54, 72].

1.2 Organization of the Thesis

The reminder of this thesis is organized as follows.

Chapter 2: To make the thesis self-contained, we introduce some fundamental concepts and methods which will be utilized in the subsequent chapters.

Chapter 3: A pure small gain approach is proposed to handle a disturbance attenuation problem for a class of feedforward systems subject to both dynamic uncertainty and disturbance.

Chapter 4: A small gain based bottom-up recursive design is developed to solve a global robust stabilization problem for a class of feedforward systems subject to both time-varying static and dynamic uncertainties.

Chapter 5: The small gain based bottom-up recursive design is further extended to handle a global robust stabilization problem for a class of feedforward systems which are approximated at the origin by a nonlinear chain of integrators and perturbed by some type of input unmodeled dynamics.

Chapter 6: Using the stabilization result obtained in Chapter 4, a global robust output regulation problem for a class of feedforward systems is solved.

Chapter 7: The result of global robust output regulation problem obtained in the previous chapter is applied to solve two trajectory tracking problems for a chain of integrators with uncertain parameters and the VTOL aircraft, respectively.

Chapter 8: The dynamic high gain scaling technique is utilized to handle an input disturbance suppression problem for a class of feedforward systems.

The thesis was typeset using LaTeX. All numerical simulations were done using MATLAB.

Chapter 2

Fundamental Properties of Nonlinear Systems

In this chapter, we introduce some fundamental concepts and properties of nonlinear systems that will be utilized in the subsequent chapters. The stability property and some fundamental methods to check the stability property of nonlinear system will be given in Section 2.1. In Section 2.2, the input-to-state stability and its local extensions are summarized. Finally, we provide two small gain theorems with restrictions adapted from [85].

Throughout the thesis, we will use (x_1, x_2) with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ to denote the vector $(x_1^T, x_2^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and let $\mathcal{L}_\infty^m$ be the set of all piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^m$ with a finite supremum norm $\|u\|_\infty = \sup_{t \geq 0} \|u(t)\|$, and let $\|u\|_a = \limsup_{t \to \infty} \|u(t)\|$ denote the asymptotic $\mathcal{L}_\infty$ norm of u, where $\|\cdot\|$ denotes the standard Euclidean norm. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a gain function if it is continuous, nondecreasing, and satisfies $\gamma(0) = 0$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If, in addition, $\alpha(r) \to \infty$ as $r \to \infty$, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$. For a function $g(u, d)$ satisfying $g(0, d) = 0$ for any d, the notation $g(u, d) = o_d(u)$ means $\lim_{\|u\| \to 0} \frac{\|g(u,d)\|}{\|u\|} = 0$. For a square matrix P, let $\bar{\lambda}(P)$ and $\underline{\lambda}(P)$ denote the maximal and minimal eigenvalue of P, respectively.

2.1 Stability of Nonlinear Systems

Consider the following nonlinear system

$$\dot{x} = f(x, d) \tag{2.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^{n_d} \to \mathbb{R}^n$ is locally Lipschitz and $f(0, d) = 0$ for any $d \in \mathbb{R}^{n_d}$, and $d : [0, \infty) \to \mathcal{D}$ is a continuous function with its range $\mathcal{D}$ a compact subset of $\mathbb{R}^{n_d}$. Let $x(t)$ denote the trajectory of system (2.1) with initial state $x(0)$ and time-varying static uncertainty d.

Remark 2.1.1 Clearly, $x = 0$ is an equilibrium point of system (2.1). Since d in (2.1) is a function of time, system (2.1) is a nonautonomous system. However, system (2.1) will be treated as an "autonomous" system in the following because d ranges in a compact set $\mathcal{D}$.

The stability concept considered in this thesis is defined as follows.

Definition 2.1.1 The equilibrium point $x = 0$ of system (2.1) is said to be:

a) stable, if for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\|x(0)\| \leq \delta(\epsilon)$ implies $\|x(t)\| \leq \epsilon$ for all $t \geq 0$.

b) unstable, if it is not stable.

c) locally asymptotically stable, if it is stable and there exists an open set X of the origin of $\mathbb{R}^n$, such that $x(0) \in X$ implies $x(t) \to 0$ as $t \to \infty$. Moreover, if $X = \mathbb{R}^n$, then the equilibrium point $x = 0$ is globally asymptotically stable.

d) exponentially stable, if there exist positive constants c, k and λ such that for any $\|x(0)\| < c$,

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}. \tag{2.2}$$

Moreover, if (2.2) holds for any $x(0)$, then the equilibrium point $x = 0$ is globally exponentially stable.

There are many useful Lyapunov stability theorems in literature. Here we only list some theorems relevant to our work. All these theorems can be found in [36] (see also [26, 75]).

We first introduce two theorems which are usually called Lyapunov's indirect or linearization method for nonautonomous and autonomous nonlinear systems respectively.

From Theorem 4.13 of [36], the exponential stability of the equilibrium point of a nonautonomous nonlinear system can be dictated by the exponential stability of its linearization about the equilibrium point.

Theorem 2.1.1 [36] Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is continuously differentiable, $D = \{x \in \mathbb{R}^n : \|x\| < r\}$, and the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \frac{\partial f(t, x)}{\partial x}|_{x=0}$$

Then, $x = 0$ is an exponential stable equilibrium point for the nonlinear system if it is an exponential stable equilibrium point for the linear system

$$\dot{x} = A(t)x.$$

From Theorem 4.7 of [36], the stability property of the equilibrium point of an autonomous nonlinear system, under certain circumstances, can be dictated by the location of the eigenvalues of its Jacobian matrix about the equilibrium point.

Theorem 2.1.2 [36] Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x) \tag{2.3}$$

where $f(x)$ is continuously differentiable in some neighborhood of $x = 0$. Let the Jacobian matrix

$$A = \frac{\partial f(x)}{\partial x}|_{x=0}. \tag{2.4}$$

Then,

1. $x = 0$ is exponentially stable if $\mathrm{Re}\lambda_i < 0$ for all eigenvalues of A;

2. $x = 0$ is unstable if $\mathrm{Re}\lambda_i > 0$ for one or more eigenvalues of A.

Remark 2.1.2 For the autonomous system (2.3), the Lyapunov's linearization method fails when $\mathrm{Re}\lambda_i \leq 0$ for all i and $\mathrm{Re}\lambda_i = 0$ for some i. In this critical case, the equilibrium point $x = 0$ could be asymptotically stable, stable or unstable (for illustration, see Example 4.14 of [36]). Nonetheless, the critical case can be handled by the center manifold theory [11],[65].

The following theorem is a sufficient condition for the uniform asymptotic stability of a nonautonomous system.

Theorem 2.1.3 [36] Let $x = 0$ be an equilibrium point for system

$$\dot{x} = f(t, x) \tag{2.5}$$

where $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain containing $x = 0$. Let $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$
$$\dot{V}(t, x) = \tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where $W_i(x), i = 1, 2, 3$, are positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable. Moreover, if $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.

The next theorem is known as Barbalat's Lemma.

Theorem 2.1.4 [36] Let $\phi : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t \to \infty} \int_0^t \phi(\tau) d\tau$ exists and is finite. Then

$$\lim_{t \to \infty} \phi(t) = 0 \tag{2.6}$$

The following theorem is an extension of LaSalle's invariance theorem to nonautonomous system.

Theorem 2.1.5 [36] Let $D \subset \mathbb{R}^n$ be a domain containing $x = 0$ and suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x, uniformly in t, on $[0, \infty) \times D$. Furthermore, suppose $f(t, 0)$ is uniformly bounded for all $t \geq 0$. Let $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$
$$\dot{V}(t, x) = \tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are positive definite functions and $W_3(x)$ is a positive semidefinite function on D. Choose $r \geq 0$ such that $B_r \subset D$ and let

$\rho < \min_{\|x\|=r} W_1(x)$. Then all solutions of $\dot{x} = f(t, x)$ with $x(t_0) \in \{x \in B_r : W_2(x) \le \rho\}$ are bounded and satisfy

$$\lim_{t\to\infty} W_3(x(t)) = 0 \tag{2.7}$$

Moreover, if all the assumptions hold globally and $W_1(x)$ is radially unbounded, the statement is true for all $x(t_0) \in \mathbb{R}^n$.

2.2 Input-to-State Stability and Its Local Extensions

Input-to-state stability (ISS) was introduced by Sontag in [76] and many results related to ISS have been obtained during the last fifteen years, see [78] for a tutorial. As a result, ISS and its extended variations, such as integral-ISS [5], together with their related notions of input-output stability and detectability have become a very useful framework—ISS framework [77] for nonlinear feedback analysis and design [38].

2.2.1 Input-to-State Stability

Consider the following system

$$\dot{x} = f(x, u) \tag{2.8}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and $f(0,0) = 0$, and input $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ is piecewise continuous. Let $x(t)$ denote the trajectory of system (2.8) with initial state $x(0)$ and input u.

Definition 2.2.1 [76] System (2.8) is input-to-state stable (ISS) if there exist class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ such that, for all $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$, the following holds

$$\|x(t)\| \le \max\{\beta(\|x(0)\|), t), \gamma(\|u\|_\infty)\}, \quad \forall t > 0 \tag{2.9}$$

Remark 2.2.1 If the equilibrium point $x = 0$ of system (2.8) with $u = 0$ is globally asymptotically stable, then system (2.8) is said to be 0-GAS [5]. Obviously, if system (2.8) is ISS, then system (2.8) is 0-GAS.

There exists a Lyapunov-like characterization of ISS, which extends the well known Lyapunov theorem for asymptotic stability.

Definition 2.2.2 [79] A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS Lyapunov function for system (2.8) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \bar{\alpha}, \alpha$ and class $\mathcal{K}$ function χ, such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, it holds that

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|) \tag{2.10}$$

$$\chi(\|u\|) \leq \|x\| \Rightarrow \frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha(\|x\|) \tag{2.11}$$

The following theorem shows that the existence of an ISS Lyapunov function is necessary as well as sufficient for the system to be ISS.

Theorem 2.2.1 [79] System (2.8) is ISS if and only if it admits an ISS Lyapunov function.

Remark 2.2.2 If one can find an ISS Lyapunov function for a given system (2.8), then the system is ISS by the above theorem and moreover, the gain function γ which appears in the estimate (2.9) can be computed from the functions $\underline{\alpha}, \bar{\alpha}, \alpha$ and χ as follows [26]:

$$\gamma(r) = \underline{\alpha}^{-1} \circ \bar{\alpha} \circ \chi(r) \tag{2.12}$$

Remark 2.2.3 Notice that, for any pair $\beta \geq 0, \gamma \geq 0$, $\max\{\beta, \gamma\} \leq \beta + \gamma \leq \max\{2\beta, 2\gamma\}$, thus an equivalent characterization of ISS is that, there exist class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ such that, for all $x(0) \in \mathbb{R}^n$ and all $u \in \mathcal{L}_\infty^m$, the following holds

$$\|x(t)\| \leq \beta(\|x(0)\|), t) + \gamma(\|u\|_\infty), \quad \forall t > 0 \tag{2.13}$$

The imposition of some restriction on the input leads to the notion of ISS with restriction on the input.

Definition 2.2.3 [4] System (2.8) is said to be ISS with restriction Δ on u, if there exist $\Delta > 0$, class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ such that, for all $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$ with $\|u\|_\infty \leq \Delta$, the following holds

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|), t), \gamma(\|u\|_\infty)\}, \quad \forall t > 0$$

Definition 2.2.4 [4] System (2.8) satisfies the ultimate boundedness (UBND) property with restriction Δ on the input u, if there exists a nondecreasing function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any initial condition $x(0) \in \mathbb{R}^n$ and any $u \in \mathcal{L}_\infty^m$ with $\|u\|_\infty \leq \Delta$, the following holds

$$\|x\|_a \leq \gamma(\|u\|_\infty) \tag{2.14}$$

Remark 2.2.4 If γ is strengthened to be of class $\mathcal{K}$ function, then UBND is strengthened to the asymptotic gain (AG) property [80]. Clearly, UBND is a concept weaker than AG.

The following fact states an equivalent characterization of ISS with restriction on the input.

Theorem 2.2.2 [4] System (2.8) is ISS with restriction on the input if and only if it is 0-GAS and satisfies UBND with suitable restriction.

2.2.2 Local Extensions of ISS

The imposition of restrictions on both of the initial state and the input leads to the local extension of ISS.

Definition 2.2.5 [85] The output y of system (2.8) is said to satisfy an a-$\mathcal{L}_\infty$ stability bound (a-LB) with restrictions X, Δ on $x(0), u$ and gains γ^0, γ respectively, if there exist open set X of the origin of $\mathbb{R}^n$, positive real number Δ, gain functions γ^0, γ, such that, for each $x(0) \in X$, $\|u\|_\infty < \Delta$, the solution of (2.8) exists for all $t \geq 0$ and

$$\|y\|_\infty \leq \max\{\gamma^0(\|x(0)\|), \gamma(\|u\|_\infty)\}, \tag{2.15}$$

$$\|y\|_a \leq \gamma(\|u\|_a). \tag{2.16}$$

Remark 2.2.5 It has been shown in [80] that, the state of system (2.8) satisfies a-LB if and only if the system is ISS. As a result, a-LB with restrictions is also called ISS with restrictions in some literature, e.g., [4],[26].

Lemma 3.3 of [85] provides a sufficient condition for a-LB with restrictions.

Theorem 2.2.3 [85] Suppose there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, globally invertible gain functions $\underline{\alpha}$ and $\bar{\alpha}$, a gain function γ, and strictly positive real numbers (or possibly ∞) δ_u and δ_x, such that (2.10) and the following inequality hold

$$\gamma(\|u\|) < \|x\| < \delta_x, \ \|u\| < \delta_u \Rightarrow \frac{\partial V}{\partial x} f(x, u) < 0. \tag{2.17}$$

Let r and Δ be strictly positive real numbers (or possibly ∞) satisfying

$$\underline{\alpha}^{-1} \circ \bar{\alpha}(r) \leq \delta_x, \ \underline{\alpha}^{-1} \circ \bar{\alpha} \circ \gamma(\Delta) \leq \delta_x, \ \Delta \leq \delta_u. \tag{2.18}$$

Then the state of system (2.8) satisfies a-LB with restriction $\{x \in \mathbb{R}^n : \|x\| < r\}$ on $x(0)$ and gain $\alpha^{-1} \circ \bar{\alpha}$, restriction Δ on u and gain $\alpha^{-1} \circ \bar{\alpha} \circ \gamma$.

Definition 2.2.6 [85] The output y of system (2.8) is said to satisfy an asymptotic bound (AB) with restriction X on $x(0)$, restriction Δ on u and gain γ, if there exist open set X of the origin of $\mathbb{R}^n$, non-negative real number Δ, gain function γ, such that, for each $x(0) \in X$ and piecewise continuous u satisfying $\|u\|_a \leq \Delta$, the solution of (2.8) exists for all $t \geq 0$ and

$$\|y\|_a \leq \gamma(\|u\|_a). \tag{2.19}$$

Remark 2.2.6 If the state of system (2.8) satisfies a-LB with restriction on $x(0)$, then the equilibrium point $x = 0$ of system (2.8) with $u = 0$ is locally asymptotically stable; if the state of system (2.8) satisfies AB with restriction no restriction on $x(0)$, then the equilibrium point $x = 0$ of system (2.8) with $u = 0$ is globally attractive. As a result, the combination of a-LB and AB can be used to study the global asymptotic stability of the equilibrium point $x = 0$ of system (2.8) with $u = 0$. However, it can be seen from (2.16) and (2.19) that, there is some overlap between a-LB and AB, and (2.16) turns out redundant in showing the asymptotic stability. This observation motivates us to introduce the $\mathcal{L}_\infty$ stability bound.

Definition 2.2.7 The output y of system (2.8) is said to satisfy a $\mathcal{L}_\infty$ stability bound (LB) with restrictions X, Δ on $x(0), u$ and gains γ^0, γ respectively, if there exist open set X of the origin of $\mathbb{R}^n$, positive real number Δ, gain functions γ^0, γ such that, for each $x(0) \in X$, $\|u\|_\infty < \Delta$, the solution of (2.8) exists for all $t \geq 0$ and

$$\|y\|_\infty \leq \max\{\gamma^0(\|x(0)\|), \gamma(\|u\|_\infty)\}. \tag{2.20}$$

Remark 2.2.7 A subtle difference between the concept a-LB and the combination of the concepts LB and AB is that AB requires that inequality (2.19) hold for all $\|u\|_a \leq \Delta$ while a-LB requires that inequality (2.16) hold for all $\|u\|_\infty < \Delta$. To be more specific, we first show that LB+AB $\Rightarrow$ a-LB. Suppose the output y of system (2.8) satisfies LB with restrictions X_s, Δ_s on $x(0), u$ and gains γ^0, γ_s respectively, and satisfies AB with restriction X_a on $x(0)$, restriction Δ_a on u and gain γ_a. Then, the output y of system (2.8) satisfies a-LB with restrictions $X_s \cap X_a, \min\{\Delta_s, \Delta_a\}$ on $x(0), u$ and gains $\gamma^0, \max\{\gamma_s, \gamma_a\}$ respectively. However, the converse is not true. Note that, the state of the system

$$\dot{x} = -x^3 + x^3 u \tag{2.21}$$

satisfies a-LB with restriction 1 on u. However, x does not satisfy AB with any restriction on $x(0)$ and u, because given any $x(0) \neq 0$ there exists $u(t)$ satisfying $\|u(t)\|_a = 0$ such that $x(t)$ diverges in finite time. For example, let $b > \frac{x(0)^{-2}}{2}$, $u(t) = 2$ for $0 \leq t \leq b$ and $u(t) = 0$ for $t > b$, then the solution of (2.21) becomes $x(t) = (x(0)^{-2} - 2t)^{-\frac{1}{2}}$ and $x(t)$ diverges as $t \to x(0)^{-2}/2$. In addition, the relation between a-LB and AB is also discussed on page 1260 of [85].

Remark 2.2.8 The combination of LB and AB can also be used to study the asymptotic stability of system (2.8) with $u = 0$. More specifically, if the state x of system (2.8) satisfies LB with restriction on $x(0)$, then the equilibrium point $x = 0$ of system (2.8) with $u = 0$ is locally stable; if the state x of system (2.8) satisfies AB with restriction X_a on $x(0)$, then the equilibrium point $x = 0$ of system (2.8) with $u = 0$ is locally attractive. As a result, the equilibrium point $x = 0$ of system (2.8) with $u = 0$ is locally asymptotically stable, and if, in addition, $X_a = \mathbb{R}^n$, then it is globally asymptotically stable.

In the following, for simplicity, if the state x of system (2.8) satisfies LB, AB or a-LB, then we will say system (2.8) satisfies LB, AB or a-LB. If the output y of system (2.8) satisfies LB with restriction on $x(0)$, restriction Δ on u and gain γ, and satisfies AB with no restriction on $x(0)$, restriction Δ on u and gain γ, then we will say y satisfies LB with restriction and AB with no restriction on $x(0)$, both with restriction Δ on u and gain γ.

Remark 2.2.9 It would be interesting to show that, if system (2.8) satisfies LB with restriction and AB with no restriction on $x(0)$, both with restriction Δ on u, then system (2.8) is ISS with restriction on u. By Theorem 2.2.2 and Remark 2.2.8, we only need to show system (2.8) satisfies UBND with restriction. Define $\mathcal{U}_1 = \{u \in \mathcal{L}_\infty^m : \|u\|_\infty < \Delta\}$ and $\mathcal{U}_2 = \{u : \|u\|_a \leq \Delta\}$. Clearly, $\mathcal{U}_1 \subset \mathcal{U}_2$. Since system (2.8) satisfies AB with restriction Δ on u, $\|x\|_a \leq \gamma(\|u\|_a)$ for all $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{U}_2$. Then, noting $\mathcal{U}_1 \subset \mathcal{U}_2$ and $\|u\|_a \leq \|u\|_\infty$ yields the UBND property as follows

$$\|x\|_a \leq \gamma(\|u\|_a), \ \forall u \in \mathcal{U}_1 \Rightarrow \|x\|_a \leq \gamma(\|u\|_\infty), \ \forall u \in \mathcal{U}_1$$

2.3 Small Gain Theorems with Restrictions

Over the years, several different versions of the small gain theorem in the framework of ISS have been established [31, 32, 85]. In the following, we introduce two versions of the small gain theorem with restrictions adapted from [85] which play a crucial role in the subsequent chapters.

Consider the following nonlinear system

$$\dot{x} = f(x, u, d), \ y = h(x, u, d), \tag{2.22}$$

where $x \in \mathbb{R}^n$ is the plant state, $y \in \mathbb{R}^p$ the output, $u \in \mathbb{R}^m$ the piecewise continuous input, $f(x, u, d)$ and $h(x, u, d)$ are locally Lipschitz functions vanishing at $(0, 0, d)$ for all $d \in \mathcal{D}$, and $d : [0, \infty) \to \mathcal{D}$ is a continuous function with its range $\mathcal{D}$ a compact subset of $\mathbb{R}^{n_d}$. Let $x(t)$ denote the solution of system (2.22) with initial state $x(0)$, input u and d.

Definition 2.3.1 The output y of system (2.22) is said to satisfy a robust $\mathcal{L}_\infty$ stability bound (RLB) with restrictions X, Δ on $x(0), u$ and gains γ^0, γ respectively, if there exist open set X of the origin of $\mathbb{R}^n$, positive real number Δ, gain functions γ^0, γ, all independent of d, such that, for each $x(0) \in X$, $d \in \mathcal{D}$, $\|u\|_\infty < \Delta$, the solution of (2.22) exists for all $t \geq 0$ and

$$\|y\|_\infty \leq \max\{\gamma^0(\|x(0)\|), \gamma(\|u\|_\infty)\}. \tag{2.23}$$

Definition 2.3.2 [85] The output y of system (2.22) is said to satisfy a robust a-$\mathcal{L}_\infty$ stability bound (Ra-LB) with restrictions X, Δ on $x(0), u$ and gains γ^0, γ respectively, if y satisfies RLB with restrictions X, Δ on $x(0), u$ and gains γ^0, γ respectively, and for each $x(0) \in X$, $d \in \mathcal{D}$, $\|u\|_\infty < \Delta$,

$$\|y\|_a \leq \gamma(\|u\|_a). \tag{2.24}$$

The output y is said to satisfy a robust asymptotic bound (RAB) with restriction X on $x(0)$, restriction Δ on u and gain γ, if there exist open set X of the origin of $\mathbb{R}^n$, non-negative real number Δ, gain function γ, all independent of d, such that, for each $x(0) \in X$, $d \in \mathcal{D}$ and piecewise continuous u satisfying $\|u\|_a \leq \Delta$, the solution of (2.22) exists for all $t \geq 0$ and

$$\|y\|_a \leq \gamma(\|u\|_a). \tag{2.25}$$

Remark 2.3.1 In both Definitions 2.3.1 and 2.3.2, the word "robust" is used to emphasize that the inequalities (2.23)-(2.25) hold regardless of the presence of the disturbance d in (2.22). For convenience, we will simply use, in the following, LB, AB and a-LB to mean RLB, RAB and Ra-LB, respectively.

Remark 2.3.2 Lyapunov's linearization method and the asymptotic small gain theorem combined approach [4, 7, 85] has been one of the standard approaches in dealing with the problem of global asymptotic stabilization of feedforward systems. In particular, the Lyapunov's linearization method is utilized to guarantee the local stability, and the asymptotic small gain theorem is used to establish the global attractiveness. The employment of the Lyapunov's linearization method necessitates that the Jacobian linearization of the closed-loop system at the origin must be exponentially stable. However, it is difficult to achieve the exponential stability of the Jacobian linearization at the origin for feedforward systems with uncertain constant parameters [4], and it is even more so if the feedforward system contains both time-varying static and dynamic uncertainty. Furthermore, when the Jacobian linearization of the given plant about the equilibrium point is not stabilizable, it is impossible to use the Lyapunov's linearization method to conclude the local stability at all. Thus, we resort to the *pure* small gain approach instead of combining Lyapunov's linearization method and asymptotic small gain theorem. Note from [85] that, a-LB can be used to study the asymptotic stability of $x = 0$ of system (2.22) with $u = 0$. Therefore, we first turn to a-LB small gain theorem, i.e. Theorem 1 in [85]. It can be seen from [85] that, given two subsystems that satisfy a-LB with no restriction on the initial state and with restriction on the input, a finite restriction on the initial state of the interconnected system may still be incurred due to the restrictions on the inputs of the individual subsystems. As a result, even if the two subsystems satisfy a-LB with no restriction on the initial state and with restriction on the input, a-LB small gain theorem alone can not conclude the global asymptotic stability but the local asymptotic stability. On the other hand, note from [4, 7, 85] that, the global attractiveness can be studied by AB property. Thus, the local stability and the global attractiveness can also be checked by a-LB and AB, and correspondingly the a-LB and asymptotic small gain theorems, respectively. However, it can be seen from (2.16) and (2.19) that, there is some overlap between a-LB and AB, and (2.16) turns out redundant in showing the asymptotic stability. This observation motivates us to introduce LB, and use LB and AB, and correspondingly, LB and AB small gain theorems as described by Propositions 2.3.1 and 2.3.2 to study the local stability and the global attractiveness respectively. As can be seen from the subsequent chapters, such setup leads to a more concise and simpler treatment, since a-LB strictly implies LB.

Now consider the following two systems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, v_1, u_1, d), y_1 = h_1(x_1, v_1, u_1, d) \\ \dot{x}_2 &= f_2(x_2, v_2, u_2, d), y_2 = h_2(x_2, v_2, u_2, d) \end{aligned} \tag{2.26}$$

subject to the interconnection

$$v_1 = y_2, v_2 = y_1 \tag{2.27}$$

where for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $v_i \in \mathbb{R}^{l_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$ with $p_1 = l_2$ and $p_2 = l_1$,

$f_i(x_i, v_i, u_i, d)$ and $h_i(x_i, v_i, u_i, d)$ are locally Lipschitz functions vanishing at $(0,0,0,d)$ for all $d \in \mathcal{D}$. Also suppose the following.

Assumption 2.3.1 The equations

$$y_1 = h_1(x_1, h_2(x_2, y_1, u_2, d), u_1, d),\ \ y_2 = h_2(x_2, h_1(x_1, y_2, u_1, d), u_2, d)$$

have unique solutions y_1 and y_2 such that, under the interconnection (2.27), the interconnected system (2.26) can be written in the form of (2.22) where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $u = (u_1, u_2)$, and the resulting f and h as in (2.22) are locally Lipschitz.

Proposition 2.3.1 Suppose Assumption 2.3.1 is satisfied. Let $y_1 = (y_{11}, y_{12})$ and $v_2 = (v_{21}, v_{22})$ with $y_{11} = v_{21}$ and $y_{12} = v_{22}$. Assume for $i = 1, 2$, the output y_{1i} satisfies LB with restrictions $\bar{X}_{s1}, \bar{\Delta}_1, \bar{\Delta}_{u1}$ on $x_1(0), v_1, u_1$ and gains $\bar{\gamma}_1^0, \bar{\gamma}_{1i}, \bar{\gamma}_{1i}^{u_1}$ respectively, and the output y_2 satisfies LB with restrictions $\bar{X}_{s2}, \bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{u2}$ on $x_2(0), v_{21}, v_{22}, u_2$ and gains $\bar{\gamma}_2^0, \bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}_2^{u_2}$ respectively. If the small gain condition holds, i.e., $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for all $s > 0$ and $i = 1, 2$, then for $i = 1, 2$, the output y_{1i} satisfies LB with restrictions $\check{X}_s, \check{\Delta}_{u1}, \check{\Delta}_{u2}$ on $x(0), u_1, u_2$ and gains $\check{\gamma}_{1i}^0, \check{\gamma}_{1i}^{u_1}, \check{\gamma}_{1i}^{u_2}$ respectively, and y_2 satisfies LB with restrictions $\check{X}_s, \check{\Delta}_{u1}, \check{\Delta}_{u2}$ on $x(0), u_1, u_2$ and gains $\check{\gamma}_2^0, \check{\gamma}_2^{u_1}, \check{\gamma}_2^{u_2}$ respectively, where $\check{X}_s = \check{X}_{s1} \times \check{X}_{s2}$, and

$$\begin{aligned}
\check{X}_{s1} = \{x_1 \in \bar{X}_{s1} : \bar{\gamma}_1^0(\|x_1\|) < \min\{\bar{\Delta}_{21}, \bar{\Delta}_{22}\}, \bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_1^0(\|x_1\|) < \bar{\Delta}_{21} \\
\bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(\|x_1\|) < \bar{\Delta}_{22}, \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(\|x_1\|) < \bar{\Delta}_1, \bar{\gamma}_{22} \circ \bar{\gamma}_1^0(\|x_1\|) < \bar{\Delta}_1\}, \\
\check{X}_{s2} = \{x_2 \in \bar{X}_{s2} : \bar{\gamma}_2^0(\|x_2\|) < \bar{\Delta}_1, \bar{\gamma}_{11} \circ \bar{\gamma}_2^0(\|x_2\|) < \bar{\Delta}_{21}, \bar{\gamma}_{12} \circ \bar{\gamma}_2^0(\|x_2\|) < \bar{\Delta}_{22}\},
\end{aligned} \tag{2.28}$$

and for $i = 1, 2$, $\check{\gamma}_{1i}^0, \check{\gamma}_{1i}^{u_1}, \check{\gamma}_{1i}^{u_2}, \check{\gamma}_2^{u_1}, \check{\gamma}_2^{u_2}$ are defined as

$$\begin{aligned}
\check{\gamma}_{11}^0(s) &= \max\{\bar{\gamma}_1^0(s), \bar{\gamma}_{11} \circ \bar{\gamma}_2^0(s), \bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_1^0(s)\}, \\
\check{\gamma}_2^0(s) &= \max\{\bar{\gamma}_2^0(s), \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(s), \bar{\gamma}_{22} \circ \bar{\gamma}_1^0(s)\}, \\
\check{\gamma}_{12}^0(s) &= \max\{\bar{\gamma}_1^0(s), \bar{\gamma}_{12} \circ \bar{\gamma}_2^0(s), \bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(s)\}, \\
\check{\gamma}_{11}^{u_1}(s) &= \max\{\bar{\gamma}_{11}^{u_1}(s), \bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(s)\}, \\
\check{\gamma}_{11}^{u_2}(s) &= \bar{\gamma}_{11} \circ \bar{\gamma}_2^{u_2}(s), \check{\gamma}_{12}^{u_1}(s) = \max\{\bar{\gamma}_{12}^{u_1}(s), \bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(s)\}, \\
\check{\gamma}_{12}^{u_2}(s) &= \bar{\gamma}_{12} \circ \bar{\gamma}_2^{u_2}(s), \check{\gamma}_2^{u_1}(s) = \max\{\bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(s), \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(s)\}, \check{\gamma}_2^{u_2}(s) = \bar{\gamma}_2^{u_2}(s),
\end{aligned} \tag{2.29}$$

$\check{\Delta}_{u1}, \check{\Delta}_{u2}$ are positive real numbers satisfying $\check{\Delta}_{u1} \leq \bar{\Delta}_{u1}, \check{\Delta}_{u2} \leq \bar{\Delta}_{u2}$ and

$$\begin{aligned}
&s \in [0, \check{\Delta}_{u1}) \Rightarrow \max\{\bar{\gamma}_{11}^{u_1}(s), \bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(s)\} < \bar{\Delta}_{21}, \\
&\quad \max\{\bar{\gamma}_{12}^{u_1}(s), \bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(s)\} < \bar{\Delta}_{22}, \max\{\bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(s), \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(s)\} < \bar{\Delta}_1, \\
&s \in [0, \check{\Delta}_{u2}) \Rightarrow \bar{\gamma}_{11} \circ \bar{\gamma}_2^{u_2}(s) < \bar{\Delta}_{21}, \bar{\gamma}_{12} \circ \bar{\gamma}_2^{u_2}(s) < \bar{\Delta}_{22}, \bar{\gamma}_2^{u_2}(s) < \bar{\Delta}_1.
\end{aligned} \tag{2.30}$$

Proof: The result can be viewed as a variant of Theorem 1 of [85] and the proof is divided into two parts as that of Theorem 1 of [85].

$\bar{\Delta}_1 = \bar{\Delta}_{21} = \bar{\Delta}_{22} = \infty$: Given a particular initial condition $x(0) \in \check{X}_{s1} \times \check{X}_{s2}$ and input signal $w = (u_1, u_2)$ with $\|u_1\|_\infty < \check{\Delta}_{u1}$ and $\|u_2\|_\infty < \check{\Delta}_{u2}$, let $[0, T)$ be the maximal interval of definition. Then, since w is piecewise continuous, by Assumption 2.3.1 we have that $\|y_{11t}\|_\infty < \infty$, $\|y_{12t}\|_\infty < \infty$ and $\|y_{2t}\|_\infty < \infty$ for all $t \in [0, T)$. Then, by casuality we have for each $t \in [0, T)$

$$\|y_{11t}\|_\infty \le \max\{\bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{11}(\|y_{2t}\|_\infty), \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty)\} \tag{2.31}$$

$$\|y_{12t}\|_\infty \le \max\{\bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{12}(\|y_{2t}\|_\infty), \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty)\} \tag{2.32}$$

$$\|y_{2t}\|_\infty \le \max\{\bar{\gamma}_2^0(\|x_2(0)\|), \bar{\gamma}_{21}(\|y_{11t}\|_\infty), \bar{\gamma}_{22}(\|y_{12t}\|_\infty), \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\} \tag{2.33}$$

Substituting (2.31) and (2.32) into (2.33), we get

$$\begin{aligned}\|y_{2t}\|_\infty \le \max\{&\bar{\gamma}_2^0(\|x_2(0)\|), \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{21} \circ \bar{\gamma}_{11}(\|y_{2t}\|_\infty), \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty),\\ &\bar{\gamma}_{22} \circ \bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{22} \circ \bar{\gamma}_{12}(\|y_{2t}\|_\infty), \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\}\end{aligned} \tag{2.34}$$

Noting $\bar{\gamma}_{11} \circ \bar{\gamma}_{21}(s) < s, \bar{\gamma}_{12} \circ \bar{\gamma}_{22}(s) < s$, gives

$$\begin{aligned}\|y_{2t}\|_\infty \le \max\{&\bar{\gamma}_2^0(\|x_2(0)\|), \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(\|x_1(0)\|),\\ &\bar{\gamma}_{22} \circ \bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\}\end{aligned} \tag{2.35}$$

Substituting (2.35) into (2.31) and (2.32) respectively yields

$$\begin{aligned}\|y_{11t}\|_\infty \le \max\{&\bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{11} \circ \bar{\gamma}_2^0(\|x_2(0)\|),\\ &\bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{11} \circ \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\}\end{aligned} \tag{2.36}$$

$$\begin{aligned}\|y_{12t}\|_\infty \le \max\{&\bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{12} \circ \bar{\gamma}_2^0(\|x_2(0)\|),\\ &\bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(\|x_1(0)\|), \bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{12} \circ \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\}\end{aligned} \tag{2.37}$$

Now, noting that the right hand sides of (2.36), (2.37) and (2.35) are independent of t, shows that $\sup_{t\in[0,T)} \|y_{11}(t)\| < \infty, \sup_{t\in[0,T)} \|y_{12}(t)\| < \infty$ and $\sup_{t\in[0,T)} \|y_2(t)\| < \infty$. Using Lemma 3.1 in [85] yields that if T is finite then the state is also bounded on $[0, T)$. This contradicts $[0, T)$ being the maximal interval of definition and thus we conclude that $T = \infty$. Letting $t \to \infty$ on the left hand sides of (2.36), (2.37) and (2.35) yields that, for $i = 1, 2$, the output y_{1i} satisfies LB with restrictions $\check{X}_s, \check{\Delta}_{u1}, \check{\Delta}_{u2}$ on $x(0), u_1, u_2$ and gains $\check{\gamma}_{1i}^0, \check{\gamma}_{1i}^{u_1}, \check{\gamma}_{1i}^{u_2}$ respectively, and y_2 satisfies LB with restrictions $\check{X}_s, \check{\Delta}_{u1}, \check{\Delta}_{u2}$ on $x(0), u_1, u_2$ and gains $\check{\gamma}_2^0, \check{\gamma}_2^{u_1}, \check{\gamma}_2^{u_2}$ respectively.

Finite $\bar{\Delta}_1$ or $\bar{\Delta}_{21}$ or $\bar{\Delta}_{22}$: Given $x(0) \in \check{X}_{s1} \times \check{X}_{s2}$ and input signal $w = (u_1, u_2)$ with $\|u_1\|_\infty < \check{\Delta}_{u1}$ and $\|u_2\|_\infty < \check{\Delta}_{u2}$, let $p(x(0), \lambda)$ represent a continuous path in $\check{X}_{s1} \times \check{X}_{s2}$ from the origin to $x(0)$ with the property that $p(x(0), 0) = 0$ and $p(x(0), 1) = x(0)$, and let $y_{11}^\lambda, y_{12}^\lambda$ and y_2^λ represent the outputs produced starting at $x^\lambda(0) = p(x(0), \lambda)$ with input λw. Note that when $\lambda = 0$, the solutions and the outputs are identically zero. Note also that the solutions are a continuous function of λ (see Theorem 3.5 in [36]). In other words, given $T > 0$ (arbitrarily large) and given $\varepsilon_{21} > 0, \varepsilon_{22} > 0$ and $\varepsilon_1 > 0$, there exists $\lambda^\star > 0$ such that $\lambda \in [0, \lambda^\star]$ implies that the solution exists on $[0, T]$ and

$$\|y_{11T}^\lambda\|_\infty \leq \varepsilon_{21}, \|y_{12T}^\lambda\|_\infty \leq \varepsilon_{22}, \|y_{2T}^\lambda\|_\infty \leq \varepsilon_1 \tag{2.38}$$

Now, let

$$\begin{aligned}
\bar{\bar{\Delta}}_{21} &= \max\{\bar{\gamma}_1^0(\|x_1^\lambda(0)\|), \bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_1^0(\|x_1^\lambda(0)\|), \\
&\bar{\gamma}_{11} \circ \bar{\gamma}_2^0(\|x_2^\lambda(0)\|), \bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{11} \circ \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\}, \\
\bar{\bar{\Delta}}_{22} &= \max\{\bar{\gamma}_1^0(\|x_1^\lambda(0)\|), \bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_1^0(\|x_1^\lambda(0)\|), \\
&\bar{\gamma}_{12} \circ \bar{\gamma}_2^0(\|x_2^\lambda(0)\|), \bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{12} \circ \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\} \\
\bar{\bar{\Delta}}_1 &= \max\{\bar{\gamma}_{21} \circ \bar{\gamma}_1^0(\|x_1^\lambda(0)\|), \bar{\gamma}_{22} \circ \bar{\gamma}_1^0(\|x_1^\lambda(0)\|), \\
&\bar{\gamma}_2^0(\|x_2^\lambda(0)\|), \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(\|u_1\|_\infty), \bar{\gamma}_2^{u_2}(\|u_2\|_\infty)\}
\end{aligned}$$

From (2.28) and (2.30), we have $\bar{\bar{\Delta}}_{21} < \bar{\Delta}_{21}, \bar{\bar{\Delta}}_{22} < \bar{\Delta}_{22}$ and $\bar{\bar{\Delta}}_1 < \bar{\Delta}_1$. Now let $T > 0$ be arbitrarily large, let $\varepsilon_{21}, \varepsilon_{22}$ and ε_1 satisfy $\bar{\bar{\Delta}}_{21} < \varepsilon_{21} < \bar{\Delta}_{21}, \bar{\bar{\Delta}}_{22} < \varepsilon_{22} < \bar{\Delta}_{22}$, and $\bar{\bar{\Delta}}_1 < \varepsilon_1 < \bar{\Delta}_1$ respectively, and let $\lambda^\star$ be the largest value belonging to the interval $(0, 1]$ such that (2.38) holds for all $\lambda \in [0, \lambda^\star]$. Suppose $\lambda^\star < 1$. Then we have, using the same calculations as for the case when $\bar{\Delta}_{21}, \bar{\Delta}_{22}$ and $\bar{\Delta}_1$ were infinite, that $\|y_{11T}^{\lambda^\star}\|_\infty \leq \bar{\bar{\Delta}}_{21} < \varepsilon_{21}, \|y_{12T}^{\lambda^\star}\|_\infty \leq \bar{\bar{\Delta}}_{22} < \varepsilon_{22}, \|y_{2T}^{\lambda^\star}\|_\infty \leq \bar{\bar{\Delta}}_1 < \varepsilon_1$. By continuity of solutions, there exists $\lambda' > \lambda^\star$ so that (2.38) holds, thus contradicting that $\lambda^\star < 1$. We conclude that $\lambda^\star = 1$ and, since T is arbitrary, the solutions are defined on $[0, \infty)$ and $\|y_{11}\|_\infty < \bar{\Delta}_{21}, \|y_{12}\|_\infty < \bar{\Delta}_{22}$ and $\|y_2\|_\infty < \bar{\Delta}_1$. The reminder of the proof of is the same as for the case where $\bar{\Delta}_{21}, \bar{\Delta}_{22}$ and $\bar{\Delta}_1$ are infinite.

Proposition 2.3.2 Suppose Assumption 2.3.1 is satisfied. Let $y_1 = (y_{11}, y_{12})$ and $v_2 = (v_{21}, v_{22})$ with $y_{11} = v_{21}$ and $y_{12} = v_{22}$. Assume for $i = 1, 2$, the output y_{1i} satisfies AB with restriction $\bar{X}_{a1}$ on $x_1(0)$, restrictions $\bar{\Delta}_1, \bar{\Delta}_{u1}$ on v_1, u_1 and gains $\bar{\gamma}_{1i}, \bar{\gamma}_{1i}^{u_1}$ respectively, and the output y_2 satisfies AB with restriction $\bar{X}_{a2}$ on $x_2(0)$, restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{u2}$ on v_{21}, v_{22}, u_2 and gains $\bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}_2^{u_2}$ respectively. Assume that, for all $(x_1(0), x_2(0)) \in \bar{X}_{a1} \times \bar{X}_{a2}$, all piecewise continuous u_1 satisfying $\|u_1\|_a \leq \check{\Delta}_{u1}$, where

$\check{\Delta}_{u1}$ is a real number satisfying $0 \le \check{\Delta}_{u1} \le \bar{\Delta}_{u1}, \bar{\gamma}_{11}^{u_1}(\check{\Delta}_{u1}) \le \bar{\Delta}_{21}$ and $\bar{\gamma}_{12}^{u_1}(\check{\Delta}_{u1}) \le \bar{\Delta}_{22}$, and all piecewise continuous u_2 satisfying $\|u_2\|_a \le \bar{\Delta}_{u2}$, $(x_1(t), x_2(t))$ is defined for all $t \ge 0$. Assume

1) $\bar{\Delta}_1 = \infty$;

2) $\lim_{s\to\infty} \bar{\gamma}_{1i}(s) < \infty$ and $\lim_{s\to\infty} \bar{\gamma}_{1i}(s) \le \bar{\Delta}_{2i}$, $i = 1, 2$;

3) the small gain condition holds, i.e., $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for all $s > 0$ and $i = 1, 2$.

Then for $i = 1, 2$, the output y_{1i} satisfies AB with restriction $\bar{X}_{a1} \times \bar{X}_{a2}$ on $x(0)$, restrictions $\check{\Delta}_{u1}, \bar{\Delta}_{u2}$ on u_1, u_2 and gains $\check{\gamma}_{1i}^{u_1}, \check{\gamma}_{1i}^{u_2}$ respectively, and y_2 satisfies AB with restriction $\bar{X}_{a1} \times \bar{X}_{a2}$ on $x(0)$, restrictions $\check{\Delta}_{u1}, \bar{\Delta}_{u2}$ on u_1, u_2 and gains $\check{\gamma}_2^{u_1}, \check{\gamma}_2^{u_2}$ respectively, where $\check{\gamma}_{1i}^{u_1}, \check{\gamma}_{1i}^{u_2}, \check{\gamma}_2^{u_1}, \check{\gamma}_2^{u_2}$ are defined as (2.29).

Proof: The proof is a modification of Theorem 2 of [85].

The bound $\|u_1\|_a \le \check{\Delta}_{u1}$ implies

$$\|y_{11}\|_a \le \max\{\bar{\gamma}_{11}(\|y_2\|_a), \bar{\gamma}_{11}^{u_1}(\|u_1\|_a)\} \le \bar{\Delta}_{21}, \tag{2.39}$$

$$\|y_{12}\|_a \le \max\{\bar{\gamma}_{12}(\|y_2\|_a), \bar{\gamma}_{12}^{u_1}(\|u_1\|_a)\} \le \bar{\Delta}_{22}. \tag{2.40}$$

This, together with $\|u_2\|_a \le \bar{\Delta}_{u2}$, implies

$$\|y_2\|_a \le \max\{\bar{\gamma}_{21}(\|y_{11}\|_a), \bar{\gamma}_{22}(\|y_{12}\|_a), \bar{\gamma}_2^{u_2}(\|u_2\|_a)\}. \tag{2.41}$$

From the definition of $\check{\gamma}_i^{u_1}, \check{\gamma}_i^{u_2}, i = 1, 2$, if either $\bar{\gamma}^{u_1}(\|u_1\|_a)$ or $\bar{\gamma}^{u_2}(\|u_2\|_a)$ is infinite then there is nothing to prove. Otherwise, both $\|y_1\|_a$ and $\|y_2\|_a$ are bounded so that substituting (2.39) and (2.40) into (2.41) and noting the small gain condition gives

$$\|y_2\|_a \le \max\{\bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(\|u_1\|_a), \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(\|u_1\|_a), \bar{\gamma}_2^{u_2}(\|u_2\|_a)\}, \tag{2.42}$$

Then, substituting (2.42) into (2.39) and (2.40) respectively yields

$$\|y_{11}\|_a \le \max\{\bar{\gamma}_{11} \circ \bar{\gamma}_{22} \circ \bar{\gamma}_{12}^{u_1}(\|u_1\|_a), \bar{\gamma}_{11}^{u_1}(\|u_1\|_a), \bar{\gamma}_{11} \circ \bar{\gamma}_2^{u_2}(\|u_2\|_a)\},$$

$$\|y_{12}\|_a \le \max\{\bar{\gamma}_{12} \circ \bar{\gamma}_{21} \circ \bar{\gamma}_{11}^{u_1}(\|u_1\|_a), \bar{\gamma}_{12}^{u_1}(\|u_1\|_a), \bar{\gamma}_{12} \circ \bar{\gamma}_2^{u_2}(\|u_2\|_a)\}.$$

Using (2.29) gives the desirable result. If either $\bar{\gamma}_{1i}^{u_1}(\|u_1\|_a)$ $(i = 1, 2)$ or $\bar{\gamma}_2^{u_2}(\|u_2\|_a)$ is infinite, then (2.29) implies either $\check{\gamma}_{1i}^{u_1}(\|u_1\|_a)$ or $\check{\gamma}_2^{u_2}(\|u_2\|_a)$ is infinite. Thus, nothing is left to prove.

Remark 2.3.3 Suppose the two subsystems satisfy LB and AB with no restriction on the initial state, i.e., $\bar{X}_{s1} = \bar{X}_{a1} = \mathbb{R}^{n_1}, \bar{X}_{s2} = \bar{X}_{a2} = \mathbb{R}^{n_2}$ and with restriction on the input, and all other assumptions of Propositions 2.3.1 and 2.3.2 are satisfied, then the interconnected system satisfies LB with restriction $\check{X}_s$ and AB with no restriction on the initial state $(x_1(0), x_2(0))$.

Remark 2.3.4 By letting the dimension of y_{12} be zero, Theorem 2 of [85] can be viewed as a special case of Proposition 2.3.2. The reason for partitioning y_1 into y_{11} and y_{12} is to allow y_{11} and y_{12} to satisfy LB and AB with different gains from v_1.

Corollary 2.3.1 Consider the following system

$$\begin{aligned} \dot{\tilde{x}}_1 &= f_1(\tilde{x}_1, \tilde{u}_1) \\ \dot{\tilde{x}}_2 &= f_2(\tilde{x}_2, \tilde{x}_1, \tilde{u}_1) \end{aligned} \tag{2.43}$$

where for $i = 1, 2$, $\tilde{x}_i \in \mathbb{R}^{n_i}, \tilde{u}_1 \in \mathbb{R}^{m_1}$, $f_1(\tilde{x}_1, \tilde{u}_1), f_2(\tilde{x}_2, \tilde{x}_1, \tilde{u}_1)$ are locally Lipschitz functions satisfying $f_1(0,0) = 0$ and $f_2(0,0,0) = 0$. Assume $\tilde{x}_1$ satisfies LB with restriction and AB with no restriction on $\tilde{x}_1(0)$, both with restriction $\bar{\Delta}_{\tilde{u}_1}$ on $\tilde{u}_1$ and linear gain; $\tilde{x}_2$ satisfies LB and AB with no restriction on $\tilde{x}_2(0)$, both with no restriction on $\tilde{x}_1, \tilde{u}_1$ and linear gains. Then, $\tilde{x}_1, \tilde{x}_2$ satisfy LB with restriction and AB with no restriction on $(\tilde{x}_1(0), \tilde{x}_2(0))$, both with restriction $\bar{\Delta}_{\tilde{u}_1}$ on $\tilde{u}_1$ and linear gain.

Proof: System (2.43) can be seen as a cascade connection $\tilde{v}_2 = \tilde{y}_1$ of the following two systems

$$\begin{aligned} \dot{\tilde{x}}_1 &= f_1(\tilde{x}_1, \tilde{u}_1),\ \tilde{y}_1 = \tilde{x}_1 \\ \dot{\tilde{x}}_2 &= f_2(\tilde{x}_2, \tilde{v}_2, \tilde{u}_1) \end{aligned}$$

Since $\tilde{x}_1$ satisfies AB with no restriction on $\tilde{x}_1(0)$ and restriction $\bar{\Delta}_{\tilde{u}_1}$ on $\tilde{u}_1$, for all $\tilde{x}_1(0)$, piecewise continuous $\tilde{u}_1$ satisfying $\|\tilde{u}_1\|_a \le \bar{\Delta}_{\tilde{u}_1}$, $\tilde{x}_1(t)$ is defined for all $t \ge 0$. Since $\tilde{x}_2$ satisfies AB with no restrictions on $\tilde{x}_2(0)$ and $\tilde{x}_1, \tilde{u}_1$, for all $(\tilde{x}_1(0), \tilde{x}_2(0))$, and all piecewise continuous $\tilde{u}_1$ satisfying $\|\tilde{u}_1\|_a \le \bar{\Delta}_{\tilde{u}_1}$, $\tilde{x}_1(t), \tilde{x}_2(t)$ are defined for all $t \ge 0$. Let $\tilde{y}_2 = \tilde{x}_2$. Noting that $\tilde{y}_2$ has no effect on $\tilde{y}_1$ shows $\bar{\gamma}_{11}(s) = \bar{\gamma}_{12}(s) = 0$, $\bar{\Delta}_1 = \infty$ in Propositions 2.3.1-2.3.2. Thus, all assumptions of Propositions 2.3.1-2.3.2 are satisfied. Finally, noting that all gains from $\tilde{u}_1$ and $\tilde{v}_2$ are linear completes the proof.

Corollary 2.3.2 Consider the cascade connection $\tilde{v}_2 = \tilde{y}_1$ of the following two systems

$$\begin{aligned} \dot{\tilde{x}}_1 &= f_1(\tilde{x}_1, \tilde{u}_1, d),\ \tilde{y}_1 = h_1(\tilde{x}_1, \tilde{u}_1, d) \\ \dot{\tilde{x}}_2 &= f_2(\tilde{x}_2, \tilde{v}_2, d) \end{aligned} \tag{2.44}$$

where for $i = 1, 2$, $\tilde{x}_i \in \mathbb{R}^{n_i}, \tilde{u}_1 \in \mathbb{R}^{m_1}, \tilde{v}_2, \tilde{y}_1 \in \mathbb{R}^{p_1}$ and f_1, h_1, f_2 are locally Lipschitz functions vanishing at $(0, 0, d)$ for all $d \in \mathcal{D}$. Assume $\tilde{y}_1$ satisfies LB with restriction and AB with no restriction on $\tilde{x}_1(0)$, both with restriction $\bar{\Delta}_{\tilde{u}_1}$ on $\tilde{u}_1$ and gain $\bar{N}_1^{\tilde{u}_1} s$; $\tilde{x}_2$ satisfies LB with restriction and AB with no restriction on $\tilde{x}_2(0)$, both with restriction $\bar{\Delta}_2$ on $\tilde{v}_2$ and gain $\bar{N}_2 s$. Then $\tilde{x}_2$ satisfies LB with restriction and AB with no restriction on $(\tilde{x}_1(0), \tilde{x}_2(0))$, both with restriction $\min\{\bar{\Delta}_{\tilde{u}_1}, \bar{\Delta}_2/\bar{N}_1^{\tilde{u}_1}\}$ on $\tilde{u}_1$ and gain $\bar{N}_2 \bar{N}_1^{\tilde{u}_1} s$.

Proof: Note that $\tilde{v}_2 = \tilde{y}_1$ and no finite escape time exists for any $\tilde{x}_1(0) \in \mathbb{R}^{n_1}, \tilde{x}_2(0) \in \mathbb{R}^{n_2}$, and for any piecewise continuous $\tilde{u}_1$ satisfying $\|\tilde{u}_1\|_a \leq \min\{\bar{\Delta}_{\tilde{u}_1}, \bar{\Delta}_2/\bar{N}_1^{\tilde{u}_1}\}$. Thus letting $\tilde{y}_2 = \tilde{x}_2$ shows that the result is a direct application of Propositions 2.3.1 and 2.3.2.

Chapter 3

Disturbance Attenuation of Feedforward Systems with Dynamic Uncertainty

3.1 Introduction

In this chapter, we consider the following system

$$\begin{aligned}
\dot{x}_i &= c_i x_{i-1} + f_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d) \\
\dot{\xi}_i &= g_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d), \;\; i = n, ..., 2 \\
\dot{x}_1 &= D_1 \xi_1 + c_1 u + f_1(\xi_1, u, d) \\
\dot{\xi}_1 &= A_1 \xi_1 + B_1 u + g_1(\xi_1, u, d)
\end{aligned} \tag{3.1}$$

where for $i = 1, ..., n$, $x_i \in \mathbb{R}$, $d \in \mathbb{R}^{n_d}$, $\xi_i \in \mathbb{R}^{n_{\xi_i}}$, $u \in \mathbb{R}$, f_i and g_i are locally Lipschitz functions satisfying $f_i(0, ..., 0) = 0$ and $g_i(0, ..., 0) = 0$, A_1, B_1, D_1 and $c_1, ..., c_n$ are constant matrices and constants, respectively. The dynamics governing $\xi_1, \xi_2, ..., \xi_n$ is called dynamic uncertainty because the state of the dynamics is not allowed for feedback. Thus we can view system (3.1) as nonlinear systems in feedforward form subject to dynamic uncertainty $\xi_1, \xi_2, ..., \xi_n$ and disturbance d. System (3.1) contains two classes of well known nonlinear systems as special cases. First, when there is no dynamic uncertainty, that is, $n_{\xi_i} = 0, i = 1, ..., n$, system (3.1) becomes a subclass of the feedforward systems studied in [4],[53],[85],[87], and second, when $n_{\xi_1} \neq 0$ and $n_{\xi_i} = 0, i = 2, ..., n$, system (3.1) becomes the feedforward system subject to input unmodeled dynamics as studied in [7]. System (3.1) is interesting on its own, on one hand, because dynamic uncertainty is ubiquitously present in real systems. On the other hand, the robust output regulation problem for nonlinear systems in strict feedforward form can be converted into a robust stabilization problem of an augmented system in the form of (3.1) where the dynamic uncertainty models the internal model [21, 22]. Thus the stabilization solution of system (3.1) also shed light on the solution of the robust output regulation problem of nonlinear systems in feedforward form.

