Better Than Classical and Dynamic Mean-Variance Policy

CUI, Xiangyu

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Thesis/Assessment Committee

Professor Nan CHEN (Chair)
Professor Duan LI (Thesis Supervisor)
Professor Shuzhong ZHANG (Committee Member)
Professor Jia-An YAN (External Examiner)

Abstract

Since Markowitz published his seminal work on mean-variance portfolio selection in 1952, almost all literatures in the past half century adhere their investigation to a *binding budget spending* assumption in static problem settings and a *self financing* assumption in dynamic settings. In the mean-variance world for a market of all risky assets, however, the common belief of monotonicity does not hold, i.e., not the larger amount you invest, the larger expected future wealth you can expect for a given risk (variance) level. We introduce in this thesis the concept of pseudo efficiency to remove from the candidates such efficient mean-variance policies which can be achieved by less initial investment level. By relaxing the binding budget spending restriction in investment, we derive an optimal scheme in managing initial wealth which dominates the traditional mean-variance efficient frontier. Moreover, as the general dynamic mean-variance portfolio selection formulation does not satisfy the principle of optimality of dynamic programming, phenomena of time inconsistency occur, i.e., investors may have incentives to deviate from the *pre-committed* optimal mean-variance portfolio policy during the investment process under certain circumstances. By introducing the concept of time inconsistency in efficiency and defining the induced trade-off, we further demonstrate in this thesis that investors behave irrationally under the *pre-committed* optimal mean-variance portfolio policy when their wealth is above certain threshold during the investment process. By relaxing the self-financing restriction to allow withdrawal of money out of the market, we develop a revised dynamic mean-variance policy for a market with a riskless asset which dominates the *pre-committed* optimal mean-variance portfolio policy in the sense that, while the two achieve the same mean-variance pair of the terminal wealth, the revised policy enables the investor to receive a free cash flow stream (FCFS) during the investment process. We further apply the concept of pseudo efficiency to a dynamic market of all risky assets and explore (better) revised dynamic mean-variance policies. By including the free cash flow stream in the total wealth, our proposed policy dominates the *pre-committed* optimal mean-variance portfolio policy in the sense that while both achieve the same total mean, the revised policy generates a smaller total variance. We reveal in this thesis that the time consistency in efficiency is closely related to the completeness of the market. We further discuss the relationship between time consistency in efficiency and the variance-optimal signed martingale measure (VSMM) of the market. Finally we show that time inconsistency in efficiency can be eliminated by enforcing no-shorting constraint for some market setting.

摘要

自从Markowitz1952年发表其均值-方差投资组合选择的开创性论文后，近半个世纪的几乎所有文献在研究静态模型时，都假设全部预算开支，而在研究动态模型时，坚持自融资假设。然而在只包含风险资产的市场中，均值-方差模型并不满足一般的单调性，即并不是投资越多就一定能够使投资者在给定的风险（方差）水平下获得更大的期望收益。我们在本论文中引入伪有效性的概念，旨在从传统的有效均值-方差投资策略中剔除那些可以由较少的初始投资水平达到的策略。通过放松投资中预算约束假设，我们推导出管理初始财富的最优机制。该机制的表现优于传统的均值-方差有效前沿。此外，由于一般的动态均值-方差投资组合选择模型并不满足动态规划的最优化原理，时间不一致现象时有发生，即投资者在某些特定条件下有着偏离其预先设定的最优投资策略的愿望。通过引入有效性意义下的时间一致性的概念，并定义诱导权重参数，我们进一步说明在投资过程中当投资者的财富水平超过某个临界值时，投资者采用预先设定的最优均值-方差投资策略，将导致不理性的行为。在放松了自融资的约束，允许投资者从市场中撤出部分财富的条件下，我们为含有无风险资产的市场设计了严格优于预先设定的最优均值-方差投资策略的修正投资策略。该修正的投资策略可以实现与预先设定的最优投资策略相同的终端财富的均值和方差，同时为投资者提供了在投资过程中获得一系列自由现金流（FCFS）的机会。我们也将伪有效性概念推广到只包含风险资产的动态市场中，并设计了更好的动态均值-方差修正投资策略。通过将全部自由现金流包含在总财富中，我们提出的修正策略可以实现和预先设定的最优投资策略相同的均值，但是拥有较小的方差。在本论文中我们还揭示了有效性意义下的时间一致性与市场的完备性有着紧密的联系。以此为基础，我们进一步讨论了有效

性意义下的时间一致性与与市场中方差最优的符号鞅测度（VSMM）之间的关系。最后，我们指出在一些市场模型中，有效性意义下的时间不一致现象可以通过添加不允许卖空的约束条件来消除。

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NOTATIONS

Usual Notations

Notation	Descriptions
$\mathbb{R}^n$	The set of n-dimensianl real vectors
$\mathbf{1}$	The vector with all its elements being one
$\mathbf{0}$	The vector with all its elements being zero
$'$	The transposition operator
x_0	The initial wealth level of the investor
$E(X)$	The expectation of any random variable X
$E(X\|\mathcal{F})$	The conditional expectation of any random variable X for given $\mathcal{F}$
$Var(X)$	The variance of any random variable X
$Cov(Y)$	The covariance matrix of any random vector Y
$(\Omega, \mathcal{F}, P)$	The probability space
$\mathcal{F}_t$	The σ-algebra generated by assets' prices until time t
1_A	The indicative function of event A
$Pr(A)$	The probability of event A

Notations of Chapter 2

Notation	Descriptions
r	The n-dimensional random total return vector of n risky assets
x	The n-dimensional decision vector with x_i being the dollar amount invested in the ith risky asset
μ	The pre-given expected future wealth
σ^2	The pre-given variance of future wealth
$x(x_0; \mu)$	The optimal policy of problem (MV)
$x^*(\hat{x}_0^*; \mu)$	The optimal policy of problem (MV_1)
$x^*(\hat{x}_0^*; \sigma^2)$	The optimal policy of problem (MV_2)
$\hat{x}_0^*$	The optimal investment level
r_f	The return of riskless asset

Notations of Chapter 3

Notation	Descriptions
s_t	The given return of the riskless asset at period t
$\mathbf{e_t}$	The vector of random returns of the n risky assets at period t
x_t	The wealth of the investor at the beginning of period t
$\mathbf{u_t}$	The n-dimensional portfolio policy at period t with u_t^i being the amount invested in the ith risky asset at period t
$\mathbf{P_t}$	The excess return vector at period t
λ	The overall trade-off between mean and variance
Γ	The risk attitude parameter of problem (MV)
$\mathbf{u_t^*}(x_t)$	The *pre-committed* optimal policy at period t of problem (MV)
λ_k	The period-k trade-off induced by the *pre-committed* optimal policy
x_k^*	The wealth threshold marking induced trade-off equal to zero
$\mathbf{\hat{u}_k^*}(\hat{x}_k)$	The revised policy at period k of problem (MV)
Γ_k	The risk attitude parameter at period k of the revised policy
$\bar{x}_k^*$	The wealth threshold at period k of the revised policy

Notations of Chapter 4

Notation	Descriptions
$\mathbf{e_t}$	The vector of random returns of the $n+1$ risky assets at period t
$\mathbf{P_t}$	The excess return vector respect to 0th risky asset at period t
$\mathbf{u_t}$	The n-dimensional portfolio policy at period t with u_t^i being the amount invested in the ith risky asset at period t
λ	The overall trade-off between mean and variance
$\mathbf{u_t^*}(x_t)$	The *pre-committed* optimal policy at period t of problem (MV)
λ_k	The period-k trade-off induced by the *pre-committed* optimal policy
x_k^*	The wealth threshold marking induced trade-off equal to zero
$\bar{\mathbf{u}}_\mathbf{t}^*(\cdot)$	The Revised Policy 4.1
$\hat{\mathbf{u}}_\mathbf{k}^*(\hat{x}_k)$	The Revised Policy 4.2 at period k of problem (MV)
Γ_k	The risk attitude parameter at period k of the Revised Policy 4.2
$\bar{x}_k^*$	The wealth threshold at period k of the Revised Policy 4.2
$\tilde{\mathbf{u}}_\mathbf{t}^*(\cdot)$	The Revised Policy 4.3
$\check{\mathbf{u}}_\mathbf{k}^*(\check{x}_k)$	The Revised Policy 4.4 at period k of problem (MV)
$\bar{\Gamma}_k$	The risk attitude parameter at period k of the Revised Policy 4.4
$\tilde{x}_k^*$	The wealth threshold at period k of the Revised Policy 4.4
$\mathbf{u}^*(t, x(t))$	The *pre-committed* optimal policy at time t of problem (MV_C)
$\hat{\mathbf{u}}^*(t, \hat{x}(t))$	The first Revised Policy at time t of problem (MV_C)
$\check{\mathbf{u}}^*(t, \check{x}(t))$	The second Revised Policy at time t of problem (MV_C)

Notations of Chapter 5

Notation	Descriptions
r_t	The given return of the riskless asset at period t
$\mathbf{e_t}$	The vector of random returns of the n risky assets at period t
$\mathbf{P_t}$	The excess return vector at period t
$E_t(X)$	The conditional expectation of any random variable (vector, matrix) X for given $\mathcal{F}_t$
$Cov_t(X)$	The conditional covariance matrix of any random vector X for given $\mathcal{F}_t$
λ	The overall trade-off between mean and variance
$\mathbf{u_t^*}(x_t)$	The *pre-committed* optimal policy at period t of problem (MV)
$\{\theta_t\}$	The adapted process appearing in the *pre-committed* optimal policy
λ_k	The period-k trade-off induced by the *pre-committed* optimal policy
x_k^*	The wealth threshold marking induced trade-off equal to zero
$E_t^Q(X)$	The conditional expectation of any random variable (vector, matrix) X under measure Q for given $\mathcal{F}_t$
$\bar{\mathbf{P}}_\mathbf{t}$	The excess return vector with auxiliary risky assets at period t
κ_t^j	The auxiliary market parameter of jth risky asset at period t
$\bar{\mathbf{P}}_\mathbf{t}^\kappa$	The excess return vector of the auxiliary market $\mathcal{M}_\kappa$ at period t

CHAPTER 1

INTRODUCTION

The seminal work on mean-variance portfolio selection by Markowitz (1952) [42] more than a half century ago laid the foundation for modern financial analysis and led a remarkable development of the mean-risk portfolio selection framework by leaps and bounds witnessed in the advancement of the theory and practice of financial economics. Note that the original setting of Markowitz (1952) [42] considers a market of all risky assets and does not allow shorting. The critical line method was developed in Markowitz (1956), (1959) [43, 41] to solve numerically the mean-variance portfolio selection problem without shorting.

Investigating further the Markowitz's mean-variance portfolio selection model without short selling, Tobin (1958) [66] revealed the famous mutual fund theorem that the optimal portfolio of a mean-variance optimizer is a combination of a riskless asset and a risky fund. Sharpe (1966), (1967) [60, 61] also discussed the performance of the mutual fund later in his papers.

Sharpe (1864) [59], Lintner (1965) [40] and Mossin (1966) [47] introduced the capital asset pricing model, independently, using different approaches. Using the equilibrium analysis, Mossin (1966) [47] found that all investors in a market allowing short selling would hold the same percentage of the total outstanding stocks of all risky assets and this percentage is positive. Furthermore, the return of any risky asset satisfies a linear relationship with respect to its risk parameter, β.

Black (1972) [7] discussed the equilibrium of a capital market allowing short selling and with no riskless asset or with no riskless borrowing. The efficient portfolio set in a market with no riskless asset is the combination of two different efficient portfolios. One of them can be chosen as a particular portfolio with a zero beta, i.e. a particular portfolio without systematic risk. By replacing the riskless return with the expected return of the zero-beta portfolio, an extension of the CAPM in this type of market can be obtained, which is termed the zero-beta capital asset pricing model. The efficient portfolio set in a market with no riskless borrowing is of two parts: One part consists of a weighted combination of two different portfolios of risky assets, and the other part consists of a weighted combination of the riskless asset with a single efficient portfolio of risky assets, market portfolio.

Merton (1972) [46] derived the analytic solution for the unconstrained mean-variance portfolio selection and found that one fund theorem holds if and only if the expected return of the minimum variance portfolio is larger than the riskless return in an equilibrium market.

The extension of the mean-variance formulation to dynamic settings, however, has been unsuccessful for many years, due to an inherent nonseparable structure of the variance minimization problem in the sense of dynamic programming. To seek an optimal dynamic portfolio policy under a mean-variance framework implies to achieve a dual balance between the expected return and the risk and between a short-term and long-term goals. Li and Ng (2000) [35] finally solved the mean-variance formulation of the multi-period portfolio selection problem by adopting an embedding scheme. In the same year, Zhou and Li (2000) [75] also solved the mean-variance formulation in continuous-time by adopting the same embedding scheme. The past eight years have witnessed numerous extensions of the mean-variance portfolio selection theory, see for examples, Li, Zhou and Lim (2002) [37], Lim and Zhou (2002) [39], Zhou and Yin (2003) [76], Hu and Zhou (2005) [23], Bielecki, Pliska and Zhou (2005) [6], Li and Zhou (2006) [36], Chiu and Li (2006) [12], Xia and Yan (2006) [69], Xiong and Zhou [70] in continuous-time settings and Leippold, Trojani and Vanini (2004) [31], Zhu, Li and Wang (2004) [78], Liang, Zhang and Li (2008) [38], Yi, Li and Li (2008) [72] in discrete-time settings. Recently, Černý and Kallsen (2007), (2009) [10, 11] studied the optimal mean-variance portfolio selection in a more general setting with a semi-martingale price process, which includes both discrete-time and continuous-time settings as its special cases.

We emphasize here that both the derived optimal policies in Li and Ng (2000) [35] and Zhou and Li (2000) [75] are a linear function of both the current wealth level and the initial trade-off between the mean and the variance and do not satisfy the principle of optimality, still due to the nonseparable property of the dynamic mean-variance formulation. Zhu, Li and Wang (2003) [77] later investigated the wealth reduction phenomena associated with the optimal multi-period mean-variance policy (termed *pre-committed* optimal dynamic mean-variance policy in [4]) derived by Li and Ng (2000) [35]. Basak and Chabakauri (2008) [4] also recognized that investors may have incentives to deviate from the *pre-committed* optimal dynamic mean-variance policy before reaching the terminal time.

Stimulated by the ground-breaking work of Markowitz (1952) [42] in measuring investment risk by a variance term, investment decision formulations under a return-risk framework have been extensively investigated in financial economics, almost independent of and parallel to the development of the utility theory in economics literatures Von Neumann and Morgenstern (1947) [68], Merton (1969), (1971) [44, 45], although the latter is considered to be more systematic and more mathematically rigorous. One reason behind this phenomenon could be that practitioners in portfolio management prefer measurable quantities to abstract terms. In a return-risk framework, risk level is explicitly measured as a real number, such as the variance or Value-at-Risk, whereas risk is only implicitly represented by the utility function which itself is very hard to determine exactly by investors. The past four decades have witnessed numerous alternative risk measures appearing in the literature, including the safety-first criterion (Roy (1952) [55]), mean-semivariance criterion (Markowitz (1952) [42], Stefani and Szegö (1976) [64]), mean-Gini measure (Shalit and Yitzhaki (1984) [58]), measure of absolute deviation (Konno and Yamazaki (1991) [28], Zenios and Kang (1993) [74], Speranza (1993) [63]), value-at-risk (VaR) (Duffie and Pan (1997) [18]), and conditional value-at-risk (CVaR) (Bawa (1978) [5], Uryasev (2000) [67]). To evaluate different risk measures, Artzner, Delbaen, Eber and Heath (1997), (1999) [2, 3] introduced the so called "coherence" axioms as the requirement for any type of appropriate risk measures in quantifying the riskiness of financial positions with a maturity at a future time. Föllmer and Schied (2002) [19], Frittelli and Rosazza Gianin (2002) [20] further extended, independently, coherent risk measures to a broader class of convex risk measure.

When dealing with risk measures for dynamic portfolio selection, an additional requirement, "time consistency", seems natural and necessary for appropriate risk measures. Although the definitions of "time consistency" introduced in Rosazza Gianin (2002b) [53], Boda and Filar (2006) [8], Artzner, Delbaen, Eber, Heath, Ku (2007) [1] and Jobert and Rogers (2008) [25] read differently, they all have their essence rooted in Bellman's dynamic programming.

Let x_t be the wealth level at the beginning of period t, π_t an admissible investment policy at period t and $\mathcal{M}_{t-T}(\pi_t, \ldots, \pi_{T-1} \mid x_t)$ a risk measure from period t to period $T-1$ under given policy $\{\pi_t, \ldots, \pi_{T-1}\}$ with a given initial wealth x_t, $t = 0, 1, \ldots, T-1$.

Definition 1.1 (time consistency). *Risk measure $\mathcal{M}_{0-T}$ is time consistent if any optimal policy for the portfolio selection problem over the entire time horizon,*

$$\pi^* = (\pi_0^*, \ldots, \pi_{T-1}^*) \in \arg\min_{\pi_0, \ldots, \pi_{T-1}} \mathcal{M}_{0-T}(\pi_0, \ldots, \pi_{T-1} \mid x_0)$$

also satisfies the local optimality conditions for all $t = 1, \ldots, T-1$,

$$(\pi_t^*, \ldots, \pi_{T-1}^*) \in \arg\min_{\pi_t, \ldots, \pi_{T-1}} \mathcal{M}_{t-T}(\pi_t, \ldots, \pi_{T-1} \mid x_t),$$

where x_t is any realizable wealth at the beginning of period t.

Almost all widely adopted risk measures in the literature, including the variance and VaR, are not time consistent according to Boda and Filar (2006) [8], with only very few exceptions, e.g., the safety-first criterion by Roy (1952) [55]. One prominent candidate for a time consistent dynamic risk measure is the nonlinear expectation ("g-expectation") introduced by Peng (1997) [48] via Backward Stochastic Differential Equation, which has been studied extensively by Rosazza Gianin (2002a), (2006) [52, 54], Peng (2004), (2005) [49, 50], Jiang (2008) [24] and Cohen and Elliott (2008), (2009) [13, 14].

The above definition of time consistency only concerns the risk measures. In investment, however, the risk criterion is always measured against a measure of the expected wealth due to the existing trade-off. In other words, the best portfolio policy is always sought with a best trade-off between the expected terminal wealth and the risk under a multi-objective framework. We thus need to consider a more general objective function in multi-period portfolio selection formulations,

$$\min \mathcal{M}_{0-T}(\pi_0, \ldots, \pi_{T-1} \mid x_0) + \lambda E(x_T \mid \pi_0, \ldots, \pi_{T-1}, x_0),$$

where E is the expectation operator and $\lambda \leq 0$ denotes the trade-off between the expected value of the terminal wealth x_T and the risk. In such a multi-objective setting, we are interested in finding a set of efficient solutions, in terms of maximization of the expected terminal wealth and minimization of the defined risk, by varying the trade-off parameter λ.

While parameter λ represents the trade-off between two conflicting objectives for the entire time horizon, the trade-off between the two objectives for any efficient portfolio policy may essentially change at intermediate periods during the investment horizon. One interesting result revealed in Li (1990) [32] and Li (2000) [33] is that, unless the vector-valued objective function is of a periodwise additive form, the trade-off for various objective functions are time-varying as the system evolves along any efficient trajectory. A multi-objective version of principle of optimality has been stated in Li and Haimes (1987) [34] and Li (1990) [32]: *The principle of optimality holds if any tail part of an efficient policy is also efficient for any realizable state at an intermediate period.* We need to extend the concept of time consistency to a version to incorporate efficiency.

Definition 1.2 (time consistency in Efficiency). *A combined risk-expected return measure* $\mathcal{M}_{0-T}(\pi_0, \ldots, \pi_{T-1} \mid x_0) + \lambda E(x_T \mid \pi_0, \ldots, \pi_{T-1}, x_0)$ *is time consistent in efficiency if any optimal policy for the portfolio selection problem over the entire time horizon,*

$$\begin{aligned} \pi^* &= (\pi_0^*, \ldots, \pi_{T-1}^*) \\ &\in \arg\min_{\pi_0, \ldots, \pi_{T-1}} \{\mathcal{M}_{0-T}(\pi_0, \ldots, \pi_{T-1} \mid x_0) + \lambda E(x_T \mid \pi_0, \ldots, \pi_{T-1}, x_0)\} \end{aligned}$$

also satisfies the local optimality conditions for all $t = 1, \ldots, T-1$,

$$\begin{aligned} &(\pi_t^*, \ldots, \pi_{T-1}^*) \\ &\in \arg\min_{\pi_t, \ldots, \pi_{T-1}} \{\mathcal{M}_{t-T}(\pi_t, \ldots, \pi_{T-1} \mid x_t) + \lambda_t E(x_T \mid \pi_t, \ldots, \pi_{T-1}, x_t)\} \end{aligned}$$

for some nonpositive λ_t *(termed trade-off induced by* π^**), where* x_t *is any realizable wealth at the beginning of period* t.

Note that time consistency implies time consistency in efficiency, but the reverse is in general untrue. Time consistency in efficiency yields time consistency only when $\lambda_t = \lambda$, for all $t = 1, \ldots, T-1$. Thus, time consistency in efficiency is a relaxed version of time consistency.

This thesis is organized as follows. After reviewing the literature in this chapter, we revisit the classical mean-variance portfolio selection problem in a market of all risky assets in Chapter 2. In Chapter 3, we study the dynamic mean-variance portfolio selection problem in a market with a riskless asset, reveal that the mean-variance formulation in general does not satisfy time consistency in efficiency, and propose a revised policy that is better than the pre-committed mean-variance policy. We discuss a similar issue in Chapter 4 for a dynamic market with only risky assets, and develop two revised policies. We investigate in Chapter 5 the relationship between time consistency in efficiency and the variance-optimal signed martingale measure and explore the possibility in eliminating time inconsistency in efficiency by adding no-shorting constraint. We finally conclude this thesis in Chapter 6 with some remarks.

CHAPTER 2

Classical Mean Variance Model Revisited: Pseudo Efficiency

2.1. Introduction

Since Markowitz published his ground-breaking work on mean-variance portfolio selection in 1952, almost all literature and textbooks in the past half century adopt an assumption of binding budget spending in their investigation on this classical investment issue. The only exception, to our knowledge, is Steinbach (2001) [65], in which Steinbach considered a market consisting of risky assets, one riskless asset and a zero-interest loss-guaranteed cash account. His model does not change the efficient frontier of the mean-variance model for markets with risky assets and a riskless asset, but improves the inefficient boundary by removing the risk completely.

We consider in this chapter the following classical mean-variance portfolio selection problem in a market of n risky assets with a random total return vector

$r = (r_1, r_2, \ldots, r_n)'$:

$$(MV) \quad \min_x \quad x'Vx \tag{2.1}$$
$$\text{s.t.} \quad x'e = \mu,$$
$$x'\mathbf{1} = x_0,$$

where x_0 is the initial wealth, $\mathbf{1}$ is the n-dimensional vector with all its components equal to 1, x is the n-dimensional decision vector with x_i being the dollar amount invested in the ith risky asset, $e = E(r)$, $V = Cov(r)$ and μ is the pre-given expected future wealth. We assume that the covariance matrix V is positive definite. In our model, we allow a negative value for x_0, which could be resulted from short selling. Note that normalizing x_0 to 1 will reduce x to the vector with x_i being the percentage invested in the ith risky asset, and problem (MV) in such a case reduces to the mean-variance model originally studied in Markowitz (1952) [42], except that we allow short selling here in (MV). To study a market consisting of only risky assets is justifiable, as in practice, i) almost no asset is one-hundred percent riskless, and ii) most fund managers are specialized in their own sectors and seldom consider bonds.

We reveal in this chapter that, in the mean-variance world for a market of all risky assets, the common belief of monotonicity: "The larger amount you invest, the larger expected future wealth you can expect for a given risk (variance) level" does not hold. We introduce in this chapter the concept of pseudo efficiency to remove from the candidates such efficient mean-variance policies which can be achieved by a less initial investment level. By relaxing the binding budget spending restriction in investment, we derive in this chapter an optimal scheme in managing initial wealth which dominates the traditional mean-variance efficient frontier.

The organization of this chapter is as follows. In Section 2.2, after reviewing the classical mean-variance portfolio selection model, we demonstrate that, for almost all reachable mean-standard deviation pairs, there exist dual realizations associated with two different initial investment levels, leading to a finding of violation of monotonicity in the mean-variance world. In Section 2.3, we introduce first the concept of pseudo efficiency (type 1) and prove that the phenomena of pseudo efficiency exist in the traditional mean-variance formulation, leading to a conclusion that investors can perform better by relaxing the unnecessary binding budget spending constraint. We then introduce the concept of pseudo efficiency (type 2) and discuss the issue of optimal wealth management when only a market of risky assets is available for investment. We also show that the phenomena of pseudo efficiency occur in situations without shorting too. We further investigate in Section 2.4 the implication of our findings to the existence of a market equilibrium. Finally, we conclude this chapter in Section 2.5 with some remarks.

2.2. Non-monotonicity in the Mean-Variance World: Dual Realizations of Mean-Variance Pair

It can be verified (for example, see Merton (1972) [46]) that the optimal policy of (MV) is given by

$$x(x_0;\mu) = \frac{x_0}{D}(BV^{-1}\mathbf{1} - AV^{-1}e) + \frac{\mu}{D}(CV^{-1}e - AV^{-1}\mathbf{1}), \tag{2.2}$$

where

$$A = \mathbf{1}'V^{-1}e = e'V^{-1}\mathbf{1},$$
$$B = e'V^{-1}e > 0,$$
$$C = \mathbf{1}'V^{-1}\mathbf{1} > 0,$$
$$D = BC - A^2 > 0.$$

The positiveness of D can be seen from the positiveness of $(Ae - B\mathbf{1})'V^{-1}(Ae - B\mathbf{1}) = BD$ (Merton (1972) [46]). A fact that has not been fully recognized in the literature is that parameter $A = e'V^{-1}\mathbf{1}$ can be positive, negative or zero. We provide a detailed analysis for situations with negative A in Appendix 2.6.

Furthermore, the *minimum variance set* of problem (MV) can be expressed as

$$\sigma^2 = \frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}. \tag{2.3}$$

As the mean-variance pair of the minimum variance portfolio (MVP) is given as $(\frac{A}{C}x_0, \frac{x_0^2}{C})$, the upper branch of the minimum variance set,

$$\{(\mu, \sigma^2) \mid \sigma^2 = \frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C} \text{ and } \mu \geq \frac{A}{C}x_0\},$$

constitutes the so-called ***mean-variance efficient frontier***.

We denote all policies corresponding to the mean-variance pairs on the minimum variance set of problem (MV) *boundary policies*, which could be either efficient or inefficient. More specifically, the set of efficient boundary policies, denoted by X^e, and the set of inefficient boundary policies, denoted by X^{ie}, can be expressed explicitly as follows:

$$\begin{aligned} X^e &= \{x(x_0;\mu) \mid x(x_0;\mu) \text{ is given in (2.2) and } \mu \geq \frac{A}{C}x_0\}, \\ X^{ie} &= \{x(x_0;\mu) \mid x(x_0;\mu) \text{ is given in (2.2) and } \mu < \frac{A}{C}x_0\}. \end{aligned}$$

For a given pair (μ, σ) in the mean-standard deviation space, we are interested in solving an inverse problem to find out which initial wealth levels enable us to achieve the given mean-standard deviation pair by adopting a boundary policy.

Solving x_0 from (2.3) yields the following two solutions when condition $|\mu| \leq \sqrt{B}\sigma$ holds:

$$x_0^+ = \frac{A\mu + \sqrt{D(B\sigma^2 - \mu^2)}}{B}, \tag{2.4}$$

$$x_0^- = \frac{A\mu - \sqrt{D(B\sigma^2 - \mu^2)}}{B}. \tag{2.5}$$

Clearly, x_0^+ and x_0^- represent two initial wealth levels which can achieve the given pair of (μ, σ), where $x_0^+ \geq x_0^-$ holds whenever $B\sigma^2 \geq \mu^2$ and they are equal only when $B\sigma^2 = \mu^2$. Note that any pair (μ, σ) that does not satisfy $|\mu| \leq \sqrt{B}\sigma$ is not achievable.

Definition 2.1. *The reachable region in the mean-standard deviation space is defined as*

$$\{(\mu, \sigma) \mid \ |\mu| \ \leq \ \sqrt{B}\sigma\},$$

while the boundary of the reachable region is given by

$$\{(\mu, \sigma) \mid \ |\mu| = \ \sqrt{B}\sigma\}.$$

Note that the reachable region is a cone, which is uniquely determined by the market parameter B. See Figure 2.1.

There are several phenomena to notice.

i) All interior points within the reachable region in the mean-standard deviation space can be realized by adopting one of the two boundary policies associated, respectively, with two different initial wealth levels. See Figure 2.1.

ii) Any boundary point of the reachable region in the mean-standard deviation space is generated by a single boundary policy associated with one specific initial wealth level. The boundary of the reachable region is expressed in Figure 2.1 by the red rays.

iii) The minimum variance sets associated with different initial wealth levels of x_0 form a family of hyperbolas. As the vertex of each individual hyperbola is the minimum variance point, whose position in the mean-standard deviation space is specified by $(\frac{A}{C}x_0, \sqrt{\frac{x_0^2}{C}})$, we can conclude that a) When A is positive, decreasing the level of x_0 moves the hyperbola downwards; and b) When A is negative, decreasing the level of x_0 moves the hyperbola upwards.

The most important finding of this section is that any given interior mean-standard deviation pair inside the reachable region can be *priced* differently by two different initial investment levels. It is evident that an optimal investment policy should prevent investors from adopting any policy associated with the higher investment level.

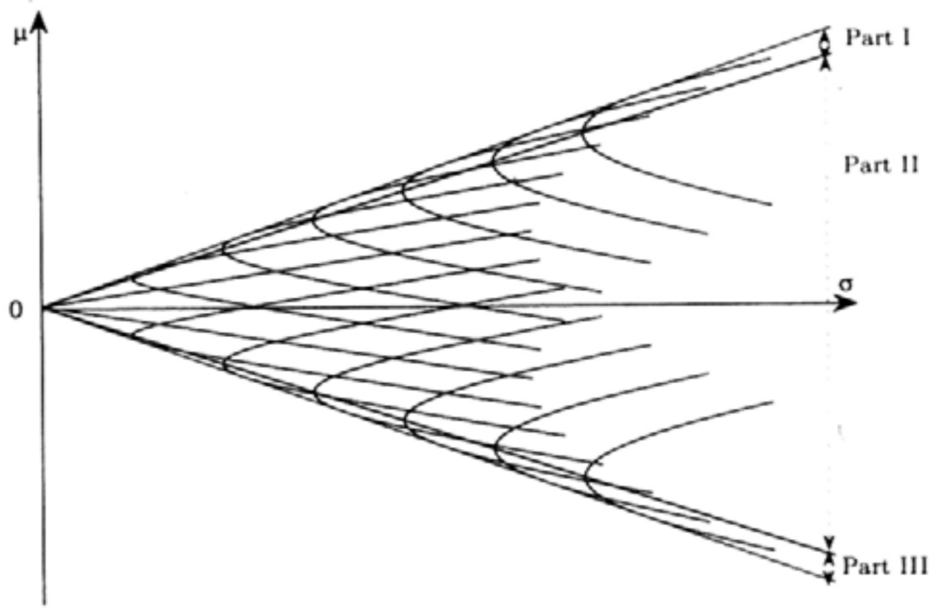


Figure 2.1: The reachable region and its partition

As the mean-standard deviation pair of the minimum variance portfolio for a given initial wealth level x_0 is given as $(\frac{A}{C}x_0, \sqrt{\frac{x_0^2}{C}})$, the set of all mean-standard deviation pairs associated with the minimum variance portfolios corresponding to different initial wealth levels is given by

$$\{(\mu, \sigma) \mid \mu = \pm\frac{|A|}{\sqrt{C}}\sigma\},$$

where $\{(\mu, \sigma) \mid \mu = \frac{|A|\sigma}{\sqrt{C}}\}$ represents all minimum variance portfolios with positive expected future wealth and $\{(\mu, \sigma) \mid \mu = -\frac{|A|\sigma}{\sqrt{C}}\}$ represents all minimum variance portfolios with negative expected future wealth.

Thus, we can further divide the reachable region into three parts (See Figure 2.1):

Part I:

$$\{(\mu, \sigma) \mid \ \frac{|A|}{\sqrt{C}}\sigma \ \leq \ \mu \ \leq \ \sqrt{B}\sigma\}.$$

For any point (μ, σ) in Part I, both $\mu \geq A/Cx_0^+$ and $\mu \geq A/Cx_0^-$ hold, where

x_0^+ and x_0^- are given in (2.4) and (2.5), respectively. Therefore, every point in Part I is achieved by two efficient boundary policies corresponding to two different initial wealth levels, except for the boundary points of the reachable region which are achieved by one efficient boundary policy.

Part II:

$$\{(\mu,\sigma) \mid \ -\frac{|A|}{\sqrt{C}}\sigma \ \leq \ \mu \ < \ \frac{|A|}{\sqrt{C}}\sigma\},$$

For any point (μ,σ) in Part II, it can be verified that $\mu \geq A/Cx_0^-$ and $\mu < A/Cx_0^+$ hold when $A > 0$, and $\mu \geq A/Cx_0^+$ and $\mu < A/Cx_0^-$ hold when $A < 0$. Therefore, any point in Part II is achieved by one efficient boundary policy and one inefficient boundary policy corresponding to two different initial wealth levels. When $A = 0$, part II vanishes.

Part III:

$$\{(\mu,\sigma) \mid \ -\sqrt{B}\sigma \ \leq \ \mu \ < -\frac{|A|}{\sqrt{C}}\sigma \ \}.$$

For any point (μ,σ) in Part III, both $\mu < A/Cx_0^+$ and $\mu < A/Cx_0^-$ hold. Therefore, every point in Part III is achieved by two inefficient boundary policies corresponding to two different initial wealth levels, except for the boundary points of the reachable region which are achieved by one inefficient boundary policy.

2.3. Better than Classical Mean-Variance

For any mean-standard deviation pair on the efficient frontier of (MV), $(\mu,\sigma \mid \mu \geq \frac{A}{C}x_0) = (\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}} \mid \mu \geq \frac{A}{C}x_0)$, it is impossible to find a better mean-standard deviation pair under constraint $x'\mathbf{1} = x_0$, i.e., under the assumption that the investor invests all his initial wealth, x_0, into the market of all risky assets. The traditional school of thinking always adheres itself to this binding budget spending restriction. Let us now change the way of thinking from the traditional school to explore a possibility whether we can achieve the same efficient mean-standard deviation pair by less initial investment level. The key point is, when the investment performance is measured by a mean-variance pair, the common assumption that *the higher investment the better* does not hold, as we have already revealed in the earlier section.

2.3.1. Pseudo efficiency (Type 1) and best investment performance

Definition 2.2. *If an efficient mean-standard deviation pair of problem (MV) associated with initial wealth x_0, $(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}})$, can be also generated or even dominated by another mean-standard deviation pair, $(\hat{\mu}, \sqrt{\frac{C}{D}\left(\hat{\mu} - \frac{A}{C}\hat{x}_0\right)^2 + \frac{\hat{x}_0^2}{C}})$, which is generated by another boundary policy associated with initial investment level $\hat{x}_0$ which is strictly less than x_0, i.e.,*

$$(\mu, -\sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \preceq (\hat{\mu}, -\sqrt{\frac{C}{D}\left(\hat{\mu} - \frac{A}{C}\hat{x}_0\right)^2 + \frac{\hat{x}_0^2}{C}}), \quad (2.6)$$

$$\hat{x}_0 < x_0, \quad (2.7)$$

then, the given mean-standard deviation pair associated with initial wealth x_0 is termed **pseudo efficient (type 1)** *and the corresponding efficient boundary policy $x(x_0; \mu)$ is called* **pseudo efficient policy (type 1)**.

In other words, if a mean-variance efficient boundary policy of (MV) with respect to a given initial wealth x_0 becomes inefficient in an expanded three dimensional objective space:

$$\begin{aligned}\{ &\min(\text{initial investment level}),\\ &\max(\text{expected future wealth}),\\ &\min(\text{variance of the future wealth})\},\end{aligned}$$

it is pseudo efficient.

One important recognition from our earlier discussion on dual realization in Section 2.2 is that, for any given mean-standard deviation pair in the interior of the reachable region, $(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}})$, associated with an initial wealth

x_0, there exists another initial investment level $\hat{x}_0$ such that

$$x_0 + \hat{x}_0 = \frac{2A\mu}{B} \tag{2.8}$$

and (2.6) becomes an equality. In other words, two boundary policies, $x(x_0;\mu)$ and $x(\hat{x}_0;\mu)$, with respect to two initial investment levels, x_0 and $\hat{x}_0$, achieve the same mean-standard deviation pair. The following two propositions give the conditions for the existence of such an $\hat{x}_0$ with $\hat{x}_0 < x_0$.

Proposition 2.1. *When $A > 0$, all mean-standard deviation pairs within*

$$\{(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \mid \frac{A}{C}x_0 \leq \mu < \frac{B}{A}x_0,\ x_0 > 0\}$$

are pseudo efficient (type 1).

Proof: When $x_0 > 0$, set $\{\mu \mid \frac{A}{C}x_0 \leq \mu < \frac{B}{A}x_0\}$ is non-empty as $D = BC - A^2 > 0$. From (2.8) and the assumption of $\mu < \frac{B}{A}x_0$, we have

$$\hat{x}_0 = -x_0 + \frac{2A}{B}\mu < -x_0 + 2x_0 = x_0.$$

□

Example 2.1. Let's consider the example in Chapter 7 of Sharpe, Alexander and Bailey (1995) [62], which is a revised version of the example on page 176 of Markowitz (1959) [41]. For this market of three risky assets with expected return vector $e = (1.162, 1.246, 1.228)'$ and covariance

$$V = \begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix},$$

the corresponding parameters can be calculated as $A = 80.6015$, $B = 93.5679$, $C = 69.8459$, and $D = 38.7199$. We consider an instance with investor's initial wealth equal to 1 and the pre-given expected future wealth equal to $\mu = 1.160$, which is greater than $\frac{A}{C}x_0 = 1.154$ and less than $\frac{B}{A}x_0 = 1.1609$. The optimal efficient policy in such an instance is specified by

$$x(x_0 = 1; \mu = 1.16) = (1.1075, -0.0471, 0.0296)'$$

and the corresponding efficient mean-standard deviation pair is given by (1.160, 0.1199). From (2.8) and Proposition 2.1, it can be verified that the following boundary policy associated with a less initial investment level $\hat{x}_0 = 0.9985$,

$$x(\hat{x}_0 = 0.9985; \mu = 1.16) = (0.9914, -0.0404, 0.0475)'$$

yields the same mean-standard deviation pair of (1.160, 0.1199). Thus, policy $x(x_0 = 1; \mu = 1.16)$ is pseudo efficient (type 1).

Proposition 2.2. *i) When $A \leq 0$ and $x_0 > 0$, all mean-standard deviation pairs of (MV),*

$$\{(\mu, \sqrt{\tfrac{C}{D}\left(\mu - \tfrac{A}{C}x_0\right)^2 + \tfrac{x_0^2}{C}}) \mid \tfrac{A}{C}x_0 \leq \mu,\ x_0 > 0\}$$

are pseudo efficient (type 1).

ii) When $A < 0$ and $x_0 \leq 0$, all mean-standard deviation pairs within

$$\{(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \mid \frac{B}{A}x_0 < \mu,\ x_0 \leq 0\}$$

are pseudo efficient (type 1).

Proof: i) When $A \leq 0$ and $x_0 > 0$, from (2.8), the assumption of $\frac{A}{C}x_0 \leq \mu$ and the fact of $BC > A^2$, we have

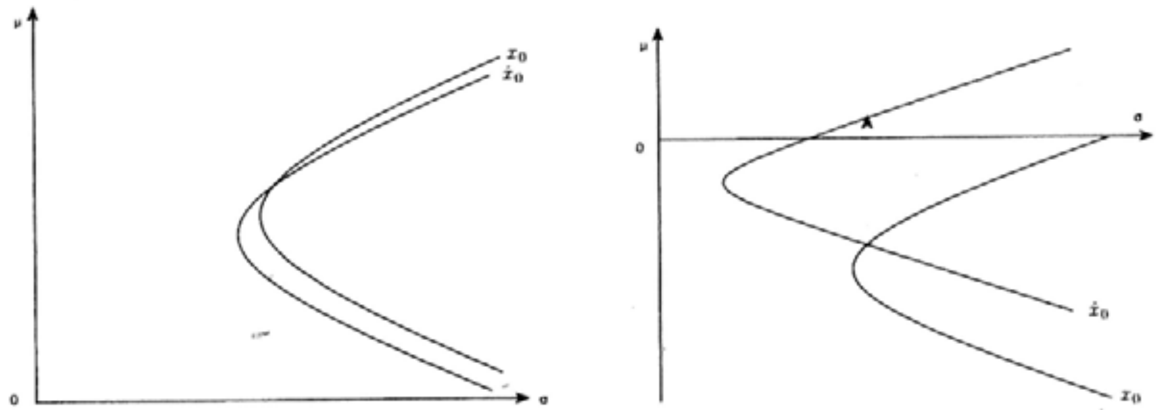
$$\hat{x}_0 = -x_0 + \frac{2A}{B}\mu \leq -x_0 + \frac{2A^2}{BC}x_0 < x_0.$$

ii) When $A < 0$ and $x_0 \leq 0$, from (2.8) and the assumption of $\frac{B}{A}x_0 < \mu$, we have

$$\hat{x}_0 = -x_0 + \frac{2A}{B}\mu < -x_0 + 2x_0 = x_0.$$

□

The phenomena of pseudo efficiency are illustrated in Figure 2.2 for situations with positive and negative A when $x_0 > 0$, respectively.



(a) $A > 0$, $x_0 > 0$ (b) $A < 0$, $x_0 > 0$

Figure 2.2: Pseudo Efficiency (Type 1)

The economic implication of the above discussion and derivation is that investors should not implement any pseudo efficient policy by investing all his initial wealth, x_0, into a market of all risky assets. He should rather consider alternative investment strategies by committing less initial investment level to the market. If we relax the assumption of binding budget spending, these pseudo efficient policies are essentially dominated by some other boundary policies associated with less initial investment level.

For a pseudo efficient policy, let us name the particular boundary policy, which achieves the same mean-standard deviation pair with less initial investment level,

$$x(\hat{x}_0; \mu) = \frac{\hat{x}_0}{D}(BV^{-1}\mathbf{1} - AV^{-1}e) + \frac{\mu}{D}(CV^{-1}e - AV^{-1}\mathbf{1}), \tag{2.9}$$

the *replacing policy*. It will be interesting to us to investigate the efficiency of the replacing policy with respect to its initial investment level $\hat{x}_0$.

Proposition 2.3. *Assume that $A \geq 0$. Then, when a replacing policy, $x(\hat{x}_0; \mu)$, exists, it is always efficient with respect to the less initial investment level, $\hat{x}_0$.*

Proof: Recall that, if parameter $A > 0$, the minimum variance set moves down when the initial investment level decreases and the pseudo efficient policy exists only when $x_0 > 0$. Thus, pseudo efficiency only occurs in Part I of the reachable region. In other words, the replacing policy is efficient in situations with $A > 0$.

If parameter $A = 0$, the pseudo efficient policy exists only when $\mu \geq 0$ and $x_0 > 0$. As the less initial investment level is $\hat{x}_0 = -x_0$, the replacing policy is efficient due to $\mu \geq 0 = \frac{A}{C}\hat{x}_0$. □

The situation with negative A is more complicated.

Proposition 2.4. *Assume that $A < 0$. i) When $D < A^2$, the replacing policy, $x(\hat{x}_0; \mu)$, is efficient with respect to the less initial investment level $\hat{x}_0$ if and only if $\frac{-AB}{D-A^2}x_0 \geq \mu > \frac{B}{A}x_0$, $x_0 \leq 0$.*

ii) When $D > A^2$, the replacing policy, $x(\hat{x}_0; \mu)$, is efficient with respect to the less initial investment level $\hat{x}_0$ if and only if $\mu \geq -\frac{AB}{D-A^2}x_0$, $x_0 > 0$ or $\mu > \frac{B}{A}x_0$, $x_0 \leq 0$.

iii) When $D = A^2$, the replacing policy, $x(\hat{x}_0; \mu)$ is efficient with respect to the less initial investment level $\hat{x}_0$ if and only if x_0 is nonpositive.

Proof: As the expected future wealth of the minimum variance portfolio associated with the less initial investment level, $\hat{x}_0$, is

$$\frac{A}{C}\hat{x}_0 = -\frac{A}{C}x_0 + \frac{2A^2}{BC}\mu,$$

the replacing policy is efficient if and only if

$$\mu \geq -\frac{A}{C}x_0 + \frac{2A^2}{BC}\mu, \tag{2.10}$$

which can be expressed in the following equivalent form,

$$(D - A^2)\mu \geq -ABx_0. \tag{2.11}$$

i) If $D < A^2$ and $x_0 > 0$, using the relationships $BC > A^2$ and $\frac{-A}{D-A^2} < \frac{1}{A}$ for all negative A, (2.11) reduces to

$$\mu \leq -\frac{AB}{D-A^2}x_0 < \frac{B}{A}x_0 < \frac{A}{C}x_0,$$

which contradicts the assumption that $x(x_0;\mu)$ is efficient.

If $D < A^2$ and $x_0 \leq 0$, the existence of replacing policy requires that $\frac{B}{A}x_0 < \mu$ and (2.11) is satisfied when $\mu \leq -\frac{AB}{D-A^2}x_0$, which together yield the conclusion in i).

ii) If $D > A^2$ and $x_0 > 0$, (2.11) holds true when

$$\mu \geq -\frac{AB}{D-A^2}x_0 > \frac{A}{C}x_0,$$

which is also the condition for the existence of a replacing policy (see item i) of Proposition 2.2).

If $D > A^2$ and $x_0 \leq 0$, the existence of a replacing policy (see item ii) of Proposition 2.2) requires $\mu > \frac{B}{A}x_0$, which further implies

$$\mu > \frac{B}{A}x_0 \geq \frac{A}{C}x_0 \geq -\frac{AB}{D-A^2}x_0, \text{ with } x_0 \leq 0,$$

which is exactly (2.11).

iii) When $D = A^2$, (2.11) reduces to $-ABx_0 \leq 0$, which is true only for nonpositive x_0. □

Table 2.1 summarizes the situations with different conditions under which the replacing policy is efficient with respect to its initial investment level $\hat{x}_0$.

Actually, the replacing policy $x(\hat{x}_0;\mu)$ is efficient with respect to the less initial investment level $\hat{x}_0$, only when the efficient mean-standard deviation pair lies in Part I of the reachable region. When the efficient mean-standard deviation pair lies in Part II of the reachable region, the original policy $x(x_0;\mu)$ is an efficient boundary policy corresponding to x_0 and the replacing policy $x(\hat{x}_0;\mu)$ is an inefficient boundary policy corresponding to $\hat{x}_0$. In the latter situation, investors can further modify his replacing policy $\hat{x}(\hat{x}_0;\mu)$ to the efficient one

Table 2.1: Efficiency situations of the replacing policy

wealth	*parameter*	*interval for* μ
$x_0 > 0$	$A > 0$	$[\frac{A}{C}x_0, \frac{B}{A}x_0)$
$x_0 > 0$	$A = 0$	$[0, +\infty)$
$x_0 \leq 0$	$A < 0,\ D < A^2$	$(\frac{B}{A}x_0, -\frac{AB}{D-A^2}x_0]$
$x_0 > 0$	$A < 0,\ D > A^2$	$[-\frac{AB}{D-A^2}x_0, +\infty)$
$x_0 \leq 0$	$A < 0,\ D > A^2$	$(\frac{B}{A}x_0, +\infty)$
$x_0 \leq 0$	$A < 0,\ D = A^2$	$(\frac{B}{A}x_0, +\infty)$

with the same standard deviation, albeit high expected return $\hat{\mu} = \frac{2A}{C}\hat{x}_0 - \mu$ ($> \mu$). The same procedure can be applied to this newly identified better mean-standard deviation pair $(\hat{\mu}, \sqrt{\frac{C}{D}(\hat{\mu} - \frac{A}{C}\hat{x}_0)^2 + \frac{\hat{x}_0^2}{C}})$ to seek further improvement with less initial investment level and higher expected future wealth. This iterative improvement process will stop when the replacing policy becomes an efficient boundary policy. In other words, the iterative improvement process will stop when the newly resulting mean-standard deviation pair falls into Part I of the reachable region.

Example 2.2. Consider a market of two risky assets with

$$e = (1.22, 1.78)'$$

and

$$V = \begin{pmatrix} 0.1200 & 0.1622 \\ 0.1622 & 0.2200 \end{pmatrix} \succ 0.$$

In such a market, $A = e'V^{-1}\mathbf{1} = -50.4607 < 0$, $B = e'V^{-1}e = 34.9820$, $C = \mathbf{1}'V^{-1}\mathbf{1} = 171.1277$, $D = BC - A^2 = 3440.1053$, and $D - A^2 = 893.8202 > 0$. Assume that the investor's initial wealth is $x_0 = 1$.

i) If the pre-given expected future wealth is set at $\mu = 2.0$, we have

$$\mu = 2.0 > -\frac{AB}{D - A^2}x_0 = 1.9749.$$

Then, the replacing policy is an efficient boundary policy corresponding to the less initial investment level $\hat{x}_0 = -x_0 + \frac{2A\mu}{B} = -6.7699$.

ii) If the pre-given expected future wealth is set at $\mu = 1.5$, we have

$$\mu = 1.5 < -\frac{AB}{D - A^2}x_0 = 1.9749.$$

The replacing policy $x(\hat{x}_0; \mu)$ now is an inefficient boundary policy corresponding to the less initial investment level $\hat{x}_0 = -x_0 + \frac{2A\mu}{B} = -5.3274$. This inefficient boundary policy can be replaced by an efficient boundary policy $x(\hat{x}_0; \hat{\mu})$ with a higher expected future wealth,

$$\hat{\mu} = \frac{2A}{C}\hat{x}_0 - \mu = 1.6418,$$

and the same variance. As the other boundary policy which also generates the mean-standard deviation pair, $(\hat{\mu}, \sqrt{\frac{C}{D}(\hat{\mu} - \frac{A}{C}\hat{x}_0)^2 + \frac{\hat{x}_0^2}{C}})$, is associated with a larger investment level,

$$\hat{\hat{x}}_0 = -\hat{x}_0 + \frac{2A\hat{\mu}}{B} = 0.5909 > \hat{x}_0,$$

the policy $x(\hat{x}_0; \hat{\mu})$ is thus not pseudo efficient and the iterative improvement process stops. Essentially, the newly generated mean-standard deviation pair, $(\hat{\mu}, \sqrt{\frac{C}{D}(\hat{\mu}_0 - \frac{A}{C}\hat{x}_0)^2 + \frac{\hat{x}_0^2}{C}})$, lies in Part I of the reachable region.

