Aspects of the Bridge Between Optimization and Game Theory

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Abstract

Both of the two major components of Game Theory, e.g., the non-cooperative game theory and the cooperative game theory, are becoming more and more closely related to the field of optimization, as the needs to study the analytical properties of games start to rise. The results presented in this thesis illustrate several connections between Optimization and Game Theory, and attempts are made to build a bridge between the cooperative game theory and the non-cooperative game theory, to characterize the co-existence of competition and cooperation in practice. We start by applying the properties of Polymatroid Optimization to the cooperative game theory, and show that both of the joint replenish game and the one warehouse multi retailer game are submodular games. In the next part, we show that the strategies promoting learning from history are convergent under certain conditions. This result can also be viewed as an efficient algorithm to compute the Nash Equilibrium of the game. Because the competitive routing game satisfies the condition, we know that if every user adapts with good enough memory, then asymptotically the system converges to Nash Equilibrium. Therefore, if the decision of cooperation is difficult to reverse, then it can be justified for the farsighted players to use the cost structure in the Nash Equilibrium point to decide if they should cooperate or not, instead of reacting to the immediate consequences as a basis to make decisions. With the optimization tools applied, we are able to show that in parallel network, the social cost and the cost of other players tend to decrease if two players cooperate. Also, the price of anarchy is higher when the flow demand of players are more evenly distributed. Using that structural result, we derive the exact upper bound of the price of anarchy for a given parallel network with fixed number of players. The exact upper bound of the price of anarchy for arbitrary parallel network with given number of players, which is independent to the network structure and parameters, can be derived consequently.

中文摘要

随着研究博弈中的解析性质的需求越来越大，博弈论的两个主要组成部分：合作博弈与非合作博弈，都与优化领域越来越紧密地联系在一起。本篇论文中的结果阐述了优化与博弈论之间的一些关联，并且成功地尝试建立了合作博弈与非合作博弈之间的一座桥梁，以用来描述实际应用中合作与竞争的共存。我们首先把广义拟阵优化的性质应用到博弈论中，证明了合作进货博弈及单仓库多零售博弈均为子模博弈。随后我们证明了鼓励从历史中学习的策略在某些情况下是收敛的。这个结果同时也可以看作一个计算博弈的Nash均衡点的有效算法。因为竞争网络博弈满足上述的条件，如果其中的每一个用户都使用以上策略并有足够好的记忆力，那么从长远来看系统会趋近于Nash均衡点。所以如果关于合作的决定很难反悔，那么对于有远见的参与者来说，使用在Nash均衡点的费用结构来作为决策基础是成为一个合理的策略。我们使用了一些优化领域的工具证明了当两者合作时，系统费用及其他参与者的费用都会减少。而且各个用户之间的流量需求分布得越平均，博弈的调和率越高。根据这个结构性的结果，我们得到了限制参与者人数的给定平行图及所有平行图的精确的调和率上界。

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This work is dedicated to my parents

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Chapter 1

An Overview

Game Theory has two major components, the non-cooperative game theory and cooperative game theory. These two components seem to be quite distant from each other on the first look, with totally different ingredients and flavors. However, both of them are becoming more and more closely connected to the field of optimization, as the needs to study the analytical properties of the games began to rise sharply. For an excellent coverage on algorithmic game theory, we refer the readers to [46].

The tools in algorithm design and optimization have in particular attracted attention from researchers in cooperative game theory, since it is necessary to show the existence of the core, and to answer other basic questions of the game, for example, how to compute the core, is the game submodular or not, etc. For example, in supply chain management, getting all parties to agree on how to share the costs and benefits has been identified as one of the major barriers to collaborative commerce in practice [17] [45], and an excellent survey can be found in Nagarajan and Sosic [42]. In the game related to the well-known joint replenishment model, this question reduces to the mathematical question whether or not the game is submodular. The submodularity captures the notion of decreasing marginal cost, and if the game is submodular then as a consequence there is a population monotonic allocation scheme, under which the cost is shared fairly among the participants and every coalition has no conflict of interest with anyone else. In a recent paper, Anily and Haviv [4] showed that the joint replenish game is submodular when the joint setup cost has the so-called first order interaction structure, and Jiawei Zhang showed that the game has population monotonic allocation scheme [64] when the joint setup cost is submodular. However, it remains open if this game is submodular or not, under the general assumption that the joint setup cost is submodular. The work in the next chapter is to answer the question, by using the tool known as polymatroid optimization.

For the non-cooperative game theory, in 1928 von Neumann established the mini-max theorem, which also shows that certain zero sum game has an equilibrium solution. Later, John Nash in 1951 [43] extended the result to show that basically all games will have equilibria. To analyze the properties of the equilibrium point and the game quantitatively, it is important to be able to find the equilibrium point efficiently first. Therefore, the question whether there exists an efficient algorithm to compute the Equilibrium point has been raised by many researchers in the field. However, even for simple games, for example, bimatrix game, this problem has remained to be unsettled for many years, until recently shown by Chen and Deng [7] that it is PPAD-complete. But for certain types of games, it is still possible to design polynomial time algorithm to find the Nash Equilibrium point. The most well known example is the two person zero-sum game, which corresponds exactly to Linear Programming. During the years, many works have been dedicated to designing efficient algorithms for special games. For example, the spectrum management game in wireless communication, was shown by Luo and Pang [37] that an iterative waterfilling algorithm does converge to the Nash Equilibrium.

There are also algorithms which can compute the Equilibrium point, but are not assumed to run within polynomial time. The typical example of it is the famous Lemke's algorithm for LCP, which corresponds to finding the equilibrium point of a bimatrix game. To design an efficient algorithm, or to rule out the possibility of doing so, it is natural to use ideas and results from the theory of algorithm design and optimization.

It is natural to assume that when people make decisions, they are primarily concerned only with their self interests, which is known as the rationality hypothesis in economics. However, usually people make decisions based not only on the current situation, but also on the history that they observed. It is therefore interesting to study the dynamics of the system, if everyone learns from history. The fictitious play has been designed naturally following this idea, as an algorithm to find a Nash Equilibrium of a game. It can be interpreted as everyone simply plays the best response to the average of the history being observed. It has been shown by Robinson [50] and Brown [6] that this algorithm does converge to the Nash Equilibrium of the zero-sum bimatrix game. However, even in the simplest case, this strategy converges very slowly. And it has been noticed that by applying the idea of fading memory, the convergence speed can be vastly improved [39] [40] [35] (if it does converge at all). In the third chapter of this thesis, we establish the converge result for this type of strategy, for games that correspond to an monotone LCP system. And we also conduct empirical studies on the tradeoff between the convergence speed and the condition of convergence, with respect to the better or worse memories.

It is also interesting to see whether or not there is a bridge between the cooperative game theory and the non-cooperative game theory. In particular, we are interested in how people decide to cooperate, and the effects of the cooperation. Specifically, we study the network routing game, which can be represented by a monotone LCP system. By the second chapter of this thesis, we know that if everyone has good enough memory and make decisions based on that, the system converges to the Nash Equilibrium. Therefore, it makes sense for the players to make a decision whether to cooperate or not, based on their costs under the Nash Equilibrium instead of the immediate consequence of the action. In the parallel network, we study the consequences of cooperation by two players, to the social cost and the costs of other players, which can be shown to be beneficial to all of them, except the two players in action. However, there is always an incentive for the biggest company to grow by grabbing partial demands from the second biggest one. Also, we derive tight upper bound of the price of anarchy for a certain network where there are k players, as well as tight upper bound among all possible networks, independent of network structure and parameters.

Now we shall say a few words about the organization of this thesis. The second chapter is based on a joint paper coauthored with Dr. Jiawei Zhang and my advisor Shuzhong Zhang. In this chapter, we focus on studying the properties of polymatroid optimization, and the application of it in cooperative game theory, showing that both the joint replenish game and the one warehouse multi retailer game are submodular games.

The third chapter is about the convergence properties for some quadratic games, with the structure that each player learns from the history. We also give empirical example which shows the connection between the convergence rate and the condition for convergence. In short, the better the memory, the higher the chance of convergence, but the worse the convergence rate (if it does converge at all). This work was originated from a project with my colleague Mr. Li Min, motivated by Prof Tom Luo's game theory class and his work on game theory models in wireless communication. The result is then extended to general game models, with the joint strong convexity property, and shows that the strategy converges linearly under this condition.

The fourth chapter is from an ongoing project with Mr. Xiaguo Wang and my advisor Shuzhong Zhang. As an attempt to study the connection between the noncooperative game theory and the cooperative game theory. We study the incentive of players to cooperate, and the influences of this cooperation to the other players as well as the whole society, by assuming everyone is far-sighted and will predict based on the Nash-Equilibrium scenario. The model is restricted to the routing game with linear unit cost functions on each arc. For parallel network, we show that if players cooperate, the social cost would decrease accordingly. Based on this result, we can identify the worst case of the "price of anarchy" for any given network, and the number of players with given total flow demand. Also we can upperbound the price of anarchy for k many players at parallel network by $\frac{4k^2}{3k^2+2k-1}$.

Chapter 2

Polymatroid, Submodularity and Joint Replenish Game

This chapter is based on a joint work with Dr. Jiawei Zhang and Dr. Shuzhong Zhang [30], which is currently under review by Operations Research. Most material presented here is as the formate submitted, with some editing to fit in the structure of the thesis. We consider the problem of maximizing a separable concave function over a polymatroid. More specifically, we study the submodularity of its optimal objective value in the parameters of the objective function. This question is interesting in its own right and is encountered in many applications. But our research has been mainly motivated by a cooperative game associated with the well-known joint replenishment model. By applying our general results on polymatroid optimization, we prove that this cooperative game is submodular, if the joint setup cost is a normalized and non-decreasing submodular function. Furthermore, the same result holds true for a more general one-warehouse multiple retailer game, which affirmatively answers an open question posed by Anily and Haviv [4] and Zhang [64]. The submodularity results regarding polymatroid optimization also motivates the use of greedy algorithms for certain NP-hard optimization problems.

2.1 Introduction

In recent years, many companies have come to realize that their performance can be improved significantly by exploring innovative collaborative strategies in supply chain management. Companies can collaborate in many different ways. For example, shippers that make small, frequent shipments that do not use the full capacity of their trucks can collaborate and consolidate their orders into hopefully cheaper, faster truckloads. It has been reported that such collaboration among shippers leads to significant reduction in transportation cost as well as inventory cost. It is also known that inventory pooling is an effective way to reduce safety stock and increase customer service [17, 45]. Thus, some companies collaborate by sharing their inventories. The cooperation usually takes the form of lateral transshipment from a location with a surplus of on-hand inventory to a location that faces a stockout.

One issue in such collaboration is to keep different parties motivated to collaborate. The willingness to collaborate often depends on the existence of mechanisms that allocate the cost or gain (from the collaboration) in such a way that is considered advantageous by all the participants. Even though collaboration often leads to overall cost reduction, it is not always the case that such mechanisms exist. Indeed, getting all parties to agree on how to share costs and benefits was identified as one of the major barriers to collaborative commerce in practice (see [17, 45]).

It is natural to apply cooperative game theory to analyze cost allocation issues. Indeed, supply chain collaborations have motivated more and more studies on cooperative games in the last few years; see Nagarajan and Sosic [42] for an excellent review in this area.

Our work is motivated by a cooperative game that is associated with the well-known joint replenishment model. In this model, there are multiple retailers which sell a single product. Constant customer demand occurs at each retailer over an infinite time horizon. The retailers replenish their inventories by ordering from an external supplier. There are two types of costs: a holding cost charged against each unit of inventory per unit time at each retailer, and a setup cost charged against each order that is a *submodular* function of the set of retailers that places the order together. We shall define submodularity in Section 2.2. Roughly speaking, submodularity captures the notion of decreasing marginal cost. For examples of submodular setup cost functions, we refer interested readers to Federgruen and Zheng [22]. The lead times are assumed to be zero, i.e., orders are delivered instantaneously. The goal of the model is to find an inventory replenishment policy for the system that minimizes the long-run average cost over an infinite time horizon. The optimal policy for this joint replenishment problem is unknown. However, it is well-known that a class of easy-to-implement policies, called power-of-two policies, are 98% effective; see Roundy [54] and Federgruen and Zheng [22].

We assume that the retailers follow an optimal power-of-two policy to replenish their inventories. We are interested in the question of how the system-wide cost should be allocated among the retailers. A proper cost allocation scheme is important particularly when the retailers belong to different firms or are decentralized divisions of an organization. For this purpose, we formulate a cooperative game (in coalitional form) denoted by (N, V) where the grand coalition N is the set of all retailers, and for any subset $S \subseteq N$, the characteristic cost function $V(S)$ is the system-wide cost under optimal power-of-two policy when the system consists only of retailers in S. We call this cooperative game the joint replenishment game.

The theoretical question that we would like to address regarding the joint replenishment game is whether the characteristic cost function $V(\cdot)$ is submodular or not. If the answer is yes, then the joint replenishment game is submodular (or concave). This question is of particular importance since a submodular game has many nice properties. We mention a few of them below. *First*, if $V(\cdot)$ is submodular, then the grand coalition is stable. That is, there exists a cost allocation under which no group of retailers would be better off by deviating from the grand coalition and acting alone. Such an allocation is often called a core allocation. *Second*, if $V(\cdot)$ is submodular, then there exist efficient (polynomial time) algorithms to find a core allocation and check whether a given allocation is a core allocation or not. This is important because for a non-submodular game, it is possible that finding a core allocation can be done in polynomial time, but the problem of deciding whether a given allocation is a core allocation or not may be NP-hard. *Third*, for a submodular game, its nucleolus can be computed in polynomial time, and it has a large core, and its stable set coincides with the core. See Peleg and Sudholter [48] for the definition of the aforementioned important concepts in cooperative game theory.

In a recent paper, Anily and Haviv [4] show that the joint replenishment game is submodular when the joint setup cost function, denoted by $K(\cdot)$, has the so-called first order interaction structure, i.e., there exist K_0 and K_i for $i \in N$ such that $K(S) = K_0 + \sum_{i \in S} K_i$ for any $S \subseteq N$. However, the submodularity of the joint replenishment game with general submodular setup cost function $K(\cdot)$ has been posted as an open question in [4] and a more recent paper on this subject [64]. The joint replenishment game is known to have a population monotonic allocation scheme [65], which typically is an indication that a game may be submodular. The population monotonicity implies that no retailer would be worse off when a new retailer joins the coalition. As we shall see in Section 2.3, the function $V(S)$ can be expressed as

$$\begin{aligned} V(S) = \quad \max \quad & \textstyle\sum_{i\in S} f_i(k_i) \\ \text{s.t.} \quad & \textstyle\sum_{i\in A} k_i \leq K(A) \quad \forall A \subseteq S \\ & k \in \mathbb{R}_+^{|S|}, \end{aligned} \tag{2.1.1}$$

where $k \in \mathbb{R}_+^{|S|}$ is the decision variable and for each $i \in N$, $f_i(k_i)$ is a concave function of k_i. Also, given our assumptions on the joint replenishment model, the feasible set of (2.1.1) turns out to be a polymatroid. Our goal is to show that the function $V(\cdot)$ defined in (2.1.1) is submodular.

This motivates us to consider the class of optimization problems of maximizing a separable concave function (or minimizing a separable convex function) over a polymatroid. Besides the joint replenishment model described above, this class of problems has many important applications in combinatorial optimization, resource allocation [31], dynamic scheduling [62], information theory [59], and many other areas. These problems can be solved by greedy algorithms; see Edmonds [18] and Federgruen and Groenevelt [20] and the references therein. We mention that this class of problems is a special case of the polynomially solvable problems studied by Hochbaum and Shanthikumar [32].

The main contributions of this work are the following. First, we show that the optimal objective value (of the polymatroid maximization problem with a separable concave objective function) as a function of the index set is submodular. This immediately implies that the joint replenishment game is submodular. We also prove the submodularity of the optimal objective value with respect to certain parameters of the objective function. This can be used to prove the submodularity of the one-warehouse multiple retailer game studied in [64], which is a generalization of the joint replenishment game.

The remainder of the chapter is organized as follows. In Section 2.2, we present our result regarding maximizing a separable concave function over a polymatroid. This result is applied, in Section 2.3, to derive the submodularity of the joint replenishment game and the one-warehouse multiple retailer game. In Section 2.4, we apply our result to the network source location problem. We conclude the chapter in Section 2.5.

2.2 A Structural Result on Polymatroid Optimization

In this section, we consider the problem of maximizing a separable concave function over a polymatroid. We study the submodularity of the optimal objective value with respect to the parameters of the objective function and the index set. In order to present our key results, we first formally introduce the necessary concepts and notations below.

Given a finite set E, let $2^E = \{A : A \subseteq E\}$ be its power set. A function $z : 2^E \to \mathbb{R}$ is said to be *submodular* if for all $A, B \subseteq E$,

$$z(A \cup B) + z(A \cap B) \leq z(A) + z(B).$$

A function $z : 2^E \to \mathbb{R}$ is said to be *supermodular* if $-z$ is submodular.

A function $z : 2^E \to \mathbb{R}$ is called a *rank funation*, if it satisfies the following conditions:

- z is normalized, i.e., $z(\emptyset) = 0$;
- z is nondecreasing, i.e., $z(A) \leq z(B)$ whenever $A \subseteq B \subseteq E$;
- z is submodular.

For a given finite set E, and a function $z : 2^E \to \mathbb{R}$, the polyhedron

$$P(z, E) = \{x \in \mathbb{R}_+^{|E|} : \sum_{i \in A} x_i \leq z(A) \quad \text{for all} \quad A \subseteq E\}$$

is called a *polymatroid*, if z is a rank function. Throughout this chapter, we denote $P(z, E)$ the polymatroid defined by the finite E and the rank function z.

A set $\mathcal{X} \subseteq \mathbb{R}^n$ is a *sublattice* if for any $x, y \in \mathcal{X}$, we have $x \vee y \in \mathcal{X}$ and $x \wedge y \in \mathcal{X}$, where $x \vee y$ and $x \wedge y$ denote, respectively, the coordinatewise maximum and minimum of x, y, i.e., $x \vee y = (\max(x_1, y_1), \cdots, \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \cdots, \min(x_n, y_n))$. If $\mathcal{X} \subseteq \mathbb{R}^n$ is a sublattice, then a function $f : \mathcal{X} \to \mathbb{R}$ is said to be *submodular*, if for all $x, y \in \mathcal{X}$,

$$f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$$

A function $f : \mathcal{X} \to \mathbb{R}$ is said to be *supermodular*, if $-f$ is submodular. Several supermodular/submodular functions that we shall refer to in this chapter are listed below.

Example 1. Let $\mathcal{X} \in \mathbb{R}^n, \mathcal{Y} \in \mathbb{R}^n$ be two sublattices. Then function $f(x, y) = x^T y : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is supermodular.

Example 2. Let $\mathcal{X} \in \mathbb{R}^n, \mathcal{Y} \in \mathbb{R}^n$ be two sublattices. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a separable convex function. Then function $f(x, y) = g(x - y) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is submodular. In particular, function $f(x, y) = \sum_{i=1}^n (x_i - y_i)^+ : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is submodular.

Here and throughout the chapter, we denote $x^+ = x \vee 0$ for any $x \in \mathbb{R}^n$.

Example 3. If function $f : \mathbb{R}^n \to \mathbb{R}$ is submodular, and function $g_i : \mathbb{R} \to \mathbb{R}$ is monotonic for each $i = 1, 2, \ldots, n$, then function $h(x) = f(y)$ with $y_i = g_i(x_i)$ is submodular too.

2.2.1 Optimizing a Linear Function

In this subsection, we start with a special case, i.e., maximizing a linear function over a polymatroid. More specifically, for any vector $a \in \mathbb{R}^{|E|}$ and subset $A \subseteq E$ consider

$$\begin{aligned} \max \quad & \textstyle\sum_{i \in A} a_i x_i \\ \text{s.t.} \quad & x \in P(z, E). \end{aligned} \tag{2.2.1}$$

We also consider a closely related problem

$$\begin{aligned} \max \quad & \textstyle\sum_{i \in A} a_i x_i \\ \text{s.t.} \quad & x \in P(z, A). \end{aligned} \tag{2.2.2}$$

It is clear that problems (2.2.2) and (2.2.1) share the same optimal objective value. Furthermore, any optimal solution to problem (2.2.2) can be extended to an optimal solution to problem (2.2.1). In particular, let x_A^* be an optimal solution to (2.2.2), and define x_E^* as follows: for any $i \in A$, $(x_E^*)_i = (x_A^*)_i$, otherwise, $(x_E^*)_i = 0$. Then x_E^* is optimal to (2.2.1).

The following is a well-known result [18] concerning an optimal solution of linear program (2.2.2). We shall refer to this result in several places of this chapter.

Lemma 2.2.1 *Assume $a \in \mathbb{R}_+^{|A|}$ and let $\tilde{\pi} = (\tilde{\pi}_1, \cdots, \tilde{\pi}_{|A|})$ be a permutation of set A, so that $a_{\tilde{\pi}_1} \geq a_{\tilde{\pi}_2} \geq \cdots \geq a_{\tilde{\pi}_{|A|}} \geq 0$. Define $x_{\tilde{\pi}} = (x_{\tilde{\pi}_i} : i \in A)$ as follows.*

$$\begin{aligned} x_{\tilde{\pi}_1} &= z(\{\tilde{\pi}_1\}) \\ x_{\tilde{\pi}_i} &= z(\{\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_i\}) - z(\{\tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_{i-1}\}), \quad i = 2, 3, \cdots, |A|. \end{aligned} \tag{2.2.3}$$

Then $x_{\tilde{\pi}}$ is an optimal solution to (2.2.2). Furthermore, if for any $i, j \in A$ with $i \neq j$, $a_i \neq a_j$, then $x_{\tilde{\pi}}$ is the unique optimal solution to (2.2.2).

We next study the property of the optimal objective value of linear program (2.2.1). We first show that the optimal objective value as a function of the set A, is submodular. To the best of our knowledge, this result was first formally stated and proved in Schulz and Uhan [56], where it is used to show that certain scheduling games are supermodular. A slightly weaker version (regarding matroid optimization) of this result had appeared in Nemhauser et al. [44]. We provide an alternative and simple proof here.

