# On Steady Compressible Flows in a Duct with Variable Sections

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Abstract of thesis entitled:

*On Steady Compressible Flows in a Duct with Variable Sections*

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First, we investigate the steady Euler flows through a general 3-D axially symmetric infinitely long nozzles without irrotationality. Global existence and uniqueness of subsonic solution are established, when the variation of Bernoulli's function in the upstream is sufficiently small and mass flux has an upper critical value.

Second, we concerns the following transonic shock phenomena in a class of de Laval nozzles with porous medium posed by Courant-Friedrichs: Given a appropriately large receiver pressure $p_r$, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes $p_r$. We investigate this problem for the full Euler equations, the stability of the transonic shock is proved when the upstream supersonic flow is a small steady perturbation of the uniform supersonic flow and the corresponding pressure at the exit has a small perturbation.

# 摘要

首先, 我們研究了三維軸對稱無窮長管道中的穩態歐拉流, 得到了全局亞音速解的存在性和唯一性结論. 這裏我們要求來流的布努利函數足夠小並且質量流有一個上臨界指標. 對於這樣一個問題, 主要的困難點之一是在於一般的穩態歐拉系統在亞音速區域, 是雙曲拋物偶合的系統. 關鍵的地方是我們引進了兩個延著流線的不變數, 使得流函數框架在這裏是可以使用的. 通過這樣的框架, 歐拉方程組等價於一個擬線性的兩階方程. 另外一個困難點是，我們先驗的並不知道方程的解是否是一致亞音速的. 所以, 我們通過截斷的方法, 使得方程變成一致橢圓的. 由於我們考慮的是軸對稱問題, 我們還需要面對的一個困難是在對稱軸附近,方稱的係數會出現奇性. 通過多次的截斷, 我們找到一個可以求解的逼近問題, 並且可以得到逼近解的詳細估計. 通過研究逼近解的極限和極限函數的漸進行為, 還有對稱軸附近的精細估計, 我們從而得到了整體一致亞音速解的存在唯一性结論.

其次，我們研究了一類de Laval管道中的跨音速激波問題. 特別的, 這裏我們考慮的是多孔介質邊界問題. Courant 和Friedrichs 在他們的書中曾經提到了如下的問題: 在給定管道出口一個大的壓力條件下, 如果來流在管喉是一致超音速的, 是否在張口管道的特定位置會有激波產生並且氣體會被壓縮, 之後速度會降低為亞音速. 這個激波的位置和強度會自動由出口壓力來调节. 我們使用整體歐拉方程作為模型, 研究了上面的問題. 對於出口壓力有一个小擾動的情況下, 我們得到了穩定性的结果. 對於這個問題, 主要的難點在於激波的位置是一個自由邊界問題, 還有复雜的相容性條件需要滿足.

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# Introduction

Many basic phenomena of natural sciences are governed by nonlinear conservation laws of the form

$$\partial_t u + divF(u) = 0, \quad u \in \mathbb{R}^N, x \in \mathbb{R}^m \tag{0.0.1}$$

by neglecting those small scale physical effects such that as dissipation, dispersion, relaxation, chemical reaction and external sources. The convection term is assumed to be hyperbolic in the sense that the $n \times n$ matrix $\xi \cdot \nabla F(u)$ has $n$ real eigenvalues for all $\xi \in \mathbb{R}^m \setminus \{0\}$. For a compressible fluid, if one neglects diffusion and heat conduction, it is described the following famous compressible Euler equations:

$$\begin{cases} \partial_t \rho + div(\rho u) = 0, \\ \partial_t(\rho u) + div(\rho u \otimes u) + \nabla p = 0, \\ \partial_t(\rho E) + div(\rho u E + up) = 0, \end{cases} \tag{0.0.2}$$

which describes the fundamental physical laws in continuum mechanics: conservation of mass, momentum and energy, where $x \in \mathbb{R}^d$, $t \in \mathbb{R}^1$; the unknowns, $\rho, u \in \mathbb{R}^d$, $p$ and $E$ denote the density, velocity, pressure and total energy respectively, and $E = e + \frac{|u|^2}{2}$ with the internal energy $e$, and $p = p(\rho, e)$. The compressible Euler system is one of the most important systems of nonlinear hyperbolic conservation laws.

Investigating in this field is fantastic, since plenty of natural phenomenon are governed by balance laws. There are lots of challenging open problems, such problems will make a big deal not only in mathematics, also in physics and engineering.

One of the most important feature of (0.0.1) is that the speed of wave depends on the wave itself, which leads to great complexity and rich phenomena in the behavior of solutions. Generally, smooth solutions will break down in finite time, so the weak solution is essential for hyperbolic conservation laws both theoretically and numerically.

A rather complete and satisfactory theory exists for scalar conservation laws in arbitrary space dimensions in terms of well-posedness theory, regularity and compactness of solution operator, large time asymptotic behavior of nonlinear waves, local structure of entropy weak solutions, and convergence of various approximate solutions generated by either physical perturbations, or various numerical schemes [42, 68, 116]. All the methods working in this case rely essentially on the availability of the maximum principle.

The most important breakthrough in the mathematical theory of shock waves for systems is the celebrated random choice method pioneered by Glimm [60], which yields not only the existence of BV weak solutions for generally one dimensional strictly hyperbolic system of conservation laws with small BV data, but also precise asymptotic structures [44, 61, 86]. In 1990's, Bressan successfully proved the uniqueness and $L^1$-continuous dependence in the class of the viscosity solutions by a semigroup approach in a weighted space [14, 16]. Also, the $L^1$-continuous dependence of Glimm's solution is shown alternatively later by Liu and Yang by constructing a different interesting and very useful functional [87]. These yield a satisfactory well-posedness theory for one dimensional hyperbolic conservation laws with small BV data [15].

At the end of 1970's, an elegant theory, compensated compactness method was developed by Murat [103] and Tartar [109] to solve nonlinear partial differential equations and Tartar and DiPerna succeeded in applying the method to solve a class of hyperbolic conservation laws [46, 109]. However, this theory seems to require the availability of some type of maximum principle and the existence of plenty of entropy-entropy flux pairs for the underlying hyperbolic systems. In general, these two hold only for scalar conservation law or one dimensional $2 \times 2$ system.

As important applications for previous theories, for the isentropic gas dynamics, when the initial data is away from the vacuum, global existence of weak solution was obtained via Glimm scheme, even for large data in particular case, [104, 105]. When the vacuum appears, global existence was established by the compensated compactness theory with various kinds of approximation solutions for the system, see [43. 45, 65, 82].

However, it seems that the success of the previous theory is somewhat limited to scalar or one dimensional system of conservation laws. Since the theory for scalar conservation law depends mainly on the maximum principle, which does not hold for general systems of conservation laws. The theory of compensated compactness does not work for general system of multidimensional conservation laws either, since, in general there are few entropies for multidimensional systems. Moreover, it seems that the celebrated Glimm's method is limited to strictly hyperbolic system in one-dimension since BV space is not a suitable space for hyperbolic systems in more than one dimension [106].

For unsteady compressible Euler equations, smooth solutions will blow up in finite time for generic smooth initial data [108]. So far, there is no suitable function space for studying the global-in-time well-posedness of compressible Euler equations. Since multidimensional full compressible Euler equations are too difficult. As a first step to understand multidimensional conservation laws, many researchers involved in the work on some important physically relevant wave patterns where a lot of experimental data, numerical simulations and asymptotic results have been done. These patterns include transonic flows, standing shock in a nozzle, shock reflection phenomena, self-similar flows and other flows with various symmetry. Since 1980's, there have been a lot of works on transonic shock for quasi-one dimensional nozzles [85], global solution for steady supersonic flows past a perturbed cone [80], and unsteady spherical flows [29], etc. Although there are some multidimensional physical phenomena involved in all of these work, the governing equations are essentially one dimensional systems with various source terms. so the powerful Glimm scheme, and even compensated compactness theory can be applied. This is the biggest restriction to describe the real multidimensional phenomena.

The first significant theoretical progress for multidimensional conservation laws is due to Majda. By reducing the problem into a nonlinear hyperbolic initial boundary value problem. he showed local structural stability for multidimensional shock fronts with the help of Kreiss theory. Moreover, he successfully applied this general theory to two important concrete examples, multidimensional isentropic Euler and full Euler equations, see [88, 89]. For the later development in this direction, please refer to recent nice book by Berzoni-Gavage and Serre [10] and references therein.

One of the difficulties for compressible Euler equations is that it may have two types of discontinuous solutions, shock waves and vortex sheets. Many important approximate models were proposed to study some important wave patterns in multidimensional fluids. Two of them are potential equation and incompressible Euler equations. Potential flows originate from the study compressible flows without vorticity, while incompressible Euler flows concentrate on the effects of vorticity. For the comprehensive study on incompressible Euler equations, please refer to [92] and references therein. For the study on the existence of steady incompressible Euler flows in a bounded domain, please refer to Troshkin [110], Alber [1], Glass [49], etc, and references therein.

Compared with Euler equations, potential equation has many advantages to help understand multidimensional conservation laws [91]. In fact, for structural stability of shock front for potential flows, the theory is not only proved much simpler [93], but also quite satisfactory, there are even some global stability results [63].

Combining some fundamental idea of Majda's with the method of partial hodograph transformations, local structural stability of shock front for a steady supersonic flow past a symmetric and non-symmetric body is established in [35] and [32] respectively. Later on, Chen, Xin and Yin established global existence for supersonic flow past a perturbed cone by a deliberate choice of multipliers for energy estimates [38]. Surprisingly, it was show that the flow is smooth except for the main shock. Recently, Xin and Yin established global existence for supersonic flow past a three dimensional body under the perforated boundary conditions [122]. For the local structural stability for unsteady supersonic flow past a wedge, please see [39].

The study on subsonic flows has a long history. For global subsonic potential flows, a significant progress was due to Bers [11], who showed that for two dimensional flows past a profile, if the Mach number of the free stream is small enough, then the whole flow field will be subsonic outside the profile; furthermore, as the freestream Mach number increases, the maximum of flow speed will tend to the sound speed. Later on, Finn and Gilbarg [54] showed uniqueness of subsonic flow past a profile by maximum principles and obtained asymptotic behavior of flows at far field. For the three dimensional flows, it was studied initially by Finn and Gilbarg [55], and then by Dong [47], the final results are quite similar to the two dimensional case, that a subsonic flow exists globally if the freestream Mach number is suitably small. Moreover, the maximum of the flow speed will tend to the sound speed if the frreestream Mach number increases to some critical value.

We note that Bers' result does not apply to the flow with the critical freestream Mach number. In fact,by the maximum principle, Gilbarg and Shiffman [58] asserted that the sonic point should occur on the profile, which presupposed the existence of the smooth critical flows. in this regard, Glibarg and Shiffman [58] remarked in footnote 8:" The actual existence of critical flows past finite profiles of bounded curvature has been proved by M. Shiffman (unpublished)". Bers also mentioned this unpublished result in [12]. There is a proof on the existence of weak solution for subsonic-sonic flows via compensated compactness method very recently by Chen et al [21].

On the other hand, for flows through an infinite ling nozzle, so far it does not have a complete theory compared with what has been obtained by Bers, et al, for flow past an obstacle. In his famous survey [12], Bers proposed the following problem, for a given nozzle. show that there is a global subsonic flow through the nozzle for an appropriately given incoming mass flux. Although it seems that this problem is quite similar to the airfoil problem physically. It does not seem to be true mathematically and few study has been carried out along these lines. However, one should note the significant result due to Gilbarg, [57], where he showed that if an subsonic plane nozzle flow or axially symmetric flow approximates to uniform subsonic flows at far fields, then the flow speed on the boundary is monotone increasing with respect to the incoming mass flux by a comparison principle. Recently, Xie and Xin [112, 113, 114] answer Bers' problem first on subsonic potential flow in both 2-D and 3-D axially symmetric nozzle and Euler flow in 2-D nozzle. One of the aims of this thesis is to give a positive answer the problem for Euler flows in 3-D axially symmetric infinitely long nozzle. Moreover, we would like to show that there exists a critical value such that a global uniform subsonic flow exists uniquely in a general nozzle as long as the incoming mass flux is less than the critical value. More importantly, we would like to investigate the properties of these uniform subsonic flows, so that we can obtain a class of subsonic-sonic flows corresponding to the critical incoming mass flux as the limits of uniform subsonic flows associated with the incoming mass fluxes which increase to the critical value.

Concerning transonic flows, some significant progress was made by Morawetz, she showed that smooth transonic flows past an airfoil are not stable by multiplier method [96, 97, 98, 99]. One of the most important features for transonic-flow problems is that the governing equations are of mixed type. A unified treatment for mixed type equations was developed by Friedrichs, he used energy method to develop the whole theory for positive symmetric systems, such as existence, uniqueness of weak solutions, admissible boundary conditions, and regularity of solutions, etc [56, 70]. Recently, this idea was used by Kuzmin to treat accelerated transonic nozzle flows and transonic nozzle flows with perforated boundary conditions [69]. Another important contribution to transonic flows by Morawetz is on the existence of a weak solution to transonic flows via the theory of compensated compactness in some special cases [100, 101].

For quasi-one-dimensional model, Liu [84, 85] studied gas flow along a duct in a one-dimension model for both the case of a duct with variable sections and the case of a duct with constant sections. He showed that the flows along the expanding portion of the nozzle are stable, while flows with shock waves along the contracting duct are dynamically unstable by Glimm scheme method. Embid, Goodman, and Majda [51] studied the existence of multiple steady states with the same far field behavior for simple one-dimensional transonic model problems by using some explicit solutions in a scalar model case. They showed that only some of these solutions in a scalar model case. They showed that only some of these solutions are dynamically stable and are accessible through physical time independent perturbations.

In [20]. the authors established the existence and the stability of a uniform planar transonic shock for the two-dimensional transonic small disturbance equation (TSDE). TSDE is the first order of an asymptotic expansion for flows around slender bodies at free-stream speeds close to sonic speed and can be written into a second-order nonlinear equation mixed type in two dimension with the coefficients depending only upon the unknown function itself. Since the coefficients of the TSDE equations are independent of the gradient of the unknown function, additional compactness of solutions can be obtained which play a key role in the analysis of [20].

It should be noted that there are some important recent works on other approximate models for compressible Euler equations, such as transonic small disturbance equation, pressure gradient system, and nonlinear wave systems, etc. The existence and stability of transonic shocks, shock reflection, and Reimann problems for there systems were studied by Canic, et al, [17, 18, 19, 20] and Zheng [125, 126].

For the numerical method for hyperbolic conservation laws, please refer to [40]. There is a very nice introduction to one dimensional viscous conservation laws in [117]. The recent progress on general existence theory for compressible Navier-Stokes equations is summarized and presented in detail in [81] and [52]. For the recent work on inviscid limit of viscous problems, please refer to [13] for general one dimensional sysstem with general small BV data, [64] for multidimensional shock data and reference therein.

For the transonic flow with shock in an infinitely long nozzle, the existence and stability of multidimensional transonic shock are also established in a series of papers [25, 26, 27]. In this case, the transonic flow is governed by the inviscid potential flow equation with supersonic upstream flow at the entrance, uniform subsonic downstream flow at the exit at the infinity, and the slip boundary condition on the nozzle wall. They find that for this problem, one can prescribe the uniformity condition of the flow but can not prescribe a velocity state at infinity in the downstream direction in general. After that, Xie and Wang [115] show that the uniform transonic shock wave in an infinite cylindrical nozzle is stable with respect to a perturbation of the incoming flow and of the nozzle wall. In a recent paper [36] of Chen and Yuan, they study the uniqueness of solutions with a transonic shock in a duct, which are not necessarily small perturbations of the background solution for steady potential equation. Their results indicate that for transonic shock solutions in semi-infinitely long ducts, the a priori assumptions on the asymptotic behavior of transonic shock solutions in an infinitely long duct in [25, 26, 27] may not be necessary. One just need the reasonable assumptions that the velocity and acceleration of the flow are bounded.

Concerning the transonic flow for full Euler equation, the authors in [23] establish the existence and uniqueness of transonic flow with a transonic shock when the flow is in a finite nozzle of slowly varying cross-sections with nearly horizontal velocity at the exist of the nozzle.

As described in [41], from the physical point of view, it is more reasonable to prescribe the pressure at the exit of the nozzle. For the steady potential equation and the slowly-varying nozzle walls, Xin-Yin [118, 121] showed that the well-posedness of the transonic shock problem can not be true for arbitrary large pressure at exit. In the case of instability, they found a class of pressures such that the transonic shock problem is stable and satisfies the given boundary conditions. The main ingredients of their analysis are a generalized hodograph transformation and multiplier methods for elliptic equation with mixed boundary conditions and corner singularities.

With the exit pressure condition, this phenomena is also true for the steady two dimensional compressible Euler equations in $(-N_1, N_2) \times (0, b)$ when the shock is assumed to go through some fixed point. In [119], the authors proved the uniqueness of the transonic shock problems in a 2-D nozzle under the assumption that the shock wave goes through a fixed point and then, based on the uniqueness result, they proved the nonexistence of transonic solution for two dimensional nozzle with flat walls without the assumption that the shock front goes through a fixed point. For a divergent nozzle with a given large pressure at the exit of the nozzle, there is also no such a transonic shock solution if the shock front is assumed to go through a fixed point.

Note that the uniqueness results in [119], [118, 121] are obtained under the additional assumption that the shock curve goes through a fixed point in advance. However, as showed in [119], this additional condition may lead the transonic shock problem to be over-determined for a divergent nozzle. In a series of paper [71, 72, 73], the authors have found a new way to determine the position of the transonic shock and remove the undesired assumption that the shock curve goes through a fixed point so that the transonic shock problem as described by Courant-Friedrich is well-posed. Compared with the results in [118, 121], the different Bernoulli's constants or entropies on two sides of the shock is closely related to the existence and the position of the shock wave. In their analysis, the transonic shock problem was reduced to solve a boundary value problem for the steady Euler system in the subsonic domain with a free boundary (the shock surface), which can be reformulated as a system consisting of an ordinary differential equation for the shock with a free initial position, a first order nonlinear elliptic system for the pressure and angular velocity, and two transport equations for the specific entropy and Bernoulli's function respectively on a fixed domain. The new key ingredients are to establish the monotonic property of the pressure along the nozzle walls and to estimate the gradients of the solution instead of the solution itself so that they can avoid the difficulties induced by the unknown position of the shock. Actually in 2-D case, they obtained the existence and uniqueness of transonic shock in the slowly-varying nozzle with variable end pressure. In 3-D case, the uniqueness is still true but the existence result is only obtained in the case of the axis-symmetric exit pressure. It should be emphasized that in almost all the previous work [71, 72, 73], the diverging part of the nozzle is assumed to change slowly so that the subsonic flow of the background transonic shock is close to a constant state, which are crucial in the procedures of analysis and related estimates. In [74], the authors have found an effective way to decompose the Euler system to a canonical form, in which the hyperbolic part and the elliptic part are only weakly coupled in their coefficients. The key issue is to solve a boundary value problem for a first $2 \times 2$ elliptic system with non-local terms and an unknown parameter. By a new elaborate iteration scheme, the authors are able to solve the transonic shock problem without some artificial boundary conditions on requirement of the divergent part of the nozzle being slowly-varying.

In this thesis, first we give an affirmative answer to Bers' problem on subsonic Euler flows in infinitely long nozzle. We will show that, for 3-D axially symmetric nozzle, if the variation of Bernoulli's constant in far field is sufficiently small, the there exist true Euler flows in the nozzles with prescribed mass flux in a suitably regime with an upper critical value. One of the main difficulties is that the governing equations are a mixed elliptic-hyperbolic system. We introduce two invariants along the stream lines, then stream function formulation can be available. By this formulation, Euler equations are equivalent to a quasilinear second order equation for a stream function so that the hyperbolicity of the particle path is already involved. Another main difficulty is that one does not know a priorly whether the flow is uniformly subsonic, we first truncate the equation to be a uniformly elliptic equation, then we need to give a priori estimate as good as possible to remove the truncation. For axially symmetric flows, besides the difficulties above, the equation contains singular coefficients near the symmetric axis. We use several truncations to have a solvable approximate problem and detailed elliptic estimates. By those, we proved the global existence of subsonic solution and obtained the asymptotic behavior. For this problem, all boundary conditions and far field conditions are carefully selected from physical observation, such that the problem itself is closed.

Second, we investigate transonic shock phenomena in a class of de Laval nozzles with porous medium posed. For the full Euler equations, the stability of the transonic shock is proved when the upstream supersonic flow is a small steady perturbation of the uniform supersonic flow and the corresponding pressure at the exit has a small perturbation. The main difficulties are the free boundary problem of the shock position and the complicated compatibility conditions because of the leakage on the boundary. The key point is to establish the monotonicity between shock position and end pressure. With this monotonic property, we proved the existence and uniqueness of a transonic shock solution to the full steady compressible Euler system in this class of nozzles.

The rest of the thesis is arranged as follows: In Chapter 1, we recall some basic definition and theorems of elliptic equation which we used in following chapters. In Chapter 2, we established the existence and uniqueness of subsonic flows through infinitely long three dimensional axially symmetric nozzles. In Chapter 3, two dimensional Transonic Shocks in Nozzle with Porous Medium is investigate. In the last Chapter, we give a brief summary and discuss about the future work.

# Chapter 1

# Preliminaries

In this Chapter we will introduce some basic definitions and theorems for elliptic equations, which will be used in this thesis

Consider a second order differential equation of the form

$$L = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u \qquad a^{ij} = a^{ji} \tag{1 0 1}$$

where $x = (x_1, \quad, x_n)$ lies in a domain $\Omega$ of $\mathbb{R}^n$, $n \geq 2$

**Definition 1.0.1** *The operator $L$ in (1 0 1) is said to be elliptic at a point $x \in \Omega$ if the coefficient matrix $[a^{ij}(x)]$ is positive, that is , if $\lambda(x)$, $\Lambda(x)$ denote respectively the minimum and maximum eigenvalues of $[a^{ij}]$, then*

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2$$

*for all $\xi = (\xi_1, \quad, \xi_n) \in \mathbb{R}^n \setminus \{0\}$ If $\lambda > 0$ in $\Omega$, then $L$ is called elliptic in $\Omega$, and strictly elliptic if $\lambda \geq \lambda_0 > 0$ for some constant $\lambda_0$ If $\Lambda/\lambda$ is bounded in $\Omega$, we shall call $L$ uniformly elliptic in $\Omega$*

The maximum principle is one of the most important properties for elliptic operator, which is the following

**Theorem 1.0.1** *(Chapter3 of [59]) Let $L$ in (1 0 1) be elliptic in the bounded domain $\Omega$ Suppose that*

$$Lu \geq 0 (\leq 0) \text{ in } \Omega, \quad c = 0 \text{ in } \Omega,$$

*with $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ Thus the maximum (minimum) of $u$ in $\overline{\Omega}$ is achieved on $\partial\Omega$, that is,*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \ (\inf_{\Omega} u = \inf_{\partial\Omega} u).$$

Next, we introduce a class of special domains:

**Definition 1.0.2** *The domain $\Omega$ is said to satisfy an interior sphere condition at $x_0 \in \partial\Omega$ if there exists a ball $B \subset \Omega$ with $x_0 \in \partial B$. $\Omega$ is said to satisfy the exterior sphere condition at $x_0 \in \partial\Omega$ if the complement of $\Omega$ satisfies an interior sphere condition at $x_0 \in \partial\Omega = \partial\Omega^c$.*

Then we state the following Hopf Lemma.

**Lemma 1.0.2** *(Chapter3 of [59]) Suppose that $L$ defined in (1.0.1) is uniformly elliptic, $c = 0$ and $Lu \geq 0$ in $\Omega$. Let $x_0 \in \partial\Omega$ be such that*

(i) *$u$ is continuous at $x_0$,*

(ii) *$u(x_0) > u(x)$ for all $x \in \Omega$,*

(iii) *$\partial\Omega$ satisfies an interior sphere condition at $x_0$.*

*Then the outer normal derivative of $u$ at $x_0$, if it exists, satisfies the strict inequality*

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

*If $c \leq 0$ and $c/\lambda$ is bounded, the same conclusion holds provided $u(x_0) \geq 0$, and if $u(x_0) = 0$, the same conclusion holds irrespective of the sign of $c$.*

By the Hopf Lemma above, we can strengthen the maximum principle:

**Theorem 1.0.3** *(Chapter3 of [59]) Let $L$ in (1.0.1) be uniformly elliptic, $c = 0$ and $Lu \geq 0 (\leq 0)$ in a domain $\Omega$ (not necessary bounded). Then if $u$ achieves its maximum (minimum) in the interior of $\Omega$, $u$ is constant . If $c \leq 0$ and $c/\lambda$ is bounded, then $u$ can not achieve a nonnegative maximum (nonpositive minimum) in the interior of $\Omega$ unless it is constant.*

In Theorem 1.0.1 and Theorem 1.0.2, $Lu$ is either nonnegative or nonpositive. If this assumption is not true, we can write $u^+ = max\{u, 0\}$, $f^- = min\{f, 0\}$ and then the following a priori estimate holds.

**Theorem 1.0.4** *(Chapter3 of [59]) Let $Lu \geq f(= f)$ in a bounded domain $\Omega$, where $L$ defined in (1.0.1) is elliptic, $c \leq 0$, and $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Then*

$$\sup_{\Omega} u(|u|) \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} \frac{|f^-|}{\lambda} \left(\frac{|f|}{\lambda}\right), \tag{1.0.2}$$

*where $C$ is a constant depending only on diam$\Omega$ and $\beta = \sup |b/\lambda|$. In particular, if $\Omega$ lies between two parallel planes with distance $d$, then (1.0.2) is satisfies with $C = e^{(\beta+1)d} - 1$.*

Now, we note that Hölder space is a suitable space to study classical solution of elliptic equations, which is introduced in many textbooks.

$f$ is said to be uniformly Hölder continuous with exponent $\alpha$ in $D$ if the quantity

$$[f]_{\alpha,D} = \sup_{x,y\in D\ x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},\ 0 < \alpha \leq 1$$

is finite.

Set

$$\begin{aligned}
[u]_{k,0,\Omega} &= |D^k u|_{0,\Omega} = \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u|;\ k = 0, 1, 2, \cdots, \\
[u]_{k,\alpha,\Omega} &= [D^k u]_{\alpha,\Omega} = \sup_{|\beta|=k} [D^\beta u]_{\alpha,\Omega}, \\
\|u\|_{C^k(\overline{\Omega})} &= |u|_{k,\Omega} = |u|_{k,0,\Omega} = \sum_{j=0}^{k} [u]_{j,0,\Omega} = \sum_{j=0}^{k} |D^j u|_{0,\Omega}, \\
\|u\|_{C^{k,\alpha}(\overline{\Omega})} &= |u|_{k,\alpha,\Omega} = |u|_{k,\Omega} + [u]_{k,\alpha,\Omega} = |u|_{k,\Omega} + [D^k u]_{\alpha,\Omega}.
\end{aligned}$$

Define

$$\begin{aligned}
C^{k,\alpha}(\overline{\Omega}) &= \{u | u \in C^k(\overline{\Omega}), |u|_{k,\alpha,\Omega} < \infty\}, \\
C^{k,\alpha}(\Omega) &= \{u | u \in C^k(\Omega), |u|_{k,\alpha,E} < \infty, \text{for each compact set} E \Subset \Omega\}.
\end{aligned}$$

The Hölder space has the following properties:

**Lemma 1.0.5** *(i) If $f \in C^{k,\alpha}(\overline{U}, \mathbb{R})$ and $g \in C^{j,\beta}(\overline{\Omega}, \overline{U})$, then $f \circ g \in C^{m,\alpha\beta}(\overline{\Omega}, \mathbb{R})$, where $m = \min(k, j)$.*

*(ii)(Chapter6 of [59])Let $\Omega$ be a $C^{k,\alpha}$ domain in $\mathbb{R}^n$ (with $k \geq 1$) and let $S$ be a bounded set in $C^{k,\alpha}(\overline{\Omega})$. Then $S$ is precompact in $C^{j,\beta}(\Omega)$ if $j + \beta < k + \alpha$.*

At this stage, we can state the global Schauder estimate for elliptic equations in terms of Hölder space.

**Theorem 1.0.6** *(Chapter6 of [59]) Let $\Omega$ be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of $Lu = f$ in $\Omega$, where $f \in C^{\alpha}(\overline{\Omega})$ and $L$ is of the form (1.0.1) and the coefficients of $L$ satisfy*

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n,$$

*and*

$$|a^{ij}|_{0,\alpha,\Omega}, |b^i|_{0,\alpha,\Omega}, |c|_{0,\alpha,\Omega} \leq \Lambda,$$

*for positive constants $\lambda$ and $\Lambda$. Let $\varphi(x) \in C^{2,\alpha}(\overline{\Omega})$, and suppose $u = \varphi$ on $\partial\Omega$. Then*

$$|u|_{2,\alpha,\Omega} \leq C(|u|_{0,\Omega} + |\varphi|_{2,\alpha,\Omega} + |f|_{0,\alpha,\Omega})$$

*where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$.*

By virtue of the Schauder estimate above, the solvability of Poisson equation and the method of continuity, we can show the existence of elliptic boundary value problems.

**Theorem 1.0.7** *(Chapter6 of [59]) Let $L$ in (1.0.1) be strictly elliptic in a bounded domain $\Omega$, with $c \leq 0$, and let $f$ and the coefficients of $L$ belong to $C^{\alpha}(\overline{\Omega})$. Suppose that $\Omega$ is a $C^{2,\alpha}$ domain and that $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Then the Dirichlet problem,*

$$Lu = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

*has a (unique) solution lying in $C^{2,\alpha}(\overline{\Omega})$.*

For the existence of quasilinear elliptic equations, we usually use the Leray-Schauder fixed point theorem as following:

**Theorem 1.0.8** *(Chapter11 of [59]) Let $T$ be a compact mapping of a Banach space $\mathcal{B}$ into itself, and suppose there exists a constant $M$ such that*

$$\|x\|_{\mathcal{B}} < M$$

*for all $x \in \mathcal{B}$ and $\sigma \in [0,1]$ satisfying $x = \sigma Tx$. Then $T$ has a fixed point.*

Then, we get the existence theorem for quasilinear elliptic equations.

**Theorem 1.0.9** *(Chapter11 of [59]) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and suppose that*

$$Qu = a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du)$$

*is elliptic in $\overline{\Omega}$ with coefficients $a^{ij}$, $b \in C^{\alpha}(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$, $0 < \alpha < 1$. Let $\partial\Omega \in C^{2,\alpha}$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Then, if for some $\beta > 0$ there exists a constant $M$, independent of $u$ and $\sigma$, such that every $C^{2,\alpha}(\overline{\Omega})$ solution of the Dirichlet problem,*

$$\begin{cases} Q_{\sigma}u = a^{ij}(x,u,Du)D_{ij}u + \sigma b(x,u,Du) = 0 \ \text{in } \Omega, \\ u = \sigma\varphi \ \text{on } \partial\Omega, \qquad 0 \le \sigma \le 1 \end{cases}$$

*satisfies*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} < M,$$

*it follows that the Dirichlet problem, $Qu = 0$ in $\Omega$, $u = \varphi$ on $\partial\Omega$, is solvable in $C^{2,\alpha}(\overline{\Omega})$.*

For elliptic equations of two variables, we have the following nice estimate and then the existence theory.

**Theorem 1.0.10** *(Chapter12 of [59]) Let $u$ be a bounded $C^2(\overline{\Omega})$ solution of*

$$Lu = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = f,$$

*where $\Omega \subset \mathbb{R}^2$ is a $C^2$ domain and that $u = 0$ on $\partial\Omega$, $L$ is uniformly elliptic, satisfying*

$$\lambda(\xi^2+\eta^2) \le a\xi^2 + 2b\xi\eta + c\eta^2 \le \gamma\lambda(\xi^2+\eta^2),$$

*where $\lambda(x, y) > 0$ and $\gamma \ge 1$ is a constant. Then for some $\alpha = \alpha(\gamma) > 0$, we have*

$$|u|_{1,\alpha,\Omega} \le C$$

*where $C = C(|u|_0, |f/\lambda|_0, |\partial\Omega|_{C^2}, \gamma, diam\Omega)$ is a continuous function satisfying $C(0, 0, |\partial\Omega|_{C^2}, \gamma, diam\Omega) = 0$. Moreover, $C$ is increasing with respect to all the above five quantities.*

**Theorem 1.0.11** *Let $\Omega$ be a $C^{2,\beta}$ domain in $\mathbb{R}^2$ and let $\varphi \in C^{2,\beta}(\overline{\Omega})$. Then if quasilinear operator*

$$Qu = a(x,y,u,u_x,u_y)u_{xx} + 2b(x,y,u_x,u_y)u_{xy} + c(x,y,u_x,u_y)u_{yy} + f(x,y,u_x,u_y)$$

*satisfies conditions*

- *The functions $a = a(x,y,u,p,q)$, $, f = f(x,y,u,p,q)$ are defined for all $(x,y,u,p,q)$ in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2$ and, in addition, $a, b, c, f \in C^\beta(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2)$ for some $\beta \in (0,1)$,*

- *The operator $Q$ is uniformly elliptic in $\Omega$ for bounded $u$, that is, the eigenvalues $\lambda = \lambda(x,y,u,p,q)$, $\Lambda = \Lambda(x,y,u,p,q)$ of the coefficient matrix satisfy*

  $$1 \le \frac{\Lambda}{\lambda} \le \gamma(|u|), \quad \forall (x,y,u,p,q) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$$

  *where $\gamma$ is non-decreasing,*

- *The function $f$ satisfies the structure conditions*

  $$\frac{|f|}{\lambda} \le \mu(|u|)(1+|p|+|q|),$$
  $$\frac{f}{\lambda} signu \le \nu(1+|p|+|q|), \quad \forall (x,y,u,p,q) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$$

  *where $\mu$ is non-decreasing and $\nu$ is a non-negative constant*

*Then the Dirichlet problem*

$$Qu = 0 \ in \ \Omega, \quad u = \varphi \ on \ \partial\Omega,$$

*has a solution* $u \in C^{2,\beta}(\overline{\Omega})$.

Finally, we state the gradient estimate for solutions to elliptic equations, which lead the $C^{1,\alpha}$ estimate.