The objective of this chapter is to design a static partial state feedback control law so that the closed-loop system is input-to-state stable (ISS) with the disturbance d as input [76, 80], with no restriction on the initial state and restriction on d. As a result, when d vanishes, the origin of the closed-loop system is globally asymptotically stable. For the second special case mentioned above, the same problem is studied in [7] under the assumption that the Jacobian linearization of the system at the origin is stabilizable. We will manage to remove this assumption. Like [7], we use the asymptotic small gain theorem to establish the global attractiveness of the closed-loop system. However, unlike [7], we cannot use the linearization technique to establish the local stability of the closed-loop system because the Jacobian linearization of the closed-loop system at the origin may have eigenvalues on the imaginary axis. To overcome this difficulty, we have to employ two versions of small gain theorem with restrictions adapted from [85]. An advantage of our result and technique is that we can handle a larger class of systems than those in [7]. It should be noted that a similar problem was studied recently for system (3.1) in [90]. However, the result is semi-global.

3.2 Robust Stabilization of Feedforward Systems

Like [7], [85], our approach will utilize saturation functions characterized as follows.

Definition 3.2.1 A locally Lipschitz function $\sigma(\cdot) : \mathbb{R} \to [-\lambda, \lambda]$ is said to be a saturation function with saturation level $\lambda > 0$, if for all $s \in \mathbb{R}$,

1) $\sigma(s) = s$ when $|s| \leq \frac{\lambda}{2}$;

2) $\frac{\lambda}{2} \leq sgn(s)\sigma(s) \leq \min\{|s|, \lambda\}$ when $|s| \geq \frac{\lambda}{2}$.

The assumptions for system (3.1) used in this chapter are introduced below.

Assumption 3.2.1 $f_1(\xi_1, u, 0) = o(\xi_1, u)$, $g_1(\xi_1, u, 0) = o(\xi_1, u)$ and $f_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, 0) = o(\xi_i, x_{i-1}, ..., x_1, \xi_1, u)$, $i = 2, ..., n$.

Assumption 3.2.2 The *dc* gain $\theta_1 = c_1 - D_1 A_1^{-1} B_1$ and $c_2, ..., c_n$ are positive.

Assumption 3.2.3 ξ_1 satisfies LB and AB with no restriction on $\xi_1(0)$, no restrictions on u, d and linear gains, and for $i = 2, ..., n$, ξ_i satisfies LB and AB with no restriction on $\xi_i(0)$, no restrictions on $x_{i-1}, \xi_{i-1}, ..., x_1, \xi_1, u, d$ and linear gains.

Lemma 3.2.1 Consider the following control system

$$\begin{aligned} \dot{z} &= \theta u + F(\xi, u, d) \\ \dot{\xi} &= G(\xi, u, d) \end{aligned} \tag{3.2}$$

where $z, u \in \mathbb{R}, \xi \in \mathbb{R}^{n_\xi}, d \in \mathbb{R}^{n_d}$, θ is a positive real number, $F(\xi, u, d), G(\xi, u, d)$ are locally Lipschitz, and $G(0, 0, 0) = 0$. Assume $F(\xi, u, 0) = o(\xi, u)$, and ξ satisfies LB with restrictions Ξ, Δ_u, Δ_d on $\xi(0), u, d$ and gains $\gamma_1^0, N_u s, N_d s$ respectively, and AB with restrictions Δ_u, Δ_d on u, d and gains $N_u s, N_d s$ respectively. Then there exist $\lambda, k > 0$ such that under the control

$$u = -\sigma(kz + kH\xi - \bar{u}) \tag{3.3}$$

where σ is a saturation function with level λ, and H is a $1 \times n_\xi$ constant matrix, z, ξ satisfy LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restrictions on $\bar{u}, d$ and linear gains.

Proof: Define $\tilde{\lambda} = \theta\lambda$ and $\tilde{k} = \theta k$. Then

$$\theta u = -\theta\sigma(k(z + H\xi - \frac{\bar{u}}{k})) = -\tilde{\sigma}(\tilde{k}(z + H\xi - \frac{\bar{u}}{k})) \tag{3.4}$$

where $\tilde{\sigma}(s) = \theta\sigma(s/\theta)$ is a saturation function with level $\tilde{\lambda}$.

With (3.4), system (3.2) can be viewed as the interconnection

$$v_1 = y_2, \quad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = y_1 \tag{3.5}$$

of the following two subsystems

$$\begin{aligned} \Sigma_1 : \quad & \dot{\xi} = G(\xi, -\sigma(kv_1), d), \; y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} H\xi \\ F(\xi, -\sigma(kv_1), d)/\tilde{k} \end{bmatrix}, \\ \Sigma_2 : \quad & \dot{z} = -\tilde{\sigma}(\tilde{k}(z + v_{21} - \frac{\bar{u}}{k})) + \tilde{k}v_{22}, y_2 = z + v_{21} - \frac{\bar{u}}{k}. \end{aligned}$$

Let us first apply Propositions 2.3.1-2.3.2 to show that the output (y_1, y_2) of Σ_1 and Σ_2 under the interconnection (3.5) satisfies LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restrictions on $\bar{u}, d$ and linear gains.

Step 1. Consider Σ_1 system viewing v_1, d as inputs. Show that for $i = 1, 2$, y_{1i} satisfies LB with restriction and AB with no restriction on $\xi(0)$ and restriction on d.

The assumption on ξ subsystem and $|\sigma(kv_1)| \leq \min\{k|v_1|, \lambda\}$ with $\lambda < \Delta_u$ implies that, for all $\xi(0)$, piecewise continuous v_1 and d satisfying $\|d\|_a \leq \Delta_d$, $\xi(t)$ exists for all $t \geq 0$, and

$$\|\xi\|_\infty \leq \max\{\gamma_1^0(\|\xi(0)\|), N_u \min\{k\|v_1\|_\infty, \lambda\}, N_d\|d\|_\infty\} \tag{3.6}$$

for all $\xi(0) \in \Xi$, $v_1 \in \mathcal{L}_\infty^1$, $\|d\|_\infty < \Delta_d$, and

$$\|\xi\|_a \leq \max\{N_u \min\{k\|v_1\|_a, \lambda\}, N_d\|d\|_a\} \tag{3.7}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, piecewise continuous v_1 and d satisfying $\|d\|_a \leq \Delta_d$.

Let $l > 0$ such that $\|H\| \leq l$. Then, from $\|y_{11}\| = \|H\xi\| \leq l\|\xi\|$, we obtain

$$\|y_{11}\|_\infty \leq \max\{l\gamma_1^0(\|\xi(0)\|), \bar{\gamma}_{11}(\|v_1\|_\infty), \bar{\gamma}_{11}^d(\|d\|_\infty)\} \tag{3.8}$$

for all $\xi(0) \in \Xi$, $v_1 \in \mathcal{L}_\infty^1$, $\|d\|_\infty < \Delta_d$, where $\bar{\gamma}_{11}(s) = lN_u \min\{ks, \lambda\}$, $\bar{\gamma}_{11}^d(s) = lN_d s$, and

$$\|y_{11}\|_a \leq \max\{\bar{\gamma}_{11}(\|v_1\|_a), \bar{\gamma}_{11}^d(\|d\|_a)\} \tag{3.9}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, piecewise continuous v_1 and d satisfying $\|d\|_a \leq \Delta_d$.

Next consider y_{12}. The Lipschitz continuity of $F(\xi, u, d)$ implies that, if ξ, u, d belong to a compact set, then there exists $L \geq 0$ such that

$$|F(\xi, u, d)| \leq |F(\xi, u, 0)| + L\|d\|. \tag{3.10}$$

Without loss of generality, by selecting Δ_u, Δ_d to be finite and Ξ to be bounded, it follows from (3.6) that, with $\|d\|_\infty < \Delta_d, \lambda < \Delta_u$ and $\xi(0) \in \Xi$, $\|\xi\|_\infty$ belongs to some compact set, and from (3.7) that, with $\|d\|_a \leq \Delta_d$ and $\lambda < \Delta_u$, $\|\xi\|_a$ also belongs to some compact set. Then from (3.6), (3.7), (3.10), and $F(\xi, u, 0) = o(\xi, u)$, there exist $L_d \geq 0$ and a gain function $\gamma_o(s) = o(s)$ such that

$$\begin{aligned}\|y_{12}\|_\infty &\leq \max\{2\gamma_o \circ \gamma_1^0(\|\xi(0)\|), 2\gamma_o(\max\{N_u, 1\} \min\{k\|v_1\|_\infty, \lambda\}, 2L_d\|d\|_\infty\}/\tilde{k} \\ &\leq \max\{2\gamma_o \circ \gamma_1^0(\|\xi(0)\|)/\tilde{k}, \bar{\gamma}_{12}(\|v_1\|_\infty), \bar{\gamma}_{12}^d(\|d\|_\infty)\} \end{aligned} \tag{3.11}$$

for all $\xi(0) \in \Xi$, $v_1 \in \mathcal{L}_\infty^1$, $\|d\|_\infty < \Delta_d$, where $\bar{\gamma}_{12}(s) = 2\gamma_o(\max\{N_u, 1\} \min\{ks, \lambda\})/\tilde{k}$ and $\bar{\gamma}_{12}^d(s) = 2L_d s/\tilde{k}$, and

$$\|y_{12}\|_a \leq \max\{\bar{\gamma}_{12}(\|v_1\|_a), \bar{\gamma}_{12}^d(\|d\|_a)\} \tag{3.12}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, piecewise continuous v_1 and d satisfying $\|d\|_a \le \Delta_d$.

Now let $\bar{\gamma}_1^0(s) = 2\max\{l\gamma_1^0(s), 2\gamma_o \circ \gamma_1^0(s)/\tilde{k}\}$. Then, it follows from (3.8) and (3.11) that, for $i = 1,2$ y_{1i} satisfies LB with restrictions $\Xi, \bar{\Delta}_1, \bar{\Delta}_d$ on $\xi(0), v_1, d$ and gains $\bar{\gamma}_1^0, \bar{\gamma}_{1i}, \bar{\gamma}_{1i}^d$ respectively, and from (3.9) and (3.12) that for $i = 1,2$, y_{1i} satisfies AB with restrictions $\bar{\Delta}_1, \bar{\Delta}_d$ on v_1, d and gains $\bar{\gamma}_{1i}, \bar{\gamma}_{1i}^d$ respectively, where $\bar{\Delta}_1 = \infty$ and $\bar{\Delta}_d = \Delta_d$.

Step 2. Consider Σ_2 system viewing $v_2, \bar{u}$ as inputs. Show that, y_2 satisfies LB with no restriction on $\bar{u}$, restriction $\frac{\lambda}{3k}$ on v_2, and linear gains respectively, and satisfies AB with no restrictions on $\bar{u}, v_{21}$, restriction $\frac{\lambda}{3k}$ on v_{22} and linear gains respectively.

We first claim that, there exists a gain function γ_2^0 such that, for all $z(0) \in \mathbb{R}$, piecewise continuous $\bar{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \le \frac{\lambda}{3k}$, $z(t)$ exists for all $t \ge 0$, and satisfies

$$\|z\|_\infty \le \max\{\gamma_2^0(|z(0)|), 3\|v_{21}\|_\infty, 3\|v_{22}\|_\infty, \frac{3}{k}\|\bar{u}\|_\infty\} \tag{3.13}$$

for all $z(0) \in \mathbb{R}$, $\bar{u}, v_{21} \in \mathcal{L}_\infty^1$, $\|v_{22}\|_\infty < \frac{\lambda}{3k}$, and

$$\|z\|_a \le \max\{3\|v_{21}\|_a, 3\|v_{22}\|_a, \frac{3}{k}\|\bar{u}\|_a\} \tag{3.14}$$

for all $z(0) \in \mathbb{R}$, piecewise continuous $\bar{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \le \frac{\lambda}{3k}$. In fact, the proof of (3.14) can be extracted from the derivation of (A.16) of [7] and the proof of (3.13) can be derived similarly.

It follows from (3.13), (3.14) and $y_2 = z + v_{21} - \frac{\bar{u}}{k}$ that, y_2 satisfies

$$\begin{aligned} \|y_2\|_\infty &\le \max\{2\gamma_2^0(|z(0)|), 6\|v_{21}\|_\infty, 6\|v_{22}\|_\infty, \frac{6}{k}\|\bar{u}\|_\infty\} \\ &= \max\{\bar{\gamma}_2^0(|z(0)|), \bar{\gamma}_{21}(\|v_{21}\|_\infty), \bar{\gamma}_{22}(\|v_{22}\|_\infty), \bar{\gamma}^{\bar{u}}(\|\bar{u}\|_\infty)\} \end{aligned} \tag{3.15}$$

for all $z(0) \in \mathbb{R}, \bar{u}, v_{21} \in \mathcal{L}_\infty^1$, $\|v_{22}\|_\infty < \frac{\lambda}{3k}$, and

$$\begin{aligned} \|y_2\|_a &\le \max\{6\|v_{21}\|_a, 6\|v_{22}\|_a, \frac{6}{k}\|\bar{u}\|_a\} \\ &= \max\{\bar{\gamma}_{21}(\|v_{21}\|_a), \bar{\gamma}_{22}(\|v_{22}\|_a), \bar{\gamma}^{\bar{u}}(\|\bar{u}\|_a)\} \end{aligned} \tag{3.16}$$

for all $z(0) \in \mathbb{R}$, piecewise continuous $\bar{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \le \frac{\lambda}{3k}$, where $\bar{\gamma}_2^0(s) = 2\gamma_2^0(s), \bar{\gamma}_{21}(s) = \bar{\gamma}_{22}(s) = 6s$, and $\bar{\gamma}^{\bar{u}}(s) = 6s/k$.

That is, y_2 satisfies LB with no restriction on $z(0)$, restriction $\bar{\Delta}_{21}, \bar{\Delta}_{21}, \bar{\Delta}_{\bar{u}}$ on $v_{21}, v_{22}, \bar{u}$ and gains $\bar{\gamma}_2^0, \bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}^{\bar{u}}$ respectively, and satisfies AB with restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{\bar{u}}$ on $v_{21}, v_{22}, \bar{u}$ and gains $\bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}^{\bar{u}}$ respectively, where $\bar{\Delta}_{\bar{u}} = \bar{\Delta}_{21} = \infty$ and $\bar{\Delta}_{22} = \frac{\lambda}{3k}$.

Step 3. Choose λ, k appropriately to satisfy the small gain conditions of Propositions 2.3.1 and 2.3.2.

Let us first consider the small gain condition $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for $s > 0$ and $i = 1, 2$ of Proposition 2.3.1. Note that $\bar{\gamma}_{12}(s)$ can be written as follows:

$$\bar{\gamma}_{12}(s) = \begin{cases} \frac{2\gamma_o(\max\{N_u,1\}ks)}{\theta k}, & 0 < s \leq \frac{\lambda}{k}, \\ \frac{2\gamma_o(\max\{N_u,1\}\lambda)}{\theta k}, & s \geq \frac{\lambda}{k}. \end{cases} \tag{3.17}$$

Since $\gamma_o(s) = o(s)$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\gamma_o(s) \leq \epsilon s$ for $0 < s \leq \delta$. Thus, letting $\delta = \max\{N_u, 1\}\lambda$ gives

$$\bar{\gamma}_{12}(s) \leq \frac{2\max\{N_u,1\}\epsilon s}{\theta}, \ s > 0. \tag{3.18}$$

Then, the small gain condition $6\bar{\gamma}_{1i}(s) < s$ for $s > 0$ and $i = 1, 2$ reduces to

$$6\max\{lN_u k, \frac{2\epsilon\max\{N_u,1\}}{\theta}\}s < s, \ s > 0.$$

It suffices to choose k and ϵ sufficiently small such that

$$6lN_u k < 1, \ 6\frac{2\epsilon\max\{N_u,1\}}{\theta} < 1. \tag{3.19}$$

Note that λ is determined by ϵ and is independent of k. Therefore, it is possible to choose sufficiently small positive numbers λ, k such that the small gain condition is satisfied.

Thus, by Proposition 2.3.1, the output y of the interconnected system satisfies

$$\|y_{1i}\|_\infty \leq \max\{(\bar{\gamma}_1^0 + \frac{1}{6}\bar{\gamma}_2^0)(\|(z(0),\xi(0))\|), 2(lN_d + 2L_d/\tilde{k})\|d\|_\infty, \|\bar{u}\|_\infty/k\} \tag{3.20}$$
$$\|y_2\|_\infty \leq 6\max\{(\bar{\gamma}_1^0 + \frac{1}{6}\bar{\gamma}_2^0)(\|(z(0),\xi(0))\|), 2(lN_d + 2L_d/\tilde{k})\|d\|_\infty, \|\bar{u}\|_\infty/k\} \tag{3.21}$$

for $i = 1, 2$ and for all $z(0) \in \{z \in \mathbb{R} : \bar{\gamma}_2^0(|z|) < \frac{2\lambda}{k}\}$, $\xi(0) \in \{\xi \in \Xi : \bar{\gamma}_1^0(\|\xi\|) < \frac{\lambda}{3k}\}$, $\|\bar{u}\|_\infty < \tilde{\Delta}_u$, and $\|d\|_\infty < \tilde{\Delta}_d$, where $\tilde{\Delta}_u = \frac{\lambda}{3}$ and $\tilde{\Delta}_d = \min\{\frac{\lambda}{6k(lN_d + 2L_d/\tilde{k})}, \Delta_d\}$.

Next consider Proposition 2.3.2. First note that the solution of the interconnected system exists for all $t \geq 0$ using the same argument as that in Lemma 3.5 of [85]. To check the second condition, note that since $\lim_{s\to\infty}\bar{\gamma}_{11}(s) < \infty$, $\lim_{s\to\infty}\bar{\gamma}_{12}(s) < \infty$ and since $\bar{\Delta}_1 = \bar{\Delta}_{21} = \infty$ and $\bar{\Delta}_{22} = \frac{\lambda}{3k}$, we only need to check $\lim_{s\to\infty}\bar{\gamma}_{12}(s) \leq \frac{\lambda}{3k}$. From (3.17), (3.18) and the second inequality of (3.19), we have

$$\lim_{s\to\infty}\bar{\gamma}_{12}(s) \leq \frac{2\epsilon\max\{N_u,1\}\lambda}{\theta k} < \frac{\lambda}{6k} < \frac{\lambda}{3k}.$$

Finally, note that the small gain condition, i.e., $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$, for $s > 0$ and $i = 1, 2$, is implied by (3.19). Thus it is satisfied when the positive numbers λ, k are so small that (3.19) holds.

Thus, by Proposition 2.3.2, the output y of the interconnected system satisfies

$$\|y_{1i}\|_a \leq \max\{2(lN_d + 2L_d/\tilde{k})\|d\|_a, \|\bar{u}\|_a/k\}, i = 1, 2, \tag{3.22}$$

$$\|y_2\|_a \leq 6\max\{2(lN_d + 2L_d/\tilde{k})\|d\|_a, \|\bar{u}\|_a/k\} \tag{3.23}$$

for all $z(0) \in \mathbb{R}$, $\xi(0) \in \mathbb{R}^{n_\xi}$, piecewise continuous $\bar{u}$ and d satisfying $\|d\|_a \leq \tilde{\Delta}_d$.

Let $\Omega = \{(z, \xi) \in \mathbb{R} \times \Xi : (\bar{\gamma}_1^0 + \frac{1}{6}\bar{\gamma}_2^0)(\|(z, \xi)\|) < \frac{\lambda}{3k}\}, \tilde{\gamma}^0(s) = \max\{(\frac{1}{l} + 3)(\bar{\gamma}_1^0(s) + \frac{1}{6}\bar{\gamma}_2^0(s)), \gamma_1^0(s), \gamma_2^0(s)\}$. For all $(z(0), \xi(0)) \in \Omega$, $\|\bar{u}\|_\infty < \tilde{\Delta}_u$ and $\|d\|_\infty < \tilde{\Delta}_d$, (3.20) implies $\|y_1\|_\infty < \frac{\lambda}{3k}$. Using (3.20), (3.13), (3.21), (3.6) and noting (3.19) yields

$$\|z\|_\infty \leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), 6(lN_d + 2L_d/\tilde{k})\|d\|_\infty, 3\|\bar{u}\|_\infty/k\}$$

$$\|\xi\|_\infty \leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), 2(N_d + 2L_d/(\tilde{k}l))\|d\|_\infty, 6N_u\|\bar{u}\|_\infty\}$$

for all $(z(0), \xi(0)) \in \Omega$, $\|\bar{u}\|_\infty < \tilde{\Delta}_u$, $\|d\|_\infty < \tilde{\Delta}_d$.

Then for all $(z(0), \xi(0))$, piecewise continuous $\bar{u}$ and d satisfying $\|\bar{u}\|_a \leq \tilde{\Delta}_u$ and $\|d\|_a \leq \tilde{\Delta}_d$ respectively, (3.22) implies $\|y_{12}\|_a \leq \frac{\lambda}{3k}$. Using (3.22), (3.14), (3.23), (3.7) and noting (3.19) yields

$$\|z\|_a \leq \max\{6(lN_d + 2L_d/\tilde{k})\|d\|_a, 3\|\bar{u}\|_a/k\}$$

$$\|\xi\|_a \leq \max\{2(N_d + 2L_d/(\tilde{k}l))\|d\|_a, 6N_u\|\bar{u}\|_a\}$$

for all $(z(0), \xi(0)) \in \mathbb{R} \times \mathbb{R}^{n_\xi}$, piecewise continuous $\bar{u}$ and d satisfying $\|\bar{u}\|_a \leq \tilde{\Delta}_u$ and $\|d\|_a \leq \tilde{\Delta}_d$ respectively.

Remark 3.2.1 Unlike [7], we do not require the Jacobian linearization of ξ dynamics be Hurwitz. Thus, we cannot use the linearization technique to conclude the local stability of system (3.2). To overcome this difficulty, we have utilized LB and AB small gain theorems to conclude the local stability and global attractiveness respectively.

Now we are ready to state the main result of this chapter.

Theorem 3.2.1 Consider system (3.1). Under Assumptions 3.2.1-3.2.3, there exist positive real numbers λ_i, k_i, $i = 1, ..., n$, such that under the control

$$u = -\sigma_1(k_1x_1 + \sigma_2(k_2x_2 + \cdots + \sigma_n(k_nx_n - u_n))) \tag{3.24}$$

where for $i = 1, ..., n$, σ_i is a saturation function with level λ_i, the closed-loop system satisfies LB with restriction and AB with no restriction on $(x_n(0), \xi_n(0), ..., x_1(0), \xi_1(0))$, both with restrictions on u_n, d and linear gains. In particular, when $u_n = 0$, the closed-loop system with $d = 0$ is globally asymptotically stable at the origin.

Proof: To make use of Lemma 3.2.1, performing on system (3.1) the following coordinate transform:

$$z_1 = x_1 - D_1 A_1^{-1}\xi_1, z_i = x_i + \frac{\theta_i}{\theta_{i-1}} z_{i-1},\ i = 2, ..., n \tag{3.25}$$

where $\theta_i = c_i/k_{i-1}, i = 2, ..., n$, gives

$$\begin{aligned}
\dot{z}_i &= \theta_i u + F_i(\xi_i, z_{i-1}, ..., z_1, \xi_1, u, d) \\
\dot{\xi}_i &= \tilde{g}_i(\xi_i, z_{i-1}, ..., z_1, \xi_1, u, d),\ i = n, ..., 2 \\
\dot{z}_1 &= \theta_1 u + F_1(\xi_1, u, d) \\
\dot{\xi}_1 &= A_1\xi_1 + B_1 u + g_1(\xi_1, u, d)
\end{aligned} \tag{3.26}$$

where $F_1(\xi_1, u, d) = f_1(\xi_1, u, d) - D_1 A_1^{-1} g_1(\xi_1, u, d)$ and for $i = 2, ..., n$, $F_i(\xi_i, z_{i-1}, ..., z_1, \xi_1, u, d) = f_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d) + \theta_i F_{i-1}(\xi_{i-1}, z_{i-2}, ..., \xi_1, u, d)/\theta_{i-1} + \theta_i k_{i-1} x_{i-1}\,|_{(3.25)}$, where z_0 is a dummy variable, and $\tilde{g}_i(\xi_i, z_{i-1}, ..., z_1, \xi_1, u, d) = g_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d)|_{(3.25)}$. Moreover, under the coordinate transform (3.25), the control (3.24) takes the following form:

$$\begin{aligned}
u = &-\sigma_1(k_1 z_1 + k_1 D_1 A_1^{-1}\xi_1 + \sigma_2(k_2 z_2 - k_2 \tfrac{\theta_2}{\theta_1} z_1 \\
&+ \cdots + \sigma_n(k_n z_n - k_n \tfrac{\theta_n}{\theta_{n-1}} z_{n-1} - u_n))).
\end{aligned} \tag{3.27}$$

Thus, the proof will be completed if we can show that, for system (3.26), under the control (3.27), the closed-loop system satisfies LB with restriction and AB with no restriction on $(z_n(0), \xi_n(0), ..., z_1(0), \xi_1(0))$, both with restrictions on u_n, d and linear gains.

Since the last two equations of (3.26) is in the form of (3.2) and satisfy all assumptions of Lemma 3.2.1, there exist $\lambda_1, k_1 > 0$ such that under the control $u = -\sigma_1(k_1 z_1 + k_1 H_1 \xi_1 - u_1)$ with $H_1 = D_1 A_1^{-1}$, z_1, ξ_1 satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restrictions on u_1, d and linear gains.

Let $\zeta_1 = (\xi_2, z_1, \xi_1)$. Then, under the control $u = -\sigma_1(k_1 z_1 + k_1 H_1 \xi_1 - u_1)$, the last four equations of (3.26) can be put into the following form:

$$\begin{aligned}
\dot{z}_2 &= \theta_2 u_1 + \tilde{F}_2(\zeta_1, u_1, d) \\
\dot{\zeta}_1 &= G_1(\zeta_1, u_1, d)
\end{aligned} \tag{3.28}$$

where $\tilde{F}_2(\zeta_1, u_1, d) = F_2(\xi_2, z_1, \xi_1 - \sigma_1(k_1 z_1 + k_1 H_1 \xi_1 - u_1), d) + \theta_2(-\sigma_1(k_1 z_1 + k_1 H_1 \xi_1 - u_1) - u_1)$. By the property of the saturation function, $-\sigma_1(k_1 x_1 - u_1) + k_1 x_1 - u_1 = o(x_1, u_1)$. Then $\tilde{F}_2(\zeta_1, u_1, 0) = o(\zeta_1, u_1)$ since $F_1(\xi_1, u, 0) = o(\xi_1, u)$, $f_2(\xi_2, x_1, \xi_1, u, 0) = o(\xi_2, x_1, \xi_1, u)$.

ζ_1 subsystem of (3.28) can be viewed as in the form of (2.43) with $\tilde{x}_1 = (z_1, \xi_1), \tilde{x}_2 = \xi_2$ and $\tilde{u}_1 = (u_1, d)$. Noting (3.25), Assumption 3.2.3 and $|\sigma_1(k_1 z_1 + k_1 H_1 \xi_1 - u_1)| \leq 3\max\{k_1|z_1|, k_1\|H_1\|\|\xi_1\|, |u_1|\}$ shows that, under the control $u = -\sigma_1(k_1 z_1 + k_1 H_1 \xi_1 - u_1)$, ξ_2 subsystem of (3.26) satisfies LB and AB with no restriction on $\xi_2(0)$, no restriction on z_1, ξ_1, u_1, d and linear gains. Thus by Corollary 2.3.1, ζ_1 satisfies LB with restriction and AB with no restriction on $\zeta_1(0)$, both with restrictions on u_1, d and linear gains. Then, since system (3.28) is in the form of (3.2) and satisfies all assumptions of Lemma 3.2.1, there exist $\lambda_2, k_2 > 0$ such that under the control $u_1 = -\sigma_2(k_2 z_2 + k_2 H_2 \zeta_1 - u_2)$ with $H_2 = [0_{1\times n_{\xi_2}} \ -\frac{\theta_2}{\theta_1} \ 0_{1\times n_{\xi_1}}]$, z_2, ζ_1 satisfy LB with restriction and AB with no restriction on $(z_2(0), \zeta_1(0))$, both with restrictions on u_2, d and linear gains.

Repeating the above procedure $(n-1)$ times leads to the following system

$$\begin{aligned} \dot{z}_n &= \theta_n u_{n-1} + \tilde{F}_n(\zeta_{n-1}, u_{n-1}, d) \\ \dot{\zeta}_{n-1} &= G_{n-1}(\zeta_{n-1}, u_{n-1}, d) \end{aligned} \tag{3.29}$$

where $\zeta_{n-1} = (\xi_n, z_{n-1}, \ldots, z_1, \xi_1)$, and $\tilde{F}_n(\zeta_{n-1}, u_{n-1}, 0) = o(\zeta_{n-1}, u_{n-1})$. ζ_{n-1} satisfies LB with restriction and AB with no restriction on $\zeta_{n-1}(0)$, both with restrictions on u_{n-1}, d and linear gains. Then applying Lemma 3.2.1 to system (3.29) shows that, there exist $\lambda_n, k_n > 0$ such that under the control $u_{n-1} = -\sigma_n(k_n z_n + k_n H_n \zeta_{n-1} - u_n)$ with $H_n = [0_{1\times n_{\xi_n}} \ \frac{-\theta_n}{\theta_{n-1}} \ 0_{1\times(n-2+n_{\xi_{n-1}}+\cdots+n_{\xi_1})}]$, z_n, ζ_{n-1} satisfy LB with restriction and AB with no restriction on $(z_n(0), \zeta_{n-1}(0))$, both with restrictions on u_n, d and linear gains.

Finally, setting $u_n = 0$ in (3.24) gives the result of global robust asymptotic stabilization for system (3.1) with $d = 0$.

3.3 An Example

Consider the following system

$$\begin{aligned} \dot{x}_2 &= x_1 + 0.2\xi_2^2 \\ \dot{\xi}_2 &= -|\xi_2|\xi_2 + 0.1\|\xi_1\|^2 + 0.1x_1^2 \\ \dot{x}_1 &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \xi_1 + u \\ \dot{\xi}_1 &= \begin{bmatrix} -1 & 0.4 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \xi_1 + \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} u + (\xi_{11}^2 - \xi_{12}^2 - \xi_{13}^2) \begin{bmatrix} 0 \\ \xi_{12} \\ \xi_{13} \end{bmatrix} \end{aligned} \tag{3.30}$$

where $\xi_1 = (\xi_{11}, \xi_{12}, \xi_{13})$. The Jacobian linearization of system (3.30) at the origin is not stabilizable. Even without the ξ_2 dynamics, system (3.30) still cannot be handled by the approach in [7]. However, since system (3.30) is in the form of (3.1) and satisfies Assumptions 3.2.1-3.2.3, Theorem 3.2.1 can be applied to design the stabilizing control law.

Let $z_1 = x_1 - D_1 A_1^{-1}\xi_1, z_2 = x_2 + \frac{\theta_2}{\theta_1} z_1$, where $\theta_1 = 1.1, \theta_2 = 1/k_1$. Then, system (3.30) can be transformed into the following form:

$$\begin{aligned}
\dot{z}_2 &= \theta_2 u + 0.2\xi_2^2 + \frac{\theta_2}{\theta_1} 0.6(\xi_{11}^2 - \xi_{12}^2 - \xi_{13}^2)\xi_{13} + \theta_2 k_1 x_1 \\
\dot{\xi}_2 &= -|\xi_2|\xi_2 + 0.1\|\xi_1\|^2 + 0.1x_1^2 \\
\dot{z}_1 &= \theta_1 u + 0.6(\xi_{11}^2 - \xi_{12}^2 - \xi_{13}^2)\xi_{13} \\
\dot{\xi}_1 &= A_1\xi_1 + B_1 u + g_1(\xi_1, u, d)
\end{aligned} \tag{3.31}$$

where for convenience, we retain the original coordinates on the righthand side of (3.31).

First, consider the last two equations of (3.31). It can be verified by the a-LB small gain theorem [85] that, ξ_1 subsystem satisfies a-LB with no restriction on $\xi_1(0)$, no restriction on u and gain $0.4s$. Thus, ξ_1 subsystem satisfies LB and AB with no restriction on $\xi_1(0)$, no restriction on u and gain $0.4s$.

Since $D_1 A_1^{-1} = [-1 \;\; 0 \;\; -0.6]$, we let $l = 1.1662$. Note that $F_1(\xi_1, u, d) = 0.6(\xi_{11}^2 - \xi_{12}^2 - \xi_{13}^2)\xi_{13}$ and it is independent of u. Then the small gain condition becomes

$$6 \times 1.1662 \times 0.4k_1 < 1,$$
$$6 \times 2 \times 0.6(0.4\min\{k_1 s, \lambda_1\})^3/(\theta_1 k_1) < s, \;\; s > 0.$$

The above inequalities are satisfied with $k_1 = 0.178$ and $\lambda_1 = 1.092$. Thus, under the control $u = -\sigma_1(k_1 z_1 + k_1 D_1 A_1^{-1}\xi_1 - u_1)$, ξ_1, z_1, x_1 satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restriction 0.36 on u_1 and gains $2.4s, 16.85s, 33.7s$, respectively.

It can be verified that, ξ_2 satisfies LB and AB with no restriction on $\xi_2(0)$, no restrictions on ξ_1, x_1 and gains $0.4472s, 0.4472s$ respectively. Now let $\zeta_1 = (\xi_2, z_1, \xi_1)$. Then ζ_1 satisfies LB with restriction and AB with no restriction on $\zeta_1(0)$, both with restriction 0.36 on u_1 and linear gain, and in particular, the gain from u_1 to ξ_2 is $15.1s$.

Next, we consider the following system

$$\begin{aligned}
\dot{z}_2 &= \theta_2 u_1 + \tilde{F}_2(\zeta_1, u_1, d) \\
\dot{\zeta}_1 &= G_1(\zeta_1, u_1, d)
\end{aligned}$$

where $\tilde{F}_2(\zeta_1, u_1, d) = 0.2\xi_2^2 + \frac{\theta_2}{\theta_1} 0.6(\xi_{11}^2 - \xi_{12}^2 - \xi_{13}^2)\xi_{13} + \theta_2 h_1(x_1, u_1)$ and $h_1(x_1, u_1) = k_1 x_1 - u_1 - \sigma_1(k_1 x_1 - u_1)$.

In this case, we have $y_{11} = -\theta_2 z_1/\theta_1$. Then $\bar{\gamma}_{11}(s) = \frac{3}{1.1k_1^2}\min\{k_2 s, \lambda_2\}$. With $\lambda_2 \leq \frac{\lambda_1}{12}$ and by the property of the saturation function, it can be seen that $h_1(x_1, u_1)$ has no contribution to $\bar{\gamma}_{12}(s)$. Then, from the expression of $\tilde{F}_2(\zeta_1, u_1, d)$ and by the gains from u_1 to ξ_1, ξ_2, we have

$$\bar{\gamma}_{12}(s) \leq \frac{0.4k_1(15.1\min\{k_2 s,\lambda_2\})^2 + \frac{1.2}{1.1}(2.4\min\{k_2 s,\lambda_2\})^3}{k_2}, \quad s > 0.$$

Then k_2 and λ_2 should satisfy

$$6 \times 3k_2/(1.1k_1^2) < 1,$$
$$6\frac{0.4k_1(15.1\min\{k_2 s,\lambda_2\})^2 + \frac{1.2}{1.1}(2.4\min\{k_2 s,\lambda_2\})^3}{k_2} < s, \quad s > 0.$$

The above inequalities are satisfied with $k_2 = 0.0096, \lambda_2 = 0.00256$.

As a result, we obtain the following nested saturation control law:

$$u = -\sigma_1(0.178x_1 + \sigma_2(0.0096x_2)) \tag{3.32}$$

where σ_1, σ_2 are saturation functions with level 1.092 and 0.00256 respectively. As an illustration, Fig. 3.1 shows the simulation result of system (3.30) under the control (3.32) with initial state $(x_2(0), \xi_2(0), x_1(0), \xi_1(0)) = (-10, 2, 15, (-10, 10, -10))$.

3.4 Conclusion

In this chapter, we have studied the disturbance attenuation problem for a class of feedforward systems subject to both dynamic uncertainty and disturbance. The system includes the feedforward system subject to input unmodeled dynamics as studied in [7] as a special case. Moreover, by utilizing two versions of small gain theorem with restrictions adapted from [85], we have removed the stabilizability assumption of the Jacobian linearization of the system at the origin needed in [7].

It might also be possible to use a combination of a-LB and AB, as mentioned in [85], to study the problem of this chapter. However, since a-LB strictly implies LB, our current approach leads to a simpler treatment of the problem.

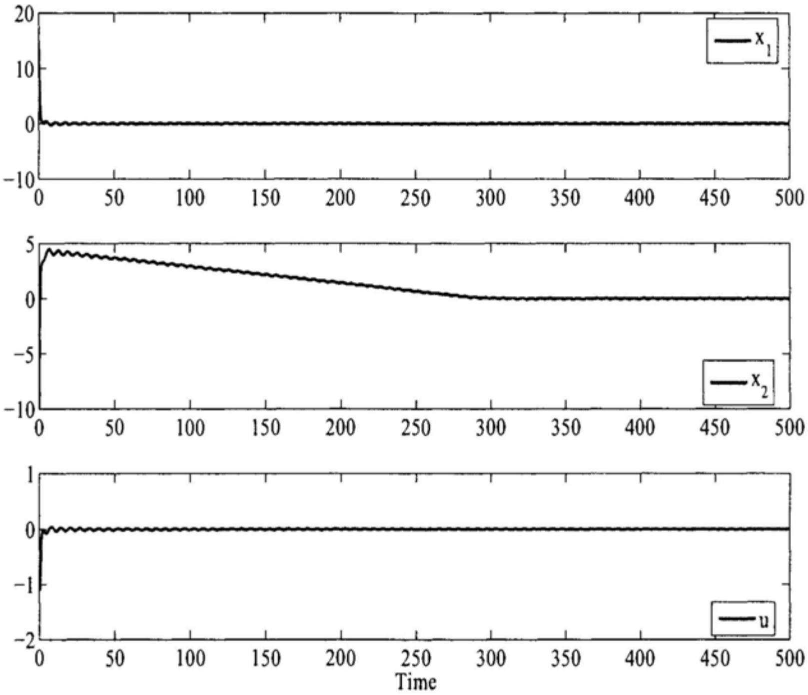

Figure 3.1: Profile of x_1, x_2 and u

Chapter 4

Global Robust Stabilization of Feedforward Systems with Both Time-Varying Static and Dynamic Uncertainties

4.1 Introduction

In this chapter, we study the global robust stabilization problem of the system described by the following equation

$$\begin{aligned} \dot{x}_i &= f_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d) \\ \dot{\xi}_i &= g_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d), \ \ i = n, ..., 2 \\ \dot{x}_1 &= f_1(\xi_1, u, d) \\ \dot{\xi}_1 &= g_1(\xi_1, u, d) \end{aligned} \tag{4.1}$$

where for $i = 1, ..., n$, $x_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^{n_{\xi_i}}$, $d \in \mathbb{R}^{n_d}$, $u \in \mathbb{R}$, f_i, g_i are globally defined smooth functions vanishing at $(0, ..., 0, d)$ for all $d \in \mathcal{D}$, n_{ξ_i} and n_d are dimensions of ξ_i and d respectively. System (4.1) contains two types of uncertainties, i.e., time-varying static uncertainty represented by the external disturbance d where $d : [0, \infty) \to \mathcal{D}$ is a continuous function with its range $\mathcal{D}$ a compact subset having a known bound, and dynamic uncertainty represented by dynamics governing $\xi_1, \xi_2, ..., \xi_n$. The dynamics governing $\xi_1, \xi_2, ..., \xi_n$ are called dynamic uncertainty because $\xi_1, \xi_2, ..., \xi_n$ are not allowed for feedback. Thus, we call system (4.1) a class of feedforward systems with both time-varying static and dynamic uncertainty.

As will be seen in Chapter 6, the global robust output regulation problem for a class of feedforward systems can be converted into a global robust stabilization problem of an augmented system in the form of (4.1). Therefore, if we can solve the global robust stabilization problem of system (4.1), then in turn, we can solve the global robust output regulation problem for a class of feedforward systems. For this purpose, we study the global robust stabilization problem of system (4.1) in this chapter.

When $n_{\xi_i} = 0, i = 1, ..., n$, i.e., the dynamic uncertainty is not present, the stabilization problem of various special cases of system (4.1) has been studied extensively, see e.g., [4, 29, 45, 53, 59, 72, 85, 87] and the references therein. On the other hand, the global robust stabilization problem of another special case of system (4.1) with $n_{\xi_1} \neq 0$ and $n_{\xi_i} = 0, i = 2, ..., n$, also gathers a lot of interest [7, 33, 45, 54, 56, 72]. Recently, [44] studied an output feedback stabilization and disturbance attenuation problem for a system of the form of (4.1) without ξ_1 dynamics. The feedforward systems considered in [44] have to satisfy some bounds which are linearly bounded in the unmeasured states and polynomially bounded in the output. In Chapter 3, a state feedback stabilization and disturbance attenuation problem for a system of the form (4.1) was studied under the assumption that for $i = 2, ..., n$, $f_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d) = c_i x_{i-1} + f_i^h(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d)$ where c_i is a nonzero constant and

$$f_i^h(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, 0) = o(\xi_i, x_{i-1}, ..., x_1, \xi_1, u). \tag{4.2}$$

To solve the global robust output regulation problem in Chapter 6, we need to study the global robust stabilization problem of system (4.1) without the condition described by (4.2) made in Chapter 3. As a result, the approach in Chapter 3 cannot be applied to the stabilization problem of system (4.1). Thus, we need to develop a approach to solve the global robust stabilization problem of system (4.1). In particular, we will present a small gain based bottom-up recursive design for constructing a nested saturation control. At each recursion, a cascade subsystem is considered and a saturation control is designed to guarantee the stability property of the subsystem by invoking two versions of the small gain theorem with restrictions adapted from [85] to establish the local stability and global attractiveness of the closed-loop system at the origin respectively. Unlike most existing results, our approach does not require the bottom dynamics at each recursion be locally exponentially stable.

4.2 A Technical Lemma

In this section, we first introduce a technical lemma and then by recursively applying the lemma, a small gain based bottom-up recursive design for the global robust stabilization problem of system (4.1) is accomplished.

Like Chapter 3, our approach will utilize saturation functions characterized in Definition 3.2.1.

In the following, we will consider the system

$$\begin{aligned} \dot{z} &= \theta(d)u + F(\xi, u, d) \\ \dot{\xi} &= G(\xi, u, d) \end{aligned} \tag{4.3}$$

where $F : \mathbb{R}^{n_\xi} \times \mathbb{R} \times \mathcal{D} \to \mathbb{R}, G : \mathbb{R}^{n_\xi} \times \mathbb{R} \times \mathcal{D} \to \mathbb{R}^{n_\xi}$ are locally Lipschitz functions vanishing at $(0, 0, d)$ for all $d \in \mathcal{D}$, and $\theta : \mathcal{D} \to \mathbb{R}$ is continuous, nonzero and does not change its sign.

Under the control

$$u = -\sigma(kz + kH(d)\xi - \tilde{u}) \tag{4.4}$$

where σ is a saturation function with level $\lambda > 0$, k is a nonzero real number with the same sign as $\theta(d)$, $H(d)$ is a $1 \times n_\xi$ matrix depending on d and satisfying $\|H(d)\| \leq \nu$ for all $d \in \mathcal{D}$ and some positive constant ν, system (4.3) takes the form

$$\begin{aligned} \dot{z} &= -\theta(d)\sigma(kz + kH(d)\xi - \tilde{u}) + F(\xi, -\sigma(kz + kH(d)\xi - \tilde{u}), d) \\ \dot{\xi} &= G(\xi, -\sigma(kz + kH(d)\xi - \tilde{u}), d) \end{aligned} \tag{4.5}$$

which can be viewed as the interconnection

$$v_1 = y_2, \quad v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = y_1 \tag{4.6}$$

of the following two subsystems

$$\Sigma_1 : \dot{\xi} = G(\xi, -\sigma(kv_1), d), \;\; y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} H(d)\xi \\ F(\xi, -\sigma(kv_1), d)/\tilde{k} \end{bmatrix} \tag{4.7}$$

$$\Sigma_2 : \dot{z} = -\tilde{\sigma}(\tilde{k}(z + v_{21} - \tfrac{\tilde{u}}{k})) + \tilde{k}v_{22}, \quad y_2 = z + v_{21} - \tfrac{\tilde{u}}{k} \tag{4.8}$$

where $\tilde{\sigma}(s) = \theta(d)\sigma(\frac{s}{\theta(d)})$ is a saturation function with level $\tilde{\lambda} = |\theta(d)|\lambda$ and $\tilde{k} = \theta(d)k > 0$.

Lemma 4.2.1 Consider system (4.3). Assume ξ subsystem satisfies LB with restriction and AB with no restriction on $\xi(0)$, both with restriction Δ on u and gain $\bar{N}s$. Then under the control (4.4), the following results hold:

1) With $\lambda < \Delta$, for $i = 1, 2$, the output y_{1i} of Σ_1 subsystem satisfies LB with restriction and AB with no restriction on $\xi(0)$, both with no restriction on v_1 and gain $\bar{\gamma}_{1i}$.

2) Further assume $\bar{\gamma}_{1i}$ satisfies

$$\lim_{s\to\infty} \bar{\gamma}_{1i}(s) = \bar{\gamma}_{1i}(\tfrac{\lambda}{|k|}), i = 1, 2 \tag{4.9}$$

and

$$6\bar{\gamma}_{11}(s) < s, 6\bar{\gamma}_{12}(s) < s, s > 0. \tag{4.10}$$

Then $z, z + H(d)\xi, \xi, u$ satisfy LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction $\frac{\lambda}{3}$ on $\tilde{u}$ and gains $\frac{3}{|k|}s$, $\frac{6}{|k|}s, 6\bar{N}s, 6s$, respectively.

Remark 4.2.1 Put the closed-loop system (4.5) in the compact form $\dot{x} = f(x, \tilde{u}, d(t))$ where $x = (z, \xi)$. Then Lemma 3.1 implies that the equilibrium point $x = 0$ of the system $\dot{x} = f(x, 0, d(t))$ is globally asymptotically stable for all $d(t) \in \mathcal{D}$. In particular, since $\dot{x} = f(x, 0, d(t))$ is time-varying, the local stability of the equilibrium point $x = 0$ cannot be implied by the global attractiveness of the equilibrium point $x = 0$ of the system $\dot{x} = f(x, 0, d(t))$. That is why we need to separately establish the LB and AB properties of the closed-loop system (4.5). In contrast, the corresponding result in the Appendix of [7] only requires that the equilibrium point $x = 0$ of the system $\dot{x} = f(x, 0, 0)$ be globally asymptotically stable. Due to the special structure of the function $f(x, 0, 0)$, the global attractiveness of the equilibrium point $x = 0$ of the system $\dot{x} = f(x, 0, 0)$ implies the local exponential stability of the equilibrium point $x = 0$ of the system $\dot{x} = f(x, 0, 0)$. Therefore, in [7], it suffices to establish the AB property of the closed-loop system (4.5).

Proof: Part 1): The assumption on ξ subsystem and $|\sigma(kv_1)| \leq \min\{|k||v_1|, \lambda\}$ with $\lambda < \Delta$ implies that, there exist an open set Ξ of the origin of $\mathbb{R}^{n_\xi}$ and a gain function γ_1^0, all independent of d, such that, for all $\xi(0) \in \mathbb{R}^{n_\xi}, d \in \mathcal{D}$ and piecewise continuous v_1, $\xi(t)$ exists for all $t \geq 0$ and satisfies

$$\|\xi\|_\infty \leq \max\{\gamma_1^0(\|\xi(0)\|), \bar{N}\min\{|k|\|v_1\|_\infty, \lambda\}\} \tag{4.11}$$

for all $\xi(0) \in \Xi, d \in \mathcal{D}$ and $v_1 \in \mathcal{L}_\infty^1$, and

$$\|\xi\|_a \leq \bar{N}\min\{|k|\|v_1\|_a, \lambda\} \tag{4.12}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}, d \in \mathcal{D}$ and piecewise continuous v_1.

Noting $|y_{11}| = |H(d)\xi| \leq \|H(d)\|\|\xi\| \leq \nu\|\xi\|$ yields

$$\begin{aligned} \|y_{11}\|_\infty &\leq \max\{\nu\gamma_1^0(\|\xi(0)\|), \nu\bar{N}\min\{|k|\|v_1\|_\infty, \lambda\}\} && (4.13) \\ &= \max\{\nu\gamma_1^0(\|\xi(0)\|), \bar{\gamma}_{11}(\|v_1\|_\infty)\} && (4.14) \end{aligned}$$

for all $\xi(0) \in \Xi, d \in \mathcal{D}$ and $v_1 \in \mathcal{L}_\infty^1$, where $\bar{\gamma}_{11}(s) = \nu\bar{N}\min\{|k|s, \lambda\}$, and

$$\|y_{11}\|_a \leq \nu\bar{N}\min\{|k|\|v_1\|_a, \lambda\} = \bar{\gamma}_{11}(\|v_1\|_a) \tag{4.15}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}, d \in \mathcal{D}$ and piecewise continuous v_1.

Next consider y_{12}. Since $F(\xi, u, d)$ is continuous and $F(0,0,d) = 0$ for all $d \in \mathcal{D}$, there exists a gain function $\alpha(s)$, independent of d, such that $|F(\xi, u, d)| \leq \alpha(\|(\xi, u)\|)$ for any $\xi \in \mathbb{R}^{n_\xi}, u \in \mathbb{R}$ and $d \in \mathcal{D}$. Then from (4.11) and (4.12),

$$\begin{aligned} \|y_{12}\|_\infty &\leq \max\{\alpha(2\gamma_1^0(\|\xi(0)\|)), \alpha(2\max\{\bar{N}, 1\}\min\{|k|\|v_1\|_\infty, \lambda\})\}/\tilde{k} \\ &= \max\{\alpha(2\gamma_1^0(\|\xi(0)\|))/\tilde{k}, \bar{\gamma}_{12}(\|v_1\|_\infty)\} && (4.16) \end{aligned}$$

for all $\xi(0) \in \Xi, d \in \mathcal{D}$ and $v_1 \in \mathcal{L}_\infty^1$, where $\bar{\gamma}_{12}(s) = \alpha(2\max\{\bar{N}, 1\}\min\{|k|s, \lambda\})/\tilde{k}$, and

$$\|y_{12}\|_a \leq \alpha(2\max\{\bar{N}, 1\}\min\{|k|\|v_1\|_a, \lambda\}) = \bar{\gamma}_{12}(\|v_1\|_a) \tag{4.17}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}, d \in \mathcal{D}$ and piecewise continuous v_1.

Letting $\bar{\gamma}_1^0(s) = \max\{\nu\gamma_1^0(s), \alpha \circ 2\gamma_1^0(s)/\tilde{k}\}$ and noting (4.14), (4.15) and (4.16), (4.17), completes Part 1). Since $y_{1i}, i = 1, 2$, satisfies LB and AB both with no restriction on v_1, let $\bar{\Delta}_1 = \infty$.

Part 2): Let us first apply Propositions 2.3.1 and 2.3.2 to show that the output (y_1, y_2) of Σ_1 and Σ_2 under the interconnection (4.6) satisfies LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction on $\tilde{u}$ and linear gain.

Step 1. Show that, for Σ_2 system viewing $v_{21}, v_{22}, \tilde{u}$ as inputs, y_2 satisfies LB with no restriction on $z(0)$ and gain $\bar{\gamma}_2^0$, restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{\tilde{u}}$ on $v_{21}, v_{22}, \tilde{u}$ and gains $\bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}_2^{\tilde{u}}$ respectively, and satisfies AB with no restriction on $z(0)$, restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{\tilde{u}}$ on $v_{21}, v_{22}, \tilde{u}$ and gains $\bar{\gamma}_{21}, \bar{\gamma}_{22}, \bar{\gamma}_2^{\tilde{u}}$ respectively,

We first claim that, there exists a gain function γ_2^0, independent of d, such that, for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, piecewise continuous $\tilde{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \leq \frac{\lambda}{3|k|}$, $z(t)$ is defined for all $t \geq 0$, and

$$\|z\|_\infty \leq \max\{\gamma_2^0(|z(0)|), 3\|v_{21}\|_\infty, 3\|v_{22}\|_\infty, \tfrac{3}{|k|}\|\tilde{u}\|_\infty\}, \tag{4.18}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, $\tilde{u}, v_{21} \in \mathcal{L}_\infty^1$ and $\|v_{22}\|_\infty < \frac{\lambda}{3|k|}$, and

$$\|z\|_a \le \max\{3\|v_{21}\|_a, 3\|v_{22}\|_a, \tfrac{3}{|k|}\|\tilde{u}\|_a\}. \tag{4.19}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, piecewise continuous $\tilde{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \le \frac{\lambda}{3|k|}$. In fact, the proof of (4.19) can be extracted from the derivation of (A.16) of [7] and the proof of (4.18) can be derived similarly.

Then it follows from (4.18), (4.19) and $y_2 = z + v_{21} - \frac{\tilde{u}}{k}$ that, y_2 satisfies

$$\begin{aligned}\|y_2\|_\infty &\le \max\{2\gamma_2^0(|z(0)|), 6\|v_{21}\|_\infty, 6\|v_{22}\|_\infty, \tfrac{6}{|k|}\|\tilde{u}\|_\infty\} \\ &= \max\{\bar{\gamma}_2^0(|z(0)|), \bar{\gamma}_{21}(\|v_{21}\|_\infty), \bar{\gamma}_{22}(\|v_{22}\|_\infty), \bar{\gamma}_2^{\tilde{u}}(\|\tilde{u}\|_\infty)\} \end{aligned} \tag{4.20}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, $\tilde{u}, v_{21} \in \mathcal{L}_\infty^1$ and $\|v_{22}\|_\infty < \frac{\lambda}{3|k|}$, and

$$\|y_2\|_a \le \max\{6\|v_{21}\|_a, 6\|v_{22}\|_a, \frac{6}{|k|}\|\tilde{u}\|_a\} \tag{4.21}$$

$$= \max\{\bar{\gamma}_{21}(\|v_{21}\|_a), \bar{\gamma}_{22}(\|v_{22}\|_a), \bar{\gamma}_2^{\tilde{u}}(\|\tilde{u}\|_a\}) \tag{4.22}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, piecewise continuous $\tilde{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \le \frac{\lambda}{3|k|}$, where $\bar{\gamma}_2^0(s) = 2\gamma_2^0(s), \bar{\gamma}_{21}(s) = \bar{\gamma}_{22}(s) = 6s, \bar{\gamma}_2^{\tilde{u}}(s) = \frac{6}{|k|}s$. Since y_2 satisfies LB and AB both with no restriction on $\tilde{u}, v_{21}$ and restriction $\frac{\lambda}{3|k|}$ on v_{22}, let $\bar{\Delta}_{\tilde{u}} = \bar{\Delta}_{21} = \infty$ and $\bar{\Delta}_{22} = \frac{\lambda}{3|k|}$.

