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Picture This: A dissertation examining the quantitative mental imagery of children

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Picture This:

A Dissertation Examining Children's Quantitative Mental Imagery

Jonathan Norris Thomas, BA; M.Ed.

The University of Cincinnati

In partial fulfillment of the requirements for:

Doctor of Education (Ed.D.) in the department of Curriculum and Instruction

Date of Defense: September 29, 2010

Dissertation Committee
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Abstract

This dissertation examines the manner in which young school-age children construct quantitative mental imagery and act upon these constructions for arithmetic ends. After conducting dyadic teaching experiments with three first grade students who were participating in mathematics intervention, I found the connection between the participants' constructed image and the different mathematical tools varied in a manner suggestive of a progression. Additionally, I explored the interactive patterns of the intervention dyad, and identified distinct themes in which the participants' imagery constructions may be situated.

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Acknowledgements

As I near the completion of this dissertation, I am reminded that this product represents much more than my own efforts. Indeed, my arrival at this point can be attributed to the unwavering support of many individuals. Prior to delving into this dissertation study, I find it necessary to acknowledge those that make such activity possible.

First and foremost, I must thank the members of my family. Specifically, I give thanks to my wife Sandy for committing to this doctoral program alongside me. Always willing to go the extra mile so that I might have more time to read or write, she cheered for me every step of the way, and I am forever grateful to her. I also thank my children, Ethan and Alissa, for inspiring me to be a better teacher. I see something of them in each child with whom I work. Also, my parents made a substantial contribution to my doctoral study. As educators, they modeled the distinct honor and fulfillment that comes with intellectual curiosity and the pursuit of understanding. As the best of teachers, they never ceased to believe in and support the passion for learning that resided within me.

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Undoubtedly, there are many others that played a part in my doctoral experience these past several years; however, the individuals above warrant special mention given their selfless participation. Again, I thank you. I thank each of you from the bottom of my heart. This has been the most challenging and rewarding experience of my life. As I begin my work as a new faculty member, I am tremendously excited by the opportunities for teaching, scholarship, and service to the discipline. I am excited by the prospect of making a difference in the lives of teachers and children. Each of you has made me a believer in the power and potential of education, and I am grateful.

Chapter One

Introduction

"The water is wide and I can't cross over. Neither have I wings to fly.

Build me a boat that can carry two and both shall row."

~Bob Dylan: The Water is Wide

"This is an alarm call, so wake up, wake up now."

~Bjork: Alarm Call

Chapter Introduction

In the fall of 2007, I began work as a mathematics intervention teacher at an elementary school in Newport, KY. After assessing a number of first grade students that were struggling with mathematics, I had the good fortune to meet William. Even at this young age, many children are keenly aware of their mathematical difficulties and William was no different. His classroom teacher informed me that he often became frustrated to the point of engaging in disruptive behaviors, and that much of William's frustration centered on his challenges working with numeral-based tasks. Indeed, his struggles were of such significance that his teacher indicated that she was considering referring William to a special education program. After a brief interview, I also concluded that numerals presented William with much difficulty. Through a series of tasks, I watched William become frustrated and confused as he attempted to identify and write different numerals; however, posing problems that involved collections of materials and that were devoid of numerals presented an entirely different picture. Through the course of addition and subtraction tasks involving materials that were both visible and concealed, William enacted several counting strategies that suggested a relatively sophisticated conception of quantity. Specifically, William was able to generate and keep track of mental images to quickly and accurately solve different tasks. As this was my first experience working with a child that demonstrated such a facility in this department, I spent a great deal of time posing quantitative arithmetic tasks and questioning him about his strategies.

I found William's strategy of invoking mental images to solve arithmetic tasks quite compelling, but unfortunately, I soon began to experience difficulty in helping him advance his understanding. As a novice intervention teacher, I sought out those with expertise in this area including the founder of the Math Recovery intervention program, Dr. Robert Wright. I was specifically interested in different tasks, materials, and contexts that might support William's use of imagery and enable him to grow conceptually. Although I was aware of a robust trajectory of instructional strategies, I found no answers regarding the extent to which different materials might support or constrain imagery use. I returned to my intervention teaching with some idea of productive tasks and tools; though, I seemed to be guided by trial and error more often than not. By the year's end, I did experience some success helping William to construct a more sophisticated understanding of quantity (as well as some knowledge of numerals); however, I have since come to believe that a better understanding of quantitative mental imagery would have greatly increased the precision of my instructional decisions. Moreover, such understanding has the potential to enhance the intervention settings of the doubtlessly many mathematically struggling children that are conceptually similar to William. Indeed, working with William was my alarm call to the importance of imagery.

The aim of this dissertation study is to better discern the nature of children's quantitative mental imagery primarily to refine mathematics intervention teaching practices, although such understanding of imagery could certainly inform classroom instruction as well. While I have appropriated the context of mathematics intervention for this inquiry, I submit that the mathematical constructions of intervention participants are, indeed, representative of children in general. Of course, this does not mean that such constructions are not unique or individualized in some way, but rather that children participating in mathematics intervention often exhibit mathematical creativity and sophistication on par with their peers when instructional conditions are optimized. In this introductory chapter, I will describe the importance of intervening to establish robust mathematical foundations as well as a particularly productive approach to intervention. In subsequent chapters, I will present literature pertinent to the study of quantitative mental imagery as well as the methodology and methods appropriated for this research. More importantly, though, I will share the experiences of three children and myself as we worked together in the intervention classroom, and what these experiences mean in terms of the quantitative image. Before I trek off on this journey, however, perhaps it useful to present the specific research questions that will guide us on our trip.

Research Questions

1) How do children in the early elementary grades (ages 5-8) describe and reveal through activity the mental images that they invoke when working with different arithmetic tasks and tools?
2) How do these descriptions and revelations change over time?

With these questions in mind, let me first address the importance of attending to such obscure conceptual components and the manner in which they contribute to mathematical understanding.

The Importance of Meaningful Mathematics

Presently, there is substantial (but not complete) agreement among mathematics educators that a meaningful understanding of mathematics (sometimes referred to as conceptual understanding) is far preferable to understandings dominated by meaningless and disconnected procedures (Erlwanger, 1973). What is meant by the term meaning, though? On this topic, Brownell (1947) writes:

> We should . . . distinguish what I shall designate the *meaning of* a thing and the *meaning of* a thing *for* something else; for the sake of brevity, between *meaning of* and *meaning for*. I know little about the meaning *of* the atomic bomb because I lack the knowledge of chemistry and physics which are requisite to accurate understanding, but I think I know a good deal about the meaning of the atomic bomb *for* other things – for peace or for the destruction of our culture, for example. . . that children have meaningful experiences when they use arithmetic in connection with real life needs, relates to meanings *for* . . . On the other hand, just as the meaning *of* the atomic bomb is to be found in the related physical sciences, so the meanings *of* arithmetic are to be found in mathematics. . .They must be sought in the mathematical relationships of the subject itself, in its concepts, generalizations, and principles. In this sense a child has a meaningful arithmetic experience when the situation he deals with "makes sense" mathematically. He behaves meaningfully with respect to a quantitative situation when he knows what to do arithmetically and when he knows how to do it [*meaning for*]; and he possesses arithmetical meanings when he understands arithmetic as mathematics [*meaning of*] (p. 256).

Indeed, both types of meaning are of great importance to mathematics educators; moreover, this focus on quantitative mental imagery necessitates consideration of both *meanings of* and *meanings for.* Specifically, a child's constructed images may allow the consideration of arithmetic principles such as hierarchical inclusion (Inhelder & Piaget, 1964) as well as the enactment of meaningful counting processes aimed at discovering the solution to a particular task. The point, here, is that learning experiences organized around the production of meaningful mathematics are essential to the practice of the discipline.

Building upon this significance of meaningful mathematics, it follows that the conceptual understanding of early constructions is of paramount importance if later constructions are to have meaning. Indeed, given the cumulative nature of mathematical constructions (Tabach, Hershkowitz, & Schwartz, 2006), a robust and meaningful understanding of whole numbers and arithmetic operations, oft referred to as numeracy (Wright, Martland, & Stafford, 2000), is absolutely crucial. Examination of the Curriculum Focal Points (NCTM, 2006) reveals that six of the nine focal points from first grade to third grade deal specifically with number and operation. Additionally, the most recent report from the National Mathematics Advisory Panel (USDE, 2008) identified a "robust sense of number [including an] understanding of place value and the ability to compose and decompose whole numbers [as well as] the meaning of the basic operations of addition, subtraction, multiplication, and division" as a critical foundation of algebra (p. 17). Lastly, the recently released Common Core State Standards for Mathematics (CCSSI, 2010) heavily emphasize the understanding of number and operations in the primary grades. All of this to say that there is considerable agreement on the importance of meaningful mathematical foundations involving number and operations.

Mathematical Struggle and Intervention

Ideally, children construct meaningful mathematical foundations within their regular classroom community; however, there are many occasions where students struggle to ascribe any manner of meaning to their mathematical practices. While some degree of mathematical struggle can be considered as healthy (and even essential) for conceptual construction (Carter, 2008; Kline, 2008), many children are unable to create sufficient mathematical meaning from their regular classroom experiences. In response to these cases, many schools identify an educator to intervene in the mathematical learning of these students and design individually tailored instructional experiences (KCM, 2009; Phillips, Leonard, Horton, Wright, & Stafford, 2003). Because of this emphasis on individualized experiences, mathematics interventions are often structured around small groups (two to six students) or, more intensively, in dyadic (or one to one) contexts. Regardless of structure, though, the primary aim of mathematics intervention remains the rapid advancement of children's conceptual understanding of mathematics with respect to their peers. Quite simply, intervention specialists are in the business of helping students 'catch up' mathematically. Towards this end, intervention specialists must possess considerable expertise including a deep understanding of how mathematical knowledge typically develops among children, and design tasks that capitalize on a child's current understanding. The practical aim of better discerning the quantitative image is that additional guidance in this area may translate into more productive experiences for struggling students.

The Kentucky Center for Mathematics

One of the primary missions of the Kentucky Center for Mathematics (KCM) deals with increasing the mathematical understanding, proficiency, and achievement of students in grades K-3 who are experiencing significant difficulties within the discipline. Referred to as the Primary Mathematics Intervention Program (PMIP), this system is somewhat congruent with the tiered Response to Intervention (RTI) model (Fuchs & Fuchs, 2006) in that intervention intensity and effectiveness are directly linked to teacher expertise and student-centered experiences. Through a competitive grant process enacted by the Kentucky Department of Education, elementary school leaders vie for funding to maintain the professional growth and activity of a Mathematics

Intervention Teacher (MIT). Currently, the KCM is associated with more than 100 MITs teaching in elementary schools across the Kentucky.

Returning to the notion that intervention effectiveness depends upon teacher expertise, the KCM has implemented a rigorous system of professional development involving summer institutes and sustained, job-embedded, collegial events aimed at developing a cadre of highly skilled, committed MITs. Although advancing student achievement is named as the primary goal of the center, the mechanism for such advancement hinges on promoting a mathematical love for learning, or *diligo erudition,* among all program participants, be they MITs or students. Moreover, there is considerable evidence for success both in terms of increasing student achievement and fostering a love for mathematical learning (KCM, 2009).

Math Recovery

Although there is considerable flexibility regarding the MITs' selection of professional development and implementation of a program of intervention, nearly all of the MITs have been trained in and have adopted frameworks (i.e., Math Recovery, Add+VantageMR, SNAP) supported by the U.S. Math Recovery Council (2009) due to their emphasis on robust teacher development and student-centered instruction. As the most sophisticated of the frameworks, Math Recovery is an intensive professional growth experience designed to bolster diagnostic and instructional effectiveness in one-on-one intervention settings. With a foundation (Wright et al., 2000) that draws heavily from Stages of Early Arithmetic Learning (SEAL) and the extensive use of video recording as a reflective tool, Math Recovery immerses intervention teachers in the developmental progressions of children as a means to promote mathematical understanding rather than assuming a programmatic (Griffin, 2004) or scripted approach (Binder & Watkins, 1990) to intervention. The point, here, is not that Math Recovery is the sole route to productive intervention experiences, but rather that this approach to intervention adopts a useful focus on helping teachers develop the prodigious expertise necessary for such a complex enterprise. For this study, I have appropriated the Math Recovery approach to intervention. Specifically, I relied upon Math Recovery diagnostic assessments and learning materials to guide the events of this research.

Chapter Summary

As we prepare for this journey into the realm of quantitative mental imagery, there are a couple of key points that I wish to make. First, that deep understanding of such imagery is essential in maximizing the mathematical experiences for young children - particularly those that struggle within the discipline. In the following chapter, I present literature supporting this truth, but perhaps more importantly, my own experiences with William bear out this fact. Second, I strongly believe that blazing a trail towards illuminating these imagery constructions could lead to significant improvements in the selection of mathematical tools, thus facilitating a far more effective learning experience both in the moment and for future understanding. Let's begin.

Chapter Two

A Review of Literature

"Let me take your picture, add it to the mixture, there it is I got you now! Really nothin' to it, anyone can do it, it's easy and we all know how. Now begins the changin', mental rearrangin', nothing's really where it's at. Now the Eiffel Tower's holdin up a flower. Can you picture that?"

~The Muppets: Can You Picture That

"Will you walk with me out on the wire?"

~Bruce Springsteen: Born to Run

Chapter Introduction

Delving into the world of quantitative mental imagery is not a trivial task. In order to meaningfully examine the manner in which children construct and use imagery in early mathematics requires a search of literary work spanning the domains of education, psychology, and philosophy. Indeed, children's understanding of quantity has been the object of relentless study by scholars from varied traditions for many decades; moreover, the mysteries of the mental image have prompted questions from great thinkers dating back to Socrates. Exploring these realms requires a winding walk across traditions and time; and, I invite you now to walk with me.

Origins of Understanding

Prior to any investigation of mathematical knowledge and understanding, one must first attend to the nature of knowledge and understanding itself. Here, I find it useful to step away, briefly, from mathematical ideas specific to this study (namely, quantitative mental imagery) and examine two differing perspectives on precisely where such knowing and understanding originates. In this section, I provide an overview of the psychological and social perspectives. I conclude this section with an argument in favor of emphasizing the psychological perspective based on the cognitive orientation of this dissertation study.

Psychological Perspective

Dig deeply enough into most any contemporary aspect of education and one will likely discover some interpretation of the psychological perspective, often termed psychological constructivism. Given the ubiquitous nature of this theory of learning and understanding, it is often employed with little description or context which, in turn, may lead to confusion (Held, 1990). Certainly, most educators would be able to narrate the generalities of Piaget's (1961, 1974) assertion that knowledge is a cognitive construction of the individual. Germaine to this review, however, is the psychological constructivist declaration that perception involves *actively* interpreting one's experiences and does not conform to a simple relationship between stimulus and response. Furthermore, the process of remembering is considered reconstructive with respect to current understandings rather than a simple recovery of stored information (Cobb, 1994). These particular tenets of psychological constructivism are worthy of mention here as they provide an elaborated context for a child to perceive and interpret stimuli involving mathematical tools and then reconstruct these interpretations with respect to current understandings to form new learning experiences. Specifically, this provides a plausible framework that describes how a sensory conception of number (and attendant counting activities) may be actively transformed into a more sophisticated conception through novel experiences and individual reflection.

Perhaps it is also useful to mention Piaget's (1937) proposal of specific and global stages that he used to frame the development of a child. Although contemporary scholars have generated substantial critique of Piagettian stage theory (Donaldson, 1978), such evidence does not refute the presence of trajectorial understanding among children, but rather suggests that introducing task variability opens the global and/or temporal nature of Piaget's proposed stages to many questions.

Subsequent discussion of numeracy development may invoke the presence of stages; however, there are typically no specific temporal assignations to the proposed stages aside from the assertion that stages exist along a temporal axis. Indeed, cognitive trajectories among children are often hypothesized as directionally congruent but asynchronous among groups of children (Wright et al., 2000). There are, however, certain features associated with the term 'stage' that are worth noting. Glasersfeld and Kelly (1982) explicate:

> [W]hen we speak of stages we intend a sequence of segments along a temporal axis, each one of which can be individually characterized by a change relative to the adjacent ones. The nature of the change is crucial. This may be the presence or absence of an item (property, state, activity, or anything that can be isolated) . . . the constitutive characteristic of a stage cannot legitimately be a merely quantitative difference but must have a qualitative component. A good example of different kinds of change is the development of the frog. From egg to adulthood there is an increase in the frog's locomotion . . .Given a suitable choice of temporal parameters, one could show a quantitative change that could be considered 'abrupt' (i.e. when the frog conquers land); but a discontinuity in the quantitative change of locomotion no matter how large or sudden, could never be said to constitute a qualitative change. On the other hand, the grown frog has several features and capabilities that are in no way manifest in the tadpole (and vice versa) and it is precisely on the respective presence or absence that the judgment of qualitative change is founded (p154-156).

The principal idea, here, is that stage must be founded on qualitative rather than quantitative terms. Elaborating on this position, Steffe and his colleagues use the term stage to describe mathematical development which implies the satisfaction of four criteria: "[1] a characteristic remains constant throughout a period of time, [2] each stage incorporates the earlier stage, [3] the stages form an invariant sequence, and [4] each new stage involves a conceptual reorganization resulting from reflection and abstraction" (Steffe, Cobb, & von Glasersfeld, 1988, p. 7-8). It is this definition that will serve as the referent for following invocations of the term stage.

While psychological constructivism may account for some aspects of individual learning, it and similar psychological perspectives fall prey to several distinct limitations. While there is not complete agreement about precise aspects of constructivism (Davis, Maher, & Noddings, 1990), proponents would likely "see the interaction with others as a part of the environment that might act as a catalyst for reflective abstraction" (Tabor, 2008, p. 10). Quite simply, there is some potential to recast social interaction as merely perturbations (Steffe et al., 1988) from the environment that prompt individual construction. Confrey (1990) accurately identified the constraints of such a position thusly, "[p]ut into simple terms, constructivism can be described as essentially a theory about the limits of human knowledge, a belief that all knowledge is necessarily a product of our own [individual] cognitive acts" (p. 108). In its ultimate form, radical (psychological) constructivism casts the individual's knowledge and understanding of reality as components of an interpretation (von Glaserfeld, 1990). Indeed, "one can no longer maintain that cognizing activity should or would produce a true representation of an objective world" (von Glaserfeld, p. 22); moreover, the positing of unique mathematical constructions that are unknowable to others is very much at odds with social phenomena such as intersubjectivity (i.e., two individuals seemingly possessing the same understanding). To surmount this significant hurdle, Stephan, Cobb, & Gravemeijer (2003) describe a process where mathematical constructions may be *taken-as-shared* by different people. Here, *taken-as-shared* refers to the manner in which individuals may construct a unique understanding that is ultimately unknowable to other individuals yet simultaneously lacking any need for discussion or justification among the learning community (Stephan et al., 2003).

Social Perspective

Holding fast to the idea that knowledge is actively constructed rather than passively received (Rogoff, 1995), the social perspective, often referred to as social constructivism, theorizes that knowledge is first produced from interactions among a learning community, and later as an individual structure. Arguably the most famous proponent of social constructivism, Vygotsky (1978) wrote the following on the nature of childhood development:

> Every function . . . appears twice: first on the social level, and later on the individual level; first; *between* people (*interpsychological*), and then *inside* the child (*intrapsychological*). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All higher functions originate as actual relations between human individuals (p. 57).

This necessarily moves the initial production of knowledge towards activity within the social collective rather than strictly within the individual.

Among the different variants of the social perspective, interactionism (Bauersfeld, Krummheuer, & Voigt, 1988; Bauersfeld, 1994) provides some initial terrain for coordinating the psychological and the social perspectives in terms of inquiry. Bauersfeld asserts that "[i]n order to understand sufficiently the individual gains and the social regularities emerging from certain classroom cultures, it was necessary to switch between both views, the psychological and the sociological, *without giving preference* to either one" (p. 138, emphasis in the original). interactionism, however, must still be classified as a social perspective in that the basic theoretical tenets position the origin of knowledge within the collective. Examining a few of these tenets, referred to as "core convictions," Bauersfeld describes the perspective thusly:

> 5. *Mathematizing* describes a practice based on social conventions rather than the applying of a universally applicable set of eternal truths. . .
>
> 6. (Internal) *representations* are taken as individual constructs, emerging through social interaction as a viable balance between the person's actual interests and realized constraints, rather than an internal one-to-one mapping of pre-given realities or a fitting reconstruction of "the" world.
>
> 7. Using *visualizations and embodiments* with the related intention of using them as didactical means depends on taken-as-shared social conventions rather than on a plain reading or the discovering of inherent or inbuilt mathematical structures and meanings (p. 139).

Clearly, this perspective emphasizes the social collective in mathematical activity pertinent to this study; however, the attention that interactionism calls to psychological processes provides some opportunity to blur the boundary between psychological and social perspectives as well as a framework for understanding the relationship between social interactions and psychological constructions.

Emphasizing the Psychological Perspective

Given the sometimes stark difference in views regarding the origins of knowledge and nature of learning, a division often arises between proponents of psychological and social perspectives. Although Cobb and Yackel (1996) pioneered the construction of an emergent perspective to coordinate psychological and social perspectives forming a conglomeration of lenses (Kieran, 2000) from which to design and conduct research, this perspective is typically appropriated to examine socio-mathematical norms and practices (Cobb 2000b; Cobb, Stephan, McClain, & Gravemeijer, 2001) rather than psychological constructions. Ultimately, given the necessarily psychological nature of the quantitative mental image (Kosslyn, 1983; Steffe, 1992), any serious attempt to illuminate the nature of these constructions must grapple with issues that are primarily of psychological quality. With that said, I must note that these constructions most certainly do not occur within a social vacuum; hence, some effort must be given to the interactive conditions in which such constructions occur – the mathematics intervention dyad in this case. Indeed, on this point, Cobb (2000b) asserts that "when conducting a psychological analysis, one analyzes the individual students' activity *as the participant in the practices of the classroom community*" suggesting that analytical focus cannot focus entirely on individual, psychological processes, but rather must also attend to these psychological processes as they are situated within a particular mathematical community (p. 174, emphasis in the original). It is here that I find the theoretical foundation for understanding as it relates to this study. Specifically, the psychological perspective (or psychological constructivism) provides a framework for the study of cognitive phenomena, while interactionism lends structure to the ancillary (but necessary) social examination of children's intervention experiences.

Summary

While the psychological perspective is of primary importance to inquiry into cognitive constructions such as quantitative mental imagery, one must note that these constructions do not occur in a social vacuum; thus, in addition to psychologically-oriented analyses, some attention must be given to the quality of the mathematical interactions.

Counting

Given the tremendous emphasis placed on counting activities in this study, it is necessary to first lay out exactly what is meant by the term counting. In this section, I will provide an overview of components, considerations, and existing frameworks associated with the foundational process of counting.

The Three Aspects of Number

Prior to delineating the nature of counting, perhaps it is wise to briefly examine numbers themselves. Numbers as conceptual and cultural entities possess three distinct components (see Figure 2): verbal, numeral, and quantity (Wright, 1994; Wright et al., 2000; Thomas, Tabor, and Wright, in press) and neurological confirmation of these different aspects may be found in Dehaene's (1992) triple code model which describes the location of neural activity as a function of number aspect. Quite simply, this model explains how different parts of the brain become active depending upon the aspect of number with which one is working. Mention of these distinct aspects of number is important here as I consider the nature of tasks, tools, and children's mathematical activity. Specifically, I find it useful to consider which aspect(s) are emphasized within a given mathematical moment. In particular, this study will primarily focus on tasks, tools, and interactions that emphasize the quantitative aspect of number.

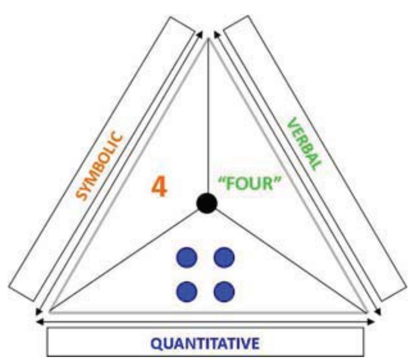

Figure 2: Thomas, Tabor, and Wright's (in press) Three Aspects of Number

Distinguishing between Subitizing and Counting

Subitizing. Any exploration of early quantitative development ultimately encounters two distinct and important phenomena among children, namely the processes of counting and subitizing (Clements, 1999; Glasersfeld, 1982). Subitizing (sometimes referred to as perceptual subitizing) derives its name from the Latin *subitos* meaning to 'arrive suddenly.' As opposed to counting, subitizing is "thought to provide information about very small numerosities much faster than counting, probably by processing the items simultaneously rather than in succession" (Sophian, 1998, p. 29). Indeed, it is this rapidity of quantitative reasoning and small range that distinguish between subitizing and counting; moreover, the process of subitizing appears as a distinctly spatial, rather than quantitative, activity. Glasersfeld writes:

> [T]hat, in subitizing, the child associates figural patterns with number words by a *semantic* connection and not because of the number of perceptual units of which they are composed. In acts of subitizing, the figural patterns that give rise to it are taken as figural wholes and not because of the number of perceptual units of which they are composed. In acts of subitizing, the figural patterns that give rise to it are taken as figural wholes and not as composites of units. In fact, they are recognized as a global configuration and not as a collection of countable items (p. 8).

What Glasersfeld suggests here is that subitizing acts are figural (imagery based), but that perceptually subitized quantities are connected semantically to number words on the basis of their spatial, rather than quantitative, attributes.

Counting. Counting may take on different meanings depending upon the context. Returning to the multi-aspect notion of number, common uses of the term might describe the production of a specified verbal sequence (e.g., "one, two, three", etc.) or, perhaps, verbally identifying a particular sequence of numerals (e.g., uttering number words when presented with a 100's chart); however, my use of the term counting is best characterized as activity in the face of problematic situations where it is assumed that the child has the particular goal of accomplishing a given task involving quantity (Steffe et al., 1983). Specifically, this type of counting, "involves the coordination of each uttered number word with the conceptual production of a unit item" (Wright et al., 2000, p. 52). Quite simply, this activity may be thought of as quantitative counting (as opposed to spatial subitizing) applied to a specific purpose or task.

Process overlap. It is important to note that as children become more experienced in terms of their counting and subitizing activity these processes appear to overlap somewhat. Clements (1999) describes conceptual subitizing as a process where individuals are able to rapidly apprehend quantities beyond the range of perceptual subitizing activity by organizing and structuring the perceived quantity into more manageable chunks. As opposed to (spatial) perceptual subitizing, this process is quantitative in orientation and such organization involves "counting and patterning abilities" (Clements, p. 401). This is significant in that conceptual subitizing processes provides additional terrain for students to engage in quantitatively oriented figural activity.

Unit Items

Returning to the notion of quantitative counting, the items that are to be counted deserve brief attention. Mentioned earlier, the heart of quantitative counting is the production of a unit item. Steffe et al. (1983) elaborate on this process:

> [C]ounting is not the simple recitation of a number word sequence . . . first, there must be items; and second, there must be a procedure to make each utterance of a number word coincide with one of the items that are to be counted. . . To begin with, these

> experiential [unit] items are perceptual: beads, marbles, fingers . . . anything that can be experienced as a succession of individual items. Regardless of whether or not we believe that such individual items have an independent ontological existence, a counter must perceive and conceptualize them as discrete and unitary before he can begin to count them (p. 2).

Here we see the origins of unit items in the perceptual entities around us. Indeed, Steffe and his colleagues assert that any perceptual entities may be appropriated as a unit item provided that each item is a discrete sensory experience within a succession of sensory experiences. Over time, the production of unit may move away from perceptual entities and transform into more abstract cognitive constructions; however, it would be wise to briefly examine extant literature related to these initial counting activities as it is here that we locate a prodigious understanding of what it means to count unit items.

Counting of Unit Items

There have been volumes written on the nature of counting perceptual unit items, but within this extensive body of literature, several seminal works arise. Worthy, perhaps, of first mention is Gelman and Gallistel's (1978) groundbreaking research which established certain domain specific principles that allow children to "reason verbally about numerosity, numerical relations, and operations that affect" (Gallistel & Gelman, 1992, p. 65). In terms of counting, the authors theorize five primary principles that guide children's behavior. These principles, summarized by Gelman and Meck (1983), consist of:

> (1) [T]he one-one principle--every item in a display should be tagged with one and only one unique tag; (2) the stable order principle- the tags must be ordered in the same

> sequence across trials; (3) the cardinal principle-the last tag used in a count sequence is the symbol for the number of items in the set; (4) the abstraction principle- any kinds of objects can be collected together for purposes of a count; and (5) the order-irrelevance principle-the objects in a set may be tagged in any sequence as long as the other counting principles are not violated. The first three of these principles define the counting procedure; the fourth determines the types of sets to which the procedure may be applied; and the fifth distinguishes counting from labeling (p. 343-344).

What Gelman and Gallistel note, here, is the presence of several distinct precepts that appear to emerge over time and structure the counting activity of children.

Building on these ideas, Fuson (1982, 1988) conducted a comprehensive examination of children's counting errors and found the emergence of common errors (i.e. pointing or touching a particular item once, but verbalizing more than one number word). Based on the presence of these errors, Fuson provides some elaboration on the five counting principles put forth by Gelman and Gallistel (1978). Specifically, Fuson questions the existence of a singular one-to-one correspondence principle in favor of multiple criteria associated with evaluating students' construction of one-to-one correspondence. Fuson's analysis of counting errors also provides evidence that verbalizing the last word in a counting sequence to the inquiry of "How many?" does not necessarily imply an understanding of cardinality as Gelman and Gallistel suggested. While Fuson raises several substantive questions regarding the five counting principles and ultimately concludes that, in terms of counting, procedure and conception be considered as separate and interacting entities (Clements, 1989).

Siegler and his colleagues (Siegler & Robinson, 1982; Siegler & Shrager, 1984) have also made substantial contributions to the counting literature. Emphasizing children's enacted strategies in arithmetic contexts, Siegler and Robinson conducted a series of videotaped observations of children ages four and five actively solving various addition problems. Subsequent analysis of these data revealed the use of four distinct strategies: *counting-fingers, finger, counting*, and *retrieval*. The *counting-fingers* strategy involved the raising and subsequent counting of fingers corresponding to each addend of an addition problem. The *fingers* strategy describes the raising, but not counting, of fingers to correspond with each addend of the problem. The *counting* strategy simply involved the utterance of a verbal sequence (with no external referent) that was congruent with the addends of the problem. Lastly, the *retrieval* strategy describes strategies that "involved no visible or audible behavior" (Siegler &Shrager, 1984, p. 236).

It is important to note here that these videotaped observations demonstrated the individual use of different strategies depending upon the problem. Indeed, a single problem solving event could be viewed as a series of strategy phases, progressing in real-time, that are dependent upon the cognition of the child (Siegler & Shrager, 1984). Specifically, when faced with an arithmetic task, a child may first attempt to mentally *retrieve* the answer. If, for some reason, this strategy is deemed insufficient, the same child may then enact one or more of the overt strategies involving counting and or fingers. This finding is of particular significance as it opens the doorway for variability in counting practices. Indeed, Siegler and his colleagues observed, quite astutely, that the same child may enact qualitatively different counting strategies depending upon the presented task.

Examining Counting from an Alternate Worldview

Up to this point, I have presented counting literature drawn from the methodological traditions of cognitive and developmental psychology. Specifically, these traditions are strongly associated with what Thompson (1982) refers to as an environmentalist paradigm where conditions are controlled and effects are measured with the aim of discerning differences among treatments. While the scholars in these respective fields have undoubtedly made seminal contributions to our collective understanding of counting, the final framework that I present in this section is borne of a different methodological worldview, namely one that privileges the constructed problem space of the participants over the controlled environment of the researcher.

Adhering to this worldview, Steffe and his colleagues enacted a series of long-term teaching experiments (in natural environments) with young children to determine different types of quantitative understanding and how such understanding may change over time (Steffe et al., 1983; Steffe, et al., 1988; Steffe, 1992; Steffe & Thompson, 2000). These experiments typically involved a sequence of teaching episodes that “include[d] a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episodes”; furthermore, effort was expended to ensure that these episodes focused on the *students’ mathematics* rather than the mathematics of the teacher (Steffe & Thompson, p. 273). Steffe and Thompson (2000) explain the importance of students’ mathematics in the following terms:

> We have found it necessary to attribute mathematical realities to students that are independent of our own mathematical realities. . . Although our attribution of mathematical realities to students is a conceptual construct, it is grounded in the mathematical realities of students as we experience them. . . So, we have to accept the student's mathematical reality as being distinct from ours. We call those mathematical realities "students' mathematics," whatever they might be. Students’ mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic

goal of the researchers in a teaching experiment is to construct models of students' mathematics (p. 267-268).

Confrey (1990) referred to this activity of seeing "a situation as perceived by another" as decentering (p. 108). In this instance, students' mathematics describe the counting solutions *developed by children* "when faced with problematic arithmetical situations" including additive and subtractive tasks presented in a variety of configurations (Steffe and Cobb, 1988, p. vii). The mention of these ideas are important here as they serve to counter certain objections regarding the progression put forth by Steffe and his colleagues. More specifically, decentered teaching experiments provide us with a ready answer to opponents who may question the role of counting in quantitative development or the necessity of moving children through a particular counting progression. Such questions are certainly warranted given the minimal emphasis placed on counting in other cultures (van den Heuvel-Panhuizen, 1996). The answers, quite simply, are that 1) counting appears integral to the manner in which children in this country, *by their own right*, come to understand quantity and 2) this progression, termed Stages of Early Arithmetic Learning (SEAL), describes several qualitatively different modes of quantitative understanding (termed 'counting types') rather than a prescriptive trajectory to be employed by a teacher (Steffe, 1992; Wright et al., 2000, 2002).

Stages of Early Arithmetic Learning

Emergent counting. The least sophisticated type of quantitative understanding that we see among school-age children is that of the emergent counter. Although this particular stage was not hypothesized in the original progression put forth by Steffe and his colleagues, Wright, (a protégé of Steffe) found it useful to describe a preceding level of understanding that he observed among school age children (Wright, 1994; Wright et al., 2000; Wright et al., 2002). At this stage of development, significant difficulties exist in the counting practice. Wright et al. (2002) remark:

> The child at the emergent stage is not able to count a [visible] collection of counters. . .The child might not know the forward number word sequence from one to ten or beyond ten. Alternatively, the child might know the number words but not be able to correctly coordinate a number word with each counter. This might involve omitting one or more counters or perhaps pointing to one or more counters twice during counting. Some emergent children seem to interpret questions such as, 'how many counters are there?', literally as an instruction to say the number word sequence from one. It is as if the literal meaning for the child is to say the words 'one, two, three, and so on' while pointing at the collection of items in question (p. 73).

This activity suggests not only that emergent counters do not have an adequate tracking mechanism to coordinate their number word sequences with actual objects, but also that such counting acts are more about the naming of objects rather than the establishment of a collection's cardinality. Indeed, among emergent counters, numbers exist as names.

Perceptual counting. Increasing in sophistication from the emergent counter, one encounters children that regard quantity as a direct sensory experience. This type of counting activity requires that, "objects need to remain in [individuals'] perceptual field in order for them to be able to count them" and is thus termed perceptual counting (Olive, 2001, p. 5); furthermore, within this stage, one finds three distinct sub-types of sophistication that each provide insight as to how the child conceptualizes quantity (Steffe et al., 1988; Wright et al., 2000). The first and least sophisticated sub-type describes children that can use counting to determine the numerosity of a single, visible collection. Such children, however, are unable to successfully use counting to produce a combined total of two visible collections. Wright et al. elaborate:

> In the case of two separate collections, for example a collection of 9 red counters and a collection of 6 blue counters, some perceptual children do not seem to alternately regard the two collections as one for the purposes of establishing the overall numerosity. This difficulty has been observed to arise even in cases involving very small numbers of counters (for example, five and two). Thus when asked how many counters in all the child answers ‘five’ or ‘two’, but does not seem to realize that they can make one count from one, to count the counters in both collections (Wright et al., 2000, p. 57).

As the reader might suspect, the next sub-type describes children that *are* able to successfully use counting to establish the numerosity of two visible collections. Indeed, what may appear to some as a relatively trivial advancement is, with all likelihood, the first occurrence of an applied arithmetic operation - addition.

The last and most sophisticated sub-type of perceptual counting represents another striking advancement in sophistication. Although children that demonstrate counting strategies of this sub-type still regard quantity as a sensory experience, they exhibit a level of abstraction not seen among other perceptual counters. Specifically, these individuals are able to count non-visible collections by substituting a *perceptual replacement* for each non-visible unit (Steffe et al., 1988; Steffe, 1992). These replacements may occur as objects within the child’s visual field (e.g., stickers on a chart), or more robustly as creations of the child (e.g., raised finger patterns). Indeed, “[i]t is one thing for children accidentally to find perceptual items to count, and quite another intentionally to create sensory motor items as substitutes for hidden perceptual items” (Steffe et al., p. 289). In any event, the use of these replacements comes with certain constraints that suggest a perceptual conceptualization of quantity. Key among these is the necessity to activate equivalent replacement collections prior to the application of counting to determine the combined numerosity of two collections. Consider the following task: two screened collections of counters (three and four) are presented and the child is asked “How many altogether?” For a child that uses her fingers as perpetual replacements, the first observed activity would be to activate the replacement collections by raising (either sequentially or simultaneously) three fingers and then counting these fingers starting at one. The child then raises four fingers in a similar manner and counts these fingers from one. Having activated the two replacement finger collections, the child then begins at one to count both groups of fingers. Indeed, it is these three distinct counts from one (two counts to activate the finger patterns and a third to determine the numerosity of the combined finger patterns) that suggest the perceptual nature of these replacements; because, for each hidden collection, an equivalent replacement collection must be at the ready prior to the establishment of numerosity. Although this activity is still classed as perceptual counting, the presence of perceptual replacements signals a child’s readiness to take one step away from the original sensory materials. While her/his perceptual counting peers may simply guess at the numerosity of screened collections, the perceptual replacement user is on the verge of quantitative abstraction and nearly ready to move forward.

Figurative counting. If perceptual counting is considered a direct sensory experience, then figurative counting may be thought of as one step removed from a direct sensory experience. At this stage of quantitative understanding, children are no longer tied to working with items that they can see or touch, but rather can begin to leverage mental replays of sensory experiences to facilitate counting acts. These mental replays have been termed *re-presentations* (as distinct from representations) to describe how children re-present the sensory experience to themselves; these re-presentations may occur in several different forms (Steffe et al., 1988; Steffe, 1992; Wright et al., 2000). The first and least sophisticated of these forms is the figural re-presentation which involves the counting of mentally visualized, figural unit items that resemble perceptual items from previous sensory experiences. As the student gains facility with producing these mental images, re-presentations increase in sophistication to include the use of verbal and motor unit items (Steffe et al., 1983; Steffe et al., 1988). In the case of verbal unit items, figurative counters of this type utter number words as they mentally examine figural unit items, and these utterances serve as a mechanism to keep track of the count. Similarly, motor unit items (i.e., sequentially raised fingers) may be employed by figurative counters to keep track of their counting. The key idea here, though, is that conceptual development beyond a sensory understanding of mathematics involves the production of a mental image. Indeed, it is here, at this figurative stage, that I ground this review.