**Theorem 1.0.12** *(Chapter13 of [59]) Let* $u \in C^2(\overline{\Omega})$ *satisfy* $|u|_{1;\Omega} = K$ *and* $Qu = 0$ *in* $\Omega$ *where* $Q$ *is elliptic in* $\overline{\Omega}$ *and is of divergence form*

$$Qu = divA(x, u, Du) + B(x, u, Du)$$

*with* $A \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $B \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. *Suppose that there exist positive constants* $\lambda_K$, $\Lambda_K$ *and* $\mu_K$ *such that*

$$D_{p_j} A^i(x, z, p)\xi_i\xi_j \geq \lambda_K |\xi|^2,$$
$$|D_{p_j} A^i(x, z, p)| \leq \Lambda_K,$$
$$|p_j D_z A^i(x, z, p) + D_{x_j} A^i(x, z, p)| + |B(x, z, p)| \leq \mu_K,$$

*for all* $x \in \Omega$, $|z| + |p| \leq K$, $i, j = 1, \cdots, n$. *Then if* $\partial\Omega \in C^2$ *and* $u = \varphi$ *on* $\partial\Omega$, *where* $\varphi \in C^2(\overline{\Omega})$ *with* $|\varphi|_{2,\Omega} = \Phi$, *we have the estimate*

$$[Du]_{\alpha;\Omega} \leq C$$

*where*

$$C = C(n, K, \Lambda_K/\lambda_K, \mu_K/\lambda_K, \Omega, \Phi), \quad \alpha = \alpha(n, \Lambda_K/\lambda_K, \Omega).$$

# Chapter 2

# Subsonic Euler Flows in Axially Symmetric Nozzle

## 2.1 Introduction and main results

In this paper, we establish the existence and uniqueness of the steady subsonic flow through an infinitely long axisymmetric nozzle. Such problems rise naturally in the physical experiments and the engineering designs. (See [12] and [41] and references cited therein)

To understand some important phenomena in Steady ideal fluids, it is nature to start from the steady Euler equations. However, the steady Euler equations themselves are not so easy to tackle. An approximate model is potential flow, which comes from the study of flows without vorticity. Since 1950's, a lot of progress has been made in understanding the potential flows and Euler flows. Subsonic potential flows were studied extensively by Shiffman [111], Bers [11], [12], Finn, Gilbarg [54], [55], Chen, et al [21] and Morawetz [96]-[101], et al. In 2006, Xie and Xin [112], [113] established the wellposedness for subsonic and supersonic-sonic potential flows through infinitely long 2-D and 3-D axially symmetric nozzles. Recently, Du, Xin and Yan [48] proved the existence and uniqueness of global subsonic potential flows through infinitely long nozzles for arbitrary dimension. For subsonic Euler flows, Xie and Xin [114] established the global existence of steady subsonic Euler flows through infinitely long nozzle by using the stream function formulation when the variation of Bernoulli's constant in the upstream is sufficiently small and mass flux is in a suitable regime with an upper critical value.

In this chapter, we establish the existence and uniqueness of global steady subsonic Euler flows though 3-D infinitely long axially symmetric nozzles.

Consider three-dimensional steady isentropic Euler equations

$$\begin{cases} (\rho u_1)_{x_1} + (\rho u_2)_{x_2} + (\rho u_3)_{x_3} = 0, \\ (\rho u_1^2)_{x_1} + (\rho u_1 u_2)_{x_2} + (\rho u_1 u_3)_{x_3} + P_{x_1} = 0, \\ (\rho u_1 u_2)_{x_1} + (\rho u_2^2)_{x_2} + (\rho u_2 u_3)_{x_3} + P_{x_2} = 0, \\ (\rho u_1 u_3)_{x_1} + (\rho u_2 u_3)_{x_2} + (\rho u_3^2)_{x_3} + P_{x_3} = 0, \end{cases} \tag{2.1.1}$$

where $\rho$ is the density, $(u_1, u_2, u_3)$ is the velocity, and $P = P(\rho)$ denotes the pressure. In general, we assume $P = P(\rho)$ is smooth with $P'(\rho) > 0$ and $P''(\rho) \geq 0$ for $\rho > 0$, $c(\rho) = \sqrt{P'(\rho)}$ is called the sound speed.For the ideal polytropic gas, the state equation is given by

$$P(\rho) = A\rho^\gamma,$$

where $A$ and $\gamma$ are positive constants with $\gamma > 1$. Since the length of the nozzles is usually much larger than their cross-sections in the practical application, then problem can be formulated mathematically into an infinite long nozzle problem. We consider the flows through an infinitely long axisymmetric nozzle as

$$\Omega_0 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \middle| 0 \leq \sqrt{x_2^2 + x_3^2} < f(x_1), \ -\infty < x_1 < +\infty \right\},$$

where $f(x_1)$ satisfies

$$\begin{aligned} &f(x_1) \to 1, \quad \text{as } x_1 \to -\infty, \qquad f(x_1) \to r_0 > 0, \quad \text{as } x_1 \to +\infty, \\ &\|f\|_{C^{2,\alpha}(\mathbb{R})} \le C \quad \text{for} \quad \text{some } \alpha > 0, \ \ C > 0 \ \text{ and } \ \inf_{\mathbb{R}} f(x_1) = b > 0. \end{aligned} \tag{2.1.2}$$

Assume that the nozzle walls are solid, then the flow satisfies no-slip boundary condition

$$(u_1, u_2, u_3) \cdot \vec{n} = 0 \quad \text{on } \partial\Omega, \tag{2.1.3}$$

where $\vec{n}$ is the unit outward normal to the nozzle walls. The continuity equation in (2.1.1) and the no-flow boundary condition (2.1.3) imply that the mass flux

$$\int_\Sigma (\rho u_1, \rho u_2, \rho u_3) \cdot \vec{l} ds \equiv m_0 \tag{2.1.4}$$

remains for some positive constant $m_0$, where $\Sigma$ is any surface transversal to the $x_1$-axis direction, $\vec{l}$ is the normal of $\Sigma$ in the positive $x_1$-axis direction.

In this paper, we focus on the axisymmetric flows, so let the fluid density and velocity be $\rho(x, r)$ and $(U(x, r), V(x, r), W(x, r))$ in cylinderical coordinates, where $U, V, W$ are axial velocity, radial velocity and swirl velocity respectively, $x = x_1$, $r = \sqrt{x_2^2 + x_3^2}$. Furthermore, we seek such a flow without swirls, one has

$$u_1 = U(x, r), \quad u_2 = V(x, r)\frac{x_2}{r}, \quad u_3 = V(x, r)\frac{x_3}{r}. \tag{2.1.5}$$

Then, instead of (2.1.1), we have

$$\begin{cases} (r\rho U)_x + (r\rho V)_r = 0, \\ \left(r\rho U^2\right)_x + (r\rho UV)_r + rP_x = 0, \\ (r\rho UV)_x + \left(r\rho V^2\right)_r + rP_r = 0. \end{cases} \tag{2.1.6}$$

The system of conservation laws (2.1.6) in the cylinderical coordinates can be written in a matrix form as

$$AF_x + BF_r + C = 0,$$

where

$$A = \begin{pmatrix} \frac{Uc^2(\rho)}{\rho} & c^2(\rho) & 0 \\ c^2(\rho) & \rho U & 0 \\ 0 & 0 & \rho U \end{pmatrix}, \quad B = \begin{pmatrix} \frac{Vc^2(\rho)}{\rho} & 0 & c^2(\rho) \\ 0 & \rho V & 0 \\ c^2(\rho) & 0 & \rho V \end{pmatrix}, \quad C = \begin{pmatrix} \frac{Vc^2(\rho)}{r} \\ 0 \\ 0 \end{pmatrix},$$

and $F = (\rho, U, V)^t$ is the unknown function. Then the eigenvalues of the symmetric system are

$$\lambda_1 = \frac{V}{U}, \quad \lambda_{2,3} = \frac{UV \pm c(\rho)\sqrt{U^2 + V^2 - c^2(\rho)}}{U^2 - c^2(\rho)},$$

which are the solutions of

$$det(\lambda A - B) = 0.$$

Clearly, if the flow is supersonic, ie., $U^2+V^2-c^2(\rho) > 0$, the system has three real eigenvalues, the steady Euler system (2.1.6) is hyperbolic; Whereas, if $U^2 + V^2 - c^2(\rho) < 0$, ie., the flow is subsonic, the system has a real eigenvalue $\lambda_1$ and two complex eigenvalues $\lambda_{2.3}$, the steady Euler system (2.1.6) is hyperbolic-elliptic coupled system, which implies we have to deal with the hyperbolic mode, even for globally subsonic flow. This is the one of main difficulties in this paper.

Rewrite the axisymmetric nozzle as

$$\Omega_0 = \{(x, r)|0 \leq r < f(x), \quad -\infty < x < +\infty\}$$

with axis and boundary of the nozzle

$$T_1 = \{(x, r)|r = 0, -\infty < x < +\infty\}, \qquad T_2 = \{(x, r)|r = f(x), -\infty < x < +\infty\}.$$

For convenience, we denote by $\Omega$ the interior of the nozzle except the axis,

$$\Omega = \Omega_0 \backslash T_1 = \{(x, r)|0 < r < f(x), \quad -\infty < x < +\infty\}.$$

Here, $f$ satisfies the condition (2.1.2).

The no-flow boundary condition (2.1.3) becomes

$$(U, V, 0) \cdot \vec{n}_1 = 0 \quad \text{on } T_2, \tag{2.1.7}$$

where $\vec{n}_1$ is the unit outer normal of nozzle walls in cylinderical coordinates.

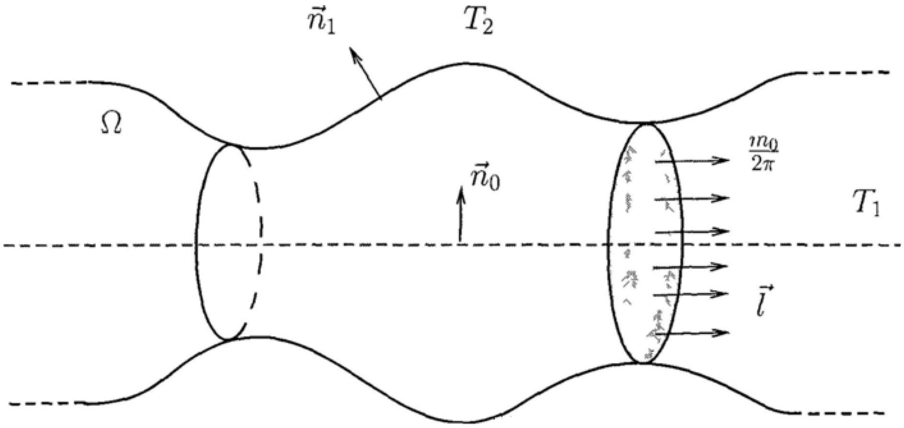
Moreover, since $T_1$ is the symmetry axis in the original problem, by symmetric property, we also have

$$(U, V, 0) \cdot \vec{n}_0 = 0 \quad \text{on } T_1, \tag{2.1.8}$$

where $\vec{n}_0$ is the unit vector which is perpendicular to the axis $T_1$. The mass flux condition (2.1.4) can be rewritten in the cylinderical coordinates as

$$\int_{\Sigma} (r\rho U, r\rho V, 0) \cdot \vec{l} dS \equiv m = \frac{m_0}{2\pi}, \tag{2.1.9}$$

where $\Sigma$ is any curve transversal to the $x$-axis direction and $\vec{l}$ is unit normal of $\Sigma$.




Due to the continuity equation in cylinderical coordinates

$$(r\rho U)_x + (r\rho V)_r = 0, \tag{2.1.10}$$

one can introduce a stream function $\psi = \psi(x, r)$ such that

$$\psi_x = -r\rho V, \quad \psi_r = r\rho U.$$

Combining the continuity equation, when the flow is away from vacuum, the momentum equations are equivalent to

$$UU_x + VU_r + h(\rho)_x = 0, \quad UV_x + VV_r + h(\rho)_r = 0, \tag{2.1.11}$$

where $h(\rho)$ is the enthalpy of the flow which satisfies $h'(\rho) = \dfrac{P'(\rho)}{\rho}$. To determine $h(\rho)$, we have to specify the integral constant. For example, we always choose

$$\begin{cases} h(0) = 0 & \text{for polytropic gas} \quad \gamma > 1, \\ h(0) = 1 & \text{for isothermal gas} \quad \gamma = 1, \end{cases} \tag{2.1.12}$$

Recalling the definition of the stream function, (2.1.11) implies the Bernoulli's law

$$\nabla^{\perp}\psi \cdot \nabla \left( h(\rho) + \frac{U^2 + V^2}{2} \right) = 0, \tag{2.1.13}$$

where $\nabla = (\partial_x, \partial_r)$ and $\nabla^{\perp} = (\partial_r, -\partial_x)$. The quantity $B(\rho, U, V) = h(\rho) + \dfrac{U^2 + V^2}{2}$ is so-called Bernoulli's function, which remains a constant along each streamline.

For the Euler flows in the axisymmetric nozzle, we assume that Bernoulli's function is given in the upstream, namely,

$$h(\rho) + \frac{U^2 + V^2}{2} \longrightarrow B(r) \text{ as } x \to -\infty, \tag{2.1.14}$$

where $B(r)$ is a smooth function defined on $[0, 1]$.

Before stating the main results in this paper, we give some notations as follows

$$B_0 = \inf_{\rho > 0} h(\rho) = \begin{cases} 0 & \text{for polytropic gas} \quad \gamma > 1, \\ -\infty & \text{for isothermal gas} \quad \gamma = 1, \end{cases}$$

$$\underline{B} = \inf_{r \in [0,1]} B(r), \quad \bar{B} = \sup_{r \in [0,1]} B(r), \quad \delta = \|B'(r)\|_{C^{0,1}([0,1])} \tag{2.1.15}$$

**Theorem 2.1.1** *(Existence) Suppose the nozzle satisfies (2.1.2) and the Bernoulli's function $B(r)$ in the upstream satisfies*

$$\underline{B} > B_0, \quad B'(r) \in C^{0,1}([0,1]), \quad B'(0) = 0, \quad B'(r) \geq 0 \quad on \quad r \in [0,1]. \tag{2.1.16}$$

*Then there exists a $\delta_0 > 0$ such that if $\delta \leq \delta_0$, then there exists $\tilde{m} \leq 2\delta_0^{\frac{\gamma}{2}}$ such that for any $m \in (\delta^\gamma, \tilde{m})$, there exists an axisymmetric subsonic flow through the nozzle with mass flux condition (2.1.9) and the asymptotic condition (2.1.14) in the upstream.*

*Furthermore, the flow is globally uniformly subsonic and the axial velocity is always positive, i.e.,*

$$\sup_{\bar{\Omega}} \left(U^2 + V^2 - c^2(\rho)\right) < 0 \quad and \quad U > 0 \quad in \quad \bar{\Omega}; \tag{2.1.17}$$

**Theorem 2.1.2** *(Properties of the Flows) Suppose the hypotheses of Theorem 2.1.1 hold. Then the subsonic flow in Theorem 2.1.1 satisfies*

$$\|\rho\|_{C^{1,\alpha}(\Omega)}, \quad \|U\|_{C^{1,\alpha}(\Omega)}, \quad \|V\|_{C^{1,\alpha}(\Omega)} \leq C \tag{2.1.18}$$

*for some constant $C > 0$, and possesses the following asymptotic behaviors in far fields*

$$\begin{aligned} &\rho \to \rho_0 > 0, \quad U \to U_0(r) > 0, \quad V \to 0 \quad as \quad x \to -\infty, \\ &\nabla\rho \to 0, \quad \nabla U \to (0, U_0'(r)) > 0, \quad \nabla V \to 0 \quad as \quad x \to -\infty, \end{aligned} \tag{2.1.19}$$

*uniformly for $r \in K_1 \subset\subset (0,1)$, and*

$$\begin{aligned} &\rho \to \rho_1 > 0, \quad U \to U_1(r) > 0, \quad V \to 0 \quad as \quad x \to +\infty, \\ &\nabla\rho \to 0, \quad \nabla U \to (0, U_1'(r)) > 0, \quad \nabla V \to 0 \quad as \quad x \to +\infty, \end{aligned} \tag{2.1.20}$$

*uniformly for $r \in K_2 \subset\subset (0, r_0)$, where $\rho_0$ and $\rho_1$ are both positive constants, and $\rho_0$, $\rho_1$, $U_0(r)$ and $U_1(r)$ can be determined by $m$, $B(r)$ and $r_0$ uniquely.*

**Theorem 2.1.3** *(Uniqueness) The axisymmetric uniformly subsonic Euler flow in Theorem 2.1.1 is unique.*

**Theorem 2.1.4** *(Critical Mass Flux) Assume that the hypotheses of Theorem 2.1.1 hold, and $B(r)$ also satisfies*

$$B'(1) = 0, \tag{2.1.21}$$

*then there exists a critical mass flux $m_c$ such that for any $m \in (\delta^{1/4}, m_c)$, there exists a unique axisymmetric subsonic flow through the nozzle with mass flux condition (2.1.9) and the asymptotic behavior (2.1.19). Moreover, $m_c$ is the upper critical mass flux for the existence of subsonic flow in the following sense: either*

$$\sup_{\bar{\Omega}} \left(U^2 + V^2 - c^2(\rho)\right) \to 0 \quad as \quad m \to m_c \tag{2.1.22}$$

*or there is no $\sigma > 0$ such that for all $m \in (m_c, m_c + \sigma)$, there are Euler flows with the mass flux $m$ through the nozzle which satisfy the asymptotic condition (2.1.14), the asymptotic behaviors (2.1.19) and*

$$\sup_{m \in (m_c, m_c+\sigma)} \sup_{\bar{\Omega}} \left(c^2(\rho) - (U^2 + V^2)\right) > 0. \tag{2.1.23}$$

**Remark 2.1.1** *In [112], [113] and [114], Xie and Xin established the above results in Theorem 2.1.1-Theorem 2.1.4 first on subsonic and subsonic-sonic potential flows in both 2-D and 3-D axially symmetric infinitely long nozzle, and then on steady subsonic Euler flows in 2-D infinitely long nozzle. Our results are the extension to steady subsonic Euler flows in case of 3-D axially symmetric nozzle.*

**Remark 2.1.2** *One of the main difficulties for the general steady Euler flows is that the governing equations are a mixed elliptic-hyperbolic system even for uniformly subsonic flows. In 2-D Euler flows, Xin and Xie used stream function formulation for subsonic-sonic flows. However, for general 3-D Euler flows, there is no such formulation. Fortunately, motivated by Xin and Xie, we use two* *invariants along the stream lines such that stream function formulation can be available for axially symmetric case. By this formulation, Euler equations are equivalent to a quasilinear second order equation for a stream function so that the hyperbolicity of the particle path is already involved.*

**Remark 2.1.3** *Another difficulty is that one does not know a priorily whether the flow is uniformly subsonic. When the flow approaches to sonic, the equation becomes degenerate. Thus we first truncate the equation to be a uniformly elliptic equation, then we need to give a more precise a priori estimate to remove the truncation. For axially symmetric flows, besides the difficulties above, the equation contains singular coefficients near the symmetric axis. Unfortunately, different from the 3-D axially symmetric potential flows, the equation does not have a variational structure, which helps to establish existence of subsonic flow in [113]. So, we use several truncations to have a solvable approximate problem and precise elliptic estimates. By those, we proved the global existence of subsonic solution and obtained the asymptotic behavior.*

This paper is organized as follows. In the next section, based on the stream function formulation in [114], we first reformulate the problem into a quasilinear second order equation for the stream function under the asymptotic assumptions in the upstream. In section 3, we consider the existence of the stream function problem. Since the problem for stream function may be degenerate near the sonic and occurs singularity on the symmetric axis, we truncate the coefficients of the equation and formulate a uniformly elliptic problem without singularities. On the other hand, the problem is in an unbounded domain, we first solve the approximating problems in the bounded nozzle with the Dirichlet boundary condition. To remove the truncation and the singularity, we establish the uniform estimates in section 3 and show the solutions of the approximating problems converge to the one of the original problem. The stream function formulation depends on the assumption of asymptotic behavior and the positivity of the axial velocity, so in section 4, we will show the subsonic flow induced in section 3 satisfies these properties. With these properties, the formulation is consistent with the original problem for steady Euler flows in the infinitely long axisymmetric nozzle. In section 5, the uniqueness of the axisymmetric uniform subsonic flow is established by energy estimate. In the last section, the existence of the critical mass flux is proved.

## 2.2 Stream function formulation

In this section, under the asymptotic assumptions in the upstream, we reformulate the original subsonic problem into a boundary value problem with a quasi-linear second order equation for the stream function. The steady Euler system is hyperbolic-elliptic coupled system even though the flow is globally subsonic, so we have to deal with the hyperbolic mode, which is the main issue in this section.

### 2.2.1 The invariants.

Set $\omega = V_x - U_r$. It follows from (2.1.11) that

$$\partial_r(UU_x + VU_r + h(\rho)_x) + \partial_x(UV_x + VV_r + h(\rho)_r) = 0, \tag{2.2.1}$$

which implies that

$$(U, V) \cdot \nabla\omega + \omega \ \mathrm{div}(U, V) = 0, \tag{2.2.2}$$

where div denotes the divergence operator respect to cylinderical coordinates $(x, r)$. The continuity equation in (2.1.6) implies that

$$\mathrm{div}(U, V) = -\frac{1}{r\rho}(U, V) \cdot \nabla(r\rho) \tag{2.2.3}$$

provided the flow is away from vacuum. Substituting (2.2.3) into (2.2.2) yields that

$$(U, V) \cdot \left(\nabla\omega - \frac{\omega}{r\rho}\nabla(r\rho)\right) = r\rho(U, V) \cdot \nabla\left(\frac{\omega}{r\rho}\right) = \nabla^{\perp}\psi \cdot \nabla\left(\frac{\omega}{r\rho}\right) = 0. \tag{2.2.4}$$

Note that, this means that the quantity $\frac{\omega}{r\rho}$ remains an invariant along the streamline and it is functional dependent on $\psi$. Similarly, Bernoulli's law (2.1.13) implies that the Bernoulli's function $h(\rho) + \frac{U^2+V^2}{2}$ is also an invariant along each streamline. Therefore, one may regard the quantities $\frac{\omega}{r\rho}$ and $h(\rho) + \frac{U^2+V^2}{2}$ as two functions of $\psi$ by

$$\frac{\omega}{r\rho} = \mathcal{W}(\psi) \tag{2.2.5}$$

and

$$h(\rho) + \frac{U^2+V^2}{2} = \mathcal{B}(\psi) \tag{2.2.6}$$

respectively.

Moreover, the no-flow boundary condition (2.1.7) implies that $T_2$ is a streamline, so $\psi$ is a constant on it. On the other hand, the symmetric property of the flows implies the axis $T_1$ is also a streamline. From the mass flux condition (2.1.9), we may assume that

$$\psi = 0 \quad \text{on} \quad T_1, \quad \textit{and} \quad \psi = m \quad \text{on} \quad T_2. \tag{2.2.7}$$

With the invariants, we can easily derive the equivalence of the Euler system between (2.1.10), (2.2.5) and (2.2.6).

**Proposition 2.2.1** *For a smooth non-vacuum flow in the nozzle $\Omega$ satisfying (2.1.2), the Euler system (2.1.6) is equivalent to the system consisting of (2.1.10), (2.2.5) and (2.2.6) provided that*

1. *the given flow satisfies no flow boundary condition,*

2. *the axial velocity is always positive, ie.,*

$$U > 0 \quad \textit{in} \quad \Omega, \tag{2.2.8}$$

3. *$U$, $\rho$ and $V_r$ are bounded,*

$$V, V_x, \rho_r \to 0 \quad \textit{as} \quad x \to -\infty. \tag{2.2.9}$$

*Proof.*

From previous analysis, we know that system (2.1.6) implies (2.1.10), (2.2.5) and (2.2.6).

On the other hand, (2.2.5) may induce

$$(U, V) \cdot \nabla \left( \frac{\omega}{r\rho} \right) = 0$$

i.e.

$$(U, V) \cdot \nabla\omega + \omega \cdot div(U, V) = 0.$$

Noting $\omega = V_x - U_r$ and $\nabla\omega = (V_{xx} - U_{rx}, V_{xr} - U_{rr})$, we have

$$\partial_r(UU_x + VU_r + h(\rho)_x) - \partial_x(UV_x + VV_r + h(\rho)_r) = 0,$$

Set $\Phi_1 = UU_x + VU_r + h(\rho)_x$ and $\Phi_2 = UV_x + VV_r + h(\rho)_r$. Then $culr(\Phi_1, \Phi_2) = 0$ implies there exists a function $\Phi$ such that

$$\Phi_x = \Phi_1, \quad \Phi_r = \Phi_2.$$

So, (2.2.6) is equivalent to

$$(U, V) \cdot \nabla\Phi = 0. \tag{2.2.10}$$

Because of no flow boundary condition, $\Phi$ is a constant along each component of nozzle. If, in addition,

$$\Phi_r \to 0 \quad \text{as} \quad x \to -\infty, \tag{2.2.11}$$

then $\Phi \to C$ as $x \to -\infty$. However, it follows from (2.2.8) that through each point in $\Omega$, there is only one streamline satisfying

$$\begin{cases} \dfrac{dx}{ds} = U(x(s), r(s)), \\ \dfrac{dr}{ds} = V(x(s), r(s)), \end{cases}$$

which can be defined globally in the nozzle. Furthermore, (2.2.8) guarantees that any streamline through some point in $\Omega$ can not touch the nozzle wall.

Thus, one can always solve (2.2.10) in the whole domain $\Omega$, which yields $\Phi \equiv C$ in $\Omega$, if (2.2.11) holds. Obviously, (2.2.11) is guaranteed by (2.2.9).

Therefore $\Phi_x \equiv \Phi_r \equiv 0$ in $\Omega$, which implies (2.1.6) immediately.

□

### 2.2.2 Asymptotic structure in the far fields

To determine the explicit forms of $\mathcal{B}$ and $\mathcal{W}$, we need to study the far field behavior of the flows. The following propositions show that there exist asymptotic states $(\rho_0, U_0)$ and $(\rho_1, U_1)$ in the upstream and downstream, respectively, provided the variation of Bernoulli's function in upstream is sufficiently small. The proof of these propositions in this subsection will be only sketched, see Section 2 in [114] for details.

We have the following propositions.

**Proposition 2.2.2** *For any given Bernoulli's constant* $s > B_0 = \inf_{\rho>0} h(\rho)$,

1. *There exists a unique* $\bar{\varrho}(s)$ *such that* $h(\bar{\varrho}(s)) = s$ *and* $\dfrac{d\bar{\varrho}(s)}{ds} > 0$.

2. *There exists a unique* $\varrho(s) \in (0, \bar{\varrho}(s))$ *such that* $q^2(\varrho(s), s) = c^2(\varrho(s))$ *and* $\dfrac{d\varrho(s)}{ds} > 0$, *where*
$$q^2(\rho, s) = 2(s - h(\rho))$$
*which is derived by the density-speed relation*
$$h(\rho) + \frac{q^2}{2} = s, \tag{2.2.12}$$
*and* $q = \sqrt{U^2 + V^2}$ *is the speed of the flow,* $c(\rho) = \sqrt{P'(\rho)}$ *is the sound speed.*

*3. Define*

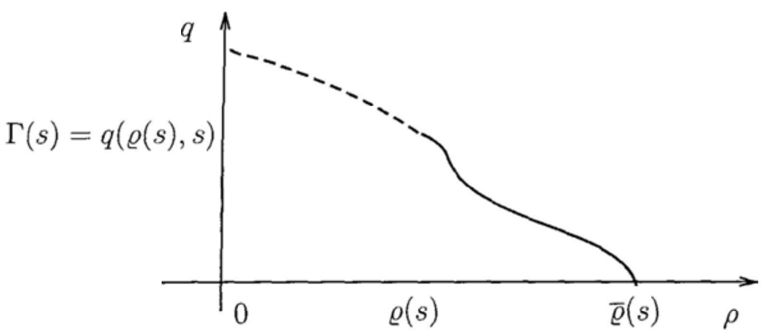
$$\Gamma(s) = q(\underline{\varrho}(s), s) = c(\underline{\varrho}(s)), \qquad \Sigma(s) = \underline{\varrho}(s)\Gamma(s). \tag{2.2.13}$$

*Then,*

$$\frac{d\Sigma(s)}{ds} > 0.$$

*4. There exists a unique* $\underline{\delta} > 2\delta = 2\|B'(r)\|_{C^{0,1}([0,1])}$ *such that*

$$\varrho(\bar{B}) \le \varrho\left(\underline{B} + \delta\right) \le \varrho\left(\underline{B} + \frac{\delta}{2}\right) < \varrho(\underline{B} + \underline{\delta}) = \bar{\varrho}(\underline{B}). \tag{2.2.14}$$




The density-speed realtion for some given $s$

**Remark 2.2.1** *In Proposition 2.2.2,* $\bar{\varrho}(s)$, $\underline{\varrho}(s)$, $\Gamma(s)$ *and* $\Sigma(s)$ *are the maximum density, the critical density, the critical speed and the critical mass flux, respectively for the states with given Bernoulli's constant* $s > B_0$.

*Proof.*

(i) Note that $P'(\rho) > 0$ for $\rho > 0$ and $P''(\rho) \ge 0$, therefore $h'(\rho) = \dfrac{P'(\rho)}{\rho}$ for $\rho > 0$ and for some $\bar{\rho} > 0$,

$$h(\rho) = h(\bar{\rho}) + \int_{\bar{\rho}}^{\rho} \frac{P'(s)}{s} ds \ge h(\bar{\rho}) + \int_{\bar{\rho}}^{\rho} \frac{P'(\bar{\rho})}{s} ds \quad \text{for} \quad \rho > \bar{\rho},$$

which implies that $h(\rho) \to \infty$ as $\rho \to \infty$. On the other hand, by the definition of $B_0$,

$$h(\rho) \to B_0 \quad \text{as} \quad \rho \to 0.$$

Thus, by the monotonicity of $h(\rho)$, for any fixed $s > B_0$, there exists a unique $\bar{\varrho} = \bar{\varrho}(s) > 0$ such that

$$h(\bar{\varrho}(s)) = s.$$

(ii) By the density-speed relation (2.2.12), for any fixed $s > B_0$, the corresponding speed $q(\rho, s)$ satisfies

$$q(\rho, s) = \sqrt{2(s - h(\rho))}.$$

Hence, for any fixed $s$, $q(\rho, s)$ is a strictly decreasing function of $\rho$ on $[0, \bar{\varrho}(s)]$. By the definition of $\bar{\varrho}(s)$, one has

$$q(\bar{\varrho}(s), s) = 0 < c(\bar{\varrho}(s)).$$

Indeed, we have $q(0, s) > c(0)$. since $c^2(\rho) = P'(\rho)$ is an increasing function of $\rho$, there exists a unique $\varrho(s) \in [0, \bar{\varrho}(s)]$, such that

$$q(\varrho(s), s) = c(\varrho(s))$$

In summary, for any given $s > B_0$, there exist $\bar{\varrho} = \bar{\varrho}(s)$, $\varrho = \varrho(s)$ and $\Gamma = \Gamma(s)$ such that

$$h(\bar{\varrho}(s)) = s, \quad h(\varrho(s)) + \frac{\Gamma^2(s)}{2} = s, \quad c^2(\varrho(s)) = \Gamma^2(s), \tag{2.2.15}$$

where $\bar{\varrho}(s)$, $\varrho(s)$ and $\Gamma(s)$ are the maximum density, the critical density and the critical speed, respectively, for the states with given Bernoulli's constant $s$.

(iii) Set the critical mass flux

$$\Sigma(s) = \varrho(s)c(\varrho(s)) = \varrho(s)\sqrt{2(s - h(\varrho(s)))}. \tag{2.2.16}$$

Then direct calculation shows that

$$\frac{d\bar{\varrho}}{ds} = \frac{\bar{\varrho}}{p'(\bar{\varrho})}, \quad \frac{d\varrho}{ds} = \frac{1}{\frac{p'(\varrho)}{\varrho} + \frac{p''(\varrho)}{2}},$$

and

$$\frac{d\Sigma}{ds} = \frac{\sqrt{2(s - h(\varrho(s)))}}{\frac{p'(\varrho)}{\varrho} + \frac{p''(\varrho)}{2}} + \varrho\frac{1 - \frac{2p'(\varrho)}{2p'(\varrho)+\varrho p''(\varrho)}}{\sqrt{2(s - h(\varrho(s)))}}.$$

Thus

$$\frac{d\bar{\varrho}(s)}{ds} > 0. \quad \frac{\varrho(s)}{ds} > 0, \quad \text{and} \quad \frac{d\Sigma(s)}{ds} > 0. \tag{2.2.17}$$

(iv) By the continuity and monotonicity of $\varrho(s)$ and $\sqrt{2(s - h(\rho))}$, there exists a $\underline{\delta}$ such that

$$\varrho(\underline{B} + \underline{\delta}) = \bar{\varrho}(\underline{B}). \tag{2.2.18}$$

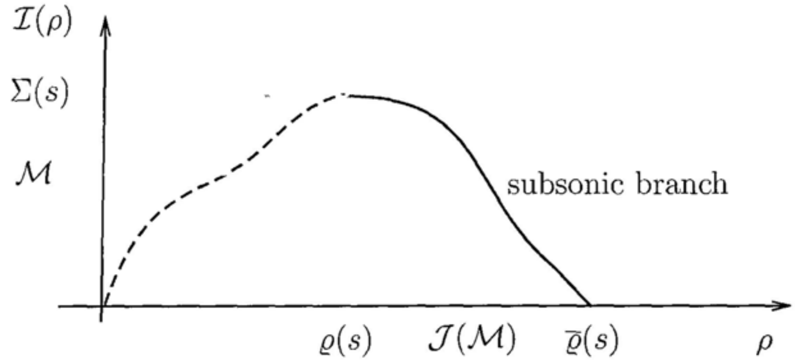
This completes the proof. □

Next, we introduce the mass flux as a function of $\rho$ for any given $s > B_0$, defined by

$$\mathcal{I}(\rho) = \rho^2 q^2(\rho, s) = 2\rho^2(s - h(\rho)). \tag{2.2.19}$$

Denote $\mathcal{M} = \mathcal{I}(\rho)$ in $(0, \bar{\varrho}(s))$, thus $\rho$ is a two-valued function of $\mathcal{M}$ for $\mathcal{M} \in [0, \Sigma^2(s))$. We denote the subsonic branch by

$$\rho = \mathcal{J}(\mathcal{M}) \quad \text{for} \quad \mathcal{M} \in (0, \Sigma^2(s)), \tag{2.2.20}$$




The density-mass flux relation for given $s$

which satisfies $\mathcal{J}(\mathcal{M}) > \underline{\varrho}(s)$.

When $s$ varies in $(B_0, +\infty)$, we denote this branch by

$$\rho = \mathcal{J}(\mathcal{M}, s) \quad \text{for} \quad (\mathcal{M}, s) \in \left\{(\mathcal{M}, s) | \mathcal{M} \in \left(0, \Sigma^2(s)\right), s > B_0\right\}, \tag{2.2.21}$$

which is the relation between mass flux and density for given Bernoulli's constant.