Step 2. Check the conditions of Propositions 2.3.1 and 2.3.2.

Obviously, condition (4.10) implies the small gain condition $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for $s > 0$ and for $i = 1, 2$. Thus, by (4.14), (4.16) and (4.20), and Proposition 2.3.1,

$$\|y_{1i}\|_\infty \le \max\{\bar{\gamma}_1^0(\|\xi(0)\|), \tfrac{1}{6}\bar{\gamma}_2^0(\|z(0)\|), \tfrac{1}{|k|}\|\tilde{u}\|_\infty\},\ i = 1, 2 \tag{4.23}$$

$$\|y_2\|_\infty \le \max\{6\bar{\gamma}_1^0(\|\xi(0)\|), \bar{\gamma}_2^0(\|z(0)\|), \tfrac{6}{|k|}\|\tilde{u}\|_\infty\} \tag{4.24}$$

for all $z(0) \in Z = \{z \in \mathbb{R} : \bar{\gamma}_2^0(|z|) < \frac{2\lambda}{|k|}\}$, $\xi(0) \in \hat{\Xi} = \{\xi \in \Xi : \bar{\gamma}_1^0(\|\xi\|) < \frac{\lambda}{3|k|}\}$, $d \in \mathcal{D}$ and $\|\tilde{u}\|_\infty < \frac{\lambda}{3}$.

Next consider Proposition 2.3.2. First note that the solution of the interconnected system exists for all $t \ge 0$ using the same argument as that in Lemma 3.5 of [85]. Then $\bar{\Delta}_1 = \infty$ implies that the first condition of Proposition 2.3.2 is satisfied. To check the second condition, note that for $i = 1, 2$ $\lim_{s\to\infty} \bar{\gamma}_{1i}(s) = \bar{\gamma}_{1i}(\frac{\lambda}{|k|}) < \infty$ by condition (4.9), and $\bar{\Delta}_{21} = \infty$, $\bar{\Delta}_{22} = \frac{\lambda}{3|k|}$. Then we only need to check $\lim_{s\to\infty} \bar{\gamma}_{12}(s) \le \frac{\lambda}{3|k|}$. From (4.10), we have

$$\lim_{s\to\infty} \bar{\gamma}_{12}(s) = \bar{\gamma}_{12}(\tfrac{\lambda}{|k|}) < \tfrac{\lambda}{6|k|} < \tfrac{\lambda}{3|k|}.$$

Finally, condition (4.10) implies the small gain condition $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for all $s > 0$ and $i = 1, 2$. Thus, by (4.15), (4.17) and (4.22), and Proposition 2.3.2,

$$\|y_{1i}\|_a \leq \tfrac{1}{|k|}\|\tilde{u}\|_a, i = 1, 2, \;\; \|y_2\|_a \leq \tfrac{6}{|k|}\|\tilde{u}\|_a \tag{4.25}$$

for all $z(0) \in \mathbb{R}, \xi(0) \in \mathbb{R}^{n_\xi}, d \in \mathcal{D}$ and piecewise continuous $\tilde{u}$.

Now we can conclude Part 2). Let $\tilde{\gamma}^0(s) = \max\{6\bar{N}|k|\bar{\gamma}_1^0(s), 3\bar{\gamma}_1^0(s), \bar{N}|k|\bar{\gamma}_2^0(s), \frac{1}{2}\bar{\gamma}_2^0(s), \gamma_1^0(s), \gamma_2^0(s)\}$. For all $(z(0), \xi(0)) \in Z \times \hat{\Xi}$, $d \in \mathcal{D}$ and $\|\tilde{u}\|_\infty < \frac{\lambda}{3}$, (4.23) implies $\|y_{12}\|_\infty < \frac{\lambda}{3|k|}$. Using (4.23), (4.18) and (4.24), (4.11) yields

$$\begin{aligned} \|z\|_\infty &\leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), \tfrac{3}{|k|}\|\tilde{u}\|_\infty\} \\ \|\xi\|_\infty &\leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), 6\bar{N}\|\tilde{u}\|_\infty\} \end{aligned} \tag{4.26}$$

for all $(z(0), \xi(0)) \in Z \times \hat{\Xi}, d \in \mathcal{D}$ and $\|\tilde{u}\|_\infty < \frac{\lambda}{3}$. For all $(z(0), \xi(0)) \in \mathbb{R} \times \mathbb{R}^{n_\xi}, d \in \mathcal{D}$ and piecewise continuous $\tilde{u}$ satisfying $\|\tilde{u}\|_a \leq \frac{\lambda}{3}$, (4.25) implies $\|y_{12}\|_a \leq \frac{\lambda}{3|k|}$. Using (4.25) and (4.19), (4.12) yields

$$\|z\|_a \leq \tfrac{3}{|k|}\|\tilde{u}\|_a, \;\; \|\xi\|_a \leq 6\bar{N}\|\tilde{u}\|_a$$

for all $(z(0), \xi(0)) \in \mathbb{R} \times \mathbb{R}^{n_\xi}, d \in \mathcal{D}$ and piecewise continuous $\tilde{u}$ satisfying $\|\tilde{u}\|_a \leq \frac{\lambda}{3}$.

Finally, noting $|z + H(d)\xi| = |z + y_{11}| \leq 2\max\{|z|, |y_{11}|\}$, $|u| = |\sigma(kv_1)| = |\sigma(ky_2)| \leq |k||y_2|$ and equations (4.23) to (4.25) completes the proof.

Remark 4.2.2 Another major difference between Lemma 4.2.1 and the result in the Appendix of [7] is that we have removed the higher order assumption $F(\xi, u, 0) = o(\xi, u)$. Without assumption $F(\xi, u, 0) = o(\xi, u)$, the LB and AB properties of the closed-loop system rely on a small gain condition (4.10). Nevertheless, as will be seen in the proof of Theorem 4.3.1, the condition (4.10) can be made satisfied by proper design of the controller.

Remark 4.2.3 Lemma 4.2.1 can also be proved by invoking Proposition 2.3.2 and a-LB small gain theorem, i.e., Theorem 1 of [85]. However, in doing so would make the proof further lengthy and complicate, because at each time when one intends to replace LB by a-LB , one redundant inequality like (2.24) will be added.

Remark 4.2.4 Suppose the assumption of Lemma 4.2.1 hold. Assume $y = h(\xi, u, d)$ is an output of ξ subsystem and moreover, y satisfies LB with restriction and AB with no restriction on $\xi(0)$, both with restriction Δ on u and gain $\tilde{N}s$. Then from $u = \sigma(kv_1) = \sigma(ky_2)$, (4.24), (4.25), it can be concluded that, y satisfies LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction $\frac{\lambda}{3}$ on $\tilde{u}$ and gain $6\tilde{N}s$.

4.3 Nested Saturation Controller Design

Now we are ready to state the main result of this section. We make two assumptions on system (4.1), under which the global robust stabilization problem of system (4.1) can be solved.

Define $A_1(d) = \frac{\partial g_1}{\partial \xi_1}|_{(0,0,d)}, B_1(d) = \frac{\partial g_1}{\partial u}|_{(0,0,d)}, c_1(d) = \frac{\partial f_1}{\partial u}|_{(0,0,d)}, D_1(d) = \frac{\partial f_1}{\partial \xi_1}|_{(0,0,d)}$ and for $i = 2, ..., n$, $A_i(d) = \frac{\partial g_i}{\partial \xi_i}|_{(0,...,0,d)}, B_i(d) = \frac{\partial g_i}{\partial x_{i-1}}|_{(0,...,0,d)}, c_i(d) = \frac{\partial f_i}{\partial x_{i-1}}|_{(0,...,0,d)}, D_i(d) = \frac{\partial f_i}{\partial \xi_i}|_{(0,...,0,d)}$. To simplify the notation, we drop the argument d in the matrices defined above, then system (4.1) can be rewritten in the following form:

$$\begin{aligned}
\dot{x}_i &= D_i\xi_i + c_i x_{i-1} + f_i^r(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d) \\
\dot{\xi}_i &= A_i\xi_i + B_i x_{i-1} + g_i^r(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d), \ i = n, ..., 2 \\
\dot{x}_1 &= D_1\xi_1 + c_1 u + f_1^r(\xi_1, u, d) \\
\dot{\xi}_1 &= A_1\xi_1 + B_1 u + g_1^r(\xi_1, u, d)
\end{aligned} \tag{4.27}$$

where f_i^r, g_i^r are suitably defined smooth functions.

Assumption 4.3.1 For $i = 1, ..., n$, $D_iA_i^{-1}$ is a constant matrix and $\mu_i = c_i - D_iA_i^{-1}B_i$ is a positive (or alternatively negative) constant.

Assumption 4.3.2 ξ_1 satisfies LB and AB with no restriction on $\xi_1(0)$, both with restriction Δ_1 on u and gain $\bar{N}_1 s$, and for $i = 2, ..., n$, ξ_i satisfies LB and AB with no restriction on $\xi_i(0)$, both with restriction Δ_i on $(x_{i-1}, \xi_{i-1}, ..., x_1, \xi_1, u)$ and gain $\bar{N}_i s$.

Theorem 4.3.1 Consider system (4.1). Under Assumptions 4.3.1-4.3.2, there exist $\lambda_i > 0$ and nonzero k_i with the same sign as θ_i where $\theta_1 = \mu_1$ and $\theta_i = \mu_i/k_{i-1}$, $i = 2, ..., n$, such that, under the control

$$u = -\sigma_1(k_1x_1 + \sigma_2(k_2x_2 + ... + \sigma_n(k_nx_n))) \tag{4.28}$$

where for $i = 1, ..., n$, σ_i is a saturation function with level $\lambda_i > 0$, the closed-loop system is globally asymptotically stable at $(0, ..., 0)$ for all $d \in \mathcal{D}$.

To prove Theorem 4.3.1, we first note some facts. Let $h_i(x_i, u_i) = k_ix_i - u_i - \sigma_i(k_ix_i - u_i)$. By the property of the saturation function, $h_i(x_i, u_i) = 0$ when $|k_ix_i - u_i| \leq \frac{\lambda_i}{2}$. Thus $h_i(x_i, u_i) = o(x_i, u_i)$. From Assumption 4.3.1 and the smoothness of f_i, g_i, there exist positive constants μ_i^L, μ_i^U, ν_i, $i = 1, \cdots, n$, such that $\mu_i^L \leq |\mu_i| \leq \mu_i^U$ and $\|D_iA_i^{-1}\| \leq \nu_i$ for all $d \in \mathcal{D}$, and moreover, there exist positive constants $L_i, i = 1, ..., n-1$, and gain functions $\gamma_i^o(s) = o(s), i = 1, ..., n$, such that, for any $\xi_n, x_{n-1}, ..., x_1, \xi_1, u$ and $d \in \mathcal{D}$, the following inequalities hold:

$$|f_1(\xi_1, u, d) - D_1 A_1^{-1} g_1(\xi_1, u, d)| \le \gamma_1^o(\|(\xi_1, u)\|) \tag{4.29}$$

$$\begin{aligned} &|f_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d) - D_i A_i^{-1} g_i(\xi_i, x_{i-1}, ..., x_1, \xi_1, u, d)| \\ &\quad \le L_{i-1}\|(\xi_{i-1}, x_{i-2}, ..., \xi_1, u)\| + \gamma_i^o(\|(\xi_i, x_{i-1}, ..., x_1, \xi_1, u)\|), i = 2, ..., n \end{aligned} \tag{4.30}$$

where x_0 is a dummy state. Since $\gamma_i^o(s) = o(s)$, for any $\varepsilon_i > 0$, there exists $\delta_i > 0$ such that

$$\gamma_i^o(s) \le \varepsilon_i s, \ 0 < s \le \delta_i, i = 1, ..., n \tag{4.31}$$

Proof: First, performing the coordinate transformation

$$z_1 = x_1 - D_1 A_1^{-1}\xi_1, z_i = x_i - D_i A_i^{-1}\xi_i + \frac{\theta_i}{\theta_{i-1}} z_{i-1}, i = 2, ..., n \tag{4.32}$$

on system (4.1) gives

$$\begin{aligned} \dot{z}_i &= \theta_i u + F_i(\xi_i, z_{i-1}, ..., z_1, \xi_1, u, d) \\ \dot{\xi}_i &= \tilde{g}_i(\xi_i, z_{i-1}, ..., z_1, \xi_1, u, d), \ i = n, ..., 2 \\ \dot{z}_1 &= \theta_1 u + F_1(\xi_1, u, d) \\ \dot{\xi}_1 &= \tilde{g}_1(\xi_1, u, d) \end{aligned} \tag{4.33}$$

where $\tilde{g}_i, i = 1, ..., n$, are suitably defined functions, $F_1(\xi_1, u, d) = f_1(\xi_1, u, d) - D_1 A_1^{-1}$ $g_1(\xi_1, u, d)$, and for $i = 2, ..., n$, $F_i(\xi_i, z_{i-1}, ..., \xi_1, u, d) = f_i(\xi_i, x_{i-1}, ..., \xi_1, u, d) - D_i A_i^{-1}$ $g_i(\xi_i, x_{i-1}, ..., \xi_1, u, d) + \frac{\theta_i}{\theta_{i-1}} F_{i-1}(\xi_{i-1}, z_{i-2}, ..., \xi_1, u, d) + \theta_i k_{i-1} x_{i-1} |_{(4.32)}$ and z_0 is a dummy state. Moreover, under the coordinate transformation (4.32), the control (4.28) becomes

$$u = -\sigma_1(k_1 z_1 + k_1 D_1 A_1^{-1}\xi_1 + ... + \sigma_n(k_n z_n + k_n D_n A_n^{-1}\xi_n - k_n \frac{\theta_n}{\theta_{n-1}} z_{n-1})). \tag{4.34}$$

The proof will be completed if we can show that, for system (4.33), under the control (4.34), the closed-loop system with $d \in \mathcal{D}$ at $(0, ..., 0)$ is globally asymptotically stable.

By recursively applying Lemma 4.2.1, we can use mathematical induction to prove the theorem. For convenience, let $(\cdot)^{(j)}$ denote the corresponding notation in the jth induction. The proof will be completed in three steps.

Step 1: Under the control $u = -\sigma_1(k_1 z_1 + k_1 H_1 \xi_1 - u_1)$ where $H_1 = D_1 A_1^{-1}$, the subsystem consisting of the last two equations of (4.33) can be viewed as the interconnection (4.6) of the two subsystems (4.7) and (4.8) where $v_1^{(1)} = y_2^{(1)} = z_1 + H_1\xi_1 - u_1/k_1$,

$v_{21}^{(1)} = y_{11}^{(1)} = H_1\xi_1$, $v_{22}^{(1)} = y_{12}^{(1)} = \frac{1}{\tilde{k}_1}F_1(\xi_1, u, d)$ with $\tilde{k}_1 = \theta_1 k_1$. By Assumptions 4.3.1 and 4.3.2, and Lemma 4.2.1, with $\lambda_1 < \Delta_1$, $y_{1i}^{(1)}, i = 1, 2$, satisfies LB with restriction and AB with no restriction on $\xi_1(0)$, both with no restriction on $v_1^{(1)}$. Moreover, it follows from the expression of $y_{11}^{(1)}$ that $\bar{\gamma}_{11}^{(1)}(s) = \nu_1 \bar{N}_1 \min\{|k_1|s, \lambda_1\}$ and from the expression of $y_{12}^{(1)}$ and (4.29) that $\bar{\gamma}_{12}^{(1)}(s) = \gamma_1^o(2\max\{1, \bar{N}_1\}\min\{|k_1|s, \lambda_1\})/\tilde{k}_1$. Clearly, $\bar{\gamma}_{11}^{(1)}$ and $\bar{\gamma}_{12}^{(1)}$ satisfy (4.9). We now further show that there exist sufficiently small $\lambda_1, |k_1|$ such that

$$6\bar{\gamma}_{11}^{(1)}(s) < s, 6\bar{\gamma}_{12}^{(1)}(s) < s, \; s > 0. \tag{4.35}$$

Noting (4.31) and substituting the expression of $\bar{\gamma}_{11}^{(1)}$ and $\bar{\gamma}_{12}^{(1)}$ into (4.35), we can obtain

$$\bar{\gamma}_{11}^{(1)}(s) \leq \nu_1 \bar{N}_1 |k_1| s < \tfrac{1}{6}s, \; \bar{\gamma}_{12}^{(1)}(s) \leq \tfrac{2\max\{1,\bar{N}_1\}\varepsilon_1}{\mu_1^L} s < \tfrac{1}{6}s, \; s > 0. \tag{4.36}$$

The first inequality of (4.36) can be satisfied with $|k_1| < \frac{1}{6\nu_1 \bar{N}_1}$. Next note that for any $0 < \tau_1 \leq \frac{1}{6}$, there exists $\varepsilon_1 > 0$ such that $\frac{2\max\{1,\bar{N}_1\}\varepsilon_1}{\mu_1^L} < \tau_1$. Corresponding to this ε_1, there exists a $\delta_1 > 0$ such that $\lambda_1 < \min\{\Delta_1, \frac{\delta_1}{2\max\{1,\bar{N}_1\}}\}$ implies the second inequality of (4.36). By Lemma 4.2.1, z_1, x_1, ξ_1, u satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restriction $\frac{\lambda_1}{3}$ on u_1 and gains $\frac{3}{|k_1|}s$, $\frac{6}{|k_1|}s$, $6\bar{N}_1 s$, $6s$, respectively.

Note that (x_1, ξ_1, u) can be seen as an output of the subsystem consisting of the last two equations of (4.33) and moreover, the last three equations of (4.33) can be seen in the form of (2.43) with $\tilde{x}_1 = (z_1, \xi_1), \tilde{x}_2 = \xi_2, y_1 = (x_1, \xi_1, u), \tilde{u}_1 = u_1$, by Assumption 4.3.2 and Corollary 2.3.2, ξ_2 satisfies LB with restriction and AB with no restriction on $(\xi_2(0), z_1(0), \xi_1(0))$, both with restriction $\tilde{\Delta}_1 = \min\{\frac{\lambda_1}{3}, \frac{\Delta_2}{18\max\{\frac{1}{|k_1|}, \bar{N}_1, 1\}}\}$ on u_1 and gain $18\bar{N}_2 \max\{\frac{1}{|k_1|}, \bar{N}_1, 1\}s$. Now let $\zeta_1 = (\xi_2, z_1, \xi_1)$. Then, both ζ_1 and (ξ_2, x_1, ξ_1, u) satisfy LB with restriction and AB with no restriction on $(\xi_2(0), z_1(0), \xi_1(0))$, both with restriction $\tilde{\Delta}_1$ and gain $\tilde{N}_1 s$ on u_1, where $\tilde{N}_1 = 36\max\{\bar{N}_2, 1\}\max\{\frac{1}{|k_1|}, \bar{N}_1, 1\}$.

Step 2: The last four equations of (4.33) can be put into the form:

$$\begin{aligned} \dot{z}_2 &= \theta_2 u_1 + \tilde{F}_2(\zeta_1, u_1, d) \\ \dot{\zeta}_1 &= G_1(\zeta_1, u_1, d) \end{aligned} \tag{4.37}$$

where $\tilde{F}_2(\zeta_1, u_1, d) = F_2(\zeta_1, u, d) + \theta_2(u - u_1) = f_2(\xi_2, x_1, \xi_1, u, d) - D_2 A_2^{-1} g_2(\xi_2, x_1, \xi_1, u, d) + \frac{\theta_2}{\theta_1}F_1(\xi_1, u, d) + \theta_2 h_1(x_1, u_1)$ and G_1 is a suitably defined function. Under the control $u_1 = -\sigma_2(k_2 z_2 + k_2 H_2 \zeta_1 - u_2)$ where $H_2 = [D_2 A_2^{-1} \; -\frac{\theta_2}{\theta_1} \; 0_{1\times n_{\xi_1}}]$, system (4.37) can be viewed as the interconnection (4.6) of the two subsystems (4.7) and (4.8) where $v_1^{(2)} = y_2^{(2)} = z_2 + H_2\zeta_1 - u_2/k_2$, $v_{21}^{(2)} = y_{11}^{(2)} = H_2\zeta_1 = D_2 A_2^{-1}\xi_2 - \theta_2 z_1/\theta_1$ and $v_{22}^{(2)} = y_{12}^{(2)} = \frac{1}{\tilde{k}_2}\tilde{F}_2(\zeta_1, u_1, d)$ with $\tilde{k}_2 = \theta_2 k_2$. By Step 1, ζ_1 subsystem satisfies LB with restriction and AB with no restriction on $\zeta_1(0)$, both with restriction $\tilde{\Delta}_1$ and gain $\tilde{N}_1 s$ on u_1. Then by Assumption 4.3.1 and Lemma 4.2.1, with $\lambda_2 < \tilde{\Delta}_1$, for $i = 1, 2$, $y_{1i}^{(2)}$ satisfies LB with restriction and AB with no restriction on $\zeta_1(0)$, both with no restriction on $v_1^{(2)}$. Moreover, it follows from the expression of $y_{11}^{(2)}$ that $\bar{\gamma}_{11}^{(2)}(s) \leq 2(18\nu_2\bar{N}_2 \max\{\frac{1}{|k_1|}, \bar{N}_1, 1\} + \frac{|\theta_2|}{\theta_1}\frac{3}{k_1})\min\{|k_2|s, \lambda_2\}$, and from the expression of $y_{12}^{(2)}, y_{12}^{(1)}$ and (4.30) that

$$|y_{12}^{(2)}| \leq \frac{L_1\|(\xi_1,u)\|}{\bar{k}_2} + \frac{\gamma_2^o(\|(\xi_2,x_1,\xi_1,u)\|)}{\bar{k}_2} + \frac{|k_1|}{|k_2|}|y_{12}^{(1)}| + \frac{1}{|k_2|}|h_1(x_1,u_1)|.$$

Noting $k_1x_1 - u_1 = k_1v_1^{(1)}$, $v_1^{(1)} = y_2^{(1)}$, (4.24), (4.25) and the property of $h_1(x_1, u_1)$ yields that $h_1(x_1, u_1)$ has no contribution to $\bar{\gamma}_{12}^{(2)}(s)$ when $\lambda_2 < \min\{\tilde{\Delta}_1, \frac{\lambda_1}{12}\}$, and moreover, from (4.16), (4.24) and (4.17), (4.25), we obtain

$$\bar{\gamma}_{12}^{(2)}(s) \leq 2[\frac{12L_1\max\{\bar{N}_1,1\}\min\{|k_2|s,\lambda_2\}}{\bar{k}_2} + \frac{\gamma_2^o(\tilde{N}_1\min\{|k_2|s,\lambda_2\})}{\bar{k}_2} + \frac{|k_1|}{|k_2|}\bar{\gamma}_{12}^{(1)}(\frac{6\min\{|k_2|s,\lambda_2\}}{|k_1|})].$$

Clearly, $\bar{\gamma}_{11}^{(2)}$ and $\bar{\gamma}_{12}^{(2)}$ satisfy (4.9). We now further show that there exist sufficiently small $\lambda_i, |k_i|, i = 1, 2$ such that

$$6\bar{\gamma}_{11}^{(2)}(s) < s, 6\bar{\gamma}_{12}^{(2)}(s) < s, \ s > 0. \tag{4.38}$$

Noting (4.31) and substituting the expression of $\bar{\gamma}_{11}^{(2)}$ and $\bar{\gamma}_{12}^{(2)}$ into (4.38), we can obtain

$$\begin{aligned} \bar{\gamma}_{11}^{(2)}(s) &\leq 2(18\nu_2\bar{N}_2\max\{\bar{N}_1, 1, \tfrac{1}{|k_1|}\} + 3\tfrac{\mu_2^U}{\mu_1^L k_1^2})|k_2|s < \tfrac{1}{6}s, \ s > 0 \\ \bar{\gamma}_{12}^{(2)}(s) &\leq 2(\tfrac{12L_1\max\{\bar{N}_1,1\}|k_1|}{\mu_2^L} + \tfrac{\tilde{N}_1|k_1|\varepsilon_2}{\mu_2^L} + 6\tau_1)s < \tfrac{1}{6}s, \ s > 0 \end{aligned} \tag{4.39}$$

The first inequality of (4.39) can be satisfied with $|k_2| < \frac{1}{12}(18\nu_2\bar{N}_2\max\{\bar{N}_1, 1, \frac{1}{|k_1|}\} + 3\frac{\mu_2^U}{\mu_1^L k_1^2})^{-1}$. Next note that for any $0 < \tau_2 \leq \frac{1}{6}$, there exist $|k_1|, \varepsilon_2 > 0, \tau_1 > 0$ such that $2(\frac{12L_1\max\{\bar{N}_1,1\}|k_1|}{\mu_2^L} + \frac{\tilde{N}_1|k_1|\varepsilon_2}{\mu_2^L} + 6\tau_1) < \tau_2$. Corresponding to this ε_2, there exists a $\delta_2 > 0$ such that $\lambda_2 < \min\{\tilde{\Delta}_1, \frac{\lambda_1}{12}, \frac{\delta_2}{\tilde{N}_1}\}$ implies the second inequality of (4.39). By Lemma 4.2.1, z_2, x_2, ζ_1, u_1 satisfy LB with restriction and AB with no restriction on $(z_2(0), \zeta_1(0))$, both with restriction $\frac{\lambda_2}{3}$ on u_2 and gains $\frac{3}{|k_2|}s, \frac{6}{|k_2|}s, 6\tilde{N}_1 s, 6s$ respectively.

Note that (ξ_2, x_1, ξ_1, u) can be seen as an output of ζ_1 subsystem of (4.37), then from the paragraph above equation (4.37) and by Remark 4.2.4, (ξ_2, x_1, ξ_1, u) satisfies LB with restriction on $(z_2(0), \zeta_1(0))$ and AB with no restriction on $(z_2(0), \zeta_1(0))$, both with restriction $\frac{\lambda_2}{3}$ on u_2 and gain $6\tilde{N}_1 s$. Since the last five equations of (4.33) can be seen in the form of (2.43) with $\tilde{x}_1 = (z_2, \zeta_1), \tilde{x}_2 = \xi_3, y_1 = (x_2, \xi_2, x_1, \xi_1, u), \tilde{u}_1 = u_2$, then by Assumption 4.3.2 and Corollary 2.3.2, ξ_3 satisfies LB with restriction and AB with no restriction on $(\xi_3(0), z_2(0), \zeta_1(0))$, both with restriction $\tilde{\Delta}_2 = \min\{\frac{\lambda_2}{3}, \frac{\Delta_3}{12\max\{\frac{1}{|k_2|}, \tilde{N}_1\}}\}$ on u_2 and gain $12\bar{N}_3\max\{\frac{1}{|k_2|}, \tilde{N}_1\}s$. Now let $\zeta_2 = (\xi_3, z_2, \zeta_1)$. Then, both ζ_2 and $(\xi_3, x_2, \xi_2, x_1, \xi_1, u)$

satisfy LB with restriction and AB with no restriction on $(\xi_3(0), z_2(0), \zeta_1(0))$, both with restriction $\tilde{\Delta}_2$ on u_2 and gain $\tilde{N}_2 s$, where $\tilde{N}_2 = 24\max\{\bar{N}_3, 1\}\max\{\frac{1}{|k_2|}, \tilde{N}_1\}$.

Step 3: For $3 \le j \le n$, put the last $2j$ equations of (4.33) into the form:

$$\begin{aligned} \dot{z}_j &= \theta_j u_{j-1} + \tilde{F}_j(\zeta_{j-1}, u_{j-1}, d) \\ \dot{\zeta}_{j-1} &= G_{j-1}(\zeta_{j-1}, u_{j-1}, d) \end{aligned} \tag{4.40}$$

where $\zeta_{j-1} = (\xi_j, z_{j-1}, \zeta_{j-2})$, G_{j-1} is a suitably defined function and

$$\begin{aligned} \tilde{F}_j(\zeta_{j-1}, u_{j-1}, d) = {} & f_j(\xi_j, x_{j-1}, ..., \xi_1, u, d) - D_j A_j^{-1} g_j(\xi_j, x_{j-1}, ..., \xi_1, u, d) \\ & + \tfrac{\theta_j}{\theta_{j-1}} \tilde{F}_{j-1}(\zeta_{j-2}, u_{j-2}, d) + \theta_j h_{j-1}(x_{j-1}, u_{j-1}). \end{aligned} \tag{4.41}$$

Under the control $u_{j-1} = -\sigma_j(k_j z_j + k_j H_j \zeta_{j-1} - u_j)$ where $H_j = [D_j A_j^{-1} \; -\frac{\theta_j}{\theta_{j-1}}, 0_{1\times(j-2+n_{\xi_1}+...+n_{\xi_{j-1}})}]$, system (4.40) can be viewed as the interconnection (4.6) of the two subsystems (4.7) and (4.8) where $v_1^{(j)} = y_2^{(j)} = z_j + H_j\zeta_{j-1} - u_j/k_j$, $v_{21}^{(j)} = y_{11}^{(j)} = H_j\zeta_{j-1} = D_j A_j^{-1}\xi_j - \frac{\theta_j}{\theta_{j-1}} z_{j-1}$ and $v_{22}^{(j)} = y_{12}^{(j)} = \frac{1}{\tilde{k}_j}\tilde{F}_j(\zeta_{j-1}, u_{j-1}, d)$ with $\tilde{k}_j = \theta_j k_j$.

Assume, $z_{j-1}, x_{j-1}, (\xi_{j-1}, x_{j-2}, ..., x_1, \xi_1, u)$ satisfy LB with restriction and AB with no restriction on $(z_{j-1}(0), \zeta_{j-2}(0))$, both with restriction $\frac{\lambda_{j-1}}{3}$ on u_{j-1} and gains $\frac{3}{|k_{j-1}|}s$, $\frac{6}{|k_{j-1}|}s, 6\tilde{N}_{j-2}s$ respectively, and $\xi_j, \zeta_{j-1}, (\xi_j, x_{j-1}, ..., x_1, \xi_1, u)$ satisfy LB with restriction and AB with no restriction on $(\xi_j(0), z_{j-1}(0), \zeta_{j-2}(0))$, both with restrictions $\tilde{\Delta}_{j-1}$ on u_{j-1} and gains $12\bar{N}_j\max\{\frac{1}{|k_{j-1}|}, \tilde{N}_{j-2}\}s, \tilde{N}_{j-1}s, \tilde{N}_{j-1}s$ respectively, where $\tilde{\Delta}_{j-1} = \min\{\frac{\lambda_{j-1}}{3}, \frac{\Delta_j}{12\max\{\frac{1}{|k_{j-1}|}, \tilde{N}_{j-2}\}}\}$ and $\tilde{N}_{j-1} = 24\max\{\bar{N}_j, 1\}\max\{\frac{1}{|k_{j-1}|}, \tilde{N}_{j-2}\}$. By this assumption, ζ_{j-1} subsystem satisfies LB with restriction and AB with no restriction on $\zeta_{j-1}(0)$, both with restriction $\tilde{\Delta}_{j-1}$ on u_{j-1} and gain $\tilde{N}_{j-1}s$. Then by Assumption 4.3.1 and Lemma 4.2.1, with $\lambda_j < \tilde{\Delta}_{j-1}$, for $i = 1, 2$, $y_{1i}^{(j)}$ satisfies LB with restriction and AB with no restriction on $\zeta_{j-1}(0)$, both with no restriction on $v_1^{(j)}$. Moreover, it follows from the expression of $y_{11}^{(j)}$ that $\bar{\gamma}_{11}^{(j)}(s) \le 2(12\nu_j\bar{N}_j\max\{\frac{1}{|k_{j-1}|}, \tilde{N}_{j-2}\} + \frac{|\theta_j|}{\theta_{j-1}}\frac{3}{k_{j-1}})\min\{|k_j|s, \lambda_j\}$, and from the expression of $y_{12}^{(j)}, y_{12}^{(j-1)}$ and (4.30) that

$$|y_{12}^{(j)}| \le \frac{L_{j-1}\|(\xi_{j-1}, x_{j-2}, ..., \xi_1, u)\|}{\tilde{k}_j} + \frac{\gamma_j^o(\|(\xi_j, x_{j-1}, ..., \xi_1, u)\|)}{\tilde{k}_j} + \frac{|k_{j-1}|}{|k_j|}|y_{12}^{(j-1)}| + \frac{1}{|k_j|}|h_{j-1}(x_{j-1}, u_{j-1})|.$$

Noting $k_{j-1}x_{j-1} - u_{j-1} = k_{j-1}v_1^{(j-1)}$, $v_1^{(j-1)} = y_2^{(j-1)}$, (4.24), (4.25) and the property of $h_{j-1}(x_{j-1}, u_{j-1})$ yields that $h_{j-1}(x_{j-1}, u_{j-1})$ has no contribution to $\bar{\gamma}_{12}^{(j)}(s)$ when $\lambda_j < \min\{\tilde{\Delta}_{j-1}, \frac{\lambda_{j-1}}{12}\}$, and moreover, from (4.16),(4.24) and (4.17), (4.25), we obtain

$$\bar{\gamma}_{12}^{(j)}(s) \le 2[\frac{6L_{j-1}\tilde{N}_{j-2}\min\{|k_j|s, \lambda_j\}}{\tilde{k}_j} + \frac{\gamma_j^o(\tilde{N}_{j-1}\min\{|k_j|s, \lambda_j\})}{\tilde{k}_j} + \frac{|k_{j-1}|}{|k_j|}\bar{\gamma}_{12}^{(j-1)}(\frac{6\min\{|k_j|s, \lambda_j\}}{|k_{j-1}|})].$$

Clearly, $\bar{\gamma}_{11}^{(j)}$ and $\bar{\gamma}_{12}^{(j)}$ satisfy (4.9). We now further show that there exist sufficiently small $\lambda_i, |k_i|, i = 1, 2, ..., j$, such that

$$6\bar{\gamma}_{11}^{(j)}(s) < s, 6\bar{\gamma}_{12}^{(j)}(s) < s, \; s > 0. \tag{4.42}$$

Noting (4.31) and substituting the expression of $\bar{\gamma}_{11}^{(j)}$ and $\bar{\gamma}_{12}^{(j)}$ into (4.42), we can obtain

$$\begin{aligned}\bar{\gamma}_{11}^{(j)}(s) &\leq 2(12\nu_j\tilde{N}_j\max\{\tfrac{1}{|k_{j-1}|},\tilde{N}_{j-2}\}+3\tfrac{\mu_j^U}{\mu_{j-1}^L}\tfrac{|k_{j-2}|}{k_{j-1}^2})|k_j|s<\tfrac{1}{6}s,\ s>0\\ \bar{\gamma}_{12}^{(j)}(s) &\leq 2(\tfrac{6L_{j-1}\tilde{N}_{j-2}|k_{j-1}|}{\mu_j^L}+\tfrac{\tilde{N}_{j-1}|k_{j-1}|\varepsilon_j}{\mu_j^L}+6\tau_{j-1})s<\tfrac{1}{6}s,\ s>0.\end{aligned} \tag{4.43}$$

The first inequality of (4.43) can be satisfied with $|k_j|<\frac{1}{12}(12\nu_j\tilde{N}_j\max\{\frac{1}{|k_{j-1}|},\tilde{N}_{j-2}\}+3\frac{\mu_j^U}{\mu_{j-1}^L}\frac{|k_{j-2}|}{k_{j-1}^2})^{-1}$. Next note that for any $0<\tau_j\leq\frac{1}{6}$, there exist $|k_{j-1}|,\varepsilon_j>0,\tau_{j-1}>0$ such that $2(\frac{6L_{j-1}\tilde{N}_{j-2}|k_{j-1}|}{\mu_j^L}+\frac{\tilde{N}_{j-1}|k_{j-1}|\varepsilon_j}{\mu_j^L}+6\tau_{j-1})<\tau_j$. Corresponding to this ε_j, there exists a $\delta_j>0$ such that $\lambda_j<\min\{\tilde{\Delta}_{j-1},\frac{\lambda_{j-1}}{12},\frac{\delta_j}{\tilde{N}_{j-1}}\}$ implies the second inequality of (4.43). By Lemma 4.2.1, $z_j,x_j,\zeta_{j-1},u_{j-1}$ satisfy LB with restriction and AB with no restriction on $(z_j(0),\zeta_{j-1}(0))$, both with restriction $\frac{\lambda_j}{3}$ on u_j and gains $\frac{3}{|k_j|}s,\frac{6}{|k_j|}s,6\tilde{N}_{j-1}s,6s$ respectively.

Note that $(\xi_j,x_{j-1},...,\xi_1,u)$ can be seen as an output of ζ_{j-1} subsystem of (4.40), then by Remark 4.2.4, $(\xi_j,x_{j-1},...,\xi_1,u)$ also satisfies LB with restriction and AB with no restriction on $(z_j(0),\zeta_{j-1}(0))$, both with restriction $\frac{\lambda_j}{3}$ on u_j and gain $6\tilde{N}_{j-1}s$. Since the last $2j+1$ equations of (4.33) can be seen in the form of (2.43) with $\tilde{x}_1=(z_j,\zeta_{j-1}),\tilde{x}_2=\xi_{j+1},y_1=(x_j,\xi_j,...,\xi_1,u),\tilde{u}_1=u_j$, then by Assumption 4.3.2 and Corollary 2.3.2, ξ_{j+1} satisfies LB with restriction and AB with no restriction on $(\xi_{j+1}(0),z_j(0),\zeta_{j-1}(0))$, both with restriction $\tilde{\Delta}_j=\min\{\frac{\lambda_j}{3},\frac{\Delta_{j+1}}{12\max\{\frac{1}{|k_j|},\tilde{N}_{j-1}\}}\}$ on u_j and gain $12\tilde{N}_{j+1}\max\{\frac{1}{|k_j|},\tilde{N}_{j-1}\}s$. Now let $\zeta_j=(\xi_{j+1},z_j,\zeta_{j-1})$. Then, both ζ_j and $(\xi_{j+1},x_j,...,\xi_1,u)$ satisfy LB with restriction and AB with no restriction on $(\xi_{j+1}(0),z_j(0),\zeta_{j-1}(0))$, both with restriction $\tilde{\Delta}_j$ on u_j and gain $\tilde{N}_js$, where $\tilde{N}_j=24\max\{\tilde{N}_{j+1},1\}\max\{\frac{1}{|k_j|},\tilde{N}_{j-1}\}$.

Therefore, the proof is completed by induction. Finally, setting $u_n=0$ gives the result of global asymptotic stabilization for system (4.33) with $d\in\mathcal{D}$.

Remark 4.3.1 The recursive design in this chapter is quite different from those in [7] and Chapter 4 where by taking advantage of the high order condition like (4.2), for each $j=1,...,n$, λ_j,k_j can be determined separately at the jth recursion. However, due to the presence of the time-varying static and dynamic uncertainties and due to the absence of the high order condition like (4.2), $\lambda_1,k_1,...,\lambda_n,k_n$, cannot be determined by the recursive designs of [7] and Chapter 4. Nevertheless, from the proof of Theorem 4.3.1, it can be seen that, $\lambda_1,k_1,...,\lambda_n,k_n$, are governed by a set of algebraic inequalities and can be determined simultaneously at the end of the whole recursive design. For example, with

$\tau_j = \frac{1}{6}\frac{1}{18^{n-j}}, j = 1, ..., n$, $\lambda_1, k_1, ..., \lambda_n, k_n$, are governed by

$$\frac{2\varepsilon_1 \max\{\bar{N}_1, 1\}}{\mu_1^L} s < \frac{1}{6}\frac{1}{18^{n-1}},$$
$$\nu_1 \bar{N}_1 |k_1| < \frac{1}{6},$$
$$2(\frac{12L_1 \max\{\bar{N}_1, 1\}|k_1|}{\mu_2^L} + \frac{\tilde{N}_1 |k_1| \varepsilon_2}{\mu_2^L}) < \frac{1}{18^{n-1}},$$
$$2(18\nu_2 \bar{N}_2 \max\{\bar{N}_1, 1, \frac{1}{|k_1|}\} + 3\frac{\mu_2^U}{\mu_1^L k_1^2})|k_2| < \frac{1}{6},$$
$$2(\frac{6L_{j-1}\tilde{N}_{j-2}|k_{j-1}|}{\mu_j^L} + \frac{\tilde{N}_{j-1}|k_{j-1}|\varepsilon_j}{\mu_j^L}) < \frac{1}{18^{n-j+1}},$$
$$2(12\nu_j \bar{N}_j \max\{\frac{1}{|k_{j-1}|}, \tilde{N}_{j-2}\} + 3\frac{\mu_j^U}{\mu_{j-1}^L}\frac{|k_{j-2}|}{k_{j-1}^2})|k_j| < \frac{1}{6}, j = 3, ..., n$$

which are extracted from the inequalities (4.36), (4.39) and (4.43), respectively. It can be seen that this set of inequalities are solvable, and $\lambda_1, k_1, ..., \lambda_n, k_n$ can be determined in order. In fact, λ_1 can be determined from the 1st inequality, k_1 from the 2nd and 3rd inequalities, and for $i = 2, ..., n$, λ_i can be determined from the $(2i-1)$th inequality and k_i from the $(2i)$th and $(2i+1)$th inequalities (k_n from the last inequality).

Remark 4.3.2 The approach in [85] cannot be applied to system (4.1). In [85], by recursively applying Theorem 4 of [85], the author proposed a recursive design for constructing a nested saturation control. From Theorem 4 on p.1263 of [85], it can be seen that, for x_{i+1} subsystem, the designed control takes the following form

$$k_{i+1}(x_{i+1}, v) = k_i(x_i, \lambda\sigma(\frac{Fx_{i+1} + \Gamma_v}{\lambda}) + \Omega_v). \tag{4.44}$$

where λ is the saturation level of σ and F is a good saturated linear controller for (A_{i+1}, B_{i+1}) which is the Jacobian linearization of the x_{i+1} subsystem. It can be seen from the proof of Theorem 3 of [85] that, given any F, the small gain condition can always be satisfied by adjusting only the saturation level λ. In fact, F can be determined in advance and two methods are given to determine F (See Lemma 4 of [85] and the remark following Lemma 4). Both of these two methods are straightforward. Therefore, the main difficulty of the solution of Theorem 5 of [85] lies in how to determine a suitable saturation level λ at each recursion. By using either of the two methods, the term Fx_{i+1} in (4.44) shows that the designed control at each recursion in general depends on all the states of the cascaded system considered at the recursion, i.e., the x_{i+1} subsystem. As evident from Theorem 5 of [85], such recursive design usually necessitates that all the states of the feedforward system should be available for feedback. When part of x_{i+1} is not available for feedback, as shown in [7], adjusting only the saturation level λ is not enough to guarantee the satisfaction of the small gain condition. As a result, F cannot be determined in advance anymore and should be chosen suitably to render the satisfaction of the small gain condition. In this case, how to determine F becomes a difficult and challenging task. It is because of this difficulty that Arcak, Teel and Kokotovic provided in [7] a different recursive design for a subclass of feedforward systems as studied in [85] in the presence of some type of input unmodeled dynamics. Note that, besides the input unmodeled dynamics ξ_1, system (4.1) also contains another $n-1$ dynamic uncertainty $\xi_2, ..., \xi_n$. Furthermore, like [7], d is treated as the disturbance to be attenuated in [85]. However, d in (4.1) is a time-varying static uncertainty to be rejected. Thus the Jacobian linearization of (4.1) with $d \in \mathcal{D}$ at $(0, ..., 0)$ in general contains certain unknown time-varying parameters. In the presence of the unknown time-varying parameters and the dynamic uncertainty $\xi_1, ..., \xi_n$, there is no clue how to guarantee the existence of a suitable good saturated linear controller at each recursion. Therefore, the approach in [85] cannot be applied to the stabilization problem of system (4.1).

Remark 4.3.3 The approach in [7] does not work for the global robust stabilization problem of system (4.1) either. By applying Lemma 1 of [7], the authors proposed a different (from [85]) recursive design. A crucial assumption of Lemma 1 is the existence of a time invariant coordinate transformation (equation (A.2) of [7]) such that the stabilization and disturbance attenuation problem for the original cascaded system can be converted to the same problem for system (4.3) with $F(\xi, u, 0) = o(\xi, u)$. However, since system (4.1) contains the time-varying uncertainty d, there does not exist any time invariant coordinate transformations such that $F(\xi, u, d)$ in (4.3) satisfies $F(\xi, u, d) = o_d(\xi, u)$. Furthermore, as already discussed in Remark 4.2.1, since d is a time-varying static uncertainty to be rejected, the arguments on p. 271 of [7] cannot be applied to prove the local exponential stability of system (4.5) with $d \in \mathcal{D}$ as time-varying static uncertainty. Therefore, one has to take extra efforts to show the local exponential stability of the Jacobian linearization of system (4.5) with $d \in \mathcal{D}$. However, there is no clue how to show the local exponential stability because system (4.1) takes a general form. As a result, the approach in [7] cannot be applied to the global robust stabilization problem of system (4.1).

4.4 An Example

Consider the global robust stabilization problem of the following system

$$
\begin{aligned}
\dot{x}_2 &= D_2\xi_2 + x_1 + 0.05(x_1 + d(t))(u + 0.1D_1\xi_1) \\
\dot{\xi}_2 &= A_2\xi_2 + B_2x_1 + 0.05B_2(x_1 + d(t))(u + 0.1D_1\xi_1) \\
\dot{x}_1 &= D_1\xi_1 + 10u \\
\dot{\xi}_1 &= A_1\xi_1
\end{aligned}
\tag{4.45}
$$

where $d(t) = 0.5[\cos(t)]^3$, $D_1 = [70.83\ -90\ \ 32.5\ -3.33], D_2 = [5\ -2], B_2 = [1\ \ 1]^T$, and

$$
A_1 = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}
$$

Clearly, system (4.45) is in the form of (4.1). It can be verified that system (4.45) satisfies Assumptions 4.3.1- 4.3.2, thus Theorem 4.3.1 can be applied to solve the global robust stabilization problem of system (4.45).

To solve the problem, performing the coordinate transformation

$$
z_1 = x_1 - D_1A_1^{-1}\xi_1, z_2 = x_2 - D_2A_2^{-1}\xi_2 + \tfrac{\theta_2}{\theta_1}z_1,
$$

on (4.45) gives (for convenience, we retain the original coordinates on the righthand side of the following equation)

$$
\begin{aligned}
\dot{z}_2 &= \theta_2 u + 0.025(x_1 + d(t))(u + 0.1D_1\xi_1) + \theta_2 k_1 x_1 \\
\dot{\xi}_2 &= A_2\xi_2 + B_2x_1 + 0.05B_2(x_1 + d(t))(u + 0.1D_1\xi_1) \\
\dot{z}_1 &= \theta_1 u \\
\dot{\xi}_1 &= A_1\xi_1
\end{aligned}
\tag{4.46}
$$

where $\theta_1 = \mu_1 = 10, \theta_2 = \mu_2/k_1 = 0.5/k_1$. Since k_i has the same sign with θ_i, k_1, k_2 are both positive in this case.

First, consider z_1, ξ_1 dynamics. Since $\bar{N}_1 = 0, \Delta_1 = \infty$, for arbitrarily positive λ_1, k_1, under the control $u = -\sigma_1(k_1z_1 + k_1D_1A_1^{-1}\xi_1 - u_1)$, z_1, x_1, u satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restriction $\frac{\lambda_1}{3}$ on u_1 and gains $\frac{3}{k_1} \cdot Id$, $\frac{6}{k_1} \cdot Id, 6 \cdot Id$ respectively.

Then consider z_2, ξ_2 dynamics. We first calculate the gain from u_1 to ξ_2. Let P_2 be a positive definite and symmetric matrix such that $A_2^T P_2 + P_2 A_2 = -2I$, and $\tilde{u}_2 =$

$x_1 + 0.05(x_1 + d(t))(u + 0.1D_1\xi_1)$. It can be verified that, ξ_2 subsystem satisfies a-LB with no restriction on $\xi_2(0)$, no restriction on $\tilde{u}_2$ and gain $\frac{\bar{\lambda}(P_2)}{\underline{\lambda}(P_2)}\|P_2B_2\| \cdot Id$. Then note that

$$\begin{aligned}(x_1 + d(t))(u + 0.1D_1\xi_1) &= x_1u + d(t)u + 0.1x_1D_1\xi_1 + 0.1d(t)D_1\xi_1 \\ &\leq 0.50005x_1^2 + 0.5u^2 + |u| + 50\|D_1\|^2\|\xi_1\|^2 + 0.1\|D_1\|\|\xi_1\|\end{aligned} \tag{4.47}$$

which implies

$$(x_1 + d(t))(u + 0.1D_1\xi_1) \leq 0.50005|x_1| + 1.5|u| + (50\|D_1\| + 0.1)\|D_1\|\|\xi_1\|$$

for $|x_1| \leq 1, |u| \leq 1$ and $\|\xi_1\| \leq 1$. Then we have

$$|\tilde{u}_2| \leq 1.026|x_1| + 0.075|u| + 35462\|\xi_1\|$$

for $|x_1| \leq 1, |u| \leq 1$ and $\|\xi_1\| \leq 1$. Thus, ξ_2 satisfies LB with restriction and AB with no restriction on $\xi_2(0)$, both with restriction $\min\{\frac{\lambda_1}{3}, \frac{k_1}{6}, \frac{1}{6}\}$ on u_1 and gain $N_{\xi_2u_1} \cdot Id$, where $N_{\xi_2u_1} = 2(\frac{1.026\times 6}{k_1} + 0.075 \times 6)$.

Now let $\zeta_1 = (\xi_2, z_1, \xi_1)$. Then (4.46) can be written in the following form

$$\begin{aligned}\dot{z}_2 &= \theta_2u_1 + \tilde{F}_2(\zeta_1, u_1, d) \\ \dot{\zeta}_1 &= G_1(\zeta_1, u_1, d)\end{aligned}$$

where $\tilde{F}_2(\zeta_1, u_1, d) = 0.05(1 - D_2A_2^{-1}B_2)(x_1 + d(t))(u + 0.1D_1\xi_1) + \theta_2h_1(x_1, u_1)$ and G_1 is a suitably defined function.

Let $u_1 = -\sigma_2(k_2z_2 + k_2D_2A_2^{-1}\xi_2 - k_2\frac{\theta_2}{\theta_1}z_1)$. Clearly, $\bar{\gamma}_{11}^{(2)}(s) \leq 2\max\{\|D_2A_2^{-1}\|N_{\xi_2u_1}, \frac{\theta_2}{\theta_1}\frac{3}{k_1}\}k_2s$. Note that $h_1(x_1, u_1)$ has no contribution to $\bar{\gamma}_{12}^{(2)}(s)$ when $\lambda_2 < \min\{\frac{\lambda_1}{12}, \frac{k_1}{6}, \frac{1}{6}\}$, then from (4.47) and the expression of $\tilde{F}_2(\zeta_1, u_1, d)$, we have $\bar{\gamma}_{12}^{(2)}(s) \leq 0.1\frac{k_1}{k_2}(6\min\{k_2s, \lambda_2\} + (\frac{0.50005\times 36}{k_1^2} + 18)\min\{k_2s, \lambda_2\}^2)$. By solving $6\max\{\bar{\gamma}_{11}^{(2)}(s), \bar{\gamma}_{12}^{(2)}(s)\} < s$ for $s > 0$, we set $k_1 = 0.2, k_2 = 0.00041, \lambda_1 = 10$ and $\lambda_2 = 0.0049$, and obtain

$$u = -\sigma_1(0.2x_1 + \sigma_2(0.00041(x_2))) \tag{4.48}$$

where σ_1, σ_2 are saturation functions with level 10 and 0.0049 respectively.

For illustration, Fig. 4.1 and Fig. 4.2 show the simulation result of system (4.45) under the control (4.48) with initial state $(\xi_2(0), x_2(0), \xi_1(0), x_1(0)) = ((0.5, 1), -1.9, (1, 1, 1, 1), 4)$.

4.5 Conclusion

In this chapter, we solve the global robust stabilization problem for a class of feedforward systems subject to both time-varying static and dynamic uncertainties. By recursively

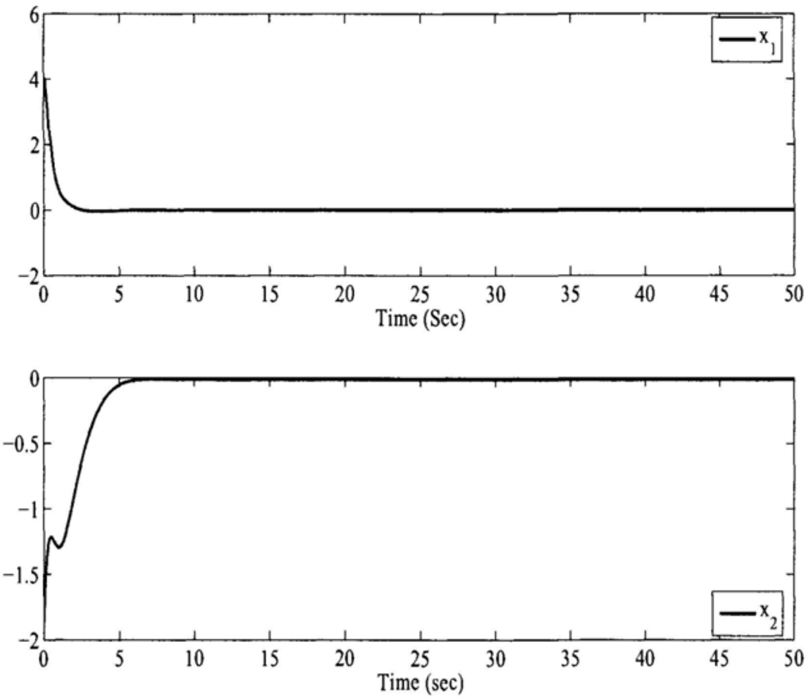

Figure 4.1: Profile of x_1 and x_2

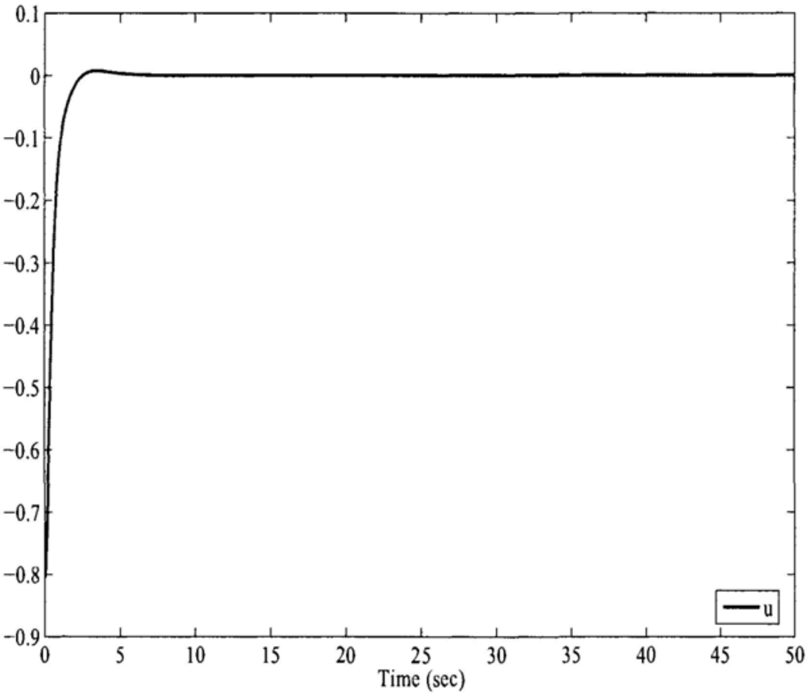

Figure 4.2: Profile of u

applying Lemma 4.2.1, a small gain based bottom-up recursive design has been developed. Unlike most existent results, the global asymptotic stability of the closed-loop system is guaranteed by employing two versions of the small gain theorem with restrictions adapted from [85] to establish the local stability and global attractiveness of the closed-loop system at the origin respectively. A specific feature of our approach is that, our approach can deal with the global robust stabilization problem for a class of feedforward systems with both time-varying static and dynamic uncertainties, and does not require the bottom dynamics at each recursion be locally exponentially stable.