We can conclude from our earlier discussion that, in a market with only risky assets, the common belief of monotonicity does not hold, i.e., not the larger amount you invest, the larger expected future wealth you can expect for a given risk (variance) level. More specifically, in certain situations, a smaller investment can achieve the same or even better performance than a larger investment, in the sense of mean-variance. Such findings lead to a clear conclusion: It is not justifiable to insist the binding budget spending assumption in a market setting with only risky assets.

We now consider the following revised formulation of problem (MV) by allowing investors the flexibility not to invest all his initial wealth into the market,

$$(MV_1) \quad \min_x \quad x'Vx \tag{2.12}$$
$$\text{s.t.} \quad x'e = \mu,$$
$$x'\mathbf{1} \le x_0.$$

As the expected future wealth of the minimum variance portfolio of (MV_1) is given by

$$\mu^* = \begin{cases} 0, & x_0 \ge 0, \\ \frac{A}{C}x_0, & x_0 < 0. \end{cases}$$

we confine parameter μ in (MV_1) within $\{\mu \mid \mu \ge 0\}$ when $x_0 \ge 0$ and within $\{\mu \mid \mu \ge \frac{A}{C}x_0\}$ when $x_0 < 0$.

Proposition 2.5. *The optimal policy of (MV_1) is given as,*

$$x^*(\hat{x}_0^*; \mu) = \frac{\hat{x}_0^*}{D}(BV^{-1}\mathbf{1} - AV^{-1}e) + \frac{\mu}{D}(CV^{-1}e - AV^{-1}\mathbf{1}), \tag{2.13}$$

where $\hat{x}_0^*$ is termed optimal investment level and is given as follows,

$$\hat{x}_0^* = \begin{cases} x_0, & A > 0, x_0 > 0, \mu \ge \frac{B}{A}x_0, \\ & \text{or } A \ge 0, x_0 \le 0, \mu \ge \frac{A}{C}x_0, \\ & \text{or } A < 0, x_0 \le 0, \frac{A}{C}x_0 \le \mu \le \frac{B}{A}x_0. \\ \frac{A}{B}\mu, & \text{otherwise.} \end{cases} \tag{2.14}$$

Furthermore, the mean-variance efficient frontier of (MV_1) is given by

$$\mu = \begin{cases} \sqrt{\frac{D}{C}\sigma^2 - \frac{D}{C^2}x_0^2} + \frac{A}{C}x_0, & A > 0, x_0 > 0, \mu \ge \frac{B}{A}x_0, \\ & \text{or } A \ge 0, x_0 \le 0, \mu \ge \frac{A}{C}x_0, \\ & \text{or } A < 0, x_0 \le 0, \frac{A}{C}x_0 \le \mu \le \frac{B}{A}x_0, \\ \sqrt{B}\sigma, & \text{otherwise.} \end{cases} \tag{2.15}$$

Proof: Solving problem (MV_1) is equivalent to solving a two-phase optimization problem: Finding first the optimal investment level, $\hat{x}_0^*$, through

$$\hat{x}_0^* = \arg\min_{x \le x_0} \quad \frac{C}{D}\left(\mu - \frac{A}{C}x\right)^2 + \frac{x^2}{C}, \tag{2.16}$$

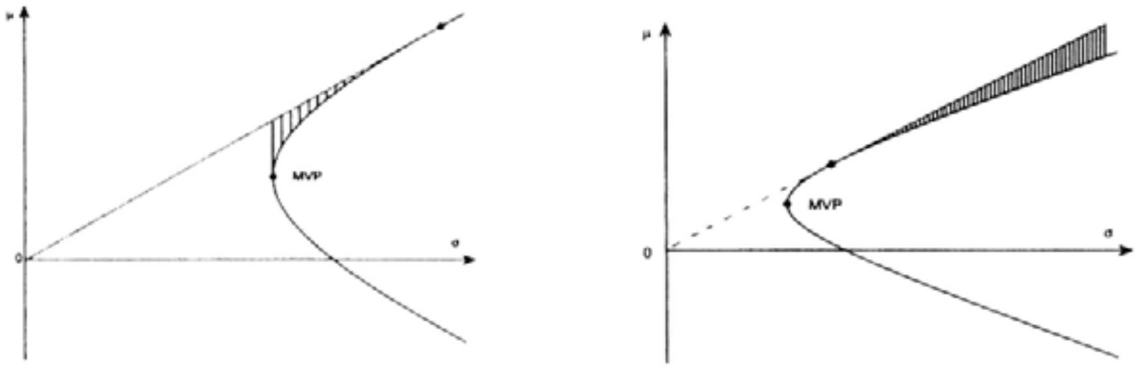
and applying then the boundary policy $x^*(\hat{x}_0^*; \mu)$. It can be verified that the optimal investment level $\hat{x}_0^*$ of problem (2.16) is given by (2.14). Applying the efficient boundary policy to $\hat{x}_0^*$ yields the efficient frontier in (2.15). □

When $A < 0$, $x_0 > 0$, the efficient frontier of problem (MV_1) is exactly the upper boundary of the reachable region. In other words, in this situation, all efficient policies in the traditional mean-variance sense are essentially pseudo efficient (type 1), i.e., all the efficient mean-standard deviation pairs of (MV) are dominated by mean-standard deviation pairs of (MV_1) corresponding to optimal investment levels.

When $A > 0$, $x_0 > 0$, the efficient frontier of problem (MV_1) is a combination of the lower segment of the upper boundary of the reachable region (for $\mu \leq \frac{B}{A}x_0$) and the upper segment of the efficient frontier of problem (MV) with initial wealth x_0 (for $\mu \geq \frac{B}{A}x_0$). In other words, in this situation, some pseudo efficient (type 1) policies always exist for relatively small value of μ.

When $A < 0$, $x_0 < 0$, the efficient frontier of problem (MV_1) is a combination of the upper segment of the upper boundary of the reachable region (for $\mu \geq \frac{B}{A}x_0$) and the lower segment of the efficient frontier of problem (MV) with initial wealth x_0 (for $\mu \leq \frac{B}{A}x_0$). In other words, in this situation, some pseudo efficient (type 1) policies always exist for relatively large value of μ.

It can be verified that the efficient frontier of (MV_1) is a continuous differentiable function. Figure 2.3 illustrates further the dominance relationship between efficient frontiers of problems (MV) and (MV_1). Any mean-standard deviation pair in the shadow area dominates at least one efficient mean-standard deviation pair of (MV) and, at the same time, is dominated by an efficient mean-standard deviation pair of (MV_1).



(a) $A > 0$, $x_0 > 0$

(b) $A < 0$, $x_0 < 0$

Figure 2.3: Efficient frontiers of (MV) and (MV_1)

2.3.2. Pseudo efficiency (Type 2) and optimal management of initial wealth

In the formulation of (MV_1), we only consider the best investment performance in the market under an investor's budget constraint. As the investor's expected future wealth essentially involves a summation of the expected future wealth from the investment in risky assets and the money the investor places aside at the initial time, $(x_0 - x'\mathbf{1})$, it seems more reasonable to study the following revised mean variance model for the optimal management of the total initial wealth,

$$
\begin{aligned}
(MV_2) \qquad \max_{x} \quad & x'e + (x_0 - x'\mathbf{1}) \\
\text{s.t.} \quad & x'Vx = \sigma^2, \\
& x'\mathbf{1} \le x_0,
\end{aligned}
\tag{2.17}
$$

where zero interest is assumed to be applied to the money the investor places aside at the initial time. For (MV_2) to have a solution, x_0 is required to be no less than $-\sqrt{C}\sigma$. Furthermore, as the standard deviation of the minimum variance portfolio of (MV_2) is given by

$$\sigma^* = \begin{cases} 0, & x_0 \geq 0, \\ -x_0\sqrt{\frac{1}{C}}, & x_0 < 0, \end{cases}$$

we confine parameter σ in (MV_2) within $\{\sigma \mid \sigma \geq 0\}$ when $x_0 \geq 0$ and within $\{\sigma \mid \sigma \geq -x_0\sqrt{\frac{1}{C}}\}$ when $x_0 < 0$.

Proposition 2.6. *For $x_0 \geq -\sqrt{C}\sigma$, the optimal policy of (MV_2) is given as,*

$$x^*(\hat{x}_0^*;\sigma^2) = \frac{\hat{x}_0^*}{D}(BV^{-1}\mathbf{1} - AV^{-1}e) + (\sqrt{\frac{\sigma^2}{CD} - \frac{(\hat{x}_0^*)^2}{C^2D}} + \frac{A\hat{x}_0^*}{CD})(CV^{-1}e - AV^{-1}\mathbf{1}), \tag{2.18}$$

where $\hat{x}_0^*$ is termed optimal investment level and is given as

$$\hat{x}_0^* = \begin{cases} x_0, & A > C, x_0 > 0, \sigma \geq \frac{\sqrt{B+C-2A}}{A-C}x_0, \\ & \text{or } A \geq C, x_0 \leq 0, \sigma \geq -x_0\sqrt{\frac{1}{C}}, \\ & \text{or } A < C, x_0 \leq 0, -x_0\sqrt{\frac{1}{C}} \leq \sigma \leq \frac{\sqrt{B+C-2A}}{A-C}x_0, \\ \sigma(A-C)\sqrt{\frac{1}{B+C-2A}}, & \text{otherwise.} \end{cases} \tag{2.19}$$

Furthermore, the mean-variance efficient frontier of (MV_2) can be expressed as,

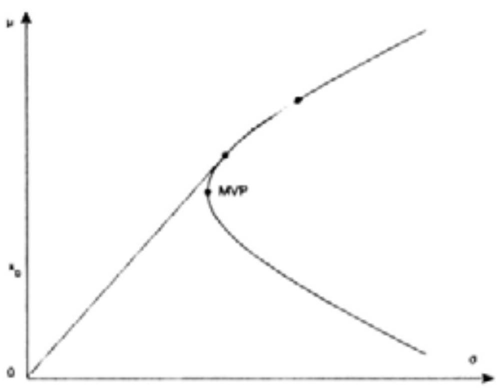
$$\mu = \begin{cases} \sqrt{\frac{D}{C}\sigma^2 - \frac{D}{C^2}x_0^2} + \frac{A}{C}x_0, & A > C, x_0 > 0, \sigma \geq \frac{\sqrt{B+C-2A}}{A-C}x_0, \\ & \text{or } A \geq C, x_0 \leq 0, \sigma \geq -x_0\sqrt{\frac{1}{C}}, \\ & \text{or } A < C, x_0 \leq 0, -x_0\sqrt{\frac{1}{C}} \leq \sigma \leq \frac{\sqrt{B+C-2A}}{A-C}x_0, \\ x_0 + \sqrt{B+C-2A}\sigma, & \text{otherwise.} \end{cases} \tag{2.20}$$

Proof: Solving problem (MV_2) is equivalent to solving a two-phase optimization problem: Finding first the optimal investment level, $\hat{x}_0^*$, through

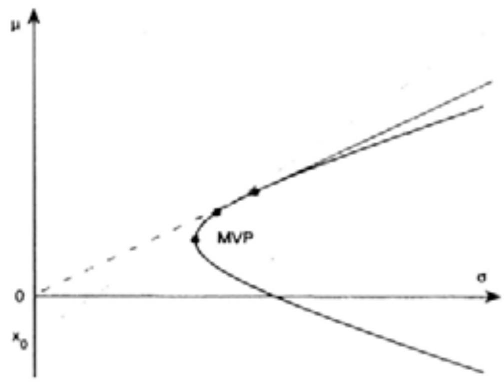
$$\hat{x}_0^* = \arg \max_{-\sqrt{C}\sigma \leq x \leq x_0} \sqrt{\frac{D}{C}\sigma^2 - \frac{Dx^2}{C^2}} + \frac{A}{C}x + x_0 - x,$$

and applying then the efficient boundary policy, $x^*(\hat{x}_0^*;\sigma^2)$. It can be verified that the optimal investment level $\hat{x}_0^*$ is given by (2.19), and the efficient policy in (2.18) and the efficient frontier in (2.20) then follow. □

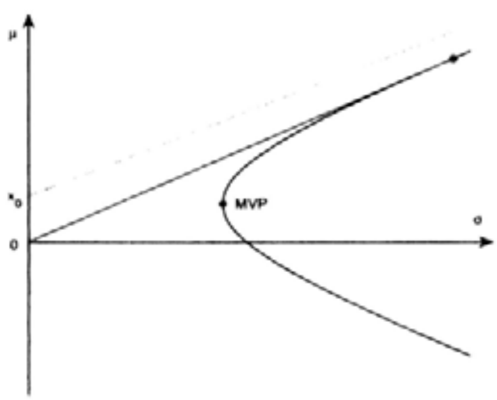
It is easy to check that the efficient frontier of (MV_2) is also a continuous differentiable function. Figure 2.4 demonstrates the relationship among three efficient frontiers of (MV), (MV_1) and (MV_2) under different situations, which are represented by the blue curve, the red curve and the green curve in the figures, respectively.



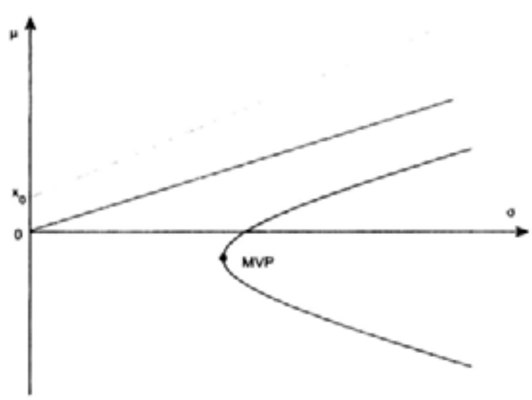
(a) $A > C,\ x_0 > 0$



(b) $A < C,\ x_0 < 0$



(c) $0 < A < C, x_0 > 0$



(d) $A < 0, x_0 > 0$

Figure 2.4: Efficient frontiers of (MV), (MV_1) and (MV_2)

Definition 2.3. *If there exists a σ such that an efficient mean-standard deviation pair of problem (MV) associated with initial wealth x_0, $(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}})$, which is not pseudo efficient (type 1), is dominated by the efficient mean-standard* *deviation pair of problem* (MV_2) *with this specific* σ, *i.e.*,

$$(\mu, -\sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \prec ((x^*)'e + (x_0 - (x^*)'\mathbf{1}), -\sqrt{(x^*)'V(x^*)}), \tag{2.21}$$

where x^* *is the optimal policy of* (MV_2) *given in (2.18), then the given mean-standard deviation pair associated with* x_0 *is called* **pseudo efficient (type 2)** *and the corresponding efficient boundary policy* $x(x_0;\mu)$ *is called* **pseudo efficient policy (type 2)**.

Proposition 2.7. *i) When* $A < 0$, *all mean-standard deviation pairs within*

$$\{(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \mid \frac{B-A}{A-C}x_0 < \mu \le \frac{B}{A}x_0,\ x_0 < 0\}$$

are pseudo efficient (Type 2).

ii) When $A = 0$, *all mean-standard deviation pairs within*

$$\{(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \mid -\frac{B}{C}x_0 < \mu,\ x_0 \le 0\}$$

are pseudo efficient (Type 2).

iii) When $0 < A < C$, *all mean-standard deviation pairs within*

$$\{(\mu, \sqrt{\tfrac{C}{D}\left(\mu - \tfrac{A}{C}x_0\right)^2 + \tfrac{x_0^2}{C}}) \mid \tfrac{B}{A}x_0 \le \mu,\ x_0 > 0\}$$

$$\text{or} \quad \{(\mu, \sqrt{\tfrac{C}{D}\left(\mu - \tfrac{A}{C}x_0\right)^2 + \tfrac{x_0^2}{C}}) \mid \tfrac{B-A}{A-C}x_0 < \mu,\ x_0 \le 0\}$$

are pseudo efficient (Type 2).

iv) When $A = C$, *all mean-standard deviation pairs within*

$$\{(\mu, \sqrt{\tfrac{C}{D}\left(\mu - \tfrac{A}{C}x_0\right)^2 + \tfrac{x_0^2}{C}}) \mid \tfrac{B}{A}x_0 \le \mu,\ x_0 > 0\}$$

are pseudo efficient (Type 2).

v) When $A > C$, *all mean-standard deviation pairs within*

$$\{(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \mid \frac{B}{A}x_0 \le \mu < \frac{B-A}{A-C}x_0,\ x_0 > 0\}$$

are (Type 2) pseudo efficient.

Proof: i) From (2.20), when $A < 0$ and $x_0 < 0$, the two sectors of the efficient frontier of (MV_2) intersect at $\mu = \frac{B-A}{A-C}x_0$. When $\mu > \frac{B-A}{A-C}x_0$, the efficient frontier of (MV_2) is a straight line, which dominates the original efficient frontier of (MV). Excluding type-1 pseudo efficient mean-standard deviation pairs, $\{(\mu, \sqrt{\frac{C}{D}\left(\mu - \frac{A}{C}x_0\right)^2 + \frac{x_0^2}{C}}) \mid \frac{B}{A}x_0 < \mu,\ x_0 < 0\}$, yields the conclusion. When $A < 0$ and $x_0 = 0$, there is no type-2 pseudo efficient solution of (MV). On the other hand, it is clear from Proposition 2.5 that all efficient solutions of (MV) are pseudo efficient (type 1) when $A < 0$ and $x_0 > 0$.

We can prove ii), iii), iv) and v) similarly. □

For situation with $A > C$ and $x_0 > 0$, the original efficient frontier can be divided into three sectors, the sector of Type-1 pseudo efficiency, the sector of Type-2 pseudo efficiency, and the remaining sector from the original mean-variance efficient frontier of (MV) (see Figure 2.4(a)). While the sector of Type 1 pseudo efficiency can be achieved by less initial investment level, the sector of Type 2 pseudo efficiency cannot be achieved by less initial investment level, but it is dominated when we consider the optimal management of the total initial wealth.

Definition 2.4. *A mean-standard deviation pair of (MV) associated with initial wealth x_0 is called* **pseudo efficient** *if it is either type-1 pseudo efficient or type-2 pseudo efficient, and its corresponding efficient boundary policy is called* **pseudo efficient policy**.

For different situations, we list in Table 2.2 the range of parameter μ in (MV) such that the solution of (MV) is pseudo efficient.

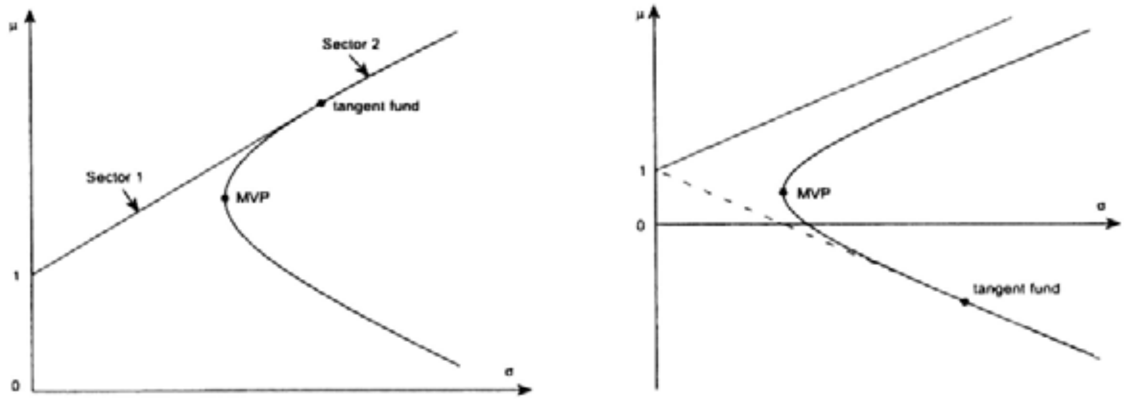
Now let us discuss the impact of our findings on the separation theorem. In this discussion, we set $x_0 = 1$ and random variable x thus reduces to the return of the portfolio. Note that the efficient frontier of (MV_2) includes all efficient mean-standard deviation pairs of return with binding budget spending assumption relaxed. Based on Proposition 2.6, when $A > C$, the efficient frontier

Table 2.2: Pseudo efficient intervals for μ

wealth	*parameter*	*type 1*	*type 2*	*efficient*
$x_0 > 0$	$A > C$	$[\frac{A}{C}x_0, \frac{B}{A}x_0)$	$[\frac{B}{A}x_0, \frac{B-A}{A-C}x_0)$	$[\frac{B-A}{A-C}x_0, +\infty)$
$x_0 > 0$	$0 < A \leq C$	$[\frac{A}{C}x_0, \frac{B}{A}x_0)$	$[\frac{B}{A}x_0, +\infty)$	NULL
$x_0 > 0$	$A \leq 0$	$[\frac{A}{C}x_0, +\infty)$	NULL	NULL
$x_0 < 0$	$A \geq C$	NULL	NULL	$[\frac{A}{C}x_0, +\infty)$
$x_0 < 0$	$0 \leq A < C$	NULL	$(\frac{B-A}{A-C}x_0, +\infty)$	$[\frac{A}{C}x_0, \frac{B-A}{A-C}x_0]$
$x_0 < 0$	$A < 0$	$(\frac{B}{A}x_0, +\infty)$	$(\frac{B-A}{A-C}x_0, \frac{B}{A}x_0]$	$[\frac{A}{C}x_0, \frac{B-A}{A-C}x_0]$
$x_0 = 0$	$A \geq C$	NULL	NULL	$[0, +\infty)$
$x_0 = 0$	$0 \leq A < C$	NULL	$(0, +\infty)$	$\mu = 0$
$x_0 = 0$	$A < 0$	$(0, +\infty)$	NULL	$\mu = 0$

of (MV_2) is the combination of a linear segment (Sector 1) and a sector of the original efficient frontier of (MV) (Sector 2), see Figure 2.5(a). For Sector 2, we need two different funds to construct the whole sector of the efficient mean-standard deviation pair, while for Sector 1, the whole line segment can be realized by varying the percentage investment of holding this particular efficient tangent fund indicated in Figure 2.5(a). Thus, in cases with $A > C$, a mix of one-fund and two-fund theorems holds. When $A < C$, the efficient frontier of (MV_2) is a straight line. There exists a unique inefficient tangent fund, i.e., $x^*(\hat{x}_0^*; \sigma^2)/\hat{x}_0^*$, see Figure 2.5(b). By varying the percentage investment of shorting this particular inefficient tangent fund yields the entire efficient frontier of (MV_2), i.e., a one fund theorem holds for cases with $A < C$. When $A = C$, the efficient frontier of (MV_2) is also a straight line. But it can only be realized by holding net zero percentage investment in different risky funds. Because the efficient frontier of (MV_2) is the asymptote of the efficient frontier of (MV) in this situation.

Example 2.3. *[Continuation of Example 2.1]* Applying the two revised mean-variance formulations, (MV_1) and (MV_2), to Example 2.1 yields the mean-



(a) $A > C$ (b) $A < C$

Figure 2.5: Separation theorem for (MV_2)

variance efficient frontier of (MV_1) as

$$\mu = \begin{cases} \sqrt{0.5544\sigma^2 - 0.0079} + 1.1540, & \mu > 1.1609, \\ 9.6730\sigma, & 0 \leq \mu \leq 1.1609, \end{cases}$$

and the mean-variance efficient frontier of (MV_2) as

$$\mu = \begin{cases} \sqrt{0.5544\sigma^2 - 0.0079} + 1.1540, & \mu > 1.2055, \\ 1 + 1.4868\sigma, & 0 \leq \mu \leq 1.2055. \end{cases}$$

It can be seen from Figure 2.6 that the mean-standard deviation pairs in

$$\{(\mu, \sqrt{1.8039(\mu - 1.1540)^2) + 0.0143}) \mid 1.1540 \leq \mu \leq 1.1609\}$$

are pseudo efficient (type 1) and the mean-standard deviation pairs in

$$\{(\mu, \sqrt{1.8039(\mu - 1.1540)^2) + 0.0143}) \mid 1.1609 \leq \mu \leq 1.2055\}$$

are pseudo efficient (type 2). In Figure 2.6, three efficient frontiers of (MV), (MV_1) and (MV_2) are represented by the blue curve, the red curve and the green curve, respectively.

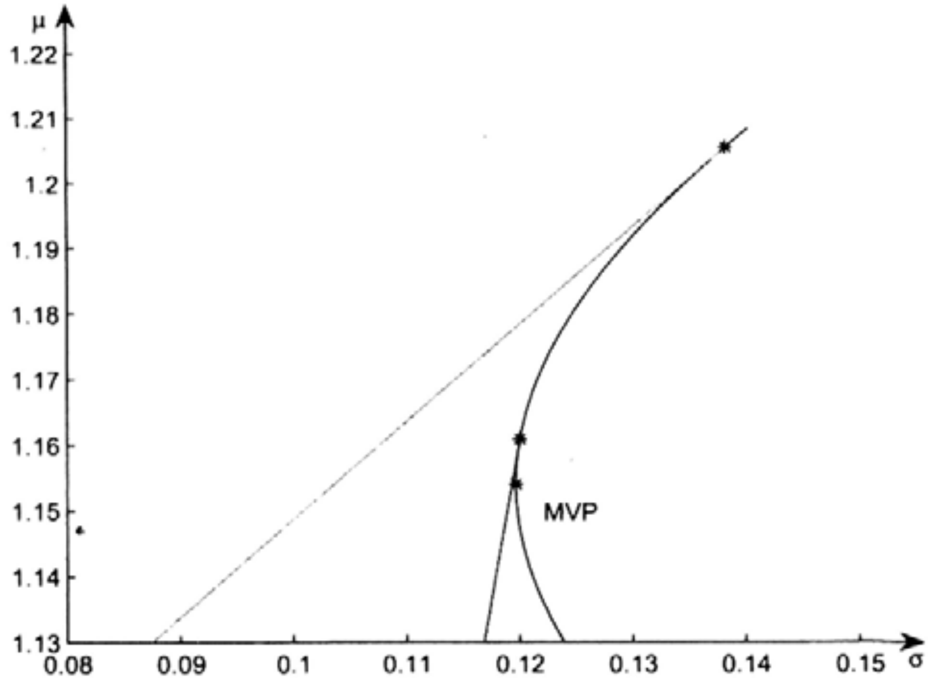


Figure 2.6: Efficient frontiers of (MV), (MV_1) and (MV_2) for Example 2.3

2.3.3. Pseudo efficiency in markets without shorting

The phenomenon of pseudo efficiency may occur in markets with no shorting constraint as well. Let's continue to consider Example 2.1 and assume that shorting is not allowed now. The mean-variance portfolio selection problem is formulated in Markowitz (1959) [41] as follows,

$$\begin{array}{llll} (MV-O) & \min\limits_{x} & x'Vx & (2.22) \\ & \text{s.t.} & x'e = \mu, & \\ & & x'\mathbf{1} = 1, & \\ & & x \geq 0, & \end{array}$$

where $x \in \mathbb{R}^n$ with x_i being the percentage invested in the ith risky asset. Relaxing the binding budget constraint yields the following revised mean-variance portfolio selection problem,

$$(MV - R) \quad \min_{x} \quad x'Vx \tag{2.23}$$
$$\text{s.t.} \quad x'e = \mu,$$
$$x'\mathbf{1} \leq 1,$$
$$x \geq 0.$$

Solving $(MV - O)$ and $(MV - R)$ of Example 2.1 numerically gives rise two efficient frontiers in Figure 2.7. It is evident that the efficient frontier of $(MV-R)$ dominates the efficient frontier of $(MV - O)$. In markets without shorting, you may also invest less to get better investment performance, compared to the solution based on the classical mean-variance formulation.

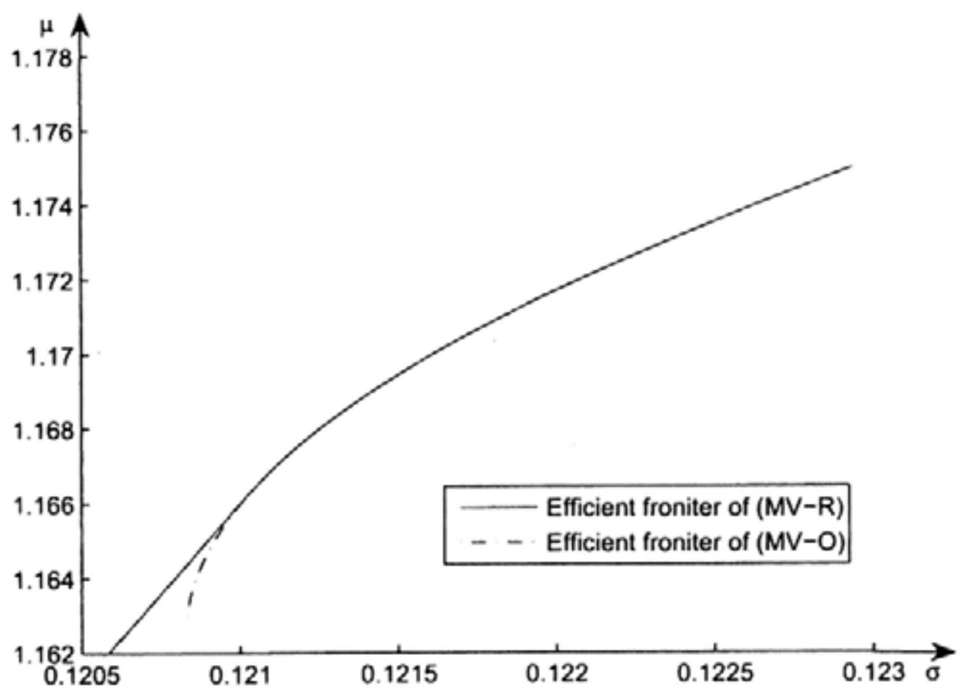


Figure 2.7: Efficient frontiers of $(MV - O)$ and $(MV - R)$

Unlike situations with shorting in which pseudo efficiency (type 1) occurs for each model with $x_0 > 0$, pseudo efficient (type 1) may not occur in some situations without shorting. If we change, for example, the return vector in the above example to $\hat{e} = (1.222, 1.246, 1.228)'$, we have observed that the two efficient frontiers of $(MV - O)$ and $(MV - R)$ become the same.

2.4. Issues Related to Market equilibrium

It is assumed in Mossin (1966) [47] that a large number of risk averse individuals in a market, whose utility functions depend on the expected yield and the variance of his/her portfolio, construct their portfolios from $(n-1)$ risky assets and one riskless asset. All individuals are assumed to have identical estimation of the random yields' distribution. Mossin (1966) [47] found that each individual in the equilibrium market would hold the same percentage of the total outstanding stock of all risky assets and this identical percentage is positive. In other words, Mossin (1966) [47] claimed that each individual should hold positive position on each risky asset in the market, although he did not consider the conditions for random yields under which the existence of the market equilibrium is ensured.

Merton (1972) [46] discussed both the efficient mean-variance sets in a market with only risky assets and in a market with multiple risky assets and a riskless asset with return r_f. The model in Merton (1972) [46] for the market with multiple risky assets is the same as formulation (MV) in this chapter, except x_0 is set at 1. He found that the one-fund theorem holds if and only if the riskless return r_f is less than the expected return of the minimum variance portfolio, A/C, and asserted that when $r_f \geq A/C$, the optimal portfolio policy is not a general equilibrium solution with homogeneous expectations, for when a riskless asset is included, it is possible to select a portfolio with nonpositive net amount on risky assets.

We consider now the mean-variance portfolio selection for a market with a riskless asset with a deterministic return $r_f > 0$:

$$(MV(r_f)) \qquad \min_{x} \quad x'Vx \tag{2.24}$$
$$\text{s.t.} \quad x'e + (x_0 - x'\mathbf{1})R = \mu,$$

where x_0 is the initial wealth and $\mu \geq r_f x_0$ is a pre-given expected future wealth.

It is well known that the optimal policy of $(MV(r_f))$ is given by

$$x(x_0;\mu) = \frac{(\mu - r_f x_0)}{H} V^{-1}(e - r_f \mathbf{1}), \tag{2.25}$$

where $H = (e - r_f\mathbf{1})'V^{-1}(e - r_f\mathbf{1}) > 0$. Furthermore, the minimum variance set of $(MV(r_f))$ is given by

$$\sigma^2 = \frac{(\mu - r_f x_0)^2}{H}. \tag{2.26}$$

Remark 2.1. *It is easy to check that there exists only one initial wealth level, $\frac{\mu - \sqrt{H}\sigma}{r_f}$, which enables us to achieve a given mean-standard deviation pair (μ,σ) by adopting an efficient boundary policy. Therefore, all efficient mean-standard deviation pairs are not pseudo efficient.*

When $A/C \le r_f$, it is also easy to check that $\mathbf{1}'x(x_0;\mu) \le 0$ holds, which implies that the net amount invested in risky assets by each investor of mean-variance type is nonpositive. The market is thus not in an equilibrium. In conclusion, $A/C > r_f$ is a necessary condition for an equilibrium of a market with a riskless asset. It is also worth to point out that when one-fund theorem holds, the market may be or may be not in an equilibrium: Each individual may hold a negative position on some risky asset, while having a positive net amount on all risky assets.

In a market of all risky assets, if binding budget spending is not enforced, any mean-variance optimizer with an initial wealth, x_0, will adopt formulation (MV_2) to determine his/her optimal investment level to the market. Recall that the optimal investment level is given by

$$\hat{x}_0^* = \begin{cases} x_0, & A > C, \sigma \ge \frac{\sqrt{B+C-2A}}{A-C} x_0, \\ & \text{or } A < C, \sigma \le \frac{\sqrt{B+C-2A}}{A-C} x_0, \\ & \text{or } A = C, x_0 \le 0. \\ \sigma(A-C)\sqrt{\frac{1}{B+C-2A}}, & \text{otherwise.} \end{cases}$$

When $A \le C$, the optimal investment level $\hat{x}_0^*$ is nonpositive regardless the initial wealth x_0, which implies any individual of a mean-variance type will invest a nonpositive net amount in risky assets. The market is thus not in an equilibrium. In conclusion, $A > C$ is a necessary equilibrium condition for a market of all risky assets when binding budget spending is relaxed.

Following Merton (1972) [46], when $A/C > r_f$ and $x_0 > 0$, one fund theorem holds and the expected return of the so called market portfolio is given as

$$\mu^* = \frac{AH - r_f D}{CH - D} x_0,$$

by solving

$$\begin{cases} \sigma^2 = \frac{(\mu - r_f x_0)^2}{H}, \\ \sigma^2 = \frac{C}{D}\left(\mu - \frac{Ax_0}{C}\right)^2 + \frac{x_0^2}{C}. \end{cases}$$

Proposition 2.8. *The mean-standard deviation pair of the market portfolio, when it exists, is never pseudo efficient.*

Proof: Note that when $A > C$, efficient policies of (MV) are pseudo efficient only when $x_0 > 0$ and $\mu < \frac{B-A}{A-C} x_0$. As $A/C > r_f > 1$ and $CH - D = (Cr_f - A)^2 > 0$, we have the following for $x_0 > 0$,

$$\begin{aligned} &\mu^* - \frac{B-A}{A-C} x_0 \\ =&\frac{(AH - r_f D)(A - C) - (CH - D)(B - A)}{(CH - D)(A - C)} x_0 \\ =&\frac{-Cr_f^2 + (A + C)r_f - A}{(CH - D)(A - C)} x_0 \\ >&0. \end{aligned}$$

□

We thus conclude that for markets with a riskless asset, the phenomena of pseudo efficiency discussed in this chapter never occur.

2.5. Conclusions

In this chapter, we have revisited the Markowitz's classical mean-variance model for markets consisting of all risky assets. One key recognition is the dual realization of mean-variance pairs, revealing a violation of the one price law and even raising concerns of arbitrage opportunities in the sense of mean-variance.

By removing the constraint of binding budget spending and reexamining this classical problem from an expanded three-objective framework: Maximizing the expected future wealth, minimizing the risk (variance) of the future wealth and minimizing the initial investment level, we have derived somehow surprising results. More specifically, we have identified the set of portfolio policies which are efficient in the original mean-variance space, and are, however, inefficient in this newly introduced three-dimensional objective space. Stimulated by the revealed non-monotonic phenomenon in the mean-variance world, we have demonstrated that we can do better than the classical mean-variance when removing the binding budget spending constraint.

Collective action of moving away from pseudo efficient solutions makes the market in transition, thus offering a possible new avenue to investigate a converging process to a new market equilibrium.

2.6. Appendix

Appendix A: Analysis for Situations where A is Negative

If the initial wealth is positive, the expected future wealth of the minimum variance portfolio (MVP), $\frac{A}{C}x_0$, is negative when the parameter A is negative. The minimum variance set in such situations is illustrated in Figure 2.8.

Example 2.4. Consider a market of two risky assets with

$$e = (1.2, 3.6)',$$

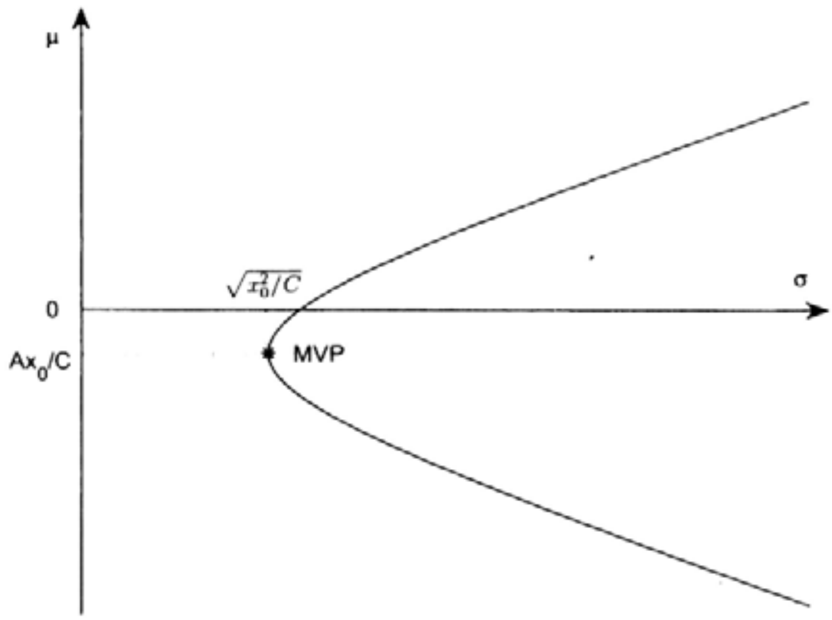


Figure 2.8: The portfolio frontier when $A < 0$

and

$$V = \begin{pmatrix} 1 & 1.6 \\ 1.6 & 3.2 \end{pmatrix} \succ 0.$$

In such a market, parameter $A = e'V^{-1}\mathbf{1} = -0.3750$, $B = e'V^{-1}e = 5.8500$, $C = \mathbf{1}'V^{-1}\mathbf{1} = 1.5625$.

One may question whether this type of market is a reasonable market setting and whether this type of market always permits an arbitrage opportunity. Let us continue with Example 2.4.

Example 2.5. *(Continuation of Example 2.4)* Assume that the market in Example 2.4 is generated by the following two risky assets, X_1 and X_2, with their returns, r_1 and r_2, being of log-normal,

$$r_1 = e^{0.2Y+0.6981Z-0.0814}, \quad r_2 = e^{0.2927Y+0.3674Z+1.1706},$$

respectively, where Y and Z are independent standard normal random variables.

It can be verified that the first and the second moments of the return vector are given exactly as in Example 2.4.

We show now that this market setting does not permit an arbitrage opportunity from the definition of arbitrage opportunity. When setting the initial wealth at zero, then the only feasible investment strategy in this market is to hold the wealth position x in asset X_1 and hold $-x$ in asset X_2 at time 0.

The random future wealth of this type portfolio is then expressed by

$$e^{0.2Y+0.6981Z-0.0814}x - e^{0.2927Y+0.3674Z+1.1706}x.$$

When $x \neq 0$, the future wealth can be positive, zero or negative, depending on the realizations of the two independent normal random variables, Y and Z, while the future wealth is zero when $x = 0$. Therefore, there does not exist an arbitrage opportunity in this exemplary market with a negative parameter A.

One important conclusion revealed from the above example is that, even in a no-arbitrage market with random returns always positive, the phenomenon of $A < 0$ may happen.

CHAPTER 3

Better Than Dynamic Mean-Variance: Time Inconsistency and Free Cash Flow Stream

3.1. Introduction

In this chapter, we will show first that the multi-period mean-variance formulation is neither time consistent nor time consistent in efficiency. In contrast, although the continuous-time mean-variance formulation is not time consistent, it is time consistent in efficiency. We find that the trade-offs induced by the multi-period efficient mean-variance policy, which reflect investor's risk attitude during the investment process, are not only time-varying but also state-dependent. One fundamental question which we try to address and to answer in this chapter is why the time consistency, or more specifically, the time consistency in efficiency, really matters. If we relax the restriction that an admissible portfolio must be a self-financing policy and allow withdrawal of positive dollar amounts out of the market during the investment process, we can then devise an investment policy which is strictly better than the optimal multi-period mean-variance policy. More specifically, we propose a revised policy which can achieve the original mean-variance pair attained by any given multi-period efficient mean-variance policy and obtain during the investment process some extra (positive) dollar amounts with a strictly positive probability under certain probability distribution assumptions.

The organization of this chapter is as follows. In Section 3.2, we summarize the current results of optimal policies for both multi-period and continuous-time mean-variance portfolio selection problems. In Section 3.3, we examine the trade-offs induced by the optimal multi-period mean-variance policy, thus concluding that the dynamic mean-variance formulation in discrete-time is neither time consistent nor time consistent in efficiency. In Section 3.4, we demonstrate that the dynamic mean-variance formulation in continuous-time is time consistent in efficiency. In Section 3.5, we develop a revised mean-variance policy which dominates the optimal multi-period mean-variance policy in the sense that, while the two achieve the same mean-variance pair of the terminal wealth, the revised policy enables the investor to receive a free cash flow stream (FCFS) during the investment process. In Section 3.6, we investigate properties of FCFS and discuss the existence probability of FCFS, leading to an introduction of an inefficiency measure for multi-period mean-variance portfolio selection. We report in Section 3.7 the results from a numerical experiment to demonstrate the features and benefits from adopting the revised mean-variance policy. Finally, we conclude this chapter in Section 3.8.

3.2. Preliminaries

3.2.1. Discrete-time dynamic mean-variance portfolio selection

The motivation behind Markowitz's pioneering mean-variance portfolio selection formulation is to strike a balance between the expected final wealth and the risk measured by the variance of the final wealth. When the single-period mean-variance portfolio selection formulation is extended to a multi-period setting, an additional dimension of the balance between a short-term and long-term goals has to be dealt with.

To be more specific about our discrete-time model (Li and Ng (2000) [35]), we consider a capital market consisting of one riskless asset and n risky assets within a time horizon T. Let s_t be the given return of the riskless asset at period t and $\mathbf{e_t} = (e_t^1, ..., e_t^n)'$ the vector of random returns of the n risky assets at period t. An investor joins the market at the beginning of period 0 with an initial wealth x_0. It is assumed that vectors $\mathbf{e_t}$, $t = 0, 1, \ldots, T-1$, are statistically independent and the only information known about the random return vector $\mathbf{e_t}$ is its first two moments, the mean and the covariance.

The investor can allocate his/her wealth among the riskless asset and n risky assets at the beginning of period 0 and reallocates his/her wealth at the beginning of each of the following $(T-1)$ consecutive periods. Let x_t be the wealth of the investor at the beginning of period t, and u_t^i, $i = 1, 2, \ldots, n$, be the amount invested in the ith risky asset at period t.

The investor seeks a best investment strategy, $\mathbf{u}_t = (u_t^1, u_t^2, ..., u_t^n)'$ for $t = 0$, $1, 2, \ldots, T-1$, to attain the optimality of the following dynamic mean-variance model:

$$\begin{aligned}(MV) \qquad \min \quad & Var(x_T \mid x_0) + \lambda E(x_T \mid x_0) \\ \text{s.t.} \quad & x_{t+1} = s_t x_t + \mathbf{P_t'}\mathbf{u_t}, \; x_0 \text{ is given,}\end{aligned} \tag{3.1}$$

where the excess return vector $\mathbf{P_t}$ is defined as

$$\mathbf{P_t} = (P_t^1, P_t^2, ..., P_t^n)' = ((e_t^1 - s_t), (e_t^2 - s_t), ..., (e_t^n - s_t))',$$

and is assumed to satisfy

$$\begin{aligned} E(\mathbf{P_t}'\mathbf{P_t}) \succ 0, \quad & \forall t = 0, 1, ..., T-1, \\ s_t^2(1 - E(\mathbf{P_t}')E^{-1}(\mathbf{P_t}'\mathbf{P_t})E(\mathbf{P_t})) > 0, \quad & \forall t = 0, 1, ..., T-1. \end{aligned}$$

Note that λ represents the overall trade-off between two objectives of maximizing the expected return and minimizing the risk. Changing λ from 0 to $-\infty$ yields the entire mean-variance efficient frontier.

As the variance operation does not satisfy the smoothing property, i.e., $Var(Var(\cdot \mid I^j) \mid I^k) \neq Var(\cdot \mid I^k)$, $\forall\, j > k$, where I^k is the information set at period k, problem (MV) is nonseparable in the sense of dynamic programming. Li and Ng (2000) [35] embed problem (MV) into a separable parametric auxiliary problem with a quadratic utility function and derive the following optimal policy for (MV):

$$\begin{aligned}\mathbf{u}_t^*(x_t) &= -s_t E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(\mathbf{P}_t)x_t + \left(\prod_{k=0}^{T-1} s_k x_0 - \frac{\lambda}{2\prod_{k=0}^{T-1}(1-B_k)}\right) \\ &\quad \times \left(\prod_{k=t+1}^{T-1}\frac{1}{s_k}\right) E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(\mathbf{P}_t), \quad t = 0, 1, \ldots, T-1, \qquad (3.2)\end{aligned}$$

where $B_t = E(\mathbf{P}_t')E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(\mathbf{P}_t)$ and $\prod_{k\in\emptyset} f_k$ is assumed to be equal to 1 for any function f_k. $\prod_{k=0}^{T-1} s_k x_0 - \frac{\lambda}{2\prod_{k=0}^{T-1}(1-B_k)}$ is also denoted by Γ, which is called risk attitude parameter in this chapter. Furthermore, Li and Ng (2000) [35] give the mean-variance efficient frontier of (MV) explicitly as follows,

$$Var(x_T \mid x_0) = \frac{\prod_{t=0}^{T-1}(1-B_t)}{1-\prod_{t=0}^{T-1}(1-B_t)}\left(E(x_T \mid x_0) - x_0\prod_{t=0}^{T-1} s_t\right)^2,$$

$$\text{for} \quad E(x_T \mid x_0) \geq \prod_{t=0}^{T-1} s_t x_0.$$

For a given trade-off between mean and variance, λ, the expected value and variance of the terminal wealth are given, respectively, as

$$\begin{aligned}E(x_T) &= \prod_{j=0}^{T-1} s_j x_0 - \frac{\lambda(1-\prod_{j=0}^{T-1}(1-B_j))}{2\prod_{j=0}^{T-1}(1-B_j)}, \\ Var(x_T) &= \frac{\lambda^2(1-\prod_{j=0}^{T-1}(1-B_j))}{4\prod_{j=0}^{T-1}(1-B_j)}.\end{aligned}$$

As the original problem (MV) is nonseparable, the derived dynamic mean-variance policy does not satisfy the principle of optimality. We will show in the next section that the solution specified in (3.2) is neither time consistent nor time consistent in efficiency. Adopting the term in Basak and Chabakauri (2007) [4], we call the solution in (3.2) as the *pre-committed* optimal mean-variance policy, which is derived to achieve the best mean-variance pair for the entire time horizon spanning from period 0 to period $(T-1)$. Note that, at any intermediate period t, (short-sighted) investors have incentives to deviate from the pre-committed policy determined at time 0 for a shorter time horizon from period t to period $(T-1)$.

3.2.2. Continuous-time mean-variance portfolio selection

In the continuous-time setting (Zhou and Li (2000) [75]), there are $n+1$ basic assets which can be traded continuously. One of the assets is a riskless bank account whose value process $S_0(t)$ is subject to the following ordinary differential equation (ODE),

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \ t \geq 0, \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r(t) > 0$ is the interest rate. The other n assets are risky securities whose price processes, $S_1(t), \ldots, S_n(t)$, satisfy the following stochastic differential equations (SDE),

$$\begin{cases} dS_i(t) = S_i(t)(b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW^j(t)), \ t \geq 0, \\ S_i(0) = s_i > 0, i = 1, 2, \ldots, n, \end{cases}$$

where $(W^1(t), \ldots, W^n(t))$ is the n-dimensional Brown motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

Assumption 3.1. *(nondegeneracy condition)*

$$\sigma(t)\sigma(t)' \geq \delta I, \quad \forall t \in [0, T],$$

where $\sigma(t) = (\sigma_{ij}(t))_{n\times n}$ *and* $\delta > 0$.