Suppose that the flow satisfies the asymptotic behavior (2.1.19) in the upstream, then the Bernoulli's law and the mass flux condition implies

$$h(\rho_0) + \frac{U_0^2(r)}{2} = B(r), \quad U_0(r) > 0, \tag{2.2.22}$$

and

$$\int_0^1 r\rho_0 U_0(r) dr = m, \tag{2.2.23}$$

i.e.

$$U_0(r) = \sqrt{2(B(r) - h(\rho_0))}, \tag{2.2.24}$$

and

$$m = \int_0^1 r\rho_0 \sqrt{2(B(r) - h(\rho_0))} dr. \tag{2.2.25}$$

For the downstream, since (2.2.8) implies a simple topological structure of streamlines, we can introduce a streamline

$$r = r(s) \quad \text{for} \quad s \in [0, 1], \quad \text{and} \quad r(0) = 0, \quad r(1) = 1. \tag{2.2.26}$$

Hence if the flow satisfies the asymptotic behavior (2 1 20) in the downstream, then it can be determined by the position in upstream, ie ,

$$h(\rho_0) + \frac{U_0^2(s)}{2} = h(\rho_1) + \frac{U_1^2(r(s))}{2} \qquad U_1(r(s)) > 0 \tag{2 2 27}$$

and

$$\int_0^s t\rho_0 U_0(t)dt = \int_0^{r(s)} r(t)\rho_1 U_1(t)dt \tag{2 2 28}$$

We can give the existence of the asymptotic structure of the flow in the far field

**Proposition 2.2.3** *Let $\underline{B} > B_0$, $\gamma \in (0, 1/3)$ There exists $\bar{\delta}_0$ such that for*

$$\|B'(r)\|_{C^{0,1}([0,1])} = \delta \leq \bar{\delta}_0 \quad \bar{m} \geq 2\bar{\delta}_0^{\frac{\gamma}{2}} \ \ and \ \ m \in (\delta^\gamma, \bar{m}) \tag{2 2 29}$$

*such that there exist solution $(\rho_0, U_0)$ to (2 2 22)-(2 2 23) and $(\rho_1, U_1)$ solving (2 2 35)-(2 2 36) with the following properties*

1. $\rho_0, \rho_1 \in (\varrho(\bar{B})\ \bar{\varrho}(\underline{B}))$,

2. *there exists a positive constant $C$, such that*

$$\overline{\varrho}(\underline{B}) - C \leq \rho_0 \leq \overline{\varrho}(\underline{B}) + C^{-1}\delta^{2\gamma}, \quad C^{-1}\delta^\gamma \leq U_0(r) \leq C \quad |U_0'(r)| \leq C\delta^{1-\gamma}, \tag{2 2 30}$$

3. *either $\rho_0 \to \varrho(\bar{B})$ or $\rho_1 \to \varrho(\bar{B})$ as $m \to \bar{m}$*

**Remark 2.2.2** *Choosing $\bar{\delta}_0 \leq \underline{\delta}/2$, where $\underline{\delta}$ is defined in Proposition 2 2 2, (2 2 9) implies that the interval $(\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$ is well-defined and non-empty*

In order to proof the proposition, we need the following lemmas

**Lemma 2.2.4** *Assume that (2 1 19) is true, and $\delta \leq \frac{1}{2}\underline{\delta}$ Then, for any $\gamma \in (0\ \frac{1}{3})$, there exits a $\tilde{\delta}_0 \in (0\ \frac{1}{2}\underline{\delta})$ such that there exits a unique $\rho_0$ with the property*

$$\rho_0 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B})), \tag{2 2 31}$$

*and*

$$\int_0^1 r\rho_0\sqrt{2((B(r)-h(\rho_0)))}dr = m, \tag{2.2.32}$$

*provided* $\delta \in (0, \tilde{\delta_0})$ *and* $m \in (\delta^\gamma, m_1)$ *with* $m_1 \geq C^{-1}\delta^{\frac{1}{2}} \geq 2\tilde{\delta_0}^{\frac{\gamma}{2}}$.

*Furthermore, there exits a* $\tilde{\tilde{\delta_0}} \in (0, \tilde{\delta_0})$ *such that if* $\delta \in (0, \tilde{\tilde{\delta_0}})$, *then*

$$C \geq \bar{\varrho}(\underline{B}) - \rho_0 \geq C\delta^{2\gamma}. \tag{2.2.33}$$

*Proof.*

Define

$$m(\rho) = \int_0^1 r\rho\sqrt{2(B(r)-h(\rho))}dr,$$

then

$$\frac{dm}{d\rho} = \int_0^1 r\frac{2(B(r)-h(\rho))-c^2(\rho)}{\sqrt{2(B(r)-h(\rho))}}dr \leq \int_0^1 r\frac{2(\bar{B}-h(\rho))-c^2(\rho)}{\sqrt{2(B(r)-h(\rho))}}dr < 0.$$

Thus, $m(\rho)$ is decreasing in $\rho$ on $(\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$.

$$m(\bar{\varrho}(\underline{B})) = \int_0^1 r\bar{\varrho}(\underline{B})\sqrt{2(B(r)-h(\bar{\varrho}(\underline{B})))}dr \leq C\delta^{\frac{1}{2}}.$$

$$\begin{aligned}
m(\varrho(\bar{B})) &= \int_0^1 r\varrho(\bar{B})\sqrt{2(B(r)-h(\varrho(\bar{B})))}dr \\
&\geq \varrho(\bar{B})\int_0^1 r\sqrt{2(\underline{B}-h(\varrho(\bar{B})))}dr \\
&= \varrho(\bar{B})\int_0^1 r\sqrt{2(h(\bar{\varrho}(\underline{B}))-h(\varrho(\bar{B})))}dr \\
&= \varrho(\bar{B})\int_0^1 r\sqrt{2(h(\varrho(\underline{B}+\underline{\delta}))-h(\varrho(\bar{B})))}dr \\
&\geq \varrho(\bar{B})\int_0^1 r\sqrt{2(h(\varrho(\underline{B}+\underline{\delta}))-h(\varrho(\underline{B}+\frac{\delta}{2})))}dr \\
&\geq c^{-1}\underline{\delta}^{\frac{1}{2}}.
\end{aligned}$$

Therefore there exists a unique solution $\rho \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$ such that (2.2.32) holds.

$$
\begin{aligned}
m = \int_0^1 r\rho_0\sqrt{2(B(r)-h(\rho_0))}dr \;&\leq \int_0^1 r\rho_0\sqrt{\delta + h(\bar{\varrho}(\underline{B}) - h(\rho_0))}dr \\
&= \frac{1}{2}\rho_0\sqrt{\delta + h(\bar{\varrho}(\underline{B}) - h(\rho_0))},
\end{aligned}
$$

so,

$$\delta + h(\bar{\varrho}(\underline{B})) - h(\rho_0) \geq C^{-1}m^2 \geq C^{-1}\delta^{2\gamma},$$

Hence

$$h(\bar{\varrho}(\underline{B})) - h(\rho_0) \geq C^{-1}\delta^{2\gamma}$$

for $\delta$ small enough. Then (2.2.33) holds. □

**Lemma 2.2.5** *Assume that (2.1.19) is true, and* $\delta \leq \frac{1}{2}\underline{\delta}$.
*Then there exists a unique* $U_0(r)$ *given by (2.2.24) and a constant* $C$*, such that*

$$
\begin{cases}
C^{-1}\delta^{\gamma} \leq U_0 \leq C, \\
|U_0'(r)| = |\dfrac{B'(r)}{2\sqrt{(B(r)-h(\rho))}}| \leq C\delta^{1-\gamma}, \\
|U_0'(r)|_{C^{0,1}([0,1])} \leq C(\delta^{1-\gamma} + \delta^{2-3\gamma}).
\end{cases}
\tag{2.2.34}
$$

*Proof.*

We need only to verify the estimates.

$$U_0(r) \geq \sqrt{2(\underline{B} - h(\rho_0))} = \sqrt{2(h(\bar{\varrho}(\underline{B})) - h(\rho_0))} \geq C^{-1}\delta^{\gamma},$$

and

$$|U_0'(r)| = \left|\frac{B'(r)}{\sqrt{2(B(r)-h(\rho_0))}}\right| = \frac{|B'(r)|}{U_0(r)} \leq \frac{\delta}{C^{-1}\delta^{\gamma}} = C\delta^{1-\gamma}.$$

□

**Lemma 2.2.6** *Assume that (2.1.19), (2.1.20) and (2.2.8) hold.*

*Then, there exist* $r = r(s)$, $s \in [0, 1]$, $\bar{\delta}_0 \in (0, \tilde{\bar{\delta}}_0)$ *and a unique* $\rho_1 \in (\underline{\varrho}(\bar{B}), \bar{\varrho}(\underline{B}))$ *and* $U_1(r)$, $r \in [0, a]$ *such that*

$$h(\rho_0) + \frac{U_0^2(s)}{2} = h(\rho_1) + \frac{U_1^2(r(s))}{2}, \tag{2.2.35}$$

$$\int_0^s t\rho U_0(t)dt = \int_0^{r(s)} r(t)\rho_1 U_1(t)dt, \tag{2.2.36}$$

*and*

$$r(s)|_{s=0} = 0, \quad r(s)|_{s=1} = a. \tag{2.2.37}$$

*provided* $\delta \in (0, \bar{\delta}_0)$ *and* $m \in (\delta^\gamma, m_2)$ *with* $m_2 \geq 2\bar{\delta}_0^{\frac{\gamma}{2}}$.

*Proof.*

If such a $r = r(s)$ exists and is smooth, then it follows from (2.2.36) that

$$s\rho_0 U_0(s) = r(s)\rho_1 U_1(r(s))r'(s),$$

$$\begin{cases} \dfrac{dr}{ds} &= \dfrac{s\rho_0 U_0(s)}{r(s)\rho_1 U_1(r(s))} = \dfrac{s\rho_0 U_0(s)}{r(s)\rho_1\sqrt{2(h(\rho_0) + \frac{U_0^2(s)}{2} - h(\rho_1))}} \\ &= \dfrac{s\rho_0 U_0(s)}{r(s)\rho_1\sqrt{2(B(s) - h(\rho_1))}}, \\ r(0) &= 0 \end{cases}$$

Thus

$$a = \int_0^1 \frac{s\rho_0 U_0(s)}{r(s)\rho_1\sqrt{2(B(s) - h(\rho_1))}}ds.$$

Define

$$G(\rho_1) = \int_0^1 \frac{s\rho_0 U_0(s)}{r(s)\rho_1\sqrt{2(B(s) - h(\rho_1))}}ds,$$

It is easy to check $G'(\rho_1) > 0$, $\rho_1 \in (\underline{\varrho}(\bar{B}), \bar{\varrho}(\underline{B}))$.

$$\frac{dG}{d\rho_1} = \int_0^1 \frac{s\rho_0 U_0(c^2(\rho_1) - q^2(B(s), \rho_1))}{r(s)\rho_1^2(\sqrt{2(B(s) - h(\rho_1))})^3}ds \geq \int_0^1 \frac{s\rho_0 U_0(c^2(\rho_1) - q^2(\bar{B}, \rho_1))}{r(s)\rho_1^2(\sqrt{2(B(s) - h(\rho_1))})^3}ds > 0.$$

$$\begin{aligned}
G(\bar{\varrho}(\underline{B})) &= \int_0^1 \frac{s\rho_0 U_0(s)}{r(s)\bar{\varrho}(\underline{B})\sqrt{2(B(s)-h(\bar{\varrho}(\underline{B}))}}ds \\
&\geq \int_0^1 \frac{s\rho_0 U_0(s)}{\bar{\varrho}(\underline{B})\sqrt{2(B(s)-\underline{B})}}ds \\
&\geq \int_0^1 \frac{sC^{-1}\delta^\gamma}{C\delta^{\frac{1}{2}}}ds = C\delta^{\gamma-\frac{1}{2}} > a.
\end{aligned}$$

Similarly, we can show $G(\varrho(\bar{B})) < a$. So one can choose $\tilde{\delta}_0$ and $m_1$ suitable, such that for any $\delta \in (0, \tilde{\delta}_0)$, $m \in (\delta^\gamma, m_1)$ that

$$G(\varrho(\bar{B})) < a, \text{ and } G(\bar{\varrho}(\underline{B})) > a,$$

there exists unique $\rho_1$ such that

$$G(\rho_1) = a = \int_0^1 \frac{s\rho_0 U_0(s)}{r(s)\rho_1\sqrt{2(B(s)-h(\rho_1))}}ds.$$

As soon as $\rho_1$ is determined, one solves the ODE above to obtain $r(s)$, and then to get $U_1(r(s))$. □

As a consequence of above lemmas, we may complete the proof of Proposition 2.2.3.

*Proof.* It remains to verify the last statement.

If $m \in (\delta^{\frac{1}{4}}, m_1)$, then both $\rho_0$ and $\rho_1$ exists and $\rho_0 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$, $\rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$.

As the proof of Lemma 3.4, for given $B(r)$, when $m$ increase $\rho_0$ must decrease. So if $m \to \tilde{m}$, $\rho_0 \to \varrho(\bar{B})$,

$$\tilde{m} = \int_0^1 r\rho(\bar{B})\sqrt{2(B(r)-h(\varrho(\bar{B}))}dr.$$

Thus, there must exists an upper bound for $m$ to ensure the existence of $\rho_0$, $\rho_1 \in (\varrho(\bar{B}).\ \bar{\varrho}(\underline{B}))$.

Define

$$\bar{m} = Sup\left\{s \,\middle|\, m \in (\delta^{\frac{1}{4}}, s) \text{ such that } \rho_0, \rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))\right\}.$$

Clearly, $\bar{m} \in [m_2, \tilde{m}]$. Now it is easy to see that $\bar{m}$ satisfies the requirement. □

### 2.2.3 The explicit form of $\mathcal{W}$ and $\mathcal{B}$

To determine the explicit from $\mathcal{W}$ and $\mathcal{B}$, it suffices to fix the expression of the two invariants, Bernoulli's function and $\frac{\omega}{r\rho}$ by $\psi$ in the upstream. Under the assumption (2.1.19), the stream function can be written as

$$\psi = \int_0^r s\rho_0 U_0(s)ds, \quad 0 \le r \le 1, \tag{2.2.38}$$

in the upstream. Furthermore, due to $s\rho_0 U_0(s) > 0$ for $s \in (0,1]$, $\psi$ is an increasing function of $r$. Thus, one can represent $r$ as a function of $\psi$, write as $r = \kappa(\psi)$.

Let the axial velocity and its derivative in the upstream be functions of the stream function

$$\Theta(\psi) = U_0(\kappa(\psi)), \quad \theta(\psi) = U_0'(\kappa(\psi)) \qquad \text{for} \quad \psi \in [0, m]. \tag{2.2.39}$$

Hence, (2.2.5) and (2.2.6) yield the explicit forms of $\mathcal{W}$ and $\mathcal{B}$, ie.,

$$\left(h(\rho) + \frac{U^2+V^2}{2}\right)\Bigg|_{x=-\infty} = h(\rho_0) + \frac{\Theta^2(\psi)}{2} = \mathcal{B}(\psi) \tag{2.2.40}$$

and

$$\frac{\omega}{r\rho}\Bigg|_{x=-\infty} = -\frac{\theta(\psi)}{\kappa(\psi)\rho_0} = \mathcal{W}(\psi), \tag{2.2.41}$$

provided that (2.2.8) holds. Furthermore, note that (2.2.8) implies that

$$0 \le \psi \le m. \tag{2.2.42}$$

**Remark 2.2.3** *In the upstream,*

$$\psi = \int_0^{\kappa(\psi)} s\rho_0 U_0(s)ds, \qquad \textit{for} \quad 0 \le \psi \le m.$$

*Differentiating the both sides with respect to $\psi$ yields*

$$\kappa'(\psi) = \frac{1}{\kappa(\psi)\rho_0 U_0(\kappa(\psi))} = \frac{1}{\kappa(\psi)\rho_0\Theta(\psi)}.$$

*This, combined with (2.2.39), shows*

$$\Theta'(\psi) = U_0'(\kappa(\psi))\kappa'(\psi) = \frac{\theta(\psi)}{\kappa(\psi)\rho_0\Theta(\psi)},$$

*that is*

$$\theta(\psi) = \kappa(\psi)\rho_0\Theta(\psi)\Theta'(\psi) \quad \text{and} \quad \mathcal{W}(\psi) = -\Theta(\psi)\Theta'(\psi). \tag{2.2.43}$$

*Furthermore, if $B(r)$ satisfies (2.1.16) with $0 \le \delta \le \bar{\delta}_0$ and $m \in (\delta^\gamma, \bar{m})$, one can derive that there exists a constant $C > 0$, such that*

$$\begin{cases} C^{-1}\delta^\gamma \le \Theta \le C, \qquad \Theta'(m) \ge 0, \quad \Theta'(0) \le 0, \\ |\Theta'(\psi)| \le C\delta^{1-2\gamma}, \quad \|\Theta'(\psi)\|_{C^{0,1}([0,1])} \le C\delta^{1-3\gamma}. \end{cases} \tag{2.2.44}$$

### 2.2.4 Formulation of the problem

Recall the density-mass flux relation (2.2.21), one has

$$\rho = \mathcal{J}\left(\left|\frac{\nabla\psi}{r}\right|^2, h(\rho_0) + \frac{\Theta^2(\psi)}{2}\right), \tag{2.2.45}$$

which can be regarded as a function of $\left|\frac{\nabla\psi}{r}\right|^2$ and $\psi$, written as

$$\rho = H\left(\left|\frac{\nabla\psi}{r}\right|^2, \psi\right). \tag{2.2.46}$$

It follows from the definition of $\omega$ and $\psi$ that

$$\omega = -\mathrm{div}\left(\frac{\nabla\psi}{r\rho}\right).$$

Furthermore, due to (2.2.41) and (2.2.43), the stream function satisfies

$$\mathrm{div}\left(\frac{\nabla\psi}{rH\left(\left|\frac{\nabla\psi}{r}\right|^2, \psi\right)}\right) = r\Theta(\psi)\Theta'(\psi)H\left(\left|\frac{\nabla\psi}{r}\right|^2, \psi\right). \tag{2.2.47}$$

Hence, we reformulate the original subsonic problem as follows.

**Problem 1.** Assume $\|B'(r)\|_{C^{0,1}([0,1])}$ be sufficiently small, find a solution $\psi(x,r)$ to the boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla\psi}{rH\left(\left|\frac{\nabla\psi}{r}\right|^2,\psi\right)}\right) = r\Theta(\psi)\Theta'(\psi)H\left(\left|\dfrac{\nabla\psi}{r}\right|^2,\psi\right) \quad \text{in} \quad \Omega, \\ \psi = 0 \quad \text{on} \quad T_1, \quad \psi = m \quad \text{on} \quad T_2, \end{cases} \tag{2.2.48}$$

and the flow fields induced by

$$\rho = H\left(\left|\frac{\nabla\psi}{r}\right|^2,\psi\right), \quad U = \frac{\psi_r}{r\rho}, \quad V = -\frac{\psi_x}{r\rho}$$

satisfy the far field behavior (2.1.19), (2.1.20) and (2.1.17).

**Remark 2.2.4** *Since Bernoulli's function is also invariant along each streamline, one has*

$$h(\rho) + \frac{\mathcal{M}}{2\rho^2} = h(\rho_0) + \frac{\Theta^2(\psi)}{2},$$

*which can determine the implicit form of* $H(\mathcal{M},\psi)$ *and* $\mathcal{M} = \left|\frac{\nabla\psi}{r}\right|^2$. *Furthermore, one has*

$$\frac{\partial H(\mathcal{M},\psi)}{\partial\mathcal{M}} = -\frac{H}{2(H^2c^2-\mathcal{M})}, \qquad \frac{\partial H(\mathcal{M},\psi)}{\partial\psi} = \frac{\Theta\Theta' H^3}{H^2c^2-\mathcal{M}}.$$

*denoted by* $H_1(\mathcal{M},\psi)$ *and* $H_2(\mathcal{M},\psi)$, *respectively.*

**Remark 2.2.5** *The quasilinear second order equation in (2.2.48) can be rewritten in non-divergence form as*

$$A_{ij}\left(\frac{\nabla\psi}{r},\psi\right)\partial_{ij}\psi = \mathcal{F}(\mathcal{M},\psi,r) + \mathcal{G}, \tag{2.2.49}$$

*where*

$$A_{ij}\left(\frac{\nabla\psi}{r},\psi\right) = H\delta_{ij} - 2H_1\frac{\psi_i}{r}\frac{\psi_j}{r}, \tag{2.2.50}$$

*and*

$$\mathcal{F}(\mathcal{M},\psi,r) = \frac{r^2\Theta\Theta' H^5 c^2}{H^2c^2-\mathcal{M}}, \quad \mathcal{G} = \left(H - 2\tilde{H}_1\mathcal{M}\right)\frac{\psi_2}{r}, \tag{2.2.51}$$

*with* $(\psi_1,\psi_2) = (\partial_x\psi,\partial_r\psi)$. *Clearly, the equation (2.2.49) may be degenerate near the sonic state and contains singularities on the axis* $r=0$.

## 2.3 Existence of solution for Problem 1

In this section, we consider the existence of the solution for Problem 1. There are three main obstacles to solve the Problem 1. First, the ellipticity of the equation (2.2.47) is not guaranteed beforehand, since the equation in Problem 1 may be degenerate elliptic near sonic state. Second, The coefficients in (2.2.47) rise the singularity near the axis. Third, the nozzle region is unbounded. In order to overcome these difficulties, we first truncate the coefficients of the equation in Problem 1 to ensure the strong ellipticity, and then, truncate the domain $\Omega$ to a series of bounded domains $\Omega_L$, with additional boundary conditions. Therefore, to solve the Problem 1 becomes to study a series of approximate strong elliptic problems in bounded domains and their uniform estimates, which ensure to pass the limit of approximate solutions to Problem 1. To remove the singularity, we construct a sequence of auxiliary regular problems and use these to approximate Problem 1, with some uniform estimates.

### 2.3.1 Extension

First, note that the function $H(\mathcal{M}, \psi)$ is not well-defined when $\mathcal{M}$ and $\psi$ are larger than some values. We introduce the following extension.

Set

$$\tilde{g}(s) = \begin{cases} \Theta'(s), & \text{if } \ 0 \le s \le m, \\ \Theta'(m)\dfrac{2m-s}{m}, & \text{if } \ m \le s \le 2m, \\ \Theta'(0)\dfrac{s+m}{m}, & \text{if } \ -m \le s \le 0, \\ 0, & \text{if } \ s \ge 2m, \quad \text{or} \quad s \le -m. \end{cases} \tag{2.3.1}$$

It is obvious that $\tilde{g} \in C^{0,1}(\mathbb{R})$ and

$$\|\tilde{g}(s)\|_{C^0(\mathbb{R})} \le \|\Theta'(s)\|_{C^0([0,m])}.$$

Moreover, it follows from the properties (2.2.44) of $\Theta'(s)$ that

$$\tilde{g}(s) \geq 0 \quad \text{if} \quad s \geq m, \qquad \tilde{g}(s) \leq 0 \quad \text{if} \quad s \leq 0,$$

and

$$\|\tilde{g}(s)\|_{C^{0,1}(\mathbb{R})} \leq C\delta^{1-3\gamma}. \tag{2.3.2}$$

Define the extension functions of $\Theta(s)$ and $\mathcal{B}(\psi)$ as

$$\tilde{\Theta}(s) = \Theta(0) + \int_0^s \tilde{g}(t)dt \quad \text{and} \quad \tilde{\mathcal{B}}(\psi) = h(\rho_0) + \frac{\tilde{\Theta}^2(\psi)}{2}, \tag{2.3.3}$$

respectively. Clearly, $\tilde{\Theta}'(s) = \tilde{g}(s)$ and $\tilde{\Theta}(s) \in C^{1,1}(\mathbb{R})$. Moreover, if $m > \delta^\gamma$, there exists a suitably small $\bar{\delta}_1$ such that for $\delta < \bar{\delta}_1$,

$$B_0 < \underline{B} - \varepsilon_0 \leq h(\rho_0) + \frac{\tilde{\Theta}^2(s)}{2} \leq \bar{B} + \varepsilon_0 \tag{2.3.4}$$

holds for some $\varepsilon_0 > 0$. Furthermore, by (2.2.44) and (2.3.2) one has

$$\| \tilde{\Theta}'(s) \|_{C^0(\mathbb{R})} \leq C\delta^{1-2\gamma} \quad \text{and} \quad \| \tilde{\Theta}'(s) \|_{C^{0,1}(\mathbb{R})} \leq C\delta^{1-3\gamma}. \tag{2.3.5}$$

### 2.3.2 Subsonic truncation

Note that the derivative $H_1(\mathcal{M}, \psi)$ in the coefficients of (2.2.49) goes to the negative infinity when the flow approaches to sonic from subsonic. Thus the equation (2.2.47) becomes degenerate elliptic near the sonic. To guarantee the uniformly ellipticity, we truncate the term $\mathcal{M}$ in $H(\mathcal{M}, \psi)$ in the following way.

Choose a smooth increasing function $\zeta_0(s)$ such that

$$\zeta_0(s) = \begin{cases} s, & \text{if} \quad s < -2\varepsilon_0, \\ -\varepsilon_0, & \text{if} \quad s > -\varepsilon_0. \end{cases} \tag{2.3.6}$$

Then define the truncation of the term $\mathcal{M} = \left|\frac{\nabla\psi}{r}\right|^2$ as

$$\tilde{\mathcal{M}}(\mathcal{M}, \psi) = \zeta_0\left(\mathcal{M} - \Sigma^2(\tilde{\mathcal{B}}(\psi))\right) + \Sigma^2(\tilde{\mathcal{B}}(\psi)). \tag{2.3.7}$$

where $\Sigma(s)$ is defined in (2.2.13). Set the truncation of $H(\mathcal{M},\psi)$ as

$$\tilde{H}(\mathcal{M},\psi) = \mathcal{J}\left(\tilde{\mathcal{M}}(\mathcal{M},\psi), \tilde{\mathcal{B}}(\psi)\right), \tag{2.3.8}$$

which can be determined by

$$h(\tilde{H}) + \frac{\tilde{\mathcal{M}}}{2\tilde{H}^2} = h(\rho_0) + \frac{\tilde{\Theta}^2(\psi)}{2}.$$

Thus, the derivatives of $\tilde{H}$ are

$$\begin{cases} \tilde{H}_1(\mathcal{M},\psi) = -\dfrac{\zeta_0'\tilde{H}}{2(\tilde{H}^2c^2 - \tilde{\mathcal{M}})}, \\ \tilde{H}_2(\mathcal{M},\psi) = \dfrac{\tilde{\Theta}\tilde{\Theta}'\tilde{H}(\tilde{H}^2 + \Sigma\Sigma'(\zeta_0' - 1))}{\tilde{H}^2c^2 - \tilde{\mathcal{M}}}. \end{cases} \tag{2.3.9}$$

Instead of (2.2.48), we first solve the following problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla\psi}{r\tilde{H}\left(\left|\frac{\nabla\psi}{r}\right|^2,\psi\right)}\right) = r\tilde{\Theta}(\psi)\tilde{\Theta}'(\psi)\tilde{H}\left(\left|\dfrac{\nabla\psi}{r}\right|^2,\psi\right) \quad \text{in} \quad \Omega, \\ \psi = 0 \quad \text{on} \quad T_1, \quad \psi = m \quad \text{on} \quad T_2. \end{cases} \tag{2.3.10}$$

The equation in (2.3.10) can be rewritten in non-divergence form as

$$\tilde{A}_{ij}\left(\frac{\nabla\psi}{r},\psi\right)\partial_{ij}\psi = \tilde{\mathcal{F}}(\mathcal{M},\psi,r) + \tilde{\mathcal{G}}, \tag{2.3.11}$$

where

$$\tilde{A}_{ij}\left(\frac{\nabla\psi}{r},\psi\right) = \tilde{H}\delta_{ij} - \frac{2\tilde{H}_1}{r^2}\psi_i\psi_j, \tag{2.3.12}$$

$$\tilde{\mathcal{F}}(\mathcal{M},\psi,r) = r^2\tilde{\Theta}\tilde{\Theta}'\tilde{H}\left(\frac{\tilde{H}^2 + \Sigma\Sigma'(\zeta_0' - 1)}{\tilde{H}^2c^2 - \tilde{\mathcal{M}}}\mathcal{M} + \tilde{H}^2\right), \tag{2.3.13}$$

and

$$\tilde{\mathcal{G}} = \frac{\tilde{H} - 2\tilde{H}_1\mathcal{M}}{r}\psi_2. \tag{2.3.14}$$

It is easy to check that away from the axis $T_1$, there exist two positive constants $\lambda$ and $\Lambda$, such that

$$\lambda|\xi|^2 \le \tilde{A}_{ij}\xi_i\xi_j \le \Lambda|\xi|^2,$$

for any $\xi \in \mathbb{R}^n$.

To the end of this section, we will show that the solution of the truncated problem (2.3.10) satisfies

$$\mathcal{M} = \left|\frac{\nabla\psi}{r}\right|^2 \leq \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0 \quad \text{for some} \quad \varepsilon_0, \tag{2.3.15}$$

as long as the incoming mass flux $m$ and the variation of Bernoulli's function is sufficiently small. Consequently, the extension and the subsonic truncation can be removed.

### 2.3.3 Truncation of the singularity on the axis

Due to the singularity at $r = 0$, we consider following approximate problems

$$\begin{cases} \tilde{A}_{ij}^{(k)}\partial_{ij}\psi = \tilde{\mathcal{F}}\left(\left|\dfrac{\nabla\psi}{r+k}\right|^2, \psi, r\right) + \tilde{\mathcal{G}}_k \quad \text{in} \quad \Omega, \\ \psi = 0 \quad \text{on} \quad T_1, \quad \psi = m \quad \text{on} \quad T_2, \end{cases} \tag{2.3.16}$$

here

$$\tilde{A}_{ij}^{(k)} = \tilde{H}\left(\left|\frac{\nabla\psi}{r+k}\right|^2, \psi\right)\delta_{ij} - 2\tilde{H}_1\left(\left|\frac{\nabla\psi}{r+k}\right|^2, \psi\right)\frac{\psi_i}{r+k}\frac{\psi_j}{r+k},$$

and

$$\tilde{\mathcal{G}}_k = \frac{\tilde{H}\left(\left|\frac{\nabla\psi}{r+k}\right|^2, \psi\right) - 2\tilde{H}_1\left(\left|\frac{\nabla\psi}{r+k}\right|^2, \psi\right)\tilde{\mathcal{M}}\left(\left|\frac{\nabla\psi}{r+k}\right|^2, \psi\right)}{r+k}\psi_2,$$

for $0 < k << 1$. Note that the estimates (2.3.5) implies

$$|\tilde{\mathcal{F}}| \leq C\delta^{1-2\gamma}. \tag{2.3.17}$$

### 2.3.4 Truncation of the domain

Our strategy to deal with the unbounded domain in Problem 1 here is to truncate the domain and to construct a sequence of truncated problems which approximate Problem 1.

For any given integer $L > 0$, choose $\Omega_L$ such that

1. $\{(x,r)|(x,r)\in\Omega,\quad -L<x<L\}\subset\Omega_L\subset\{(x,r)|(x,r)\in\Omega,\quad -4L<x<4L\}$.

2. $\Omega_L\in C^{2,\kappa_1}(0<\kappa_1<\kappa)$ for some constant $0<\kappa<1$ satisfies the uniform exterior sphere condition with uniform radius $R_0$, for all $L>L_0$ with some $L_0$ sufficiently large.

For the explicit construction of such $\Omega_L$, please refer to Appendix in [112]. Thus, we formulate the approximated truncated problems as

$$\begin{cases} \tilde{A}^{(k)}_{ij}\partial_{ij}\psi=\tilde{\mathcal{F}}\left(\left|\dfrac{\nabla\psi}{r+k}\right|^2,\psi,r\right)+\tilde{\mathcal{G}}_k \quad \text{in} \quad \Omega_L, \\ \psi=\dfrac{r^2}{f^2(x)}m \quad \text{on} \quad \partial\Omega_L, \end{cases} \tag{2.3.18}$$

which are Dirichlet problems for uniformly elliptic equations with two variables in bounded domains $\Omega_L$.

Applying Theorem 12.5 in [59] that there exists a solution $\psi^k_L\in C^{2,\mu}(\Omega_L)\cap C^0(\bar{\Omega}_L)$ to the problem (2.3.18), for any fixed $k>0$ and $L>0$. Note that the estimate (2.3.17) implies the linear growth condition (12.27) in [59] holds. Furthermore, by Theorem 3.7 in [59], we can obtain the apriori bound for the solution

$$|\psi^k_L|\le\sup_{T_2}|\psi^k_L|+C\sup_{\Omega_L}\frac{|\tilde{\mathcal{F}}|}{\lambda}, \tag{2.3.19}$$

where $C=e^{2d}-1$ with $d=\sup f(x)$. Thus,

$$-\frac{|\tilde{\mathcal{F}}|}{\lambda}\le\psi^k_L\le m+C\sup_{\Omega_L}\frac{|\tilde{\mathcal{F}}|}{\lambda}. \tag{2.3.20}$$

By (2.3.17), one has

$$-C\delta^{1-2\gamma}\le\psi^k_L\le m+C\delta^{1-2\gamma}, \tag{2.3.21}$$

where $C=C(k,|f|,m)$.

### 2.3.5 A priori estimates in $\Omega$ for fixed $k$

Now, we derive some precise apriori estimates of the approximated solution $\psi_L^k$ and to show that it converges to the solution to Problem 1 in original domain $\Omega$ as $L \to \infty$.