Chapter 5

A Small Gain Approach to Global Stabilization of Feedforward Systems with Input Unmodeled Dynamics

5.1 Introduction

In this chapter, we study the global robust stabilization problem of strict feedforward systems described by

$$\begin{aligned} \dot{x}_i &= g_i(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, v, d), \ i = n, ..., 2 \\ \dot{x}_1 &= g_1(v, d) \end{aligned} \tag{5.1}$$

subject to the following input unmodeled dynamics

$$\dot{\xi}_1 = q(\xi_1, u, d), v = p(\xi_1, u, d) \tag{5.2}$$

where $p_1, ..., p_{n-1}$ are odd positive integers satisfying $p_1 \leq p_2 \leq ... \leq p_{n-1}$, $x_i \in \mathbb{R}, i = 1, ..., n$, $\xi_1 \in \mathbb{R}^{n_{\xi_1}}$, $u, v \in \mathbb{R}$, $d \in \mathbb{R}^{n_d}$ is a uncertain constant vector ranging within a compact set $\mathcal{D}$ having a known bound, and g_i, $i = 1, ..., n$, p, q are locally Lipschitz, and vanish at $(0, ..., 0, d)$ for all $d \in \mathcal{D}$.

The robust stabilization problem of nonlinear systems subject to input unmodeled dynamics has been studied for over fifteen years, see, e.g., [2, 6, 7, 30, 32, 37, 45, 48, 54, 67, 71, 72, 89] and the references therein. Among them, [7, 45, 54, 72, 72] studied various special cases of system (5.1) with $p_1 = ... = p_{n-1} = 1$. For example, it is assumed in [7] that, for $i = 2, ..., n$, $g_i(x_{i-1}, ..., x_1, v, d) = x_{i-1} + \tilde{g}_i(x_{i-1}, ..., x_1, v, d)$ where $\tilde{g}_i(x_{i-1}, ..., x_1, v, 0) = o(x_{i-1}, ..., x_1, v)$. A common assumption of these papers is the stabilizability of the Jacobian linearization of system (5.1) at $(0, ..., 0, d)$. However, this assumption is not satisfied by system (5.1) when some of the p_i's are greater than one. As a result, the approaches in [7, 45, 54, 72] do not work for our problem. In particular, the Lyapunov linearization technique cannot be used to establish the local stability of the closed-loop system as what was done in [7]. In this chapter, we will adopt the small gain approach to handle the global robust stabilization problem of system (5.1) subject to the input unmodeled dynamics (5.2), and to design a nested saturation controller recursively to guarantee the global robust asymptotic stability in the presence of the input unmodeled dynamics. Over the years, several different versions of the small gain theorem in the framework of input-to-state stability [51, 76, 80] have been established [3, 13, 32, 85]. More specifically, we will employ two versions of small gain theorem with restrictions adapted from [85] to establish the local stability and global attractiveness of the closed-loop system at the origin respectively.

It is noted that when the input unmodeled dynamics (5.2) is not present, the problem reduces to the global robust stabilization problem of system (5.1) viewing v as the input. This special case is also treated in [50, 58, 69, 86, 88] under various assumptions. The approaches in [50, 58, 69, 86, 88] are Lyapunov based. In contrast, ours is a small gain approach which leads to the well-known nested saturation controller. It will be seen later that even for this special case, the results in [50, 58, 69, 86, 88] do not contain ours because the functions g_i's in this chapter only need to satisfy Assumption 5.3.1 to be given in Section 5.3, while in [50, 58, 69, 86, 88], the functions g_i's are subject to some other assumptions. For example, a problem similar to ours was studied in [88] under the assumption that, for $i = 2, ..., n$, $\dot{x}_i = x_{i-1}^{p_{i-1}} + \hat{g}_i(x, v, d)$ where $|\hat{g}_i(x, v, d)| \leq a_i(1 + |x_i|)(x_1^{p_1+1} + ... + x_{i-1}^{p_{i-1}+1} + v^2)\chi_i(x_1, ..., x_{i-1}, v)$ with $a_i \geq 0$ being an unknown constant and $\chi_i(x_1, ..., x_{i-1}, v) \geq 0$ being a known function. In the case when $\hat{g}_i(x, v, d)$ is a polynomial in $x_1, ..., x_{i-1}, v$, the above assumption implies that the degree of each x_j $(j = 1, ..., i-2)$ and v has to be greater than p_j and 1 respectively. However, we allow the degree of each x_j $(j = 1, ..., i-2)$ and v to be equal to p_j and 1 respectively. As an illustration of this point, a simple example that cannot be handled by the approaches in [50, 58, 69, 86, 88] will be given in Section 5.4.

Finally, we note that, like [7], the major technique used in this chapter is the small gain technique. However, due to the general form of our system, the small gain condition cannot be made satisfied by a "naive" calculation of the gain functions. A deliberated calculation taking advantage of the special structure of various nonlinear functions is needed. This point will be made clear in the proof of Theorem 4.2 and Remark 4.2.

The rest of the chapter is organized as follows: In Section 5.2, we give the main technical lemma of this chapter. The main result of this chapter is contained in Section 5.3: we first design a nested saturation controller for system (5.1) and then show how to redesign the controller in the presence of the input unmodeled dynamics (5.2). An example is elaborated in Section 5.4 to show that the input unmodeled dynamics can be destabilizing, and thus a redesign of the controller is necessary to guarantee the global robust asymptotic stability in the presence of the input unmodeled dynamics.

5.2 A Technical Lemma

Like the previous chapters, our approach will utilize saturation functions characterized in 3.2.1.

As in Section 4.2, we will consider the system

$$\begin{aligned} \dot{z} &= \theta(d)u + F(\xi, u, d) \\ \dot{\xi} &= G(\xi, u, d) \end{aligned} \tag{5.3}$$

where $z, u \in \mathbb{R}, \xi \in \mathbb{R}^{n_\xi}$ and $d \in \mathbb{R}^{n_d}$ is a uncertain constant vector ranging within a compact set $\mathcal{D}$ having a known bound, $F(\xi, u, d), G(\xi, u, d)$ are locally Lipschitz and vanish at $(0, 0, d)$ for all $d \in \mathcal{D}$, and $\theta : \mathcal{D} \to \mathbb{R}$ is continuous, nonzero and does not change its sign.

Under the control

$$u = -\sigma(k(z + H(d)\xi)^p - \bar{u}) \tag{5.4}$$

where σ is a saturation function with level $\lambda > 0$, k is a nonzero real number with the same sign as $\theta(d)$, $H(d)$ is a $1 \times n_\xi$ matrix depending on d satisfying $\|H(d)\| \leq \nu$ for all $d \in \mathcal{D}$ and some positive constant ν, and p is an odd positive integer, system (5.3) takes the form

$$\begin{aligned} \dot{z} &= -\theta(d)\sigma(k(z + H(d)\xi)^p - \bar{u}) + F(\xi, -\sigma(k(z + H(d)\xi)^p - \bar{u}), d) \\ \dot{\xi} &= G(\xi, -\sigma(k(z + H(d)\xi)^p - \bar{u}), d) \end{aligned}$$

which can be viewed as the interconnection

$$v_1 = y_2, \quad v_2 = (v_{21}, v_{22}) = y_1 \tag{5.5}$$

of the following two subsystems

$$\Sigma_1: \ \dot{\xi} = G(\xi, -\sigma(kv_1), d), \ y_1 = \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} H(d)\xi \\ F(\xi, -\sigma(kv_1), d)/\tilde{k} \end{bmatrix} \tag{5.6}$$

$$\Sigma_2: \ \dot{z} = -\tilde{\sigma}(\tilde{k}[(z+v_{21})^p - \tfrac{\bar{u}}{k}]) + \tilde{k}v_{22}, \ y_2 = (z+v_{21})^p - \tfrac{\bar{u}}{k} \tag{5.7}$$

where $\tilde{\sigma}(s) = \theta(d)\sigma(s/\theta(d))$ is a saturation function with level $\tilde{\lambda} = |\theta(d)|\lambda$, and $\tilde{k} = \theta(d)k > 0$.

Lemma 5.2.1 Consider system (5.3). Assume ξ subsystem satisfies LB with restriction and AB with no restriction on $\xi(0)$, both with restriction Δ on u and gain $\gamma(s)$. Then under the control (5.4), the following results hold:

a) With $\lambda < \Delta$, for $i = 1, 2$, the output y_{1i} of Σ_1 subsystem satisfies LB with restriction and AB with no restriction on $\xi(0)$, both with no restriction on v_1 and gain $\bar{\gamma}_{1i}(s)$.

b) Further, assume $\bar{\gamma}_{1i}(s)$ satisfies

$$\lim_{s\to\infty} \bar{\gamma}_{1i}(s) = \bar{\gamma}_{1i}(\tfrac{\lambda}{|k|}), i = 1, 2, \tag{5.8}$$

and

$$2 \cdot 6^p(\bar{\gamma}_{11}(s))^p < s, 2 \cdot 6^p \bar{\gamma}_{12}(s) < s, s > 0. \tag{5.9}$$

Then, $z, z + H(d)\xi, u, \xi$ satisfy LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction $\frac{\lambda}{3}$ on $\bar{u}$ and gains $3(\frac{s}{|k|})^{\frac{1}{p}}, 6(\frac{s}{|k|})^{\frac{1}{p}}, 2 \cdot 6^p s, \gamma(2 \cdot 6^p s)$, respectively.

Remark 5.2.1 Let $\alpha > 0$ and $\bar{\gamma}(s)$ be a gain function. Since for $s > 0$, $\alpha(\bar{\gamma}(s))^p < s \Leftrightarrow \bar{\gamma}(\alpha s^p) < s$, we have

$$\begin{aligned} 2 \cdot 6^p(\bar{\gamma}_{11}(s))^p < s &\Leftrightarrow \bar{\gamma}_{11}(2 \cdot 6^p s^p) < s, s > 0, \\ 2 \cdot 6^p \bar{\gamma}_{12}(s) < s &\Leftrightarrow \bar{\gamma}_{12}(2 \cdot 6^p s) < s, \ s > 0. \end{aligned} \tag{5.10}$$

Remark 5.2.2 Lemma 5.2.1 is an extension of Lemma 4.2.1 in two aspects: first, ξ dynamics is allowed to satisfy a nonlinear gain γ rather than the linear gain and second, the control (5.4) takes a more general form than (4.4).

Proof: Part a): The assumption on ξ subsystem and $|\sigma(kv_1)| \leq \min\{|k||v_1|, \lambda\}$ with $\lambda < \Delta$ implies that, there exist an open set Ξ of the origin of $\mathbb{R}^{n_\xi}$ and a gain function $\gamma_1^0(s)$, all independent of d, such that, for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$, piecewise continuous v_1, $\xi(t)$ exists for all $t \geq 0$, and

$$\|\xi\|_\infty \leq \max\{\gamma_1^0(\|\xi(0)\|), \gamma(\min\{|k|\|v_1\|_\infty, \lambda\})\} \tag{5.11}$$

for all $\xi(0) \in \Xi$, $d \in \mathcal{D}$, $v_1 \in \mathcal{L}_\infty^1$, and

$$\|\xi\|_a \leq \gamma(\min\{|k|\|v_1\|_a, \lambda\}) \tag{5.12}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous v_1.

Noting $|y_{11}| = |H(d)\xi| \leq \|H(d)\|\|\xi\| \leq \nu\|\xi\|$ yields

$$\|y_{11}\|_\infty \leq \max\{\nu\gamma_1^0(\|\xi(0)\|), \bar{\gamma}_{11}(\|v_1\|_\infty)\} \tag{5.13}$$

for all $\xi(0) \in \Xi$, $d \in \mathcal{D}$, $v_1 \in \mathcal{L}_\infty^1$, where $\bar{\gamma}_{11}(s) = \nu\gamma(\min\{|k|s, \lambda\})$ and

$$\|y_{11}\|_a \leq \nu\gamma(\min\{|k|\|v_1\|_a, \lambda\}) = \bar{\gamma}_{11}(\|v_1\|_a) \tag{5.14}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous v_1.

Next consider y_{12}. Since $F(\xi, u, d)$ is continuous and $F(0,0,d) = 0$ for all $d \in \mathcal{D}$, there exists a gain function $\alpha(s)$, independent of d, such that,

$$|F(\xi, u, d)| \leq \alpha(\|(\xi, u)\|), \tag{5.15}$$

for $\xi \in \mathbb{R}^{n_\xi}, u \in \mathbb{R}$ and $d \in \mathcal{D}$. Then, using (5.11) and (5.12) gives

$$\|y_{12}\|_\infty \leq \max\{\alpha(2\gamma_1^0(\|\xi(0)\|))/\tilde{k}, \bar{\gamma}_{12}(\|v_1\|_\infty)\} \tag{5.16}$$

for all $\xi(0) \in \Xi$, $d \in \mathcal{D}$, $v_1 \in \mathcal{L}_\infty^1$, where $\bar{\gamma}_{12}(s) = \max\{\alpha(2\gamma(\min\{|k|s, \lambda\}))$, $\alpha(2\min\{|k|s, \lambda\})\}/\tilde{k}$, and

$$\|y_{12}\|_a \leq \bar{\gamma}_{12}(\|v_1\|_a) \tag{5.17}$$

for all $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous v_1.

Letting $\bar{\gamma}_1^0(s) = \max\{\nu\gamma_1^0(s), \alpha \circ 2\gamma_1^0(s)/\tilde{k}\}$ and noting (5.13), (5.14) and (5.16), (5.17), completes Part a). In addition, note that $y_{1i}, i = 1, 2$, satisfies LB and AB both with no restriction on v_1, let $\bar{\Delta}_1 = \infty$.

Part b): Let us first apply Propositions 2.3.1-2.3.2 to show that the output (y_1, y_2) of Σ_1 and Σ_2 under the interconnection (5.5) satisfies LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction on $\bar{u}$.

Step 1. Show that, for Σ_2 system viewing $v_{21}, v_{22}, \bar{u}$ as inputs, y_2 satisfies LB with no restriction on $z(0)$ and gain $\bar{\gamma}_2^0(s)$, restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{\bar{u}}$ on $v_{21}, v_{22}, \bar{u}$ and gains

$\bar{\gamma}_{21}(s), \bar{\gamma}_{22}(s), \bar{\gamma}_2^{\bar{u}}(s)$ respectively, and satisfies AB with no restriction on $z(0)$, restrictions $\bar{\Delta}_{21}, \bar{\Delta}_{22}, \bar{\Delta}_{\bar{u}}$ on $v_{21}, v_{22}, \bar{u}$ and gains $\bar{\gamma}_{21}(s), \bar{\gamma}_{22}(s), \bar{\gamma}_2^{\bar{u}}(s)$ respectively.

Let $V(z) = \frac{1}{2}z^2$. Then its time derivative along the trajectory of Σ_2 subsystem satisfies

$$\dot{V} = -(\tilde{\sigma}(\tilde{k}[(z+v_{21})^p - \tfrac{\bar{u}}{k}]) - \tilde{k}v_{22})z$$

Now consider the following three cases:

(1) $\tilde{k}|(z+v_{21})^p - \frac{\bar{u}}{k}| \leq \frac{\tilde{\lambda}}{2}$: We have $\dot{V} = -\tilde{k}((z+v_{21})^p - \frac{\bar{u}}{k} - v_{22})z$. Thus,

$$\begin{aligned}
|z| > 3\max\{|v_{21}|, |\tfrac{\bar{u}}{k}|^{\frac{1}{p}}, |v_{22}|^{\frac{1}{p}}\} &\geq |v_{21}| + |\tfrac{\bar{u}}{k}|^{\frac{1}{p}} + |v_{22}|^{\frac{1}{p}} \\
\Rightarrow |z| &> |v_{21}| + |\tfrac{\bar{u}}{k} + v_{22}|^{\frac{1}{p}} \\
&\left.\begin{array}{l} \Rightarrow |z+v_{21}|^p > |\frac{\bar{u}}{k} + v_{22}| \\ |z| > 3|v_{21}| \end{array}\right\} \Rightarrow \dot{V} < 0
\end{aligned} \tag{5.18}$$

(2) $\tilde{k}|(z+v_{21})^p - \frac{\bar{u}}{k}| > \frac{\tilde{\lambda}}{2}$ and $z > 0$: We have

$$\begin{aligned}
z = |z| > 2\max\{|v_{21}|, |\tfrac{\bar{u}}{k}|^{\frac{1}{p}}\} &\geq -v_{21} + (\tfrac{\bar{u}}{k})^{\frac{1}{p}} \\
\Rightarrow \tilde{k}|(z+v_{21})^p - \tfrac{\bar{u}}{k}| &= \tilde{k}((z+v_{21})^p - \tfrac{\bar{u}}{k}) > \tfrac{\tilde{\lambda}}{2} \\
\Rightarrow \tilde{\sigma}(\tilde{k}[(z+v_{21})^p - \tfrac{\bar{u}}{k}]) &> \tfrac{\tilde{\lambda}}{2} \\
\Rightarrow \dot{V} < -z(\tfrac{\tilde{\lambda}}{2} - \tilde{k}|v_{22}|) &< 0
\end{aligned} \tag{5.19}$$

for all $|v_{22}| < \frac{\tilde{\lambda}}{2\tilde{k}} = \frac{\lambda}{2|k|}$.

(3) $\tilde{k}|(z+v_{21})^p - \frac{\bar{u}}{k}| > \frac{\tilde{\lambda}}{2}$ and $z < 0$: We have

$$\begin{aligned}
-z = |z| > 2\max\{|v_{21}|, |\tfrac{\bar{u}}{k}|^{\frac{1}{p}}\} &\geq v_{21} - (\tfrac{\bar{u}}{k})^{\frac{1}{p}} \\
\Rightarrow \tilde{k}|(z+v_{21})^p - \tfrac{\bar{u}}{k}| &= -\tilde{k}((z+v_{21})^p - \tfrac{\bar{u}}{k}) > \tfrac{\tilde{\lambda}}{2} \\
\Rightarrow \tilde{\sigma}(\tilde{k}[(z+v_{21})^p - \tfrac{\bar{u}}{k}]) &< -\tfrac{\tilde{\lambda}}{2} \\
\Rightarrow \dot{V} < -z(-\tfrac{\tilde{\lambda}}{2} + \tilde{k}|v_{22}|) &< 0
\end{aligned} \tag{5.20}$$

for all $|v_{22}| < \frac{\tilde{\lambda}}{2\tilde{k}} = \frac{\lambda}{2|k|}$.

Noting (5.18) to (5.20), we claim that, there exists a gain function $\gamma_2^0(s)$, independent of d, such that, for all $z(0) \in \mathbb{R}$, $d \in \mathcal{D}$, piecewise continuous $\bar{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \leq \frac{\lambda}{3|k|}$, $z(t)$ exists for $t \geq 0$, and satisfies

$$\|z\|_\infty \leq \max\{\gamma_2^0(|z(0)|), \|v_{21}\|_\infty, 3(\|v_{22}\|_\infty)^{\frac{1}{p}}, 3(\tfrac{\|\bar{u}\|_\infty}{|k|})^{\frac{1}{p}}\} \tag{5.21}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}, \bar{u}, v_{21} \in \mathcal{L}_\infty^1, \|v_{22}\|_\infty < \frac{\lambda}{3|k|}$, and

$$\|z\|_a \leq \max\{3\|v_{21}\|_a, 3(\|v_{22}\|_a)^{\frac{1}{p}}, 3(\tfrac{\|\bar{u}\|_a}{|k|})^{\frac{1}{p}}\} \tag{5.22}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, piecewise continuous $\bar{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \leq \frac{\lambda}{3|k|}$. In fact, the proof of (5.21) and (5.22) follows from Lemma 3.3 in [85] and the derivation of (A.16) of [7] respectively.

Then, from (5.21), (5.22) and $|y_2| = |(z+v_{21})^p - \frac{\bar{u}}{k}| \leq \max\{2^{p+1}|z|^p, 2^{p+1}|v_{21}|^p, 2|\frac{\bar{u}}{k}|\}$, we obtain

$$\|y_2\|_\infty \leq \max\{\bar{\gamma}_2^0(|z(0)|), \bar{\gamma}_{21}(\|v_{21}\|_\infty), \bar{\gamma}_{22}(\|v_{22}\|_\infty), \bar{\gamma}_2^{\bar{u}}(\|\bar{u}\|_\infty)\} \tag{5.23}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}, \bar{u}, v_{21} \in \mathcal{L}_\infty^1, \|v_{22}\|_\infty < \bar{\Delta}_{22}$, and

$$\|y_2\|_a \leq \max\{\bar{\gamma}_{21}(\|v_{21}\|_a), \bar{\gamma}_{22}(\|v_{22}\|_a), \bar{\gamma}_2^{\bar{u}}(\|\bar{u}\|_a\}) \tag{5.24}$$

for all $z(0) \in \mathbb{R}, d \in \mathcal{D}$, piecewise continuous $\bar{u}, v_{21}$ and v_{22} satisfying $\|v_{22}\|_a \leq \bar{\Delta}_{22}$, where $\bar{\gamma}_2^0(s) = 2^{p+1}(\gamma_2^0(s))^p, \bar{\gamma}_{21}(s) = 2 \cdot 6^p s^p, \bar{\gamma}_{22}(s) = 2 \cdot 6^p s, \bar{\gamma}_2^{\bar{u}}(s) = \frac{2 \cdot 6^p}{|k|} s$ and $\bar{\Delta}_{22} = \frac{\lambda}{3|k|}$. Since y_2 satisfies LB and AB both with no restriction on $\bar{u}, v_{21}$, let $\bar{\Delta}_{\bar{u}} = \bar{\Delta}_{21} = \infty$.

Step 2. Check the conditions of Propositions 2.3.1-2.3.2.

Clearly, condition (5.9) implies the small gain condition $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for $s > 0$ and $i = 1, 2$. By (5.23), (5.13), (5.16) and (5.10), and Proposition 2.3.1,

$$\|y_{11}\|_\infty \leq \max\{\check{\gamma}_{11}^0(\|(z(0), \xi(0))\|), (\tfrac{\|\bar{u}\|_\infty}{|k|})^{\frac{1}{p}}\} \tag{5.25}$$

$$\|y_{12}\|_\infty \leq \max\{\check{\gamma}_{12}^0(\|(z(0), \xi(0))\|), \tfrac{\|\bar{u}\|_\infty}{|k|}\} \tag{5.26}$$

$$\|y_2\|_\infty \leq \max\{\check{\gamma}_2^0(\|(z(0), \xi(0))\|), 2 \cdot 6^p \tfrac{\|\bar{u}\|_\infty}{|k|}\} \tag{5.27}$$

for all $z(0) \in Z = \{z \in \mathbb{R} : \bar{\gamma}_2^0(|z|) < \frac{2 \cdot 6^p \lambda}{3|k|}\}$, $\xi(0) \in \hat{\Xi} = \{\xi \in \Xi : \max\{\bar{\gamma}_1^0(\|\xi\|), (\bar{\gamma}_1^0(\|\xi\|))^p\} < \frac{\lambda}{3|k|}\}$, $d \in \mathcal{D}$ and $\|\bar{u}\|_\infty < \frac{\lambda}{3}$, where the gain functions $\check{\gamma}_{11}^0, \check{\gamma}_{12}^0, \check{\gamma}_2^0$ are defined by (2.29).

Next consider Proposition 2.3.2. First note that the solution of the interconnected system exists for all $t \geq 0$ using the same argument as that in Lemma 3.5 of [85]. Then $\bar{\Delta}_1 = \infty$ implies that the first condition of Proposition 2.3.2 is satisfied. To check the second condition, note that, for $i = 1, 2$, $\lim_{s\to\infty} \bar{\gamma}_{1i}(s) = \bar{\gamma}_{1i}(\frac{\lambda}{|k|}) < \infty$ by condition (5.8) and $\bar{\Delta}_{21} = \infty$, $\bar{\Delta}_{22} = \frac{\lambda}{3|k|}$. Then we only need to check $\lim_{s\to\infty} \bar{\gamma}_{12}(s) \leq \frac{\lambda}{3|k|}$. From (5.9), we have

$$\lim_{s\to\infty} \bar{\gamma}_{12}(s) = \bar{\gamma}_{12}(\tfrac{\lambda}{|k|}) < \tfrac{\lambda}{2 \cdot 6^p |k|} < \tfrac{\lambda}{3|k|}$$

Clearly, condition (5.9) implies the small gain condition $\bar{\gamma}_{2i} \circ \bar{\gamma}_{1i}(s) < s$ for $s > 0$ and $i = 1, 2$. By (5.24), (5.14), (5.17) and (5.10), and Proposition 2.3.2,

$$\|y_{11}\|_a \leq (\tfrac{\|\bar{u}\|_a}{|k|})^{\frac{1}{p}}, \|y_{12}\|_a \leq \tfrac{\|\bar{u}\|_a}{|k|}, \tag{5.28}$$

$$\|y_2\|_a \leq 2 \cdot 6^p \tfrac{\|\bar{u}\|_a}{|k|} \tag{5.29}$$

for all $z(0) \in \mathbb{R}$, $\xi(0) \in \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $\bar{u}$.

Now we can conclude Part b). Let $\tilde{\gamma}^0(s) = \max\{\gamma_1^0(s), \gamma_2^0(s), 3\tilde{\gamma}_{11}^0(s), 3(\tilde{\gamma}_{12}^0(s))^{\frac{1}{p}}, \gamma(k\tilde{\gamma}_2^0(s))\}$. For all $(z(0), \xi(0)) \in Z \times \hat{\Xi}, d \in \mathcal{D}$, and $\|\bar{u}\|_\infty < \frac{\lambda}{3}$, (5.26) implies $\|y_{12}\|_\infty < \frac{\lambda}{3|k|}$. Using (5.21), (5.25), (5.26) and (5.11), (5.27), yields

$$\begin{aligned}\|z\|_\infty &\leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), 3(\tfrac{\|\bar{u}\|_\infty}{|k|})^{\frac{1}{p}}\}\\ \|\xi\|_\infty &\leq \max\{\tilde{\gamma}^0(\|(z(0), \xi(0))\|), \gamma(2 \cdot 6^p\|\bar{u}\|_\infty)\}\end{aligned}$$

for all $(z(0), \xi(0)) \in Z \times \hat{\Xi}$, $d \in \mathcal{D}$ and $\|\bar{u}\|_\infty < \frac{\lambda}{3}$.

For all $(z(0), \xi(0)) \in \mathbb{R} \times \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $\bar{u}$ satisfying $\|\bar{u}\|_a \leq \frac{\lambda}{3}$, (5.28) implies $\|y_{12}\|_a \leq \frac{\lambda}{3|k|}$. Using (5.22), (5.28), and (5.12), (5.29), yields

$$\|z\|_a \leq 3(\tfrac{\|\bar{u}\|_a}{|k|})^{\frac{1}{p}}, \|\xi\|_a \leq \gamma(2 \cdot 6^p\|\bar{u}\|_a)$$

for all $(z(0), \xi(0)) \in \mathbb{R} \times \mathbb{R}^{n_\xi}$, $d \in \mathcal{D}$ and piecewise continuous $\bar{u}$ satisfying $\|\bar{u}\|_a \leq \frac{\lambda}{3}$.

Then, noting $|z + H(d)\xi| = |z + y_{11}| \leq 2\max\{|z|, |y_{11}|\}$, $|u| = |\sigma(kv_1)| = |\sigma(ky_2)| \leq |k||y_2|$ and equations (5.25) to (5.29) completes the proof.

Remark 5.2.3 Suppose the assumption of Lemma 3.1 holds. Assume $y = h(\xi, u, d)$ is an output of ξ subsystem of system (5.3), and satisfies LB with restriction and AB with no restriction on $\xi(0)$, both with restriction Δ on u and gain $\tilde{\gamma}(s)$. Then, from $u = \sigma(kv_1) = \sigma(ky_2)$, (5.27),(5.29), it can be verified that y satisfies LB with restriction and AB with no restriction on $(z(0), \xi(0))$, both with restriction $\frac{\lambda}{3}$ on $\bar{u}$ and gain $\tilde{\gamma}(2 \cdot 6^p s)$.

Remark 5.2.4 Although we have obtained the general expressions of $\bar{\gamma}_{1i}(s), i = 1, 2$, in the proof of Lemma 5.2.1, these expressions are not specific enough to render the satisfaction of the small gain condition of Propositions 2.3.1-2.3.2. Therefore, in the statement of Part b) of Lemma 3.1, we have to expediently impose conditions (5.8) and (5.9) on $\bar{\gamma}_{1i}(s), i = 1, 2$, for rendering the satisfaction of the small gain condition. It will be seen later from the proof of Theorem 5.3.2 that conditions (5.8) and (5.9) can always be made satisfied when the more specific form of $\bar{\gamma}_{1i}(s), i = 1, 2$, are obtained by taking advantage of the specific expressions of y_{11} and y_{12}.

5.3 Nested Saturation Controller Design

We first design a nested saturation controller for system (5.1) and then show how to redesign the controller when the input unmodeled dynamics (5.2) is present.

Let us make the following assumption.

Assumption 5.3.1 Assume, for $i = 1, ..., n$, g_i is continuously differentiable at $(0, ..., 0, d)$ for $d \in \mathcal{D}$, and $c_i(d)$ is nonzero and does not change its sign for all $d \in \mathcal{D}$, where $c_1(d) = \frac{\partial g_1}{\partial v}|_{(0,d)}$, $c_i(d) = \frac{\partial g_i}{\partial x_{i-1}^{p_{i-1}}}|_{(0,...,0,d)}, i = 2, ..., n$.

As a result of this assumption, system (5.1) can be rewritten in the following form:

$$\begin{aligned} \dot{x}_i &= c_i x_{i-1}^{p_{i-1}} + g_i^r(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, v, d), \ i = n, ..., 2 \\ \dot{x}_1 &= c_1 v + g_1^r(v, d) \end{aligned} \tag{5.30}$$

where g_i^r, $i = 1, ..., n$, are suitably defined functions vanishing at $(0, ..., 0, d)$, and for simplicity, we have dropped the argument d in c_i, $i = 1, ..., n$.

Theorem 5.3.1 Consider system (5.30). Under Assumption 5.3.1, there exist $\lambda_i > 0$ and nonzero k_i where k_1 has the same sign with c_1 and k_i has the same sign with c_i/k_{i-1}, $i = 2, ..., n$, such that under the control

$$v = -\sigma_1(k_1 x_1^{p_1} + ... + \sigma_n(k_n x_n^{p_n} - u_n)) \tag{5.31}$$

where, for $i = 1, ..., n$, σ_i is a saturation function with level λ_i and $p_n \geq p_{n-1}$ is an odd positive integer, the closed-loop system satisfies LB with restriction and AB with no restriction on $(x_n(0), ..., x_1(0))$, both with restriction on u_n. In particular, when $u_n = 0$, the closed-loop system is globally asymptotically stable at the origin for all $d \in \mathcal{D}$.

Proof: The proof is a special case of the proof of Theorem 5.3.2 with $v = u$ and $n_{\xi_1} = 0$, i.e., when the input unmodeled dynamics (5.2) is not present.

As will be shown in Section 5.4, the control law (5.31) can be destabilizing when the input unmodeled dynamics (5.2) is present. So we have to *redesign*. Let us make the following assumption on the input unmodeled dynamics (5.2).

Assumption 5.3.2 Assume, ξ_1 subsystem satisfies LB with restriction and AB with no restriction on $\xi_1(0)$, both with restriction Δ_1 on u and gain $\bar{N}_1 s$, and moreover, the functions $p(\xi_1, u, d), q(\xi_1, u, d)$ are continuously differentiable at $(0, 0, d)$ for $d \in \mathcal{D}$ and $A_1(d) = \frac{\partial q}{\partial \xi_1}|_{(0,0,d)}, B_1(d) = \frac{\partial q}{\partial u}|_{(0,0,d)}$, $D_1(d) = \frac{\partial p}{\partial \xi_1}|_{(0,0,d)}$, $e_1(d) = \frac{\partial p}{\partial u}|_{(0,0,d)}$ are such that $e_1(d) - D_1(d)A_1^{-1}(d)B_1(d)$ is nonzero and does not change its sign for all $d \in \mathcal{D}$.

To simplify the notation, we drop the argument d in the above defined matrices and numbers. Then, system (5.1) subject to (5.2) can be written as follows:

$$\begin{aligned} \dot{x}_i &= c_i x_{i-1}^{p_{i-1}} + f_i(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, \xi_1, u, d), \ i = n, ..., 2 \\ \dot{x}_1 &= c_1 D_1 \xi_1 + c_1 e_1 u + f_1(\xi_1, u, d) \\ \dot{\xi}_1 &= A_1 \xi_1 + B_1 u + f_0(\xi_1, u, d) \end{aligned} \tag{5.32}$$

where $f_0(\xi_1, u, d) = q(\xi_1, u, d) - A_1\xi_1 - B_1 u$, $f_1(\xi_1, u, d) = g_1^r(p(\xi_1, u, d), d) + c_1(p(\xi_1, u, d) - D_1\xi_1 - e_1 u)$, and $f_i(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, \xi_1, u, d) = g_i^r(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, p(\xi_1, u, d), d)$, $i = 2, ..., n$.

Theorem 5.3.2 Consider system (5.32). Under Assumptions 5.3.1-5.3.2, there exist $\lambda_i > 0$ and nonzero k_i with the same sign as θ_i where $\theta_1 = c_1(e_1 - D_1 A_1^{-1} B_1)$ and $\theta_i = c_i/k_{i-1}$, $i = 2, ..., n$, such that under the control

$$u = -\sigma_1(k_1 x_1^{p_1} + ... + \sigma_n(k_n x_n^{p_n} - u_n)) \tag{5.33}$$

where for $i = 1, ..., n$, σ_i is a saturation function with level λ_i, and $p_n \geq p_{n-1}$ is an odd positive integer, the closed-loop system satisfies LB with restriction and AB with no restriction on $(x_n(0), ..., x_1(0), \xi_1(0))$, both with restriction on u_n. In particular, when $u_n = 0$, the closed-loop system at the origin is globally asymptotically stable for all $d \in \mathcal{D}$.

To prove Theorem 5.3.2, we first note some facts. Let $h_i(x_i^{p_i}, u_i) = -\sigma_i(k_i x_i^{p_i} - u_i) + k_i x_i^{p_i} - u_i$. By the property of the saturation function, $h_i(x_i^{p_i}, u_i) = 0$ when $|k_i x_i^{p_i} - u_i| \leq \frac{\lambda_i}{2}$. Thus $h_i(x_i^{p_i}, u_i) = o(x_i^{p_i}, u_i)$. From Assumptions 5.3.1-5.3.2, there exist positive constants c_i^L, c_i^U, ν_1 such that $c_1^L \leq |c_1(e_1 - D_1 A_1^{-1} B_1)| \leq c_1^U$, $c_i^L \leq |c_i| \leq c_i^U, i = 2, ..., n$, and $\|c_1 D_1 A_1^{-1}\| \leq \nu_1$ for all $d \in \mathcal{D}$, and there also exist positive constants $L_i, i = 1, ..., n-1$, and gain functions $\gamma_i^o(s) = o(s), i = 1, ..., n$ such that, for any $x_{n-1}, ..., x_1, \xi_1, u$ and $d \in \mathcal{D}$, the following inequalities hold:

$$|f_1(\xi_1, u, d) - c_1 D_1 A_1^{-1} f_0(\xi_1, u, d)| \leq \gamma_1^o(\|(\xi_1, u)\|) \tag{5.34}$$

$$\begin{aligned} |f_i(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, \xi_1, u, d)| &\leq L_{i-1}\|(x_{i-2}^{p_{i-2}}, ..., \xi_1, u)\| \\ &+\gamma_i^o(\|(x_{i-1}^{p_{i-1}}, ..., x_1^{p_1}, \xi_1, u)\|), i = 2, \cdots, n \end{aligned} \tag{5.35}$$

where x_0 is a dummy variable. Since $\gamma_i^o(s) = o(s)$, given any $\varepsilon_i > 0$, there exists a $\delta_i > 0$ such that

$$\gamma_i^o(s) \leq \varepsilon_i s, \ 0 < s \leq \delta_i, \ i = 1, ..., n. \tag{5.36}$$

Proof: First, performing the coordinate transform:

$$z_1 = x_1 - c_1 D_1 A_1^{-1}\xi_1, z_i = x_i + \tfrac{\theta_i}{\theta_{i-1}} z_{i-1}, \tag{5.37}$$

where $i = 2, ..., n$, on system (5.32) gives

$$\begin{aligned} \dot{z}_i &= \theta_i u + F_i(z_{i-1}, ..., z_1, \xi_1, u, d), \ i = n, ..., 2 \\ \dot{z}_1 &= \theta_1 u + F_1(\xi_1, u, d) \\ \dot{\xi}_1 &= A_1\xi_1 + B_1 u + f_0(\xi_1, u, d) \end{aligned} \tag{5.38}$$

where $F_1(\xi_1,u,d) = f_1(\xi_1,u,d) - c_1D_1A_1^{-1}f_0(\xi_1,u,d)$ and for $i = 2,...,n$, $F_i(z_{i-1},...,z_1,\xi_1,u,d) = f_i(x_{i-1}^{p_{i-1}},...,x_1^{p_1},\xi_1,u,d) + \frac{\theta_i}{\theta_{i-1}}F_{i-1}(z_{i-2},...,\xi_1,u,d) + \theta_i k_{i-1}x_{i-1}^{p_{i-1}}\mid_{(5.37)}$, where z_0 is a dummy variable. Moreover, under the coordinate transform (5.37), the control (5.33) becomes

$$u = -\sigma_1(k_1(z_1 + c_1D_1A_1^{-1}\xi_1)^{p_1} + ... + \sigma_n(k_n(z_n - \frac{\theta_n}{\theta_{n-1}}z_{n-1})^{p_n} - u_n)). \qquad (5.39)$$

The proof will be completed if we can show that, for system (5.38), under the control (5.39), the closed-loop system at the origin is globally asymptotically stable for all $d \in \mathcal{D}$.

By recursively applying Lemma 5.2.1, we can use mathematical induction to prove the theorem. The proof will be divided into three parts and $(\cdot)^{(j)}$ will be used to denote the corresponding notation in the jth induction.

Step 1: Under the control $u = -\sigma_1(k_1(z_1 + H_1\xi_1)^{p_1} - u_1)$ where $H_1 = c_1D_1A_1^{-1}$, the subsystem consisting of the last two equations of (5.38) can be viewed as the interconnection (5.5) of the two subsystems (5.6) and (5.7) where $v_1^{(1)} = y_2^{(1)} = (z_1 + H_1\xi_1)^{p_1} - \frac{u_1}{k_1}$, $v_{21}^{(1)} = y_{11}^{(1)} = H_1\xi_1$ and $v_{22}^{(1)} = y_{12}^{(1)} = \frac{1}{\tilde{k}_1}F_1(\xi_1,u,d)$ with $\tilde{k}_1 = \theta_1k_1$. By Assumption 5.3.2 and Lemma 5.2.1, with $\lambda_1 < \Delta_1$, $y_{1i}^{(1)}, i = 1,2$, satisfies LB with restriction and AB with no restriction on $\xi_1(0)$, both with no restriction on $v_1^{(1)}$. Moreover, it follows from the expression of $y_{11}^{(1)}$ that $\bar{\gamma}_{11}^{(1)}(s) = \nu_1\bar{N}_1\min\{|k_1|s,\lambda_1\}$, and from the expression of $y_{12}^{(1)}$ and (5.34) that $\bar{\gamma}_{12}^{(1)}(s) = \gamma_1^o(\tilde{N}_1\min\{|k_1|s,\lambda_1\})/\tilde{k}_1$ where $\tilde{N}_1 = 2\max\{1,\bar{N}_1\}$. Clearly, $\bar{\gamma}_{11}^{(1)}(s)$ and $\bar{\gamma}_{12}^{(1)}(s)$ satisfy (5.8). We now further show that there exist sufficiently small $\lambda_1, |k_1|$ such that

$$2\cdot 6^{p_1}(\bar{\gamma}_{11}^{(1)}(s))^{p_1} < s, 2\cdot 6^{p_1}\bar{\gamma}_{12}^{(1)}(s) < s,\ s > 0. \qquad (5.40)$$

In fact, substituting the expression of $\bar{\gamma}_{11}^{(1)}(s)$ into the first inequality of (5.40) gives

$$2\cdot 6^{p_1}(\nu_1\bar{N}_1\min\{|k_1|s,\lambda_1\})^{p_1} < s, s > 0. \qquad (5.41)$$

It can be verified that (5.41) is satisfied with $|k_1| < \frac{1}{12\nu_1\bar{N}_1}$ when $p_1 = 1$, and with $\lambda_1 < |k_1|2^{-\frac{1}{p_1-1}}(6\nu_1\bar{N}_1|k_1|)^{-\frac{p_1}{p_1-1}}$ when $p_1 > 1$. Next note that, for any $0 < \tau_1 \leq \frac{1}{2\cdot 6^{p_1}}$, there exists an $\epsilon_1 > 0$ such that

$$\frac{\tilde{N}_1\varepsilon_1}{c_1^L} < \tau_1. \qquad (5.42)$$

From the expression of $\bar{\gamma}_{12}^{(1)}(s)$, (5.36), and (5.42), corresponding to this ε_1, there exists a $\delta_1 > 0$ such that $\lambda_1 < \min\{\Delta_1, \frac{\delta_1}{\tilde{N}_1}\}$ implies the second inequality of (5.40). By Lemma

5.2.1, z_1, x_1, u, ξ_1 satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restriction $\frac{\lambda_1}{3}$ on u_1 and gains $3(\frac{s}{|k_1|})^{\frac{1}{p_1}}, 6(\frac{s}{|k_1|})^{\frac{1}{p_1}}, 2\cdot 6^{p_1}s, 2\bar{N}_1 6^{p_1}s$, respectively.

Step 2: Let $\xi_2 = (z_1, \xi_1)$. Then the last three equations of (5.38) can be put into the form:

$$\begin{aligned} \dot{z}_2 &= \theta_2 u_1 + \tilde{F}_2(\xi_2, u_1, d) \\ \dot{\xi}_2 &= \tilde{G}_1(\xi_2, u_1, d) \end{aligned} \tag{5.43}$$

where $\tilde{F}_2(\xi_2, u_1, d) = F_2(z_1, \xi_1, u, d) + \theta_2(u - u_1) = f_2(x_1^{p_1}, \xi_1, u, d) + \frac{\theta_2}{\theta_1}F_1(\xi_1, u, d) + \theta_2 h_1(x_1^{p_1}, u_1)$. Under the control $u_1 = -\sigma_2(k_2(z_2 + H_2\xi_2)^{p_2} - u_2)$ where $H_2 = [-\frac{\theta_2}{\theta_1}\; 0_{1\times n_{\xi_1}}]$, (5.43) can be viewed as the interconnection (5.5) of the two subsystems (5.6) and (5.7) where $v_1^{(2)} = y_2^{(2)} = (z_2 + H_2\xi_2)^{p_2} - \frac{u_2}{k_2}, v_{21}^{(2)} = y_{11}^{(2)} = H_2\xi_2 = -\frac{\theta_2}{\theta_1}z_1$ and $v_{22}^{(2)} = y_{12}^{(2)} = \frac{1}{\tilde{k}_2}\tilde{F}_2(\xi_2, u_1, d)$ with $\tilde{k}_2 = \theta_2 k_2$. By *Step 1*, ξ_2 subsystem satisfies LB with restriction and AB with no restriction on $\xi_2(0)$ and both with restriction $\frac{\lambda_1}{3}$ on u_1. Then, by Lemma 5.2.1, with $\lambda_2 < \frac{\lambda_1}{3}$, $y_{1i}^{(2)}, i = 1, 2$, satisfies LB with restriction and AB with no restriction on $\xi_2(0)$, both with no restriction on $v_1^{(2)}$. Moreover, it follows from the expression of $y_{11}^{(2)}$ that $\bar{\gamma}_{11}^{(2)}(s) \le 3\frac{|\theta_2|}{\theta_1}(\frac{1}{k_1}\min\{|k_2|s, \lambda_2\})^{\frac{1}{p_1}}$, and from the expression of $y_{12}^{(2)}, y_{12}^{(1)}$, and (5.35) that $|y_{12}^{(2)}| \le \frac{L_1\|(\xi_1,u)\|}{\tilde{k}_2} + \frac{\gamma_2^o(\|(x_1^{p_1},\xi_1,u)\|)}{\tilde{k}_2} + |\frac{k_1}{k_2}||y_{12}^{(1)}| + \frac{1}{|k_2|}|h_1(x_1^{p_1}, u_1)|$. Noting $k_1 x_1^{p_1} - u_1 = k_1 v_1^{(1)}$, $v_1^{(1)} = y_2^{(1)}$, (5.27), (5.29) and the property of $h_1(x_1^{p_1}, u_1)$ yields that, $h_1(x_1^{p_1}, u_1)$ has no contribution to $\bar{\gamma}_{12}^{(2)}(s)$ when $\lambda_2 \le \frac{\lambda_1}{4\cdot 6^{p_1}}$, and moreover, from (5.16), (5.27), and (5.17), (5.29), $\bar{\gamma}_{12}^{(2)}(s) \le 2[\frac{2L_1\bar{N}_1 6^{p_1}\min\{|k_2|s,\lambda_2\}}{\tilde{k}_2} + \frac{\gamma_2^o(\bar{N}_2\min\{|k_2|s,\lambda_2\})}{\tilde{k}_2} + |\frac{k_1}{k_2}|\bar{\gamma}_{12}^{(1)}(2\cdot 6^{p_1}\frac{\min\{|k_2|s,\lambda_2\}}{|k_1|})]$ where $\bar{N}_2 = 2\max\{\frac{1}{|k_1|}, 2\bar{N}_1\}6^{p_1}$. Clearly, $\bar{\gamma}_{11}^{(2)}(s)$ and $\bar{\gamma}_{12}^{(2)}(s)$ satisfy (5.8). We now further show that there exist sufficiently small $\lambda_i, |k_i|, i = 1, 2$, such that

$$2\cdot 6^{p_2}(\bar{\gamma}_{11}^{(2)}(s))^{p_2} < s, 2\cdot 6^{p_2}\bar{\gamma}_{12}^{(2)}(s) < s,\; s > 0. \tag{5.44}$$

Substituting the expression of $\bar{\gamma}_{11}^{(2)}(s)$ into the first inequality of (5.44) gives

$$2(\frac{18c_2^U}{|k_1|c_1^L})^{p_2}(\frac{\min\{|k_2|s, \lambda_2\}}{|k_1|})^{\frac{p_2}{p_1}} < s, s > 0. \tag{5.45}$$

It can be verified that (5.45) is satisfied with $|k_2| < 2^{-1}\cdot|k_1|(\frac{c_1^L|k_1|}{18c_2^U})^{p_2}$ when $p_2 = p_1$, and with $\lambda_2 < 2^{-\frac{p_1}{p_2-p_1}}|k_2|\cdot|\frac{k_1}{k_2}|^{\frac{p_2}{p_2-p_1}}(\frac{c_1^L|k_1|}{18c_2^U})^{\frac{p_1p_2}{p_2-p_1}}$ when $p_2 > p_1$. Next, note that for any $0 < \tau_2 \le \frac{1}{2\cdot 6^{p_2}}$, there exist $|k_1|, \varepsilon_2 > 0, \tau_1 > 0$, such that

$$2(\frac{2L_1\bar{N}_1 6^{p_1}|k_1|}{c_2^L} + \frac{\bar{N}_2|k_1|\varepsilon_2}{c_2^L} + 2\tau_1 6^{p_1}) < \tau_2. \tag{5.46}$$

From the expression of $\bar{\gamma}_{12}^{(2)}(s)$, (5.36), and (5.46), corresponding to this ε_2, there exists a $\delta_2 > 0$ such that $\lambda_2 \le \min\{\frac{\lambda_1}{4\cdot 6^{p_1}}, \frac{\delta_2}{\bar{N}_2}\}$ implies the second inequality of (5.44). By Lemma

5.2.1, z_2, x_2, u_1, ξ_2 satisfy LB with restriction and AB with no restriction on $(z_2(0), \xi_2(0))$, both with restriction $\frac{\lambda_2}{3}$ on u_2, and the gains from u_2 to z_2, x_2, u_1 are $3(\frac{s}{|k_2|})^{\frac{1}{p_2}}, 6(\frac{s}{|k_2|})^{\frac{1}{p_2}}, 2\cdot 6^{p_2}s$ respectively. Note that $(x_1^{p_1}, \xi_1, u)$ can be seen as an output of ξ_2 subsystem of (5.43), by Remark 5.2.3, $(x_1^{p_1}, \xi_1, u)$ satisfies LB with restriction and AB with no restriction on $(z_2(0), \xi_2(0))$, both with restriction $\frac{\lambda_2}{3}$ on u_2 and gain $2\tilde{N}_2 6^{p_2}s$.

Step 3: For $3 \le j \le n-1$, put the last $j+1$ equations of (5.38) into the form:

$$\begin{aligned}\dot{z}_j &= \theta_j u_{j-1} + \tilde{F}_j(\xi_j, u_{j-1}, d)\\ \dot{\xi}_j &= \tilde{G}_{j-1}(\xi_j, u_{j-1}, d)\end{aligned} \tag{5.47}$$

where $\xi_j = (z_{j-1}, \xi_{j-1})$ and

$$\begin{aligned}\tilde{F}_j(\xi_j, u_{j-1}, d) &= f_j(x_{j-1}^{p_{j-1}}, ..., x_1^{p_1}, \xi_1, u, d)\\ &+\frac{\theta_j}{\theta_{j-1}}\tilde{F}_{j-1}(\xi_{j-1}, u_{j-2}, d) + \theta_j h_{j-1}(x_{j-1}^{p_{j-1}}, u_{j-1}).\end{aligned} \tag{5.48}$$

Under the control $u_{j-1} = -\sigma_j(k_j(z_j + H_j\xi_j)^{p_j} - u_j)$ where $H_j = [-\frac{\theta_j}{\theta_{j-1}}, 0_{1\times(j-2+n_{\xi_1})}]$, (5.47) can be viewed as the interconnection (5.5) of the two subsystems (5.6) and (5.7) where $v_1^{(j)} = y_2^{(j)} = (z_j + H_j\xi_j)^{p_j} - \frac{u_j}{k_j}, v_{21}^{(j)} = y_{11}^{(j)} = H_j\xi_j = -\frac{\theta_j}{\theta_{j-1}}z_{j-1}$ and $y_{12}^{(j)} = \frac{1}{\tilde{k}_j}\tilde{F}_j(\xi_j, u_{j-1}, d)$ with $\tilde{k}_j = \theta_j k_j$.