Theorem 2.2.2 *For fixed $a \in \mathbb{R}^{|E|}$, let $h(A)$ denote the optimal objective value of problem (2.2.1). Then function $h : 2^E \to \mathbb{R}$ is submodular.*

Proof. We first assume $a \geq 0$. Let $\pi = (\pi_1, \cdots, \pi_{|E|})$ be a permutation of set E, so that

$$a_{\pi_1} \geq a_{\pi_2} \geq \cdots \geq a_{\pi_{|E|}} \geq 0.$$

By Lemma 2.2.1, we know that $x_\pi = (x_{\pi_i} : i \in E)$ with

$$x_{\pi_i} = z(\{\pi_1, \pi_2, \cdots, \pi_i\} \cap A) - z(\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} \cap A)$$

is an optimal solution to problem (2.2.1). By definition, $x_{\pi_i} = 0$ if $\pi_i \notin A$. Therefore, the optimal objective value of (2.2.1) is

$$\begin{aligned} h(A) &= \sum_{i \in A} a_{\pi_i} x_{\pi_i} \\ &= \sum_{i \in E} a_{\pi_i} x_{\pi_i} \\ &= \sum_{i \in E} a_{\pi_i} \left(z(\{\pi_1, \pi_2, \cdots, \pi_i\} \cap A) - z(\{\pi_1, \pi_2, \cdots, \pi_{i-1}\} \cap A) \right) \\ &= \sum_{i \in E} (a_{\pi_i} - a_{\pi_i + 1})\, z(\{\pi_1, \pi_2, \cdots, \pi_i\} \cap A) \end{aligned}$$

where we define $a_{\pi_n+1} = 0$. It follows immediately that $h(A)$ is submodular, because for each i, $a_{\pi_i} - a_{\pi_i+1} \geq 0$, $z(\{\pi_1, \pi_2, \cdots, \pi_i\} \cap A)$ is submodular in A, and non-negative linear combination of submodular functions is also submodular.

When $a \notin \mathrm{Re}_+^{|E|}$, notice that the optimal objective value $h(A)$ is equal to the optimal objective value of the following problem

$$\max \quad \sum_{i \in A} a_i^+ x_i$$
$$s.t. \quad x \in P(z, E),$$

which is submodular in A. This completes the proof. □

Theorem 2.2.2 can be extended to the case where there are both upper and lower bounds for the decision variables. We shall need this result in the next subsection.

Corollary 2.2.3 *For fixed $a, \underline{\omega}, \overline{\omega} \in \mathbb{R}^{|E|}$, let $h(A)$ denote the optimal objective value of the following linear program,*

$$\begin{aligned} \max \quad & \textstyle\sum_{i\in A} a_i x_i \\ s.t. \quad & x \in P(z, E) \\ & \underline{\omega} \leq x \leq \overline{\omega}. \end{aligned} \tag{2.2.4}$$

then function $h : 2^E \to \mathbb{R}$ is submodular if (2.2.4) is feasible.

Proof. Without loss of generality, we assume that $\underline{\omega} \geq 0$. Define another set function $z' : 2^E \to \mathrm{Re}$, such that for any $S \subseteq E$,

$$z'(S) = \min_{R \subseteq S} z(S \setminus R) + \sum_{i \in R} \overline{\omega}_i.$$

It is known from [18] that z' is a rank function, i.e., $P(z', E)$ is a polymatroid, and that furthermore,

$$P(z', E) = P(z, E) \cap \{x : x \leq \overline{\omega}\}.$$

Therefore, linear program (2.2.4) is equivalent to

$$\begin{aligned} \max \quad & \textstyle\sum_{i\in A} a_i x_i \\ \text{s.t.} \quad & x \in P(z', E) \\ & x \geq \underline{\omega}, \end{aligned}$$

which in turn is equivalent to

$$\begin{aligned} \max \quad & \textstyle\sum_{i\in A} a_i y_i + \sum_{i\in A} a_i \underline{w}_i \\ s.t. \quad & \textstyle\sum_{i\in S} y_i \leq \overline{z}(S) := z'(S) - \sum_{i\in S} \underline{\omega}_i \quad \forall S \subseteq E \\ & y \geq 0. \end{aligned} \tag{2.2.5}$$

Define $\hat{z}(S) = \min_{S'\supseteq S} \overline{z}(S')$. Let y be a feasible solution to problem (2.2.5). Notice that for any $S' \supseteq S$,

$$\sum_{i\in S} y_i \leq \sum_{i\in S'} y_i \leq \bar{z}(S').$$

Thus,

$$\sum_{i\in S} y_i \leq \hat{z}(S).$$

On the other hand, if $\sum_{i\in S} y_i \leq \hat{z}(S)$, then $\sum_{i\in S} y_i \leq \overline{z}(S)$. Thus, problem (2.2.5) is equivalent to

$$\begin{aligned} \max \quad & \sum_{i\in A} a_i y_i + \sum_{i\in A} a_i \underline{w}_i \\ s.t. \quad & \sum_{i\in S} y_i \leq \hat{z}(S) \quad \forall S \subseteq E \\ & y \geq 0. \end{aligned}$$

Now we show that $\hat{z}$ is a rank function. It is clear that $\hat{z}$ is normalized and non-decreasing. We need only to show that $\hat{z}$ is submodular.

Notice that $\overline{z}$ is submodular. For any sets S_1 and S_2, there exist $S_1' \supseteq S_1$ and $S_2' \supseteq S_2$ such that

$\hat{z}(S_1) = \overline{z}(S_1')$ and $\hat{z}(S_2) = \overline{z}(S_2')$. Therefore, we have that

$$\begin{aligned} & \hat{z}(S_1) + \hat{z}(S_2) \\ = \; & \overline{z}(S_1') + \overline{z}(S_2') \\ \geq \; & \overline{z}(S_1' \cap S_2') + \overline{z}(S_1' \cup S_2') \\ \geq \; & \hat{z}(S_1 \cap S_2) + \hat{z}(S_1 \cup S_2) \end{aligned}$$

where the first inequality follows from the fact that $\overline{z}$ is submodular, and the second inequality follows from the fact that $S_1' \cap S_2' \supseteq S_1 \cap S_2$ and $S_1' \cup S_2' \supseteq S_1 \cup S_2$. This completes the proof. □

Let g be the optimal objective value of (2.2.2) with $A = E$ and $n = |E|$. In particular,

$$\begin{aligned} g(a) \quad := \quad & \max \quad a^{\mathrm{T}} x \\ & \text{s.t.} \quad x \in P(z, E). \end{aligned} \tag{2.2.6}$$

Clearly, g is a convex function of a. There are actually two different ways to interpret the function g. First, g is a natural extension of the set function z: for any $S \subseteq E$ and denoting 1_S to be the indicator vector of S, we always have $g(1_S) = z(S)$. Second, for any $a \in \mathbb{R}^n_+$ there is a unique decomposition

$$a = \lambda_1 1_{S_1} + \cdots + \lambda_m 1_{S_m}$$

where $\lambda_i > 0$, $i = 1, ..., m$, and $S_1 \supset S_2 \supset \cdots \supset S_m$. Then,

$$g(a) = \lambda_1 z(S_1) + \cdots + \lambda_m z(S_m). \tag{2.2.7}$$

Lovász [36] showed that the definitions (2.2.6) and (2.2.7) are equivalent if and only if z is submodular.

For any $a \in \mathbb{R}^n_+$, an explicit way to write g is to introduce a permutation of set E, denoted by $\pi(a)$, so that

$$a_{\pi_1(a)} \geq a_{\pi_2(a)} \geq \cdots \geq a_{\pi_n(a)}.$$

Furthermore, we define the index sets

$$\Pi_i(a) = \{\pi_1(a), ..., \pi_i(a)\},\ i = 1, 2, ..., n.$$

As a convention, denote $\Pi_0(a) := \emptyset$. Then, for any $a \in \mathbb{R}^n_+$

$$g(a) = \sum_{i=1}^{n} a_{\pi_i(a)} \left(z(\Pi_i(a)) - z(\Pi_{i-1}(a)) \right). \tag{2.2.8}$$

Our main result of this section is to show the submodularity of the optimal objective value of problem (2.2.1) with respect to the objective parameter vector.

Theorem 2.2.4 *Consider problem (2.2.1) with $A = E$ and let $g(a)$ be defined as in (2.2.6). Then, (i) g is homogeneous, i.e. $g(\lambda a) = \lambda g(a)$ for any $\lambda \geq 0$; (ii) $g(a)$ is a convex function; (iii) $g(a)$ is submodular, i.e.*

$$g(a \vee b) + g(a \wedge b) \leq g(a) + g(b) \tag{2.2.9}$$

for all $a, b \in \mathbb{R}^n$. These results hold even if there are lower and upper bounds on the decision variables as in problem (2.2.4).

Proof. The properties (i) and (ii) are rather straightforward; they follow directly from the definition of g in (2.2.6). Let us now focus on (iii).

We first show that it is sufficient to prove the submodularity of g in $\mathrm{Re}_+^{|E|}$. Assume that g is submodular in $\mathrm{Re}_+^{|E|}$. For any $a \notin \mathrm{Re}_+^{|E|}$, notice that $g(a) = g(a^+)$. Thus, for any $a, b \in \mathrm{Re}^{|E|}$, we have

$$\begin{aligned} & g(a \vee b) + g(a \wedge b) \\ = \; & g((a \vee b)^+) + g((a \wedge b)^+) \\ = \; & g(a^+ \vee b^+) + g(a^+ \wedge b^+) \\ \leq \; & g(a^+) + g(b^+) \\ = \; & g(a) + g(b) \end{aligned}$$

where the inequality follows from the submoduarity of g in $\mathrm{Re}_+^{|E|}$.

Therefore, in the rest of the proof, we prove (2.2.9) for $a, b \in \mathrm{Re}_+^n$. It suffices to show: for any $a, u, v \in \mathrm{Re}_+^n$, and $u^\mathrm{T} v = 0$, it holds that

$$g(a+v) - g(a) \geq g(a+u+v) - g(a+u). \tag{2.2.10}$$

Notice that $u^\mathrm{T} v = 0$ means that $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$ where we denote $\mathrm{supp}(u)$ to be the support of u, i.e., $\mathrm{supp}(u) = \{i \in E : u_i > 0\}$. Furthermore, since g is clearly a continuous function, we need only to show that (2.2.10) holds for the vectors a, u, v whose positive parts are in general geometric positions, i.e., the positive coordinates of $a, u, v, a+u, a+v, u+v$ are all different.

To that end, we define, for any $i \in E$,

$$s_{\pi_i(a)} = z(\Pi_i(a)) - z(\Pi_{i-1}(a)).$$

By (2.2.8), it is clear that $s(a) \in \partial g(a)$, i.e., $s(a)$ is a subgradient for the convex function g at a. Clearly, as long as $\pi(a)$ remains unchanged, $s(a)$ is a constant vector.

Furthermore, we note that since a, u, v are generally positioned on their supports, the permutations $\pi(a+tv)$ and $\pi(a+u+tv)$ are uniquely determined on $\mathrm{supp}(a) \cup \mathrm{supp}(v)$ and $\mathrm{supp}(a) \cup \mathrm{supp}(u) \cup \mathrm{supp}(v)$

respectively, for almost all t in $[0, 1]$ except for no more than $O(n^2)$ discrete values of t. By (2.2.8), it follows that, $g(a + tv)$ and $g(a + u + tv)$ as functions of t are everywhere differentiable, except for at most $O(n^2)$ points.

Therefore,

$$g(a + v) - g(a) = \int_0^1 s(a + tv)^{\mathrm{T}} v dt$$

and

$$g(a + u + v) - g(a + u) = \int_0^1 s(a + u + tv)^{\mathrm{T}} v dt,$$

(see e.g. Corollary 24.2.1 of Rockafellar [51]). It follows that, in order to prove (2.2.10), it will be sufficient to show

$$s(a + tv)^{\mathrm{T}} v \geq s(a + u + tv)^{\mathrm{T}} v \tag{2.2.11}$$

for almost all $t \in [0, 1]$ (except for at most $O(n^2)$ points).

Now, consider a general t value such that $\pi(a + tv)$ and $\pi(a + u + tv)$ are uniquely determined for the parts $\mathrm{supp}(a) \cup \mathrm{supp}(v)$ and $\mathrm{supp}(a) \cup \mathrm{supp}(u) \cup \mathrm{supp}(v)$ respectively, and consider a given $i \in \mathrm{supp}(v)$. Since $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$, for any j, if $(a + tv)_j > (a + tv)_i$, then $(a + u + tv)_j > (a + u + tv)_i$. This is to say, if

$$i = \pi_k(x + tv) = \pi_{k'}(x + u + tv),$$

then $k \leq k'$, and

$$\Pi_{k-1}(x + tv) \subseteq \Pi_{k'-1}(x + u + tv).$$

Consequently,

$$\begin{aligned} s_i(x+tv) &= z(\Pi_{k-1}(x+tv)\cup\{i\}) - z(\Pi_{k-1}(x+tv)) \\ &\geq z(\Pi_{k'-1}(x+u+tv)\cup\{i\}) - z(\Pi_{k'-1}(x+u+tv)) \\ &= s_i(x+u+tv), \end{aligned}$$

where the inequality is due to the submodularity of z. Thus, (2.2.11) holds for almost all $t \in [0,1]$, which proves (2.2.10), hence the theorem. (For the case where there are both lower and upper bounds on the decision variables, we can follow exactly the same proof as in Corollary 2.2.3.) □

Theorem 2.2.4 immediately implies the following result, which shall be useful in the next subsection.

Corollary 2.2.5 *Consider the problem:*

$$\begin{aligned} \max \quad & \textstyle\sum_{i\in E} \alpha_i a_i x_i + \beta_i a_i + \gamma_i x_i + \delta_i \\ \text{s.t.} \quad & x \in P(z,E) \\ & \underline{\omega} \leq x \leq \overline{\omega}. \end{aligned} \qquad (2.2.12)$$

Let $g : \mathbb{R}^{|E|} \to \mathbb{R}$ denote the optimal objective value, as a function of a. Then function g is submodular if $\alpha_i \geq 0$ for all $i \in E$.

Proof. Since the sum of submodular functions is still submodular, we can safely assume that $\beta_i = \delta_i = 0$ for all $i \in E$. Let

$$\begin{aligned} h(b) = \max \quad & \textstyle\sum_{i\in E} b_i x_i \\ \text{s.t.} \quad & x \in P(z,E) \\ & \underline{\omega} \leq x \leq \overline{\omega}. \end{aligned} \qquad (2.2.13)$$

Then by Theorem 2.2.4 we conclude that $h(b)$ is submodular with respect to b. Since $b(a)_i := \alpha_i a_i + \gamma_i$ is monotonically increasing function of a_i for all $i \in E$, we know that function $g(a) = h(b(a))$ is submodular with respect to a. □

The application of Theorem 2.2.4 to the joint replenishment game will be described in Section 2.3. Here we provide two simple examples where Theorem 2.2.4 can be applied.

Example 3. Let $c \in \mathbb{R}^n$ and p be an integer such that $1 \leq p \leq n$. Denote the sum of the p largest coordinates of c by $p(c)$. It is shown in [61] (Proposition 4) that $p(c)$ as a function of c is submodular in $\mathbb{R}^n$. This can be seen by applying Theorem 2.2.4 directly. Notice that

$$\begin{aligned} p(c) = \quad \max \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = p \\ & 0 \leq x_i \leq 1. \end{aligned}$$

It is clear that the linear program above can be cast in the form of problem (2.2.1), using the rank function of the so-called uniform matroid of rank p. Thus it follows from Theorem 2.2.4 that $p(c)$ is submodular.

Example 4. Let $\lambda \in \mathbb{R}^n$ so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For any $c \in \mathbb{R}^n$, let $c_{[i]}$ be the ith largest component of c. Define $f(c) = \sum_{i=1}^{n} \lambda_i c_{[i]} : \mathbb{R}^n \to \mathbb{R}$. It is shown in [49] (Theorem 4.1) that $f(c)$ is a submodular function. This can be seen by noticing that

$$f(c) = \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) k(c)$$

where $\lambda_{n+1} = 0$ and $k(c)$ is the sum of the k largest coordinates of c. By the result of Example 3, we know that $f(c)$ is submodular.

2.2.2 Maximizing a Separable Concave Function

In this subsection, we generalize the results in the previous subsection to the case where the objective function is separable concave. Theorems 2.2.6 and 2.2.7 generalize Theorems 2.2.2 and 2.2.4, respectively. The key idea underlying the proofs is to linearize the objective function.

Theorem 2.2.6 *Fix a finite set E and a polymatroid $P(z, E)$. For each $i \in E$, let $f_i : \mathbb{R} \to \mathbb{R}$ be a concave function. For any $A \subseteq E$, define*

$$h(A) = \max \sum_{i \in A} f_i(x_i) \quad \text{s.t.} \quad x \in P(z, E). \tag{2.2.14}$$

Then $h : 2^E \to \mathbb{R}$ is submodular.

Proof. For any $A \subseteq E$, denote $x(A) \in \text{Re}_+^{|E|}$ as an optimal solution of problem (2.2.14). Also, for ease of presentation, we rewrite $h(A)$ as $h(A, f, P(z, E))$ to highlight its dependence on the polymatroid $P(z, E)$ and the objective function $f = (f_i : i \in E)$.

Consider any $A, B \subseteq E$. For each $i \in E$, there exists a linear function, denoted by $\hat{f}_i$, such that

$$\hat{f}_i(x(A \cup B)_i) = f_i(x(A \cup B)_i)$$

and

$$\hat{f}_i(x(A \cap B)_i) = f_i(x(A \cap B)_i).$$

And the concavity of f_i implies that $f_i(x_i) \geq \hat{f}_i(x_i)$ for $\min(x(A \cup B)_i, x(A \cap B)_i) \leq x_i \leq \max(x(A \cup B)_i, x(A \cap B)_i)$. We define

$$\Omega_{A,B} = \{x \in \text{Re}_+^{|E|} : x(A \cup B) \wedge x(A \cap B) \leq x \leq x(A \cup B) \vee x(A \cap B)\}.$$

Thus, for any $x \in \Omega_{A,B}$, $f_i(x_i) \geq \hat{f}_i(x_i)$ for all $i \in E$.

Then we have

$$\begin{aligned} h(A) + h(B) &= h(A, f, P(z, E)) + h(B, f, P(z, E)) \\ &\geq h(A, f, P(z, E) \cap \Omega_{A,B}) + h(B, f, P(z, E) \cap \Omega_{A,B}) \\ &\geq h(A, \hat{f}, P(z, E) \cap \Omega_{A,B}) + h(B, \hat{f}, P(z, E) \cap \Omega_{A,B}) \\ &\geq h(A \cup B, \hat{f}, P(z, E) \cap \Omega_{A,B}) + h(A \cap B, \hat{f}, P(z, E) \cap \Omega_{A,B}) \end{aligned}$$

where the last inequality follows from Corollary 2.2.3. Notice that, $x(A \cup B), x(A \cap B) \in P(z, E) \cap \Omega_{A,B}$. Thus, by the definition of $\hat{f}$, we have

$$\begin{aligned} & h(A \cup B, \hat{f}, P(z, E) \cap \Omega_{A,B}) + h(A \cap B, \hat{f}, P(z, E) \cap \Omega_{A,B}) \\ \geq & \sum_{i \in A \cup B} \hat{f}_i(x(A \cup B)_i) + \sum_{i \in A \cap B} \hat{f}_i(x(A \cap B)_i) \\ = & \sum_{i \in A \cup B} f_i(x(A \cup B)_i) + \sum_{i \in A \cap B} f_i(x(A \cap B)_i) \\ = & h(A \cup B, f, P(z, E)) + h(A \cap B, f, P(z, E)) \\ = & h(A \cup B) + h(A \cap B). \end{aligned}$$

This completes the proof. □

Theorem 2.2.7 *Fix a finite set E and a polymatroid $P(z, E)$. For any $i \in E$, let $f_i : \mathbb{R}^2 \to \mathbb{R}$ be a supermodular function. For any $a \in \mathbb{R}^{|E|}$, define*

$$\begin{aligned} g(a) = \quad & \max \quad \textstyle\sum_{i \in E} f_i(x_i, a_i) \\ & \text{s.t.} \quad x \in P(z, E). \end{aligned} \tag{2.2.15}$$

Then $g : \mathbb{R}^{|E|} \to \mathbb{R}$ is submodular if $f_i(x_i, a_i)$ is concave in both x_i and a_i, for all $i \in E$.

To see that Theorem 2.2.7 generalizes Theorem 2.2.4, we notice that $f_i(x_i, a_i) = a_i x_i : \mathbb{R}^2 \to \mathbb{R}$ is supermodular in (a_i, x_i) and concave in both a_i and x_i. Similar to the proof of Theorem 2.2.6, in order to prove Theorem 2.2.7, we need to carefully linearize the functions $f_i(x_i, a_i)$. To that end, we need the following lemma.

Lemma 2.2.8 *If $\psi(y, b)$ is supermodular in $(y, b) \in \mathbb{R}^2$ and concave in both y and b, then for any $y_1 < y_2$ and $b_1 < b_2$, there exists a function*

$$L(y, b) = \alpha by + \beta b + \gamma y + \delta$$

such that

- $\alpha \geq 0$;
- $L(y, b) \leq \psi(y, b)$ *for any* $(y, b) \in [y_1, y_2] \times [b_1, b_2]$; *and*
- $L(y_i, b_j) = \psi(y_i, b_j)$ *for any* $i, j \in \{1, 2\}$.