It follows from the techniques developed in [59] that, one can get the following estimate for Hölder semi-norm of

$$[u]_{1,\alpha} \leq C(\gamma, \Omega)\left(1 + |Du|_0 + \frac{1}{\lambda}|f|_0\right). \tag{2.3.22}$$

Actually, $C(\gamma, \Omega)$ depends only on the diam $\Omega$ and $C^2$ norm of the boundary of $T_2$. Applying it to problem (2.3.18) shows that there exists $\mu = \mu(\Lambda/\lambda) > 0$, such that for any $x^0 \in \bar{\Omega}_L$ with $K \geq 4L$, one has

$$\left[\psi_K^k\right]_{1,\mu;B_1(x^0)\cap\Omega_L} \leq C(\Lambda/\lambda, k, f, m)\left(1 + |D\psi_K^k|_{0;B_1(x^0)\cap\Omega_L} + \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right). \tag{2.3.23}$$

This, together with interpolation inequality and (2.3.21), yields

$$\begin{aligned}\left\|\psi_K^k\right\|_{1;B_1(x^0)\cap\Omega_L} &\leq \eta\left[\psi_K^k\right]_{1,\mu;B_1(x^0)\cap\Omega_L} + C_\eta\left|\psi_K^k\right|_0 \\ &\leq \eta C\left(\frac{\Lambda}{\lambda}, k, f, m\right)\left(1 + \left|D\psi_K^k\right|_{0;B_1(x^0)\cap\Omega_L} + \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) + C_\eta\left(m + \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right).\end{aligned} \tag{2.3.24}$$

Taking $\eta_0$ sufficiently small so that $\eta C(\Lambda/\lambda, k, f, m) \leq \frac{1}{2}$, if $\eta \leq \eta_0$, one has

$$\left\|\psi_K^k\right\|_{1;B_1(x^0)\cap\Omega_L} \leq \eta C(\Lambda/\lambda, k, f, m)\left(1 + \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) + C_\eta\left(m + \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right). \tag{2.3.25}$$

Thus, the local Hölder estimate (2.3.23) becomes

$$\begin{aligned}
\left\|\psi_K^k\right\|_{1,\mu;B_1(x^0)\cap\Omega_L} &\leq \left\|\psi_K^k\right\|_{1;B_1(x^0)\cap\Omega_L} + \left[\psi_K^k\right]_{1,\mu;B_1(x^0)\cap\Omega_L} \\
&\leq (1+C(\Lambda/\lambda,k,f,m))\left\|\psi_K^k\right\|_{1;B_1(x^0)\cap\Omega_L} + C(\Lambda/\lambda,k,f,m)\left(1+\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) \\
&\leq C(\Lambda/\lambda,k,f,m)\left[\eta_0 C(\Lambda/\lambda,k,f,m)\left(1+\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) + C_{\eta_0}\left(m + C\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right)\right. \\
&\quad + C(\Lambda/\lambda,k,f,m)\left(1+\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) \\
&\leq C(\Lambda/\lambda,k,f,m)\left(1+m+\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right).
\end{aligned} \tag{2.3.26}$$

Note that, for any $x, y \in \bar{\Omega}_L$,

$$\frac{\left|\nabla\psi_K^k(x) - \nabla\psi_K^k(y)\right|}{|x-y|^\mu} \leq \begin{cases} \left\|\psi_K^k\right\|_{1,\mu;B_1(x^0)\cap\Omega_L}, & \text{if} \quad y \in B_1(x)\cap\bar{\Omega}_L, \\ 2\left\|\psi_K^k\right\|_{1;\Omega_L}, & \text{if} \quad y \bar{\in} B_1(x)\cap\bar{\Omega}_L. \end{cases} \tag{2.3.27}$$

This together with (2.3.25) and (2.3.26), yields the global Hölder estimate

$$\left[\psi_K^k\right]_{1,\mu;\Omega_L} \leq C(\Lambda/\lambda,k,f,m)\left(1+m+\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right). \tag{2.3.28}$$

Furthermore, it follows from (2.3.26), the Schauder estimate and the bootstrap argument that one has

$$\left\|\psi_K^k\right\|_{2,\alpha;B_{\frac{1}{2}}(x^0)\cap\Omega_L} \leq C\left(\Lambda/\lambda,k,f,m,\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) \quad \text{for some } 0<\alpha<\mu. \tag{2.3.29}$$

Similar to the argument as (2.3.28), one has

$$\left\|\psi_K^k\right\|_{2,\alpha;\Omega_L} \leq C\left(\Lambda/\lambda,k,f,m,\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right), \tag{2.3.30}$$

which is a uniformly bound to $L$. Hence, using Arzela-Ascoli lemma and diagonal procedure, we see that there exists a subsequence $\{\psi_{K_l}^k\}_{l=1}^\infty$ for fixed $k$, such that

$$\psi_{K_l}^k \to \psi^k \quad \text{in} \quad C^{2,\beta}(U) \quad \text{for any compact set} \quad U\subset\bar{\Omega} \quad \text{and} \quad 0<\beta<\alpha.$$

Furthermore, $\psi^k$ satisfies the problem

$$\begin{cases} \tilde{A}_{ij}^{(k)}\partial_{ij}\psi = \tilde{\mathcal{F}}\left(\left|\dfrac{\nabla\psi}{r+k}\right|^2, \psi, r\right) + \tilde{\mathcal{G}}_k \quad \text{in} \quad \Omega, \\ \psi = 0 \quad \text{on} \quad T, \qquad \psi = m \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.3.31}$$

and the estimate

$$\left\|\psi^k\right\|_{1,\Omega} \leq \eta C\left(\Lambda/\lambda, k, f, m\right)\left(1+\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) + C_\eta\left(m + C\frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right), \tag{2.3.32}$$

where $\eta \in (0, \eta_0)$. Thanks to the estimate (2.3.17) for $\tilde{\mathcal{F}}$, one has

$$\left\|\psi^k\right\|_{1,\Omega} \leq \eta C\left(\Lambda/\lambda, k, f, m\right)(1 + C\delta^{1-2\gamma}) + C_\eta(m + C\delta^{1-2\gamma}), \tag{2.3.33}$$

where $C$ depends only on $\bar{\delta}_0, \bar{m}, \Lambda, \lambda$.

### 2.3.6 Removal of the singularity on axis

Set

$$\bar{\psi}(r) = \frac{m}{b^2}(r+k)^2 \quad \text{with} \quad b = \min_{x\in\mathbb{R}} f(x) > 0.$$

A direct calculation yields that

$$\nabla\bar{\psi}(r) = \left(0, \frac{2m(r+k)}{b^2}\right) \quad \text{and} \quad \left|\frac{\nabla\bar{\psi}(r)}{r+k}\right|^2 = \frac{4m^2}{b^4}.$$

Furthermore,

$$\operatorname{div}\left(\frac{\nabla\bar{\psi}(r)}{(r+k)\tilde{H}(\frac{4m^2}{b^4}, \bar{\psi}(r))}\right) = \partial_r\left(\frac{2m}{b^2\tilde{H}(\frac{4m^2}{b^4}, \bar{\psi}(r))}\right) = -\frac{4m^2(r+k)}{b^4\tilde{H}^2}\tilde{H}_2\left(\frac{4m^2}{b^4}, \bar{\psi}(r)\right). \tag{2.3.34}$$

Recalling the expression of $\tilde{H}$, the right hand side of (2.3.34) is non-positive if the flow is subsonic. On another side, $\bar{\psi} \geq \dfrac{r^2}{f^2(x)}m$ on $\Omega_L$, then by the standard comparison principle, $\bar{\psi}(r)$ is a super-solution to (2.3.18), and

$$0 \leq \psi_K^k \leq \frac{m}{b^2}(r+k)^2 \quad \text{in} \quad \Omega_L,$$

for any $L > 0$. Moreover, one gets

$$0 \leq \psi^k \leq \frac{m}{b^2}(r+k)^2 \quad \text{in } \Omega, \tag{2.3.35}$$

which implies

$$\psi^k(x,r) \to 0 \quad \text{as } k \to 0 \quad \text{on the axis } T_1. \tag{2.3.36}$$

Away from $T_1$, we denote $\Omega_{L,\varepsilon} = \{(x,r) \| x| \leq L, \varepsilon < r < f(x)\}$ for any $L > 0$ and $0 < \varepsilon << 1$. Using Caccioppoli's inequality, both in interior and on the boundary, one can obtain

$$\|\nabla \psi^k\|_{L^2(\bar{\Omega}_{L,\varepsilon})} \leq C(\Lambda/\lambda, \varepsilon, f, m). \tag{2.3.37}$$

Furthermore, one gets the Hölder estimate for the gradient

$$[\nabla \psi^k]_{C^{\alpha_1}(\bar{\Omega}_{L,\varepsilon})} \leq C\left(\Lambda/\lambda, \varepsilon, f, m, \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right) \quad \text{for some } 0 < \alpha_1 < \alpha.$$

It follows from the Schauder estimate

$$\|\psi^k\|_{C^{1,\alpha_1}(\bar{\Omega}_{L,\varepsilon})} \leq C\left(\Lambda/\lambda, \varepsilon, f, m, \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right). \tag{2.3.38}$$

Due to a diagonal process and Arzela-Ascoli Lemma again, there exists a subsequence $\{k_j\} \to 0$ as $j \to \infty$ and $\psi \in C^{1,\nu}(\Omega)$ such that

$$\psi^{k_j} \to \psi \quad \text{in} \quad C^{1,\nu}(\bar{\Omega}_{L,\varepsilon}) \qquad \text{for some } 0 < \nu < \alpha \tag{2.3.39}$$

as $j \to \infty$ for any $L > 0$, $0 < \varepsilon << 1$. In particular,

$$\psi^{k_j} \to \psi \quad \text{pointwisely in} \quad \Omega \quad \text{as} \quad j \to \infty. \tag{2.3.40}$$

It follows from (2.3.36), (2.3.39)-(2.3.40) that $\psi$ is a solution to (2.3.16) with $k = 0$. It follows bootstrap arguments that $\psi \in C^{2,\nu}(\Omega) \cap C^{1,\nu}(\Omega \cup T_2) \cap C^0(\bar{\Omega})$.

### 2.3.7 Removal of the extension and truncation

To get rid of the extension and the subsonic truncation, we need some more precise estimates for $\mathcal{M} = \left|\frac{\nabla \psi}{r}\right|^2$.

Step 1. The estimate away from the axis $T_1$. For any given

$$(x_0, r_0) \in \Omega_{\infty, b/4} = \{(x,r)|(x,r) \in \Omega,\ r > b/4\},$$

it follows from (2.3.33) and (2.3.40) that

$$\left|\nabla\psi^k(x_0, r_0)\right| \le \eta C\left(\Lambda/\lambda, k, f, m\right)(1 + C\delta^{1-2\gamma}) + C_\eta(m + C\delta^{1-2\gamma}).$$

Therefore

$$\left|\frac{\nabla\psi(x_0, r_0)}{r_0}\right| \le \eta C\left(\Lambda/\lambda, k, f, m\right)(1 + C\delta^{1-2\gamma}) + C_\eta(m + C\delta^{1-2\gamma}), \qquad (2.3.41)$$

for $r_0 > b/4$.

Step 2. The estimate near the axis $T_1$. For any fixed point $(x_0, r_0) \in \mathbb{R} \times (0, b/2)$, set

$$\psi_0(x,r) = \frac{1}{\xi^2}\psi(x_0 + x\xi, r_0 + r\xi), \quad \text{with } \xi = \frac{r_0}{2},$$

which is well-defined in $B_1(0,0)$. Moreover, direct calculations yield that

$$\frac{\nabla\psi(x_0 + x\xi, r_0 + r\xi)}{r_0 + r\xi} = \frac{\nabla\psi_0}{2 + r},$$

and $\psi_0$ satisfies

$$\operatorname{div}\left(\frac{\nabla\psi_0}{(2+r)\tilde{H}\left(|\frac{\nabla\psi_0}{2+r}|^2, \xi^2\psi_0\right)}\right) = \xi(r_0 + r\xi)\Theta\Theta'\tilde{H}\left(\left|\frac{\nabla\psi_0}{2+r}\right|^2, \xi^2\psi_0\right). \quad (2.3.42)$$

Due to (2.3.35) and (2.3.40), one gets

$$0 \le \psi_0(x,r) = \frac{4}{r_0^2}\psi(x_0 + \xi x, r_0 + \xi r) \le \frac{9m}{b^2}.$$

Applying Moser's iteration in the interior, we can obtain the estimate of the gradient

$$|\nabla\psi_0| \le C(b)m \qquad \text{in } B_{1/2}(0,0).$$

In particular,

$$\left|\frac{\nabla\psi(x_0, r_0)}{r_0}\right| = |\nabla\psi_0(0,0)| \le C(b)m \quad \text{for } 0 < r_0 < b/2. \qquad (2.3.43)$$

Step 3. Hölder continuity of $\mathcal{M}$ near the axis. As in step 2, we set

$$\psi_0(x,r) = \frac{1}{\xi^2}\psi(x_0 + x\xi, r_0 + r\xi), \quad \text{with } \xi = r_0^{\frac{1}{2}},$$

defined in $B_a(0,0)$ for any fixed point $(x_0, r_0) \in \mathbb{R}\times(0, b/2)$, where $a$ is a constant satisfying $r_0 + ar_0^{\frac{1}{2}} > 0$. It holds that

$$\left|\frac{\nabla\psi(x_0,r_0)}{r_0}\right| = |\nabla\psi_0(0,0)| \leq C(b)mr_0 \quad \text{for } 0 < r_0 < b/2. \tag{2.3.44}$$

Then by the same argument, we conclude that $\dfrac{\nabla\psi}{r}$ is uniformly Hölder continuous up to $r = 0$. Moreover,

$$\lim_{(x,r)\to(x_0,0)} \frac{\psi_x}{r}(x,r) = 0 \quad \forall x_0 \in \mathbb{R}^1. \tag{2.3.45}$$

**Proposition 2.3.1** *(Existence of subsonic flows of Problem 1) Suppose the hypotheses of Theorem 2.1.1 hold. Then, there exists a positive constant $\delta_1 \leq \bar{\delta}_0$, where $\bar{\delta}_0$ is defined in Proposition 2.2.3, such that if $\delta \leq \delta_1$ and $m \in (\delta^\gamma, \tilde{m})$ with $\tilde{m} = 2\delta_1^{\frac{\gamma}{2}} \leq \bar{m}$, then the problem (2.2.48) has a solution $\psi \in C^{2,\alpha}(\bar{\Omega})$ satisfying*

$$0 \leq \psi \leq m, \quad \left|\frac{\nabla\psi}{r}\right|^2 \leq \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0 \quad \textit{for some} \quad \varepsilon_0 > 0. \tag{2.3.46}$$

*Proof.* Obviously, there exist $\eta_1 \in (0, \eta_0)$ and $\delta_1 \in (0, \bar{\delta}_0)$ such that

$$\eta_1 C(\gamma, m, f)(1 + C\bar{\delta}_0^{1-2\gamma}) \leq \sqrt{(\Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0)}/2,$$
$$C_{\eta_1}(2\delta_1^{\gamma/2} + C\delta_1^{1-2\gamma}) \leq \sqrt{(\Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0)}/2,$$

and

$$C(b)2\delta_1^{\gamma/2} \leq \sqrt{\Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0}.$$

Therefore, for any $\delta \in (0, \delta_0)$ and $\tilde{m} \in (\delta^\gamma, 2\delta_1^{\gamma/2})$, the estimates (2.3.41) and (2.3.43) imply that the solution $\psi$ satisfies

$$\begin{cases} \left|\dfrac{\nabla\psi}{r}\right|^2 \leq \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0, & \text{for } r > \dfrac{b}{4}, \\ \left|\dfrac{\nabla\psi}{r}\right|^2 \leq \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0, & \text{for } r < \dfrac{b}{2}. \end{cases} \tag{2.3.47}$$

Hence, $\tilde{\mathcal{M}}(\mathcal{M},\psi) = \mathcal{M}$, and

$$\tilde{\mathcal{F}}(\mathcal{M},\psi,r) = \frac{r^2\tilde{\Theta}\tilde{\Theta}'\tilde{H}^5c^2}{\tilde{H}^2c^2-\mathcal{M}}, \quad \text{and } \tilde{\mathcal{G}} = \left(\tilde{H} - 2\tilde{H}_1\mathcal{M}\right)\frac{\psi_2}{r}.$$

To remove the extension, we first consider the domain $\Omega_e = \{(x,r) \in \Omega | \psi(x,r) \geq m\}$. Owing to $\Theta'(\psi) \geq 0$ in $\Omega_e$,

$$\tilde{A}_{ij}\left(\frac{\nabla\psi}{r},\psi\right)\partial_{ij}\psi - \left(\tilde{H} - 2\tilde{H}_1\mathcal{M}\right)\frac{\psi_2}{r} = \tilde{\mathcal{F}}(\mathcal{M},\psi,r) \geq 0, \quad \text{in } \Omega_e.$$

Therefore, by maximum principle $\psi$ achieves its maximum on the boundary of $\Omega_e$, so $\psi \leq m$ in $\Omega$. Similarly, one can show $\psi \geq 0$. Thus, $\psi$ satisfies

$$0 \leq \psi \leq m \quad \text{in } \Omega, \tag{2.3.48}$$

which implies $\tilde{\Theta}'(\psi) = \Theta'(\psi)$ and then the all extensions disappear naturally. Hence, $\psi$ solves the original problem (2.2.48) and satisfies (2.3.46). □

Therefore, it follows from (2.3.28), (2.3.30) and (2.3.45) that $\psi$ satisfies the following higher order estimates

$$\|\psi\|_{1,\nu;\bar{\Omega}} \leq C(\Lambda/\lambda, f, m)\left(1 + m + \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right), \tag{2.3.49}$$

and

$$\|\psi\|_{2,\alpha;\bar{\Omega}} \leq C\left(\Lambda/\lambda, f, m, \frac{1}{\lambda}|\tilde{\mathcal{F}}|_0\right). \tag{2.3.50}$$

Thus, there exist $\delta_1 \in (0, \tilde{\delta}_1)$ such that (2.3.47) is true for all $r$.

Therefore, combining (2.3.41) and (2.3.43), we can remove both extension and truncation appeared in (2.3.10).

## 2.4 Properties of the subsonic Euler flows

In this section, we will consider some properties of the subsonic Euler flows obtained in section 3. We obtain the asymptotic behavior in the far fields and the positivity of the axial velocity, which are not only of importance themselves but also crucial for our formulation. The stream function formulation is consistent with the steady Euler system in the infinitely long axisymmetric nozzle, as long as the flow induced by a solution of (2.2.48) satisfies the asymptotic behavior (2.1.19) in the upstream and the positivity of the axial velocity (2.2.8).

To obtain the profile of the solution in the upstream, we first investigate the flow in the cylinderical nozzle $\overline{D}$, then show the flow in the upstream converges to the one in cylinder nozzle.

Set $\overline{D}$ be an infinite long cylinderical nozzle with

$$D = \{(x,r)| -\infty < x < +\infty, \quad 0 < r < 1\},$$

and $\bar{\psi}$ be the solution to the following problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla\bar{\psi}}{rH(|\frac{\nabla\bar{\psi}}{r}|^2,\bar{\psi})}\right) = r\Theta(\bar{\psi})\Theta'(\bar{\psi})H\left(\left|\dfrac{\nabla\bar{\psi}}{r}\right|^2,\bar{\psi}\right) \quad \text{in} \quad D, \\ \bar{\psi} = 0 \quad \text{on} \quad r=0, \qquad \bar{\psi} = m \quad \text{on} \quad r=1. \end{cases} \tag{2.4.1}$$

**Proposition 2.4.1** *There exists $\delta_2 \in (0,\bar{\delta}_0]$ such that if*

1. *$\delta \leq \delta_2$, $m \in (0,\bar{m})$, where $\bar{m}$ is defined in Proposition 2.2.3,*

2. *there exists $\epsilon \leq \varepsilon_0$ such that $\bar{\psi} \in C^{2,\alpha}(\bar{D})$ solves the problem (2.4.1) and satisfies*

$$\|\bar{\psi}\|_{C^{2,\alpha}(D)} \leq C(\epsilon,\delta), \qquad \left|\frac{\nabla\bar{\psi}}{r}\right|^2 - \Sigma^2(\underline{B}-\epsilon) \leq \epsilon, \tag{2.4.2}$$

*then $\bar{\psi}$ is independent of $x$, moreover,*

$$\bar{\psi}(x,r) = \bar{\psi}(r) = \int_0^r s\rho_0 U_0(s)ds. \tag{2.4.3}$$

*Proof.* The proof is divided into two steps. First, it will be shown that $\bar{\psi}$ is independent of $x$. Then we will prove that $\bar{\psi}$ is of explicit form (2.4.3).

Step 1. Set $\omega = \bar{\psi}_x$ and $\bar{\mathcal{M}} = \left|\frac{\nabla\bar{\psi}}{r}\right|^2$. Differentiating the equation in (2.4.1) with respect to $x$ yields

$$\begin{aligned}&\partial_i\left(\frac{\bar{A}_{ij}}{rH^2(\bar{\mathcal{M}},\bar{\psi})}\partial_j\omega\right) - \partial_i\left(\frac{H_2(\bar{\mathcal{M}},\bar{\psi})\partial_i\bar{\psi}}{rH^2\left(\bar{\mathcal{M}},\bar{\psi}\right)}\omega\right)\\ &= r\mathcal{D}\left(\bar{\mathcal{M}},\bar{\psi}\right)\omega + d\left(\bar{\mathcal{M}},\bar{\psi}\right)\frac{\partial_i\bar{\psi}}{r}\partial_i\omega,\end{aligned} \tag{2.4.4}$$

where $\bar{A}_{ij}$, $\mathcal{D}(\bar{\mathcal{M}},\bar{\psi})$ and $d(\bar{\mathcal{M}},\bar{\psi})$ are defined as

$$\begin{cases} \bar{A}_{ij} &= H(\bar{\mathcal{M}},\bar{\psi})\delta_{ij} - 2H_1(\bar{\mathcal{M}},\bar{\psi})\dfrac{\bar{\psi}_i\bar{\psi}_j}{r^2},\\ \mathcal{D}(\bar{\mathcal{M}},\bar{\psi}) &= \left(\Theta''(\bar{\psi})\Theta(\bar{\psi}) + (\Theta'(\bar{\psi}))^2\right)H(\bar{\mathcal{M}},\bar{\psi}) + \Theta'(\bar{\psi})\Theta(\bar{\psi})H_2(\bar{\mathcal{M}},\bar{\psi}),\\ d(\bar{\mathcal{M}},\bar{\psi}) &= 2\Theta(\bar{\psi})\Theta'(\bar{\psi})H_1(\bar{\mathcal{M}},\bar{\psi}).\end{cases} \tag{2.4.5}$$

for $\nabla\bar{\psi} = (\bar{\psi}_1,\bar{\psi}_2)$, here $\Theta''(\bar{\psi}) \in L^\infty(\mathbb{R})$ due to (2.2.44). It follows from (2.4.2) that there exists a positive constant $\Lambda$ which depends only on $\epsilon$ such that

$$\left|\bar{A}_{ij}\right| \leq \Lambda(\epsilon).$$

Although we don't know whether $\bar{\psi} \in C^3(D)$, the equation (2.4.4) holds in weak sense. Moreover, $\omega$ satisfies the boundary conditions

$$\omega = 0 \text{ on } r = 0, 1.$$

Let $\eta$ be a $C_0^\infty(\mathbb{R})$ function satisfying

$$\eta = 1 \quad \text{for} \quad |s| < l, \quad \eta = 0 \quad \text{for} \quad |s| > l+1, \quad \text{and} \quad |\eta'(s)| \leq 2. \tag{2.4.6}$$

Now multiplying $\eta^2(x)\omega$ on both sides of (2.4.4), and integrating it over $D$ yields

$$\begin{aligned}&\iint_D\left[\frac{\bar{A}_{ij}}{rH^2\left(\bar{\mathcal{M}},\bar{\psi}\right)}\partial_j\omega\partial_i(\eta^2\omega) - \frac{H_2\left(\bar{\mathcal{M}},\bar{\psi}\right)\partial_i\bar{\psi}}{rH^2\left(\bar{\mathcal{M}},\bar{\psi}\right)}\omega\partial_i(\eta^2\omega)\right]dxdr\\ &= -\iint_D\left[r\mathcal{D}\left(\bar{\mathcal{M}},\bar{\psi}\right)\eta^2\omega^2 + d\left(\bar{\mathcal{M}},\bar{\psi}\right)\frac{\partial_i\bar{\psi}}{r}\partial_i\omega\eta^2\omega\right]dxdr.\end{aligned}$$

Substituting the explicit forms of $\bar{A}_{ij}$, $H_1\left(\bar{\mathcal{M}},\bar{\psi}\right)$ and $H_2\left(\bar{\mathcal{M}},\bar{\psi}\right)$ into the above equality and noting that $\bar{\psi}$ satisfies (2.4.2), gives

$$
\begin{aligned}
&\iint_D \frac{\eta^2|\nabla\omega|^2}{rH\left(\bar{\mathcal{M}},\bar{\psi}\right)}dxdr \\
= &\iint_D \frac{2H_1\left(\bar{\mathcal{M}},\bar{\psi}\right)}{rH^2\left(\bar{\mathcal{M}},\bar{\psi}\right)}\left|\frac{\nabla\bar{\psi}}{r}\cdot\nabla\omega\right|^2\eta^2dxdr - \iint_D 2\frac{\bar{A}_{ij}}{rH^2\left(\bar{\mathcal{M}},\bar{\psi}\right)}\partial_j\omega\partial_i\eta\eta\omega dxdr \\
&+\iint_D \frac{H_2\left(\bar{\mathcal{M}},\bar{\psi}\right)\partial_i\bar{\psi}}{rH^2\left(\bar{\mathcal{M}}.\bar{\psi}\right)}(\eta^2\omega\partial_i\omega+2\eta\partial_i\eta\omega^2)dxdr \\
&-\iint_D r\left(\Theta''(\bar{\psi})\Theta(\bar{\psi})+(\Theta'(\bar{\psi}))^2\right)H\left(\bar{\mathcal{M}},\bar{\psi}\right)\eta^2\omega^2dxdr \\
&-\iint_D r\Theta'(\bar{\psi})\Theta(\bar{\psi})H_2\left(\bar{\mathcal{M}},\bar{\psi}\right)\eta^2\omega^2dxdr - \iint_D 2\Theta'(\bar{\psi})\Theta(\bar{\psi})H_1\left(\bar{\mathcal{M}},\bar{\psi}\right)\eta^2\omega\frac{\nabla\bar{\psi}}{r}\cdot\nabla\omega dxdr \\
= &-\iint_D \frac{|\frac{\nabla\bar{\psi}}{r}\cdot\nabla\omega|^2\eta^2}{rH\left(\bar{\mathcal{M}},\bar{\psi}\right)\left(H^2\left(\bar{\mathcal{M}},\bar{\psi}\right)c^2-\bar{\mathcal{M}}\right)}dxdr - \iint_D 2\frac{\bar{A}_{ij}}{rH^2\left(\bar{\mathcal{M}},\bar{\psi}\right)}\partial_j\omega\partial_i\eta\eta\omega dxdr \\
&+\iint_D \frac{2H_2\left(\bar{\mathcal{M}},\bar{\psi}\right)}{H^2\left(\bar{\mathcal{M}},\bar{\psi}\right)}\eta\omega^2\frac{\nabla\bar{\psi}}{r}\cdot\nabla\eta dxdr + \iint_D \frac{2\Theta(\bar{\psi})\Theta'(\bar{\psi})H(\bar{\mathcal{M}},\bar{\psi})}{H^2\left(\bar{\mathcal{M}},\bar{\psi}\right)c^2-\bar{\mathcal{M}}}\eta^2\omega\frac{\nabla\bar{\psi}}{r}\cdot\nabla\omega dxdr \\
&-\iint_D r(\Theta''(\bar{\psi})\Theta(\bar{\psi})+(\Theta'(\bar{\psi}))^2)H\left(\bar{\mathcal{M}},\bar{\psi}\right)\eta^2\omega^2dxdr \\
&-\iint_D \frac{r(\Theta(\bar{\psi})\Theta'(\bar{\psi}))^2H^3\left(\bar{\mathcal{M}},\bar{\psi}\right)}{H^2\left(\bar{\mathcal{M}},\bar{\psi}\right)c^2-\bar{\mathcal{M}}}\eta^2\omega^2dxdr \\
= &\sum_{i=1}^{6} I_i
\end{aligned}
$$

Note that

$$
\begin{aligned}
&I_1+I_4+I_6 \\
= &\frac{-\eta^2}{rH(H^2c^2-\bar{\mathcal{M}})}\left(\left|\frac{\nabla\bar{\psi}}{r}\cdot\nabla\omega\right|^2-2\Theta\Theta'H^2\omega\nabla\bar{\psi}\cdot\nabla\omega+(\Theta\Theta')^2H^4r^2\omega^2\right)\leq 0,
\end{aligned}
$$

here $H=H(\bar{\mathcal{M}},\bar{\psi})$ for simplicity. Moreover, due to (2.2.44), one has

$$
|I_5|\leq C\delta^{1-3\gamma}\int_{-l-1}^{l+1}\int_0^1 r\omega^2drdx.
$$

Finally, since $H \leq \bar{\varrho}(\bar{B})$, thus if $\delta_2$ is sufficiently small, one gets from above that

$$\begin{aligned}
&\int_{-l}^{l} dx \int_0^1 \frac{|\nabla\omega|^2}{r} dr \\
\leq\ & C(\bar{B},\epsilon)\left(\int_{-l-1}^{-l} dx + \int_l^{l+1} dx\right)\int_0^1 \frac{|\nabla\omega\omega|}{r} + \omega^2 + r\omega^2 dr + C(\bar{B})\delta^{1-3\gamma}\int_{-l}^{l}\int_0^1 r\omega^2 dr dx \\
\leq\ & C(\bar{B},\epsilon)\left(\int_{-l-1}^{-l} dx + \int_l^{l+1} dx\right)\int_0^1 \left|\frac{\omega}{r}\right|^2 + \omega^2 + |\nabla\omega|^2 dr + C(\bar{B})\delta^{1-3\gamma}\int_{-l}^{l}\int_0^1 \omega^2 dr dx.
\end{aligned} \tag{2.4.7}$$

Since $\omega = 0$ on $r = 0$, Poincaré inequality implies that

$$\int_0^1 \omega^2 dr \leq \int_0^1 |\nabla\omega|^2 dr.$$

Combining this with (2.4.7) yields that

$$\begin{aligned}
\int_{-l}^{l}\int_0^1 |\nabla\omega|^2 dr dx \leq\ & \int_{-l}^{l}\int_0^1 \frac{|\nabla\omega|^2}{r} dr dx \\
\leq\ & C\left(\int_{-l-1}^{-l} dx + \int_l^{l+1} dx\right)\int_0^1 \left(2 + \frac{1}{r^2}\right)|\nabla\omega|^2 dr + C\delta^{1-3\gamma}\int_{-l}^{l}\int_0^1 |\nabla\omega|^2 d
\end{aligned}$$

Consequently,

$$\int_{-l}^{l}\int_0^1 |\nabla\omega|^2 dr dx \leq C\left(\int_{-l-1}^{-l}\int_0^1 |\nabla\omega|^2 dr dx + \int_l^{l+1}\int_0^1 |\nabla\omega|^2 dr dx\right) \tag{2.4.8}$$

for some small $\delta_2$ and any $l > 0$. It follows from (2.4.2) that

$$\int_{-l-1}^{-l}\int_0^1 |\nabla\omega|^2 dr dx + \int_l^{l+1}\int_0^1 |\nabla\omega|^2 dr dx \leq C$$

for some uniform constant $C$ which is independent of $l$. Therefore,

$$\int_{-l}^{l}\int_0^1 |\nabla\omega|^2 dr dx \leq C \tag{2.4.9}$$

for any $l > 0$ and some constant $C > 0$. Letting $l \to \infty$ in (2.4.9) yields

$$\int_{-\infty}^{\infty}\int_0^1 |\nabla\omega|^2 dr dx \leq C.$$

Hence

$$\int_{-l-1}^{-l}\int_0^1 |\nabla\omega|^2 dr dx + \int_l^{l+1}\int_0^1 |\nabla\omega|^2 dr dx \to 0 \text{ as } l \to \infty.$$

In view of (2.4.8), this implies

$$\int_{-\infty}^{\infty}\int_0^1 |\nabla\omega|^2 dr dx = 0.$$

Recalling the Poincaré inequality, we can conclude that

$$\omega = \bar{\psi}_r = 0 \qquad \text{in } D.$$

Therefore, $\bar{\psi}(x,r)$ is independent of $x$, we still denote it as $\bar{\psi}(r)$. Thus $\bar{\psi}(r)$ satisfies the following boundary value problem for a second order ordinary differential equation

$$\begin{cases} \dfrac{d}{dr}\left(\dfrac{\bar{\psi}'}{rH(|\frac{\bar{\psi}'}{r}|^2, \bar{\psi})}\right) = r\Theta'(\bar{\psi})\Theta(\bar{\psi})H\left(\left|\dfrac{\bar{\psi}'}{r}\right|^2, \bar{\psi}\right), \\ \bar{\psi}(0) = 0, \qquad \bar{\psi}(1) = m. \end{cases} \tag{2.4.10}$$

Step 2. Uniqueness of the solution to the boundary value problem (2.4.10).

Suppose that there are two solutions $\bar{\psi}_1$ and $\bar{\psi}_2$ to (2.4.10). Let $\bar{\phi} = \bar{\psi}_1 - \bar{\psi}_2$. Then $\bar{\phi}$ satisfies

$$\begin{cases} (\bar{a}\bar{\phi}' + \bar{b}\bar{\phi})' = \dfrac{\bar{c}}{r}\bar{\phi}' + r\bar{h}\bar{\phi}, \\ \bar{\phi}(0) = \bar{\phi}(1) = 0, \end{cases} \tag{2.4.11}$$

where

$$\bar{a} = \int_0^1 \frac{H\left(|\frac{\tilde{\psi}'}{r}|^2, \tilde{\psi}\right) - 2H_1\left(|\frac{\tilde{\psi}'}{r}|^2, \tilde{\psi}\right)|\frac{\tilde{\psi}'}{r}|^2}{rH^2\left(|\frac{\tilde{\psi}'}{r}|^2, \tilde{\psi}\right)} ds, \quad \bar{b} = \int_0^1 \frac{-H_2\left(|\frac{\tilde{\psi}'}{r}|^2, \tilde{\psi}\right)\tilde{\psi}'}{rH^2\left(|\frac{\tilde{\psi}'}{r}|^2, \tilde{\psi}\right)} ds,$$

$$\bar{c} = \int_0^1 d\left(\left|\frac{\tilde{\psi}'}{r}\right|^2, \tilde{\psi}\right)\tilde{\psi}' ds, \qquad \bar{h} = \int_0^1 \mathcal{D}\left(\left|\frac{\tilde{\psi}'}{r}\right|^2, \tilde{\psi}\right) ds,$$

with $\tilde{\psi} = s\bar{\psi}_1 + (1-s)\bar{\psi}_2$ $(0 \le s \le 1)$, where $\mathcal{D}$ and $d$ are defined in (2.4.5).