Assume, $z_{j-1}, \xi_{j-1}, x_{j-1}, (x_{j-2}^{p_{j-2}}, ..., x_1^{p_1}, \xi_1, u)$ satisfy LB with restriction and AB with no restriction on $(z_{j-1}(0), \xi_{j-1}(0))$, both with restriction $\frac{\lambda_{j-1}}{3}$ on u_{j-1}, and the gains from u_{j-1} to $z_{j-1}, x_{j-1}, (x_{j-2}^{p_{j-2}}, ..., x_1^{p_1}, \xi_1, u)$ are $3(\frac{s}{|k_{j-1}|})^{\frac{1}{p_{j-1}}}, 6(\frac{s}{|k_{j-1}|})^{\frac{1}{p_{j-1}}}, 2\tilde{N}_{j-1}6^{p_{j-1}}s$ respectively, where $\tilde{N}_{j-1}$ is a positive real number dependent on $k_1, ..., k_{j-2}$. By this assumption, ξ_j subsystem satisfies LB with restriction and AB with no restriction on $\xi_j(0)$ and both with restriction $\frac{\lambda_{j-1}}{3}$ on u_{j-1}. Then by Lemma 5.2.1, with $\lambda_j < \frac{\lambda_{j-1}}{3}$, $y_{1i}^{(j)}, i = 1, 2$, satisfies LB with restriction and AB with no restriction on $\xi_j(0)$, both with no restriction on $v_1^{(j)}$. Moreover, it follows from the expression of $y_{11}^{(j)}$ that $\bar{\gamma}_{11}^{(j)}(s) \le 3\frac{|\theta_j|}{\theta_{j-1}}(\frac{1}{k_{j-1}}\min\{|k_j|s, \lambda_j\})^{\frac{1}{p_{j-1}}}$, and from the expression of $y_{12}^{(j)}, y_{12}^{(j-1)}$, and (5.35) that

$$\begin{aligned}|y_{12}^{(j)}| \le &\frac{L_{j-1}\|(x_{j-2}^{p_{j-2}}, ..., x_1^{p_1}, \xi_1, u)\|}{\tilde{k}_j} + \frac{\gamma_j^o(\|(x_{j-1}^{p_{j-1}}, ..., x_1^{p_1}, \xi_1, u)\|)}{\tilde{k}_j}\\ &+\frac{|k_{j-1}|}{|k_j|}|y_{12}^{(j-1)}| + \frac{1}{|k_j|}|h_{j-1}(x_{j-1}^{p_{j-1}}, u_{j-1})|.\end{aligned}$$

Noting $k_{j-1}x_{j-1}^{p_{j-1}} - u_{j-1} = k_{j-1}v_1^{(j-1)}$, $v_1^{(j-1)} = y_2^{(j-1)}$, (5.27), (5.29) and the property of $h_{j-1}(x_{j-1}^{p_{j-1}}, u_{j-1})$ yields that, $h_{j-1}(x_{j-1}^{p_{j-1}}, u_{j-1})$ has no contribution to $\bar{\gamma}_{12}^{(j)}(s)$ when $\lambda_j \le \frac{\lambda_{j-1}}{4\cdot 6^{p_{j-1}}}$, and moreover, from (5.16), (5.27) and (5.17), (5.29),

$$\begin{aligned}\bar{\gamma}_{12}^{(j)}(s) \le 2[&\frac{2L_{j-1}\tilde{N}_{j-1}6^{p_{j-1}}\min\{|k_j|s, \lambda_j\}}{\tilde{k}_j} + \frac{\gamma_j^o(\tilde{N}_j\min\{|k_j|s, \lambda_j\})}{\tilde{k}_j}\\ &+\frac{|k_{j-1}|}{|k_j|}\bar{\gamma}_{12}^{(j-1)}(\frac{2\cdot 6^{p_{j-1}}}{|k_{j-1}|}\min\{|k_j|s, \lambda_j\})]\end{aligned} \tag{5.49}$$

where $\tilde{N}_j = 2\max\{\frac{1}{|k_{j-1}|}, 2\tilde{N}_{j-1}\}6^{p_{j-1}}$. Clearly, $\bar{\gamma}_{11}^{(j)}(s)$ and $\bar{\gamma}_{12}^{(j)}(s)$ satisfy (5.8). We now further show that there exist sufficiently small $\lambda_i, |k_i|, i = 1, ..., j$, such that

$$2 \cdot 6^{p_j}(\bar{\gamma}_{11}^{(j)}(s))^{p_j} < s, 2 \cdot 6^{p_j}\bar{\gamma}_{12}^{(j)}(s) < s, s > 0. \tag{5.50}$$

Substituting the expression of $\bar{\gamma}_{11}^{(j)}(s)$ into the first inequality of (5.50) gives

$$2(\frac{18c_j^U|k_{j-2}|}{|k_{j-1}|c_{j-1}^L})^{p_j}(\frac{\min\{|k_j|s, \lambda_j\}}{|k_{j-1}|})^{\frac{p_j}{p_{j-1}}} < s, s > 0. \tag{5.51}$$

It can be verified that (5.51) is satisfied with $|k_j| < 2^{-1}|k_{j-1}||\frac{c_{j-1}^L k_{j-1}}{18c_j^U k_{j-2}}|^{p_j}$ when $p_j = p_{j-1}$, and with $\lambda_j < 2^{-\frac{p_{j-1}}{p_j - p_{j-1}}}|k_j||\frac{k_{j-1}}{k_j}|^{\frac{p_j}{p_j - p_{j-1}}}(\frac{c_{j-1}^L|k_{j-1}|}{18c_j^U|k_{j-2}|})^{\frac{p_{j-1}p_j}{p_j - p_{j-1}}}$ when $p_j > p_{j-1}$. Next, note that for any $0 < \tau_j \le \frac{1}{2 \cdot 6^{p_j}}$, there exist $|k_{j-1}|, \varepsilon_j > 0, \tau_{j-1} > 0$ such that

$$2(\frac{2L_{j-1}\tilde{N}_{j-1}6^{p_{j-1}}|k_{j-1}|}{c_j^L} + \frac{\tilde{N}_j|k_{j-1}|\varepsilon_j}{c_j^L} + 2\tau_{j-1}6^{p_{j-1}}) < \tau_j. \tag{5.52}$$

From the expression of $\bar{\gamma}_{12}^{(j)}(s)$, (5.36), and (5.52), corresponding to this ε_j, there exists a $\delta_j > 0$ such that $\lambda_j \le \min\{\frac{\lambda_{j-1}}{4 \cdot 6^{p_{j-1}}}, \frac{\delta_j}{\tilde{N}_j}\}$ implies the second inequality of (5.50). By Lemma 5.2.1, z_j, x_j, u_{j-1}, ξ_j satisfy LB with restriction and AB with no restriction on $(z_j(0), \xi_j(0))$, both with restriction $\frac{\lambda_j}{3}$ on u_j, and the gains from u_j to z_j, x_j, u_{j-1} are $3(\frac{s}{|k_j|})^{\frac{1}{p_j}}, 6(\frac{s}{|k_j|})^{\frac{1}{p_j}}, 2 \cdot 6^{p_j}s$ respectively. Note that $(x_{j-1}^{p_{j-1}}, ..., x_1^{p_1}, \xi_1, u)$ can be seen as an output of ξ_j subsystem of (5.47), by Remark 5.2.3, $(x_{j-1}^{p_{j-1}}, ..., x_1^{p_1}, \xi_1, u)$ satisfies LB with restriction and AB with no restriction on $(z_j(0), \xi_j(0))$, both with restriction $\frac{\lambda_j}{3}$ on u_j and gain $2\tilde{N}_j 6^{p_j}s$.

Therefore, the proof is completed by induction. Finally, setting $u_n = 0$ in (5.33) gives the result of global asymptotic stabilization for system (5.32).

Remark 5.3.1 The design parameters $\lambda_i, k_i, i = 1, ..., n$, are governed by a set of algebraic inequalities and can be determined simultaneously at the end of the whole recursive design. For illustration, we consider the special case $1 < p_1 < \cdots < p_n$. Then, if we let $k_0 = 1$ and choose $\tau_i = \frac{1}{2^{n-i+1}6^{p_i+\cdots+p_n+n-i}}, i = 1, \cdots, n$, $\lambda_i, k_i, i = 1, ..., n$, are governed by the following set of inequalities:

$$\frac{\tilde{N}_1\varepsilon_1}{c_1^L} < \frac{1}{2^n 6^{p_1+\cdots+p_n+n-1}},$$

$$\lambda_1 < \min\{2^{-\frac{1}{p_1-1}}|k_1|(6\nu_1\bar{N}_1|k_1|)^{-\frac{p_1}{p_1-1}}, \Delta_1, \frac{\delta_1}{\tilde{N}_1}\},$$

$$2(\frac{2L_{j-1}\tilde{N}_{j-1}6^{p_{j-1}}|k_{j-1}|}{c_j^L} + \frac{\tilde{N}_j|k_{j-1}|\varepsilon_j}{c_j^L}) < \frac{1}{2^{n-j-1}6^{p_j+\cdots+p_n+n-j+1}},$$

$$\lambda_j < \min\{2^{\frac{-p_{j-1}}{p_j-p_{j-1}}}|k_j||\frac{k_{j-1}}{k_j}|^{\frac{p_j}{p_j-p_{j-1}}}|\frac{c_{j-1}^L k_{j-1}}{18c_j^U k_{j-2}}|^{\frac{p_{j-1}p_j}{p_j-p_{j-1}}}\frac{\lambda_{j-1}}{4 \cdot 6^{p_{j-1}}}, \frac{\delta_j}{\tilde{N}_j}, \}, j = 2, ..., n,$$

which are extracted from (5.42),(5.41), (5.52) and (5.51). It can be seen that this set of inequalities are solvable and $k_1, \lambda_1, ..., k_n, \lambda_n$ can be determined in order. More specifically, for $j = 1, ..., n-1$, k_j can be determined from the $(2j+1)$th inequality, λ_j can be determined from the $(2j-1)$th and $(2j)$th inequalities, k_n can be arbitrary nonzero real number with the same sign as θ_n, and λ_n can be determined from the $(2n-1)$th and the $(2n)$th inequalities.

Remark 5.3.2 Like the proof of Lemma 3.1, we could have obtained a much simpler expression for the gain function $\bar{\gamma}_{12}^{(j)}(s), j = 2, ..., n$, by making use of the inequality $|\tilde{F}_j(\xi_j, u_{j-1}, d)| \leq \alpha_j(\|(\xi_j, u_{j-1})\|)$ where $\alpha_j(s)$ is some gain function independent of d. However, doing so will lead to a gain function which cannot guarantee the satisfaction of the condition $2 \cdot 6^{p_j} \bar{\gamma}_{12}^{(j)}(s) < s, s > 0$, no matter how the parameters $\lambda_i, |k_i|, i = 1, ..., j$, are adjusted. Therefore, we have to take into account the specific expression of $\tilde{F}_j(\xi_j, u_{j-1}, d)$, i.e., (5.48) in order to calculate appropriate $\bar{\gamma}_{12}^{(j)}(s), j = 2, ..., n$. In particular, from (5.48), for $j = 2, ..., n$, we note that the contributions to the gain function $\bar{\gamma}_{12}^{(j)}(s)$ come from the first two terms of (5.48). In calculating the contribution of the first term of (5.48) to $\bar{\gamma}_{12}^{(j)}(s)$, we have taken advantage of the inequality (5.35), and in calculating the contribution of the second term of (5.48), we have made use of the fact that the gain function from $y_{12}^{(j-1)}$ to $v_1^{(j-1)}$ is already given by $\bar{\gamma}_{12}^{(j-1)}(s)$ calculated at the $(j-1)$th step. Thus we can obtain $\bar{\gamma}_{12}^{(j)}(s)$ in the form of (5.49). It is this specific expression of the gain function that enables the satisfaction of the second inequality of (5.50) by adjusting the parameters of the controller. Similarly, for $j = 2, ..., n$, using the fact that $y_{11}^{(j)} = H_j \xi_j = -\frac{\theta_j}{\theta_{j-1}} z_{j-1}$ allows the gain being calculated from u_{j-1} to z_{j-1} instead of being calculated from u_{j-1} to ξ_j, which leads to a much simpler expression for $\bar{\gamma}_{11}^{(j)}(s)$.

5.4 An Example

Consider the system

$$
\begin{aligned}
\dot{x}_2 &= x_1^3 + av \\
\dot{x}_1 &= v
\end{aligned}
\tag{5.53}
$$

where a is a positive real number. Clearly, system (5.53) satisfies Assumption 5.3.1 although it does not satisfy the assumptions needed in [50, 58, 69, 86, 88].

Let $z_1 = x_1$ and $z_2 = x_2 + \frac{\theta_2}{\theta_1} z_1$, where $\theta_1 = 1, \theta_2 = \frac{1}{k_1}$. Since k_i has the same sign with $\theta_i,\ i=1,2,\ k_1,k_2$ are positive in this case. Then system (5.53) becomes

$$\dot{z}_2 = \theta_2 v + av + \theta_2 k_1 x_1^3$$
$$\dot{z}_1 = \theta_1 v.$$

Note that $z_1 = x_1$ and no high order terms appear in the z_1 dynamics, i.e., $v_2^{(1)} = y_1^{(1)} = 0$, then it can be verified that, for arbitrary positive λ_1, k_1, under the control $v = -\sigma_1(k_1 z_1^3 - u_1)$, z_1, v satisfy LB and AB both with no restriction on $z_1(0)$, and no restriction on u_1 and gains $(\frac{s}{k_1})^{\frac{1}{3}}, 2s$, respectively.

Next, consider the z_2 dynamics. Let $u_1 = -\sigma_2(k_2(z_2 - \frac{\theta_2}{\theta_1}z_1)^3)$. Since $u_2 = 0$, it can be verified using the same arguments as that in Lemma 5.2.1 that $\bar{\gamma}_{21}^{(2)}(s) = 2^6 s^3$ and $\bar{\gamma}_{22}^{(2)}(s) = 2^6 s$ in this case. Thus, the small gain condition becomes $2^6(\bar{\gamma}_{11}^{(2)}(s))^3 < s, 2^6\bar{\gamma}_{12}^{(2)}(s) < s, s > 0$. Note that $\bar{\gamma}_{11}^{(2)}(s) = \frac{\theta_2}{\theta_1}(\frac{1}{k_1}\min\{k_2 s, \lambda_2\})^{\frac{1}{3}}$, then $2^6(\bar{\gamma}_{11}^{(2)}(s))^3 < s$ reduces to $2^6(\frac{\theta_2}{\theta_1})^3\frac{k_2}{k_1} < 1$ and hence we obtain $k_2 < \frac{k_1^4}{2^6}$. On the other hand, note that $\tilde{F}_2(\xi_2, u_1, d) = av + \theta_2 h_1(x_1^3, u_1)$ and $h_1(x_1^3, u_1)$ has no contribution to $\bar{\gamma}_{12}^{(2)}(s)$ when $\lambda_2 \le \frac{\lambda_1}{4}$, thus $2^6\bar{\gamma}_{12}^{(2)}(s) < s$ reduces to $2^7 a k_1 < 1$. To avoid too small design parameters, setting $a = 2^{-5}, \lambda_1 = 10$ and solving the corresponding inequalities yields $\lambda_2 = 2.5, k_1 = 0.24, k_2 = 6.10 \times 10^{-5}$. Thus the designed nested saturation control law is

$$v = -\sigma_1(0.24x_1^3 + \sigma_2(6.10 \times 10^{-5}x_2^3)) \tag{5.54}$$

where σ_1, σ_2 are saturation functions with level 10 and 2.5 respectively.

However, it can be shown that, for system (5.53), in the presence of the following non-minimum phase input unmodeled dynamics

$$v(s) = \frac{s-1}{s+1}u(s) \tag{5.55}$$

and under the control

$$u = -\sigma_1(k_1 x_1^3 + \sigma_2(k_2 x_2^3))$$

where σ_1, σ_2 are saturation functions with level λ_1, λ_2 respectively, and $\lambda_i, k_i, i = 1, 2$, are arbitrary positive real numbers, the resulting closed-loop system is not only unstable at the origin, but also has unbounded solutions.

Note that the state space equation of (5.55) is

$$\dot{\xi}_1 = -\xi_1 + u$$
$$v = -2\xi_1 + u$$

then the resulting closed-loop system becomes

$$\begin{aligned} \dot{x}_2 &= x_1^3 + a(-2\xi_1 - \sigma_1(k_1x_1^3 + \sigma_2(k_2x_2^3))) \\ \dot{x}_1 &= -2\xi_1 - \sigma_1(k_1x_1^3 + \sigma_2(k_2x_2^3)) \\ \dot{\xi}_1 &= -\xi_1 - \sigma_1(k_1x_1^3 + \sigma_2(k_2x_2^3)) \end{aligned} \tag{5.56}$$

Suppose $x_1(0), x_2(0)$ are any positive real numbers such that

$$\sigma_1(k_1(x_1(0))^3 + \sigma_2(k_2(x_2(0))^3)) = \lambda_1$$

Let $\xi_1(0) = -\lambda_1$. Then, $\xi_1(t) = -\lambda_1$ for all $t \geq 0$ and $x_1(t), x_2(t)$ are strictly increasing and diverge to infinity as $t \to \infty$, because $\dot{x}_1(t) = \lambda_1 > 0, \dot{x}_2(t) = (x_1(t))^3 + a\lambda_1 > 0$ for all $t \geq 0$.

To show the instability of the origin of (5.56), let $\phi = (\phi_2, \phi_1)$ where $\phi_2 = x_2 - ax_1$ and $\phi_1 = x_1 - 2\xi_1$. Then, system (5.56) can be written as follows:

$$\dot{\phi} = f(\phi, \xi_1) = \begin{bmatrix} \bar{f}_2(\phi, \xi_1) \\ \bar{f}_1(\phi, \xi_1) \end{bmatrix}, \ \dot{\xi}_1 = -\xi_1 + g(\phi, \xi_1) \tag{5.57}$$

where $\bar{f}_2(\phi, \xi_1) = (\phi_1 + 2\xi_1)^3$ and $\bar{f}_1(\phi, \xi_1) = \sigma_1(k_1(\phi_1 + 2\xi_1)^3 + \sigma_2(k_2(a\phi_1 + \phi_2 + 2a\xi_1)^3))$, and $g(\phi, \xi_1) = -\sigma_1(k_1(\phi_1 + 2\xi_1)^3 + \sigma_2(k_2(a\phi_1 + \phi_2 + 2a\xi_1)^3))$.

By the property of the saturation function and the Local Center Manifold Theorem [65] (see also [11]), there exists a local center manifold $\xi_1 = h(\phi)$ for sufficiently small $\|\phi\|$, where h is $\mathcal{C}^2$, $h(0) = 0$, $\frac{\partial h}{\partial \phi}(0) = 0$ and satisfies

$$\frac{\partial h(\phi)}{\partial \phi} f(\phi, h(\phi)) + h(\phi) - g(\phi, h(\phi)) = 0.$$

In turn, by Theorem 2 in Chap. 1 of [11], the equation which determines the stability of the origin of (5.57) is

$$\dot{\phi} = f(\phi, h(\phi)). \tag{5.58}$$

Now, let $\chi(\phi) = -k_1\phi_1^3 - k_2(a\phi_1 + \phi_2)^3$. For sufficiently small $\|\phi\|$, we obtain

$$\frac{\partial \chi(\phi)}{\partial \phi} f(\phi, \chi(\phi)) + \chi(\phi) - g(\phi, \chi(\phi)) = O(\|\phi\|^5).$$

Then, by Theorem 3 in Chap. 1 of [11], $h(\phi) = \chi(\phi) + O(\|\phi\|^5)$ for sufficiently small $\|\phi\|$. Thus, (5.58) becomes

$$\begin{aligned} \dot{\phi} &= f(\phi, \chi(\phi) + O(\|\phi\|^5)) \\ &= \begin{bmatrix} \phi_1^3 \\ k_1\phi_1^3 + k_2(a\phi_1 + \phi_2)^3 \end{bmatrix} + O(\|\phi\|^5) \end{aligned} \tag{5.59}$$

We further let $\varphi = (\varphi_2, \varphi_1)$ where $\varphi_2 = \phi_1 - k_1\phi_2, \varphi_1 = \phi_2$. Then (5.59) can be rewritten as follows:

$$\dot{\varphi} = \begin{bmatrix} k_2[a\varphi_2 + (ak_1+1)\varphi_1]^3 \\ (\varphi_2 + k_1\varphi_1)^3 \end{bmatrix} + O(\|\varphi\|^5) \tag{5.60}$$

If we can show the instability of the origin of system (5.60) for arbitrary positive $a, \lambda_1, \lambda_2, k_1, k_2$, then the origin of (5.56) is unstable for arbitrary positive $a, \lambda_1, \lambda_2, k_1, k_2$.

Let $\varphi(t)$ denote the solution of (5.60) starting from $\varphi(0)$. We will show the instability of the origin of (5.60) by definition, that is, given some $\varepsilon > 0$, there exist some $\varphi(0)$ ($\|\varphi(0)\|$ can be arbitrarily small) and a finite $T > 0$ such that $\|\varphi(T)\| \geq \varepsilon$. To show this, note that if $\varphi_i > 0,\ i = 1, 2$, then

$$k_2[a\varphi_2 + (ak_1+1)\varphi_1]^3 \geq q_1(\varphi_2+\varphi_1)^3 > q_1(\varphi_2^2+\varphi_1^2)^{\frac{3}{2}} = q_1\|\varphi\|^3$$
$$(\varphi_2 + k_1\varphi_1)^3 \geq q_2(\varphi_2+\varphi_1)^3 > q_2(\varphi_2^2+\varphi_1^2)^{\frac{3}{2}} = q_2\|\varphi\|^3$$

where $q_1 = k_2(\min\{a, ak_1+1\})^3, q_2 = (\min\{k_1, 1\})^3$. By the definition of $O(\|\varphi\|^5)$, given any $0 < q_3 < \frac{1}{2}\min\{q_1, q_2\}$, there exists $q_4 > 0$ such that $\|O(\|\varphi\|^5)\| \leq q_3\|\varphi\|^3$ for all $\|\varphi\| \leq q_4$. Let $\varepsilon = \frac{1}{2}q_4$. Then, for any positive φ_1, φ_2 satisfying $\|\varphi\| \leq 2\varepsilon$, we have

$$\dot{\varphi}_i > (q_i - q_3)\|\varphi\|^3 > 0, i = 1, 2. \tag{5.61}$$

Thus, the solutions $\varphi_1(t), \varphi_2(t)$ starting from any positive $\varphi_1(0), \varphi_2(0)$ satisfying $\|\varphi(0)\| < \varepsilon$, are strictly increasing. In turn, (5.61) becomes

$$\dot{\varphi}_i > (q_i - q_3)\|\varphi\|^3 \geq (q_i - q_3)\|\varphi(0)\|^3, i = 1, 2,$$

which shows that, there must exist a finite $T > 0$ such that $\|\varphi(T)\| \geq \varepsilon$. Hence, the origin of system (5.60) is unstable for arbitrary positive $a, \lambda_1, \lambda_2, k_1, k_2$.

As a result, when (5.55) is present, we have to redesign the control (5.54) by Theorem 5.3.2. In the presence of (5.55), system (5.53) becomes

$$\begin{aligned} \dot{x}_2 &= x_1^3 + a(-2\xi_1 + u) \\ \dot{x}_1 &= -2\xi_1 + u \\ \dot{\xi}_1 &= -\xi_1 + u. \end{aligned} \tag{5.62}$$

Let $z_1 = x_1 - 2\xi_1, z_2 = x_2 + \frac{\theta_2}{\theta_1}z_1$, where $\theta_1 = -1, \theta_2 = \frac{1}{k_1}$. Since k_i has the same sign with $\theta_i,\ i = 1, 2,\ k_1, k_2$ are negative in this case. Then system (5.62) becomes

$$\begin{aligned} \dot{z}_2 &= \theta_2 u + a(-2\xi_1 + u) + \theta_2 k_1 x_1^3 \\ \dot{z}_1 &= \theta_1 u \\ \dot{\xi}_1 &= -\xi_1 + u. \end{aligned} \tag{5.63}$$

Let $u = -\sigma_1(k_1(z_1 + 2\xi_1)^3 - u_1)$. Note that no high order terms appear in z_1 dynamics, i.e., $v_{22}^{(1)} = y_{12}^{(1)} = 0$, then it can be verified using the same arguments as that in Lemma 5.2.1 that $\bar{\gamma}_{21}^{(1)}(s) = 2^7 s^3, \bar{\gamma}_{22}^{(1)}(s) = 0$ in this case. Thus, the small gain condition becomes $2^7(\bar{\gamma}_{11}^{(1)}(s))^3 < s, s > 0$. Note that ξ_1 subsystem satisfies LB and AB both with no restriction on $\xi_1(0)$, no restriction on u and gain $\bar{N}_1 s = s$, $\bar{\gamma}_{11}^{(1)}(s) \leq 2\min\{|k_1|s, \lambda_1\}$. Thus, the small gain condition is satisfied with $\lambda_1 < 2^{-5}|k_1|^{-\frac{1}{2}}$. As a result, z_1, ξ_1, u satisfy LB with restriction and AB with no restriction on $(\xi_1(0), z_1(0))$, both with restriction $\frac{\lambda_1}{3}$ on u_1 and gains $2(\frac{s}{|k_1|})^{\frac{1}{3}}, 2^7 s, 2^7 s$, respectively.

Next, consider the z_2 dynamics. Let $u_1 = -\sigma_2(k_2(z_2 - \frac{\theta_2}{\theta_1}z_1)^3)$. Since $u_2 = 0$, it can be verified using the same arguments as that in Lemma 5.2.1 that $\bar{\gamma}_{21}^{(2)}(s) = 2^6 s^3$ and $\bar{\gamma}_{22}^{(2)}(s) = 2^6 s$. Thus, the small gain condition becomes $2^6(\bar{\gamma}_{11}^{(2)}(s))^3 < s, 2^6\bar{\gamma}_{12}^{(2)}(s) < s, s > 0$. Note that $\bar{\gamma}_{11}^{(2)}(s) = 2\frac{|\theta_2|}{\theta_1}(\frac{1}{k_1}\min\{|k_2|s, \lambda_2\})^{\frac{1}{3}}$, then $2^6(\bar{\gamma}_{11}^{(2)}(s))^3 < s$ reduces to $2^6(2\frac{\theta_2}{\theta_1})^3\frac{k_2}{k_1} < 1$ and hence we obtain $|k_2| < \frac{k_1^4}{2^9}$. On the other hand, note that $\tilde{F}_2(\xi_2, u_1, d) = a(-2\xi_1 + u) + \theta_2 h_1(x_1^3, u_1)$ and $h_1(x_1^3, u_1)$ has no contribution to $\bar{\gamma}_{12}^{(2)}(s)$ when $\lambda_2 \leq \frac{\lambda_1}{2\cdot 2^7}$, thus $2^6\bar{\gamma}_{12}^{(2)}(s) < s$ reduces to $2a(2^8 + 2^7)|k_1| < 2^{-6}$. Setting $a = 2^{-5}$ and solving the corresponding inequalities yields $k_1 = -6.5 \times 10^{-4}, k_2 = -3.5 \times 10^{-16}, \lambda_1 = 1.2247$ and $\lambda_2 = 0.0047$. Thus, the redesigned nested saturation control law is

$$u = -\sigma_1(-6.5 \times 10^{-4}x_1^3 + \sigma_2(-3.5 \times 10^{-16}x_2^3)) \tag{5.64}$$

where σ_1, σ_2 are saturation functions with level 1.2247 and 0.0047 respectively.

For illustration, Fig. 5.1, Fig. 5.2 and Fig. 5.3 show the simulation results with the initial condition $(\xi_1(0), x_1(0), x_2(0)) = (0.1, 0.2, -5)$ for system (5.53) under the control (5.54), and for system (5.53) subject to the input unmodeled dynamics (5.55) under the control (5.54), and for system (5.53) subject to the input unmodeled dynamics (5.55) under the control (5.64), respectively.

5.5 Conclusion

In this chapter, we have addressed the global robust stabilization problem for feedforward system (5.1) subject to some type of input unmodeled dynamics (5.2). A specific difficulty in dealing with this problem is that the Jacobian linearization of (5.1) is not stabilizable. We have overcome this difficulty by employing two versions of small gain theorem with restrictions adapted from [85] to establish the local stability and global attractiveness of the closed-loop system at the origin respectively.

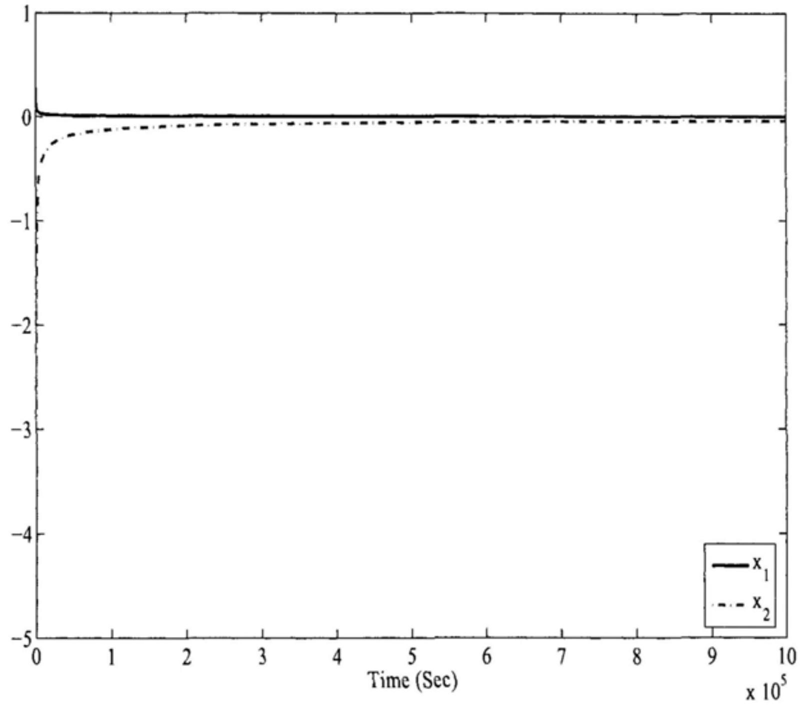

Figure 5.1: Profile of x: original design without the presence of input unmodeled dynamics

It is noted that, even in the special case where the input unmodeled dynamics (5.2) is not present, our result cannot be covered by the existing results in [50, 58, 69, 86, 88] because in this chapter the functions g_i's in (5.1) do not have to satisfy the structural constraints needed in [50, 58, 69, 86, 88]. In particular, the functions g_i's are allowed to be linear in its arguments.

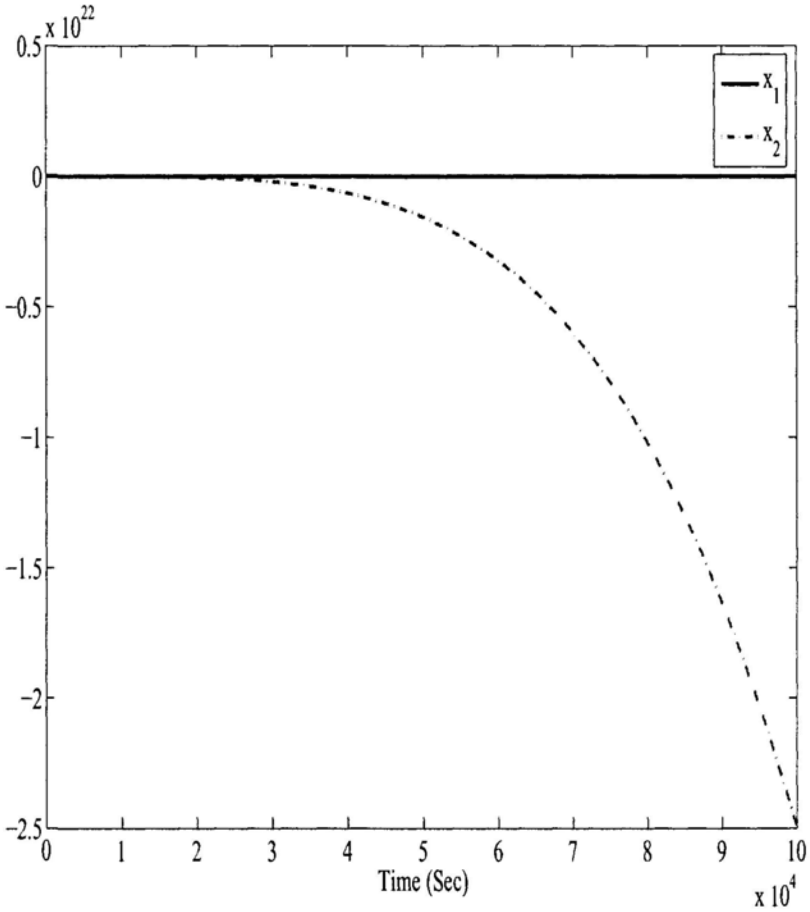

Figure 5.2: Profile of x: original design in the presence of input unmodeled dynamics

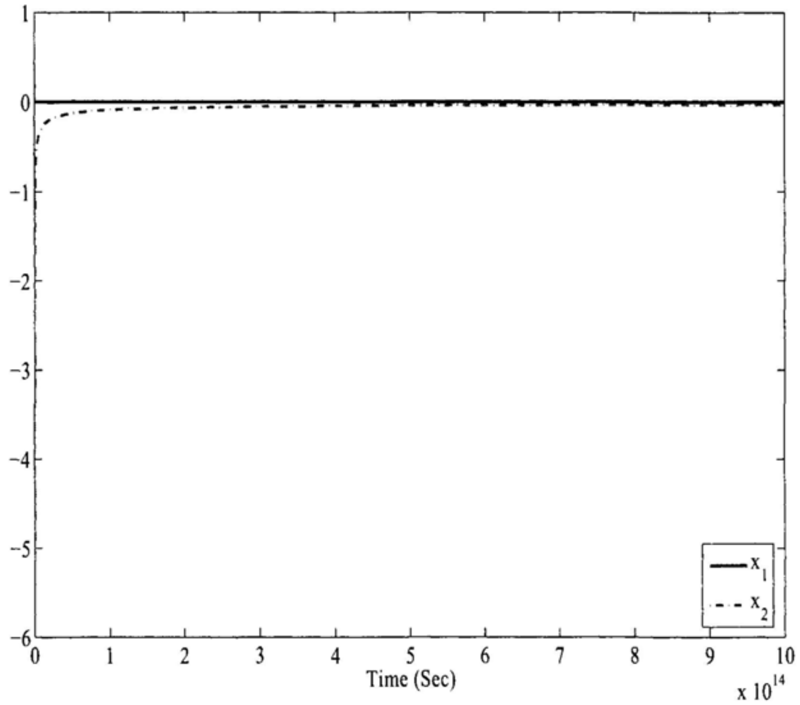

Figure 5.3: Profile of x: redesign in the presence of input unmodeled dynamics

Chapter 6

Global Robust Output Regulation of Nonlinear Systems in Strict Feedforward Form

6.1 Introduction

In this chapter, we study the global robust output regulation problem of nonlinear systems in strict feedforward form:

$$\begin{aligned} \dot{x}_i &= f_i(x_{i-1}, ..., x_1, u, v, w), \ \ i = n, ..., 2 \\ \dot{x}_1 &= cu + f_1(v, w) \\ e &= x_1 - q_d(v, w) \end{aligned} \tag{6.1}$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the control, $e \in \mathbb{R}$ the tracking error, $w \in \mathbb{R}^{n_w}$ the uncertain constant parameter, $v \in \mathbb{R}^q$ the state of the exosystem

$$\dot{v} = Sv \tag{6.2}$$

where all eigenvalues of the matrix S are simple with zero real parts, c is a known nonzero constant, and for $i = 1, ..., n$, the functions f_i and q_d are globally defined smooth functions satisfying $f_i(0, ..., 0, w) = 0$ and $q_d(0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$.

Global robust output regulation problem (GRORP): For any compact set $V_0 \subset \mathbb{R}^q$ with a known bound and any compact set $W \subset \mathbb{R}^{n_w}$ with a known bound, we will design for system (6.1) a dynamic state feedback controller in the following form

$$u = \mathcal{K}(\eta, x, e), \ \ \dot{\eta} = \mathcal{F}(\eta, x, e) \tag{6.3}$$

where η is the compensator state and $\mathcal{K}, \mathcal{F}$ are locally Lipschitz functions vanishing at the origin, such that the closed-loop system composed of (6.1) and (6.3) has the following properties:

(a) For all $v(0) \in V_0, w \in W$ and for all initial state $x(0), \eta(0)$, the trajectory of the closed-loop system exists and is bounded for all $t \geq 0$;

(b) The tracking error converges to zero as t tends to infinity, i.e., $\lim_{t\to\infty} e(t) = 0$.

To our knowledge, the only chapters that are relevant to the problem described above are [9] and [56]. An approximate and restricted tracking problem for a class of block feedforward systems is studied in [9] via dynamic output feedback control. The term approximate refers to the approximate regulation which is achieved by utilizing the k-fold internal model [20]. The term restricted refers to the fact that the state of the exosystem should be sufficiently small. In [56], the authors deal with the input disturbance suppression problem (IDSP) via dynamic state feedback control for the following system

$$\begin{aligned} \dot{x}_i &= w_{i-1}x_{i-1} + g_i(\dot{x}_{i-1}, ..., \dot{x}_1, w), \ \ i = n, ..., 2 \\ \dot{x}_1 &= u - g_1(v), \end{aligned} \tag{6.4}$$

where $w_i, i = 1, ..., n-1$, are possibly time varying. The goal of IDSP is to achieve property (a) of GRORP and $\lim_{t\to\infty} x(t) = 0$. There is distinct difference between IDSP and GRORP. Roughly speaking (See Remark 6.2.4 for more specific comparison between IDSP and GRORP), for the IDSP, only one internal model associated with the input u needs to be constructed. The IDSP of system (6.4) can be converted into a global robust stabilization problem of a class of feedforward systems subject to input unmodeled dynamics. Several results about this robust stabilization problem have been reported, see e.g., [7, 45, 54, 72]. In contrast, for the GRORP, n internal models associated with $x_2, ..., x_n$ and the input u need to be constructed. The GRORP of system (6.1) will be converted into a global robust stabilization problem of a class of feedforward systems subject to both time-varying static and dynamic uncertainty described by equation (4.1) below. The global robust stabilization problem for this class of systems has not been studied so far. Therefore, even if we succeed in converting the output regulation problem of system (6.1) into the global robust stabilization problem of system (4.1), how to stabilize system (4.1) is still a challenging problem.

We will present a set of solvability conditions on the GRORP of strict feedforward system (6.1). In order to obtain our results, we first identify the structural properties of the functions q_d and f_i in (6.1) so that an internal model candidate exists. Then, by looking for a suitable internal model and performing appropriate coordinate and input transformations on the augmented system consisting of (6.1) and the internal model, the GRORP of system (6.1) can be converted into a global robust stabilization problem for a class of feedforward systems subject to both time varying static uncertainty and dynamic uncertainty in the form of (4.1). Therefore, by applying the stabilization result obtained in Chapter 4, i.e., Theorem 4.3.1, the global robust output regulation problem of system (6.1) will be solved.

6.2 Main Result

As pointed out in Introduction, the GRORP of system (6.1) can be converted into a global robust stabilization problem of a well defined augmented system. To introduce this conversion, we make following assumptions.

Assumption 6.2.1 There exist smooth functions $\mathbf{x}(v,w) = (\mathbf{x}_1(v,w), ..., \mathbf{x}_n(v,w))$ and $\mathbf{u}(v,w)$ with $\mathbf{x}(0,0) = 0$ and $\mathbf{u}(0,0) = 0$ satisfying for all $v \in \mathbb{R}^q, w \in \mathbb{R}^{n_w}$,

$$
\begin{aligned}
\dot{\mathbf{x}}_i(v,w) &= f_i(\mathbf{x}_{i-1}(v,w), ...\mathbf{x}_1(v,w), \mathbf{u}(v,w), v, w), \;\; i = n, \cdots, 2 \\
\dot{\mathbf{x}}_1(v,w) &= c\mathbf{u}(v,w) + f_1(v,w) \\
\mathbf{x}_1(v,w) &= q_d(v,w)
\end{aligned}
\tag{6.5}
$$

Assumption 6.2.2 There exist sufficiently smooth functions $\tau_i : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}^{r_i}$, $i = 1, ..., n$, vanishing at $(0,0)$, such that

$$\dot{\tau}_i(v,w) = \Phi_i \tau_i(v,w), \;\; \pi_i(v,w) = \Psi_i \tau_i(v,w) \tag{6.6}$$

where the pair (Ψ_i, Φ_i) is observable and all the eigenvalues of Φ_i are simple with zero real parts, and $\pi_1(v,w) = \mathbf{u}(v,w)$, $\pi_i(v,w) = \mathbf{x}_i(v,w)$, $i = 2, ..., n$.

Remark 6.2.1 Equation (6.5) is called regulator equations and the solvability of these equations is necessary but not sufficient for the solvability of the robust output regulation problem [10, 21, 22, 27]. Assumption 6.2.2 is made for the existence of appropriate linear internal models. Both Assumption 6.2.1 and 6.2.2 are quite standard in literature though Assumption 6.2.2 can be relaxed when nonlinear internal models are employed [21, 22]. Under Assumption 6.2.2, given a pair of controllable matrices (M_i, N_i) with $M_i \in \mathbb{R}^{r_i \times r_i}$ Hurwitz and $N_i \in \mathbb{R}^{r_i}$, there exists a unique and nonsingular matrix $T_i \in \mathbb{R}^{r_i \times r_i}$ satisfying the Sylvester equation

$$T_i\Phi_i - M_iT_i = N_i\Psi_i \tag{6.7}$$

since the spectra of M_i and Φ_i are disjoint and the pair (Ψ_i, Φ_i) is observable.

We define the following system

$$\begin{aligned} \dot{\eta}_1 &= M_1\eta_1 + N_1u - \tfrac{M_1N_1}{c}e, \\ \dot{\eta}_i &= M_i\eta_i + N_ix_i, \ i = 2, ..., n. \end{aligned} \tag{6.8}$$

as the internal model of (6.1) with output $(u, x_2, ..., x_n)$.

Next, we will convert the GRORP for system (6.1) into a global robust stabilization problem for the augmented system composed of the original plant (6.1) and the internal model (6.8). Performing the following coordinate and input transformation

$$\begin{aligned} \bar{x}_1 &= x_1 - \mathbf{x}_1(v, w) = e \\ \bar{x}_i &= x_i - \Psi_iT_i^{-1}\eta_i, \ i = 2, ..., n \\ \bar{\eta}_i &= \eta_i - T_i\tau_i, \ i = 1, ..., n \\ \hat{u} &= u - \Psi_1T_1^{-1}\eta_1 \end{aligned} \tag{6.9}$$

on the augmented system gives

$$\begin{aligned} \dot{\bar{x}}_i &= -\Psi_iT_i^{-1}[(M_i + N_i\Psi_iT_i^{-1})\bar{\eta}_i + N_i\bar{x}_i] + \hat{f}_i(\bar{x}_{i-1}, \bar{\eta}_{i-1}, ..., \bar{x}_1, \bar{\eta}_1, \hat{u}, v, w) \\ \dot{\bar{\eta}}_i &= (M_i + N_i\Psi_iT_i^{-1})\bar{\eta}_i + N_i\bar{x}_i, \ i = n, ..., 2 \\ \dot{\bar{x}}_1 &= c\Psi_1T_1^{-1}\bar{\eta}_1 + c\hat{u} \\ \dot{\bar{\eta}}_1 &= (M_1 + N_1\Psi_1T_1^{-1})\bar{\eta}_1 + N_1\hat{u} - \frac{M_1N_1}{c}\bar{x}_1 \end{aligned} \tag{6.10}$$

where $\hat{f}_2(\bar{x}_1, \bar{\eta}_1, \hat{u}, v, w) = -f_2(\mathbf{x}_1, \mathbf{u}, v, w) + f_2(\bar{x}_1 + \mathbf{x}_1, \hat{u} + \Psi_1T_1^{-1}\bar{\eta}_1 + \mathbf{u}, v, w)$ and $\hat{f}_i(\bar{x}_{i-1}, \bar{\eta}_{i-1}, ..., \bar{x}_1, \bar{\eta}_1, \hat{u}, v, w) = -f_i(\mathbf{x}_{i-1}, ..., \mathbf{x}_1, \mathbf{u}, v, w) + f_i(\bar{x}_{i-1} + \Psi_{i-1}T_{i-1}^{-1}\bar{\eta}_{i-1} + \mathbf{x}_{i-1}, ..., \bar{x}_1 + \mathbf{x}_1, \hat{u} + \Psi_1T_1^{-1}\bar{\eta}_1 + \mathbf{u}, v, w), i = 3, ..., n$.

It is known from [21, 22] that the GRORP of system (6.1) will be solved if we can make the equilibrium of system (6.10) at $(\bar{x}, \bar{\eta}) = (0, 0)$ globally asymptotically stable for all trajectories $v(t)$ starting from V_0 and all $w \in W$. A system of the form (6.10) has never been encountered and there is no clue whether or not the equilibrium of this system at the origin is stabilizable. Nevertheless, by performing some further coordinate and input transformation on (6.10), it is possible to convert (6.10) to the form of (4.1) with all desirable properties. For this purpose, we introduce two more assumptions.

Assumption 6.2.3 For $i = 2, ..., n$, Φ_i is invertible.

Assumption 6.2.4 For $i = 2, \cdots, n$, $\frac{\partial f_i}{\partial x_{i-1}}|_{(x_{i-1},\cdots,x_1,u)=(\mathbf{x}_1(v,w),\cdots,\mathbf{x}_{i-1}(v,w),\mathbf{u}(v,w))}$ is a positive (or alternative negative) constant.

Now define the following coordinate and input transformation

$$\begin{aligned} \xi_1 = c\bar{\eta}_1 - N_1\bar{x}_1, \xi_i = (M_i + N_i\Psi_i T_i^{-1})\bar{\eta}_i + N_i\bar{x}_i, \ i = 2, ..., n, \\ \bar{u} = \hat{u} + \tfrac{\Psi_1 T_1^{-1} N_1}{c}\bar{x}_1. \end{aligned} \tag{6.11}$$

From (6.7), $M_i + N_i\Psi_i T_i^{-1} = T_i\Phi_i T_i^{-1}$, and then from Assumption 6.2.3 and $c \neq 0$, the transformation (6.11) is globally invertible. Performing the transformation (6.11) on (6.10) yields,

$$\begin{aligned} \dot{\bar{x}}_i &= -\Psi_i T_i^{-1}\xi_i + \tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, ..., \bar{x}_1, \xi_1, \bar{u}, d) \\ \dot{\xi}_i &= M_i\xi_i + N_i\tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, ..., \bar{x}_1, \xi_1, \bar{u}, d), \ i = n, ..., 2 \\ \dot{\bar{x}}_1 &= \Psi_1 T_1^{-1}\xi_1 + c\bar{u} \\ \dot{\xi}_1 &= M_1\xi_1 \end{aligned} \tag{6.12}$$

where $d = (v, w)$, $\tilde{f}_2(\bar{x}_1, \xi_1, \bar{u}, v, w) = -f_2(\mathbf{x}_1, \mathbf{u}, v, w) + f_2(\bar{x}_1 + \mathbf{x}_1, \bar{u} + \frac{1}{c}\Psi_1 T_1^{-1}\xi_1 + \mathbf{u}, v, w)$, and $\tilde{f}_i(\bar{x}_{i-1}, \xi_{i-1}, ..., \bar{x}_1, \xi_1, \bar{u}, v, w) = -f_i(\mathbf{x}_{i-1}, ..., \mathbf{x}_1, \mathbf{u}, v, w) + f_i((1 - \Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}N_{i-1})$ $\bar{x}_{i-1} + \Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}\xi_{i-1} + \mathbf{x}_{i-1}, ..., \bar{x}_1 + \mathbf{x}_1, \bar{u} + \frac{1}{c}\Psi_1 T_1^{-1}\xi_1 + \mathbf{u}, v, w), i = 3, ..., n$

Without loss of generality, assume $d \in \mathcal{D} = V \times W$ where V is a compact set containing all trajectories of (6.2) starting from V_0. Since V_0 is compact, and all eigenvalues of the matrix S in (6.2) are simple with zero real parts, V exists. Thus $\mathcal{D}$ is compact. Clearly, if we can globally asymptotically stabilize the origin of system (6.12) with $d \in \mathcal{D}$, the GRORP of system (6.1) will be solved.

Theorem 6.2.1 Suppose system (6.1) satisfies Assumptions 6.2.1 to 6.2.4. Then, the GRORP can be solved by a dynamic state feedback controller of the form

$$\begin{aligned} u = \Psi_1 T_1^{-1}(\eta_1 - \tfrac{1}{c}N_1 e) - \sigma_1(k_1 e + ... + \sigma_n(k_n x_n - k_n\Psi_n T_n^{-1}\eta_n)), \\ \dot{\eta}_1 = M_1\eta_1 + N_1 u - \tfrac{1}{c}M_1 N_1 e, \ \dot{\eta}_i = M_i\eta_i + N_i x_i, \ i = 2, ..., n. \end{aligned} \tag{6.13}$$

Proof: Since system (6.12) is in the form of (4.1), by Theorem 4.3.1, it suffices to show that, system (6.12) satisfies Assumptions 4.3.1 and 4.3.2.

Let us first verify that system (6.12) satisfies Assumption 4.3.1. Rewrite (6.12) in the form of (4.27) as follows:

$$\begin{aligned}
\dot{\bar{x}}_i &= -\Psi_i T_i^{-1}\xi_i + \bar{c}_i\bar{x}_{i-1} + \bar{f}_i^r(\bar{x}_{i-1},\xi_{i-1},\cdots,\bar{x}_1,\xi_1,\bar{u},d)\\
\dot{\xi}_i &= M_i\xi_i + N_i\bar{c}_i\bar{x}_{i-1} + N_i\bar{f}_i^r(\bar{x}_{i-1},\xi_{i-1},\cdots,\bar{x}_1,\xi_1,\bar{u},d),\ i=n,\cdots,2\\
\dot{\bar{x}}_1 &= \Psi_1 T_1^{-1}\xi_1 + \bar{c}_1\bar{u}\\
\dot{\xi}_1 &= M_1\xi_1
\end{aligned} \tag{6.14}$$

where $\bar{f}_i^r, i=2,...,n$, are suitably defined smooth functions, and $\bar{c}_1 = c, \bar{c}_2 = \frac{\partial \tilde{f}_2}{\partial \bar{x}_1}|_{(0,0,0,d)} = \frac{\partial f_2}{\partial x_1}|_{(x_1,u)=(\mathbf{x}_1,\mathbf{u})}$ and for $i=3,...,n$, $\bar{c}_i = \frac{\partial \tilde{f}_i}{\partial \bar{x}_{i-1}}|_{(0,...,0,d)} = (1-\Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}N_{i-1})\cdot \frac{\partial f_i}{\partial x_{i-1}}|_{(x_{i-1},...,x_1,u)=(\mathbf{x}_{i-1},...,\mathbf{x}_1,\mathbf{u})}$. Then noting (6.14) yields

$$\mu_1 = c,\ \mu_2 = (1+\Psi_2T_2^{-1}M_2^{-1}N_2)\frac{\partial f_2}{\partial x_1}|_{(x_1,u)=(\mathbf{x}_1,\mathbf{u})},$$
$$\mu_i = (1+\Psi_iT_i^{-1}M_i^{-1}N_i)(1-\Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}N_{i-1})\frac{\partial f_i}{\partial x_{i-1}}|_{(x_{i-1},...,x_1,u)=(\mathbf{x}_{i-1},...,\mathbf{x}_1,\mathbf{u})}, i=3,...,n$$

We claim that, $1+\Psi_iT_i^{-1}M_i^{-1}N_i \neq 0, i=2,...,n$, and $1-\Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}N_{i-1}\neq 0, i=3,...,n$. From (6.7), Assumption 6.2.3 and the identity $\det(I_n - PQ) = \det(I_m - QP)$ where P, Q are $n\times m$ and $m \times n$ matrices respectively, we have

$$\begin{aligned}
&1+\Psi_iT_i^{-1}M_i^{-1}N_i = \det(I_{r_i}+N_i\Psi_iT_i^{-1}M_i^{-1}) = \det(T_i\Phi_iT_i^{-1}M_i^{-1})\neq 0,\\
&1-\Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}N_{i-1} = \det(I_{r_{i-1}}-N_{i-1}\Psi_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1}) = \det(M_{i-1}T_{i-1}\Phi_{i-1}^{-1}T_{i-1}^{-1})\neq 0.
\end{aligned}$$

Then noting Assumption 6.2.4 shows that Assumption 4.3.1 is satisfied.

Next, we show that system (6.12) also satisfies Assumption 4.3.2. For $i=1$, the specific form of the last equation of (6.12) immediately implies that ξ_1 satisfies Assumption 4.3.2 with $\bar{N}_1 = 0$ and $\Delta_1 = \infty$. For $i=2,...,n$, let $\tilde{u}_i = \tilde{f}_i(\bar{x}_{i-1},\xi_{i-1},...,\bar{x}_1,\xi_1,\bar{u},d)$. Then ξ_i subsystem in (6.12) is rewritten as $\dot{\xi}_i = M_i\xi_i + N_i\tilde{u}_i$. Since M_i is Hurwitz, ξ_i satisfies a-LB with no restriction on $\xi_i(0)$, no restriction on $\tilde{u}_i$ and linear gain J_is, where J_i is an appropriate nonnegative constant. Since $\tilde{f}_i(\bar{x}_{i-1},\xi_{i-1},...,\bar{x}_1,\xi_1,\bar{u},d)$ is smooth and $\tilde{f}_i(0,...,0,d) = 0$ for $d\in\mathcal{D}$, there exist positive constants $\bar{L}_i, \bar{\delta}_i$ such that $\|\tilde{u}_i\| \leq \bar{L}_i\|(\bar{x}_{i-1},\xi_{i-1},...,\bar{x}_1,\xi_1,\bar{u})\|$ for $\|(\bar{x}_{i-1},\xi_{i-1},...,\bar{x}_1,\xi_1,\bar{u})\| \leq \bar{\delta}_i$ and $d\in\mathcal{D}$. Thus, Assumption 4.3.2 is satisfied with $\bar{N}_i = J_i\bar{L}_i$ and $\Delta_i = \bar{\delta}_i$, $i=2,...,n$.

By Theorem 4.3.1, there exist $\lambda_i > 0$ and nonzero k_i such that, the following control

$$\bar{u} = -\sigma_1(k_1\bar{x}_1 + \sigma_2(k_2\bar{x}_2 + ... + \sigma_n(k_n\bar{x}_n))) \tag{6.15}$$

can globally asymptotically stabilize system (6.12). Noting (6.9), (6.11) and (6.15) yields the controller (6.13), which solves the GRORP of system (6.1).

Remark 6.2.2 For the class of nonlinear strict feedforward systems which only involve polynomial nonlinearities, Assumptions 6.2.1 to 6.2.3 can be easily testified. Let us first review some facts which can be found in [21]. Let $v^{[1]} = v = (v_1, ..., v_q) \in \mathbb{R}^q$ and for $l \geq 2$, $v^{[l]} = (v_1^l, v_1^{l-1}v_2, ..., v_1^{l-1}v_q, \; v_1^{l-2}v_2^2, ..., v_1^{l-2}v_2v_q, ..., v_q^l)$. Then from Section 4.2 of [21], there exists a matrix denoted by $S^{[l]}$ such that

$$\frac{\partial v^{[l]}}{\partial v} S v = S^{[l]} v^{[l]} \tag{6.16}$$

Moreover, all the eigenvalues of $S^{[l]}$ are given by

$$\lambda = l_1\lambda_1 + ... + l_q\lambda_q, l_1 + ... + l_q = l, \; l_1, ..., l_q = 0, 1, ..., l$$

where $\lambda_1, ..., \lambda_q$ are eigenvalues of S. As a result, when all eigenvalues of S are simple with zero real parts, $S^{[l]}$ is nonsingular if and only if q is even and l is odd. Moreover, the roots of the minimal polynomial of $S^{[l]}$ coincide with all distinct eigenvalues of $S^{[l]}$.

In the following, we call $f : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}$ a polynomial in v, if it takes the form

$$f(v, w) = \sum_{l=1}^{\kappa} F_l(w) v^{[l]} \tag{6.17}$$

where $F_l(w)$, $l = 1, ..., \kappa$, are row vectors of appropriate dimensions.

Proposition 6.2.1 Assume $f : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}$ is a polynomial function in v and takes the form (6.17). If q is even and $F_l(w) = 0$ when l is even, then there exists a polynomial solution $x(v, w)$ in v for the following partial differential equation

$$\frac{\partial x(v, w)}{\partial v} S v = f(v, w) \tag{6.18}$$

Moreover, there exist an integer r and matrices $\Phi \in \mathbb{R}^{r\times r}, \Psi \in \mathbb{R}^{1\times r}$, where Φ is nonsingular with all its eigenvalues simple and on the imaginary axis and the pair (Ψ, Φ) is observable, such that $\tau(v, w) = (x(v, w), \dot{x}(v, w), ..., \frac{d^{(r-1)}x(v,w)}{dt^{(r-1)}})$ satisfies

$$\dot{\tau}(v, w) = \Phi\tau(v, w), \;\; x(v, w) = \Psi\tau(v, w). \tag{6.19}$$

Proof: Assume $x(v, w)$ takes the following form

$$x(v, w) = \sum_{l=1}^{\kappa} X_l(w) v^{[l]} \tag{6.20}$$

where $X_l(w)$, $l = 1, ..., \kappa$, are suitable row vectors to be determined.