Proof. Define

$$\begin{cases} \alpha' &= \frac{\psi(y_2,b_2)+\psi(y_1,b_1)-\psi(y_2,b_1)-\psi(y_1,b_2)}{(b_2-b_1)(y_2-y_1)}, \\ \beta' &= \frac{\psi(y_1,b_2)-\psi(y_1,b_1)}{b_2-b_1}, \\ \gamma' &= \frac{\psi(y_2,b_1)-\psi(y_1,b_1)}{y_2-y_1}, \\ \delta' &= \psi(y_1, b_1) \end{cases}$$

and

$$L(y, b) = \alpha'(y - y_1)(b - b_1) + \beta'(b - b_1) + \gamma'(y - y_1) + \delta'.$$

By supermodularity of $\psi(y, b)$, we know that

$$\psi(y_2, b_2) + \psi(y_1, b_1) = \psi((y_2, b_1) \vee (y_1, b_2)) + \psi((y_2, b_1) \wedge (y_1, b_2)) \geq \psi(y_2, b_1) + \psi(y_1, b_2)$$

and thus $\alpha = \alpha' \geq 0$. It is also easy to verify that $L(y_i, b_j) = \psi(y_i, b_j)$ for any $i, j \in \{1, 2\}$. Finally, for any $(y, b) \in [y_1, y_2] \times [b_1, b_2]$, (y, b) can be expressed as a convex combination of two points (y_1, b) and (y_2, b), i.e., there exists $\lambda_i \geq 0, i = 1, 2$ such that $\lambda_1 + \lambda_2 = 1$ and $(y, b) = \lambda_1(y_1, b) + \lambda_2(y_2, b)$. Since $\psi(y, b)$ is concave in y, we have

$$\psi(y, b) \geq \lambda_1 \psi(y_1, b) + \lambda_2 \psi(y_2, b).$$

Similarly, there exists $\mu_i \geq 0, i = 1, 2$ such that $\mu_1 + \mu_2 = 1$ and $b = \mu_1 b_1 + \mu_2 b_2$. Since $\psi(y, b)$ is concave in b, we have

$$\psi(y_1, b) \geq \mu_1 \psi(y_1, b_1) + \mu_2 \psi(y_1, b_2)$$

and

$$\psi(y_2, b) \geq \mu_1 \psi(y_2, b_1) + \mu_2 \psi(y_2, b_2).$$

It then follows that

$$\psi(y, b) \geq \lambda_1 \mu_1 \psi(y_1, b_1) + \lambda_1 \mu_2 \psi(y_1, b_2) + \lambda_2 \mu_1 \psi(y_2, b_1) + \lambda_2 \mu_2 \psi(y_2, b_2).$$

On the other hand, one can verify that

$$L(y, b) = \lambda_1 \mu_1 L(y_1, b_1) + \lambda_1 \mu_2 L(y_1, b_2) + \lambda_2 \mu_1 L(y_2, b_1) + \lambda_2 \mu_2 L(y_2, b_2).$$

Therefore, by the fact that $L(y_i, b_j) = \psi(y_i, b_j)$ for any $i, j \in \{1, 2\}$, we get $L(y, b) \leq \psi(y, b)$ for any $(y, b) \in [y_1, y_2] \times [b_1, b_2]$. □

It can be easily seen from the proof that the supermodularity of $\psi(y, b)$ is used to guarantee the non-negativity of the coefficient of yb in the function $L(y, b)$. The concavity of $\psi(y, b)$ is used to ensure that $L(y, b)$ is a lower bound of $\psi(y, b)$. Now we are ready to prove Theorem 2.2.7. The proof is similar in spirit to that of Theorem 2.2.6.

Proof of Theorem 2.2.7. For any $a \in \mathbb{R}_+^{|E|}$, let $x(a)$ denote an optimal solution to problem (2.2.15). Now for any $b, d \in \mathbb{R}_+^{|E|}$, by Lemma 2.2.8, for any $i \in E$, there exists a linear function $L_i(a_i, x_i) = \alpha_i a_i x_i + \beta_i x_i + \gamma_i a_i + \delta_i$ such that $\alpha_i \geq 0$, and for any

$$(a_i, x_i) \in [b_i \wedge d_i, b_i \vee d_i] \times [x(b \vee d)_i \vee x(b \wedge d)_i, x(b \vee d)_i \wedge x(b \wedge d)_i],$$

we have $f_i(a_i, x_i) \geq L_i(a_i, x_i)$, and the inequality holds as an equality when (a_i, x_i) is an extreme point of the set $[b_i \wedge d_i, b_i \vee d_i] \times [x(b \vee d)_i \wedge x(b \wedge d)_i, x(b \vee d)_i \vee x(b \wedge d)_i]$. We further denote

$$\Omega(b, d) := \left\{ x \in \mathbb{R}_+^{|E|} : x_i \in [x(b \vee d)_i \wedge x(b \wedge d)_i, x(b \vee d)_i \vee x(b \wedge d)_i], \quad \forall i \in E \right\}$$

$$F(a, x) = \sum_{i \in E} f_i(a_i, x_i), \quad \text{and} \quad L(a, x) = \sum_{i \in E} L_i(a_i, x_i).$$

These constructions and definitions, together with Theorem 2.2.4, lead to

$$
\begin{aligned}
& g(b \vee d) + g(b \wedge d) && (2.2.16)\\
= \; & F(b \vee d, x(b \vee d)) + F(b \wedge d, x(b \wedge d)) && (2.2.17)\\
= \; & L(b \vee d, x(b \vee d)) + L(b \wedge d, x(b \wedge d)) && (2.2.18)\\
\leq \; & \max_{x \in P(z,E) \cap \Omega(b,d)} L(b \vee d, x) + \max_{x \in P(z,E) \cap \Omega(b,d)} L(b \wedge d, x) && (2.2.19)\\
\leq \; & \max_{x \in P(z,E) \cap \Omega(b,d)} L(b, x) + \max_{x \in P(z,E) \cap \Omega(b,d)} L(d, x) && (2.2.20)\\
\leq \; & \max_{x \in P(z,E) \cap \Omega(b,d)} F(b, x) + \max_{x \in P(z,E) \cap \Omega(b,d)} F(d, x) && (2.2.21)\\
\leq \; & \max_{x \in P(z,E)} F(b, x) + \max_{x \in P(z,E)} F(d, x) && (2.2.22)\\
= \; & g(b) + g(d). && (2.2.23)
\end{aligned}
$$

Equality (2.2.17) holds because of the definition of $x(b \vee d)$ and $x(b \wedge d)$. Equality (2.2.18) and inequality (2.2.21) hold by the construction of function L. Inequality (2.2.19) holds because $x(b \vee d)$

and $x(b \wedge d)$ are in $P(z, E) \cap \Omega(b, d)$. Inequality (2.2.20) follows from Corollary 2.2.5. Inequality (2.2.22) holds since $P(z, E) \cap \Omega(b, d) \subseteq P(z, E)$. □

To conclude this section, we show the Theorems 2.2.7 is stronger than Theorem 2.2.6. To see this, we consider problem (2.2.14).

For each $i \in E$, we define a function $\tilde{f}_i$ as follows. Recall that f_i is concave. If f_i is non-decreasing, then let $\tilde{f}_i = f_i$. Otherwise, there must exist $x_i^* \in \mathbb{R}$ such that $f_i(x_i^*) = \max_{x_i \in \mathbb{R}} f_i(x_i)$. In this case, define

$$\tilde{f}_i(x_i) = \begin{cases} f_i(x_i) & \text{if } x_i \leq x_i^* \\ f_i(x_i^*) & \text{otherwise.} \end{cases}$$

It is clear that $\tilde{f}_i$ is non-decreasing and concave. Furthermore, problem (2.2.14) is equivalent to

$$\begin{aligned} h(A) = \quad & \max \quad \textstyle\sum_{i \in A} \tilde{f}_i(x_i) \\ & s.t. \quad x \in P(z, E). \end{aligned} \tag{2.2.24}$$

For each $i \in E$, define $\bar{f}_i : \mathbb{R}^2 \to \mathbb{R}$ such that $\bar{f}_i(x_i, a_i) = \tilde{f}_i(x_i)a_i$. It is straightforward to verify that $\bar{f}_i$ is submodular in (x_i, a_i) and concave in both x_i and a_i. Therefore, by Theorem 2.2.7, if we define, for each $a \in \mathbb{R}^{|E|}$,

$$\begin{aligned} g(a) = \quad & \max \quad \sum_{i \in E} \bar{f}_i(x_i, a_i) \\ & s.t. \quad x \in P(z, E), \end{aligned}$$

then $g : \mathbb{R}^{|E|} \to \mathbb{R}$ is submodular. Let $a^A \in \mathbb{R}^{|E|}$ such that for each $i \in E$,

$$a_i^A = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $g(a^A) = h(A)$, and for any $A, B \subseteq E$, we have $a^{A\cup B} = a^A \vee a^B$ and $a^{A\cap B} = a^A \wedge a^B$. Therefore,

$$\begin{aligned} h(A\cup B) + h(A\cap B) &= g(a^{A\cup B}) + g(a^{A\cap B}) \\ &= g(a^A \vee a^B) + g(a^A \wedge a^B) \\ &\leq g(a^A) + g(a^B) \\ &= h(A) + h(B), \end{aligned}$$

which implies that $h : 2^E \to \mathbb{R}$ is submodular. Theorem 2.2.6 follows.

2.3 One-Warehouse Multiple Retailer Game

In this section, we consider the one-warehouse multiple retailer game studied by Zhang [64]; it is a generalization of the joint replenishment game studied by Anily and Haviv [4] and Zhang [65]. The presentation of the model follows close to that in [64]. In this model, we are given a set of n retailers, denoted by $N = \{1, 2, \cdots, n\}$. The demand that retailer i faces is continuous and deterministic at a fixed rate $d_i > 0$. The retailers place orders to a single warehouse to satisfy customer demands. These orders generate demands at the warehouse, which holds inventory and is replenished from an external supplier. Backlogging is not allowed in this model. The lead time is assumed to be zero, i.e., orders arrive instantaneously.

For ease of presentation, the warehouse is denoted by 0. Also, any $i \in N \cup \{0\}$ is called a facility, i.e., a facility can be a warehouse or a retailer.

For each $i \in N \cup \{0\}$, there is a per unit holding cost rate h_i. For simplicity we denote, $H_i = \frac{1}{2}h_i d_i$ and $H_i^w = \frac{1}{2}h_0 d_i$, for each $i \in N$. We also assume that $0 < h_0 < h_i$ and thus $0 < H_i^w < H_i$, for any $i \in N$. This assumption is common in the literature; see, e.g., Roundy [54], and Federgruen et al.

[21]. When a subset $S \subseteq N \cup \{0\}$ of facilities places an order together, a joint setup cost is incurred, which is denoted by $K(S)$. We assume that $K(S)$ is a rank function.

We restrict ourselves to the so-called power-of-two inventory policies, which can be characterized by an $(n+1)$-tuple, $(T_0, T_i : i \in N)$, where T_i is the replenishment interval at facility i for $i \in N \cup \{0\}$. That is, the replenishment epoches of facility i are $0, T_i, 2T_i, \cdots$. Furthermore, we require that for all $i \in N \cup \{0\}$, $T_i = 2^{m_i}\mathcal{L}$ where $\mathcal{L} > 0$ is a constant called the base planning period, and m_i is an integer that can be negative. Denote

$$\Gamma_{\mathcal{L}} = \{t : t > 0 \quad \text{and} \quad t = 2^m\mathcal{L} \quad \text{for some} \quad m \in \mathbb{Z}\}.$$

The effectiveness of power-of-two policies has been discussed in Federgruen et al. [21]. If the base planning period $\mathcal{L}$ is chosen arbitrarily, then the optimal power-of-two policy yields an average cost that is at most 6% higher than the optimal cost, and thus is 94% effective. By choosing the best $\mathcal{L}$, the optimal power-of-two policy is 98% effective.

Now we consider a cooperative game associated with this inventory model. We denote this game by $(N, V_{\Gamma_{\mathcal{L}}})$. Here N is the grand coalition of n retailers and $V_{\Gamma_{\mathcal{L}}}$ is the characteristic cost function defined for every coalition $S \subseteq N$. In particular, $V_{\Gamma_{\mathcal{L}}}(\emptyset) = 0$ and for $\emptyset \neq S \subseteq N$, $V_{\Gamma_{\mathcal{L}}}(S)$ is the long-run average cost, under an optimal power-of-two policy, of the system that consists of the warehouse and the retailers in S. The game (N, V_Γ) is called a concave game if the set function $V_\Gamma(\cdot)$ is submodular.

Federgruen et al. [21] have shown that

$$V_{\Gamma_{\mathcal{L}}}(S) := \min_{\mathbb{T}_S \in \Gamma_{\mathcal{L}}^{s+1}} \max_{k \in P(K, S\cup\{0\})} \frac{k_0}{T_0} + \sum_{i \in S} \left(\frac{k_i}{T_i} + H_i T_i + H_i^{w} \max\{T_0 - T_i, 0\} \right) \tag{2.3.1}$$

where $\Gamma_{\mathcal{L}}^{s+1} = \{t = (t_1, t_2, \cdots, t_{s+1}) : t_i \in \Gamma_{\mathcal{L}}, \quad i = 1, 2, \cdots, s+1\}$.

2.3.1 Submodularity of the Joint Replenishment Game

Now we consider a special case of the one-warehouse multiple retailer game when there is no warehouse. This reduces to the joint replenishment game studied in Zhang [65]. We denote the game by $(N, V_{J\Gamma_{\mathcal{L}}})$ where the characteristic cost function $V_{J\Gamma_{\mathcal{L}}}(\cdot)$ is defined as, for any $S \subseteq N$,

$$V_{J\Gamma_{\mathcal{L}}}(S) := \min_{\mathbb{T}_S \in \Gamma^s_{\mathcal{L}}} \max_{k \in P(K,S)} \sum_{i \in S} \left(\frac{k_i}{T_i} + H_i T_i \right). \tag{2.3.2}$$

(This can be obtained by setting $k_0 = 0$ and $H_i^w = 0$ in (2.3.1).) It is known that we can change the order of the optimization of (2.3.2) from min-max to max-min without changing the optimal objective value [64]. That is,

$$V_{J\Gamma_{\mathcal{L}}}(S) := \max_{k \in P(K,S)} \sum_{i \in S} \min_{\mathbb{T}_i \in \Gamma_{\mathcal{L}}} \left(\frac{k_i}{T_i} + H_i T_i \right). \tag{2.3.3}$$

In [65], an analytical solution to problem (2.3.3) was derived, which is in turn used to propose a population monotonic allocation scheme for the joint replenishment game. As most of the cooperative games that admit a population monotonic allocation scheme are submodular, Zhang [65] conjecture that the joint replenishment game is submodular. Here we show that this is indeed the case.

Theorem 2.3.1 *The joint replenishment game $(N, V_{J\Gamma_{\mathcal{L}}})$ is submodular.*

Proof. For each fixed T_i, the function $\frac{k_i}{T_i} + H_i T_i$ is linear in k_i. Then it is clear that, for any $i \in S$,

$$\min_{\mathbb{T}_i \in \Gamma_{\mathcal{L}}} \left(\frac{k_i}{T_i} + H_i T_i \right)$$

is a concave function of k_i, denoted by $f_i(k_i)$. Thus, from (2.3.3),

$$V_{J\Gamma_{\mathcal{L}}}(S) := \max_{k \in P(K,S)} \sum_{i \in S} f_i(k_i)$$

By Theorem 2.2.6, $V_{J\Gamma_{\mathcal{L}}}(S)$ is submodular. This completes the proof. □

Anily and Haviv [4] proved that Theorem 2.3.1 holds for a special case of the joint replenishment game where the joint setup cost function has the first order interaction structure. Theorem 2.3.1 generalizes their main result.

2.3.2 Submodularity of the One-Warehouse Multiple Retailer Game

Now we consider the submodularity of the one-warehouse multiple retailer game $(N, V_{\Gamma_{\mathcal{L}}})$, where the function $V_{\Gamma_{\mathcal{L}}}(S)$ is defined by (2.3.1). It is tempting to prove the submodularity of $V_{\Gamma_{\mathcal{L}}}(S)$ by following the approach used in the proof of Theorem 2.3.1. The dual problem of (2.3.1) can be formulated as follows [64]:

$$\max_{k\in P(K,S\cup\{0\}),0\leq u_i\leq H_i^w:i\in S} \quad \min_{\mathbb{T}_S\in\Gamma_{\mathcal{L}}^{s+1}} \frac{k_0}{T_0} + \left(\sum_{i\in S} u_i\right) T_0 + \sum_{i\in S}\left((H_i - u_i)T_i + \frac{k_i}{T_i}\right). \qquad (2.3.4)$$

It is known that this pair of primal-dual problems (2.3.1) and (2.3.4) do not have duality gap. However, the objective function of (2.3.4) is not separable. Therefore, the results developed in Section 2 are not directly applicable to problem (2.3.4). In order to prove the submodularity of $V_{\Gamma_{\mathcal{L}}}(S)$, we focus on the primal formulation (2.3.1) and apply Theorem 2.2.4 to the inner maximization problem of (2.3.1).

Theorem 2.3.2 *The one-warehouse multiple retailer game* $(N, V_{\Gamma_{\mathcal{L}}})$ *is submodular.*

Proof. For any $S \subseteq N$, let $\mathbb{T}_S^*$ be an optimal solution to the outer minimization problem of (2.3.1). Denote $\bar{\Gamma}_{\mathcal{L}} = \Gamma_{\mathcal{L}} \cup \{+\infty\}$. Then for any $S \subseteq N$, we have

$$V_{\Gamma_{\mathcal{L}}}(S) = \min_{\mathbb{T}_S \in \Gamma_{\mathcal{L}}^{s+1}} \max_{k \in P(K, S \cup \{0\})} \frac{k_0}{T_0} + \sum_{i \in S} \left(\frac{k_i}{T_i} + H_i T_i + H_i^w \max\{T_0 - T_i, 0\} \right) \quad (2.3.5)$$

$$= \min_{\mathbb{T}_N \in \bar{\Gamma}_{\mathcal{L}}^{n+1}} \max_{k \in P(K, N \cup \{0\})} \frac{k_0}{T_0} + \sum_{i \in N} \frac{k_i}{T_i} + \sum_{i \in S} (H_i T_i + H_i^w \max\{T_0 - T_i, 0\}). \quad (2.3.6)$$

In particular, if we define $\mathbb{T}_{S,N}^*$ such that $(\mathbb{T}_{S,N}^*)_i = (\mathbb{T}_S^*)_i$ for $i \in S$ and $(\mathbb{T}_{S,N}^*)_i = +\infty$ otherwise, then $\mathbb{T}_{S,N}^*$ is an optimal solution to the outer minimization problem of (2.3.6).

Notice that the feasible set of the outer minimization problem of (2.3.6), $\bar{\Gamma}_{\mathcal{L}}^{n+1}$, is a sublattice of $\mathrm{Re}^{\,n+1}$. That is, for any coalitions $A, B \subseteq N$, and for any i, $(\mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*)_i \in \bar{\Gamma}_{\mathcal{L}}$ and $(\mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*)_i \in \bar{\Gamma}_{\mathcal{L}}$. Then for any coalitions $A, B \subseteq N$, we define $\mathbb{T}_N^{A \cap B} = \mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*$ and $\mathbb{T}_N^{A \cup B} = \mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*$, which must be feasible solutions to problem (2.3.6) for $S = A \cap B$ and $S = A \cup B$ respectively. Recall the definition of $g_N(\mathbb{T}_N)$ and $h_S(\mathbb{T}_S)$. We denote

$$G_S(\mathbb{T}_N) = \max_{k \in P(K, N \cup \{0\})} \frac{k_0}{T_0} + \sum_{i \in N} \frac{k_i}{T_i} + \sum_{i \in S} (H_i T_i + H_i^w \max\{T_0 - T_i, 0\})$$

$$= g_N(\mathbb{T}_N) + h_S((\mathbb{T}_N)_S)$$

where $(\mathbb{T}_N)_S$ is the projection of $\mathbb{T}_N$ to the subset S. Then we have

$$V_{\Gamma_{\mathcal{L}}}(A \cup B) + V_{\Gamma_{\mathcal{L}}}(A \cap B)$$

$$\leq G_{A \cup B}(\mathbb{T}_N^{A \cup B}) + G_{A \cap B}(\mathbb{T}_N^{A \cap B})$$

$$= G_{A \cup B}(\mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*) + G_{A \cap B}(\mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*)$$

$$= g_N(\mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*) + g_N(\mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*) + h_{A \cup B}((\mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*)_{A \cup B}) + h_{A \cap B}((\mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*)_{A \cap B}).$$

By Theorem 2.2.4, $g_N(\mathbb{T}_N)$ is submodular in $\mathbb{T}_N$. Therefore,

$$g_N(\mathbb{T}_{A,N}^* \wedge \mathbb{T}_{B,N}^*) + g_N(\mathbb{T}_{A,N}^* \vee \mathbb{T}_{B,N}^*) \leq g_N(\mathbb{T}_{A,N}^*) + g_N(\mathbb{T}_{B,N}^*). \quad (2.3.7)$$

Also,

$$\begin{aligned} h_{A\cup B}((\mathbb{T}^*_{A,N}\wedge\mathbb{T}^*_{B,N})_{A\cup B}) &= h_{A\cap B}((\mathbb{T}^*_{A,N}\wedge\mathbb{T}^*_{B,N})_{A\cap B}) \\ &\quad+\sum_{i\in A\setminus B}\left(H_i(\mathbb{T}^*_{A,N})_i+H_i^w\max\{(\mathbb{T}^*_{A,N})_0-(\mathbb{T}^*_{A,N})_i,0\}\right) \\ &\quad+\sum_{i\in B\setminus A}\left(H_i(\mathbb{T}^*_{B,N})_i+H_i^w\max\{(\mathbb{T}^*_{B,N})_0-(\mathbb{T}^*_{B,N})_i,0\}\right). \end{aligned}$$

We know that $\max\{T_0-T_i,0\}$ is submodular in (T_0,T_i) for any $i\in S$. Therefore, $h_S(\mathbb{T}_S)=\sum_{i\in S}(H_iT_i+H_i^w\max\{T_0-T_i,0\})$ is submodular in $\mathbb{T}_S$. Thus,

$$h_{A\cap B}((\mathbb{T}^*_{A,N}\wedge\mathbb{T}^*_{B,N})_{A\cap B})+h_{A\cap B}((\mathbb{T}^*_{A,N}\vee\mathbb{T}^*_{B,N})_{A\cap B})\le h_{A\cap B}((\mathbb{T}^*_{A,N})_{A\cap B})+h_{A\cap B}((\mathbb{T}^*_{B,N})_{A\cap B}).$$

It follows that

$$\begin{aligned} & h_{A\cup B}((\mathbb{T}^*_{A,N}\wedge\mathbb{T}^*_{B,N})_{A\cup B})+h_{A\cap B}((\mathbb{T}^*_{A,N}\vee\mathbb{T}^*_{B,N})_{A\cap B}) \\ \le\ & h_{A\cap B}((\mathbb{T}^*_{A,N})_{A\cap B})+\sum_{i\in A\setminus B}\left(H_i(\mathbb{T}^*_{A,N})_i+H_i^w\max\{(\mathbb{T}^*_{A,N})_0-(\mathbb{T}^*_{A,N})_i,0\}\right) \\ & +h_{A\cap B}(\mathbb{T}^*_{B,N})_{A\cap B})+\sum_{i\in B\setminus A}\left(H_i(\mathbb{T}^*_{B,N})_i+H_i^w\max\{(\mathbb{T}^*_{B,N})_0-(\mathbb{T}^*_{B,N})_i,0\}\right) \\ =\ & h_A((\mathbb{T}^*_{A,N})_A)+h_B((\mathbb{T}^*_{B,N})_B). \end{aligned}$$

This, together with (2.3.7), implies that

$$\begin{aligned} & V_{\Gamma_{\mathcal{L}}}(A\cup B)+V_{\Gamma_{\mathcal{L}}}(A\cap B) \\ \le\ & g_N(\mathbb{T}^*_{A,N})+h_A((\mathbb{T}^*_{A,N})_A)+g_N(\mathbb{T}^*_{B,N})+h_B((\mathbb{T}^*_{B,N})_B) \\ =\ & V_{\Gamma_{\mathcal{L}}}(A)+V_{\Gamma_{\mathcal{L}}}(B) \end{aligned}$$

which shows that $V_{\Gamma_{\mathcal{L}}}(S)$ is submodular. □

We remark that Theorem 2.3.2 can be generalized to the case where there are upper and lower bounds on the replenishment intervals of the retailers and the warehouse. The reason is that, with this additional constraint, the feasible set for the replenishment intervals is still a sublattice, and so the proof of Theorem 2.3.2 will still go through.