Multiplying $\bar{\phi}$ on the both sides of the equation in (2.4.11), and integrating it over $[0,1]$, we have

$$\int_0^1 \frac{|\bar{\phi}'|^2}{rH\left(|\frac{\tilde{\psi}'}{r}|^2, \tilde{\psi}\right)} dr \le -\int_0^1 r\left(\Theta''(\tilde{\psi})\Theta(\tilde{\psi}) + (\Theta'(\tilde{\psi}))^2\right) H\left(\left|\frac{\tilde{\psi}'}{r}\right|^2, \tilde{\psi}\right)\bar{\phi}^2 dr.$$

By (2.2.44), thanks to the smallness of $\delta$ and the Poincaré inequality, one has

$$\int_0^1 |\bar{\phi}'|^2 dr \le \int_0^1 \frac{|\bar{\phi}'|^2}{r} dr \le 0.$$

Therefore, $\bar{\phi}' = 0$. So the solution to (2.4.10) is unique.

On the other hand, by the definition of $H$ and $\Theta$, we have that problem (2.4.10) has a solution

$$\bar{\psi} = \bar{\psi}(r) = \int_0^r s\rho_0 U_0(s) ds.$$

This completes the proof of the Proposition. □

In the following, we will show that the solution of Problem 1 converges to $\bar{\psi}$ in the upstream.

For $x \le n$, define $\psi^{(n)}(x,r) = \psi(x-n,r)\chi_{\{0<r<f(x-n)\}}$. For any compact set $K \subset\subset D$, it follows from (2.3.50) that

$$\|\psi^{(n)}\|_{C^{2,\alpha}(K)} \le C \quad \text{for } n \text{ sufficient large.}$$

Therefore, by Arzela-Ascoli lemma and the diagonal procedure, there exists a subsequence $\psi^{(n_k)}$, such that

$$\psi^{(n_k)} \to \bar{\psi} \quad \text{in} \quad C^{2,\beta}(K) \tag{2.4.12}$$

for any $K \subset\subset D$ and $\beta \in (0,\alpha)$. However, $\bar{\psi}$ solves the problem (2.4.1) and satisfies

$$0 \le \bar{\psi} \le m, \qquad \left|\frac{\nabla\bar{\psi}}{r}\right| \le \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0 \quad \text{for some} \quad \varepsilon_0 > 0.$$

It follows from Proposition 2.4.1 and (2.4.12) that the flow induced by the stream function satisfies (2.1.19).

The asymptotic behavior in the downstream can be obtained by a similar argument.

**Proposition 2.4.2** *Suppose that* $\Omega$ *satisfies the assumptions (2.1.2). Then there exists* $\delta_3 \in (0, \bar{\delta}_0]$ *such that if* $\|B'(r)\|_{C^{0,1}([0,1])} = \delta \le \delta_3$*, and the mass flux* $m \in (\delta^\gamma, \bar{m})$*,* $\psi$ *satisfies (2.3.46) and solves (2.2.48) then*

$$0 < \psi < m \quad in \quad \Omega, \tag{2.4.13}$$

*and*

$$\psi_r > 0 \quad in \quad \bar{\Omega}. \tag{2.4.14}$$

*Proof.* It follows from (2.3.46) that $\psi$ achieves its minimum on $T_1$ and maximum on $T_2$, hence

$$\psi_r \ge 0 \quad \text{on } T_2.$$

On the other hand, set $\mathcal{M} = \left|\frac{\nabla\psi}{r}\right|^2$ and $\nu = \psi_r$ satisfies

$$\begin{aligned} &\partial_i \left( \frac{A_{ij}}{rH^2(\mathcal{M},\psi)} \partial_j \nu \right) - \partial_i \left( \frac{H_2(\mathcal{M},\psi)\partial_i \psi}{rH^2(\mathcal{M},\psi)} \nu \right) + \partial_i \left( \frac{\beta(\mathcal{M},\psi)}{H^2(\mathcal{M},\psi) r^2} \partial_i \psi \right) \\ &= r\mathcal{D}(\mathcal{M},\psi)\,\nu + d(\mathcal{M},\psi)\,\frac{\partial_i \psi}{r} \partial_i \nu - \Theta'(\psi)\Theta(\psi)\beta(\mathcal{M},\psi) \end{aligned} \tag{2.4.15}$$

in the weak sense, where

$$\beta(\mathcal{M},\psi) = 2H_1(\mathcal{M},\psi)\mathcal{M} - H(\mathcal{M},\psi),$$

$A_{ij}$, $\mathcal{D}$ and $d$ are defined similar as in Proposition 2.4.1 except we replace $\tilde{\Theta}$, $\tilde{H}$ and $\bar{\psi}$ by $\Theta$, $H$ and $\psi$, respectively. We first claim that

$$\nu \ge 0 \quad \text{in } \Omega.$$

Indeed, it follows from Proposition 2.4.1 that $\nu(x, r) > 0$ when $|x| > L$ for some $L$ sufficiently large. Multiplying (2.4.15) by $\nu^- = \min(\nu, 0)$, one may get that

$$
\begin{aligned}
&\iint_{\{\nu\le 0\}} \frac{|\nabla\nu|^2}{rH(\mathcal{M},\psi)}dxdr \\
=& \iint_{\{\nu\le 0\}} \frac{2H_1(\mathcal{M},\psi)}{rH^2(\mathcal{M},\psi)}|\nabla\psi\cdot\nabla\nu|^2dxdr + \iint_{\{\nu\le 0\}} \frac{H_2(\mathcal{M},\psi)}{H^2(\mathcal{M},\psi)}\frac{\partial_\imath\psi}{r}\nu\partial_\imath\nu dxdr \\
&- \iint_{\{\nu\le 0\}} \frac{\beta(\mathcal{M},\psi)}{r^2H^2}\partial_\imath\psi\partial_\imath\nu dxdr - \iint_{\{\nu\le 0\}} r\mathcal{D}(\mathcal{M},\psi)\nu^2dxdr \\
&- \iint_D d(\mathcal{M},\psi)\nu\frac{\nabla\psi}{r}\cdot\nabla\nu dxdr + \iint_{\{\nu\le 0\}} \Theta'(\psi)\Theta(\psi)\beta(\mathcal{M},\psi)\nu dxdr \\
\le& -\iint_{\{\nu\le 0\}} r(\Theta''(\psi)\Theta(\psi)+(\Theta'(\psi))^2)H(\mathcal{M},\psi)\nu^2dxdr \\
\le& C\delta^{1-3\gamma}\iint_{\{\nu\le 0\}} r\nu^2dxdr.
\end{aligned}
$$

Since $r$ is bounded above, we have

$$
\iint_{\{\nu\le 0\}} \frac{|\nabla\nu|^2}{H(\mathcal{M},\psi)}dxdr \le C\iint_{\{\nu\le 0\}} \frac{|\nabla\nu|^2}{rH(\mathcal{M},\psi)}dxdr \le C\delta^{1-3\gamma}\iint_{\{\nu\le 0\}} \nu^2dxdr.
$$

Define $K_x = \{r|0\le r\le f(x),\ \nu(x,r)<0\}$, then $K_x$ is an open set for each $x$. Let $K_x = \bigcup_{\imath\in\mathcal{A}} I_\imath^\imath$, where $I_x^\imath$ are connected components of $K_x$. For each $r\in I_x^\imath$,

$$
\nu(x,r) = \int_{\min I_x^\imath}^r \partial_r\nu(x,s)ds.
$$

Therefore,

$$\begin{aligned}
\iint_{\{\nu\le 0\}} \nu^2(x,r)dxdr &= \int_{-l}^{l} dx \sum_{i\in\mathcal{A}} \int_{I_x^i} \nu^2(x,r)dr \\
&= \int_{-l}^{l} dx \sum_{i\in\mathcal{A}} \int_{I_x^i} \left(\int_{\min I_x^i}^{r} \partial_r\nu(x,s)ds\right)^2 dr \\
&\le \int_{-l}^{l} dx \sum_{i\in\mathcal{A}} \int_{I_x^i} \int_{\min I_x^i}^{\max I_x^i} (\partial_r\nu(x,s))^2 ds(\max I_x^i - \min I_x^i)dr \\
&= \int_{-l}^{l} dx \sum_{i\in\mathcal{A}} (\max I_x^i - \min I_x^i)^2 \int_{\min I_x^i}^{\max I_x^i} (\partial_r\nu(x,s))^2 ds \\
&\le \max_{x\in\mathbb{R}} |f(x)|^2 \int_{-l}^{l} dx \sum_{i\in\mathcal{A}} \int_{\min I_x^i}^{\max I_x^i} (\partial_r\nu(x,s))^2 ds \\
&\le \max_{x\in\mathbb{R}} |f(x)|^2 \iint_{\{\nu\le 0\}} |\nabla\nu|^2 dxdr
\end{aligned}$$

Hence,

$$\iint_{\{\nu\le 0\}} \frac{|\nabla\nu|^2}{H(\mathcal{M},\psi)} dxdr \le C\delta^{1-3\gamma} \iint_{\{\nu\le 0\}} |\nabla\nu|^2 dxdr.$$

this implies

$$\iint_{\{\nu\le 0\}} |\nabla\nu|^2 dxdr \le 0,$$

so $\nu \ge 0$ in $\Omega$.

Now, we use the argument similar to the proof of Lemma 1 in Section 9.5.2 in [49] to show that

$$\psi_r = \nu > 0 \quad \text{in } \Omega \tag{2.4.16}$$

holds for any weak solutions $\nu$ to (2.4.15).

Indeed, let $\tilde{\nu} = e^{-\sigma r}\nu$, where $\sigma > 0$ will be selected later. Then $\tilde{\nu}$ is a nonnegative weak solution to

$$\partial_i\left(\frac{A_{ij}}{rH^2}e^{\sigma r}\partial_j\tilde{\nu}\right) + \left(\frac{A_{i2}}{rH^2}\sigma - \frac{H_2}{H^2}\frac{\partial_i\psi}{r} - d(\mathcal{M},\psi)\frac{\partial_i\psi}{r}\right)e^{\sigma r}\partial_i\tilde{\nu} + G_1 e^{\sigma r}\tilde{\nu} + G_2 = 0,$$

where $A_{ij}$ and $d$ are defined similarly in Proposition 2.4.1, and

$$G_1 = \frac{A_{22}}{rH^2}\sigma^2 + \left(\partial_i\left(\frac{A_{i2}}{rH^2}\right) - \frac{H_2}{H^2}\frac{\partial_2\psi}{r} - d(\mathcal{M},\psi)\frac{\partial_2\psi}{r}\right)\sigma - \partial_i\left(\frac{H_2}{H^2}\frac{\partial_i\psi}{r}\right) - r\mathcal{D}(\mathcal{M},\psi),$$

and

$$G_2 = \partial_i \left( \frac{\beta(\mathcal{M},\psi)}{r^2H^2} \partial_i \psi \right) + \Theta'\Theta\beta(\mathcal{M},\psi),$$

with $\mathcal{D}$ and $\beta$ defined as before. Choose $\sigma > 0$ sufficiently large such that $G_1 > 0$. Thus

$$\partial_i \left( \frac{A_{ij}}{rH^2} e^{\sigma r} \partial_j \tilde{\nu} \right) + \left( \frac{A_{i2}}{rH^2}\sigma - \frac{H_2}{H^2}\frac{\partial_i \psi}{r} - d(\mathcal{M},\psi)\frac{\partial_i \psi}{r} \right) e^{\sigma r}\partial_i \tilde{\nu} + G_2 \leq 0.$$

It follows from maximal principle (see Theorem 8.19 in [59]) that (2.4.16) holds.

Now, (2.4.13) follows directly from (2.2.42) and (2.4.16).

Next, we consider the positivity of $\psi_r$ on the boundary $T_2$ and the axis $T_1$.

Since $\psi = m$ on $T_2$, if $\Theta'(m) > 0$, then for any $(x^0, f(x^0)) \in T_2$, there exists a small disk $B \subset \Omega$ satisfying $\bar{B} \cap \bar{\Omega} = (x^0, f(x^0))$ such that $\Theta'(\psi) \geq 0$ in $B$, therefore,

$$A_{ij}\left( \frac{\nabla \psi}{r}, \psi \right) \partial_{ij}\psi - (H - 2H_1\mathcal{M})\frac{\psi_r}{r} = \mathcal{F}(\mathcal{M},\psi,r) \geq 0, \qquad \text{in } B.$$

Moreover, by (2.4.13) $\psi < m$ in $B$. Thus, by Hopf Lemma, one has

$$\psi_r(x^0, f(x^0)) > 0.$$

On the other hand, in the case $\Theta'(m) = 0$, also by Hopf Lemma, we have $\partial_r\psi > 0$ on $T_2$.

Similarly, on the axis $T_1$, one can show that $\psi_r(x, 0) > 0$ for any $x \in \mathbb{R}$.

This finishes the proof of the Proposition. □

**Proof of Theorem 2.1.1-2.1.2.** Choose $\delta_0 = \min\{\delta_1, \delta_2\}$, then $\delta_0 > 0$. If $\delta \leq \delta_0$, for any $m \in (\delta^\gamma, 2\delta_0^{\frac{\gamma}{2}})$, it follows from Proposition 2.3.1 and Proposition 2.4.2 that there exists a solution to the problem (2.2.48) with asymptotic condition (2.1.14), (2.1.19), mass flux condition (2.1.9).

□

## 2.5 Uniqueness of the uniformly subsonic flow

**Proposition 2.5.1** *(Uniqueness) Suppose that $\Omega$ satisfies the assumptions (2.1.2). Then there exists $\delta_3 \in (0, \bar{\delta}_0]$ such that if $\|B'(r)\|_{C^{0,1}([0,1])} = \delta \leq \delta_3$, and the mass flux $m \in (\delta^\gamma, \bar{m})$, then there exists at most one solution $\psi$ to (2.2.48) satisfying*

$$0 \leq \psi \leq m, \quad \left|\frac{\nabla\psi}{r}\right|^2 - \Sigma^2(\mathcal{B}(\psi)) \leq \epsilon \quad \textit{for some} \quad \epsilon > 0. \qquad (2.5.1)$$

*Proof.* Let $\psi_1$ and $\psi_2$ be two solutions to (2.2.48). Set $\psi = \psi_1 - \psi_2$. Then $\psi$ satisfies

$$\begin{cases} \partial_i \left(a_{ij}\partial_j\psi\right) + \partial_i(b_i\psi) = \dfrac{c_i}{r}\partial_i\psi + rh\psi, & \text{in } \Omega, \\ \psi = 0 & \text{on } T_1 \cup T_2, \end{cases} \qquad (2.5.2)$$

where

$$a_{ij} = \int_0^1 \frac{A_{ij}(\frac{\nabla\tilde{\psi}}{r}, \tilde{\psi})}{rH^2(|\frac{\nabla\tilde{\psi}}{r}|^2, \tilde{\psi})}ds, \qquad b_i = \int_0^1 \frac{-H_2(|\frac{\nabla\tilde{\psi}}{r}|^2, \tilde{\psi})\partial_i\tilde{\psi}}{rH^2(|\frac{\nabla\tilde{\psi}}{r}|^2, \tilde{\psi})}ds,$$
$$c_i = \int_0^1 d\left(\left|\frac{\nabla\tilde{\psi}}{r}\right|^2, \tilde{\psi}\right)\partial_i\tilde{\psi}ds, \quad h = \int_0^1 \mathcal{D}\left(\left|\frac{\nabla\tilde{\psi}}{r}\right|^2, \tilde{\psi}\right)ds,$$

here $\tilde{\psi} = s\psi_1 + (1-s)\psi_2$ $(0 \leq s \leq 1)$, $A_{ij}$, $\mathcal{D}$ and $d$ are defined similar to Proposition 2.4.1.

Set $\eta$ be a cut-off function defined in (2.4.6), multiplying both sides of equation (2.5.2) by $\eta^2\psi^+(x,r)$ and $\psi^+(x,r) = \max(\psi(x,r), 0)$, then the similar to the proof of Proposition 2.4.1, one has

$$\iint_{\Omega\cap\{|x|\leq l\}\cap\{\psi\geq 0\}} \left|\frac{\nabla\psi}{r}\right|^2 dxdr \leq C(\underline{B}, \epsilon)\iint_{\Omega\cap\{l\leq|x|\leq l+1\}\cap\{\psi\geq 0\}} \left|\frac{\nabla\psi}{r}\right|^2 + |\nabla\psi|^2 dxdr.$$

Since $r$ has upper bound. we have

$$\iint_{\Omega\cap\{|x|\leq l\}\cap\{\psi\geq 0\}} |\nabla\psi|^2 dxdr \leq C(\underline{B}, \epsilon)\iint_{\Omega\cap\{l\leq|x|\leq l+1\}\cap\{\psi\geq 0\}} \left|\frac{\nabla\psi}{r}\right|^2 dxdr.$$

It follows from Theorem 2.1.2 that $\psi_1$ and $\psi_2$ have the same far fields behavior, thus $|\psi|$ and $\nabla\psi \to 0$ as $|x| \to \infty$. And also note that $\left|\dfrac{\nabla\psi}{r}\right| \to 0$ as $|x| \to \infty$.

Thus

$$\iint_{\Omega\cap\{\psi\geq 0\}} |\nabla\psi|^2 dxdr = 0,$$

so $\psi \leq 0$. Similarly, one can show that $\psi \geq 0$. Therefore, $\psi = 0$. This finishes the proof of the proposition. □

**Lemma 2.5.2** *The swirl component of any smooth subsonic axisymmetric flow must be zero, provided that it satisfies the asymptotic conditions.*

*Proof.* Let the fluid density and velocity be $\rho(x,r)$ and $(U(x,r), V(x,r), W(x,r))$ in cylindrical coordinates, where $U, V, W$ are axial velocity, radial velocity and swirl velocity respectively, $x = x_1$, $r = \sqrt{x_2^2 + x_3^2}$. Then, instead of (2.1.1), we have

$$\begin{cases} (r\rho U)_x + (r\rho V)_r = 0, \\ \left(r\rho U^2\right)_x + (r\rho UV)_r + rP_x = 0, \\ (r\rho UV)_x + \left(r\rho V^2\right)_r - \rho W^2 + rP_r = 0, \\ (r\rho UW)_x + (r\rho VW)_r + \rho VW = 0. \end{cases} \tag{2.5.3}$$

It follows from the first and the fourth equation in (2.5.3) that

$$rUW_x + rVW_r + VW = 0. \tag{2.5.4}$$

First, it is easy to see that the axial velocity $U$ in the nozzle is positive.

Secondly, on the axis $r = 0$, note that the swirl velocity $W$ must be zero due to the axisymmetricity of the flow.

For $r \neq 0$, it follows from (2.5.4) that

$$W_x + \frac{V}{U}W_r + \frac{V}{rU}W = 0.$$

Due to the positivity of $U$, for any point in the inlet, there is one and only one streamline satisfying

$$\frac{dr(x)}{dx} = \frac{V}{U}(x, r(x)), \quad r(x = -\infty) = r_0 \neq 0.$$

Obviously, it can be defined globally in the nozzle. Furthermore, any streamline can not touch the axis for $r_0 \neq 0$. Thus,

$$\frac{d}{dx}W(x,r(x)) + \frac{1}{r(x)}\frac{VW}{U}(x,r(x)) = 0, \quad W(x,r(x))|_{x=-\infty} = 0,$$

which is a linear ordinary differential equation to $W$. Hence, we have $W = 0$. □

As a direct consequence of Proposition 2.5.1, Lemma 2.5.2, it completes the proof of Theorem 2.1.3.

## 2.6 Existence of critical mass flux

Now, we have shown that for given Bernoulli's function $B(r)$ in the upstream satisfying (2.1.16), there exists a unique uniformly subsonic Euler flow in an axisymmetric nozzle as long as the mass flux $m \in (\delta^\gamma, 2\delta_0^{\frac{\gamma}{2}})$. Finally, we will show that there exists a critical value of mass flux, the subsonic Euler flow exists if the incoming mass flux is less than the critical value.

**Proof of Theorem 2.1.4.** Recall the definitions (2.2.22) (2.2.23) of $\rho_0$ and $U_0(r)$ in the upstream, we can find the relationships between $\rho_0$, $U_0(r)$ and $m$,

$$\int_0^1 s\rho_0\sqrt{2(B(r) - h(\rho_0))}dr = m, \qquad U_0(r) = \sqrt{2(B(r) - h(\rho_0))}.$$

Thus, for the given Bernoulli's function $B(r)$ in the upstream satisfying (2.1.16) and $m \in (\delta^\gamma, \bar{m})$, $\rho_0$ and $U_0(r)$ can be regarded as the functions of $m$, denoted by $\rho_0(m)$ and $U_0(r;m)$, respectively. $\Theta(\psi)$ also depends on $m$ by definition (2.2.39), we denote it by $\Theta(\psi; m)$.

Set

$$M(m) = \sup_{\bar{\Omega}}\left(\rho^2(U^2+V^2) - c^2(\rho)\right) = \sup_{\bar{\Omega}}\left(\left|\frac{\nabla\psi}{r}\right|^2 - \Sigma^2\left(\mathcal{B}(\psi)\right)\right).$$

The condition (2.1.21) implies

$$\Theta'(m) = \Theta'(0).$$

Thus, we can make the zero-extension of $\Theta'(s)$ as

$$\tilde{\Theta}'(s) = \begin{cases} \Theta'(s), & \text{if } 0 \le s \le m, \\ \\ 0, & \text{if } s < 0 \quad \text{or} \quad s > m. \end{cases} \tag{2.6.1}$$

Then the extension $\tilde{\Theta}(s)$, defined by $\tilde{\Theta}(s) = \Theta(0) + \int_0^s \tilde{\Theta}'(s)ds$, satisfies

$$B_0 < \underline{B} - \varepsilon_0 \le h(\rho_0) + \frac{\tilde{\Theta}^2(s)}{2} \le \bar{B} \quad \text{and} \quad \|\tilde{\Theta}'\|_{C^{0,1}(\mathbb{R}^1)} \le C\delta^{1-2\gamma}. \tag{2.6.2}$$

Set a strictly decreasing positive sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ satisfies

$$\varepsilon_1 \le \varepsilon_0/4, \quad \text{and} \quad \lim_{n\to\infty} \varepsilon_n = 0.$$

One can define the truncation of $H$ associated with $\varepsilon_n$ similar to Section 3. Set a sequence of smooth increasing function $\zeta_n$ such that

$$\zeta_n(s) = \begin{cases} s, & \text{if} \quad s < -2\varepsilon_n, \\ \\ -\varepsilon_n, & \text{if} \quad s > -\varepsilon_n. \end{cases} \tag{2.6.3}$$

Truncate $\mathcal{M} = \left|\frac{\nabla\psi}{r}\right|^2$ as

$$\tilde{\mathcal{M}}_n(\mathcal{M}, \psi; m) = \zeta_n\left(\mathcal{M} - \Sigma^2(\tilde{\mathcal{B}}(\psi; m))\right) + \Sigma^2(\tilde{\mathcal{B}}(\psi; m)). \tag{2.6.4}$$

where

$$\tilde{\mathcal{B}}(\psi; m) = h(\rho_0(m)) + \frac{\tilde{\Theta}^2(\psi; m)}{2}. \tag{2.6.5}$$

Furthermore, we can define the truncation of $H$ associate with $\varepsilon_n$ as

$$\tilde{H}^{(n)}(\mathcal{M}, \psi; m) = \mathcal{J}\left(\tilde{\mathcal{M}}_n(\mathcal{M}, \psi; m), \tilde{\mathcal{B}}(\psi; m)\right). \tag{2.6.6}$$

where $\mathcal{J}$ is defined in (2.2.21). Hence, we obtain the subsonic truncated problem associated with $\varepsilon_n$

$$\begin{cases} \tilde{A}_{ij}^{(n)}\partial_{ij}\psi = \tilde{\mathcal{F}}_n(\mathcal{M}, \psi, r; m) + \tilde{\mathcal{G}}_n & \text{in} \quad \Omega, \\ \\ \psi = \dfrac{r^2}{f^2(x)}m & \text{on} \quad T_1 \cap T_2, \end{cases} \tag{2.6.7}$$

where

$$\tilde{A}_{ij}^{(n)} = \tilde{H}^{(n)}(\mathcal{M}, \psi; m)\delta_{ij} - 2\tilde{H}_1^{(n)}(\mathcal{M}, \psi; m)\frac{\psi_i}{r}\frac{\psi_j}{r}, \tag{2.6.8}$$

.

$$\tilde{\mathcal{F}}_n(\mathcal{M}, \psi, r; m) = r^2\tilde{\Theta}\tilde{\Theta}'\tilde{H}^{(n)}\left(\frac{\left(\tilde{H}^{(n)}\right)^2 + \Sigma\Sigma'(\zeta_n' - 1)}{(\tilde{H}^{(n)})^2c^2 - \tilde{\mathcal{M}}_n}\tilde{\mathcal{M}}_n + \left(\tilde{H}^{(n)}\right)^2\right), \tag{2.6.9}$$

and

$$\tilde{\mathcal{G}}_n = \left(\tilde{H}^{(n)} - 2\tilde{H}_1^{(n)}\tilde{\mathcal{M}}_n\right)\frac{\psi_2}{r}. \tag{2.6.10}$$

After the subsonic truncation, it is easy to check that there exist two positive constants $\lambda^{(n)}$ and $\Lambda^{(n)}$ such that

$$\lambda^{(n)}|\xi|^2 \le \tilde{A}_{ij}^{(n)}\xi_i\xi_j \le \Lambda^{(n)}|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^2.$$

Applying the argument above, for any $m \in (\delta^\gamma, \bar{m})$, there exists a solution $\psi^{(n)}(x, r; m)$ to the problem (2.6.7). Moreover, if

$$\left|\frac{\nabla\psi^{(n)}}{r}\right|^2 - \tilde{\mathcal{B}}(\psi^{(n)}; m) \le -2\varepsilon_n, \tag{2.6.11}$$

then $\zeta_n' = 1$ and the subsonic truncation disappears, one has

$$0 \le \psi^{(n)}(x, r; m) \le m.$$

Since the bounds of $\tilde{\Theta}(\psi; m)$ is independent of $\varepsilon_n$, one can estimate the integration term $I_5$ in Proposition 2.4.1. Furthermore, it follows from the same argument in Proposition 2.4.1 that the solution to (2.6.7) satisfying (2.6.11) has far fields behavior as (2.4.3). In addition, such a solution is unique among the class of solution satisfying (2.4.3).

Note that in general, we do not know uniqueness of solutions to problem (2.6.7). Let the set of the solution of problem (2.6.7) as

$$S_n(m) = \left\{\psi^{(n)}(x, r; m) | \psi^{(n)}(x, r; m) \text{ solves problem (2.6.7)}\right\}. \tag{2.6.12}$$

Then define

$$M_n(m) = \inf_{\psi^{(n)} \in S_n(m)} \sup_{\bar{\Omega}} \left( \left| \frac{\nabla \psi^{(n)}(x,r;m)}{r} \right|^2 - \Sigma^2(\tilde{\mathcal{B}}(\psi^{(n)};m)) \right) \qquad (2.6.13)$$

and

$$T_n = \{ s | \delta_0^\gamma \le s \le \bar{m} \text{ and } M_n(m) \le -4\varepsilon_n \text{ if } m \in (\delta^\gamma, s) \} . \qquad (2.6.14)$$

It follows from the existence theorem above, we have that $[\delta_0^\gamma, 2\delta_0^{\gamma/2}] \subset T_n$, therefore, $T_n$ is not an empty set and define $m_n = \sup T_n$. Clearly, $\{m_n\}_{n=1}^\infty$ is an increasing sequence. due to the monotonicity of $\varepsilon_n$.

We claim that $M_n(m)$ is left continuous for $m \in (\delta^\gamma, m_n]$.

In fact, for any $m \in (\delta^\gamma, m_n]$, choose an increasing sequence $\{m_n^{(k)}\}_{k=1}^\infty \subset (\delta^\gamma, m)$ with $\lim_{k\to\infty} m_n^k = m$. Since $M_n(m_n^{(k)}) \le -4\varepsilon_n$, we can obtain the following estimate from Section 3

$$\left\| \psi^{(n)}\left(x, r; m_n^{(k)}\right) \right\|_{C^{2,\alpha}(\bar{\Omega})} \le C,$$

here $C$ is independent of $k$. Therefore, there exists a subsequence $\psi^{(n)}\left(x,r;m_n^{(k_l)}\right)$ such that $\psi^{(n)}\left(x,r;m_n^{(k_l)}\right) \to \psi$, moreover, $\psi$ solves problem (2.6.7). Thus $M_n(m) \le \lim M_n \left(m_n^{(k}\right.$ So $M_n(m) \le -4\varepsilon_n$. Note that all these solution satisfy the far field behavior as (2.4.3), by uniqueness of solution in this class

$$M_n\left(m_n^{(k)}\right) \to M_n(m), \quad \text{as } m_n^{(k)} \to m,$$

which implies the left continuous at $m$. For the arbitrariness of $m \in (\delta^\gamma, m_n]$, we prove the claim.

Furthermore, we can claim that $m_n < \bar{m}$. Indeed, suppose on the contrary $m_n = \bar{m}$. By the definition of $m_n$, one has $\bar{m} \in T_n$. It follows from the left continuity of $M_n$ at $\bar{M}$ that $M_n(\bar{m}) \le -4\varepsilon_n$. Thus by means of the proof of Proposition 2.4.1, $\psi^{(n)}(x,r;\bar{m})$ has far field behavior as in (2.4.3). However, it follows from the definition of $\bar{m}$ that

$$\sup_{(x,r)\in\bar{\Omega}} \left( \left| \frac{\nabla\psi^{(n)}(x,r;\bar{m})}{r} \right|^2 - \Sigma^2(\tilde{\mathcal{B}}(\psi^{(n)}(x,r;\bar{m}))) \right)$$

$$\geq \sup_{r\in[0,1]} \max\left\{ |\rho_0(\bar{m})U_0(r;\bar{m})|^2 - \Sigma^2(B(r)),\ |\rho_1(\bar{m})U_1(r(s);\bar{m})|^2 - \Sigma^2(B(r)) \right\}$$

$$= 0$$

where $r = r(s)$ is defined in (2.2.26). Thus $M_n(\bar{m}) \geq 0$, which leads a contradiction. Therefore $m_n < \bar{m}$.

Hence, $\{m_n\}_{n=1}^{\infty}$ is a bounded increasing sequence, we can define $m_c = \lim_{n\to\infty} m_n$ and $m_c \leq \bar{m}$.

Note that for any $m \in (\delta^\gamma, m_c)$, there exists $m_n > m$, therefore $M_n(m) \leq -4\varepsilon_n$. Thus $\psi = \psi^{(n)}(x,r;m)$ solves (2.2.48) and

$$\sup_{\bar{\Omega}} \left( \left| \frac{\nabla\psi}{r} \right|^2 - \Sigma^2(\mathcal{B}(\psi)) \right) = M_n(m) \leq -4\varepsilon_n.$$

If $\sup_{m\in(\delta^\gamma,m_c)} M(m) < 0$, then there exists $n$ such that $\sup_{m\in(\delta^\gamma,m_c)} M(m) < -4\varepsilon_n$. As the same as the proof for the left continuity of $M_n(m)$ on $(\delta^\gamma, m_n]$, $M_n(m_c) \leq -4\varepsilon$. Suppose that there exists $\sigma > 0$ such that (2.2.48) always has a solution $\psi$ for $m \in (m_c, m_c + \sigma)$, and

$$\sup_{m\in(m_c,m_c+\sigma)} M(m) = \sup_{m\in(m_c,m_c+\sigma)} \sup_{\bar{\Omega}} \left( \left| \frac{\nabla\psi}{r} \right|^2 - \Sigma^2(\mathcal{B}(\psi)) \right) < 0.$$

Then there exists $k > 0$ such that

$$\sup_{m\in(m_c,m_c+\sigma)} M(m) = \sup_{m\in(m_c,m_c+\sigma)} \sup_{\bar{\Omega}} \left( \left| \frac{\nabla\psi}{r} \right|^2 - \Sigma^2(\mathcal{B}(\psi)) \right) < -4\varepsilon_{n+k}.$$

This yields that $m_{n+k} \geq m_c + \sigma$, which contradicts with the definition of $m_c$. The contradiction implies that either $M(m) \to 0$, or there does not exist $\sigma > 0$ such that (2.2.48) has solution for all $m \in (m_c, m_c + \sigma)$ and

$$\sup_{m\in(m_c,m_c+\sigma)} M(m) < 0.$$

This finishes the proof of Theorem 2.1.4. □

# Chapter 3

# Transonic shocks in 2-D nozzles with porous medium

Concerning to the transonic flows, Xin and Yin [118] proved the wellposedness of a transonic shock for the steady potential flows through a general 2-D nozzle with variable section. For compressible Euler fluids, an important phenomena for transonic flow is posed by Courant-Friedrichs [41]: Given a appropriately large receiver pressure $p_e$, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes $p_e$. As indicated by Courant-Friedrichs, Xin and Yin [120] showed that for a symmetric incoming flow in a symmetric nozzle and for a nontrivial range of exit pressure, there exists a symmetric transonic shock. Recently, Li, Xin and Yin [71] solve this Courant-Friedrichs problem for the 2-D steady Euler system with a variable exit pressure in a nozzle whose divergent part is an angular sector.

In this chapter, motivated by the iteration scheme developed in [74], we establish the stability of a transonic shock solution to the full steady compressible Euler system in a class of de Laval nozzles under the $C^{2,\alpha}$ perturbation of the supersonic incoming flow.

## 3.1 Introduction and main results

The 2-D full steady Euler system is

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0, \\ \partial_1(\rho u_1^2 + P) + \partial_2(\rho u_1 u_2) = 0, \\ \partial_1(\rho u_1 u_2) + \partial_2(\rho u_2^2 + P) = 0, \\ \partial_1((\rho e + \frac{1}{2}\rho|u|^2 + P)u_1) + \partial_2((\rho e + \frac{1}{2}\rho|u|^2 + P)u_2) = 0, \end{cases} \tag{3.1.1}$$

where $u = (u_1, u_2)$ is the velocity, $\rho$ is the density, $P = P(\rho, S)$ is the pressure, $e$ is the internal energy and $S$ is the special entropy respectively. Moreover, the pressure function $P = P(\rho, S)$ and the internal energy function $e = e(\rho, S)$ are smooth in their arguments. In particular, $\partial_\rho P(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$, and $c(\rho, S) = \sqrt{\partial_\rho P(\rho, S)}$ stands for the local sound speed.

For ideal polytropic gases, the equations of states are given by

$$P = A\rho^\gamma e^{\frac{S}{c_v}} \qquad \text{and} \qquad e = \frac{P}{(\gamma - 1)\rho},$$

here $A, c_v$ and $\gamma$ $(1 < \gamma < 3)$ are positive constants.