Substituting (6.17) and (6.20) into (6.18), and using (6.16) gives

$$\sum_{l=1}^{\kappa} X_l(w) S^{[l]} v^{[l]} = \sum_{l=1}^{\kappa} F_l(w) v^{[l]} \tag{6.21}$$

Equating the coefficients of $v^{[l]}$ on both sides of (6.21) gives

$$X_l(w)S^{[l]} = F_l(w),\ l = 1, ..., \kappa \tag{6.22}$$

By Remark 6.2.2, $S^{[l]}$ is nonsingular when l is odd and q is even, and thus equation (6.22) has a solution $X_l(w)$ as follows:

$$X_l(w) = \begin{cases} F_l(w)(S^{[l]})^{-1} & l = 1, 3, ..., \\ 0, & l = 2, 4, ..., \end{cases}$$

Thus we can write $x(v, w)$ as follows

$$x(v, w) = \sum_{i=0}^{k} X_{2i+1}(w)v^{[2i+1]} \tag{6.23}$$

for some integer k and for all $v \in \mathbb{R}^n, w \in \mathbb{R}^{n_w}$.

By Remark 6.2.2, it can be deduced that the minimal polynomial of $S^{[2i+1]}$ divides the minimal polynomial of $S^{[2j+1]}$ whenever $i \leq j$. Without loss of generality, let $\kappa = 2k + 1$ and $F_\kappa(w) \neq 0$. Denote the minimal polynomial of the matrix $S^{[\kappa]}$ by $P(\lambda) = \lambda^r - a_1 - a_2\lambda - ... - a_r\lambda^{r-1}$ for some real numbers $a_1, ..., a_r$. Then, the roots of $P(\lambda)$ are non-repeated with zero real parts. By the Cayley-Hamilton Theorem, $P(S^{[2i+1]}) = 0,\ i = 1, ..., k$. Thus we obtain

$$\frac{d^r x(v(t), w)}{dt^r} - a_1 x(v(t), w) - a_2\frac{dx(v(t), w)}{dt} - ... - a_r\frac{d^{(r-1)}x(v(t), w)}{dt^{(r-1)}} = 0 \tag{6.24}$$

for all trajectories $v(t)$ of the exosystem and $w \in \mathbb{R}^{n_w}$.

Next, let $\tau(v, w) = (x(v, w), \dot{x}(v, w), ..., \frac{d^{(r-1)}x(v,w)}{dt^{(r-1)}})$. Then it can be verified that (6.19) holds with

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & ... & 1 \\ a_1 & a_2 & a_3 & ... & a_r \end{bmatrix}, \ \Psi = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix}.$$

It can be seen that the pair (Ψ, Φ) is observable. Moreover, since the characteristic polynomial of Φ is the minimal polynomial of $S^{[\kappa]}$, by Remark 6.2.2, Φ is nonsingular, and all its eigenvalues are simple and on the imaginary axis.

Remark 6.2.3 We call $x(v, w)$ an odd polynomial in v if it takes the special form (6.23). Assume q is even and $q_d(v, w)$ is an odd polynomial in v. Then by Proposition 6.2.1, it can be concluded that, Assumptions 6.2.1 to 6.2.3 are satisfied if $f_1(v, w)$ is an odd polynomial in v and for $i = 2, ..., n$, $f_i(x_{i-1}, ..., x_1, u, v, w)$ is an odd polynomial in $(x_{i-1}, ..., x_1, u, v)$.

Remark 6.2.4 When $q_d(v,w) = 0$ and the functions $f_i, i = 2, ..., n$, in (6.1) are independent of u and vanishing at $(0, ..., 0, v, w)$, the GRORP of system (6.1) reduces to the IDSP studied in [56]. For this special case, $\mathbf{u}(v,w) = -\frac{1}{c}f_1(v,w), \mathbf{x}(v,w) = 0$ and thus Assumptions 6.2.1 is satisfied automatically. Moreover, since $\mathbf{x}(v,w) = 0$, there is no need to estimate $\mathbf{x}(v,w)$. It suffices to use one single system $\dot{\eta}_1 = M_1\eta_1 + N_1 u - \frac{M_1 N_1}{c}x_1$ to define the internal model which is essentially the same as what has been done in [56]. Clearly, Assumption 6.2.3 is not needed anymore and thus Assumption 6.2.2 with $i = 1$ and Assumption 6.2.4 become the assumptions to the IDSP of system (6.1). The IDSP of system (6.1) can be converted into a global robust stabilization problem of a class of feedforward systems with input unmodeled dynamics. On the other hand, when $q_d(v,w) \neq 0$, in general $\mathbf{x}(v,w) \neq 0$. To estimate $\mathbf{u}(v,w)$ and $\mathbf{x}_2(v,w), ..., \mathbf{x}_n(v,w)$, we define the internal model (6.8). If Assumptions 6.2.1-6.2.4 are satisfied, then the GRORP of system (6.1) can be solved by converting it into a global robust stabilization problem of a class of feedforward systems with both time varying static and dynamic uncertainty. Thus, the IDSP can be seen as a special case of the GRORP.

6.3 An Example

We study the GRORP of the following system

$$\begin{aligned} \dot{x}_2 &= 0.05w^2 x_1 - 0.05u^3 + 3wv_1v_2^2 \\ \dot{x}_1 &= u - wv_1(3v_1v_2 + 1) \\ \dot{v}_1 &= -v_2, \ \dot{v}_2 = v_1 \\ e &= x_1 - wv_1^3 \end{aligned} \tag{6.25}$$

where $0.8 \leq |w| \leq 1$ is the uncertain parameter and $v_1^2(t) + v_2^2(t) \leq 1$ for all $t \geq 0$.

System (6.25) is in the form of (6.1). Let us first verify that (6.25) satisfies Assumptions 6.2.1 to 6.2.4. Firstly, Assumption 6.2.1 is satisfied with

$$\mathbf{x}_1(v,w) = wv_1^3, \mathbf{x}_2(v,w) = wv_2^3, \mathbf{u}(v,w) = wv_1$$

which also implies that Assumption 6.2.2 is satisfied. Simple calculation shows that

$$\Phi_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{bmatrix}, \Psi_1 = [1\ 0], \ \Psi_2 = [1\ 0\ 0\ 0]. \tag{6.26}$$

which implies the satisfaction of Assumption 6.2.3. From the form of (6.25), Assumption 6.2.4 is also satisfied.

To design the internal model, let

$$M_1 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, M_2 = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, N_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, N_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solving the Sylvester equation (6.7) gives

$$T_1 = \begin{bmatrix} 0.4 & -0.2 \\ 0.5 & -0.5 \end{bmatrix}, T_2 = \begin{bmatrix} 0.2447 & -0.0612 & 0.0094 & -0.0024 \\ 0.3167 & -0.1056 & 0.0167 & -0.0056 \\ 0.4308 & -0.2154 & 0.0308 & -0.0154 \\ 0.5500 & -0.5500 & 0.0500 & -0.0500 \end{bmatrix}.$$

The internal model takes the following form:

$$\dot{\eta}_1 = M_1\eta_1 + N_1 u - M_1 N_1 e, \ \dot{\eta}_2 = M_2\eta_2 + N_2 x_2. \tag{6.27}$$

Then Theorem 6.2.1 can be applied to solve the GRORP for (6.25). First, using the coordinate and input transformations (6.9), (6.11) and (4.32) with $i = 1, 2$ and $D_1 = \Psi_1 T_1^{-1}, D_2 = -\Psi_2 T_2^{-1}, A_1 = M_1, A_2 = M_2$, the augmented system consisting of (6.25) and (6.27) is put into the following form (for convenience, we retain the original coordinates on the righthand side of the following equation)

$$\begin{aligned} \dot{z}_2 &= \theta_2\bar{u} - 0.01875[(\bar{u} + \Psi_1 T_1^{-1}\xi_1 + wv_1)^3 - w^3 v_1^3] + \theta_2 k_1 \bar{x}_1 \\ \dot{\xi}_2 &= M_2\xi_2 + 0.05N_2[w^2\bar{x}_1 - (\bar{u} + \Psi_1 T_1^{-1}\xi_1 + wv_1)^3 + w^3 v_1^3] \\ \dot{z}_1 &= \theta_1\bar{u} \\ \dot{\xi}_1 &= M_1\xi_1 \end{aligned} \tag{6.28}$$

where $\theta_1 = 1, \theta_2 = 0.01875w^2/k_1$. Clearly, k_1, k_2 are both positive in this case.

First, consider z_1, ξ_1 dynamics. Since $\bar{N}_1 = 0, \Delta_1 = \infty$, for arbitrarily positive λ_1, k_1, under the control $\bar{u} = -\sigma_1(k_1 z_1 + k_1\Psi_1 T_1^{-1} M_1^{-1}\xi_1 - u_1)$, $z_1, \bar{x}_1, \bar{u}$ satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restriction $\frac{\lambda_1}{3}$ on u_1 and gains $\frac{3}{k_1}s, \frac{6}{k_1}s, 6s$ respectively.

Then consider ξ_2 dynamics. We first calculate the gain from u_1 to ξ_2. Let P_2 be a positive definite and symmetric matrix such that $M_2^T P_2 + P_2 M_2 = -2I$, and $\tilde{u}_2 = 0.05[w^2\bar{x}_1 - (\bar{u} + \Psi_1 T_1^{-1}\xi_1 + wv_1)^3 + w^3 v_1^3]$. It can be verified that, ξ_2 subsystem satisfies a-LB with no restriction on $\xi_2(0)$, no restriction on $\tilde{u}_2$ and gain $\frac{\bar{\lambda}(P_2)}{\underline{\lambda}(P_2)}\|P_2N_2\|s$. Note that for $|\bar{u}| \leq 0.9$ and $\|\Psi_1 T_1^{-1}\xi_1\| \leq 0.1$,

$$|\tilde{u}_2| \leq 0.05|\bar{x}_1| + 0.35|\bar{u}| + 0.35\|\Psi_1 T_1^{-1}\|\|\xi_1\|.$$

Thus, ξ_2 satisfies LB with restriction and AB with no restriction on $\xi_2(0)$, both with restriction $\min\{\frac{\lambda_1}{3}, \frac{0.9}{6}\}$ on u_1 and gain $N_{\xi_2 u_1}s$, where $N_{\xi_2 u_1} = 2\max\{\frac{0.3}{k_1}, 2.1\}$.

Now let $\zeta_1 = (\xi_2, z_1, \xi_1)$. Then (6.28) can be written in the following form

$$\begin{aligned}\dot{z}_2 &= \theta_2 u_1 + \tilde{F}_2(\zeta_1, u_1, d)\\ \dot{\zeta}_1 &= G_1(\zeta_1, u_1, d)\end{aligned}$$

where $\tilde{F}_2(\zeta_1, u_1, d) = -0.01875[(\bar{u} + \Psi_1 T_1^{-1}\xi_1 + wv_1)^3 - w^3v_1^3] + \theta_2 h_1(\bar{x}_1, u_1)$ and G_1 is a suitably defined function.

Let $u_1 = -\sigma_2(k_2 z_2 - k_2\Psi_2 T_2^{-1}M_2^{-1}\xi_2 - k_2\frac{\theta_2}{\theta_1}z_1)$. Clearly, $\bar{\gamma}_{11}^{(2)}(s) \leq 2\max\{\|\Psi_2 T_2^{-1}M_2^{-1}\| N_{\xi_2 u_1}, \frac{\theta_2}{\theta_1}\frac{3}{k_1}\}k_2 s$. Since $h_1(\bar{x}_1, u_1)$ has no contribution to $\bar{\gamma}_{12}^{(2)}(s)$ when $\lambda_2 < \min\{\frac{\lambda_1}{12}, \frac{0.9}{6}\}$, we have $\bar{\gamma}_{12}^{(2)}(s) \leq \frac{2k_1}{w^2k_2}(3\min\{k_2 s, \lambda_2\} + 3\min\{k_2 s, \lambda_2\}^2 + \min\{k_2 s, \lambda_2\}^3)$. By solving $4\max\{\bar{\gamma}_{11}^{(2)}(s), \bar{\gamma}_{12}^{(2)}(s)\} < s$ for $s > 0$, we set $k_1 = 0.024, k_2 = 0.000129, \lambda_1 = 10$ and $\lambda_2 = 0.083$, and obtain

$$u = \Psi_1 T_1^{-1}(\eta_1 - N_1 e) - \sigma_1(0.024e + \sigma_2(0.000129(x_2 - \Psi_2 T_2^{-1}\eta_2))), \tag{6.29}$$

where σ_1, σ_2 are saturation functions with level 10 and 0.083 respectively.

As an illustration, Fig. 6.1, Fig. 6.2 and Fig. 6.3 show the simulation result of system (6.25) under the control (6.29) with initial state $(x_1(0), x_2(0), v(0), \eta_1(0), \eta_2(0)) = (2.1, -1.15, 0.5, -0.6, 0.5, 1, 2.5, -2, 2.5, 2.5)$ and $w = 0.9$.

6.4 Conclusion

In this chapter, we have presented the solvability conditions of the GRORP for nonlinear systems in strict feedforward form. The problem is approached in two steps. In the first step, the GRORP of the feedforward system is converted into a global robust stabilization problem of an augmented system. In the second step, the stabilization problem of the augmented system is further addressed by applying Theorem 4.3.1. Both of these two steps involve some nontrivial technical difficulties. In particular, for the success of the first step, a suitable internal model and appropriate transformations have to be found so that the augmented system takes some special form and is stabilizable.

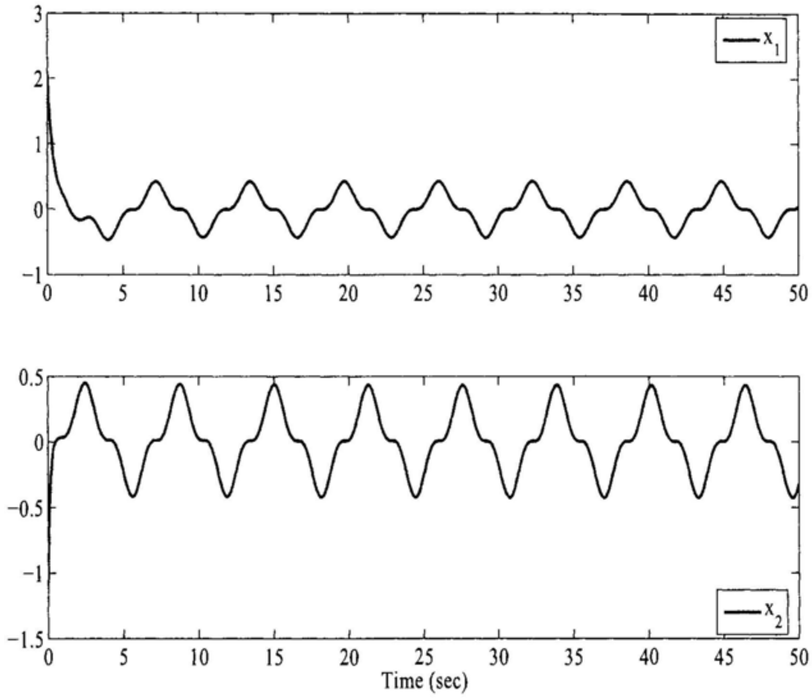

Figure 6.1: Profile of x

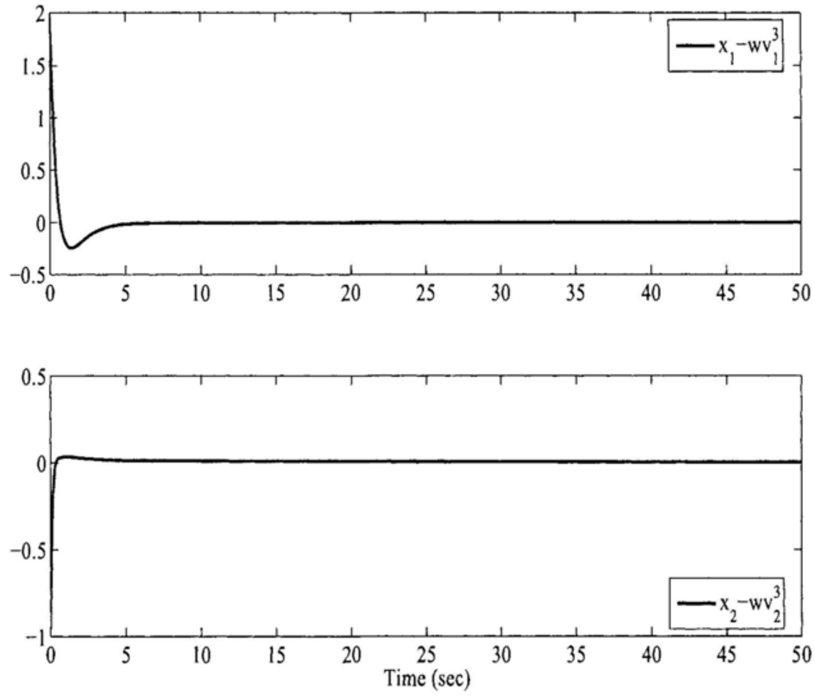

Figure 6.2: Profile of $x_1 - wv_1^3$ and $x_2 - wv_2^3$

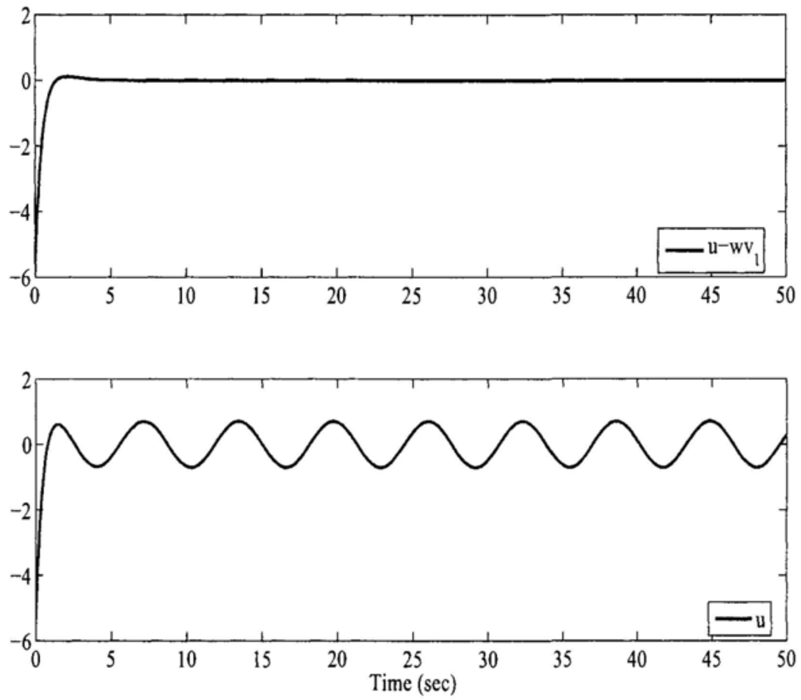

Figure 6.3: Profile of u and $u - wv_1$

Chapter 7

Applications of Global Robust Output Regulation

7.1 Trajectory Tracking of a Chain of Integrators

Teel [85] studied a "restricted" tracking problem for a chain of integrators described by the following equation

$$\begin{aligned} \dot{x}_i &= x_{i-1}, \ i = n, \cdots, 2 \\ \dot{x}_1 &= u \end{aligned} \tag{7.1}$$

The task therein is to let x_n track a reference trajectory $q_r(t)$. To accomplish the task, it is assumed that the trajectory $q_r(t)$ as well as the time derivative of $q_r(t)$ up to the order n, i.e., $\dot{q}_r(t), \ddot{q}_r(t), ..., q_r^{(n)}(t)$, had to be known, and moreover, $|q_r^{(n)}(t)| \leq \lambda - \varepsilon$ for some positive constants λ and ε. Under the above assumptions, Teel designed the following nested saturation control

$$u = q_r^{(n)}(t) - \sigma_n(h_n(\tilde{x}) + \sigma_{n-1}(h_{n-1}(\tilde{x}) + \cdots + \sigma_1(h_1(\tilde{x})))) \tag{7.2}$$

where $\tilde{x}$ is defined as $\tilde{x}_i = x_i - q_r^{(n-i)}(t)$, $i = 1, ..., n$, such that $|u(t)| \leq \lambda$ for all $t \geq 0$, and the closed-loop system satisfies the following property

$$\lim_{t \to \infty} [x_i(t) - q_r^{(n-i)}(t)] = 0, i = 1, ..., n \tag{7.3}$$

In this section, we first study the trajectory tracking problem for a chain of integrators with uncertain parameters by formulating it into a global robust output regulation problem with full order internal model. Then, we further show that, the trajectory tracking problem for system (7.1) can also be solved by converting it into a global output regulation problem with only one internal model.

7.1.1 Full Order Internal Model Design

In this section, we study the trajectory tracking problem for a chain of integrators with uncertain parameters described by the following equation

$$\begin{aligned} \dot{x}_i &= w_{i-1}x_{i-1}, \; i = n, \cdots, 2 \\ \dot{x}_1 &= u \end{aligned} \tag{7.4}$$

where $w = (w_1, ..., w_{n-1})$ is the uncertain parameter and for $i = 1, ..., n-1$, $|w_i|$ is bounded from below and from above by known positive real numbers, i.e., $0 < w_i^L \leq |w_i| \leq w_i^U$.

Same as [83], we aim to let x_n track a reference trajectory $q_r(t)$. Different from [85], instead of assuming the knowledge of $q_r(t)$ as well as the time derivative of $q_r(t)$ up to the order n, we assume that $q_r(t) = q_d(v(t), w)$ where $q_d(v(t), w)$ is a smooth function satisfying $q_d(0, w) = 0$ for all $w \in \mathbb{R}^{n-1}$, and $v \in \mathbb{R}^q$ is generated by the following exosystem

$$\dot{v} = Sv \tag{7.5}$$

where all eigenvalues of S are simple with zero real parts.

This trajectory tracking problem can be solved by formulating it as a global robust output regulation problem of system (7.4) with tracking error

$$e = x_1 - \frac{1}{w_1 \cdots w_{n-1}} q_r^{(n-1)}(t) = x_1 - \frac{1}{w_1 \cdots w_{n-1}} q_d^{(n-1)}(v(t), w) \tag{7.6}$$

Since system (7.4) is in the form of (6.1), we can solve by Theorem 6.2.1 the global robust output regulation problem of system (7.4) with the tracking error (7.6).

To solve the problem, we first determine the solution of the regulator equation. Due to the specific form of system (7.4), the solution of the regulator equation can be derived straightforward as follows:

$$\begin{aligned} \mathbf{x}_i(v, w) &= \frac{1}{w_i \cdots w_{n-1}} q_d^{(n-i)}(v(t), w), i = 1, ..., n-1, \\ \mathbf{x}_n(v, w) &= q_d(v(t), w), \; \mathbf{u}(v, w) = \frac{1}{w_1 \cdots w_{n-1}} q_d^{(n)}(v(t), w) \end{aligned} \tag{7.7}$$

Assumption 7.1.1 Assume there exists a sufficiently smooth function $\tau : \mathbb{R}^q \times \mathbb{R}^{n-1} \to \mathbb{R}^r$ vanishing at $(0, 0)$, such that

$$\dot{\tau}(v, w) = \Phi\tau(v, w), \; \mathbf{u}(v, w) = \Psi\tau(v, w) \tag{7.8}$$

where the pair (Ψ, Φ) is observable, and Φ is nonsingular and all its eigenvalues are simple with zero real parts.

Given a pair of controllable pair (M, N) with $M \in \mathbb{R}^r \times \mathbb{R}^r$ Hurwitz and $N \in \mathbb{R}^r$, there exists a unique and nonsingular solution T to the following Sylvester equation

$$T\Phi - MT = N\Psi \tag{7.9}$$

because the spectra of M and Φ are disjoint and the pair (M, N) is controllable and the pair (Ψ, Φ) is observable.

Under the above assumption, the global robust output regulation problem of system (7.4) with the tracking error (7.6), and in turn, the trajectory tracking problem of system (7.4) with the reference trajectory $q_r(t) = q_d(v(t), w)$ can be solved.

Proposition 7.1.1 Consider system (7.4). Under Assumption 7.1.1, there exist sufficiently small $\lambda_i, |k_i|, i = 1, ..., n$, such that, under the control

$$\begin{aligned} u &= \Psi T^{-1}(\eta_1 - Ne) - \sigma_1(k_1 e + ... + \sigma_n(k_n x_n - k_n \Psi T^{-1}\eta_n)), \\ \dot{\eta}_1 &= M\eta_1 + Nu - MNe, \ \dot{\eta}_i = M\eta_i + Nx_i, \ i = 2, ..., n, \end{aligned} \tag{7.10}$$

the trajectory tracking problem of system (7.4) with the reference trajectory $q_r(t) = q_d(v(t), w)$ can be solved, i.e., the closed-loop system satisfies the property

$$\lim_{t\to\infty} [x_i(t) - \kappa_i q_r^{(n-i)}(t)] = 0, i = 1, ..., n \tag{7.11}$$

where $\kappa_i = \frac{1}{w_i \cdots w_{n-1}}, i = 1, ..., n-1$ and $\kappa_n = 1$.

Proof: Note from (7.7) and $0 < w_i^L \leq |w_i| \leq w_i^U, i = 1, ..., n$, that Assumptions 6.2.1 and 6.2.4 are satisfied, thus we only need to check Assumptions 6.2.2 and 6.2.3. From (7.7) and Assumption 7.1.1, Assumption 6.2.2 is satisfied with

$$\begin{aligned} \tau_1(v, w) &= \tau(v, w), \ \Phi_1 = \Phi, \Psi_1 = \Psi \\ \tau_i(v, w) &= w_1 \cdots w_{i-1} \cdot \tau(v, w), \ \Phi_i = \Phi, \ \Psi_i = \Psi\Phi^{-i}, i = 2, ..., n \end{aligned} \tag{7.12}$$

Note that Φ is nonsingular by assumption, thus Assumption 6.2.3 is also satisfied. As a result, the global robust output regulation problem of system (7.4) with the tracking error (7.6) can be solved by Theorem 6.2.1 and thus a controller in the form of (6.13) can be obtained.

Furthermore, we choose for convenience $M_i = M, N_i = N, i = 1, ..., n$. Then the Sylvester equation (6.7) with $i = 2, ..., n$, becomes

$$T_i\Phi - MT_i = N\Psi\Phi^{-i} \Leftrightarrow T_i\Phi^i\Phi - MT_i\Phi^i = N\Psi$$

From (7.9) and the above equation, we have $T_i = T\Phi^{-i}, i = 2, ..., n$. Let $T_1 = T$. Then substituting the above defined M_i, N_i, Ψ_i, T_i, $i = 1, ..., n$, into (6.13) yields the controller (7.10).

Finally, note from (7.11) and (6.9) that $\lim_{t\to\infty}[x_1(t) - \kappa_1 q_r^{(n-1)}(t)] = \lim_{t\to\infty} e(t) = 0$ and for $i = 2, ..., n$,

$$\begin{aligned}
\lim_{t\to\infty}[x_i(t) - \kappa_i q_r^{(n-i)}(t)] &= \lim_{t\to\infty}[x_i(t) - \kappa_i q_d^{(n-i)}(v(t), w)] \\
&= \lim_{t\to\infty}[x_i(t) - \mathbf{x}_i(v, w)] \\
&= \lim_{t\to\infty}[\bar{x}_i(t) + \Psi T^{-1}\bar{\eta}_i(t)] = 0
\end{aligned}$$

Thus the trajectory tracking problem of system (7.4) with the reference trajectory $q_r(t) = q_d(v(t), w)$ is solved.

7.1.2 Reduced Order Internal Model Design

In this section, we further show that the trajectory tracking problem for system (7.1), i.e., a chain of integrators without uncertain parameters can be solved by converting it into a global output regulation problem with only one internal model.

Same as the previous section, we aim to let x_n track a reference trajectory $q_r(t) = q_d(v(t), w)$. Similarly, this trajectory tracking problem can be solved by formulating it as a global output regulation problem of system (7.1) with tracking error

$$e = x_1 - q_r^{(n-1)}(t) = x_1 - q_d^{(n-1)}(v(t), w) \tag{7.13}$$

Under Assumption 7.1.1, the global output regulation problem of system (7.1) with the tracking error (7.13), and in turn, the trajectory tracking problem of system (7.1) with the reference trajectory $q_r(t) = q_d(v(t), w)$ can be solved.

Proposition 7.1.2 Consider system (7.1). Under Assumption 7.1.1, there exist sufficiently small $\lambda_i, |k_i|, i = 1, ..., n$, such that, under the control

$$\begin{aligned}
u &= \Psi T^{-1}(\eta - Ne) - \sigma_1(k_1 e + ... + \sigma_n(k_n x_n - \Psi\Phi^{-i}T^{-1}\eta)) \\
\dot{\eta} &= M\eta + Nu - MNe
\end{aligned} \tag{7.14}$$

the trajectory tracking problem of system (7.1) with the reference trajectory $q_r(t) = q_d(v(t), w)$ can be solved, i.e., the closed-loop system satisfies the property

$$\lim_{t\to\infty}[x_i(t) - q_r^{(n-i)}(t)] = 0, i = 1, ..., n \tag{7.15}$$

Proof: Due to the specific form of system (7.1), the solution of the regulator equation can be derived straightforward as follows:

$$\mathbf{x}_i(v,w) = q_d^{(n-i)}(v(t),w), i = 1, ..., n, \ \mathbf{u}(v,w) = q_d^{(n)}(v(t),w) \tag{7.16}$$

From the above equation and Assumption 7.1.1,

$$\mathbf{x}_i(v,w) = \Psi\Phi^{-i}\tau(v,w), i = 1, ..., n \tag{7.17}$$

which shows that all the solutions of the regulator equation can be generated by the linear autonomous system (7.8). This observation implies that, the output regulation problem for a chain of integrators (7.1) can be solved by using only the first internal model in (6.8), i.e.,

$$\dot{\eta} = M\eta + Nu - MNe \tag{7.18}$$

and the other $n-1$ internal models in (6.8) with $i = 2, \cdots, n$, turn out redundant and thus can be removed. As a result, the global output regulation problem for system (7.1) can be converted into a global robust stabilization problem for a class of feedforward systems subject to an input unmodeled dynamics.

First, performing the following coordinate and input transformation

$$\bar{x}_1 = x_1 - \mathbf{x}_1(v,w) = e, \ \bar{x}_i = x_i - \Psi\Phi^{-i}T^{-1}\eta, \ \bar{\eta} = \eta - T\tau, \ \hat{u} = u - \Psi T^{-1}\eta \tag{7.19}$$

on (7.1) and (7.18) yields,

$$\begin{aligned}
\dot{\bar{x}}_i &= \bar{x}_{i-1} - \Psi\Phi^{-i}T^{-1}(N\hat{u} - MN\bar{x}_1), \ i = n, \cdots, 3 \\
\dot{\bar{x}}_2 &= \bar{x}_1 - \Psi\Phi^{-2}T^{-1}((M + N\Psi T^{-1})\bar{\eta} + N\hat{u} - MN\bar{x}_1) \\
\dot{\bar{x}}_1 &= \Psi T^{-1}\bar{\eta} + \hat{u}, \\
\dot{\bar{\eta}} &= (M + N\Psi T^{-1})\bar{\eta} + N\hat{u} - MN\bar{x}_1
\end{aligned} \tag{7.20}$$

Further performing the coordinate and input transformation

$$\xi = \bar{\eta} - N\bar{x}_1, \ \bar{u} = \hat{u} + \Psi T^{-1}N\bar{x}_1 \tag{7.21}$$

on (7.20) gives

$$\begin{aligned}
\dot{\bar{x}}_i &= \bar{x}_{i-1} + \Psi\Phi^{-i+1}T^{-1}N\bar{x}_1 - \Psi\Phi^{-i}T^{-1}N\bar{u}, \ i = n, \cdots, 3 \\
\dot{\bar{x}}_2 &= \bar{x}_1 - \Psi\Phi^{-2}T^{-1}((M + N\Psi T^{-1})\xi + N\bar{u}), \\
\dot{\bar{x}}_1 &= \Psi T^{-1}\xi + \bar{u} \\
\dot{\xi} &= M\xi
\end{aligned} \tag{7.22}$$

Therefore, the global output regulation of system (7.1) has been converted into a global robust stabilization of system (7.22) with ξ dynamics as the input unmodeled dynamics.

Since system (7.22) is in the form of (4.1) and satisfies Assumptions 4.3.1 to 4.3.2, the global robust stabilization problem of system (7.22) can be solved by Theorem 4.3.1. Thus we can design a stabilizing controller in the form of (4.28) for system (7.22). Then, noting (7.19), (7.21) and the internal model (7.18) yields the controller (7.14).

Finally, note from (7.15) and (7.19) that that $\lim_{t\to\infty}[x_1(t)-q_r^{(n-1)}(t)] = \lim_{t\to\infty} e(t) = 0$ and for $i = 2, ..., n$,

$$\begin{aligned}\lim_{t\to\infty}[x_i(t) - q_r^{(n-i)}(t)] &= \lim_{t\to\infty}[x_i(t) - q_d^{(n-i)}(v(t), w)] \\ &= \lim_{t\to\infty}[x_i(t) - \mathbf{x}_i(v, w)] \\ &= \lim_{t\to\infty}[\bar{x}_i(t) + \Psi\Phi^{-i}T^{-1}\bar{\eta}_i(t)] = 0\end{aligned}$$

Thus the trajectory tracking problem of system (7.1) with the reference trajectory $q_r(t) = q_d(v(t), w)$ is solved.

Remark 7.1.1 Under the assumption that $q_r(t) = q_d(v(t), w)$, we solve the trajectory tracking problem [83] for a chain of integrators by formulating it into a global output regulation problem. In contrast with [83], we do not need to know the reference trajectory $q_r(t)$ exactly and what we need to know is Assumption 7.1.1.

Remark 7.1.2 Given any $\lambda > 0$, the control (7.2) designed in [83] satisfies $|u(t)| \leq \lambda$ for all $t \geq 0$. However, it is worthy to note that this feature is achieved by assuming $|q_r^{(n)}(t)| \leq \lambda - \varepsilon$ for some positive constant ε. This is the reason why the tracking problem studied in [83] is called a "restricted" tracking problem. In the following, we further show that, if the reference trajectory $q_r(t)$ is known exactly and $|q_r^{(n)}(t)| \leq \lambda-\varepsilon$ for $\lambda > 0, \varepsilon > 0$, the control (7.14) will satisfy $|u(t)| \leq \lambda$ for all $t \geq 0$ as well. Noting (7.19), (7.21) and $q_r^{(n)}(t) = q_d^{(n)}(v(t), w)$ yields

$$\Psi T^{-1}(\eta - Ne) = \Psi T^{-1}(\bar{\eta} + T\tau - N\bar{x}_1) = \Psi T^{-1}\xi + \mathbf{u}(v, w) = \Psi T^{-1}\xi + q_r^{(n)}(t)$$

Then from the last equation of system (7.22) and noting the fact that the Hurwitz matrix M can always be chosen such that $\|\xi(t)\| = \|\exp(Mt)\xi(0)\| \leq \|\xi(0)\|$ for any $\xi(0)$ and $t \geq 0$, we have

$$|\Psi T^{-1}(\eta - Ne)| \leq \|\Psi T^{-1}\|\|\xi(0)\| + |q_r^{(n)}(t)| \tag{7.23}$$

Note that

$$\xi(0) = \eta(0) - Nx_1(0) + N\mathbf{x}_1(v(0), w) - T\tau(v(0), w)$$

thus if $\mathbf{x}_1(v(0), w)$ and $T\tau(v(0), w)$ are known, i.e., the reference trajectory $q_r(t)$ is known exactly, then we can choose $\eta(0) = Nx_1(0) - N\mathbf{x}_1(v(0), w) + T\tau(v(0), w)$ so that $\xi(0) = 0$ and thus (7.23) becomes

$$|\Psi T^{-1}(\eta - Ne)| \leq |q_r^{(n)}(t)|$$

From the above equation and the expression of (7.14), we obtain

$$|u(t)| \leq |q_r^{(n)}(t)| + \lambda_1$$

where λ_1 is the saturation level of σ_1. Finally, note that $|q_r^{(n)}(t)| \leq \lambda - \varepsilon$ and λ_1 can be arbitrarily small, thus choosing $\lambda_1 \leq \varepsilon$ yields $|u(t)| \leq \lambda$ for all $t \geq 0$.

7.1.3 An Example

For illustration, consider the following system

$$\begin{aligned} \dot{x}_2 &= w_1 x_1 \\ \dot{x}_1 &= u \end{aligned} \tag{7.24}$$

We aim to let $x_2(t)$ track a reference trajectory $q_r(t) = 2v_1v_2$ where v_1, v_2 are generated by

$$\dot{v}_1 = 0.5v_2, \ \dot{v}_2 = -0.5v_1$$

Full Order Internal Model Design

In this case, assume w_1 is unknown and $1 \leq w_1 \leq 2$.

Clearly, $\mathbf{x}_1(v, w) = (v_2^2 - v_1^2)/w_1$, $\mathbf{x}_2(v, w) = 2v_1v_2$ and $\mathbf{u}(v, w) = -2v_1v_2/w_1$. It can be verified that Assumption 7.1.1 is satisfied with

$$\Psi = [1 \ 0], \ \Phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, Proposition 7.1.1 can be applied to solve the trajectory tracking problem for system (7.24). To design the internal model, let

$$M = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving the Sylvester equation (7.9) gives

$$T = \begin{bmatrix} 0.4 & -0.2 \\ 0.5 & -0.5 \end{bmatrix} \tag{7.25}$$

The internal model takes the following form

$$\dot{\eta}_1 = M\eta_1 + Nu - MNe, \ \dot{\eta}_2 = M\eta_2 + Nx_2 \tag{7.26}$$

Using the coordinate and input transformation (6.9) and (6.11), the augmented system consisting of (7.24) and (7.26) is put into the following form:

$$\begin{aligned} \dot{\bar{x}}_2 &= -\Psi T^{-1}\xi_2 + w_1\bar{x}_1 \\ \dot{\xi}_2 &= M\xi_2 + Nw_1\bar{x}_1 \\ \dot{\bar{x}}_1 &= \Psi T^{-1}\xi_1 + \bar{u} \\ \dot{\xi}_1 &= M\xi_1 \end{aligned} \tag{7.27}$$

Performing the coordinate transformation

$$z_1 = \bar{x}_1 - \Psi T^{-1}M^{-1}\xi_1, \ z_2 = \bar{x}_2 + \Psi T^{-1}M^{-1}\xi_2 + \frac{\theta_2}{\theta_1}z_1$$

on (7.27) gives (for convenience, we left the righthand side in the original coordinates)

$$\begin{aligned} \dot{z}_2 &= \theta_2\bar{u} + \theta_2 k_1\bar{x}_1 \\ \dot{\xi}_2 &= M\xi_2 + w_1N\bar{x}_1, \\ \dot{z}_1 &= \theta_1\bar{u} \\ \dot{\xi}_1 &= M\xi_1 \end{aligned} \tag{7.28}$$

where $\theta_1 = \mu_1 = 1, \theta_2 = \mu_2/k_1 = 0.5w_1/k_1$. Note that k_i has the same sign with θ_i, thus k_1 and k_2 are both positive in this case.

Since $\bar{N} = 0, \Delta_1 = \infty$, for arbitrarily positive λ_1, k_1, under the control $\bar{u} = -\sigma_1(k_1z_1 + k_1\Psi T^{-1}M^{-1}\xi_1 - u_1)$, $z_1, \bar{x}_1, u$ satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi_1(0))$, both with restriction $\lambda_1/3$ on u_1 and gains $2s/k_1, 4s/k_1, 4s$, respectively.

Now consider z_2, ξ_2 dynamics. We first calculate the gain from u_1 to ξ_2. Let P be a positive definite and symmetric matrix such that $M^TP + PM = -2I$. It can be verified that, ξ_2 satisfies a-LB with no restriction on $\xi_2(0)$, no restriction on $\bar{x}_1$ and gain $2\frac{\bar{\lambda}(P)}{\underline{\lambda}(P)}\|PN\|s$. Then, ξ_2 satisfies LB with restriction and AB with no restriction on $(\xi_2(0), z_1(0), \xi_1(0))$, both with restriction $\lambda_1/3$ on u_1 and gain $N_{\xi_2u_1}s$, where $N_{\xi_2u_1} = \frac{\bar{\lambda}(P)}{\underline{\lambda}(P)}\|PN\|\frac{8}{k_1}$.

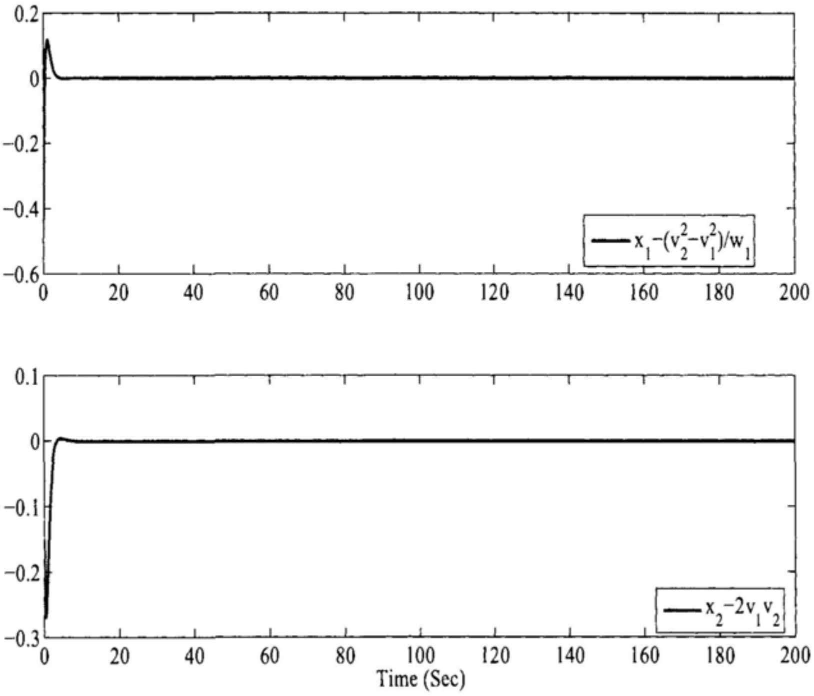

Figure 7.1: Profile of $x_1 - (v_2^2 - v_1^2)/w_1$ and $x_2 - 2v_1v_2$

Under the control $\bar{u} = -\sigma_1(k_1 z_1 + k_1 \Psi T^{-1} M^{-1} \xi_1 - u_1)$, system (7.28) becomes

$$\begin{aligned} \dot{z}_2 &= \theta_2 u_1 + \tilde{F}_2(\zeta_1, u_1, d) \\ \dot{\zeta}_1 &= G(\zeta_1, u_1, d) \end{aligned} \tag{7.29}$$

where $\zeta_1 = (\xi_2, z_1, \xi_1)$, $\tilde{F}_2(\zeta_1, \bar{u}, d) = \theta_2 h_1(\bar{x}_1, u_1)$ and G is a suitably defined function. Since $h_1(\bar{x}_1, u_1)$ has no contribution to $\bar{\gamma}_{12}(s)$ with $\lambda_2 \le \frac{\lambda_1}{8}$, $\bar{\gamma}_{12}^{(2)}(s) = 0$. Note that $\bar{\gamma}_{21}^{(2)}(s) = \bar{\gamma}_{22}^{(2)}(s) = 4s$ in this case, then the small gain condition reduces to

$$\bar{\gamma}_{11}^{(2)}(s) \le 2\max\{\|\Psi T^{-1} M^{-1}\| N_{\xi_2 u_1}, \tfrac{\theta_2}{\theta_1}\tfrac{2}{k_1}\} \min\{k_2 s, \lambda_2\} < \tfrac{1}{4}s$$

Solving the small gain condition gives $k_1 = 1, k_2 = 0.0021, \lambda_1 = 1$ and $\lambda_2 = 0.125$.

Thus, the designed controller takes the following form

$$u = \Psi T^{-1}(\eta_1 - Ne) - \sigma_1(e + \sigma_2(0.0021 x_2 - 0.0021 \Psi T^{-1} \eta_2)) \tag{7.30}$$

where σ_1, σ_2 are saturation functions with level 1 and 0.125 respectively.

Fig. 7.1 and Fig. 7.2 show the simulation result of system (7.24) under the control (7.30) with $w_1 = 1.5$ and initial condition $(x_1(0), x_2(0), v(0), \eta_1(0), \eta_2(0)) = (0.2, -0.15, (0,1), (0,0), (0,0))$.

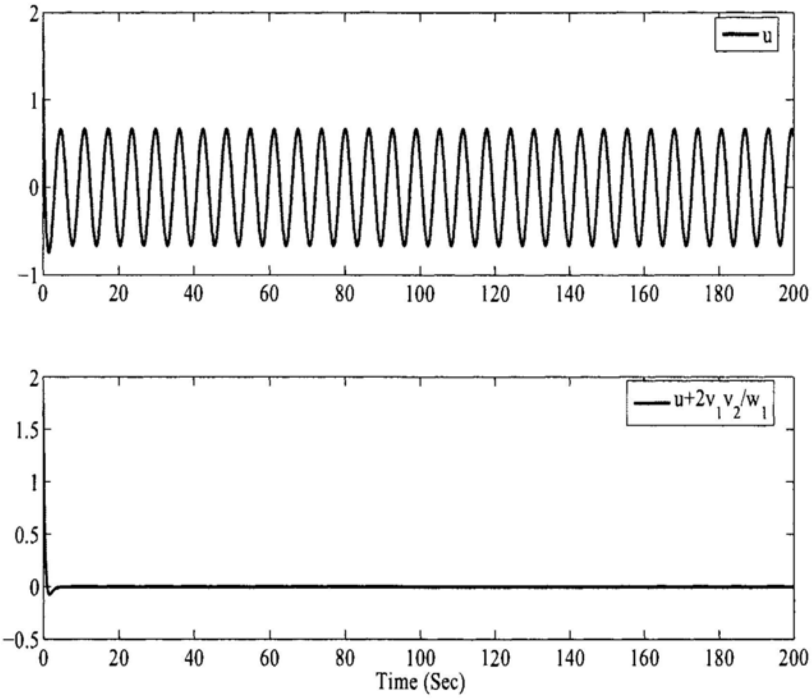

Figure 7.2: Profile of u and $u + 2v_1v_2/w_1$

Reduced Order Internal Model Design

In this case, we assume $w_1 = 1$ in (7.24).

For convenience, we choose M, Ψ, Φ, T same as the previous section. Performing the coordinate transformation

$$z_1 = \bar{x}_1 - \Psi T^{-1} M^{-1}\xi, \;\; z_2 = \bar{x}_2 + \frac{\theta_2}{\theta_1} z_1,$$

on (7.22) with $n = 2$ gives (for convenience, we retain the righthand side in the original coordinates)

$$\begin{aligned} \dot{z}_2 &= \theta_2 \bar{u} + \theta_2 k_1 \bar{x}_1 - \Psi\Phi^{-2}T^{-1}((M + N\Psi T^{-1})\xi + N\bar{u}) \\ \dot{z}_1 &= \theta_1 \bar{u} \\ \dot{\xi} &= M\xi \end{aligned} \tag{7.31}$$

where $\theta_1 = \mu_1, \theta_2 = \mu_2/k_1$. Since $\mu_1 = \mu_2 = 1$ and k_i has the same sign with θ_i, k_1 and k_2 are both positive in this case.

Since ξ dynamics is globally exponentially stable, for arbitrarily positive λ_1, k_1, under the control $\bar{u} = -\sigma_1(k_1 z_1 + k_1 \Psi T^{-1} M^{-1} \xi - u_1)$, $z_1, \bar{x}_1, u$ satisfy LB with restriction and AB with no restriction on $(z_1(0), \xi(0))$, both with restriction $\lambda_1/3$ on u_1 and gains $\frac{2}{k_1}s$, $\frac{4}{k_1}s$, $4s$ respectively.

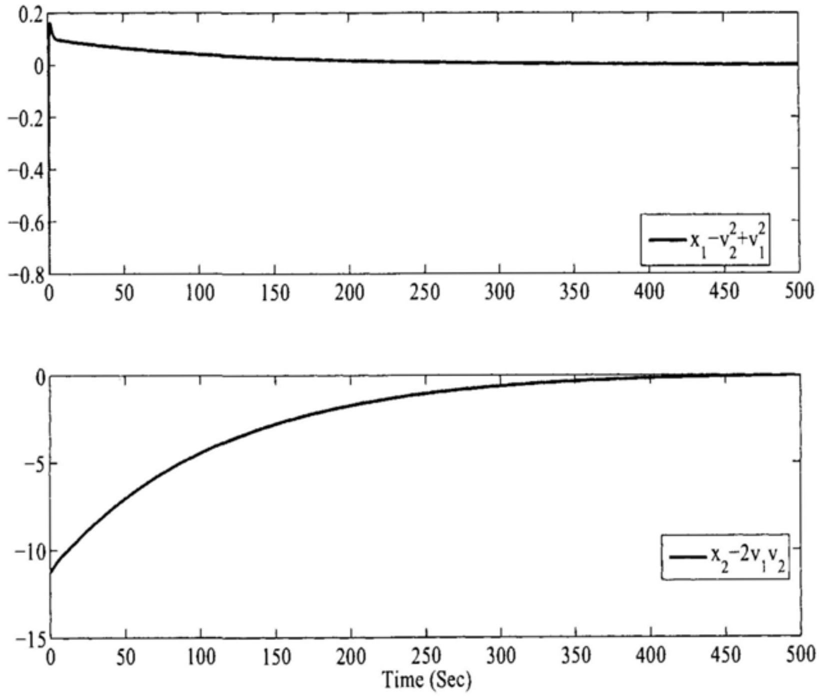

Figure 7.3: Profile of $x_1 - v_2^2 + v_1^2$ and $x_2 - 2v_1v_2$

Under the control $\bar{u} = -\sigma_1(k_1 z_1 + k_1 \Psi T^{-1} M^{-1}\xi - u_1)$, system (7.31) becomes

$$\begin{aligned} \dot{z}_2 &= \theta_2 u_1 + \tilde{F}_2(\zeta_1, u_1, d) \\ \dot{\zeta}_1 &= G(\zeta_1, u_1, d) \end{aligned} \tag{7.32}$$

where $\zeta_1 = (z_1, \xi)$, $\tilde{F}_2(\zeta_1, \bar{u}, d) = -\Psi\Phi^{-2}T^{-1}((M + N\Psi T^{-1})\xi + N\bar{u}) + \theta_2 h_1(\bar{x}_1, u_1)$ and G is a suitably defined function. Note that $h_1(\bar{x}_1, u_1)$ has no contribution on $\bar{\gamma}_{12}(s)$ with $\lambda_2 \le \frac{\lambda_1}{8}$, then $\bar{\gamma}_{12}^{(2)}(s) = \frac{24\min\{k_2 s, \lambda_2\}}{\bar{k}_2}$. In this case, the small gain condition becomes

$$\bar{\gamma}_{11}^{(2)}(s) \le \frac{\theta_2}{\theta_1}\frac{2}{k_1}\min\{k_2 s, \lambda_2\} < \frac{1}{4}s, \ \bar{\gamma}_{12}^{(2)}(s) = \frac{24\min\{k_2 s, \lambda_2\}}{\bar{k}_2} < \frac{1}{4}s$$

Solving the small gain condition gives $k_1 = 0.0104, \lambda_1 = 1, k_2 = 1.3563 \times 10^{-5}$ and $\lambda_2 = 0.125$.

Thus, the designed controller takes the following form

$$u = \Psi T^{-1}(\eta - Ne) - \sigma_1(0.0104e + \sigma_2(1.3563 \times 10^{-5}(x_2 - \Psi\Phi^{-2}T^{-1}\eta))) \tag{7.33}$$

Fig. 7.3 and Fig. 7.4 show the simulation result of system (7.24) under the control (7.33) with initial condition $(x_1(0), x_2(0), v(0), \eta_1(0)) = (0.2, -11, (0, 1), (0, 0))$.

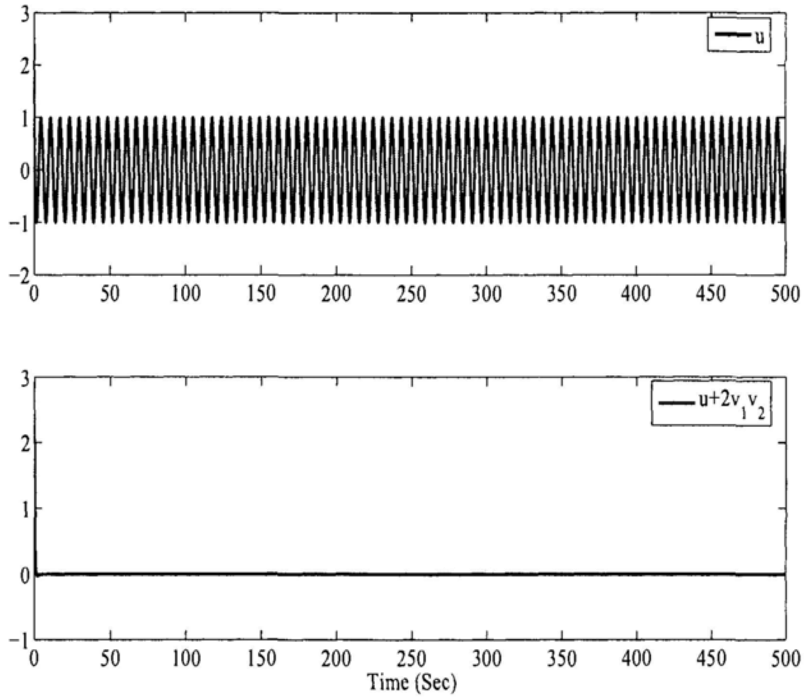

Figure 7.4: Profile of u and $u + 2v_1v_2$

7.2 Trajectory Tracking of the VTOL Aircraft

In this section, we study a robust trajectory tracking problem of the Vertical Take-Off and Landing (VTOL) aircraft. As in [18, 49, 55, 57], a simplified model governing the dynamics of the aircraft in the vertical lateral plane is described by the following equation

$$\begin{aligned} M\ddot{y} &= \cos(\theta)T + 2\sin(\theta)\sin(\alpha)F - gM \\ M\ddot{x} &= -\sin(\theta)T + 2\cos(\theta)\sin(\alpha)F \\ J\ddot{\theta} &= 2l\cos(\alpha)F \end{aligned} \tag{7.34}$$

where, as shown in Fig. 7.5, x, y denote the horizontal and vertical position of the center of mass C, θ the roll angle of the aircraft with respect to the horizon, M the mass of the aircraft, J the moment of inertia about the center of mass C, l the distance between the wingtips, and g the gravitational acceleration. The control inputs are the thrust T directed out the bottom of the aircraft and the rolling moment produced by a couple of equal forces F acting at the wingtips respectively. The direction of F is not perpendicular to the horizontal body axis, but tilted by some fixed angle α.

Similar to [55, 85], we also take into account some parameter uncertainties in (7.34). In particular, we assume $|\alpha| < \frac{\pi}{2}$ and J range in some compact set with a known bound.