Initial number sequence and beyond. As children become robust users of imagery in the activity of counting, they tend to develop the ability to produce an abstract numerical composite (Olive, 2001; Steffe, 1992; Wright et al., 2000). This composite consists of the simultaneous understanding that a collection of units also exists as a single unit item. Quite simply, a collection of five counters is no longer conceived as only five unit items, but rather as five unit items *and* a single unit item “five”. We find evidence of this understanding in additive tasks when children *count-on* rather than attending to each unit item of the task and beginning the count at one. For example: when determining the numerosity of two collections (8 and 5), the child at this stage may still produce a figural re-presentation; however, he/she trusts that the collection of eight unit items exists also as the single unit item “eight” which may serve as a starting point for the count. If this student re-presents verbally, the count will likely sound like "nine, ten eleven, twelve, *thirteen*." Although children at this stage still engage in unitary counting (count-by-ones), the presence of a single numerical composite indicates the first attempt to treat quantity as something that may be utilized in terms of chunks of units (Steffe, 1992). Moving further up the developmental progression observed by Steffe and his colleagues, we see students gain facility, and ultimately automaticity, in their ability to work with multiple composite units and consider these units both in terms of their composite and component attributes. Indeed, the sophistication to automatically decompose and recompose multiple composites flexibly according to a specific task provides a worthy benchmark for what it means to be numerate. Steffe goes on to suggest that this process of counting abstraction "strips the figurative unit items of their sensorimotor quality and creates [an] abstract unit item that contains the record of the counting acts" (1992, p. 93). Indeed, while the re-presentational capacity remains and continues to be used, the re-presented unit items begin to disconnect from the sensory experiences on which they were founded.

More on figural re-presentations. Given that this review is focused on the current understanding of the production of quantitative mental imagery, perhaps some additional discussion of figural re-presentation (the act of creating collections of countable figural unit items) is in order as this activity exemplifies the first instance of useful imagery in mathematical applications. First and foremost, it is important to note the significant advancement in thinking as children make the transition away from perceptual counting and towards the use of figural re-presentations (Steffe, 1992). More specifically, at this stage of development "the child's counting scheme can be activated when the items to be counted are not in the range of action or perception" (Steffe, p. 88). Such counting schemes depend upon the creation of figural unit items or mental images of previous perceptual experiences.

It is at this point that Steffe recognized both the certainty of quantitative mental imagery in counting acts and his own lack of understanding regarding the process of this creation. Steffe elaborates:

> Although I am not certain of all the processes involved in establishing the ability to recreate an experience of an object in its immediate absence – a re-presentation of the object – the categorizing act of reprocessing the items of a collection must be involved . . .There is at least incipient visualization of the common sensory material used in the categorization because what the activated records point to is also in a state of having just been used more than once in the immediate past (1992, p. 86).

What Steffe is suggesting here is that some manner of mental imagery (newly emergent or otherwise) lies at the heart of the figurative re-presentation and that these images are a result of preceding sensory experiences; moreover, the process in which children produce these images remains uncertain. What Steffe does make clear is the fact that the use of imagery in quantitative contexts is much more difficult for some children than many adults suspect. Steffe writes the following on this point:

> These children may have constructed object concepts, but they are yet to use them to create figurative collections. Although they might know that there are, say, cookies in a cookie jar because they can imagine what a cookie might look like, they do not have an awareness of a figurative plurality of cookies because they do not use their cookie template to "run through" a production of more than one cookie (p. 88).

In the final analysis of figural re-presentations, we see that the production of mental imagery is both a necessary and sometimes difficult step for children to take; furthermore, we seem to know little of how this important step is taken.

Summary

Considerable theoretical groundwork has been laid by psychologists. In particular, Fuson's (1988) assertion that conceptual and procedural aspects of counting are both separable and interrelated as well as the observance of strategy variability observed by Siegler and his colleagues provide key ideas for the study of quantity and counting (Siegler & Robinson, 1982; Siegler & Shrager, 1984). However, the model put forth by Steffe and his colleagues, steeped in longitudinal examinations of students' mathematics, ultimately provides far greater explanatory power of counting types and the emergence of quantitative mental imagery. Indeed, Steffe and his colleagues have demonstrated, through decentered teaching experiments that the progression of quantitative understanding occurs is directional and typically involves counting activity and, at times, re-presentations of past sensory experiences.

Additionally, a significant milestone in progression put forth by Steffe and his colleagues (Steffe et al., 1983; 1988) involves the production of mental imagery that is useful in the activity of counting. At this point, I reach the functional limits of the SEAL framework in that little hypothesis, let alone description, of the processes is provided regarding the production of mental imagery. Certainly, these assertions do not stand to indict the progression proposed by Steffe and his colleagues. Indeed, this framework is profoundly robust and has provided the basis for increased mathematical attainment for thousands of students (KCM, 2008). Instead, I intend to illustrate a significant opportunity to extend this framework and our understanding of how children describe and reveal their imagery construction. Towards this end, I find it useful now to venture beyond the formal boundaries of mathematics education and into territories claimed by philosophy and neuroscience in order to search for the elusive foundations of the mental image. Let's continue our walk.

Mental Imagery

Sometimes referred to as seeing with the mind's eye, the phenomenon of the mental image is both transcendent and perplexing. The visual manner in which we may revisit previous experiences can inspire awe and emotion among the individual; however, attempts to pin down the nature of these images have frustrated great thinkers for millennia (Thomas, 2008). While these historic considerations of the mental image are certainly germane to this review, I find it useful to lay out a working definition prior to launching headlong into the philosophical fray. Toward this end, the production of mental imagery may be considered as both a representational process (Kosslyn, 1980, 1983) and a quasi-perceptual experience (Sarbin, 1972). The first half of this definition deals with the directed nature of imagery. Indeed, we see that the mental image is almost always produced intentionally on the basis of a particular subject or perceptual experience. In fact, "nearly all serious discussions of mental imagery take it for granted that it bears intentionality in the sense of being *of, about*, or *directed at* something" (Thomas, 2008, p. 8). The second, experiential, half of my definition refers to the manner in which the mental image, once produced, seems to behave in a similar manner to the subject or perceptual experience upon which it is based. Quite simply, the mental image is a cognitive experience (absence of related/appropriate external stimuli) that feels perceptual, hence, the descriptor of quasi-perceptual (Thomas). To begin this exploration of the mental image, I submit that this dichotomous definition will provide the torchlight for a pathway through history. In this section, I will describe the pre-scientific roots of the mental image as well as both sides of the contemporary imagery debate. Specifically, attention will be given to the competing arguments that posit imagery as a distinct visual phenomena or a non-distinct epiphenomena of a larger propositional structure (Kosslyn, 1980) or in simpler terms, whether or not the mental image we think we see is actually an image. Lastly, I will explicate certain aspects of contemporary imagery theory that likely pertain to mathematics intervention and the figurative counter.

Pre-scientific Conceptions of Mental Imagery

The quest to understand the nature of the mental image has fascinated philosophers for much of recorded time. Although there is some speculation that mental imagery may have been the topic of pre-Socratic thought (Sarbin & Juhasz, 1970), Plato provided us with some of the earliest ventures into this philosophical realm. Describing a dialogue between Theaetetus and Socrates, Plato (trans. 1921) wrote:

> Please assume, then, for the sake of argument, that there is in our souls a block of wax, in one case larger, in another smaller, in one case the wax is purer, in another more impure and harder, in some cases softer, and in some of proper quality. Let us, then, say that this is the gift of Memory, the mother of the Muses, and that whenever we wish to remember anything we see or hear or think of in our own minds, we hold this wax under the perceptions and thoughts and imprint them upon it, just as we make impressions from seal rings; and whatever is imprinted we remember and know as long as its image lasts, but whatever is rubbed out or cannot be imprinted we forget and do not know (Plato, Theaetetus, p. 191 c, d, e).

What Plato appears to have hypothesized in the context of this wax metaphor is the visual nature of recording and remembering perceptual experiences; however, this passage appears more as an allusion to imagery rather than a deliberate exploration.

In what may be the original comprehensive cognitive theory (Thomas, 2008), Aristotle (trans. 2007a) provided us with discussion of imagery that was quite intentional and explicit when he wrote, “to imagine is therefore (on this view) identical with the thinking of exactly the same as what one in the strictest sense perceives" (Book 3, ¶ 24). Elaborating on the inextricable nature of perception and imagery, Aristotle (trans. 2007b) wrote, "Now, whether the presentative faculty of the soul be identical with, or different from, the faculty of sense-perception, in either case the illusion does not occur without our actually seeing or [otherwise] perceiving something" (¶ 4). In these remarks, we see an early supposition of mental imagery and the deep connection that this imagery has with our experiences.

Many centuries later, Renè Descartes proposed the presence of physiological structures that facilitated the production of imagery and elaborated on a proposed connection between perception and image. From Descartes' *Meditations*, Gueroult (1985) interpreted the following:

> Descartes understands two different things by imagination: imagination as the mental faculty, which is the soul exercising an action on the brain, and corporeal imagination, which consists of the body to preserve traces of actions exercised on it, either from within or from without. This capacity resides in the pineal gland . . . (p. 29).

It is this speculation of corporeal imagination that appears particularly prescient for any contemporary discussion of mathematical imagery in that such processes would allow experiences to pave the way to figurative counting.

Regarding Descartes' hypothesized pineal gland, depicted (see Figure 3) in *Treatise of Man* (1664), this ultimately fictional organ was believed to be an acorn-like structure centralized within the mind that transmitted visual stimuli to the soul (Thomas, 2008).

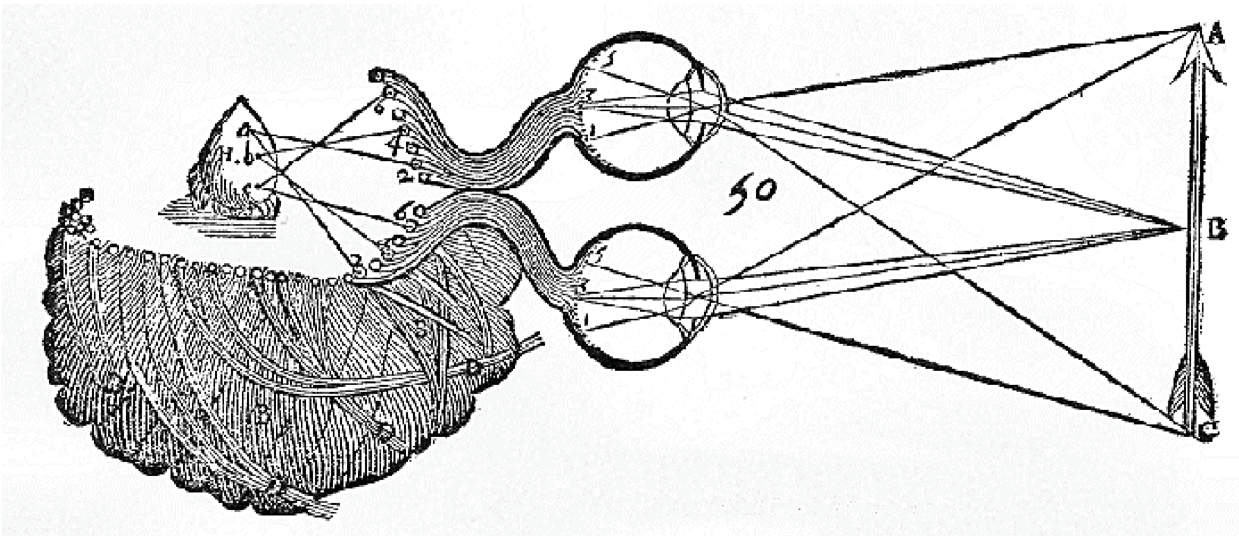

Figure 3: Diagram from Descartes' *Treatise of Man* (1664) depicting the pineal gland

Descartes also proposed the involvement of the phantasia in this process as “a crucial organ of imagination in the brain. . .it is the place where images occur, whether they derive from the senses, from memory, or from the activity of the intellect” (Sepper, 1993). Like the pineal gland, this phantasia was ultimately a fictional construction; however, Descartes ideas about mental mechanisms for imagery production and his elaborations on the relationship between experience and image set the stage for contemporary work in the field.

Imagery Wars

Although the search for mental imagery has continued in earnest for millennia with different interpretations and postulates put forth along the way, interest in psychotherapy and psychosomatic medication in the 1960’s led to a full-scale imagery revival that has lasted from the early 1970’s to present day (Thomas, 2008). Principally, this revival has been motivated by fervent proponents of two opposing camps: the first vehemently arguing in favor of the pictorial image, and the second hypothesizing a language-based phenomena. Supporters of the pictorial image typically refer to such occurrences as *analog* in that mental images seem to have spatial qualities similar to pictures and are thus analogous to visual stimuli (Kosslyn, 1980, 1983).

Conversely, opponents of this position argue that such representations are, in fact, more linguistic or *propositional* in nature in that they describe rather than depict. Although many scholars have contributed to this long-standing feud, Kosslyn (1973, 2005) and Pylyshyn (1973, 2002) stand as the champions for the analog and propositional camps respectively in that their research and writing appears foundational to this ongoing war. While it would take volumes to give justice to the nuances of each position, I have enlisted the seminal works of these two authors and their respective colleagues to provide a very general lay of the land.

Propositional Imagery

Beginning with the premise that the evidence to support the presence of analog imagery is insufficient, Pylyshyn (1973) described a propositional system of representation that eschews a visual format in favor of some manner of mental language. For example, a propositional representation of a ball on top of a box might consist of the phrase 'ball on box' (in a mental language) which would then be available for inspection by the individual. It is important to note that this language of thought or *mentalese* (Fodor 1975) is not necessarily similar to natural, spoken language. Pylyshyn described the nature of propositions in the following passage:

> Thus propositions [representations] are to be found in the deep structures of language and not in its surface form. But this is still not sufficiently abstract for our purposes since it might be taken to imply that each proposition is expressible by *some* sentence in a natural language. This is not, however, a necessary condition for our use of the term. We claim that it is still useful to think of propositional knowledge even when the concepts and predicates in such propositions do not correspond to available words in our vocabulary . . .Such view implies that we can have mental concepts or ways of abstracting from our

> sense data which are beyond the reach of our current stock of words, but for which we *could* develop a vocabulary if communicating such concepts became important (p. 7)

What Pylyshyn described here is a *mentalese* based system of representation that, when decoupled from natural language, becomes sophisticated enough to produce functionally detailed representations of sensory experiences.

Failure to reject the null hypothesis. Because there seems to be little or no dispute regarding the existence of propositional structures of the mind which may be invoked for the purposes of speaking and writing (Kosslyn, 1980), the arguments in favor of propositional representation, presented by Pyslyshyn (2002) as the null hypothesis, deal with the failure of analog theory to reject this hypothesis.

Perhaps the most substantial argument against analog imagery as a distinct cognitive mechanism rests on such a position's seeming invocation of an internal homunculus (Latin for 'little man') that interprets the images. Consider Kosslyn's (1983) description:

> Do you remember Jiminy Cricket? He was a Walt Disney character, a cheerful little green fellow, who figured in a cartoon I saw twenty or thirty years ago and still remember vividly. Jiminy was in a quandary. The camera flashed to a scene inside Jiminy's head, where there was a tiny control room with a little man seated before a console. The little man scratched *his* head, made a decision, and pulled a lever; and then we saw Jiminy smile and go off on his way. Jiminy Cricket may be a Disney fantasy, but the concept represented in the cartoon is centuries old. It is the literal extreme of the mind existing within, but separate from, the body. The little man inside of Jiminy's head, known to philosophers as a homunculus, is trouble for any theory of the mind. How does he decide what to do? Is there another little man inside *his* head? Are we stuck with an

> endless series of homunculi nested inside each other's heads like Russian egg-dolls? (p. 12).

Pylyshyn (1973) asserted that any visual theory of imagery ultimately relies on some internal homunculus to make sense of and interpret these images into propositions thus resting on a cognitive fallacy.

Pylyshyn (2002) also put forth the notion of cognitive penetrability to describe a particular flaw in analog theory. Cognitive penetrability describes the extent to which a representation's innate form is infiltrated and transformed by the beliefs and attitudes of the individual. Pylyshyn elaborated:

> The distinction between effects attributable to the intrinsic nature of mental mechanisms and those attributable to more transitory states, such as people's beliefs, utilities, habits, or interpretation of the task at hand, is central not only for understanding the nature of mental imagery, but for understanding mental processes in general (p. 159-160).

This attitudinal invasion is important given the reliance on an "early visual processing module" (Thomas, 2008, p. 36) of the analog perspective which suggests that the generated representations should be inherently impenetrable in that they remain deeply connected to their perceptual counterpart. Quite simply, if one accepts that perceptual experiences are impenetrable (we cannot mentally influence what we perceive) because of a modularized cognitive process, then it follows that products of this process should be impenetrable as well. Indeed, we do experience cognitively impenetrable representations from time to time. Consider the Müller-Lyer (1889) illusion involving two lines that appear to be different in length, but are actually the same length (see Figure 4).

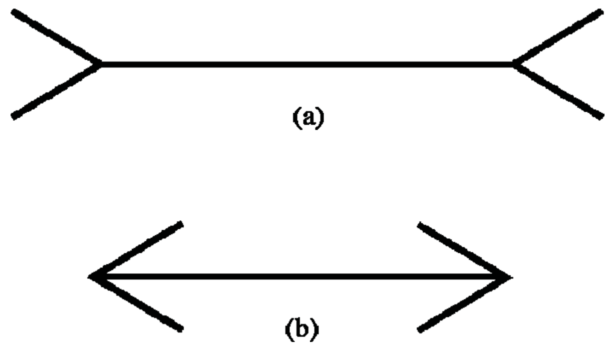

Figure 4: Müller-Lyer illusion

Although we know that lines a and b are the same length and that this illusion is prompted by the different structures at the end of each line, most are still ‘tricked’ by their own mental representations of this figure thus rendering the image impenetrable to personal knowledge and beliefs.

Conversely, cognitively penetrable images are those representations that are affected by the individual. Perhaps this phenomenon is best exemplified in a brief task. Generate, if you will, an image of the trademark Coca-Cola emblem. Could it be that you visualized this emblem on the side of a can or bottle? Perhaps your image involved a large delivery truck or billboard. Why should a diversity of seemingly superfluous items enter into a representation, particularly when one is able to just as easily visualize only the emblem when specifically prompted to think of it in isolation? Indeed, this exercise illustrates the manner in which the individual actively penetrates a representation with a unique and personal interpretation. It is this penetration, Pylyshyn (2002) stridently argues, that effectively dispels the notion of an analog imagery system distinct from existing propositional systems effectively moving visual experiences of imagery back into the role of epiphenomena. Quite simply, the visual image may seem real to the individual, but it is actually a representation stemming from an existing propositional system that does the heavy lifting.

Analog Imagery

Aside from the proponents of propositional imagery, a trip back through time reveals a strong privileging of analog imagery. From Plato's (trans. 1921) description of images imprinted onto a mental wax to the visualization of a Coca Cola emblem moments ago, our collective experiences strongly suggest a pictorial type of representation that transcends a syntactic proposition. This transcendence lies, in part, with our ability to examine a mental image much like a physical picture. Imagine, if you will, a Wal-Mart store sign. Now scan to the space between 'Wal' and 'Mart' to determine what symbol lies there. Chances are you 'saw' a star or, perhaps, a hyphen from one of the older signs. At any rate, the point is not whether or not you were able to accurately image the sign, but rather, that you were able to inspect a specific part of your image much like a picture. Kosslyn (1980, 1983) sought to explicate this profundity of the analog image in the light of formidable arguments put forth by Pylshyn and his colleagues.

Rebutting the propositional perspective. Before one can engage in any honest discussion of analog imagery, attention must first be given to the points of critique. Given Pylyshyn's (2002) claim that our experiences should produce impenetrable images because such images are a product of component systems of perception (e.g. eyes, nose, ears) that inform but operate independently of the mind. Countering this assertion, Kosslyn and his colleagues presented significant overlap in cognitive systems of perception and representation (Kosslyn, Ganis, & Thompson, 2001; Kosslyn, 2005) which provide the context for significant interplay among systems. While such evidence does not necessarily support an analog position, it does damage the penetrability portion of the propositional argument.

Returning to the propositionalist charge that analog theories of imagery commit a homunculus fallacy by tacitly incorporating a mind within the mind that views the images, Kosslyn and his colleagues turned to technology for the answer. Hypothesizing a cognitive process similar to that of a computer interpreting a coded matrix to produce an image, Kosslyn (1983) wrote:

> The computer approach to understanding the mind also provides the answer to the apparent paradox of the "mind'e eye": That is, if an image is a picture in the head, who is looking at the picture? The eyes that scrutinize the outside world obviously cannot be turned inward, and we know there is no homunculus watching an inner screen. The computer approach suggests that the mind's eye is comparable to the processes that allow a computer to interpret visual display. When a computer stores pictorial information, it translates it into a set of points, with each point being stored in a cell in an imaginary matrix. To see if a straight line (or any other visual feature) appears on the matrix, the computer treats the points as being organized spatially and checks which cells have points . . . The presence of a line is thus determined by two things: the stored information and the tests carried out to interpret it (p. 25).

What Kosslyn described here was not the production of a pure image that matches a perceptual experience, but rather a matrix of perceptual data, spatial in its own right, coded such that its interpretation occurs much like the running of a computer program. Like a computer, Kosslyn posited, the mind needs no homunculus to operate in this fashion.

Lastly, Kosslyn (1983) argued against the logistics of the propositional perspective. If asked to mentally represent an elephant and a mouse at a distance, the elephant will be larger in almost every image. If these representations are propositional, "does this mean that somewhere in the mind is stored a statement that an elephant is bigger than a mouse? If so, our mental file of such observations must be enormous! A horse is bigger than a robin; an airplane is bigger than a breadbox – the list is almost infinite and therefore not a practical possibility" (p. 163).

Quasi-pictorial imagery in two dimensions. While useful in refuting the need for a homunculus, Kosslyn's (1983) theory requires, perhaps, some additional attention to further distinguish coded mental matrices from propositional representations. Indeed, at this point, one could assert that the invocation of mental code appears more in keeping with a propositional rather than analog perspective; however, the key to this position lies in the spatial arrangement of the code that is pictorial. Kosslyn and Schwartz (1977) used a computer to simulate what an image of an automobile might look like according to this hypothesis (see Figure 5).

Figure 5: Kosslyn and Schwartz's simulation of a mental image: car

Although still analog in the sense that such a representation holds spatial qualities, Kosslyn (1980) recognized the perceptual dissimilarity and accurately described this position as quasi-pictorial. Kosslyn elaborated, "[t]he key distinction here rests on the difference between *description* – as it is accomplished with propositions [e.g., "car"] – and *depiction* – which is achieved with pictorial representations" (p. 31). It is important to recognize that the coarse matrix of the simulated quasi-pictorial image does not necessarily represent the sophistication of the mind. Note that this particular simulation was generated using 1970's technology. Indeed, while each image will have an irreducible grain, the resolution of mental matrices could be staggeringly high. Lastly and perhaps most importantly, the proposal of a matrix necessarily grounds the mental image in two-dimensions or 2D (Kosslyn & Schwartz). In this model, three-dimensional perceptual objects may be represented isometrically much like a perspective drawing mimics the dimensionality of an actual box.

Evidence for depiction. While proponents of quasi-pictorial analog imagery have provided considerable rebuttal of propositional critiques (Kosslyn, 1980), the dismissal of a homunculus and connections between perceptual and representational systems hardly supports a strictly depictive system of imagery. In order to truly sustain an analog perspective, evidence must be provided that illustrates that mental images are distinctly spatial qualities that are analogous to our perceptions. Kosslyn and his colleagues assert that confirmation of these properties is best illustrated by an individual's ability to examine these representations much like a physical image. Although research involving imagery rotation (Shepard & Metzler, 1971) has produced some suggestive results in favor of an analog perspective, this work lacked conclusivity in that such rotations could simply be the product of many nuanced propositions (Kossyln, 1983). In an effort to unearth specific depictive properties of imagery, distance across an image was identified as unique to pictorial representations. Kosslyn (1983) described his hypothesis:

> The best way to do this seemed to be to use time as a measure of distance in a mental image scanning task. If subjects took more time to scan a long distance across an object in an image than a short distance, then we could reasonably conclude they really were *using* a depictive representation, not just [epiphenomenally] experiencing the properties of one. Distance is a "privileged property" of depictions, so it should affect scanning

times only when subjects use imagery, provided that images actually do depict information (p. 41-42).

In this particular experiment, 49 subjects were asked to memorize a sample drawing from a collection (Kosslyn, 1973). One group of individuals was then asked to close their eyes and visualize a specific feature of the drawing (focal point). From this visualization, subjects were then asked to locate another feature that may or may not have been on the original drawing. These features varied in location around the image. For example, one individual may have visualized a picture of a speedboat and then been asked to find the motor. Subjects were to push one button if they could find the feature in their mental image and another if they could not. As quickly as the button was pushed, individuals were prompted with another possible feature at a different location. Kosslyn found that the distance between these features significantly increased the time that it took for individuals to scan across the image. Returning to the speedboat example (see Figure 6), individuals took longer to determine the subsequent presence of a motor and an anchor than a motor and the porthole.

Figure 6: Kosslyn's (1973) speedboat image

In order to ensure that this difference in scanning time was not attributable to some peculiar quality of the image features, another group of subjects was asked to visualize the entire object (rather than beginning with a focal point) and then prompted to determine the presence of specific features. As hypothesized, the scanning times for this group remained constant and did not vary with the distance between features. Kosslyn concluded, "[t]he results clearly indicated that people can selectively retrieve information from preset spatial locations on a generative image. Furthermore, retrieval of items from an image is a function of actual physical distance from the point of initial focus" (p. 93).

Subsequent research provided additional evidence supporting the analog perspective's claim of depictive imagery. In an experiment similar to the one described earlier, Kosslyn demonstrated that the size of the features impacted the time in which individuals took to determine their presence with larger features being apprehended more rapidly (Kosslyn, 1976). Perhaps even more interesting, however, are the claims that an individual's internal 'visual angle' may impact the facility with which one examines an image and that overall image space may be limited (Kosslyn, 1978). More specifically, when prompted to mentally 'move toward' an object, individuals reported that the image seemed to overflow and not remain visible as whole and that overflow seemed to happen around the periphery of the object in a manner that suggested a particular viewing angle. Although Kosslyn (1980) conceded that it is possible to describe these experimental phenomena from a propositional perspective, in his point-by-point analysis of the propositional argument, attention is called to the ad hoc nature of such explanations compared to the much more natural derivation of explanations from analog theory.

Summary

Although the foundations of imagery remain in contention and Pylyshyn continues to argue that the analog case has yet to be made convincingly, Kosslyn and his colleagues provide substantial evidence that mental imagery is, in some manner, spatial and depictive of our sensory experiences. Indeed, depictive mental images seem congruent with individual experiences of pictorial representations; furthermore, the analog perspective provides a tremendously useful framework in which we may understand how sensory perceptions of a mathematical tool may be visually re-presented for inspection and action by the child. Quite simply, if we assume an analog perspective (although well-supported, it remains an assumption), then the visual nature of the mathematical tool used becomes quite important as the figural re-presentation will depict, in some manner, what was previously perceived. With this in mind, let us examine the contents of our numeracy toolbox to determine how the appearance and presentation of mathematical materials could potentially influence imagery production.

Mathematical Tools

Tools and Models

Like a hammer hanging by a workbench, a mathematical tool requires activity in order to promote growth. The mathematics of the intervention classroom (and any classroom for that matter) is not to be found in the blocks and computers, but in the children themselves as they actively appropriate different tools to represent their thinking. When used in this manner, a mathematical tool may be considered a model. Fosnot and Dolk (2001) wrote:

> To mathematize, one sees, organizes, and interprets the world through and with mathematical models. Like language, these models often begin as simply representations of situations, or problems, by learners. For example, learners may initially represent a situation with Unifix cubes or with a drawing. . . These models of situations eventually become generalized as learners explore connections between and across them (p. 12).

In this example, the authors describe how a particular tool (i.e., unifix cube, drawing) becomes a model when it is employed for a mathematical purpose by the student; moreover, individuals may connect different situational representations to construct a more sophisticated model for more generalized applications. Fosnot and Dolk termed these situational and general constructions as *models of thinking* and *models for thinking,* respectively. From this perspective, the educator may only present the tools, and the child must then use those tools to construct the model. Fosnot and Dolk (2002) explained that, "models cannot be transmitted any more than strategies or big ideas can be; learners must construct them. Just because we plan a context with a certain model in mind does not mean that all learners will interpret, or assimilate, the context that way" (p. 79). Quite simply, tools are necessary but insufficient for the creation of a model.

This perspective has several implications for the intervention classroom. First, given that the mathematical model is produced by the individual, then anything that may be appropriated by children for mathematical activity has the potential to become a model. Or, from a different perspective, there are virtually no limits on what may be considered a mathematical tool. Second, the notion that models transition from situational to general suggests that mathematical tools may be employed with different degrees of sophistication. Lastly, and perhaps most importantly, as educators exercise much control over the learning environment, some informed judgment must be made regarding which tools have the most potential to become models for children. Indeed, teachers may employ tools as settings or contexts for a particular task with the aim that the student will, after some consideration, act mathematically on the materials. For example, an emergent counter may be presented with a certain number of chips (potentially a model in which the child considers a collection quantitatively) and asked to determine the numerosity of the collection. Ideally, tool use of this sort should exist at the cutting edge of a child's knowledge (Vygotskty, 1978; Wright, 2002); however, it is important that such initiations not be construed as overly prescriptive. In describing how the Dutch method of Realistic Mathematics Education (RME) employs potential models, Gravemeijer (1999) wrote, "[a]ctually, the students are to *discover* the mathematics that is concretized by the designer" (p. 159, emphasis added). Quite simply, educators set the stage for growth through selection and deployment of cutting edge tools (with respect to an individual's understanding) which may transform into models when/if they are appropriated by the student for mathematical purposes. Of great importance, though, is the necessity that tools be judiciously selected on their transformative potential.

Tool vs. Strategy

Having distinguished between tool and model, it is useful to draw some distinctions between a tool and a strategy as there appears to be some historic muddying of these waters. Fosnot and Dolk (2002) wrote:

> Teachers often confuse tools and strategies. Unifix cubes or fraction bars or paper and pencil are not different strategies. They are different tools. Representing [a problem involving money] with stacks of unifix cubes, or with fraction bars, or by drawing twenty-four dollars and circling eighteen of them are all the same mathematically. No benefit is derived by changing tools unless the new tool helps the child develop a higher level of schematizing (p. 27).

What Fosnot and Dolk suggested, first, is that there should be clear lines drawn between a tool and its use. For example, stating that a child used her fingers to solve a particular problem only gives information about a tool. One would have to determine how the child was using her fingers to ascertain the strategy. Secondly, the authors asserted that simply using different tools does not necessarily indicate advances in student strategies. Similarly, continued use of a particular tool does not necessarily indicate stagnation. Consider the dramatic shift between a perceptual counter using his/her fingers as a replacement and a figurative counter using his/her fingers as motor re-presentations. In the first instance, the strategy relies on fingers as countable objects whereas the second instance positions fingers as a tracking mechanism for a visual image. Quite simply, tools and strategies do not have one-to-one correspondence. With respect to the theoretical position of this review, tools are the medium through which students will engage in mathematical activity (thus transforming tool into model) and their strategies provide insights into their current understanding. Or from a different perspective, models (internal activity) are the product of strategic tool use (external activity), and we may gain glimpses of an individual's internal mathematical conceptions through observing her external work with mathematical tools.

Counting Apparatus for Imagery

Continuing with our focus on imagery production in the context of applied counting necessitates the deployment of particular tools. Just as a carpenter and painter will use different implements in working to complete a house, the tools employed by mathematics educators will likely differ depending upon the quantitative development of the child. When working towards advancing the emerging imagery of perceptual counters, teachers may rely on apparatus that avoid numerals and deal specifically with quantity (Wright, 2002). Specifically, the tool itself may vary wildly as both plastic chips and sea shells could serve as useful settings for applied counting; however, it is important to the advancement of the early quantitative understanding (i.e., emergent, perceptual, figurative) that selected materials often feature countable unit items. With this in mind, using tools where quantity is presented implicitly, such as a calculator, will likely have minimal potential for model adoption given the distance between the child's conception of number and that represented by the tool; however, there are times when employing numeral-based tasks can provide a window into the quantitative thinking of children and should not be discarded outright. Although well-chosen materials rely to some extent on tension between current understanding and some specified increase in sophistication, there cannot be an inordinate amount of distance between current and past experiences. Indeed, the "introduction of tools should be sequenced to ensure that they continually build on students' prior experience with other tools" (Stephen, Cobb, & Gravemeijer, 2001, p. 75). In terms of counting tools, this suggests that materials should look both forward and backwards.

Returning to the idea that good quantitative tools feature explicit quantities, consider the rekenrek or "bead-rack." This particular tool features either one or two rows of beads with ten or twenty beads per row (Fosnot & Dolk, 2001). Children moving from an emergent to a perceptual understanding of number may slide individual beads from one side of the rack to the other in order to accurately determine the numerosity of one or more bead collections. Children moving from a perceptual to a figurative understanding of number may be tasked with determining the numerosity of two bead collections on the rack, only, in this instance, the entire rack or parts of the rack may be deliberately presented and then concealed before the child is able to use the tool perceptually. Referred to as *screening*, this particular type of tool use is specifically designed to both assess and promote figural re-presentations (Steffe, 1992; Wright 2000, 2002). Remembering that figurative counting is one step removed from a direct sensory experience, it makes sense that facilitating such activity would involve concealing (but not removing) the sensory object. Because a single apparatus has many different uses, it is easy to see how a particular tool may simultaneously evoke past experiences (perceptual counting activity) while promoting more sophisticated understanding (figurative counting activity). In terms of promoting figural counting, such an evolution in tool use would likely involve screening of some sort.

Visual Aspects of Counting Tools

If we adopt a framework of depictive imagery based on perceptual experiences, then it stands to reason that the visual nature of mathematical tools likely holds tremendous significance in terms of promoting figural re-presentations; however, it is, perhaps, natural to question the importance of a tool's appearance if the object is only to be concealed from the child's view. The answer to this dilemma lies in the initial presentation prior to concealment which provides a link to previous perceptual experiences as well as a scaffold for the generation of an immediate image (Steffe et al., 1988). Rather than removing the tool outright, this technique relies on the continued presence within the task (albeit concealed) to better facilitate the figural re-presentation (Wright, 2002). Recalling the analog proposal of a 2D mental matrix (Kosslyn, 1983) on which we examine our mental images, perhaps an examination of tool appearance is in order. As we near the focal point of this review, my goal is to unearth extant knowledge related to differences in the relationship between the appearances of mathematical tools and the practices of children. Although this remains a developing literary domain, research related to tool dimensionality may illuminate this relationship.

Dimensionality

The dimensional status of a particular tool is undoubtedly fundamental to its presentation. Indeed, a physical rekenrek and a picture of a rekenrek are hardly the same thing even when presented in an identical configuration. Although many adults can move fluidly between dimensions when engaging in mental activity, assuming that children will also possess similar facility will likely prove problematic to their development. Rather than selecting tools based on assumptions, I propose that such decisions be based upon empirical evidence, particularly in the intervention classroom where time is of the essence.

Orthographic versus isometric. To my knowledge, research examining the dimensional impact of counting tools has yet to be conducted; however, several studies of dimensionality outside of this specific area provide some illumination of the matter. Although

Kosslyn's (1977) mental matrix suggests that imagery will necessarily be grounded in two dimensions, such constructions could be either an orthographic or an isometric image (see Figure 7). Cooper (1990) sought to explore this type of dimensionality with two related

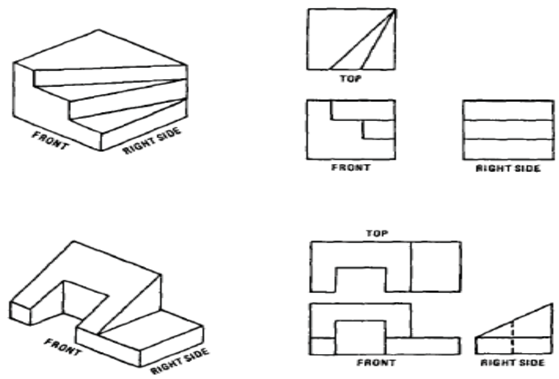

Figure 7: Cooper's (1990) isometric (left) and orthographic (right) projections

experiments in which subjects were presented with a series of problems consisting of two orthographic pictures that "uniquely determined the structure of a single three dimensional object" (p. 1098). These individuals were then asked to determine whether a third, presented orthographic image was compatible with the first two. In a later image recognition phase of these experiments, subjects were presented with two isometric images and a previous orthographic set. Individuals were asked to select the isometric image that coordinated with the previous orthographic set. Given the considerable precision (76.5% correct) with which individuals were able to correctly identify the compatibility of orthographic images as well as the high level of accuracy (85.5% correct) in the discrimination between target and distracter isometric images suggest some manner of dimensionality in the mental representation. Cooper elaborated:

> Two central features of the results are consistent with the notion that solving spatial problems based on orthographic projections of objects is accomplished by construction of an internal representation preserving properties of three-dimensional structure. First, the

> generally high level of recognition for previously unseen isometric views of objects supports this conclusion. This result demonstrates that—whatever the form of spatial information retained from problem solving—that information permitted successful recognition of object-like views of three-dimensional structures, even though the problems themselves depicted only separated, flat views of individual sides of visual objects. More importantly, the significantly superior later recognition of isometrics based on problems that had been solved *correctly* over those solved *incorrectly,* provides strong evidence for the claim that the solution process was mediated by a mental model of three-dimensional structure, rather than by the orthographic views presented in the problems (p. 1100).

While this suggests that individuals likely craft isometric images rather than string together an orthographic slideshow, one could fairly wonder at this point, why? Cooper pointed to the tremendous cognitive economy of such a representation along with the relative fragility that would accompany strictly orthographic images as one potential answer. Regardless of reason, though, it appears that individuals will attempt to attach isometric dimensionality to an image when possible and appropriate.

Difficulties with dimensionality. While Cooper's (1990) work with isometric and orthographic images implies that adults may have a proclivity towards dimensional imagery production, additional research suggests that this tendency may not yet be present in children and adolescents. Furthermore, reproducing dimensional images may be problematic for these individuals. Deregowski and Bentley (1987) provided some insight into the matter with their study of 'impossible' isometric images and the construction of physical models to represent those images. Among a group of school-age South African youth (ages 7-17), subjects were presented briefly with a dimensionally ambiguous figure and asked to reconstruct this image using plasticine and bamboo (see Figure 8). This initial task was aimed at discovering whether or not children and adolescents would construct such a figure in an orthographic or isometric manner, and, presumably, in the absence of a perceptual stimulus, individuals would rely on some degree of mental image to assist with this construction.

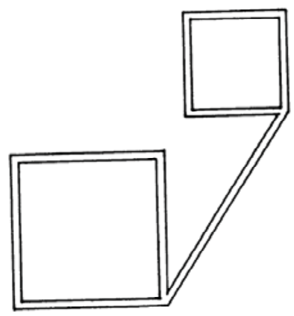

Figure 8: Deregowski and Bentley's (1987) construction task drawing

These physical models were judged to be either two-dimensional or three-dimensional depending upon whether or not the construction was confined to a single plane, and subdivided within these categories according to image integration. Deregowski and Bentley explained:

> The properly integrated 3D models were those which had two squares in clearly parallel planes, connected by a single splint running between the corresponding corners. All other 3D models were judged to lack proper integration. Of the 2D models, those which had co-planar squares connected by a single splint running between the corresponding corners were judged to be well integrated; all other 2D models were judged to lack integration (see Figure 9)(p. 92).

Immediately apparent to the researchers were the clear differences in the manner in which the drawing was perceived, mentally represented, and subsequently constructed. Among the tested subjects, 59% constructed a 3D model of some type while 41% produced a 2D model. While it is unfortunate that these results were not disaggregated by age or gender, this clearly indicates that three-dimensional facility with isometric imagery is not a certainty.