Assume that the nozzle walls $\Gamma_1$ and $\Gamma_2$ are $C^{3,\alpha}-$ regular for $X_0 - 1 < \sqrt{x_1^2 + x_2^2} < X_0 + 1$ (here $0 < \alpha < 1$, and $X_0 > 1$ is a fixed constant) and $\Gamma_i$ consists of two curves $\Gamma_i^1$ and $\Gamma_i^2$ with $\Gamma_1^1$ and $\Gamma_2^1$ including the walls for the converging part of the nozzle, while $\Gamma_1^2$ and $\Gamma_2^2$ being the straight line segments so that the divergent part of the nozzle is part of a symmetric angular sector. Assume that $\Gamma_i^2$ is represented by $x_2 = (-1)^i x_1 tan\theta_0$ with $x_1 > 0$ and $X_0 < r < X_0 + 1$, where $0 < \theta_0 < \frac{\pi}{2}$.

Let the uniform supersonic incoming flow $U_- = (u_{1,0}^-(x), u_{2,0}^-(x), P_0^-(x), S_0^-(x))$ is $C^\infty-$ smooth and symmetric near $r = X_0$ so that $u_{i,0}^-(x) = \frac{U_0^-(r)x_i}{r}(i = 1, 2), P_0^-(x) = P_0^-(r)$ and $S_0^-(x) = S_0^-$ ($S_0^-$ is a constant). It is noted that this assumption can be easily realized by the hyperbolicity of the supersonic incoming flow and the symmetric property of the nozzle walls for $X_0 < r < X_0 + 1$, one can see [67].

Suppose the supersonic incoming flow at the inlet $r = X_0$ is given by

$$\Phi_p = (u_1^-, u_2^-, P^-, S^-), \tag{3.1.2}$$

which is close to the uniform supersonic flow in the following sense

$$\|(\Phi_p - \Phi_b)(X_0 \cos\theta, X_0 \sin\theta)\|_{C^{2,\alpha}[-\theta_0,\theta_0]} \leq \varepsilon \tag{3.1.3}$$

and satisfying the following compatibility conditions:

$$\frac{d}{d\theta}(B, S)(X_0 \cos\theta, X_0 \sin\theta)|_{\theta=\pm\theta_0} = 0. \tag{3.1.4}$$

where $\Phi_b = (u_{1\,0}^-(x), u_{2,0}^-(x), P_0^-(x), S_0^-(x))$, and $B = \frac{1}{2}(u_1^2 + u_2^2) + \frac{\gamma}{\gamma - 1}e(P, S)$ is the Bernoulli's function.

If the transonic shock curve $\Sigma : x_1 = \eta(x_2)$ is formed, and denote the flow behind $\Sigma$ by $(u_1^+(x), u_2^+(x), P^+(x), S^+(x))$. Then it follows from the Rankine-Hugoniot conditions on $\Sigma$ that

$$\begin{cases} [\rho u_1] - \eta'(x_2)[\rho u_2] = 0, \\ [\rho u_1^2 + P] - \eta'(x_2)[\rho u_1 u_2] = 0, \\ [\rho u_1 u_2] - \eta'(x_2)[\rho u_2^2 + P] = 0, \\ [(\rho e + \frac{1}{2}\rho|u|^2 + P)u_1] - \eta'(x_2)[(\rho e + \frac{1}{2}\rho|u|^2 + P)u_2] = 0. \end{cases} \tag{3.1.5}$$

In addition, the pressure $P(x)$ satisfies the physical entropy condition :

$$P^+(x) > P^-(x) \qquad on \qquad x_1 = \eta(x_2). \tag{3.1.6}$$

On the exit of the nozzle, the end pressure is prescribed by

$$P^+(\theta) = P_e + \varepsilon P_0(\theta) \qquad on \qquad r = X_0 + 1, \tag{3.1.7}$$

here $\varepsilon > 0$ is suitable small, $\theta = arctan\frac{x_2}{x_1}$, $P_0(\theta) \in C^{2,\alpha}([-\theta_0, \theta_0])$ with

$$P_0'(\pm\theta_0) = 0, \qquad \|P_0(\theta)\|_{C^{2,\alpha}[-\theta_0,\theta_0]} \leq C, \tag{3.1.8}$$

the constant $P_e$ denotes by the end pressure for which a symmetric transonic shock lies at the position $r = r_0$ with $r_0 \in (X_0, X_0 + 1)$ and the supersonic incoming flow is given by $(U_0^-(r), P_0^-(r), S_0^-)$ in the domain $\{r : X_0 \le r \le X_0 + 1\}$. For more details, one can see Section 147 of [41] or Theorem 1.1 of [118].

We assume upper nozzle wall is porous medium means for given function $f > 0$, $||f||_{C^{2,\alpha}} \le \varepsilon$ and $supp(f) \subset (\tilde{r}_0, X_0 + 1) \subset (\frac{r_0 + X_0 + 1}{2}, X_0 + 1)$, on the upper boundary

$$(u_1, u_2) \cdot \overrightarrow{n} = f(\sqrt{x_1^2 + x_2^2}). \tag{3.1.9}$$

Since the flow is tangent to the nozzle wall $x_2 = (-1)x_1 \tan\theta_0$, then

$$u_2^+ = (-1)u_1^+ \tan\theta_0 \qquad on \qquad x_2 = (-1)x_1 \tan\theta_0. \tag{3.1.10}$$

As been stated in Section 147 of [41] (see also Theorem 1.1 of [118]), under the above assumptions on the nozzle and the symmetric supersonic incoming flow near the throat of the nozzle, there exists a unique symmetric transonic shock solution for the given constant end pressure $P_e$. Furthermore, the position of the shock, $r = r_0$, depends monotonically on the given end pressure. This solution will be called the background solution in this paper. Let $(U_0^+(r), P_0^+(r), S_0^+)$ ($S_0^+$ is a constant) be the subsonic part of the background solution for $r_0 < r < X_0+1$, which can be extended into the domain $\{r : r_0 - \delta_0 \le r \le X_0+1\}$ ($\delta_0 > 0$ is some constant depending only on the supersonic incoming flow) (see Theorem 1.1 of [118]). The corresponding extension will be denoted by $(\hat{U}_0^+(r), \hat{P}_0^+(r), S_0^+)$.

Now we can state the main result in this chapter

**Theorem 3.1.1** *(Existence and Uniqueness) Under the assumptions above, there exists a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, if (3.1.2)-(3.1.4)hold, the problem (3.1.1),(3.1.5)-(3.1.10) has a unique transonic shock solution*

$$(U_-, U_+; \Sigma)$$

*which satisfies the following properties:*

*(1). $U_-$ is supersonic, $U_+$ is subsonic, and $\Sigma$ is the transonic shock front separating $U_-$ and $U_+$ and satisfies the entropy condition.*

*(2) Let the equation of $\Sigma$ be $x_1 = \eta(x_2)$ and define the supersonic region $\Omega_-$ and subsonic $\Omega_+$ region as follows:*

$$\Omega_- = \{(x_1, x_2) : \sqrt{X_0^2 - x_2^2} < x_1 < \eta(x_2), |x_2| < x_1 tan\theta_0\}$$

*and*

$$\Omega_+ = \{(x_1, x_2) \quad \eta(x_2) < x_1 < \sqrt{(X_0+1)^2 - x_2^2}, |x_2| < x_1 tan\theta_0\},$$

*then the following estimates hold*

*(i). $\Phi_- = (u_1^-(x), u_2^-(x), P^-(x), S^-(x)) \in C^{2,\alpha}(\bar{\Omega}_-)$, and*

$$\|(u_1^-, u_2^-, P^-, S^-) - (\hat{u}_{1,0}^-, \hat{u}_{2,0}^-, \hat{P}_0^-, S_0^-)\|_{C^{2,\alpha}(\bar{\Omega}_-)} \leq C_0\varepsilon, \tag{3.1.11}$$

*and the generic constant $C_0$ is a positive constant depending only on $\alpha$ and the supersonic incoming flow.*

*Also for $(r\cos\theta_0, (-1)^i r\sin\theta_0) \in \partial\Omega_-$ we have:*

$$\frac{\partial}{\partial\theta}(U_1^-, P^-, S^-)(r\cos\theta_0, (-1)^i r\sin\theta_0) = 0, \qquad \frac{\partial^2}{\partial\theta^2}U_2^-(r\cos\theta_0, (-1)^i r\cos\theta_0) = 0. \tag{3.1.12}$$

*(ii). $\eta(x_2) \in C^{3,\alpha}[x_2^1, x_2^2]$, and*

$$\|\eta(x_2) - \sqrt{r_0^2 - x_2^2}\|_{C^{3,\alpha}[x_2^1, x_2^2]} \leq C_0\varepsilon, \tag{3.1.13}$$

*where $(x_1^i, x_2^i)(i = 1,2)$ stands for the intersection points of $x_1 = \eta(x_2)$ with $x_2 = (-1)^i x_1 tan\theta_0$ for $i = 1,2$.*

*(iii). $\Phi_+ = (u_1^+(x), u_2^+(x), P^+(x), S^+(x)) \in C^{1,\alpha}(\bar{\Omega}_+)$, and*

$$\|(u_1^+, u_2^+, P^+, S^+) - (\hat{u}_{1,0}^+, \hat{u}_{2,0}^+, \hat{P}_0^+, S_0^+)\|_{C^{1,\alpha}(\bar{\Omega}_+)} \leq C_0\varepsilon, \tag{3.1.14}$$

*where $(\hat{u}_{1,0}^+, \hat{u}_{2,0}^+, \hat{P}_0^+) = (\hat{U}_0^+(r)\frac{x}{r}, \hat{P}_0^+(r))$.*

**Remark 3.1.1** *In [119], Z. P. Xin, W. Yan and H. C. Yin, they investigate the problem on the well-posedness of a transonic shock to the steady compressible Euler flow through a 2-D slowly variable nozzle. If the upstream flow remains supersonic behind the throat of the nozzle, then at a certain place in the nozzle, a shock front intervenes and the flow is compressed and slowed down to subsonic speed, and the position and the strength of the shock front are automatically adjusted so that the end pressure at exit becomes $P_r$ by [41]. They showed that, the uniqueness of such a transonic shock solution if it exists and the shock front goes through a fixed point. Moreover, they proved that there is no such transonic shock solution for flat nozzles with some large pressure given at exit. Motivated by above work, we would like to investigate porous medium boundary condition instead of solid wall condition and expect the existence of transonic shock solutions with perturbed end pressure. However, there are essential difficulties, we only could deal with divergent nozzles in Theorem 3.1.1. If we can have the uniform estimate respect to $\theta$, we may use a limit argument to conclude the desired result. Unfortunately, we did not have such estimate yet.*

To solve this transonic shock problem, we first establish the existence and uniqueness of supersonic flows in the whole nozzle $\Omega$ via the method of characteristic.

**Theorem 3.1.2** *(Existence and Uniqueness for supersonic flow) If (3.1.3)-(3.1.4) hold, the problem (3.1.1),(3.1.2), (3.1.9) and (3.1.10) has a unique supersonic solution in domain $\tilde{\Omega} = \{(x,r)|X_0 < r < \tilde{r}_0, -\theta_0 \le \theta \le \theta_0\}$,*

$$\Phi_- = (u_1^-(x), u_2^-(x), P^-(x), S^-(x)) \in C^{2,\alpha}(\tilde{\Omega}),$$

*satisfies the following properties:*

$$\|(u_1^-, u_2^-, P^-, S^-) - (\hat{u}_{1,0}^-, \hat{u}_{2,0}^-, \hat{P}_0^-, S_0^-)\|_{C^{2,\alpha}(\tilde{\Omega})} \le C_0\varepsilon, \tag{3.1.15}$$

*and the generic constant $C_0$ is a positive constant depending only on $\alpha$ and the supersonic incoming flow*

*Furthermore, we have*

$$\frac{\partial}{\partial\theta}(U_1^- \; P^-, S^-)(r\cos\theta_0, (-1)^i r\sin\theta_0) = 0, \qquad \frac{\partial^2}{\partial\theta^2}U_2^-(r\cos\theta_0, (-1)^i r\sin\theta_0) = 0 \tag{3 1 16}$$

Indeed, this theorem has been essentially proved in [119], [23], we will give an outline of the proof of this theorem in Appendix A

With Theorem 3 1 2, we can transform the transonic shock problem into the following one-phase free boundary value problem

**FBP:** *Given a supersonic solution $\Phi_- = (u_1^-(x), u_2^-(x), P^-(x)\; S^-(x))$ of (3 1 1),(3 1 2), (3 1 9) and (3 1 10) satisfying (3 1 11)-(3 1 12) for some small constants $\varepsilon > 0$, find a subsonic flow $U_+$ in the downstream separated by a transonic shock front $x_1 = \eta(x_2)$ satisfying (3 1 5)-(3 1 10)*

Then we only need to prove the following theorem

**Theorem 3.1.3** *Let $\varepsilon > 0$ be small, and the supersonic incoming flow obtained in Theorem 3 1 2, the problem* **FBP** *has a subsonic solution*

$$\Phi_+ = (u_1^+(x), u_2^+(x)\; P^+(x), S^+(x)) \in C^{1\,\alpha}(\bar{\Omega}_+)$$

*and a shock front*

$$x_1 = \eta(x_2)$$

*which satisfy the Rankine-Hugoniot condition, the entropy condition and the estimates (3 1 13)-(3 1 14)*

**Remark 3.1.2** *Compared with the results in [71]- [73], we do not need to require that the diverging part of the nozzle wall changes slowly The key ingredient in the analysis of [71]- [73] is to establish the monotonic property of the shock position along the nozzle wall with respect to the exit pressure so that one can* *avoid the difficulties caused by the unknown position of the shock. After some modification of the new elaborate scheme developed in [74], we can determine the shock position together with the solution in each iteration step. The key issue is to solve a boundary value problem for a first* $2 \times 2$ *elliptic system with non-local terms and an unknown parameter. Our results show that the background transonic shock solution with arbitrarily changing subsonic flow is structurally stable under small perturbation of the supersonic incoming flow and the exit pressure.*

**Remark 3.1.3** *One should note that the main difference between our case and the one in [74] is the Bernoulli's function is not a constant any more. Actually, the Bernoulli's function is not conserved across the shock, hence we have to deal with the Rankine-Hugoniot condition in a different way. The monotonicity of pressure for the background solution in the subsonic region plays an important role in the well-posedness of the elliptic system.*

**Remark 3.1.4** *By the results in [119], we know that the shock curve is perpendicular to the nozzle wall. To guarantee the* $C^1$ *regularity of transonic shock solution in the downstream region (up to the boundary), the Bernoulli's constant and entropy of supersonic incoming flow and the curvature of the nozzle wall is required to satisfy some compatibility conditions. And the compatibility condition (2.1.4) is a sufficient condition. This condition is also necessary in the isentropic case. One can see* §3.3 *for more detailed explanation.*

**Remark 3.1.5** *One can expect the existence and uniqueness results for such a question is still true without this condition. In this case, singularity will be developed and propagated along the nozzle wall and may affect the regularity of the interior. Another more essential difficulty lies in the loss of regularity on dealing with the hyperbolic mode. Hence one has to employ the Lagrangian transformation. It is easy to prove the equivalence of the weak solution in Euler and*

*Lagrangian coordinates since the Lagrangian transformation is Lipschitz. However, it is quite difficult to prove the norm equivalence due to the formation of shock. Recently, Li, Xin and Yin have solved this difficult problem.*

Now we explain the main idea of the iteration scheme. Actually, the problem can be reformulated as a system consisting of an ordinary differential equation for the shock with a free initial position, a first order nonlinear elliptic system for the pressure and angular velocity, and two transport equations for the specific entropy and Bernoulli's function respectively on a fixed domain. Linearizing the nonlinear equation and the nonlinear boundary condition, one can obtain a new iteration scheme which involve a boundary value problem for a first order $2 \times 2$ elliptic system with non-local terms and an unknown parameter. The non-local terms arise from the Rankine-Hugoniot condition and hyperbolic modes, the unknown parameter denotes the unknown shock position on the nozzle wall.

The rest of this chapter will be organized as follows. In §3.2, following [74], we reformulate the 2-D problem (3.1.1) with the boundary conditions (3.1.5)-(3.1.10) so that one can obtain a $2 \times 2$ first order elliptic system on $\omega = \dfrac{U_2^+}{U_1^+}$ and pressure $P^+$ together with the shock curve equation and the entropy $S^+$, two first order hyperbolic equation on $S^+$ and $B$ along the streamline. In §3.3, using the decomposition techniques in §3.2, we linearize the resulted nonlinear equations and construct a suitable iteration scheme, especially, a linear $2 \times 2$ first order elliptic system with the nonlocal terms and an unknown constant is derived. In §3.4, we establish some a priori estimates on the linearized equations derived in §3.3 and further complete the proof on Theorem 3.1.1.

## 3.2 Reformulation of the problem

In this section, we reformulate the nonlinear problem (3.1.1) with (3.1.5)-(3.1.10) so that we can obtain an a first order elliptic system for the pressure $P^+(x)$ and the angular velocity $U_2^+(x)$, two first order partial differential equations for the Bernoulli's function and the special entropy $S^+(x)$ respectively.

Due to the angular geometric structure of the nozzle walls, it is more convenient to use the polar coordinates

$$\begin{cases} x_1 = rcos\theta, \\ x_2 = rsin\theta, \end{cases} \tag{3.2.1}$$

and decompose the velocity $(u_1^+, u_2^+)$ into the radial speed $U_1^+$ and angular speed $U_2^+$ as follows

$$\begin{cases} u_1^+ = U_1^+ cos\theta - U_2^+ sin\theta, \\ u_2^+ = U_1^+ sin\theta + U_2^+ cos\theta. \end{cases} \tag{3.2.2}$$

Under the polar coordinate transformation (3.2.1), the domains

$$\Omega = \{(x_1, x_2) : X_0 < \sqrt{x_1^2 + x_2^2} < X_0 + 1, |x_2| < x_1 tan\theta_0\}$$

and

$$\Omega_+ = \{(x_1, x_2) : \eta(x_2) < x_1 < \sqrt{(X_0 + 1)^2 - x_2^2}, |x_2| < x_1 tan\theta_0\}$$

are changed into

$$R = \{(r, \theta) : X_0 < r < X_0 + 1, -\theta_0 < \theta < \theta_0\}, \tag{3.2.3}$$

and

$$R_+ = \{(r, \theta) : \xi(\theta) < r < X_0 + 1, -\theta_0 < \theta < \theta_0\} \tag{3.2.4}$$

respectively, where $r = \xi(\theta)$ stands for the equation of shock curve $\Sigma$ in the polar coordinate $(r, \theta)$.

It follows a direct computation that (3.1.1) and (3.1.5) become respectively

$$\begin{cases} \partial_r(\rho^+ U_1^+) + \frac{1}{r}\partial_\theta(\rho^+ U_2^+) + \frac{\rho^+ U_1^+}{r} = 0, \\ \partial_r(\rho^+ (U_1^+)^2 + P^+) + \frac{1}{r}\partial_\theta(\rho^+ U_1^+ U_2^+) + \frac{\rho^+((U_1^+)^2 - (U_2^+)^2)}{r} = 0, \\ \partial_r(\rho^+ U_1^+ U_2^+) + \frac{1}{r}\partial_\theta(\rho^+ (U_2^+)^2 + P^+) + \frac{2}{r}\rho^+ U_1^+ U_2^+ = 0, \\ \partial_r((\rho^+ e^+ + \frac{1}{2}\rho^+|U^+|^2 + P^+)U_1^+) + \frac{1}{r}\partial_\theta((\rho^+ e^+ + \frac{1}{2}\rho^+|U^+|^2 + P^+)U_2^+) \\ \quad + \frac{1}{r}(\rho^+ e^+ + \frac{1}{2}\rho^+|U^+|^2 + P^+)U_1^+ = 0 \end{cases} \tag{3.2.5}$$

and

$$\begin{cases} [\rho U_1] - \dfrac{\xi'(\theta)}{\xi(\theta)}[\rho U_2] = 0, \\ [\rho U_1^2 + P] - \dfrac{\xi'(\theta)}{\xi(\theta)}[\rho U_1 U_2] = 0, \\ [\rho U_1 U_2] - \dfrac{\xi'(\theta)}{\xi(\theta)}[\rho U_2^2 + P] = 0, \\ [(\rho e + \dfrac{1}{2}\rho|U|^2 + P)U_1] - \dfrac{\xi'(\theta)}{\xi(\theta)}[(\rho e + \dfrac{1}{2}\rho|U|^2 + P)U_2] = 0, \end{cases} \tag{3.2.6}$$

where $U = (U_1, U_2)$.

Meanwhile. (3.1.7), (3.1.9) and (3.1.10) are converted into

$$P^+(X_0 + 1, \theta) = P_e + \varepsilon P_0(\theta), \tag{3.2.7}$$

and

$$U_2^+(r, \theta_0) = f(r) \quad U_2^+(r, -\theta_0) = 0. \tag{3.2.8}$$

From now on, for notational conveniences, the superscripts "+" will be neglected. Then for any $C^1$ solution, (3.2.5) is actually equivalent to

$$\begin{cases} \partial_r(\rho U_1) + \dfrac{1}{r}\partial_\theta(\rho U_2) + \dfrac{\rho U_1}{r} = 0, \\ U_1\partial_r U_1 + \dfrac{U_2}{r}\partial_\theta U_1 + \dfrac{\partial_r P}{\rho} - \dfrac{U_2^2}{r} = 0, \\ U_1\partial_r U_2 + \dfrac{U_2}{r}\partial_\theta U_2 + \dfrac{1}{r}\dfrac{\partial_\theta P}{\rho} + \dfrac{U_1 U_2}{r} = 0, \\ U_1\partial_r S + \dfrac{U_2}{r}\partial_\theta S = 0. \end{cases} \tag{3.2.9}$$

Now we reformulate the boundary conditions on the shock line. By Rankine-Hugoniot conditions, we have

$$\begin{cases} G_1(U, U_-) = [\rho U_1][\rho U_2^2 + P] - [\rho U_1 U_2][\rho U_2] = 0, \\ G_2(U, U_-) = ([\rho U_1 U_2])^2 - [\rho U_1^2 + P][\rho U_2^2 + P] = 0, \\ G_3(U, U_-) = [\rho(e + \dfrac{1}{2}|U|^2 + \dfrac{P}{\rho})U_1][\rho U_2^2 + P] - [\rho U_1 U_2][\rho(e + \dfrac{1}{2}|U|^2 + \dfrac{P}{\rho})U_2] = 0, \end{cases} \tag{3.2.10}$$

As in [118], by implicit function theorem, one has on $r = \xi(\theta)$

$$\begin{cases} U_1 - U_0(r_0) = \tilde{g}_1\big(U_2^2, U_1^- - U_0^-(r_0), P^- - P_0^-(r_0), S^- - S_0^-, (U_2^-)^2, U_2^- U_2\big), \\ P - P_0^+(r_0) = \tilde{g}_2\big(U_2^2, U_1^- - U_0^-(r_0), P^- - P_0^-(r_0), S^- - S_0^-, (U_2^-)^2, U_2^- U_2\big), \\ S - S_0^+(r_0) = \tilde{g}_3\big(U_2^2, U_1^- - U_0^-(r_0), P^- - P_0^-(r_0), S^- - S_0^-, (U_2^-)^2, U_2^- U_2\big), \end{cases} \tag{3.2.11}$$

here $\tilde{g}_i(0,0,0,0,0,0) = 0$ for $i = 1,2,3$. An important property of $g_i$ is

$$\tilde{g}_i = O(U_2^2) + O(U_1^- - U_0^-(r_0)) + O(P^- - P_0^-(r_0)) + O(S^- - S_0^-) + O((U_2^-)^2) \\ + O(U_2^- U_2), \ \ for\ i = 1,2,3.$$

Actually, we can give a more detailed description of $\tilde{g}_i$. Set $\Phi = (U_1, U_2, P, S)$ and denote $\Phi_b^+, \Phi_b^-$ for the subsonic and supersonic state of the background solution respectively. Since $G_i(\Phi_b^+, \Phi_b^-) = 0$ hold for $i = 1,2,3$, we have

$$\begin{aligned} &\nabla_+ G_i(\Phi_b^+(r_0), \Phi_b^-(r_0))(\Phi(\xi(\theta),\theta) - \Phi_b^+(r_0)) \\ &= -\{(\nabla_+ G_i(\Phi_b^+(\xi(\theta)), \Phi_b^-(\xi(\theta))) - \nabla_+ G_i(\Phi_b^+(r_0), \Phi_b^-(r_0)))(\Phi(\xi(\theta),\theta) - \Phi_b^+(r_0))\} \\ &+ \{\nabla_+ G_i(\Phi_b^+(\xi(\theta)), \Phi_b^-(\xi(\theta)))(\Phi(\xi(\theta),\theta) - \Phi_b^+(r_0)) - (G_i(\Phi(\xi(\theta),\theta), \Phi_-(\xi(\theta),\theta)) \\ &- G_i(\Phi_b^+(\xi(\theta)), \Phi_-(\xi(\theta),\theta)))\} - \{G_i(\Phi_b^+(\xi(\theta)), \Phi_-(\xi(\theta),\theta)) \\ &- G_i(\Phi_b^+(\xi(\theta)), \Phi_b^-(\xi(\theta)))\} - \{G_i(\Phi_b^+(\xi(\theta)), \Phi_b^-(\xi(\theta))) - G_i(\Phi_b^+(r_0), \Phi_b^-(r_0))\}. \end{aligned}$$

The terms in the first two brackets $\{\}$ in the above formula are all high order term. The third term has the form $O(\Phi^-(\xi(\theta),\theta) - \Phi_b^-(\xi(\theta)))$, which is not a so important term. We only need to calculate the fourth term.

$$\begin{aligned} &\{G_i(\Phi_b^+(\xi(\theta)), \Phi_b^-(\xi(\theta))) - G_i(\Phi_b^+(r_0), \Phi_b^-(r_0))\} \\ &= \Big(\nabla_+ G(\Phi_b^+(r_0), \Phi_b^-(r_0)) \cdot \frac{d\Phi_b^+}{dr}(r_0) + \nabla_- G(\Phi_b^+(r_0), \Phi_b^-(r_0)) \cdot \frac{d\Phi_b^-}{dr}(r_0)\Big)(\xi - r_0) \\ &\quad + O((\xi - r_0)^2). \end{aligned}$$

Careful calculations show that

$$\begin{cases} \nabla_+ G_1(\Phi_b^+(r_0), \Phi_b^-(r_0))(\Phi(\xi(\theta), \theta) - \Phi_b^+(r_0)) = O((\xi - r_0)^2) + O(\Phi^- - \Phi_b^-) \\ \qquad + O((\Phi(\xi(\theta), \theta) - \Phi_b^+(\xi(\theta), \theta))^2), \\ \nabla_+ G_2(\Phi_b^+(r_0), \Phi_b^-(r_0))(\Phi(\xi(\theta), \theta) - \Phi_b^+(r_0)) = -\frac{[P_0]}{r_0}(\xi - r_0) + O((\xi - r_0)^2) \\ \qquad + O(\Phi^- - \Phi_b^-) + O((\Phi(\xi(\theta), \theta) - \Phi_b^+(\xi(\theta), \theta))^2), \\ \nabla_+ G_3(\Phi_b^+(r_0), \Phi_b^-(r_0))(\Phi(\xi(\theta), \theta) - \Phi_b^+(r_0))(r_0))\} = O((\xi - r_0)^2) \\ \qquad + O(\Phi^- - \Phi_b^-) + O((\Phi(\xi(\theta), \theta) - \Phi_b^+(\xi(\theta), \theta))^2), \end{cases}$$

By implicit function theorem, we obtain

$$\begin{cases} U_1 - U_0^+(r_0) = B_1(\xi(\theta) - r_0) + R_1((\xi(\theta) - r_0)^2, U_2^2, \Phi^- - \Phi_b^-), \\ P - P_0^+(r_0) = B_2(\xi - r_0) + R_2((\xi(\theta) - r_0)^2, U_2^2, \Phi^- - \Phi_b^-), \\ S - S_0^+ = B_3(\xi - r_0) + R_3((\xi(\theta) - r_0)^2, U_2^2, \Phi^- - \Phi_b^-), \end{cases} \tag{3.2.12}$$

where

$$\begin{aligned} B_1 &= -\frac{\gamma U_0^+(r_0)}{\rho_0^+(r_0)(c^2(\rho_0^+(r_0), S_0^+) - (U_0^+(r_0))^2)} \frac{[P_0]}{r_0} < 0. \\ B_2 &= -\frac{(\gamma - 1)(U_0^+(r_0))^2 + \gamma c^2(\rho_0^+(r_0), S_0^+)}{c^2(\rho_0^+(r_0), S_0^+) - (U_0^+(r_0))^2} \frac{[P_0]}{r_0} < 0, \\ B_3 &= -\frac{(\gamma - 1)\rho_0^+}{\gamma \partial_S \rho_0^+ P_0^+} \frac{[P_0]}{r_0} > 0, \end{aligned}$$

Furthermore, it follows from the third equation in (3.2.6) that $r = \xi(\theta)$ satisfies

$$\xi'(\theta) = \xi(\theta)\left(\frac{\rho U_1^2 \omega - \rho^- U_1^- U_2^-}{[P] + \rho U_1^2 \omega^2 - \rho^-(U_2^-)^2}\right)(\xi(\theta), \theta), \tag{3.2.13}$$

where $\omega = \dfrac{U_2}{U_1}$.

We now decompose the elliptic-hyperbolic system (3.2.10) by its elliptic and hyperbolic modes.

$U_1\times$ {the first equation}$-\rho\times${the second equation} and $U_2\times$ {the first equation} $-\rho\times$ {the third equation} in (3.2.10) respectively, together with (3.2.11) and

(3.2.7)-(3.2.8), we know that $\omega = \dfrac{U_2}{U_1}$ and $P$ satisfy:

$$\begin{cases} \partial_r \omega + \dfrac{1}{r}\dfrac{U_1^2\omega}{U_1^2 - c^2(\rho, S)}\partial_\theta \omega + \dfrac{1}{r}\left(\dfrac{c^2(\rho, S) - U_1^2\omega^2}{\rho U_1^2 c^2(\rho, S)} + \dfrac{U_1^2\omega^2}{\rho c^2(\rho, S)(U_1^2 - c^2(\rho, S))}\right)\partial_\theta P \\ \qquad + \dfrac{1}{r}\dfrac{U_1^2\omega}{U_1^2 - c^2(\rho, S)}(1 + \omega^2) = 0 \qquad in \qquad R_+, \\ \partial_r P + \dfrac{1}{r}\dfrac{\rho c^2(\rho, S)U_1^2}{U_1^2 - c^2(\rho, S)}\partial_\theta \omega + \dfrac{1}{r}\dfrac{U_1^2\omega}{U_1^2 - c^2(\rho, S)}\partial_\theta P + \dfrac{1}{r}\dfrac{\rho c^2(\rho, S)U_1^2}{U_1^2 - c^2(\rho, S)}(1 + \omega^2) = 0 \qquad in \qquad R \\ P - P_0^+(r_0) = B_2(\xi - r_0) + R_2(\theta) \qquad on \qquad r = \xi(\theta), \\ \omega(r, \theta_0) = \dfrac{f(r)}{U_1} \quad \omega(r, -\theta_0) = 0, \\ P = P_e + \varepsilon P_0(\theta) \qquad on \qquad r = X_0 + 1. \end{cases} \tag{3.2.14}$$

In addition, it follows from the fourth equation in (3.2.10), (3.2.11) and the Bernoulli's law that

$$\begin{cases} U_1 \partial_r S + \dfrac{U_2}{r}\partial_\theta S = 0 \qquad in \qquad R_+, \\ S - S_0^+ = B_3(\xi(\theta) - r_0) + R_3(\theta), \qquad on \qquad r = \xi(\theta) \end{cases} \tag{3.2.15}$$

and

$$\begin{cases} U_1 \partial_r B + \dfrac{U_2}{r}\partial_\theta B = 0 \qquad in \qquad R_+, \\ B = \dfrac{1}{2}((U_0^+(r_0) + B_1(\xi(\theta) - r_0) + R_1(\theta))^2 + U_2^2) \\ + \dfrac{\gamma}{\gamma - 1}e(P_0^+(r_0) + B_2(\xi(\theta) - r_0) + R_2(\theta), S_0^+ + B_3(\xi(\theta) - r_0) + R_2(\theta)) \\ \qquad on \qquad r = \xi(\theta). \end{cases} \tag{3.2.16}$$

Thus, to prove Theorem 3.1.3, it suffices to solve the problems (3.2.14)-(3.2.16). Furthermore, it is more convenient to reduce the free boundary problem into a fixed boundary value problem by setting

$$\begin{cases} z_1 = \dfrac{r - \xi(\theta)}{X_0 + 1 - \xi(\theta)}(X_0 + 1 - r_0), \\ z_2 = \theta. \end{cases} \tag{3.2.17}$$

Then the domain $R_+$ defined in (3.2.4) is transformed into

$$E_+ = \{(z_1, z_2) : 0 < z_1 < X_0 + 1 - r_0, -\theta_0 < z_2 < \theta_0\}. \tag{3.2.18}$$

A direct computation yields

$$\partial_r = \frac{X_0+1-r_0}{X_0+1-\xi(z_2)}\partial_{z_1}, \qquad \partial_\theta = \frac{z_1-(X_0+1-r_0)}{X_0+1-\xi(z_2)}\xi'(z_2)\partial_{z_1}+\partial_{z_2}. \tag{3.2.19}$$

Thus, in the new coordinate, the problems (3.2.13)-(3.2.16) can be rewritten respectively as

$$\xi'(\theta) = \xi(\theta)\bigg(\frac{\rho U_1^2\omega - \rho^- U_1^- U_2^-}{[P]+\rho U_1^2\omega^2-\rho^-(U_2^-)^2}\bigg)(\xi(\theta),\theta) \qquad in \qquad [-\theta_0,\theta_0] \tag{3.2.20}$$

and

$$\begin{cases} \partial_{z_1}\omega + \frac{1}{r}\frac{X_0+1-\xi(z_2)}{(X_0+1-r_0)\rho U_1^2}\partial_{z_2}P + \frac{1}{r}\frac{(X_0+1-\xi(z_2))U_1^2}{(X_0+1-r_0)(U_1^2-c^2(\rho,S))}\omega \\ \qquad +\frac{1}{r}(\frac{z_1}{X_0+1-r_0}-1)\frac{\partial_{z_1}P}{\rho U_1^2}\xi'(z_2) = F_1(U_1,\omega,P,S,\xi) \qquad in \qquad E_+, \\ \partial_{z_1}P + \frac{X_0+1-\xi(z_2)}{r(X_0+1-r_0)}\frac{\rho c^2(\rho,S)U_1^2}{U_1^2-c^2(\rho,S)}\partial_{z_2}\omega + \frac{\gamma}{r}\frac{(X_0+1-\xi(z_2))U_1^2}{(X_0+1-r_0)(U_1^2-c^2(\rho,S))}P \\ \qquad\qquad = F_2(U_1,\omega,P,S,\xi) \qquad in \qquad E_+, \\ P-P_0^+(r_0) = B_2(\xi-r_0)+R_2(\theta) \qquad on \qquad z_1=0, \\ \omega(z_1,\theta_0) = \frac{f(z_1+r_0)}{U_1} \quad \omega(z_1,-\theta_0)=0, \\ P = P_e+\varepsilon P_0(z_2) \qquad on \qquad z_1 = X_0+1-r_0 \end{cases} \tag{3.2.21}$$

and

$$\begin{cases} \Big((X_0+1-r_0)\xi(z_2)+(X_0+1-\xi(z_2))z_1+\big(z_1-(X_0+1-r_0)\big)\xi'(z_2)\omega\Big)\partial_{z_1}S \\ \qquad +\Big(X_0+1-\xi(z_2)\Big)\omega\partial_{z_2}S = 0 \qquad in \qquad E_+, \\ S-S_0^+ = B_3(\xi-r_0)+R_3(z_2), \qquad on \qquad z_1=0, \end{cases} \tag{3.2.22}$$

and

$$\begin{cases} \Big((X_0+1-r_0)\xi(z_2)+(X_0+1-\xi(z_2))z_1+\big(z_1-(X_0+1-r_0)\big)\xi'(z_2)\omega\Big)\partial_{z_1}B \\ \quad +\Big(X_0+1-\xi(z_2)\Big)\omega\partial_{z_2}B=0 \qquad in \qquad E_+, \\ B=\dfrac{1}{2}((U_0^+(r_0)+B_1(\xi-r_0)+R_1)^2+U_2^2) \\ \quad +\dfrac{\gamma}{\gamma-1}e(P_0^+(r_0)+B_2(\xi-r_0)+R_2,S_0^+ +B_3(\xi-r_0)+R_3) \qquad on \qquad z_1=0. \end{cases} \tag{3.2.23}$$

where

$$\begin{aligned} &F_1(U_1,\omega,P,S,\xi) \\ &=-\frac{1}{r}\frac{U_1^2\omega}{U_1^2-c^2(\rho,S)}\bigg(\big(\frac{z_1}{X_0+1-r_0}-1\big)\xi'(z_2)\partial_{z_1}\omega+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}\partial_{z_2}\omega\bigg) \\ &+\frac{1}{r}\bigg(\frac{\omega^2}{\rho c^2(\rho,S)}-\frac{U_1^2\omega^2}{\rho c^2(\rho,S)\big(U_1^2-c^2(\rho,S)\big)}\bigg)\bigg(\big(\frac{z_1}{X_0+1-r_0}-1\big)\xi'(z_2)\partial_{z_1}P \\ &+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}\partial_{z_2}P\bigg)-\frac{1}{r}\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}\frac{U_1^2\omega^3}{U_1^2-c^2(\rho,S)}, \end{aligned}$$

and

$$\begin{aligned} &F_2(U_1,\omega,P,S;\xi) \\ &=\frac{1}{r}\big(1-\frac{z_1}{X_0+1-r_0}\big)\frac{\rho c^2(\rho,S)U_1^2}{U_1^2-c^2(\rho,S)}\xi'(z_2)\partial_{z_1}\omega-\frac{\gamma}{r}\frac{(X_0+1-\xi(z_2))PU_1^2}{(X_0+1-r_0)(U_1^2-c^2(\rho,S))}\omega^2 \\ &-\frac{1}{r}\frac{U_1^2\omega}{U_1^2-c^2(\rho,S)}\bigg(\big(\frac{z_1}{X_0+1-r_0}-1\big)\xi'(z_2)\partial_{z_1}P+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}\partial_{z_2}P\bigg) \end{aligned}$$

with

$$r=\xi(z_2)+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1, \tag{3.2.24}$$

here one should note that the functions $F_1(P,U_1,\omega,\xi)$ and $F_2(P,U_1,\omega,\xi)$ both are error terms of second order in $\varepsilon$, if Theorem 3.1.3 holds.