Let $u_i(t)$, $i = 1, \ldots, n$, be the dollar amount which an investor invests in the ith risky asset at time t. The wealth of the investor, $x(t)$, then satisfies the following stochastic differential equation,

$$\begin{cases} dx(t) = (r(t)x(t) + \sum_{i=1}^{n}(b_i(t) - r(t))u_i(t))\,dt + \sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}(t)u_i(t)dW^j(t), \\ x(0) = x_0 > 0. \end{cases} \tag{3.3}$$

Similar to the discrete-time situation, the investor seeks a best investment strategy, $\mathbf{u}(t)$, which is an adapted vector random process, to attain the optimality of the following continuous-time mean-variance model:

$$\begin{array}{lll} (MV_C) & \min & Var(x(T)) + \lambda E(x(T)) \\ & \text{s.t.} & \begin{cases} \mathbf{u}(.) \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^n), \\ (x(.), \mathbf{u}(.)) \text{ satisfy (3.3)}. \end{cases} \end{array} \tag{3.4}$$

Using the same embedding scheme as in Li and Ng (2000) [35], Zhou and Li (2000) [75] derive the following optimal policy $\mathbf{u}^*(t, x(t))$ for (MV_C),

$$\mathbf{u}_t^*(x(t)) = -(\sigma(t)\sigma(t)')^{-1}B(t)'\left(x(t) - \gamma e^{-\int_t^T r(s)ds}\right), \tag{3.5}$$

where

$$\begin{aligned} B(t) &= (b_1(t) - r(t), \ldots, b_n(t) - r(t)), \\ \gamma &= x_0 e^{\int_0^T r(t)dt} - \frac{\lambda e^{\int_0^T \rho(t)dt}}{2}, \\ \rho(t) &= B(t)(\sigma(t)\sigma(t)')^{-1}B(t)', \end{aligned}$$

γ is also called risk attitude parameter and express the efficient frontier of (MV_C) explicitly as follows,

$$Var(x(T)) = \frac{e^{-\int_0^T \rho(t)dt}}{1 - e^{-\int_0^T \rho(t)dt}}\left(E(x(T)) - x_0 e^{\int_0^T r(s)ds}\right)^2, \quad \text{for } E(x(T)) \geq x_0 e^{\int_0^T r(s)ds}.$$

3.3. Induced trade-offs and Preference Switching

Substituting the optimal policy in (3.2) into the wealth dynamics in (3.1), performing some algebraic operations and taking the expected value give rise the following (Li and Ng (2000) [35]),

$$\begin{aligned} E(x_{t+1} \mid \mathbf{u_0^*}, \ldots, \mathbf{u_t^*}, x_0) &= s_t(1-B_t)E(x_t \mid \mathbf{u_0^*}, \ldots, \mathbf{u_{t-1}^*}, x_0) \\ &\quad + \frac{B_t}{\prod_{k=t+1}^{T-1} s_k}\left(\prod_{k=0}^{T-1} s_k x_0 - \frac{\lambda}{2\prod_{k=0}^{T-1}(1-B_k)}\right). \end{aligned}$$

Solving the above equation recursively yields the following expression of the conditional expectation,

$$E(x_k \mid \mathbf{u_0^*}, \ldots, \mathbf{u_{k-1}^*}, x_0) = \prod_{j=0}^{k-1} s_j x_0 - \frac{\left(1-\prod_{j=0}^{k-1}(1-B_j)\right)\lambda}{2\prod_{j=k}^{T-1} s_j\left(\prod_{j=0}^{T-1}(1-B_j)\right)}.$$

Lemma 3.1. *The pre-committed optimal mean-variance policy specified in (3.2) satisfies time consistency only when its wealth process follows a particular path,*

$$x_0 \to E(x_1 \mid \mathbf{u_0^*}, x_0) \to E(x_2 \mid \mathbf{u_0^*}, \mathbf{u_1^*}, x_0) \to \cdots \to E(x_{T-1} \mid \mathbf{u_0^*}, \ldots, \mathbf{u_{T-2}^*}, x_0).$$

Proof: Consider the following truncated multi-period mean-variance problem from period k to period $(T-1)$ with a given x_k and the same trade-off parameter λ as given in (MV):

$$\begin{aligned} (MV_{k-T}^{\lambda}) \quad \min \quad & Var(x_T \mid x_k) + \lambda E(x_T \mid x_k) \\ \text{s.t.} \quad & x_{t+1} = s_t x_t + \mathbf{P_t'u_t}, \ x_k \text{ is given.} \end{aligned}$$

Similar to the solution to (MV), the optimal policy of (MV_{k-T}^{λ}) at period t, $t = k, k+1, \ldots, T-1$, can be derived as

$$\begin{aligned} \mathbf{u_t^{k-T}}(x_t, \lambda) = &- s_t E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t})x_t + \\ &\left(\prod_{j=k}^{T-1} s_j x_k - \frac{\lambda}{2\prod_{j=k}^{T-1}(1-B_j)}\right)\left(\prod_{j=t+1}^{T-1} \frac{1}{s_j}\right) E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t}). \end{aligned} \tag{3.6}$$

For the pre-committed optimal mean-variance policy to be time consistent, the policies in (3.2) and (3.6) must be the same due to the uniqueness of the solution. Note that $\mathbf{u_t^*}$ is equal to $\mathbf{u_t^{k-T}}$ for $t = k, k+1, \ldots, T-1$ only when

$$x_k = \prod_{j=0}^{k-1} s_j x_0 - \frac{\left(1 - \prod_{j=0}^{k-1}(1-B_j)\right)\lambda}{2\prod_{j=k}^{T-1} s_j \left(\prod_{j=0}^{T-1}(1-B_j)\right)},$$

which is exactly the expectation of the x_k conditional on x_0 under the pre-committed optimal mean-variance policy. □

Note when the returns of risky assets are continuous random variables, the probability that the wealth process follows the path of its expected value is equal to zero. Thus, the discrete-time dynamic mean-variance formulation is not time consistent.

Now we further consider the following inverse optimization problem: For $k = 1, 2, \ldots, T-1$, find a trade-off parameter λ_k between $E(x_T|x_k)$ and $Var(x_T|x_k)$ such that the pre-committed optimal mean-variance policy $\mathbf{u_t^*}(x_t)$ $(t = k, k+1, \ldots, T-1)$ specified in (3.2) solves

$$(MV_{k-T}^{\lambda_k}) \quad \min \quad Var(x_T \mid x_k) + \lambda_k E(x_T \mid x_k)$$
$$\text{s.t.} \quad x_{t+1} = s_t x_t + \mathbf{P_t'u_t},\ x_k \text{ is given.}$$

Let a threshold x_k^* be defined as follows at the beginning of period k:

$$x_k^* = -\frac{\lambda}{2\prod_{l=k}^{T-1} s_l \prod_{l=0}^{T-1}(1-B_l)} + \prod_{l=0}^{k-1} s_l x_0, \tag{3.7}$$

which is constant and just the discounted risk attitude parameter at the beginning of period k.

Proposition 3.1. *The pre-committed optimal mean-variance policy,* $\mathbf{u_t^*}(x_t)$ $(t = k, k+1, \ldots, T-1)$, *specified in (3.2), solves* $(MV_{k-T}^{\lambda_k})$ *with* λ_k *satisfying*

$$\lambda_k = 2\left(x_k - \prod_{l=0}^{k-1} s_l x_0\right)\prod_{l=k}^{T-1} s_l(1-B_l) + \frac{\lambda}{\prod_{l=0}^{k-1}(1-B_l)}. \tag{3.8}$$

Furthermore, $\lambda_k < 0$ *when* $x_k < x_k^*$, $\lambda_k = 0$ *when* $x_k = x_k^*$, *and* $\lambda_k > 0$ *when* $x_k > x_k^*$.

Proof: Similar to the solution to (MV), the optimal policy of $(MV_{k-T}^{\lambda_k})$ at period t, $t = k, k+1, \ldots, T-1$, is given by

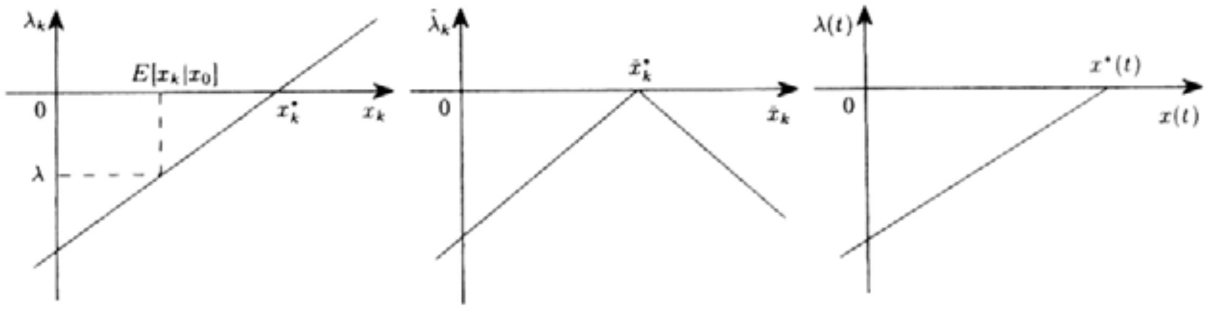
$$\mathbf{u_t^{k-T}}(x_t, \lambda_k) = - s_t E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t})x_t + \left(\prod_{j=k}^{T-1} s_j x_k - \frac{\lambda_k}{2\prod_{j=k}^{T-1}(1-B_j)}\right)\left(\prod_{j=t+1}^{T-1}\frac{1}{s_j}\right)E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t}). \tag{3.9}$$

Equalizing the policies in (3.2) and (3.9) yields the relationship between λ_k and λ given in (3.8). Form the assumptions of the model, $s_t^2(1-B_t) > 0$, $\forall\, t = 0, 1, \ldots, T-1$. It is evident now that $\lambda_k = 0$ when $x_k = x_k^*$ defined in (3.7), $\lambda_k < 0$ when $x_k < x_k^*$, and $\lambda_k > 0$ when $x_k > x_k^*$. □

The trade-off induced by the pre-committed optimal mean-variance policy at state x_k, λ_k, can not be guaranteed to be nonpositive under many returns assumptions, for example, under the assumption of a normal distribution for the returns of risky assets. Thus, the pre-committed optimal mean-variance policy specified in (3.2) is, in general, not time consistent in efficiency.

Remark 3.1. *If the realizations of a return distribution are all small enough such that the wealth* x_k *is never bigger than the threshold* x_k^*, *the trade-off induced by the pre-committed optimal mean-variance policy at state* x_k, λ_k, *can be assured to be nonpositive.*

It is interesting to note from (3.8) that, at a given state x_k, the trade-off induced by the pre-committed optimal mean-variance policy, λ_k , is a linear function of both the initial overall trade-off λ and the current wealth x_k. On one hand, the larger the overall trade-off λ, the larger the induced trade-off λ_k. On the other hand, the higher the current wealth level x_k, the higher the induced trade-off λ_k, which implies that the investor will place less weight on maximizing his/her expected return when he/she gets richer. Figure 3.1(a) presents the linear relationship between λ_k and x_k. The intersection of the line of λ_k and the horizontal line pin points the threshold x_k^*, which is larger than $E(x_k \mid \mathbf{u}_0^*, \ldots, \mathbf{u}_{k-1}^*, x_0)$.



(a) Induced trade-off under the pre-committed MV policy (b) Induced trade-off under the revised policy (c) Induced trade-off under the continuous-time MV policy

Figure 3.1: Relationship between induced trade-off λ and wealth x

When $x_k < x_k^*$, the dynamic mean-variance policy $\mathbf{u}_t^*$ $(t = k, k+1, \ldots, T-1)$ specified in (3.2) can be generated by $(MV_{k-T}^{\lambda_k})$ with a $\lambda_k < 0$, i.e., the pre-committed optimal mean-variance policy $\mathbf{u}_t^*$ $(t = k, k+1, \ldots, T-1)$ remains to be mean-variance efficient at period k, although the trade-off parameter λ_k between $E(x_T|x_k)$ and $Var(x_T|x_k)$ differs, in general, from λ, the initial trade-off parameter between $E(x_T|x_0)$ and $Var(x_T|x_0)$ in (MV).

When $x_k = x_k^*$, the pre-committed optimal mean-variance policy $\mathbf{u}_t^*$ $(t = k, k+1, \ldots, T-1)$ can be generated by $(MV_{k-T}^{\lambda_k})$ with $\lambda_k = 0$, i.e., the investor only cares about minimization of the variance of the terminal wealth, and the pre-committed optimal mean-variance policy $\mathbf{u}_t^*$ $(t = k, k+1, \ldots, T-1)$ becomes the least variance policy.

When $x_k > x_k^*$, the pre-committed optimal mean-variance policy $\mathbf{u}_t^*$ $(t = k, k+1, \ldots, T-1)$ can be generated by $(MV_{k-T}^{\lambda_k})$ with a $\lambda_k > 0$, i.e., the pre-committed optimal mean-variance policy $\mathbf{u}_t^*$ $(t = k, k+1, \ldots, T-1)$ is no longer mean-variance efficient at period k. More specifically, the investor switches his/her risk attitude at period k to minimize both $E(x_T \mid x_k)$ and $Var(x_T \mid x_k)$ and this trade-off is x_k-dependent.

The above analysis leads us to conclude that, in order to achieve a mean-variance efficiency for the overall T-period problem, an investor needs to adjust his/her "induced" trade-off from period to period, to sacrifice his/her "local" interests, or even to behave irrationally in certain circumstances. The root of all these surprising phenomena is the inherent nonseparability in the dynamic mean-variance formulation.

3.4. Nonpositive Induced Trade-Offs from Continuous-Time Mean-Variance Policy

We now define the time consistency in efficiency for continuous-time portfolio selection. Let

$$\pi_{t-T} = \{\pi_s \mid t \leq s \leq T\}.$$

Definition 3.1 (time consistency in Efficiency (Continuous-time)). *A combined risk expected return measure $\mathcal{M}_{0-T}(\pi_{0-T} \mid x_0) + \lambda E(x_T \mid \pi_{0-T}, x_0)$ is time consistent in efficiency if any optimal policy for the portfolio selection problem over the entire time horizon,*

$$\pi^*_{0-T} \in \arg\min_{\pi_{0-T}} \{\mathcal{M}_{0-T}(\pi_{0-T}) \mid x_0) + \lambda E(x_T \mid \pi_{0-T}, x_0)\},$$

also satisfies the local optimality conditions at any $0 \leq t \leq T$,

$$\pi^*_{t-T} \in \arg\min_{\pi_{t-T}} \{\mathcal{M}_{t-T}(\pi_{t-T}) \mid x_t) + \lambda_t E(x_T \mid \pi_{t-T}, x_t)\}$$

for some nonpositive λ_t, where x_t is any reachable wealth level at time t.

Similar to the discrete-time case, we also consider an inverse optimization problem for the continuous-time mean-variance portfolio selection: For any given $0 < t \leq T$, find a time-t trade-off $\lambda(t)$ between $E(x(T)|x(t))$ and $Var(x(T)|x(t))$

such that the pre-committed continuous-time mean-variance policy $\mathbf{u}_s^*(x(s))$ $(t \leq s \leq T)$ specified in (3.5) solves

$$(MV_{C_{t-T}}^{\lambda(t)}) \quad \min \quad Var(x(T)) + \lambda(t)E(x(T))$$
$$\text{s.t.} \quad \begin{cases} \mathbf{u}(.) \in L^2_{\mathcal{F}}(t, T; \mathbf{R}^n), \\ (x(.), \mathbf{u}(.)) \text{ satisfy the wealth dynamics with initial } x(t). \end{cases}$$

Let a threshold $x^*(t)$ be defined as follows at time t:

$$x^*(t) = -\frac{\lambda e^{\int_0^T \rho(s)ds}}{2e^{\int_t^T r(s)ds}} + x_0 e^{\int_0^t r(s)ds}, \tag{3.10}$$

which is just the discounted risk attitude parameter at time t.

Proposition 3.2. *The pre-committed continuous-time mean-variance policy, $\mathbf{u}_s^*(x(s))$ $(t \leq s \leq T)$, specified in (3.5), solves $(MV_{C_{t-T}}^{\lambda(t)})$ with $\lambda(t)$ satisfying*

$$\lambda(t) = 2\left(x(t) - x_0 e^{\int_0^t r(s)ds}\right) e^{\int_t^T (r(s)-\rho(s))ds} + \lambda e^{\int_0^t \rho(s)ds}. \tag{3.11}$$

Furthermore, $\lambda(t) < 0$ when $x(t) < x^*(t)$, $\lambda(t) = 0$ when $x(t) = x^*(t)$, and $\lambda(t) > 0$ when $x(t) > x^*(t)$.

Proof: Similar to the proof of Proposition 3.1. □

Proposition 3.3. *The optimal wealth $x(t)$ in continuous-time under the pre-committed optimal mean-variance policy specified in (3.5) never exceeds the threshold $x^*(t)$ given in (3.10). In other words, the trade-off induced by the pre-committed continuous-time optimal policy, $\lambda(t)$, is always nonpositive.*

Proof: Substituting $\mathbf{u}_t^*(x_t)$ specified in (3.5) into the wealth dynamics in (3.3) yields the following SDE,

$$\begin{cases} dx(t) = (b(t)x(t) + a(t))dt + \sum_{j=1}^n (d_j(t)x(t) + c_j(t))dW^j(t), \\ x(0) = x_0 > 0, \end{cases}$$

where $a(t)$, $b(t)$, $c_j(t)$, $d_j(t)$, $j = 1, \ldots, n$, are deterministic time dependent functions, which implies that the optimal wealth process is continuous. Note also that the threshold $x^*(t)$ is a function of t with $x^*(0) \geq x_0$, which implies that the optimal wealth process starts from a point below the threshold trajectory. There are two possible situations for the optimal wealth process $x(t)$.

i) $x(t) < x^*(t), \forall 0 \leq t \leq T$, i.e., the optimal wealth never exceeds the threshold. The induced trade-off is thus always negative.

ii) At some time t, the optimal wealth reaches the threshold, $x(t) = x^*(t)$. Due to the continuity of the wealth process $x(t)$ and the form of the pre-committed optimal policy $\mathbf{u}_t^*(x(t))$, the optimal policy $\mathbf{u}_s^*(x(s)) = 0$, for all s such that $t \leq s \leq T$, leading $x(s) = x^*(s)$, $\forall\, t \leq s \leq T$. The induced trade-off for the remaining time, $\lambda(s)$, is zero for all s such that $t \leq s \leq T$. □

The trade-off induced by the pre-committed continuous-time optimal mean-variance policy at any state $x(t)$, $\lambda(t)$, is nonpositive all the time. Thus, the continuous-time mean-variance formulation is time consistent in efficiency. Figure 3.1(c) illustrates the relationship between $\lambda(t)$ and $x(t)$. The prominent feature of the continuous-time case is that the wealth level never exceeds the threshold.

3.5. A Strategy Better than the Pre-committed Optimal Multi-Period Mean-Variance Policy

Stimulated by the recognition of an irrational behavior of an investor when he/she follows the pre-committed optimal mean-variance policy in discrete-time situations, we demonstrate in this section that any pre-committed optimal mean-variance policy specified in (3.2) is dominated by another better policy that generates the same mean-variance pair, while having a positive probability to receive a free cash flow stream.

3.5.1. Two-period case

We first consider the following two-period mean-variance portfolio selection problem to better explain the motivation of our approach,

$$\begin{aligned} \min \quad & Var(x_2 \mid x_0) + \lambda E(x_2 \mid x_0) \\ \text{s.t.} \quad & x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \ t = 0, 1, \ x_0 \text{ is given.} \end{aligned}$$

Applying (3.2) to the above two-period problem yields its pre-committed optimal mean-variance policy,

$$\begin{aligned} \mathbf{u}_0^*(x_0) &= -s_0 E^{-1}(\mathbf{P_0}\mathbf{P_0}')E(\mathbf{P_0})x_0 + \Gamma \frac{1}{s_1} E^{-1}(\mathbf{P_0}\mathbf{P_0}')E(\mathbf{P_0}), \\ \mathbf{u}_1^*(x_1) &= -s_1 E^{-1}(\mathbf{P_1}\mathbf{P_1}')E(\mathbf{P_1})x_1 + \Gamma E^{-1}(\mathbf{P_1}\mathbf{P_1}')E(\mathbf{P_1}), \end{aligned}$$

where $\Gamma = s_0 s_1 x_0 - \frac{\lambda}{2(1-B_0)(1-B_1)}$. From Proposition 3.1, policy $\mathbf{u}_1^*(x_1)$ is no longer a single-period mean-variance efficient policy when $x_1 > x_1^* = \frac{\Gamma}{s_1}$.

The above recognized irrational behavior motivates us to modify the pre-committed optimal dynamic mean-variance policy when $x_1 > x_1^*$. For x_1 greater than x_1^*, our new strategy is to divide x_1 into two parts: Investing only a partial amount $\hat{x}_1$ under the optimal dynamic mean-variance policy and taking the amount $x_1 - \hat{x}_1$ out of the market. Clearly, our new policy is of a partial stopping nature and goes beyond the class of self-financing policies. An immediate question is whether or not we are able to ensure the modified policy to achieve the same mean-variance pair of the terminal wealth as the original pre-committed two-period optimal mean-variance policy, while having a possibility to take "free cash" out of the market at the beginning of period 1.

We propose the following revised portfolio policy for the two-period problem under consideration: At period 0,

$$\hat{\mathbf{u}}_0^*(x_0) = \mathbf{u}_0^*(x_0);$$

At period 1, when $x_1 \le x_1^*$,

$$\hat{\mathbf{u}}_1^*(x_1) = \mathbf{u}_1^*(x_1);$$

when $x_1 > x_1^*$, take a positive amount, $x_1 - \hat{x}_1$, out of the market, where

$$\hat{x}_1 = x_1 - 2(x_1 - x_1^*)B_1 \tag{3.12}$$

is the remaining amount left in the market. Let the portfolio policy at $\hat{x}_1$ be

$$\hat{\mathbf{u}}_1^*(\hat{x}_1) = -s_1 E^{-1}(\mathbf{P}_1\mathbf{P}_1')E(\mathbf{P}_1)\hat{x}_1 + \hat{\Gamma}E^{-1}(\mathbf{P}_1\mathbf{P}_1')E(\mathbf{P}_1), \tag{3.13}$$

where

$$\hat{\Gamma} = 2s_1x_1 - 2s_1(x_1 - x_1^*)B_1 - \Gamma. \tag{3.14}$$

Theorem 3.1. *The revised policy with $\hat{x}_1$ and $\hat{\Gamma}$ specified in (3.12) and (3.14), respectively, achieves the same mean-variance pair, $E(x_2 \mid x_0)$ and $Var(x_2 \mid x_0)$, as does the pre-committed optimal mean-variance policy, while having a possibility to take free cash out of the market.*

Proof: When $x_1 > x_1^*$, the original pre-committed optimal mean-variance policy $\mathbf{u}_1^*$ yields

$$\begin{aligned} E(x_2|x_1) &= s_1x_1 + (\Gamma - s_1x_1)B_1, \\ Var(x_2|x_1) &= (\Gamma - s_1x_1)^2 B_1(1 - B_1). \end{aligned}$$

At a wealth level $\hat{x}_1$ which is strictly less than x_1, we require that the one-period portfolio policy yield the same conditional expected value and conditional variance as the pair of $E(x_2|x_1)$ and $Var(x_2|x_1)$. Note that one-period mean-variance policy at $\hat{x}_1$ specified in (3.13) gives rise

$$\begin{aligned} E(x_2|\hat{x}_1) &= s_1\hat{x}_1 + (\hat{\Gamma} - s_1\hat{x}_1)B_1, \\ Var(x_2|\hat{x}_1) &= (\hat{\Gamma} - s_1\hat{x}_1)^2 B_1(1 - B_1). \end{aligned}$$

Equalizing $E(x_2|x_1)$ and $E(x_2|\hat{x}_1)$ and equalizing $Var(x_2|x_1)$ and $Var(x_2|\hat{x}_1)$ simultaneously yield

$$\begin{cases} s_1x_1 + (\Gamma - s_1x_1)B_1 = s_1\hat{x}_1 + (\hat{\Gamma} - s_1\hat{x}_1)B_1, \\ (\Gamma - s_1x_1)^2 B_1(1 - B_1) = (\hat{\Gamma} - s_1\hat{x}_1)^2 B_1(1 - B_1). \end{cases}$$

Solving the above system of two equations for $\hat{x}_1$ and $\hat{\Gamma}$ leads to two solutions, one given in (3.12) and (3.14) and the other one, $(\hat{x}_1 = x_1,\ \hat{\Gamma} = \Gamma)$, which is rejected.

We now demonstrate that $E(x_2|x_0)$ and $Var(x_2|x_0)$ are the same under both the original pre-committed optimal mean-variance policy $\mathbf{u}^*$ and the revised policy $\hat{\mathbf{u}}^*$:

$$\begin{aligned}
E(x_2|x_0)\ |_{\mathbf{u}^*} &= E(E(x_2|x_1)|x_0) \\
&= \int E(x_2|x_1) f(x_1) dx_1 \\
&= \int_{x_1^*}^{\infty} E(x_2|x_1) f(x_1) dx_1 + \int_{-\infty}^{x_1^*} E(x_2|x_1) f(x_1) dx_1 \\
&= \int_{x_1^*}^{\infty} E(x_2|\hat{x}_1) f(x_1) dx_1 + \int_{-\infty}^{x_1^*} E(x_2|x_1) f(x_1) dx_1 \\
&= E(x_2|x_0)\ |_{\hat{\mathbf{u}}^*}
\end{aligned}$$

$$\begin{aligned}
Var(x_2|x_0)\ |_{\mathbf{u}^*} =& E(Var(x_2|x_1)|x_0) + Var(E(x_2|x_1)|x_0) \\
=& \int Var(x_2|x_1) f(x_1) dx_1 + \int (E(x_2|x_1) - E(x_2|x_0))^2 f(x_1) dx_1 \\
=& \int_{x_1^*}^{\infty} Var(x_2|\hat{x}_1) f(x_1) dx_1 + \int_{-\infty}^{x_1^*} Var(x_2|x_1) f(x_1) dx_1 \\
&+ \int_{x_1^*}^{\infty} (E(x_2|\hat{x}_1) - E(x_2|x_0))^2 f(x_1) dx_1 \\
&+ \int_{-\infty}^{x_1^*} (E(x_2|x_1) - E(x_2|x_0))^2 f(x_1) dx_1 \\
=& Var(x_2|x_0)\ |_{\hat{\mathbf{u}}^*},
\end{aligned}$$

where $f(x_1)$ is the probability density function of random variable x_1, which depends on the initial wealth x_0 and period-0 policy $\mathbf{u}_0^*(x_0)$. □

Figure 3.2 explains the solution concept behind Theorem 3.1. When $x_1 > x_1^*$, policy $\mathbf{u}_1^*$ places point $(E(x_2|x_1), Var(x_2|x_1))$ on the lower branch (thick red curve) of the (quadratic) minimum variance set associated with x_1, of which only the upper branch is mean-variance efficient. By taking some positive amount

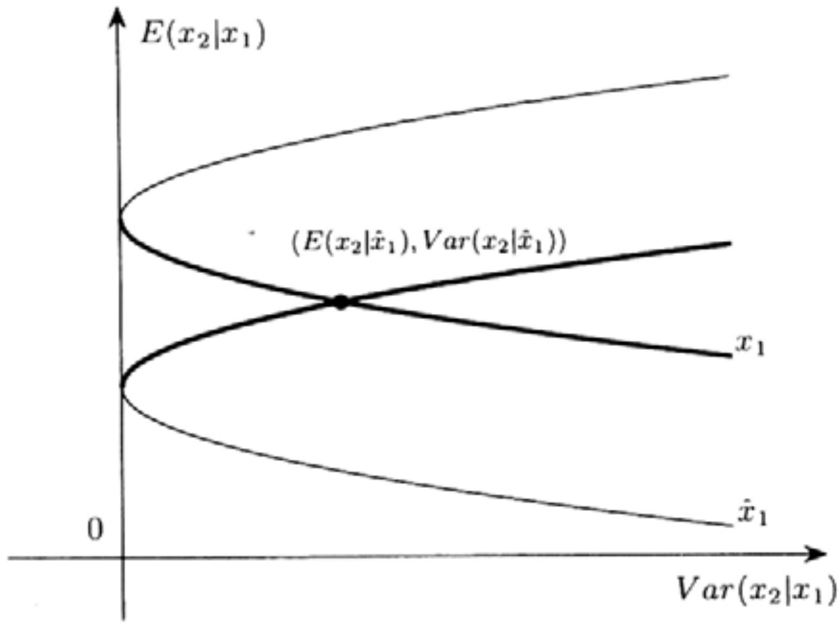


Figure 3.2: Two minimum variance sets associated with x_1 and $\hat{x}_1$

out of the market, we relocate the mean-variance point $(E(x_2|\hat{x}_1), Var(x_2|\hat{x}_1))$ corresponding to the remaining amount of $\hat{x}_1$ on the upper branch (thick blue curve) of the minimum variance set associated with $\hat{x}_1$. By choosing a suitable $\hat{x}_1$, we can make two thick curves intersect exactly at $(E(x_2|x_1), Var(x_2|x_1))$, which implies that the revised policy retains the same conditional expectation and the conditional variance of the wealth, while achieving an efficiency corresponding to the remaining amount of $\hat{x}_1$.

The revised policy is better than the pre-committed two-period mean-variance policy in the sense while the two achieve the same mean and variance for the terminal wealth, the revised policy generates additional positive cash flow with probability of $Pr(x_1 > x_1^*)$ during the investment process. Whenever $x_1 > x_1^*$, we withdraw amount of $2(x_1 - x_1^*)B_1$ out of the market and continue to invest the remaining amount $\hat{x}_1$ in the market. From Theorem 3.1, the expected value of the free cash flow taken out at the beginning of period 1 is given by

$$\int_{x_1^*}^{+\infty} 2(x_1 - x_1^*)B_1 f(x_1)dx_1,$$

where $f(x_1)$ is the density function of wealth x_1.

The following fact is interesting. When $x_1 > x_1^*$, after taking the amount $2(x_1 - x_1^*)B_1$ out of the market, the revised portfolio policy at the reduced wealth level $\hat{x}_1$ essentially takes the negative value of the original pre-committed two-period mean-variance policy at x_1,

$$\begin{aligned}\hat{\mathbf{u}}_1^*(\hat{x}_1) &= -s_1 E^{-1}(\mathbf{P}_1\mathbf{P}_1')E(\mathbf{P}_1)\hat{x}_1 + \hat{\Gamma}E^{-1}(\mathbf{P}_1\mathbf{P}_1')E(\mathbf{P}_1)\\ &= s_1 E^{-1}(\mathbf{P}_1\mathbf{P}_1')E(\mathbf{P}_1)x_1 - \Gamma E^{-1}(\mathbf{P}_1\mathbf{P}_1')E(\mathbf{P}_1)\\ &= -\mathbf{u}_1^*(x_1).\end{aligned}$$

For the proposed revised policy, we investigate further its corresponding induced trade-off at period 1, $\hat{\lambda}_1$, under which the revised policy, $\hat{\mathbf{u}}_1^*$, solves the second-period problem.

When $x_1 \leq x_1^*$, the truncated second-period portfolio selection problem, $(MV_{1-2}^{\hat{\lambda}_1})$, is

$$\begin{aligned}(MV_{1-2}^{\hat{\lambda}_1}) \qquad \min \quad & Var(x_2 \mid x_1) + \hat{\lambda}_1 E(x_2 \mid x_1)\\ \text{s.t.} \quad & x_2 = s_1 x_1 + \mathbf{P}_1'\mathbf{u}_1,\ x_1 \text{ is given.}\end{aligned}$$

Equalizing the optimal policy of $(MV_{1-2}^{\hat{\lambda}_1})$ and $\hat{\mathbf{u}}_1^*(x_1)$ yields

$$s_1 x_1 - \frac{\hat{\lambda}_1}{2(1 - B_1)} = \Gamma,$$

which implies

$$\hat{\lambda}_1 = 2s_1(1 - B_1)(x_1 - x_1^*) \leq 0.$$

When $x_1 > x_1^*$, the truncated second-period portfolio selection problem, $(MV_{1-2}^{\hat{\lambda}_1})$, is

$$\begin{aligned}(MV_{1-2}^{\hat{\lambda}_1}) \qquad \min \quad & Var(x_2 \mid \hat{x}_1) + \hat{\lambda}_1 E(x_2 \mid \hat{x}_1)\\ \text{s.t.} \quad & x_2 = s_1 \hat{x}_1 + \mathbf{P}_1'\mathbf{u}_1,\ \hat{x}_1 = x_1 - 2(x_1 - x_1^*)B_1.\end{aligned}$$

Equalizing the optimal policy of $(MV_{1-2}^{\hat{\lambda}_1})$ and $\hat{\mathbf{u}}_1^*(\hat{x}_1)$ yields

$$s_1 \hat{x}_1 - \frac{\hat{\lambda}_1}{2(1 - B_1)} = \hat{\Gamma},$$

which implies

$$\hat{\lambda}_1 = 2s_1(1 - B_1)(x_1^* - x_1) < 0.$$

In summary, the trade-off induced by the revised policy at any x_1 is

$$\hat{\lambda}_1 = -2s_1(1 - B_1)|x_1 - x_1^*|, \tag{3.15}$$

which is always nonpositive, i.e., the revised policy remains to be efficient for x_1.

3.5.2. General T-period problem

We now extend the revised policy presented in the previous subsection for the two-period case to a general multi-period setting. Let the wealth process, $\{x_k\}$, under the pre-committed optimal mean-variance policy, $\mathbf{u}_\mathbf{k}^*(x_k)$ $(k = 0, 1, \ldots, T-1)$, be generated by the following recursive equation,

$$x_{k+1} = s_k x_k + \mathbf{P}_k' \mathbf{u}_\mathbf{k}^*(x_k),$$

with initial wealth x_0.

Similar to the two-period case, we propose the following revised portfolio policy, $\hat{\mathbf{u}}_\mathbf{k}^*(\hat{x}_k)$, for the T-period problem (MV):

At period 0: $\hat{\mathbf{u}}_\mathbf{0}^*(\hat{x}_0) = \mathbf{u}_\mathbf{0}^*(x_0)$;

At period k for $k = 1, \ldots T-1$, implement a revised policy according to the following recursions:

$$\hat{\mathbf{u}}_\mathbf{k}^*(\hat{x}_k) = -s_k E^{-1}(\mathbf{P_k P_k'})E(\mathbf{P_k})\hat{x}_k + \Gamma_k \left(\prod_{j=k+1}^{T-1} \frac{1}{s_j} \right) E^{-1}(\mathbf{P_k P_k'})E(\mathbf{P_k}); \tag{3.16}$$

$$\hat{x}_k = \begin{cases} \bar{x}_k, & \text{if } \bar{x}_k \le \bar{x}_k^*, \\ \bar{x}_k - 2(\bar{x}_k - \bar{x}_k^*)\left(1 - \prod_{j=k}^{T-1}(1 - B_j)\right), & \text{if } \bar{x}_k > \bar{x}_k^*, \end{cases} \tag{3.17}$$

$$\hat{x}_0 = x_0$$

$$\bar{x}_{k+1} = s_k \hat{x}_k + \mathbf{P}_k' \hat{\mathbf{u}}_\mathbf{k}^*(\hat{x}_k), \tag{3.18}$$

$$\Gamma_k = \begin{cases} \Gamma_{k-1} & \text{if } \bar{x}_k \le \bar{x}_k^*, \\ -\Gamma_{k-1} + \prod_{j=k}^{T-1} s_j \left(2\bar{x}_k - 2(\bar{x}_k - \bar{x}_k^*)\left(1 - \prod_{j=k}^{T-1}(1-B_j)\right)\right) & \text{if } \bar{x}_k > \bar{x}_k^*, \end{cases} \tag{3.19}$$

$$\Gamma_0 = \prod_{j=0}^{T-1} s_j \hat{x}_0 - \frac{\lambda}{2\prod_{j=0}^{T-1}(1-B_j)}$$

$$\bar{x}_k^* = \frac{\Gamma_{k-1}}{\prod_{j=k}^{T-1} s_j}. \tag{3.20}$$

Note that both risk attitude parameter Γ_k and the wealth threshold $\bar{x}_k^*$ ($k = 1, 2, \ldots, T-1$) are path-dependent. Thus, $\bar{x}_k^*$ is different from threshold x_k^* discussed in Section 3, as $\bar{x}_k^*$ is a path-dependent threshold for a wealth process in which cash withdrawals may occur. One major feature of this revised policy is that, when the wealth level $\bar{x}_k > \bar{x}_k^*$, we withdraw a positive free cash flow, $2(\bar{x}_k - \bar{x}_k^*)\left(1 - \prod_{j=k}^{T-1}(1-B_j)\right)$, out of the market and apply the mean-variance policy for the remaining amount in the market,

$$\hat{x}_k = \bar{x}_k - 2(\bar{x}_k - \bar{x}_k^*)\left(1 - \prod_{j=k}^{T-1}(1-B_j)\right). \tag{3.21}$$

Theorem 3.2. *The revised policy with Γ_k and $\bar{x}_k^*$ defined in (3.19) and (3.20), respectively, achieves the same mean-variance pair of the terminal wealth as does the original pre-committed optimal T-period mean-variance policy, while having possibility to take free cash flow stream out of the market during the investment process.*

Proof: The case with $T = 2$ has been already proved in Theorem 3.1. We assume that the theorem is true for $T = k$ with $k \ge 2$. We now proceed to prove that the theorem is also true for $T = k+1$. The following is clear from Proposition 3.1 and its discussion.

When $\bar{x}_1 = x_1 \le \bar{x}_1^* = x_1^*$, the truncated pre-committed optimal policy $\mathbf{u}_t^*(x_t)$ ($t = 1, 2, ..., k$) specified in (3.2) is k-period mean-variance efficient policy which solves problem $(MV_{1-(k+1)}^{\lambda_1})$ with $\Gamma_0 = \prod_{j=1}^{k} s_j x_1 - \frac{\lambda_1}{2\prod_{j=1}^{k}(1-B_j)}$ and yields a k-period mean-variance pair $(E(x_{k+1}|x_1)|_{\mathbf{u}^*}, Var(x_{k+1}|x_1)|_{\mathbf{u}^*})$ with trade-off parameter $\lambda_1 \leq 0$ and initial wealth x_1.

Apply the k-period revised policy $\hat{\mathbf{u}}_t^*(\hat{x}_t)$ $(t = 1, 2, ..., k)$ at $\hat{x}_1 = \bar{x}_1 = x_1$ which possesses the risk attitude parameter $\Gamma_1 = \Gamma_0 = \prod_{j=1}^k s_j \hat{x}_1 - \frac{\lambda_1}{2\prod_{j=1}^k (1-B_j)}$ to $(MV_{1-(k+1)}^{\lambda_1})$. From the assumption of the mathematical induction, we have the following for $\bar{x}_1 = x_1 \leq \bar{x}_1^* = x_1^*$,

$$\begin{aligned} E(x_{k+1}|x_1)|_{\mathbf{u}^*} &= E(\bar{x}_{k+1}|\hat{x}_1)|_{\hat{\mathbf{u}}^*}, \\ Var(x_{k+1}|x_1)|_{\mathbf{u}^*} &= Var(\bar{x}_{k+1}|\hat{x}_1)|_{\hat{\mathbf{u}}^*}. \end{aligned}$$

When $\bar{x}_1 = x_1 > \bar{x}_1^* = x_1^*$, the trancated pre-committed optimal policy $\mathbf{u}_t^*(x_t)$ $(t = 1, 2, ..., k)$ is no longer k-period mean-variance efficient and gives rises to

$$\begin{aligned} E(x_{k+1}|x_1)|_{\mathbf{u}^*} &= \prod_{j=1}^k s_j(1-B_j)x_1 + \Gamma_0 \left(1 - \prod_{j=1}^k (1-B_j)\right), \\ Var(x_{k+1}|x_1)|_{\mathbf{u}^*} &= \frac{\prod_{j=1}^k (1-B_j)}{1-\prod_{j=1}^k (1-B_j)} \left(E(x_{k+1}|x_1)|_{\mathbf{u}^*} - \prod_{j=1}^k s_j x_1\right)^2. \end{aligned}$$

Let us consider problem $(MV_{1-(k+1)}^{\tilde{\lambda}_1})$ with initial wealth $\hat{x}_1$, which is strictly less than x_1, trade-off parameter $\tilde{\lambda}_1 < 0$ and risk attitude parameter $\Gamma_1 = \prod_{j=1}^k s_j \hat{x}_1 - \frac{\tilde{\lambda}_1}{2\prod_{j=1}^k (1-B_j)}$. The efficient mean-variance pair of $(MV_{1-(k+1)}^{\tilde{\lambda}_1})$ can be expressed as

$$\begin{aligned} E(x_{k+1}|\hat{x}_1) &= \prod_{j=1}^k s_j \hat{x}_1 + \left(1 - \prod_{j=1}^k (1-B_j)\right)\left(\Gamma_1 - \prod_{j=1}^k s_j \hat{x}_1\right), \\ Var(x_{k+1}|\hat{x}_1) &= \frac{\prod_{j=1}^k (1-B_j)}{1-\prod_{j=1}^k (1-B_j)} \left(E(x_{k+1}|\hat{x}_1) - \prod_{j=1}^k s_j \hat{x}_1\right)^2. \end{aligned}$$

Equalizing $E(x_{k+1}|x_1)|_{\mathbf{u}^*}$ and $E(x_{k+1}|\hat{x}_1)$ and equalizing $Var(x_{k+1}|x_1)|_{\mathbf{u}^*}$ and $Var(x_{k+1}|\hat{x}_1)$ at the same time yield

$$\begin{cases} E(x_{k+1}|x_1)|_{\mathbf{u}^*} = E(x_{k+1}|\hat{x}_1), \\ \left(E(x_{k+1}|x_1)|_{\mathbf{u}^*} - \prod_{j=1}^k s_j x_1\right)^2 = \left(E(x_{k+1}|\hat{x}_1) - \prod_{j=1}^k s_j \hat{x}_1\right)^2. \end{cases}$$

Solving the above system of two equations for $\hat{x}_1$ and Γ_1 leads to two solutions, one that satisfies both (3.19) and (3.21) and the other one, $(\hat{x}_1 = x_1, \Gamma_1 = \Gamma_0)$, which is rejected.

Applying the k-period revised policy to $(MV_{1-(k+1)}^{\hat{\lambda}_1})$, we have the following for $\bar{x}_1 = x_1 > \bar{x}_1^* = x_1^*$,

$$\begin{aligned} E(x_{k+1}|x_1)|_{\mathbf{u}^*} &= E(\bar{x}_{k+1}|\hat{x}_1)|_{\hat{\mathbf{u}}^*}, \\ Var(x_{k+1}|x_1)|_{\mathbf{u}^*} &= Var(\bar{x}_{k+1}|\hat{x}_1)|_{\hat{\mathbf{u}}^*}. \end{aligned}$$

Carrying out similar steps as in Theorem 3.1, we can further obtain

$$\begin{aligned} E(x_{k+1}|x_0)|_{\mathbf{u}^*} &= E(\bar{x}_{k+1}|x_0)|_{\hat{\mathbf{u}}^*}, \\ Var(x_{k+1}|x_0)|_{\mathbf{u}^*} &= Var(\bar{x}_{k+1}|x_0)|_{\hat{\mathbf{u}}^*}. \end{aligned}$$

□

Our newly proposed T-period revised policy is better than the pre-committed mean-variance policy in the sense while the two achieve the same mean and variance of the terminal wealth, the revised policy enables investors to receive a free cash flow stream during the investment process.

In the two-period problem, when $x_1 > x_1^*$, the revised policy at $\hat{x}_1^* = x_1 - 2(x_1 - x_1^*)B_1$ satisfies $\hat{\mathbf{u}}_1^*(\hat{x}_1) = -\mathbf{u}_1^*(x_1)$. Furthermore, the trade-off induced by the revised policy at period 1, $\hat{\lambda}_1$, is always nonpositive. We now extend the results to the T-period setting.

Without loss of generality, we assume that event $\{\bar{x}_k > \bar{x}_k^*\}$ occurs at $0 < t_1 < t_2 < ... < t_S < T$ with $S \leq T - 1$.

Definition 3.2. *For the revised policy, $\hat{\lambda}_k$ is the induced trade-off at period k such that the truncated revised policy, $\hat{\mathbf{u}}_t^*(\hat{x}_t)$ $(t = k, k+1, ..., T-1)$, is also the revised policy of the following truncated portfolio selection problem, $(MV_{k-T}^{\hat{\lambda}_k})$,*

$$\begin{aligned} (MV_{k-T}^{\hat{\lambda}_k}) \qquad \min \quad & Var(x_T \mid \hat{x}_k) + \hat{\lambda}_k E(x_T \mid \hat{x}_k) \\ s.t. \quad & x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \; x_k = \hat{x}_k \textit{ is given.} \end{aligned}$$

Remark 3.2. *The revised policy of* $(MV^{\hat{\lambda}_k}_{k-T})$ *is governed by a risk attitude parameter,* $\prod_{j=k}^{T-1} s_j\hat{x}_k - \frac{\hat{\lambda}_k}{2\prod_{j=k}^{T-1}(1-B_j)}$*, and wealth at the beginning of period* k*,* $\hat{x}_k$*, while the truncated revised policy is determined by risk attitude parameter* Γ_k *and wealth at the beginning of period* k*,* $\hat{x}_k$*, which evolve from* Γ_0 *and* x_0*, respectively, according to (3.17) - (3.20). Therefore, the trade-off induced by the revised policy at time* k*,* $\hat{\lambda}_k$*, satisfies* $\prod_{j=k}^{T-1} s_j\hat{x}_k - \frac{\hat{\lambda}_k}{2\prod_{j=k}^{T-1}(1-B_j)} = \Gamma_k$.

Lemma 3.2. *When* $\bar{x}_{t_1} > \bar{x}^*_{t_1}$*, the revised policy satisfies* $\hat{\mathbf{u}}^*_{\mathbf{k}}(\hat{x}_k) = -\mathbf{u}^*_{\mathbf{k}}(x_k)$, $k = t_1, t_1+1, \ldots, t_2-1$.

Proof: Please see Appendix A. □

Proposition 3.4. *The revised policy satisfies*

$$\hat{\mathbf{u}}^*_{\mathbf{k}}(\hat{x}_k) = (-1)^{\alpha_k}\mathbf{u}^*_{\mathbf{k}}(x_k),$$

and the trade-off induced by the revised policy satisfies

$$\hat{\lambda}_k = (-1)^{\alpha_k}\lambda_k = -2\prod_{j=k}^{T-1} s_j(1-B_j)|\bar{x}_k - \bar{x}^*_k|,$$

where $\alpha_k = \sum_{t=1}^{k} I_{\{\bar{x}_t > \bar{x}^*_t\}}$.

Proof: Please see Appendix B. □

Figure 3.1(b) illustrates the relationship between the trade-off induced by the revised policy and the wealth level. Comparison between Figures 3.1(a) and 3.1(b) reveals that when the wealth level is higher than the threshold and the investor switches his/her risk attitude, by taking free cash flow out, we can keep the trade-off between the mean and the variance for the remaining periods negative. In other words, by taking certain free cash flow out of the market, investors remain a rational risk attitude for the entire time horizon.

Independent of what the actual return distribution is, the revised policy can achieve the same optimal mean-variance pair of the terminal wealth as the pre-committed optimal mean-variance policy does and obtain a free cash flow stream with a positive probability when the wealth process has a positive probability to exceed the threshold. The revised policy reduces to the original pre-committed optimal mean-variance policy only when the wealth level is guaranteed not to exceed the defined threshold.

Example 3.1. Consider Example 2 in Li and Ng (2000) [35], in which there exist i) three risky assets, A, B, and C with their expected return vector given by $E(\mathbf{e_t}) = (E(e_t^A), E(e_t^B), E(e_t^C))' = (1.162, 1.246, 1.228)'$, $t = 0, 1, 2, 3$, and their covariance given by

$$Cov(\mathbf{e_t}) = \begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix}, \quad t = 0, 1, 2, 3,$$

and ii) a riskless asset with a sure return of 1.04. The above given data give rise,

$$E(\mathbf{P_t}) = E(e_t^A - s_t, e_t^B - s_t, e_t^C - s_t)' = (0.122, 0.206, 0.188)', \quad t = 0, 1, 2, 3,$$

$$E(\mathbf{P_t}\mathbf{P_t'}) = Cov(\mathbf{e_t}) + E(\mathbf{P_t})E(\mathbf{P_t'}) = \begin{pmatrix} 0.0295 & 0.0438 & 0.0374 \\ 0.0438 & 0.1278 & 0.0491 \\ 0.0374 & 0.0491 & 0.0642 \end{pmatrix},$$
$$t = 0, 1, 2, 3,$$

and $B_t = E(\mathbf{P_t'})E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(\mathbf{P_t}) = 0.593817$, $t = 0, 1, 2, 3$.

Assume that an investor with initial wealth $x_0 = 1$ would like to minimize the mean-variance objective of $Var(x_4) - \frac{1}{2}E(x_4)$. We now assume that the random return takes a particular sample path, $\mathbf{P_t} = (0.5, 0.5, 0.5)'$, $t = 0, 1, 2, 3$, and compare the revised policy with the original pre-committed optimal mean-variance policy for such an instance.

For this particular sample path, according to (3.2), the pre-committed optimal mean-variance portfolio is given as,

$$\begin{aligned}
\mathbf{u_0^*}(x_0) &= (3.1436, 5.0998, 18.1618)', \\
\mathbf{u_1^*}(x_1) &= (-2.0171, -3.2724, -11.6537)', \\
\mathbf{u_2^*}(x_2) &= (1.2943, 2.0997, 7.4777)', \\
\mathbf{u_3^*}(x_3) &= (-0.8305, -1.3473, -4.7982)',
\end{aligned}$$

and the corresponding wealth trajectory is $\{x_0 = 1, x_1 = 14.2426, x_2 = 6.3407, x_3 = 12.0302, x_4 = 9.0234\}$. The corresponding revised portfolio for this particular sample path can derived according to (3.16) - (3.20).