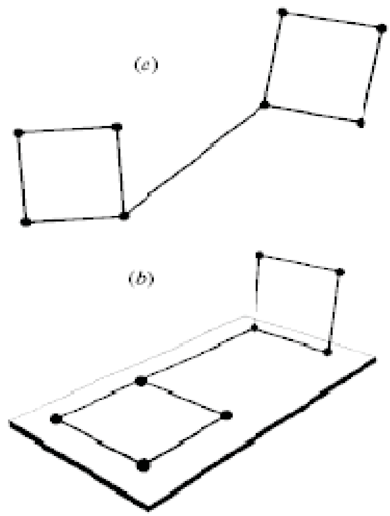

Figure 9: Deregowski and Bentley's examples of a nonintegrated

2D response (a) and a nonintegrated 3D response (b)

Dergowski and Bentley (1987) also conducted a second experiment where subjects, after some acclimation tasks, were presented with a depiction of a possible (3-prong) or impossible (2-prong) trident (see Figure 10) and then asked to draw each figure. Indeed, individuals could look at a particular figure as long and as often as he/she liked but were not allowed to draw during these times.

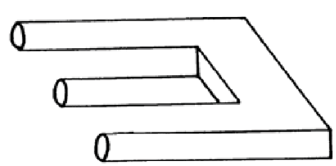

Figure 10: Deregowski and Bentley's (1987) impossible trident

When examining the amount of time that subjects needed to examine the figures as they produced a drawing, the author's found that individuals who produced 2D responses and non-integrated 3D responses in the previous experiment needed less viewing time that those that produced integrated 3D models. Deregowski and Bentley concluded that this difference is a function of two factors: "the ability to perceive pictorial depth of the elements of a figure and the ability to construct a correct mental representation of the entire figure" (p. 96). Quite simply, the dimensional impossibility of the figure appeared to be lost on a significant portion of children and adolescents. Although, individuals appear to gravitate towards isometric mental imagery (Cooper, 1990), this evidence suggests that educators cannot assume that children and adolescents possess facility in the generation of accurate imagery.

Accuracy of the image. The question of accuracy is an interesting one. Specifically, if figural re-presentations are ultimately about functionality, then does it matter whether or not the re-presentation looks exactly like the tool from previous experiences so long as the integral quantity is produced? Indeed, the idea that mental images necessarily and accurately resemble the stimuli upon which they are based has come under attack in recent decades on the grounds that such images do not seem to hold the symmetrical features of true resemblance (Fodor, 1975; Thomas, 2008). If we are to accept such a notion, then would not any discussion of mathematical tool appearance be rendered moot? I answer this question with a resounding, no. If the functionality of a figural re-presentation hinges on the successful perception and subsequent reimaging of some integral unit quantity, then it is absolutely imperative that the appearance of the mathematical tool facilitate this apprehension.

While Deregowski and Bentley (1987) illustrated that imaging accuracy may prove problematic, Battista and Clements' (1996) examination of how children interacted with 3D cube arrays illustrates how inaccurate imagery production could confound the child's apprehension of some integral image quantity. When 123 third and fifth grade students engaged in many diverse tasks involving the perception and representation of different cube arrays (presented isometrically), the authors discovered a progression of interaction with these dimensional materials beginning with "students [that] initially conceive of a 3-D rectangular array of cubes as an uncoordinated set of faces" (Battista & Clements, p. 290). In this context, students at the early stages of this progression had difficulty establishing the numerosity of different cube arrays because they were unable to apprehend the integral unit quantity of a cube.

Given the paucity of study in this area, I have presented but a few key investigations of isometric stimuli which may faintly illuminate the issue. From this research, we may glean the following points: 1) individuals appear to have some proclivity towards dimensional imagery; 2) children and adolescents may perceive and image the same tool with different degrees of accuracy; 3) these variances in perception and reimaging could have real impact on the functionality of an image. Consider the ubiquitous unifix cube. Often, these cubes are depicted isometrically on a workbook page or computer screen with the assumption that such pictures will be perceived identically to their physical counterparts. If such a tool is then appropriated for a particular screened task, a child may attempt and succeed at dimensionally re-presenting the tool in some manner; however, difficulties with the integration of different dimensional components of the image may hinder the apprehension of the fundamental quantity necessary to perform a counting task. A child that fails to coordinate the faces of isometrically presented cubes could attend to the multiple faces, rather than the individual cubes, in the act of counting. In such instances, non-isometric tools would appear to have a clear advantage.

Here, it is important to note that this review has dealt exclusively with isometric depictions of dimensionality. To my knowledge, there is virtually no understanding of the effects of true 3D or 2D mathematical tools on the mental imagery production of young children. Given the real and potential obstacles of isometric interactions, it appears that further study regarding the dimensionality of mathematical tools is, indeed, warranted if we are to better understand which materials will best support the struggling child. For this particular study, though, it bears repeating that studies of isometric tool presentation suggest a real possibility that tool appearance may either enhance or constrain the functionality of a particular child's mental image. Quite simply, the look of the tool, even when screened, matters.

Summary

In this exposition of mathematical tools, we see that these implements may be considered on their varying potential for adoption as models for thinking (Fosnot & Dolk, 2001). Here, I am most concerned with certain aspects of the potentially transformative counting tools and how these aspects may impact a child's mathematical activity. Specifically, literature related to tool dimensionality provides insight into the manner in which certain tools may influence the construction of imagery. Here, perhaps the evidence related to isometric materials suggests that educators employ only two-dimensional or nominally three-dimensional tools (e.g. counters) to promote mental imagery and avoid potential re-presentational pitfalls. Conversely, maybe this evidence suggests that greater impact is to be found in truly three-dimensional tools (e.g. a physical rekenrek). The point here is that without further study into the nature of imagery across a range of mathematical materials, tool selection in this aspect of intervention teaching will continue to be dominated by guesswork; moreover, such study will almost necessarily involve the manner in which student's describe and reveal (via mathematical practices) their imagery constructions.

Chapter Summary

Perhaps the most deserving points for emphasis in this chapter deal with the importance of the mental image in the mathematical activity of children, and that our understanding of how children construct and use quantitative imagery in these practices is woefully inadequate. More specifically, this knowledge rift impacts struggling students through the necessarily imprecise selection of mathematical tools. Towards this end, perhaps it is useful to revisit a few key components examined throughout the chapter.

The progression (SEAL) put forth by Steffe et al. (1983, 1988) provides a rich context for understanding how children consider quantity via their counting practices and provides a detailed explication of emergent mathematical imagery. Looking closely at individuals engaging in figurative counts, we see the first instance of willingness to step away from perceptual materials and begin to internalize their mathematical practices and the catalyst for such activity is the formation of robust and functional re-presentations. Indeed, at this stage, children rely on mental rather than material objects for mathematical scrutiny. While there exists some lingering disagreement regarding the nature and structure of the mental image, there is considerable evidence to support re-presentations that are depictive (rather than descriptive) of previous perceptual experiences. This has tremendous implications for intervention teachers as the appearance of mathematical tools now becomes a factor in the facilitation of mathematical practices involving mental imagery. Without explicit guidance in the area of tool selection for children moving from perceptual to figurative counting, educators have relied on personal experience and intuition to make instructional choices. Certainly, the considerable successes of mathematics intervention teachers in this area (KCM, 2008) are a testament to the prodigious expertise of this community; but, there remains much ambiguity surrounding the nature of the child's quantitative mental image and the relationship between this image and the mathematical tools, tasks, and teaching practices enacted by the intervention specialist.

Chapter Three

Methodology

"It's a subject we rarely mention. But when we do, we have this little invention, by pretending they're a different world from me."

~Sting: One World (Not Three)

"Workin on a mystery, going wherever it leads . . ."

~Tom Petty: Runnin Down a Dream

Chapter Introduction

Investigating the quantitative mental imagery of children is not a trivial task. Given the inherently veiled nature of this cognitive activity, it is important that decisions related to appropriated methodologies and aspects of study design reflect the complexity of examining this mathematical conception. In this chapter, I present my worldview for conducting this research as well as a methodological framework for this dissertation study. I conclude this chapter with description of a preliminary inquiry into quantitative mental imagery as well as specific details regarding the methods of this study. Prior to launching into an examination of worldviews, perhaps the specific research questions associated with this study bear repeating.

Research Questions

1) How do children in the early elementary grades (ages 5-8) describe and reveal through activity, the mental images that they invoke when working with different arithmetic tasks and tools?
2) How do these descriptions and revelations change over time?

Research Worldviews

Throughout any educational enterprise, it is important to distinguish between the mathematics that is taught or presented and the mathematics that is constructed by the student. Such an assertion likely seems quite sensible to those teachers (myself included) that have experienced teaching events that appear decoupled from learning. Indeed, the occurrence of novel counting strategies (Steffe, von Glasersfeld, Richards, & Cobb, 1983), and invented algorithms (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998) are but two examples of an autonomous students' mathematics. For those that engage in study of mathematical conceptions, the consideration of research worldview is of great importance as it is the worldview of the researcher(s) that lends definition to the strategy, tactics, and interpretations of a study. Indeed, an individual necessarily views all research activity through a particular worldview.

The Influence of Worldviews

While there exist, undoubtedly myriad, nuanced perspectives that influence methodological decisions, the notion of pervasive worldviews, sometimes referred to as 'paradigms' (Moschkovich & Brenner, 2000), that may both guide and describe entire bodies of scholarship is a useful construct for this dissertation and is well characterized by Whorf (1956) in his study of time conceptualization among the Hopi tribe of Native Americans, Whorf remarked:

> [T]he world is presented in a kaleidoscopic flux of impressions which has to be organized by our minds...We cut nature up, organize it into concepts, and ascribe significances as we do largely because we are parties to an agreement to organize it in this way. . . the agreement is, of course, an implicit and unstated one, BUT ITS TERMS ARE ABSOLUTELY OBLIGATORY. . . This fact is very significant for modern science; for it means that no individual is free to describe nature with absolute impartiality but is constrained by modes of interpretation even while he thinks himself most free. (p. 213-214).

From this assertion, we may begin to understand how the worldview to which one subscribes will influence strategic and tactical research choices, even in domains long associated with or aspiring to objectivity including educational research and mathematics itself.

Environmentalism and Constructivism

Within this discussion of worldviews, the divergent perspectives of *environmentalism* and *constructivism* emerge as dominant traditions of historic and present operation (Thompson, 1979, 1982). Confusing as it may be, constructivism, in this instance, refers to a global research outlook rather than a distinct assemblage of tenets associated with the psychological perspective (see Chapter Two). Like any worldview, there exist many gradations; however, there are some fundamental commonalities to each position. Environmentalism, in general terms, may be ascribed to the positivist traditions of the controlled experiment. Grounded in what Steffe and Thompson (2000) term the "agricultural paradigm" (p. 271), environments are controlled and effects are measured with the aim of discerning differences among treatments. Conversely, the primary aim of the constructivist researcher is to look at the world through the eyes of another. Contrary to environmentalism, such a position privileges the constructed problem space of the participant over the controlled environment of the researcher. Such a fundamental difference in the locus of mathematics and of truth in general has the potential to produce considerable disagreement. Consider Thompson's (1982) exchange with a colleague:

> I recently discussed with a prominent problem solving researcher a taped interview of a child solving a missing addend problem, and commented that his behavior led me to believe that it was an ill-structured problem. The other party couldn't understand how I could say that, since a missing addend problem is certainly well-structured. I remarked that from his perspective it may be, but the problem that this child was solving seemed to

> be ill-structured. The discussion soon turned to epistemology: his argument was that if we don't maintain the objectivity of problems (meaning, I assume, that a problem has an independent ontological status), then we lose all hope of being scientific— we've nothing of interest left to control. My argument was that (1) it is tremendously egocentric for anyone to think that his vision is so clear, and (2) that there are tremendous epistemological difficulties inherent in the position that problems are objective—if we were to take it seriously, then I would see little hope of ever explaining how individual children ever come to see what the problem "is". Having located the source of disagreement, we saw the fruitlessness of the debate and agreed to disagree. Had we not uncovered this fundamental divergence of views, we would have continued speaking past one another. (p. 155).

From this conversation, one may gather a sense of the deep divide between the environmental and constructivist positions. Indeed, where the environmentalist sees truth in the form of treatment and effect, the constructivist questions the relationship among input and output. Conversely, constructivist models that are created by "interloper observers" (Thompson, 1982, p. 159) decimate the level of variable control required for validity by the environmentalist. With each of these worldviews come specific affordances and constraints in terms of the methods that researchers deem appropriate for the pursuit of truth.

Affordances and Constraints of the Environmentalist Worldview

As one might suspect, for any inquiry into the mind of a child, the subscribers to this worldview privilege the environment surrounding the subject. Specifically, the environmentalist researcher is concerned with varying factors external to the child and measuring the effects of these actions to make some determination about the nature of reality. Thompson (1979) described this position thusly:

> The treatments themselves arise from the researcher's logical analysis of the students' environment -- that is, from the researcher's attempt to make sense of the various possible factors in the students' environment that, through manipulation, might cause a corresponding variation in the target population. The manipulation of the variables through treatments is apparently an attempt by the researcher to observe the effects of various environments upon the students' behavior (p. 2).

From this position, one may study the effects of any variable that might be controlled. Examples of such work run the gamut from measuring the impact of technology provided scaffolds (Davis, 2003; Rittle-Johnson & Koedinger, 2005) to the effectiveness of specific features in teacher education programs (Quinn, 1997; Timmerman, 2004). Quite simply, effect measurement through the variable control provides an expansive landscape for the environmentalist; however, the validity of such measurement is based upon two distinct assumptions: "[t]he predominance of the students' environment as a determiner of their behavior, and . . . that the students' behavior is structurally determined by the structure of their environment" (Thompson, 1979, p. 1). The obvious constraint with this position is the potential for a methodological unraveling should this lineal cause and effect relationship be disrupted. To encounter evidence contradictory to these assumptions would likely imperil conclusions drawn from any study informed by the environmentalist worldview as constructs such as constant stimuli and effect size lose their meaning with respect to the learning of a child.

Towards a Constructivist Worldview

With an initial condition that the environment is a unique construction of the individual, the constructivist seeks to ascertain the understanding of another with respect to certain experiences “without assuming a God’s eye view of those experiences” (Steffe & Thompson, 2000, p. 273). Quite simply, the constructivist acknowledges “the inaccessibility (to us) of the child’s environment as seen from the child’s point of view, while at the same time acknowledging that we have no grounds for assuming it is the same as one’s own” (Thompson, 1979, p. 6). On the surface, such a position looks hopeless for any study related to the learning and development of children as the researcher remains forever separated from the subject; however, the constructivist maintains that model building allows the scholar to hypothesize what mathematics looks like through the eyes of a child. Generally, the term model refers to a “conceptual system held by the modeler which provides an explanation of the phenomenon of interest, in this case a student's behavior within some portion of mathematics” (Thompson, 1982, p. 160). More specifically, though, models allow the constructivist to surmount the hurdle of autonomy by leveraging tenets of information theory (MacKay, 1969) including the creation of cognitive maps to hypothesize the structure of another’s cognition through the medium of interaction. Quite simply, to the extent that we consider children as thinking individuals, we may create models of their thinking by interacting with them. Teaching, then, may be characterized as attempts to enact a dynamic version of the hypothesized cognitive map of the student *with the student*.

Of course, there is a significant hazard that the hypothesized map may be invalid (or unconnected, if you will) with respect to the actual cognition of the student, and therein lies the primary constraint of the constructivist worldview; however, this potential for invalid hypotheses is mitigated by iterative researcher processes of reflective abstraction (Piaget, 1952, 1970) and retrospective analysis (Cobb, 2000). Indeed, these models are not capricious constructions, but rather products of intense and iterated consideration. Even in light of such activity, though, some threat to model validity remains. The constructivist argues, however, that this is an acceptable risk, for the creation of models (borne of hypothesized cognitive maps) is the only viable avenue towards describing the understanding of another.

Summary

Of utmost importance to any research endeavor is the identification of the particular worldview from which the activity is founded. Here, we see a stark contrast between the controlled aspects of environmentalism and the constructivist emphasis on ill-structured problem-spaces that are created among researchers and participants. While each worldview presents certain affordances and constraints, it bears noting that the research of this dissertation is conducted through the lens of a constructivist worldview which undoubtedly influences every subsequent decision, interpretation, and conclusion.

Methodology

Having established a particular worldview in which this study is situated, we must now attend to the specific methods appropriated for this research and how these methods fit within a larger methodological framework. In this section, I will present the methodological framework for this study, namely the teaching experiment.

The Teaching Experiment

I find the teaching experiment highly congruent with a constructivist worldview given their mutual emphasis on students' mathematics and model building (Steffe & Thompson, 2000). As a rich methodological tradition and a specific method of inquiry, the teaching experiment is uniquely situated to enable the study of quantitative mental imagery located within a psychological perspective, and provides the primary mode of inquiry for the study at hand. Prior to discussing specific applications, however, I find it necessary to provide a history and overview of this unique methodology.

Origins. The teaching experiment has a long history in educational research. Originating in the Soviet Union as a methodology to more realistically examine pedagogy and problem solving (Wirszup & Kilpatrick, 1975-1978), applications of the teaching experiment began to emerge in the United States as early as 1970. Steffe and Thompson (2000) cite two distinct causes for this emergence. First, efforts to adapt learning models from other disciplines had proved less than effective for the study of mathematical development. Researchers “could not simply borrow models from the fields of genetic epistemology, philosophy, or psychology and use them with the expectation that they could be used to explain students’ mathematical learning” (Steffe & Thompson, p. 270). A methodology was needed for specific inquiries into the mathematics of students. The second, related cause dealt with the growing chasm between the practices of research and teaching as agriculturally rooted experimental methodologies (Campbell & Stanley, 1966) grew increasingly distant from the instructional landscape of the typical classroom. Moreover, these purely experimental methodologies, grounded in the manipulation and control of variables, “suppressed conceptual analysis in the conduct of research. . .[and] inhibited efforts to investigate students’ sense-making constructs” (Steffe & Thompson, p270-271). Hence, the teaching experiment has been adopted by many scholars over the past 40 years as a means to examine cognitive development.

Description. Although the teaching experiment is a continuously evolving methodology, several distinct characteristics emerge. Thompson (1979) provided a succinct list of typical features.

> 1) an orientation toward uncovering processes by which students learn school subject matter; (2) the longitudinal nature of the investigation; (3) intervention by the researcher into the students' learning processes; (4) the constant interaction between observations gathered up to the current point of the investigation and the planning of future activities in the investigation; and (5) data that is [*sic*] qualitative rather than quantitative; whenever quantitative data is [*sic*] gathered it is [*sic*] used mainly in a descriptive manner. (p. 1-2).

Regarding the longitudinal nature (item 2 above) and aims of these investigations, Cobb (2000a) remarked that "[e]xperiments of this type can vary in duration from a few weeks to an entire school year and typically have as one of their goals the development of instructional activities for students" (p. 308). Perhaps most interesting among these qualities, though, are the researcher/subject interactions and the manner in which these interactions influence subsequent activity (items 3 and 4 above). Quite simply, in the teaching experiment, the researcher aims to better understand the subject by conforming and dynamically reforming the inquiry around the individual through a series of interactions. Indeed, these dynamic interactions are primary to the teaching experiment and serve as drivers for this research endeavor. Steffe and Thompson (2000) elaborated:

> Teaching actions occur in a teaching experiment in the context of interacting with students. However, interaction is not taken as a given--learning how to interact with students is a central issue in any teaching experiment. The nuances of how to act and how

> to ask questions after being surprised are among, in our experience, the most central issues in conducting a teaching experiment. The researchers may have research hypotheses to test at the beginning of a teaching experiment, but even researchers experienced in teaching may not know well enough what progress students will make or know well enough their mathematical thinking and power of abstraction to formulate learning environments prior to teaching. Wholly unexpected possibilities may open up to the teacher-researcher in the course of the teaching experiment. (p. 277-278)

From this perspective, the interactions of the teaching experiment provide potential for discovery regarding the mathematical understanding of children. Indeed, the teaching experiment is primarily aimed at illuminating the understanding of others; however, it must be noted that such endeavors are social enterprises founded upon dynamic interactions between teacher and student.

Modeling. Mentioned earlier, the teaching experiment is aimed at the production of models, and practitioners of this methodology provide guidance on how such models should be constructed through a process of analysis termed retrospective interpretation. Retrospective analysis "activate the records of past experience with students and brings them into conscious awareness" to facilitate meaningful interpretation (Steffe & Thompson, 2000, p. 296). These interpretations may then be leveraged to construct a model for understanding the identified mathematical event. Thompson (1982) elaborated:

> One constructs a model just as any other conceptual system— by reflectively abstracting and relating operations which serve to connect experientially derived states. Here I am applying Piaget's notion of reflective abstraction to the researcher. As he or she watches a student ease through some problems and stumble over others, or successively ease and blunder through parts of a problem, the researcher asks himself "What can this person be

> thinking so that his actions make sense from his perspective? What organization does the student have in mind so that his actions seem, to him, to form a coherent pattern?" This is the ground floor of modeling a student's understanding. The researcher puts himself into the position of the student and attempts to examine the operations that he (the researcher) would need and the constraints he would have to operate under in order to (logically) behave as the student did (p. 160-161).

Indeed, what Thompson describes is a process of systematically leveraging collected data to assume the students' perspective for the purposes of model building. Given that quantitative mental images are cognitive constructions necessarily concealed from the observer, any methodology appropriated for the study of these constructions must attempt to view the world through eyes of the student.

Summary

The teaching experiment methodology is highly congruent with a constructivist worldview, and several distinct features arise which privilege this methodology for the study of quantitative mental imagery. Specifically, the manner in which the teaching experiment dynamically conforms and reforms to the student's mathematics is conducive to making determinations regarding mathematical conceptions. Additionally, the social nature of these experiments (enhanced by their inherent interactional longevity) provides additional context for the psychological inquiry. Because of these features, the teaching experiment methodology provides the primary framework for my research activities.

Preliminary Inquiry into Quantitative Mental Imagery

Having established a methodological location for my research activities, now I will turn features specific to the study at hand. In this section, I will describe exploratory teaching experiences and preliminary inquiry into the area of quantitative mental imagery.

Exploratory Teaching

As a staff member of the KCM, I often work with primary grades intervention students as a means to better support MITs' work within PMIP. Throughout 2007-2008, one particular first-grade student, William, demonstrated considerable difficulty with numerals in the range of 1-20; however, he was able to successfully add and subtract quantities well beyond that range through the use of figural and motor re-presentations. Given the student's facility with figurative counting strategies, I spent much time with him working with screened quantities as a means to develop a robust numerical composite. Although the primary purpose of this work was, obviously, to allow for the advancement of this student's understanding of quantity, there were also many opportunities for me to explore the phenomenon of figurative counting and re-presentations. With each passing lesson, working with screened counters became increasingly tedious for William, so I introduced new and novel materials such as seashells and rubber frogs. Through these many instances of figurative counting, I raised questions regarding the nature of mathematical tools and the manner in which these tools might influence one's re-presentations.

This work with a student can be thought of as exploratory teaching (Steffe & Thompson, 2000). As a component of a teaching experiment, exploratory teaching allows the researcher to become "thoroughly acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest" (Steffe &

Thompson, p. 274). Towards this end, my work with this student allowed me to develop a rich and contextual understanding of figurative counts and re-presentations prior to this investigation.

Pilot Study Parameters

The spring of 2009, I conducted a pilot study of children's quantitative mental imagery aimed at answering a single research question. Specifically, how do children describe and reveal through activity, the mental images they invoke when working with different tasks and tools?

Participants. Initial invitations to recruit participants for this study were sent to eight MITs in northern and central portions of Kentucky. MITs were asked to identify intervention students who provided evidence of a figurative counting strategy or a robust perceptual counting strategy (i.e. using perceptual replacements). Three MITs from rural elementary schools in the central portion of the state responded that they were working with a total of five students that fit these parameters. Additionally, these responding MITs were asked to provide their rationale for recommending a particular intervention student along with any relevant contextual information relating to the student's mathematical experiences. Interestingly, all of the responding MITs were in their first year of intervention teaching, although each had between 5-15 years of experience as a classroom teacher.

Student interviews. Although I have, thus far, emphasized the methodological power of the teaching experiment, data collection for this preliminary inquiry occurred during a single lesson, and is more in keeping with a clinical interview than a teaching experiment. While, "the clinical interview is aimed at understanding students' current knowledge, the teaching experiment is directed toward understanding the progress students make over extended periods" (Steffe & Thompson, p. 273). Given the preliminary nature of this study, I deemed the clinical interview sufficient for providing the initial lay of the land.

Structure and tactics. I conducted the student interviews in the context of a 30 minute, video-recorded Math Recovery intervention lesson. In each instance, the MIT worked with the student for the first portion of the lesson, and the researcher posed particular arithmetic tasks during the latter portion. This was advantageous as it allowed me to gain a contextualized sense of the students' mathematics and make more informed tactical interview decisions. Indeed, I chose to privilege flexibility in terms of task and tool selection in order to facilitate a more accessible construction space for each child. This necessarily resulted in a series of interview tasks that were not identical across the participants but conceptually similar (i.e. screened arithmetic tasks).

Tasks and tools. Students were presented with 5-10 addition or subtraction tasks involving either screened collections of different materials (e.g., counters, seashells, plastic bears) or numerals. Returning to my initial goal of better discerning the nature of students' counting imagery, diverse types of mathematical tools were used to determine the extent to which a figural re-presentation may be connected to the color or dimensionality of a particular mathematical tool. Numeral-based tasks were included based on their potential to be apprehended similarly to a screened collection. Indeed, numerals could be interpreted by the student as symbolic representations of a quantity; however, much like a screened collection, the perceptual nature of the quantity may be unavailable to the student. Tasks were tactically micro-adjusted within the interview to accommodate perceived needs for conceptual support. For example, a child that experienced considerable difficulty determining the numerosity of two collections of screened materials was then presented with partially screened tasks. At the conclusion of each task, I asked the students several questions aimed to evoke the nature of their counting strategies and/or re-presentations (e.g., "How did you work that out? What number did you start with? What were you thinking about when you were counting?) Responses to these questions, the video record of mathematical activity, along with selection and contextual information from the MITs provided the terrain for subsequent analysis and discussion.

Conclusions

Although the short duration of the clinical interview coupled with the low number of participants limits conclusive strength, analysis of student video yielded two distinct themes worthy of discussion: 1) the diversity and longevity of re-presentations among children; and, 2) the variable connection between a student's constructed image and the mathematical tool.

Re-presentational diversity and longevity. Dealing first with re-presentational diversity and longevity, as I observed the mathematical activity of the children involved in this study and analyzed their imagery descriptions, each individual presented a unique manner of re-presenting. Indeed, analysis of the mathematical activity of five children provided five distinctive constructions of counting imagery. Additionally, re-presentations appeared to linger well into the development of the initial number sequence. Students appeared to construct and work off of a numerical composite; however, each student also described some manner of imagery that facilitated her or his counting act. Here, it is important to note that re-presentations do not appear to be the exclusive province of the figurative stage. Although Steffe et al. (1983, 1988) strongly associate re-presentations with the figurative counting type, the observed re-presentational longevity of this preliminary study seems congruent with the construct of stage put forth by Steffe et al. (1988) which implies the satisfaction of four criteria (cited previously in Chapter Two): "a characteristic remains constant throughout a period of time, each stage incorporates the earlier stage, the stages form an invariant sequence, and each new stage involves a conceptual reorganization resulting from reflection and abstraction" (p. 7-8). From this perspective, figural re-presentations are *incorporated into* rather than *displaced by* subsequent conceptual development; moreover, subsequent inquiry has revealed that older children likely invoke figural re-presentations to negotiate multiplicative tasks (Heirdsfield, Cooper, Mulligan, & Irons, 1999). Indeed, these later re-presentations appear to involve managing images of numerical composites rather than individual unit items. For example, a child tasked with determining the numerosity of four separate, concealed groups of three objects might construct and count composite re-presentations (three, in this case) to determine the numerosity of the total collection. Such a count could involve the number sequence "three, six, nine, twelve" as well as the motor act of touching the screen of each group but once. Such findings suggest the need for children to become facile with re-presenting quantity, though it appears that there exist many different (likely individualized) pathways by which one may develop such facility.

Connection between image and tool. Turning now to observed connections between the imagery of students and the mathematical tools at hand, I anticipated the existence of such a connection given Steffe's (1992) assertion that re-presentations involve "visualization of the common sensory material used . . . more than once in the immediate past [which] replicates certain sensory characteristics . . . on the level of imaginary experience" (p. 86). This statement suggests that the experiences with tools (i.e., presenting and then screening a collection) immediately preceding the act of counting is formative to the creation of figural unit items and, indeed, one student demonstrated this type of connection between re-presentation and tool as his motor activity occurred in very close proximity to a given mathematical tool, and the student attempted to replicate spatial aspects of particular tools with figurative motor acts (i.e. touching a screen in a spatial pattern similar to that of the concealed objects beneath). Consider this exchange around a task (9+4) involving seashells.

RESEARCHER: "Nine shells under there." [places screen on top of collection of 9 seashells]

RESEARCHER: "And one, two, three, four." [places 4 shells to the right of the screened collection] "Four shells there, how many altogether?"

STUDENT 1: "How many are under here?" [touches screen]

RESEARCHER: [lifts screen to reveal 9 seashells]

STUDENT 1: [attempts to touch the seashells]

RESEARCHER: [replaces screen] "Nine shells under there."

STUDENT 1: [touches screen once, pauses and then 9 times in a spatial pattern indicative of the shells below with index finger, then touches the 4 unscreened seashells]

STUDENT 1: "Thirteen." [smiles]

RESEARCHER: "How did you know it was thirteen?"

STUDENT 1: "I don't know."

Here, student 1 produces motor acts that connect to the mathematical tool, namely the touching of the screen in a spatial pattern in a manner consistent with the concealed materials below. Indeed, this type of motor activity suggests that a rigid connection remains between the student and the mathematical tool.

Conversely, the imagery that other participants revealed and described was somewhat disconnected from the mathematical tools. These students did not need to work in close proximity to the mathematical tools and described imagery that was incongruent with the materials of the task. For example, one child was presented with screened (white) seashells and then numerals in a subsequent task. For both of these tasks, the student provided only very subtle evidence of motor activity, but purported to count using color patterns. Consider this brief exchange around a task (14+5) again involving concealed seashells:

RESEARCHER: “So fourteen.” [lifts screen to reveal collection of 14 seashells] “and five” [lifts screen to reveal collection of 5 seashells]

RESEARCHER: “How many altogether?” [waives hand over both screened collections]

STUDENT 2: (pause 10 seconds) [student shifts left and right in chair]

STUDENT 2: “Twenty.”

RESEARCHER: “Twenty? How did you figure that out?”

STUDENT 2: “Um, I counted how many over here” [motions at screened collection of 14 seashells] “and then how many over there.” [motions at collection of 5 seashells]

RESEARCHER: “Okay, so you counted how many were here. How many were here?” [lifts screen to reveal 14 seashells and then replaces screen]

STUDENT 2: “Fourteen.”

RESEARCHER: “Fourteen and then what did you do?”

STUDENT 2: “And I counted those.” [looks at screened collection of 5 seashells)

RESEARCHER: [lifts screen to reveal 5 seashells and then replaces screen] “Five more.”

RESEARCHER: “So what did that count sound like?”

STUDENT 2: “One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen.”

RESEARCHER: “Nineteen? How did you keep track of this side over here?” [points at screened collection of 5 seashells]

STUDENT 2: “I did it in a pattern”

RESEARCHER: “You did it in a pattern? What did that pattern look like?”

STUDENT 2: [looks downward] “Red, blue, red, blue, red.”

RESEARCHER: “Red, blue, red, blue, red. Were there seashells in that pattern or was it just the colors?”

STUDENT 2: “Um, colors.”

Here, student 2 was presented with screened (white) seashells (as well as numerals in a subsequent task) and, for both tasks, attached the descriptors ‘red’ and ‘blue’ to her re-presentations (“patterns” according to the student). Although there were perceptual items of these general colors available in the classroom at the time, student 2’s re-presentation, clearly, was unrelated to either the presented (screened) white seashells (or numerals). There seemed to be two possibilities regarding this disconnection. Either the student’s re-presentations were influenced by perceptually available objects in the classroom or they were constructed independent of perceptually available objects. In the first supposition, her re-presentations could be construed as connected to materials, just not the immediate tools of the task; however, the latter supposition necessarily separated student 2’s figural re-presentation from any of the classroom tools. Unlike the student in the first exchange, student 2 demonstrates a flexible connection between image and tool in that the quantity of the task materials was built into a re-presentation rooted in imagery from another source.

Summary

Exploratory intervention teaching conducted during the 2007-2008 school year, provided terrain in which I became experientially well acquainted with the manner in which re-presentations occur across a range of tools and tasks. Building on these experiences, I conducted a series of clinical interviews with several students who had very recently demonstrated either robust perceptual or figurative counting strategies. From these interviews, I found that re-presentations appear to be an individualized act and persist into subsequent counting types. Additionally, the connection between re-presentation and mathematical tool varies as some students demonstrated a more rigid connection while the activity of other students was more flexibly connected to the mathematical tools. Given these provocative findings, I concluded that a more substantial study of quantitative mental imagery was in order.

Dissertation Study Methods

Research Site

The site for this dissertation research (as well as prior exploratory teaching) was an elementary school (grades K-5) located in an impoverished urban section of a mid-western metropolitan area. The most recent data available for this school (Commonwealth of Kentucky, 2008) indicate that the average length of teaching experience was 14.1 years with 100% of teachers participating in content-focused professional development that year; moreover, 72% of teachers in the school possess a graduate degree (either M.A. or M.S.) Additionally, the attendance rate for this school was 96% and the retention rate (percentage of students repeating a grade) is 2.6%, both better than state averages of 94% and 3.0% respectively. Performance on a state-mandated measure of achievement indicated that 65% of students at this school attained a mathematics score in the top two categories, proficient or distinguished (compared to 70% statewide).

Participants

In the winter of 2010, I asked the MIT at the research site to identify up to five students, each qualifying for intervention based on standardized testing, but not yet receiving services, that she observed enacting either robust perceptual or figurative arithmetic strategies. The MIT identified two students (pseudonyms-Allan and James) who met this description. Additionally, I was presently serving as the intervention teacher for a third student (pseudonym-Amy) that had demonstrated robust perceptual strategies for a period of time prior to this investigation. Based on MIT recommendation and my experience, Allan, James, and Amy were asked to participate in a diagnostic interview aimed at determining capacities with number word sequences, numeral identification as well as counting and collections-based arithmetic strategies (Wright et al., 2000). The interview format is based on the Math Recovery Learning Framework in Number (mentioned in Chapter One) and results in some determination of predominant mathematical strategies. In terms of SEAL, such a determination is referred to as one's *arithmetic stage* while spatial/collections-based strategies (i.e., work with subitizing-type tasks) are referred to as *structuring level.* During these interviews, both Allan and James demonstrated apparent figurative activity (i.e., pointing at or touching imagined materials) thus confirming the MIT observations. It should be noted, though, that Allan, at times, also enacted arithmetic strategies indicative of the initial number sequence (i.e., counting-on from a numerical composite). Additionally, Amy continued to demonstrate strong perceptual strategies (i.e., use of perceptual replacements) throughout the assessment. Given that I would be serving as the daily intervention teacher for these students over a period of time, it was important to generate an initial profile of Allan, Amy, and James prior to the onset of the teaching experiments. While, undoubtedly, quantitative mental imagery is the focal point of this study, my work with these students could not neglect other aspects of mathematical practice, particularly those involving conventions (i.e., number word sequences, numeral identification). Although I had worked with Amy from the fall 2009, I also generated an initial study profile for her in order to establish some manner of research congruence with Allan and James. Specific information of each student's profile is found below (see Figure 11).

Intervention Teaching Experiments

The 1-1 intervention teaching experiments took place in addition to the participants' regular classroom mathematics experiences and over the course of four weeks from early April to early May of 2010. Unfortunately, scheduled school vacation (Spring Break) and personal travel prevented this experiment from occurring over contiguous weeks. Rather, intervention teaching activities occurred for spans of one-week, two-weeks, and one-week, with week-long breaks between each span. Additionally, James's family moved from the school district in mid-April truncating his participation in the teaching experiment. School functions also impeded upon intervention teaching on occasion. Ultimately, Allan participated in 16 intervention lessons, Amy participated in 14 intervention lessons, and James participated in 11 intervention lessons. The typical duration for a PMIP intervention lesson was 30 minutes including student transit, and the average lesson duration across the three participants was approximately 24 minutes (leaving roughly 6 minutes for travel between classrooms).

MATH RECOVERY LEARNING FRAMEWORK IN NUMBER INITIAL STUDENT PROFILE		
Student	**Arithmetic Stage**	**Structuring Level**
Allan	**STAGE 3-INITIAL NUMBER SEQUENCE:** Student uses counting-on rather than counting from "one" to solve addition or missing addend tasks (e.g. 6+x=9). The student may use a count-down-from strategy to solve a removed item task (e.g. 17-3 as 16,15,14-answer 14) but not count-down-to strategies to solve missing subtrahend tasks (e.g. 17-14 as 16,15,14-answer 3).	**LEVEL 0-EMERGENT:** The student can subitize only small quantities (up to 3) and relies on counting to quantify larger groups. The student builds finger patterns by raising fingers sequentially.
James	**STAGE 2: FIGURATIVE COUNTING:** Can count the items in a screened collection but counting typically included what adults might regard as redundant activity. For example, when presented with two screened collections, told how many in each collection and asked how many counters in all, the student will count from "one" instead of counting on.	**LEVEL 0-EMERGENT:** The student can subitize only small quantities (up to 3) and relies on counting to quantify larger groups. The student builds finger patterns by raising fingers sequentially.
Amy	**STAGE 1-PERCEPTUAL COUNTING:** Can count perceived items but not those in screened (i.e. concealed) collections [without identifying or generating perceptual replacements]. This may involve seeing, hearing, or feeling items.	**LEVEL 1-FACILE STRUCTURES TO 5:** The student can subitize regular spatial patterns to 6 and irregular spatial patterns to 5. The student can create finger patterns in the range of 1-5 by raising fingers simultaneously. The student is able to combine and partition numbers in the range of 1-5 without counting.

Figure 11: Math Recovery Initial Student Profiles-Arithmetic-Stage, Structuring-Level

Based on the initial Math Recovery profile for each student (and past experiences in the case of Amy) initial intervention lessons were created for each student that featured tasks aimed at facilitating development across the different aspects of number (i.e., quantitative, symbolic, and verbal); however, emphasis was placed on quantitative experiences given the aims of this investigation. Each intervention lesson was video-recorded and I spent time daily reviewing lesson outcomes as I planned the experiences for the next day. Indeed, this dynamic process of planning and enacting learning experiences is the hallmark of the intervention teaching

Student	Forward Number Word Sequence (FNWS)	Backward Number Word Sequence (BNWS)
Allan	**LEVEL 3: FACILE WITH FNWS UP TO TEN:** The student can produce the FNWS from "one" to "ten". The student can produce the number word just after a given number in the ranger of "one" to "ten" without dropping back. The student has difficulty producing the number word just after a given number word for numbers beyond "ten".	**LEVEL 3: FACILE WITH BNWS UP TO TEN:** The student can produce the BNWS from "ten" to "one". The student can produce the number word just before a given number word in the range of "one" to "ten" without dropping back. The student has difficulty producing the number word just before a number word for numbers beyond "ten".
James	**LEVEL 4: FACILE WITH FNWS UP TO THIRTY:** The student can produce the FNWS from "one" to "thirty". The student can produce the number word just after a given number in the ranger of "one" to "thirty" without dropping back. Students at this level may be able to produce the FNWS beyond "thirty".	**LEVEL 3: FACILE WITH BNWS UP TO TEN:** The student can produce the BNWS from "ten" to "one". The student can produce the number word just before a given number word in the range of "one" to "ten" without dropping back. The student has difficulty producing the number word just before a number word for numbers beyond "ten".
Amy	**LEVEL 3: FACILE WITH FNWS UP TO TEN:** The student can produce the FNWS from "one" to "ten". The student can produce the number word just after a given number in the ranger of "one" to "ten" without dropping back. The student has difficulty producing the number word just after a given number word for numbers beyond "ten"	**LEVEL 3: FACILE WITH BNWS UP TO TEN:** The student can produce the BNWS from "ten" to "one". The student can produce the number word just before a given number word in the range of "one" to "ten" without dropping back. The student has difficulty producing the number word just before a number word for numbers beyond "ten".
Student	**Numeral Identification**	
Allan	**LEVEL 2-NUMERALS TO 20:** Can identify numerals in the range "1" to "20"	
James	**LEVEL 3-NUMERALS TO 100:** Can identify one and two digit numerals	
Amy	**LEVEL 2-NUMERALS TO 20:** Can identify numerals in the range "1" to "20"	

Figure 12: Math Recovery Initial Student Profiles-FNWS, BNWS, NID

experiment, and it should be noted that these experiences emphasized individualized activities rather than a series of identical, control-type lessons across the participants. With that said, similar tasks and tools were often used with each of the students given their relative conceptual resemblance in terms of quantitative mental imagery.