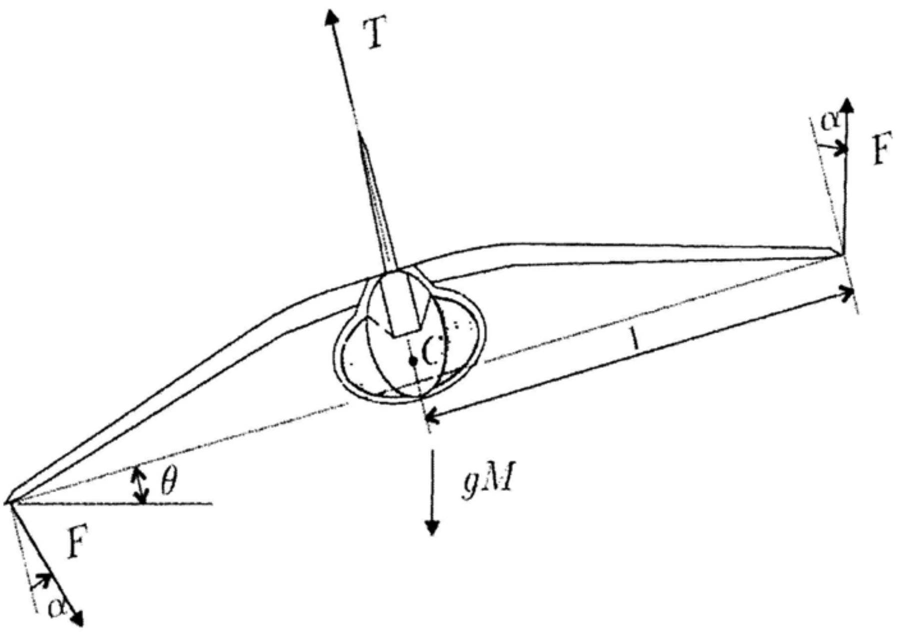

Figure 7.5: The vertical take-off and landing aircraft [56]

For convenience, let $w = (J, \alpha)$ represent the parameter uncertainty in the following. Like [55, 85], we choose the control inputs T, F as follows:

$$T = \frac{M}{\cos(a\text{sgn}(\frac{\theta}{a})\min\{|\frac{\theta}{a}|, 1\})}(g + u_2), F = \frac{J_0}{2l\cos(\alpha_0)}u_1 \tag{7.35}$$

where u_1, u_2 are new control inputs, and a is a positive constant satisfying $a < \pi/2$.

Robust trajectory tracking problem of the VTOL aircraft: Assume $e = \dot{y} - \dot{q}_d(v(t))$ and $y, x, \dot{x}, \theta, \dot{\theta}$ are available for feedback. We aim to design appropriate controls u_1, u_2 such that, the vertical position y of the aircraft asymptotically tracks an unknown reference signal $q_d(v(t))$, while the horizontal position x and roll angle θ asymptotically converge to the origin, i.e.,

$$\lim_{t\to\infty}(y(t) - q_d(v(t)), x(t), \theta(t)) = 0 \tag{7.36}$$

where $q_d(v)$ is a polynomial of v and $q_d(0) = 0$, and $v \in \mathbb{R}^q$ is generated by the following known exosystem

$$\dot{v} = Sv \tag{7.37}$$

where all eigenvalues of S are simple with zero real parts and the initial condition $v(0)$ is unknown but ranging in some compact set with a known bound.

Note that the exosystem (7.37) does not contain any unknown parameters. This is the major difference between the problem studied in this section and the one studied in [55] where the exosystem can be unknown in the sense that it is perturbed by some unknown constant parameter ranging in some compact set with a known bound. In [55], Marconi, Isidori and Serrani studied first a robust (non-adaptive) and in turn an adaptive, trajectory tracking problem for the VTOL aircraft. To address the problem, they proposed a high gain and low gain combined design. In particular, a high gain design is provided for both the robust output regulation problem of the vertical dynamics with known exosystem and the adaptive output regulation problem of the vertical dynamics with unknown exosystem, and a low gain design is provided for the stabilization of the horizontal-angular dynamics. In contrast, we study a robust trajectory tracking problem for the VTOL aircraft. In particular, by employing the output regulation result obtained in Chapter 6, we provide a low gain design for a robust output regulation problem of the vertical dynamics with known exosystem. As for the stabilization of the horizontal-angular dynamics, we resort directly to the low gain design in [55]. As a result, we obtain a complete low gain design for the robust trajectory tracking problem of the VTOL aircraft.

7.2.1 Main Result

Since $q_d(v), \ddot{q}_d(v)$ are polynomials of v and vanishing at $v = 0$, then there exist sufficiently smooth functions $\tau_i : \mathbb{R}^q \to \mathbb{R}^{r_i}$ vanishing at the origin, such that

$$\dot{\tau}_i(v) = \Phi_i \tau_i(v), \;\; \pi_i(v) = \Psi_i \tau_i(v), \;\; i = 1, 2 \tag{7.38}$$

where the pair (Ψ_i, Φ_i) is observable and all the eigenvalues of Φ_i are simple with zero real parts, and $\pi_1(v) = \ddot{q}_d(v), \pi_2(v) = q_d(v)$.

For $i = 1, 2$, given a pair of controllable matrices (M_i, N_i) with $M_i \in \mathbb{R}^{r_i \times r_i}$ Hurwitz and $N_i \in \mathbb{R}^{r_i}$, there exits a nonsingular matrix T_i such that

$$T_i \Phi_i - M_i T_i = N_i \Psi_i. \tag{7.39}$$

Then, we can define the following system

$$\begin{aligned} \dot{\eta}_1 &= M_1 \eta_1 + N_1 u_2 - M_1 N_1 e \\ \dot{\eta}_2 &= M_2 \eta_2 + N_2 y \end{aligned} \tag{7.40}$$

as the internal model of (7.34).

The main result of this section is given as follows.

Proposition 7.2.1 Consider system (7.34). Assume $q_d(v)$ is such that the matrix Φ_2 is nonsingular and $|\ddot{q}_d(v)| < g$. Then there exist positive real numbers $\lambda_i, k_i, i = 1, ..., 6$, such that, under the control

$$\begin{aligned} u_1 &= -\lambda_1\sigma(\tfrac{k_1}{\lambda_1}(\dot{\theta} + \lambda_2\sigma(\tfrac{k_2}{\lambda_2}(\theta - \lambda_3\sigma(\tfrac{k_3}{\lambda_3}(\dot{x} + \lambda_4\sigma(\tfrac{k_4}{\lambda_4}x)))))))\\ u_2 &= \Psi_1 T_1^{-1}(\eta_1 - N_1 e) - \sigma_5(k_5 e + \sigma_6(k_6(y - \Psi_2 T_2^{-1}\eta_2)))\\ \dot{\eta}_1 &= M_1\eta_1 + N_1 u_2 - M_1 N_1 e, \dot{\eta}_2 = M_2\eta_2 + N_2 y \end{aligned} \tag{7.41}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is any differentiable function satisfying $\sigma(0) = 0$, $s\sigma(s) > 0$ for all $s \neq 0$, $|s| < |\sigma(s)| < 1$ for all $0 < |s| < 1$, $\sigma(s) = \text{sgn}(s)$ for $|s| \geq 1$, $\sigma'(s) \leq 2$ for all s, and σ_5 and σ_6 are saturation functions with level λ_5 and λ_6, respectively, the robust trajectory tracking problem of the VTOL aircraft is solved, i.e. (7.36) is achieved.

Remark 7.2.1 u_1 is the stabilizing control for the horizontal-angular dynamics and it is constructed by applying the low gain design provided in [55]. u_2 is the dynamic state feedback controller for the robust output regulation problem of the vertical dynamics with known exosystem and it is constructed by applying the low gain design provided in Chapter 4. As a result, our design is a complete low gain design and different from the high gain and low gain combined design provided in [55]. Another difference between our design and the one in [55] is that the measurement of $y - q_d(v)$ is not needed for feedback because of the particular form of the internal model (7.40). In addition, the mass M of the VTOL is assumed to be unknown in [55], which can be viewed as a benefit of the high gain technique.

7.2.2 Design Procedure

We will first convert the robust trajectory tracking problem of the VTOL aircraft into a robust stabilization problem of an augmented system.

Problem Conversion

Like [55, 85], define the following functions

$$\phi_a(s) = \frac{\sin(s)}{\cos(a\text{sgn}(\frac{s}{a})\min\{|\frac{s}{a}|, 1\})}, \psi_a(s) = \frac{\cos(s)}{\cos(a\text{sgn}(\frac{s}{a})\min\{|\frac{s}{a}|, 1\})} - 1 \tag{7.42}$$

Clearly, $|s| \leq a$ implies $\phi_a(s) = \tan(s)$ and $\psi_a(s) = 0$.

Let $p_w = \frac{J_0 \sin(\alpha)}{l\cos(\alpha_0)}$, $q_w = \frac{J_0\cos(\alpha)}{J\cos(\alpha_0)}$, and

$$\theta_1 = \dot{\theta}, \theta_2 = \theta, y_1 = \dot{y}, y_2 = y, x_1 = \dot{x}, x_2 = x \tag{7.43}$$

Then, system (7.34) can be rewritten as follows:

$$\begin{aligned}
\dot{y}_2 &= y_1 \\
\dot{y}_1 &= u_2 + (g + u_2)\psi_a(\theta_2) + \frac{p_w}{M}\sin(\theta_2)u_1 \\
\dot{x}_2 &= x_1 \\
\dot{x}_1 &= -(g + u_2)\phi_a(\theta_2) + \frac{p_w}{M}\cos(\theta_2)u_1 \\
\dot{\theta}_2 &= \theta_1 \\
\dot{\theta}_1 &= q_w u_1
\end{aligned} \tag{7.44}$$

Performing the following coordinate and input transformation

$$\begin{gathered}
\bar{y}_1 = y_1 - \dot{q}_d(v) = e,\ \bar{y}_2 = y_2 - \Psi_2 T_2^{-1}\eta_2, \bar{\eta}_i = \eta_i - T_i\tau_i,\ i = 1, 2, \\
\hat{u}_2 = u_2 - \Psi_1 T_1^{-1}\eta_1
\end{gathered} \tag{7.45}$$

on the augmented system composed of (7.44) and (7.40), yields

$$\begin{aligned}
\dot{\bar{y}}_2 &= \bar{y}_1 - \Psi_2 T_2^{-1}[(M_2 + N_2\Psi_2 T_2^{-1})\bar{\eta}_2 + N_2\bar{y}_2] \\
\dot{\bar{\eta}}_2 &= (M_2 + N_2\Psi_2 T_2^{-1})\bar{\eta}_2 + N_2\bar{y}_2 \\
\dot{\bar{y}}_1 &= \hat{u}_2 + \Psi_1 T_1^{-1}\bar{\eta}_1 + [g + \hat{u}_2 + \Psi_1 T_1^{-1}\bar{\eta}_1 + \ddot{q}_d(v)]\psi_a(\theta_2) + \frac{p_w}{M}\sin(\theta_2)u_1 \\
\dot{\bar{\eta}}_1 &= (M_1 + N_1\Psi_1 T_1^{-1})\bar{\eta}_1 + N_1\hat{u}_2 - M_1 N_1\bar{y}_1 \\
\dot{x}_2 &= x_1 \\
\dot{x}_1 &= -[g + \hat{u}_2 + \Psi_1 T_1^{-1}\bar{\eta}_1 + \ddot{q}_d(v)]\phi_a(\theta_2) + \frac{p_w}{M}\cos(\theta_2)u_1 \\
\dot{\theta}_2 &= \theta_1 \\
\dot{\theta}_1 &= q_w u_1
\end{aligned} \tag{7.46}$$

It is known from [21, 22] that the robust trajectory tracking problem of the VTOL aircraft (7.34) will be solved if we can make the origin of system (7.46) asymptotically stable for all trajectories $v(t)$ starting from V_0 and all $w \in W$ where V_0 and W are compact sets with known bounds.

Noting from (7.39) that $M_2 + N_2\Psi_2 T_2^{-1} = T_2\Phi_2 T_2^{-1}$ and noting that Φ_2 is nonsingular by assumption, the coordinate and input transformation

$$\begin{gathered}
\xi_1 = \bar{\eta}_1 - N_1\bar{y}_1, \xi_2 = (M_2 + N_2\Psi_2 T_2^{-1})\bar{\eta}_2 + N_2\bar{y}_2 \\
\bar{u}_2 = \hat{u}_2 + \Psi_1 T_1^{-1} N_1\bar{y}_1
\end{gathered} \tag{7.47}$$

is globally invertible.

Performing the transformation (7.47) on (7.46) yields

$$\begin{aligned}
\dot{\bar{y}}_2 &= \bar{y}_1 - \Psi_2 T_2^{-1}\xi_2 \\
\dot{\xi}_2 &= M_2\xi_2 + N_2\bar{y}_1 \\
\dot{\bar{y}}_1 &= \bar{u}_2 + \Psi_1 T_1^{-1}\xi_1 + [g + \bar{u}_2 + \Psi_1 T_1^{-1}\xi_1 + \ddot{q}_d(v)]\psi_a(\theta_2) + \frac{p_w}{M}\sin(\theta_2)u_1 \\
\dot{x}_2 &= x_1 \\
\dot{x}_1 &= -[g + \ddot{q}_d(v) + \bar{u}_2]\phi_a(\theta_2) - \Psi_1 T_1^{-1}\xi_1\phi_a(\theta_2) + \frac{p_w}{M}\cos(\theta_2)u_1 \\
\dot{\xi}_1 &= M_1\xi_1 - N_1[g + \bar{u}_2 + \Psi_1 T_1^{-1}\xi_1 + \ddot{q}_d(v)]\psi_a(\theta_2) - N_1\frac{p_w}{M}\sin(\theta_2)u_1 \\
\dot{\theta}_2 &= \theta_1 \\
\dot{\theta}_1 &= q_w u_1
\end{aligned} \tag{7.48}$$

Controller Design

The design will be divided into two steps. In the first step, we will utilize the low gain design proposed in [55] for constructing the controller u_1 in (7.41). In the second step, we will utilize the low gain design proposed in Chapter 4 for constructing the controller u_2 in (7.41).

Step 1: Design of the controller u_1 in (7.41)

Consider the subsystem consisting of the last five equations of (7.48). We first introduce a lemma which is a combination of Propositions 1 and 2 of [55].

Lemma 7.2.1 [55] Consider the following system

$$\begin{aligned}
\dot{x}_2 &= x_1 \\
\dot{x}_1 &= -d_1(t)\phi_a(\theta_2) - \frac{\phi_a(\theta_2)}{M}C_1\xi_1 + \frac{p_w}{M}\cos(\theta_2)u_1 \\
\dot{\xi}_1 &= M_1\xi_1 - N_1\frac{p_w}{M}\sin(\theta_2)u_1 - N_1(D_1\xi_1 + d_2(t))\psi_a(\theta_2) \\
\dot{\theta}_2 &= \theta_1 \\
\dot{\theta}_1 &= q_w u_1
\end{aligned} \tag{7.49}$$

where $x_1, x_2, \theta_1, \theta_2 \in \mathbb{R}, \xi_1 \in \mathbb{R}^{n_{r_1}}$, and M_1, N_1, C_1, D_1 are constant matrices with appropriate dimensions, $d_1(t), d_2(t)$ are unknown bounded time varying functions, and q_w, p_w are unknown bounded uncertain parameters. Assume that

$$0 < d^L \leq d_1(t) \leq d^U,\ 0 < q^L \leq q_w,\ 0 < p_w \leq p^U,\ 1 \leq |\frac{\tan(\theta_2)}{\theta_2}| \leq \phi^U \tag{7.50}$$

for some positive constants $d^L, d^U, q^L, p^U, \phi^U$, and M_1 is Hurwitz, and the functions ϕ_a, ψ_a are as defined in (7.42). Then, there exist positive real numbers $\lambda_i, k_i, i = 1, ..., 4$, such that, under the control

$$u_1 = -\lambda_1\sigma(\frac{k_1}{\lambda_1}(\theta_1 + \lambda_2\sigma(\frac{k_2}{\lambda_2}(\theta_2 - \lambda_3\sigma(\frac{k_3}{\lambda_3}(x_1 + \lambda_4\sigma(\frac{k_4}{\lambda_4}x_2))))))) \tag{7.51}$$

system (7.49) is asymptotically stable at $(x_2, x_1, \xi_1, \theta_2, \theta_1) = (0, 0, 0, 0, 0)$.

Proof: The proof can be found in Section 4 of [55].

Clearly, the subsystem consisting of the last five equations of (7.48) is in the form of (7.49) with

$$d_1(t) = d_2(t) = g + \ddot{q}_d(v) + \bar{u}_2,\ C_1 = M\Psi_1T_1^{-1},\ D_1 = \Psi_1T_1^{-1} \tag{7.52}$$

Note that $|\bar{u}_2| \leq \lambda_5$ for all $t \geq 0$, thus if $|\ddot{q}_d(v)| < g$, we can always choose a sufficiently small λ_5 such that $0 < d^L \leq d_1(t) \leq d^U$ for some positive constants d^L, d^U. As a result, Lemma 7.2.1 can be applied to the global robust stabilization problem of the subsystem consisting of the last five equations of (7.48) viewing u_1 as the control input and $\bar{u}_2$ as the time-varying static uncertainty.

To derive the stabilizing controller (7.51) for the subsystem consisting of the last five equations of (7.48), i.e., system (7.49) with $d_1(t), d_2(t), C_1, D_1$ that are as defined in (7.52), we need to determine the parameters $\lambda_i, k_i, i = 1, ..., 4$.

According to Proposition 1 of [55], there exist positive real numbers r^* and c^*, positive numbers c_1, c_2, c_3 and $\delta_1, \delta_2, \delta_4$ such that, if

$$k_3 = c_3k_4,\ k_2 = c_2k_4,\ k_1 = c_1k_4 \tag{7.53}$$

and

$$\lambda_4 = \frac{\delta_4}{k_4}\lambda_3,\ \lambda_2 = \delta_2k_4\lambda_3, \lambda_1 = \delta_1k_4^2\lambda_3 \tag{7.54}$$

then under the control (7.51), the following system

$$\begin{aligned}\dot{x}_2 &= x_1\\ \dot{x}_1 &= -d_1(t)\tan(\theta_2) + \ell\\ \dot{\theta}_2 &= \theta_1\\ \dot{\theta}_1 &= q_wu_1\end{aligned} \tag{7.55}$$

where

$$\ell = -\tan(\theta_2)\Psi_1T_1^{-1}\xi_1 + \frac{p_w}{M}\cos(\theta_2)u_1$$

satisfies ISS with restriction $r^*\lambda_3$ on the input ℓ and moreover, u_1 satisfies AB with restriction $r^*\lambda_3$ on ℓ and gain $c^*k_4^2s$.

Further by Lemma 1 of [55],

$$\begin{aligned}\delta_4 < \tfrac{d^L}{8},\ c_3 > \tfrac{1}{\delta_4},\ \delta_2 > \tfrac{m_1}{m_2} + 4d^U\phi^U c_3, \\ c_2 > \max\{2\delta_2, 32d^U\phi^U c_3 + 72\},\ \delta_1 = m_1c_2,\ c_1 = m_2c_2 \\ c^* = \tfrac{8c_2c_3}{q^L}(3 + 6\tfrac{d^U}{d^L}),\ r^* = \tfrac{d^L}{4}\end{aligned} \tag{7.56}$$

where m_1 and m_2 are arbitrary positive constants such that $m_1 > 4/q^L$ and $m_2 > 16/q^L$.

In what follows, we determine λ_3 and k_4. By Proposition 2 of [55], there exists $\lambda_3^* > 0$ such that $\lambda_3 \leq \lambda_3^*$ implies that, there exists $T_a > 0$ such that $|\theta_2(t)| \leq a$ for all $t \geq T_a$. λ_3^* can be determined by the following inequality

$$\lambda_3^*[1 + \frac{\delta_2}{c_2} + (\frac{\delta_1 k_4}{c_1} + 2\delta_2 k_4)\frac{3\delta_2}{\delta_1 k_4 q^L}] \leq a \tag{7.57}$$

When $|\theta_2(t)| \leq a$, $\psi_a(\theta_2(t)) = 0$. In this case, ξ_1 subsystem of (7.49) becomes

$$\dot{\xi}_1 = M_1\xi_1 - N_1\frac{p_w}{M}\sin(\theta_2)u_1 \tag{7.58}$$

and ℓ becomes

$$\ell = -\tan(\theta_2)\Psi_1T_1^{-1}\xi_1 + \frac{p_w}{M}\cos(\theta_2)u_1 \tag{7.59}$$

Since M_1 is Hurwitz, let P_1 be a positive definite and symmetric matrix such that $M_1^TP_1 + P_1M_1 = -2I$. Then, it can be verified that ℓ satisfies AB with no restriction on u_1 and gain R^*s where

$$R^* = 2\max\{|\tan(a)|\|\Psi_1T_1^{-1}\|\frac{\bar{\lambda}(P_1)}{\underline{\lambda}(P_1)}\|P_1N_1\|\frac{p_w}{M}|\sin(a)|, \frac{p_w}{M}\} \tag{7.60}$$

Then according to equations (43) and (44) of [55], k_4 should satisfy

$$R^*c^*k_4^2 < 1, R^*\delta_1k_4^2 < r^* \tag{7.61}$$

Step 2: Design of the controller u_2 in (7.41).

Hereinafter, we will design the control u_2 by applying Lemma 4.2.1 recursively. To apply the lemma, performing the following coordinate transformation

$$z_5 = \bar{y}_1 - \Psi_1T_1^{-1}M_1^{-1}\xi_1, z_6 = \bar{y}_2 + \Psi_2T_2^{-1}M_2^{-1}\xi_2 + \tfrac{\vartheta_6}{\vartheta_5}z_5 \tag{7.62}$$

where $\vartheta_5 = 1, \vartheta_6 = \frac{1+\Psi_2 T_2^{-1} M_2^{-1} N_2}{k_5}$, on (7.48) yields

$$\begin{aligned}
\dot{z}_6 &= \vartheta_6 \bar{u}_2 + \vartheta_6 k_5 \bar{y}_1 + \frac{\vartheta_6}{\vartheta_5} F(\zeta, \bar{u}_2, d) \\
\dot{\xi}_2 &= M_2 \xi_2 + N_2 \bar{y}_1 \\
\dot{z}_5 &= \vartheta_5 \bar{u}_2 + F(\zeta, \bar{u}_2, d) \\
\dot{\zeta} &= G(\zeta, \bar{u}_2, d)
\end{aligned} \tag{7.63}$$

where $d = (v, w), \zeta = (x_2, x_1, \xi_1, \theta_2, \theta_1)$, $F(\zeta, \bar{u}_2, d) = (1 + \Psi_1 T_1^{-1} M_1^{-1} N_1)[\frac{p_w}{M} \sin(\theta_2) u_1 + (g + \bar{u}_2 + \Psi_1 T_1^{-1} \xi_1 + \ddot{q}_d(v))\psi_a(\theta_2)]$, and $G(\zeta, \bar{u}_2, d)$ is a suitably defined function and $G(0, \bar{u}_2, d) = 0$.

Under the coordinate and input transformations (7.43), (7.45), (7.47) and (7.62), u_2 in (7.41) becomes

$$\bar{u}_2 = -\sigma_5(k_5(z_5 + \Psi_1 T_1^{-1} M_1^{-1} \xi_1) + \sigma_6(k_6(z_6 - \Psi_2 T_2^{-1} M_2^{-1} \xi_2 - \tfrac{\vartheta_6}{\vartheta_5} z_5))) \tag{7.64}$$

Clearly, if we can show that, there exist sufficiently small $\lambda_i, k_i, i = 5, 6$, such that under the control (7.64), system (7.63) is globally asymptotically stable, then the robust trajectory tracking problem of the VTOL aircraft (7.34) will be solved.

First, consider the last two equations of (7.63). Since $\vartheta_5 > 0$, let $k_5 > 0$. Under the control $\bar{u}_2 = -\sigma_5(k_5(z_5 + H_4\zeta) - u_{21})$ where $H_4 = [0_{1\times 2}\ \Psi_1 T_1^{-1} M_1^{-1}\ 0_{1\times 2}]$, the last two equations of (7.63) can be viewed as the interconnection (4.6) of the two subsystems (4.7) and (4.8) where $v_1^{(5)} = y_2^{(5)} = z_5 + H_4\zeta - \frac{u_{21}}{k_5}, v_{21}^{(5)} = y_{11}^{(5)} = H_4\zeta = \Psi_1 T_1^{-1} M_1^{-1} \xi_1$ and $v_{22}^{(5)} = y_{12}^{(5)} = \frac{1}{\tilde{k}_5} F(\zeta, \bar{u}_2, d)$ with $\tilde{k}_5 = \vartheta_5 k_5$. Note from *Step 1* that ζ subsystem of (7.63) is globally asymptotically stable at the origin $\zeta = 0$, then by Lemma 4.2.1, $y_{1i}^{(5)}, i = 1, 2$, satisfy LB with restriction and AB with no restriction on $\zeta(0)$, both with no restriction on $v_1^{(5)}$ and gain $\bar{\gamma}_{1i}^{(5)}(s) = 0$. Clearly, the small gain condition is satisfied trivially for any positive k_5, λ_5. Thus $z_5, \bar{y}_1, \bar{u}_2, \zeta$ satisfy LB with restriction and AB with no restriction on $(z_5(0), \zeta(0))$, both with restrictions $\frac{\lambda_5}{3}$ on u_{21} and the gains from $z_5, \bar{y}_1, \bar{u}_2$ to u_{21} are $\frac{3}{k_5}s, \frac{6}{k_5}s, 6s$, respectively. However, it should be noted from *Step 1* that λ_5 has to be chosen sufficiently small such that $g + \ddot{q}_d(v) + \bar{u}_2 > 0$ for all $t \geq 0$.

Next consider ξ_2 dynamics. Let P_2 be a positive definite and symmetric matrix such that $M_2^T P_2 + P_2 M_2 = -2I$. Then, it can be verified that ξ_2 satisfies LB with restriction and AB with no restriction on $(\xi_2(0), z_5(0), \zeta(0))$, both with restriction $\frac{\lambda_5}{3}$ on u_{21} and gain $N_{\xi_2 u_{21}} s$, where $N_{\xi_2 u_{21}} = \frac{\bar{\lambda}(P_2)}{\underline{\lambda}(P_2)} \|P_2 N_2\| \frac{6}{k_5}$.

Now let $\zeta_1 = (\xi_2, z_5, \zeta)$. Then (7.63) can be written in the following form

$$\begin{aligned} \dot{z}_6 &= \vartheta_6 u_{21} + F_1(\zeta_1, u_{21}, d) \\ \dot{\zeta}_1 &= G_1(\zeta_1, u_{21}, d) \end{aligned} \tag{7.65}$$

where $F_1(\zeta_1, u_{21}, d) = \vartheta_6 h_5(\bar{y}_1, u_{21}) + \frac{\vartheta_6}{\vartheta_5} F(\zeta, \bar{u}_2, d)$, $h_5(\bar{y}_1, u_{21}) = k_5\bar{y}_1 - u_{21} - \sigma_5(k_5\bar{y}_1 - u_{21})$, and G_1 is a suitably defined function. By assumption, Φ_2 is nonsingular and all eigenvalues of Φ_2 is simple with zero real parts, thus Φ_2 is an even dimensional square matrix and so is M_2. Further, note that the determinant of a matrix is equal to the multiplication of all the eigenvalues of the matrix, then $1 + \Psi_2 T_2^{-1} M_2^{-1} N_2 > 0$ because $1 + \Psi_2 T_2^{-1} M_2^{-1} N_2 = \det(T_2 \Phi_2 T_2^{-1} M_2^{-1}) = \frac{\det(\Phi_2)}{\det(M_2)}$ and M_2 is Hurwitz. Since $\vartheta_6 = \frac{1 + \Psi_2 T_2^{-1} M_2^{-1} N_2}{k_5} > 0$, let $k_6 > 0$.

Under the control $u_{21} = -\sigma_6(k_6(z_6 + H_5\zeta_1))$ where $H_5 = [-\Psi_2 T_2^{-1} M_2^{-1} \; -\frac{\vartheta_6}{\vartheta_5} \; 0_{1\times(4+r_1)}]$, (7.65) can be viewed as the interconnection (4.6) of the two subsystems (4.7) and (4.8) where $v_1^{(6)} = y_2^{(6)} = z_6 + H_5\zeta_1, v_{21}^{(6)} = y_{11}^{(6)} = H_5\zeta_1 = -\Psi_2 T_2^{-1} M_2^{-1}\xi_1 - \frac{\vartheta_6}{\vartheta_5} z_5$ and $v_{22}^{(6)} = y_{12}^{(6)} = \frac{1}{\tilde{k}_6} F_1(\zeta_1, u_{21}, d)$ with $\tilde{k}_6 = \vartheta_6 k_6$. Then, by Lemma 4.2.1, $y_{1i}^{(6)}, i = 1, 2$, satisfy LB with restriction and AB with no restriction on $\zeta_1(0)$, both with no restriction on $v_1^{(6)}$. Moreover, it follows from the expression of $y_{11}^{(6)}$ that $\bar{\gamma}_{11}^{(6)}(s) = 2\max\{\|\Psi_2 T_2^{-1} M_2^{-1}\| N_{\xi_2 u_{21}}, \frac{\vartheta_6}{\vartheta_5}\frac{3}{k_5}\}$ $\min\{k_6 s, \lambda_6\}$ and from the expression of $y_{12}^{(6)}$ that

$$y_{12}^{(6)} = \frac{1}{k_6} h_5(\bar{y}_1, u_{21}) + \frac{k_5}{k_6} y_{12}^{(5)} \tag{7.66}$$

From the property of $h_5(\bar{y}_1, u_{21})$ that $h_5(\bar{y}_1, u_{21}) = 0$ when $\lambda_6 \leq \frac{\lambda_5}{12}$, $\bar{\gamma}_{12}^{(6)}(s)$ only depends on the last term of (7.66) when $\lambda_6 \leq \frac{\lambda_5}{12}$. Furthermore, $\bar{\gamma}_{12}^{(6)}(s) = 0$ because $\bar{\gamma}_{12}^{(5)}(s) = 0$. Clearly, $\bar{\gamma}_{1i}^{(6)}(s), i = 1, 2$ satisfy (4.9). Note that the small gain condition $4\bar{\gamma}_{1i}^{(6)}(s) < s, s > 0, i = 1, 2$, is satisfied with appropriate $\lambda_i, k_i, i = 5, 6$, satisfying

$$2\max\{\|\Psi_2 T_2^{-1} M_2^{-1}\| N_{\xi_2 u_{21}}, \tfrac{\vartheta_6}{\vartheta_5}\tfrac{3}{k_5}\} k_6 < \tfrac{1}{4}, \lambda_6 \leq \tfrac{\lambda_5}{12} \tag{7.67}$$

Thus z_6, ζ_1 satisfies LB with restriction and AB with no restriction on $(z_6(0), \zeta_1(0))$.

7.2.3 Simulation Result

Same as [28], we choose

$M = 4 \times 10^4$ Kg, $l = 5$ m, $a = \frac{\pi}{3}$, $J_0 = 1.25 \times 10^4$ Kg m^2, $\alpha_0 = 2°$

and the uncertainties in J and α can change up to 50 percent of the nominal values, and we aim to let the vertical position y track the reference trajectory

$$q_d(t) = A_1 \cos(0.5t + \phi_1) + A_2 \cos(t + \phi_2) \tag{7.68}$$

where the amplitudes A_i and phase ϕ_i are unknown constants ranging within fixed closed intervals. Further assume $|\ddot{q}_d(t)| \leq g/3$.

Simple calculation shows that $\mathbf{y}_2(v) = q_d(v) = v_1 + v_3$, $\mathbf{y}_1(v) = \dot{q}_d(v) = 0.5v_2 + v_4$ and $\mathbf{u}_2(v) = \ddot{q}_d(v) = -0.25v_1 - v_3$ where v is the generated by the following exosystem

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Then we have

$$\Psi_1 = [-0.25\ 0\ -1\ 0],\ \Psi_2 = [1\ 0\ 1\ 0],\ \Phi_1 = \Phi_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

To design the internal model (7.40), let

$$M_1 = M_2 = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},\ N_1 = N_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving the Sylvester equation (7.39) gives

$$T_1 = \begin{bmatrix} -0.0615 & 0.0154 & -0.2353 & 0.0588 \\ -0.0811 & 0.0270 & -0.3000 & 0.1000 \\ -0.1176 & 0.0588 & -0.4000 & 0.2000 \\ -0.2000 & 0.2000 & -0.5000 & 0.5000 \end{bmatrix}, T_2 = \begin{bmatrix} 0.2462 & -0.0615 & 0.2353 & -0.0588 \\ 0.3243 & -0.1081 & 0.3000 & -0.1000 \\ 0.4706 & -0.2353 & 0.4000 & -0.2000 \\ 0.8000 & -0.8000 & 0.5000 & -0.5000 \end{bmatrix}$$

Then, by Proposition 7.2.1, we can construct a controller in the form of (7.41) which solves the robust trajectory tracking problem of the VTOL aircraft. The design parameters $\lambda_i, k_i, i = 1, ..., 6$ involved in (7.41) can be calculated according to the steps described in the previous section. In particular, we have

$$\begin{gathered} \lambda_1 = 0.39317, \lambda_2 = 0.09771, \lambda_3 = 0.71019, \lambda_4 = 10, \lambda_5 = 0.12, \lambda_6 = 0.01, \\ k_1 = 10, k_2 = 0.98012, k_3 = 0.0093125, k_4 = 0.00093125, k_5 = 100, k_6 = 0.021865. \end{gathered} \tag{7.69}$$

For illustration, the following figures show the simulation result with $J = 10^4$, $\alpha = 2^\circ$ and $\dot{\theta}(0) = 0, \theta(0) = 0.5, \dot{x}(0) = 5, x(0) = 55, \dot{y}(0) = 0.17, y(0) = 48, v(0) = (2, 2.2, 1, 2.2)$, $\eta_1(0) = (-3.512, -3.5124, -3.4306, -2.88), \eta_2(0) = (168.5305, 224.6097, 336.6424, 672.12)$.

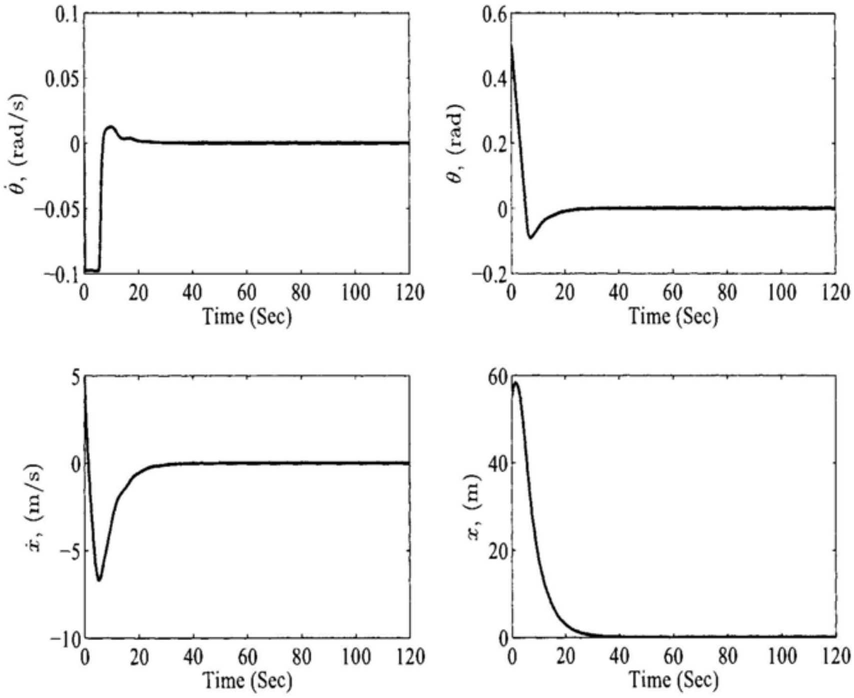

Figure 7.6: Profile of $\dot{\theta}, \theta, \dot{x}$ and x

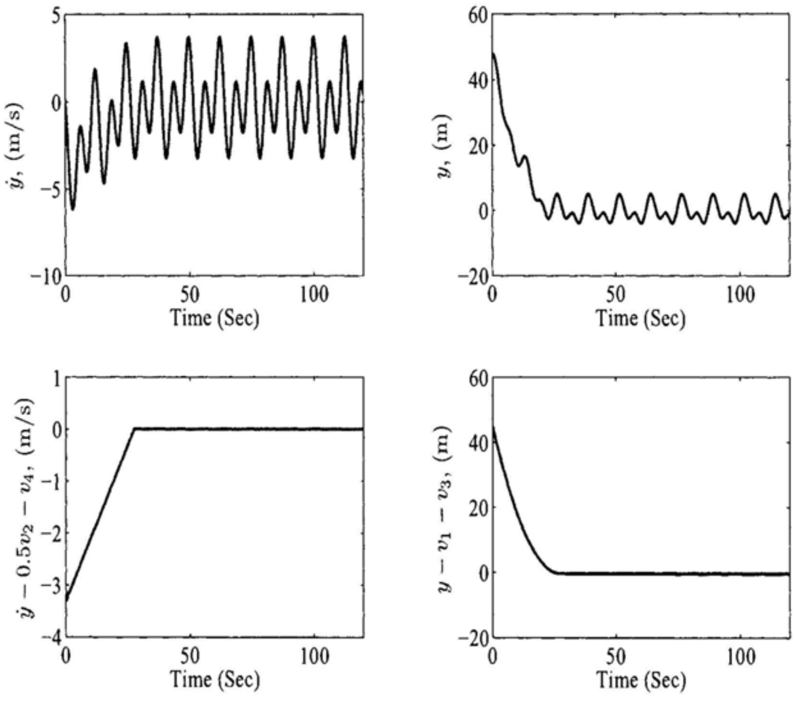

Figure 7.7: Profile of $\dot{y}, y$, $\dot{y} - 0.5v_2 - v_4$ and $y - v_1 - v_3$

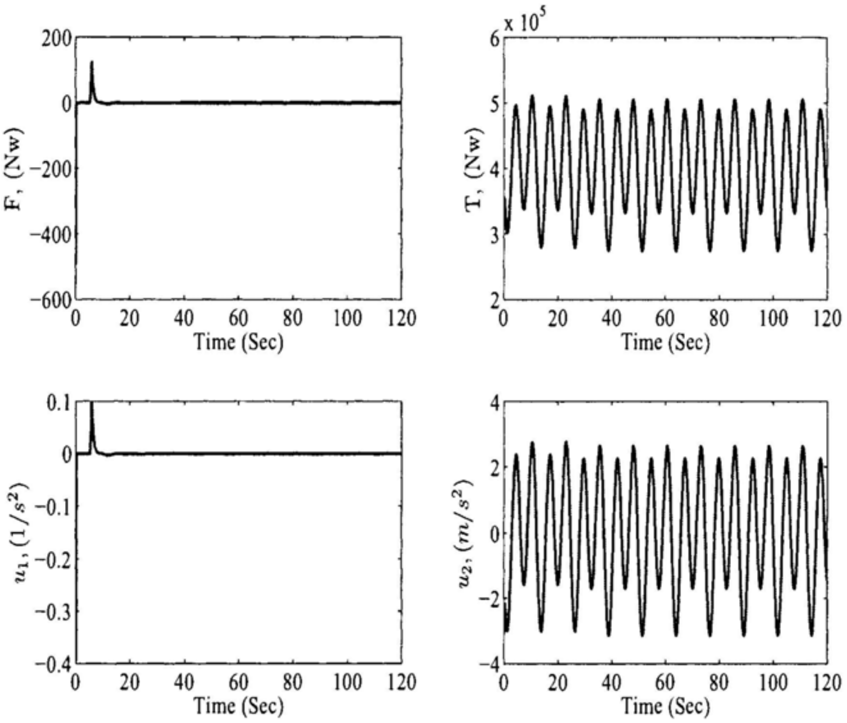

Figure 7.8: Profile of F, T, u_1 and u_2

Chapter 8

Input Disturbance Suppression for a Class of Feedforward Systems

8.1 Introduction

In this chapter, we study the input disturbance suppression problem for feedforward systems in the following form:

$$\begin{aligned}
\dot{x}_i &= c_{i-1}x_{i-1} + f_{i-1}(x_{i-2}, ..., x_1, u - d(v, w), v, w), \;\; i = n, ..., 3, \\
\dot{x}_2 &= c_1 x_1 + f_1(x_1, u - d(v, w), v, w), \\
\dot{x}_1 &= u - d(v, w), \\
y &= (x_1, x_n),
\end{aligned} \tag{8.1}$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the control, $y \in \mathbb{R}^2$ the output, $d(v, w)$ the input disturbance to be suppressed, $w \in \mathbb{R}^{n_w}$ the uncertain parameter, $v \in \mathbb{R}^q$ the state of the exosystem

$$\dot{v} = S(\sigma)v \tag{8.2}$$

where all the eigenvalues of $S(\sigma)$ are simple with zero real parts and $\sigma \in \mathbb{R}^{n_\sigma}$ ranges over some compact subset with a known bound, and c_i, $i = 1, ..., n-1$, are known nonzero real numbers, and the functions $f_i, i = 1, ..., n-1$, and $d(v, w)$ are locally Lipschitz functions satisfying $f_i(0, ..., 0, v, w) = 0, i = 1, ..., n-1$, and $d(0, w) = 0$ for all $v \in \mathbb{R}^q, w \in \mathbb{R}^{n_w}$, and the following assumption:

Assumption 8.1.1

$$|f_1(x_1, u - d(v,w), v, w)| \leq \gamma_1(x_1)|u - d(v,w)|,$$

$$|f_i(x_{i-1}, ..., x_1, u - d(v,w), v, w)| \leq \gamma_1(x_1)(\sum_{j=1}^{i-1} |x_j| + |u - d(v,w)|), i = 2, ..., n-1$$

where $\gamma_1(x_1)$ is a known positive continuous function and $\gamma_1(x_1) \leq a_1 + a_2|x_1|^{p_1}$ for some positive real numbers a_1, a_2, p_1.

The input disturbance suppression problem is defined as follows: given arbitrary fixed compact sets $V_0 \subset \mathbb{R}^q$ and $W \subset \mathbb{R}^{n_w}$, find a controller in the following form

$$u = \mathcal{K}(\zeta, z), \ \dot{\zeta} = \mathcal{F}(\zeta, z) \tag{8.3}$$

where $z = x$ or y, ζ is the compensator state, and $\mathcal{K}, \mathcal{F}$ are locally Lipschitz and $\mathcal{K}(0,0) = 0, \mathcal{F}(0,0) = 0$, such that the closed-loop system composed of (8.1) and (8.3) has the following properties:

1) For all $v(0) \in V_0, w \in W$ and for all initial state $x(0)$, the trajectory of the closed-loop system exists and is bounded for all $t \geq 0$;

2) $\lim_{t\to\infty} x(t) = 0$.

In particular, when $z = x$ or y, (8.3) is called dynamic state or dynamic output feedback controller, respectively.

The input disturbance suppression problem for a more general class of feedforward systems than (8.1) was studied by Marconi, Isidori and Serrani [56]. Therein, the input disturbance to be suppressed was assumed to be generated by a known exosystem and the authors solved the problem via dynamic state feedback control. In this chapter, we extend the results of [56] in two aspects. When the exosystem is known, we solve the problem via dynamic output feedback control. When the exosystem is unknown, we solve the problem via adaptive dynamic state feedback control and we also give the conditions under which an estimated parameter vector can converge to the true parameter vector.

Like [56], the input disturbance suppression problem will be converted into a robust stabilization problem for a class of feedforward systems subject to an input unmodeled dynamics. But different from the small gain approach used in [56], the robust stabilization problem will be solved in this chapter by the dynamic high gain scaling technique introduced recently by Krishnamurthy and Khorrami [39, 40]. The dynamic high gain scaling technique has been employed to solve the robust state and output feedback stabilization problem for feedforward systems [41, 43, 44]. In particular, [44] studied the output feedback stabilization and disturbance attenuation problem for a class of feedforward systems subject to some dynamic uncertainty satisfying certain ISS property. However, the dynamic uncertainty considered in [44] does not contain the input unmodeled dynamics and therefore, the approach in [44] cannot be applied directly to the stabilization problem to be studied in this chapter. In what follows, we will use the dynamic high gain scaling technique to solve the global robust stabilization problem for a class of feedforward systems subject to an input unmodeled dynamics and in turn, to solve the input disturbance suppression problem for system (8.1).

8.2 Internal Model Design

To solve the input disturbance suppression problem for system (8.1), we first make the following assumption on the input disturbance $d(v, w)$.

Assumption 8.2.1 There exists a sufficiently smooth function $\tau : \mathbb{R}^q \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_\tau}$ such that, $\tau(0,0) = 0$ and

$$\dot{\tau}(v, w) = \Phi_\sigma \tau(v, w),\ \ d(v, w) = \Psi \tau(v, w) \tag{8.4}$$

where the pair (Ψ, Φ_σ) is observable and all the eigenvalues of Φ_σ are simple with zero real parts.

Under Assumption 8.2.1, given a pair of controllable matrices (M, N) with $M \in \mathbb{R}^{n_\tau \times n_\tau}$ Hurwitz and $N \in \mathbb{R}^{n_\tau}$, there exists a unique and nonsingular matrix $T_\sigma \in \mathbb{R}^{n_\tau \times n_\tau}$ satisfying the Sylvester equation

$$T_\sigma \Phi_\sigma - M T_\sigma = N \Psi \tag{8.5}$$

since the spectra of M and Φ_σ are disjoint and the pair (Ψ, Φ_σ) is observable.

The following system

$$\dot{\eta} = M\eta + Nu - MNx_1 \tag{8.6}$$

is defined as the internal model of system (8.1).

Remark 8.2.1 The internal model in this chapter is different from the one in [56]. It will be shown later that such internal model is instrumental for the proof of the global asymptotic stability of the closed-loop system. In addition, since Φ_σ, T_σ usually depend on σ, Φ_σ, T_σ cannot be used to construct the controller if σ is unknown.

8.3 Dynamic Output Feedback Control

In this section, we assume that σ in the exosystem (8.2) is known. As a result, Φ_σ in (8.4) and T_σ in (8.5) are known as well.

Theorem 8.3.1 Consider system (8.1). Under Assumptions 8.1.1 and 8.2.1, there exist a locally Lipschitz function γ, and real numbers $\alpha_1 > 0, \alpha_2 > 0$ and $k_i, g_i, i = 1, ..., n$, such that the input disturbance suppression problem is solved by the following dynamic output feedback controller

$$\begin{aligned} u &= \Psi T_\sigma^{-1}(\eta - Nx_1) + k_1 \frac{\hat{x}_1}{r} + ... + k_n \frac{\hat{x}_n}{r^n} \\ \dot{\eta} &= M\eta + Nu - MNx_1, \\ \dot{\hat{x}}_i &= c_{i-1}\hat{x}_{i-1} + r^{i-n-1} g_i(\hat{x}_n - x_n), i = n, ..., 2, \\ \dot{\hat{x}}_1 &= \bar{u} + r^{-n} g_1(\hat{x}_n - x_n) \\ \dot{r} &= -\alpha_1 + \frac{\gamma(x_1)}{\alpha_2 r}, r(0) \geq 1, \end{aligned} \tag{8.7}$$

The input disturbance suppression problem for system (8.1) will be converted into a global robust stabilization problem of a feedforward system subject to an input unmodeled dynamics.

Performing the following coordinate and input transformation

$$\xi = \eta - T_\sigma \tau - Nx_1, \bar{u} = u - H_\sigma(\eta - Nx_1) \tag{8.8}$$

where $H_\sigma = \Psi T_\sigma^{-1}$, on the augmented system gives

$$\begin{aligned} \dot{x}_i &= c_{i-1}x_{i-1} + f_{i-1}(x_{i-2}, ..., x_1, \bar{u} + H_\sigma \xi, v, w), \ i = n, ..., 3 \\ \dot{x}_2 &= c_1 x_1 + f_1(x_1, \bar{u} + H_\sigma \xi, v, w), \\ \dot{x}_1 &= \bar{u} + H_\sigma \xi, \\ \dot{\xi} &= M\xi. \end{aligned} \tag{8.9}$$

Since $\tau(v, w)$ are not available, from (8.8), ξ cannot be used for feedback. Thus, system (8.9) can be viewed as a feedforward system subject to an input unmodeled dynamics. Clearly, $(x, \xi) = (0, 0)$ is an equilibrium point of system (8.9) with $\bar{u} = 0$. Then from (8.8), the input disturbance suppression problem for system (8.1) has been converted into a global robust stabilization problem of system (8.9) with $\bar{u}$ as the new control input. In the following, we will solve the global robust stabilization problem via output feedback control.

Like [42], we use the following dynamic high gain observer

$$\dot{\hat{x}}_i = c_{i-1}\hat{x}_{i-1} + r^{i-n-1}g_i(\hat{x}_n - x_n), i = n, ..., 2,$$
$$\dot{\hat{x}}_1 = \bar{u} + r^{-n}g_1(\hat{x}_n - x_n) \tag{8.10}$$
$$\dot{r} = -\alpha_1 + \frac{\gamma(x_1)}{\alpha_2 r}, r(0) \geq 1$$

where $g_i, i = 1, ..., n$, and $\alpha_1, \alpha_2 > 0$ are design parameters to be specified, and $\gamma(\cdot)$ is a positive locally Lipschitz function to be specified. Clearly, $r(t) \geq 1$ for all $t \geq 0$ if $r(0) \geq 1$ and $\gamma(x_1) \geq \alpha_1\alpha_2$ for all x_1.

We first consider the scaled error dynamics. Let

$$\varepsilon_i = \frac{\hat{x}_i - x_i}{r^{i-1+b}}, i = 1, ..., n, \tag{8.11}$$

where b is a positive design parameter to be specified. Then, we obtain

$$\begin{aligned}
\dot{\varepsilon}_i &= \frac{c_{i-1}\hat{x}_{i-1} + r^{i-n-1}g_i(\hat{x}_n - x_n) - c_{i-1}x_{i-1} - f_{i-1}}{r^{i-1+b}} - (i-1+b)\frac{\dot{r}}{r}\frac{\hat{x}_i - x_i}{r^{i-1+b}} \\
&= \frac{1}{r}c_{i-1}\varepsilon_{i-1} + \frac{1}{r}g_i\varepsilon_n - (i-1+b)\frac{\dot{r}}{r}\varepsilon_i - \frac{f_{i-1}}{r^{i-1+b}}, i = n, ..., 2, \\
\dot{\varepsilon}_1 &= \frac{\bar{u} + r^{-n}g_1(\hat{x}_n - x_n) - \bar{u} - H_\sigma\xi}{r^b} - b\frac{\dot{r}}{r}\frac{\hat{x}_1 - x_1}{r^b} \\
&= \frac{1}{r}g_1\varepsilon_n - b\frac{\dot{r}}{r}\varepsilon_1 - \frac{H_\sigma\xi}{r^b},
\end{aligned}$$

where for simplicity, we use $f_i, i = 1, ..., n-1$, to denote the corresponding functions in (8.9). Let $\varepsilon = (\varepsilon_n, ..., \varepsilon_1)$. The scaled error dynamics can be rewritten as follows:

$$\dot{\varepsilon} = \frac{1}{r}A_o\varepsilon - \frac{\dot{r}}{r}D\varepsilon - F \tag{8.12}$$

where

$$A_o = \begin{bmatrix} g_n & c_{n-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ g_2 & 0 & \dots & c_1 \\ g_1 & 0 & \dots & 0 \end{bmatrix}, \quad \begin{aligned} D &= \mathrm{diag}(n-1+b, ..., i-1+b, ..., b), \\ F &= (\tfrac{f_{n-1}}{r^{n-1+b}}, ..., \tfrac{f_{i-1}}{r^{i-1+b}}, ..., \tfrac{H_\sigma\xi}{r^b}). \end{aligned}$$

Next, we consider the scaled observer dynamics. Let

$$\chi_i = \frac{\hat{x}_i}{r^{i-1+b}}, i = 1, ..., n. \tag{8.13}$$

Then, we obtain

$$\begin{aligned}
\dot{\chi}_i &= \frac{c_{i-1}\hat{x}_{i-1} + r^{i-n-1}g_i(\hat{x}_n - x_n)}{r^{i-1+b}} - (i-1+b)\frac{\dot{r}}{r}\frac{\hat{x}_i}{r^{i-1+b}} \\
&= \frac{1}{r}c_{i-1}\chi_{i-1} + \frac{1}{r}g_i\varepsilon_n - (i-1+b)\frac{\dot{r}}{r}\chi_i, i = n, ..., 2 \\
\dot{\chi}_1 &= \frac{\bar{u} + r^{-n}g_1(\hat{x}_n - x_n)}{r^b} - b\frac{\dot{r}}{r}\frac{\hat{x}_1}{r^b} \\
&= \frac{1}{r}\frac{\bar{u}}{r^{b-1}} + \frac{1}{r}g_1\varepsilon_n - b\frac{\dot{r}}{r}\chi_1,
\end{aligned}$$

By further letting

$$\bar{u} = \frac{k\chi}{r^{1-b}} \tag{8.14}$$

where $k = (k_n, ..., k_1)^T$ is a design parameter vector to be specified. Let $\chi = (\chi_n, ..., \chi_1)$. The scaled observer dynamics can be rewritten as follows:

$$\dot{\chi} = \frac{1}{r}A_c\chi - \frac{\dot{r}}{r}D\chi + \frac{1}{r}G\varepsilon_n \tag{8.15}$$

where

$$A_c = \begin{bmatrix} 0 & c_{n-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c_1 \\ k_n & k_2 & \dots & k_1 \end{bmatrix}, G = (g_n, ..., g_1).$$

By now, we have obtained the closed-loop system as follows

$$\begin{aligned} \dot{\varepsilon} &= \frac{1}{r}A_o\varepsilon - \frac{\dot{r}}{r}D\varepsilon - F \\ \dot{\chi} &= \frac{1}{r}A_c\chi - \frac{\dot{r}}{r}D\chi + \frac{1}{r}G\varepsilon_n \\ \dot{\xi} &= M\xi \end{aligned} \tag{8.16}$$

Hereinafter, we will introduce three Lyapunov functions for the subsystems in (8.16), respectively, and then use a linear combination of them as the overall Lyapunov function for the closed-loop system (8.16).

By Theorem A1 in [39], given any positive real number b and nonzero real numbers c_i, $i = 1, ..., n-1$, there exist positive definite matrices P_o, P_c, and k_i, g_i, $i = 1, ..., n$, such that, the following Lyapunov inequalities

$$\begin{aligned} A_o^T P_o + P_o A_o \le -q_o I, \underline{q}_o I \le P_o D + D P_o \le \bar{q}_o I \\ A_c^T P_c + P_c A_c \le -q_c I, \underline{q}_c I \le P_c D + D P_c \le \bar{q}_c I \end{aligned} \tag{8.17}$$

hold for some positive real numbers $q_o, \underline{q}_o, \bar{q}_o, q_c, \underline{q}_c, \bar{q}_c$.