Period 0:

$$\hat{x}_0 = 1,\ \Gamma_0 = 10.3544,\ \hat{\mathbf{u}}_\mathbf{0}^*(\hat{x}_0) = \mathbf{u_0^*}(x_0).$$

Period 1:

$$\begin{aligned}
\bar{x}_1 &= s\bar{x}_0 + \mathbf{P}_\mathbf{0}'\hat{\mathbf{u}}_\mathbf{0}^* = 14.2426 > \bar{x}_1^* = \frac{\Gamma_0}{s^3} = 9.2050, \\
\hat{x}_1 &= \bar{x}_1 - 2(\bar{x}_1 - \bar{x}_1^*)\left(1 - \prod_{j=1}^{4-1}(1 - B_j)\right) = 4.8425, \\
\Gamma_1 &= -\Gamma_0 + \prod_{j=1}^{4-1} s_j \left(2\bar{x}_1 - 2(\bar{x}_1 - \bar{x}_1^*)\left(1 - \prod_{j=1}^{4-1}(1 - B_j)\right)\right) = 11.1138, \\
\hat{\mathbf{u}}_\mathbf{1}^*(\hat{x}_1) &= (2.0171, 3.2724, 11.6537)' = -\mathbf{u_1^*}(x_1).
\end{aligned}$$

Period 2:

$$\begin{aligned}
\bar{x}_2 &= s\hat{x}_1 + \mathbf{P}_\mathbf{1}'\hat{\mathbf{u}}_\mathbf{1}^* = 13.5078 > \bar{x}_2^* = \frac{\Gamma_1}{s^2} = 10.2754, \\
\hat{x}_2 &= \bar{x}_2 - 2(\bar{x}_2 - \bar{x}_2^*)\left(1 - \prod_{j=2}^{4-1}(1 - B_j)\right) = 8.1095, \\
\Gamma_2 &= -\Gamma_1 + \prod_{j=2}^{4-1} s_j \left(2\bar{x}_2 - 2(\bar{x}_2 - \bar{x}_2^*)\left(1 - \prod_{j=2}^{4-1}(1 - B_j)\right)\right) = 12.2675, \\
\hat{\mathbf{u}}_\mathbf{2}^*(\hat{x}_2) &= (1.2943, 2.0997, 7.4777)' = \mathbf{u_2^*}(x_2).
\end{aligned}$$

Period 3:

$$\begin{aligned}
\bar{x}_3 &= s\hat{x}_2 + \mathbf{P}_2'\hat{\mathbf{u}}_2^\bullet = 13.8698 > \bar{x}_3^\bullet = \frac{\Gamma_2}{s^1} = 11.7956, \\
\hat{x}_3 &= \bar{x}_3 - 2(\bar{x}_3 - \bar{x}_3^\bullet)\left(1 - \prod_{j=3}^{4-1}(1 - B_j)\right) = 11.4065, \\
\Gamma_3 &= -\Gamma_2 + \prod_{j=3}^{4-1} s_j \left(2\bar{x}_3 - 2(\bar{x}_3 - \bar{x}_3^\bullet)\left(1 - \prod_{j=3}^{4-1}(1 - B_j)\right)\right) = 14.0198, \\
\hat{\mathbf{u}}_3^\bullet(\hat{x}_3) &= [0.8305, 1.3473, 4.7982]' = -\hat{\mathbf{u}}_3^\bullet(x_3).
\end{aligned}$$

For this particular sample path, the revised policy generates a wealth trajectory $\{\hat{x}_0 = 1, \bar{x}_1 = 14.2426, \bar{x}_2 = 13.5078, \bar{x}_3 = 13.8698, \bar{x}_4 = 15.3507\}$ and a free cash flow stream $\{\bar{x}_1 - \hat{x}_1 = 9.4001, \bar{x}_2 - \hat{x}_2 = 5.3983, \bar{x}_3 - \hat{x}_3 = 2.4633\}$.

3.6. Properties associated with Free Cash Flow Stream

Following the proposed revised policy, an investor is able to withdraw a positive dollar amount from the market at the beginning of period k ($k = 1, 2, \ldots, T-1$) when $\bar{x}_k > \bar{x}_k^\bullet$. We name this positive cash flow stream taken out as the *free cash flow stream* (FCFS). We are interested in finding out the existence probability of such a free cash flow stream, or equivalently, its non-existence probability given by

$$Pr(\bar{x}_1 \le \bar{x}_1^\bullet, \bar{x}_2 \le \bar{x}_2^\bullet, ..., \bar{x}_{T-1} \le \bar{x}_{T-1}^\bullet).$$

We are also interested in finding out the expected value of the free cash flow stream. Although we only need the first two moments of the returns in the discrete-time mean-variance formulation, we require more specific return distribution information for calculating the above non-existence probability and the expected value.

Let us set up a filtration for the discrete-time mean-variance portfolio selection model. We assume that we know at the beginning of period 0 the distributions of $\mathbf{P}_k$ $(k = 0, 1, \ldots, T-1)$. At the beginning of period k, the information set is given by $\mathcal{F}_k = \sigma(\mathcal{F}_0 \vee \sigma(\mathbf{P}_0) \vee \sigma(\mathbf{P}_1) \ldots \vee \sigma(\mathbf{P}_{k-1}))$. To simplify the writing, we define $\mathbf{H_j}' \triangleq E(\mathbf{P_j}')E^{-1}(\mathbf{P_j}\mathbf{P_j}')$ in this section.

Assumption 3.2. *The investor's initial trade-off parameter, λ, is assumed to be negative.*

Note that, when $\lambda = 0$, the pre-committed optimal mean-variance policy is $\mathbf{u_k^*}(x_k) = 0$ $(k = 0, 1, \ldots, T-1)$, i.e., the investor puts all his/her wealth in the riskless asset. There is thus no chance of receiving FCFS for any return distribution. Under Assumption 3.2, the initial risk attitude parameter of the revised policy, Γ_0, satisfies $\Gamma_0 > \prod_{j=0}^{T-1} s_j x_0$.

3.6.1. Nonexistence Probability of the Free Cash Flow Stream

At the beginning of any period k, the risk attitude parameter Γ_k and the threshold $\bar{x}_k^*$ can be obtained using (3.19) and (3.20), respectively. Furthermore, the proposed revised policy at the reduced wealth level $\hat{x}_k$ can be calculated using (3.16).

Lemma 3.3. *For $k = 1, 2, \ldots, T-2$, if $\boldsymbol{A} \in \mathcal{F}_k$ and $\boldsymbol{A} \subseteq \{\bar{x}_k < \bar{x}_k^*\}$, then*

$$
\begin{aligned}
Pr(\bar{x}_{k+1} < \bar{x}_{k+1}^* | \boldsymbol{A}) &= Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right), \\
Pr(\bar{x}_{k+1} = \bar{x}_{k+1}^* | \boldsymbol{A}) &= Pr\left(\mathbf{H_k}'\mathbf{P_k} = 1\right).
\end{aligned}
$$

For $k = 1, 2, \ldots, T-2$, if $\boldsymbol{B} \in \mathcal{F}_k$ and $\boldsymbol{B} \subseteq \{\bar{x}_k = \bar{x}_k^*\}$, then $Pr(\bar{x}_{k+1} = \bar{x}_{k+1}^* | \boldsymbol{B}) = 1$.

In addition, under Assumption 3.2,

$$
\begin{aligned}
Pr(\bar{x}_1 < \bar{x}_1^*) &= Pr\left(\mathbf{H_0}'\mathbf{P_0} < 1\right), \\
Pr(\bar{x}_1 = \bar{x}_1^*) &= Pr\left(\mathbf{H_0}'\mathbf{P_0} = 1\right).
\end{aligned}
$$

Proof: Our construction of the revised policy assures process $\{\bar{x}_k\}$ to be Markovian , i.e., $\bar{x}_{k+1}$ depends only on $\bar{x}_k$ and Γ_{k-1}, or equivalently, depends only on $\bar{x}_k$ and $\bar{x}_k^\bullet$. Note when $\bar{x}_k \le \bar{x}_k^\bullet$,

$$\bar{x}_{k+1}^\bullet = s_k \bar{x}_k^\bullet,$$
$$\bar{x}_{k+1} = s_k \hat{x}_k + \mathbf{P_k}' \hat{\mathbf{u}}_\mathbf{k}^\bullet(\hat{x}_k) = s_k \bar{x}_k (1 - \mathbf{H_k}'\mathbf{P_k}) + \bar{x}_{k+1}^\bullet \mathbf{H_k}'\mathbf{P_k}.$$

As $\mathbf{A}$ implies $\{\bar{x}_k < \bar{x}_k^\bullet\}$,

$$Pr(\bar{x}_{k+1} < \bar{x}_{k+1}^\bullet|\mathbf{A}) = Pr\left((s_k\bar{x}_k - \bar{x}_{k+1}^\bullet)(1 - \mathbf{H_k}'\mathbf{P_k}) < 0|\mathbf{A}\right) = Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right),$$
$$Pr(\bar{x}_{k+1} = \bar{x}_{k+1}^\bullet|\mathbf{A}) = Pr\left((s_k\bar{x}_k - \bar{x}_{k+1}^\bullet)(1 - \mathbf{H_k}'\mathbf{P_k}) = 0|\mathbf{A}\right) = Pr\left(\mathbf{H_k}'\mathbf{P_k} = 1\right).$$

Similarly, we have $Pr(\bar{x}_{k+1} = \bar{x}_{k+1}^\bullet|\mathbf{B}) = 1$.

At period 0, we have

$$\bar{x}_1^\bullet = s_0 \frac{\Gamma_0}{\prod_{j=0}^{T-1} s_j},$$
$$\bar{x}_1 = s_0 x_0 + \mathbf{P_0}' \hat{\mathbf{u}}_\mathbf{0}^\bullet(x_0) = s_0 x_0 (1 - \mathbf{H_0}'\mathbf{P_0}) + \bar{x}_1^\bullet \mathbf{H_0}'\mathbf{P_0}.$$

As $x_0 < \frac{\Gamma_0}{\prod_{j=0}^{T-1} s_j}$ is assured by Assumption 3.2, carrying out similar steps as above gives rise the last part of the lemma. □

The above lemma also applies to $k = T - 1$ with $\bar{x}_T^\bullet = \Gamma_{T-1}$, although there is no concern of free cash flow at terminal time T.

A *signed variance-optimal martingale measure* is introduced in Černý and Kallsen (2007), (2009) [10, 11] for the mean-variance hedging. Applying their particular martingale measure to our discrete-time market setting gives rise the following density,

$$Z_k^Q := \prod_{j=0}^{k} \frac{1 - \mathbf{H_j}'\mathbf{P_j}}{1 - B_j}. \tag{3.22}$$

When the wealth is always less than the threshold, i.e., $\bar{x}_k < \bar{x}_k^\bullet$, $k = 1, 2, \ldots, T$, the variance-optimal martingale measure defined in (3.22) becomes a probability measure.

Proposition 3.5. *The nonexistence probability of the free cash flow stream is given by*

$$\prod_{k=0}^{T-2} Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right) + \sum_{k=0}^{T-2}\prod_{i=0}^{k-1} Pr\left(\mathbf{H_i}'\mathbf{P_i} < 1\right) Pr\left(\mathbf{H_k}'\mathbf{P_k} = 1\right). \qquad (3.23)$$

Proof: Note that, for any $1 \le j \le T-2$, $\{\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{j-1} < \bar{x}_{j-1}^\bullet, \bar{x}_j = \bar{x}_j^\bullet, \bar{x}_{j+1} \le \bar{x}_{j+1}^\bullet, ..., \bar{x}_{T-1} \le \bar{x}_{T-1}^\bullet\}$ is a subset of $\{\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{j-1} < \bar{x}_{j-1}^\bullet, \bar{x}_j = \bar{x}_j^\bullet\}$. On the other hand, based on Lemma 3.3, set $\{\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{j-1} < \bar{x}_{j-1}^\bullet, \bar{x}_j = \bar{x}_j^\bullet\}$ implies set $\{\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{j-1} < \bar{x}_{j-1}^\bullet, \bar{x}_j = \bar{x}_j^\bullet, \bar{x}_{j+1} = \bar{x}_{j+1}^\bullet ..., \bar{x}_{T-1} = \bar{x}_{T-1}^\bullet\}$. Therefore, sets $\{\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{j-1} < \bar{x}_{j-1}^\bullet, \bar{x}_j = \bar{x}_j^\bullet, \bar{x}_{j+1} \le \bar{x}_{j+1}^\bullet ..., \bar{x}_{T-1} \le \bar{x}_{T-1}^\bullet\}$ and $\{\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{j-1} < \bar{x}_{j-1}^\bullet, \bar{x}_j = \bar{x}_j^\bullet\}$ are equal.

For $k = 0, 1, \ldots, T-2$,

$$\begin{aligned}
& Pr(\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{k-1} < \bar{x}_{k-1}^\bullet, \bar{x}_{k+1} = \bar{x}_{k+1}^\bullet) \\
= \; & Pr(\bar{x}_1 < \bar{x}_1^\bullet) Pr(\bar{x}_2 < \bar{x}_2^\bullet | \bar{x}_1 < \bar{x}_1^\bullet) \cdots Pr(\bar{x}_{k+1} = \bar{x}_{k+1}^\bullet | \bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_k < \bar{x}_k^\bullet) \\
= \; & \prod_{i=0}^{k-1} Pr\left(\mathbf{H_i}'\mathbf{P_i} < 1\right) Pr\left(\mathbf{H_k}'\mathbf{P_k} = 1\right).
\end{aligned}$$

Similarly,

$$Pr(\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{k-1} < \bar{x}_{k-1}^\bullet, \bar{x}_{T-1} < \bar{x}_{T-1}^\bullet) = \prod_{i=0}^{T-2} Pr\left(\mathbf{H_i}'\mathbf{P_i} < 1\right).$$

The nonexistence probability of the free cash flow stream is

$$\begin{aligned}
& Pr(\bar{x}_1 \le \bar{x}_1^\bullet, \bar{x}_2 \le \bar{x}_2^\bullet, ..., \bar{x}_{T-1} \le \bar{x}_{T-1}^\bullet) \\
= & Pr(\bar{x}_1 = \bar{x}_1^\bullet) + Pr(\bar{x}_1 < \bar{x}_1^\bullet, \bar{x}_2 = \bar{x}_2^\bullet) + \cdots \\
& + Pr(\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_k < \bar{x}_k^\bullet, \bar{x}_{k+1} = \bar{x}_{k+1}^\bullet) + \cdots \\
& + Pr(\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{T-2} < \bar{x}_{T-2}^\bullet, \bar{x}_{T-1} = \bar{x}_{T-1}^\bullet) \\
& + Pr(\bar{x}_1 < \bar{x}_1^\bullet, ..., \bar{x}_{T-2} < \bar{x}_{T-2}^\bullet, \bar{x}_{T-1} < \bar{x}_{T-1}^\bullet) \\
= & \sum_{k=0}^{T-2}\prod_{i=0}^{k-1} Pr\left(\mathbf{H_i}'\mathbf{P_i} < 1\right) Pr\left(\mathbf{H_k}'\mathbf{P_k} = 1\right) + \prod_{k=0}^{T-2} Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right).
\end{aligned}$$

□

Remark 3.3. *When the returns of risky assets are continuous random variables,* $Pr(\mathbf{H_k}'\mathbf{P_k} = 1) = 0$, $k = 0, 1, \ldots, T-1$. *The nonexistence probability of the free cash flow stream then reduces to* $\prod_{k=0}^{T-2} Pr(\mathbf{H_k}'\mathbf{P_k} \leq 1)$.

For any $k = 0, 1, \ldots, T-1$, since $E(\mathbf{H_k}'\mathbf{P_k}) = B_k < 1$, we have $0 < Pr(\mathbf{H_k}'\mathbf{P_k} < 1) \leq 1$. As the probability of receiving a free cash flow stream over T periods is

$$1 - \prod_{k=0}^{T-2} Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right) - \sum_{j=0}^{T-2}\prod_{i=0}^{j-1} Pr\left(\mathbf{H_i}'\mathbf{P_i} < 1\right) Pr\left(\mathbf{H_j}'\mathbf{P_j} = 1\right), \qquad (3.24)$$

we introduce the following new measure.

Definition 3.3. *For the formulation of a given market setting, the term (3.23) is defined as the degree of time consistency in efficiency.*

When a model is time consistent in efficiency, the degree of time consistency in efficiency is equal to 1. Under various return distributions, the mean-variance portfolio selection models in the discrete-time setting can be time consistent in efficiency or time inconsistent in efficiency.

Proposition 3.6. *When a market is complete for the first $T-1$ periods, its degree of time consistency in efficiency equals 1.*

Proof: Page 133 of Pliska (1997) [51] states that "*The multiperiod model is complete if and only if every underlying single period model is complete.*" We consider now a single period k, $k = 0, 1, \ldots, T-2$.

Denote by $(\Omega, \mathcal{F}_{k+1}, P)$ the probability space. Assume that there are d risky assets and one riskless asset in the market. Denote by e_k^j the random return of risky asset j at period k and s_k the sure return of riskless asset at period k. Note that the excess return vector is given by

$$\mathbf{P_k} = \left((e_k^1 - s_k), \cdots, (e_k^d - s_k)\right)'.$$

Define a signed measure Q_k,

$$\frac{dQ_k}{dP} = \frac{1 - \mathbf{H_k}'\mathbf{P_k}}{1 - B_k},$$

which is absolutely continuous to probability P. We have

$$Q_k(\Omega) = E\left(\frac{dQ_k}{dP}\right) = E\left(\frac{1 - \mathbf{H_k}'\mathbf{P_k}}{1 - B_k}\right) = 1. \tag{3.25}$$

We further have

$$\begin{aligned} E^{Q_k}(\mathbf{P_k}) &= E\left(\mathbf{P_k}\frac{dQ_k}{dP}\right) \\ &= E\left(\frac{\mathbf{P_k}(1 - \mathbf{H_k}'\mathbf{P_k})}{1 - B_k}\right) \\ &= \frac{E\left(\mathbf{P_k} - \mathbf{P_k}\mathbf{P_k}'E^{-1}(\mathbf{P_k}\mathbf{P}_\mathbf{k}')E(\mathbf{P_k})\right)}{1 - B_k} \\ &= \frac{E(\mathbf{P_k}) - E(\mathbf{P_k}\mathbf{P_k}')E^{-1}(\mathbf{P_k}\mathbf{P}_\mathbf{k}')E(\mathbf{P_k})}{1 - B_k} \\ &= 0, \end{aligned} \tag{3.26}$$

where the third equality is due to the symmetry of $E^{-1}(\mathbf{P_k}\mathbf{P_k}')$. As, from (3.26),

$$E^{Q_k}\left(\frac{e_k^j}{s_k}\right) = 1, \quad j = 1, \cdots, d, \tag{3.27}$$

Q_k is a signed martingale measure for the single period k.

As stated on Page 25 of Pliska (1997) [51] that *"The model is complete if and only if* $\mathbb{M}$ *consists of exactly one risk neutral probability measure"*, there exists a **unique** probability measure $Q^\bullet$, under which the discounted returns of assets equals to 1.

To make the market complete, we have $\Omega = \{\omega_1, \cdots, \omega_{d+1}\}$. Let $q_i^\bullet$ be the risk neutral probability for state ω_i. Then, $\{q_i^\bullet \geq 0, \ i = 1, \cdots, d+1\}$ is the unique solution of

$$\begin{cases} q_1 s_k + \cdots + q_{d+1} s_k = s_k, \\ q_1 e_k^1 + \cdots + q_{d+1} e_k^1 = s_k, \\ \cdots \\ q_1 e_k^d + \cdots + q_{d+1} e_k^d = s_k. \end{cases}$$

It is also the unique solution in $\mathbb{R}^{d+1}$. Therefore, $Q^\bullet$ is also the **unique** signed martingale measure. The uniqueness of the signed martingale measure implies

$$q_i^\bullet = \frac{dQ_k}{dP}(\omega_i)p_i, \quad i = 1, \cdots, d+1.$$

Thus, $\frac{dQ_k}{dP}(\omega_i)$ is nonnegative due to the nonnegativeness of q_i^* and the positiveness of p_i, $i = 1, \cdots, d+1$, which in turn implies

$$Pr((1 - \mathbf{H_k}'\mathbf{P_k}) \geq 0) = 1.$$

Furthermore,

$$\begin{aligned}
& \sum_{j=0}^{T-2}\prod_{k=0}^{j-1} Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right) Pr\left(\mathbf{H_j}'\mathbf{P_j} = 1\right) + \prod_{k=0}^{T-2} Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right) \\
= & \sum_{j=0}^{T-3}\prod_{k=0}^{j-1} Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right) Pr\left(\mathbf{H_j}'\mathbf{P_j} = 1\right) + \prod_{k=0}^{T-3} Pr\left(\mathbf{H_k}'\mathbf{P_k} < 1\right) \\
& \cdots \\
= & Pr\left(\mathbf{H_0}'\mathbf{P_0} \leq 1\right) \\
= & 1.
\end{aligned}$$

□

If a market is incomplete, the degree of time consistency in efficiency can often be less than 1.

Example 3.2. Assume that there are one risky asset and one riskless asset in a market with a time horizon of 2. Assume further that, at period 0, the random return of risky asset, e_0, takes three possible values 1.52, 0.62 and 11.02 with corresponding probabilities of 0.5, 0.49 and 0.01, while the riskless return is 1.02. It is obvious that this market is incomplete. It can be verified that $E(P_0)$ $= E(e_0 - s_0) = 0.154$, $E(P_0^2) = E((e_0 - s_0)^2) = 1.2034$, $E(P_0)E^{-1}(P_0^2)P_0(\omega_1)$ $= 0.0640$, $E(P_0)E^{-1}(P_0^2)P_0(\omega_2) =$-0.0512, $E(P_0)E^{-1}(P_0^2)P_0(\omega_3) = 1.2797$. Thus, the degree of of time consistency in efficiency, $Pr(E(P_0)E^{-1}(P_0^2)P_0) \leq 1) = 0.99$. To make the degree of of time consistency equal to 1 in this model, all the realizations of e_0 must be no bigger than $\frac{E(P_0^2)}{E(P_0)}$ when $E(P_0) > 0$ or be no less than $\frac{E(P_0^2)}{E(P_0)}$ when $E(P_0) < 0$.

3.6.2. Expected value of Free Cash Flow Stream

When $\bar{x}_{k+1} > \bar{x}^{\bullet}_{k+1}$ at the beginning of period $k+1$, the free cash flow received according to the proposed revised policy, $2(\bar{x}_{k+1} - \bar{x}^{\bullet}_{k+1})\left(1 - \prod_{j=k+1}^{T-1}(1-B_j)\right)$, is a random variable with respect to $\mathcal{F}_0$. We assume that the free cash flow taken out at the beginning of period $k+1$ earns riskless return for the remaining periods and are interested in finding the expected value of the entire free cash flow stream,

$$E\left(\sum_{k=0}^{T-2} 2(\bar{x}_{k+1} - \bar{x}^{\bullet}_{k+1})\left(1 - \prod_{j=k+1}^{T-1}(1-B_j)\right)\prod_{j=k+1}^{T-1} s_j 1_{\{\bar{x}_{k+1} > \bar{x}^{\bullet}_{k+1}\}} | \mathcal{F}_0\right).$$

Lemma 3.4. *Under Assumption 3.2,*

$$\{\bar{x}_{k+1} > \bar{x}^{\bullet}_{k+1}\} = \{\mathbf{H_k}'\mathbf{P_k} > 1,\ \bar{x}_k \neq \bar{x}^{\bullet}_k\},\ k = 1, 2, \ldots, T-2,$$
$$\{\bar{x}_{k+1} < \bar{x}^{\bullet}_{k+1}\} = \{\mathbf{H_k}'\mathbf{P_k} < 1,\ \bar{x}_k \neq \bar{x}^{\bullet}_k\},\ k = 1, 2, \ldots, T-2,$$
$$\{\bar{x}_1 > \bar{x}^{\bullet}_1\} = \{\mathbf{H_0}'\mathbf{P_0} > 1\},$$
$$\{\bar{x}_1 < \bar{x}^{\bullet}_1\} = \{\mathbf{H_0}'\mathbf{P_0} < 1\}.$$

Proof: Based on Lemma 3.3, $Pr(\bar{x}_{k+1} = \bar{x}^{\bullet}_{k+1} | \bar{x}_k = \bar{x}^{\bullet}_k) = 1$ implies $\{\bar{x}_{k+1} > \bar{x}^{\bullet}_{k+1}, \bar{x}_k = \bar{x}^{\bullet}_k\} = \emptyset$ and $\{\bar{x}_{k+1} > \bar{x}^{\bullet}_{k+1}, \bar{x}_k \neq \bar{x}^{\bullet}_k\} = \{\bar{x}_{k+1} > \bar{x}^{\bullet}_{k+1}\}$.

As, when $\bar{x}_k < \bar{x}^{\bullet}_k$,

$$\bar{x}^{\bullet}_{k+1} = s_k \bar{x}^{\bullet}_k,$$
$$\bar{x}_{k+1} = s_k \hat{x}_k + \mathbf{P}'_{\mathbf{k}}\hat{\mathbf{u}}^{\bullet}_{\mathbf{k}}(\hat{x}_k) = s_k \bar{x}_k(1 - \mathbf{H_k}'\mathbf{P_k}) + \bar{x}^{\bullet}_{k+1}\mathbf{H_k}'\mathbf{P_k},$$

we then have

$$\bar{x}_{k+1} > \bar{x}^{\bullet}_{k+1} \Leftrightarrow (s_k \bar{x}_k - s_k \bar{x}^{\bullet}_k)(1 - \mathbf{H_k}'\mathbf{P_k}) > 0 \Leftrightarrow \mathbf{H_k}'\mathbf{P_k} > 1.$$

As, when $\bar{x}_k > \bar{x}^*_k$,

$$\bar{x}^*_{k+1} = -s_k\bar{x}^*_k + s_k\left(2\bar{x}_k - 2(\bar{x}_k - \bar{x}^*_k)\left(1 - \prod_{j=k}^{T-1}(1 - B_j)\right)\right),$$

$$\begin{aligned}\bar{x}_{k+1} &= s_k\hat{x}_k + \mathbf{P}'_\mathbf{k}\hat{\mathbf{u}}^*_\mathbf{k}(\hat{x}_k)\\ &= s_k\left(\bar{x}_k - 2(\bar{x}_k - \bar{x}^*_k)\left(1 - \prod_{j=k}^{T-1}(1 - B_j)\right)\right)(1 - \mathbf{H}'_\mathbf{k}\mathbf{P}_\mathbf{k}) + \bar{x}^*_{k+1}\mathbf{H_k}'\mathbf{P_k},\end{aligned}$$

we then have

$$\bar{x}_{k+1} > \bar{x}^*_{k+1} \Leftrightarrow (s_k\bar{x}^*_k - s_k\bar{x}_k)(1 - \mathbf{H_k}'\mathbf{P_k}) > 0 \Leftrightarrow \mathbf{H_k}'\mathbf{P_k} > 1.$$

Thus, $\{\bar{x}_{k+1} > \bar{x}^*_{k+1}\} = \{\mathbf{H_k}'\mathbf{P_k} > 1,\ \bar{x}_k \neq \bar{x}^*_k\}$ follows. The second equation in the lemma can be derived by similar steps.

At period 0, $x_0 < \frac{\Gamma_0}{\prod_{j=0}^{T-1} s_j}$ is assured by Assumption 3.2, carrying out similar steps as above, we can also derive the last two equations in the lemma. □

Lemma 3.5. *The time-k conditional expected value of the free cash flow taken out at the beginning of period $k+1$ ($k = 1, 2, \ldots, T-2$) is given by*

$$\begin{aligned}&E\left(2(\bar{x}_{k+1} - \bar{x}^*_{k+1})\left(1 - \prod_{j=k+1}^{T-1}(1 - B_j)\right)\prod_{j=k+1}^{T-1} s_j 1_{\{\bar{x}_{k+1} > \bar{x}^*_{k+1}\}}|\mathcal{F}_k\right)\\ =&2\left(1 - \prod_{j=k+1}^{T-1}(1 - B_j)\right)\left(\Gamma_k - \prod_{j=k}^{T-1} s_j\hat{x}_k\right)E\left((\mathbf{H_k}'\mathbf{P_k} - 1)_+\right)1_{\{\bar{x}_k \neq \bar{x}^*_k\}},\end{aligned} \quad (3.28)$$

where $(x)_+ := \max\{x, 0\}$.

Proof:

$$\begin{aligned}&E\left(2(\bar{x}_{k+1} - \bar{x}^*_{k+1})\left(1 - \prod_{j=k+1}^{T-1}(1 - B_j)\right)\prod_{j=k+1}^{T-1} s_j 1_{\{\bar{x}_{k+1} > \bar{x}^*_{k+1}\}}|\mathcal{F}_k\right)\\ =&2(1 - \prod_{j=k+1}^{T-1}(1 - B_j))E((s_k\hat{x}_k + \mathbf{P}'_\mathbf{t}\hat{\mathbf{u}}^*_\mathbf{k}(\hat{x}_k) - \frac{\Gamma_k}{\prod_{j=k+1}^{T-1} s_j})\prod_{j=k+1}^{T-1} s_j 1_{\{\bar{x}_{k+1} > \bar{x}^*_{k+1}\}}|\mathcal{F}_k)\\ =&2\left(1 - \prod_{j=k+1}^{T-1}(1 - B_j)\right)E\left(\left(\prod_{j=k}^{T-1} s_j\hat{x}_k - \Gamma_k\right)(1 - \mathbf{H_k}'\mathbf{P_k})1_{\{\bar{x}_{k+1} > \bar{x}^*_{k+1}\}}|\mathcal{F}_k\right)\\ =&2\left(1 - \prod_{j=k+1}^{T-1}(1 - B_j)\right)\left(\Gamma_k - \prod_{j=k}^{T-1} s_j\hat{x}_k\right)E\left((\mathbf{H_k}'\mathbf{P_k} - 1)1_{\{\bar{x}_{k+1} > \bar{x}^*_{k+1}\}}|\mathcal{F}_k\right).\end{aligned}$$

As we have the following from Lemma 3.4,

$$\begin{aligned}
&E\left((\mathbf{H_k}'\mathbf{P_k}-1)1_{\{\bar{x}_{k+1}>\bar{x}^{\bullet}_{k+1}\}}|\mathcal{F}_k\right)\\
=&E\left((\mathbf{H_k}'\mathbf{P_k}-1)1_{\{\mathbf{H_k}'\mathbf{P_k}>1\}}|\mathcal{F}_k\right)1_{\{\bar{x}_k\neq\bar{x}^{\bullet}_k\}}\\
=&E\left((\mathbf{H_k}'\mathbf{P_k}-1)_+\right)1_{\{\bar{x}_k\neq\bar{x}^{\bullet}_k\}},
\end{aligned}$$

we conclude with (3.28). □

Lemma 3.6. *The unconditional expected value of the free cash flow taken out at the beginning of period $k+1$ $(k=1,2,\ldots,T-2)$ is given by*

$$\begin{aligned}
&E\left(2(\bar{x}_{k+1}-\bar{x}^{\bullet}_{k+1})\left(1-\prod_{j=k+1}^{T-1}(1-B_j)\right)\prod_{j=k+1}^{T-1}s_j1_{\{\bar{x}_{k+1}>\bar{x}^{\bullet}_{k+1}\}}|\mathcal{F}_0\right)\\
&=-\left(1-\prod_{j=k+1}^{T-1}(1-B_j)\right)\frac{\lambda}{\prod_{j=0}^{T-1}(1-B_j)}E\left((\mathbf{H_k}'\mathbf{P_k}-1)_+\right)\prod_{j=0}^{k-1}\gamma_j, \qquad (3.29)
\end{aligned}$$

where

$$\begin{aligned}
-\frac{\lambda}{\prod_{j=0}^{T-1}(1-B_j)} &= 2\left(\Gamma_0-\prod_{j=0}^{T-1}s_jx_0\right),\\
\gamma_j &= 1-B_j+2E\left((\mathbf{H_j}'\mathbf{P_j}-1)_+\right).
\end{aligned}$$

Proof: Please see Appendix C. □

The following proposition is immediate from Lemma 3.6.

Proposition 3.7. *The unconditional expected value of the entire free cash flow stream is*

$$-\frac{\lambda}{\prod_{j=0}^{T-1}(1-B_j)}\sum_{k=0}^{T-2}\left(1-\prod_{j=k+1}^{T-1}(1-B_j)\right)\prod_{j=0}^{k-1}\gamma_jE(\mathbf{H_k}'\mathbf{P_k}-1)_+, \qquad (3.30)$$

where $\gamma_j=(1-B_j)+2E(\mathbf{H_j}'\mathbf{P_j}-1)_+$ and $\prod_{j=0}^{-1}\gamma_j=1$.

Remark 3.4. *When all the excess returns $\mathbf{P_k}$ $(k=0,1,\ldots,T-1)$, are independent identically distributed, the unconditional expected value of the entire free cash flow stream reduces to*

$$-\frac{\lambda}{(1-B)^T}\sum_{k=0}^{T-2}\left(1-(1-B)^{T-k-1}\right)\left((1-B)+2C\right)^kC, \qquad (3.31)$$

where $B = E(\mathbf{P}')E^{-1}(\mathbf{PP}')E(\mathbf{P})$ *and* $C = E(E(\mathbf{P}')E^{-1}(\mathbf{PP}')\mathbf{P} - 1)_+$.

3.6.3. Case with normal returns

Corollary 3.1. *If all the excess returns* $\mathbf{P_k}$ $(k = 0, 1, \ldots, T-1)$, *follow independent normal distributions with mean* $E(\mathbf{P_k})$ *and variance* $E(\mathbf{P_k P_k}') - E(\mathbf{P_k})E(\mathbf{P_k}')$, *the probability of receiving a free cash flow stream over the* T *periods is*

$$1 - \prod_{k=0}^{T-2} \Phi\left(\sqrt{\frac{1-B_k}{B_k}}\right)$$

and the unconditional expected value of the entire free cash flow stream is

$$-\frac{\lambda}{\prod_{j=0}^{T-1}(1-B_j)} \sum_{k=0}^{T-2}\left(1 - \prod_{j=k+1}^{T-1}(1-B_j)\right)\prod_{j=0}^{k-1}(1 - B_j + 2\mathbf{Q}_j)\mathbf{Q_k},$$

where $\mathbf{Q_j} = \sqrt{B_j(1-B_j)}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(q_j^*)^2}{2}} - q_j^*(1-\Phi(q_j^*))\right)$ *and* $q_j^* = \sqrt{\frac{1-B_j}{B_j}}$.

Proof: As $\mathbf{H_k}'\mathbf{P_k} \sim N(B_k, B_k(1-B_k))$, we have

$$\begin{aligned} Pr(\mathbf{H_k}'\mathbf{P_k} \le 1) &= Pr\left(\frac{\mathbf{H_k}'\mathbf{P_k} - B_k}{\sqrt{B_k(1-B_k)}} \le \frac{1-B_k}{\sqrt{B_k(1-B_k)}}\right) \\ &= \Phi\left(\sqrt{\frac{1-B_k}{B_k}}\right), \\ E((\mathbf{H_j}'\mathbf{P_j} - 1)_+) &= \int_1^{+\infty}(x-1)\frac{1}{\sqrt{2\pi B_j(1-B_j)}}e^{-\frac{(x-B_j)^2}{2B_j(1-B_j)}}dx \\ &= \sqrt{B_j(1-B_j)}\int_{q_j^*}^{+\infty}(y-q_j^*)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= \sqrt{B_j(1-B_j)}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(q_j^*)^2}{2}} - q_j^*(1-\Phi(q_j^*))\right). \end{aligned}$$

The corollary then follows from Proposition 3.5, Remark 3.3 and Proposition 3.7.

□

Example 3.3. We reconsider Example 2 in Li and Ng (2000) [35] with the initial trade-off λ equal to -0.5. As in Li and Ng (2000) [35], the pre-committed optimal policy is given as follows from (3.2):

$$\mathbf{u}_t^*(x_t) = -\mathbf{K}_t x_t + \mathbf{v}_t,$$

where

$$\mathbf{K}_t = \begin{pmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{pmatrix}, \qquad t = 0,1,2,3, \qquad \mathbf{v}_0 = \begin{pmatrix} 3.5440 \\ 5.7494 \\ 20.4751 \end{pmatrix},$$

$$\mathbf{v}_1 = \begin{pmatrix} 3.6858 \\ 5.9794 \\ 21.2941 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3.8332 \\ 6.2185 \\ 22.1459 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3.9865 \\ 6.4673 \\ 23.0317 \end{pmatrix}.$$

The corresponding expected terminal wealth and the risk level are $E(x_4) = 10.1043$ and $Var(x_4) = 2.2336$, respectively.

The revised policy is given as follows according to the results in Section 5.2:

$$\hat{\mathbf{u}}_t^*(\hat{x}_t) = -\mathbf{K}_t \hat{x}_t + \frac{1}{1.04^{(4-t)}} \Gamma_t \mathbf{L}_t$$

where

$$\mathbf{K}_t = \begin{pmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{pmatrix}, \quad \mathbf{L}_t = \begin{pmatrix} 0.3850 \\ 0.6246 \\ 2.2244 \end{pmatrix}, \quad t = 0,1,2,3,$$

and Γ_t and $\hat{x}_t$ follow (3.16) - (3.20) with $\Gamma_0 = \prod_{j=0}^{T-1} s_j x_0 - \frac{\lambda}{2\prod_{j=0}^{T-1}(1-B_j)} = 10.3544$ and $\hat{x}_0 = x_0 = 1$. The corresponding expected terminal wealth and the risk level are $E(\bar{x}_4) = 10.1043$ and $Var(\bar{x}_4) = 2.2336$, respectively.

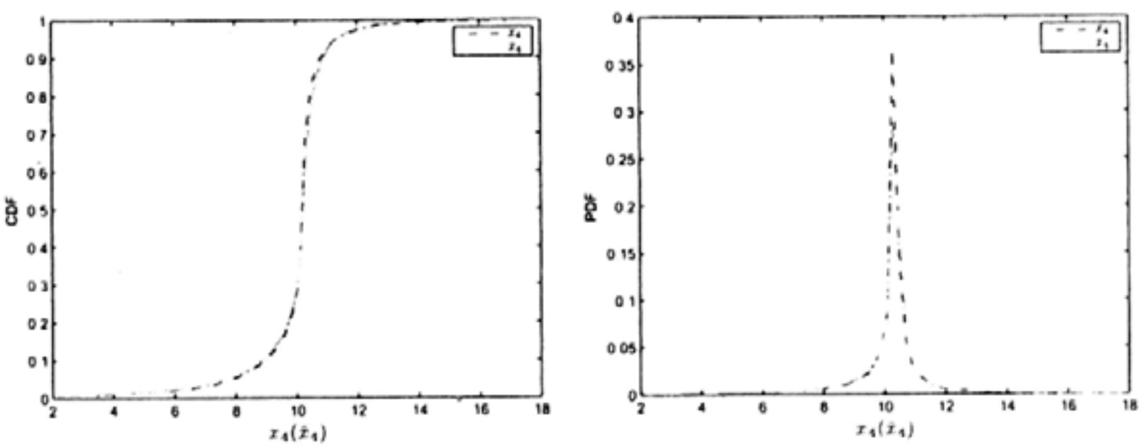
We assume now that the random returns of risky assets, $\mathbf{e}_t$ $(t = 0,1,2,3)$, are normal vectors with the given mean and covariance matrix in the example. Applying Corollary 3.1 gives rise the probability of receiving FCFS over the 4 periods equal to 0.4958 and the unconditional expected value of FCFS equal to 1.5773.

We run 5000, 20000, and 50000 sample pathes, respectively, in our simulation to verify the theoretical value of the mean-variance pair of the terminal wealth under the pre-committed optimal policy and the revised policy, i.e. x_4 and $\bar{x}_4$ and to estimate the probability of receiving FCFS and the unconditional expected value of FCFS, with the following results given in Table 1.

Table 3.1: The simulation results

Samples	$E(x_4)$	$Var(x_4)$	$E(\bar{x}_4)$	$Var(\bar{x}_4)$	Probability	$E(FCFS)$
5000	10.1485	1.8992	10.1559	1.9220	0.4970	1.5255
20000	10.1049	2.2071	10.0990	2.1792	0.4994	1.5758
50000	10.0910	2.2800	10.0959	2.2857	0.4950	1.5691

Using the simulation data, we can also estimate the distributions of the terminal wealth under the pre-committed optimal policy and the revised policy, respectively. From Figure 3.3, where the distributions of x_4 and $\bar{x}_4$ are approximated from 20000 samples, we find that the distribution of terminal wealth under the revised policy, $\bar{x}_4$, has somewhat fatter tails in both directions.



(a) The cumulative distribution functions of x_4 and $\bar{x}_4$

(b) The probability density functions of x_4 and $\bar{x}_4$

Figure 3.3: The distributions of x_4 and $\bar{x}_4$

3.7. Numerical Experiment

We consider a continuous-time market model with constant parameters and investment horizon T. There are two assets in the market: A riskless asset with its price process $S_0(t)$ following

$$\begin{cases} dS_0(t) = rS_0(t)dt, \ t \geq 0, \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r > 0$ is the interest rate, and a risky asset with its price process $S_1(t)$ satisfying

$$\begin{cases} dS_1(t) = S_1(t)\left(bdt + \sigma dW(t)\right), t \geq 0, \\ S_1(0) = s_1 > 0, \end{cases}$$

where $W(t)$ is a 1-dimensional Brown Motion. We consider both the continuous-time and discrete-time trading strategies for the mean-variance portfolio selection problem.

When adopting the continuous-time optimal mean-variance policy, the efficient frontier is given below as specified in Section 2.2,

$$Var(x(T)) = \frac{e^{-\rho T}}{1-e^{-\rho T}}\left(E(x(T)) - x_0 e^{rT}\right)^2 \quad \text{for} \quad E(x(T)) \geq x_0 e^{rT}, \tag{3.32}$$

where $\rho = \frac{(b-r)^2}{\sigma^2}$.

Now we confine ourselves to trade assets only at discrete-time points $i\Delta t$, $i = 0, \cdots, N-1$, where $\Delta t = \frac{T}{N}$. In a corresponding formulation for the discrete-time mean-variance portfolio selection problem, we can assume the stock's returns $P_i + e^{r\Delta t}$, $i = 0, \cdots, N-1$, to be i.i.d. log normal random variables,

$$\ln\left(P_i + e^{r\Delta t}\right) \sim N\left(\left(b - \frac{1}{2}\sigma^2\right)\Delta t, \sigma^2 \Delta t\right), \quad \forall i.$$

Furthermore, we have

$$\begin{aligned} E(P_i) &= e^{b\Delta t} - e^{r\Delta t}, \quad \forall i, \\ E(P_i^2) &= e^{(2b+\sigma^2)\Delta t} - 2e^{(b+r)\Delta t} + e^{2r\Delta t}, \quad \forall i, \\ B &= E(P_i)E^{-1}(P_i^2)E(P_i) = \frac{(e^{b\Delta t} - e^{r\Delta t})^2}{e^{(2b+\sigma^2)\Delta t} - 2e^{(b+r)\Delta t} + e^{2r\Delta t}}, \quad \forall i. \end{aligned}$$

We can verify the following,

$$Pr(E(P_i)E^{-1}(P_i^2)P_i \le 1) = \Phi(p^*), \quad \forall i, \tag{3.33}$$

where

$$p^* = \frac{1}{\sigma\sqrt{\Delta t}}\left(\ln(e^{(2b+\sigma^2)\Delta t} - e^{(b+r)\Delta t}) - \ln(e^{b\Delta t} - e^{r\Delta t}) - b\Delta t + \frac{1}{2}\sigma^2\Delta t\right). \tag{3.34}$$

Please refer to Appendix D for the derivation of (3.33). For this constant-parameter discrete-time model, the mean-variance efficient frontier is given by

$$Var(x_T) = \frac{(1-B)^N}{1-(1-B)^N}\left(E(x_T) - e^{rT}x_0\right)^2 \quad \text{for} \quad E(x_T) \ge e^{rT}x_0. \tag{3.35}$$

When the trade-off between the mean and the variance, λ, is selected, the corresponding expected value and variance of the terminal wealth are expressed, respectively, by

$$\begin{aligned} E(x_T) &= e^{rT}x_0 - \frac{\lambda(1-(1-B)^N)}{2(1-B)^N}, \\ Var(x_T) &= \frac{\lambda^2(1-(1-B)^N)}{4(1-B)^N}. \end{aligned} \tag{3.36}$$

When adopting the proposed revised policy, the expected value of the free cash flow stream associated with a given trade-off λ is given as

$$-\frac{\lambda}{(1-B)^N}\sum_{k=0}^{N-2}\left(1-(1-B)^{N-k-1}\right)((1-B)+2C)^k C,$$

where for all i,

$$\begin{aligned} C &= E\left((E(P_i)E^{-1}(P_i^2)P_i - 1)_+\right) \\ &= E(P_i)E^{-1}(P_i^2)e^{b\Delta t}(1-\Phi(p')) - (E(P_i)E^{-1}(P_i^2)e^{r\Delta t} + 1)(1-\Phi(p^*)), \end{aligned} \tag{3.37}$$

with p^* being given in (3.34) and

$$p' = p^* - \sigma\sqrt{\Delta t}.$$

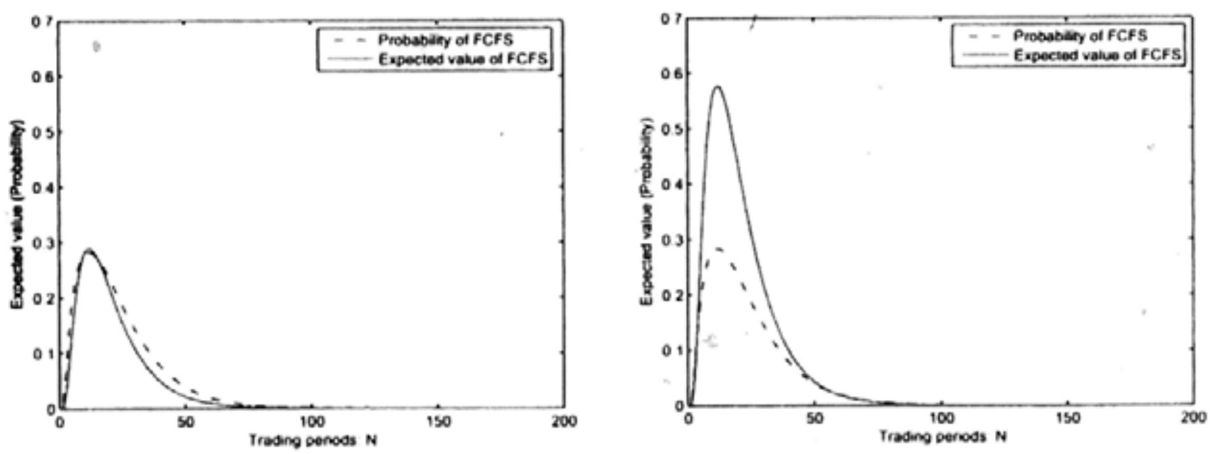
Please refer to Appendix D for the derivation of (3.37).

If we consider the total expected wealth under the revised mean-variance policy as the summation of the expected wealth from the original mean-variance pair and the expected value of the free cash flow stream, the efficient mean-variance frontier under the revised mean-variance policy is given by

$$Var(x_T) = \frac{(1-B)^N(1-(1-B)^N)}{((1-(1-B)^N)+D)^2}\left(E(x_T) - e^{rT}x_0\right)^2 \quad \text{for} \quad E(x_T) \geq e^{rT}x_0, \tag{3.38}$$

where $D = 2\sum_{k=0}^{N-2}(1-(1-B)^{N-k-1})((1-B)+2C)^k C$.

We now consider a specific continuous-time market with parameters $r = 0.6$, $b = 3.2$, $\sigma = 1.5$ and $T = 1$. We assume $x_0 = 1$ and let N, the number of discrete-time periods, equal to $1, 2, 3, 8, 9, 30, 100$ and ∞. Note that the two situations with N equal to 1 and ∞ lead, respectively, to single-period (static) and continuous-time mean-variance formulations.



(a) $\lambda = -1$ (b) $\lambda = -2$

Figure 3.4: Probability of the occurrence and the expected value of the free cash flow stream

Figures 3.4 depicts the occurrence probability and the expected value of the free cash flow streams associated with λ equal to -1 and -2, respectively. It is

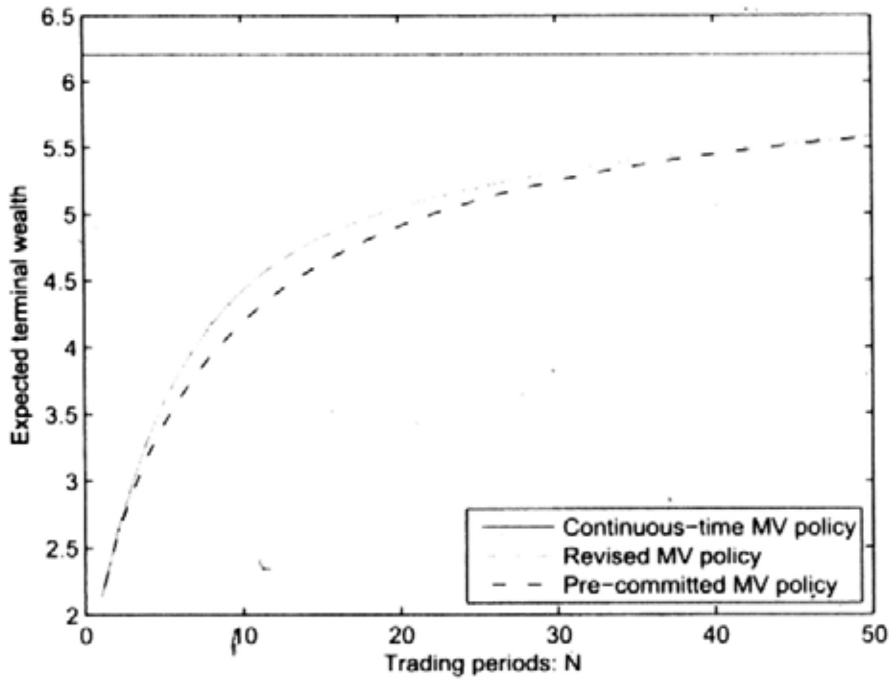


Figure 3.5: Dominance relationship for $Var(x_T) = 1$

clear that the shapes of the occurrence probability and the expected value have a similar pattern and both converge to zero when N goes to infinite.