We now set for $z\in E_+$

$$\begin{cases} U_1(z)=U_1\big(\xi(z_2)+\dfrac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1,z_2\big), \qquad \tilde{U}_0^+(z_1)=U_0^+(r_0+z_1), \\ \omega(z)=\omega\big(\xi(z_2)+\dfrac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1,z_2\big), \\ P(z)=P\big(\xi(z_2)+\dfrac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1,z_2\big), \qquad \tilde{P}_0^+(z_1)=P_0^+(r_0+z_1), \\ S(z)=S\big(\xi(z_2)+\dfrac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1,z_2\big) \end{cases} \tag{3.2.25}$$

and

$$\begin{cases} W_1(z) = U_1(z) - \tilde{U}_0^+(z_1), \\ W_2(z) = \omega(z), \\ W_3(z) = P(z) - \tilde{P}_0^+(z_1), \\ W_4(z) = S(z) - S_0^+, \\ W_5(z_2) = \xi(z_2) - r_0, \\ W = (W_1, W_2, W_3, W_4, W_5). \end{cases} \tag{3.2.26}$$

It is noted that the corresponding background solution $(\tilde{U}_0^+(z_1), \tilde{P}_0^+(z_1), S_0^+)$ satisfies

$$\partial_{z_1} \tilde{P}_0^+ + \frac{\gamma}{r_0 + z_1} \frac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)} \tilde{P}_0^+ = 0. \tag{3.2.27}$$

Then in terms of the notations in (3.2.25)-(3.2.26) and a direct computation, we can derive from the equations (3.2.20)-(3.2.23) and (3.2.15) that

$$W_5'(z_2) = \frac{\xi(z_2)\rho(0, z_2)(U_1(0, z_2))^2 W_2(0, z_2) - (\rho^- U_1 U_2)(\xi(z_2), z_2)}{P(0, z_2) - P_0^-(\xi(z_2)) + \rho(0, z_2) U_1^2(0, z_2)\omega^2(0, z_2) - (\rho^-(U_2^-)^2)(\xi(z_2), z_2)} \tag{3.2.28}$$

and

$$\begin{cases} \partial_{z_1} W_2 + \frac{1}{(r_0 + z_1)\tilde{\rho}_0^+ (\tilde{U}_0^+)^2} \partial_{z_2} W_3 + \frac{1}{r_0 + z_1} \frac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)} W_2 \\ \quad - \frac{1}{r_0 + z_1}\left(1 - \frac{z_1}{X_0 + 1 - r_0}\right) \frac{\partial_{z_1} \tilde{P}_0^+}{\tilde{\rho}_0^+ (\tilde{U}_0^+)^2} W_5'(z_2) = F_3(W, \nabla W) \qquad in \qquad E_+, \\ \partial_{z_1} W_3 + \frac{1}{r_0 + z_1} \frac{\tilde{\rho}_0^+ c^2(\tilde{\rho}_0^+, S_0^+)(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)} \partial_{z_2} W_2 + \frac{\gamma}{r_0 + z_1} \frac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)} W_3 \\ \quad + \gamma\Bigg(\frac{(X_0 + 1 - \xi(z_2)) U_1^2}{(X_0 + 1 - r_0)\left(\xi(z_2) + \frac{X_0 + 1 - \xi(z_2)}{X_0 + 1 - r_0} z_1\right)\left(U_1^2 - c^2(\rho, S)\right)} - \\ \quad \frac{(\tilde{U}_0^+)^2}{(r_0 + z_1)\left((\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)\right)}\Bigg) \tilde{P}_0^+ = F_4(W, \nabla W) \qquad in \qquad E_+, \\ W_3(0, z_2) = B_2 W_5(z_2) + R_2(z_2), \\ W_3(X_0 + 1 - r_0, z_2) = \varepsilon P_0(z_2), \\ W_2(z_1, \theta_0) = \frac{f(z_1 + r_0)}{U_1} \quad W_2(z_1, -\theta_0) = 0, \end{cases} \tag{3.2.29}$$

and

$$\begin{cases} \Big((X_0+1-r_0)\xi(z_2)+(X_0+1-\xi(z_2))z_1+\big(z_1-(X_0+1-r_0)\big)\xi'(z_2)\omega\Big)\partial_{z_1}W_4 \\ \qquad +\Big(X_0+1-\xi(z_2)\Big)\omega\partial_{z_2}W_4=0 \qquad in \qquad E_+, \\ W_4=B_3W_5+R_3(z_2), \qquad on \qquad z_1=0. \end{cases} \tag{3.2.30}$$

and

$$\begin{cases} \Big((X_0+1-r_0)\xi(z_2)+(X_0+1-\xi(z_2))z_1+\big(z_1-(X_0+1-r_0)\big)\xi'(z_2)\omega\Big) \\ \qquad \partial_{z_1}(B-B_0)+\Big(X_0+1-\xi(z_2)\Big)\omega\partial_{z_2}(B-B_0)=0 \qquad in \qquad E_+, \\ B-B_0=\dfrac{1}{2}((U_0^+(r_0)+B_1W_5+R_1)^2+U_2^2) \\ \qquad +\dfrac{\gamma}{\gamma-1}e(P_0^+(r_0)+B_2W_5+R_2,S_0^++B_3W_5+R_3)-B_0 \qquad on \qquad z_1=0. \end{cases} \tag{3.2.31}$$

here

$$\begin{aligned} &F_3(W,\nabla W)= \\ &\Big(\frac{1}{(r_0+z_1)\tilde{\rho}_0^+(\tilde{U}_0^+)^2}-\frac{X_0+1-\xi(z_2)}{\big((X_0+1-r_0)\xi(z_2)+(X_0+1-\xi(z_2))z_1\big)\rho U_1^2}\Big)\partial_{z_2}W_3+ \\ &\Big(\frac{(\tilde{U}_0^+)^2}{(r_0+z_1)\big((\tilde{U}_0^+)^2-c^2(\tilde{\rho}_0^+,S_0^+)\big)} \\ &-\frac{(X_0+1-\xi(z_2))U_1^2}{\big(\xi(z_2)+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1\big)(X_0+1-r_0)\big(U_1^2-c^2(\rho,S)\big)}\Big)W_2 \\ &+\big(1-\frac{z_1}{X_0+1-r_0}\big)\Big(\frac{1}{\xi(z_2)+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1}-\frac{1}{r_0+z_1}\Big)\frac{\partial_{z_1}P}{\rho U_1^2}W_5'(z_2) \\ &+\frac{1}{r_0+z_1}\big(1-\frac{z_1}{X_0+1-r_0}\big)\Big(\frac{\partial_{z_1}P}{\rho U_1^2}-\frac{\partial_{z_1}\tilde{P}_0^+}{\tilde{\rho}_0^+(\tilde{U}_0^+)^2}\Big)W_5'(z_2) \\ &+F_1(U_1,\omega,P,S,\xi)-F_1(\tilde{U}_0^+,0,\tilde{P}_0^+,S_0^+,r_0), \end{aligned}$$

and

$$
\begin{aligned}
F_4(W,\nabla W) = &\Big(\frac{1}{r_0+z_1}\frac{\tilde{\rho}_0^+ c^2(\tilde{\rho}_0^+,S_0^+)(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2-c^2(\tilde{\rho}_0^+,S_0^+)}\\
&-\frac{X_0+1-\xi(z_2)}{(X_0+1-r_0)(\xi(z_2)+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1)}\frac{\rho c^2(\rho,S)U_1^2}{U_1^2-c^2(\rho,S)}\Big)\partial_{z_2}W_2\\
&-\gamma\Big(\frac{\big(X_0+1-\xi(z_2)\big)U_1^2}{(X_0+1-r_0)\big(\xi(z_2)+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1\big)\big(U_1^2-c^2(\rho,S)\big)}\\
&-\frac{(\tilde{U}_0^+)^2}{\big(r_0+z_1\big)\big((\tilde{U}_0^+)^2-c^2(\tilde{\rho}_0^+,S_0^+)\big)}\Big)W_3+F_2(\rho,U_1,\omega,\xi)-F_2(\tilde{\rho}_0^+,\tilde{U}_0^+,0,r_0).
\end{aligned}
$$

and $B_0=\frac{1}{2}\tilde{U}^+(0)^2+\frac{\gamma}{\gamma-1}e(\tilde{P}_0^+(0),S_0^+)$.

Set

$$
\begin{aligned}
A=\gamma\tilde{P}_0^+\Big(&\frac{\big(X_0+1-\xi(z_2)\big)U_1^2}{(X_0+1-r_0)\big(\xi(z_2)+\frac{X_0+1-\xi(z_2)}{X_0+1-r_0}z_1\big)\big(U_1^2-c^2(\rho,S)\big)}\\
&-\frac{(\tilde{U}_0^+)^2}{\big(r_0+z_1\big)\big((\tilde{U}_0^+)^2-c^2(\tilde{\rho}_0^+,S_0^+)\big)}\Big).
\end{aligned}
$$

Computations show that

$$
\begin{aligned}
A=&\frac{\gamma(X_0+1)(\tilde{U}_0^+)^2\tilde{P}_0^+}{\big(X_0+1-r_0\big)\big(r_0+z_1\big)^2(c^2(\tilde{\rho}_0^+,S_0^+)-(\tilde{U}_0^+)^2)}W_5\\
&+\frac{\gamma\tilde{P}_0^+(z)}{\big(r_0+z_1\big)\big(c^2(\tilde{\rho}_0^+,S_0^+)-(\tilde{U}_0^+)^2\big)^2}\big((\tilde{U}_0^+)^2c^2(\rho,S)-U_1^2c^2(\tilde{\rho}_0^+,S_0^+)\big)+O(|W|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&(\tilde{U}_0^+)^2c^2(\rho,S)-U_1^2c^2(\tilde{\rho}_0^+,S_0^+)\\
&=-2\tilde{U}_0^+c^2(\tilde{\rho}_0^+,S_0^+)W_1+(\tilde{U}_0^+)^2(\frac{1}{\tilde{\rho}_0^+}W_3+\frac{1}{c_v}\frac{\tilde{P}_0^+}{\tilde{\rho}_0^+}W_4)+O(|W|^2).
\end{aligned}
$$

Finally, we obtain

$$
A=B_4(z_1)W_1+B_5(z_1)W_3+B_6(z_1)W_4+B_7(z_1)W_5+R_4(W)
$$

where

$$\begin{cases} B_4(z_1) = \dfrac{-2\gamma\tilde{U}_0^+(z_1)\tilde{P}_0^+(z_1)c^2(\tilde{\rho}_0^+(z_1), S_0^+)}{(r_0+z_1)(c^2(\tilde{\rho}_0^+(z_1), S_0^+) - (\tilde{U}_0^+(z_1))^2)^2} < 0, \\ B_5(z_1) = \dfrac{(\gamma-1)(\tilde{U}_0^+(z_1))^2 c^2(\tilde{\rho}_0^+(z_1), S_0^+)}{(r_0+z_1)(c^2(\tilde{\rho}_0^+(z_1), S_0^+) - (\tilde{U}_0^+(z_1))^2)^2} > 0 \\ B_6(z_1) = \dfrac{\gamma(\tilde{U}_0^+(z_1))^2(\tilde{P}_0^+(z_1))^2}{c_v\tilde{\rho}_0^+(z_1)(r_0+z_1)(c^2(\tilde{\rho}_0^+(z_1), S_0^+) - (\tilde{U}_0^+(z_1))^2)^2} > 0, \\ B_7(z_1) = \dfrac{\gamma(X_0+1)(\tilde{U}_0^+(z_1))^2\tilde{P}_0^+(z_1)}{(X_0+1-r_0)(r_0+z_1)^2(c^2(\tilde{\rho}_0^+(z_1), S_0^+) - (\tilde{U}_0^+(z_1))^2)} > 0. \end{cases}$$

Hence (3.2.29) can be rewritten as

$$\begin{cases} \partial_{z_1}W_2 + \dfrac{1}{(r_0+z_1)\tilde{\rho}_0^+(\tilde{U}_0^+)^2}\partial_{z_2}W_3 + \dfrac{1}{r_0+z_1}\dfrac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)}W_2 \\ \quad -\dfrac{1}{r_0+z_1}\big(1-\dfrac{z_1}{X_0+1-r_0}\big)\dfrac{\partial_{z_1}\tilde{P}_0^+}{\tilde{\rho}_0^+(\tilde{U}_0^+)^2}W_5'(z_2) = F_3(z_1, W, \nabla W) \qquad in \qquad E_+, \\ \partial_{z_1}W_3 + \dfrac{1}{r_0+z_1}\dfrac{\tilde{\rho}_0^+c^2(\tilde{\rho}_0^+, \hat{S}_0^+)(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)}\partial_{z_2}W_2 + \dfrac{\gamma}{r_0+z_1}\dfrac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)}W_3 + B_4(z_1))W_1 \\ +B_5(z_1)W_3 + B_6(z_1)W_4 + B_7(z_1)W_5 = F_4(z_1, W, \nabla W) + R_4(W) \qquad in \qquad E_+, \\ W_3(0, z_2) = B_1W_5(z_2) + R_1(z_2), \\ W_3(X_0+1-r_0, z_2) = \varepsilon P_0(z_2), \\ W_2(z_1, \theta_0) = \dfrac{f(z_1+r_0)}{U_1} \quad W_2(z_1, \theta_0) = 0, \end{cases} \tag{3.2.32}$$

With these, Theorem 3.1.3 can be derived from the following theorem:

**Theorem 3.2.1** *Under the assumptions of Theorem 3.1.3, there exists a positive constant $C$ depending only $\alpha$ and the uniform supersonic incoming flow such that the system (3.2.28), (3.2.30)-(3.2.32) has a unique solution $W$ with the following estimates*

$$\|W_5\|_{C^{3,\alpha}} \leq C\varepsilon \tag{3.2.33}$$

*and*

$$\sum_{i=1}^{4}\|W_i\|_{C^{1,\alpha}} \leq C\varepsilon \tag{3.2.34}$$

## 3.3 Iteration Scheme

In this section, we will follow [74] to develop a new iteration scheme which is suitable for us to obtain the existence theorem. To find the suitable iteration space, we firstly derive some compatibility conditions.

**Lemma 3.3.1** *If the system (3.2.5)-(3.2.6) with (3.2.7)-(3.2.8), has a solution*

$$(U_1(r,\theta), U_2(r,\theta), P(r,\theta), S(r,\theta)) \in C^{2,\alpha}$$

*and* $\xi(\theta) \in C^{3,\alpha}$, *then the following compatible conditions at the corners hold for* $\xi(\theta) \leq r \leq \xi(\theta) + \tilde{r_0}$,

$$\begin{cases} \partial_\theta U_1(r, \pm\theta_0) = 0, \\ \partial_\theta P(r, \pm\theta_0) = \partial_\theta S(r, \pm\theta_0) = 0, \\ U_2(r, \pm\theta_0) = 0, \partial_\theta^2 U_2(r, \pm\theta_0) = 0, \\ \xi'(\pm\theta_0) = 0, \xi^{(3)}(\pm\theta_0) = 0. \end{cases} \tag{3.3.1}$$

**Proof.** It follows from boundary condition (3.2.8), the jumping condition (3.2.6) that

$$U_2(r, \theta_0) = \frac{f(r)}{U_1}, U_2(r, -\theta_0) = 0, \partial_\theta P(r, \pm\theta_0) = 0, \xi'(\pm\theta_0) = 0.$$

Applying $\xi'(\theta)\partial_r + \partial_\theta$ to the first, the second and the fourth equations in (3.2.6) and $U_2(r, \theta_0) = 0$ when $r \in (\xi(\theta), \xi(\theta) + \tilde{r_0})$, then evaluate at the $(\xi(\pm\theta_0), \pm\theta_0)$

$$\begin{cases} \partial_\theta(\rho U_1) = \partial_\theta(\rho^- U_1^-), \\ \partial_\theta(\rho(U_1)^2 + P) = \partial_\theta(\rho(U_1^-)^2 + P^-), \\ \partial_\theta(\rho(e + |U|^2 + \frac{P}{\rho})U_1) = \partial_\theta(\rho(e^- + |U^-|^2 + \frac{P^-}{\rho^-})U_1^-). \end{cases}$$

By Theorem 3.1.2, one has $\partial_\theta U_1^-(r, \pm\theta_0) = \partial_\theta P^-(r, \pm\theta_0) = \partial_\theta S^-(r, \pm\theta_0) = 0$. Hence we obtain

$$\begin{cases} \partial_\theta(\rho U_1) = 0, \\ \partial_\theta(\rho(U_1)^2 + P) = 0, \\ \partial_\theta(\rho(e + |U|^2 + \frac{P}{\rho})U_1) = 0. \end{cases}$$

That is

$$\begin{cases} U_1\partial_P\rho\partial_\theta P + U_1\partial_S\rho\partial_\theta S + \rho\partial_\theta U_1 = 0, \\ ((U_1)^2\partial_P\rho + 1)\partial_\theta P + (U_1)^2\partial_S\rho\partial_\theta S + 2\rho U_1\partial_\theta U_1 = 0, \\ ((U_1)^3\partial_P\rho + \frac{\gamma}{\gamma-1}U_1)\partial_\theta P + \frac{(U_1)^3}{2}\partial_S\rho\partial_\theta S + (\frac{3}{2}\rho(U_1)^2\partial_P\rho + \frac{\gamma}{\gamma-1}P)\partial_\theta U_1 = 0. \end{cases}$$

Since the determinant of the coefficient matrix is not zero, we obtain that

$$\partial_\theta U_1(\xi(\pm\theta_0), \pm\theta_0) = \partial_\theta P(\xi(\pm\theta_0), \pm\theta_0) = \partial_\theta S(\xi(\pm\theta_0), \pm\theta_0) = 0$$

Apply $\partial_\theta$ to the second and the fourth equation in (3.2.5), and then evaluate at $\pm\theta_0$, we find that $\partial_\theta U_1(r, \pm\theta_0)$ *and* $\partial_\theta S(r, \pm\theta_0)$ satisfies

$$\begin{cases} U_1\partial_r(\partial_\theta U_1) + (\partial_r U_1 + \frac{1}{r}\partial_\theta U_2 - \frac{2U_1}{r})\partial_\theta U_1 - \frac{\partial_r P\partial_S\rho}{\rho^2}\partial_\theta S = 0 \quad \text{on} \quad \theta = \pm\theta_0, \\ U_1\partial_r(\partial_\theta S) + (\partial_r U_1 + \frac{1}{r}\partial_\theta U_2)\partial_\theta S + \partial_r S\partial_\theta U_1 = 0 \quad \text{on} \quad \theta = \pm\theta_0, \\ \partial_\theta U_1(\xi(\pm\theta_0), \pm\theta_0) = 0, \\ \partial_\theta S(\xi(\pm\theta_0), \pm\theta_0) = 0. \end{cases}$$

which implies $\partial_\theta U_1(r, \pm\theta_0) = \partial_\theta S(r, \pm\theta_0) = 0$.

In addition, differentiating the first equation of (3.2.5) with respect to $\theta$, one can get

$$\partial_\theta^2 U_2(r, \pm\theta_0) = 0.$$

And taking $\xi'(\theta)\partial_r + \partial_\theta$ on the third equation of (3.2.6) twice yields

$$\partial_\theta^3\xi(\pm\theta_0) = 0.$$

We have finished the proof of Lemma 3.3.1.

**Remark 3.3.1** *For isentropic flow, one can find that* $\partial_\theta B^-(z_1, \pm\theta_0) = 0$ *is a necessary condition to guarantee the compatibility condition.*

*Indeed, in this case at* $(\xi(\pm\theta_0), \pm\theta_0)$ *we have*

$$\begin{cases} \partial_\theta[\rho U_1] = 0, \\ \partial_\theta[\rho(U_1)^2 + P] = 0. \end{cases}$$

*From this equation, we can derive that*

$$\begin{cases} \partial_\theta U_1 = \dfrac{1}{\rho}\{\partial_\theta(\rho^- U_1^-) - \partial_\theta \rho U_1\}, \\ \partial_\theta U_1 = \dfrac{1}{2\rho U_1}\{\partial_\theta(\rho^-(U_1^-)^2 + P^-) - ((U_1)^2 + c^2(\rho))\partial_\theta\rho\}. \end{cases}$$

*Using the boundary condition, one can derive that*

$$\partial_{z_1}\rho^-(r, \pm\theta_0) = 0 \qquad \textit{and} \qquad \partial_{z_2}U_1^-(r, \pm\theta_0) = \frac{1}{U_1^-}\partial_{z_2}B^-(r, \pm\theta_0).$$

*Hence it follows from the above equation that*

$$(c^2(\rho) - (U_1)^2)\partial_\theta\rho = \frac{\rho(U_1^- - U_1^+)}{U_1^-}\partial_\theta B^-.$$

*Using Rankine-Hugoniot conditions and entropy condition, one has*

$$(U_1^- - U_1^+)(\xi(\pm\theta_0), \pm\theta_0) \neq 0.$$

*To guarantee $\partial_\theta\rho(\xi(\pm\theta_0), \pm\theta_0) = 0$, we need the condition $\partial_\theta B^-(\xi(\pm\theta_0), \pm\theta_0) =$*

0

Next, we construct an iteration scheme to solve the nonlinear problem (3.2.28) and (3.2.30)-(3.2.31).

To this end, we introduce an iteration space as follows

$$\Xi_\delta = \{W : \sum_{i=1}^{4} \|W_i\|_{C^{1,\alpha}(\bar{E}_+)} + \|W_5\|_{C^{3,\alpha}[-\theta_0,\theta_0]} \leq \delta; \partial_{z_2}W_j(z_1, \pm\theta_0) = 0, j = 1, 3, 4;$$
$$W_2(z_1, \pm\theta_0) = \partial_{z_2}^2 W_2(z_1, \pm\theta_0) = 0, 0 \leq z_1 < \tilde{r_0}; W_2(z_1, \pm\theta_0) \geq 0; W_5'(\pm\theta_0) = W_5^{(3)}(\pm\theta_0) = ( \tag{3.3.2}$$

where the constant $\delta > 0$ will be determined later on.

In terms of the notations in (3.2.26), each $\hat{W} \in \Xi_\delta$ has the following expression

$$(\hat{U}_1(z), \hat{\omega}(z), \hat{P}(z), \hat{S}(z), \hat{\rho}(z); \hat{\xi}(z_2)). \tag{3.3.3}$$

We now define the linearized scheme to the problem (3.2.28) and (3.2.30)-(3.2.31) and determine its corresponding solution as follows

$$\bar{W}(z) = (\bar{W}_1(z), \bar{W}_2(z), \bar{W}_3(z), \bar{W}_4(z), \bar{W}_5(z_2)).$$

### 3.3.1 Determination of $\bar{W}_5$

Due to (2.2.28), $\bar{W}_5$ is defined as

$$\bar{W}_5'(z_2) = \frac{r_0\tilde{\rho}_0^+(0)(\tilde{U}_0^+(0))^2}{\tilde{P}_0^+(0) - P_0^-(r_0)}\bar{W}_2(0, z_2) + F_6(z_2), \tag{3.3.4}$$

where

$$\begin{aligned}F_6(z_2) \equiv &\Bigg(\frac{\hat{\xi}(z_2)\hat{\rho}(0,z_2)(\hat{U}_1(0,z_2))^2}{\hat{P}(0,z_2) - P_0^-(\hat{\xi}) + \big(\hat{\rho}(\hat{U}_1)^2(\hat{\omega})^2(0,z_2) - \rho^-(U_2^-)^2(\hat{\xi}(z_2),z_2)\big)}\\ &- \frac{r_0\tilde{\rho}_0^+(0)(\tilde{U}_0^+(0))^2}{\tilde{P}_0^+(0) - P_0^-(r_0)}\Bigg)\hat{W}_2(0,z_2)\\ &+ \frac{\hat{\xi}(\rho^-U_1^-U_2^-)(\hat{\xi}(z_2),z_2)}{\hat{P}(0,z_2) - P_0^-(\hat{\xi}) + (\hat{\rho}(\hat{U}_1)^2(\hat{\omega})^2(0,z_2) - \rho^-(U_2^-)^2(\hat{\xi}(z_2),z_2))}.\end{aligned}$$

Since $\hat{W} \in \Xi_\delta$ and (3.1.16), one checks easily that

$$\begin{cases} F_6(\pm\theta_0) = F_6''(\pm\theta_0) = 0, \\ \|F_6\|_{C^{k,\alpha}[-\theta_0,\theta_0]} \le C((\delta+\varepsilon)\|\hat{W}_2\|_{C^{k,\alpha}(\bar{E}_+)} + \varepsilon), \qquad k = 0,1,2, \end{cases} \tag{3.3.5}$$

here and below the generic positive constant $C$ is independent of $\delta, \varepsilon$.

### 3.3.2 Determination of $\bar{W}_4$

From (3.2.31), $\bar{W}_4$ is required to satisfy

$$\begin{aligned}\big((X_0+1-r_0)\hat{\xi}(z_2) + (X_0+1-\hat{\xi}(z_2))z_1 + \big(z_1 - (X_0+1-r_0)\big)\hat{\xi}'(z_2)\hat{W}_2\big)\partial_{z_1}\bar{W}_4 \\ + \big(X_0+1-\hat{\xi}(z_2)\big)\hat{W}_2\partial_{z_2}\bar{W}_4 = 0 \qquad in \qquad E_+.\end{aligned} \tag{3.3.6}$$

with the initial data $\bar{W}_4(0, z_2)$ being chosen in terms of the expression of $W_4(0, z_2)$ in (3.2.30).

Let $z_2(s;\beta)$ be the characteristics going through $z = (z_1, z_2)$ with $z_2(0;\beta) = \beta$ for the first order differential operator

$$\partial_{z_1} + \frac{\big(X_0+1-\hat{\xi}(z_2)\big)\hat{W}_2}{(X_0+1-r_0)\hat{\xi}(z_2) + (X_0+1-\hat{\xi}(z_2))z_1 + \big(z_1 - (X_0+1-r_0)\big)\hat{\xi}'(z_2)\hat{W}_2}\partial_{z_2},$$

namely,

$$\begin{cases} \dfrac{dz_2(s;\beta)}{ds} = \left( \dfrac{\big(X_0+1-\hat{\xi}(z_2)\big)\hat{W}_2}{(X_0+1-r_0)\hat{\xi}(z_2)+(X_0+1-\hat{\xi}(z_2))z_1+\big(z_1-(X_0+1-r_0)\big)\hat{\xi}'(z_2)\hat{W}_2} \right)\Big|_{z=} \\ z_2(z_1;\beta)=z_2, \qquad z_2(0,\beta)=\beta, \qquad \beta\in[-\theta_0,\theta_0]. \end{cases} \tag{3.3.7}$$

Due to (3.3.7), the variable $\beta$ can be regarded as the function of $z=(z_1,z_2)$, which is denoted by

$$\beta=\beta(z). \tag{3.3.8}$$

It follows from (3.3.7) that

$$z_2-\beta= \left( \frac{\big(X_0+1-\hat{\xi}(z_2)\big)\hat{W}_2}{(X_0+1-r_0)\hat{\xi}(z_2)+(X_0+1-\hat{\xi}(z_2))z_1+\big(z_1-(X_0+1-r_0)\big)\hat{\xi}'(z_2)\hat{W}_2} \right)\Big|_{z=(s,z_2(s;\beta))} ds. \tag{3.3.9}$$

It follows from $\hat{W}_2(z_1,\pm\theta_0)=0$, (3.3.7) and (3.3.9) that

$$\beta(z_1,\pm\theta_0)=\pm\theta_0, \qquad \|\beta-z_2\|_{C^{k,\alpha}(\bar{E}_+)}\le C_0\|\hat{W}_2\|_{C^{k,\alpha}(\bar{E}_+)}, \qquad k=0,1,2. \tag{3.3.10}$$

It is noted that (3.3.6) is a first order linear partial differential equation of $\bar{W}_4$, then it follows from the characteristics method and the expression of $W_4(0,z_2)$ in (3.2.31) that

$$\bar{W}_4(z)=\bar{W}_4(0,\beta(z))=B_3\bar{W}_5(z_2)+F_7(z), \tag{3.3.11}$$

where

$$F_7(z)\equiv F_7(\hat{W})(z)=B_3\int_{z_2}^{\beta(z)}\hat{W}_5'(s)ds+R_3(\hat{W}(0,\beta(z))).$$

Due to (3.1.16), one can check that $\partial_{z_2}\tilde{g}_3(z_1,\pm\theta_0)=0$, which implies

$$\partial_{z_2}R_3(z_1,\pm\theta_0)=0.$$

$$\begin{cases} \partial_{z_2}F_7(z_1,\pm\theta_0)=0, \\ \|F_7\|_{C^{k,\alpha}(\bar{E}_+)}\le C(\delta+\varepsilon)\big(\sum_{l=1}^{4}\|\hat{W}_l\|_{C^{k,\alpha}(\bar{E}_+)}+\|\hat{W}_5\|_{C^{k+1,\alpha}[-\theta_0,\theta_0]}\big)+C\varepsilon. \end{cases} \tag{3.3.12}$$

where $k=0,1,2$.

### 3.3.3 Determinations of $\bar{W}_1$

From (3.2.31), $\bar{B}$ is required to satisfy

$$\big((X_0+1-r_0)\hat{\xi}(z_2)+(X_0+1-\hat{\xi}(z_2))z_1+\big(z_1-(X_0+1-r_0)\big)\hat{\xi}'(z_2)\hat{W}_2\big)\partial_{z_1}(\bar{B}-B_0)$$
$$+\big(X_0+1-\hat{\xi}(z_2)\big)\hat{W}_2\partial_{z_2}(\bar{B}-B_0)=0 \qquad in \qquad E_+. \tag{3.3.13}$$

A similar analysis shows that

$$(\bar{B}-B_0)(z)=(\bar{B}-\bar{B}_0)(0,\beta(z)).$$

Using the mean value theorem and after some tedious calculations. we have

$$\bar{W}_1(z)=\frac{\tilde{U}_1^+(0)}{\tilde{U}_1^+(z_1)}\bar{W}_1(0,\beta(z))+\frac{1}{\tilde{\rho}_0^+(0)\tilde{U}_0^+(z_1)}\bar{W}_3(0,\beta(z))-\frac{1}{\tilde{\rho}_0^+(z_1)\tilde{U}_0^+(z_1)}\bar{W}_3(z)$$
$$+\frac{1}{\gamma c_v\tilde{U}_0^+(z_1)}(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)}\bar{W}_4(0,\beta(z))-\frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)}\bar{W}_4(z))+\tilde{F}_8(z).$$

where $\tilde{F}_8=O(|\hat{W}|^2)$ has same properties as $F_7$.