Define $V_o = \varepsilon^T P_o \varepsilon$ and $V_c = \chi^T P_c \chi$. The time derivative of V_o along the trajectory of ε subsystem in (8.16) and the time derivative of V_c along the trajectory of χ subsystem in (8.16) are

$$\begin{aligned} \dot{V}_o &\le -\frac{q_o}{r}\|\varepsilon\|^2 - \frac{\dot{r}}{r}(P_o D + D P_o)\|\varepsilon\|^2 - 2\varepsilon^T P_o F, \\ \dot{V}_c &\le -\frac{q_c}{r}\|\chi\|^2 - \frac{\dot{r}}{r}(P_c D + D P_c)\|\chi\|^2 + \frac{2}{r}\chi^T P_c G \varepsilon_n \end{aligned} \tag{8.18}$$

Noting $x_j = r^{j-1+b}(\chi_j - \varepsilon_j)$, (8.14), $r \geq 1$ and Assumption 8.1.1 yields that, for $i = 2, ..., n-1$,

$$\frac{|f_i(x_{i-1}, ..., x_1, \bar{u} + H_\sigma \xi, v, w)|}{r^{b+i}}$$
$$\leq \frac{\gamma_1(x_1)}{r^2}[\sum_{j=1}^{i-1} r^{j+1-i}(|\chi_j| + |\varepsilon_j|) + r^{1-i}\|k\|\|\chi\|] + \frac{\|H_\sigma\|\gamma_1(x_1)}{r^{b+i}}\|\xi\|$$
$$\leq \frac{\gamma_1(x_1)}{r^2}[(\sqrt{n-2} + \|k\|)\|\chi\| + \sqrt{n-2}\|\varepsilon\|] + \frac{\|H_\sigma\|\gamma_1(x_1)}{r^{b+1}}\|\xi\|$$

In turn, we obtain

$$\|F\| \leq \frac{\sqrt{n-1}(\sqrt{n-2} + \|k\|)\gamma_1(x_1)}{r^2}[\|\chi\| + \|\varepsilon\|] + \frac{\sqrt{n-1}\|H_\sigma\|\gamma_1(x_1)}{r^{b+1}}\|\xi\| + \frac{\|H_\sigma\|}{r^b}\|\xi\|$$

which implies

$$\|\varepsilon\|\|F\| \leq \frac{\sqrt{n-1}(\sqrt{n-2} + \|k\|)\gamma_1(x_1) + \kappa(\gamma_1(x_1))^2}{2r^2}(\|\chi\|^2 + 3\|\varepsilon\|^2) + \frac{q_o}{8\bar{\lambda}(P_o)r}\|\varepsilon\|^2$$
$$+\frac{(n-1)\|H_\sigma\|^2}{2\kappa r^{2b}}\|\xi\|^2 + \frac{2\bar{\lambda}(P_o)\|H_\sigma\|^2}{q_o r^{2b-1}}\|\xi\|^2 \quad (8.19)$$

where κ is any positive real number. Also note that

$$2\chi^T P_c G \varepsilon_n \leq 2\bar{\lambda}(P_c)\|\chi\|\|G\|\|\varepsilon\| \leq \frac{1}{2}q_c\|\chi\|^2 + \frac{2(\bar{\lambda}(P_c))^2\|G\|^2}{q_c}\|\varepsilon\|^2. \quad (8.20)$$

Let $V_x = \rho V_o + V_c$ where $\rho \geq \frac{8(\bar{\lambda}(P_c))^2\|G\|^2}{q_o q_c}$. Noting equations (8.17) to (8.20), $r \geq 1$, $b > 0$ and the dynamics of r in (8.10) yields

$$\dot{V}_x \leq -\frac{\rho q_o}{2r}\|\varepsilon\|^2 - \frac{q_c}{2r}\|\chi\|^2 + \frac{(n-1)\rho\bar{\lambda}(P_o)\|H_\sigma\|^2}{\kappa}\|\xi\|^2 + \frac{4\rho(\bar{\lambda}(P_o))^2\|H_\sigma\|^2}{q_o r^{2b-1}}\|\xi\|^2$$
$$+3\rho\bar{\lambda}(P_o)\frac{\sqrt{n-1}(\sqrt{n-2} + \|k\|)\gamma_1(x_1) + \kappa(\gamma_1(x_1))^2}{r^2}(\|\chi\|^2 + \|\varepsilon\|^2)$$
$$+\frac{\alpha_1}{r}(\rho\bar{q}_o\|\varepsilon\|^2 + \bar{q}_c\|\chi\|^2) - \frac{\gamma(x_1)}{\alpha_2 r^2}(\rho\underline{q}_o\|\varepsilon\|^2 + \underline{q}_c\|\chi\|^2) \quad (8.21)$$

Let $\alpha_1 = \frac{1}{4}\min\{\frac{q_o}{\bar{q}_o}, \frac{q_c}{\bar{q}_c}\}$, $\alpha_2 = \min\{\rho\underline{q}_o, \underline{q}_c\}$. By Assumption 8.1.1, there exists a locally Lipschitz function $\gamma(x_1)$ such that

$$\gamma(x_1) \geq \max\{3\rho\bar{\lambda}(P_o)[\sqrt{n-1}(\sqrt{n-2} + \|k\|)\gamma_1(x_1) + \kappa(\gamma_1(x_1))^2], \alpha_1\alpha_2\} \quad (8.22)$$

Then, (8.21) becomes

$$\dot{V}_x \leq -\frac{\rho q_o}{4r}\|\varepsilon\|^2 - \frac{q_c}{4r}\|\chi\|^2 + \frac{(n-1)\rho\bar{\lambda}(P_o)\|H_\sigma\|^2}{\kappa}\|\xi\|^2 + \frac{4\rho(\bar{\lambda}(P_o))^2\|H_\sigma\|^2}{q_o r^{2b-1}}\|\xi\|^2 \quad (8.23)$$

Now consider the ξ dynamics in (8.9). Since M is Hurwitz, there exist positive definite and symmetric matrices P, Q such that $M^T P + PM = -Q$. Define $V_\xi = \xi^T P \xi$. The time derivative of V_ξ along the trajectory of ξ subsystem in (8.9) is

$$\dot{V}_\xi = \xi^T(M^T P + PM)\xi = -\xi^T Q \xi \leq -\underline{\lambda}(Q)\|\xi\|^2 \quad (8.24)$$

Further define the overall Lyapunov function for the closed-loop system (8.16) as follows

$$V = V_x + \varrho \frac{(n-1)\rho\bar{\lambda}(P_o)\|H_\sigma\|^2}{\kappa\underline{\lambda}(Q)} V_\xi$$

where $\varrho > 1$ is a real number to be specified. Noting (8.23) and (8.24) yields, the time derivative of V along the trajectory of the closed-loop system (8.16) is

$$\begin{aligned}
\dot{V} &\leq -\frac{\rho q_o}{4r}\|\varepsilon\|^2 - \frac{q_c}{4r}\|\chi\|^2 - \frac{(\varrho-1)(n-1)\rho\bar{\lambda}(P_o)\|H_\sigma\|^2}{\kappa}\|\xi\|^2 + \frac{4\rho(\bar{\lambda}(P_o))^2\|H_\sigma\|^2}{q_o r^{2b-1}}\|\xi\|^2 \\
&= -\frac{\rho q_o}{4r}\|\varepsilon\|^2 - \frac{q_c}{8r}\|\chi\|^2 - \frac{(\varrho-1)(n-1)\rho\bar{\lambda}(P_o)\|H_\sigma\|^2}{\kappa}\|\xi\|^2 \\
&\quad -\frac{1}{r}\left(\frac{q_c}{8}\|\chi\|^2 - \frac{4\rho(\bar{\lambda}(P_o))^2\|H_\sigma\|^2}{q_o r^{2(b-1)}}\|\xi\|^2\right)
\end{aligned}$$

Consider the following two cases:

1) When $\frac{q_c}{8}\|\chi\|^2 - \frac{4\rho(\bar{\lambda}(P_o))^2\|H_\sigma\|^2}{q_o r^{2(b-1)}}\|\xi\|^2 \geq 0$, i.e., $\|\xi\| \leq \frac{1}{4\bar{\lambda}(P_o)\|H_\sigma\|}\sqrt{\frac{q_o q_c}{2\rho}}\frac{\|\chi\|}{r^{1-b}}$, $\dot{V} \leq 0$ and thus $\varepsilon(t), \chi(t)$ are bounded for all $t \geq 0$. From (8.22) and Assumption 8.1.1, there exist positive real numbers b_1, b_2 such that $\gamma(x_1) \leq b_1 + b_2|x_1|^{2p_1}$ for all x_1. Then from $x_1 = r^b(\chi_1 - \varepsilon_1)$ and the dynamics of r in (8.10), we have

$$\dot{r} = -\alpha_1 + \frac{\gamma(x_1)}{\alpha_2 r} \leq -\alpha_1 + \frac{b_1 + b_2 r^{2bp_1}|\chi_1 - \varepsilon_1|^{2p_1}}{\alpha_2 r}.$$

Let $0 < b < \frac{1}{2p_1}$. Then the above inequality implies that $r(t)$ is bounded for all $t \geq 0$.

2) When $\frac{q_c}{8}\|\chi\|^2 - \frac{4\rho(\bar{\lambda}(P_o))^2\|H_\sigma\|^2}{q_o r^{2(b-1)}}\|\xi\|^2 \leq 0$, i.e., $\frac{\|\chi\|}{r^{1-b}} \leq 4\bar{\lambda}(P_o)\|H_\sigma\|\sqrt{\frac{2\rho}{q_o q_c}}\|\xi\|$, it follows from (8.14) that

$$|\bar{u}| \leq 4\|k\|\bar{\lambda}(P_o)\|H_\sigma\|\sqrt{\frac{2\rho}{q_o q_c}}\|\xi\|.$$

Then, from the above inequaltiy, from the x_1 subsystem of (8.9) and from the exponential stability of ξ subsystem of (8.9), $x_1(t)$ is bounded for all $t \geq 0$. In turn, from the r subsystem of (8.10), $r(t)$ is bounded for all $t \geq 0$ as well.

In either of the above two cases, $r(t)$ is bounded for all $t \geq 0$ and thus we can assume $1 \leq r(t) \leq r_m$ for some finite r_m and for $t \geq 0$. As a result, we can always choose a sufficiently large ϱ such that

$$-\frac{(\varrho-1)(n-1)\rho\bar{\lambda}(P_o)}{\kappa} + \frac{4\rho(\bar{\lambda}(P_o))^2}{q_o r^{2b-1}} \leq -1$$

and in turn, the time derivative of V is bounded as follows:

$$\dot{V} \leq -\frac{\rho q_o}{4r_m}\|\varepsilon\|^2 - \frac{q_c}{4r_m}\|\chi\|^2 - \|H_\sigma\|^2\|\xi\|^2 \tag{8.25}$$

which by Theorem 2.1.3 implies the global asymptotic stability of $(\varepsilon, \chi, \xi) = (0, 0, 0)$. Finally, noting $x_j = r^{j-1+b}(\chi_j - \varepsilon_j)$ and $1 \leq r(t) \leq r_m$ yields $x(t) \to 0$ as $t \to \infty$, and then noting (8.8), (8.11), and (8.13) yields the boundedness of the trajectory of the closed-loop system composed of (8.1) and the dynamic controller (8.7).

8.4 Adaptive Dynamic State Feedback Control

In this section, we assume that the parameter σ in the exosystem (8.2) is unknown but ranges over some compact subset with a known bound. As a result, Φ_σ in (8.4) and T_σ in (8.5) cannot be used to construct the controller.

To solve the problem for system (8.1), we further make two more assumptions.

Assumption 8.4.1 The functions $f_i, i = 1, ..., n-1$, can be written into the following form

$$\begin{aligned} f_1(x_1, u - d(v,w), v, w) &= f_1^a(x_1)(u - d(v,w)), \\ f_i(x_{i-1}, ..., x_1, u - d(v,w), v, w) &= f_i^a(x_1)(u - d(v,w)) \\ &\quad + f_i^b(x_{i-1}, ..., x_1, v, w), i = 2, ..., n-1 \end{aligned} \tag{8.26}$$

where f_i^a, f_i^b are continuously differentiable functions. The positive real number p_1 defined in Assumption 8.1.1 satisfies $p_1 \leq 0.5$.

Assumption 8.4.2 The smooth function $\tau(v,w)$ defined in Assumption 8.2.1 is persistent exciting (PE), i.e., there exist $\alpha, \delta > 0$ such that, for all $t_0 > 0$,

$$\int_{t_0}^{t_0+\delta} \tau(t)\tau^T(t)dt \geq \alpha I \tag{8.27}$$

Remark 8.4.1 We assume that $f_i, i = 1, ..., n-1$, are continuously differentiable, because we will invoke Barbalat's Lemma to prove the parameter convergence. The expansion of $f_i, i = 1, ..., n-1$, into the form of (8.26) is to guarantee the applicability of the adaptive control method. In addition, note that the assumption $0 \leq p_1 \leq 0.5$ is not restrict in the literature of output feedback stabilization of feedforward systems. For example, it is assumed $p_1 = 0$ in [14].

Theorem 8.4.1 Consider system (8.1). Under Assumptions 8.1.1, 8.2.1, 8.4.1 and 8.4.2, there exist continuously differentiable functions φ, γ, and real numbers $\Gamma > 0, \alpha_1 > 0, \alpha_2 > 0$ and $k_i, i = 1, ..., n$, such that the input disturbance suppression problem is solved by the following dynamic state feedback controller

$$
\begin{aligned}
u &= \hat{H}_\sigma(\eta - Nx_1) + k_1 \frac{x_1}{r} + \ldots + k_n \frac{x_n}{r^n} \\
\dot{\eta} &= M\eta + Nu - MNx_1, \\
\dot{\hat{H}}_\sigma &= \Gamma\varphi(x, \eta, r), \\
\dot{r} &= -\alpha_1 + \frac{\gamma(x_1)}{\alpha_2 r}, r(0) \geq 1,
\end{aligned} \tag{8.28}
$$

where $\hat{H}_\sigma$ is the estimate of $H_\sigma = \Psi T_\sigma^{-1}$, and moreover, $\hat{H}_\sigma(t) \to H_\sigma$ as $t \to \infty$.

The input disturbance suppression problem for system (8.1) will be converted into a global robust stabilization problem for a class of feedforward systems subject to an input unmodeled dynamics.

Performing the following coordinate and input transformation

$$
\xi = \eta - T_\sigma \tau - Nx_1, \ \tilde{H} = \hat{H}_\sigma - H_\sigma, \ \bar{u} = u - \hat{H}_\sigma \eta + \hat{H}_\sigma N x_1 \tag{8.29}
$$

on the augmented system composed of system (8.1), the internal model (8.6), and $\hat{H}_\sigma$ dynamics in (8.28) and noting that H_σ is a unknown constant vector, yields

$$
\begin{aligned}
\dot{x}_i &= c_{i-1}x_{i-1} + f_{i-1}(x_{i-2}, \ldots, x_1, \bar{u} + H_\sigma\xi + (\eta - Nx_1)^T \tilde{H}^T, v, w), \ i = n, \ldots, 3 \\
\dot{x}_2 &= c_1 x_1 + f_1(x_1, \bar{u} + H_\sigma\xi + (\eta - Nx_1)^T \tilde{H}^T, v, w), \\
\dot{x}_1 &= \bar{u} + H_\sigma\xi + (\eta - Nx_1)^T \tilde{H}^T \\
\dot{\xi} &= M\xi, \\
\dot{\tilde{H}}^T &= \Gamma\varphi^T(x, \eta, r)
\end{aligned} \tag{8.30}
$$

Since $\tau(v, w)$ and H_σ are not available, from (8.29), ξ and $\tilde{H}^T$ cannot be used for feedback. Thus, system (8.30) can be viewed as a feedforward system subject to the input unmodeled dynamics composed of the ξ subsystem and $\tilde{H}^T$ subsystem. Clearly, $(x, \xi, \tilde{H}^T) = (0, 0, 0)$ is an equilibrium point of system (8.30) with $\bar{u} = 0$. The input disturbance suppression problem for system (8.1) has been converted into a global robust stabilization problem of system (8.30) with $\bar{u}$ as the new control input. In the following, the global robust stabilization will be further pursued.

From Assumption 8.4.1, system (8.30) can be rewritten as follows

$$
\begin{aligned}
\dot{x}_i &= c_{i-1}x_{i-1} + f_{i-1}(x_{i-1}, ..., x_1, \bar{u} + H_\sigma \xi, v, w) + f_{i-1}^a(x_1)(\eta - Nx_1)^T \tilde{H}^T, i = n, ..., 2, \\
\dot{x}_2 &= c_1 x_1 + f_1(x_1, \bar{u} + H_\sigma \xi, v, w) + f_1^a(x_1)(\eta - Nx_1)^T \tilde{H}^T \\
\dot{x}_1 &= \bar{u} + H_\sigma \xi + (\eta - Nx_1)^T \tilde{H}^T \\
\dot{\xi} &= M\xi, \\
\dot{\tilde{H}}^T &= \Gamma \varphi^T(x, \eta, r)
\end{aligned}
\tag{8.31}
$$

Now let

$$
\chi_i = \frac{x_i}{r^{i-1+b}}, i = 1, ..., n, \tag{8.32}
$$

then we have

$$
\begin{aligned}
\dot{\chi}_i &= \frac{c_{i-1}x_{i-1} + f_{i-1} + f_{i-1}^a(x_1)(\eta - Nx_1)^T \tilde{H}^T}{r^{i-1+b}} - (i-1+b)\frac{\dot{r}}{r}\frac{x_i}{r^{i-1+b}} \\
&= \frac{1}{r}c_{i-1}\chi_{i-1} + \frac{1}{r^{i-1+b}}[f_{i-1} + f_{i-1}^a(x_1)(\eta - Nx_1)^T \tilde{H}^T] - (i-1+b)\frac{\dot{r}}{r}\chi_i, \ i = n, ..., 2 \\
\dot{\chi}_1 &= \frac{\bar{u} + H_\sigma \xi + (\eta - Nx_1)^T \tilde{H}^T}{r^b} - b\frac{\dot{r}}{r}\frac{x_1}{r^b} \\
&= \frac{1}{r}\frac{\bar{u}}{r^{b-1}} + \frac{1}{r^b}[H_\sigma \xi + (\eta - Nx_1)^T \tilde{H}^T] - b\frac{\dot{r}}{r}\chi_1,
\end{aligned}
$$

where for simplicity, we let $f_i, i = 1, ..., n-1$, denote the corresponding functions in (8.31).

By further letting

$$
\bar{u} = \frac{k\chi}{r^{1-b}} \tag{8.33}
$$

where $k = (k_n, ..., k_1)^T$ and $\chi = (\chi_n, ..., \chi_1)$, system (8.31) can be rewritten into the following form:

$$
\begin{aligned}
\dot{\chi} &= \frac{1}{r}A_c\chi - \frac{\dot{r}}{r}D\chi + F + E(x_1, r, \eta)\tilde{H}^T, \\
\dot{\xi} &= M\xi, \\
\dot{\tilde{H}}^T &= \Gamma \varphi^T(x, \eta, r)
\end{aligned}
\tag{8.34}
$$

where

$$
A_c = \begin{bmatrix} 0 & c_{n-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c_1 \\ k_n & k_2 & \dots & k_1 \end{bmatrix}, \quad
\begin{aligned}
D &= \mathrm{diag}(n-1+b, ..., i-1+b, ..., b), \\
F &= \left(\tfrac{f_{n-1}}{r^{n-1+b}}, ..., \tfrac{f_{i-1}}{r^{i-1+b}}, ..., \tfrac{H_\sigma \xi}{r^b}\right), \\
E(x_1, r, \eta) &= \left(\tfrac{f_{n-1}^a(x_1)}{r^{n-1+b}}, ..., \tfrac{f_{i-1}^a(x_1)}{r^{i-1+b}}, ..., \tfrac{1}{r^b}\right)(\eta - Nx_1)^T
\end{aligned}
\tag{8.35}
$$

Note that system (8.34) is the closed-loop system with $(\chi, \xi, \tilde{H}^T)$ as state. Hereinafter, we will introduce three Lyapunov functions for the subsystems in (8.34), respectively, and then use a linear combination of them as the overall Lyapunov function for the closed-loop system (8.34).

By Theorem A1 in [39], given any positive real number b and nonzero real numbers c_i, $i = 1, ..., n-1$, there exist positive definite matrix P_c and k_i, $i = 1, ..., n$, such that, the following Lyapunov inequalities

$$A_c^T P_c + P_c A_c \leq -q_c I, \underline{q}_c I \leq P_c D + D P_c \leq \bar{q}_c I \tag{8.36}$$

hold for some positive real numbers $q_c, \underline{q}_c, \bar{q}_c$.

Define

$$V_x = \chi^T P_c \chi + \frac{1}{2\Gamma} \tilde{H} \tilde{H}^T$$

Noting (8.36) and the $\tilde{H}^T$ dynamics in (8.31) yields that the time derivative of V_x along the trajectory of χ and $\tilde{H}^T$ subsystems in (8.34) is

$$\begin{aligned}
\dot{V}_x &= \frac{1}{r}\chi^T(A_c^T P_c + P_c A_c)\chi - \frac{\dot{r}}{r}(P_c D + D P_c)\|\chi\|^2 + 2\chi^T P_c F + [2\chi^T P_c E(x_1, r, \eta) + \varphi]\tilde{H}^T \\
&\leq -\frac{q_c}{r}\|\chi\|^2 - \frac{\dot{r}}{r}(P_c D + D P_c)\|\chi\|^2 + 2\bar{\lambda}(P_c)\|\chi\|\|F\|
\end{aligned} \tag{8.37}$$

where we have set

$$\varphi(x, \eta, r) = -2\chi^T P_c E(x_1, r, \eta). \tag{8.38}$$

Then, noting $x_j = r^{j-1+b}\chi_j$, (8.33), $r \geq 1$ and Assumption 8.1.1 yields that, for $i = 2, ..., n-1$,

$$\begin{aligned}
&\frac{|f_i(x_{i-1}, ..., x_1, \bar{u} + H_\sigma \xi, v, w)|}{r^{i+b}} \\
&\leq \frac{\gamma_1(x_1)}{r^2}[\sum_{j=1}^{i-1} r^{j+1-i}|\chi_j| + r^{1-i}\|k\|\|\chi\|] + \frac{\|H_\sigma\|\gamma_1(x_1)}{r^{b+i}}\|\xi\| \\
&\leq \frac{\gamma_1(x_1)}{r^2}(\sqrt{n-2} + \|k\|)\|\chi\| + \frac{\|H_\sigma\|\gamma_1(x_1)}{r^{b+1}}\|\xi\|
\end{aligned}$$

Thus, we have

$$\|F\| \leq \frac{\sqrt{n-1}(\sqrt{n-2} + \|k\|)\gamma_1(x_1)}{r^2}\|\chi\| + \frac{\sqrt{n-1}\|H_\sigma\|\gamma_1(x_1)}{r^{b+1}}\|\xi\| + \frac{\|H_\sigma\|}{r^b}\|\xi\|$$

which implies

$$\begin{aligned}
\|\chi\|\|F\| &\leq \frac{2\sqrt{n-1}(\sqrt{n-2} + \|k\|)\gamma_1(x_1) + \kappa(\gamma_1(x_1))^2}{2r^2}\|\chi\|^2 + \frac{(n-1)\|H_\sigma\|^2}{2\kappa r^{2b}}\|\xi\|^2 \\
&\quad + \frac{q_c}{8\bar{\lambda}(P_c)r}\|\chi\|^2 + \frac{2\bar{\lambda}(P_c)\|H_\sigma\|^2}{q_c r^{2b-1}}\|\xi\|^2
\end{aligned} \tag{8.39}$$

where κ is any positive real number.

Then, from $r \geq 1$, $b > 0$ and the dynamics of r in (8.10), (8.37) becomes

$$\begin{aligned}\dot{V}_x \leq& -\frac{3q_c}{4r}\|\chi\|^2 + \frac{(n-1)\bar{\lambda}(P_c)\|H_\sigma\|^2}{\kappa r^{2b}}\|\xi\|^2 + \frac{4(\bar{\lambda}(P_c))^2\|H_\sigma\|^2}{q_c r^{2b-1}}\|\xi\|^2 + \frac{\alpha_1}{r}\bar{q}_c\|\chi\|^2 \\ &+\bar{\lambda}(P_c)\frac{2\sqrt{n-1}(\sqrt{n-2}+\|k\|)\gamma_1(x_1)+\kappa(\gamma_1(x_1))^2}{r^2}\|\chi\|^2 - \frac{\gamma(x_1)}{\alpha_2 r^2}\underline{q}_c\|\chi\|^2 \quad (8.40)\end{aligned}$$

Let $\alpha_1 = \frac{q_c}{4\bar{q}_c}$, $\alpha_2 = \underline{q}_c$. By Assumptions 8.1.1 and 8.4.1, there exists a continuously differentiable function $\gamma(x_1)$ such that

$$\gamma(x_1) \geq \max\{\bar{\lambda}(P_c)[2\sqrt{n-1}(\sqrt{n-2}+\|k\|)\gamma_1(x_1)+\kappa(\gamma_1(x_1))^2], \alpha_1\alpha_2\} \quad (8.41)$$

Then, (8.40) becomes

$$\dot{V}_x \leq -\frac{q_c}{2r}\|\chi\|^2 + \frac{(n-1)\bar{\lambda}(P_c)\|H_\sigma\|^2}{\kappa}\|\xi\|^2 + \frac{4(\bar{\lambda}(P_c))^2\|H_\sigma\|^2}{q_c r^{2b-1}}\|\xi\|^2 \quad (8.42)$$

Define the overall Lyapunov function for the closed-loop system (8.16) as follows

$$V = V_x + \varrho\frac{(n-1)\bar{\lambda}(P_c)\|H_\sigma\|^2}{\kappa\underline{\lambda}(Q)}V_\xi$$

where $\varrho > 1$ is a real number to be specified. Noting (8.42) and (8.24) yields that, the time derivative of V along the trajectory of the closed-loop system (8.34) is

$$\begin{aligned}\dot{V} &\leq -\frac{q_c}{2r}\|\chi\|^2 - \frac{(\varrho-1)(n-1)\bar{\lambda}(P_c)\|H_\sigma\|^2}{\kappa}\|\xi\|^2 + \frac{4(\bar{\lambda}(P_c))^2\|H_\sigma\|^2}{q_c r^{2b-1}}\|\xi\|^2 \quad (8.43)\\ &= -\frac{q_c}{4r}\|\chi\|^2 - \frac{(\varrho-1)(n-1)\bar{\lambda}(P_c)\|H_\sigma\|^2}{\kappa}\|\xi\|^2 - \frac{1}{r}(\frac{q_c}{4}\|\chi\|^2 - \frac{4(\bar{\lambda}(P_c))^2\|H_\sigma\|^2}{q_c r^{2(b-1)}}\|\xi\|^2)\end{aligned}$$

Now we consider the following two cases:

1) When $\frac{q_c}{4}\|\chi\|^2 - \frac{4(\bar{\lambda}(P_c))^2\|H_\sigma\|^2}{q_c r^{2(b-1)}}\|\xi\|^2 \geq 0$, i.e., $\|\xi\| \leq \frac{q_c}{4\bar{\lambda}(P_c)\|H_\sigma\|}\frac{\|\chi\|}{r^{1-b}}$, $\dot{V} \leq 0$ and thus $\chi(t)$ is bounded for all $t \geq 0$. From (8.41) and Assumption 8.1.1, there exist positive real numbers b_1, b_2 such that, $\gamma(x_1) \leq b_1 + b_2|x_1|^{2p_1}$ for all x_1. Then from $x_1 = r^b\chi_1$ and the dynamics of r in (8.28), we have

$$\dot{r} = -\alpha_1 + \frac{\gamma(x_1)}{\alpha_2 r} \leq -\alpha_1 + \frac{b_1 + b_2 r^{2bp_1}|\chi_1|^{2p_1}}{\alpha_2 r}. \quad (8.44)$$

Let $0 < b < \frac{1}{2p_1}$. Then (8.44) implies that $r(t)$ is bounded for all $t \geq 0$.

2) When $\frac{q_c}{4}\|\chi\|^2 - \frac{4(\bar{\lambda}(P_c))^2\|H_\sigma\|^2}{q_c r^{2(b-1)}}\|\xi\|^2 \leq 0$, i.e., $\frac{\|\chi\|}{r^{1-b}} \leq \frac{4\bar{\lambda}(P_c)\|H_\sigma\|}{q_c}\|\xi\|$, $\frac{\|\chi\|}{r^{1-b}} \to 0$ as $t \to \infty$ because $\xi(t)$ does. In this case, (8.44) becomes

$$\dot{r} \leq -\alpha_1 + \frac{b_1 + b_2 r^{2bp_1}|\chi_1|^{2p_1}}{\alpha_2 r} = -\alpha_1 + \frac{b_1}{\alpha_2 r} + \frac{b_2}{\alpha_2}(\frac{|\chi_1|}{r^{1-b}})^{2p_1}\frac{1}{r^{1-2p_1}} \quad (8.45)$$

Since $p_1 \leq 0.5$ by Assumption 8.4.1, (8.45) implies that $r(t)$ is bounded for all $t \geq 0$.

In either of the above two cases, $r(t)$ is bounded for all $t \geq 0$ and thus we can assume $1 \leq r(t) \leq r_m$ for some finite r_m and for all $t \geq 0$. As a result, we can always choose a sufficiently large ϱ such that

$$-\frac{(\varrho-1)(n-1)\bar{\lambda}(P_c)}{\kappa} + \frac{4(\bar{\lambda}(P_c))^2}{q_c r^{2b-1}} \leq -1$$

and in turn, (8.43) becomes

$$\dot{V} \leq -\frac{q_c}{2r_m}\|\chi\|^2 - \|\xi\|^2$$

which by Theorem 2.1.5 implies the trajectory $(\chi(t), \tilde{H}^T(t), \xi(t))$ is bounded for all $t \geq 0$, and moreover, $\chi(t) \to 0$ as $t \to \infty$. Noting $x_j = r^{j-1+b}\chi_j$ and $1 \leq r(t) \leq r_m$ yields $x(t) \to 0$ as $t \to \infty$ and further noting (8.29) yields the boundedness of the trajectory of the closed-loop system composed of system (8.1) and dynamic state feedback controller (8.28).

It remains to show that $\tilde{H}(t) \to 0$ as $t \to \infty$. To this end, we first show by Barbalat's lemma that $\dot{\chi}(t) \to 0$ as $t \to \infty$. Since $\chi(t) \to 0$ as $t \to \infty$, to show $\dot{\chi}(t) \to 0$ as $t \to \infty$, we only need to show $\dot{\chi}(t)$ is uniformly continuous. Note that $f_i, i = 1, ..., n$, φ, and γ are all continuously differentiable, then from (8.34), $\dot{\chi}(t)$ is continuously differentiable, i.e., $\ddot{\chi}(t)$ exists. Furthermore, note that $\ddot{\chi}(t)$ is bounded because of the boundedness of the trajectory of the closed-loop system. Then, it follows that $\dot{\chi}(t)$ is uniformly continuous.

In turn by the following lemma [62], we can conclude that $\tilde{H}(t) \to 0$ as $t \to \infty$.

Lemma 8.4.1 [62] Consider the vector functions $\tau, \tilde{H}^T : [0, \infty) \to \mathbb{R}^{n_\tau}$. Suppose $\tilde{H}^T(t)$ is continuously differentiable, and $\dot{\tilde{H}}^T(t) \to 0$ as $t \to \infty$. Assume further that $\tau(t)$ is uniformly bounded and persistently exciting. Then $\tilde{H}^T(t) \to 0$ as $t \to \infty$ provided that $\tilde{H}(t)\tau(t) \to 0$ as $t \to \infty$.

From $(\dot{\chi}(t), \chi(t), x(t), \xi(t)) \to 0$ as $t \to \infty$, and the $\chi, \tilde{H}^T$ subsystems in (8.34), we have $\dot{\tilde{H}}^T(t) \to 0$ as $t \to \infty$ and

$$\lim_{t\to\infty} E(0, r(t), \eta(t))\tilde{H}^T(t) = 0 \tag{8.46}$$

Noting (8.8) and (8.35), (8.46) can be written explicitly as follows:

$$\begin{aligned}
0 &= \lim_{t\to\infty} E(0, r(t), \eta(t))\tilde{H}^T(t) = \lim_{t\to\infty}\left(\frac{f_{n-1}^a(0)}{r^{n+1}(t)}, ..., \frac{f_{i-1}^a(0)}{r^{i+1}(t)}, ..., 1\right)\eta^T(t)\tilde{H}^T(t) \\
&= \lim_{t\to\infty}\left(\frac{f_{n-1}^a(0)}{r^{n+1}(t)}, ..., \frac{f_{i-1}^a(0)}{r^{i+1}(t)}, ..., 1\right)(\xi(t) + T_\sigma\tau(t) + Nx_1(t))^T\tilde{H}^T(t) \\
&= \lim_{t\to\infty}\left(\frac{f_{n-1}^a(0)}{r^{n+1}(t)}, ..., \frac{f_{i-1}^a(0)}{r^{i+1}(t)}, ..., 1\right)(\tilde{H}(t)T_\sigma\tau(t))^T
\end{aligned} \tag{8.47}$$

Since the last element of $(\frac{f_{n-1}^a(0)}{r^{n+1}(t)}, ..., \frac{f_{i-1}^a(0)}{r^{i+1}(t)}, ..., 1)$ is 1 and nonzero,

$$\lim_{t\to\infty} \tilde{H}(t)T_\sigma\tau(t) = 0 \tag{8.48}$$

Since $\tau(t)$ is PE, by Lemma 8.4.1, $\tilde{H}(t)T_\sigma \to 0$ as $t \to \infty$. Finally, note that T_σ is nonsingular, then it follows that $\tilde{H}(t) \to 0$ as $t \to \infty$.

8.5 An Example

Consider the following feedforward system

$$\begin{aligned}
\dot{x}_3 &= x_2 - 0.001(x_1^2 + 0.1^2)^{\frac{1}{4}}(wx_1 + u - wv_1v_2) \\
\dot{x}_2 &= x_1 - 0.001(x_1^2 + 0.1^2)^{\frac{1}{4}}(u - wv_1v_2) \\
\dot{x}_1 &= u - wv_1v_2 \\
y &= (x_1, x_3)
\end{aligned} \tag{8.49}$$

where $|w| \leq 1$, and v_1, v_2 are governed by the following exosystem

$$\dot{v}_1 = -\sigma v_2, \dot{v}_2 = \sigma v_1 \tag{8.50}$$

where the initial state $v(0)$ satisfies $\|v(0)\| \leq 0.5$. In the following, we will study the input disturbance suppression problem of system (8.49) via dynamic output and dynamic state feedback control, respectively.

8.5.1 Dynamic Output Feedback Control

In this section, assume $\sigma = 0.2$. On one hand, note that system (8.49) is in the form of (8.1) and satisfies Assumption 8.1.1 with

$$\gamma_1(x_1) = 0.001(x_1^2 + 0.1^2)^{\frac{1}{4}} \leq a_1 + a_2|x_1|^{p_1} \tag{8.51}$$

where $a_1 = a_2 = 0.001(1 + 0.1^2)^{\frac{1}{4}}$ and $p_1 = 0.5$. On the other hand, note that $d(v, w) = wv_1v_2$, therefore Assumption 8.2.1 holds as well. As a result, Theorem 8.3.1 can be applied to solve the input disturbance suppression problem. To derive the controller (8.7), we have to determine c_1, c_2, b, and M, N, H_σ, and $g_1, g_2, g_3, k_1, k_2, k_3$, and $\alpha_1, \alpha_2, \gamma(x_1)$.

First, we determine c_1, c_2, b. From (8.49), $c_1 = c_2 = 1$, and from $0 < b < \frac{1}{2p_1} = 1$, let $b = 0.9$.

Then, we determine M, N, H_σ. Since $d(v, w) = wv_1v_2$, it can be verified that Assumption 8.2.1 is satisfied with

$$\Phi_\sigma = \begin{bmatrix} 0 & 1 \\ -0.16 & 0 \end{bmatrix}, \Psi = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

To design the internal model, we let

$$M = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solving the Sylvester equation (8.5) gives

$$T_\sigma = \begin{bmatrix} 0.48077 & -0.24038 \\ 0.86207 & -0.86207 \end{bmatrix}$$

and thus $H_\sigma = \Psi T_\sigma^{-1} = \begin{bmatrix} 4.16 & -1.16 \end{bmatrix}$.

Next, we determine $g_1, g_2, g_3, k_1, k_2, k_3$ and $\alpha_1, \alpha_2, \gamma$. By Theorem A1 in [39], given any positive real number b and nonzero real numbers c_1, c_2, there exist positive definite matrices P_o, P_c, and k_i, g_i, $i = 1, ..., 3$, such that (8.17) is satisfied. Here we set

$$k_1 = -3.6, k_2 = -4.32, k_3 = -1.728, g_1 = -0.512, g_2 = -1.92, g_3 = -2.4$$

Under the above setting,

$$A_o = \begin{bmatrix} -2.4 & 1 & 0 \\ -1.92 & 0 & 1 \\ -0.512 & 0 & 0 \end{bmatrix}, A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.728 & -4.32 & -3.6 \end{bmatrix}. \tag{8.52}$$

Furthermore, let

$$P_o = \begin{bmatrix} 0.86638 & -0.5 & -1.2096 \\ -0.5 & 1.2096 & -0.5 \\ -1.2096 & -0.5 & 6.5683 \end{bmatrix}, P_c = \begin{bmatrix} 1.0259 & 1.1148 & 0.11574 \\ 1.1148 & 2.7251 & 0.30434 \\ 0.11574 & 0.30434 & 0.40398 \end{bmatrix}$$

then the coupled Lyapunov inequalities (8.17) is satisfied with

$$q_o = 1, \underline{q}_o = 0.72139, \bar{q}_o = 14.144, q_c = 0.4, \underline{q}_c = 0.65195, \bar{q}_c = 14.008.$$

From $\rho \geq \frac{8(\bar{\lambda}(P_c))^2 \|G\|^2}{q_o q_c}$, set $\rho = 2131.5$. Then, we have

$$\alpha_1 = \frac{1}{4}\min\{\frac{q_o}{\bar{q}_o}, \frac{q_c}{\bar{q}_c}\} = 0.007139, \alpha_2 = \min\{\rho\underline{q}_o, \underline{q}_c\} = 0.65195$$

Noting (8.22) and (8.51) yields the expression of γ. Finally, we set $\kappa = 10^{-10}$.

As an illustration, Fig. 8.1 and Fig. 8.2 show the simulation result of system (8.49)

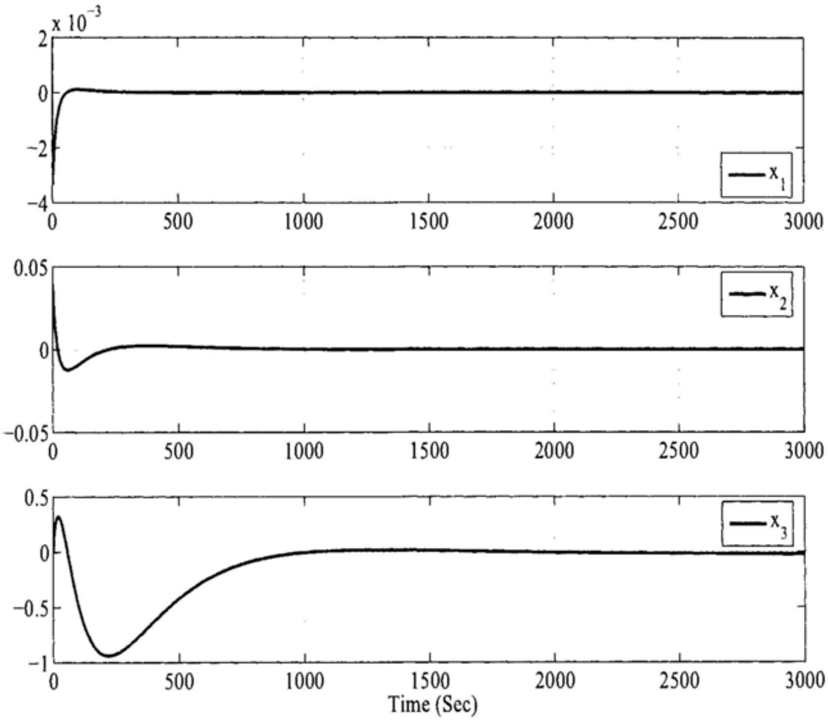

Figure 8.1: Profile of x

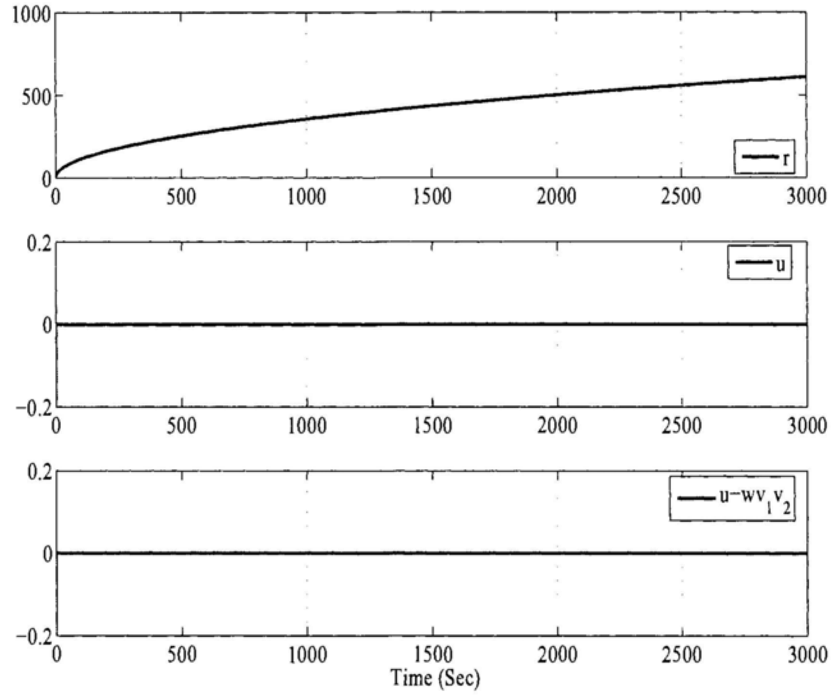

Figure 8.2: Profile of r, u and $u - wv_1v_2$

under the control

$$
\begin{aligned}
u &= \Psi T_{\sigma}^{-1}(\eta - Nx_1) + k_1\frac{\hat{x}_1}{r} + k_2\frac{\hat{x}_2}{r^2} + k_3\frac{\hat{x}_3}{r^3} \\
\dot{\eta} &= M\eta + Nu - MNx_1, \\
\dot{\hat{x}}_3 &= \hat{x}_2 + r^{-1}g_3(\hat{x}_3 - x_3), \\
\dot{\hat{x}}_2 &= \hat{x}_1 + r^{-2}g_2(\hat{x}_3 - x_3), \\
\dot{\hat{x}}_1 &= \bar{u} + r^{-3}g_1(\hat{x}_3 - x_3) \\
\dot{r} &= -\alpha_1 + \frac{\gamma(x_1)}{\alpha_2 r}, r(0) \geq 1,
\end{aligned}
\tag{8.53}
$$

with initial state $(x_1(0), x_2(0), x_3(0), v(0), \eta(0), \hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0), r(0)) = (0.002, 0.04, 0.00125, (-0.01, 0), (0.002, 0.002), 0.002, 0.04, 0.002, 1)$ and $w = 0.5$.

8.5.2 Adaptive Dynamic State Feedback Control

In this section, we still let $\sigma = 0.2$ but assume σ is not known beforehand. Clearly, system (8.49) satisfies Assumptions 8.1.1, 8.2.1, and 8.4.1. In particular, $f_1^a(x_1) = f_2^a(x_1) = -0.001(x_1^2 + 0.1^2)^{\frac{1}{4}}$. Thus Theorem 8.4.1 can be applied to solve the input disturbance suppression problem. To derive the controller (8.28), we have to determine M, N, and b, k_1, k_2, k_3, and $\alpha_1, \alpha_2, \gamma$, and Γ, φ.

First, we determine M, N. For convenience, let M, N, H_σ be the same as the above section. Note that H_σ is not available for feedback.

Then, we determine b, k_1, k_2, k_3 and $\alpha_1, \alpha_2, \gamma$. For convenience, we set b, k_1, k_2, k_3 and A_c, P_c same as the previous section. The coupled Lyapunov inequalities (8.36) is satisfied with

$$q_c = 0.4, \underline{q}_c = 0.65195, \bar{q}_c = 14.008.$$

Furthermore, we have

$$\alpha_1 = \frac{q_c}{4\bar{q}_c} = 0.007139, \alpha_2 = \underline{q}_c = 0.65195.$$

Noting (8.41) and (8.51), and (8.38) yields the expression of γ and φ, respectively. Finally, we set $\Gamma = 10^4$ and $\kappa = 10^{-10}$.

As an illustration, Fig. 8.3, Fig. 8.4 and Fig. 8.5 show the simulation result of system

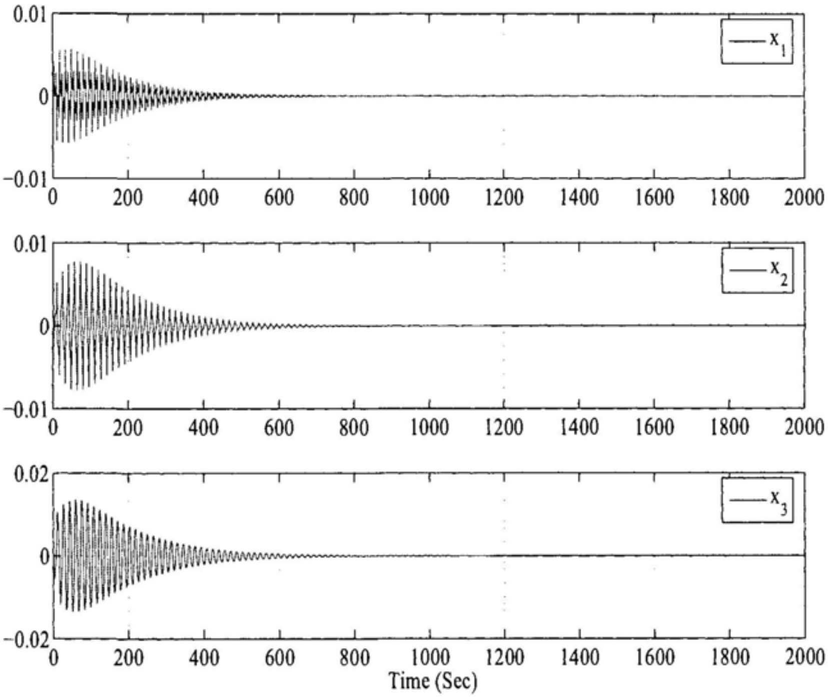

Figure 8.3: Profile of x

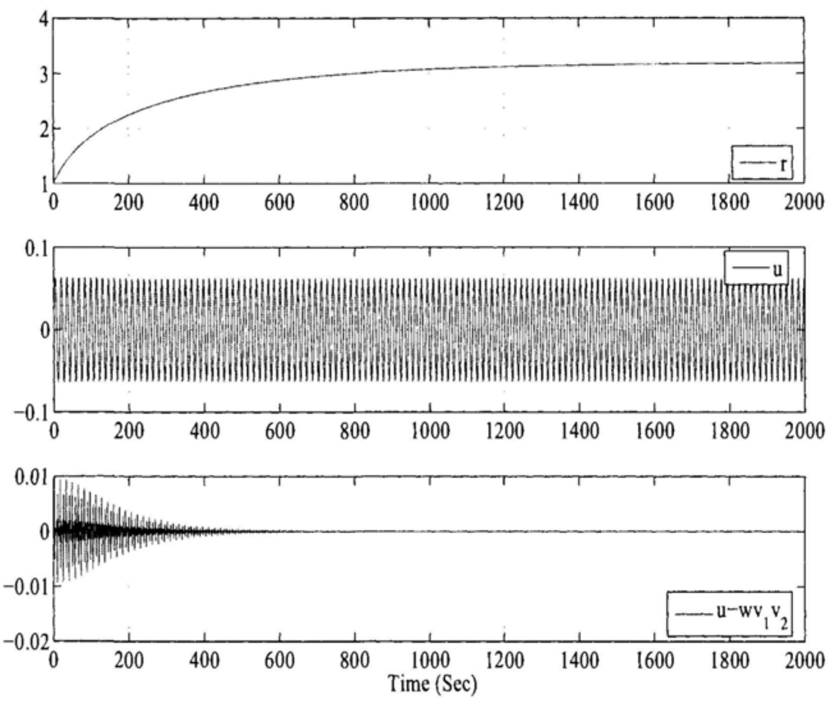

Figure 8.4: Profile of r, u and $u - wv_1v_2$

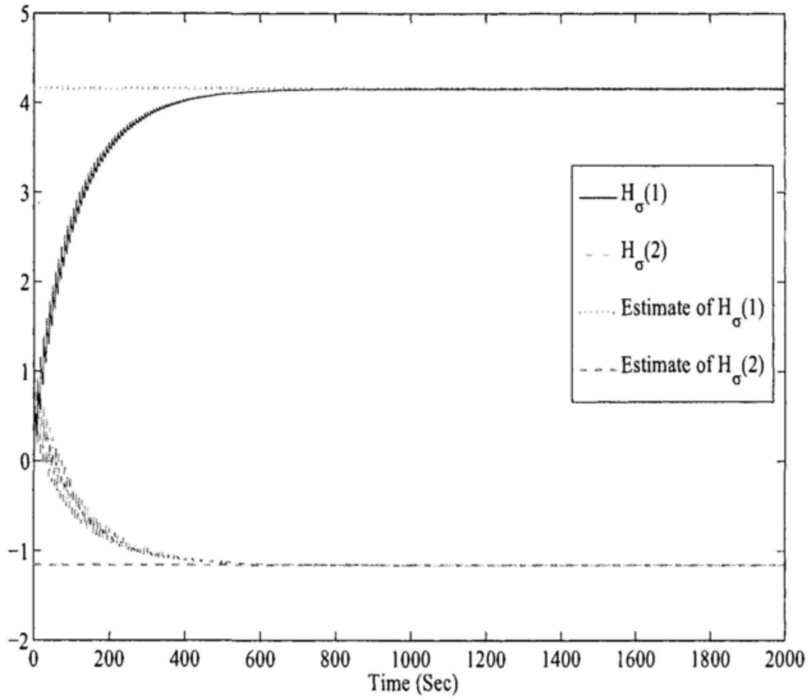

Figure 8.5: Profile of H_σ and its estimate

(8.49) under the control

$$
\begin{aligned}
u &= \hat{H}_\sigma(\eta - Nx_1) + k_1\frac{x_1}{r} + k_2\frac{x_2}{r^2} + k_3\frac{x_3}{r^3} \\
\dot{\eta} &= M\eta + Nu - MNx_1, \\
\dot{\hat{H}}_\sigma &= \Gamma\varphi(x, \eta, r), \\
\dot{r} &= -\alpha_1 + \frac{\gamma(x_1)}{\alpha_2 r}, r(0) \geq 1,
\end{aligned}
$$

with initial state $(x_1(0), x_2(0), x_3(0), v(0), r(0), \eta(0), \hat{H}_\sigma(0)) = (0.005, 0, 0, (0.5, 0), 1, (0.001, 0.02), (-0.003, 0.008))$ and $w = 0.5$.

8.6 Conclusion

In this chapter, we have utilized the dynamic high gain scaling technique to address the input disturbance suppression problem for a class of feedforward systems. In contrast with [56], we take into account two critical cases respectively: the state of the given system is not available and the exosystem is unknown. Especially in the second case, we have proved that the estimated parameter vector can converge to the true parameter vector if some PE condition is satisfied.

Conclusion

In this thesis, we have investigated the global robust stabilization problem and the global robust output regulation problem of feedforward systems. In what follows, we will conclude this thesis with some remarks.

In the first part of this thesis, we have studied the global robust stabilization problem for various classes of feedforward systems containing both time-varying static and dynamic uncertainties. Different from most of the existing approaches, we proposed a pure small gain approach to solve the problem. In contrast with the Lyapunov's linearization method and the asymptotic small gain theorem combined approach, our approach removes two restrictions brought by the Lyapunov's linearization method. On one hand, we do not assume the Jacobian linearization of the given system at the origin be stabilizable. The unstabilizability may come from two aspects: the Jacobian linearization of the dynamic uncertainty is not stabilizable or the Jacobian linearization of the feedforward system itself is not stabilizable. Both of these two cases have been addressed in this thesis. On the other hand, we do not require the Jacobian linearization of the bottom dynamics at each recursion be exponentially stable. It has been shown in the literature that it is difficult to achieve the exponential stability of the Jacobian linearization at the origin for feedforward systems with uncertain constant parameters. It is even more so if the feedforward system contains both time-varying static and dynamic uncertainties. The removal of these two restrictions is made possible because the LB small gain theorem, instead of the Lyapunov's linearization method, is employed to guarantee the local stability of the given system at the origin. Furthermore, the recursive design procedure proposed in this thesis is also quite different from the existing ones, and particularly works for feedforward systems subject to both time-varying static and dynamic uncertainties. Even without the dynamic uncertainty, applying our recursive design procedure can still yield some new results which cannot be handled by the existing approaches.

In the second part of this thesis, we have studied the global robust output regulation problem for a class of feedforward systems. To solve the problem, we first construct a suitable internal model so that the augmented system is stabilizable. Then by performing appropriate coordinate and input transformations on the augmented system, the global robust output regulation problem is converted into a global robust stabilization problem for a class of feedforward systems subject to both time-varying static and dynamic uncertainties. As a result, the global robust stabilization results obtained in the first part of this thesis has been used to solve the global robust output regulation problem. We have applied the results of global robust output regulation problem to solve two trajectory tracking problems for a chain of integrators with uncertain parameters and the VTOL aircraft, respectively. In contrast with the existing designs, for the chain of integrators with uncertain parameters, our design is low gain and does not need the exact knowledge of the reference trajectory, and for the VTOL aircraft, our design is a complete low gain design and thus is more cost effective. Finally, to complete the thesis, we have studied a special case of output regulation problem, the input disturbance suppression problem for a class of feedforward systems by Lyapunov approach. We have designed an adaptive dynamic state feedback controller which can handle the uncertain parameters not only in the plant but also in the exosystem. Furthermore, we have also given the conditions under which an estimated parameter vector can converge to the true parameter vector. In the particular case where the exosystem is known, a dynamic output feedback controller has been designed.

To conclude this thesis, we will depict some future research stemming from the work in this thesis. On one hand, the feedforward systems considered in this thesis are all comprised of n scalar subsystems. Thus it would be meaningful to study the stabilization and output regulation problem for feedforward systems in more general form. On the other hand, to study the stabilization problem of feedforward systems subject to dynamic uncertainty via Lyapunov approach is also a potential issue. However, since the feedforward system does not lend itself easily to the Lyapunov design, research in this direction would be challenging.

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Biography

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[1] T. Chen and J. Huang, "Disturbance attenuation of feedforward systems with dynamic uncertainty," *IEEE Trans. Auto. Cont.*, vol. 53, no. 7, pp. 1711–1717, August, 2008.

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[10] ——, "Adaptive input disturbance suppression for a class of feedforward systems," to be submitted.