Fixing $Var(x_T)$ at 1, Figure 3.5 illustrates the dominance relationship between the revised mean-variance policy and the pre-committed mean-variance policy, while the horizontal line on the top represents the expected terminal wealth level under the continuous-time trading.

The efficient frontiers, associated with different N, in the mean-standard deviation space are given in Figure 3.6. The solid lines from bottom to top are corresponding to N =1, 2, 3, 8, 9, 30, 100, ∞ under the pre-committed mean-variance policy, while the dash-dot lines from bottom to top are corresponding to N = 2, 3, 8, 9, 30, 100 under the proposed revised mean-variance policy. It is clear that, under either the pre-committed mean-variance policy or the revised mean-variance policy, the N_1-period efficient frontier dominates the N_2-period efficient frontier when $N_1 > N_2$. When N is increasing, the efficient frontier

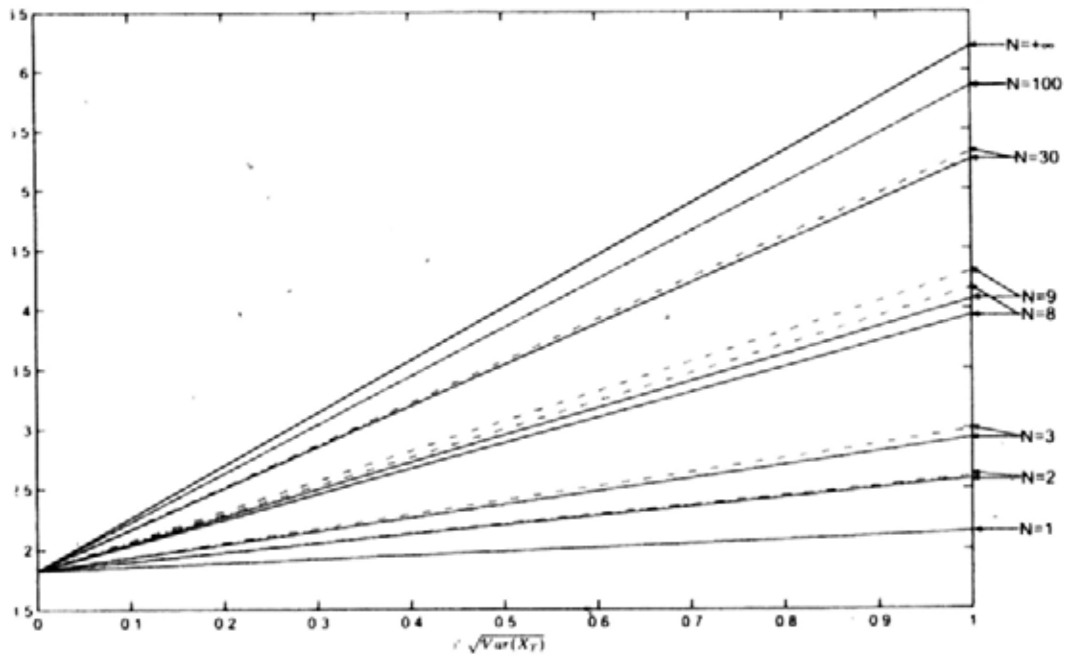


Figure 3.6: Mean-Standard Deviation efficient frontiers

under the revised mean-variance policy converges back to the one under the pre-committed mean-variance policy, and both converge to the continuous-time efficient frontier. It is interesting to observe that the 8-period efficient frontier under the revised mean-variance policy dominates the 9-period efficient frontier under the pre-committed mean-variance policy. In this sense, the revised mean-variance policy could offer a better way to improve the mean-variance efficiency without suffering a burden of increasing the trading frequency.

As evidenced from (3.36), the contribution to the expected final wealth from investing in risky assets under the pre-committed mean-variance policy is given by $-\frac{\lambda\left(1-(1-B)^N\right)}{2(1-B)^N}$. For given N, we define the ratio of the expected value of the free cash flow stream over this amount as the improvement ratio of the revised mean-variance policy over the pre-committed mean-variance policy. Furthermore, the *best improvement ratio* is found as follows,

$$\alpha^* = \max_N \frac{2\sum_{k=0}^{N-2}\left(1-(1-B)^{N-k-1}\right)((1-B)+2C)^k C}{1-(1-B)^N}. \tag{3.39}$$

Note that the effect of the investor's risk attitude, λ, is eliminated in this ratio. The computational results of the best improvement ratio for various market settings are given in Table 2.

Table 3.2: The best improvement ratio under various market settings

r	b	σ	$\rho = \frac{(b-r)^2}{\sigma^2}$	α^*	$N(\alpha^*)$
1.1	1.2	1.5	0.0044	0.00067%	2
1.1	1.2	0.8	0.0156	0.00031%	2
0.6	1.2	1.5	0.1600	0.27%	3
0.6	1.2	0.8	0.5625	1.33%	3
1.1	3.2	1.5	1.9600	7.09%	6
0.6	3.2	1.5	3.0044	11.05%	7
1.1	3.2	0.8	6.8906	21.48%	6
0.6	3.2	0.8	10.5625	26.11%	7

It is clear from the table that the best improvement ratio varies significantly under different market settings. Note that parameter ρ, to certain degree, represents the position of the risky asset. It seems from the table that a positive correlation exists between parameter ρ and the best improvement ratio, which makes sense as the free cash flow stream is generated from the risky asset. When parameter ρ is very small, the best improvement ratio is also very small, and in such a situation, the variance of the return becomes the dominating force.

3.8. Conclusion

The mean-variance framework in dynamic portfolio selection is not time consistent, due to the inherent nonseparable nature of the involved variance term. The trade-off between the two conflicting objectives, the expected value and the variance of the terminal wealth, is time-varying and state-dependent. In some situations where the wealth level exceeds some threshold, the trade-off may change its sign, which implies that the investor changes his/her risk attitude towards the objectives, leading to time inconsistency in efficiency and irrational trading behaviors for the remaining investment periods.

By devising a revised mean-variance policy, we retain the efficiency of the portfolio policy for all time periods. While achieving the same mean-variance pair of the original pre-committed optimal mean-variance policy, the revised mean-variance policy enables investors to receive a free cash flow stream. Note that the probability of receiving free cash flow stream and its expected value are both path-independent.

We emphasize that the distribution-free discrete-time mean-variance model in Li and Ng (2000) [35], in general, corresponds to an incomplete market, while the Brown motion driven continuous-time mean-variance model in Zhou and Li (2000) [75] represents a complete market. The existence of the free cash flow stream seems to be related to the market completeness. Recall from Proposition 3.6 that the existence of the free cash flow stream disappears in a complete discrete-time market. We tend to conclude that the market incompleteness is the source of the existence of the free cash flow stream. Finally, the fact that the optimal dynamic mean-variance policy is dominated, revealed in this chapter, could have a profound impact on the theory of portfolio selection and asset pricing.

3.9. Appendix

Appendix A: Proof of Lemma 3.2

Proof: Let

$$Z_{t_1} = 2\bar{x}_{t_1} - 2(\bar{x}_{t_1} - \bar{x}^*_{t_1})\left(1 - \prod_{j=t_1}^{T-1}(1 - B_j)\right).$$

We use mathematical induction to prove

$$\hat{x}_k = \prod_{j=t_1}^{k-1} s_j Z_{t_1} - x_k,\ k = t_1, t_1+1, \ldots, t_2-1$$

$$\hat{\mathbf{u}}_{\mathbf{k}}^{*}(\hat{x}_k) = -\mathbf{u}_{\mathbf{k}}^{*}(x_k),\ k = t_1, t_1+1, \ldots, t_2-1.$$

Since $\bar{x}_{t_1} > \bar{x}_{t_1}^{*}$, we have

$$\begin{aligned}
\hat{x}_{t_1} &= Z_{t_1} - \bar{x}_{t_1} = Z_{t_1} - x_{t_1}, \\
\Gamma_{t_1} &= -\Gamma_0 + \prod_{j=t_1}^{T-1} s_j Z_{t_1}, \\
\hat{\mathbf{u}}_{\mathbf{t_1}}^{*}(\hat{x}_{t_1}) &= -s_{t_1} E^{-1}(\mathbf{P}_{\mathbf{t_1}}\mathbf{P}_{\mathbf{t_1}}')E(\mathbf{P}_{\mathbf{t_1}})\hat{x}_{t_1} + \Gamma_{t_1}\left(\prod_{j=t_1+1}^{T-1}\frac{1}{s_j}\right)E^{-1}(\mathbf{P}_{\mathbf{t_1}}\mathbf{P}_{\mathbf{t_1}}')E(\mathbf{P}_{\mathbf{t_1}}) \\
&= s_{t_1} E^{-1}(\mathbf{P}_{\mathbf{t_1}}\mathbf{P}_{\mathbf{t_1}}')E(\mathbf{P}_{\mathbf{t_1}})x_{t_1} - \Gamma_0\left(\prod_{j=t_1+1}^{T-1}\frac{1}{s_j}\right)E^{-1}(\mathbf{P}_{\mathbf{t_1}}\mathbf{P}_{\mathbf{t_1}}')E(\mathbf{P}_{\mathbf{t_1}}) \\
&= -\mathbf{u}_{\mathbf{t_1}}^{*}(x_{t_1}).
\end{aligned}$$

Thus, the statement holds true for $k = t_1$. We assume now that the statement holds true for $k = t$: $\hat{x}_t = \prod_{j=t_1}^{t-1} s_j Z_{t_1} - x_t$ and $\hat{\mathbf{u}}_{\mathbf{t}}^{*}(\hat{x}_t) = -\mathbf{u}_{\mathbf{t}}^{*}(x_t)$. When $k = t+1$, we have

$$\begin{aligned}
\hat{x}_{t+1} &= s_t\hat{x}_t + \mathbf{P}_t'\hat{\mathbf{u}}_{\mathbf{t}}^{*}(\hat{x}_t) \\
&= s_t\left(\prod_{j=t_1}^{t-1} s_j Z_{t_1} - x_t\right) - \mathbf{P}_t'\mathbf{u}_{\mathbf{t}}^{*}(x_t) \\
&= \prod_{j=t_1}^{t} s_j Z_{t_1} - x_{t+1},
\end{aligned}$$

and

$$
\begin{aligned}
&\hat{\mathbf{u}}^{*}_{\mathbf{t+1}}(\hat{x}_{t+1})\\
&=-s_{t+1}E^{-1}(\mathbf{P_{t+1}P'_{t+1}})E(\mathbf{P_{t+1}})\hat{x}_{t+1}+\Gamma_{t+1}\left(\prod_{j=t+2}^{T-1}\frac{1}{s_j}\right)E^{-1}(\mathbf{P_{t+1}P'_{t+1}})E(\mathbf{P_{t+1}})\\
&=-s_{t+1}E^{-1}(\mathbf{P_{t+1}P'_{t+1}})E(\mathbf{P_{t+1}})\hat{x}_{t+1}+\Gamma_{t_1}\left(\prod_{j=t+2}^{T-1}\frac{1}{s_j}\right)E^{-1}(\mathbf{P_{t+1}P'_{t+1}})E(\mathbf{P_{t+1}})\\
&=s_{t+1}E^{-1}(\mathbf{P_{t+1}P'_{t+1}})E(\mathbf{P_{t+1}})x_{t+1}-\Gamma_{0}\left(\prod_{j=t+2}^{T-1}\frac{1}{s_j}\right)E^{-1}(\mathbf{P_{t+1}P'_{t+1}})E(\mathbf{P_{t+1}})\\
&=-\mathbf{u}^{*}_{\mathbf{t+1}}(x_{t+1}).
\end{aligned}
$$

We complete the proof. □

Appendix B: Proof of Proposition 3.4

Proof: For $t_s \in \{t_1, t_2, \ldots, t_S\}$, let

$$
Z_{t_s} = 2\bar{x}_{t_s} - 2(\bar{x}_{t_s} - \bar{x}^{*}_{t_s})\left(1 - \prod_{j=t_s}^{T-1}(1-B_j)\right).
$$

To prove the theorem, we will use mathematical induction to prove the following statement for $t_s \in \{t_1, t_2, \ldots, t_S\}$ and $t = t_s, t_s+1, \ldots, t_{s+1}-1$,

$$
\begin{aligned}
\hat{x}_t &= (-1)^s x_t + \sum_{j=1}^{s}(-1)^{(s-j)}\left(\prod_{k=t_j}^{t-1} s_k Z_{t_j}\right),\\
\Gamma_t &= (-1)^s \Gamma_0 + \sum_{j=1}^{s}(-1)^{(s-j)}\left(\prod_{k=t_j}^{T-1} s_k Z_{t_j}\right),\\
\hat{\mathbf{u}}^{*}_{\mathbf{t}}(\hat{x}_t) &= (-1)^s \mathbf{u}^{*}_{\mathbf{t}}(x_t).
\end{aligned}
$$

Note that the above statement holds true for $t_s = t_1$ as proved in Lemma 3.2. Assume that the statement is true for $t_s = t_p$. We consider now the case with t_s

$= t_{p+1}$. As

$$\begin{aligned}
\hat{x}_{t_{p+1}} &= Z_{t_{p+1}} - \bar{x}_{t_{p+1}} \\
&= Z_{t_{p+1}} - s_{t_{(p+1)}-1}\hat{x}_{t_{(p+1)}-1} + \mathbf{P}'_{t_{(p+1)}-1}\hat{\mathbf{u}}^{*}_{\mathbf{t}_{(p+1)}-1}(\hat{x}_{t_{(p+1)}-1}) \\
&= Z_{t_{p+1}} + \sum_{j=1}^{p}(-1)^{(p-j+1)}\left(\prod_{k=t_j}^{t_{p+1}-1} s_k Z_{t_j}\right) + (-1)^{(p+1)}x_{t_{p+1}}, \\
&= (-1)^{(p+1)}x_{t_{p+1}} + \sum_{j=1}^{p+1}(-1)^{(p-j+1)}\left(\prod_{k=t_j}^{t_{p+1}-1} s_k Z_{t_j}\right), \\
\Gamma_{t_{p+1}} &= (-1)^{p+1}\Gamma_0 + \sum_{j=1}^{p+1}(-1)^{(p-j+1)}\left(\prod_{k=t_j}^{T-1} s_k Z_{t_j}\right),
\end{aligned}$$

we have

$$\begin{aligned}
\hat{\mathbf{u}}^{*}_{\mathbf{t_{p+1}}}(\hat{x}_{t_{p+1}}) =& - s_{t_{p+1}}E^{-1}(\mathbf{P_{t_{p+1}}}\mathbf{P}'_{\mathbf{t_{p+1}}})E(\mathbf{P_{t_{p+1}}})\hat{x}_{t_{p+1}} \\
&+ \Gamma_{t_{p+1}}\left(\prod_{j=t_{p+1}+1}^{T-1}\frac{1}{s_j}\right)E^{-1}(\mathbf{P_{t_{p+1}}}\mathbf{P}'_{\mathbf{t_{p+1}}})E(\mathbf{P_{t_{p+1}}}) \\
=&(-1)^{p+1}\mathbf{u}^{*}_{\mathbf{t_{p+1}}}(x_{t_{p+1}}).
\end{aligned}$$

It is clear that the statement is true for $t = t_{p+1}$. Assume that the statement is true for $t = t_{p+1} + k$. We consider the case with $t = t_{p+1} + k + 1$ ($k = 0, 1, \ldots, t_{p+2} - t_{p+1} - 2$),

$$\begin{aligned}
\hat{x}_{t_{(p+1)}+k+1} &= s_{t_{(p+1)}+k}\hat{x}_{t_{(p+1)}+k} + \mathbf{P}'_{\mathbf{t}_{(p+1)}+\mathbf{k}}\hat{\mathbf{u}}^{*}_{\mathbf{t}_{(p+1)}+\mathbf{k}}(\hat{x}_{t_{(p+1)}+k}) \\
&= (-1)^{(p+1)}x_{t_{(p+1)}+k+1} + \sum_{j=1}^{p+1}(-1)^{(p-j+1)}\left(\prod_{k=t_j}^{t_{(p+1)}+k} s_k Z_{t_j}\right), \\
\Gamma_{t_{(p+1)}+k+1} &= (-1)^{(p+1)}\Gamma_0 + \sum_{j=1}^{p+1}(-1)^{(p-j+1)}\left(\prod_{k=t_j}^{T-1} s_k Z_{t_j}\right).
\end{aligned}$$

We thus have

$$
\begin{aligned}
&\hat{\mathbf{u}}^{*}_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\left(\hat{x}_{t_{(p+1)}+k+1}\right)\\
=&-s_{t_{(p+1)}+k+1}E^{-1}\left(\mathbf{P}_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\mathbf{P}'_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\right)E\left(\mathbf{P}_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\right)\hat{x}_{t_{(p+1)}+k+1}\\
&+\Gamma_{t_{(p+1)}+k+1}\left(\prod_{j=t_{(p+1)}+k+2}^{T-1}\frac{1}{s_j}\right)E^{-1}\left(\mathbf{P}_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\mathbf{P}'_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\right)E\left(\mathbf{P}_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\right)\\
=&(-1)^{p+1}\mathbf{u}^{*}_{\mathbf{t}_{(\mathbf{p}+1)}+\mathbf{k}+1}\left(x_{t_{(p+1)}+k+1}\right).
\end{aligned}
$$

Now we consider the induced trade-off under the revised policy. Note that

$$
\prod_{j=t}^{T-1}s_j\hat{x}_t-\frac{\hat{\lambda}_t}{2\prod_{j=t}^{T-1}(1-B_j)}=\Gamma_t.
$$

Substituting the expressions of Γ_t and $\hat{x}_t$ into the above equation yields

$$
\hat{\lambda}_t=(-1)^s 2\left(\prod_{j=t}^{T-1}s_jx_t-\Gamma_0\right)\prod_{j=t}^{T-1}(1-B_j)=(-1)^s\lambda_t.
$$

Furthermore, when $\bar{x}_t\le\bar{x}_t^*$,

$$
\hat{\lambda}_t=2\prod_{j=t}^{T-1}s_j(1-B_j)\left(\bar{x}_t-\frac{\Gamma_{t-1}}{\prod_{j=t}^{T-1}s_j}\right)=2\prod_{j=t}^{T-1}s_j(1-B_j)(\bar{x}_t-\bar{x}_t^*)\le 0;
$$

when $\bar{x}_t>\bar{x}_t^*$,

$$
\hat{\lambda}_t=2\prod_{j=t}^{T-1}s_j(1-B_j)\left(\frac{\Gamma_{t-1}}{\prod_{j=t}^{T-1}s_j}-\bar{x}_t\right)=2\prod_{j=t}^{T-1}s_j(1-B_j)(\bar{x}_t^*-\bar{x}_t)<0.
$$

□

Appendix C: Proof of Lemma 3.6

Proof: Based on Lemma 3.5, we only need to prove

$$
\begin{aligned}
&E\left(\left(\Gamma_k-\prod_{j=k}^{T-1}s_j\hat{x}_k\right)1_{\{\bar{x}_k\neq\bar{x}_k^*\}}|\mathcal{F}_0\right)\\
=&-\frac{\lambda}{2\prod_{j=0}^{T-1}(1-B_j)}\prod_{j=0}^{k-1}\gamma_j=\left(\Gamma_0-\prod_{j=0}^{T-1}s_jx_0\right)\prod_{j=0}^{k-1}\gamma_j.
\end{aligned}
$$

For $k = 2, \dots, T-2$, we know

$$\begin{aligned}
&E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \hat{x}_k\right) 1_{\{\bar{x}_k \neq \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \bar{x}_k\right) 1_{\{\bar{x}_k < \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) + E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \hat{x}_k\right) 1_{\{\bar{x}_k > \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right),
\end{aligned}$$

while the first and second parts of the above expression can be expressed, respectively, as

$$\begin{aligned}
&E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \bar{x}_k\right) 1_{\{\bar{x}_k < \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&E\left(\left(\Gamma_{k-1} - \prod_{j=k}^{T-1} s_j (s_{k-1}\hat{x}_{k-1} + \mathbf{P}'_{\mathbf{k-1}} \mathbf{\hat{u}}^*_{\mathbf{k-1}}(\hat{x}_{k-1}))\right) 1_{\{\bar{x}_k < \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&E\left(\left(\Gamma_{k-1} - \prod_{j=k-1}^{T-1} s_j \hat{x}_{k-1}\right) (1 - \mathbf{H_{k-1}}'\mathbf{P_{k-1}}) 1_{\{\bar{x}_k < \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&\left(\Gamma_{k-1} - \prod_{j=k-1}^{T-1} s_j \hat{x}_{k-1}\right) E\left((1 - \mathbf{H_{k-1}}'\mathbf{P_{k-1}}) 1_{\{\bar{x}_k < \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&\left(\Gamma_{k-1} - \prod_{j=k-1}^{T-1} s_j \hat{x}_{k-1}\right) E\left((1 - \mathbf{H_{k-1}}'\mathbf{P_{k-1}})_+\right) 1_{\{\bar{x}_{k-1} \neq \bar{x}_{k-1}^*\}},
\end{aligned}$$

and

$$\begin{aligned}
&E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \hat{x}_k\right) 1_{\{\bar{x}_k > \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&E\Bigg(\Bigg(-\Gamma_{k-1} + \prod_{j=k}^{T-1} s_j \left(2\bar{x}_k - 2(\bar{x}_k - \bar{x}_k^*)\left(1 - \prod_{j=k}^{T-1}(1 - B_j)\right)\right) \\
&- \prod_{j=k}^{T-1} s_j \left(\bar{x}_k - 2(\bar{x}_k - \bar{x}_k^*)\left(1 - \prod_{j=k}^{T-1}(1 - B_j)\right)\right)\Bigg) 1_{\{\bar{x}_k > \bar{x}_k^*\}} | \mathcal{F}_{k-1}\Bigg) \\
=&E\left(\left(-\Gamma_{k-1} + \prod_{j=k}^{T-1} s_j \left(s_{k-1}\hat{x}_{k-1} + \mathbf{P}'_{\mathbf{k-1}} \mathbf{\hat{u}}^*_{\mathbf{k-1}}(\hat{x}_{k-1})\right)\right) 1_{\{\bar{x}_k > \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&E\left(\left(\Gamma_{k-1} - \prod_{j=k-1}^{T-1} s_j \hat{x}_{k-1}\right) (\mathbf{H_{k-1}}'\mathbf{P_{k-1}} - 1) 1_{\{\bar{x}_k > \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \\
=&\left(\Gamma_{k-1} - \prod_{j=k-1}^{T-1} s_j \hat{x}_{k-1}\right) E\left((\mathbf{H_{k-1}}'\mathbf{P_{k-1}} - 1)_+\right) 1_{\{\bar{x}_{k-1} \neq \bar{x}_{k-1}^*\}}.
\end{aligned}$$

Therefore,

$$E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \hat{x}_k\right) 1_{\{\bar{x}_k \neq \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) = \left(\Gamma_{k-1} - \prod_{j=k-1}^{T-1} s_j \hat{x}_{k-1}\right) \gamma_{k-1} 1_{\{\bar{x}_{k-1} \neq \bar{x}_{k-1}^*\}}.$$

Furthermore,

$$\begin{aligned} & E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \hat{x}_k\right) 1_{\{\bar{x}_k \neq \bar{x}_k^*\}} | \mathcal{F}_0\right) \\ = \; & E\left(E\left(\cdots E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \hat{x}_k\right) 1_{\{\bar{x}_k \neq \bar{x}_k^*\}} | \mathcal{F}_{k-1}\right) \cdots | \mathcal{F}_1\right) | \mathcal{F}_0\right) \\ = \; & E\left(\left(\Gamma_1 - \prod_{j=1}^{T-1} s_j x_1\right) 1_{\{\bar{x}_1 \neq \bar{x}_1^*\}} | \mathcal{F}_0\right) \prod_{j=1}^{k-1} \gamma_j. \end{aligned}$$

At period 0, by following the discussion above and noticing Lemma 3.4, we have

$$E\left(\left(\Gamma_k - \prod_{j=k}^{T-1} s_j \hat{x}_k\right) 1_{\{\bar{x}_k \neq \bar{x}_k^*\}} | \mathcal{F}_0\right) = \left(\Gamma_0 - \prod_{j=0}^{T-1} s_j x_0\right) \prod_{j=0}^{k-1} \gamma_j.$$

□

Appendix D: Proof of Eqs. (3.33) and (3.37)

Proof: As P_i can be represented by

$$P_i = e^{b\Delta t - \frac{1}{2}\sigma^2 \Delta t + \sigma\sqrt{\Delta t} x} - e^{r\Delta t},$$

where x is the standard normal random variable, we have

$$\begin{aligned} C =& E\left((E(P_i)E^{-1}(P_i)^2 P_i - 1)_+\right) \\ =& \int_{p^*}^{+\infty} \left(E(P_i)E^{-1}(P_i)^2 \left(e^{b\Delta t - \frac{1}{2}\sigma^2 \Delta t + \sigma\sqrt{\Delta t} x} - e^{r\Delta t}\right) - 1\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ =& E(P_i)E^{-1}(P_i)^2 e^{b\Delta t} \int_{p^*}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sigma\sqrt{\Delta t})^2}{2}} dx \\ & - \left(E(P_i)E^{-1}(P_i)^2 e^{r\Delta t} + 1\right) \int_{p^*}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ =& E(P_i)E^{-1}(P_i)^2 e^{b\Delta t}(1 - \Phi(p')) - \left(E(P_i)E^{-1}(P_i)^2 e^{r\Delta t} + 1\right)(1 - \Phi(p^*)) \end{aligned}$$

and

$$
\begin{aligned}
& Pr\left(E(P_1)E^{-1}(P_1)^2 P_1 \le 1\right) \\
=& Pr\left(P_1 \le E(P_1)^2 E^{-1}(P_1)\right) \\
=& Pr\left(b\Delta t - \frac{1}{2}\sigma^2\Delta t + \sigma\sqrt{\Delta t}x \le \ln\left(\frac{e^{(2b+\sigma^2)\Delta t} - e^{(b+r)\Delta t}}{e^{b\Delta t} - e^{r\Delta t}}\right)\right) \\
=& \Phi(p^\bullet),
\end{aligned}
$$

where $p^\bullet$ is the solution of equation

$$
E(P_1)E^{-1}(P_1)^2\left(e^{b\Delta t - \frac{1}{2}\sigma^2\Delta t + \sigma\sqrt{\Delta t}x} - e^{r\Delta t}\right) - 1 = 0.
$$

□

CHAPTER 4

Better than Dynamic Mean-Variance Policy in Market with ALL Risky Assets

4.1. Introduction

In this chapter, we first demonstrate that the conventional multi-period mean-variance portfolio selection in a market with all risky assets also does not satisfy time consistency in efficiency. By relaxing the assumption of self financing at the beginning of period s, we extend the concept of pseudo efficiency (type 1 or type 2) in Chapter 2 to a dynamic setting. We then propose two different revised policies for dealing with two types of pseudo efficiency. While being able to achieve the same mean-variance pair attained by a pre-committed optimal mean-variance policy, the first revised policy is also able to generate during the investment process a positive cash flow stream or free cash flow stream (FCFS) with a strictly positive probability under certain probability distribution assumptions. While being able to achieve the same total mean as the one by a pre-committed optimal mean-variance policy, the second revised policy ensures a total variance no bigger than the one by the pre-committed optimal mean-variance policy by including the free cash flow stream in the total wealth. Similar to the discrete-time case, we also derive two revised versions of the continuous-time optimal mean-variance policy in a market with all risky assets.

The organization of this chapter is as follows. In Section 4.2, we summarize the current results of the optimal policy for multi-period mean-variance portfolio selection in a market with all risky assets and discuss some prominent properties of the parameters. In Section 4.3, we examine the trade-offs induced by the pre-committed optimal mean-variance policy, thus concluding that the discrete-time mean-variance formulation in a market with all risky assets is not time consistent in efficiency. In Section 4.4, we extend the concept of pseudo efficiency (type 1 or type 2) to a dynamic setting and develop two revised policies which dominate the pre-committed optimal mean-variance policy. In Section 4.5, we investigate the continuous-time optimal mean-variance policy in a market with all risky assets and construct two similar revised policies in the continuous-time setting. Finally, we conclude this chapter in Section 4.6.

4.2. Discrete-time Mean-Variance Portfolio Selection

We consider a capital market consisting of $n+1$ risky securities within a finite time horizon T. Let $\mathbf{e_t} = (e_t^0, e_t^1, ..., e_t^n)'$ be the vector of random total return rates of the $n+1$ risky securities at time period t. We assume in this chapter that vectors $\mathbf{e_t}$, $t = 0, 1, \ldots, T-1$, are statistically independent, and the only information known about the random return vector, $\mathbf{e_t}$, is its first two moments, the mean and the covariance, which are assumed to be finite for all t. We also assume that the covariance matrix is positive definite for all t.

An investor joins the market at time 0 with an initial wealth x_0. He can allocate his wealth among $n+1$ risky securities at time 0 and reallocates his wealth at the beginning of each of the following $(T-1)$ consecutive time periods. Let x_t be the wealth of the investor at the beginning of period t, and u_t^i, $i = 1, 2, \ldots, n$, be the amount invested in the ith risky asset at the beginning of period t.

Then the amount invested in the 0th risky asset at the beginning of the period t is equal to $x_t - \sum_{i=1}^{n} u_t^i$.

The investor is seeking a best investment policy, $\mathbf{u_t^*} = ((u_t^1)^*, (u_t^2)^*, ..., (u_t^n)^*)'$ for $t = 0, 1, 2, \ldots, T-1$, to attain the optimality of the following multi-period mean-variance portfolio selection model:

$$\begin{array}{lll} (MV) & \min & Var(x_T|x_0) + \lambda E(x_T|x_0) \\ & \text{s.t.} & x_{t+1} = e_t^0 x_t + \mathbf{P_t'} \mathbf{u_t}, \quad t = 0, 1, ..., T-1, \\ & & x_0 \text{ is given,} \end{array} \tag{4.1}$$

where

$$\mathbf{P_t} = (P_t^1, P_t^2, ..., P_t^n)' = ((e_t^1 - e_t^0), (e_t^2 - e_t^0), ..., (e_t^n - e_t^0))'$$

satisfies

$$E(\mathbf{P_t}\mathbf{P_t'}) \succ 0, \quad \forall t = 0, 1, ..., T-1,$$

$$E((e_t^0)^2) - E(e_t^0\mathbf{P_t'})E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(e_t^0\mathbf{P_t}) > 0, \quad \forall t = 0, 1, ..., T-1,$$

due to the positive definiteness assumption of the covariance matrix of $\mathbf{e_t}$ for all t. Note that λ represents the overall trade-off between two objectives of maximizing the expected return and minimizing the risk. Changing λ from 0 to $-\infty$ yields the entire mean-variance efficient frontier.

Problem (MV) is nonseparable in sense of dynamic programming. Li and Ng (2000) [35] solve problem (MV) analytically using an embedding scheme and derive the following pre-committed optimal policy for (MV):

$$\mathbf{u_t^*}(x_t) = -E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(e_t^0\mathbf{P_t})x_t + \frac{1}{2}\left(b_0 x_0 - \frac{\nu_0 \lambda}{2a_0}\right)\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right)E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(\mathbf{P_t}),$$

$$t = 0, 1, \ldots, T-1, \tag{4.2}$$

where the parameters are defined as

$$
\begin{aligned}
B_t &= E(\mathbf{P}_t')E^{-1}(\mathbf{P_t}\mathbf{P}_t')E(\mathbf{P_t}) > 0,\\
A_t^1 &= E(e_t^0) - E(\mathbf{P}_t')E^{-1}(\mathbf{P_t}\mathbf{P}_t')E(e_t^0\mathbf{P_t}),\\
A_t^2 &= E((e_t^0)^2) - E(e_t^0\mathbf{P}_t')E^{-1}(\mathbf{P_t}\mathbf{P}_t')E(e_t^0\mathbf{P_t}) > 0,\\
B_t^1 &= B_t\frac{\prod_{k=t+1}^{T-1}A_k^1}{2\prod_{k=t+1}^{T-1}A_k^2}, \quad B_t^2 = B_t\left(\frac{\prod_{k=t+1}^{T-1}A_k^1}{2\prod_{k=t+1}^{T-1}A_k^2}\right)^2,\\
\mu_t &= \prod_{k=t}^{T-1}A_t^1, \quad \nu_t = \sum_{k=t}^{T-1}\left(\prod_{j=k+1}^{T-1}A_j^1\right)B_k^1, \quad \tau_t = \prod_{k=t}^{T-1}A_k^2,\\
a_t &= \frac{\nu_t}{2} - (\upsilon_t)^2, \quad b_t = \frac{\mu_t\nu_t}{a_t} = \frac{2\mu_t}{1-2\nu_t},\\
c_t &= \tau_t - (\mu_t)^2 - a_t(b_t)^2 = \frac{\tau_t(1-2\nu_t)-\mu_t^2}{1-2\nu_t}.
\end{aligned}
$$

Note that $\prod_{k\in\emptyset} f_k$ is assumed to be equal to 1 and $\sum_{k\in\emptyset} f_k$ is assumed to be equal to 0 for any function f_k.

For a given overall trade-off between mean and variance, λ, the expected value and the variance of the optimal terminal wealth are given, respectively, in Li and Ng (2000) [35] as

$$E(x_T|x_0) = (\mu_0 + b_0\nu_0)x_0 - \frac{\nu_0^2\lambda}{2a_0}, \quad Var(x_T|x_0) = \frac{\lambda^2\nu_0^2}{4a_0} + c_0x_0^2.$$

Furthermore, Li and Ng (2000) [35] give the minimum variance set of (MV) explicitly as follows,

$$Var(x_T|x_0) = \frac{a_0}{\nu_0^2}(E(x_T|x_0) - (\mu_0 + b_0\nu_0)x_0)^2 + c_0x_0^2. \tag{4.3}$$

All the mean-variance pairs on the minimum variance set are called *boundary mean-variance pairs* in this chapter. It is easy to verify that, when $E(x_T|x_0) \geq (\mu_0 + b_0\nu_0)x_0$, the mean-variance pair is efficient. We denote all policies corresponding to the boundary mean-variance pairs on the minimum variance set of problem (MV) *boundary policies*, which could be either efficient or inefficient.

Remark 4.1. *Parameter A_t^1 may be negative.*

Example 4.1. Consider an instance with an expected return vector $E(\mathbf{e_t}) = (1.162, 18.246, 4.228)'$ and a positive definite covariance matrix

$$Cov(\mathbf{e_t}) = \begin{pmatrix} 1.46 & 1.87 & 1.45 \\ 1.87 & 8.54 & 1.04 \\ 1.45 & 1.04 & 2.89 \end{pmatrix}.$$

We can calculate

$$\begin{aligned} E(\mathbf{P'_t}) &= E(e_t^1 - e_t^0, e_t^2 - e_t^0)' = (17.0840, 3.0660)', \\ E(\mathbf{P_t P'_t}) &= E\begin{pmatrix} (e_t^1)^2 - 2e_t^0 e_t^1 + (e_t^0)^2 & e_t^1 e_t^2 - e_t^0 e_t^1 - e_t^0 e_t^2 + (e_t^0)^2 \\ e_t^1 e_t^2 - e_t^0 e_t^1 - e_t^0 e_t^2 + (e_t^0)^2 & (e_t^2)^2 - 2e_t^0 e_t^2 + (e_t^0)^2 \end{pmatrix} \\ &= \begin{pmatrix} 298.1231 & 51.5595 \\ 51.5595 & 10.8504 \end{pmatrix}, \\ E(e_t^0 \mathbf{P_t}) &= E(e_t^1 e_t^0 - (e_t^0)^2, e_t^2 e_t^0 - (e_t^0)^2)' = (20.2616, 3.5527)'. \end{aligned}$$

We thus have $B_t = 0.9854 > 0$, $A_t^1 = -0.0019 < 0$, and $A_t^2 = 1.4320 > 0$.

Lemma 4.1. *For any $0 \le t \le T-1$, the parameters A_t^1, A_t^2 and B_t satisfy*

$$A_t^2(1 - B_t) > (A_t^1)^2. \tag{4.4}$$

Proof: Let

$$M \triangleq \begin{pmatrix} Var(e_t^0) & E(e_t^0 \mathbf{P'_t}) - E(e_t^0)E(\mathbf{P'_t}) \\ E(e_t^0 \mathbf{P_t}) - E(e_t^0)E(\mathbf{P_t}) & E(\mathbf{P_t P'_t}) - E(\mathbf{P_t})E(\mathbf{P'_t}) \end{pmatrix}.$$

Since

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -1 & 0 & \dots & 1 \end{pmatrix} Cov(\mathbf{e_t}) \begin{pmatrix} 1 & -1 & \dots & -1 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

M is positive definite.

Due to the positiveness of $E^{-1}(\mathbf{P_t P_t'})$ and $E(\mathbf{P_t P_t'}) - E(\mathbf{P_t})E(\mathbf{P_t'})$,

$$\begin{pmatrix} 1 & 0 \\ 0 & [E^{-1}(\mathbf{P_t P_t'})]^{\frac{1}{2}} \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & [E^{-1}(\mathbf{P_t P_t'})]^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} Var(e_t^0) & q' \\ q & Q \end{pmatrix} \succ 0, \tag{4.5}$$

$$\begin{pmatrix} 1 & E(\mathbf{P_t'}) \\ E(\mathbf{P_t}) & E(\mathbf{P_t P_t'}) \end{pmatrix} \succ 0, \tag{4.6}$$

where

$$\begin{aligned} q' &= p'[E^{-1}(\mathbf{P_t P_t'})]^{\frac{1}{2}}, \\ p' &= E(e_t^0 \mathbf{P_t'}) - E(e_t^0)E(\mathbf{P_t'}), \\ Q &= I - [E^{-1}(\mathbf{P_t P_t'})]^{\frac{1}{2}} E(\mathbf{P_t})E(\mathbf{P_t'})[E^{-1}(\mathbf{P_t P_t'})]^{\frac{1}{2}}. \end{aligned}$$

Applying Sherman-Morrison formula gives rise

$$Q^{-1} = I + \frac{[E^{-1}(\mathbf{P_t P_t'})]^{\frac{1}{2}} E(\mathbf{P_t})E(\mathbf{P_t'})[E^{-1}(\mathbf{P_t P_t'})]^{\frac{1}{2}}}{1 - E(\mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t})}.$$

As the two matrices in (4.5) and (4.6) are positive definite, the product of their determinants is positive, i.e.,

$$\begin{aligned} & (1 - B_t)(Var(e_t^0) - q'Q^{-1}q) \\ = & (1 - B_t)\left[Var(e_t^0) - q'q - \frac{p'E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t})E(\mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})p}{1 - B_t}\right] \\ = & (1 - B_t)Var(e_t^0) - q'q + q'qB_t - p'E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t})E(\mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})p \\ > & 0. \end{aligned} \tag{4.7}$$

The last two terms in the above expression can also be rewritten as,

$$\begin{aligned} & q'qB_t - p'E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t})E(\mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})p \\ = & E(e_t^0 \mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})E(e_t^0 \mathbf{P_t})B_t - 2E(e_t^0)E(\mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})E(e_t^0 \mathbf{P_t})B_t \\ & + E^2(e_t^0)(B_t)^2 + 2E(e_t^0)E(\mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})E(e_t^0 \mathbf{P_t})B_t \\ & - [E(\mathbf{P_t'})E^{-1}(\mathbf{P_t P_t'})E(e_t^0 \mathbf{P_t})]^2 - E^2(e_t^0)(B_t)^2 \\ = & (A_t^2 - E((e_t^0)^2))B_t - (A_t^1 - E(e_t^0))^2. \end{aligned} \tag{4.8}$$

Finally, from Eqs. (4.7) and (4.8), we have

$$
\begin{aligned}
& A_t^2(1-B_t)-(A_t^1)^2 \\
=& (1-B_t)E((e_t^0)^2)-E(e_t^0\mathbf{P}_t')E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(e_t^0\mathbf{P}_t)+(A_t^2-E((e_t^0)^2))B_t \\
& -E^2(e_t^0)+2E(e_t^0)E(\mathbf{P}_t')E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(e_t^0\mathbf{P}_t)-(A_t^1-E(e_t^0))^2 \\
=& (1-B_t)Var(e_t^0)-q'q+(A_t^2-E((e_t^0)^2))B_t-(A_t^1-E(e_t^0))^2 \\
>& 0.
\end{aligned}
$$

□

Lemma 4.2.

$$0<\nu_t\le\nu_0<\frac{1}{2}, \quad \textit{and} \quad \mu_t^2<\tau_t(1-2\nu_t).$$

Proof: First,

$$\nu_0=\frac{1}{2}\sum_{k=0}^{T-1}\left(\prod_{j=k+1}^{T-1}\frac{(A_j^1)^2}{A_j^2}\right)B_k\ge\nu_t=\frac{1}{2}\sum_{k=t}^{T-1}\left(\prod_{j=k+1}^{T-1}\frac{(A_j^1)^2}{A_j^2}\right)B_k>0.$$

Second, based on Lemma 4.1,

$$
\begin{aligned}
& 1-2\nu_t \\
=& 1-\sum_{k=t}^{T-1}\left(\prod_{j=k+1}^{T-1}\frac{(A_j^1)^2}{A_j^2}\right)B_k>1-\sum_{k=t}^{T-1}\prod_{j=k+1}^{T-1}(1-B_j)B_k=\prod_{j=t}^{T-1}(1-B_j)>0,
\end{aligned}
$$

$$\mu_t^2=\prod_{j=t}^{T-1}(A_j^1)^2<\prod_{j=t}^{T-1}(A_j^2)(1-B_j)=\tau_t\prod_{j=t}^{T-1}(1-B_j)<\tau_t(1-2\nu_t).$$

□

4.3. Induced Stage Trade-offs and Preference Switching

Substituting the pre-committed optimal policy given in (4.2) into the wealth dynamics described in (4.1), performing some algebraic operations and taking the expected value give rise the following as shown in Li and Ng (2000) [35],

$$E(x_{t+1}) = A_t^1 E(x_t) + \frac{1}{2}\left(b_0 x_0 - \frac{\lambda \nu_0}{2a_0}\right)\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right) B_t.$$

Solving the above equation recursively yields the following expression of the conditional expectation,

$$\begin{aligned} E(x_t|x_s)|_{\mathbf{u}^*} &\triangleq E(x_t|\mathbf{u}_s^*, \ldots, \mathbf{u}_{t-1}^*, x_s) \\ &= \left(\prod_{j=s}^{t-1} A_j^1\right) x_s + \left(b_0 x_0 - \frac{\lambda\nu_0}{2a_0}\right)\left[\sum_{j=s}^{t-1}\left(\prod_{k=j+1}^{t-1} A_k^1\right) B_j^1\right] \\ &= \frac{\mu_s}{\mu_t} x_s + \left(b_0 x_0 - \frac{\lambda\nu_0}{2a_0}\right)\left(\frac{\nu_s - \nu_t}{\mu_t}\right), \quad t \geq s. \end{aligned} \tag{4.9}$$

Lemma 4.3. *The pre-committed optimal mean-variance policy given in (4.2) satisfies time consistency only when its wealth process follows a particular path,*

$$x_0 \to E(x_1|\mathbf{u}_0^*, x_0) \to E(x_2|\mathbf{u}_0^*, \mathbf{u}_1^*, x_0) \to \ldots \to E(x_{T-1}|\mathbf{u}_0^*, \ldots, \mathbf{u}_{T-2}^*, x_0).$$

Proof: Consider the following truncated multi-period mean-variance portfolio selection problem from time k to T with a given x_k and the same trade-off parameter λ as given in (MV):

$$\begin{aligned} (MV_{k-T}^{\lambda}) \quad \min \quad & Var(x_T|x_k) + \lambda E(x_T|x_k) \\ \text{s.t.} \quad & x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = k, k+1, ..., T-1, \\ & x_k \text{ is given.} \end{aligned}$$

Similar to the solution of (MV), the optimal policy of (MV_{k-T}^{λ}) at period t can be derived as

$$\begin{aligned} &\mathbf{u}_t^{k-T}(x_t, \lambda) \\ &= -E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(e_t^0\mathbf{P}_t)x_t + \frac{1}{2}\left(b_k x_k - \frac{\lambda\nu_k}{2a_k}\right)\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right)E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(\mathbf{P}_t), \\ &\qquad t = k, k+1, \ldots, T-1. \end{aligned} \tag{4.10}$$

For the pre-committed optimal mean-variance policy to be time consistent, the policies in (4.2) and (4.10) must be the same due to the uniqueness of the solution.

Note that $\mathbf{u_t^*}$ is equal to $\mathbf{u_t^{k-T}}$ only when

$$x_k = \frac{b_0}{b_k}x_0 - \frac{\lambda\nu_0}{2a_0b_k} + \frac{\lambda\nu_k}{2a_kb_k} = \frac{\mu_0}{\mu_t}x_0 + \left(b_0x_0 - \frac{\lambda\nu_0}{2a_0}\right)\left(\frac{\nu_0-\nu_t}{\mu_t}\right),$$

which is exactly the expectation of x_k conditional on x_0 under the pre-committed optimal mean-variance policy. □

Now we consider the following inverse optimization problem: For $0 < k \le T-1$, find a trade-off parameter λ_k between $E(x_T|x_k)$ and $Var(x_T|x_k)$ such that the pre-committed optimal mean-variance policy $\mathbf{u_t^*}(x_t)$, $t = k, k+1, \cdots, T-1$, specified in (4.2) solves

$$
\begin{array}{lll}
(MV_{k-T}^{\lambda_k}) & \min & Var(x_T|x_k) + \lambda_k E(x_T|x_k) \\
 & \text{s.t.} & x_{t+1} = e_t^0 x_t + \mathbf{P}_t'\mathbf{u_t}, \quad t = k, k+1, ..., T-1, \\
 & & x_k \text{ is given.}
\end{array}
$$

When $\mu_k \neq 0$, we define the following threshold x_k^* at stage k:

$$x_k^* = \frac{a_k}{\mu_k\nu_k}\left(b_0x_0 - \frac{\lambda\nu_0}{2a_0}\right). \tag{4.11}$$

Proposition 4.1. *The pre-committed truncated optimal mean-variance policy at* x_t*,* $\mathbf{u_t^*}(x_t), (t = k, k+1, ..., T-1)$ *specified in (4.2), solves* $(MV_{k-T}^{\lambda_k})$ *with* λ_k *satisfying*

$$\lambda_k = 2\mu_k x_k + \frac{2a_k}{\nu_k}\left(-b_0x_0 + \frac{\lambda\nu_0}{2a_0}\right). \tag{4.12}$$

We then have i) $\lambda_k < 0$ *when* $\mu_k > 0$ *and* $x_k < x_k^*$*, ii)* $\lambda_k < 0$ *when* $\mu_k < 0$ *and* $x_k > x_k^*$*, iii)* $\lambda_k = 0$ *when* $x_k = x_k^*$*, iv)* $\lambda_k > 0$ *when* $\mu_k > 0$ *and* $x_k > x_k^*$*, and v)* $\lambda_k > 0$ *when* $\mu_k < 0$ *and* $x_k < x_k^*$*. Furthermore, when* $\mu_k = 0$*, we have* $\lambda_j < 0$ $(j \le k)$*.*

Proof: Similar to the solution to (MV), the optimal policy of $(MV_{k-T}^{\lambda_k})$ at stage k is given by

$$\begin{aligned} &\mathbf{u}_{\mathbf{t}}^{\mathbf{k-T}}(x_t, \lambda_k) \\ &= -E^{-1}(\mathbf{P_t P_t'})E(e_t^0 \mathbf{P_t})x_t + \frac{1}{2}\left(b_k x_k - \frac{\lambda_k \nu_k}{2a_k}\right)\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right)E^{-1}(\mathbf{P_t P_t'})E(\mathbf{P_t}), \\ &\qquad t = k, k+1, \dots, T-1. \end{aligned}$$

Equalizing the policy above and the policy in (4.2) yields the relationship between λ_k and λ given in (4.12). As

$$0 < \nu_k \le \nu_0 < \frac{1}{2},$$

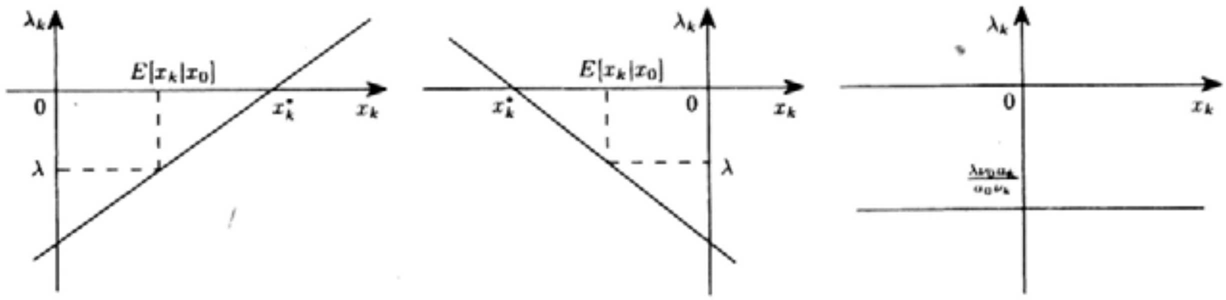
we have $a_k > 0$. Since $\mu_k = 0$ implies $\mu_j = 0$ $(j \le k)$ and $b_j = 0$ $(j \le k)$, (4.12) in such a case reduces to

$$\lambda_j = \frac{\lambda \nu_0 a_k}{a_0 \nu_k} < 0, \quad j \le k.$$

Other conclusions in the proposition can be obtained directly from the linear relationship between λ_k and x_k. □

Remark 4.2. *We cannot ensure λ_k, the trade-off induced by the pre-committed optimal policy at state x_k, to be nonpositive all the stages under many return distribution assumptions, for example, under the assumption of a normal distribution for the returns. Thus, the pre-committed optimal mean-variance policy $\mathbf{u}_{\mathbf{k}}^{*}(x_k)$ specified in (4.2) is, in general, not time consistent in efficiency.*

It is interesting to note from (4.12) that the stage trade-off λ_k at state x_k induced by the pre-committed optimal mean-variance policy is a linear function of both the initial trade-off λ and the current wealth x_k. Figures 4.1(a), 4.1(b) and 4.1(c) illustrate the linear relationship between λ_k and x_k for situations with different μ_ks. The intersection of the straight line of λ_k and the horizontal line is the threshold x_k^*.