2.4 Locating Sources in a Network

In this section, we illustrate how the submodularity result regarding polymatroid optimization can be used to obtain near optimal solution to NP-hard problems. We focus on the problem of locating a fixed number of sources in a network to maximize a function of the outflows from the sources to a single sink. The notations used in this section is independent of those used in Section 2.3.

Consider a directed network $N = (V, A)$ with the node set V and the arc set A. Let $c : A \to R^+$ define a capacity on each arc, i.e., c_{ij} is the capacity defined on arc $(i, j) \in A$. Let $\mathcal{X} \subseteq V$ be the set of nodes where sources can be located, and $t \in V$ but $t \notin \mathcal{X}$ be the single sink. For each $i \in \mathcal{X}$, a non-negative concave function function h_i is defined. For each $S \subseteq \mathcal{X}$, Denote G_S as the set of all feasible flows assuming that S is selected as the set of source nodes. Also, for any $g_S \in G_S$, let $v_i(g_S)$ be the netflow out of node $i \in N$. Define

$$F(S) := \max_{g_S \in G_S} \sum_{i \in S} h_i(v_i(g_S)) \tag{2.4.1}$$

Let $k \leq |\mathcal{X}|$ be a given integer. We are interested in the following maximization problem

$$\max_{S \subseteq \mathcal{X}: |S| = k} F(S). \tag{2.4.2}$$

We refer to this problem as the Max-Source-Location problem (MSLP).

Notice that for any given S, $F(S)$ can be computed efficiently; see for example Federgruen and

Groenevelt [20]. In fact, if we denote $v(g_S) = (v_i(g_S) : i \in S)$, then it is clear that

$$\{v(g) | g \in G_N\} = \{(v_i : i \in S) | \sum_{i \in R} v_i \leq \Upsilon(R), \ \forall R \subseteq S\}$$

where

$$\Upsilon(R) := \max_{g_S \in G_S} \sum_{i \in R} v_i(g_S).$$

It is known (Meggido [38]) that $\Upsilon(R)$ is a rank function. Therefore, $\{v(g) | g \in G_N\}$ is a polymatroid. It follows that the problem of computing $F(S)$ for a given S is to maximizing a separable concave function over a polymatroid, which can be solved by a greedy algorithm [20].

However, as we shall show that the problem of choosing S of a given size to maximize $F(S)$, i.e., the MSLP, is in general an NP-hard problem. Therefore, we will settle with an approximation algorithm for the MSLP.

Theorem 2.4.1 *The MSLP can be approximated in polynomial time by a factor of* $1 - \frac{1}{e}$, *and this ratio can not be improved unless* $P = NP$.

Proof. By definition, and the discussion above, $F(S)$ is the optimal objective value of maximizing a sparable concave function over a polymatroid. By Theorem 2.2.6, $F(S)$ is a submodular function. Therefore, the MSLP is a special case of the problem of maximizing a submodular function with cardinality constraint. It is well known that the latter can be approximated by a factor of $1 - 1/e$ via a greedy algorithm [44]. Therefore, there is a greedy algorithm for the MSLP with an approximation ratio of $1 - 1/e$. Clearly, this greedy algorithm can be implemented in polynomial time.

In order to show that the factor $1 - 1/e$ is the best possible unless $P = NP$, we reduce the MSLP to the so-called maximum coverage problem, which is stated as follows. We are given a positive integer k, a grand set U, and $\mathcal{S} = \{S_1, S_2, \cdots, S_m\}$ of subsets of U. We want to select a subset $B \subseteq \{1, 2, \cdots, m\}$ such that $|B| \leq k$ and $|\cup_{l \in B} S_l|$ is maximized. It is shown by Feige [23] that the maximum coverage problem can not be approximated with a factor better than $1 - 1/e$ unless $P = NP$.

Now given an instance of the maximum converge problem, we can construct an instance of the MSLP. In particular, we construct a network $G = (N, A)$ with the node set $N = N_1 \cup N_2 \cup \{s\}$, where each node in N_1 corresponds to a subset in $\mathcal{S}$, each node in N_2 corresponds to an element in U. We call each node in N_1 a subset node, and each node in N_2 an element node. Furthermore, we let the set of potential source nodes $\mathcal{X} = N_1$ and let s be the single sink. For each subset node $i \in N_1$ and each element node $j \in N_2$, there is an arc from i to j if the subset corresponding to node i contains the element corresponding to node j; let the capacity of this arc be 1. For each element node $j \in N_2$, there is an arc from j to the sink s; let the capacity of such an arc be 1 as well. Furthermore, for each $i \in N_1$, we define a concave function $h_i(x) = x$.

Let a feasible solution to the maximum coverage problem be determined by a subset $B^* \subseteq \{1, 2, \cdots, m\}$. This gives a feasible solution to the MSLP as follows. For all subsets $S_l : l \in B^*$, we select the corresponding subset nodes in N_1 as source nodes. It is clear that, given this set of source nodes, the maximum total flow to the sink is exactly equal to $|\cup_{l \in B^*} S_l|$, due to the capacity constraint from each element node to the sink.

On the other hand, assume $\hat{B} \subseteq N_1$ is a feasible solution to the MSLP problem with an optimal objective value of $F(\hat{B})$. This gives a feasible solution to the maximum coverage problem. In particular, we just select the subsets of $\mathcal{S}$ that correspond to the subset nodes in $\hat{B}$. It is left to show that the number of elements covered by these subsets is equal to $F(\hat{B})$. To that end, we notice that when $\hat{B}$ is fixed, the problem of computing $F(\hat{B})$ is a simple network flow problem with a linear objective function. Furthermore, all the capacities are ones. Therefore, there exists an optimal flow where the flow on each arc is either zero or one. Since the total flow is $F(\hat{B})$, then there are exactly these many element nodes in N_2 have flows to the sink, and these elements must be covered by the subsets corresponding to $\hat{B}$.

It then follows that any ρ-approximation algorithm for the MSLP is also a ρ-approximation algorithm for the maximum converge problem, and vice versa. This completes the proof. □

2.5 Concluding Remarks

In this chapter, we have obtained some structural results regarding polymatroid optimization. We identify conditions so that the optimal objective function is a submodular function in the index set and the objective parameters. In the most general version, for each $a \in \mathbb{R}^{|E|}$, we consider the following problem,

$$\max_{x \in P(z,E)} \sum_{i \in E} f_i(x_i, a_i)$$

where $P(z,E)$ is a polymatroid, and for each $i \in E$, $f_i : \mathbb{R}^2 \to \mathbb{R}$ is submodular in (x_i, a_i), and concave in both x_i and a_i. We prove that the optimal objective value as a function of parameter a is submodular. This result and its variants have been applied to analyze the joint replenishment game and the one-warehouse multiple retailer game.

The submodularity results regarding polymatroid optimization may find other applications as well, given the wide range of applications of polymatroid optimization. One possible area is for problems of scheduling multiclass queueing systems that satisfy strong conservation laws; see for example Garbe and Glazebrook [26], where the objective function of the optimization problem is linear, but the feasible set is slightly more general than a polymatroid.

Chapter 3

Modified Iterative Waterfilling Algorithm and Fictitious Play

3.1 Introduction

Game theory has been developed and studied extensively in the last century [46]. Originally the study of game theory mainly focused on either pure theoretical models or empirical experiments to better understand human behaviors. However, with the development of the theory of algorithms and the advancements of computational powers, the question of how to compute the Nash Equilibrium gradually has come to the attention of researchers in various areas. Since finding the Nash Equilibrium in general is PPAD-hard [10] even for the simple case of bimatrix games [7], efforts are mainly focusing on finding approximation algorithms [8] [12], polynomial time algorithms under condition [63] [37], or heuristics like fictitious play [6] [40] [50]. Also, the studies can lead to efficient distributive algorithms for solving large scale optimization systems.

The idea of learning has been used extensively in game theory, both as an attempt to achieve efficient algorithms for finding Nash Equilibrium, and to model the human behavior of learning, as well as the biological evolution. From the algorithmic point of view, the fictitious play was first introduced in 1951 by Robinson [50] and Brown [6], which used the idea of learning from the history (average of all history plays). Robinson showed that it is convergent for zero-sum matrix game. Many papers, e.g. ([41],[27]) followed and established the convergence of fictitious play for special types of games. Also, this simple idea of learning from history has been extended, by introducing the idea of fading memory, limiting memory length, introducing stochastic behavior, etc, to create various types of modified fictitious play [39] [40] [35]. However, most of these results need to assume the non-cycling property or similar conditions. For general cases, it has been shown to be not convergent, for example, the famous Shapley game [57]. On the other hand, the evolutionary game theory has been developed to provide models of disequilibrium behavior in strategic settings, in which players are assumed to be learning from history and play best response to the observed history. Sandholm [55] provides a survey of the evolutionary game theory.

In the DSL communication model, a noncooperative game arises when each user wants to maximize his/her own transmitting power, under the power budget constraint. The previous studies are mainly based on the widely used IWFA method, in which each user updates his/her strategy according to the current observation of the system. Under some special conditions the convergence of this update rule has been proved [63] [37], and it was conjectured that it will converge in any cases. The IWFA method, can be viewed as the simple strategy that everyone plays the best response to the current action of other players. It is therefore natural to extend the idea of learning (e.g., fictitious play) to this setting. We show that when memory fades fast enough, the algorithm converges to the Nash

Equilibrium under the unique Nash Equilibrium condition as given in [37].

The contribution of this chapter is listed below:

1. We show a counter example where the standard IWFA does not converge, which is against the common belief in the community.

2. We propose two iterative algorithms with global linear convergence rate, under the condition in Luo and Pang's setting [37].

3. Both of the algorithms need only local noise information at each step and are easy to implement, thus can be viewed as a distributive algorithm to calculate the NE of the game.

4. The modified IWFA algorithm has a game interpretation, which can be viewed as a modified fictitious play. Also, based on our convergence result, we establish the result that if everyone has been educated to learn from history with good enough memory, then the system will eventually move to the stable point (Nash Equilibrium).

5. We extend our result to an even more general setting, and introduce the notion of jointly convex game. Also, we discuss the connection to monotone LCP systems.

6. We believe that our work can enhance the connection between optimization and the game theory, so as to justify the behavior of learning from history with appropriate mathematical models. In addition to this, our work motivates to solve large scale optimization problems with distributive computation.

The structure of the rest of this chapter is as follows. In Section 3.2 we establish the mathematical model for the problem and introduce the commonly used IWFA method. In Section 3.3 we propose the new iterative learning based algorithm, and then proceed to analyze the properties of this new algorithm. Section 3.4 extends the algorithm to a more general setting, and introduces the notion of jointly strong convex game. The last section discusses the future research directions.

3.2 Model and Existing Result

3.2.1 The Model

In many communication systems, such as the DSL networks, and cable networks, there are a number of users accessing to a limited number of communication channels or tones. Although people can share the same tones, the cross-talks between different users will create interferences among the users and reduce their transmission capabilities. In general, the users are not cooperative, and are only keen to their own needs. Also, sometimes they are only aware of the local information, such as the noises to himself, instead of the other players decisions. Therefore, we study the behavior of the users in the system, assuming each user acts greedily.

Suppose there are m DSL users communicating with a central office in an uplink multi-access channel, and there are n frequency tones that can be accessed by the users. Each user has a power budget constraint as:

$$(PC)^i = \left\{p^i \in \mathbb{R}^n \mid 0 \le p^i \le CAP^i, e^T p^i \le P^i_{max}\right\},$$

where P^i denote the power allocation of user i, and e is the all one vector. The objective for each user is to maximize his/her own transmission rate (Shannon Capacity) in a noncooperative game environment, which is:

$$\begin{aligned} \max \quad & u_i(p^1, \cdots, p^m) \equiv \sum_{k=1}^{n} \log\left(1 + \frac{p_k^i}{\sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} p_k^j}\right) \\ \text{s.t.} \quad & p^i \in (PC)^i, \end{aligned} \tag{3.2.1}$$

where σ_k^i are positive scalars and α_k^{ij} are nonnegative scalars for all $i \neq j$ and k, which represent the noise power spectra and channel crosstalk coefficients respectively. Notice that the utility function is monotonically increasing with respect to p_k^i, the problem remains equivalent if we restrict the feasible region to

$$P^i = \left\{p^i \in \mathbb{R}^n \mid 0 \leq p^i \leq CAP^i, e^T p^i = P_{max}^i\right\}.$$

3.2.2 The IWFA method

The IWFA method allows each user to greedily reallocate their own power allocation across different spectrums to optimize their transmission rate at each iteration, repeatedly. It has been conjectured to be convergent in all cases, although only been proven under certain conditions.

The IWFA is the strategy of playing best response to the current action of the other players. Denote $B_k = (\alpha_k^{ij})$, $B = \text{diag}(B_k)$, $b_k = (\sigma_k^i)_{1\times m}$, and $b = (b_1, b_2, \ldots, b_n)^T$. For player i, if the current joint action vector of other players is x^{-i} and the responding decision vector is noted as $R_i(x^{-i})$, by the KKT condition that $y = R_i(x^{-i})$ if and only if there exist t and $s_k \geq 0$, $(k = 1, 2, \ldots, n)$ such that:

$$\begin{cases} 0 &= s_k^T y_k \qquad \text{for all } k = 1, 2, \ldots, n \\ t &= s_k + \frac{1}{y_k^i + \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} x_k^j} \qquad \text{for all } k = 1, 2, \ldots, n, \end{cases}$$

or equivalently,

$$\begin{cases} t &\geq \frac{1}{y_k^i + \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} x_k^j} \qquad \text{for all } k = 1, 2, \ldots, n \\ t &= \frac{1}{y_k^i + \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} x_k^j} \qquad \text{if } y_k > 0. \end{cases}$$

Let $t' = -1/t$, then it changes to:

$$\begin{cases} -t' & \leq y_k^i + \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} x_k^j \quad \text{for all } k = 1, 2, \ldots, n \\ -t' & = y_k^i + \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} x_k^j \quad \text{if } y_k > 0, \end{cases}$$

which is equivalent to:

$$\begin{cases} 0 & = {s'}_k^T y_k \quad \text{for all } k = 1, 2, \ldots, n \\ -t' + s' & = y_k^i + \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} x_k^j \quad \text{for all } k = 1, 2, \ldots, n. \end{cases}$$

Notice that this is exactly the KKT condition of the problem

$$\begin{aligned} \min \quad & f_i(y^i, x^{-i}) \equiv \sum_{k=1}^n \left[\tfrac{1}{2}(y_k^i)^2 + x_k^i(\sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} x_k^j) \right] \\ \text{s.t.} \quad & y^i \in P^i. \end{aligned} \tag{3.2.2}$$

Therefore the best respond dynamics of the original game is exactly the same as the quadratic cost game specified by:

$$\begin{aligned} \min \quad & f_i(x) \equiv \tfrac{1}{2}(x^i)^T x^i + (x^i)^T (b + Bx)_i, \\ \text{s.t.} \quad & x^i \in P^i. \end{aligned}$$

Also the Nash Equilibrium of both games are exactly the same, and are equivalent to the fix point of $x = [-Bx - b]_P$, where $P = \prod_{i=1}^n P^i$ is the feasible region, a closed convex compact set, and $[\cdot]_P$ is the mapping signifies the orthogonal projection onto P. It has been shown in [37] that this point exists and is unique under the condition that $\frac{-B-B^T}{2} \prec I$. All our proposed iteration methods will be based on this condition. The existence and uniqueness of this point will be proven in the process too.

Throughout this chapter, we assume that all players respond simultaneously, and assume that the condition $\frac{-B-B^T}{2} \prec I$ holds if not specified otherwise.

3.2.3 The Counter Example

We start with showing an example in which the standard IWFA method fails to converge. Consider the following settings for the DSL game: There are three users, denoted by $1, 2, 3$, and two channels $1, 2$, the positive noise scalar σ_k^i for each user i in each channel k is identical as a constant c, the interference is defined as

$$\alpha_k^{ij} = \begin{cases} 0, & (i,j) \in \{(1,2),(2,3),(3,1)\}; \\ 1, & \text{otherwise.} \end{cases}$$

Each user has power budge $P_{max}^i = 1$ and capacity $CAP_k^i \geq 1$. The iteration starts at the point where users $1, 2$ use full power in channel 1 while user 3 use full power in channel 2.

By solving the fixed point equation $x = [-Bx - b]_P$, we can verify that this game only has a unique Nash Equilibrium, in which all players equally distribute their own power budget across the two channel.

We show that if everyone follows the IWFA method, either simultaneously or taking turns, the state of the game would always be that two users allocate full power budget in one channel while the other user allocate full power budget in the other. If at an iteration the game is in this state, then among the three users, two of them are satisfied and have no incentive to move, while the other user wants to change full power budget to another channel, thus no matter which users take action, or all three of them do it simultaneously, the game would still be in such a state. Thus the IWFA method would loop inside such states, which never converge to any Nash Equilibrium as there is always one (and only one) player who is not satisfied in any of these states.

Also since the Υ matrix defined in [37] is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in this case, and the matrix spectrum $\rho(\Upsilon) = 1$, it shows that the second condition for the convergence of IWFA in Proposition 2 [37] is essentially tight.

This counter example can be generalized to k channels and $k+1$ users each with power budget 1:

$$\alpha_k^{ij} = \begin{cases} 0, & i = j+1; \\ 1, & \text{otherwise.} \end{cases}$$

In this example, if the system starts in a situation in which user l and $l+1$ (note that player $k+2$ is player 1) allocate their full power on the same channel, and the other users each occupy a full channel. If everyone follows the IWFA method, then user l receives interference from user $l-1$ on this channel, and no interference on the channel where user $l-1$ is in, therefore it will change decision and allocate full power budget in that channel. The other users will not change their decisions because they receive no interference in their current channel and interference 1 from any other channels. Notice that the result of this best response action, actually moves the system from the status corresponding to l to $l-1$, therefore it loops and will never converge.

Therefore, in general we should not expect the IWFA method converge to the Nash Equilibrium of the game.

3.3 The Best Response to the Discounted History Iteration

Usually, the decision of a person/corporation is not only based on the current status of the world, but also the history observed, and the memory usually fades as time passes by. It's natural to ask whether the system would be more stable, if everyone updates its decision according to the history. In particular, we study the case in which memory fades geometrically with a fading rate s, i.e., if the current history at time t is h_t and the play is x_t, then the updated history is $h_{t+1} = (1-s)h_t + sx_t$. The smaller the s is, the better the memory is. If at each step t every player plays the best response to the current history h_t, then the new combined decision vector is $x_t = [-Bh_t - b]_K$ and the new history is $h_{t+1} = (1-s)h_t + s[-Bh_t - b]_K$.

Define the affine mapping

$$\mathcal{A}x = -Bx - b,$$

and for each $x \in K$ and $1 > s > 0$, define

$$\begin{cases} y(x) = [\mathcal{A}x]_K \\ d_x = y - x \\ d_y(s) = [\mathcal{A}(x + sd_x)]_K - y. \end{cases}$$

Denote $N = \|B\|_2$ and assume that there exists $\delta, M > 0$, such that $(1-\delta)I \succeq \frac{-B-B^T}{2} \succeq (1-M)I$, then by the property of projection, we have:

Lemma 3.3.1 *For any $x \in K$ and $s > 0$, the following properties hold:*

1. $d_x^T(\mathcal{A}x - x) \geq \|d_x\|^2$,

2. $d_x^T(\mathcal{A}^T x - \mathcal{A}^T y) \geq (\delta - 1)\|d_x\|^2$,

3. $d_y(s)^T(y - \mathcal{A}x) \geq 0$.

4. $\|d_y(s)\| \leq sN\|d_x\|$.

Proof: By the definition of projection we know that for any $z \in K$, $(z - y)^T(\mathcal{A}x - y) \leq 0$, taking $z = x$ this implies $d_x^T(\mathcal{A}x - y) \geq 0$, and taking $z = [\mathcal{A}(x + sd_x)]_K$ it implies $d_y(s)^T(y - \mathcal{A}x) \geq 0$. Thus

$$d_x^T(\mathcal{A}x - x) = d_x^T(\mathcal{A}x - y) + d_x^T(y - x) \geq d_x^T d_x = \|d_x\|^2.$$

Also

$$d_x^T(\mathcal{A}^T x - \mathcal{A}^T y) = d_x^T \frac{B + B^T}{2} d_x \geq (\delta - 1)\|d_x\|^2.$$

where the last inequality is because of

$$\|d_y(s)\| \leq \|\mathcal{A}(x + sd_x) - \mathcal{A}x\| = \| - sBd_x\| \leq sN\|d_x\|.$$

□

Now we define a potential function

$$f(x) = 2x^T\mathcal{A}x - x^T x - 2y^T\mathcal{A}x + y^T y.$$

This can be viewed as (negative of) the sum of maximum possible improvement of each user, if each of them updates its decision assuming all others do play h_t, or in other words, update its decision based on h_t (instead of x_t).