Intervention Tasks and Tools

Given my aim to facilitate both quantitative development and students' capacity to work with mathematical conventions, I appropriated a wide range of mathematical tasks and tools for the design of intervention teaching experiences. In very general terms, mathematical tools could be considered as either convention-oriented or quantitatively-oriented (see Figure 13). Conventionally-oriented tools often featured numerals and students were tasked with identifying a particular number or putting a particular group of numbers in order. Additionally, these tools may be aimed at the promotion of number word sequences as sequences of numerals are presented as scaffolds for the production of such verbal sequences. At times, though, tasks aimed at the promotion of verbal sequences will not include a mathematical tool, but rather only a verbal prompt from the teacher (e.g., "What number comes before 23?").

Quantitative-oriented tools, on the other hand, often feature some manner of discrete or segmented-continuous (i.e., paperclip chains) units. It is worth noting that such tools do not necessarily exclude numerals as students may interact arithmetically with numeral cards or a number line in a quantitative sense. In any event, quantitative tasks for students of these particular profiles often involve the presentation and subsequent screening of materials (described in Chapter Two). Associated tasks range from drawing a briefly presented dot pattern

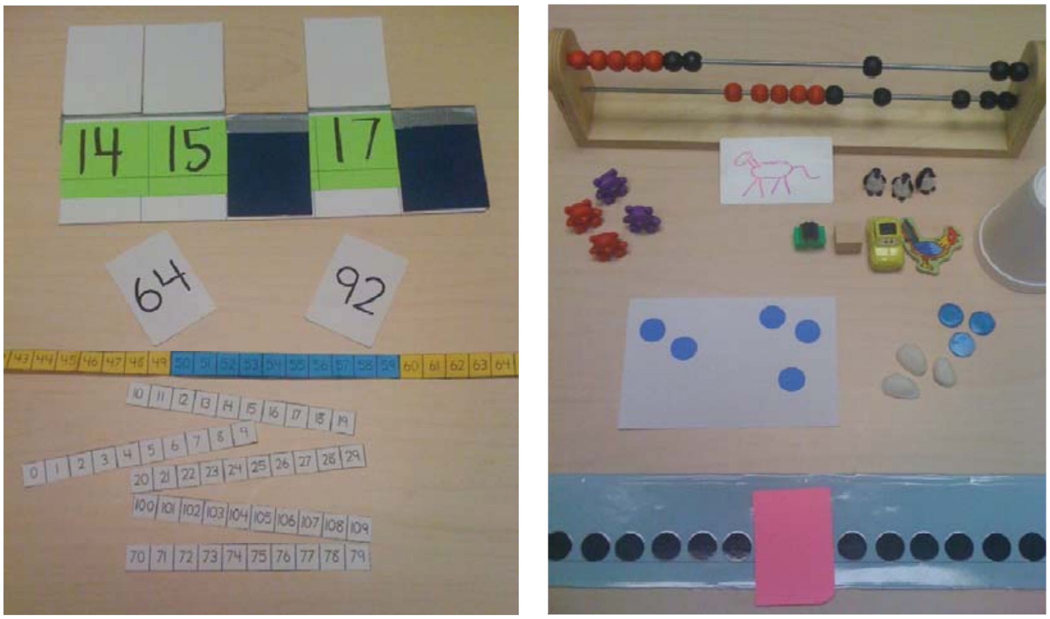

Figure 13: Convention-oriented Tools (left) and Quantitatively-oriented Tools (right)

to determining the total numerosity of two concealed collections of seashells. Additionally, I had the good fortune to collaborate with Steffe on a recent conference presentation (Steffe, Thomas, & Kinsey, 2010), and, as a result of this collaboration, tested several experimental tasks aimed at increasing re-presentational capacity. These experimental tasks included screened, linear arrangements of random items. Lastly, high-quality intervention experiences often (but not always) feature probing questions at the conclusion of particular tasks. Thomas et al. (in press) elaborate:

> As a general rule, it is often a good idea to ask students to describe or clarify their strategies by asking simple questions such as:
>
> *"How did you work that out?"*
>
> *"What where you doing with your fingers?"*
>
> *"What number did you start with?"*
>
> *"Do that out loud for me."*

> *"What were you saying in your head when you worked that out?"*
>
> It is important to note, however, that there may be a few occasions (when students exhibit overt strategic cues on their own accord or become visibly frustrated) when probing for strategies may actually prove counterproductive. Additionally, students may confound us by unwittingly describing a strategy different than the one that they actually used, and for this reason, we feel strongly that teachers must consider and reflect upon all of the data (verbal, quasi-verbal, and non-verbal) when making determinations of student understanding.

Taken as a whole, an individualized assemblage of convention-oriented and quantitatively-oriented tasks/tools along with the incorporation of strategic post-task probes comprises a comprehensive intervention experience and the video recording and reflection upon such experiences lends a dynamic power to subsequent intervention teaching and provides the mathematical landscape for the intervention teaching experiments of this investigation.

Analysis

Data analysis may be considered across two domains. The core domain related to psychological constructions provides the foundation for this inquiry into quantitative mental imagery, and the associated domain regarding the patterns of interaction within the intervention dyad lends context to the core domain.

Constructing a Model for Quantitative Mental Imagery

Recalling that the aim of the teaching experiment is the production of a model, two distinct phases of analysis were enacted with respect to the activities of the intervention teaching experiment. These phases are: 1) initial construction, 2) model testing and refining.

Phase 1: Initial construction. During this first phase of analysis, I reviewed all of the collected video-recordings and field notes from the intervention and classroom teaching experiments to identify manifest content. Here, manifest content is defined as the "math of interest or the science of interest or the literacy skill of interest, as that content is manifested in talk and perhaps also in written symbolization . . . [and to] the extent that subject matter content is manifested in physical objects, or gestures, include nonverbal as well as verbal phenomena in transcription and analysis" (Erickson, 2006, p. 187). Here, I looked for mathematical activity and verbalizations that provided *robust evidence* for the construction and use of imagined quantities hereafter referred to as figural occurrences.

Having performed the first iteration of identifying manifest content, I constructed a content database (see Figure 14) with hyperlinks to video clips from which I could review, sort, and organize student practices according to emerging themes (Glaser & Strauss, 1967). Specifically, I used this database to make retrospective interpretations of imagery-related mathematical activity. These interpretations were then leveraged to create an initial model for understanding the participants' construction of quantitative mental imagery. After this creation of an initial model, I returned to the intervention video recordings and classroom field notes to perform a second iteration of manifest content identification and this content was added to the database. I, then, examined all of the manifest content and made revisions to the themes established after the initial review as well as the preliminary model.

Phase 2: Model testing and refining. The second phase of data analysis consisted of testing the model. Here, I worked with another researcher, Dr. Shelly Harkness of the University of Cincinnati, to review the organization and themes of the content database with the aim of generating observations that would force a modification of the model. The aim here was to

A	B	C	D	E	F	G	H	I	J	K	L
Clip #	Date	Trnscrpt	Segment	random select 1	Imagery (Y/N)	Behavior Eliciting/Suggesting/ Replicating (E/S/R or N[none])	Student	Tool	Task	Connectivity (H/L/D/na)	Motor Ac
17-02-422C	422	0	44		N	N	CC	Bead Rack	7(6/1)	na	f-patterns
03-413C	413	1	5		N	R	CC	Bears/Shells	15comp9	na	
04-413J	413	1	8		N	N	JN	Counters	15-5	na	
22-505J	505	1	55		N	R	JN	Numeral Cards	9+6	na	F-Pattern
03-413J	413	1	7		N	R	JN	Numeral Cards	12+3	na	
12-420J	420	1	26		N	na	JN	POST-TASK Row Task	POST TASK-(12to16)	na	
05-415J	415	1	9		N	R	JN	Shells/Cups	3shells/cup-4cups	na	F-Pattern
01-330Q	330	1	1		N	R	QR	Counters	11+3	na	
01-401C	401	0	2		Y	N	CC	10-frame (pairwise)	7 (3 top 4 bottom)	D	none
26-503C	503	1	61		Y	N	CC	Bears/Shells	12comp5	D	f-patterns at end
02-415C	415	0	15		Y	S	CC	Dot Cards-array	3X3	D	taps table in sar
02-401C	401	1	3		Y	N	CC	Dot Cards-random	7	D	head movements
02-01-401C	413	0	4		Y	N	CC	Dot Cards-random	7	D	none
25-503C	503	1	60		Y	S	CC	Random Objects	5 spatial [quant-questions]	D	none
24-07-503C	503	0	59	X	Y	S	CC	Random Objects	4spatial (quant questions)	D	none
14-420C	420	0	31		Y	N	CC	Row Task	11to16	D	none
24-503C	503	1	52		Y	N	CC	Animal Cards	5cards [STACKED]	H	touches figura
18-422C	422	1	45		Y	N	CC	Animal Cards	4cards	H	touches figura
24-01-503C	503	0	53		Y	R	CC	Animal Cards	3cards[Stacked and then Row]	H	F-patterns/touches t
19-422C	422	1	46		Y	R	CC	Animal Cards	6cards [STACKED]	H	touches table next to
14-01-421C	421	0	34		Y	R	CC	Animal Cards	6cards	H	touches figura
01-421C	421	0	32	X	Y	R	CC	Animal Cards	4cards	H	air-points' towar

Figure 14: Screenshot of Content Database

identify and address features of the proposed model that were incongruent with the manifest content.

Given that the model was constructed on emergent themes, some manner of ensuring thematic stability was needed. Towards this end, I randomly sampled 20% of the figural occurrences and these video segments were reviewed by a colleague. We then co-reviewed these sample occurrences, negotiated our interpretations, and made minor refinements to the language of the emergent themes. Subsequently, I returned to the remaining figural occurrences and reassessed these exchanges according to the revised themes. Lastly, I randomly sampled 10% of the remaining figural occurrences and these video segments were, again, reviewed by the same colleague to establish a final degree of reliability. During process, we were able to negotiate agreement on all of the sampled video segments. Interestingly, we agreed often on the nature of the image, and the relatively small number of negotiations that occurred centered on the quality of interaction between teacher and student. Ultimately, though, this iterative review process resulted in reasonably stable themes which provided a basis for a refined model.

Finally, I examined the structural aspects (i.e., spatial presentation) of the constructed model in terms of their representative power with respect to the established themes and other aspects of the manifest content.

Having completed a rigorous process of model testing where interpretations that force refinements have been addressed, I may consider this model to be “temporarily viable” (Steffe & Thompson, 2000, p. 300). It is important to note, even if sustained testing never provided any instances for modification of the model, this would not indicate objective description of cognitive structures. Using a watch as a metaphor for the universe, Einstein and Inheld (1967) wrote:

> Physical concepts are free creations of the human mind, and are not, however it may seem, uniquely determined by the external world. In our endeavor to understand reality we are somewhat like a man trying to understand the mechanism of a closed watch. He sees the face and the moving hands, even hears its ticking, but has no way of opening the case. If he is ingenious he may form some pictures of a mechanism which could be responsible for all of the things that he observes, but he may never be quite sure his picture is the only one which could explain his observations. He will never be able to compare his picture with the real mechanism and he cannot even imagine the possibility or meaning of such a comparison (p. 31).

With that in mind, however, scientists continue to interpret our physical world; thus, it is no less scientific for educational researchers to "posit conceptual structures and mental operations and to investigate their fit with whatever seems relevant in the students' observable behavior" (Steffe & Thompson, p. 301). From this perspective, a temporarily viable model is both a powerfully descriptive and scientifically valid research product.

Analytical believing. The last item for discussion regarding this portion of the analytical process relates to the philosophy of believing (Elbow, 1986; Elbow, 2006; Harkness, 2008). I appropriated this philosophy in the model construction process. As opposed to analytical doubt which is steeped in critique, deficits, and disproving, analytical believing may be characterized as "an effort to find virtues and strengths no matter how unlikely an idea might seem to the listener or reader" (Harkness, 2009, p. 246). In practical terms, incorporating this philosophy of belief into my analysis compelled me to accept as truths *on some level*, student descriptions of imagery and re-presentations even in the occasional presence of conflicting student mathematical activity. This manner of analytical believing is typified by a subscription to Elbow's claim that "there is liable to be some rightness in what [a student] says or there's liable to be a sense in which it is right" (email correspondence, October 2006). Although in instances of conflicting data (e.g., verbal descriptions that are incongruous with observed activities), appropriating a philosophy of analytical believing may complicate model building and testing, I assert that appropriating this philosophy and working through such potential complications facilitates the construction of truly robust models. Indeed, models built directly upon the notion of inherent truth within each child's response allow us to more clearly understand something that the student is "particularly good at seeing but the rest of us are ill suited to see" (Elbow 1986, p. 259).

Patterns of Interactivity within the Dyad

In addition to enacting an iterative process of model construction, testing and refinement, I also examined the interactive patterns of the intervention classroom. Here, I returned to the content database and examined the interactions between the participants and myself. These patterns were organized according to emergent themes (Glaser & Strauss, 1967) and included in the model-testing phase conducted with a fellow researcher (described in the preceding section). The aim with this analysis is to provide some surrounding terrain from which the primary psychological model may be interpreted.

Chapter Summary

Throughout this chapter, I have endeavored to present a portrait of research design that honors the autonomous mathematical thinking of children. With the adoption of a constructivist worldview (as distinct from the constructivism of the psychological perspective), I assert that this dissertation research emphasizes the constructed problem space that exists between teacher and student rather than an environment of variables to be manipulated and controlled. Turning to appropriate methodologies for such a worldview, the teaching experiment emerges as a framework for viewing the world through the eyes of another; moreover, this methodology is easily located within the psychological

Having selected a methodological framework for this study, I examined a preliminary inquiry into quantitative mental imagery, which revealed some degree of re-presentational longevity and diversity as well as variable connectivity between the students' imagery and the mathematical tool. Building upon these conclusions, I designed teaching experiments that incorporated interactions within the mathematics intervention classrooms, and leveraged the data from these experiments to enact an iterative process of model construction. Additionally, I examined the interactive patterns within the intervention dyad. I invite you now to continue your walk with me as we learn about the quantitative mental imagery of children.

Chapter Four

Findings

"Can you see it? With my imagination, I can see it. With my pencil and crayons, I can draw it. On a piece of paper I can show it to you. I can show it to you."

~Barenaked Ladies: Drawing

"We look at each other wondering what the other is thinking."

~Dave Matthews Band: Ants Marching

Chapter Introduction

Throughout this dissertation, I have emphasized the veiled nature of the quantitative mental image. Turning our attention to the mathematical conceptions and practices of the participants, it is worth remembering that evidence for these conceptions comes in extraordinarily subtle forms. Indeed, even the smallest of gestures and utterances deserve scrutiny; moreover, attention must also be given to interactions that may have influenced the acts of the participants. In this chapter, I have organized my findings in a manner that moves from the individual to the community. In terms of a psychological perspective conjoined with interactionism, I will first present the participants' mathematical practices within the intervention teaching experiment, and provide some relation to individual mathematical conceptions. Also, I will portray certain normative aspects of the intervention teaching experiment.

Research Questions

Although these questions have been presented in preceding chapters, perhaps it is appropriate to list them, yet again, prior to launching into an exploration of findings.

1) How do children in the early elementary grades (ages 5-8) describe and reveal through activity, the mental images that they invoke when working with different arithmetic tasks and tools?
2) How do these descriptions and revelations change over time?

With these questions in mind, I may now set the stage for an examination of the students' work in the intervention setting. Here one finds a child and a teacher sitting at the table with a particular mathematical tool in front of them. Each individual is intently focused on the words and actions of the other. A task is posed and they both begin their work.

Manifest Content: Figural Occurrences

Mentioned in the previous chapter, considerable effort was expended to determine the manifest content appropriate for this study. Specifically, I aimed to identify robust evidence for figural occurrences within participants' mathematical practices. Here, figural occurrences were defined as activities or behaviors indicative of a sensory experience (i.e., touching, pointing, etc.) in the absence of corresponding sensory materials. Activities or behaviors of this type featured either an absence of mathematical materials or materials that *are not* sustained in the individual's perceptual field throughout the entire activity. Recalling that analog mental imagery is considered a quasi-perceptual experience (see Chapter Two), it makes sense that children would enact perceptual behaviors as they consider imaginary entities. It is also important to emphasize the role evidence plays in this process as students may, indeed, construct and act upon mental imagery without displaying any outward cues. Undoubtedly, this happened with some frequency throughout this inquiry; however, there were also many instances where participants did provide some outward indication of their thinking. This is the evidentiary basis for the nature of quantitative mental imagery.

Evidence for Imagery

Examining the words and actions of the participants within the intervention setting yields considerable evidence for the participants' construction of quantitative imagery. At this point I would like to call attention to the exchange prompt TEACHER used in this chapter. As opposed to the exchange prompt RESEARCHER of the preliminary clinical interviews (see Chapter Three), it is important to note that the relationship in this context is that of teacher and pupil rather than researcher and subject. Consider the following exchange with Allan.

TEACHER: “One, two, three, four, five cats” [counts out five animal cards (see Figure 15) and places them in a stack, *face-down*, in front of student]

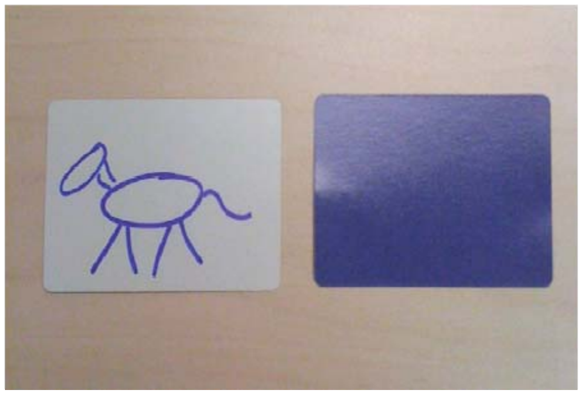

Figure 15: Animal Card Face-Up and Face-Down

TEACHER: “How many cat legs do I have?

ALLAN: “Two, four” [touches the top card two times with a pair of fingers in the approximate location of legs and then attempts to fan out the cards]

TEACHER: [places hand over stack] “See if you can do it in a pile like this” [restacks cards]

ALLAN: [touches the top card ten times in a rhythmic back-and-forth pattern with a pair of fingers in the approximate location of legs] “Two, four, six, eight, ten, twelve, thirteen, fourteen, sixteen, eighteen”

TEACHER: “You think there are eighteen cat legs?”

ALLAN: “Yeah”

TEACHER: [lays five cards out in a row, *face-down*]

ALLAN: [points at the first four cards twice with a pair of fingers in the approximate location of legs] “Two, four, six, eight, ten, twelve, thirteen, fourteen”

ALLAN: [starts over]

ALLAN: [points at each card twice with a pair of fingers in the approximate location of legs] “Two, four, six, eight, ten, twelve, fourteen, sixteen, eighteen, ***twenty***”

TEACHER: “Twenty cat legs”

Here, Allan interacted quantitatively with the materials; however, the materials themselves provided no sustained perceptual information regarding the task. Indeed, Allan appeared to count imaginary ‘cat legs’. Similarly, Amy also appeared to count imaginary items in a different task involving paperclips.

TEACHER: “Let’s say we have this chain of seven paperclips” [slides a chain of seven paperclips in front of student]

TEACHER: “And I wanted to make it eleven, how many more paperclips would I need?”

AMY: [touches each paperclip once] (audibly whispers) “One, two, three, four, five, six, seven”

AMY: [touches the table four times in a linear pattern at the end of the chain while sequentially raising four fingers] “Four more”

Again, the participant appeared to count imaginary items; however, in this instance, Amy also constructed a finger pattern to accompany her counting act.

Even in the shortened time that we worked together, James also provided similar evidence for figural occurrences on many occasions. Below is an exchange featuring screened dots on a row task (see Figure 16).

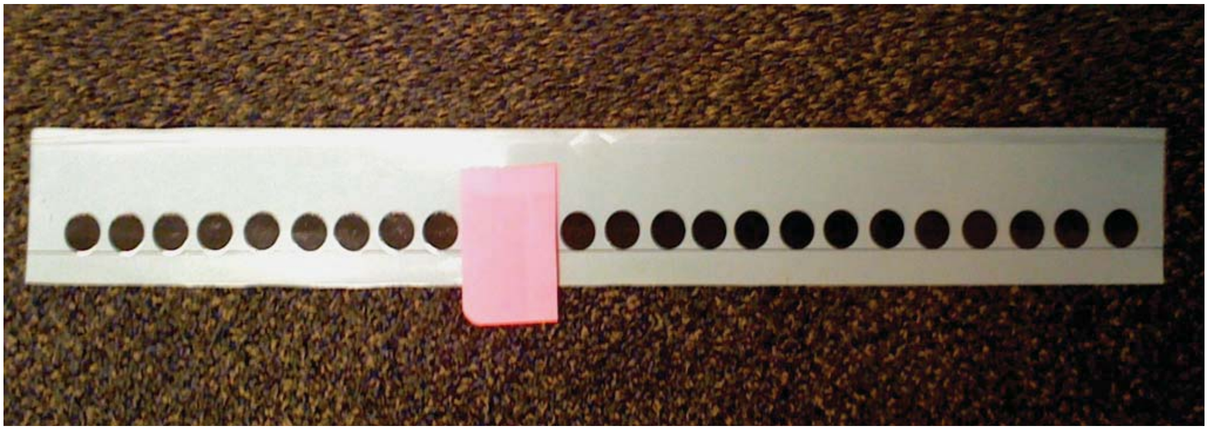

Figure 16: Row Task and Screen

TEACHER: [places translucent red counter over 13th dot]

TEACHER: "Now I have covered up four" [places screen over 14th, 15th, 16th, and 17th dot]

TEACHER: "What's this guy right here?" [places translucent red counter over 18th dot]

JAMES: [touches the first thirteen dots] (whispers audibly) "one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen"

TEACHER: "Thirteen" [touches counter on top of 13th dot] "and I covered up four" [slides screen slightly to reveal tops of four dots and then replaces screen]

TEACHER: "What's this guy?" [touches counter on top of 18th dot]

JAMES: [touches screen four times in a linear pattern] (whispers audibly) "fourteen, fifteen, sixteen, seventeen"

JAMES: "Eighteen" [touches counter covering 18th dot]

Here, the four linear touches of the screen coupled with an audible verbal sequence provided evidence of a figural occurrence. Similar to Allan and Amy, James appeared to count imaginary items. It is important to note that, thus far, I have presented exchanges that featured relatively overt physical cues suggestive of quantitative imagery. At times, though, the participants' behaviors during quantitative tasks were considerably more subtle. Consider the following exchange with Allan involving screened counters.

TEACHER: "eighteen" [places screen over a collection of 18 counters]

TEACHER: "I sneak some out" [Partially lifting the screen for concealment, slides 4 counters from underneath the screen and places an additional screen over this collection of 4 counters]

TEACHER: "Now there are fourteen" [lifts the original screen to reveal 14 counters and then replaces screen]

ALLAN: "fourteen"

TEACHER: "How many did I sneak out of here?"

ALLAN: [taps table three times with left hand and then simultaneously raises 4 fingers] "four?"

TEACHER: "How do you know it was four?"

ALLAN: "Because"

TEACHER: [lifts second screen to reveal four counters]

ALLAN: "one, two, three, *four*" [laughs]

In this exchange, Allan enacted a series of three taps (occurring at some distance from the tool) on the table and simultaneously raised four fingers. While not overtly indicative of imaginary items, these quantitative actions suggest some quasi-perceptual conception of quantity. Indeed, in previous instances, the children's imagery appeared rigidly connected to the mathematical tool (i.e., imaginary dots, paperclips), whereas in this instance, the image appeared somewhat flexibly connected to the tool.

Challenges with Indiscernible Imagery

During the iterative reviews of the intervention video recordings, there were also many instances of possible figurative activity that captured my attention; however, I was unable to identify sufficient evidence in favor of imagery.

Below is an exchange with Allan involving screened plastic bears and seashells, and in this task, he aimed to provide each little bear with a single seashell.

TEACHER: “We’ve got twelve little bears” [looks at screened collection of 12 bears in front of student]

TEACHER: “But we’ve got fifteen shells” [gathers 15 shells and holds them in two hands]

TEACHER: “How many extra shells are we going to have?”

ALLAN: “Twelve take away. . . fifteen take away twelve”

ALLAN: (pause 9 seconds) “Three” [taps the table with three fingers]

TEACHER: “How do you know three?”

ALLAN: “Because I was counting, because”

TEACHER: “Talk me through it”

ALLAN: “Fifteen take away twelve, to twelve I counted to cut in half” [raises hand in a chop-like fashion] “and three”

TEACHER: “So you took twelve and cut it in half?”

ALLAN: “Yeah”

TEACHER: “How do you mean?”

ALLAN: “Because twelve take away fifteen equals three”

TEACHER: “Twelve take away fifteen equals three?”

ALLAN: "Actually twelve" [begins to rock back and forth in chair]

TEACHER: "Which is it?"

ALLAN: "Fifteen take away twelve equals three" [continues rocking in chair]

TEACHER: "Fifteen take away twelve equals three? How do you know that?"

ALLAN: "I just know that one in my head"

In this instance, Allan considered the task and materials (absent of sustained sensory quantities) quantitatively; however, he provided no evidence as to the nature of this consideration. Note his generation of a three-fingered table tap occurred simultaneously with his production of the verbal response "three." This suggests that some manner of mental quantitative activity (perhaps involving an image) occurred prior to this production; however, lack of behavioral or verbal cues resulted in activity that is exceedingly difficult to describe in terms of imagery. Also, unfortunately, his descriptions did not provide any additional insight. Here, Allan described cutting a quantity in half (accentuated with his 'chop-like' motion) and described some counting activity; however, his conclusion that he just "knows that one in [his] head" confounds any conclusion regarding hypothetical imagery.

In a related exchange that involved a briefly presented dot-card, Amy appeared to grasp at some manner of quantitative conception. Consider the following.

TEACHER: [presents dot card with pattern of 6 dots (one diamond shaped group of 4 blue dots and diagonal pattern of 2 red dots) for one second and then conceals card]

AMY: "Four dots and I can't remember the . . .the"

TEACHER: [presents dot card again for one second and then conceals card]

AMY: "Two!"

TEACHER: "Two reds and?"

AMY: “Four blues”

Amy appeared to immediately apprehend the diamond shaped arrangement of blue dots and then struggled to construct the remaining portion of the pattern. Similar to the previous exchange, though, a lack of overt cues (either verbal or behavioral) left me guessing as to whether this activity truly involved an image. Again, imagery in this instance was certainly possible (perhaps even likely), but without behavioral or verbal evidence, I had little upon which to base a conclusion. Here, it was possible that a post-task question (e.g., “What were you thinking about just then?”) might have resulted in useful evidence.

James also demonstrated mathematical activity of this inscrutable type as well. Below is an exchange featuring an experimental task (Steffe, Thomas, & Kinsey, 2010) involving a concealed linear arrangement of seemingly random objects (see Figure 17).

Figure 17: Screen and Random Objects

TEACHER: “Alright, here we go” [places screen over linear arrangement of five objects (from left to right): rooster, penguin, block, bear, car]

TEACHER: “What’s over here on this end?” [touches far left-hand side of screen]

JAMES: “Rooster”

TEACHER: “A rooster, then what?” [touches screen to the right of previous touch]

JAMES: "A block"

TEACHER: "A block?

JAMES: "I mean a penguin"

TEACHER: "A rooster, a penguin, then what?" [touches the middle of the screen]

JAMES: "A block"

TEACHER: "A block, then what" [touches screen to the right of previous touch]

JAMES: [shrugs]

TEACHER: "A rooster, a penguin, a block" [touches the screen in a linear pattern from left to right] "then what?"

JAMES: "A car?"

TEACHER: "A car? Then what?"

JAMES: "A bear and then a car"

TEACHER: "A bear and then a car?"

TEACHER: [removes screen] "A rooster, a penguin, a block, a bear, and a car"

In this exchange, James engaged this task by producing verbalizations indicative of the materials beneath the screen. In particular, his revision of a "a bear and then a car" suggests that some attention was given to the materials; however, without evidentiary motor acts or verbal descriptions, I was unable to draw any real conclusion as to whether or not James was working from a constructed image or the nature of his image if, indeed, it existed.

Based on observations of these and similar practices, I categorized such occurrences as indiscernible as they appeared to ascribe to either category 1 or 2 below:

1) Student actions are suggestive of non-perceptual mathematical activity (i.e., whispered/mouthed number words, closing eyes tightly, etc.) and individual exhibits no discernable motor activity.

2) Motor activity (i.e., pointing, tapping, touching), if present, does not occur in close proximity to tool nor is it evocative (spatially or quantitatively) of the tool.

Although I made some initial attempts to construct a model that accounted for these indiscernible occurrences of suspected imagery, ultimately, the conceivability of alternate explanations that did not involve imagery (i.e., fact recall) required the dismissal of such occurrences from the manifest content of this study.

Counting-On Strategies and the Numerical Composite

On a few occasions, each of the participants enacted an additive strategy of counting-on from a particular addend suggestive of a numerical composite; however, both Amy and James provided some evidence that these counting-on strategies were simply mimicked procedures rather than indicative of a constructed numerical composite. Consider the following exchange with Amy that involved numeral cards.

TEACHER: “How about this one” [places three cards in front of student: (12) (+) (3)]

AMY: “I can’t do the hard ones”

TEACHER: “You think this one is hard?”

AMY: [flashes all 10 fingers simultaneously then sequentially raises three fingers] (audibly whispers) “Thirteen, fourteen, fifteen”

AMY: “Fifteen”

TEACHER: “Fifteen? How did you know it was fifteen?”

TEACHER: “So you did this” [raises two fingers sequentially]

AMY: “Wait, I made a mistake with the twelve”

TEACHER: “Okay, show me what you did”

AMY: “Thirteen, fourteen, fifteen” [sequentially raises three fingers]

TEACHER: “Okay, I agree. I think it’s fifteen”

TEACHER: “So you did thirteen, fourteen, fifteen?” [raises three fingers sequentially]

AMY: “uh huh” [nods] “and if you switch it around it will still be fifteen” [transposes the 12 and 3 cards]

TEACHER: “Oh, three plus twelve is fifteen?”

AMY: “My uncle always says do the biggest number and then go to a little number”

TEACHER: “What about this one” [places three cards in front of student: (4) (+) (17)]

AMY: “I need to still do the biggest number” [sequentially raises four fingers] (audibly whispers) “eighteen, nineteen, twenty, twenty-one”

AMY: “Twenty-one”

Recall that numeral cards may, at times, be appropriated by students for quantitative thinking in much the same manner as a screened collection; however, in this instance, Amy apparently recalled a specific procedure imparted to her by her uncle. Perhaps the presence of symbolic tools guided Amy towards a more procedural solution, but it is worth noting that, on a few occasions, Amy counted-on in the context of quantitative tools as well.

Of course, the above exchange certainly does not refute the construction of a numerical composite. For such refutation, an examination of an additional exchange that occurred later *in the same lesson* is necessary.

TEACHER: “Fifteen” [lifts screen in front of student to reveal collection of 15 blue counters and slides five counters to the right of the original collection] “and I am going to sneak out five” [places another screen over collection of five counters]

AMY: [simultaneously raises 10 fingers and then sequentially lowers five fingers] (audibly whispers) “Fourteen, thirteen, sixteen, seventeen, eighteen” [looks at other hand]

AMY: “Eighteen”

TEACHER: “So there are eighteen under here?” [taps screen covering collection of 10 counters]

AMY: “Yes, I am right”

TEACHER: [removes screen revealing collection of 10 counters]

AMY: [touching each counter] “One, two, three, four, five, six, seven, eight, nine, ten”

TEACHER: “What happened? So I had fifteen and then I snuck five out and how many were left?”

AMY: “Ten”

TEACHER: “Yeah, it gets smaller when I take them out”

AMY: “Sometimes it gets bigger”

TEACHER: “It can get bigger? So if I have a number and take some out, it can get bigger?”

AMY: “Yeah, sometimes”

TEACHER: “Show me one where it gets bigger” [slides a collection of chips over to student]

AMY: [counts out collection of 22 counters] “Twenty-two”

TEACHER: “Okay, I’ve got twenty-two and how many do I take out?”

AMY: “Two”

TEACHER: “If I take two out, then this is going to be bigger than twenty-two?” [places hand over collection of 22 counters]

AMY: “Yeah” [nods]

TEACHER: “Okay, I take two out” [removes two counters from the collection of twenty-two counters and places them to the right of the original collection]

TEACHER: “How many are left?”

AMY: [touches each counter once] “one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, *twenty*”

TEACHER: “Twenty, huh, smaller than twenty-two”

TEACHER: “I don’t know if there is ever one where you can take some away and it get bigger?”

AMY: “Oh! See, every time you put bigger and then it gets bigger and I get them right.”

AMY: “And every time it gets bigger and you take some out it gets smaller” [exaggerated motion of sliding hand across the table]

TEACHER: “Yeah, I think you are right there. If I add to it, it gets bigger, but if I take some away it gets smaller”

AMY: “Yeah, it gets smaller”

Having worked through that lengthy exchange, one sees how Amy began to develop a concept for subtraction as the task occurred. Recall that the hallmark of constructing a numerical composite is such that a construction includes component quantities (e.g., that ‘5’ is simultaneously a single unit ‘five’ and comprised of five ‘ones’) (Steffe et al., 1992; Wright et al., 2000). Indeed, for this reason, children that have constructed a numerical composite no longer need to attend to each unit item in an arithmetic situation. Similarly, these children also have some sense that quantities may contain other smaller quantities; thus, subtractive operations make conceptual sense (Wright et al., 2000). In the second exchange, Amy only began to realize the nature of subtraction; moreover, she continued to struggle with 'take-away' type subtractive tasks. Interestingly, Amy appeared to be somewhat aware of this difficulty with her remark that "every time you put bigger and then it gets bigger and I get them right."

Regarding the practices of James and Allan, both students, on occasion, demonstrated counting-on procedures in additive tasks, and similar to Amy, James experienced considerable difficulty with 'take-away' subtractive tasks suggesting either a nonexistent or fragilely constructed numerical composite. Allan, on the other hand, was considerably more comfortable with 'take-away' subtractive tasks. Returning to an exchange mentioned earlier which involved a relatively sophisticated missing subtrahend task involving screened counters.

> TEACHER: "eighteen" [places screen over a collection of 18 counters]
>
> TEACHER: "I sneak some out" [Partially lifting the screen for concealment, slides 4 counters from underneath the screen and places an additional screen over this collection of 4 counters]
>
> TEACHER: "Now there are fourteen" [lifts the original screen to reveal 14 counters and then replaces screen]
>
> ALLAN: "fourteen"
>
> TEACHER: "How many did I sneak out of here?"
>
> ALLAN: [taps table three times with left hand and then simultaneously raises 4 fingers] "four?"

TEACHER: “How do you know it was four?”

ALLAN: “Because”

TEACHER: [lifts second screen to reveal four counters]

ALLAN: “one, two, three, *four*” [laughs]

As addressed in a previous section, Allan’s mathematical activity was likely quantitative given the associated motor activity; moreover, he seemed to have little difficulty with the notion that 18 is comprised of smaller quantities. I must emphasize, though, that occurrences of counting-on and back (whether conceptual or procedural) were quite rare. Far more often, each child enacted figurative counting practices (i.e., attending to each unit item in the task) or, to a lesser degree, perceptual counting practices (i.e., counting sensory items).

Finalizing Manifest Content and Summary

Maintaining a standard of robust behavioral/motor/verbal evidence and removing indiscernible and perceptual counting exchanges considerably winnowed the data set of figural occurrences (see Figure 18); however, a sizable group of data remain for analysis. Here, a figural task group was defined as a group of tasks that often (but not always) involved the concealment or screening of mathematical tools/materials that are quantitative in nature. Examples include:

- Briefly presenting and then concealing a dot card and asking the student to draw or quantify the pattern
- Briefly presenting and concealing collections when presenting arithmetic tasks (i.e., how many altogether, how many more/less, etc.)
- Briefly presenting and concealing spatial arrangements of items and posing spatial or quantitative questions regarding the concealed arrangements

- Presenting the student with temporal patterns of items (i.e., dropping a succession of objects into a cup) and posing ordinal or quantitative questions regarding the items
- Presenting the student with arithmetic tasks involving numerals

At this point, I would like to emphasize that the aim here was not one of formal quantitative analysis; however, examining the quantities in a descriptive sense allows a more complete understanding of these figural occurrences as they happened in the intervention setting.

Student	Number of Figural Task Groups	Number of Figural Tasks	Number Observed Figural Occurrences	Percent of Figural Occurrences per Figural Tasks	Number of Figural Occurrences per Task Group
Allan	45	195	52	26.7%	1.16
Amy	45	174	49	28.2%	1.09
James	32	151	33	21.9%	1.03

Figure 18: Figural Occurrences by Student

Returning to the students' figural occurrences, one sees that this number of observed occurrences is a relatively small portion of the total number of figural tasks. This is not to say that students only constructed and used quantitative mental imagery in these instances, but rather, holding my analysis to a robust evidentiary standard necessarily reduced the number of viable occurrences. With that said, even this relatively small number of occurrences was sufficient to illustrate certain aspects of the participants' imagery as well as calling attention to different interactional modalities that likely influenced such imagery production. Let us now examine what is, perhaps, the most striking of these imagery aspects, the connection between image and tool.

Tool Connectivity

In prior sections, I described and presented exchanges and activities that did not contain sufficient evidence for analysis in terms of quantitative mental imagery. Turning attention to participant practices that did contain such evidence requires the consideration of two distinct types of imagery construction with respect to the mathematical tool: both rigidly and flexibly-connected images.

Rigidly-connected imagery defined. Regarding the operationalizing of these terms, rigidly-connected imagery meets *either* of the following criteria:

1) Motor activity occurs in close proximity to mathematical tool (i.e., touching the tool, touching near the tool, pointing at or near the tool), *and* motor activity is evocative of the spatial attributes of the tool (i.e., touching or pointing in a manner that spatially resembles the tool).

2) In instances where proximity is constrained by the task or task-presentation, motor activity is only evocative of the spatial (rather than quantitative) attributes of the tool.

Flexibly-connected imagery defined. Flexibly-connected imagery, on the other hand, is typified by *any of the following three* criteria:

1) Motor activity may or may not occur in close proximity to mathematical tool (i.e. touching the tool, touching near the tool, pointing at or near the tool), *and* motor activity is only loosely evocative or not evocative of the tool (i.e., finger patterns when counting a spatial dot pattern, touching, tapping, or scratching the side of a screen in the same location, spatial gestures that occur at some distance from the tool).

2) In instances where proximity is constrained by the task or task-presentation (i.e., flashed dot patterns), motor activity replicates the quantitative aspect of the tool and may or may not be spatially evocative of the tool.