(a) $\mu_k > 0$ (b) $\mu_k < 0$ (c) $\mu_k = 0$

Figure 4.1: Relationship between induced trade-off λ_k and wealth x_k

4.4. Pseudo Efficiency and Policies Better than the Pre-committed Optimal Mean-Variance Policy

4.4.1. Achievement of the same mean-variance pair by lower funding level

Let us investigate the T-period mean-variance portfolio selection problem (MV), of which the pre-committed optimal policy is given in (4.2). We denote $\frac{1}{2}\left(b_0x_0 - \frac{\lambda\nu_0}{2a_0}\right)$ by *risk attitude parameter*, Γ_{-1}, and define

$$\bar{x}_0^* = \frac{\Gamma_{-1}\mu_0}{\tau_0},$$

$$\Gamma_0 = \begin{cases} \Gamma_{-1}, & \text{if } x_0 \le \bar{x}_0^*, \\ \Gamma_{-1} + \dfrac{2\mu_0\tau_0(x_0 - \bar{x}_0^*)}{2\nu_0\tau_0 + \mu_0^2}, & \text{if } x_0 > \bar{x}_0^*. \end{cases}$$

We now propose the following revised portfolio policy for problem (MV).

Revised Policy 4.1.

$$\bar{\mathbf{u}}_0^*(\hat{x}_0) = -E^{-1}(\mathbf{P_0 P_0'})E(e_0^0\mathbf{P_0})\hat{x}_0 + \Gamma_0\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right)E^{-1}(\mathbf{P_0P_0'})E(\mathbf{P_0}),$$
$$\bar{\mathbf{u}}_t^*(x_t) = -E^{-1}(\mathbf{P_t P_t'})E(e_t^0\mathbf{P_t})x_t + \Gamma_0\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right)E^{-1}(\mathbf{P_tP_t'})E(\mathbf{P_t}),$$
$$t = 1,\cdots,T-1,$$

where the wealth dynamics still follows (4.1) with initial wealth $\hat{x}_0$ and

$$\hat{x}_0 = \begin{cases} x_0, & \text{if } x_0 \le \bar{x}_0^*, \\ -x_0 + \dfrac{2\mu_0(\mu_0 x_0 + 2\nu_0\Gamma_{-1})}{2\nu_0\tau_0 + \mu_0^2}, & \text{if } x_0 > \bar{x}_0^*, \end{cases}$$

is termed the revised lower funding level.

When $x_0 > \bar{x}_0^*$, the amount of money taken out termed *free cash flow*, $x_0 - \hat{x}_0$, is positive.

Proposition 4.2. *Revised policy 4.1 achieves the same mean-variance pair as does the pre-committed optimal mean-variance policy.*

Proof: When $x_0 \le \bar{x}_0^*$, the conclusion follows. When $x_0 > \bar{x}_0^*$, from (4.9), the mean-variance pair attained by the pre-committed optimal policy is given by

$$E(x_T|x_0) = \frac{\mu_0}{\mu_T}x_0 + 2\Gamma_{-1}\frac{\nu_0-\nu_T}{\mu_T} = \mu_0 x_0 + 2\nu_0\Gamma_{-1},$$
$$Var(x_T|x_0) = \frac{a_0}{\nu_0^2}(E(x_T|x_0) - (\mu_0 + b_0\nu_0)x_0)^2 + c_0x_0^2,$$

and the mean-variance pair achieved by the revised policy and the revised lower funding level is given by

$$E(x_T|\hat{x}_0) = \frac{\mu_0}{\mu_T}\hat{x}_0 + 2\Gamma_0\frac{\nu_0-\nu_T}{\mu_T} = \mu_0\hat{x}_0 + 2\nu_0\Gamma_0,$$
$$Var(x_T|\hat{x}_0) = \frac{a_0}{\nu_0^2}(E(x_T|\hat{x}_0) - (\mu_0 + b_0\nu_0)\hat{x}_0)^2 + c_0\hat{x}_0^2.$$

Equalizing $E(x_T|x_0)$ and $E(x_T|\hat{x}_0)$ and equalizing $Var(x_T|x_0)$ and $Var(x_T|\hat{x}_0)$ simultaneously yield

$$\begin{cases} \mu_0 x_0 + 2\Gamma_{-1}\nu_0 = \mu_0\hat{x}_0 + 2\Gamma_0\nu_0, \\ \dfrac{a_0}{\nu_0^2}(E(x_T|x_0) - (\mu_0+b_0\nu_0)x_0)^2 + c_0x_0^2 = \dfrac{a_0}{\nu_0^2}(E(x_T|\hat{x}_0) - (\mu_0+b_0\nu_0)\hat{x}_0)^2 + c_0\hat{x}_0^2. \end{cases}$$

Substituting the first equation into second equation gives rise,

$$\left(\tau_0 + \frac{\mu_0^2}{2\nu_0}\right)\hat{x}_0^2 - \frac{\mu_0(\mu_0 x_0 + 2\Gamma_{-1}\nu_0)}{\nu_0}\hat{x}_0 + C = 0,$$

where

$$C = \frac{a_0(\mu_0 x_0 + 2\Gamma_{-1}\nu_0)^2}{\nu_0^2} - a_0(2\Gamma_{-1} - b_0 x_0)^2 - c_0 x_0^2.$$

Solving the above quadratic equations yields two roots,

$$\hat{x}_0 = x_0 \text{ (rejected)} \quad \text{and} \quad \hat{x}_0 = -x_0 + \frac{2\mu_0(\mu_0 x_0 + 2\nu_0\Gamma_{-1})}{2\nu_0\tau_0 + \mu_0^2}.$$

We further have

$$\Gamma_0 = \Gamma_{-1} + \frac{\mu_0}{2\nu_0}\left(2x_0 - \frac{2\mu_0(\mu_0 x_0 + 2\Gamma_{-1}\nu_0)}{2\nu_0\tau_0 + \mu_0^2}\right) = \Gamma_{-1} + \frac{2\mu_0\tau_0(x_0 - \bar{x}_0^*)}{2\nu_0\tau_0 + \mu_0^2}.$$

Note that the condition, $x_0 > \bar{x}_0^*$, holds if and only if

$$-x_0 + \frac{2\mu_0(\mu_0 x_0 + 2\Gamma_{-1}\nu_0)}{2\nu_0\tau_0 + \mu_0^2} < x_0,$$

which implies that $(x_0 - \hat{x}_0)$, the free cash flow, is positive. □

The implication of the above proposition is clear. When condition $x_0 > \bar{x}_0^*$ holds, a revised funding level which is strictly less than x_0 still enables you to achieve the same mean-variance pair which you are aiming for, while you can use the released "extra" money, free cash flow, to catch other investment opportunities. Note further that the threshold $\bar{x}_0^*$ depends on investor's trade-off parameter λ. The smaller the absolute value λ, the larger degree of risk aversion, and a higher chance that $x_0 > \bar{x}_0^*$ holds.

Example 4.2. Let us consider an instance with $E(\mathbf{e}_t) = (1.162, 1.246, 1.228)'$, $t = 0, 1$, and

$$Cov(\mathbf{e}_t) = \begin{pmatrix} 1.46 & 1.87 & 1.45 \\ 1.87 & 8.54 & 1.04 \\ 1.45 & 1.04 & 2.89 \end{pmatrix}, \quad t = 0, 1.$$

We can calculate

$$E(\mathbf{P}_t') = E(e_t^1 - e_t^0, e_t^2 - e_t^0)' = (0.0840, 0.0660)', \quad t = 0, 1,$$

$$E(\mathbf{P}_t\mathbf{P}_t') = E\begin{pmatrix} (e_t^1)^2 - 2e_t^0 e_t^1 + (e_t^0)^2 & e_t^1 e_t^2 - e_t^0 e_t^1 - e_t^0 e_t^2 + (e_t^0)^2 \\ e_t^1 e_t^2 - e_t^0 e_t^1 - e_t^0 e_t^2 + (e_t^0)^2 & (e_t^2)^2 - 2e_t^0 e_t^2 + (e_t^0)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 6.2671 & -0.8145 \\ -0.8145 & 1.4544 \end{pmatrix}, \quad t = 0, 1,$$

$$E(e_t^0 \mathbf{P}_t) = E(e_t^1 e_t^0 - (e_t^0)^2, e_t^2 e_t^0 - (e_t^0)^2)' = (0.5076, 0.0667)', \quad t = 0, 1.$$

Furthermore we have $B_t = 0.0055 > 0$, $A_t^1 = 1.1476 > 0$, $A_t^2 = 2.7561 > 0$, $(t = 0, 1)$, $\mu_0 = 1.3171$, $\tau_0 = 7.5960$, $a_0 = 0.0020$, and $b_0 = 2.6557$. The condition $x_0 > \bar{x}_0^*$ holds when

$$x_0 > -1.3098\lambda.$$

We use Figure 4.2 to further explain our derived result. When the condition, $x_0 > \bar{x}_0^*$, is satisfied, a given efficient mean-variance pair $(E(x_T|x_0), Var(x_T|x_0))$ can be also produced by the upper branch of another minimum variance curve corresponding to a smaller funding level $\hat{x}_0$. Note that when the two efficient frontiers cross each other, the associated trade-off parameters must be different. More specifically, risk attitude parameter Γ_0 corresponding to the revised policy is larger than the original risk attitude parameter Γ_{-1} (in situations with positive μ_0).

It becomes clear now that, for a given initial wealth $x_0 > 0$, if the absolute value of trade-off parameter λ is large enough, the condition of $x_0 > \bar{x}_0^*$ will not be satisfied. Note that when $\mu_0 = 0$, the condition of taking free cash flow out of the market at time 0 always holds, as evidenced from $x_0 > \bar{x}_0^* = \dfrac{\Gamma_{-1}\mu_0}{\tau_0} = 0$.

Proposition 4.3. *For a given positive initial wealth x_0, condition $x_0 > \bar{x}_0^*$ does not hold when*

$$\lambda = \begin{cases} \leq \dfrac{2(\mu_0^2 - (1 - 2\nu_0))}{\mu_0} x_0 < 0, & \text{if } \mu_0 > 0, \\ \geq \dfrac{2(\mu_0^2 - (1 - 2\nu_0))}{\mu_0} x_0 > 0, & \text{if } \mu_0 < 0. \end{cases}$$

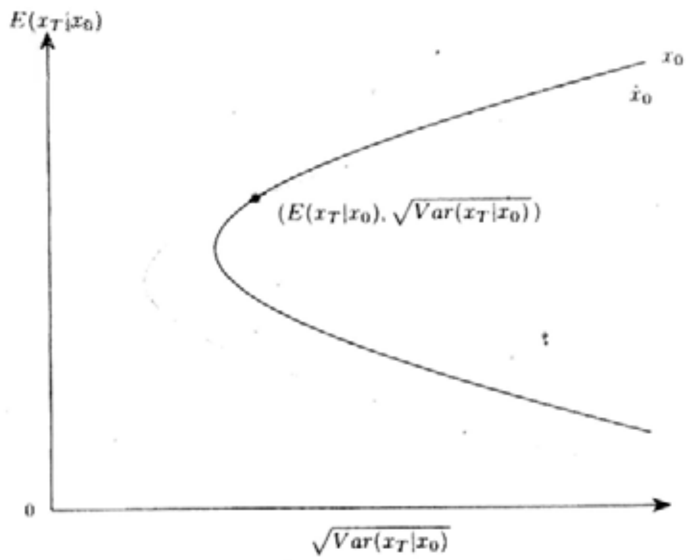


Figure 4.2: Minimum variance sets corresponding to $\hat{x}_0$ and x_0

Proof: It is the direct conclusion with the help of $\mu_0^2 < \tau_0(1 - 2\nu_0)$. □

The above proposition reveals a financial finding: In a market with $\mu_0 > 0$, you'd better to look for a portfolio with relatively high risk (relatively large variance), if you have to invest only in the risky assets. Otherwise, you can use a lower wealth level to achieve the same efficient mean-variance pair while increasing the absolute value of λ, your initial trade-off parameter.

4.4.2. Pseudo Efficiency (Type 1) and the First Type of Revised Policy

Based on the discussion in Section 4.3, the truncated pre-committed optimal mean-variance policy at the beginning of period s may be efficient or inefficient, although it must be a boundary policy. Now let us consider the truncated mean-variance portfolio selection starting from time $(s-1)$,

$$\begin{aligned}(MV_{(s-1)-T}) \qquad \min \quad & Var(x_T \mid x_{s-1}) + \lambda E(x_T|x_{s-1}) \\ \text{s.t.} \quad & x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = s-1, s, \ldots, T-1, \\ & x_{s-1} \text{ is given,}\end{aligned}$$

of which the optimal policy is given by

$$\mathbf{u}_t^*(x_t) = -E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(e_t^0\mathbf{P}_t)x_t + \Gamma_{s-1}\left(\frac{\mu_{t+1}}{\tau_{t+1}}\right)E^{-1}(\mathbf{P}_t\mathbf{P}_t')E(\mathbf{P}_t),$$
$$t = s-1, \ldots, T-1, \qquad (4.13)$$

where risk attitude parameter Γ_{s-1} is given by $\frac{1}{2}\left(b_{s-1}x_{s-1} - \frac{\nu_{s-1}\lambda}{2a_{s-1}}\right)$. At the time when the investor arrives at the beginning of period s, the truncated optimal policy, $\mathbf{u}_t^*(x_t)$, $t = s, s+1, \cdots, T-1$, specified in (4.13), leads to the following conditional mean and variance of the terminal wealth, which is a boundary mean-variance pair for (MV_{s-T}),

$$E(x_T|x_s) = \mu_s x_s + 2\nu_s\Gamma_{s-1}, \; Var(x_T|x_s) = \frac{a_s}{\nu_s^2}(E(x_T|x_s) - (\mu_s + b_s\nu_s)x_s)^2 + c_s x_s^2. \qquad (4.14)$$

Definition 4.1. *For a given wealth level x_s at period s, if an efficient mean-variance pair for the truncated $(T-s)$-period problem, (MV_{s-T}), can be also generated or is even dominated by another $(T-s)$-period mean-variance pair generated by another $(T-s)$-period boundary policy associated with a strictly less wealth level $\hat{x}_s$, i.e.,*

$$(E(x_T|x_s), -Var(x_T|x_s)) \preceq (E(x_T|\hat{x}_s), -Var(x_T|\hat{x}_s)), \text{ with } \hat{x}_s < x_s, \qquad (4.15)$$

the given $(T-s)$-period mean-variance pair is termed **pseudo efficient (type 1)**.

Proposition 4.4. *The boundary mean-variance pair specified in (4.14) can be produced by a lower wealth level, $\hat{x}_s$, if and only if*

$$x_s > \frac{\Gamma_{s-1}\mu_s}{\tau_s}.$$

Furthermore, if the boundary mean-variance pair is efficient, it is pseudo efficient (type 1).

Proof: The boundary mean-variance pair corresponding to a new wealth level, $\hat{x}_s$, is given by

$$\begin{aligned} E(x_T|\hat{x}_s) &= \mu_s\hat{x}_s + 2\nu_s\Gamma_s, \\ Var(x_T|\hat{x}_s) &= \frac{a_s}{\nu_s^2}(E(x_T|\hat{x}_s) - (\mu_s + b_s\nu_s)\hat{x}_s)^2 + c_s\hat{x}_s^2, \end{aligned}$$

where Γ_s is a revised risk attitude parameter associated with $\hat{x}_s$. Equalizing the above mean-variance pair and the mean-variance pair specified in (4.14) yield

$$\begin{cases} \mu_s x_s + 2\Gamma_{s-1}\nu_s = \mu_s\hat{x}_s + 2\Gamma_s\nu_s, \\ \frac{a_s}{\nu_s^2}(E(x_T|x_s) - (\mu_s + b_s\nu_s)x_s)^2 + c_s x_s^2 = \frac{a_s}{\nu_s^2}(E(x_T|\hat{x}_s) - (\mu_s + b_s\nu_s)\hat{x}_s)^2 + c_s\hat{x}_s^2. \end{cases}$$

Solving the above quadratic equations yields two roots,

$$\hat{x}_s = x_s \text{ (rejected)} \quad \text{and} \quad \hat{x}_s = -x_s + \frac{2\mu_s(\mu_s x_s + 2\nu_s\Gamma_{s-1})}{2\nu_s\tau_s + \mu_s^2}.$$

Note that the condition, $x_s > \frac{\Gamma_{s-1}\mu_s}{\tau_s}$, holds if and only if

$$-x_s + \frac{2\mu_s(\mu_s x_s + 2\Gamma_{s-1}\nu_s)}{2\nu_s\tau_s + \mu_s^2} < x_s.$$

□

We now propose a general T-period revised portfolio policy, $\hat{\mathbf{u}}_k^*(\hat{x}_k)$, $k = 0, \ldots, T-1$, for the T-period problem (MV).

Revised Policy 4.2.

$$\hat{\mathbf{u}}_{\mathbf{k}}^{*}(\hat{x}_k) = -E^{-1}(\mathbf{P_k}\mathbf{P_k}')E(e_k^0\mathbf{P_k})\hat{x}_k + \Gamma_k\left(\frac{\mu_{k+1}}{\tau_{k+1}}\right)E^{-1}(\mathbf{P_k}\mathbf{P_k}')E(\mathbf{P_k}); \quad (4.16)$$

$$\hat{x}_k = \begin{cases} \bar{x}_k, & \text{if } \bar{x}_k \le \bar{x}_k^*, \\ -\bar{x}_k + \dfrac{2\mu_k(\mu_k\bar{x}_k + 2\nu_k\Gamma_{k-1})}{2\nu_k\tau_k + \mu_k^2}, & \text{if } \bar{x}_k > \bar{x}_k^*, \end{cases} \quad (4.17)$$

$$\bar{x}_0 = x_0$$

$$\bar{x}_{k+1} = e_k^0\hat{x}_k + \mathbf{P}_k'\hat{\mathbf{u}}_{\mathbf{k}}^{*}(\hat{x}_k), \quad (4.18)$$

$$\Gamma_k = \begin{cases} \Gamma_{k-1}, & \text{if } \bar{x}_k \le \bar{x}_k^*, \\ \Gamma_{k-1} + \dfrac{2\mu_k\tau_k(\bar{x}_k - \bar{x}_k^*)}{2\nu_k\tau_k + \mu_k^2}, & \text{if } \bar{x}_k > \bar{x}_k^*, \end{cases} \quad (4.19)$$

$$\Gamma_{-1} = \frac{1}{2}\left(b_0x_0 - \frac{\lambda_0\nu_0}{2a_0}\right)$$

$$\bar{x}_k^* = \frac{\Gamma_{k-1}\mu_k}{\tau_k}. \quad (4.20)$$

Note that both risk attitude parameter Γ_k and the wealth threshold $\bar{x}_k^*$ ($k = 0, 1, \ldots, T-1$) in Revised Policy 4.2 are path-dependent. More specifically, $\bar{x}_k^*$ in Revised Policy 4.2 is different from threshold x_k^* discussed in Section 3, as $\bar{x}_k^*$ is a path-dependent threshold for a wealth process in which cash withdrawals may occur when $\bar{x}_k > \bar{x}_k^*$, i.e., when pseudo efficiency (type 1) appears. One major feature of Revised policy 4.2 is that, when the wealth level $\bar{x}_k > \bar{x}_k^*$, we withdraw a positive free cash flow, $\bar{x}_k - \hat{x}_k = \dfrac{4\nu_k\tau_k}{2\nu_k\tau_k + \mu_k^2}(\bar{x}_k - \bar{x}_k^*)$, out of the market and apply a different mean-variance policy to $\hat{x}_k$, the remaining amount in the market.

Theorem 4.1. *Revised policy 4.2 achieves the same mean-variance pair as does the pre-committed optimal mean-variance policy of the T-period problem (MV), while having possibility to take positive free cash flow stream, $\{\bar{x}_k - \hat{x}_k\}$, out of the market during the investment process.*

Proof: The case with $T = 1$ has been already proved in Proposition 4.2. We assume that the theorem is true for $T = k$ with $k \ge 1$. We now proceed to prove that the theorem is also true for $T = k + 1$.

Proposition 4.2 also proves that

$$E(x_{k+1}|x_0)|_{\mathbf{u}^*} = E(x_{k+1}|\hat{x}_0)|_{\bar{\mathbf{u}}^*},$$
$$Var(x_{k+1}|x_0)|_{\mathbf{u}^*} = Var(x_{k+1}|\hat{x}_0)|_{\bar{\mathbf{u}}^*},$$

where $\bar{\mathbf{u}}^*$ is the revised policy specified in Revised Policy 4.1. Then at time 1, with the wealth level $\bar{x}_1$, the policy $\bar{\mathbf{u}}_t^*, t = 1, \cdots, k$, achieves

$$(E(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*}, Var(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*}),$$

which is a k-period boundary mean-variance pair with initial wealth $\bar{x}_1$ and trade-off $\lambda_{1-} = -\dfrac{2a_1}{\nu_1}(2\Gamma_0 - b_1\bar{x}_1)$.

On the other hand, $(E(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*}, Var(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*})$ can be achieved by a k-period revised policy with initial wealth $\hat{x}_1 = \bar{x}_1$ and parameter $\Gamma_1 = \Gamma_{1-} = \dfrac{1}{2}\left(b_1 x_1 - \dfrac{\nu_1\lambda_{1-}}{2a_1}\right)$ from the assumption of the induction, i.e.,

$$E(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*} = E(\bar{x}_{k+1}|\hat{x}_1)|_{\hat{\mathbf{u}}^*},$$
$$Var(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*} = Var(\bar{x}_{k+1}|\hat{x}_1)|_{\hat{\mathbf{u}}^*}.$$

We combine the k-period revised policy and the time 0 policy to form our $(k+1)$-period revised policy. We demonstrate now the conclusion of reaching the same mean-variance pair by presenting the following facts:

$$\begin{aligned}
&E(x_{k+1}|x_0)|_{\mathbf{u}^*} \\
&= E(x_{k+1}|\hat{x}_0)|_{\bar{\mathbf{u}}_0^*, \bar{\mathbf{u}}^*} \\
&= \int E(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*} f(\bar{x}_1) d\bar{x}_1 \\
&= \int E(\bar{x}_{k+1}|\bar{x}_1)|_{\hat{\mathbf{u}}^*} f(\bar{x}_1) d\bar{x}_1 \\
&= E(\bar{x}_{k+1}|x_0)|_{\hat{\mathbf{u}}^*},
\end{aligned}$$

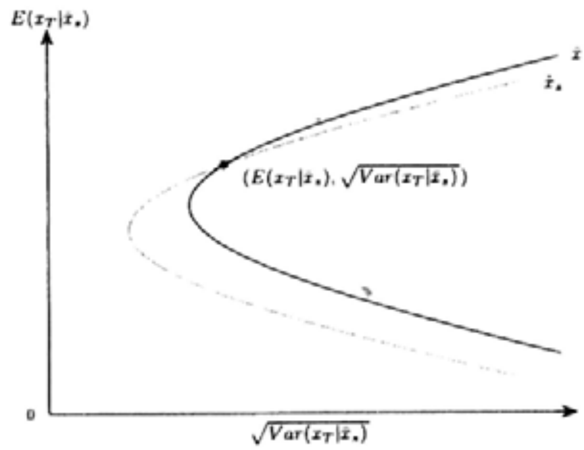
$$
\begin{aligned}
& Var(x_{k+1}|x_0)|_{\mathbf{u}^*} \\
&= Var(x_{k+1}|\hat{x}_0)|_{\bar{\mathbf{u}}_0^*,\bar{\mathbf{u}}^*} \\
&= E(Var(x_{k+1}|\bar{x}_1)|\hat{x}_0)|_{\bar{\mathbf{u}}_0^*,\bar{\mathbf{u}}^*} + Var(E(x_{k+1}|\bar{x}_1)|\hat{x}_0)|_{\bar{\mathbf{u}}_0^*,\bar{\mathbf{u}}^*} \\
&= \int \left[Var(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*} + (E(x_{k+1}|\bar{x}_1)|_{\bar{\mathbf{u}}^*} - E(x_{k+1}|\hat{x}_0)|_{\bar{\mathbf{u}}_0^*,\bar{\mathbf{u}}^*})^2\right] f(\bar{x}_1)d\bar{x}_1 \\
&= \int \left[Var(\bar{x}_{k+1}|\bar{x}_1)|_{\hat{\mathbf{u}}^*} + (E(\bar{x}_{k+1}|\bar{x}_1)|_{\hat{\mathbf{u}}^*} - E(x_{k+1}|\hat{x}_0)|_{\bar{\mathbf{u}}_0^*,\bar{\mathbf{u}}^*})^2\right] f(\bar{x}_1)d\bar{x}_1 \\
&= Var(\bar{x}_{k+1}|x_0)|_{\hat{\mathbf{u}}^*},
\end{aligned}
$$

where $f(\bar{x}_1)$ is the probability density function of random variable $\bar{x}_1$, which depends on the initial wealth $\hat{x}_0$ and period-0 policy $\bar{\mathbf{u}}_0^*(\hat{x}_0)$. □

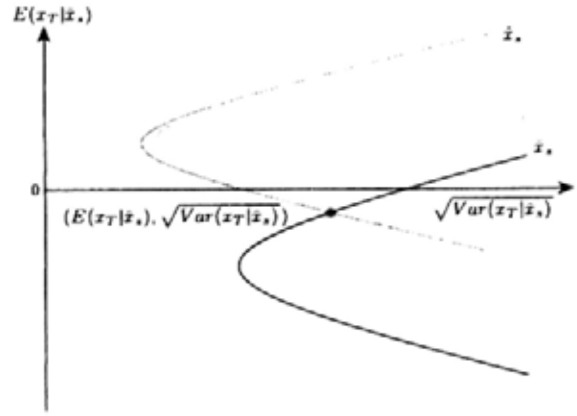
Our newly proposed Revised policy 4.2 is better than the pre-committed optimal mean-variance policy in the sense while the two achieve the same mean and variance of the terminal wealth, the revised policy enables investors to receive additional (nonnegative) free cash flow stream during the investment process. The scheme of Revised policy 4.2 can be explained by Figure 4.3. If the conditional boundary mean-variance pair attained by the truncated pre-committed optimal policy can be produced by lower wealth level, we take positive free cash flow out of the market and apply the revised policy to achieve the same conditional boundary mean-variance pair. In this revised policy, we not only relax the self-financing assumption like the revised policy introduced in a market with a riskless asset (see Chapter 3), but also increase the feasible region of investment policy by allowing a no-borrowing riskless asset.

4.4.3. Achievement of better total mean-variance pair

Based on the discussion in Section 4.4.1, in certain situations, we'd better invest lower initial wealth in the risky assets and use the extra money in a free cash flow for other investment opportunity, or just save the free cash flow in the pocket. Then, the total wealth from both the risky assets and in the free cash flow



(a) $\mu_k > 0$



(b) $\mu_k < 0$

Figure 4.3: The scheme of first revised policy

stream together forms the total wealth at the terminal time. Let we consider now a revised T-period mean-variance portfolio selection problem (RMV) by taking into the consideration the free cash flow at time 0,

$$\begin{aligned} (RMV) \qquad \min \quad & Var[x_T + (x_0 - y_0)|x_0] + \lambda E[x_T + (x_0 - y_0)|x_0] \\ \text{s.t.} \quad & x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 1, \dots, T-1, \\ & x_1 = e_0^0 y_0 + \mathbf{P}_0' \mathbf{u}_0, \\ & y_0 \leq x_0. \end{aligned} \tag{4.21}$$

Remark 4.3. *If we consider the free cash flow account as a riskless asset with total return of* 1 *and no borrowing, the concept behind dynamic mean-variance portfolio selection model* (RMV) *is quite similar to the static mean-variance portfolio selection model with no riskless borrowing proposed in Black (1972) [7].*

An equivalent formulation of (RMV) is given as follows,

$$
\begin{aligned}
(RMV(\epsilon)) \qquad \min \quad & Var(x_T + x_0 - y_0 | x_0) \\
\text{s.t.} \quad & E(x_T + x_0 - y_0 | x_0) = \epsilon, \\
& x_{t+1} = e_t^0 x_t + \mathbf{P_t'} \mathbf{u_t}, \quad t = 1, \ldots, T-1, \\
& x_1 = e_0^0 y_0 + \mathbf{P_0'} \mathbf{u_0}, \\
& y_0 \leq x_0.
\end{aligned}
\tag{4.22}
$$

The minimum variance set of the total wealth can be identified by solving (RMV) or $(RMV(\epsilon))$ with λ or ϵ changing from $-\infty$ to $+\infty$.

We denote $\frac{1}{2}\left(b_0 x_0 - \frac{\lambda \nu_0}{2a_0}\right)$ by *risk attitude parameter* $\bar{\Gamma}_{-1}$ and define

$$
\bar{\Gamma}_0 = \begin{cases} \bar{\Gamma}_{-1}, & \text{if } (\tau_0 - \mu_0)x_0 \leq (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}, \\ \dfrac{(\tau_0 - \mu_0)[(\mu_0 - 1)x_0 + 2\nu_0 \bar{\Gamma}_{-1}]}{2\nu_0(\tau_0 - 1) + (\mu_0 - 1)^2}, & \text{if } (\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}. \end{cases}
$$

We now propose the following revised portfolio policy.

Revised Policy 4.3.

$$
\begin{aligned}
\bar{\mathbf{u}}_0^*(\check{x}_0) &= -E^{-1}(\mathbf{P_0 P_0'}) E(e_0^0 \mathbf{P_0}) \check{x}_0 + \bar{\Gamma}_0 \left(\frac{\mu_{t+1}}{\tau_{t+1}}\right) E^{-1}(\mathbf{P_0 P_0'}) E(\mathbf{P_0}), \\
\bar{\mathbf{u}}_t^*(x_t) &= -E^{-1}(\mathbf{P_t P_t'}) E(e_t^0 \mathbf{P_t}) x_t + \bar{\Gamma}_0 \left(\frac{\mu_{t+1}}{\tau_{t+1}}\right) E^{-1}(\mathbf{P_t P_t'}) E(\mathbf{P_t}), \\
& \qquad\qquad\qquad\qquad\qquad\qquad t = 1, \cdots, T-1,
\end{aligned}
$$

where

$$
\check{x}_0 = \begin{cases} x_0, & \text{if } (\tau_0 - \mu_0)x_0 \leq (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}, \\ \dfrac{(\mu_0 - 1 + 2\nu_0)[(\mu_0 - 1)x_0 + 2\nu_0 \bar{\Gamma}_{-1}]}{2\nu_0(\tau_0 - 1) + (\mu_0 - 1)^2}, & \text{if } (\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}. \end{cases}
$$

is the revised initial funding level in the risky assets.

Theorem 4.2. *Revised policy 4.3 is optimal to $(RMV(\epsilon))$ with $\epsilon = \mu_0 x_0 + 2\nu_0 \bar{\Gamma}_{-1}$.*

Proof: Problem $(RMV(\epsilon))$ is equivalent to

$$\min_{\{y_0 \le x_0\}} F(y_0),$$

where $F(y_0)$ is the optimal objective value of following T-period self-financing mean-variance portfolio selection problem with initial wealth y_0,

$$\begin{array}{lll} (MV(\epsilon - (x_0 - y_0))) & \min & Var(x_T|y_0) \\ & \text{s.t.} & E(x_T|y_0) = \epsilon - (x_0 - y_0), \\ & & x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 1, \ldots, T-1, \\ & & x_1 = e_0^0 y_0 + \mathbf{P}_0' \mathbf{u}_0. \end{array}$$

The mean-variance pair, $(E(x_0|y_0), Var(x_T|y_0))$, must lie on the minimum variance set. Thus,

$$\begin{aligned} F(y_0) &= Var(x_T|y_0) \\ &= \frac{a_0}{\nu_0^2}(E(x_T|y_0) - (\mu_0 + b_0\nu_0)y_0)^2 + c_0 y_0^2 \\ &= \frac{a_0}{\nu_0^2}(\epsilon - (x_0 - y_0) - (\mu_0 + b_0\nu_0)y_0)^2 + c_0 y_0^2 \\ &= \frac{2\nu_0(\tau_0 - 1) + (\mu_0 - 1)^2}{2\nu_0} y_0^2 - \frac{(\mu_0 - 1 + 2\nu_0)[(\mu_0 - 1)x_0 + 2\nu_0\bar{\Gamma}_{-1}]}{\nu_0} y_0 + C_0, \end{aligned}$$

where C_0 is a constant not depending on y_0.

The best funding level invested in the risky asset, $\check{x}_0$, is given by

$$\check{x}_0 = \arg\min_{\{y_0 \le x_0\}} \{F(y_0)\}.$$

Note $\check{x}_0 < x_0$ holds if and only if $(\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}$.

The optimal policy for the best funding level can be further got by identifying the risk attitude parameter, $\bar{\Gamma}_0$. When $(\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}$, with the help of (4.9), we have

$$\begin{aligned} E(x_T|y_0) &= \epsilon - (x_0 - \check{x}_0) \\ \mu_0 \check{x}_0 + 2\nu_0 \bar{\Gamma}_0 &= \mu_0 x_0 + 2\nu_0 \bar{\Gamma}_{-1} - (x_0 - \check{x}_0), \end{aligned}$$

furthermore,

$$\bar{\Gamma}_0 = \bar{\Gamma}_{-1} + \frac{(\mu_0 - 1)(x_0 - \check{x}_0)}{2\nu_0} = \frac{(\tau_0 - \mu_0)[(\mu_0 - 1)x_0 + 2\nu_0\bar{\Gamma}_{-1}]}{2\nu_0(\tau_0 - 1) + (\mu_0 - 1)^2}.$$

□

Noticing the fact of $\epsilon = E(x_T|x_0)|_{\mathbf{u}^*} = \mu_0 x_0 + 2\nu_0\bar{\Gamma}_{-1}$, we have the following corollary.

Corollary 4.1. *While revised policy 4.3 solves $(RMV(\epsilon))$ with $\epsilon = E(x_T|x_0)|_{\mathbf{u}^*}$, i.e, it achieves the same total mean as does the pre-committed optimal mean-variance policy, it attains a no bigger total variance than the pre-committed optimal mean-variance policy.*

Since both $\tau_0 < \mu_0$, $\tau_0 = \mu_0$ and $\tau_0 > \mu_0$ could happen, we are unable to derive a well-defined threshold for the wealth process as we do in revised policy 4.2. When the condition of $(\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}$ is satisfied, a given efficient mean-variance pair $(E(x_T|x_0), Var(x_T|x_0))$ of (MV) will be dominated by a boundary mean-variance pair of (RMV), which possesses the same mean as the given efficient mean-variance pair.

When $\mu_0 = 1 - 2\nu_0$, we have $\tau_0 > \mu_0$ due to $\tau_0(1 - 2\nu_0) > \mu_0^2$. Thus, the condition of $(\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}$ always holds in such a situation for positive initial wealth level $(x_0 > 0)$.

Proposition 4.5. *For a given positive initial wealth level x_0, condition $(\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\bar{\Gamma}_{-1}$ does not hold when*

$$\lambda = \begin{cases} \leq \dfrac{2(\mu_0^2 - \tau_0(1 - 2\nu_0))}{\mu_0 - 1 + 2\nu_0} x_0 < 0, & \text{if } \mu_0 > 1 - 2\nu_0, \\ \geq \dfrac{2(\mu_0^2 - \tau_0(1 - 2\nu_0))}{\mu_0 - 1 + 2\nu_0} x_0 > 0, & \text{if } \mu_0 < 1 - 2\nu_0. \end{cases}$$

Proof: This conclusion follows due to $\mu_0^2 < \tau_0(1 - 2\nu_0)$. □

4.4.4. Pseudo Efficiency (Type 2) and the Second Type of Revised Policy

We now introduce a definition of "pseudo efficiency (type 2)". Let us consider the revised truncated mean-variance portfolio selection problem,

$$
\begin{aligned}
(RMV_{s-T}(\epsilon)) \qquad \min \quad & Var(x_T + x_s - y_s|x_s) \\
\text{s.t.} \quad & E(x_T + x_s - y_s|x_s) = \epsilon, \\
& x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = s+1, \ldots, T-1, \\
& x_{s+1} = e_s^0 y_s + \mathbf{P}_s' \mathbf{u}_s, \\
& y_s \le x_s.
\end{aligned}
$$

Definition 4.2. *For a given wealth level x_s at period s, if an efficient mean-variance pair for the truncated $(T-s)$-period problem, (MV_{s-T}), is not pseudo efficient (type 1) and is, however, dominated by a total mean-variance pair of problem (RMV_{s-T}), i.e.,*

$$(E(x_T|x_s), -Var(x_T|x_s)) \prec (E(x_T + x_s - y_s|x_s), -Var(x_T + x_s - y_s|x_s)), \quad \text{with } y_s \le x_s, \tag{4.23}$$

then the given $(T-s)$-period mean-variance pair associated with wealth x_s is called **pseudo efficient (type 2)**.

Remark 4.4. *It is easy to see that any pseudo efficient (type 1) mean-variance pair must satisfy (4.23).*

Proposition 4.6. *A boundary $(T-s)$-period mean-variance pair*

$$E(x_T|x_s) = \mu_s x_s + 2\nu_s \bar{\Gamma}_{s-1}, \; Var(x_T|x_s) = \frac{a_s}{\nu_s^2}(E(x_T|x_s) - (\mu_s + b_s\nu_s)x_s)^2 + c_s x_s^2,$$

with risk attitude parameter $\bar{\Gamma}_{s-1}$ and wealth level x_s is dominated by another boundary total mean-variance pair of (RMV_{s-T}) with the same mean, if and only if

$$(\tau_s - \mu_s)x_s > (\mu_s - 1 + 2\nu_s)\bar{\Gamma}_{s-1}.$$

Furthermore, if this boundary mean-variance pair is efficient but not pseudo efficient (type 1), it must be pseudo efficient (type 2).

Proof: Similar to the proof of Theorem 4.2, we solve $(RMV_{s-T}(\epsilon))$ with $\epsilon = \mu_s x_s + 2\nu_s \bar{\Gamma}_{s-1}$. $(RMV_{s-T}(\epsilon))$ is equivalent to

$$\min_{\{y_s \le x_s\}} F(y_s),$$

where $F(y_s)$ is the optimal objective value of following $(T-s)$-period self-financing mean-variance portfolio selection problem with initial wealth y_s,

$$\begin{array}{lll} (MV_{s-T}(\epsilon - (x_s - y_s))) & \min & Var(x_T|y_s) \\ & \text{s.t.} & E(x_T|y_s) = \epsilon - (x_s - y_s), \\ & & x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = s+1, \ldots, T-1, \\ & & x_{s+1} = e_s^0 y_s + \mathbf{P}_s' \mathbf{u}_s. \end{array}$$

The mean-variance pair, $(E(x_T|y_s), Var(x_T|y_s))$, must lie on the minimum variance set. Thus,

$$\begin{aligned} F(y_s) &= Var(x_T|y_s) \\ &= \frac{a_s}{\nu_s^2}(E(x_T|y_s) - (\mu_s + b_s\nu_s)y_s)^2 + c_s y_s^2 \\ &= \frac{2\nu_s(\tau_s - 1) + (\mu_s - 1)^2}{2\nu_s} y_s^2 - \frac{(\mu_s - 1 + 2\nu_s)[(\mu_s - 1)x_s + 2\nu_s\bar{\Gamma}_{s-1}]}{\nu_s} y_s + C_s, \end{aligned}$$

where C_s is a constant not depending on y_s. The best funding level invested in the risky asset at the beginning of period s, $\check{x}_s$, is given by

$$\check{x}_s = \arg\min_{\{\bar{y}_s \le x_s\}} \{F(y_s)\}.$$

Note that $\check{x}_s < x_s$ holds if and only if $(\tau_s - \mu_s)x_s > (\mu_s - 1 + 2\nu_s)\bar{\Gamma}_{s-1}$. □

We now propose a general T-period revised portfolio policy, $\check{\mathbf{u}}_k^*(\check{x}_k)$, $k = 0, \ldots, T-1$, for the T-period problem (MV).

Revised Policy 4.4.

$$\check{\mathbf{u}}_{\mathbf{k}}^*(\check{x}_k) = -E^{-1}(\mathbf{P_k P_k'})E(e_k^0 \mathbf{P_k})\check{x}_k + \bar{\Gamma}_k \left(\frac{\mu_{k+1}}{\tau_{k+1}}\right) E^{-1}(\mathbf{P_k P_k'})E(\mathbf{P_k}); \tag{4.24}$$

$$\check{x}_k = \begin{cases} \tilde{x}_k, & \text{if } (\tau_k - \mu_k)\tilde{x}_k \le (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}, \\ \dfrac{(\mu_k - 1 + 2\nu_k)[(\mu_k - 1)\tilde{x}_k + 2\nu_k \bar{\Gamma}_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if } (\tau_k - \mu_k)\tilde{x}_k > (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}, \end{cases} \tag{4.25}$$

$$\tilde{x}_0 = x_0$$

$$\tilde{x}_{k+1} = e_k^0 \check{x}_k + \mathbf{P}_k' \check{\mathbf{u}}_{\mathbf{k}}^*(\check{x}_k), \tag{4.26}$$

$$\bar{\Gamma}_k = \begin{cases} \bar{\Gamma}_{k-1}, & \text{if } (\tau_k - \mu_k)\tilde{x}_k \le (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}, \\ \dfrac{(\tau_k - \mu_k)[(\mu_k - 1)\tilde{x}_k + 2\nu_k \bar{\Gamma}_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if } (\tau_k - \mu_k)\tilde{x}_k > (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}, \end{cases} \tag{4.27}$$

$$\bar{\Gamma}_{-1} = \frac{1}{2}\left(b_0 x_0 - \frac{\lambda_0 \nu_0}{2a_0}\right),$$

Note that risk attitude parameter $\bar{\Gamma}_k$ is path-dependent. One major feature of this type of revised policy is that, when the condition $(\tau_k - \mu_k)\tilde{x}_k > (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}$ holds, we place a positive free cash flow into the pocket,

$$\Delta\tilde{x}_k = \tilde{x}_k - \check{x}_k = \frac{2\nu_k(\tau_k - \mu_k)\tilde{x}_k + 2\nu_k(\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2},$$

and apply the revised mean-variance policy for the remaining amount in the market, $\check{x}_k$.

Theorem 4.3. *Revised policy 4.4 achieves the same total mean as the pre-committed optimal mean-variance policy of the T-period problem (MV) does, while having smaller total variance than the pre-committed optimal policy does, i.e.,*

$$E(\tilde{x}_T + \sum_{j=0}^{T-1} \Delta\tilde{x}_j | x_0)|_{\check{\mathbf{u}}^*} = E(x_T|x_0)|_{\mathbf{u}^*},$$

$$Var(\tilde{x}_T + \sum_{j=0}^{T-1} \Delta\tilde{x}_j | x_0)|_{\check{\mathbf{u}}^*} < Var(x_T|x_0)|_{\mathbf{u}^*}.$$

Proof: The case with $T = 1$ has been already proved in Proposition 4.1. We assume that the theorem is true for $T = k$ with $k \geq 1$. We now proceed to prove that the theorem is also true for $T = k + 1$.

As the following is true from Proposition 4.1,

$$E(x_{k+1} + \Delta x_0 | x_0)|_{\tilde{\mathbf{u}}^*} = E(x_{k+1}|x_0)|_{\mathbf{u}^*},$$

$$Var(x_{k+1} + \Delta x_0 | x_0)|_{\tilde{\mathbf{u}}^*} < Var(x_{k+1}|x_0)|_{\mathbf{u}^*},$$

then at time 1, with the wealth level $\tilde{x}_1$, the policy $\tilde{\mathbf{u}}_t^*, t = 1, \cdots, k$, yields

$$(E(x_{k+1}|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*}, Var(x_{k+1}|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*}),$$

which is the k-period boundary mean-variance pair with initial wealth $\tilde{x}_1$ and trade-off $\lambda_{1-} = -\dfrac{2a_1}{\nu_1}(2\bar{\Gamma}_0 - b_1\tilde{x}_1)$.

On the other hand, $(E(x_{k+1}|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*}, Var(x_{k+1}|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*})$ is dominated by a k-period second revised policy with initial wealth $\tilde{x}_0 = \tilde{x}_1$ and parameter $\bar{\Gamma}_{-1} = \bar{\Gamma}_0 = \dfrac{1}{2}\left(b_1 x_1 - \dfrac{\nu_1\lambda_{1-}}{2a_1}\right)$ from the assumption of the induction, i.e.,

$$E(\tilde{x}_{k+1} + \sum_{j=1}^{k} \Delta\tilde{x}_j | \tilde{x}_1)|_{\tilde{\mathbf{u}}^*} = E(x_{k+1}|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*},$$

$$Var(\tilde{x}_{k+1} + \sum_{j=1}^{k} \Delta\tilde{x}_j | \tilde{x}_1)|_{\tilde{\mathbf{u}}^*} < Var(x_{k+1}|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*}.$$

The k-period revised policy and the time-0 policy are then combined to form our $(k + 1)$-period revised policy. The conclusion follows as evidenced from the following two relationships,

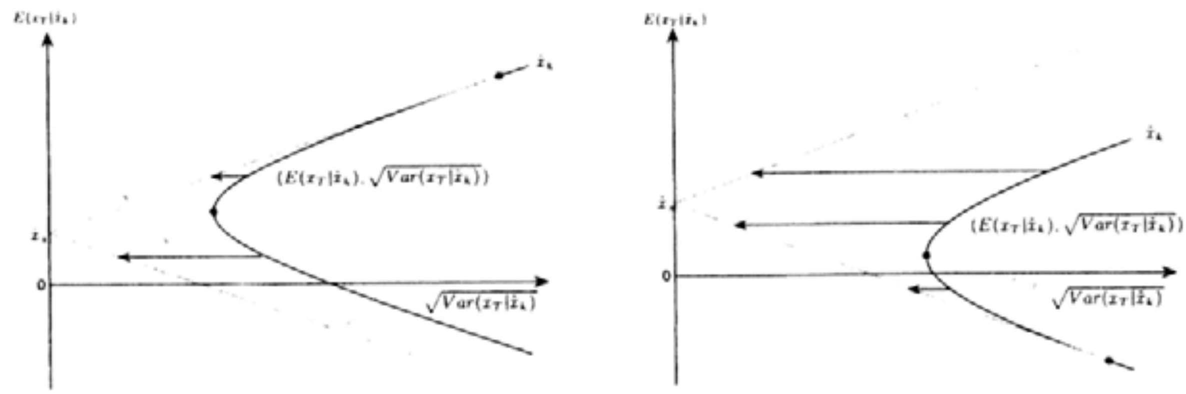
$$\begin{aligned}
& E(x_{k+1}|x_0)|_{\mathbf{u}^*} \\
&= E(x_{k+1} + \Delta x_0|x_0)|_{\tilde{\mathbf{u}}^*} \\
&= \int E(x_{k+1}|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*} f(\tilde{x}_1) d\tilde{x}_1 + \Delta x_0 \\
&= \int E(\tilde{x}_{k+1} + \sum_{j=1}^{k} \Delta\tilde{x}_j|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*} f(\tilde{x}_1) d\tilde{x}_1 + \Delta\tilde{x}_0 \\
&= E(\tilde{x}_{k+1} + \sum_{j=0}^{k} \Delta\tilde{x}_j|\tilde{x}_0)|_{\tilde{\mathbf{u}}^*},
\end{aligned}$$

$$
\begin{aligned}
& Var(x_{k+1}|x_0)|_{\mathbf{u}^*} \\
>& Var(x_{k+1} + \Delta x_0|x_0)|_{\tilde{\mathbf{u}}^*} \\
=& Var(E(x_{k+1} + \Delta x_0|\tilde{x}_1)|x_0)|_{\tilde{\mathbf{u}}^*} + E(Var(x_{k+1} + \Delta x_0|\tilde{x}_1)|x_0)|_{\tilde{\mathbf{u}}^*} \\
=& \int (E(x_{k+1} + \Delta x_0|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*} - E(x_{k+1} + \Delta x_0|x_0)|_{\tilde{\mathbf{u}}^*})^2 f(\tilde{x}_1) d\tilde{x}_1 \\
& + \int Var(x_{k+1} + \Delta x_0|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*} f(\tilde{x}_1) d\tilde{x}_1 \\
>& \int (E(\tilde{x}_{k+1} + \sum_{j=0}^{k} \Delta\tilde{x}_j|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*} - E(x_{k+1} + \Delta x_0|x_0)|_{\tilde{\mathbf{u}}^*})^2 f(\tilde{x}_1) d\tilde{x}_1 \\
& + \int Var(\tilde{x}_{k+1} + \sum_{j=0}^{k} \Delta\tilde{x}_j|\tilde{x}_1)|_{\tilde{\mathbf{u}}^*} f(\tilde{x}_1) d\tilde{x}_1 \\
=& Var(\tilde{x}_{k+1} + \sum_{j=0}^{k} \Delta\tilde{x}_j|\tilde{x}_0)|_{\tilde{\mathbf{u}}^*},
\end{aligned}
$$

where $f(\tilde{x}_1)$ is the probability density function of random variable $\tilde{x}_1$, which depends on the initial wealth $\tilde{x}_0$ and period-0 policy $\tilde{\mathbf{u}}_0^*(\tilde{x}_0)$. □

The rational of Revised policy 4.4 can be better explained by Figure 4.4. If the conditional boundary mean-variance pair attained by the truncated pre-committed optimal policy at time k is dominated by the boundary total mean-variance pair with the same mean, we place a positive free cash flow into the pocket and apply the boundary policy of (RMV_{k-T}). In Revised policy 4.4, we actually introduce a free cash flow account into the investment choice and consider the wealth in risky assets and free cash flow account as a whole. Different from previous revised policies, we do not relax self-financing assumption, while increasing the investment flexibility.