Lemma 3.3.2 *For any $x \in K$ and $0 \leq s \leq 1$, $f(x + sd_x) - f(x) \geq 2sd_x^T(\mathcal{A}x - x) + (c(s) - 2s)\|d_x\|^2$, where $c(s) = 2s(\delta - Ms - N^2 s)$.*

Proof.

$$
\begin{aligned}
& f(x+sd_x)-f(x) \\
= \; & sd_x^T(-2B-2B^T-2I)x+s^2d_x^T(-2B-I)d_x-2d_y(s)^T(-B)x-2sd_x^T(-B)^Ty-2sd_y(s)^T(-B)d_x \\
& +2d_y(s)^Ty+d_y(s)^Td_y(s)-2d_x^Tb+2d_y(s)^Tb \\
= \; & 2sd_x^T(-Bx-b-x)+2sd_x^T(-B)^T(x-y)+2d_y(s)^T(y+Bx+b)+s^2d_x^T(-2B-I)d_x \\
& -2sd_y(s)^T(-B)d_x+d_y(s)^Td_y(s) \\
\geq \; & 2sd_x^T(\mathcal{A}x-x)+2s(\delta-1)\|d_x\|^2+0-2s^2M\|d_x\|^2-2s^2N^2\|d_x\|^2+0 \\
= \; & 2sd_x^T(\mathcal{A}x-x)+(c(s)-2s)\|d_x\|^2
\end{aligned}
$$

□

Notice that if we take $s=\frac{\delta}{2(M+N^2)}$, then $c=c(s)=\frac{\delta^2}{2(M+N^2)}>0$. We will use this specific value of s in this section without further mentioned.

Lemma 3.3.3 *There exists a unique solution x^* of $\max_{x\in K} f(x)$, satisfying $[\mathcal{A}x^*]_K=x^*$.*

Proof. Because $f(x)$ is Lipschitz in a compact set and upper bounded by 0, the set of global optimal solutions of x in K, $\text{Argmax}f(x)$ is nonempty. For any $x\in\text{Argmax}f(x)$, we have $d_x=0$ by the previous lemma, and thus $[\mathcal{A}x]_K=x$. Notice that if there are two different $x_1,x_2\in\text{Argmax}f(x)$, then we have $(x_2-x_1)^T(\mathcal{A}x_1-y(x_1))\leq 0$ and $(x_1-x_2)^T(\mathcal{A}x_2-y(x_2))\leq 0$. Because $y(x_1)=x_1$ and $y(x_2)=x_2$, we have $(x_2-x_1)^T(-B-I)(x_1-x_2)\leq 0$. However,

$$(x_2-x_1)^T(-B-I)(x_1-x_2)=(x_2-x_1)^T(I+\frac{B+B^T}{2})(x_2-x_1)\geq\delta\|x_2-x_1\|^2>0,$$

which contradicts $(x_2 - x_1)^T(-B - I)(x_1 - x_2) \leq 0$. Thus the set Argmax$f(x)$ is a unique point x^* satisfying $[\mathcal{A}x^*]_K = x^*$. □

Also noticing that

$$(\mathcal{A}x - x)^T(x^* - x) = (\mathcal{A}x^* - x^*)^T(x^* - x) + (x - x^*)^T(-B - I)^T(x^* - x) \geq \delta\|x - x^*\|^2,$$

and we have

$$\begin{aligned} & 2d_x^T(\mathcal{A}x - x) - \|dx\|^2 \\ = \; & x^Tx - y^Ty + 2y^TAx - 2x^TAx \\ = \; & \|\mathcal{A}x - x\|^2 - \|\mathcal{A}x - y\|^2 \\ \geq \; & \|\mathcal{A}x - x\|^2 - \|\mathcal{A}x - z\|^2 \qquad \text{for all } z \in K. \end{aligned}$$

Take $z = [\mathcal{A}x]_B$, where $B = Conv(x, x^*)$ is the line section formed by x and x^*, a subset of feasible region K, we have

$$2d_x^T(\mathcal{A}x - x) - \|dx\|^2 \;\geq\; (\min\{\|x - x^*\|, (\mathcal{A}x - x)^T(x^* - x)/\|x^* - x\|\})^2 \geq (\min(\delta, 1))^2\|x - x^*\|^2$$

Thus we have the following lemma:

Lemma 3.3.4 *For any $x \in K$, we have*

$$2d_x^T(\mathcal{A}x - x) - \|dx\|^2 \geq \max(\min(\delta, 1)\|x - x^*\|, \|d_x\|)^2.$$

For the convergence rate, notice that $f(x)$ is Lipschitz, there exists a constant $L > 0$ such that $\|f(x) - f(x')\| \leq L\|x - x'\|$ for any $x, x' \in K$. Note $\Delta_x = x - x^*$ and $\Delta_y = y(x) - y^*$, where $y^* = x^*$ is the equilibrium point.

Now we have that

$$\begin{aligned} & f(x) - f(x^*) \\ = & f(x^* + \Delta_x) - f(x^*) \\ = & 2(\Delta_x^T - \Delta_y^T)(\mathcal{A}x^* - y^*) + \Delta_x^T(-2B - I)\Delta_x + 2\Delta_y^T B\Delta_x + \Delta_y^T\Delta_y \\ = & 2(\Delta_x - \Delta_y)^T(\mathcal{A}x^* - x^*) + O(\|\Delta_x\|^2) \\ = & -2d_x^T(\mathcal{A}x - x) + O(\|\Delta_x\|^2) + O(\|\Delta_x\|\|d_x\|) \end{aligned}$$

Thus there exist a large enough constants $a, b > 0$, such that

$$f(x^*) - f(x) \leq 2d_x^T(\mathcal{A}x - x) + a\|\Delta_x\|^2 + b\|\Delta_x\|\|d_x\|,$$

for all $x \in K$.

And in the Lemma 3.3.2 we also proved that

$$f(x + sd_x) - f(x) \geq 2sd_x^T(\mathcal{A}x - x) + (c - 2s)\|d_x\|^2.$$

If $\|dx\| \geq \min(\delta, 1)\|\Delta_x\|$, denote $\min(\delta, 1) = \mu^{-1}$, and let $w = d_x^T(\mathcal{A}x - x) - \|d_x\|^2 \geq 0$, we have that

$$\begin{aligned} & \frac{f(x + sd_x) - f(x)}{f(x^*) - f(x)} \\ \geq & \frac{2sw + c\|d_x\|^2}{2w + 2\|d_x\|^2 + a\|\Delta_x\|^2 + b\|\Delta_x\|\|d_x\|} \\ \geq & s\frac{2w + \delta\|d_x\|^2}{2w + (2 + a\mu^2 + b\mu)\|d_x\|^2} \\ \geq & s\min\left(1, \frac{\delta}{2 + a\mu^2 + b\mu}\right). \end{aligned}$$

If $\|d_x\| \leq \min(\delta, 1)\|\Delta_x\|$, let $w' = 2d_x^T(\mathcal{A}x - x) - \|d_x\|^2 - \mu^{-2}\|\Delta_x\|^2 \geq 0$, then

$$\begin{aligned} & \frac{f(x+sd_x) - f(x)}{f(x^*) - f(x)} \\ \geq \quad & \frac{sw + (\delta - 1)s\|d_x\|^2 + s\mu^{-2}\|\Delta_x\|^2}{w + \|d_x\|^2 + \mu^{-2}\|\Delta_x\|^2 + a\|\Delta_x\|^2 + b\|\Delta_x\|\|d_x\|} \\ \geq \quad & s\frac{w + \delta\mu^{-2}\|\Delta_x\|^2}{w + (2\mu^{-2} + a + b\mu^{-1})} \\ \geq \quad & s\min(1, \frac{\delta}{2 + a\mu^2 + b\mu}) > 0. \end{aligned}$$

Thus we have proved the following theorem:

Theorem 3.3.5 *For any $x \in K$, we have*

$$f(x^*) - f(x + sd_x) \leq \left[1 - s\min\left(1, \frac{\delta}{2 + a\mu^2 + b\mu}\right)\right][f(x^*) - f(x)].$$

Theorem 3.3.5 automatically guarantees the global linear convergence rate of $f(h_k)$. From the fact that

$$f(x^*) - f(h_k) \geq f(h_{k+1}) - f(h_k) \geq c\|d_x\|^2,$$

we also know $\|d_x\|$ converges to 0 linearly. By Lemma 3.3.4, we know that $\|x - x^*\|$ converges linearly to 0, thus x converges linearly to x^*. In other words, we have the following theorem:

Theorem 3.3.6 *For the DSL game with the assumption $I + \frac{B+B^T}{2} \succ I$ with $N = \|B\|_2$ and $M = N+1$, if everyone has a good enough memory (e.g., with fading rate $s \leq \frac{\delta}{2(M+N^2)}$) and plays the best response to the history of the actions taken by other players, then the system converges to the unique Nash Equilibrium linearly.*

To show how our algorithm runs in practice, we shall present the simulation results in the next subsection:

3.3.1 Simulation Results

In our simulation experiments, we use $\log \|d_{h_k}\|$, $f(h_k)$ and $\log(f(x^*)-f(x))$ to evaluate the algorithm. The reason to use $\|d_{h_k}\|$ is because of Lemma 3.3.4. And for $\log(f(x^*)-f(x))$, we used the maximum value of $f(x)$ from all iterations, then plus a small $\epsilon \sim 10^{-10}$ to avoid $\log 0$.

For randomly generated matrix, it seems that for all $0 < s < 1$ the iteration points converges extremely fast and even more surprisingly, the function $f(h_k)$ is monotonically nondecreasing, converging to $f(x^*) = 0$ exponentially fast, even though in our study we only managed to show this for small s. For both of the two log graphs, they are almost lines, which confirms our theoretical results about the linear convergence rate.

For randomly generated matrices, usually the equilibrium point is reached within a few iteration steps for big s. This means in practice IWFA is very efficient, and iterations with chosen step length are not necessary. However, this could also be linked directly to the way we generate this random matrix, so we still need more simulation tests, or maybe run some tests with real data from practice. In our simulation, we use $-f(h_l)$, and $\log(f(h_l)+\epsilon))$ to evaluate the algorithm, where the small $\epsilon \sim 10^{-10}$ is added to avoid $\log 0$.

Also, for the cases (labeled as *counter example*) where the condition $I + \frac{B+B^T}{2} \succ 0$ is violated, from the graphs it is clear that the algorithm does not converge. However, with smaller s (better memory), the algorithm tends to range of approximate Nash Equilibria with a much smaller error, and this error drops drastically as s decreases.

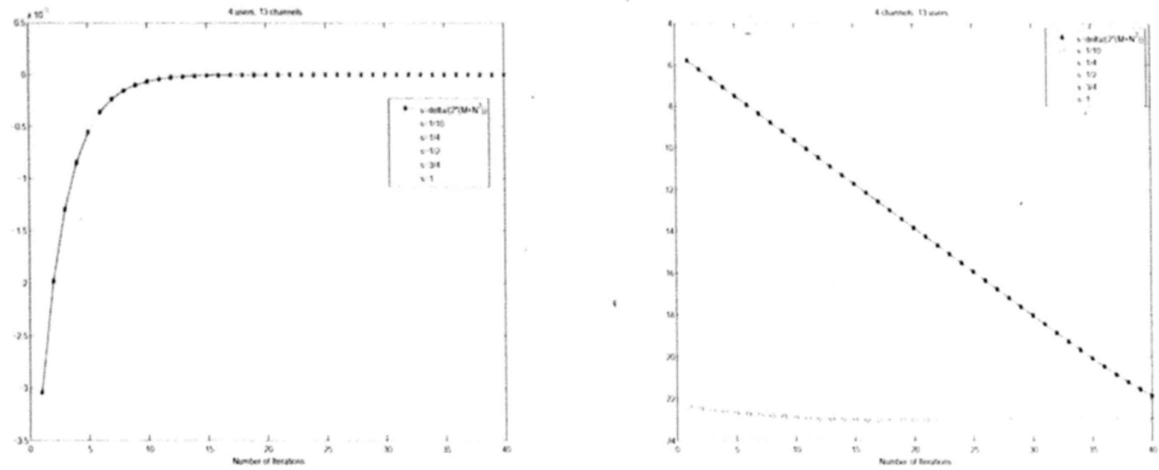


Figure 3.1: $-f(h_t)$ and $\log(T(h_t)+\epsilon))$ for randomly generated case

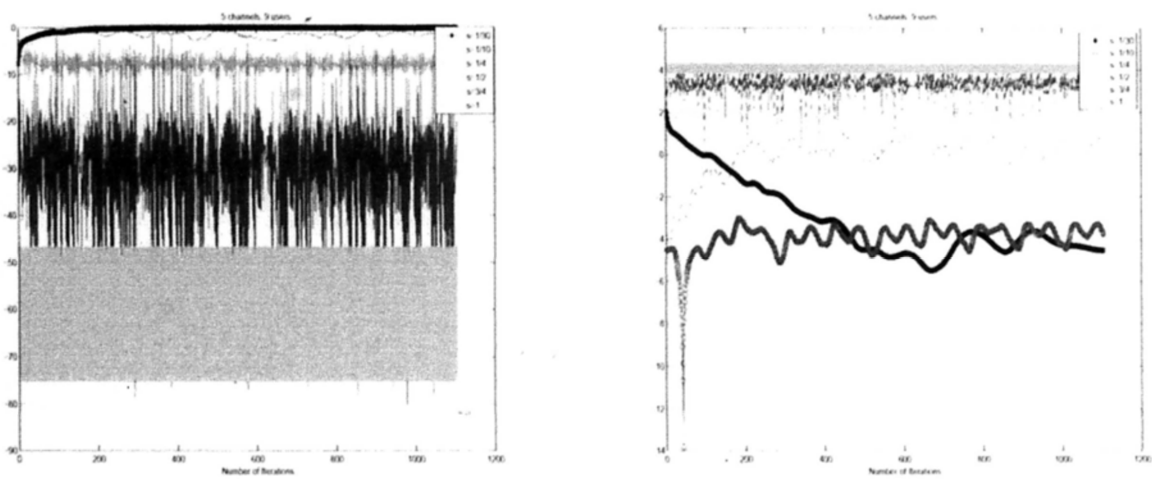


Figure 3.2: $-f(h_t)$ and $\log(T(h_t)+\epsilon))$ for counter example

3.4 Extensions and Future Directions

3.4.1 Another Iteration Scheme with Linear Global Convergence Rate

We introduce another iterative algorithm which converges linearly. The iteration step is as follows: $x^{k+1} = [(1-s)x^k + s\mathcal{A}x^k]_K$, then

$$(y-x)^T(\mathcal{A}x - x) = (y-x)^T(y-x) + (y-x)^T(\mathcal{A}x - y) \geq \|y-x\|^2,$$

thus

$$\begin{aligned}
& \|[(1-s)x + s\mathcal{A}x]_K - x^*\|^2 \\
\leq\ & \|(1-s)x + s\mathcal{A}x - [(1-s)x^* + s\mathcal{A}x^*]\|^2 \\
=\ & (1-s)^2\|x - x^*\|^2 + s^2\|\mathcal{A}x - \mathcal{A}x^*\|^2 + 2s(1-s)(x-x^*)^T(\mathcal{A}x - \mathcal{A}x^*) \\
\leq\ & [(1-s)^2 + s^2N^2 + 2s(1-s)(1-\delta)]\|x - x^*\|^2 \\
=\ & [1 - 2\delta s + (N^2 - 1 + 2\delta)s^2]\|x - x^*\|^2.
\end{aligned}$$

If we take $s = \frac{\delta}{N^2-1+2\delta}$, and define $\mu = \sqrt{1 - s\delta} < 1$, we have that

$$\|[(1-s)x + s\mathcal{A}x]_K - x^*\| \leq \mu\|x - x^*\|$$

for any $x \in K$, which guarantees global linear convergence rate for this iteration method.

However, this iteration step does not have a good game theoretical interpretation as the previous one, even though it is still a good distributive algorithm with only local information needed. Also, the lack of game intuition makes this algorithm not as good as a candidate for further generalization.

3.4.2 Connection to Monotonicity of LCP System

The idea of the algorithm above can be applied to any quadratic game in general and the key for the convergence proof is the condition that $\frac{B+B^T}{2} + I \succ 0$. If we consider the following general quadratic (cost) games: There are n players with the joint closed compact feasible region $X = \prod_{i=1}^n X_i$. Player i's objective is to minimize the cost function

$$c_i(x) = \frac{1}{2} x_i^T Q_i x_i + \sum_{j \neq i} x_i^T A_{ij} x_j.$$

Define the joint Hessian matrix H as $H_{ii} = Q_{ii}$ and $H_{ij} = A_{ij}$ if $i \neq j$. If the quadratic game is jointly strong convex, that is, the matrix $H \succ 0$, then by Theorem 3.3.6 we know that the best responds to history algorithm will converge to the unique Nash Equilibrium linearly as long as everyone has good enough memory. Notice quadratic games have one to one correspondence to LCP systems, and the condition $I + \frac{B+B^T}{2} \succ 0$ corresponds to the monotonicity of the LCP system. It would be interesting to merge the idea of learning with the existing interior point methods for solving monotone LCP problems [33] [9].

3.4.3 Extension to Non-Quadratic Game

The previous models are all based on quadratic game models. However we can approximate any game by a quadratic game, by locally approximate each cost utility function with the second order Taylor expansion. Therefore, it is natural to extend the idea to general games.

We now consider the game with the following structure:

1. There are n players each with cost function $c_i(x_i, x_{-i})$, where the decision variable of the other players is x_{-i}. The objective of each player is to minimize the cost, and c_i is assumed to be

convex with respect to x_i.

2. For each player i, $c_i(x_i, x_{-i})$ is assumed to be strongly convex with respect to x_i. Note the $\text{Argmin}c_i(x_i, x_{-i})$ as $R_i(x_{-i})$, which is assumed to be within a compact set X_i. The best response function $R(x)$ which is defined by $R(x)_i = R_i(x_{-i})$ which is the best response if everyone assumes the others play according to the joint decision vector x, but wants to deviate from it by himself.

3. Define the joint Hessian matrix of the game as:

$$H(x, y) = \left(\frac{\partial^2}{\partial x_i \partial y_j} c_i(x_i, y_{-i}) \right)_{i,j}.$$

We say the game is jointly strong convex, if $(H(R(x), x) + H^T(R(x), x))/2 \succeq \delta I_n$ with $\delta > 0$ for all $x \in X = \prod_{i=1}^n X_i$.

4. Assume the set X to have a diameter L.

5. Note the block diagonal matrix of $H(x, y)$ as $D(x, y)$, and $d_x = R(x) - x$. The Hessian matrix satisfies

$$\|d^T \nabla_d D(R(x), x) d\| \leq 2(d^T H(R(x), x) d)^{3/2},$$

and

$$\|d_x\| \leq \theta \sqrt{d_x^T D(R(x), x) d_x}.$$

6. We assume the block diagonal Hessian matrix $D(R(x), x)$ and the joint Hessian matrix are also upper bounded uniformly, that is, $(D(x, y) + D^T(x, y)) \preceq M I_n$ and $(H(x, y) + H^T(x, y)) \preceq M I_n$ for all $x, y \in X$. Also, since we can be achieved by scaling all the utility functions down uniformly, we may assume that $L^2 M \leq 1/2$.

7. The block diagonal Hessian matrix $D(R(x), x)$ is L^1 continuous, that is,

$$-N(\|u\| + \|v\|)I \preceq (D(x,y) - D(x+u, y+v)) + (D(x,y) - D(x+u, y+v))^T \preceq N(\|u\| + \|v\|)I.$$

We claim that for a jointly strong convex game, the strategy of learning of the (weighted discounted) history converges linearly.

Theorem 3.4.1 *For the jointly strong convex game with each $X_i \in \mathbb{R}^{d_i}$, consider the following algorithm: Define $C(x,h) = \sum_{i=1}^{n} c_i(x_i, h_{-i})$.*

1. *Start with an initial point $x^0 \in X$. Let $h^1 = x^0$. Initial with $k = 1$.*
2. *At each step k, each player i plays best response assuming all the other players are playing the history h^k_{-i}. The new joint decision is $x^{k+1} = R(h^k)$, where $R(\cdot)$ is the best response function. Update the history as $h^{k+1} = sx^{k+1} + (1-s)h^k$.*
3. *If $-C(x^k, h^k) + C(h^k, h^k) \leq \epsilon$, the algorithm stops. If not, then set $k = k+1$, go to step 2.*

Then the algorithm stops at most $\frac{2M}{s\delta}(\ln d_{x_0}^T D(R(x_0), x_0) d_{x_0} - \ln \epsilon)$ steps.