3) Evidence of mathematical activity (i.e., whispered counting) without discernable motor activity *and* post-task motor activity/description that connects to or invokes the tool.

Contrasting these two definitions, the nature of the figural occurrence becomes clear. In terms of connectivity to particular mathematical tools, participants' quantitative imagery was observed to be either rigidly connected in the sense that the participant provided overt cues suggestive of the tool, or flexible in the sense that the participant provided cues indicative of a quasi-perceptual experience (but not necessarily suggestive of the presented tool). Additionally, the third criterion of flexibly-connected imagery accommodates occasions in which participants did not exhibit any behavioral/motor cues, but then, subsequently, described a mathematical process or image that was deemed connected to the tool.

Rigidly Connected to Tool

Allan. At times, each of the three children demonstrated rigidly-connected, quantitative mental imagery. I will now examine several exchanges from each of the students, beginning with Allan, that typify this rigid connection. Consider the following exchanges involving screened counters, paperclips, seashells, and cups.

Exchange one.

TEACHER: "We start with fifteen" [briefly raises left-hand screen to reveal fifteen blue counters and then replaces screen]

TEACHER: "And then we put five with it" [briefly raises right-hand screen to reveal five red counters and then replaces screen]

ALLAN: [places arm on left-hand screen] "Fifteen"

ALLAN: [touches right-hand screen five times in approximate spatial pattern as the concealed counters] "sixteen, seventeen, eighteen, nineteen, *twenty*"

Exchange two.

TEACHER: “I’ve only got two paperclips. How could I measure this piece of paper?” [touches 12” paper strip] “Is there any way I could do that?”

ALLAN: [places chain of two paperclips next to the left-most edge of the paper strip as if to measure]

ALLAN: [touching each of the two paperclips] “One, two”

ALLAN: [touches the table in linear pattern next to the paper strip] “three, four, five, six, seven, eight, nine” [the final touch occurs at the end of the 12” paper strip]

TEACHER: “Let me ask you this. While you were doing that. . . ” [touches the table in a linear pattern beneath the paper strip] “. . . how do you know that each one of those was the same as a paperclip?”

ALLAN: “Because, I just know”

TEACHER: “Wait, wait. I am not convinced”

ALLAN: [touching the paperclips and then the table next to the paper strip] “one tall, two tall, three tall, four tall, five tall, six tall, seven tall, eight tall, nine tall”

TEACHER: “Let’s check” [lays a 9-paperclip chain next to the paper strip. The paper strip extends halfway between the 8^{th} and 9^{th} paperclip]

ALLAN: [touches each paperclip] “one, two, three, four, five, six, seven, eight, nine”

TEACHER: “You are sure that it is all the way nine?”

ALLAN: “I think it’s eight”

Exchange three.

TEACHER: “We’ve got four cups with three in them, right?” [gathers four cups each containing three shells and nests them such that only one complete cup and the three lips of the nested cups are visible]

TEACHER: “How many shells altogether?”

ALLAN: “one shell, two shell, three shell” [hooks finger over the lip of the first cup] “four shell, five, six” [taps lip of the second cup three times in a spatial pattern] “seven, eight, nine” [taps lip of third cup three times in a spatial pattern] “ten, eleven, ***twelve***” [taps lip of fourth cup three times in a spatial pattern]

In exchanges one and two, Allan apparently counted imaginary unit items suggestive of the tool in question. Specifically, his motor activity occurred in close proximity to the tool and the spatial nature of this activity was suggestive of the tool (i.e., replicating the spatial pattern of the concealed counters). In the third exchange, Allan’s motor acts occurred in contact with the cups (close proximity), and the spatial manner in which he touched the lip of each cup was suggestive of the collection within (albeit on the rather linear surface of the cup lips).

Amy. Similarly, analyses of Amy’s figural occurrences provided many examples of imagery that appeared rigidly connected to the mathematical tool. Consider the following exchanges involving animal cards, screened random objects, and dot cards.

Exchange one.

TEACHER: “Six ponies out there, OK” [motions across row of six animal cards face-down in front of student]

TEACHER: “How many legs”

AMY: [touches the table next to each card] “one, two, three, four, five, six”

TEACHER: “That’s six ponies, how many pony legs?”

AMY: [raises two fingers simultaneously and quickly lowers them]

AMY: [touches each card four times in the approximate location of the legs] “One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, *twenty-four"*

TEACHER: "Twenty-four?"

AMY: "uh huh" [nods]

TEACHER: "What were you thinking about when you were counting those?" [touches one of the animal cards]

AMY: "Nothing"

Exchange two.

TEACHER: "Okay, ready" [places screen over linear arrangement of five objects (from left to right): rooster, car, frog, block, shell]

TEACHER: "What's on this end?" [taps the far right-hand side of the screen]

AMY: "Shell"

TEACHER: "Shell? What's on this end?" [taps the far left-hand side of the screen]

AMY: "A rooster"

TEACHER: "What's next to that rooster?" [taps screen to the right of the previous tap]

AMY: "Car"

TEACHER: "Over here you said it was a shell" [taps the far right-hand side of the screen]

TEACHER: "What's next to that shell?" [taps the screen to the left of the previous tap]

AMY: "A frog"

TEACHER: "A frog? What's in the middle?"

AMY: (pause 3 seconds) "I forgot"

TEACHER: "Okay, let's see if we can figure it out"

TEACHER: “What’s over here?” [taps far left-hand side of screen]

AMY: “Rooster” [taps screen to the left of teacher’s tap and then taps the screen two more times in a linear pattern moving left] “car, frog, shell. . . oh, the shell is right at the end”

TEACHER: “The shell is at the end? What’s between that frog and that shell?” [taps the middle of the screen and then the far right-hand side of the screen]

AMY: “A block!”

Exchange three.

TEACHER: “Let’s see if we can do this with one look, OK?”

AMY: “Yeah”

TEACHER: “Focus hard” [presents dot card with pattern of 7 dots (see Figure 19) for one second and then conceals card]

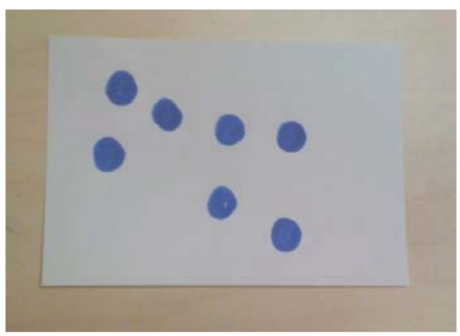

Figure 19: Unstructured 7 Dot Card

TEACHER: “Draw” [motions towards marker and note card in front of student]

AMY: [begins drawing dot pattern on note card]

AMY: [completes drawing that is spatially similar to presented pattern but contains 12 dots (see Figure 20)]

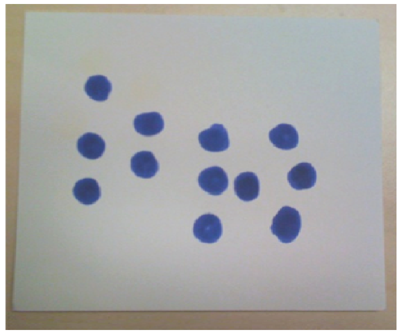

Figure 20: Amy's Drawing of 7 Dot Card (Recreated from Video Record)

TEACHER: "I am going to give you another look and you tell me if you need another card or not"

TEACHER: [presents dot card with unstructured pattern of 7 dots for one second and then conceals card]

AMY: [touches seven of the dots on her drawing] "Yes, I do. I've got too many"

TEACHER: "Okay, try again" [hands student a new blank note card]

AMY: "It was seven blues"

TEACHER: "It was seven blues? What did they look like?" [motions towards blank note card in front of student]

AMY: [student draws seven dots on note card in a spatially similar pattern to the presented card] "one, two, three, four, five, six, seven, *there*" [slides card towards teacher]

TEACHER: [places presented dot card next to student's drawing]

TEACHER: "Four up top and three at the bottom" [motions back and forth from presented card to student's drawing in an effort to emphasize spatial and quantitative similarity]

In exchanges one and two, Amy's motor activity occurred in close proximity to the tool and seemed to replicate spatial features of the concealed tool as though she was touching imaginary items. The third exchange, though, presented Amy with an interactive constraint in that the card was not present (even face-down) throughout the task. Rather, the card was briefly presented and Amy was asked, presumably, to draw the card as she 'saw' it. In this first iteration of the task Amy apparently apprehended the spatial nature of the dots and was able to replicate this nature in her initial drawing (i.e., downward sloping top row); however, this drawing failed to capture the quantitative aspect of the presented card. Indeed, in such instances where proximal motor acts were precluded, Amy appeared to work from a rigid image of the tool's shape and attention to the quantitative aspects were considered secondarily, if at all.

James. James also presented activities suggestive of a rigid connection between his constructed image and the mathematical tool. Below are example exchanges involving an array, shells, cups, and a dot card.

Exchange one.

TEACHER: "We've got a row across the top with three, right?" [places 3X3 array card (top row of three dots exposed and the bottom two rows screened) in front of student]

TEACHER: "And, there are two more rows just like this one" [motions towards row of 3 unscreened dots and then drags finger across dot card and screen in two linear motions]

TEACHER: "How many dots are on this card?"

JAMES: [touches each of the three unscreened dots]

JAMES: [touches the screen three times in a linear pattern]

JAMES: [touches the screen two times in a linear pattern beneath his previous touched pattern]

JAMES: [touches the screen once near the bottom]

JAMES: “Nine”

TEACHER: “Nine? What were you thinking about when you were pointing at those things?” [taps the screen]

JAMES: [touches each of the three unscreened dots] “One, two, three”

JAMES: [touches the screen three times in a linear pattern] “Four, five, six”

JAMES: [touches the screen two times in a linear pattern beneath his previous touched pattern] “Seven, eight”

JAMES: [touches the screen once near the bottom] “Nine” [raises hand in the air]

TEACHER: [replicates student’s spatial pattern of touches on the screen] “One, two, three, four, five, six, seven, eight. . .”

TEACHER: “Where is number nine? Is he down here?” [touches screen once near the bottom] “or, is he right here? [touches approximate location of 9th dot in the array]

JAMES: “Right there” [touches approximate location of 9th dot in the array]

Exchange two.

TEACHER: “Each one of the cups has four” [lifts a cup with printed dots on it containing four shells]

TEACHER: “How many altogether?” [motions across four cups in front of student (each containing four shells]

JAMES: [touches table beside first cup 4 times in spatial pattern] “One, two, three, four”

JAMES: [touches table beside second cup 4 times in spatial pattern] “Five, six, seven, eight”

JAMES: [touches table beside third cup 4 times in spatial pattern] “Nine, ten, eleven, twelve”

JAMES: [touches table beside fourth cup 3 times in spatial pattern] “Thirteen, fourteen, *fifteen***”**

TEACHER: “Fifteen?”

TEACHER: “OK, so there were one, two, three, four” [taps top of first cup with hand] “Talk me through this”

JAMES: [touches second cup four times] “Five, six, seven, eight”

JAMES: [touches third cup two times] “nine, ten”

JAMES: [touches fourth cup four times] “Eleven, twelve, thirteen, fourteen”

TEACHER: “How many does this one have?” [touches third cup]

JAMES: [touches four printed dots on side of first cup]

TEACHER: “Do these things tell you how many shells are in each cup?” [touches dots on side of first cup]

JAMES: “No”

JAMES: [touches the first, second, and third cup four times and the fourth cup three times] “Fifteen”

TEACHER: “Fifteen? Let’s check it.” [removes cups revealing four piles each containing four shells]

Exchange three.

TEACHER: [presents card with eight blue dots (structured in a dice patterns of 6 and 2) for one second and then conceals card]

JAMES: [makes eight pointing gestures toward card in approximately the same location] (whispers audibly) “One, two, three, four, five, six, seven, eight”

JAMES: (whispers) “eight”

TEACHER: (whispers) “eight”

JAMES: [begins drawing dot pattern on index card in front of him]

JAMES: [draws two columns of dots side by side, the first column with 4 dots and the second column with 5 dots]

JAMES: “There are some more.” [makes two quick pointing gestures]

TEACHER: “Oh, there are some more? So, how many are there?” [points at student’s drawing]

JAMES: [touches each of the drawn dots] (whispers audibly) “One, two, three, four, five, six, seven, eight, nine”

JAMES: [draws three more dots in a column on the other side of the card]

TEACHER: “Okay, so how many dots are on your card?”

JAMES: [points at each drawn dot] (whispers audibly) “One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve”

TEACHER: “Okay, so twelve dots. . . I am going to show it to you again and you tell me if you need a new card or not”

TEACHER: [presents dot card for one second and then conceals card]

JAMES: [makes eight pointing gestures in approximately the same location] (whispers audibly) “One, two, three, four, five, six, seven, eight”

JAMES: “Eight. . .I need a new card”

TEACHER: [places blank index card in front of student] “Okay, what did it look like?”

JAMES: [draws eight dots structured in 6 and 2 dice patterns]

Similar to the example exchanges involving Allan and Amy, James apparently counted imaginary items in close proximity to the screened tool. In the third exchange where this proximal operation was constrained, James apparently apprehended the spatial nature of the presented pattern (in the first iteration of the task). Interestingly, though, his whispered “eight” suggested some apprehension of the pattern’s quantity as well; however, this seemed to lose out as he constructed his drawing of 12 dots in a similar spatial arrangement as the presented card. Again, this was in keeping with a rigid image which emphasized shape.

Flexibly-Connected to Tool

Allan. During the intervention teaching experiments, each child also presented motor acts or verbalizations that appeared qualitatively different from those associated with a rigidly connected image. I deemed these actions indicative of a flexibly-connected image. Consider the following exchanges with Allan involving penguins, cups, a row task, and animal cards.

Exchange one.

TEACHER: “So, if I put four in this cup” [drops four penguins into cup (see Figure 21) and places hand over the top of the cup]

Figure 21: Penguins and Cup

TEACHER: “How many penguin feet are there?”

ALLAN: [taps table in a rhythmic, back-and-forth motion while sequentially raising eight fingers] “One, two, three, four, five, six, seven, ***eight”***

TEACHER: “Eight penguin feet? Sure . . . do you want to check?” [dumps penguins out onto table]

Exchange two.

TEACHER: “Sixteen” [places a blue translucent counter over the 16th dot] “I’m going to cover up five” [places screen over the next five dots (17th, 18th, 19th, 20th, 21st) and motions across the screen]

ALLAN: [points towards screen] “Seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two”

TEACHER: “This guy is twenty-two?” [touches the 22nd dot]

TEACHER: “So, this guy is 16, and I covered up five.” [touches 16th dot and motions across the screen]

TEACHER: “What’s this guy?” [touches 22nd dot]

ALLAN: (audibly whispers) “seventeen, eighteen”

ALLAN: “twenty-one” [touches 22nd dot] “twenty-two”

Exchange three.

TEACHER: “There is a cat, there’s a cat, and there’s a cat.” [places three animal cards *face-down* in a row in front of student]

TEACHER: “How many cat legs?” [motions across all three cards]

ALLAN: “Twelve” [glancing at the cards]

TEACHER: “Twelve? How do you know twelve?”

ALLAN: “Because, two, four, six, eight, ten, twelve” [touching each card with two fingers in the approximate location of the animal legs]

In the first two exchanges, Allan appeared to engage in some manner of quantitative thinking and presented some motor activity associated with this thinking (i.e., table taps, pointing); however, neither of these motor acts occurred in close proximity to the tools or was overtly suggestive of the concealed units. The student’s back-and-forth table taps during the ‘penguin task’ could be construed as counting imaginary penguin legs; except, that these taps occurred in the same repeated locations on the table. One working from a more rigidly-connected image would likely tap the table in close proximity to the cup in either a linear or spatial pattern suggestive of a line or group of penguins, respectively. Interestingly, Allan also raised eight fingers sequentially during this task; however, these motor acts were likely not aimed at perceptually replacing the penguin feet. Recall that collections of perceptual replacements are constructed prior to determining the numerosity of a collection (see Chapter Two); thus, Allan’s counting activity was almost certainly not perceptual in nature because his fingers were not serving as substitute unit items, but rather, his fingers were providing a mechanism, or motor re-presentation (Steffe et al., 1983), that help him keep track of his figural counting.

In the third exchange, Allan demonstrated no discernible motor activity, but provided a post-task description indicative of an imaginary tool (i.e., touching the face-down card in the approximate location of the animal legs]. Here, it was not necessary for Allan to perform such proximal motor acts during the task itself, but this description suggested that he was, indeed, thinking about the concealed image of a 'cat' as he worked on this task. In this instance, rather, the image was connected flexibly such that within-task proximal motor acts were not a necessity for the child.

Amy. Amy also presented activity indicative of imagery that was more flexibly connected to the mathematical tool. Consider the following exchanges involving screened random objects and a dot card.

Exchange one.

TEACHER: "Ready, I'm not going to say the words this time."

TEACHER: [presents a bear and drops it into cup, presents a block and drops it into cup, presents a penguin and drops it into cup, presents a frog and drops it into cup and then places hand over top of cup (see Figure 22)]

Figure 22: Cup and Random Objects

TEACHER: “What did I put in first?”

AMY: “It’s easy, the bear.”

TEACHER: “What did I put in last?”

AMY: [looks around] “The block”

TEACHER: “What did I put in before the block?”

AMY: “Oh! The block was second.” [moves both hands in a back-and-forth linear motion]

Exchange two.

TEACHER: “Alright, got it?” [places screen over linear arrangement of four objects (from left to right): car, block, penguin, bear]

TEACHER: “What’s over on this end?” [taps the far right-hand side of the screen]

AMY: “A bear” [nods]

TEACHER: “What was next to that bear?” [taps screen just to the left of the previous tap]

AMY: [moves hand in a deliberate left to right motion on the table] “A penguin”

TEACHER: “What was on this end?” [taps the far left-hand side of the screen]

AMY: “A car”

Exchange three.

TEACHER: “Here we go”

TEACHER: [presents dot card with pattern of 6 blue dots (two triangular groups of 3 dots) for one second and then conceals card (see Figure 23)]

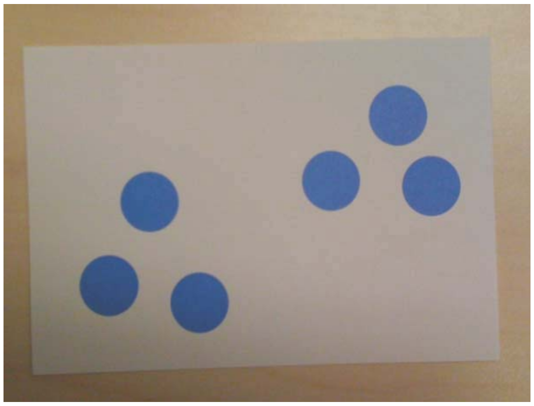

Figure 23: 6 (3-3) Dot Card

AMY: [begins drawing dot pattern on note card]

AMY: [completes drawing that contains 6 dots and is somewhat spatially similar to presented card (see Figure 24)]

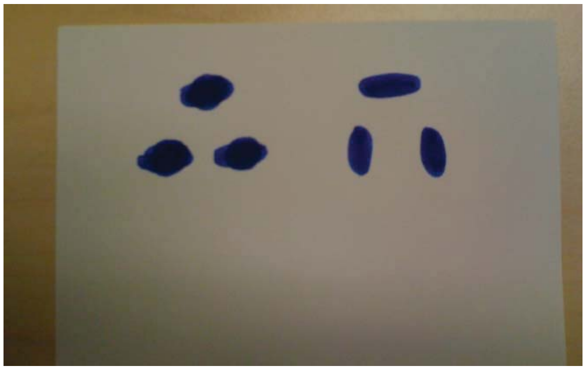

Figure 24: Amy's Drawing of 6 Dot Card (Recreated from Video Record)

TEACHER: "Let's see how we did. So how many dots were there?"

AMY: [touches each of her drawn dots with the tip of her pen] "One, two, three, four, five, six"

TEACHER: [places dot card next to student's drawing] "Yeah?"

AMY: [Looks at both cards and smiles]

In the first two exchanges, Amy moved her hand deliberately back and forth in conjunction with her mathematical thinking. Although this movement did not occur in close proximity to the mathematical tool, the reoccurrence of such gestures suggested that they hold some significance to the student's mathematical conception. Indeed, this linear gesture was congruent with the linear arrangement of materials in the second exchange, though, again, the gesture itself occurred at some distance from the tool suggesting something less than a proximally rigid connection. In the context of the first exchange, the linear gesture appeared even more flexibly connected as the arrangement of screened materials at the bottom of the cup almost certainly was not linear.

Regarding the third exchange, Amy obviously was able to apprehend the quantitative aspect of the presented dot pattern; however, in this instance, she took some apparent liberties with the spatial aspect of her representation. Notice the manner in which Amy placed the two collections of three dots side by side (versus the original diagonal presentation). Additionally, Amy's dots were also somewhat elliptical compared to the circular dots of the original presentation. All of this is to suggest that, in this instance, Amy was not working from a rigidly connected image of the tool's shape, but rather a more personalized image of the tool's core quantity that was flexibly-connected spatially.

James. As you might suspect, James also, at times, presented motor activity suggestive of this flexible connection between image and tool. Below are three exemplary exchanges involving bears, shells, a bead rack, and animal cards.

Exchange one.

TEACHER: "Twelve little bears" [motions towards row of 12 red bears in front of student]

TEACHER: "But now I've still got way too many shells" [slides collection of 17 shells next to bears] "I've got seventeen shells."

TEACHER: “How many am I going to have left over?” [places hand over collection of shells]

JAMES: [touches table next to each of the bears] (whispers audibly) “one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve”

JAMES: [slowly raises five fingers sequentially while looking at them]

JAMES: “Five”

TEACHER: “We are going to have five left over? How did you get that? I think you are right, but how did you get that?”

JAMES: “I counted these again,” [drags finger along table beside bears] “and then I counted my fingers again.”

TEACHER: “How did you know to stop at five though?”

JAMES: [shrugs]

Exchange two.

TEACHER: [presents 2 row bead rack to student with 12 beads screened (5red-1black top row and 5red-1black bottom row)]

TEACHER: [removes screen for one second and then replaces screen]

JAMES: (whispers) “Five” [raises five fingers simultaneously]

JAMES: “Five down at the bottom and five up at the top”

TEACHER: “Okay, so how many?”

JAMES: “Ten”

TEACHER: “Ten? Are you sure?” [removes screen for one second and then replaces screen]

JAMES: [taps table six times in a linear pattern] “Twelve”

Exchange three.

TEACHER: “I have three” [presents three animal cards *face-down* in stack]

TEACHER: “How many legs?”

JAMES: [looks across table and begins taping the table with fist in a rhythmic 2-beats] (whispers audibly) “one-two, three-four, five-six, seven-eight, nine-ten, eleven-twelve”

TEACHER: “How many?”

JAMES: “Because I saw these” [reaches across teacher and touches an animal card laying *face-up* on the table]

JAMES: “Twelve”

TEACHER: “Oh, because you saw those” [motions across table] “Twelve, I gotcha” [removes all other animal cards from table] “Let’s try one more”

TEACHER: [gathers five animal cards and stacks them *face-down*] “Five ponies”

TEACHER: “How many pony legs?”

JAMES: (whispers audibly) “Five”

JAMES: [taps table with index finger in a rhythmic 2-beats] (whispers audibly) “one-two, three-four, five-six, seven-eight, nine-ten, eleven-twelve, thirteen-fourteen”

JAMES: [taps the table two time quickly] (whispers audibly) “Fifteen”

JAMES: [slides index finger across table a short distance] (mouths words-indecipherable)

JAMES: [taps the table one time] (whispers audibly) “Sixteen”

JAMES: “Sixteen”

TEACHER: “Sixteen pony legs? Let’s check it” [lays five cards out in a row, *face-down*, in front of student]

JAMES: [touches each card in the approximate location of the legs] “One-two, three-four, five-six, seven-eight, nine-ten, eleven-twelve, thirteen-fourteen, fifteen-sixteen, seventeen-eighteen, nineteen-twenty”

In the first two exchanges, James leveraged finger patterns to negotiate the mathematical tasks, and similar to Allan’s work on the penguin task, these finger patterns were almost certainly not constructed as perceptual replacements. Rather, James raised these finger patterns at some distance from the tool and ‘on the fly’. Considering the task involving bears and shells, a counter of perceptual unit items would likely need sustained access to two perceptual collections of units (12 and 17) in order to draw a comparison. Further bolstering this position is the manner in which James, in exchange two, ultimately tapped the table six times (in a linear pattern) to determine the numerosity of a collection that contained twelve units. The point here, though, is not that James was likely counting figural unit items, but rather that his constructed figural unit items did not appear rigidly connected to the tool.

Turning our attention to the third exchange, James, again, presented motor acts at a distance from the tool; however, he later purported to have been influenced by a visible animal card on the other side of the table. (Note, this infiltration of surrounding sensory materials will be explored further in Appendix A.) First off, this influence does not suggest that the act was perceptual as there would need to be three visible cards (12 legs) for the counter of perceptual unit items to negotiate this task. More importantly, though, is that James did not need to produce motor acts in close physical proximity from the tool, but rather acted from some distance. This is suggestive of a more flexible connection between image and tool.

Within-Task Changes in Connectivity

Very often, the motor acts indicative of a particular connection between image and tool seemed to remain constant throughout a particular task; however, on certain occasions, the child's motor acts varied within the task. Although there were but five observed instances of this variable connectivity, even a single observation of such an occurrence would suggest a need for some scrutiny in this area.

Flexible to rigid. In this instance, James provided an example of motor activity that was suggestive of a transition from a flexibly-connected image to one that was more rigidly-connected. Consider the following exchange involving a 10-frame.

TEACHER: "Ready" [presents ten-frame with a pair-wise 7 (see Figure 25) for one second]

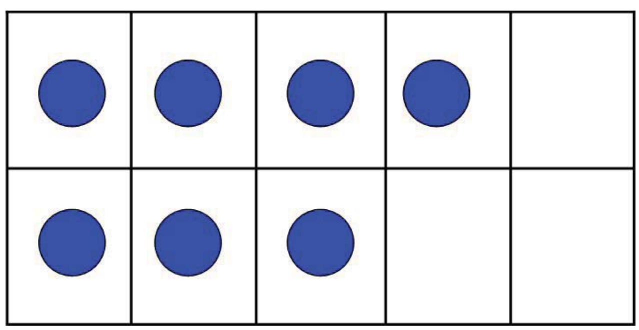

Figure 25: 10 Frame with Pair-wise 7

JAMES: [wiggles index finger on right hand three times and pauses]

JAMES: [makes three quick pointing gestures towards floor in a linear pattern and then three more quick pointing gestures towards floor in a linear pattern]

JAMES: "Six dots"

Although it was quite subtle, James' first attempt to act on this task apparently involved wiggling a single finger three times. Presumably, this did not provide a sufficiently robust platform for considering his image of the pair-wise 7; and, he then produced two linear pointing patterns far more suggestive of the tool. Although these finger points did not necessarily occur in close proximity to the tool, James' change in motor activity suggested something of a transition in the connectivity between image and tool. Specifically, his flexibly-connected finger wiggle transformed into a more rigidly-connected spatially suggestive series of pointing acts.

Rigid to flexible. At times, participants' motor acts were suggestive of a transition in the other direction, namely from a rigidly-connected image to one that was more flexibly-connected. Consider Amy's work with paperclips below.

TEACHER: [places a chain of 11 paperclips in front of student] "So how long is this?"

AMY: [touches table next to each paperclip] "Eleven"

TEACHER: "So let's say I wanted to make this eleven chain a fifteen long chain. How many more would I need?"

AMY: [places hand at the end of the chain, sequentially raises four fingers while moving her hand in a linear spatial pattern] (whispers) "Twelve, thirteen, fourteen, fifteen"

AMY: "Four"

TEACHER: "Four more?"

TEACHER: "Okay, let's say I wanted to make it a twenty long chain."

AMY: [places hand at the end of the chain, sequentially raises five fingers while moving her hand in a linear spatial pattern] "Twelve, thirteen, fourteen, fifteen, sixteen"

AMY: [lifts other hand off of the table away from the paperclip chain, sequentially raises four fingers] "Seventeen, eighteen, nineteen, twenty"

AMY: "Nine"

TEACHER: "Nine? Okay, do you need to check it or are you pretty sure?"

AMY: "Yeah, I am pretty sure."

Here, Amy apparently worked from rigidly-connected images in the first two comparisons. Confining our analysis to only the third comparison (11 to 20), though, she first acted in close proximity to the tool by placing her hand at the end of the chain (similar to the prior two tasks) and then presented motor acts that occurred at some distance from the tool. Perhaps this was a result of increasing image familiarity, but there was unquestionably a qualitative difference in the manner in which Amy initially and ultimately interacted with the mathematical tool indicating a likely change in how she was conceiving the unit items of the task. Indeed, the increasing distance as well as the change in suggestiveness (i.e., linear hand movements to finger patterns) indicated movement from a rigidly-connected image to one that was more flexibly-connected.

Problems with Quantitative Analyses

Although I alluded to the problematic nature of applying quantitative analyses to the teaching experiment methodology, again, I must note the difficulties of such practice. Indeed, the dynamic, reflexive, and individualized nature of such experiments often thwart attempts to make sense of resultant data using quantitatively oriented means. With that said, I do find it useful to present these data in terms of frequency of occurrence such that the reader may interpret these events from a more informed vantage point. Quite simply, the data presented in the coming sections are intended only to inform the reader, in the most general sense, of what participants did, how often, and when.

Construction Preference

Having established qualitatively distinct degrees of connection between image and tool (as evidenced by the motor acts and verbalizations of the participants), I organized these different image types by student according to their frequency of occurrence (see Figure 26). Here, Allan's motor acts and verbalizations were apportioned slightly differently in terms of

Student	% (#) Occurrences Rigid Connected	% (#) Occurrences Flexibly Connected
Allan	60% (31)	40% (21)
Amy	71% (35)	29% (14)
James	79% (26)	21% (7)

Figure 26: Occurrence Type by Student

connectivity; however, little can be made of this other than Allan appeared to construct and work from flexibly-connected imagery (and less from rigidly-connected imagery) slightly more often than Amy or James.

Change over Time

Thus far, I have presented single occurrences categorized with respect to the connectivity between image and tool. Recall, though, that I also endeavored to discern how these constructed quantitative images may change over time. Towards this end, I found it useful to leverage differences in image connectivity to examine such change. Each of the four weeks of the teaching experiment was assigned a phase (i.e., the first week was labeled phase 1), and participants' imagery connectivity was organized according to these phases (see Figure 27). Additionally, in the few instances where imagery appeared to vary within the task, the final connectivity type was considered. Looking across the students, there appeared to be no clear linear trajectory from either rigid to flexibly-connected imagery or vice versa. Only Allan presented something of a linear progression, but this was mitigated considerably by the paucity of figural occurrences in phase one. Even one observed occurrence of flexibly-connected

	ALLAN		AMY		JAMES	
	% (#) Occurrences Rigidly Connected	% (#) Occurrences Flexibly Connected	% (#) Occurrences Rigidly Connected	% (#) Occurrences Flexibly Connected	% (#) Occurrences Rigidly Connected	% (#) Occurrences Flexibly Connected
PHASE 1	0% (0)	100% (1)	0% (0)	0% (0)	25% (1)	75% (3)
PHASE 2	33% (5)	67% (10)	80% (12)	20% (3)	90% (19)	10% (2)
PHASE 3	65% (17)	35% (9)	57% (13)	43% (10)	75% (6)	25% (2)
PHASE 4	90% (9)	10% (1)	91% (10)	9% (1)	N/A	N/A

Figure 27: Imagery Connectivity by Phase

imagery during this phase would have shifted his percentages dramatically. Amy and James, however, appeared to waver between rigidly connected and flexibly-connected images as they progressed through the phases of the teaching experiment. Ultimately, the best that could be said of these data, with respect to change over time, was that participants' imagery was variably connected to the mathematical tool throughout the four-week teaching experiments.

Minor Figurative Themes

In addition to the manner in which participants' images appeared variably connected to the mathematical tool of the moment, two additional minor themes emerged for consideration. Specifically, several occasions of sensory interference occurred as students appeared to incorporate perceptually available objects beyond the immediate task space into their re-presentations. Additionally, there were a few instances where there appeared to be some manner of conflict between the spatial and quantitative apprehension and subsequent re-presentation of a particular mathematical tool. Given that these occurrences were quite infrequent, I elected to discuss them in the context of an appendix (see Appendix A.) rather than in this chapter; however, these events are, indeed, thought provoking and, likely, deserving of additional attention and study.

Summary

In the preceding section, I presented data supporting two qualitatively distinct forms of quantitative mental imagery. Analyses of participants' motor acts and verbalizations suggested that students constructed and worked from an image that was either rigidly-connected to the mathematical tool or flexibly-connected to the mathematical tool. Moreover, in a few instances, the nature of this image may have changed within the task. Here, I must emphasize that these observations occurred across a wide range of quantitative tasks involving many different tools, and for this reason, I was able to ascribe some stability to these different imagery types. Regarding the participants, Allan appeared to work from flexibly-connected imagery slightly more often than his peers; however, analyses of figural occurrences by phase revealed that each participants' imagery constructions were variable across the duration of the teaching experiments.

Imagery Descriptions

Although participants' descriptions of their own mathematical conceptions were incorporated into the analysis of imagery type, given my focus on determining the nature of quantitative mental imagery, I find these descriptions worthy of additional examination.

During the teaching experiments, I concluded many of the quantitative tasks with probing questions regarding the nature of the child's thinking (i.e., "What were you thinking about just then?" "What were you picturing in your head when you touched that card?", etc.). In most cases, the participants were unable to provide any description of their mathematical imagery or thinking. Consider the following exchange with Amy.

AMY: [touches each of the 12 visible dots] "one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve" [touches the screen three times in a pattern that completes the array] "thirteen, fourteen, fifteen"

TEACHER: "Fifteen, what were you thinking about when you did thirteen, fourteen, fifteen" [touches the screen three times in a pattern that completes the array]

AMY: "Nothing"

Indeed, this exchange was indicative of the many occasions where participants were either unable or chose not to provide additional description of their mathematical thinking.

Allan. Each of the participants, however, did provide some verbal description of their thinking (eight total occasions) which were worthy of examination. Consider the exchanges with Allan below.

Exchange one.

ALLAN: "Thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, *twenty-four*" [touches each *face-down* animal card four times in approximate location of legs]

TEACHER: "Yeah? What were you thinking about when you were counting these things?" [taps one of the animal cards] "You were pointing at something and counting."

ALLAN: "The legs"

Exchange two.

TEACHER: "Twelve, right?" [places translucent red counter on top of 12^{th} dot]

TEACHER: "I am going to hide three" [places screen over the 13^{th}, 14^{th}, 15^{th} dots]

TEACHER: "What number is this guy?" [places red translucent counter on top of 16^{th} dot and taps it]

ALLAN: [touches the counter on top of the 12^{th} dot, the screen three times in a linear pattern, and the counter on top of the 16^{th} dot] “Twelve, thirteen, fourteen, fifteen, *sixteen*”

TEACHER: “What were you thinking about over here when you were pointing at that thing?” [touches screen]

ALLAN: “The dots” [touches the counter on top of the 12^{th} dot]

Exchange three.

TEACHER: [presents a card with a 3X3 array of green dots to student, the bottom two rows are screened]

TEACHER: “So there are three in that row, right?” [runs finger across the top, exposed row of 3 dots]

TEACHER: “What if I told you that below this row, there are two more rows just like it” [taps screen two times in a downward linear pattern]

TEACHER: “How many dots would there be altogether?”

ALLAN: [with arms crossed, wiggles index finger] “Five?”

TEACHER: “Five dots altogether if there are two more rows of three like that?” [runs finger across the screen two times in a linear motion]

ALLAN: “Yes”

TEACHER: “Okay, let’s see” [removes screen]

ALLAN: “What the?”

TEACHER: “How many dots are there?”

ALLAN: [touches each of the dots] “Nine”

TEACHER: “What made you think five?”

ALLAN: "I saw a five, but it's nine"

TEACHER: "How did you see a five?"

ALLAN: "I saw a five or three, but I messed it up"

TEACHER: "Yeah, but what did you see when you said that you saw a five?"

ALLAN: (pause 10 seconds) [laughs]

TEACHER: "Tough to say?"

In the first two exchanges, Allan purported to have thought about the respective screened materials as he worked on the task. Indeed, these descriptions (coupled with Allan's motor acts) provided support for analog imagery; moreover, they were in keeping with the assertion that figural re-presentations are "visualization[s] of the common sensory material" (Steffe, 1992, p. 86). At times, though, Allan's descriptions were more confounding. In the third exchange, Allan described 'seeing a five' but was unable to elaborate on what he saw.

Amy. On occasion, Amy also provided descriptions of her thinking that were similar to Allan's first two exchanges in that she purportedly thought about some aspect of the concealed tool. On two occasions, though, her descriptions took on a more elaborative flavor. Consider the following exchanges involving bears, shells, and the row task.

Exchange one.

AMY: "Well, I had ten bears and then thirteen" [places hand on screened collection of 13 shells]

AMY: "Then, I counted three more."

TEACHER: "Yeah, three more, but what were you counting?" [touches table three times in a linear pattern]

AMY: “I was thinking about, like, there were ten bears and then there were thirteen shells and then you needed three more”

TEACHER: “Three more what?”

AMY: “Three more bears”

TEACHER: “So these were bears?” [touches table just beyond the linear arrangement of bears]

AMY: “Yeah”

TEACHER: “What color were they?”

AMY: “Purple”

TEACHER: “Purple?” [secures three purple bears from a visible collection near the student’s workspace and places them in a linear pattern extending the arrangement of bears to 13 (10 red/3 purple)]

Exchange two.

TEACHER: “Can you put that on number five?” [hands student a translucent blue counter]

AMY: [touches the first five dots on the tool] “one, two, three, four, five” [places counter on the fifth dot]

TEACHER: “There’s five.”

TEACHER: [places a screen over four dots]

TEACHER: “This is ten.” [places a translucent blue counter over the 10^{th} dot]

TEACHER: “How many dots are under here?” [drags finger horizontally across screen]

AMY: [touches screen four times in a linear pattern] “Six, seven, eight, nine” [touches counter on top of 10^{th} dot] “ten”

TEACHER: "Okay, so how many dots?" [taps screen]

AMY: "Four, you can't trick me."

TEACHER: "What were you thinking about when you were counting that?" [taps screen four times in a linear pattern]

AMY: "I can see under it."

TEACHER: "You can see under it? Tell me about that. You can see the dots through here?" [taps screen] "I can't see the dots through here." [taps screen]

AMY: "I can, I can see under it."

TEACHER: "So when you are pointing at that" [touches screen] "you see a dot, that's what you're telling me."

AMY: "uh huh" [nods head]

In these two exchanges, Amy provided some significant elaboration on her mathematical thinking. Regarding the first exchange, I recalled being surprised by her assertion that she was counting (apparently imaginary) purple bears. Similarly, her assertion that she could 'see through' an opaque screen to the dots beneath was also surprising. In both of these instances, a perspective of analytical believing compelled me to assign some truth to her description. Given the almost certain impossibility of actually seeing through the screen, a more likely conclusion is that Amy indeed 'saw' the dots, but as a mental image experienced in a quasi-perceptual manner.

James. James also provided some imagery-related description of his mathematical activity. Returning to his work on a task involving a partially screened array of dots, consider the following.

TEACHER: "We've got a row across the top with three, right?" [places 3X3 (green dot) array card with dots screened in front of student]

TEACHER: “And there are two more rows just like this one” [motions towards row of 3 unscreened dots and then drags finger across dot card and screen in two linear motions]

TEACHER: “How many dots are on this card?”