Example 4.3. We consider Example 1 in Li and Ng (2000) [35] again, in which there exist three risky assets, A, B, and C with their expected return vector given by $E(\mathbf{e_t})=(E(e_t^A),\ E(e_t^B),\ E(e_t^C))' = (1.162, 1.246, 1.228)'$, $t = 0, 1, 2, 3$, and their



(a) $\mu_k > 1 - 2\nu_k$ (b) $1 - 2\nu_k > \mu_k > 0$

Figure 4.4: Illustration of Revised policy 4.4

covariance given by

$$Cov(\mathbf{e_t}) = \begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix}, \quad t = 0, 1, 2, 3.$$

Taking asset A as the reference asset gives rise

$$E(\mathbf{P_t}) = E(e_t^B - e_t^A, e_t^C - e_t^A)' = (0.084, 0.066)', \ t = 0, 1, 2, 3,$$

$$E(\mathbf{P_t}\mathbf{P_t'}) = E\begin{pmatrix} (e_t^B)^2 - 2e_t^Ae_t^B + (e_t^A)^2 & e_t^Be_t^C - e_t^Ae_t^C - e_t^Ae_t^B + (e_t^A)^2 \\ e_t^Be_t^C - e_t^Ae_t^C - e_t^Ae_t^B + (e_t^A)^2 & (e_t^C)^2 - 2e_t^Ae_t^C + (e_t^A)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0697 & -0.0027 \\ -0.0027 & 0.0189 \end{pmatrix}, \quad t = 0, 1, 2, 3,$$

$$E(e_t^A\mathbf{P_t'}) = \left(E\left(e_t^Ae_t^B\right) - E\left((e_t^A)^2\right), E\left(e_t^Ae_t^C\right) - E\left((e_t^A)^2\right)\right)$$

$$= (0.1017, 0.0766), \quad t = 0, 1, 2, 3.$$

Furthermore, we have $B_t = 0.3566$, $A_t^1 = 0.7424$, $A_t^2 = 0.8711$, $t = 0, 1, 2, 3$. We assume that an investor with initial wealth $x_0 = 1$ would like to minimize the mean-variance objective of $Var(x_4) - 2E(x_4)$.

The pre-committed optimal mean-variance policy is given in Li and

Ng (2000) [35] as:

$$\mathbf{u_t^*}(x_t) = -\mathbf{K_t}x_t + \mathbf{v_t}, \quad t = 0, 1, 2, 3,$$

where

$$\mathbf{K_t} = \begin{pmatrix} 1.6238 \\ 4.2907 \end{pmatrix}, \qquad t = 0, 1, 2, 3, \qquad \mathbf{v_0} = \begin{pmatrix} 5.8919 \\ 16.1444 \end{pmatrix},$$
$$\mathbf{v_1} = \begin{pmatrix} 6.9128 \\ 18.9418 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 8.1106 \\ 22.2240 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 9.5160 \\ 26.0749 \end{pmatrix}.$$

Revised policy 4.2 can be derived as follows,

$$\hat{\mathbf{u}}_\mathbf{t}^*(\hat{x}_t) = -\mathbf{K_t}\hat{x}_t + \left(\frac{A^1}{A^2}\right)^{3-t} \Gamma_t \mathbf{L_t}, \quad t = 0, 1, 2, 3,$$

where

$$\mathbf{K_t} = \begin{pmatrix} 1.6238 \\ 4.2907 \end{pmatrix}, \quad \mathbf{L_t} = \begin{pmatrix} 1.3466 \\ 3.6899 \end{pmatrix}, \quad t = 0, 1, 2, 3,$$

and $\hat{x}_t$ and Γ_t follow (4.17) - (4.20) with $\bar{x}_0 = x_0 = 1$ and $\Gamma_{-1} = \frac{1}{2}\left(b_0 x_0 - \frac{\nu_0 \lambda}{2a_0}\right) = 7.0666$.

Revised policy 4.4 can be obtained as follows,

$$\check{\mathbf{u}}_\mathbf{t}^*(\check{x}_t) = -\mathbf{K_t}\check{x}_t + \left(\frac{A^1}{A^2}\right)^{3-t} \bar{\Gamma}_t \mathbf{L_t}, \quad t = 0, 1, 2, 3,$$

where $\mathbf{K_t}$ and $\mathbf{L_t}$ are the same as in Revised Policy 4.2, and $\check{x}_t$ and $\bar{\Gamma}_t$ follow (4.25) - (4.27) with $\check{x}_0 = x_0 = 1$ and $\bar{\Gamma}_{-1} = \frac{1}{2}\left(b_0 x_0 - \frac{\nu_0 \lambda}{2a_0}\right) = 7.0666$.

While the expected terminal wealth and the risk level of the pre-committed policy are $E(x_4) = 6.0666$ and $Var(x_4) = 4.4954$, respectively, the expected terminal wealth and the risk level of Revised Policy 4.2 are the same with $E(\bar{x}_4) = 6.0666$ and $Var(\bar{x}_4) = 4.4954$. For Revised Policy 4.4, we only know that its expected total terminal wealth and the risk level satisfy $E(\check{x}_4 + \sum_{j=0}^{3} \Delta\check{x}_j) = 6.0666$ and $Var(\check{x}_4 + \sum_{j=0}^{3} \Delta\check{x}_j) < 4.4954$.

We assume now that the random returns of risky assets, $\mathbf{e_t}$ $(t = 0, 1, 2, 3)$, are normal vectors with the given mean and covariance matrix in the example.

We run 5000, 20000, and 50000 sample pathes, respectively, in our simulation experiments to verify the above theoretical value of the mean-variance pair of the terminal wealth under the pre-committed optimal policy and the two revised policies, i.e., x_4, $\bar{x}_4$ and $\tilde{x}_4+\sum_{j=0}^{3}\Delta\tilde{x}_j$ and to estimate the probability of receiving FCFS and the unconditional expected value of FCFS for Revised Policy 4.2, and the probability of placing free cash flow in the pocket for Revised Policy 4.4, with the results given in the following Table 4.1.

Table 4.1: The simulation results

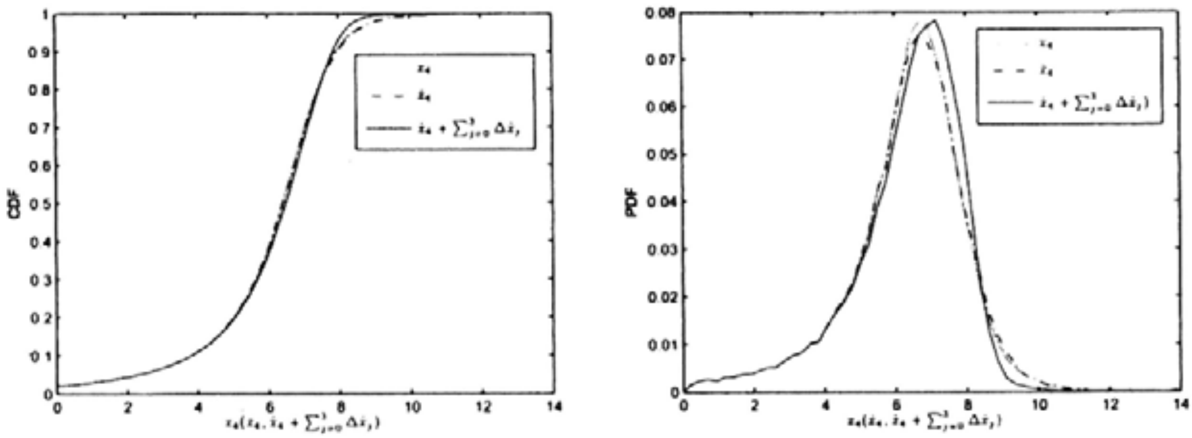
Samples	$E(x_4)$	$Var(x_4)$	$E(\bar{x}_4)$	$Var(\bar{x}_4)$	$E(\tilde{x}_4+\sum_{j=0}^{3}\Delta\tilde{x}_j)$
5000	6.0867	4.6073	6.0876	4.5846	6.0903
20000	6.0718	4.4307	6.0749	4.4494	6.0757
50000	6.0599	4.4657	6.0624	4.4669	6.0631

Simulation results of Example 4.2 (Cont'd)

Samples	$Var(\tilde{x}_4+\sum_{j=0}^{3}\Delta\tilde{x}_j)$	$Pr(FCFS)$	$E(FCFS)$	$Pr(\Delta\tilde{x}_j>0)$
5000	4.3443	0.3176	0.2506	0.5070
20000	4.2157	0.3219	0.2479	0.5178
50000	4.2403	0.3172	0.2442	0.5097

Using the simulation data, we can also estimate the distributions of the terminal wealth under the pre-committed optimal policy and the two revised policies, respectively. From Figure 4.5, where the distributions of x_4, $\bar{x}_4$ and $\tilde{x}_4+\sum_{j=0}^{3}\Delta\tilde{x}_j$ are approximated from 50000 samples, we find that the distribution of the terminal wealth under Revised Policy 4.2, $\bar{x}_4$, has somewhat fatter tails in both directions, while the distribution of terminal wealth under Revised Policy 4.4 has thinner tails.

Furthermore, we represent three efficient frontiers under the pre-committed optimal policy, Revised Policy 4.2 and Revised Policy 4.4 together in Figure 4.6. For most initial trade-offs, Revised Policy 4.2 gives the best efficient frontier,



(a) Cumulative distribution functions (b) Probability density functions

Figure 4.5: The distributions of x_4, $\bar{x}_4$ and $\tilde{x}_4 + \sum_{j=0}^{3} \Delta\tilde{x}_j$

since we consider to add the expected value of FCFS into the return, but not to consider to combine the variance of FCFS with the risk of the terminal wealth.

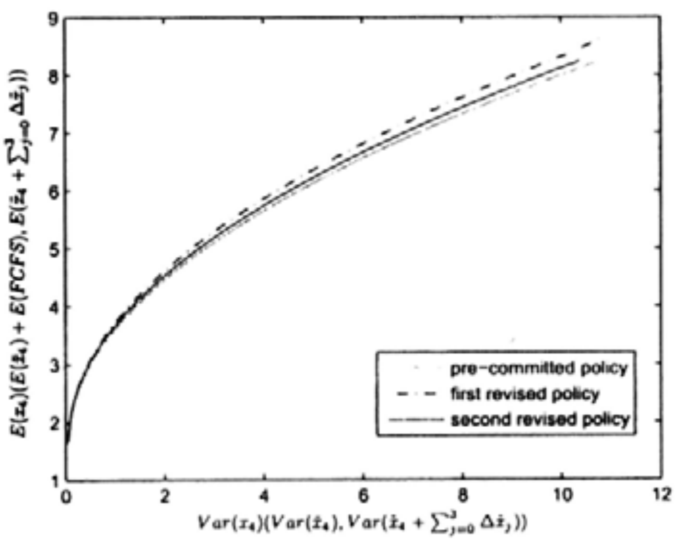


Figure 4.6: Three efficient frontiers

4.5. Continuous Time Mean-Variance Portfolio Selection in Market with ALL Risky Assets

Although Zhou and Li (2000) [75] derive the analytical solution for the mean-variance portfolio selection problem in a continuous-time market with one riskless asset and multiple risky assets, to our knowledge, there is no parallel result in the literature derived for the mean-variance formulation of a continuous-time market with all risky assets. We will fill in this gap first in this section and then apply the same concepts which we have developed in the previous sections of this chapter to the continuous-time market with all risky assets.

We assume that there are $(n+1)$ riksy assets in a continuous-time market, and their price processes, $\{S_0(t)\}$, $\{S_1(t)\}$, $\cdots$, $\{S_n(t)\}$, satisfy the following SDEs,

$$\begin{cases} dS_i(t) &= b_i(t)S_i(t)dt + \sum_{j=1}^{m+1} \sigma_{ij}(t)S_i(t)dW_j, \\ S_i(0) &= s_i, \end{cases} \quad i = 0, 1, \ldots, n$$

where W_j, $j = 1, \ldots, m+1$, $m \geq n$, are $m+1$ independent Brown motions in the filtrated probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and all $b_i(t), \sigma_{ij}(t)$ are deterministic functions.

Assumption 4.1. *(nondegeneracy condition)*

$$\bar{\sigma}(t)\bar{\sigma}(t)' \geq \delta I, \quad \textit{for some } \delta > 0,$$

$$\textit{where } \bar{\sigma}(t) = \begin{pmatrix} \sigma_{01}(t) & \cdots & \sigma_{0(m+1)}(t) \\ \cdots & \cdots & \cdots \\ \sigma_{n1}(t) & \cdots & \sigma_{n(m+1)}(t) \end{pmatrix}_{(n+1)\times(m+1)}.$$

Remark 4.5. *The nondegeneracy condition ensures that*

$$\sigma(t)\sigma(t)' \succ 0, \tag{4.28}$$

$$\mathbf{C}(t)'\mathbf{C}(t) - \mathbf{C}(t)'\sigma(t)' \left[\sigma(t)\sigma(t)'\right]^{-1} \sigma(t)\mathbf{C}(t) > 0, \tag{4.29}$$

hold true, where

$$\mathbf{C}(t) = (\sigma_{01}(t), \cdots, \sigma_{0(m+1)}(t))',$$

$$\sigma(t) = \begin{pmatrix} \sigma_{11}(t) - \sigma_{01}(t) & \cdots & \sigma_{1(m+1)}(t) - \sigma_{0(m+1)}(t) \\ \cdots & \cdots & \cdots \\ \sigma_{n1}(t) - \sigma_{01}(t) & \cdots & \sigma_{n(m+1)}(t) - \sigma_{0(m+1)}(t) \end{pmatrix}_{n\times(m+1)}.$$

Let $\mathbf{u}(t) = (u^1(t), \cdots, u^n(t))$ be the dollar amount which an investor invests in the risky assets, $S_1, \cdots, S_n$, at time t. The wealth of the investor, $x(t)$, is then governed by the following stochastic differential equation,

$$\begin{cases} dx(t) = [b_0(t)x(t) + \mathbf{B}(t)'\mathbf{u}(t)]\,dt + \sum_{j=1}^{m+1} [\sigma_{0j}(t)x(t) + \mathbf{D_j}(t)'\mathbf{u}(t)]\,dW_j(t), \\ x(0) = x_0 > 0, \end{cases} \tag{4.30}$$

where

$$\mathbf{B}(t) = (b_1(t) - b_0(t), \cdots, b_n(t) - b_0(t))',$$

$$\mathbf{D_j}(t) = (\sigma_{1j}(t) - \sigma_{0j}(t), \cdots, \sigma_{nj}(t) - \sigma_{0j}(t))'.$$

Similar to the discrete-time situation, the investor seeks a best investment strategy, $\mathbf{u}^*(t)$, which is an adapted random process, to attain the optimality of the following continuous-time mean-variance model:

$$\begin{array}{lll} (MV_C) & \min & Var(x(T)) + \lambda E(x(T)) \\ & \text{s.t.} & \begin{cases} \mathbf{u}(.) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n), \\ (x(.), \mathbf{u}(.)) \text{ satisfies (4.30)}, \end{cases} \end{array} \tag{4.31}$$

where the trade-off between two objectives, λ, is assumed to be nonpositive.

Theorem 4.4. *The optimal policy and the optimal mean variance pair of* (MV_C) *are given by*

$$\mathbf{u}^*(t, x(t)) = -\mathbf{\Phi}(t)x(t) + \frac{1}{2}\left(b_0x_0 - \frac{\nu_0\lambda}{2a_0}\right)\frac{\mu_t}{\tau_t}\mathbf{\Psi}(t), \tag{4.32}$$

$$E(x(T)) = (\mu_0 + b_0\nu_0)x_0 - \frac{\nu_0^2\lambda}{2a_0}, \tag{4.33}$$

$$Var(x(T)) = \frac{\lambda^2\nu_0^2}{4a_0} + c_0x_0^2, \tag{4.34}$$

where

$$\Phi(t) = \left[\sigma(t)\sigma(t)'\right]^{-1}\left[\mathbf{B}(t) + \sigma(t)\mathbf{C}(t)\right], \quad \Psi(t) = \left[\sigma(t)\sigma(t)'\right]^{-1}\mathbf{B}(t),$$
$$\mu_t = e^{\int_t^T \alpha_\mu(s)ds}, \quad \tau_t = e^{\int_t^T \alpha_\tau(s)ds}, \quad \nu_t = \frac{1}{2}\int_t^T \mathbf{B}(s)'\Psi(s)e^{-\int_s^T \alpha_\nu(w)dw}ds,$$
$$a_t = \frac{\nu_t}{2} - (v_t)^2, \quad b_t = \frac{\mu_t\nu_t}{a_t} = \frac{2\mu_t}{1-2\nu_t}, \quad c_t = \tau_t - (\mu_t)^2 - a_t(b_t)^2,$$

and

$$\alpha_\mu(t) = b_0(t) - \mathbf{B}(t)'\left[\sigma(t)\sigma(t)'\right]^{-1}\left[\mathbf{B}(t) + \sigma(t)\mathbf{C}(t)\right],$$
$$\alpha_\tau(t) = 2b_0(t) + \mathbf{C}(t)'\mathbf{C}(t) - \left[\mathbf{B}(t)' + \mathbf{C}(t)'\sigma(t)'\right]\left[\sigma(t)\sigma(t)'\right]^{-1}\left[\mathbf{B}(t) + \sigma(t)\mathbf{C}(t)\right],$$
$$\alpha_\nu(t) = \alpha_\tau(t) - 2\alpha_\mu(t) > \mathbf{B}(t)'\Psi(t).$$

Furthermore, the efficient frontier can be expressed by

$$Var(x(T)) = \frac{a_0}{\nu_0^2}(E(x(T)) - (\mu_0 + b_0\nu_0)x_0)^2 + c_0x_0^2, \text{ for } E(x(T)) \geq (\mu_0 + b_0\nu_0)x_0. \tag{4.35}$$

Proof: Using the same embedding scheme in Li and Ng (2000) [35] and Zhou and Li (2000) [75], we consider the following auxiliary problem $A(\mu, \lambda)$,

$$A(\mu,\lambda) \quad \min \quad E\left[\mu x(T)^2 + \lambda x(T)\right]$$
$$\text{s.t.} \quad \begin{cases} \mathbf{u}(.) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n), \\ (x(.), \mathbf{u}(.)) \text{ satisfies (4.30)}. \end{cases}$$

By defining the following variables,

$$y(t) = x(t) - \Gamma, \quad \text{and} \quad \Gamma = -\lambda/2\mu,$$

we reformulate problem $A(\mu, \lambda)$ as the following equivalent stochastic LQ form,

$$\min \quad E\left[\frac{1}{2}\mu y(T)^2\right]$$
$$\text{s.t.} \quad \begin{cases} dy(t) = & [b_0(t)y(t) + \mathbf{B}(t)'\mathbf{u}(t) + b_0(t)\Gamma]dt \\ & + \sum_{j=1}^{m+1}[\sigma_{0j}(t)y(t) + \mathbf{D_j}(t)'\mathbf{u}(t) + \sigma_{0j}\Gamma]dW_j(t) \\ y(0) = & x_0 - \Gamma. \end{cases}$$

Based on Theorem 7.10 in Chapter 6 of Yong and Zhou (1999) [73], the optimal investment policy of $A(\mu, \lambda)$ is

$$\bar{\mathbf{u}}(t, y(t)) = -\mathbf{\Phi}(t)y(t) - \mathbf{\Psi}(t)\frac{\varphi(t)}{P(t)} - (\mathbf{\Phi}(t) - \mathbf{\Psi}(t))\Gamma, \tag{4.36}$$

where $P(t)$ and $\varphi(t)$ satisfy

$$\begin{cases} \dot{P}(t) + \alpha_\tau(t)P(t) = 0, \\ P(T) = \mu, \\ [\sigma(t)\sigma(t)']\,P(t) > 0, \end{cases}$$

and

$$\begin{cases} \dot{\varphi}(t) + \alpha_\mu(t)\varphi(t) + (\alpha_\tau(t) - \alpha_\mu(t))\Gamma P(t) = 0, \\ \varphi(T) = 0, \end{cases}$$

respectively. Setting $h(t) \triangleq \dfrac{\varphi(t)}{P(t)}$ gives rise

$$\dot{h}(t) = \frac{P(t)\dot{\varphi}(t) - \dot{P}(t)\varphi(t)}{P(t)^2} = (\alpha_\tau(t) - \alpha_\mu(t))h(t) - (\alpha_\tau(t) - \alpha_\mu(t))\Gamma.$$

Since $h(T) = 0$, we solve $h(t)$ as

$$h(t) = \frac{\varphi(t)}{P(t)} = \Gamma\left[1 - e^{-\int_t^T(\alpha_\tau(s) - \alpha_\mu(s))ds}\right].$$

The optimal investment policy can be now simplified to

$$\bar{\mathbf{u}}(t, x(t)) = -\mathbf{\Phi}(t)x(t) + \Gamma\frac{\mu_t}{\tau_t}\mathbf{\Psi}(t).$$

Under this optimal investment policy, the wealth dynamic in (4.30) evolves as follows accordingly,

$$\begin{cases} dx(t) = & \left\{[b_0(t) - \mathbf{B}(t)'\mathbf{\Phi}(t)]\,x(t) + \mathbf{B}(t)'\mathbf{\Psi}(t)\Gamma\frac{\mu_t}{\tau_t}\right\}dt \\ & + \sum_{j=1}^{m+1}\left\{[\sigma_{0j}(t) - \mathbf{D}_j(t)'\mathbf{\Phi}(t)]\,x(t) + \mathbf{D}_j(t)'\mathbf{\Psi}(t)\Gamma\frac{\mu_t}{\tau_t}\right\}dW_j(t), \\ x(0) = & x_0 > 0. \end{cases} \tag{4.37}$$

We can further find out that $E(x(t))$ and $E(x(t)^2)$ satisfy the following ordinary differential equations, respectively,

$$\begin{cases} dE(x(t)) = & \left[\alpha_\mu(t)E(x(t)) + \mathbf{B}(t)'\Psi(t)\Gamma e^{-\int_t^T(\alpha_\tau(s)-\alpha_\mu(s))ds}\right]dt \\ E(x(0)) = & x_0, \end{cases} \tag{4.38}$$

and

$$\begin{cases} dE(x(t)^2) = & \left[\alpha_\tau(t)E(x(t)^2) + \mathbf{B}(t)'\Psi(t)\Gamma^2 e^{-2\int_t^T(\alpha_\tau(s)-\alpha_\mu(s))ds}\right]dt \\ E(x(0)^2) = & x_0^2. \end{cases} \tag{4.39}$$

Solving (4.38) and (4.39) yields

$$\begin{aligned} E(x(T)) =& x_0\mu_0 + 2\nu_0\Gamma, \\ E(x(T)^2) =& x_0^2\tau_0 + 2\nu_0\Gamma^2. \end{aligned}$$

From Zhou and Li (2000) [75] and Theorem 8.2 in Chapter 6 of Yong and Zhou (1999) [73], the optimal policy of (MV_C) can be found by selecting $\bar{\Gamma}$ such that

$$\bar{\Gamma} = \frac{-\lambda}{2} + E(x(T)) = \frac{-\lambda}{2} + x_0\mu_0 + 2\nu_0\bar{\Gamma},$$

which yields

$$\bar{\Gamma} = \frac{\mu_0}{1-2\nu_0}x_0 - \frac{\lambda}{2(1-2\nu_0)} = \frac{1}{2}\left(b_0x_0 - \frac{\nu_0\lambda}{2a_0}\right).$$

Equations (4.32) and (4.33) then follow and

$$Var(x(T)) = E(x(T)^2) - (Ex(T))^2 = \frac{\lambda^2\nu_0^2}{4a_0} + c_0x_0^2$$

holds. Furthermore, the efficient frontier in (4.35) can be obtained.

Besides, noting that the nondegeneracy condition in (4.29), we have

$$\alpha_\nu(t) = \mathbf{C}(t)'\mathbf{C}(t) - \mathbf{C}(t)'\sigma(t)'\left[\sigma(t)\sigma(t)'\right]^{-1}\sigma(t)\mathbf{C}(t) + \mathbf{B}(t)'\Psi(t) > \mathbf{B}(t)'\Psi(t),$$

which implies $\nu_t < \frac{1}{2}$ for all $0 \le t \le T$. □

Remark 4.6. *The structure of the optimal policy and the efficient frontier of the continuous-time setting with all risky assets are the same as its counterpart in the discrete-time setting. When applying the control at discrete time instants to the continuous-time market, the parameters of the discrete-time formulation will converge to the corresponding parameters in the continuous-time setting as the time interval of discrete-time control goes to zero.*

Lemma 4.4.

$$\mu_t > 0, \;\; \textit{and,} \;\; (1-2\nu_t)\tau_t > \mu_t^2.$$

Proof:

$$\begin{aligned}(1-2\nu_t)\tau_t =& \left(1 - \int_t^T \mathbf{B}(s)'\mathbf{\Psi}(s)e^{-\int_s^T \alpha_\nu(w)dw}ds\right)e^{\int_t^T \alpha_\tau(s)ds} \\ >& \left(1 - \int_t^T \alpha_\nu(s)e^{-\int_s^T \alpha_\nu(w)dw}ds\right)e^{\int_t^T \alpha_\tau(s)ds} \\ =& \mu_0^2.\end{aligned}$$

□

We now discuss two similar revised policies for the continuous-time mean-variance portfolio selection problem, (MV_C). Let $0 = t_0 < t_1 < \cdots < t_N = T$ be $N-1$ time points between 0 and T.

The first revised policy, $\hat{\mathbf{u}}^*(t, \hat{x}(t))$, applied at time t for $t_k \leq t < t_{k+1}$, $k = 0$,

$1, \cdots, N-1$, is carried out according to the following recursions:

$$\hat{\mathbf{u}}^*(t, \hat{x}(t)) = -\mathbf{\Phi}(t)\hat{x}(t) + \Gamma_k \frac{\mu_t}{\tau_t}\mathbf{\Psi}(t); \tag{4.40}$$

$$\hat{x}(t) = \begin{cases} \bar{x}(t_k), & \text{if } \bar{x}(t_k) \le \bar{x}^*(t_k), \\ -\bar{x}(t_k) + \dfrac{2\mu_k(\mu_k \bar{x}(t_k) + 2\nu_k \Gamma_{k-1})}{2\nu_k \tau_k + \mu_k^2}, & \text{if } \bar{x}(t_k) > \bar{x}^*(t_k), \\ \bar{x}(t), & \text{if } t_k < t < t_{k+1}, \end{cases} \tag{4.41}$$

$$\bar{x}_0 = x_0$$

$$(\bar{x}(t), \hat{\mathbf{u}}^*(t, \hat{x}(t))) \text{ satisfies (4.30) with inital wealth } \hat{x}(t_k), \tag{4.42}$$

$$\Gamma_k = \begin{cases} \Gamma_{k-1}, & \text{if } \bar{x}(t_k) \le \bar{x}^*(t_k), \\ \Gamma_{k-1} + \dfrac{2\mu_k \tau_k(\bar{x}(t_k) - \bar{x}^*(t_k))}{2\nu_k \tau_k + \mu_k^2}, & \text{if } \bar{x}(t_k) > \bar{x}^*(t_k), \end{cases} \tag{4.43}$$

$$\Gamma_{-1} = \frac{1}{2}\left(b_0 x_0 - \frac{\lambda_0 \nu_0}{2a_0}\right)$$

$$\bar{x}^*(t_k) = \frac{\Gamma_{k-1}\mu_k}{\tau_k}. \tag{4.44}$$

When the wealth level $\bar{x}(t_k) > \bar{x}^*(t_k)$, we withdraw a positive free cash flow,

$$\bar{x}(t_k) - \hat{x}(t_k) = \frac{4\nu_k \tau_k}{2\nu_k \tau_k + \mu_k^2}(\bar{x}(t_k) - \bar{x}^*(t_k)),$$

out of the market.

The second revised policy, $\check{\mathbf{u}}^*(t, \check{x}(t))$, applied at time t for $t_k \le t < t_{k+1}$, k

=0, 1, $\cdots$, $N-1$, is carried out according to the following recursions:

$$\check{\mathbf{u}}^*(t, \check{x}(t)) = -\mathbf{\Phi}(t)\check{x}(t) + \bar{\Gamma}_k \frac{\mu_t}{\tau_t}\mathbf{\Psi}(t); \tag{4.45}$$

$$\check{x}(t) = \begin{cases} \tilde{x}(t_k), & \text{if} \quad \begin{array}{l}(\tau_k - \mu_k)\tilde{x}(t_k) \\ \leq (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1},\end{array} \\ \dfrac{(\mu_k - 1 + 2\nu_k)[(\mu_k - 1)\tilde{x}(t_k) + 2\nu_k\bar{\Gamma}_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if} \quad \begin{array}{l}(\tau_k - \mu_k)\tilde{x}(t_k) \\ > (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1},\end{array} \\ \tilde{x}(t), & \text{if } t_k < t < t_{k+1}, \end{cases} \tag{4.46}$$

$$\tilde{x}_0 = x_0$$

$$(\tilde{x}(t), \check{\mathbf{u}}^*(t, \check{x}(t))) \text{ satisfies (4.30) with inital wealth } \check{x}(t_k), \tag{4.47}$$

$$\bar{\Gamma}_k = \begin{cases} \bar{\Gamma}_{k-1}, & \text{if} \quad \begin{array}{l}(\tau_k - \mu_k)\tilde{x}(t_k) \\ \leq (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1},\end{array} \\ \dfrac{(\tau_k - \mu_k)[(\mu_k - 1)\tilde{x}(t_k) + 2\nu_k\bar{\Gamma}_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if} \quad \begin{array}{l}(\tau_k - \mu_k)\tilde{x}(t_k) \\ > (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1},\end{array} \end{cases} \tag{4.48}$$

$$\bar{\Gamma}_{-1} = \frac{1}{2}\left(b_0 x_0 - \frac{\lambda_0 \nu_0}{2a_0}\right).$$

When the condition $(\tau_k - \mu_k)\tilde{x}(t_k) > (\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}$ holds, we put a positive free cash flow into the pocket,

$$\Delta\tilde{x}(t_k) = \tilde{x}(t_k) - \check{x}(t_k) = \frac{2\nu_k(\tau_k - \mu_k)\tilde{x}(t_k) + 2\nu_k(\mu_k - 1 + 2\nu_k)\bar{\Gamma}_{k-1}}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}.$$

The parameters, μ_k, ν_k and τ_k are defined in Theorem 4.4 for time t_k. The only difference between discrete time revised policies and continuous-time revised policies is that the derived continuous-time policy is continuous during each time period from t_k to t_{k+1}.

4.6. Conclusion

The dynamic mean-variance portfolio selection in market with all risky assets is not time consistent in efficiency, due to the inherent nonseparable nature of the involved variance term. The induced trade-off of pre-committed optimal mean-variance policy is time-varying and state-dependent. In some situations, the trade-off may change its sign, which implies that the investor changes his/her risk attitude towards the objectives, leading to irrational trading behaviors for the remaining investment periods.

By relaxing binding budget spending at the beginning of period s, the concept of pseudo efficiency (type 1 or type 2) in Chapter 2 has been extended to a dynamic setting in this chapter. Two revised policies have been proposed accordingly. The first revised policy eliminates possible phenomenon of type-1 pseudo efficiency and achieves the same mean-variance pair attained by the original pre-committed optimal mean-variance policy. Furthermore, it enables investors to receive a free cash flow stream. The second revised policy eliminates possible phenomenon of type-2 pseudo efficiency and achieves the same total mean and less total variance, when compared to the original pre-committed optimal mean-variance policy.

The continuous-time optimal mean-variance policy in a market with all risky assets has been derived in this chapter, which is a linear function of current wealth and the initial trade-off. Two revised policies, which are similar to their counterparts in the discrete-time setting have been proposed to the continuous time setting to attain better performance than the original pre-committed optimal mean-variance policy.

CHAPTER 5

Time Consistency in Efficiency and Variance-optimal Signed Martingale Measure

5.1. Introduction

For the constrained portfolio selection problems, Cvitanic and Karatzas (1992, 1993) [15, 16] discussed them using convex duality. Li, Zhou and Lim (2002) [37] investigated the continuous time mean-variance portfolio selection without shorting. Furthermore, Labbé and Heunis (2007) [30] extended the duality approach for mean-variance portfolio selection with general convex constraints and random market parameters in continuous time. Yi (2009) [71] derived the mean-variance portfolio selection without shorting in multi-period setting by extending the idea proposed in Pliska (1997) [51]. As the mean-variance portfolio selection problem is closely related to variance-optimal hedging problem, Schweizer (1996) [57] discussed the connection between variance-optimal hedging problem and variance-optimal signed martingale measure. Recently Černý and Kallsen (2007, 2009) [10, 11] sstudied the optimal mean-variance portfolio selection and variance-optimal hedging in a more general setting with a semi-martingale price process, which includes both discrete-time and continuous-time settings as its special cases.

Discussion on no-arbitrage opportunity in a frictionless market has been widely studied in the literature, for example, by Harrison and Kreps (1979) [21], Harrison and Pliska (1981) [22], Kreps (1981) [29], Dalang, Merton and Willinger (1990) [17], Kabanov and Kramkov (1994) [27].The famous fundamental theorem of asset pricing (FTAP) states that *There is no arbitrage opportunity if and only if there exists an equivalent probability measure that turns the discounted stock price process into a martingale.* Jouini and Kallal (1995) [26], Schürger (1996) [56] and Carassus, Pham and Touzi (2001) [9] provided an extension of FATP to the case where trading strategies are subject to different constraints.

In this chapter, we will show first that the satisfaction of time consistency in efficiency of the general multi-period mean-variance formulation in a frictionless market is equivalent to the nonnegativeness of the conditional density process of the variance-optimal signed martingale measure of this market. If the market is complete, time consistency in efficiency holds. We then prove when the state space is finite, the optimal mean-variance portfolio policy in a frictional market without shorting can be achieved by the optimal mean-variance portfolio policy in an (unconstrained) auxiliary frictionless market, $\mathcal{M}_{\kappa^*}$. Thus, verifying the time consistency in efficiency in the frictional market without shorting can be carried out by checking the time consistency in efficiency in the auxiliary frictionless market. By adding no-shorting constraint, the time inconsistency in efficiency can be eliminated for some market setting, although this does not always work for all settings. At last, we proceed to give a sufficient condition, under which time-consistency in efficiency holds in markets without shorting.

The organization of this chapter is as follows. In Section 5.2, we first derive the pre-committed optimal mean-variance policy in a frictionless market, when the returns of risky assets at different periods are correlated. We then examine the relationship between the time consistency in efficiency of the optimal policy and the variance-optimal signed martingale measure. In Section 5.3, we develop an assertion that the optimal no-shorting mean-variance policy is identical to the optimal mean-variance policy in an auxiliary frictionless market. In Section 5.4, we show that time inconsistency in efficiency can be eliminated by adding no-shorting constraint for some market setting, and provide a sufficient condition for assuring the success of such a scheme. Finally, we conclude our chapter in Section 5.5.

5.2. Time Consistency in Efficiency of Mean-Variance Portfolio Selection in Frictionless Market

We consider a frictionless capital market consisting of one riskless asset and n risky assets within a time horizon T, under an assumption of no arbitrage opportunity. Let $r_t > 0$ be the given deterministic return of the riskless asset at period t and $\mathbf{e_t} = (e_t^1, ..., e_t^n)'$ the vector of random returns of the n risky assets at period t defined on the probability space $(\Omega, \mathcal{F}, P)$, whose distribution depends on the realizations of $\mathbf{e_{t-1}}, \cdots, \mathbf{e_0}$. We use $\mathcal{F}_t$ to denote the smallest sigma algebra generated by $\mathbf{e_{t-1}}, \cdots, \mathbf{e_0}$ and $E_t(\cdot)$ to denote the conditional expectation $E(\cdot|\mathcal{F}_t)$.

An investor joins the market at the beginning of period 0 with an initial wealth x_0. He/She can allocate his/her wealth among the riskless asset and n risky assets at the beginning of period 0 and reallocates his/her wealth at the beginning of each of the following $(T-1)$ consecutive periods. Let x_t be the wealth of the investor at the beginning of period t, and u_t^i, $i = 1, 2, \ldots, n$, be the amount invested in the ith risky asset at period t.

The investor seeks a best investment strategy, $\mathbf{u_t^*} = (u_t^1, u_t^2, ..., u_t^n)'$ for $t = 0, 1, \ldots, T-1$, to attain the optimality of the following dynamic mean-variance model:

$$\begin{aligned} (MV) \qquad & \min \ Var_0(x_T) + \lambda E_0(x_T) \\ & \text{s.t.} \ x_{t+1} = r_t x_t + \mathbf{P}_t' \mathbf{u}_t, \\ & x_0 \text{ is given,} \end{aligned} \tag{5.1}$$

where the excess return vector $\mathbf{P_t}$ is defined as

$$\mathbf{P_t} = (P_t^1, P_t^2, ..., P_t^n)' = ((e_t^1 - r_t), (e_t^2 - r_t), ..., (e_t^n - r_t))',$$

and λ represents the overall trade-off between two objectives of maximizing the expected return and minimizing the risk. Changing λ from 0 to $-\infty$ yields the entire mean-variance efficient frontier.

We assume that conditional covariance matrix of $\mathbf{e_t}$ is finite and positive definite, $Cov_t(\mathbf{e_t}) \succ 0$, $P-$a.s., which implies the conditional second moment of $(r_t, \mathbf{e_t}')$ is also positive definite almost surely, i.e.,

$$E_t((r_t, \mathbf{e_t}')'(r_t, \mathbf{e_t}')) = \begin{pmatrix} 0, & \mathbf{0}_n' \\ \mathbf{0}_n, & Cov_t(\mathbf{e_t}) \end{pmatrix} + E_t((r_t, \mathbf{e_t}')')E_t((r_t, \mathbf{e_t}')) \succ 0, \ P-\text{a.s.}$$

Furthermore, we have

$$E_t(\mathbf{P_t}\mathbf{P_t}') \succ 0, \quad \forall t = 0, 1, ..., T-1, \quad P-\text{a.s.}$$

$$r_t^2(1 - E_t(\mathbf{P_t}')E_t^{-1}(\mathbf{P_t}\mathbf{P_t}')E_t(\mathbf{P_t})) > 0, \quad \forall t = 0, 1, ..., T-1, \quad P-\text{a.s.}$$

Proposition 5.1. *Using the embedding scheme in [35], we can derive the optimal policy of (MV) as*

$$\begin{aligned} \mathbf{u}_t^*(x_t) = & - r_t x_t E_t^{-1}(\theta_{t+1}\mathbf{P_t}\mathbf{P}_t')E_t(\theta_{t+1}\mathbf{P_t}) \\ & + \left[\prod_{j=0}^{T-1} r_j x_0 - \frac{\lambda}{2\theta_0}\right]\left(\prod_{j=t+1}^{T-1}\frac{1}{r_j}\right)E_t^{-1}(\theta_{t+1}\mathbf{P_t}\mathbf{P}_t')E_t(\theta_{t+1}\mathbf{P_t}), \end{aligned} \tag{5.2}$$

where random variable θ_t is $\mathcal{F}_t$-measurable and positive, which is calculated by the following backward recursion,

$$\theta_t = E_t(\theta_{t+1}) - E_t(\theta_{t+1}\mathbf{P}_t')E_t^{-1}(\theta_{t+1}\mathbf{P_t}\mathbf{P}_t')E_t(\theta_{t+1}\mathbf{P_t}), \quad t = T-1, \cdots, 0, \quad P-a.s. \tag{5.3}$$

with boundary condition $\theta_T = 1$.

Proof: Please refer to the Appendix 5.6. □

Let $\left(\prod_{j=0}^{T-1} r_j x_0 - \frac{\lambda}{2\theta_0}\right)$ be denoted by Γ, which is termed risk attitude parameter in this paper.

Assumption 5.1. *Without loss of generality, we assume $E_{T-1}(\mathbf{P_{T-1}}) \neq \mathbf{0}_n$, $P-$ a.s.*

Due to the almost surely positive definiteness of $E_{T-1}^{-1}(\mathbf{P_{T-1}}\mathbf{P_{T-1}}')$, $E_{T-1}(\mathbf{P_{T-1}}) = \mathbf{0}_n$ for some possible wealth level x_{T-1} implies the optimal policy at period $T-1$, $\mathbf{u}^*_{\mathbf{T-1}}(x_{T-1}) \equiv \mathbf{0}_n$. In such a situation, we only need to consider the mean-variance portfolio selection problem ending up at the end of period $T-2$, as only the riskless asset is active at period $T-1$.

Remark 5.1. *The adapted process, $\{\theta_t\}_{t=0,\ldots,T-1}$, is actually the same as the opportunity process, $\{L_t\}_{t=0,\ldots,T-1}$, introduced for mean-variance hedging problem in Černý and Kallsen (2009) [11].*

Now we consider the following inverse optimization problem: For $k = 1, 2, \ldots, T-1$, find an induced trade-off parameter λ_k between $E_k(x_T|x_k)$ and $Var_k(x_T|x_k)$ such that the optimal mean-variance policy $\mathbf{u}^*_\mathbf{t}(x_t)$ $(t = k, k+1, \ldots, T-1)$ specified in (5.2) solves

$$\begin{aligned} (MV_{k-T}^{\lambda_k}) \qquad \min \quad & Var_k(x_T \mid x_k) + \lambda_k E_k(x_T \mid x_k) \\ \text{s.t.} \quad & x_{t+1} = r_t x_t + \mathbf{P}_\mathbf{t}' \mathbf{u}_\mathbf{t}, \\ & x_k \text{ is given.} \end{aligned}$$

Let a threshold x_k^* be defined as follows at the beginning of period k:

$$x_k^* = -\frac{\lambda}{2\theta_0 \prod_{j=k}^{T-1} r_j} + \prod_{j=0}^{k-1} r_j x_0 = \frac{\Gamma}{\prod_{j=k}^{T-1} r_j}, \tag{5.4}$$

which is a constant and represents the discounted risk attitude parameter at the beginning of period k.

Proposition 5.2. *The optimal mean-variance policy,* $\mathbf{u}_t^*(x_t)$ $(t = k, k+1, \ldots, T-1)$, *specified in (5.2), solves* $(MV_{k-T}^{\lambda_k})$ *with* λ_k *satisfying*

$$\lambda_k = 2\left(x_k - \prod_{j=0}^{k-1} r_j x_0\right) \prod_{j=k}^{T-1} r_j \theta_k + \frac{\lambda \theta_k}{\theta_0}. \tag{5.5}$$

Furthermore, $\lambda_k < 0$ *when* $x_k < x_k^*$, $\lambda_k = 0$ *when* $x_k = x_k^*$, *and* $\lambda_k > 0$ *when* $x_k > x_k^*$.

Proof: Similar to the solution to (MV), the optimal policy of $(MV_{k-T}^{\lambda_k})$ at period t, $t = k, k+1, \ldots, T-1$, is given by

$$\begin{aligned}\mathbf{u}_t^{\mathbf{k-T}}(x_t, \lambda_k) = & - r_t E_t^{-1}(\theta_{t+1}\mathbf{P}_t\mathbf{P}_t')E_t(\theta_{t+1}\mathbf{P}_t)x_t \\ & + \left[\prod_{j=k}^{T-1} r_j x_k - \frac{\lambda_k}{2\theta_k}\right]\left(\prod_{j=t+1}^{T-1}\frac{1}{r_j}\right) E_t^{-1}(\theta_{t+1}\mathbf{P}_t\mathbf{P}_t')E_t(\theta_{t+1}\mathbf{P}_t).\end{aligned} \tag{5.6}$$

Equalizing the policies in (5.2) and (5.6) and noticing Assumption 5.1, we can find the relationship between λ_k and λ given in (5.5). From the positiveness of θ_k, it is evident now that $\lambda_k = 0$ when $x_k = x_k^*$ defined in (5.4), $\lambda_k < 0$ when $x_k < x_k^*$, and $\lambda_k > 0$ when $x_k > x_k^*$. □

If all the possible wealth processes under the optimal policy, $\{x_t\}_{t=1,\ldots,T-1}$, never exceed the threshold, $\{x_t^*\}_{t=1,\ldots,T-1}$, the truncated optimal policy is efficient for the corresponding truncated portfolio selection problem, and time consistency in efficiency is, thus, satisfied.

Assumption 5.2. *The investor's initial trade-off parameter,* λ, *is assumed to be negative.*

Note that, when $\lambda = 0$, the optimal policy is $\mathbf{u}_k^*(x_k) = 0$, $k = 0, 1, \ldots, T-1$, i.e., the investor places all his/her wealth in the riskless asset. It is obvious that the time consistency in efficiency holds in such a case. Under Assumption 5.2, the initial risk attitude parameter of the *pre-committed* policy, Γ, satisfies $\Gamma > \prod_{j=0}^{T-1} r_j x_0$.

Lemma 5.1. *For* $k = 1, 2, \ldots, T-2$,

$$Pr(x_{k+1} < x^*_{k+1} | x_k < x^*_k) = Pr\left(\mathbf{P_k}' E_k^{-1}(\theta_{k+1}\mathbf{P_k}\mathbf{P'_k}) E_k(\theta_{k+1}\mathbf{P_k}) < 1\right),$$
$$Pr(x_{k+1} = x^*_{k+1} | x_k < x^*_k) = Pr\left(\mathbf{P_k}' E_k^{-1}(\theta_{k+1}\mathbf{P_k}\mathbf{P'_k}) E_k(\theta_{k+1}\mathbf{P_k}) = 1\right),$$
$$Pr(x_{k+1} = x^*_{k+1} | x_k = x^*_k) = 1.$$

In addition, under Assumption 5.2,

$$Pr(x_1 < x^*_1) = Pr\left(\mathbf{P_0}' E_0^{-1}(\theta_1\mathbf{P_0}\mathbf{P'_0}) E_0(\theta_1\mathbf{P_0}) < 1\right),$$
$$Pr(x_1 = x^*_1) = Pr\left(\mathbf{P_0}' E_0^{-1}(\theta_1\mathbf{P_0}\mathbf{P'_0}) E_0(\theta_1\mathbf{P_0}) = 1\right).$$

Proof: We have $x^*_{k+1} = r_k x^*_k$ and

$$\begin{aligned}
& x_{k+1} \\
=& r_k x_k + \mathbf{P'_k}\mathbf{u^*_k}(x_k) \\
=& r_k x_k (1 - \mathbf{P_k}' E_k^{-1}(\theta_{k+1}\mathbf{P_k}\mathbf{P'_k}) E_k(\theta_{k+1}\mathbf{P_k})) + x^*_{k+1}\mathbf{P_k}' E_k^{-1}(\theta_{k+1}\mathbf{P_k}\mathbf{P'_k}) E_k(\theta_{k+1}\mathbf{P_k}).
\end{aligned}$$

Then,

$$\begin{aligned}
& Pr(x_{k+1} < x^*_{k+1} | x_k < x^*_k) \\
=& Pr\left((r_k x_k - x^*_{k+1})(1 - \mathbf{P_k}' E_k^{-1}(\theta_{k+1}\mathbf{P_k}\mathbf{P'_k}) E_k(\theta_{k+1}\mathbf{P_k})) < 0 | x_k < x^*_k\right) \\
=& Pr\left(\mathbf{P_k}' E_k^{-1}(\theta_{k+1}\mathbf{P_k}\mathbf{P'_k}) E_k(\theta_{k+1}\mathbf{P_k}) < 1\right).
\end{aligned}$$

The other two equations in the lemma can be obtained in similar ways.

At period 0, we have

$$\begin{aligned}
x_1 &= r_0 x_0 + \mathbf{P'_0}\mathbf{u^*_0}(x_0) \\
&= r_0 x_0 (1 - \mathbf{P_0}' E_0^{-1}(\theta_1\mathbf{P_0}\mathbf{P'_0}) E_0(\theta_1\mathbf{P_0})) + x^*_1\mathbf{P_0}' E_0^{-1}(\theta_1\mathbf{P_0}\mathbf{P'_0}) E_0(\theta_1\mathbf{P_0}).
\end{aligned}$$

As Assumption 5.2 assures $\prod_{j=0}^{T-1} r_j x_0 < \Gamma$, carrying out similar steps as above gives rise the last part of the lemma. □

Now let us go back to the assumption of no arbitrage opportunity. Based on the classical FTAP, there exists a nonempty probability measure set that contains all equivalent martingale probability measures of the market. We consider here a larger set that contains all signed martingale measures (Schweizer (1996) [57]). We use notation $Q << P$ in the following to denote that Q is absolutely continuous to P.

Definition 5.1 (Schweizer (1996) [57]). *We call a signed measure $Q << P$ with $Q(\Omega) = 1$ absolutely continuous signed σ-martingale measure (SσMM) if* $\mathbf{P}Z^Q$ *is P-σ-martingale for the density process*

$$Z_t^Q := E\left(\frac{dQ}{dP}|\mathcal{F}_t\right)$$

of Q.

Definition 5.2 (Schweizer (1996) [57]). *A signed σ-martingale measure Q^* is called* variance-optimal *if Q^* minimizes*

$$E\left[\left(\frac{dQ}{dP}\right)^2\right]$$

over all $Q \in \mathcal{Q}$, where the closed set $\mathcal{Q}$ contains all SσMMs.

Applying the above definition to our discrete-time market setting, the variance-optimal signed martingale measure, Q^*, is given by Černý and Kallsen (2007, 2009) [10, 11].

$$\frac{dQ^*}{dP} = \prod_{j=0}^{T-1} \frac{\theta_{j+1}}{E_j(\theta_{j+1})} \prod_{j=0}^{T-1} \frac{E_j(\theta_{j+1})(1 - \mathbf{P_j}' E_j^{-1}(\theta_{j+1}\mathbf{P_j}\mathbf{P_j}')E_j(\theta_{j+1}\mathbf{P_j}))}{\theta_j}. \quad (5.7)$$

Based on Lemma 5.1, whether the first $(T-1)$ periods wealth level exceed the threshold can be checked by whether conditional density process of variance-optimal signed martingale measure is a nonnegative process. We thus get the following proposition.