Proof. By the first order condition we know that $\nabla_{x_i} c_i(R(y)_i, y_{-i}) = 0$ for all $1 \leq i \leq n$ and $y \in X$. Therefore,

$$\frac{\partial^2}{\partial x_i \partial x_i} c_i(R(y)_i, y_{-i}) \frac{\partial}{\partial y} R(y)_i + \frac{\partial^2}{\partial x_i \partial x_{-i}} c_i(R(y)_i, y_{-i}) A^i = 0.$$

Here A^i is defined to be the matrix whose (k,j)-th block is I_{d_k} if $k = j$ and 0 if else, with $k = 1, \ldots, i-1, i+1, \ldots, n$ and $j = 1, 2, \ldots, n$, then we have that

$$\frac{\partial}{\partial y} R(y) = -D^{-1}(H - D)(R(y), y).$$

Consider the $P(x,d)$ defined as $d^T D(R(x),x)d$, $d_x = R(x) - x$ and the potential function at any point $x \in X$ as $P(x) = P(x, R(x) - x)$. Then if $\|d_x - d\| \le t\|d_x\|$, we have that

$$\begin{aligned}
\nabla_d P(x) &= 2d_x^T D(R(x),x)(\frac{\partial}{\partial y}R(x) - I)d + d_x^T \nabla_d D(R(x),x)d_x \\
&\le 2Mt\theta\|d_x\|^2 - 2d_x^T H(x)d_x + d_x^T \nabla_{d_x} D(R(x),x)d_x + tN\|d_x\|^3 \\
&\le 2Mt\theta\|d_x\|^2 - 2d_x^T H(x)d_x + (d_x^T H(x)d_x)^{3/2} + tN\|d_x\|^3 \\
&\le -d_x^T H(x)d_x + 2Mt\theta\|d_x\|^2 + tN\|d_x\|^3 \\
&\le \left(-\frac{\delta}{M} + \frac{2Mt\theta + tN\|d_x\|}{\delta}\right) P(x).
\end{aligned}$$

Notice that for any $s > 0$ and $y = x + sd_x$, we have

$$\|d_x - d_y\| = \left\| \int_{s'=0}^{s} \frac{\partial}{\partial y} R(x + s'd_x)d_x \right\| \le s\|d_x\|\sigma_{max}(I - D^{-1}H)(R(x + s'd_x), x + s'd_x) \le \frac{sM}{\delta}\|d_x\|.$$

Therefore, if we select $t = \frac{\delta^2}{2M(2M\theta + LN)}$ and $s = \frac{\delta}{M}t$, then we have that $(\ln P(x + zd_x)'_z) \le -\frac{\delta}{2M}$ for all $0 \le z \le s$. Therefore,

$$P(x + sd_x) \le P(x)e^{-s\delta/(2M)}.$$

Notice that because each player's utility function is convex, $-C(x^k, h^k) + C(h^k, h^k)$ is upper bounded by $(h^k - x^k)^T D(R(h^k), h^k)(h^k - x^k) = P(h^k)$, the algorithm terminates in at most $\frac{2M}{s\delta}(\ln P(x_0) - \ln \epsilon)$ many steps.

□

3.5 Future Directions and Plans

The IWFA method, from the algorithmic point of view, can also be seen as an updating to the best response direction (of the observed history), while smoothing with a step length. This is one of the common techniques used in optimization theory to construct algorithms with good convergence behavior. Actually, most recent main stream algorithms, for example, Newton's Method, Interior Point Method, etc, all adopt similar principles. The main advantage of this approach is that it is only based on the local properties, which are easier to verify and quantify than the global ones. However, it is unreasonable to force constraint on the player's step lengths, since each player is acting independently on his own. Meanwhile, in real life there are all kinds of cost for changing decisions, for example, tax, administration fees, etc. Intuitively, these friction types of cost could make the whole system more stable. It would be interesting to analyze how the system would behave with frictions involved.

And it is also natural that people make decisions based on the history which they observe. In the standard fictitious play everyone simply plays best response to the history, however the memory never fades. That is, a historic event occurred long time ago has the same weight as a recent event. It has been proved that under fictitious play, it converges for two person zero sum games [6] [50] and in other various settings, however there is no convergence rate even in very simple settings. It has also been shown to be not convergent in general by Shapley [57]. Also, even when it converges, the rate of convergence is very slow. It would be interesting to analyze the trade off between convergence rate and convergence condition via tuning the memory fading parameter, that is, analyze the behavior difference between strong and weak memory. It is also interesting whether the result can be extended to PTAS for the Bimatrix Game.

From the point of optimization theory itself, there is a strong need to solve large scale problems in application, for example, by applying of semidefinite programming in wireless communication and signal processing. Distributive algorithms have draw more and more attentions recently since it can efficiently trade resource for speed, and sometimes greatly speedup the whole optimization process. It has similarity to the game environment, as each machine corresponds to a single player, but it is in centralized environment. Our modified IWFA method can also be viewed as a distributive algorithm to efficiently solve for the Nash Equilibrium. When each players' objective coincides with social objective, it converges to the social optimum in a distributive manner. Therefore, it's natural to consider possible directions to extend the idea.

It can also be seen from the discussion in Section 4.3 that the competitive routing game with linear unit cost, is exactly a quadratic game (each player has a convex quadratic objective function) with compact convex feasible region, and furthermore the game is jointly convex. Therefore if each player has good enough memory and uses it in decision making, the system tends to the unique Nash Equilibrium of the system.

Finally, for the general game model, there are many questions left open. For instance, whether or not we can replace the Hessian matrix at the point $(R(x), x)$ with simply the one at (x, x), since it is much easier to evaluate. Also, replacing the strong convexity condition by simply the convexity condition is interesting, since it would cover many existing known solvable cases, for example, zero-sum bimatrix game, monotone LCP and so on.

Chapter 4

Bounding the Price of Anarchy in Competitive Routing Game

4.1 Introduction

Routing model has been widely used in various fields including traffic, internet, telecommunication and so on. In a routing problem setting, given a general directed network, each user has a given amount of flow to ship from user-specific source to user-specific destination and may split the flow through the paths between the two nodes. In the context of road traffic networks, for example, users can be viewed as a transportation company which need to ship a flow of vehicles. For telecommunication network, players corresponding to those users who are to send their signals through a shared network.

Applying the game theory to the routing model is a natural and promising approach. Specifically, let the individual users compete with one another to optimize their choices of the paths and an equilibrium may be obtained. The concept of Nash equilibrium is often used when considering a noncooperative model.

The research along this direction has started since 1950's. Wardrop [60] formalized the notions of a flow at Nash equilibrium and of a minimum-latency flow for one particular case of the routing game, in which all of users are infinitesimal and hence every network user controls a negligible fraction of the overall traffic. There are a rich literature on the analysis of Wardrop equilibrium. Beckmann et al. [5] found that a flow at Nash equilibrium is an optimal solution to a related convex program and obtained existence and uniqueness results for the equilibria.

However, our focus here is competitive routing, where there are only finite number of users with each controls a nonnegligible amount of flow. The problem has recently become a topic of intensive research in the routing game community. Orda, Rom and Shimkin [47] showed the uniqueness of the Nash equilibrium of a two-node multiple-links system under reasonable convexity conditions and meanwhile they proposed a counter example indicating uniqueness may fail for general networks. However, Eitan et al. [2] showed that the Nash equilibrium was unique for general network when a class of polynomial link cost functions was adopted.

The efficiency of an equilibrium has also attracted much attention. One quantitative criteria is the so-called price of anarchy, which is the ratio between the total social cost at the Nash Equilibrium and the optimal social cost, i.e., assuming that all resources is completed controlled by a single administrative domain. It is well known that the price of anarchy in routing game can be arbitrarily large. In particular, the Braess's paradox shows adding a link to the network could lead to an increased cost to all users. R. El Azouzi, E. Altman and O. Pourtallier [19] provided some guidelines for avoiding the Braess paradox when upgrading the network. T. Roughgarden in his thesis [53] established a bicriteria bound for the price of anarchy in this setting.

Most of the papers in the literature discussed about the routing game problem in the noncooperative frame. In this chapter, however, we allow users to cooperate to some extent and study the change of the user's cost at Nash equilibrium when users choose to cooperate with some others in the context of road traffic routing. We find in the parallel graph with affine cost function, most users can benefit from the cooperation. As a consequence, the price of anarchy is monotone with the extent of cooperation of users, although it fails when relaxing the assumptions.

The chapter is organized as follows: in the next section, we introduce the general model considered in this chapter. In Section 4.3, we establish some useful properties including continuity and uniqueness of the problem. In Section 4.4, we consider a special network topology and draw some conclusions there. We prove some monotonicity properties and give a tight upper bound for the price of anarchy in our setting. Next, some counter examples relaxing the assumptions are presented to show that the assumptions are necessary.

4.2 The Model and Formulation

We consider a directed graph $G = (V, L)$ with vertex set V, link set L. Assume that $|V| = n, |L| = m$ and multiple parallel links are allowed but no self-loop exists. Furthermore, let us denote $A \in \Re^{n\times m}$ to be the node-to-arc incidence matrix. Specifically, each row of A represents a node and each column represents an arc. Consider an arc connecting node i to node j. Then, the corresponding column in A has all 0 elements except for the i-th element, where it is $+1$, and the j-th element, where it is -1.

Suppose there are K players in the game and the corresponding source-destination pairs $\{s^1, d^1\}, \{s^2, d^2\}, \ldots, \{s^K, d^K\}$. Let r denote a vector in $\Re^K$, where r^k represents the value of flow that player k is required to transport from s^k to d^k. A flow of player k is a function $x^k : L \to \Re_+$, which can also be viewed as a vector in $\Re^m$. Because the inflow and outflow at each middle node has to match exactly, with the inflow at source and outflow at the end equal to r^k, which is the exactly flow demand for player k. We know that a flow is feasible if and only if $Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k}$ where $\delta_{s^k}(\delta_{d^k})$ is the vector in $\Re^n$ in which s^k-th(d^k-th) element is one and others are all zero.

For each link l, we define the total flow through it as $f_l = \sum_{k=1}^K x_l^k$. The cost of per unit of flow on the link l is given by a function $c_l : f_l \hookrightarrow c_l(f_l)$, which is called *unit cost* on link l. Then (G, r, c) specifies an instance of the routing game problem.

In the instance (G, r, c), if considering the Nash equilibrium, player k is to find a feasible flow to minimize its total cost function, which is defined as

$$C^k(x^k, x^{-k}) = \sum_{l \in L} x_l^k c_l(f_l).$$

We define the social cost as the simple sum of all players:

$$SC(x) = \sum_{k=1}^{K} C^k(x^k),$$

Let x_{Nash} denote the flow when the game reaches the Nash equilibrium, i.e. each player minimizes its cost. Let x^* denote the minimal solution of the function SC over the feasible field of x. Clearly, x_{Nash} and x^* are different vectors in general. To quantify the difference, we use the price of anarchy defined by the ratio between the social cost at the two states, i.e.,

$$\alpha(G, r, c) = \frac{SC(x_{Nash})}{SC(x^*)}.$$

Through out this chapter, we only consider the case where the unit cost function is affine linear. Specifically, the unit cost on link l is given by $c_l(f_l) = a_l f_l + b_l$. Then each player k will face the following optimization problem:

$$\begin{aligned}(P_k)\quad \min\quad & \textstyle\sum_{l\in L}(a_l f_l + b_l)x_l^k \\ \text{s.t.}\quad & Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k} \\ & x^k \geq 0.\end{aligned}$$

Substituting $f_l = \sum_{k=1}^{K} x_l^k$, we have

$$\begin{aligned}(P'_k)\quad \min\quad & \textstyle\sum_{l\in L}\left\{b_l x_l^k + a_l\left[\left(\sum_{i\neq k} x_l^i\right)x_l^k + (x_l^k)^2\right]\right\} \\ \text{s.t.}\quad & Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k} \\ & x^k \geq 0.\end{aligned}$$

4.3 Continuity and Uniqueness under Linear Cost Condition

We first establish the uniqueness of the Nash Equilibrium and the continuity result for the cost of each player in the Nash Equilibrium, with respect to the input vectors r and c, where c is specified by $a, b \in \mathbb{R}^L$. Although these results have been shown independently by liteatures, and can be directly seen from the analysis in following sections, the LCP formulation we show here is of independent interest.

Let $y^k \in \Re^m$ be the Lagrangian multiplier associated with the equality constraint $Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k}$.

The Karush-Kuhn-Tucker optimality condition for (P'_k) is:

$$\begin{cases} Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k} \\ x^k \geq 0 \\ b_l + a_l\sum_{i=1}^K x_l^i + d_l x_l^k + (A^T y^k)_j - s_l^k = 0,\ l = 1, ..., m \\ s_l^k \geq 0,\ l = 1, ..., m \\ x_l^k s_l^k = 0,\ l = 1, ..., m. \end{cases}$$

Denote s^k to be the vector whose l-th component is s_l^k, $l = 1, ..., n$. A Nash equilibrium for the routing game is attained if and only if the above condition holds simultaneously for all $k = 1, ..., K$; i.e.

$$\begin{cases} Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k} \\ x^k \geq 0, \\ b + Diag(a)\sum_{i=1}^K x^i + Diag(a)x^k + A^T y^k - s^k = 0 \\ s^k \geq 0, \\ (x^k)^T s^k = 0, \end{cases}$$

where $Diag(a)$ is the diagonal matrix whose j-th diagonal is a_l, $l = 1, ..., n$.

We can explicitly write the KKT optimality condition using the block matrix notation. Let x (respectively y, and s, and R) be the vector consisting of $x^1, ..., x^K$ (respectively $y^1, ..., y^K$, and $s^1, ..., s^K$, and $(r^1\delta_{s^1} - r^1\delta_{d^1}), ..., (r^K\delta_{s^K} - r^K\delta_{d^K})$) stacking on top of each other. The equations for the Nash equilibrium solutions are:

$$(NE)\ \begin{cases} (I_K \otimes A)x = R \\ e \otimes c + (E_K \otimes D)x + (I_K \otimes D)x + (I_K \otimes A)^T y - s = 0 \\ x \geq 0,\ s \geq 0 \\ x^T s = 0, \end{cases}$$

where '⊗' stands for the Kronecker product between two matrices, e is the (K by 1) all-one vector, E_K is the (K by K) all-one matrix, and I_K is the (K by K) identity matrix. The so-expressed Nash equilibrium is a mixed linear complementarity problem, and it specifies an if and only if condition for an point to be Nash Equilibrium.

Notice that the closed feasible set of (x, s) specified by the (NE) condition above is continuous with respect to the input variables a, b and r. If we see each player k' cost C^k as the function of them, then it is a continuous function.

To show the uniqueness, let us turn to consider the following standard mixed linear complementarity problem:

$$(LCP)\begin{cases} s &= q + Mx + L^{\mathrm{T}}y \\ Lx &= b \\ x &\geq 0 \\ s &\geq 0 \\ s^{\mathrm{T}}x &= 0, \end{cases}$$

where the dimensions of all the matrices are compatible. An important notion for solving such LCP problem is the so-called *monotonicity*.

Definition 4.3.1 *The (LCP) is called monotone if* $(\Delta x)^T M \Delta x \geq 0$ *for all* Δx *satisfying* $L\Delta x = 0$. *That is,* $M + M^T$ *is a positive semidefinite matrix over the null space of* L.

Definition 4.3.2 *We call the mixed LCP problem (LCP) feasible if there exist* x *and* y *satisfying*

$$q + Mx + L^T y \geq 0,\ x \geq 0,\ \text{and } Lx = b.$$

The following result is adapted from Theorems 3.1.2 and 3.1.7 in Cottle, Pang and Stone [9].

Theorem 4.3.3 *Suppose that (LCP) is monotone and is feasible. Then (LCP) has a solution, and the solution is unique in the sense that there is an index set α such that x is a solution to (LCP) iff x is feasible and the support of x is contained in α.*

We shall see that the mixed LCP problem arising from the Nash equilibrium of the minimum cost transportation game is monotone.

Indeed, for any Δx satisfying

$$(I_K \otimes A)\Delta x = 0,$$

the corresponding

$$\Delta s = (E_K \otimes D)\Delta x + (I_K \otimes D)\Delta x + (I_K \otimes A)^{\mathrm{T}}\Delta y.$$

Therefore

$$(\Delta x)^{\mathrm{T}}\Delta s = (\Delta x)^{\mathrm{T}}(E_K \otimes D)\Delta x + (\Delta x)^{\mathrm{T}}(I_K \otimes D)\Delta x = (\Delta x)^{\mathrm{T}}((E_K + I_K) \otimes D)\Delta x \geq 0, \quad (4.3.1)$$

since $(E_K + I_K) \otimes D$ is a positive semidefinite matrix. This shows that (NE) is a monotone mixed LCP. Clearly, it is also feasible by noting $c \geq 0$. It follows from Theorem 4.3.3 that (NE) has a unique Nash solution.

To sum up, we have the following theorem:

Theorem 4.3.4 *For the linear cost competitive routing game (G, r, c), the Nash Equilibrium is unique and when observing the cost of each player at the Nash Equilibrium as function of the input variables (r, c), it is continuous.*

There are further consequences of the monotonicity of the LCP system arises from the competitive routing game. Since this game is indeed a quadratic game, it follows from the first chapter that if every acts greedily according to the history play of the others, with a good enough memory, then the system converges linearly to the Unique Nash Equilibrium of the game. Therefore, it does make sense for players to consider their costs under the (unique) Nash Equilibrium instead of the immediate cost, when they decide to cooperate or not.

4.4 Two-nodes Multiple-links Graph with Affine Linear Cost

In this part, we consider a special network, where there are only two nodes denoted by s and t and multiple links between them. All players have the same source-destination pair $\{s, t\}$.

Then we have the following monotonicity result:

Theorem 4.4.1 *If we denote the cost of user k when the game gets the (unique) Nash equilibrium as function $C^k(r)$ of flow vector r, if $r^{k_1} > r^{k_2}$, then for any $0 \leq t \leq r^{k_2}$, the following holds true:*

$$SC(r + t\delta_{k_1} - t\delta_{k_2}) \leq SC(r).$$

Moreover,

$$C^k(r + t\delta_{k_1} - t\delta_{k_2}) \leq C^k(r) \text{ for } k \neq k_1, k_2,$$

where δ_j is the all zero vector with the exception that of value 1 at the j-th entry. The case where $t = r^{k_2}$ represents when the two players agree to merge together to form a new player to play this competitive routing game.

The theorem implies that if any two players choose to partially cooperate (e.g., the player with bigger flow demand seize more flow demands from the player with lesser demand), all the other players in the game will benefit, i.e., there will be less costs incurring to them from the cooperation, and the social cost decreases consequently. It also characterizes the worst possible scenario of the price of anarchy for a fixed network and given number of players, when all the players have exactly the same flow demand.

4.4.1 Characterization of the LCP System.

To prove the theorem, we need to begin with characterization of the LCP system (or equivalently, the KKT condition), then proceed to introducing some equations and inequalities that will be used in the following sections.

Denote the support set of player k as $S^k = \{l | x_l^k > 0\}$, i.e., the set of links which player k uses. Without losing generality we assume that $a_l > 0$ for all $l = 1, 2, \ldots, m$ (we can use continuity argument to deal with the case that $a_l = 0$). Also, without loss of generality, we can rearrange the indices and assume that $r^1 \geq r^2 \geq \ldots \geq r^K$, $b_1 \leq b_2 \leq \ldots \leq b_m$.

Let c^k denote the marginal cost of player k. By the KKT condition, it follows that

$$\begin{cases} c^k = a_l(f_l + x_l^k) + b_l, & \text{if } x_l^k > 0 \\ c^k \leq a_l f_l + b_l, & \text{if } x_l^k = 0. \end{cases}$$

In other words,

$$x_l^k = \left[\frac{c^k - b_l}{a_l} - f_l \right]_+ \text{ for all } k \text{ and } l. \tag{4.4.1}$$

Therefore

$$r^k = \sum_{l=1}^{m} x_l^k = \sum_{l=1}^{m} \left[\frac{c^k - b_l}{a_l} - f_l \right]_+,$$

which is monotonically increasing with respect to c^k, and we know that

$$c^1 \geq c^2 \geq \ldots \geq c^K. \tag{4.4.2}$$

Consequently we have that

$$x_l^1 \geq x_l^2 \geq \ldots \geq x_l^K, \text{ for all } l = 1, 2, \ldots, m, \tag{4.4.3}$$

and

$$S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K. \tag{4.4.4}$$

The relationship above shows that there are exactly k players using the link l for $l \in S^k - S^{k+1}$ and $f_l = \sum_{i=1}^k x_l^i$. Furthermore, $f_l + x_l^i = \frac{c^i - b_l}{a_l}$ for $i \leq k$. Taking the sum of the equations for $i = 1, 2, \ldots, k$, we get

$$(k+1)f_l = \frac{E^k - kb_l}{a_l} \quad \text{for all } l \in S^k - S^{k+1}, \tag{4.4.5}$$

where $E^k = \sum_{i=1}^k c^i$. Combine the above equation with equation (4.4.1), the following equation holds:

$$x_l^i = \left[\frac{(k+1)c^i - E^k - b_l}{(k+1)a_l}\right]_+ \quad \text{for } l \in S^k - S^{k+1}. \tag{4.4.6}$$

And consequently, for any $i > k$ and $l \in S^k - S^{k+1}$,

$$(k+1)c^i - E^k - b_l \leq 0.$$

Also,

$$x_l^i - x_l^k = \frac{c^i - c^k}{a_l} \quad \text{for all } l \in S^k \text{ and } i \leq k.$$

Taking the sum of $f_l + x_l^k$ for $l \in S^k$, we have

$$\begin{aligned}
\sum_{l\in S^k} \frac{e^k - b_l}{a_l} &= \sum_{l\in S^k} (f_l + x_l^k) \\
&= \sum_{l\in S^k} \left[x_l^k + \sum_{i>k} x_l^i + \sum_{i\leq k} (x_l^i + \frac{e^i - e^k}{a_l}) \right] \\
&= \sum_{l\in S^k} \left[(k+1)x_l^k + \sum_{i>k} x_l^i + \frac{E^k - ke^k}{a_l} \right] \\
&= \sum_{i>k} r^i + (k+1)r^k + \sum_{l\in S^k} \frac{E^k - ke^k}{a_l}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i>k} r^i + (k+1)r^k &= \sum_{l\in S^k} \frac{(k+1)e^k - E^k - b_l}{a_l} \\
&= \sum_{l\in S^k} \frac{kE^k - (k+1)E^{k-1} - b_l}{a_l}. \qquad (4.4.7)
\end{aligned}$$

Hence, we can get the following difference equation:

$$\frac{E^k}{k+1} - \frac{E^{k-1}}{k} = g^k(r), \qquad (4.4.8)$$

where

$$g^k(r) = \frac{\sum_{i>k} r^i + (k+1)r^k + B^k}{k(k+1)A^k},$$

whose value depends only on r and the support set S^k. For convenience the following notations are employed:

$$A^k = \sum_{l\in S^k} 1/a_l, \;\; B^k = \sum_{l\in S^k} b_l/a_l.$$

Also, it follows from equation (4.4.8) that

$$E^k = (k+1) \sum_{i=1}^{k} g^i(r). \qquad (4.4.9)$$

For any $l \in S^j - S^{j+1}$ and $j < k$,

$$(k+1)e^k - E^k = (k+1)e^k - E^j - (E^k - E^j) \leq (k+1)e^k - E^j - (k-j)e^k = (j+1)e^k - E^j \leq b_l.$$

For any $l \in S^i - S^{i+1}$ with $i > k$,

$$(k+1)e^k - E^k = (k+1)e^k - E^i + (E^i - E^k) \geq (k+1)e^i - E^i + (i-k)e^i = (i+1)e^i - E^i > b_l.$$

Consequently,

$$b_l > b_{l'} \text{ for any } l \in S^{k-1} - S^k \text{ and } l' \in S^k - S^{k+1}.$$

By checking the KKT condition, we can show that: For given support sets $S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K$, E^k can be calculated by equation (4.4.9) as $E^k = (k+1)\sum_{i=1}^{k} g^i(r)$. And conversely, if the calculated e^i's are in decreasing order and for all $l \in S^k - S^{k+1}$,

$$(k+1)e^k - E^k > b_l \geq (k+1)e^{k+1} - E^k,$$

then the KKT condition has been satisfied. Therefore, we have the following lemmas:

Lemma 4.4.2 *For the given support sets $S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K$, and cost margin $e^1 \geq e^2 \geq \ldots \geq e^K$, the condition for this support set and cost margin to be indeed corresponding to the unique Nash Equilibrium is*

$$(k+1)e^k - E^k > b_l \geq (k+1)e^{k+1} - E^k, \text{ for all } k = 1, 2, \ldots, K,$$

where e^{K+1} is defined as $\frac{E^K}{K+1}$.