JAMES: [touches each of the three unscreened dots]

JAMES: [touches the screen three times in a linear pattern]

JAMES: [touches the screen two times in a linear pattern beneath his previous touched pattern]

JAMES: [touches the screen once near the bottom]

JAMES: “Nine”

TEACHER: “Nine? What were you thinking about when you were pointing at those things?” [taps the screen]

JAMES: [touches each of the three unscreened dots] “One, two, three”

JAMES: [touches the screen three times in a linear pattern] “Four, five, six”

JAMES: [touches the screen two times in a linear pattern beneath his previous touched pattern] “Seven, eight”

JAMES: [touches the screen once near the bottom] “Nine” [raises hand in the air]

TEACHER: [replicates student’s spatial pattern of touches on the screen] “One, two, three, four, five, six, seven, eight. . .”

TEACHER: “Where is number nine? Is he down here?” [touches screen once near the bottom] “or is he right here? [touches approximate location of 9th dot in the array]

JAMES: “Right there” [touches approximate location of 9th dot in the array]

TEACHER: “So what were you picturing in your head when you were thinking about that?”

JAMES: “Green dots”

Similar to the first two exchanges involving Allan, James described thinking about the tool in question.

Summary

An examination of the participants’ descriptions of their thinking or mathematical activity provided some insight into the nature of their respective imagery. Although such descriptive instances were scarce within the larger collection of figural occurrences, the few occasions in which students did provide an imagery-oriented description, some connection between their mathematical thinking and the tool was often clear, and in certain instances, held surprising details.

Intervention Patterns of Interaction

While the primary aim of this study was to ascertain the nature of children’s quantitative mental imagery, I must emphasize that the mathematical practices (and their associated conceptions) occurred within a community of sorts: namely, the interactive environment of intervention dyad. For this reason, attention must be given to the quality of these interactions. Here, I aimed to identify interactive themes that likely influenced the observed mathematical practices of the participants.

Post-Task Questioning

While each figural occurrence necessarily involved the posing of a particular task and subsequently waiting for the student to act on the task, I also concluded many (but not all) of these occurrences with certain probing questions aimed at better discerning the students’ mathematical conceptions associated with the observed practices. Indeed, this almost certainly established a normative pattern of task, activity, and probe. Indeed, there were instances towards the end of the intervention cycle where participants, anticipating a probing question, would offer up a response. Consider the following exchange.

TEACHER: [presents a stack of 3 animal cards *face-down* to student] “Now, I’ve got three, how many pony legs?”

JAMES: [looks across table, taps the table 12 times in rhythmic patterns of two’s] (whispers) “one-two, three-four, five-six, seven-eight, nine-ten, eleven-twelve”

JAMES: “Because I saw these” [reaches across table and touches a single *face-up* animal card laying with other materials]

TEACHER: “Oh you saw those?” [reaches across table and removes animal card]

JAMES: “Twelve!”

TEACHER: “Gotcha”

Here, James, likely anticipating a question about his thinking, voluntarily elaborated on the nature of his strategy without any probe on the part of the teacher. Indeed, this particular instance was fortuitous as I may not have noticed the *face-up* animal card had James not pointed it out. In several instances, though, participants would simply state outright at the conclusion of a task, without prompting, that they were not thinking about anything. In fact, Amy’s apparent favorite phrase for such situations was “I got nothing.” At any rate, it is worth noting that participants became uniquely socialized to probing questions throughout the intervention teaching experiments.

Behavior Eliciting

Turning to potentially influencing actions on the part of the teacher, three distinct themes emerged. The first was behavior eliciting which was aimed at directing students’ in-task mathematical behaviors and/or tool interactions. Specifically, behavior eliciting occured after a particular task had been posed but prior to the student's mathematical activity and they were teaching practices that focused "on having the child respond with a particular behavior that is predetermined by the teacher" (Wright et al., 2002, p. 36). Examples included:

- Instructing the student to enact a specific strategy for solving a task (i.e., "begin your count with the larger addend").
- Directing the student to interact with a mathematical tool in a specific way (i.e., "place your finger on the third bead").

Typically such practices are regarded as undesirable. Wright et al. (2002) wrote:

> As a general rule, this kind of teaching is regarded as inappropriate because of its emphasis on the child behaving in a particular way rather than the child advancing his or her knowledge by developing a new strategy appropriate for the problem. . . In these cases, the child undergoes a cognitive shift from attempting to solve the problem to attempting to figure out the kind of behavior required by the teacher (p. 36-37).

Given the objectionable nature of such teaching practices, I endeavored to avoid instances of behavior eliciting within the teaching experiments; however, reflection upon the video record with a colleague revealed figural occurrences that likely involved such practices. Consider the following exchanges with Amy and Allan.

Exchange one.

TEACHER: "Okay, you've got them how you want them. Ready?" [places screen over linear arrangement of four objects (from left to right): car, bear, rooster, frog]

TEACHER: "What's on this end?" [touches far right-hand side of screen]

AMY: (pause 8 seconds) "Frog"

Exchange two.

TEACHER: “Okay, ready” [places screen over linear arrangement of five objects (from left to right): bear, frog, car, rooster, block]

TEACHER: “What’s in the very middle” [taps middle of screen]

ALLAN: “Car”

TEACHER: “What’s over on this end?” [taps left-hand side of screen]

ALLAN: “Bear”

TEACHER: “What’s next to the bear?” [taps screen to the right of the previous tap]

ALLAN: “Frog”

After much deliberation, my colleague and I concluded that these instances typified some form of behavior eliciting. In both of these exchanges, I directed the student to interact with the mathematical tool in a very specific manner and not necessarily the manner in which the student might have chosen. Indeed, this form of behavior eliciting is confounded by the manner in which the teaching practice is bound up with the task itself; however, we ultimately concluded that the overly directive practices of these and similar exchanges could rightly be typified as behavior eliciting. Perhaps a better tactic might have been to ask the student to simply describe the order of the objects under the screen.

Behavior Suggestion

Another thematic mode of interaction was behavior suggesting. If behavior eliciting is classed as explicitly directive teaching practices, one may consider behavior suggesting as its somewhat more subtle cousin. Specifically, behavioral suggestions occured after a particular task has been posed but prior to the student’s mathematical activity**,** and were (either intentional or unintentional) teaching practices that modeled (either tacitly or explicitly) certain behaviors within a particular task. Examples included:

- Touching a screen concealing random objects in a manner suggestive of the objects.
- Pointing at a concealed mathematical tool in a spatial pattern in a manner suggestive of the tool.
- Tapping the work surface in a spatial or temporal pattern in a manner suggestive of or connected to the task or tool.

As opposed to the explicit telling or guiding of behavior eliciting, here, subtle suggestions or cues embedded into the interaction provided some indication to the student as to how he or she might have physically interacted with a particular tool. Consider the following exchanges with Amy and James.

Exchange one.

TEACHER: [places a chain of 11 paperclips in front of student] “So how long is this?”

AMY: [touches table next to each paperclip] “Eleven”

TEACHER: “So let’s say I wanted to make this eleven chain a fifteen long chain. How many more would I need?” [makes a linear pointing motion over the paperclip chain that extends past the end of the chain]

Exchange two.

TEACHER: “Thirteen” [places a translucent red counter on top of 13^{h} dot]

TEACHER: [places a screen over the 14^{th}, 15^{th}, 16^{th}, 17^{th} dots]

TEACHER: “This guy is eighteen.” [places a translucent red counter on top of 18^{th} dot]

TEACHER: “How many did I cover up?” [runs finger across screen in a linear motion]

JAMES: “How many did you cover up?” [touches screen]

TEACHER: “Uh huh, how many did I cover up?”

In both exchanges, I posed a particular task and then enacted a behavioral cue that was potentially suggestive regarding a particular manner in which the child may interact with the mathematical tool. Quite simply, my gestures (i.e., linear pointing motions, running a finger across a screen) could very easily have been appropriated by the student as he/she looked to me for guidance. Interestingly, in several instances, my behavioral suggestions were ‘taken’ by the student; however, far more often, participants’ interactions with the mathematical tools appeared different from that which I ‘suggested’. Consider a continuation of the second exchange above.

Exchange two (continued).

JAMES: [touches the counter on top of the 13th dot and then taps the screen 4 times in a linear pattern] “Thirteen, fourteen, fifteen, thirteen . . .”

JAMES: [touches the counter on top of the 13th dot and then taps the screen 3 times in a linear pattern, and then touches the counter on top of the 18th dot] “Thirteen, fourteen, fifteen, sixteen, *seventeen*”

TEACHER: “No, this guy is eighteen.” [touches the counter on top of the 18th dot]

JAMES: [taps the screen 4 times in a linear pattern] “One, two, three, four” [shakes head]

In this instance, James elected to tap the screen rather than running his finger along the screen in a linear fashion as I ‘suggested’. This makes sense as his discrete movements were more congruent with the screened materials and likely better representative of the imagery from which he was working.

Behavior Replication

The finale emergent interactive theme was that of behavior replication. While eliciting and suggesting occurred prior to the students’ mathematical practices, replicating occurred in conjunction with or after such practices. Specifically, behavior replications are teaching practices aimed at summarizing the mathematical practices of the student. Behavioral replicating typically consisted of the teacher (me) reenacting students' mathematical behaviors (i.e., motor activity) to either probe for additional explanation (post-task) or support the negotiation of a particular mathematical impasse (within-task). Below are exchanges with Amy and James typical of behavior replication.

Exchange one.

AMY: [touches the table three times in a linear pattern just beyond the 10^{th} bear as if to continue the arrangement] "Three"

TEACHER: "How did you know three? What were you picturing when you did that?" [touches table three times in a linear pattern]

AMY: "Well, I had ten bears and then thirteen." [places hand on screened collection of 13 shells]

AMY: "Then, I counted three more."

TEACHER: "Yeah, three more, but what were you counting?" [touches table three times in a linear pattern]

Exchange two.

JAMES: [touches the side of the cup in rhythmic two-patterns] "one-two, three-four, five-six, seven-eight, nine-ten, eleven-twelve"

JAMES: [touches the side of the cup one more time] "Thirteen, yeah, thirteen"

TEACHER: "Okay, so you did one-two." [touches side of cup two times] "What were you counting on here?"

JAMES: [touches the side of the cup in rhythmic two-patterns] "one-two, three-four"

TEACHER: [touches the side of the cup in rhythmic two-patterns] “one-two, three-four”

Here, I explicitly attempted to replicate the motor acts of Amy and James (i.e., touching the table or cup) as I invited them to elaborate on their thinking within particular tasks. Interestingly, examination of the manifest content revealed a few instances where my behavior replication did not match the participants’ actual motor act. On one such occasion, a participant raised his fingers sequentially to monitor his figural counting; however, I ‘replicated’ this behavior by touching the screen in a manner indicative of the screened tools beneath.

Multiple Modes of Interaction

Given the inherently complex patterns of interaction between the intervention teacher and student, the presence of multiple modes of interaction should be expected, and this was certainly true of the interactive themes presented here. In certain instances, I would either elicit or suggest behaviors and then attempt to replicate the observed behaviors of the participants. Consider the following exchanges with Amy and Allan.

Exchange one, eliciting and replicating.

TEACHER: “Okay, ready” [places screen over linear arrangement of five objects (from left to right): rooster, car, frog, block, shell]

TEACHER: “What’s on this end?” [taps the far right-hand side of the screen]

AMY: “Shell”

TEACHER: “Shell? What’s on this end?” [taps the far left-hand side of the screen]

AMY: “A rooster”

TEACHER: “What’s next to that rooster” [taps screen to the right of the previous tap]

AMY: “Car”

TEACHER: “Over here, you said it was a shell.” [taps the far right-hand side of the screen]

TEACHER: “What’s next to that shell?” [taps the screen to the left of the previous tap]

AMY: “A frog”

TEACHER: “A frog? What’s in the middle?”

AMY: (pause 3 seconds) “I forgot”

TEACHER: “Okay, let’s see if we can figure it out.”

TEACHER: “What’s over here?” [taps far left-hand side of screen]

AMY: “Rooster” [taps screen to the left of teacher’s tap]

TEACHER: “Rooster” [taps screen close to student’s tap]

AMY: “Frog” [taps screen to the left of teacher’s tap]

TEACHER: “Frog” [taps screen close to student’s tap]

Exchange two, suggesting and replicating.

TEACHER: [places a blue translucent counter over the 15th dot] “Fifteen and now I cover up four dots” [places screen over the next four dots (16th, 17th, 18th, 19th) and runs finger across screen in a linear motion]

TEACHER: “This is fifteen.” [points at 15th dot] “What is this number?” [places a blue translucent counter over the 20th dot]

ALLAN: [touches the 15th dot] (audibly whispers) “fifteen”

ALLAN: [raises index finger towards screen] “Twenty”

TEACHER: “Twenty? How do you know it’s twenty?”

ALLAN: "Because you counted fifteen" [touches the 15th dot] "sixteen, seventeen, eighteen, nineteen" [touches the screen 4 times in a linear pattern] "twenty" [touches the 20th dot]

TEACHER: "What are you counting under here?" [taps the screen two times in the same location] "What are you thinking about?"

ALLAN: "One and two and three and four" [touches the screen 4 times in a linear pattern]

TEACHER: "What did those things look like?"

ALLAN: "Like these" [points at unscreened dots]

In both of the exchanges above, I acted in either an eliciting or suggestive manner and then concluded the interaction with some attempt to replicate or summarize the child's behavior. In the first exchange, this replicating act was aimed more at supporting within-task activities, while the replicating act of the second exchange was directed towards prompting additional post-task explanation.

Lack of Behavior Eliciting, Suggesting, and Replicating

Although instances of eliciting, suggesting, and replicating were observed with some frequency, there were also occasions in which these teaching acts were noticeably absent. Below are exchanges with Allan and James that typify these types of interactions.

Exchange one.

TEACHER: "Okay" [presents 2 row bead rack to student with 12 beads screened (6top/6bottom)]

TEACHER: [removes screen for 1 second to display 12 beads and then replaces screen]

ALLAN: [sequentially raises five fingers and then taps table with hand] "Twelve"

TEACHER: “How do you know twelve?”

ALLAN: “Because five black beads on the bottom” [simultaneously raises five fingers] “five black beads on top” [simultaneously raises other five fingers] “one red bead on the top” [lowers all ten fingers and raises index finger] “ and one red bead on bottom” [points at screened beads on bottom row]

TEACHER: “So five, five, one, and one make twelve?”

ALLAN: “Yeah”

Exchange two.

TEACHER: “So, now you have seven little bears.” [motions towards row of seven red bears in front of student]

TEACHER: “But, now I have too many shells.” [slides collection of 11 shells next to bears] “I have eleven shells.”

TEACHER: “How many shells am I going to have left over?” [places hand over collection of shells to conceal them]

JAMES: “You’ve got eleven?” [simultaneously raises seven fingers]

TEACHER: “Yeah”

JAMES: [touches the table next to each of the bears]

JAMES: [While maintaining an upward gaze away from hands, slowly, sequentially raises four fingers]

JAMES: “Four”

TEACHER: “I’m going to have four left over? What were you thinking about when you were counting?”

JAMES: "I was like one, two, three, four, five, six, seven" [touches the table next to each of the bears]

JAMES: "eight nine ten eleven" [sequentially raises four fingers]

TEACHER: "Were you thinking about shells or bears or anything?"

JAMES: "Shells and bears"

In both cases, I did not present any behaviors prior to the mathematical activity that could have been construed as either eliciting or suggestive acts; moreover, my post-task questions and comments were also devoid of any associated attempt to replicate or summarize the participants' motor acts.

Frequency of Occurrence

Here again, I must caution against the overemphasis of quantitative analyses regarding dynamic and individualized teaching experiments; however, as with imagery-oriented themes, I find it useful to present, in a general sense, what happened and how often with respect to these interactive themes (see Figure 28). The majority of teacher-initiated interactions may be typified as either replicating acts or devoid of eliciting, suggesting, or replicating acts. This is useful to know as it provides some basis for the assertion that any production of participants' behaviors via eliciting or suggesting was likely a relatively rare occurrence. Ultimately, though, additional analysis of interactions across tasks would be needed to better support that assertion. Certainly, it is possible, for example, that behavioral suggestions from a prior task could have reverberated in all subsequent, similar tasks; however, this was mitigated somewhat by the manner in which participants often did not appropriate a particular suggested behavior.

ALLAN				
% (#) Occurrences **Behavior Eliciting**	% (#) Occurrences **Behavior Suggesting**	% (#) Occurrences **Behavior Replicating**	% (#) Occurrences **Eliciting/Replicating** or **Suggesting/Replicating**	% (#) Occurrences **None**
4% (2)	4% (2)	48% (25)	10% (5)	35% (18)
AMY				
% (#) Occurrences **Behavior Eliciting**	% (#) Occurrences **Behavior Suggesting**	% (#) Occurrences **Behavior Replicating**	% (#) Occurrences **Eliciting/Replicating** or **Suggesting/Replicating**	% (#) Occurrences **None**
12% (6)	2% (1)	29% (14)	14% (7)	43% (21)
JAMES				
% (#) Occurrences **Behavior Eliciting**	% (#) Occurrences **Behavior Suggesting**	% (#) Occurrences **Behavior Replicating**	% (#) Occurrences **Eliciting/Replicating** or **Suggesting/Replicating**	% (#) Occurrences **None**
0% (0)	9% (3)	42% (14)	9% (3)	39% (13)

Figure 28: Interaction Type by Student

Summary

An examination of the interactions within the intervention teaching experiment reveal repeated cycles of posing tasks, mathematical practices, and post-task probing. Within these very general task phases, I sometimes influenced the quality of the interaction in three distinct ways: behavior eliciting, suggesting, and/or replicating. Indeed, my actions prior to the students' mathematical activity could be construed as highly-directive eliciting or more subtly-suggestive teaching moves. Moreover, in many cases, I entered the task after the student had begun some manner of mathematical practices and attempted to replicate his/her behaviors. Although these acts, undoubtedly, impacted the observed practices of the students, the relatively low frequency of pre-task eliciting or suggesting acts lend plausibility to the claim that the observed student behaviors, in many cases, were not overtly shaped by my behaviors.

Chapter Summary

Looking back over the findings, we see two distinct domains of analysis, namely the psychological imagery constructions of the students and the interactive patterns in which these constructions were situated. In terms of illuminating the quality of the participants' quantitative mental imagery, I observed behaviors and descriptions that suggested that such images may be variably connected to a particular mathematical tool. Turning to my analysis of social practices, three distinct patterns of interaction emerge (behavior eliciting, behavior suggesting, and behavior replicating) as categorical themes from which to describe the mathematical exchanges between student and teacher.

Chapter Five

Conclusions

"Made a note of it, did you write it in your hand? Put a name on it, to help you understand."

~Zero 7: Futures

"A connecting principle, linked to the invisible"

~The Police: Synchronicity I

Chapter Introduction

In the previous chapter, I endeavored to lay out the events of the teaching experiments in great detail, and I provided a robust foundation upon which certain conclusions will be based. This chapter is primarily aimed at organizing emergent themes from the teaching experiments into conclusive models. Recall that a model is defined as a "conceptual system held by the modeler which provides an explanation of the phenomenon of interest, in this case a student's behavior within some portion of mathematics" (Thompson, 1982, p. 160), and that the primary aim of the teaching experiment is the construction of such a model (see Chapter Three). In this chapter, I will present a series of models regarding students' observed counting practices to illustrate the iterative design process culminating with some discussion of a final model deemed "temporarily viable" (Steffe & Thompson, 2000, p. 300). Lastly, I will provide some location for this model within the intervention classroom and present a correlated model to explain the observed patterns of interaction as they relate to counting practices. Prior to launching into discussion of specific models, however, I will first attend to the variability among participants' mathematical practices.

Mathematical Practices and Strategy Variation

Perhaps the most significant observation within the intervention teaching experiments was related to variability of the participants' mathematical practices. Recall the definition of stage established in Chapter Two that implies the satisfaction of four criteria: "a characteristic remains constant throughout a period of time, each stage incorporates the earlier stage, the stages form an invariant sequence, and each new stage involves a conceptual reorganization resulting from reflection and abstraction" (Steffe et al., 1988, p. 7-8). In particular, the invocation of temporal constancy in the first criterion was problematic for this study. Certainly, though, my observations do not automatically invalidate stage-oriented models such as SEAL or stage theory in general. Indeed, the extensive teaching experiments conducted by Steffe and his colleagues provide considerable support for qualitatively distinct conceptions of unit (evidenced by different counting types) (Steffe et al., 1983; Steffe et al., 1988; Steffe, 1992). Rather, what I am suggesting here is that some additional theoretical layering is necessary in order to incorporate this observed variability into any model founded upon SEAL.

Here, it is useful to return to the work of Siegler and his colleagues in the area of arithmetic variability. Recall from Chapter Two, that Siegler and Robinson (1982) conducted a series of videotaped clinical interviews, referred to as behavioral sampling, of children ages 4 and 5 actively solving various addition problems. Subsequent analysis of these data revealed the use of four distinct strategies: 1) *counting-fingers*; 2) *finger*; 3) *counting*; and 4) *retrieval*. Here, Siegler and his colleagues do not describe associated mathematical conceptions in the exhaustive detail of Steffe et al. (1983); however, there is some correspondence between the two theories regarding hypothesized cognition. Indeed, Siegler and Shrager (1984) associate the forming of mental images with certain arithmetic behaviors including finger movements (see p. 242 for explication).

Returning to the notion of strategy variation, Siegler and Shrager (1984) observed many instances where a single participant might use different strategies for seemingly similar tasks.

Indeed, a single problem-solving event was viewed as a series of strategy phases, progressing in real-time, that was dependent upon the cognition of the child (Siegler & Shrager, 1984). Specifically, when faced with an arithmetic task, Siegler and Shrager concluded that a child may first attempt to mentally *retrieve* the answer. If, for some reason, this strategy is deemed insufficient, the same child may then enact one or more behaviorally overt strategies involving counting and/or fingers. In these instances, variability may be viewed as something of a process of arithmetic change as students experiment with different mathematical practices.

Synthesizing SEAL and Cognitive Variability

A structural limitation of most stage theories (including SEAL) is the minimizing (or non address) of change processes. Siegler (1994) elaborated on such a perspective in general terms:

> [M]ost theories place static states at center stage and change processes either in the wings or offstage altogether. Thus, 3-year-olds are said to have nonrepresentational theories of mind and 5-year-olds representational ones; 5-year-olds to have absolute views about justice and 10-year-olds relativistic ones; 10-year-olds to be incapable and 15-year-olds capable of true scientific reasoning. The emphasis in almost all cognitive developmental theories has been on identifying sequences of one-to-one correspondences between ages and ways of thinking or acting, rather than on specifying how the changes occur (p. 1).

Although the stages of SEAL are not bound to specific ages, the framework does emphasize variation regularity. Steffe (1992) wrote:

> The nature of the counting scheme varied widely within a particular child over time as well as across children of the same age. Having established regularity in this variability, I isolated five learning stages in the construction of the number sequence (p. 84).

Here, Steffe described perceived, natural regularities among strategy variations; however, Siegler and Shrager were most concerned with the irregularity of strategy variation regarding an individual's approach to similar tasks. Towards this end, Siegler (1994) put forth a theoretical framework of cognitive variability that would explain this type of activity. Whereas SEAL hypothesizes variability only among different conceptual stages, Siegler proposes variability "in children's thinking [that] exists at every level-not just between children of different ages, or between different children of the same age, but also within an individual solving a set of related problems, within an individual solving the same problem twice, and even within an individual on a single trial" (p. 1). While such a perspective certainly allows for the understanding of diverse strategy use, perhaps more importantly, cognitive variability is also better able to explain cognitive change than stage-based theories. Specifically, an individual's capacity to enact diverse practices sets the stage for conceptual change as the child is able to quantitatively experiment and reflect using a variety of methods and subsequently learn from the outcomes of these activities. Interestingly, though, there appear to be some constraints on the degree of strategy variability. Siegler (1994) elaborated:

> A striking empirical finding about the variability in children's thinking, and one that is important for its ability to contribute to cognitive development is the constrained quality of the variations that children generate. Far from conforming to a trial-and-error model, in which all types of variations might be expected, the new approaches that children attempt consistently conform to the principles that define legal strategies in the domain (p. 4).

Here, Siegler describes a process variability that is constrained by domain specific principles. It must be noted, though, that these domain specific principles (or "principles that define legal strategies" in Siegler's terms) exist in the *current* practices and conceptions of the individual. Quite simply, the constraints a child places on her strategy variation are a function of her present understanding of mathematics; thus, as her understanding of mathematics changes over time, the manner in which she may vary her practices (and conceptions) also changes. Here, in this notion of moveable constraints, I find the common ground from which SEAL may be joined with the change processes of cognitive variability. Specifically, the assertion that "each stage incorporates the earlier stage" suggests that a child could, for example, hold both a perceptual and figural understanding of unit items (Steffe et al., 1988, p. 7-8). Such a child might exhibit strategies that vary between counting perceptual and figurative units depending upon the presentation of the task. Of course, this explains strategy variation in a retrospective sense as children are only capable of varying among current and prior strategies. How, then, might one explain variations that appear to typify a stage that we believe a child has not yet cognitively realized? Or more succinctly, how do we explain strategies that appear to *reach forward*?

For this question, we must consider the interrelatedness among the conceptual structures of SEAL. Germaine to this study is the manner in which the construction of a numerical composite begins with the child's awareness (or monitoring) of a figurative plurality and subsequent unitizing of this plurality into a composite. Or, in more simple terms, developing the awareness that the seven individual, figural units that I just counted can be 'chunked' into a single unit called *seven.* Steffe (1992) wrote of this developing awareness:

> What monitoring might consist of was illustrated by a child with a figurative counting scheme who counted a collection of items covered by two cloths, one covering seven and the other five items, in the following way . . .He touched the first cloth seven times synchronous with subvocally uttering "1, 2, 3, 4, 5, 6, 7" and then proceeded to touch the

> second cloth six times in a row while whispering: 8, 9, 10, 11, 12, 13." As a result of touching the second cloth, he had an awareness of plurality, but he realized he had not made it definite; he had not reached his goal. So, he independently started over from *one*. In the midst of touching the second cloth again, he lost track, so he started over once again. This time, he deliberately touched the second cloth five times in a row and looked at me and said "13,14," indicating that he was not sure that he had touched the second cloth five times. After he uttered "13, 14," I asked how many were hidden under the second cloth; he said "5," and proceeded to count one more time. This time, he stopped at 12 with conviction, and while continuing to count past 7, he stared into space while touching the cloth synchronous with uttering the number words. . . To monitor counting activity intentionally in the way described, there has to be a re-presentation of the results of counting. That is, after saying "8, 9," and making two pointing acts, the child must have re-presented these counted items and "held them at a distance" while reflecting on them (p. 92-93).

Here, one sees a child seemingly develop some awareness of a figurative plurality within a task; and, it is this awareness that provides the foundation for constructing a numerical composite. Of paramount importance, here, is the manner in which a child may reflect upon his or her practices and conceptions *as he or she practices and conceives.* I argue that it is this within-practice reflection that allows children to *reach forward* in a strategic sense. At this point, I must note that such occurrences do not occur in a social vacuum, but, rather, are located within a particular community - the intervention dyad in this case.

Stage, Strategic Preponderance, and Tethers

Of course, such a notion moving strategically across a range of stages forces us to question the viability of such stages. Specifically, if children are able to enact strategies across a number of SEAL stages, then the notion that practices and conceptions would remain "constant throughout a period of time" appears to have been violated (Steffe et al.,1988, p. 7). Indeed such a charge may be true; however, returning to the notion of variation constraint, I propose that the practices observed by Steffe, Siegler, and their respective colleagues were those of children throwing strategic '*tethers*' along a particular progression (see Figure 29). In this model, an individual might enact a set of varied strategies; however, these variations are constrained to particular segment of a progression. For example, it is possible but less likely that

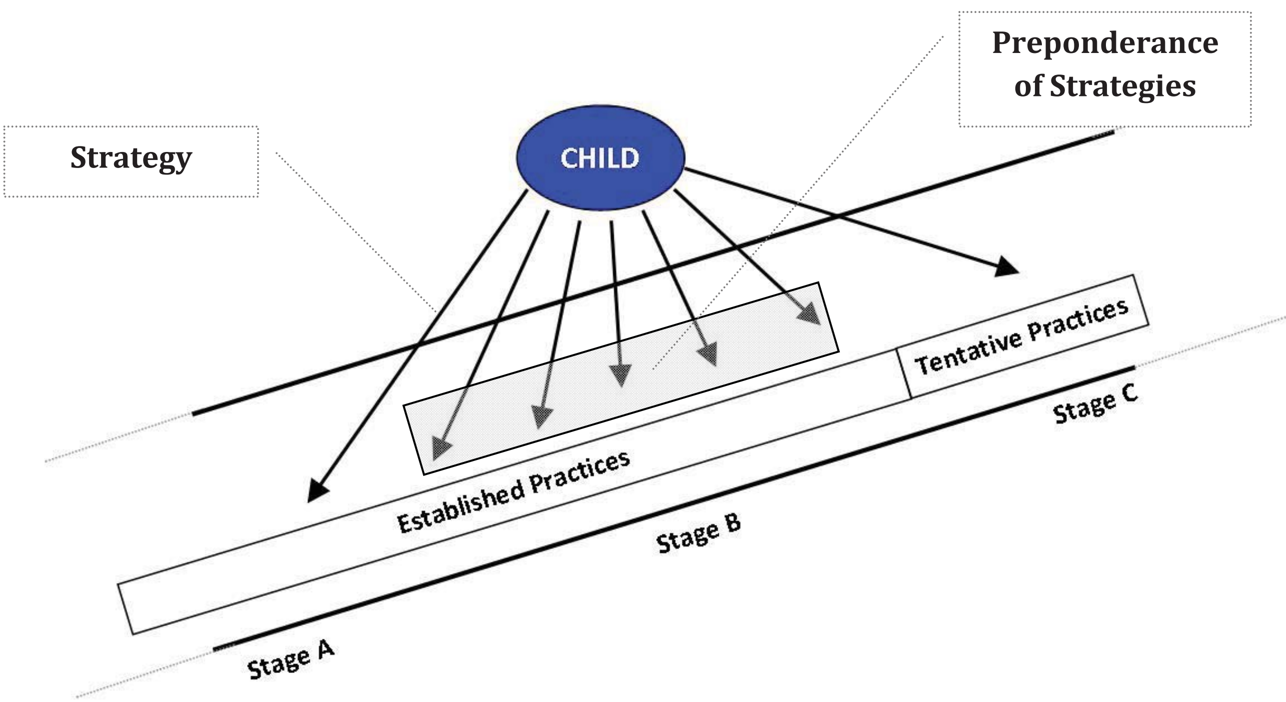

Figure 29: Strategic Variation

Amy would enact emergent counting practices (i.e., omitting perceptual unit items) as these were ostensibly discarded through a process of reflection for more reliable perceptual practices.

Similarly, Amy may, in the very near future, enact practices that suggest the construction of a numerical composite (i.e., adding and subtracting without attending to each figural unit item) based on her capacity to reflect upon and work with figural pluralities. Recall Amy's reflection upon the mechanics of subtraction in the beginning of Chapter Four. While she had yet to become fully aware of and monitor a figural plurality, given her capacity to reflect within the subtractive task, such awareness is likely not far away. On the other hand, the incipiency of her subtractive reflections suggest that it is highly unlikely that Amy would have enacted non-count-by-ones strategies involving multiple abstract composite units indicative of the tacitly and explicitly nested number sequences (Steffe, 1992). In between her perceptual activity and her potential to enact counting-on practices, Amy exhibited a preponderance of strategies that, while still somewhat variable, are more similar in quality than those located at the extremes. I believe that it is this preponderance of strategies that Steffe and his colleagues chose to emphasize when establishing their regularities of variation and subsequent construction of SEAL (Steffe et al., 1983; Steffe et al., 1988; Steffe, 1992). Indeed, Wright et al. (2002) described this strategic preponderance as "robustness" in that strategies of similar quality are enacted often across a range of tasks (p. 38). Quite simply, when faced with tasks aimed at engendering the child's most sophisticated strategy, these are his 'go to' practices when, mathematically speaking, push comes to shove. Ultimately, this model provides us with a *pax exemplar* in that it accommodates both a range of strategic variation and a concentration of somewhat similar strategies that could be considered indicative of stage.

Strategic Preponderances of Allan, Amy, and James

Examining the mathematical practices of the participants does, indeed, reveal considerable variability; however, for each child there does seem to be some strategic preponderance. At times, each child claimed to experience difficulty with screened materials and, subsequently, enacted perceptual counting practices when screens were removed; moreover, on some occasions, both James and Amy constructed perceptual replacements when I carelessly posed tasks that were within finger range (i.e., additive task with sums ≤ 10). Additionally, Amy and James, on rare occasions, enacted counting-on strategies; however, these practices seemed indicative of either a fragile conception or solely procedural understanding. For the most part, Amy and James exhibited a preponderance of practices organized around figural unit items. That is, each participant was often able to successfully negotiate tasks where materials were not sustained in the individual's perceptual field throughout the entire activity. Here, I must note that the rigorous evidentiary standards necessary for manifest content resulted in a relatively small number of identified figural occurrences with respect to the number of figural tasks (see Figure 17); however, there were many rejected instances where figural activity was strongly suspected, but ultimately deemed indiscernible due to a lack of overt behavioral or verbal evidence.

Returning to the preponderance of figural strategies enacted by James and Amy, I must emphasize that the mathematical understandings of these two children are, most certainly, not identical, but, rather, that a sizable portion of each child's practices could rightly be classed as figurative in quality. However, the extent to which their respective practices were rigidly or flexibly connected to the mathematical tool differed somewhat. Thus, a singular label of *figurative* is something of a misnomer for both students.

Allan, also exhibited a tendency to construct and act upon figural unit items, but he provided the most frequent and robust practices indicative of the initial number sequence as well. Arguably, Allan's range of strategic variation is shifted somewhat from that of James and Amy. Interestingly, Allan's practices were categorized as typical of the initial number sequence in both his initial and final diagnostic assessment; however, I also observed many practices typical of the figurative stage as well (i.e., counting from one and continuing the count). Viewing Allan's strategies through the lens of SEAL, it is quite possible that I observed Allan as he was conceptually straddling the figurative stage and the initial number sequence.

Models for Understanding Quantitative Mental Imagery

This notion of strategic preponderance provided the explanatory foundation for subsequent modeling. In this section, I present a series of models that led to the construction of a final (temporarily) viable model.

Scope and Strength of the Temporarily Viable Model

Prior to proposing any hypothetical models of children's mathematical thinking, I must first define the scope and strength of such models. Readers will note that this dissertation research involves but three participants; thus, any attempt to model mathematical thinking as well as subsequent conclusions and implications must be filtered through a lens of this reduced number of cases. In strict terms, the models presented here are descriptive of the participants' mathematical practices and conceptualizations. Undoubtedly, enacting similar teaching experiments with larger numbers of children and of increased duration would provide additional context from which one might consider the viability of the proposed model; however, even with a small number of participants, the act of modeling remains a significant research endeavor.

Indeed, one may consider the activities of Allan, Amy, and James as existence proofs that certain types of practices and conceptualizations occur among children.

Initial Design

Throughout this dissertation, the models constructed by Steffe and his colleagues are an obvious influence. Specifically, SEAL was presented as framework from which I might begin to understand quantitative mental imagery (see Chapter Two). Thus, it is appropriate that SEAL would provide some manner of foundation for the model building activities of this study. On this point, Steffe and Thompson (2000) wrote:

> [W]hen the researcher's goals change sufficiently to warrant conducting another teaching experiment, the current model can be used as input – as conceptual material to be reorganized. However, the primary emphasis should be on constructing superseding models. One model is said to supersede another . . . if it solves problems that the [preceding] model did not solve" (p. 303).

In this particular instance, the models presented here are aimed at describing the nature of participants' quantitative mental imagery and address the problem of qualitatively distinct figurative counting practices beyond the re-presentative mechanisms (i.e., figural, motor, verbal) described by Steffe et al. (1983, 1988).

With this in mind, the initial model (see Figure 30) began with rudimentary outline of SEAL and then attempted to zoom into the figurative stage such that this portion of SEAL was partitioned according to the emergent theme of imagery connectivity. The aim for this initial model was to simply provide some terrain for the consideration of the participants' practices and conceptions. During this construction, I determined that loosely-connected imagery (later termed 'flexibly-connected') existed closer to the construction of the numerical composite than did highly connected imagery (later termed ‘rigidly-connected’). This positioning is in keeping with Steffe’s (1992) assertion that holding figural objects at a distance for reflective inspection (i.e., monitoring figural pluralities) begins to “strip the figurative unit items of their sensorimotor

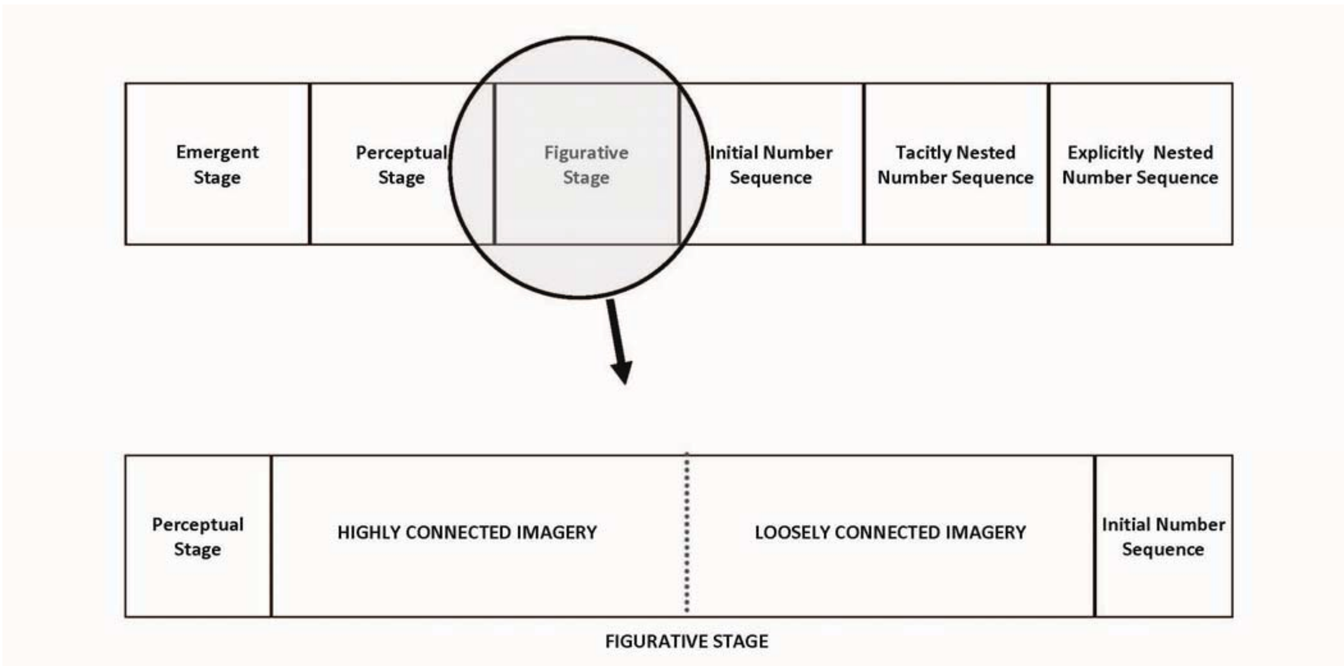

Figure 30: Counting Practices Model: Initial Design

qualities” (p. 93). At this point, children’s constructions begin a transformation from quasi-perceptual images towards abstract units that are not depictive of particular tools.

Also, note the terms for the model ‘sub-stages’, *highly-connected* imagery and *loosely-connected* imagery. At this initial phase of model building, I was still very much engaged in data analysis. While I had identified a qualitative difference between particular figurative practices, I had yet to fully ascertain the nature of this difference. Thus, these fairly undeveloped terms (high and loose) were employed.