Proposition 5.3. *Time consistency in efficiency of mean-variance portfolio selection in frictionless market holds if and only if the conditional density process of variance-optimal signed martingale measure is a nonnegative process, i.e.,*

$$E\left(\frac{dQ^*}{dP} \mid \mathcal{F}_t\right) \geq 0, \quad P-a.s., \quad for\ t = 1, 2, \ldots, T-1.$$

Proposition 5.4. *When the market is complete, time consistency in efficiency of mean-variance portfolio selection holds true.*

Proof: Let us consider the *signed martingale measure*, Q, defined in (5.7). It is easy to check that

$$Q(\Omega) = 1, \tag{5.8}$$

$$E_k^Q\left(\frac{\mathbf{e_k}}{r_k}\right) = \mathbf{1}, \quad k = 0, \cdots, T-1. \tag{5.9}$$

As stated on Page 25 of Pliska (1997) [51] that *"The model is complete if and only if $\mathcal{M}$ consists of exactly one risk neutral probability measure"*, there exists a **unique** probability measure Q^*, under which the discounted returns of assets all equal to 1.

Furthermore, page 133 of Pliska (1997) [51] states that *"The multiperiod model is complete if and only if every underlying single period model is complete."* We consider now a single period k, $k = 0, 1, \ldots, T-2$. To make the market complete, the return of risky asset must have a tree structure with $(n+1)$ branches. At period k, the states space is finite, $\Omega_k = \{\omega_1^k, \cdots, \omega_{n+1}^k\}$. Let $\{q_i^* \geq 0,\ i = 1, \cdots, n+1\}$ be the conditional risk neutral probability for state ω_i^k, which is the unique solution of

$$\begin{cases} q_1 r_k + \cdots + q_{d+1} r_k = r_k, \\ q_1 e_k^1 + \cdots + q_{d+1} e_k^1 = r_k, \\ \cdots \\ q_1 e_k^n + \cdots + q_{d+1} e_k^n = r_k. \end{cases}$$

It is also the unique solution in $\mathbb{R}^{n+1}$. Therefore, $\{q_i^* \geq 0,\ i = 1, \cdots, n+1\}$ is also the **unique** conditional signed martingale measure of period k. Furthermore, Q^* is the signed martingale measure. The uniqueness of the signed martingale measure implies

$$\frac{dQ^*}{dP}(\omega) = \frac{dQ}{dP}(\omega), \quad \omega \in \Omega.$$

Thus, $\frac{dQ}{dP}(\omega)$ is nonnegative, which in turn implies

$$Pr(\mathbf{P_j}'E_j^{-1}(\theta_{j+1}\mathbf{P_j}\mathbf{P_j}')E_j(\theta_{j+1}\mathbf{P_j}) < 1) = 1, \quad j = 0, \cdots, T-1.$$

Based on Lemma 5.1,

$$Pr(x_1 < x_1^*, \cdots, x_{T-1} < x_{T-1}^*) = 1,$$

holds. Thus, the optimal wealth process never hits the threshold and the time consistency in efficiency holds true. □

5.3. Mean-Variance Portfolio Selection without Shorting

Assume that the states of the world is finite, $\Omega = \{\omega_1, \cdots, \omega_{(m+1)^T}\}$, and the return of risky assets is of a tree structure with $(m+1)$ branches at each intermediate node. If $m > n$, the market is incomplete due to the existence of multiple martingale probability measures. Furthermore,

(i) there exist $l_t := (m+1)^t$ elements $A_t^1, \cdots, A_t^{l_t}$ at time t, which forms a partition of Ω and satisfies $\mathcal{F}_t = \sigma(A_t^1, \cdots, A_t^{l_t})$;

(ii) $\forall\ i \leq l_t$, $A_{t+1}^{(i-1)(m+1)+j} \subset A_t^i$ for $j = 1, \cdots, m+1$.

In the remaining of our paper, we consider the following mean-variance portfolio selection without shorting, $(MV-C1)$,

$$\begin{aligned} (MV-C1) \quad \min \ & Var_0(x_T) + \lambda E_0(x_T) \\ \text{s.t. } & x_{t+1} = r_t x_t + \mathbf{P_t'}\mathbf{u_t}, \\ & \mathbf{u_t} \geq \mathbf{0}_n, \ x_0 \text{ is given.} \end{aligned}$$

For any incomplete market, we can always introduce $(m-n)$ auxiliary risky assets, whose returns, $e_t^{n+1}, \cdots, e_t^m$, are specified arbitrarily to make the market with m risky assets and one riskless asset complete. The original mean-variance portfolio selection without shorting, $(MV-C1)$, can be now reformulated as

$$
\begin{aligned}
(MV-C2) \quad \min \quad & Var_0(x_T) + \lambda E_0(x_T) \\
\text{s.t.} \quad & x_{t+1} = r_t x_t + \mathbf{\bar{P}}_t' \mathbf{\bar{u}_t}, \\
& \mathbf{u_t} \geq \mathbf{0}_n, \quad \mathbf{v_t} = \mathbf{0}_{m-n}, \\
& x_0 \text{ is given,}
\end{aligned}
$$

where the excess return vector $\mathbf{\bar{P}_t}$ is defined as

$$
\mathbf{\bar{P}_t} = (P_t^1, P_t^2, ..., P_t^m)' = ((e_t^1 - r_t), (e_t^2 - r_t), ..., (e_t^m - r_t))',
$$

$\mathbf{\bar{u}_t} = (\mathbf{u_t'}, \mathbf{v_t'})'$, and $\mathbf{v_t}$ denotes the amount invested in the auxiliary risky assets at period t.

Yi (2009) [71] extended Pliska (1997) [51]'s dual approach for utility maximization problem and performed detailed analysis on mean-variance portfolio selection problem $(MV-C2)$. Following Yi (2009) [71], problem $(MV-C2)$ is first embedded into the following auxiliary problem,

$$
\begin{aligned}
\mathcal{A}_C(\beta) \quad \max \quad & E_0(-x_T^2 + \beta x_T), \\
\text{s.t.} \quad & x_{t+1} = r_t x_t + \mathbf{\bar{P}}_t' \mathbf{\bar{u}_t}, \\
& \mathbf{u_t} \geq \mathbf{0}_n, \quad \mathbf{v_t} = \mathbf{0}_{m-n}, \\
& x_0 \text{ is given.}
\end{aligned}
$$

Based on Lemma 5.2, the solution of $\mathcal{A}_C(\beta)$ also solves $(MV-C2)$ when

$$
\beta = 2E_0(x_T)|_{\mathbf{\bar{u}}^*} - \lambda.
$$

As we know, $\mathcal{A}_C(\beta)$ can be tackled as a static optimization problem, where all the realizations of $\mathbf{\bar{u}_t}$ are considered separately based on our discrete financial model. More specifically, the objective function can be expressed explicitly as follows,

$$
\sum_{\omega \in \Omega} P(\omega) U \left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \mathbf{\bar{P}}_t' \mathbf{\bar{u}_t} \prod_{j=t+1}^{T-1} r_j \right),
$$

where $U(x) = -x^2 + \beta x$. Notice that $\bar{\mathbf{u}}_t(\omega) = \bar{\mathbf{u}}_t(A_t^i)$ if $\omega \in A_t^i$, due to the tree structure of the market. The decision vector in the above static formulation consists of $\bar{\mathbf{u}}_t(A_t^i)$, $t = 0, 1, \cdots, T-1$ and $i = 1, \cdots, l_t$.

Note that the Lagrangian dual of problem $\mathcal{A}_C(\beta)$ is given as

$$(D_L)\ \min_{\xi} \max_{\bar{\mathbf{u}}} \sum_{t=0}^{T-1} \sum_{i=1}^{l_t} (\xi_t(A_t^i))' \bar{\mathbf{u}}_t(A_t^i) + \sum_{\omega \in \Omega} P(\omega) U\left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \bar{\mathbf{P}}_t' \bar{\mathbf{u}}_t \prod_{j=t+1}^{T-1} r_j \right),$$

where $\xi = \{(\xi_t^1, \ldots, \xi_t^n, \xi_t^{n+1}, \ldots, \xi_t^m)'\}_{t=0,1,\cdots,T-1}$ is an adapted process and $\xi_t^i \geq 0$ for $i = 1, \ldots, n$. As problem $\mathcal{A}_C(\beta)$ is convex, there is no duality gap between problems $\mathcal{A}_C(\beta)$ and (D_L) from the strong duality theorem. Furthermore, the process pair $(\bar{\mathbf{u}}^*, \xi^*)$ that satisfies the following first-order condition,

$$E_0 \left[1_{A_t^i} U' \left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \bar{\mathbf{P}}_t' \bar{\mathbf{u}}_t^* \prod_{j=t+1}^{T-1} r_j \right) \bar{\mathbf{P}}_t \prod_{j=t+1}^{T-1} r_j \right] + \xi_t^*(A_t^i) = 0, \quad (5.10)$$

$$\xi_t^{*j}(A_t^i) \bar{\mathbf{u}}_t^{*j}(A_t^i) = 0, \quad (5.11)$$

for all A_t^i and $j = 1, \ldots, m$, is optimal to $\mathcal{A}_C(\beta)$.

On the other hand, let us introduce an auxiliary market indexed by a adapted process, $\kappa = \{(\kappa_t^1, \ldots, \kappa_t^m)'\}$, $\mathcal{M}_\kappa$, where the period-t return of assets is adjusted to

$$\begin{aligned} r_t &\rightarrow r_t; \\ e_t^j &\rightarrow e_t^j + \kappa_t^j, \quad j = 1, \cdots, m. \end{aligned}$$

Consider the following unconstrained utility maximization problem in $\mathcal{M}_\kappa$,

$$\begin{aligned} \mathcal{A}_C(\beta, \kappa) \quad & \min \ E_0(-x_T^2 + \beta x_T) \\ & \text{s.t.} \ x_{t+1} = r_t x_t + (\bar{\mathbf{P}}_t^\kappa)' \bar{\mathbf{u}}_t, \\ & \quad x_0 \text{ is given,} \end{aligned}$$

where $\bar{\mathbf{P}}_t^\kappa = ((e_t^1 + \kappa_t^1 - r_t), (e_t^2 + \kappa_t^2 - r_t), \ldots, (e_t^m + \kappa_t^m - r_t))'$.

Proposition 5.5. *The optimal policy of* $\mathcal{A}_C(\beta)$, $\bar{\mathbf{u}}_t^*$, *is also the optimal policy of* $\mathcal{A}_C(\beta, \kappa^*)$ *in a frictionless auxiliary market,* $\mathcal{M}_{\kappa^*}$, *with*

$$\kappa_t^*(A_t^i) = \frac{\xi_t^*(A_t^i)}{E_0\left[1_{A_t^i} U'\left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \bar{\mathbf{P}}_t' \bar{\mathbf{u}}_t^* \prod_{j=t+1}^{T-1} r_j\right) \prod_{j=t+1}^{T-1} r_j\right]}. \quad (5.12)$$

Proof: It is clear from (5.11) that $\kappa_t^{*j}(A_t^i)\bar{\mathbf{u}}_t^{*j}(A_t^i) = 0$ holds for all t and j.

We next show that $\bar{\mathbf{u}}_t^*$ satisfies the optimality condition of problem $\mathcal{A}_C(\beta, \kappa^*)$. From (5.10) and (5.12), we have

$$\begin{aligned}
& E_0\left[1_{A_t^i} U'\left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} (\bar{\mathbf{P}}_t + \kappa_t^*)' \bar{\mathbf{u}}_t^* \prod_{j=t+1}^{T-1} r_j\right) (\bar{\mathbf{P}}_t + \kappa_t^*) \prod_{j=t+1}^{T-1} r_j\right] \\
=& E_0\left[1_{A_t^i} U'\left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \bar{\mathbf{P}}_t' \bar{\mathbf{u}}_t^* \prod_{j=t+1}^{T-1} r_j\right) (\bar{\mathbf{P}}_t + \kappa_t^*) \prod_{j=t+1}^{T-1} r_j\right] \\
=& E_0\left[1_{A_t^i} U'\left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \bar{\mathbf{P}}_t' \bar{\mathbf{u}}_t^* \prod_{j=t+1}^{T-1} r_j\right) \bar{\mathbf{P}}_t \prod_{j=t+1}^{T-1} r_j\right] \\
& + E_0\left[1_{A_t^i} U'\left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \bar{\mathbf{P}}_t' \bar{\mathbf{u}}_t^* \prod_{j=t+1}^{T-1} r_j\right) \kappa_t^* \prod_{j=t+1}^{T-1} r_j\right] \\
=& -\xi_t^*(A_t^i) + \xi_t^*(A_t^i) \\
=& 0.
\end{aligned}$$

□

Remark 5.2. *When* $E_0\left[1_{A_t^i} U'\left(x_0 \prod_{t=0}^{T-1} r_t + \sum_{t=0}^{T-1} \bar{\mathbf{P}}_t' \bar{\mathbf{u}}_t^* \prod_{j=t+1}^{T-1} r_j\right) \prod_{j=t+1}^{T-1} r_j\right] = 0$, $\kappa_t^*(A_t^i)$ *could be* $\pm\infty$. *Under such a circumstance, the optimal policy of* $\mathcal{A}_C(\beta, \kappa^*)$ *also exists. More specifically,* $\bar{\mathbf{u}}_t^* = 0$.

From Lemma 5.2 again, the solution of $\mathcal{A}_C(\beta, \kappa^*)$ also solves the mean-variance portfolio selection problem in auxiliary market $\mathcal{M}_{\kappa^*}$, $(MV(\kappa^*))$

$$\begin{aligned}
(MV(\kappa^*)) \quad \min \quad & Var_0(x_T) + \lambda E_0(x_T) \\
\text{s.t.} \quad & x_{t+1} = r_t x_t + (\bar{\mathbf{P}}_t^{\kappa^*})' \bar{\mathbf{u}}_t, \\
& x_0 \text{ is given},
\end{aligned}$$

with the same overall trade-off parameter λ as in $(MV - C1)$.

5.4. Time Consistency in Efficiency of Mean-Variance Portfolio Selection without Shorting

Let us consider now the truncated mean-variance portfolio selection problem without shorting:

$$\begin{aligned}(MV - C1^{\lambda_k}_{k-T}) \quad \min \ & Var_k(x_T)(A^j_k) + \lambda_k E_k(x_T)(A^j_k)\\ \text{s.t. } & x_{t+1} = r_t x_t + \mathbf{P}'_\mathbf{t}\mathbf{u}_\mathbf{t},\\ & \mathbf{u}_\mathbf{t} \geq \mathbf{0}_n, \ x_k \text{ is given,}\end{aligned}$$

where A^j_k is the current starting node after adopting optimal policy $\bar{\mathbf{u}}^*_\mathbf{0}, \ldots, \bar{\mathbf{u}}^*_{\mathbf{k-1}}$. Similarly, we consider the following auxiliary problem,

$$\begin{aligned}\mathcal{A}_C(\beta)^{\lambda_k}_{k-T} \quad \max \ & E_k(-x_T^2 + \beta x_T)(A^j_k),\\ \text{s.t. } & x_{t+1} = r_t x_t + \bar{\mathbf{P}}'_\mathbf{t}\bar{\mathbf{u}}_\mathbf{t},\\ & \mathbf{u}_\mathbf{t} \geq \mathbf{0}_n, \quad \mathbf{v}_\mathbf{t} = \mathbf{0}_{m-n},\\ & x_k \text{ is given.}\end{aligned}$$

Carrying out the similar analysis as in Section 5.3, we can conclude that the optimal policy of $(MV - C1^{\lambda_k}_{k-T})$ also solves the following unconstrained portfolio selection problem,

$$\begin{aligned}(MV(\hat{\kappa}^*)^{\lambda_k}_{k-T}) \quad \min \ & Var_k(x_T)(A^j_k) + \lambda_k E_k(x_T)(A^j_k)\\ \text{s.t. } & x_{t+1} = r_t x_t + (\bar{\mathbf{P}}^{\hat{\kappa}^*}_\mathbf{t})'\bar{\mathbf{u}}_\mathbf{t},\\ & x_k \text{ is given,}\end{aligned}$$

where

$$\hat{\kappa}^*_t(A^i_t) = \frac{\hat{\xi}^*_t(A^i_t)}{E_k\left[1_{A^i_t} U'\left(x_k \prod_{t=k}^{T-1} r_t + \sum_{t=k}^{T-1} \bar{\mathbf{P}}'_\mathbf{t}\hat{\mathbf{u}}_\mathbf{t}{}^* \prod_{j=t+1}^{T-1} r_j\right) \prod_{j=t+1}^{T-1} r_j\right](A^j_k)},$$

and process pair $(\hat{\mathbf{u}}^*, \hat{\xi}^*)(A^j_k)$ satisfies the first-order condition of $\mathcal{A}_C(\beta)^{\lambda_k}_{k-T}$.

Proposition 5.6.

$$\kappa_t^*(A_t^i) = \hat{\kappa}_t^*(A_t^i).$$

Proof: The Lagrangian dual of problem $\mathcal{A}_C(\beta)_{k-T}^{\lambda_k}$ is given as follows,

$(D_L)_{k-T}$

$$\min_{\xi}\max_{\bar{\mathbf{u}}}\sum_{t=k}^{T-1}\sum_{i=1}^{l_t}(\xi_t(A_t^i))'\bar{\mathbf{u}}_{\mathbf{t}}(A_t^i) + \sum_{\omega\in\Omega}\frac{P(\omega)}{P(A_k^j)}U\left(x_k\prod_{t=k}^{T-1}r_t + \sum_{t=k}^{T-1}\bar{\mathbf{P}}_{\mathbf{t}}'\bar{\mathbf{u}}_{\mathbf{t}}\prod_{j=t+1}^{T-1}r_j\right),$$

where $\xi = \{(\xi_t^1,\ldots,\xi_t^n,\xi_t^{n+1},\ldots,\xi_t^m)'\}_{t=k,k+1,\cdots,T-1}$ is an adapted process and $\xi_t^i \geq 0$ for $i = 1,\ldots,n$. Furthermore, the process pair $(\hat{\bar{\mathbf{u}}}^*, \hat{\xi}^*)$ that satisfies the first order condition,

$$E_k\left[1_{A_t^i}U'\left(x_k\prod_{t=k}^{T-1}r_t + \sum_{t=k}^{T-1}\bar{\mathbf{P}}_{\mathbf{t}}'\hat{\mathbf{u}}_{\mathbf{t}}^*\prod_{j=t+1}^{T-1}r_j\right)\bar{\mathbf{P}}_{\mathbf{t}}\prod_{j=t+1}^{T-1}r_j\right](A_k^j) + \hat{\xi}_t^*(A_t^i) = 0, \tag{5.13}$$

$$\hat{\xi}_t^{*j}(A_t^i)\hat{\mathbf{u}}_{\mathbf{t}}^{*j}(A_t^i) = 0, \tag{5.14}$$

for all A_t^i and $j = 1,\ldots,m$, is optimal.

Comparing (5.10)-(5.11) to (5.13)-(5.14), and noticing that both x_0 and $\{\bar{\mathbf{u}}_{\mathbf{0}}^*,\ldots,\bar{\mathbf{u}}_{\mathbf{k-1}}^*\}$ lead to x_k, we have

$$\bar{\mathbf{u}}_{\mathbf{t}}^* = \hat{\mathbf{u}}_{\mathbf{t}}^*,\ t = k, k+1,\ldots,T-1,$$

$$P(A_t^j)\hat{\xi}_t^*(A_t^i) = \xi_t^*(A_t^i),\quad t = k,k+1,\ldots,T-1.$$

We completes the proof. □

Thus the time consistency in efficiency issue for $(MV-C1)$, the mean-variance portfolio selection problem without shorting, is equivalent to the time consistency in efficiency issue for $(MV(\kappa^*))$, the mean-variance portfolio selection problem in a suitable auxiliary market, $\mathcal{M}_{\kappa^*}$.

Furthermore, when the unique signed martingale measure in the optimal auxiliary market $\mathcal{M}_{\kappa^*}$ with the density defined by (5.7) satisfies the requirement in Proposition 5.3, the time consistency in efficiency holds in the auxiliary market, $\mathcal{M}_{\kappa^\bullet}$. So does the mean-variance portfolio selection without shorting, $(MV-C1)$.

Remark 5.3. *The other direction in Proposition 5.3 does not hold true, as the condition*

$$E_{T-1}(\mathbf{P}^{\kappa^\bullet}_{\mathbf{T-1}}\mathbf{P}^{\kappa^\bullet\,'}_{\mathbf{T-1}})E_{T-1}(\mathbf{P}^{\kappa^\bullet}_{\mathbf{T-1}}) \neq \mathbf{0}_n, \quad P-a.s.$$

may not hold in the auxiliary market, $\mathcal{M}_{\kappa^\bullet}$, even if it holds in the original market, $\mathcal{M}_0$. In such situations, there may exist some states at the $(T-1)$th period such that risky assets do not play a role in the last period after setting $\bar{\mathbf{u}}^\bullet_{\mathbf{T-1}} = \mathbf{0}_m$, and the induced trade-off at the last time period can be thus arbitrarily assigned.

We use the following example to illustrate that, by adding the no-shorting constraint, we can make the optimal policy to satisfy time consistency in efficiency, while the *pre-committed* optimal policy in the original frictionless market is not time consistent in efficiency.

Example 5.1. Assume that there are one risky asset, S_1, and one riskless asset, S_0, in the market with a time horizon of $T = 2$. The random total return of risky asset follows a trinomial tree structure, which is given in Figure 5.1, while the riskless returns, r_0 and r_1, are both 1.02. It is obvious that this market is arbitrage free and incomplete. The investor's initial trade-off, λ, is assumed to be -2 and the initial wealth is assumed to be 1.

As the random total returns at different periods are independent in this example, it is not difficult to derive the following solution for this unconstrained

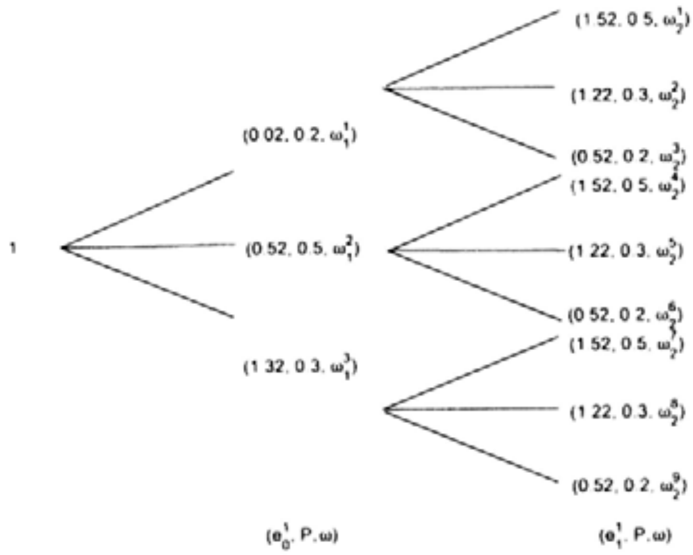


Figure 5.1: Random return of risky asset

mean-variance portfolio selection problem,

$$\begin{aligned}
\theta_1 &= 0.7642, \quad \theta_0 = 0.4828, \\
u_0^{\bullet}(1) &= -\frac{-2E_0(\theta_1 P_0)}{2r_1\theta_0 E_0(\theta_1 P_0^2)} = -2.0767, \\
x_1(\omega_1^1) &= 3.0807, \quad x_1(\omega_1^2) = 2.0503, \quad x_1(\omega_1^3) = 0.4018, \\
u_1^{\bullet}(x_1(\omega_1^1)) &= -r_1 x_1(\omega_1^1)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^1) + \left(r_0 r_1 x_0 - \frac{-2}{2\theta_0}\right)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^1) = -0.0529, \\
u_1^{\bullet}(x_1(\omega_1^2)) &= -r_1 x_1(\omega_1^2)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^2) + \left(r_0 r_1 x_0 - \frac{-2}{2\theta_0}\right)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^2) = 1.1365, \\
u_1^{\bullet}(x_1(\omega_1^3)) &= -r_1 x_1(\omega_1^3)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^3) + \left(r_0 r_1 x_0 - \frac{-2}{2\theta_0}\right)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^3) = 3.0396.
\end{aligned}$$

It can be checked that the optimal policy for the truncated mean-variance problem at ω_1^1 is $u_1^{1-2}(x_1(\omega_1^1), \lambda_1) \geq 0$ for all nonpositive λ_1. As $u_1^{\bullet}(x_1(\omega_1^1)) = -0.0529$ from the above calculation, we can conclude that the truncated *pre-committed* optimal policy at ω_1^1 is no longer efficient for the second period. Thus, time consistency in efficiency does not hold in this example. Furthermore, the $\mathcal{F}_1$ conditional density process of the variance-optimal signed martingale measure given by (5.7) is

$$E\left(\frac{dQ^{\bullet}}{dP}|\omega_1^1\right) = -0.036, \; E\left(\frac{dQ^{\bullet}}{dP}|\omega_1^2\right) = 0.7734, \; E\left(\frac{dQ^{\bullet}}{dP}|\omega_1^3\right) = 2.0683,$$

which is NOT nonnegative almost surely. (An example of Proposition 5.3)

Now we consider to enforce a no shorting constraint of the risky asset, S_1. It is clear that the above derived optimal policy is no longer admissible. We first introduce an auxiliary risky asset S_2 with the random return following a trinomial tree structure, which is given by Figure 5.2. It is easy to check that there exists a unique martingale probability measure in this market. We solve next the corresponding problem $(MV - C2)$ directly by using quadratic programming.

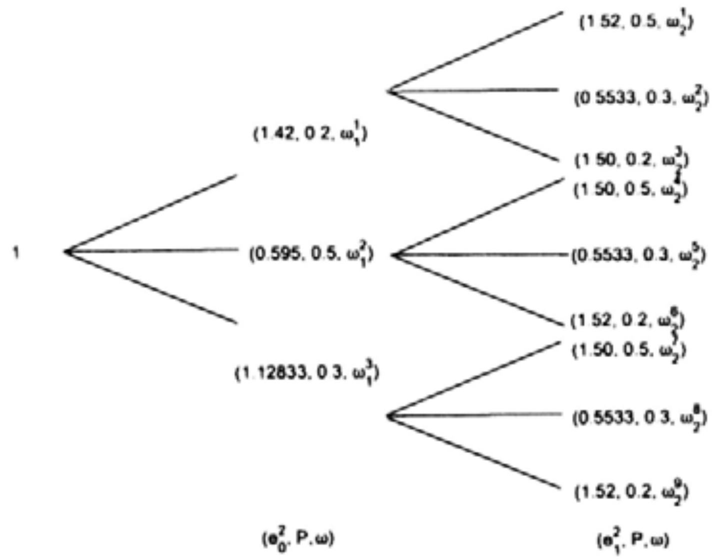


Figure 5.2: Random return of auxiliary risky asset

The optimal policy of the mean-variance portfolio selection problem without shorting is

$$
\begin{aligned}
u_0^*(1) &= 0, \\
x_1(\omega_1^1) &= x_1(\omega_1^2) = x_1(\omega_1^3) = 1.02, \\
u_1^*(x_1(\omega_1^1)) &= u_1^*(x_1(\omega_1^2)) = u_1^*(x_1(\omega_1^3)) = 1.4696.
\end{aligned}
$$

Note that u_1^* is also the optimal policy for the second period with trade-off -2. Thus, time consistency in efficiency does hold.

Accordingly, the suitable market parameter process, κ^*, is given as

$$\kappa_0^* = (0.36, 0.1)',$$
$$\kappa_1^*(\omega_1^1) = (0, -0.2)',\ \kappa_1^*(\omega_1^2) = (0, -0.2)',\ \kappa_1^*(\omega_1^3) = (0, -0.2)'.$$

It is easy to compute the unique signed martingale measure in $\mathcal{M}_{\kappa^*}$, whose density is given by

$$\begin{aligned}
\frac{dQ^{\kappa^*}}{dP}(\omega_2^1) &= 0.5738,\ \frac{dQ^{\kappa^*}}{dP}(\omega_2^2) = 1.0147,\ \frac{dQ^{\kappa^*}}{dP}(\omega_2^3) = 2.0434,\\
\frac{dQ^{\kappa^*}}{dP}(\omega_2^4) &= 0.5738,\ \frac{dQ^{\kappa^*}}{dP}(\omega_2^5) = 1.0147,\ \frac{dQ^{\kappa^*}}{dP}(\omega_2^6) = 2.0434,\\
\frac{dQ^{\kappa^*}}{dP}(\omega_2^7) &= 0.5738,\ \frac{dQ^{\kappa^*}}{dP}(\omega_2^8) = 1.0147,\ \frac{dQ^{\kappa^*}}{dP}(\omega_2^9) = 2.0434.
\end{aligned}$$

The $\mathcal{F}_1$ conditional density process of the variance-optimal signed martingale measure is

$$E\left(\frac{dQ^*}{dP}|\mathcal{F}_1\right) = 1,\ P - \text{a.s.}$$

The above scheme of adding the no-shorting constraint does not guarantee elimination of the time inconsistency in efficiency in general. The following problem serves as a counterexample.

Example 5.2. Assume that the basic market setting is the same as Example 5.1, except for some different random total returns. Please refer to Figure 5.3. It is obvious that this market is arbitrage free and incomplete.

Based on (5.2) and (5.3), the *pre-committed* optimal policy in frictionless

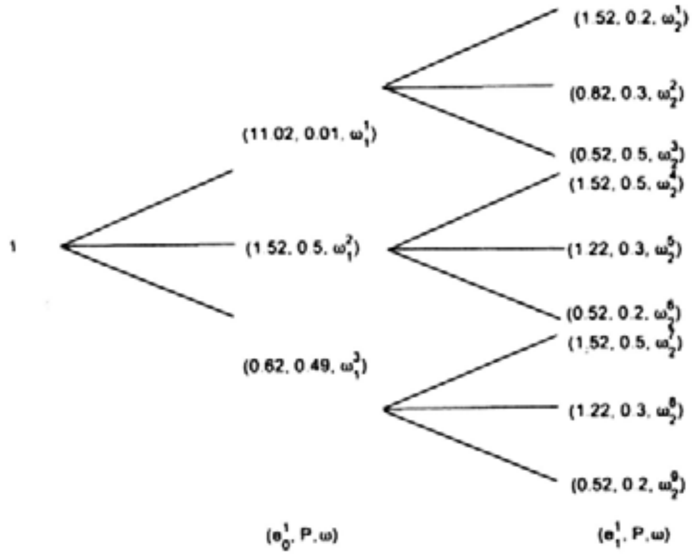


Figure 5.3: Random return of risky asset

market can be derived as

$$
\begin{aligned}
\theta_1 &= 0.7642, \quad \theta_0 = 0.7491, \\
u_0^*(1) &= -\frac{-2E_0(\theta_1 P_0)}{2r_1\theta_0 E_0(\theta_1 P_0^2)} = 0.1675, \\
x_1(\omega_1^1) &= 2.6950, \quad x_1(\omega_1^2) = 1.1038, \quad x_1(\omega_1^3) = 0.9530, \\
u_1^*(x_1(\omega_1^1)) &= -r_1 x_1(\omega_1^1)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^1) + \left(r_0 r_1 x_0 - \frac{-2}{2\theta_0}\right)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^1) = 0.4195, \\
u_1^*(x_1(\omega_1^2)) &= -r_1 x_1(\omega_1^2)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^2) + \left(r_0 r_1 x_0 - \frac{-2}{2\theta_0}\right)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^2) = 1.4031, \\
u_1^*(x_1(\omega_1^3)) &= -r_1 x_1(\omega_1^3)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^3) + \left(r_0 r_1 x_0 - \frac{-2}{2\theta_0}\right)\frac{E_1(\theta_2 P_1)}{E_1(\theta_2 P_1^2)}(\omega_1^3) = 1.5759.
\end{aligned}
$$

It can be checked that the optimal policy for the truncated mean-variance problem at ω_1^1 is $u_1^{1-2}(x_1(\omega_1^1), \lambda_1) \le 0$ for all negative λ_1. As $u_1^*(x_1(\omega_1^1)) = 0.4195$ from the above calculation, we can conclude the time consistency in efficiency does not hold in this simple example.

It is easy to see that the *pre-committed* optimal policy also solves the mean-variance portfolio selection problem without shorting. As the optimal policy for truncated mean-variance problem without shorting at ω_1^1 is $u_1^{1-2}(x_1(\omega_1^1), \lambda_1) = 0$ for all negative λ_1, the time consistency in efficiency does not hold even when no-shorting constraint is added.

If we introduce an auxiliary risky asset S_2 with the random return following a trinomial tree structure given in Figure 5.4, it is easy to check the uniqueness of the martingale measure in this market.

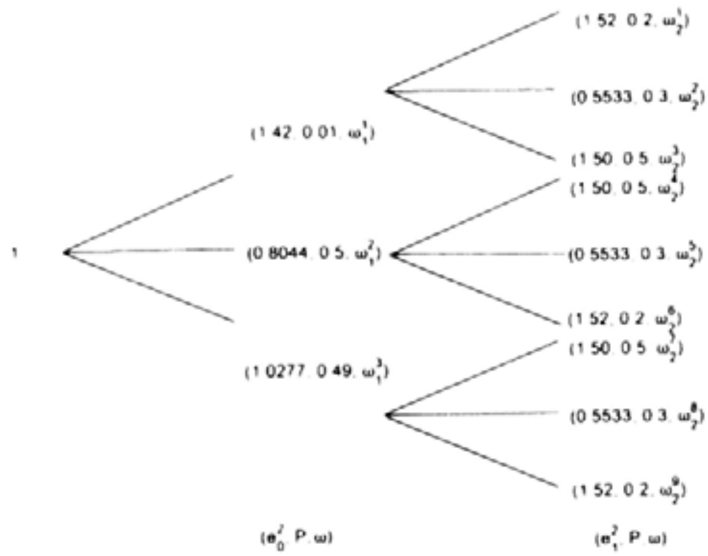


Figure 5.4: Random return of auxiliary risky asset

Since the suitable market parameter process, $\kappa^\bullet$, is given as

$$\kappa_0^\bullet = (0, -0.1)',$$
$$\kappa_1^\bullet(\omega_1^1) = (0, 0.2)',\ \kappa_1^\bullet(\omega_1^2) = (0, 0.2)',\ \kappa_1^\bullet(\omega_1^3) = (0, 0.2)'.$$

we can compute the unique signed martingale measure in $\mathcal{M}_{\kappa^\bullet}$, with its density given by

$$\frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^1) = -0.5836,\ \frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^2) = -0.2898,\ \frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^3) = -0.1639,$$
$$\frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^4) = 0.5479,\ \frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^5) = 0.9688,\ \frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^6) = 1.9510,$$
$$\frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^7) = 0.6153,\ \frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^8) = 1.0880,\ \frac{dQ^{\kappa^\bullet}}{dP}(\omega_2^9) = 2.1911.$$

The $\mathcal{F}_1$ conditional density process of the variance-optimal signed martingale measure is

$$E\left(\frac{dQ^\bullet}{dP}|\omega_1^1\right) = -0.2766,\ E\left(\frac{dQ^\bullet}{dP}|\omega_1^2\right) = 0.9548,\ E\left(\frac{dQ^\bullet}{dP}|\omega_1^3\right) = 1.0723.$$

We now proceed to find some sufficient conditions, under which time consistency in efficiency can be achieved when forcing a no-shorting constraint.

Proposition 5.7. *When $E_t(\mathbf{P}_t) > \mathbf{0}_n$, $P-a.s.$, time consistency in efficiency holds in the mean-variance portfolio selection without shorting.*

Proof: Consider the truncated mean-variance portfolio selection problem without shorting that starts from period s,

$$\begin{aligned}(MV-C1_{s-T}) \quad \min \quad & Var_s(x_T) + \lambda E_s(x_T) \\ \text{s.t.} \quad & x_{t+1} = r_t x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = s, \ldots, T-1, \\ & \mathbf{u}_t \geq \mathbf{0}_n, \; x_s \text{ is given.}\end{aligned}$$

The possible value of $E_s(x_T)$ is $[\prod_{j=s}^{T-1} r_j x_s, +\infty)$. Assume that the T-period optimal policy is given by $\mathbf{u}_t^*(x_t)$, $t = 0, \ldots, T-1$. Then we can prove that the truncated optimal policy starting from period s, $\mathbf{u}_t^*(x_t)$, $t = s, \ldots, T-1$, must satisfy the following conditions:

$$E_s(x_T)|_{\mathbf{u}^*} \geq \prod_{j=s}^{T-1} r_j x_s,$$

$$Var_s(x_T)|_{\mathbf{u}^*} = \min_{\mathbf{u}_t \geq \mathbf{0}_n} \{Var_s(x_T)|_{\mathbf{u}}, \text{ s.t. } E_s(x_T)|_{\mathbf{u}} = E_s(x_T)|_{\mathbf{u}^*}\},$$

i.e., the truncated optimal policy at the beginning of period s is also efficient for the truncated problem $(MV-C1_{s-T})$.

If it is not true, there must exist another feasible policy for $(T-s)$-period problem, $\hat{\mathbf{u}}_t(x_t)$, $t = s, \ldots, T-1$, such that

$$E_s(x_T)|_{\hat{\mathbf{u}}} = E_s(x_T)|_{\mathbf{u}^*} \geq \prod_{j=s}^{T-1} r_j x_s,$$

$$Var_s(x_T)|_{\hat{\mathbf{u}}} < Var_s(x_T)|_{\mathbf{u}^*}.$$

Combining $\mathbf{u}_t^*(x_t)$, $t = 0, \ldots, s-1$ with $\hat{\mathbf{u}}_t(x_t)$, $t = s, \ldots, T-1$, yields another T-period policy. Based on the following two relationships

$$E_0(x_T) = E_0[E_s(x_T)],$$

$$Var_0(x_T) = E_0[Var_s(x_T)] + Var_0[E_s(x_T)],$$

it can be verified that the new combined policy generates a mean-variance pair which is strictly better than $\mathbf{u}_t^*(x_t)$, $t = 0, \ldots, T-1$. It is a contradiction to the optimality of $\mathbf{u}_t^*(x_t)$, $t = 0, \ldots, T-1$. Therefore, the truncated pre-committed optimal policy is also efficient for the truncated problem, $(MV - C1_{s-T})$. Time consistency in efficiency holds. □

5.5. Conclusion

The mean-variance portfolio selection does not satisfy the multi-objective version of Bellman's principle, i.e., it does not satisfy time consistency in efficiency. When the market is frictionless, the satisfaction of time consistency in efficiency is equivalent to the nonnegativeness of the conditional density process of the variance-optimal signed martingale measure in this market. A specific corollary is that time consistency in efficiency holds when the market is complete, even the return of the risky assets are correlated during the time horizon, which represents an extension of the result in Chapter 3.

When no shorting constraint is added, the optimal policy of mean-variance portfolio selection is the same as the optimal policy of mean-variance portfolio selection in an suitable auxiliary frictionless market $\mathcal{M}_{\kappa^*}$. So does the time consistency in efficiency. We demonstrate via some simple examples that, by adding a no-shorting constraint, time inconsistency in efficiency can be eliminated in some situations. At last we derive some sufficient conditions, under which time consistency in efficiency holds in the market without shorting.

5.6. Appendix

Proof of Proposition 5.1

To solve problem (MV), we consider the auxiliary problem $\mathcal{A}(\beta)$.

$$\begin{aligned} \mathcal{A}(\beta) \quad \max \quad & E_0(-x_T^2 + \beta x_T), \\ \text{s.t.} \quad & x_{t+1} = r_t x_t + \mathbf{P_t}'\mathbf{u_t}, \quad x_0 \text{ is given.} \end{aligned}$$

Define

$$\Pi_{(MV)} \triangleq \{\mathbf{u}(\cdot) | \mathbf{u}(\cdot) \text{ is an optimal control of } (MV)\},$$

$$\Pi_{\mathcal{A}(\beta)} \triangleq \{\mathbf{u}(\cdot) | \mathbf{u}(\cdot) \text{ is an optimal control of } \mathcal{A}(\beta)\}.$$

Lemma 5.2. *For any* $\lambda < 0$, *if* $\bar{\mathbf{u}}(\cdot) \in \Pi_{(MV)}$, *then* $\bar{\mathbf{u}}(\cdot) \in \Pi_{\mathcal{A}(\beta^*)}$ *with* $\beta^* = 2E_0(\bar{x}_T) - \lambda$.

Proof: Assume that $\bar{\mathbf{u}}(\cdot) \notin \Pi_{\mathcal{A}(\beta^*)}$, then there exists $\mathbf{u}(\cdot)$ such that

$$\left(E_0(x_T^2) - E_0(\bar{x}_T^2)\right) - \beta^* \left(E_0(x_T) - E_0(\bar{x}_T)\right) < 0.$$

Set a function

$$\pi(x, y) = x - y^2 + \lambda y.$$

It is a concave function in (x, y) and

$$\pi(E_0(x_T^2), E_0(x_T)) = Var_0(x_T) + \lambda E_0(x_T),$$

which is the objective function of the problem (MV). The concavity of π implies

$$\begin{aligned} & \pi(E_0(x_T^2), E_0(x_T)) \\ \leq & \pi(E_0(\bar{x}_T^2), E_0(\bar{x}_T)) + \left(E_0(x_T^2) - E_0(\bar{x}_T^2)\right) - \left(2E_0(\bar{x}_T) - \lambda\right)\left(E_0(x_T) - E_0(\bar{x}_T)\right) \\ < & \pi(E_0(\bar{x}_T^2), E_0(\bar{x}_T)). \end{aligned}$$

Thus, $\bar{\mathbf{u}}(\cdot)$ is not optimal for the problem (MV), leading to a contradiction. $\square$

Remark 5.4. *The Lemma 5.2 also holds when $\mathbf{u_t}(\cdot)$ is constrained in the closed subset of $\mathbb{R}^n$.*

Lemma 5.3. *The optimal policy of $\mathcal{A}(\beta)$ is given by*

$$\mathbf{u_t^*}(x_t) = (\frac{\beta}{2}\prod_{j=t+1}^{T-1} r_j - r_t x_t)E_t(\theta_{t+1}\mathbf{P_t}\mathbf{P_t'})^{-1}E_t(\theta_{t+1}\mathbf{P_t}), \tag{5.15}$$

where positive θ_t can be compute recursively as,

$$\theta_t = E_t(\theta_{t+1}) - E_t(\theta_{t+1}\mathbf{P_t}')E_t(\theta_{t+1}\mathbf{P_t}\mathbf{P_t'})^{-1}E_t(\theta_{t+1}\mathbf{P_t}), \quad t = T-1,\cdots,0,$$

with boundary condition $\theta_T = 1$.

Proof: The cost-to-go function of problem $\mathcal{A}(\beta)$ is defined as,

$$J_t(x_t) \triangleq \max_{\mathbf{u}_\tau, \tau > t} E\left(-x_T^2 + \beta x_T | \mathcal{F}_t\right).$$

Clearly, it has recursion,

$$J_t(x_t) = \max_{\mathbf{u_t}} E_t\left(J_{t+1}(x_{t+1})\right),$$

$$J_T(x_T) = -x_T^2 + \beta x_T.$$

At period $t = T-1$,

$$\begin{aligned} & J_{T-1}(x_{T-1}) \\ &= \max_{\mathbf{u_{T-1}}} E_{T-1}\big\{-(r_{T-1}x_{T-1} + \mathbf{P'_{T-1}}\mathbf{u_{T-1}})^2 + \beta(r_{T-1}x_{T-1} + \mathbf{P'_{T-1}}\mathbf{u_{T-1}})\big\} \\ &= -r_{T-1}^2\theta_{T-1}x_{T-1}^2 + \beta r_{T-1}\theta_{T-1}x_{T-1} + \frac{\beta^2}{4}(1-\theta_{T-1}), \end{aligned}$$

and

$$\mathbf{u_{T-1}^*}(x_{T-1}) = (\frac{\beta}{2} - r_{T-1}x_{T-1})E_{T-1}^{-1}(\mathbf{P_{T-1}}\mathbf{P'_{T-1}})E_{T-1}(\mathbf{P_{T-1}}),$$

$$\theta_{T-1} = 1 - E_{T-1}(\mathbf{P'_{T-1}})E_{T-1}^{-1}(\mathbf{P_{T-1}}\mathbf{P'_{T-1}})E_{T-1}(\mathbf{P_{T-1}}) > 0, \quad P\text{-a.s.}$$

Note that $\theta_{T-1} > 0$ is the conclusion of the positive definite of the conditional covariance matrix of $\mathbf{e'_{T-1}}$.

At period $t = T - 2$,

$$\begin{aligned}J_{T-2}(x_{T-2}) &= \max_{\mathbf{u_{T-2}}} E_{T-2}\{J_{T-1}(x_{T-1})\} \\ &= -r_{T-2}^2 r_{T-1}^2 \theta_{T-2} x_{T-2}^2 + \beta r_{T-2} r_{T-1} \theta_{T-2} x_{T-2} + \frac{\beta^2}{4}(1 - \theta_{T-2}), \\ \mathbf{u}_{\mathbf{T-2}}^{\bullet}(x_2) &= (\frac{\beta}{2r_{T-1}} - r_{T-2} x_{T-2}) E_{T-2}^{-1}(\theta_{T-1} \mathbf{P_{T-2}} \mathbf{P}'_{\mathbf{T-2}}) E_{T-2}(\theta_{T-1} \mathbf{P_{T-2}}).\end{aligned}$$

We can define an equivalent probability measure Q with density

$$\frac{dQ}{dP} = \frac{\theta_{T-1}}{E_{T-2}(\theta_{T-1})}.$$

Then

$$\theta_{T-2} = E_{T-2}(\theta_{T-1})[1 - E_{T-2}^{Q}(\mathbf{P}'_{\mathbf{T-1}})(E_{T-2}^{Q}(\mathbf{P_{T-1}} \mathbf{P}'_{\mathbf{T-1}}))^{-1} E_{T-2}^{Q}(\mathbf{P_{T-1}})].$$

It is easy to see that the conditional covariance matrix of $\mathbf{e}'_{\mathbf{T-2}}$ under the equivalent probability measure Q is also positive definite, which further implies

$$\theta_{T-2} > 0, \quad P\text{-a.s.}$$

by noticing $E_{T-2}(\theta_{T-1}) > 0$.

Solving the problem dynamically, the conclusion follows. □

Applying the optimal policy (5.15), we have

$$E_0(x_T^{\bullet}) = \prod_{j=0}^{T-1} r_j \theta_0 x_0 + \frac{\beta}{2}(1 - \theta_0).$$

Based on Lemma 5.2, the optimal auxiliary parameter,

$$\beta^{\bullet} = 2 \prod_{j=0}^{T-1} r_j x_0 - \lambda/\theta_0,$$

is the solution of $\beta = 2E_0(x_T^{\bullet}) - \lambda$.

CHAPTER 6

CONLCUSION

Revisiting Markowitz's classical mean-variance model for markets consisting of all risky assets, one key recognition in our investigation is the dual realization of mean-variance pairs. The implication of this finding could be profound. We essentially reveal a violation of the one price law in the mean-variance world and even raise concerns of arbitrage opportunities in the sense of mean-variance. By removing the constraint of binding budget spending and reexamining the classical mean-variance problem under an expanded three-objective framework: Maximizing the expected future wealth, minimizing the risk (variance) of the future wealth and minimizing the initial investment level, we have identified the set of portfolio policies which are efficient in the original mean-variance space, and are, however, inefficient in this newly introduced three-dimensional objective space. Stimulated by the revealed non-monotonic phenomenon in the mean-variance world, we introduce the concepts of pseudo efficiency (type 1) and (type 2) and have demonstrated that we can do better than the classical mean-variance when removing the binding budget spending constraint.

The mean-variance framework in dynamic portfolio selection is not time consistent, due to the inherent nonseparable nature of the involved variance term. The trade-off between the two conflicting objectives, the expected value and the variance of the terminal wealth, is time-varying and state-dependent. In some situations where the wealth level exceeds some threshold, the trade-off may change its sign, which implies that the investor changes his/her risk attitude towards the objectives, leading to time inconsistency in efficiency and irrational trading behaviors for the remaining investment periods.

In a market with riskless asset, we retain the efficiency of the portfolio policy for all time periods by devising a revised mean-variance policy. While achieving the same mean-variance pair of the original pre-committed optimal mean-variance policy, the revised mean-variance policy enables investors to receive a free cash flow stream. Note that the probability of receiving free cash flow stream and its expected value are both path-independent when the returns of risky assets at different periods are independent.

Moreover, in a market with all risky assets, by relaxing binding budget spending at the beginning of period s, we extend the concept of pseudo efficiency (type 1 or type 2) to a dynamic setting. Two kinds of revised policies have been proposed accordingly. The first revised policy eliminates possible phenomenon of type-1 pseudo efficiency and achieves the same mean-variance pair attained by the original pre-committed optimal mean-variance policy. Furthermore, it enables investors to receive a free cash flow stream. The second revised policy eliminates possible phenomenon of type-2 pseudo efficiency and achieves the same total mean and less total variance, when compared to the original pre-committed optimal mean-variance policy. Furthermore, the continuous-time optimal mean-variance policy in a market with all risky assets has also been derived. Two revised policies, which are similar to their counterparts in the discrete-time setting have been proposed to the continuous time setting to attain better performance than the original pre-committed optimal mean-variance policy.

When the market is frictionless, the satisfaction of time consistency in efficiency is equivalent to the nonnegativeness of the conditional density process of variance-optimal signed martingale measure (VSMM) in this market. A specific corollary is that time consistency in efficiency holds when the market is complete, even when the returns of the risky assets are correlated during the time horizon.

However, when no shorting constraint is added, the optimal policy of mean-variance portfolio selection is the same as the optimal policy of mean-variance portfolio selection in an optimal auxiliary frictionless market $\mathcal{M}_{\kappa^*}$. So does the time consistency in efficiency. We illustrate by two simple examples that it is possible to eliminate time inconsistency in efficiency by enforcing a no-shorting constraint in some situations. We then give the sufficient conditions, under which time consistency in efficiency will hold in the market without shorting.

Looking into the future, the fact that we can do better than both the classical and pre-committed optimal dynamic mean-variance policies, revealed in this thesis, could have a profound impact on the theory of asset pricing and provide some possible answers to some paradoxes which have puzzled us for so many years. We understand that this further step could be very challenging, as we switch our investigation from an individual decision making framework to an equilibrium market setting.

Up to this stage, the revised policies introduced in this thesis only consider the possibility of taking money out from the portfolio. It is also reasonable to think over the possibility of adding part of the free cash flow or external money back to the portfolio. This consideration is more realistic and may further improve the mean-variance investment performance.

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