Lemma 4.4.3 *For the given support sets $S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K$, the condition for this support set to be indeed corresponding to the unique Nash Equilibrium is*

$$\begin{cases} (k^2+2k)g^{k+1}(r) - \sum_{i=1}^{k} g^i(r) \leq b_l < (k^2+k)g^k(r) \\ (k+2)g^{k+1}(r) \leq kg^k(r), \end{cases} \quad \text{for all } k = 1, 2, \ldots, K.$$

Lemma 4.4.4 *For the given support sets $S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K$, and cost margin $c^1 \geq c^2 \geq \ldots \geq c^K$, the condition for this support set and cost margin to be indeed corresponding to the unique Nash Equilibrium of a game formed by the group of K players is*

$$(k+1)c^k - E^k > b_l \geq (k+1)c^{k+1} - E^k, \text{ for all } k = 1, 2, \ldots, K.$$

And the demand of each player can be calculated from the equation (4.4.8)

$$\frac{E^k}{k+1} - \frac{E^{k-1}}{k} = g^k(r).$$

Also, we can establish the monotonicity of $\frac{\sum_{l\in S^k}(a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l}$ and $\frac{\sum_{l\in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l}$ which is shown in the following lemma:

Lemma 4.4.5 *For any $k > i$,*

$$\frac{\sum_{l\in S^k}(a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l} \leq \frac{\sum_{l\in S^i}(a_l f_l + b_l)/a_l}{\sum_{l\in S^i} 1/a_l}$$
$$\frac{\sum_{l\in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l} \geq \frac{\sum_{l\in S^i}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^i} 1/a_l}$$

Proof. For any $l_1 \in S^i - S^{i+1}$ and $l_2 \in S^k - S^{k+1}$, from $l_1 \notin S^k$, we have

$$a_{l_1} f_{l_1} + b_{l_1} \geq c^k = a_{l_2}(f_{l_2} + x^k_{l_2}) + b_{l_2} \geq a_{l_2} f_{l_2} + b_{l_2}.$$

And by $l_1 \notin S^{i+1}, l_2 \in S^k$,

$$(i+1)c^{i+1} - E^i \leq b_{l_1}, \qquad \text{and} \quad (k+1)c^k - E^k \geq b_{l_2}.$$

Furthermore, according to the monotonicity of c^k,

$$\left[E^k - (k-1)c^k\right] - \left[E^i - (i-1)c^{i+1}\right] = \sum_{j=i+1}^{k} c^j + (i-1)c^{i+1} - (k-1)c^k \geq 0,$$

i.e.,

$$E^k - (k-1)e^k \geq E^i - (i-1)e^{i+1}.$$

Therefore,

$$\begin{aligned}
2a_{l_2}f_{l_2} + b_{l_2} &= \frac{2E^k - (k-1)b_{l_2}}{k+1} \\
&\geq \frac{2E^k - (k-1)[(k+1)e^k - E^k]}{k+1} \\
&= E^k - (k-1)e^k \geq E^i - (i-1)e^{i+1} \\
&= \frac{2E^i - (i-1)[(i+1)e^{i+1} - E^i]}{i+1} \\
&\geq \frac{2E^i - (i-1)b_{l_1}}{i+1} = 2a_{l_1}f_{l_1} + b_{l_1}.
\end{aligned}$$

Due to the arbitrariness of k and i, for all $l_1 \in S^i - S^k, l_2 \in S^k$, we have

$$\begin{cases}
a_{l_2}f_{l_2} + b_{l_2} \leq a_{l_1}f_{l_1} + b_{l_1} \\
2a_{l_2}f_{l_2} + b_{l_2} \geq 2a_{l_1}f_{l_1} + b_{l_1}.
\end{cases}$$

Consider $\frac{\sum_{l\in S^k}(a_lf_l+b_l)/a_l}{\sum_{l\in S^k}1/a_l}$ and $\frac{\sum_{l\in S^k}(2a_lf_l+b_l)/a_l}{\sum_{l\in S^k}1/a_l}$ as the weighted average,

$$\begin{aligned}
a_{l_1}f_{l_1} + b_{l_1} &\geq \frac{\sum_{l\in S^k}(a_lf_l+b_l)/a_l}{\sum_{l\in S^k}1/a_l}, \\
2a_{l_1}f_{l_1} + b_{l_1} &\leq \frac{\sum_{l\in S^k}(2a_lf_l+b_l)/a_l}{\sum_{l\in S^k}1/a_l}.
\end{aligned}$$

The inequalities hold for any $l_1 \in S^i - S^k$, consequently, for $i < k$,

$$\begin{aligned}
\frac{\sum_{l\in S^i}(a_lf_l+b_l)/a_l}{\sum_{l\in S^i}1/a_l} &\geq \frac{\sum_{l\in S^k}(a_lf_l+b_l)/a_l}{\sum_{l\in S^k}1/a_l}, \\
\frac{\sum_{l\in S^i}(2a_lf_l+b_l)/a_l}{\sum_{l\in S^i}1/a_l} &\leq \frac{\sum_{l\in S^k}(2a_lf_l+b_l)/a_l}{\sum_{l\in S^k}1/a_l},
\end{aligned}$$

Because

$$e^kA^k - \sum_{l\in S^k}\frac{a_lf_l+b_l}{a_l} = \sum_{l\in S^k}\frac{e^k - a_lf_l - b_l}{a_l} = \sum_{l\in S^k}x_l^k = r^k,$$

we can also conclude that

$$e^i - \frac{r^i}{A^i} \geq e^k - \frac{r^k}{A^k} \text{ for all } i < k. \tag{4.4.10}$$

□

4.4.2 Directional Derivative Formulations and Continuity Argument

To study the change of costs for players when two players cooperate with each other, we start by looking at the subgradient of costs when the two players partially cooperate with each other, e.g., the player with smaller flow demand "gives" ϵ-small demand to the player with higher flow demand.

The cost for player k is

$$C^k = \sum_{l \in S^k} (a_l f_l + b_l) x_l^k.$$

Substitute (4.4.5) and (4.4.6) into it,

$$C^k = \sum_{i=k}^{K} \left[\sum_{l \in S^i - S^{i+1}} \frac{E^i + b_l}{i+1} \cdot \frac{(i+1)e^k - E^i - b_l}{(i+1)a_l} \right].$$

From the results in the previous section (the LCP form), we know C^k is continuous with respect to r.

For given support sets vector $S = \{S^i, i = 1, 2, \ldots, K, S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K\}$, define the corresponding 'feasible' set $r(S)$ of r by the two conditions in Lemma 4.4.3. And denote the closure of $r(S)$ bs $R(S)$. Since all the conditions are linear with respect to r, for a fixed S, we know that each $R(S)$ is a polyhedron. There are only finitely many of them since the choice of support sets are finite, and in the interior of each of these polyhedrons, all the values can be fully differentiated with respect to r.

First, we give a continuity argument, which shows that we only need to prove Theorem 4.4.1 in the special that when r moves locally inside the interior of each $R(S)$.

Lemma 4.4.6 *Suppose $X_i, i = 1, 2, \ldots, n$ are n closed polyhedrons in R^K whose union is the whole space, e.g., $\bigcup_{i=1}^{n} X_i = R^K$, and $F(x)$ is a continuous value function on R^K which is differentiable in the interior X_i° of each X_i. For any direction d, if the directional derivative $\nabla_d F(x) \leq 0$ for all $x \in X_i^\circ$, then $F(x + td) \leq F(x)$ for any $x \in R^K$ and $t \geq 0$.*

Proof. If both x and $x + td$ are in the interior of the same X_i, then by the directional derivative we know that $F(x + td) \leq F(x)$. Therefore, by continuity the property holds if both x and $x + td$ are in the same X_i, regardless if they are in the interior or on the boundary. For any $x \in R^k$, $t > 0$ and $s > 0$, we randomly (uniformly) choose a v with $\|v\| \leq s$. For each facet of any X_i, the probability that the line $[x + v, x + v + td]$ lies in the facet is 0. Because there are at most finitely many facets, with probability 1 the line cross each facet only once. Therefore there exists a vector v with $\|v\| \leq s$, such that the line $[x + v, x + v + td]$ cross the boundary of all X_i's at most finitely many times. For each fraction of the line, the directional property holds, therefore it holds for the whole line, that is, $F(x) \geq F(x + td)$. □

Based on the above lemma, we only need to worry about the directional derivatives in the interior of each $R(S)$ for given S. Take the direction $d = \delta_{k_1} - \delta_{k_2}$, where δ_k is the vector in R^k with value 1 at the k-th index, and 0 on all the other indices. We now consider the directional derivatives, with direction d. We note the directonal derivative of the player k's cost with respect as $\nabla_d C^k$ and similarly, the directional derivative of g^k as $\nabla_d g^k$.

Notice that

$$\nabla_d g^k = \begin{cases} \frac{1}{(k_1+1)A^{k_1}}, & \text{if } k = k_1 \\ -\frac{1}{k(k+1)A^k}, & \text{if } k_1 < k < k_2 \\ -\frac{1}{k_2 A^{k_2}}, & \text{if } k = k_2 \\ 0, & \text{if else} \end{cases} \tag{4.4.11}$$

Hence,

$$\begin{aligned} \nabla_d C^k &= \sum_{i=k}^{K} \sum_{l\in S^i - S^{i+1}} \sum_{j=1}^{i} \nabla_d g^j \cdot \frac{(i+1)e^k - E^i - b_l}{(i+1)a_l} \\ &+ \sum_{i=k}^{K} \sum_{l\in S^i - S^{i+1}} \frac{E^i + b_l}{(i+1)a_l} \cdot \left((k+1)\sum_{j=1}^{k} \nabla_d g^j - k\sum_{j=1}^{k-1} \nabla_d g^j - \sum_{j=1}^{i} \nabla_d g^j \right) \\ &= \sum_{i<k} \nabla_d g^i \left(e^k \sum_{l\in S^k} 1/a_l - \sum_{j=k}^{K} \sum_{l\in S^j - S^{j+1}} \frac{E^j + b_l}{(j+1)a_l} \right) \\ &+ \nabla_d g^k \left(e^k \sum_{l\in S^k} 1/a_l + (k-1)\sum_{j=k}^{K} \sum_{l\in S^j - S^{j+1}} \frac{E^j + b_l}{(j+1)a_l} \right) \\ &+ \sum_{i>k} \nabla_d g^i \left(e^k \sum_{l\in S^i} 1/a_l - 2\sum_{j=i}^{K} \sum_{l\in S^j - S^{j+1}} \frac{E^j + b_l}{(j+1)a_l} \right) \end{aligned}$$

By substituting (4.4.5) in, we get

$$\begin{aligned} \nabla_d C^k = &\textstyle\sum_{i<k} \nabla_d g^i \left(e^k A^k - \sum_{l\in S^k} \frac{a_l f_l + b_l}{a_l} \right) + \nabla_d g^k \left(e^k A^k + (k-1)\sum_{l\in S^k} \frac{a_l f_l + b_l}{a_l} \right) \\ &+ \textstyle\sum_{i>k} \nabla_d g^i \left(e^k A^i - 2\sum_{l\in S^i} \frac{a_l f_l + b_l}{a_l} \right) \end{aligned} \tag{4.4.12}$$

Because

$$e^k A^k - \sum_{l\in S^k} \frac{a_l f_l + b_l}{a_l} = \sum_{l\in S^k} \frac{e^k - a_l f_l - b_l}{a_l} = \sum_{l\in S^k} x_l^k = r^k.$$

we can rewrite equation (4.4.12) as

$$\nabla_d C^k = \sum_{i<k} \nabla_d g^i r^k + \nabla_d g^k (k e^k A^k - (k-1) r^k) + \sum_{i>k} \nabla_d g^i (e^k A^i - 2e^i A^i + 2r^i). \tag{4.4.13}$$

With the derivatives obtained above, we now start to prove for the main theorem (4.4.1) by looking at the directional derivative.

4.4.3 Directional Derivatives of The Social Cost and Individual Cost

Directional Derivative of The Social Cost

We start with the social cost. Note that the total social cost is:

$$\begin{aligned} SC &= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} a_l f_l^2 + b_l f_l \\ &= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \frac{E^k - kb_l}{(k+1)a_l} \cdot \frac{E^k + b_l}{k+1} \end{aligned}$$

Hence the directional derivative of the total social cost is

$$\begin{aligned} \nabla_d SC &= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \frac{E^k + b_l}{(k+1)a_l} + \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \frac{E^k - kb_l}{(k+1)a_l} \\ &= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \frac{2E^k - (k-1)b_l}{(k+1)a_l} \\ &= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \left[\frac{b_l}{a_l} + \frac{2E^k - 2kb_l}{(k+1)a_l} \right] \end{aligned}$$

Note that according to (4.4.5), $\frac{2E^k - 2kb_l}{(k+1)a_l} = 2f_l$ and then

$$\nabla_d SC = \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i (2a_l f_l + b_l)/a_l \ .$$

Regroup the items, we get

$$\nabla_d SC = \sum_{k=1}^{K} \nabla_d g^k \sum_{l \in S^k} (2a_l f_l + b_l)/a_l = \sum_{k=k_1}^{k_2} \nabla_d g^k A^k \frac{\sum_{l \in S^k} (2a_l f_l + b_l)/a_l}{\sum_{l \in S^k} 1/a_l} \ .$$

With Lemma (4.4.5),

$$\begin{aligned}
\nabla_d SC &= \nabla_d g^{k_1} A^{k_1} \frac{\sum_{l\in S^{k_1}}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^{k_1}} 1/a_l} + \sum_{k=k_1+1}^{k_2} \nabla_d g^k A^k \frac{\sum_{l\in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l} \\
&\leq \nabla_d g^{k_1} A^{k_1} \frac{\sum_{l\in S^{k_1}}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^{k_1}} 1/a_l} + \sum_{k=k_1+1}^{k_2} \nabla_d g^k A^k \frac{\sum_{l\in S^{k_1}}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^{k_1}} 1/a_l} \\
&= \sum_{k=k_1}^{k_2} \nabla_d g^k A^k \cdot \frac{\sum_{l\in S^{k_1}}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^{k_1}} 1/a_l} = 0.
\end{aligned}$$

Directional Derivatives of Individual Players

We first check the consequence of (partial) cooperation to the other players. For player k which is not part of the cooperation, there are three cases:

Case 1: For player $k < k_1$, e.g., player with big flow demands.

Note that it is followed from (4.4.11) that $\sum_{i=k_1}^{k_2} \nabla_d g^{i'} A^i = 0$ and $\nabla_d g^{i'} = 0$ for all $i < k_1$ or $i > k_2$.

Then

$$\begin{aligned}
\nabla_d C^k &= \sum_{i=k_1}^{k_2} \nabla_d g^i (e^k A_i - 2e^i A^i + 2r^i) \\
&= e^k \sum_{i=k_1}^{k_2} \nabla_d g^i A^i - 2\sum_{i=k_1}^{k_2} \nabla_d g^i \sum_{l\in S^i} (e^i A^i - r^i) \\
&= -2\sum_{i=k_1}^{k_2} \nabla_d g^i A^i (e^i - r^i/A^i) \\
&= 2\sum_{i=k_1+1}^{k_2-1} \frac{1}{i(i+1)}[(e^i - \frac{r^i}{A^i}) - (e^{k_1} - \frac{r^{k_1}}{A^{k_1}})] + 2\frac{1}{k_2}[(e^{k_2} - \frac{r^{k_2}}{A^{k_2}}) - (e^{k_1} - \frac{r^{k_1}}{A^{k_1}})] \\
&\leq 0
\end{aligned}$$

Case 2: For player $k > k_2$, e.g., player with small flow demands.

$$\begin{aligned}
\nabla_d C^k &= \sum_{i=k_1}^{k_2} \nabla_d g^i r^k \\
&= \nabla_d g^{k_1} A^{k_1} \frac{r^k}{A^{k_1}} + \sum_{i=k_1+1}^{k_2} \nabla_d g^i A^i \frac{r^k}{A^k} \\
&\leq \nabla_d g^{k_1} A^{k_1} \frac{r^k}{A^{k_1}} + \sum_{i=k_1+1}^{k_2} \nabla_d g^i A^i \frac{r^k}{A_1^k} \\
&= 0
\end{aligned}$$

Case 3: For player $k_1 < k < k_2$, e.g., player with flow demands neither too small nor too big.

First consider the special case that $k = k_1 + 1$:

$$\begin{aligned}
\nabla_d C^k &= \nabla_d g^{k_1} \left(c^{k_1+1} A^{k_1+1} - \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} \right) + \nabla_d g^{k_1+1} \left(c^{k_1+1} A^{k_1+1} + k \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} \right) \\
&\quad + \sum_{i > k_1+1} \nabla_d g^{i'} \left(c^{k_1+1} A^i - 2 \sum_{l \in S^i} \frac{a_l f_l + b_l}{a_l} \right) \\
&= c^{k_1+1} \sum_{i=k_1}^{k_2} \nabla_d g^i A^i + c^{k_1+1} \nabla_d g^{k_1} (A^{k_1+1} - A^{k_1}) \frac{r^k}{A^{k_1}} - \nabla_d g^{k_1} \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} \\
&\quad + k_1 \nabla_d g^{k_1+1} \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} - 2 \sum_{i > k_1+1} \nabla_d g^i \sum_{l \in S^i} \frac{a_l f_l + b_l}{a_l} \\
&\leq 0 + c^{k_1+1} \nabla_d g^{k_1} (A^{k_1+1} - A^{k_1}) \frac{r^k}{A^{k_1}} - \nabla_d g^{k_1} \sum_{l \in S^{k_1+1}} (a_l f_l + b_l)/a_l \\
&\quad + (k_1 \nabla_d g^{k_1+1} A^{k_1+1} - 2 \sum_{i > k_1+1} \nabla_d g^i A^i) \frac{\sum_{l \in S^{k_1+1}} (a_l f_l + b_l)/a_l}{A^{k_1+1}} \\
&= \frac{c^{k_1+1}}{(k_1+1) A^{k_1}} (A^{k_1+1} - A^{k_1}) + \left(-\frac{1}{A^{k_1}} + \frac{1}{A^{k_1+1}}\right) \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{(k_1+1) a_l} \\
&= \frac{A^{k_1+1} - A^{k_1}}{A^{k_1+1} A^{k_1}} \left(c^{k_1+1} A^{k_1+1} - \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{(k_1+1) a_l} \right) \\
&= \frac{A^{k_1+1} - A^{k_1}}{A^{k_1+1} A^{k_1}} r^{k_1+1} \leq 0, \quad \text{noting that} \quad A^{k_1+1} \leq A^{k_1}.
\end{aligned}$$

When $k > k_1 + 1$, notice that the definition of $\nabla_d C^k$ is dependent on the parameters k_1 and k_2, thus we can consider it as a function of k_1 and k_2, which can be written as $\nabla_d C^k(k_1, k_2)$. We also write the directional gradient of g^k as function of k_1 and k_2, as, $\nabla_d g^k(k_1, k_2)$. Note that $\nabla_d g^i(k_1, k_2) = \nabla_d g^i(k_1 + 1, k_2)$ for all $i \neq k_1$ and $i \neq k_1 + 1$ by (4.4.11), hence the change of $\nabla_d C^{k'}$ is

$$\begin{aligned}
&\nabla_d C^k(k_1 + 1, k_2) - \nabla_d C^k(k_1, k_2) \\
= \;& \nabla_d g^{k_1+1}(k_1 + 1, k_2)r^k - \nabla_d g^{k_1+1}(k_1, k_2)r^k - \nabla_d g^{k_1}(k_1, k_2)r^k \\
= \;& r^k \left(\frac{1}{(k_1 + 2)A^{k_1+1}} + \frac{1}{(k_1 + 1)(k_1 + 2)A^{k_1+1}} - \frac{1}{(k_1 + 1)A^{k_1}} \right) \\
= \;& \frac{r^k}{k_1 + 1} \left(\frac{1}{A^{k_1+1}} - \frac{1}{A^{k_1}} \right) \geq 0.
\end{aligned}$$

It follows that

$$\nabla_d C^k(k_1, k_2) \leq \nabla_d C^k(k_1 + 1, k_2) \leq \ldots \leq \nabla_d C^k(k - 1, k_2) \leq 0.$$

From these three cases above, we know that when two players cooperate with each other (partially or fully), any other player would benefit from this cooperation. It also completes the proof of the theorem.

However, it is not provable using the idea of directional derivative that two players always have the incentive to cooperate. Notice that for player k_1 and k_2, between which we consider the cooperation, it only makes sense to consider the total cost by these two players, because the flow demand of them changes but the total flow demand inside this subgroup does not change. We can construct a simple example to show that the directional derivative of as follows:

Example 4.4.7 *Consider a two-nodes two-links network with the cost parameters*

$$a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 2.$$

Suppose there are 3 players P^1, P^2, P^3 with the demand flow vector $(1, 1, 1/2)^T$.

Let $k_1 = 2, k_2 = 3$, then the directional derive of $C^{k_1} + C^{k_2}$ is positive.