Model Re-Design

Having established some manner of foundation for considering these different figurative practices, my next aim was to incorporate these practices into the model (see Figure 31) and several distinct features are worthy of discussion.

First, SEAL was recast as a continuous entity rather than a series of discrete stages. This modification seemed quite necessary to synthesize SEAL with the change processes of

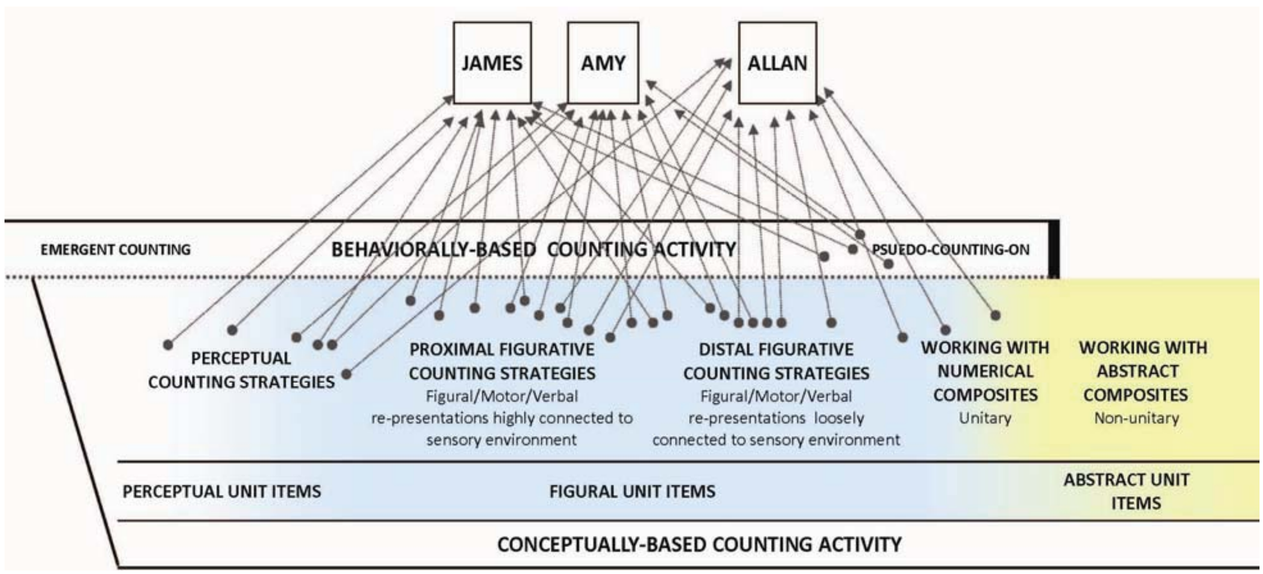

Figure 31: Counting Practices Model: Redesign

cognitive variability. Additionally, SEAL has been layered in terms of observed mathematical practices and the underlying conceptual unit items. Notice, here, that the terminology has changed from high/loose to proximal and distal. At this point, I was focused on the apparent distance between the child's image and the mathematical tool. Also, an additional layer, termed behaviorally-based counting activity, was placed on top of the recast SEAL model. The aim here was to provide a location for observed practices that appeared conceptually fragile or procedurally based (i.e., pseudo counting-on practices of Amy and James).

Second, I introduced the notions of strategic 'tethering' and preponderance. Here, the strategic 'tethers' are not indicative of a single or specific number of enacted strategies, but rather provide a simple graphic display of the participants' strategic range as well as the participants' strategic preponderance within a particular range of SEAL. Another aspect of these strategic tethers is the depth of penetration. Indeed, certain strategies only achieve a 'behavioral depth' while others penetrate to the more conceptually-founded practices. Also, note the placement of the emergent stage in the behavioral layer. Given that such practices appear primarily aimed at approximating the acts of counting and not based in key counting concepts (i.e., cardinality, 1-1 correspondence, etc.) this placement seems appropriate. Lastly, a solid barrier was placed at the end of the behavioral layer approximate with pseudo counting-on practices. After much reflection, I determined that movement beyond such practices and into non-count-by-ones practices requires some conceptual understanding of numbers as composite units. This necessarily categorizes behavioral counting activities as a terminal progression that ends with pseudo counting-on which seems in keeping with manner in which children seem to become 'stuck' on this particular practice (Kinsey & Thomas, 2010; Van de Walle, 2004).

Although this model does, in my estimation, provide a somewhat accurate depiction of the data related to participants' figural practices of differing quality, subsequent testing with a colleague revealed a few distinct flaws. Chief among these was the manner in which behaviorally-based counting activities were presented as joined with more conceptually founded practices. After much discussion, my colleague and I felt that such practices should be separated in terms of their model presentation due to radically different associated understandings. An additional problem is the manner in which each child is depicted as somewhat separated from the mathematics in question. Indeed, this depiction implies something of a representational view of mind (Cobb et al., 1992) that is not in keeping with a constructivist worldview where mathematics exists as a unique creation of an autonomous individual. Because of these substantial criticisms, refinements to the model were necessary.

Counting Practices Continuum: A Temporarily Viable Model

Address of the aforementioned criticisms required a geometric reconfiguring of the model which resulted in the *Counting Practices Continuum* (CPC) (see Figure 32). Indeed, moving the model from two to three-dimensional space allowed for the construction of a polyhedron that better depicted the differences in figural counting practices.

First, note that the behavioral/procedural and conceptual counting foundations exist as parallel 'wings' in the model. This particular structure was constructed to imply the manner in which pseudo counting-on and work with a numerical composite may feature analogous practices, but are ultimately disconnected mathematical activities. Second, the depiction of the participants is such that they are directly connected to the mathematical plane. In fact, these peg-like representations are considered to penetrate this plane into the mathematical understanding beneath. Lastly, the strategic tethers indicate potential movement along the mathematical plane. Here, each participant may enact strategies that *reach forward* towards a different understanding of mathematics, and, consequently, is windlassed in a particular direction by that act. On this point, I must emphasize that this depiction of movement in three-dimensional space via variable, strategic tethers allows for individualized practical and conceptual pathways. For example, James and Amy, on a few occasions, enacted strategies indicative of behavioral/procedural counting foundations while Allan's actions, for the most part, appeared more conceptually-

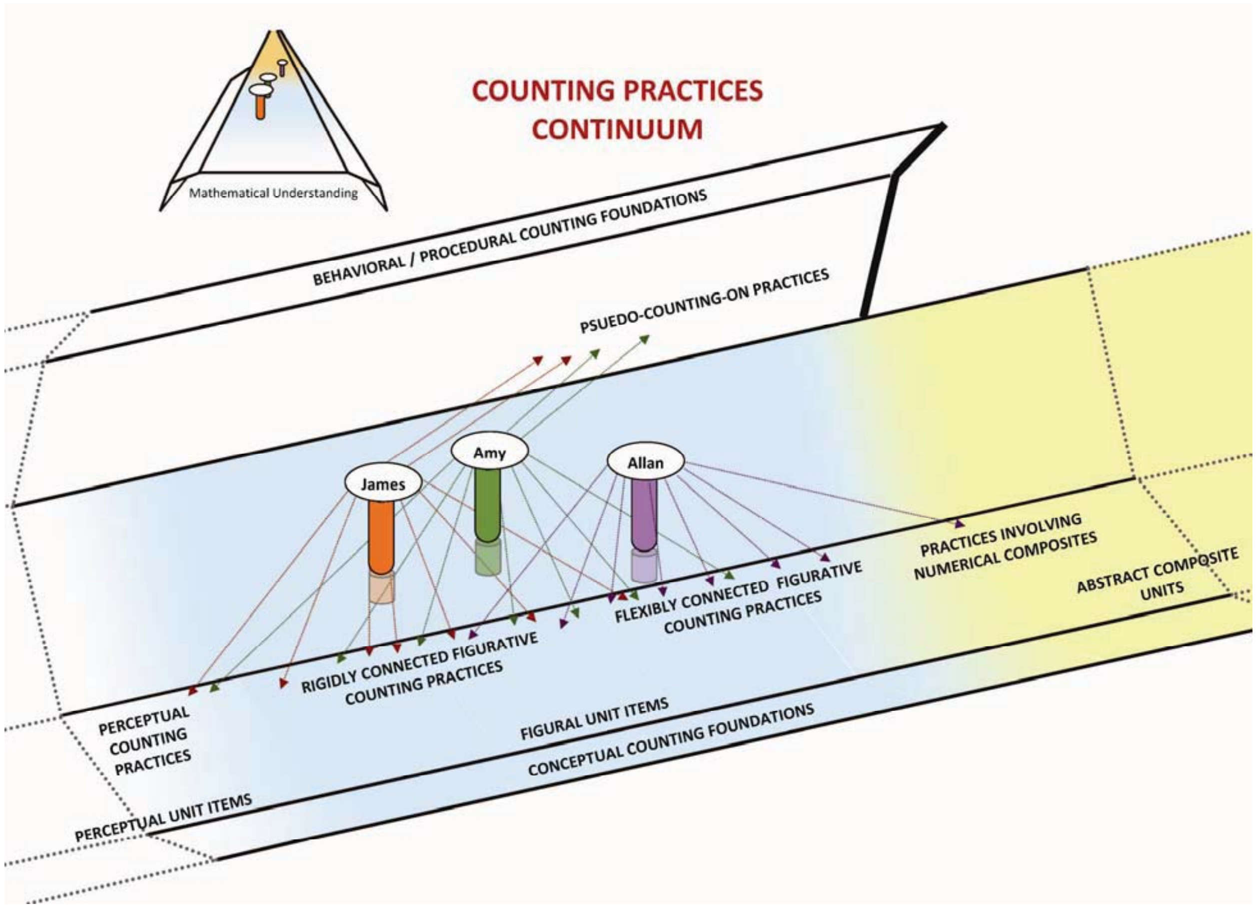

Figure 32: Counting Practices Continuum (CPC)

oriented. Thus, James and Amy's pathway (and associated mathematical understanding) might exist slightly towards the behavioral/procedural side of the model compared to Allan's more conceptually oriented pathway.

Counting Practices Continuum and the Landscape of Learning

Given the existence of many practical/conceptual pathways, the CPC may also be considered an elaboration on the Landscape of Learning (LOL) model (see Figure 33) put forth by Fosnot and Dolk (2001) where "learning and teaching are more a journey across a landscape than a trajectory, or learning line" (Fosnot, 2005, p. 11). Moreover, the proposal of strategic tethering as a mechanism for conceptual advancement is certainly congruent with a psychological constructivist worldview. Indeed, Fosnot and Dolk (2001) propose a similar mechanism of progressive schematization through mathemetizing acts for movement through their landscape.

At this point, though, similarities between the two models end. First off, the primary aim of the CPC was to provide a more detailed view of the participants' transition from work with

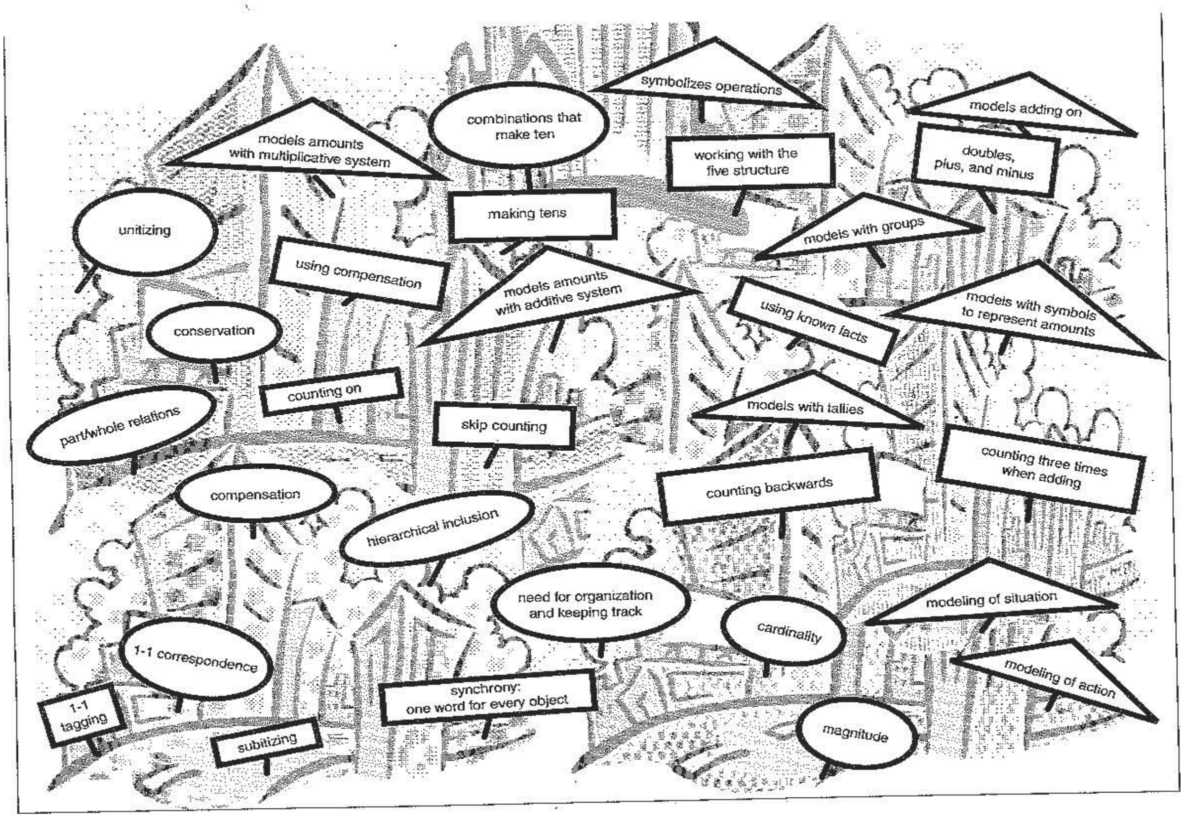

Figure 33: Fosnot and Dolk's Landscape of Learning (Early Number)

perceptual unit items and towards unit items of increasing degrees of abstraction (i.e., figural units, numerical composites). While the LOL model does include many different conceptualizations (denoted by ovals) and practices (denoted by rectangles) related to counting, there is no attention given to the nature of unit that a particular child may be counting; moreover, the LOL somewhat confounds the different aspects of number (see Chapter Two). For example, the practice of “skip counting” is placed lower on the landscape than the concept of “unitizing” (constructing a composite unit) suggesting that “skip counting”, in this instance, should be regarded only as the production of a verbal sequence. Indeed, in order to quantitatively count by units of more than one, one must first be able to construct such a unit. Quite simply, if we were to consider the LOL model as a photograph of mathematical understanding, the resolution would not provide sufficient detail for this study.

Secondly, the CPC needed to account for observed occurrences of fragile or procedurally based practices. Mentioned earlier, a three-dimensional model is useful to position such practices in locations analogous to their more conceptually founded counterparts. On this point, the LOL does not distinguish between conceptually or procedurally founded practices. Returning to the example of “skip counting” presented in the LOL, such a practice could indicate the production of multiple composite units, or simply exist as a rote recitation of a particular number word sequence. While it is true that such a recitation does denote a particular understanding, it is undeniably different than one who considers such a count in terms of composite units. On this point, there are two ideas I must emphasize. First, this distinction is of great importance to interventionists (and all educators) in their struggle to ascertain the mathematical constructions of their students. Second, any model of mathematical practices and concepts should accommodate these diverse foundations for students’ mathematical activity. Ultimately, the LOL fails to provide us with the necessary detail for these types of considerations.

The final and perhaps most important distinction between the two models deals with the location of the individual children. The LOL is presented as a preexisting assemblage of conceptions and practices from which, presumably, a child will select a particular path. Conversely, the CPC does present something of a pre-existing practical/conceptual pathway;

however, individual children are firmly established in the terrain of the model as the constructors of mathematical practices. Moreover, these mathematical practices are not depicted as preexisting entities as in the LOL, but, rather, emerge as spontaneous constructions of the participants in the form of tethers. I submit that this structure is more in keeping with the nature of the participants' mathematical practices.

Locating the Teacher

Thus far, I have presented the CPC in terms of the practices and conceptions of the student; however, I have neglected to discuss a key participant in this enterprise - the teacher. Towards this end, I have provided a closer view of the interactions between participant and teacher (see Figure 34). For the sake of clarity, I did not depict the teacher on the CPC. In this particular study, the model would have appeared to represent three teachers, something I deemed somewhat confusing. Here, we see the student and teacher existing in close proximity and connected by tasks (and tools by implication), questions, and practices. For these teaching experiments, these connections were reflexive of one another. Consider the following exchange with Allan:

> TEACHER: [places a blue translucent counter over the 15^{th} dot] "Fifteen and now I cover up four dots" [places screen over the next four dots (16^{th}, 17^{th}, 18^{th}, 19^{th}) and runs finger across screen in a linear motion]
>
> TEACHER: "This is fifteen." [points at 15^{th} dot] "What is this number?" [places a blue translucent counter over the 20^{th} dot]
>
> ALLAN: [touches the 15^{th} dot] (audibly whispers) "fifteen"
>
> ALLAN: [raises index finger towards screen] "Twenty"

TEACHER: “Twenty? How do you know it’s twenty?”

ALLAN: “Because you counted fifteen” [touches the 15th dot] “sixteen, seventeen, eighteen, nineteen” [touches the screen 4 times in a linear pattern] “twenty” [touches the 20th dot]

TEACHER: “What are you counting under here?” [taps the screen two times in the same location] “What are you thinking about?”

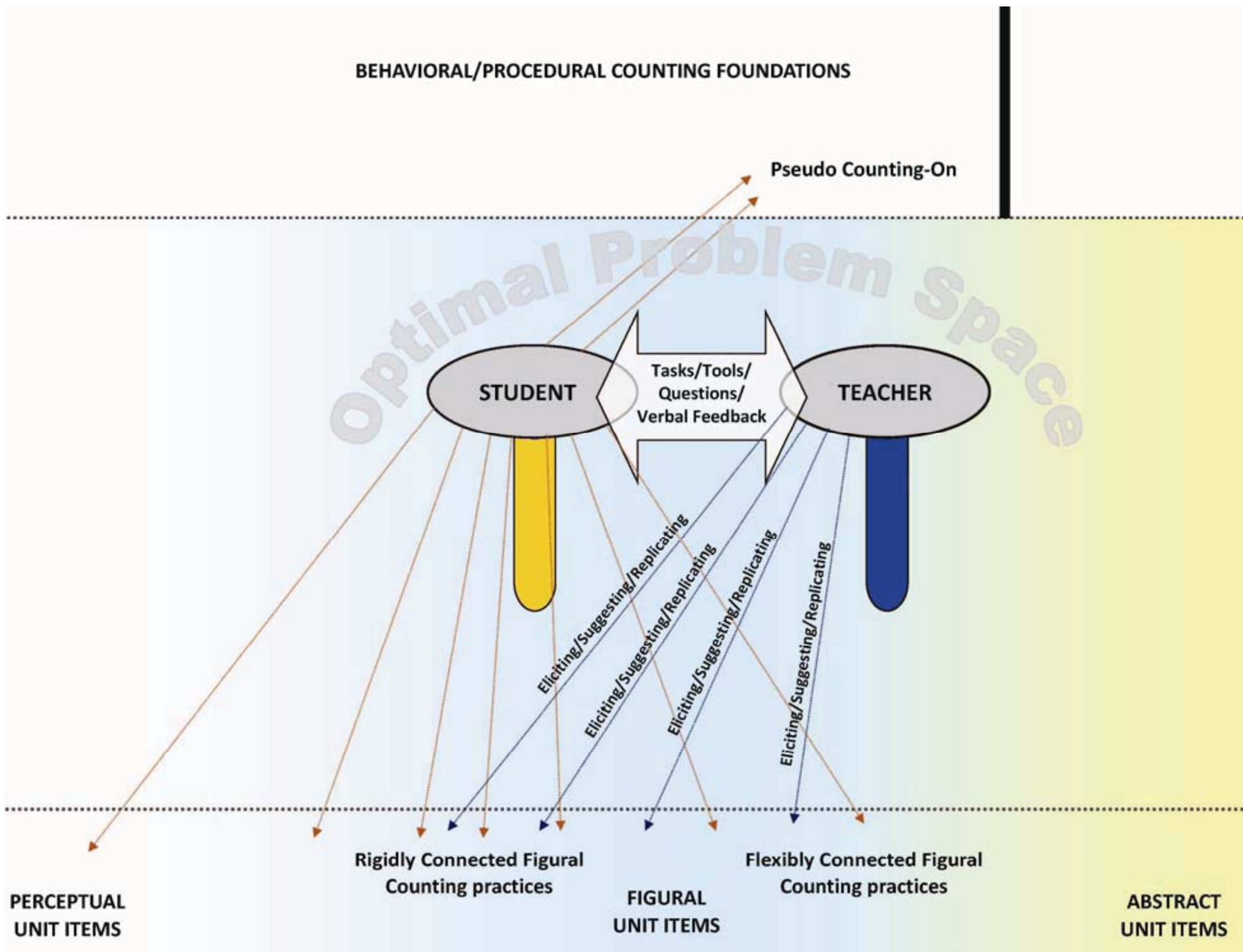

Figure 34: Intervention Interactivity Model

ALLAN: “One and two and three and four” [touches the screen 4 times in a linear pattern]

TEACHER: “What did those things look like?”

ALLAN: “Like these” [points at unscreened dots]

Here, I posed a particular task which was almost certainly founded on an interpretation of Allan’s prior practices and offered some subtle suggestion of how he might interact with the mathematical tool in question. After acknowledging that a particular dot was “fifteen,” Allan then engaged in certain mathematical practices that included audible utterances resulting in a response of “twenty.” Because of this response, I posed a probing question and attempted to reenact the behavior that I just observed. Allan then responded verbally and reenacted his own motor acts which, then, prompted me to ask another probing question. The point, here, is that we were operating reflexively (or interdependently) towards one another and this exchange could easily be continued into the next task (and the next), and presented as a shared, mathematical experience that was in the process of continuous negotiation. This type of interaction is congruent with purist interpretations of the zone of proximal development that emphasize the dynamic power of an instructional dyad consisting only of teacher and pupil (Puntambekar & Hubscher, 2005).

One may recall, though, that, there were many instances in which I did not elicit, suggest, or replicate particular behaviors; however, each of these instances is necessarily centered on a particular task that had been negotiated by student and teacher to some degree. Here, I will freely admit that significant work remains to fully explicate the interactive processes of the dynamic 1-1 intervention experience; however, for the purposes of this study, I considered the participant and teacher as reflexively connected co-actors in the enterprise of practicing and conceiving mathematics.

The Nature of the Image

Perhaps at this point, it is useful to return to the first research question of this dissertation, "How do children in the early elementary grades (ages 5-8) describe and reveal through activity, the mental images that they invoke when working with different arithmetic tasks and tools?" Having reached this point, what may we conclude regarding the nature of a particular child's quantitative mental image? Perhaps the strongest conclusion that may be drawn from this research is that individual's *may* construct mental imagery of varying connectivity with respect to the mathematical tool. Indeed, Allan, Amy, and James each enacted practices indicative of mental imagery that was either rigidly or flexibly-connected to the involved tools; moreover, there were a few occasions where this connection appeared to shift within the context of a single task. Additionally, there is some evidence to suggest that the quantitative and spatial aspects of a constructed image may sometimes conflict with one another, and also that sensory materials outside of the immediate task space may infiltrate the mental imagery (see Appendix A); however, these themes remain somewhat elusive and would benefit from further exploration.

How Does Imagery Change over Time?

Any evidentiary basis for any progressive change in imagery is simply not present in this study. Recalling the analysis of imagery type by phase (see Figure 27), one may only conclude that the participants' images varied in terms of their connectivity, and, strictly speaking, may not prescribe any particular trajectory to these variations. From this, we may conclude either that imagery constructions (in terms of their connectivity) do not adhere to any manner of trajectory or that the duration of the teaching experiments was insufficient to observe such a change. On this point, I argue that the latter explanation is more plausible, and, the longer-term teaching experiments conducted by Steffe et al. (1983, 1988) provide some insight into this matter.

Indeed, those teaching experiments often took place over the course of an entire year and documented many instances of practical/cognitive progression among individual children. Moreover, this observed progression was such that children appeared to first construct perceptual unit items, and as time passed, constructed unit items of increasing abstraction. For this reason, I have organized the CPC such that imagery first exists as a rigidly-connected entity and then as a more flexibly-connected entity as the child begins to "strip the figurative unit items of their sensorimotor qualities" (Steffe, 1992, p. 93).

Implications

Implications for Teaching and Learning

Turning our attention to the meaning of such conclusions in terms of teaching practices, two particular themes emerge for discussion.

Diagnosing with precision. First, the conclusions of this dissertation could affect the manner in which intervention teachers (and all teachers, for that matter) appraise the mathematical conceptions of their students. While practices denoting rigidly-connected images may be considered prototypical of figurative activity, teachers must also attune to the less overt practices indicative of flexibly-connected imagery. Indeed, both types of imagery construction correspond to the production of figural unit items; however, it is easy to see how one might misdiagnose or be confounded by the flexibly-connected image. Additionally, the notion of strategic variation and preponderance has the potential to shift how teachers think about and organize ideas such as stage. For example, referring to Allan as a 'figurative child' is something of a misnomer as he did, on a few occasions, appear to construct and work from numerical composites. Perhaps a better way to describe Allan is that he acts robustly in a flexibly-figural manner within a particular range (perceptual to the initial number sequence). Conversely, Student 1 from the imagery pilot study (see Chapter Three) might be described as acting robustly perceptual within a particular range (emergent to rigidly-figurative). Indeed, introducing the range of variation into how we describe students' activity provides a more accurate description of the individual child in terms of his/her mathematical present capabilities, and, perhaps more importantly, greater insight into where teachers might enact productive teaching experiences

Supporting individual imagery constructions. The second idea worthy of attention is the manner in which teachers flexibly accommodate the different imagery constructions of their students. Hypothesizing that quantitative mental imagery progresses from rigidly to more flexibly-connected images, it follows that intervention experiences would be designed to engender such changes in imagery construction. For example, a teacher working with children considered to be robust counters of rigidly-connected imagery, such as James or Amy, would likely want to design experiences that would engender imagery constructions that were more flexibly connected to the mathematical tools, and, ostensibly, this would require some understanding of the different ways the child re-presents when working with particular tools. This more nuanced instructional tack is considerably different from one where the same child (likely classified as a 'figurative counter' based on his prototypical activity) would then participate in experiences designed to engender the construction of a numerical composite (Wright et al, 2002).

While this may seem a fairly minor point, considering the potential differences in tasks illuminates the importance of this issue. In supporting the construction of a numerical composite, one powerful task to help students curtail their propensity to attend to each figural unit item is to modify the quantities such that this activity becomes cumbersome. For example, I might pose the screened additive task 34+3. Notice, here, that the large first addend is designed to create a situation where attending to each figural unit of the task (counting from one and continuing the count) is not impossible, but somewhat unwieldy. Similarly, the second addend is designed to be tantalizingly small so that the child might reflect on the task and conclude that starting her count at one is unnecessary.

Considering robust counters of rigidly-connected figural unit items, though, we find that the ‘counting-on’ task above is somewhat less than ideal. From the perspective of the CPC, my aim would be to reduce the rigidity of the student’s imagery construction by simply providing experiences where he may enact many different types of figural practices. Note, the manner in which the quantities are apportioned in the ‘counting-on’ task described above may be considered discouraging of figurative activity as the bulk of the figural unit items occur in the first unwieldy addend. Here, the student might refuse to engage the task (perhaps asking for perceptual items to count) or enact a strategy founded solely on some procedural understanding (i.e., pseudo counting-on). Given the need to maximize each and every moment of the intervention experience, I argue that such instructional distinctions are of great importance.

In addition to the potential for refinements in terms of instructional frameworks and progressions, analysis of the different patterns of interaction between the student and teacher provide some terrain to consider the manner in which micro-level teaching tactics influence the mathematical practices of children. Specifically, the proposal of distinct categories (i.e., behavior-eliciting, behavior-suggestion, behavior-replication) provide intervention teachers with a useful perspective from which to consider and reflect upon their own practices with children.

Implications for Future Research

Nature of mathematical tasks and tools. In the previous section, I described the manner in which a task, previously deemed to be highly individualized, might constrain a student's imagery construction according this the more nuanced CPC model. What I am unable to conclude, at this point, is the nature in which particular tools influence these experiences other than providing the terrain for variable constructions. Here, a detailed analysis of mathematical practices (and associated conceptions) with respect to particular tools is needed to better understand whether certain tools are more supportive of particular image types or, conversely, (and more likely, in my opinion) that each child has an individual grouping of tools, uniquely associated with his/her experiences, that are leveraged for greater reflective abstraction (compared to other tools outside of this grouping). In any event, more study is needed to comment definitively on the matter.

Definitive change over time. Another area that lacks definition deals with how imagery changes over time. Indeed, Steffe et al. (1983, 1988) provide some guidance on this topic and their results suggest that imagery constructions likely move from rigid to flexible in terms of tool connectivity. With that said, the results of this study are inconclusive, and to strengthen the CPC as a temporarily viable model, longer-term teaching experiments are likely necessary to ascertain the manner in which children's imagery constructions change over time.

Interactive quality of mathematics intervention. An important aspect of the intervention teaching experiments deals with the interactive quality between teacher and student. In this study, I endeavored to provide some categorical descriptions of these interactions; however, more detailed inquiries that emphasize interactions (rather than mathematical practices and their associated conceptions) would likely provide a more nuanced understanding of this social location. First and foremost, I am certain that some additional work in this area would likely lead to useful refinements in the categories of eliciting, suggesting, and replicating. While some measure of categorical stability was achieved for this study regarding figural task interactions, examination of interactions involving different types of numeracy tasks (i.e., verbal, symbolic) would likely provide greater insight into this area. Also, a longitudinal analysis of interactions could be useful to better understand how different teaching acts influence students' practices beyond the individual task. For example, one might examine the extent to which initial suggesting and replicating acts reverberate throughout a mathematics intervention cycle. In any event, study of social interactivity in the context of 1-1 mathematics intervention remains quite rare. Given the increasing emphasis on intensive intervention experiences in the early elementary grades (Ardoin, Witt, Connell, & Koenig, 2005; KCM, 2009; KDE, 2007), a more nuanced understanding of such interactive qualities would likely prove beneficial.

Chapter Summary and Conclusion

First off, I have worked to present an iterative process of model design that attends to the observed mathematical practices in terms of their variation as well as their foundations in particular unit items (or learned procedures). Here, a strategic preponderance within a certain self-constrained range provides the opportunity to consider the child's practices and conceptions in more detail than can be provided by the construct of *stage*. Returning to the research questions of this dissertation, the participants appeared to construct imagery that varied in its connection to the mathematical tool in question, and the CPC provides a model for understanding this strategic variability (and preponderance) with respect to the construction of particular unit items. Additionally, while I was unable to detect any stable changes in this imagery over the course of the teaching experiments, Steffe and his colleagues provide some insight into this matter.

Ultimately, though, teaching experiments conducted over longer periods of time would be necessary for definitive comment.

At this point, I must, again, emphasize that these matters are of no small importance to the many thousands of mathematics intervention specialists working with individual students around the world. Indeed, these individuals, including myself, must attune their teaching practices to the greatest possible detail, and increased insight into the obscure realm of quantitative mental imagery may result in more productive and efficient intervention experiences for both teachers and students. Let's begin.

"I'm waiting for a sign. I'm standing on the road. My mind outstretched to you.
I'm picking something up. I'm letting something go."

~Neil Young: Pictures in my Mind

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Appendix A

Additional Figurative Themes

"What you don't know, you can feel it somehow"

~U2: Beautiful Day

Sensory Interference

In a few instances, perceptually available materials that were not intended to serve as mathematical tools appeared to infiltrate the practices and conceptions of the participants. Indeed, what seemed to have occurred was some degree of unplanned sensory interference from surrounding objects. Consider the following exchanges with Amy and James.

Exchange one.

TEACHER: "Okay, you've got them how you want them. Ready?" [places screen over linear arrangement of four objects (from left to right): car, bear, rooster, frog]

TEACHER: "What's on this end?" [touches far right-hand side of screen]

AMY: (pause 8 seconds) "Frog"

TEACHER: "How many things came before that frog?" [drags finger across screen from right to left]

AMY: [touches screen three times in a linear pattern (from right to left)]

AMY: "I think I put five things"

TEACHER: "You think so? You want to have another look?" [raises screen]

AMY: "one, two, three, four, oh"

TEACHER: [lowers screen]

TEACHER: "So there is a frog" [touches far right-hand side of screen] "How many things came before the frog?" [drags finger across screen from right to left]

AMY: [looks across table towards shell and block (placed visibly in front of teacher) and touches screen in a linear pattern (from right to left)] “Rooster, shell . . .Oh! I didn’t get the shell” [motions towards shell in front of teacher]

Exchange two.

TEACHER: [presents a stack of 3 animal cards *face-down* to student] “Now, I’ve got three, how many pony legs?”

JAMES: [looks across table, taps the table 12 times in rhythmic patterns of two’s] (whispers) “one-two, three-four, five-six, seven-eight, nine-ten, eleven-twelve”

JAMES: “Because I saw these” [reaches across table and touches a single *face-up* animal card laying with other materials]

TEACHER: “Oh you saw those?” [reaches across table and removes animal card]

JAMES: “Twelve!”

TEACHER: “Gotcha”

Exchange three.

TEACHER: “We don’t need shells for this one. I want you to pretend for me”

TEACHER: “Pretend each one of these cups has five shells under it” [places 5 cups in a row in front of the student] “How many shells in all?”

JAMES: [touches side of first cup 2 times] (whispers inaudibly)

JAMES: [touches 4 of the printed dots on side of second cup] (whispers inaudibly)

JAMES: [touches 5 of the printed dots on the side of third cup] (whispers inaudibly)

JAMES: [touches 5 of the printed dots on the side of the fourth cup] (whispers) “. . . seventeen, eighteen, nineteen, twenty”

JAMES: [touches five of the printed dots on the side of the fifth cup] (whispers inaudibly)

JAMES: “Twenty-seven”

In the first two exchanges, the participants appeared to (or purported to) notice a perceptually available tool in another location and appropriate this tool into their mathematical task space (and, likely, their imagery). Particularly noteworthy is the second exchange where James leveraged a single animal card to construct a figural plurality (Steffe, 1992) of animal legs. This instance (as well as Amy’s incorporation of the shell into her figural task space) reinforced the notion that re-presentations are malleable acts that may be influenced by one’s immediate surroundings.

In the third exchange, James interacted with the tool in an unintended fashion (i.e., counting the printed dots on the side of the cup). Interestingly, there must have been some manner of unit tracking as there were far more than five printed dots on each cup; however, the presence of perceptual items in this instance confounds any conclusion regarding the involvement of figural imagery. Perhaps the only conclusion to be drawn from the third exchange is that James elected to count perceptual unit items when given the opportunity.

Spatial/Quantitative Conflict

Another supplementary figural theme dealt with observed conflicts between spatial and quantitative apprehension (and subsequent re-presentation) of a particular tool. Specifically, on a few occasions, participants’ appeared to simultaneously enact quantitative and spatial practices of distinctly different quality. Below are exchanges involving Allan and James.

Exchange one.

TEACHER: “Here are twelve.” [places blue translucent counter on the 12^{th} dot]

TEACHER: “And, I am going to cover up five.” [places screen over the 13^{th}, 14^{th}, 15^{th}, 16^{th}, and 17^{th} dot]

TEACHER: “What’s this guy?” [places blue translucent counter on the 18^{th} dot and taps it]

ALLAN: “That is what?” [touches the counter on the 12^{th} dot]

TEACHER: “That is twelve.”

ALLAN: “Twelve” [touches counter on the 12^{th} dot] “Thirteen, fourteen, fifteen, sixteen” [points at the screen four times in a linear pattern] “Seventeen?” [touching the counter on the 18^{th} dot]

TEACHER: “This is seventeen?” [touching the counter on the 18^{th} dot] “How many were under here?” [motions towards the screen]

ALLAN: “Five”

TEACHER: “Okay, so this was twelve.” [touches the counter on the 12^{th} dot] “Then, what did you do?”

ALLAN: “Thirteen, fourteen, fifteen, sixteen, seventeen”

TEACHER: [touches the screen in a linear pattern as student counts ensuring that the touch coinciding with the utterance “seventeen” occurs near the edge of the screen next to the unscreened 18^{th} dot]

ALLAN: “That guy is seventeen” [touches counter on the 18^{th} dot]

TEACHER: “You think that guy is seventeen?” [touches counter on the 18^{th} dot]

TEACHER: “Okay, let’s check it. How many dots were under here?” [touches screen]

ALLAN: “Five”

TEACHER: [removes screen] “So this is twelve.” [taps the counter on the 12^{th} dot]

ALLAN: “thirteen, fourteen, fifteen, sixteen, seventeen” [touching the 13th, 14th, 15th, 16th, 17th dot] ***“eighteen”*** [touching the counter on top of the 18th dot]

Exchange two.

TEACHER: [presents card with eight blue dots (structured in a dice patterns of 6 and 2) for one second and then conceals card]

JAMES: [makes eight pointing gestures in approximately the same location] (whispers audibly) “One, two, three, four, five, six, seven, eight”

JAMES: (whispers) “eight”

TEACHER: (whispers) “eight”

JAMES: [begins drawing dot pattern on index card in front of him]

JAMES: [draws two columns of dots side by side, the first column with 4 dots and the second column with 5 dots]

JAMES: “There are some more.” [makes two quick pointing gestures]

TEACHER: “Oh, there are some more? So, how many are there?” [points at student’s drawing]

JAMES: [touches each of the drawn dots] (whispers audibly) “One, two, three, four, five, six, seven, eight, nine”

JAMES: [draws three more dots in a column on the other side of the card]

TEACHER: “Okay, so how many dots are on your card?”

JAMES: [points at each drawn dot] (whispers audibly) “One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve”

TEACHER: “Okay, so twelve dots. . . I am going to show it to you again and you tell me if you need a new card or not”

TEACHER: [presents dot card for one second and then conceals card]

JAMES: [makes eight pointing gestures in approximately the same location] (whispers audibly) “One, two, three, four, five, six, seven, eight”

JAMES: “Eight. . .I need a new card”

TEACHER: [places blank index card in front of student] “Okay, what did it look like?”

JAMES: [draws eight dots structured in 6 and 2 dice patterns]

In the first exchange, Allan acknowledged that five dots had been concealed; however, his spatial assessment of the length of the screen (compared to the surrounding dots) appeared to result in the creation and subsequent counting of only four figural unit items. Similarly, when James was briefly presented with a pattern of eight dots, he seems to quickly apprehend the quantitative nature of this pattern; however, when he attempts to replicate this pattern (ostensibly from a figural re-presentation), he appeared to emphasize the spatial rather than quantitative aspects of his image.

Summary

Indeed, I must note that the occurrences that typify these supplementary figural themes were infrequent; however, the mere existence of such practices (and associated mathematical conceptions) likely provides some terrain for additional study.

Appendix B

Figures

"So remember, every picture tells a story, don't it?"

~Rod Stewart: Every Picture Tells a Story

SOCIAL PERSPECTIVE	PSYCHOLOGICAL PERSPECTIVE
Classroom social norms	**Beliefs about own role, others' roles, and the general nature of mathematical activity in school.**
Socio-mathematical norms	**Mathematical beliefs and values**
Classroom mathematical practices	**Mathematical Conceptions**

Figure 1: Cobb and Yackel's Interpretive Framework

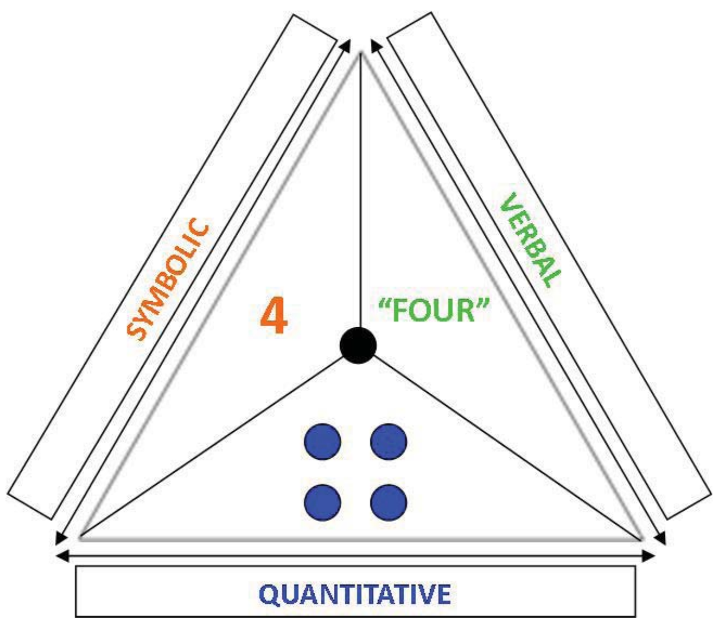

Figure 2: Thomas, Tabor, and Wright's (in press) Three Aspects of Number

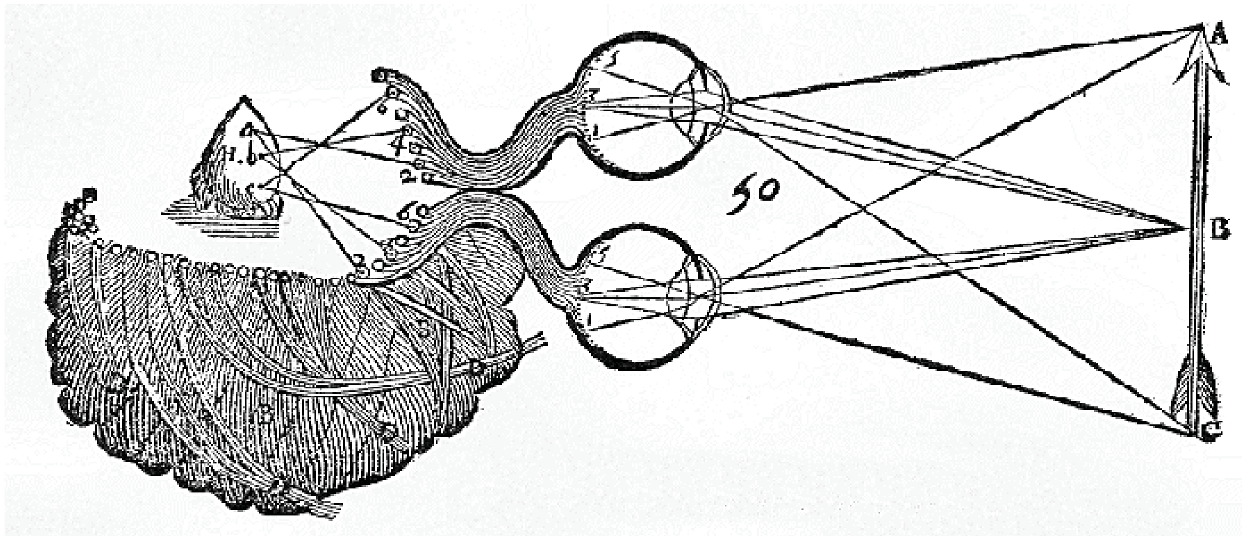

Figure 3: Diagram from Descartes' *Treatise of Man* (1664) depicting the pineal gland

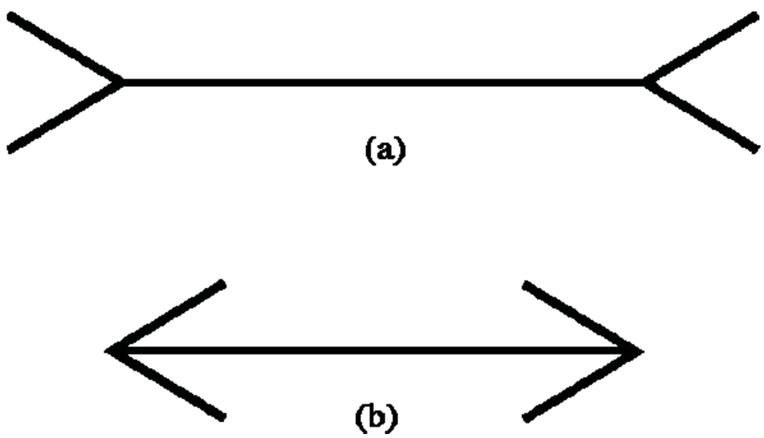

Figure 4: Müller-Lyer illusion

Figure 5: Kosslyn and Schwartz’s simulation of a mental image: car

Figure 6: Kosslyn's (1973) speedboat image

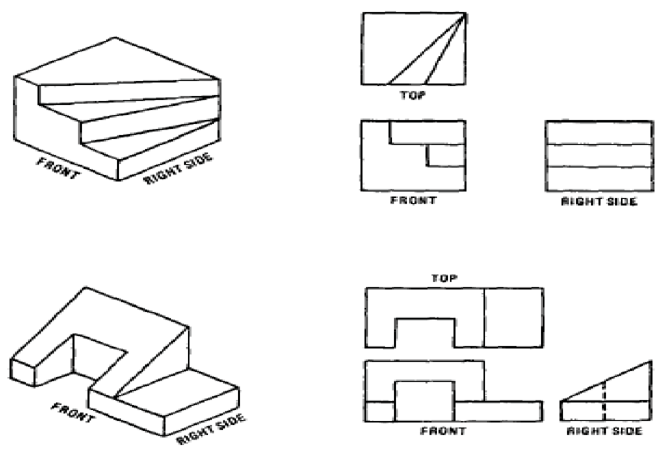

Figure 7: Cooper's (1990) isometric (left) and orthographic (right) projections

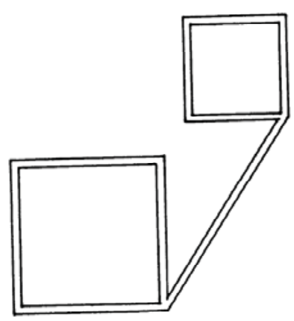

Figure 8: Deregowski and Bentley's (1987) construction task drawing

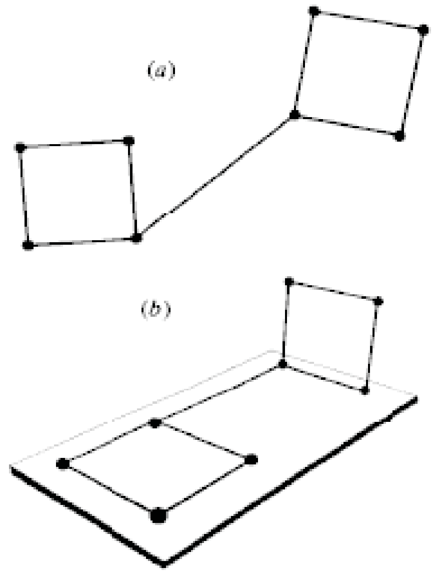

Figure 9: Deregowski and Bentley's examples of a nonintegrated 2D response (a) and a nonintegrated 3D response (b)

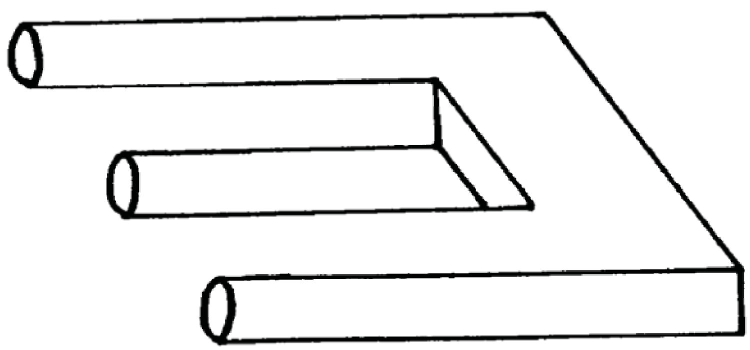

Figure 10: Deregowski and Bentley's (1987) impossible trident

MATH RECOVERY LEARNING FRAMEWORK IN NUMBER INITIAL STUDENT PROFILE		
Student	**Arithmetic Stage**	**Structuring Level**
Allan	**STAGE 3-INITIAL NUMBER SEQUENCE:** Student uses counting-on rather than counting from "one" to solve addition or missing addend tasks (e.g. 6+x=9). The student may use a count-down-from strategy to solve a removed item task (e.g. 17-3 as 16,15,14-answer 14) but not count-down-to strategies to solve missing subtrahend tasks (e.g. 17-14 as 16,15,14-answer 3).	**LEVEL 0-EMERGENT:** The student can subitize only small quantities (up to 3) and relies on counting to quantify larger groups. The student builds finger patterns by raising fingers sequentially.
James	**STAGE 2: FIGURATIVE COUNTING:** Can count the items in a screened collection but counting typically included what adults might regard as redundant activity. For example, when presented with two screened collections, told how many in each collection and asked how many counters in all, the student will count from "one" instead of counting on.	**LEVEL 0-EMERGENT:** The student can subitize only small quantities (up to 3) and relies on counting to quantify larger groups. The student builds finger patterns by raising fingers sequentially.
Amy	**STAGE 1-PERCEPTUAL COUNTING:** Can count perceived items but not those in screened (i.e. concealed) collections [without identifying or generating perceptual replacements]. This may involve seeing, hearing, or feeling items.	**LEVEL 1-FACILE STRUCTURES TO 5:** The student can subitize regular spatial patterns to 6 and irregular spatial patterns to 5. The student can create finger patterns in the range of 1-5 by raising fingers simultaneously. The student is able to combine and partition numbers in the range of 1-5 without counting.

Figure 11: Math Recovery Initial Student Profiles-Arithmetic-Stage, Structuring-Level

Student	Forward Number Word Sequence (FNWS)	Backward Number Word Sequence (BNWS)
Allan	**LEVEL 3: FACILE WITH FNWS UP TO TEN:** The student can produce the FNWS from "one" to "ten". The student can produce the number word just after a given number in the ranger of "one" to "ten" without dropping back. The student has difficulty producing the number word just after a given number word for numbers beyond "ten".	**LEVEL 3: FACILE WITH BNWS UP TO TEN:** The student can produce the BNWS from "ten" to "one". The student can produce the number word just before a given number word in the range of "one" to "ten" without dropping back. The student has difficulty producing the number word just before a number word for numbers beyond "ten".
James	**LEVEL 4: FACILE WITH FNWS UP TO THIRTY:** The student can produce the FNWS from "one" to "thirty". The student can produce the number word just after a given number in the ranger of "one" to "thirty" without dropping back. Students at this level may be able to produce the FNWS beyond "thirty".	**LEVEL 3: FACILE WITH BNWS UP TO TEN:** The student can produce the BNWS from "ten" to "one". The student can produce the number word just before a given number word in the range of "one" to "ten" without dropping back. The student has difficulty producing the number word just before a number word for numbers beyond "ten".
Amy	**LEVEL 3: FACILE WITH FNWS UP TO TEN:** The student can produce the FNWS from "one" to "ten". The student can produce the number word just after a given number in the ranger of "one" to "ten" without dropping back. The student has difficulty producing the number word just after a given number word for numbers beyond "ten"	**LEVEL 3: FACILE WITH BNWS UP TO TEN:** The student can produce the BNWS from "ten" to "one". The student can produce the number word just before a given number word in the range of "one" to "ten" without dropping back. The student has difficulty producing the number word just before a number word for numbers beyond "ten".
Student	**Numeral Identification**	
Allan	**LEVEL 2-NUMERALS TO 20:** Can identify numerals in the range "1" to "20"	
James	**LEVEL 3-NUMERALS TO 100:** Can identify one and two digit numerals	
Amy	**LEVEL 2-NUMERALS TO 20:** Can identify numerals in the range "1" to "20"	

Figure 12 Math Recovery Initial Student Profiles-FNWS, BNWS, NID

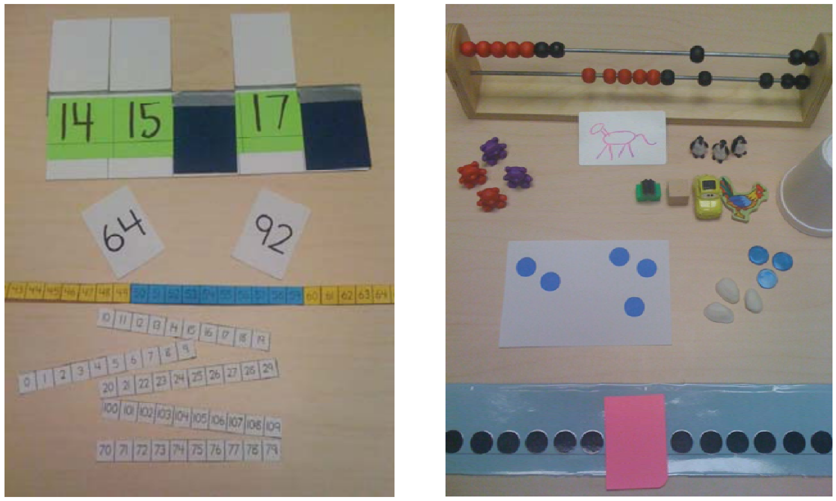

Figure 13: Convention-oriented Tools (left) and Quantitatively-oriented Tools (right)

A	B	C	D	E	F	G	H	I	J	K	L
Clip #	Date	Trnscrpt	Segment	random select 1	Imagery (Y/N)	Behavior Eliciting/Suggesting/ Replicating (E/S/R or N[none])	Student	Tool	Task	Connectivity (H/L/D/na)	Motor Ac
17-02-422C	422	0	44		N	N	CC	Bead Rack	7(6/1)	na	f-patterns
03-413C	413	1	5		N	R	CC	Bears/Shells	15comp9	na	
04-413J	413	1	8		N	N	JN	Counters	15-5	na	
22-505J	505	1	55		N	R	JN	Numeral Cards	9+6	na	F-Pattern
03-413J	413	1	7		N	R	JN	Numeral Cards	12+3	na	
12-420J	420	1	26		N	na	JN	POST-TASK Row Task	POST TASK-(12to16)	na	
05-415J	415	1	9		N	R	JN	Shells/Cups	3shells/cup-4cups	na	F-Pattern
01-330Q	330	1	1		N	R	QR	Counters	11+3	na	
01-401C	401	0	2		Y	N	CC	10-frame (pairwise)	7 (3 top 4 bottom)	D	none
26-503C	503	1	61		Y	N	CC	Bears/Shells	12comp5	D	f-patterns at end
02-415C	415	0	15		Y	S	CC	Dot Cards-array	3X3	D	taps table in sar
02-401C	401	1	3		Y	N	CC	Dot Cards-random	7	D	head movements
02-01-401C	413	0	4		Y	N	CC	Dot Cards-random	7	D	none
25-503C	503	1	60		Y	S	CC	Random Objects	5 spatial [quant-questions]	D	none
24-07-503C	503	0	59	X	Y	S	CC	Random Objects	4spatial (quant questions)	D	none
14-420C	420	0	31		Y	N	CC	Row Task	11to16	D	none
24-503C	503	1	52		Y	N	CC	Animal Cards	5cards [STACKED]	H	touches figura
18-422C	422	1	45		Y	N	CC	Animal Cards	4cards	H	touches figura
24-01-503C	503	0	53		Y	R	CC	Animal Cards	3cards[Stacked and then Row]	H	F-patterns/touches
19-422C	422	1	46		Y	R	CC	Animal Cards	6cards [STACKED]	H	touches table next to
14-01-421C	421	0	34		Y	R	CC	Animal Cards	6cards	H	touches figura
01-421C	421	0	32	X	Y	R	CC	Animal Cards	4cards	H	air-points' towar

Figure 14: Screenshot of Content Database

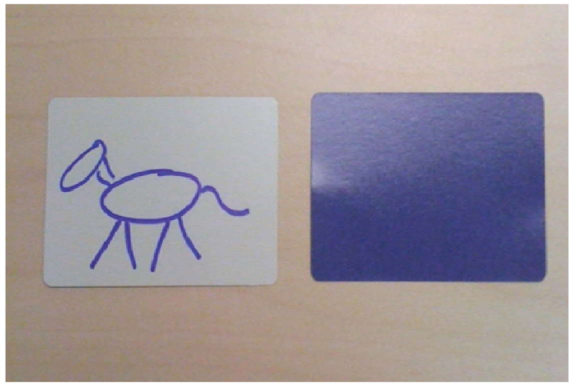

Figure 15: Animal Card Face-Up and Face-Down

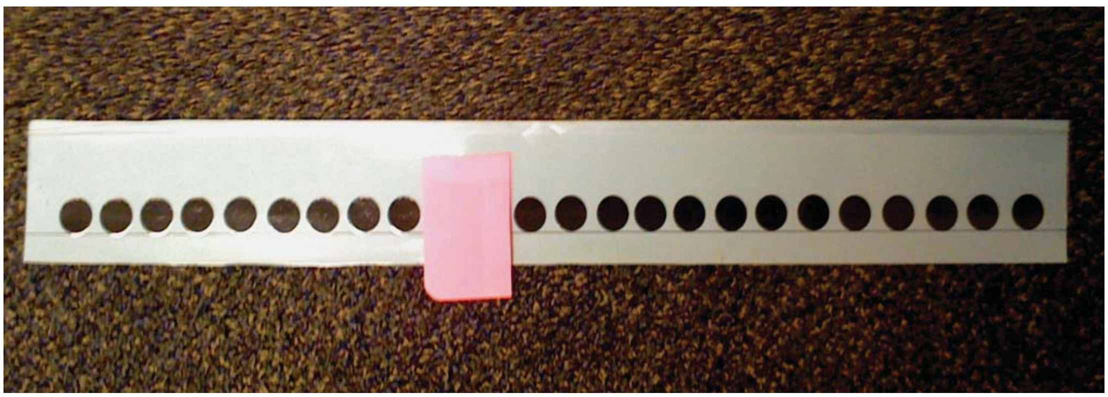

Figure 16: Row Task with Screen

Student	Number of Figural Task Groups	Number of Figural Tasks	Number Observed Figural Occurrences	Percent of Figural Occurrences per Figural Tasks	Number of Figural Occurrences per Task Group
Allan	45	195	52	26.7%	1.16
Amy	45	174	49	28.2%	1.09
James	32	151	33	21.9%	1.03

Figure 17: Figural Occurrences by Student

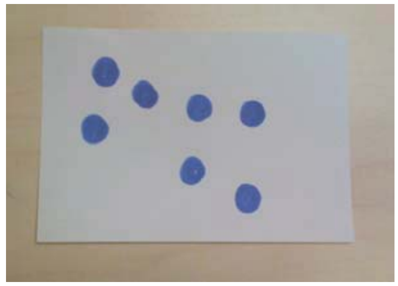

Figure 18: Unstructured 7 Dot Card

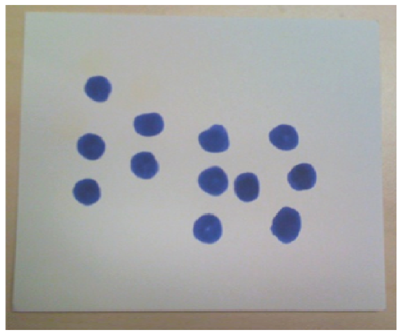

Figure 19: Amy's Drawing of 7 Dot Card (Recreated from Video Record)

Figure 20: Penguins and Cup

Figure 21: Cup and Random Objects

Figure 22: Screen and Random Objects

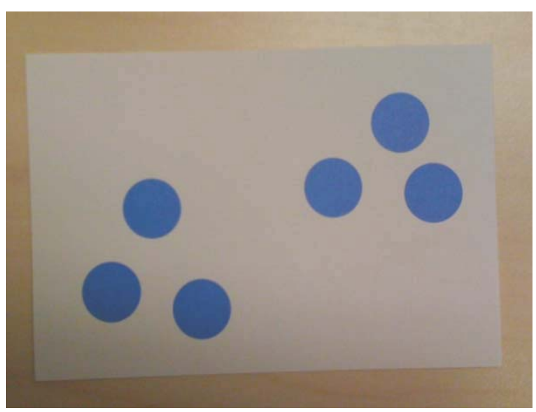

Figure 23: 6 (3-3) Dot Card

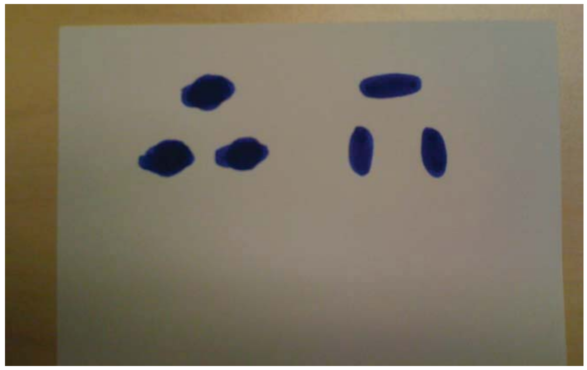

Figure 24: Amy's Drawing of 6 Dot Card (Recreated from Video Record)

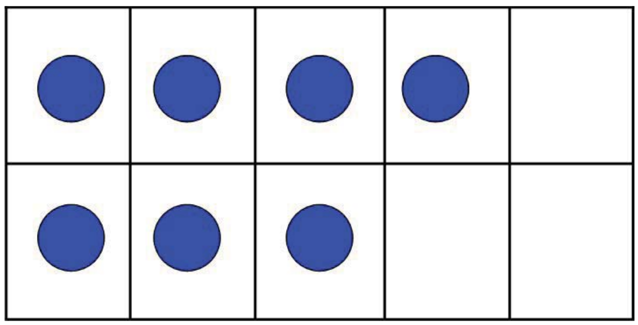

Figure 25: 10 Frame with Pair-wise 7

Student	% (#) Occurrences Rigid Connected	% (#) Occurrences Flexibly Connected
Allan	60% (31)	40% (21)
Amy	71% (35)	29% (14)
James	79% (26)	21% (7)

Figure 26: Occurrence Type by Student

	ALLAN		AMY		JAMES	
	% (#) Occurrences Rigidly Connected	**% (#) Occurrences Flexibly Connected**	**% (#) Occurrences Rigidly Connected**	**% (#) Occurrences Flexibly Connected**	**% (#) Occurrences Rigidly Connected**	**% (#) Occurrences Flexibly Connected**
PHASE 1	0% (0)	100% (1)	0% (0)	0% (0)	25% (1)	75% (3)
PHASE 2	33% (5)	67% (10)	80% (12)	20% (3)	90% (19)	10% (2)
PHASE 3	65% (17)	35% (9)	57% (13)	43% (10)	75% (6)	25% (2)
PHASE 4	90% (9)	10% (1)	91% (10)	9% (1)	N/A	N/A

Figure 27: Imagery Connectivity by Phase

ALLAN				
% (#) Occurrences **Behavior Eliciting**	% (#) Occurrences **Behavior Suggesting**	% (#) Occurrences **Behavior Replicating**	% (#) Occurrences **Eliciting/Replicating** or **Suggesting/Replicating**	% (#) Occurrences **None**
4% (2)	4% (2)	48% (25)	10% (5)	35% (18)
AMY				
% (#) Occurrences **Behavior Eliciting**	% (#) Occurrences **Behavior Suggesting**	% (#) Occurrences **Behavior Replicating**	% (#) Occurrences **Eliciting/Replicating** or **Suggesting/Replicating**	% (#) Occurrences **None**
12% (6)	2% (1)	29% (14)	14% (7)	43% (21)
JAMES				
% (#) Occurrences **Behavior Eliciting**	% (#) Occurrences **Behavior Suggesting**	% (#) Occurrences **Behavior Replicating**	% (#) Occurrences **Eliciting/Replicating** or **Suggesting/Replicating**	% (#) Occurrences **None**
0% (0)	9% (3)	42% (14)	9% (3)	39% (13)

Figure 28: Interaction Type by Student

constrained to particular segment of a progression. For example, it is possible but less likely that

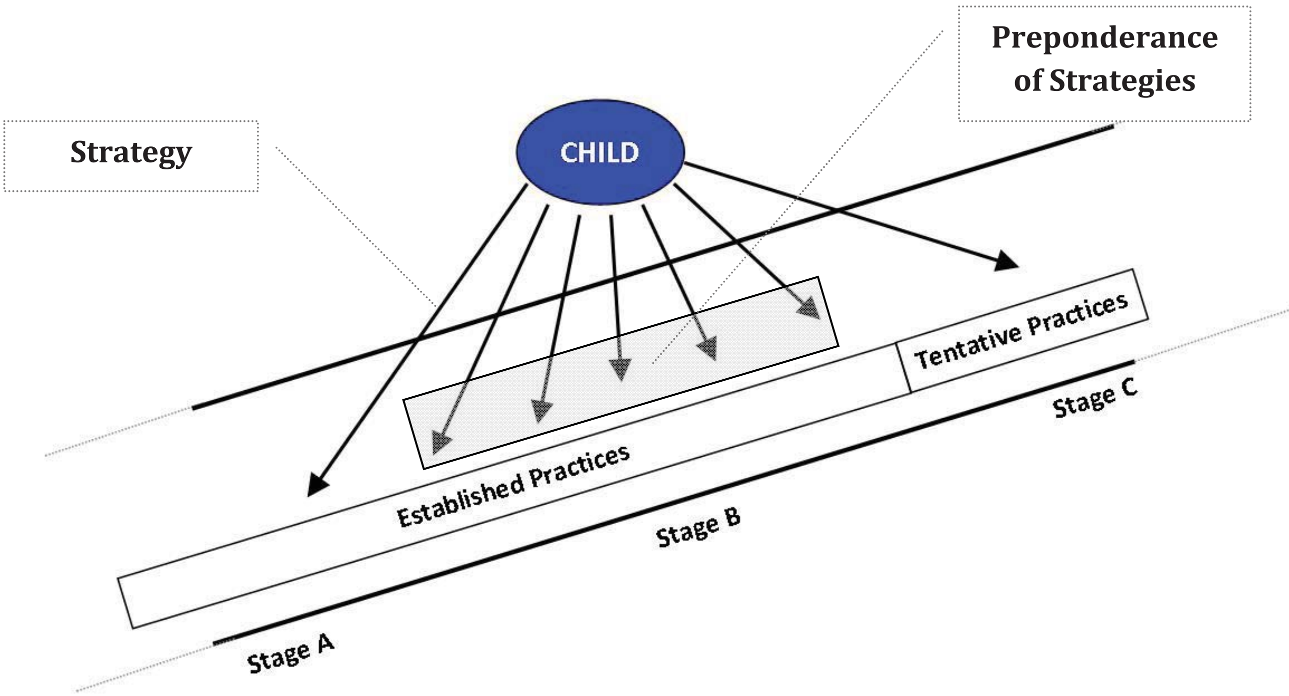

Figure 29: Strategic Variation

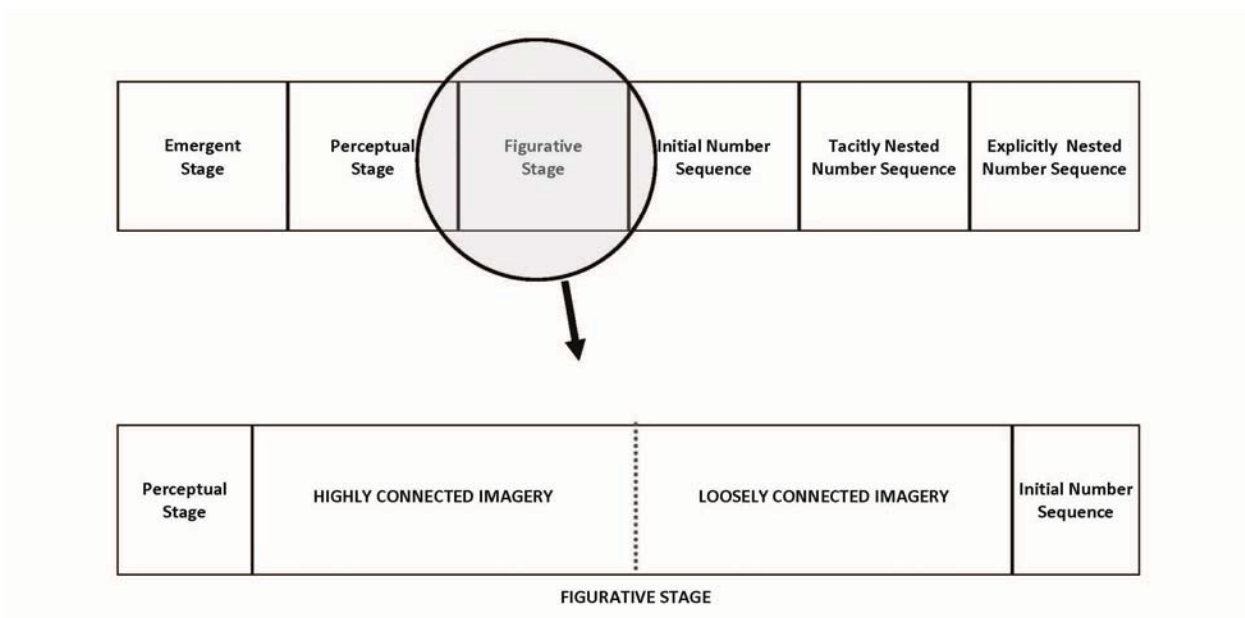

Figure 30: Counting Practices Model: Initial Design

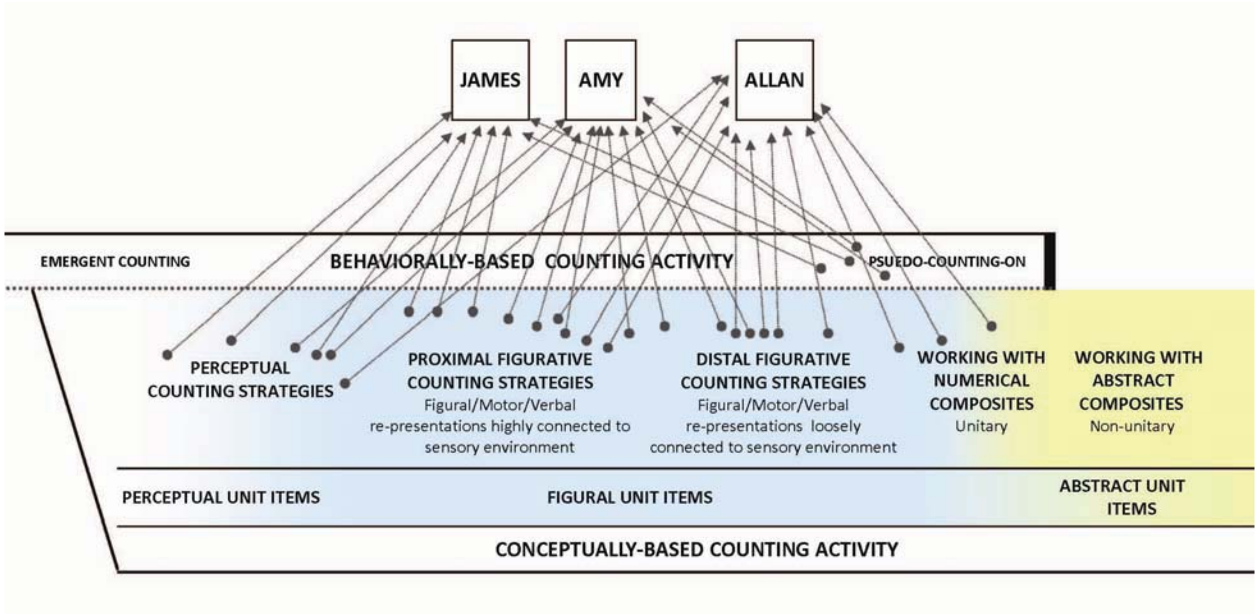

Figure 31: Counting Practices Model: Redesign

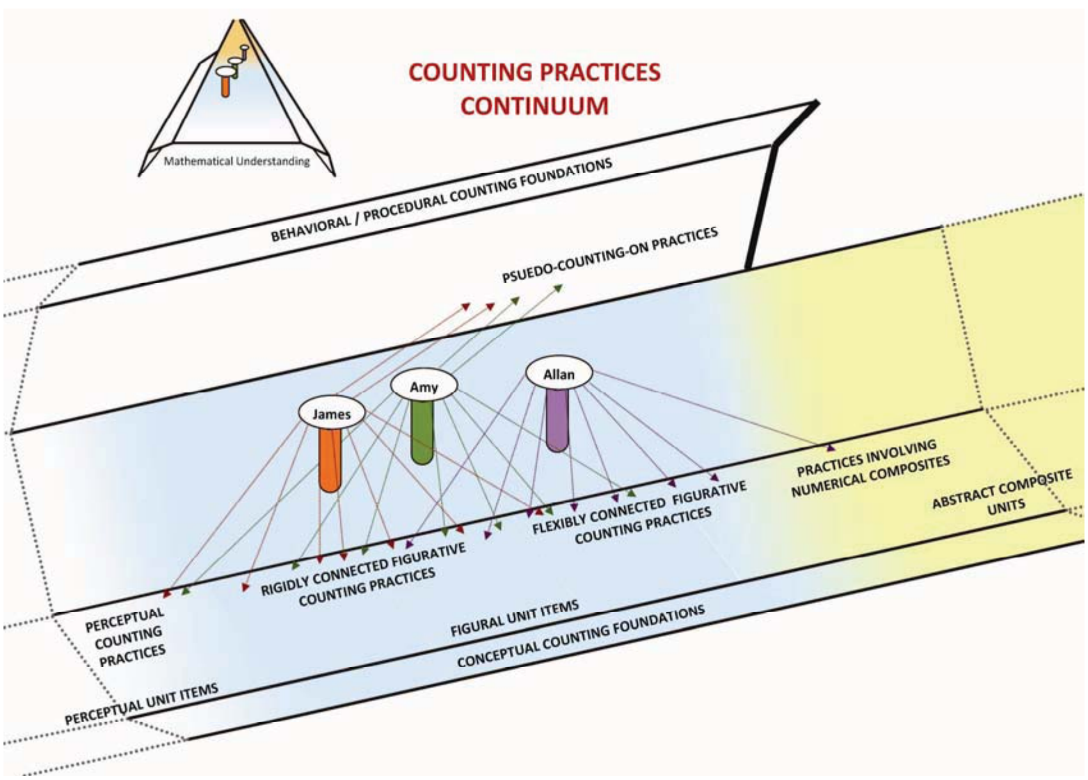

Figure 32: Counting Practices Continuum

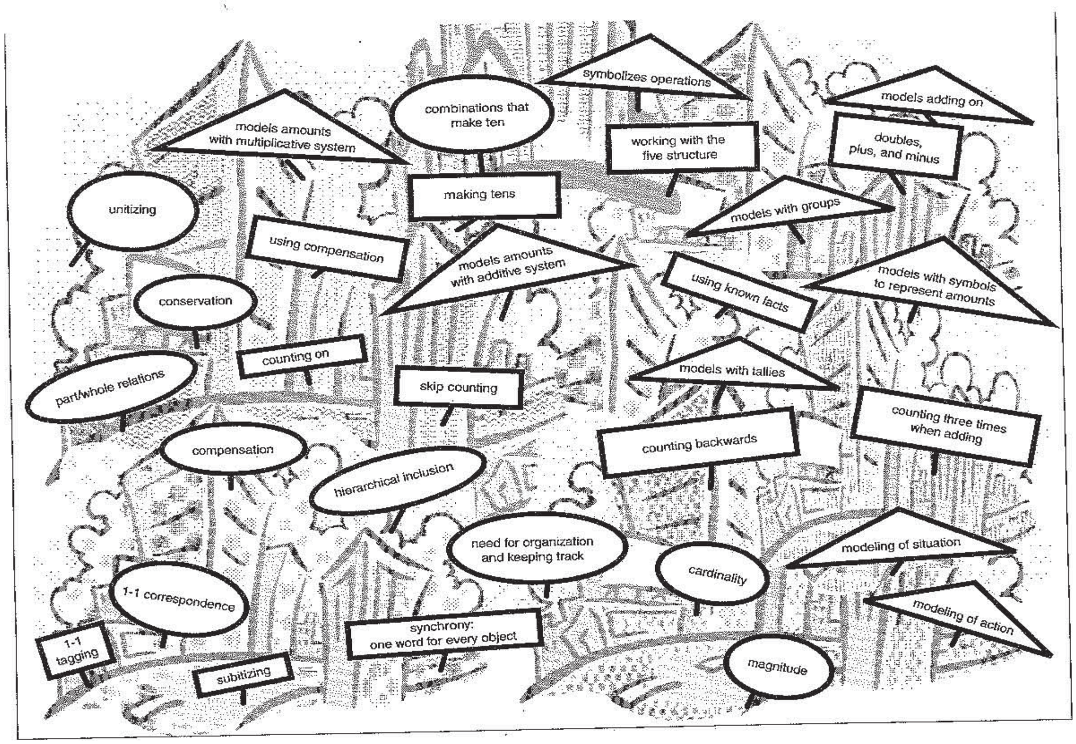

Figure 33: Fosnot and Dolk's Landscape of Learning (Early Number)

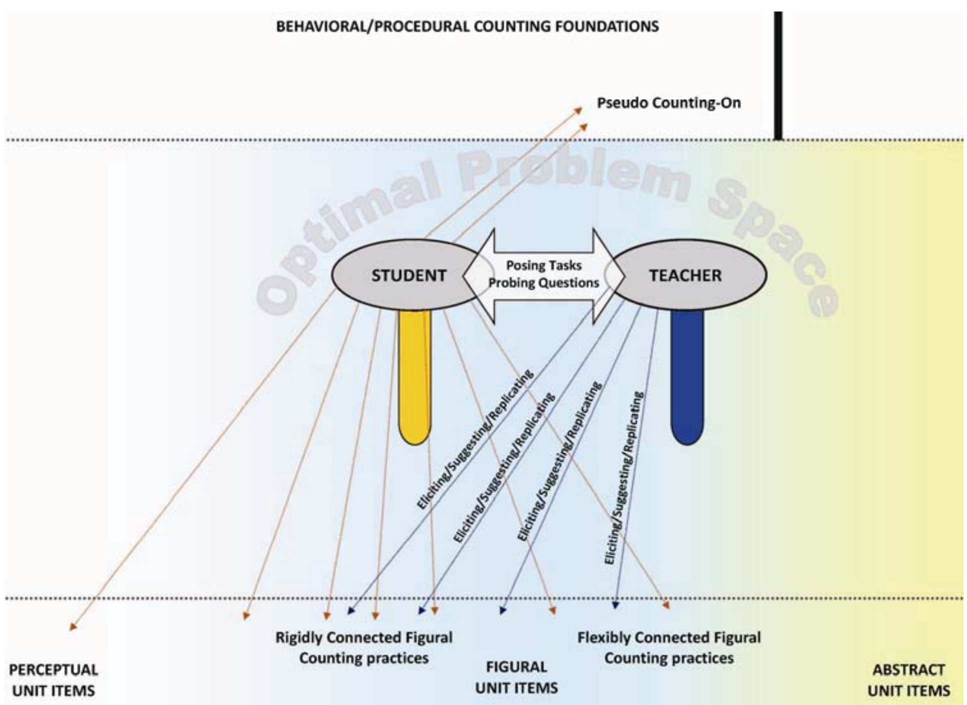

Figure 34: Intervention Interactivity Model

"Whatever I say, the Hammer will back. Twice as strong, it's goin' on and I will turn this mutha out."

~MC Hammer: Turn this Mutha Out