Further Discussion

Although there is no monotonicity of the cooperation group's cost for general case, we still observe some interesting phenomenon in some special case. Consider $k_1 = 1$ and $k_2 = 2$, we have that:

$$\begin{aligned} DC(k_1, k_2) &= \frac{1}{2A^1}(c^1A^1 + c^2A^2 - D^2) - \frac{1}{2A^2}(c^1A^2 - 2D^2 + c^2A^2 + D^2) \\ &= r^2\left(\frac{1}{2A^1} - \frac{1}{2A^2}\right) \\ &\leq 0. \end{aligned}$$

This implies the two players, who have the two biggest required flow in the system, do have the incentive to cooperate *partially*. Also, as a consequence, if the biggest player continuously seize flow demand from the second biggest player (which changes when this proceeds), everyone benefits from it and therefore so does the social value, which gives the incentive for all the players to fully cooperate at the end.

Therefore in some cases two players might resist the idea of cooperation because it might hurt them in the long run (cost at the Nash Equilibrium), the other players can compensate them for doing so and make the whole system more efficient.

4.4.4 The Upper Bound of the Price of Anarchy

The theorem 4.4.1 leads to the following:

Corollary 4.4.8 *In a routing game with two-nodes multi-links network and linear unit flow cost, the price of anarchy for K players happens only when all the K players have exactly the same required* *flow, and consequently, identical flow on all edges at the Nash Equilibrium.*

Consider the worst case when all players have the same required flow, denoted by R, which is a scalar. If the setting is centrally controlled, the required flow is KR.

Then by symmetry, we assume all players have the same support set S^{Na}, the same flow vector x^{Na}, and the same marginal cost c^{Na} when the game reaches the Nash equilibrium. From (4.4.6), we have

$$f_l^{Na} = \frac{K(c^{Na} - b_l)}{(K+1)a_l}, \qquad \text{and} \qquad x_l^{Na} = \frac{c^{Na} - b_l}{(K+1)a_l}.$$

Since $\sum_{l \in S^{Na}} x_l^{Na} = R$,

$$\sum_{l \in S^{Na}} \frac{c^{Na} - b_l}{a_l} = (K+1)R. \tag{4.4.14}$$

And the total social cost

$$\begin{aligned} SC^{Na} &= \sum_{l \in S^{Na}} (a_l f_l + b_l) f_l \\ &= \sum_{l \in S^{Na}} \frac{Kc^{Na} + b_l}{K+1} \cdot \frac{K(c^{Na} - b_l)}{(K+1)a_l} \\ &= \sum_{l \in S^{Na}} \frac{K[K(c^{Na})^2 - (K-1)b_l c^{Na} - b_l^2]}{(K+1)^2 a_l}. \end{aligned} \tag{4.4.15}$$

For the case which is centrally controlled, denote the support set as S^O, the marginal cost as c^O and the flow on the link l as f_l^O. Then the problem is converted to an optimization problem:

$$\begin{aligned} \min \quad & \sum_{l \in S^O} (a_l f_l^O + b_l) f_l^O \\ \text{s.t.} \quad & \sum_{l \in S^O} f_l^O = KR \\ & f_l^O \geq 0 \; \forall \, l. \end{aligned}$$

Note that $f_l^O = \frac{c^O - b_l}{2a_l}$ for $l \in S^O$. It is clear to see $S^{Na} \subseteq S^O$ using the similar idea with the proof of (4.4.4). Hence,

$$\sum_{l \in S^O} \frac{c^O - b_l}{2a_l} = \sum_{l \in S^O - S^{Na}} \frac{c^O - b_l}{2a_l} + \sum_{l \in S^{Na}} \frac{c^O - b_l}{2a_l} = KR. \tag{4.4.16}$$

And the total social cost

$$\begin{aligned} SC^O &= \sum_{l \in S^O} (a_l f_l + b_l) f_l = \sum_{l \in S^O} \frac{c^O + b_l}{2} \cdot \frac{c^O - b_l}{2a_l} = \sum_{l \in S^O} \frac{(c^O)^2 - b_l^2}{4a_l} \\ &= \sum_{l \in S^O - S^{Na}} \frac{(c^O)^2 - b_l^2}{4a_l} + \sum_{l \in S^{Na}} \frac{(c^O)^2 - b_l^2}{4a_l}. \end{aligned} \tag{4.4.17}$$

Now we construct a new network with two nodes and three links, in which the social cost of both the Nash Equilibrium and social optimal one, are the same as the original network, hence the price of anarchy of them. Formally, we propose the following lemma:

Lemma 4.4.9 *For the routing game with an arbitrary two-nodes multi-links network, there exists a two-nodes three-links network to replace the original network with the same price of anarchy. Furthermore, $b_l' = 0$ for some link l in these three links.*

Proof. We add a mark " ′ " to the letters to denote the corresponding parameters in the new network. Note that c^O and c^{Na} denote the original marginal cost, which we will keep it the same in the game with the new network.

Then we construct the two-nodes three-links network link by link.

First, if $S^O = S^{Na}$, let

$$b_1' = c^O, \qquad a_1' \text{ be an arbitrary positive number,}$$

otherwise,

$$\begin{cases} \frac{c^O - b_1'}{2a_1'} & = \sum_{l \in S^O - S^{Na}} \frac{c^O - b_l}{2a_l} \\ \\ \frac{(c^O)^2 - b_1'^2}{4a_1'} & = \sum_{l \in S^O - S^{Na}} \frac{(c^O)^2 - b_l^2}{4a_l} \end{cases}$$

i.e.,

$$b_1' = \frac{\sum_{l \in S^O - S^{Na}} (c^O - b_l) b_l / a_l}{\sum_{l \in S^O - S^{Na}} (c^O - b_l)/a_l}, \qquad \text{and} \quad a_1' = \frac{c^O - b_1'}{\sum_{l \in S^O - S^{Na}} (c^O - b_l)/a_l}.$$

Note that $b_l \leq c^O$ for $l \in S^O$. Hence

$$0 \leq b_1' \leq \frac{\sum_{l \in S^O - S^{Na}} (c^O - b_l) b_l / a_l}{\sum_{l \in S^O - S^{Na}} (c^O - b_l)/a_l} \leq c^O.$$

Also, $a_1 \geq 0$.

To construct the other two links, let $b_3 = 0$ and

$$\begin{cases} \frac{c^O - b_2'}{2a_2'} + \frac{c^O}{2a_3'} & = \sum_{l \in S^{Na}} \frac{c^O - b_l}{2a_l} \\ \\ \frac{(c^O)^2 - b_2'^2}{4a_2'} + \frac{(c^O)^2}{4a_3'} & = \sum_{l \in S^{Na}} \frac{(c^O)^2 - b_l^2}{4a_l} \\ \\ \frac{c^{Na} - b_2'}{(K+1)a_2'} + \frac{c^{Na}}{(K+1)a_3'} & = \sum_{l \in S^{Na}} \frac{c^{Na} - b_l}{(K+1)a_l} \end{cases}$$

After some basic mathematical reduction, we get

$$b_2' = \frac{\sum_{l \in S^{Na}} b_l^2 / a_l}{\sum_{l \in S^{Na}} b_l / a_l}, \qquad a_2' = \frac{\sum_{l \in S^{Na}} b_l^2 / a_l}{(\sum_{l \in S^{Na}} b_l / a_l)^2},$$

and

$$a_3' = \frac{(\sum_{l \in S^{Na}} b_l^2 / a_l)(\sum_{l \in S^{Na}} 1/a_l) - (\sum_{l \in S^{Na}} b_l / a_l)^2}{\sum_{l \in S^{Na}} b_l^2 / a_l}$$

Similarly, $b_l \leq c^{Na}$ for $l \in S^{Na}$,

$$0 \leq b_2' = \frac{\sum_{l \in S^{Na}} b_l^2 / a_l}{\sum_{l \in S^{Na}} b_l / a_l} \leq c^{Na}, \qquad \text{and} \quad a_2' \geq 0.$$

Furthermore, by the Cauchy-Schwartz Inequality, $a'_3 \geq 0$.

Now we have a two-nodes three-links network satisfying

$$0 = b_3 \leq b_2 \leq c^{Na} \leq b_1 \leq c^O.$$

Namely, we converge the links in $S^O - S^{Na}$ to a link with parameters (a'_1, b'_1) and the links in S^{Na} to two links with (a'_2, b'_2) and (a'_3, b'_3). The remaining task is to show that after this operation the total social cost is the same as the original one.

According to (4.4.15) and (4.4.17), the total social cost when the game with the new network reaches Nash equilibrium

$$\begin{aligned}
SC^{Na'} &= \frac{K}{(K+1)^2}\left[\frac{K(c^{Na})^2 - (K-1)b'_2 c^{Na} - b'^2_2}{a'_2} + \frac{K(c^{Na})^2}{a'_3}\right] \\
&= \frac{K}{(K+1)^2}\left[K(c^{Na})^2\left(\frac{1}{a'_2} + \frac{1}{a'_3}\right) - (K-1)c^{Na}\frac{b'_2}{a'_2} - \frac{b'^2_2}{a'_2}\right] \\
&= \frac{K}{(K+1)^2}\left[K(c^{Na})^2 \sum_{l \in S^{Na}} \frac{1}{a_l} - (K-1)c^{Na} \sum_{l \in S^{Na}} \frac{b_l}{a_l} - \sum_{l \in S^{Na}} \frac{b_l^2}{a_l}\right] \\
&= SC^{Na}.
\end{aligned}$$

And the optimal total social cost is

$$\begin{aligned}
SC^{O'} &= \frac{(c^O)^2 - b_1^2}{4a_1} + \frac{(c^O)^2 - b_2^2}{4a_2} + \frac{(c^O)^2}{4a_3} \\
&= \sum_{l \in S^O - S^{Na}} \frac{(c^O)^2 - b_l^2}{4a_l} + \sum_{l \in S^{Na}} \frac{(c^O)^2 - b_l^2}{4a_l} \\
&= SC^O.
\end{aligned}$$

□

Then we have the following upper bound for the price of anarchy:

Theorem 4.4.10 *The price of anarchy of K-players routing game in the two-nodes multi-links network with linear unit flow cost is upper bounded by* $\frac{4K^2}{3K^2+2K-1}$.

Proof. With the lemma above, we just need to consider a two-nodes three-links network satisfying

$$0 = b_3 \le b_2 \le c^{Na} \le b_1 \le c^O.$$

And

$$\frac{c^{Na} - b_2}{a_2} + \frac{c^{Na}}{a_3} = (K+1)R, \tag{4.4.18}$$

$$\frac{c^O - b_1}{a_1} + \frac{c^O - b_2}{a_2} + \frac{c^O}{a_3} = 2KR. \tag{4.4.19}$$

Rewrite the total social cost,

$$SC^{Na} = \frac{K}{(K+1)^2}\left[\frac{K(c^{Na})^2 - (K-1)b_2c^{Na} - b_2^2}{a_2} + \frac{K(c^{Na})^2}{a_3}\right] \tag{4.4.20}$$

$$SC^O = \frac{(c^O)^2 - b_1^2}{4a_1} + \frac{(c^O)^2 - b_2^2}{4a_2} + \frac{(c^O)^2}{4a_3}. \tag{4.4.21}$$

From (4.4.19),

$$\begin{aligned} SC^O &= \frac{(c^O + b_1)(c^O - b_1)}{4a_1} + \frac{(c^O)^2 - b_2^2}{4a_2} + \frac{(c^O)^2}{4a_3} \\ &= \frac{c^O + b_1}{2}\left(2KR - \frac{c^O - b_2}{a_2} - \frac{c^O}{a_3}\right) + \frac{(c^O)^2 - b_2^2}{4a_2} + \frac{(c^O)^2}{4a_3} \end{aligned}$$

Fixed other parameters, SC^O attains the minimum value when $b_1 = c^{Na}$.

Substitute $b_1 = c^{Na}$ in (4.4.19), and (4.4.18)–(4.4.19),

$$c^O = c^{Na} + \frac{(K-1)R}{1/a_1 + 1/a_2 + 1/a_3},$$

and from (4.4.18),

$$c^{Na} = \frac{(K+1)R + b_2/a_2}{1/a_2 + 1/a_3}.$$

Substitute them in (4.4.20) and (4.4.21), we get

$$SC^{Na} = \frac{K^2R^2}{1/a_2+1/a_3} + \frac{KR\,a_3b_2}{a_2+a_3} - \frac{Kb_2^2}{(K+1)^2(a_2+a_3)}$$

and

$$\begin{aligned} SC^O &= \frac{R^2}{4}\left[\frac{3K^2+2K-1}{1/a_2+1/a_3} + \frac{(K-1)^2}{1/a_1+1/a_2+1/a_3}\right] + \frac{KR\,a_3b_2}{a_2+a_3} - \frac{b_2^2}{4(a_2+a_3)} \\ &\geq \frac{R^2}{4}\cdot\frac{3K^2+2K-1}{1/a_2+1/a_3} + \frac{KR\,a_3b_2}{a_2+a_3} - \frac{b_2^2}{4(a_2+a_3)}. \end{aligned}$$

The equality holds when $a_1 \to 0$.

Then

$$\begin{aligned} &\frac{4K^2}{3K^2+2K-1}SC^O - SC^{Na} \\ \geq\;& \left(\frac{4K^2}{3K^2+2K-1}-1\right)\frac{KR\,a_3b_2}{a_2+a_3} - \left(\frac{K^2}{3K^2+2K-1} - \frac{K}{(K+1)^2}\right)\frac{b_2^2}{a_2+a_3} \\ =\;& \frac{K(K-1)^2}{3K^2+2K-1}\cdot\frac{R\,a_3b_2}{a_2+a_3} - \frac{K(K-1)^2}{(3K^2+2K-1)(K+1)}\cdot\frac{b_2^2}{a_2+a_3} \\ =\;& \frac{K(K-1)^2}{3K^2+2K-1}\cdot\frac{b_2}{a_2+a_3}\cdot[(K+1)R\,a_3 - b_2] \\ =\;& \frac{K(K-1)^2}{3K^2+2K-1}\cdot\frac{b_2}{a_2+a_3}\cdot\left[\frac{(e^{Na}-b_2)a_3}{a_2} + e^{Na} - b_2\right] \\ =\;& \frac{K(K-1)^2}{3K^2+2K-1}\cdot\frac{b_2}{a_2}\cdot(e^{Na}-b_2) \geq 0. \end{aligned}$$

When $b_2 = e^{Ma}$, the equality holds in the last inequality.

Finally, the price of anarchy

$$\alpha(G,r,c) = \frac{SC^{Na}}{SC^O} \leq \frac{4K^2}{3K^2+2K-1}$$

□

Note that the upper bound is tight. From the proof, the conditions under which the bound is tight are as follows:

$$b_1 = b_2 = e^{Na}, \qquad \text{and} \quad a_1 \to 0.$$

With this information, we can construct the following example in which the price of anarchy attains the upper bound.

Example 4.4.11 *Consider a two-nodes two-links network in which:*

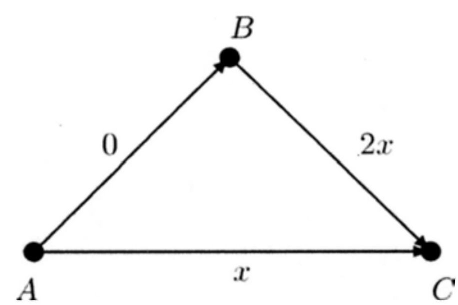
$$a_1 = 0, \quad b_1 = 1, \quad a_2 = \frac{K}{K+1}, \quad b_2 = 0.$$

Let $R = \frac{1}{K}$, *then the price of anarchy in this case is exactly* $\frac{4K^2}{3K^2+2K-1}$.

4.5 Counter Examples

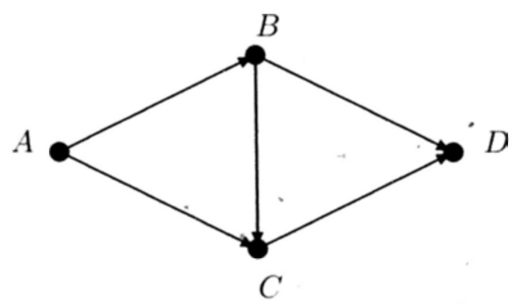
The following example shows that the monotonicity may not hold any more for general graphs.

Example 4.5.1 *Considering the directed graph below, there are 3 players* P^1, P^2 *and* P^3 *in the routing game each with one unit of flow to transport. Both* P^1 *and* P^2 *have the same starting node* A *and terminal node* C, *while* P^3 *starts at node* B *and ends at node* C. *The cost functions for* $\overrightarrow{AC}, \overrightarrow{AB}, \overrightarrow{BC}$ *are* $x, 0, 2x$, *respectively.*



Then in this example: when P^1, P^2 are two independent players, the social cost at the unique Nash equilibrium is 4; however, when P^1, P^2 cooperate, the corresponding value becomes 9/2, which is greater than the cost before.

Example 4.5.2 *The conclusion fails even if we strengthen the assumption as that all players have the same source and destination nodes, as shown in the following example:*

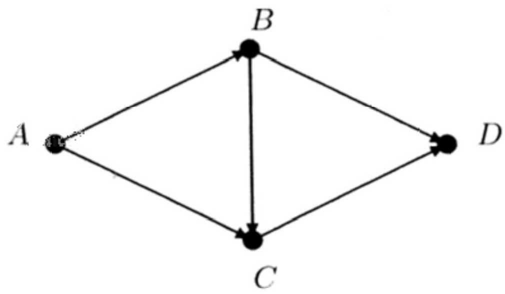


In the directed graph above, there are 3 players P^1, P^2 and P^3 in the routing game, who need to ship $1/10, 1/10$ *and* $6/10$ *units of flow from A to D, respectively. The cost functions for* $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BC}, \overrightarrow{BD}, \overrightarrow{CD}$ *are* $2 + 10x, 4 + x, 2 + 10x, 9 + 20x, 1 + 6x$*, respectively.*

In this example: when P^1, P^2 are two independent players, the social cost at the unique Nash equilibrium is $514826875/62410000 \approx 8.249108717$*; however, when P^1, P^2 cooperate, the corresponding value becomes* $950694196/115240225 \approx 8.249673202$*, which is greater than the cost before.*

4.5.1 Open Problems

1. Does there always exist a kind of allocation plan such that all players in coalition can be better off? Note that the cost is nontransferable. The following example shows that the conclusion fails to hold at least for general graph.



Given the directed graph above, 2 players, P^1 and P^2 in the game need to ship $1/10$ and $9/10$ units of flow from A to D, respectively. The cost functions for $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BC}, \overrightarrow{BD}, \overrightarrow{CD}$ are $x, 1, 0, 1, x$, respectively.

Then the flow vectors at the Nash equilibrium are $x^1 = (1/10, 0, 1/10, 0, 1/10)$ and $x^2 = (9/20, 9/20, 0, 9/20, 9/20)$. When they cooperate, the optimal flow is $x^* = (1/2, 1/2, 0, 1/2, 1/2)$. Then clearly, there exists no allocation plan such that P^1 can benefit from the cooperation.

2. Compute a tighter bound for the price of anarchy of the routing game in the general network with linear cost functions. Furthermore, for a fixed type of network, find the worst case when players employ the selfish strategies for arbitrary number of players.

3. Built a general model with transportation game as a special case to reflect the concept of "partial core", "price of monarchy", "fairness" and so on. A related stable state between noncooperative and cooperative game can be defined as following:

The players are divided into several coalitions $\{S_1, \ldots, S_p\}$, which satisfies the following three conditions:

- the game reaches the Nash equilibrium by treating coalitions as players;

- in each coalition, there exists a kind of core allocation;
- no more cooperations among players is possible.

With the definition above, under what conditions does the stable state exist?

4. If the management cost is taken into the considerations of the coalition, does a threshold for the number of members of the coalition exist? In this thesis, we study the several open questions in the cooperative game theory and non-cooperative game theory, by utilizing tools and methods from optimization theory. We first use the technique of polymatroid optimization to establish the submodularity of the joint replenishment game and one warehouse multi-retailer game. The general theoretical results on the submodularity of the objective function for certain optimization problems with polymatroid constraints are of independent interest and have further consequences in both optimization and game theory. Then by defining a potential function, we show that the strategy of learning from history does converge to the Nash Equilibrium of the game under certain conditions. With empirical results we show that the strong the memory is, the more likely it is to converge, however the convergence speed is slower as the tradeoff. Specifically, the competitive routing game falls into this category, because of a monotonic LCP formulation of the Nash Equilibrium of the game. Assuming in a traffic system everyone do act greedily and take history into account while making decisions in general, it is therefore reasonable to argue that players should use the cost in the Nash Equilibrium instead of the immediate consequence, to decide if they should cooperate or not, if the decision of "cooperation" is hard to reverse. It is interesting to notice that when two players cooperate, all the other players do benefit , as well as the social system. However, the two players might be worse off, even though the player with the biggest share of flow always has incentive to obtain more (partial) flows from its strongest competitor. Furthermore, for parallel network, the

tight bound of price of anarchy with given k-number of players has been established.

All these results for different aspects of game theory, are obtained by applying tools and methodologies from the field of optimization. As we could see, to analyze the properties of a game or to solve the core/Nash Equilibriua of the game efficiently, these quantitative methods are necessary. Game theory itself is an emerging field with many open problems while optimization theory has been well established and studied, if those well developed tools and methodologies can be fully utilized in game theory, many amazing things are bound to happen. Also, game theoretical concepts can sometimes lead to better understanding of algorithms in optimization theory, for example, the idea of learning naturally leads to efficient distributive algorithms for solving optimization problems, which is a promising direction for solving large scale optimization problems and utilizing the power of parallel computing. The two components of game theory, the cooperative game theory and the non-cooperative game theory, although seems far apart from each other since in one setting everyone fully cooperates and works together, and in the other setting everyone is greedy and acts on his/her own. However practically, cooperation and competition both exists at the same time, in the same system. It is therefore interesting to understand how partial cooperation forms and the dynamics and consequences of it, as shown in our study about competitive routing game. It would be interesting to establish a middle group between cooperative game theory and non-cooperative game theory, in a more general setting.

Since both optimization and game theory contain enormous subfields, this thesis could only cover several aspects of them as an attempt to show the bridge between optimization and game theory. Hopefully, it could inspire more interest on this topic and bring more inputs.

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