Hence we have

$$\bar{W}_1(z)=-\frac{1}{\tilde{\rho}_0^+(z_1)\tilde{U}_0^+(z_1)}\bar{W}_3(z)+\frac{1}{\tilde{U}_0^+(z_1)}\big(B_1\tilde{U}_0^+(0)+\frac{B_2}{\tilde{\rho}_0^+(0)\tilde{U}_0^+(z_1)}$$
$$+\frac{B_3}{\gamma c_v}(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)}-\frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)})\big)\bar{W}_5(z_2)+F_8(z). \tag{3.3.14}$$

where

$$F_8=\tilde{F}_8+\frac{\tilde{U}_0^+(0)}{\tilde{U}_0^+(z_1)}R_1(0,\beta)+\frac{1}{\tilde{\rho}_0^+(0)\tilde{U}_0^+(z_1)}R_2(0,\beta)+\frac{1}{\gamma c_v}(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)}$$
$$-\frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)})R_3(0,\beta)+\frac{1}{\tilde{U}_0^+(z_1)}\big(B_1\tilde{U}_0^+(0)+\frac{B_2}{\tilde{\rho}_0^+(0)\tilde{U}_0^+(z_1)}$$
$$+\frac{B_3}{\gamma c_v}(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)}-\frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)})\big)\int_{z_2}^{\beta(z)}\hat{W}_5(s)ds.$$

has the same properties as $F_7$ which can be checked in a similar way as above:

$$\begin{cases}\partial_{z_2}F_8(z_1,\pm\theta_0)=0,\\ \|F_8\|_{C^{k,\alpha}(\bar{E}_+)}\le C(\delta+\varepsilon)\big(\sum_{l=1}^{4}\|\hat{W}_l\|_{C^{k,\alpha}(\bar{E}_+)}+\|\hat{W}_5\|_{C^{k+1,\alpha}[-\theta_0,\theta_0]}\big)+C\varepsilon.\end{cases} \tag{3.3.15}$$

where $k=0,1,2$.

### 3.3.4 Determinations of $\bar{W}_2, \bar{W}_3$ and $\bar{W}_5$

By (3.2.30), (3.3.4), (3.3.11) and (3.3.14), in terms of the unknown shock position $\bar{W}_5(-\theta_0)$ at the nozzle wall $\theta = -\theta_0$ (it should be noted that $\bar{W}_5(-\theta_0)$ will be determined together with the solution $\bar{W}_2$ and $\bar{W}_3$ of the linearized equations), we define $\bar{W}_2$ and $\bar{W}_3$ by solving the following problem

$$\begin{cases}
\partial_{z_1}\bar{W}_2 + \frac{1}{(r_0+z_1)\tilde{\rho}_0^+(\tilde{U}_0^+)^2}\partial_{z_2}\bar{W}_3 + \frac{1}{r_0+z_1}\frac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)}\bar{W}_2 \\
\quad -\frac{1}{r_0+z_1}(1-\frac{z_1}{X_0+1-r_0})\frac{\partial_{z_1}\tilde{P}_0^+}{\tilde{\rho}_0^+(\tilde{U}_0^+)^2}\frac{r_0\tilde{\rho}_0^+(0)(\tilde{U}_0^+(0))^2}{\tilde{P}_0^+(0) - P_0^-(r_0)}\bar{W}_2(0,z_2) \\
\quad = F_3(z_1,\hat{W},\nabla\hat{W}) + \frac{1}{r_0+z_1}(1-\frac{z_1}{X_0+1-r_0})\frac{\partial_{z_1}\tilde{P}_0^+}{\tilde{\rho}_0^+(\tilde{U}_0^+)^2}F_6(z_2) \qquad in \qquad E_+, \\
\partial_{z_1}\bar{W}_3 + \frac{1}{r_0+z_1}\frac{\tilde{\rho}_0^+c^2(\tilde{\rho}_0^+, S_0^+)(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)}\partial_{z_2}\bar{W}_2 + \Big(\frac{\gamma}{r_0+z_1}\frac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)} + B_5(z_1) - \frac{B_4(z}{\tilde{\rho}_0^+\tilde{U}_{(}} \\
\quad +\Big(\frac{1}{\tilde{U}_0^+(z_1)}(B_1B_4\tilde{U}_0^+(0) + \frac{B_2B_4}{\tilde{\rho}_0^+(0)} + \frac{1}{\gamma c_v}B_3B_4(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)} - \frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)})) + B_3B_6 \\
\qquad +B_7\Big)\Big(\bar{W}_5(-\theta_0) + \frac{r_0\tilde{\rho}_0^+(0)(\tilde{U}_0^+(0))^2}{\tilde{P}_0^+(0) - P_0^-(r_0)}\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds\Big) \\
= F_4(z_1,\hat{W},\nabla\hat{W}) - R_4(\hat{W}) - B_4F_8(z) - B_6F_7 - \Big(\frac{1}{\tilde{U}_0^+(z_1)}(B_1B_4\tilde{U}_0^+(0) + \frac{B_2B_4}{\tilde{\rho}_0^+(0)} \\
\quad +\frac{1}{\gamma c_v}B_3B_4(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)} - \frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)})) + B_3B_6 + B_7\Big)\int_{-\theta_0}^{z_2}F_6(s)ds \qquad \text{in } E_+, \\
\bar{W}_3(0,z_2) = B_2\Big(\bar{W}_5(-\theta_0) + \frac{r_0\tilde{\rho}_0^+(0)(\tilde{U}_0^+(0))^2}{\tilde{P}_0^+(0) - P_0^-(r_0)}\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds\Big) \\
\qquad +B_2\int_{-\theta_0}^{z_2}F_6(s)ds + R_1(\hat{W}(0,z_2)), \\
\bar{W}_3(X_0+1-r_0, z_2) = \varepsilon P_0(z_2), \\
\bar{W}_2(z_1,\theta_0) = \frac{f(z_1+r_0)}{U_1}, \quad W_2(z_1,-\theta_0) = 0.
\end{cases} \tag{3.3.16}$$

To write the first and second equations of (3.3.16) in divergence forms and for notational conveniences, we define

$$
\begin{cases}
\lambda_1(z_1) = \exp\left(\displaystyle\int_0^{z_1} \frac{1}{r_0+s}\frac{(\tilde{U}_0^+)^2(s)}{(\tilde{U}_0^+)^2(s) - c^2(\tilde{\rho}_0^+, S_0^+)(s)}ds\right) > 0,\\
\lambda_2(z_1) = \dfrac{1}{(r_0+z_1)\tilde{\rho}_0^+(\tilde{U}_0^+)^2}\lambda_1(z_1) > 0,\\
\lambda_3(z_1) = \dfrac{1}{r_0+z_1}\left(1 - \dfrac{z_1}{X_0+1-r_0}\right)\dfrac{\partial_{z_1}\tilde{P}_0^+}{\tilde{\rho}_0^+(\tilde{U}_0^+)^2}\dfrac{r_0\tilde{\rho}_0^+(0)(\tilde{U}_0^+(0))^2}{\tilde{P}_0^+(0) - P_0^-(r_0)}\lambda_1(z_1) \geq 0\\
\qquad \text{with } \lambda_3(X_0+1-r_0) = 0,\\
\lambda_4(z_1) = \exp\left(\displaystyle\int_0^{z_1}\left(\frac{\gamma}{r_0+s}\frac{(\tilde{U}_0^+)^2(s)}{(\tilde{U}_0^+)^2(s) - c^2(\tilde{\rho}_0^+, S_0^+)(s)} + B_5(s) - \frac{B_4(s)}{\tilde{\rho}_0^+\tilde{U}_0^+}\right)ds\right) > 0,\\
\lambda_5(z_1) = \dfrac{1}{r_0+z_1}\dfrac{\tilde{\rho}_0^+c^2(\tilde{\rho}_0^+, S_0^+)(\tilde{U}_0^+)^2}{c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2}\lambda_4(z_1) > 0,\\
\lambda_6(z_1) = \left(\dfrac{1}{\tilde{U}_0^+(z_1)}\left(B_1B_4\tilde{U}_0^+(0) + \dfrac{B_2B_4}{\tilde{\rho}_0^+(0)} + \dfrac{1}{\gamma c_v}B_3B_4\left(\dfrac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)} - \dfrac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)}\right)\right)\right.\\
\qquad\qquad \left. + B_3B_6 + B_7\right)\lambda_4(z_1) > 0,\\
\lambda_7 = \dfrac{r_0\tilde{\rho}_0^+(0)(\tilde{U}_0^+(0))^2}{\tilde{P}_0^+(0) - P_0^-(r_0)} > 0.
\end{cases}
\tag{3.3.17}
$$

and

$$
\begin{cases}
G_1(z) = \lambda_1(z_1)\left(F_3(z_1, \hat{W}, \nabla\hat{W}) + \dfrac{X_0+1-r_0-z_1}{(r_0+z_1)(X_0+1-r_0)}\dfrac{\partial_{z_1}\tilde{P}_0^+}{\tilde{\rho}_0^+(\tilde{U}_0^+)^2}F_6(z_2)\right),\\
G_2(z) = \lambda_4(z_1)\left(F_4(z_1, \hat{W}, \nabla\hat{W}) - R_4(\hat{W}) - B_4(z_1)F_8(z) - B_6(z_1)F_7(z)\right) - \left(\dfrac{1}{\tilde{U}_0^+(z_1)}\right.\\
\left.\left(B_1B_4\tilde{U}_0^+(0) + \dfrac{B_2B_4}{\tilde{\rho}_0^+(0)} + \dfrac{1}{\gamma c_v}B_3B_4\left(\dfrac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)} - \dfrac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)}\right)\right) + B_3B_6 + B_7\right)\displaystyle\int_{-\theta_0}^{z_2} F_6(s)ds,\\
G_3(z_2) = B_2\displaystyle\int_{-\theta_0}^{z_2} F_6(s)ds + R_2(\hat{W}(0, z_2)).
\end{cases}
\tag{3.3.18}
$$

It follows from the expressions of $F_3(\hat{W}, \nabla\hat{W})$ and $F_4(\hat{W}, \nabla\hat{W})$ together with (3.3.5), (3.3.12), that

$$
\begin{cases}
G_1(z_1, \pm\theta_0) = 0, \qquad \partial_{z_2}G_2(z_1, \pm\theta_0) = 0,\\
\|G_1\|_{C^{k-1,\alpha}(\bar{E}_+)} + \|G_2\|_{C^{k-1,\alpha}(\bar{E}_+)} \leq C(\delta+\varepsilon)\|\hat{W}\|_{C^{k,\alpha}(\bar{E}_+)} + C\varepsilon, \qquad k = 1, 2
\end{cases}
\tag{3.3.19}
$$

and

$$\begin{cases} \partial_{z_2} G_3(\pm\theta_0) = 0, \\ \|G_3\|_{C^{k,\alpha}[-\theta_0,\theta_0]} \leq C(\delta+\varepsilon)\|\hat{W}\|_{C^{k,\alpha}(\bar{E}_+)} + C\varepsilon, \qquad k = 0, 1, 2 \end{cases} \tag{3.3.20}$$

Then, direct computation shows that (3.3.13) can be easily rewritten as

$$\begin{cases} \partial_{z_1}\Big(\lambda_1(z_1)\bar{W}_2\Big) + \partial_{z_2}\Big(\lambda_2(z_1)\bar{W}_3\Big) - \lambda_3(z_1)\bar{W}_2(0,z_2) = G_1(z), \\ \partial_{z_1}\Big(\lambda_4(z_1)\bar{W}_3\Big) - \partial_{z_2}\Big(\lambda_5(z_1)\bar{W}_2\Big) + \lambda_6(z_1)\Big(\bar{W}_5(-\theta_0) + \lambda_7\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds\Big) \\ \qquad = G_2(z), \\ \bar{W}_3(0,z_2) = B_2\Big(\bar{W}_5(-\theta_0) + \lambda_7\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds\Big) + G_3(z_2), \\ \bar{W}_3(X_0+1-r_0,z_2) = \varepsilon P_0(z_2), \\ \bar{W}_2(z_1,\theta_0) = \dfrac{f(z_1+r_0)}{U_1}, \quad W_2(z_1,-\theta_0) = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \partial_{z_1}\Big(\lambda_1(z_1)\bar{W}_2\Big) + \partial_{z_2}\Big(\lambda_2(z_1)\bar{W}_3 - \lambda_3(z_1)(\dfrac{\bar{W}_5(-\theta_0)}{\lambda_7} + \int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds) \\ \qquad - \int_{-\theta_0}^{z_2} G_1(z_1,s)ds\Big) = 0, \\ \partial_{z_1}\Big(\lambda_4(z_1)\bar{W}_3\Big) - \partial_{z_2}\Big(\lambda_5(z_1)\bar{W}_2\Big) + \lambda_6(z_1)\Big(\bar{W}_5(-\theta_0) + \lambda_7\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds\Big) \\ \qquad = G_2(z), \\ \bar{W}_3(0,z_2) = B_2\Big(\bar{W}_5(-\theta_0) + \lambda_7\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds\Big) + G_3(z_2), \\ \bar{W}_3(X_0+1-r_0,z_2) = \varepsilon P_0(z_2), \\ \bar{W}_2(z_1,\theta_0) = \dfrac{f(z_1+r_0)}{U_1}, \quad W_2(z_1,-\theta_0) = 0. \end{cases} \tag{3.3.21}$$

By the first equation in (3.3.20), one can set

$$\begin{cases} \partial_{z_1}\phi = \lambda_2(z_1)\bar{W}_3 - \lambda_3(z_1)(\dfrac{\bar{W}_5(-\theta_0)}{\lambda_7} + \int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds) - \int_{-\theta_0}^{z_2} G_1(z_1,s)ds, \\ \partial_{z_2}\phi = -\lambda_1(z_1)\bar{W}_2, \\ \phi(0,-\theta_0) = 0. \end{cases} \tag{3.3.22}$$

It is easy to see that (3.3.20) is equivalent to the following problem for a second order non-local elliptic equation for $\phi(z)$ with the unknown constant $\bar{W}_5(-\theta_0)$

$$\begin{cases} \partial_{z_1}(\frac{\lambda_4(z_1)}{\lambda_2(z_1)}\partial_{z_1}\phi) + \partial_{z_2}(\frac{\lambda_5(z_1)}{\lambda_1(z_1)}\partial_{z_2}\phi) - \Big(\lambda_6(z_1)\lambda_7 + \frac{d}{dz_1}(\frac{\lambda_3(z_1)\lambda_4(z_1)}{\lambda_2(z_1)})\Big) \\ \big(\phi(0,z_2) - \frac{\bar{W}_5(-\theta_0)}{\lambda_7}\big) = G_2(z) - \partial_{z_1}\Big(\frac{\lambda_4(z_1)}{\lambda_2(z_1)}\int_{-\theta_0}^{z_2} G_1(z_1,s)ds\Big) \quad in \quad E_+, \\ \partial_{z_1}\phi(0,z_2) + \big(B_2\lambda_7\lambda_2(0) - \lambda_3(0)\big)\big(\phi(0,z_2) - \frac{\bar{W}_5(-\theta_0)}{\lambda_7}\big) \\ \qquad = \lambda_2(0)G_3(z_2) - \int_{-\theta_0}^{z_2} G_1(0,s)ds, \\ \partial_{z_1}\phi(X_0+1-r_0,z_2) = \varepsilon\lambda_2(X_0+1-r_0)P_0(z_2) - \int_{-\theta_0}^{z_2} G_1(X_0+1-r_0,s)ds, \\ \partial_{z_2}\phi(z_1,\theta_0) = -\lambda_1(z_1)\frac{f(z_1+r_0)}{\bar{W}_1(z_1)+\tilde{U}_0^+(z_1)}, \quad \partial_{z_2}\phi(z_1,-\theta) = 0 \\ \phi(0,-\theta_0) = 0. \end{cases} \tag{3.3.23}$$

## 3.4 A priori estimates and proofs

In this section, we establish some key a priori estimates on the linearized problems given in §3 to define a contractible mapping from $\Xi_\delta$ into $\Xi_\delta$ so that Theorem 3.1.1 can be shown. To this end, we first derive some useful a priori estimates on (3.3.4), (3.3.11) and (3.3.18).

### 3.4.1 Estimates on $\bar{W}_2, \bar{W}_3$ and $\bar{W}_5$

Due to (3.1.8), (3.3.19)-(3.3.20), if one can verify

$$\lambda_6(z_1)\lambda_7 + \frac{d}{dz_1}\Big(\frac{\lambda_3(z_1)\lambda_4(z_1)}{\lambda_2(z_1)}\Big) > 0, \qquad B_2\lambda_7\lambda_2(0) - \lambda_3(0) < 0 \tag{3.4.1}$$

then, the conditions of Proposition 4.4 in [74] will be fulfilled. Thus, the solvability and the estimates on $\bar{W}_2, \bar{W}_3$ and $\bar{W}_5(-\theta_0)$ can be subsequently obtained by Proposition 4.4 in [74].

We now verify (3.4.1).

Since the subsonic background solution $(\tilde{P}_0^+(z_1), \tilde{U}_0^+(z_1), \tilde{\rho}_0^+(z_1), S_0^+)$ satisfies

$$\begin{cases} \partial_{z_1}\tilde{P}_0^+ = \dfrac{\tilde{\rho}_0^+ c^2(\tilde{\rho}_0^+, S_0^+)(\tilde{U}_0^+)^2}{(r_0+z_1)\big(c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2\big)}, \\ \partial_{z_1}\tilde{U}_0^+ = -\dfrac{c^2(\tilde{\rho}_0^+, S_0^+)\tilde{U}_0^+}{(r_0+z_1)\big(c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2\big)}, \end{cases} \tag{3.4.2}$$

then by a direct computation, one has

$$\begin{aligned} \partial_{z_1}^2\tilde{P}_0^+ = -\frac{\tilde{\rho}_0^+(\tilde{U}_0^+)^2 c^2(\tilde{\rho}_0^+, S_0^+)}{(r_0+z_1)^2\big(c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2\big)^3}\big((\gamma+1)(\tilde{U}_0^+)^4 - 3(\tilde{U}_0^+)^2 c^2(\tilde{\rho}_0^+, S_0^+) \\ + 3c^4(\tilde{\rho}_0^+, S_0^+)\big). \end{aligned} \tag{3.4.3}$$

In addition,

$$\begin{aligned} &\lambda_6(z_1)\lambda_7 + \frac{d}{dz_1}\Big(\frac{\lambda_3(z_1)\lambda_4(z_1)}{\lambda_2(z_1)}\Big) \\ &= \lambda_7\Big(\lambda_6(z_1) + \frac{d}{dz_1}\big((1 - \frac{z_1}{X_0+1-r_0})\partial_{z_1}\tilde{P}_0^+\lambda_4(z_1)\big)\Big) \\ &= \lambda_7\lambda_4(z_1)\Big\{\Big(\frac{1}{\tilde{U}_0^+(z_1)}(B_1B_4\tilde{U}_0^+(0) + \frac{B_2B_4}{\tilde{\rho}_0^+(0)} + \frac{1}{\gamma c_v}B_3B_4(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)} - \frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)})) \\ &\quad + B_3B_6 + B_7\Big) - \frac{1}{X_0+1-r_0}\partial_{z_1}\tilde{P}_0^+ + \big(1 - \frac{z_1}{X_0+1-r_0}\big)\Big(\partial_{z_1}^2\tilde{P}_0^+ \\ &\quad + \partial_{z_1}\tilde{P}_0^+(\frac{\gamma}{r_0+z_1}\frac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)} + B_5(z_1) - \frac{B_4(s)}{\tilde{\rho}_0^+(z_1)\tilde{U}_0^+(z_1)})\Big)\Big\}. \end{aligned} \tag{3.4.4}$$

It is noted that

$$\begin{aligned} &B_7(z_1) - \frac{1}{X_0+1-r_0}\partial_{z_1}\tilde{P}_0^+ \\ &= \frac{\tilde{\rho}_0^+ c^2(\tilde{\rho}_0^+, S_0^+)(\tilde{U}_0^+)^2}{(X_0+1-r_0)(r_0+z_1)\big(c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2\big)}(\frac{X_0+1}{r_0+z_1} - 1) \\ &= \frac{1}{X_0+1-r_0}\partial_{z_1}\hat{P}_0^+(\frac{X_0+1}{r_0+z_1} - 1) \end{aligned} \tag{3.4.5}$$

and

$$
\begin{aligned}
&\partial_{z_1}^2 \tilde{P}_0^+ + \partial_{z_1}\tilde{P}_0^+\left(\frac{\gamma}{r_0+z_1}\frac{(\tilde{U}_0^+)^2}{(\tilde{U}_0^+)^2 - c^2(\tilde{\rho}_0^+, S_0^+)} + B_5(z_1) - \frac{B_4(z_1)}{\tilde{\rho}_0^+\tilde{U}_0^+}\right)\\
&= \partial_{z_1}\tilde{P}_0^+\Bigg(\frac{(\gamma-1)c^2(\tilde{\rho}_0^+, S_0^+)(\tilde{U}_0^+)^2}{(r_0+z_1)^2\big(c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2\big)^2} - \frac{-2\gamma c^4(\tilde{\rho}_0^+, S_0^+)}{(r_0+z_1)\big(c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2\big)^2}\\
&- \frac{1}{(r_0+z_1)\big(c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2\big)^2}\big((\gamma+1)(\tilde{U}_0^+)^4 - 3(\tilde{U}_0^+)^2c^2(\tilde{\rho}_0^+, S_0^+) + 3c^4(\tilde{\rho}_0^+, S_0^+)\big)\\
&- \frac{\gamma}{r_0+z_1}\frac{(\tilde{U}_0^+)^2}{c^2(\tilde{\rho}_0^+, S_0^+) - (\tilde{U}_0^+)^2}\Bigg)\\
&= -\frac{1}{r_0+z_1}\partial_{z_1}\tilde{P}_0^+.
\end{aligned}
$$

Then substituting the expressions above into (3.4.4) and noting that $B_1 < 0, B_2 < 0$, $B_3(z_1) > 0, B_4(z_1) < 0, B_6(z_1) > 0$ and $\lambda_i > 0$ $(i = 4, 7)$ hold true,

$$
\begin{aligned}
\lambda_6(z_1)\lambda_7 + \frac{d}{dz_1}\left(\frac{\lambda_3(z_1)\lambda_4(z_1)}{\lambda_2(z_1)}\right) &= \lambda_7\lambda_4(z_1)\{\frac{1}{\tilde{U}_0^+(z_1)}(B_1B_4\tilde{U}_0^+(0) + \frac{B_2B_4}{\tilde{\rho}_0^+(0)}\\
&+ \frac{1}{\gamma c_v}B_3B_4(\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)} - \frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)})) + B_3B_6\} > 0.
\end{aligned}
\tag{3.4.6}
$$

where we have used the property of background solution $\frac{d}{dz_1}\tilde{P}_0^+(z_1) > 0$, so we have

$$
\frac{\tilde{P}_0^+(0)}{\tilde{\rho}_0^+(0)} - \frac{\tilde{P}_0^+(z_1)}{\tilde{\rho}_0^+(z_1)} < 0.
$$

Furthermore, since we have $B_1 < 0, \partial_{z_1}\tilde{P}_0^+(0) > 0$, one has

$$
\begin{aligned}
&B_1\lambda_7\lambda_2(0) - \lambda_3(0)\\
&= \lambda_7\lambda_2(0)\big(B_1 - \partial_{z_1}\tilde{P}_0^+(0)\big) < 0.
\end{aligned}
$$

Combining this with (3.4.6) yields (3.4.1). Thus, by Proposition 4.4 in [74], (3.3.21) has a unique solution $(\bar{W}_2, \bar{W}_3, \bar{W}_5(-\theta_0))$ satisfying

$$
\begin{aligned}
&\|\bar{W}_2\|_{C^{2,\alpha}(\bar{E}_+)} + \|\bar{W}_3\|_{C^{2,\alpha}(\bar{E}_+)} + |\bar{W}_5(-\theta_0)|\\
&\le C\big(\|G_1(z)\|_{C^{1,\alpha}(\bar{E}_+)} + \|G_2(z)\|_{C^{1,\alpha}(\bar{E}_+)} + \|G_3(z)\|_{C^{2,\alpha}[-\theta_0,\theta_0]} + \varepsilon\|P_0\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\big)\\
&\le C(\varepsilon + \delta^2 + \varepsilon\delta).
\end{aligned}
\tag{3.4.7}
$$

**Remark 3.4.1** *One can solve the system (3.3.21) in the following way. First, it is easy to derive that $W_2$ satisfies the following equation:*

$$\begin{cases} \partial_{z_1}(\frac{\lambda_4(z_1)}{\lambda_2(z_1)}\partial_{z_1}(\lambda_1(z_1)\bar{W}_2)) + \partial_{z_2}^2(\lambda_5(z_1)\bar{W}_2) - \\ \qquad (\lambda_6(z_1)\lambda_7 + \frac{d}{dz_1}\left(\frac{\lambda_3(z_1)\lambda_4(z_1)}{\lambda_2(z_1)}\right))\bar{W}_2(0,z_2) = \partial_{z_1}(\frac{\lambda_4(z_1)}{\lambda_2(z_1)G_1(z)}) - \partial_{z_2}G_2(z), \\ \partial_{z_1}(\lambda_1\bar{W}_2)(0,z_2) + (\lambda_7\lambda_2(0)B_1(0) - \lambda_3(0))\bar{W}_2(0,z_2) = G_1(0,z_2) - \lambda_2(0)G_3'(z_2), \\ \partial_{z_1}(\lambda_1\bar{W}_2)(X_0+1-r_0,z_2) = G_1(X_0+1-r_0,z_2) - \varepsilon\lambda_2(X_0+1-r_0)P_0'(z_2), \\ \bar{W}_2(z_1,\theta_0) = \frac{f(z_1+r_0)}{U_1}, \quad W_2(z_1,-\theta_0) = 0. \end{cases} \tag{3.4.8}$$

*As in [74], we can develop a similar theory to obtain the existence and uniqueness of $\bar{W}_2$ for (3.4.8). With $\bar{W}_2$ being solved, one can solve the following equation to obtain $\bar{W}_3$:*

$$\begin{cases} \lambda_2(z_1)\partial_{z_2}\bar{W}_3 = G_1(z) + \lambda_3(z_1)\bar{W}_2(0,z_2) - \partial_{z_1}(\lambda_1(z_1)\bar{W}_2), \\ \partial_{z_1}(\lambda_4(z_1)\bar{W}_3) = \partial_{z_2}(\lambda_5(z_1)\bar{W}_2) - \lambda_6(z_1)(\bar{W}_5(-\theta_0) + \lambda_7\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds) + G_2(z). \\ \bar{W}_3(0,z_2) = B_1(\bar{W}_5(-\theta_0) + \lambda_7\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds) + G_2(z_2), \\ \bar{W}_3(X_0+1-r_0,z_2) = \varepsilon P_0(z_2). \end{cases} \tag{3.4.9}$$

*The solvability of (3.4.9) takes the following form, which can be used to determine the shock position $\bar{W}_5(\theta_0)$.*

$$\int_0^{X_0+1-r_0}\left(\lambda_5(z_1)\partial_{z_2}\bar{W}_2(z_1,z_2) + G_2(z_1,z_2)\right)dz_1 + \left(\lambda_4(0)B_1(0) - \int_0^{X_0+1-r_0}\lambda_6(z_1)dz_1\right)\cdot \left(\bar{W}_5(-\theta_0) + \lambda_7\int_{-\theta_0}^{z_2}\bar{W}_2(0,s)ds\right) + G_3(z_2) - \varepsilon P_0(z_2)\lambda_4(X_0+1-r_0) = 0. \tag{3.4.10}$$

*Once $\bar{W}_5(\theta_0)$ is known, one can solve $\bar{W}_3$ and obtain the corresponding estimates.*

### 3.4.2 Estimate on $\bar{W}_5(z_2)$

By the estimates for $\bar{W}_2$ and $\bar{W}_5(-\theta_0)$ in (3.4.7), the unique solution $\bar{W}_5(z_2)$ of (3.3.4) satisfies

$$\|\bar{W}_5\|_{C^{3,\alpha}[-\theta_0,\theta_0]} \leq C\big(|\bar{W}_5(-\theta_0)|+\|\bar{W}_2\|_{C^{2,\alpha}(\bar{E}_+)}+\|F_6(z)\|_{C^{2,\alpha}(\bar{E}_+)}\big) \leq C(\varepsilon+\delta^2+\varepsilon\delta). \tag{3.4.11}$$

### 3.4.3 Estimate on $\bar{W}_4(z)$

It follows from (3.3.10)-(3.3.11), (3.4.11) that

$$\|\bar{W}_4\|_{C^{2,\alpha}(\bar{E}_+)} \leq C\big(\|\bar{W}_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]} + \|F_7(z)\|_{C^{2,\alpha}(\bar{E}_+)}\big) \leq C(\varepsilon + \delta^2 + \varepsilon\delta). \tag{3.4.12}$$

### 3.4.4 Estimate on $\bar{W}_1(z)$

It follows from (3.3.14), (3.3.15) and (3.4.7)-(**??**) that

$$\|\bar{W}_1\|_{C^{2,\alpha}(\bar{E}_+)} \leq C\big(\|\bar{W}_3\|_{C^{2,\alpha}(\bar{E}_+)}+\|\bar{W}_5\|_{C^{2,\alpha}(\bar{E}_+)}+\|F_8(z)\|_{C^{2,\alpha}(\bar{E}_+)}\big) \leq C(\varepsilon+\delta^2+\varepsilon\delta) \tag{3.4.13}$$

Based on the estimates above, we are now ready to show Theorem 3.2.1.

## 3.5 Proof of Theorem 3.2.1

Based on the iteration scheme and the estimates (3.4.7)-(3.4.13), if $\delta = O(1)\varepsilon$ is properly chosen, then one can show that

$$\sum_{i=1}^{4} \|W_i\|_{C^{1,\alpha}(\bar{E}_+)+\|W_5\|_{C^{3,\alpha}[-\theta_0,\theta_0]}} \leq \delta.$$

Hence we can define a mapping $T$ from $\Xi_\delta$ into itself as follows

$$T(\hat{W}) = \bar{W}, \tag{3.5.1}$$

where $\hat{W} = (\hat{W}_1, \hat{W}_2, \hat{W}_3, \hat{W}_4, \hat{W}_5)$ and $\bar{W} = (\bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4, \bar{W}_5)$.

It remains to show that the mapping $T$ is contractible.

For any given two states

$$\hat{W}^1 = (\hat{W}_1^1, \hat{W}_2^1, \hat{W}_3^1, \hat{W}_4^1, \hat{W}_5^1) \qquad \textit{and} \qquad \hat{W}^2 = (\hat{W}_1^2, \hat{W}_2^2, \hat{W}_3^2, \hat{W}_4^2, \hat{W}_5^2)$$

in $\Xi_\delta$ with the corresponding fluid variables $(\hat{U}_{11}, \hat{\omega}_1, \hat{P}_1, \hat{S}_1, \hat{\xi}_1)$ and $(\hat{U}_{12}, \hat{\omega}_2, \hat{P}_2, \hat{S}_2, \hat{\xi}_2)$ respectively, we set

$$T(\hat{W}^1) = \bar{W}^1, \qquad\qquad T(\hat{W}^2) = \bar{W}^2$$

with $\bar{W}^i = (\bar{W}_1^i, \bar{W}_2^i, \bar{W}_3^i, \bar{W}_4^i, \bar{W}_5^i)$ for $i = 1, 2$.

Let

$$\hat{Y}(z) = (\hat{Y}_1(z), \hat{Y}_2(z), \hat{Y}_3(z), \hat{Y}_4(z), \hat{Y}_5(z_2)),$$

$$\bar{Y}(z) = (\bar{Y}_1(z), \bar{Y}_2(z), \bar{Y}_3(z), \bar{Y}_4(z), \bar{Y}_5(z_2))$$

with $\hat{Y}_i(z) = \hat{W}_i^1 - \hat{W}_i^2$ and $\bar{Y}_i(z) = \bar{W}_i^1 - \bar{W}_i^2$ $(1 \le i \le 5)$.

In order to obtain the contractibility of $T$ in the Banach space $\Xi_\delta$, we establish some estimates on $\bar{Y}_i$ for $1 \le i \le 5$, which will be provided by the following four steps.

### 3.5.1 The estimate of shock location

It follows from (3.3.3) and a direct computation that

$$\bar{Y}_5'(z_2) = O(1)\bar{Y}_2(0, z_2) + O(\varepsilon)\hat{Y}. \tag{3.5.2}$$

This implies

$$\|\bar{Y}_5'\|_{C^{1,\alpha}[-\theta_0,\theta_0]} \le C\|\bar{Y}_2\|_{C^{1,\alpha}(\bar{E}_+)} + C\varepsilon\left(\sum_{i=1}^{4} \|\hat{Y}_i\|_{C^{1,\alpha}(\bar{E}_+)} + \|\hat{Y}_5\|_{C^{1,\alpha}[-\theta_0,\theta_0]}\right). \tag{3.5.3}$$

### 3.5.2 The estimate of the entropy difference

First, define the characteristics $z_2^i(s;\beta_i)$ going through $(z_1, z_2)$ with $z_2^i(0;\beta_i) = \beta_i$ as

$$\begin{cases} \dfrac{dz_2^i(s;\beta_i)}{ds} = \dfrac{\big(X_0 + 1 - \hat{\xi}_i(z_2^i)\big)\hat{W}_2^i(s, z_2^i)}{(X_0 + 1 - r_0)\hat{\xi}_i(z_2^i) + (X_0 + 1 - \hat{\xi}_i(z_2^i))s + \big(s - (X_0 + 1 - r_0)\big)\hat{\xi}_i'(z_2^i)\hat{W}_2^i(s, z_2^i)}, \\ z_2^i(z_1;\beta_i) = z_2, \qquad z_2^i(0,\beta_i) = \beta_i, \qquad \beta_i \in [-\theta_0, \theta_0] \end{cases} \tag{3.5.4}$$

for $i = 1, 2$.

Set $l(s) = z_2^1(s;\beta_1) - z_2^2(s;\beta_2)$. Then it follows from (3.5.4) and a simple computation that

$$\begin{cases} \dfrac{dl}{ds} = O(\varepsilon)l + O(1)\hat{Y}_2(s; z_2^2(s;\beta_2)) + O(\varepsilon)\big(\hat{Y}_5(z_2^2(s;\beta_2)), \hat{Y}_5'(z_2^2(s;\beta_2))\big), \\ l(z_1) = 0, \qquad l(0) = \beta_1 - \beta_2, \end{cases} \tag{3.5.5}$$

where the quantity $O(\varepsilon)$ in (3.5.5) belongs to $C^{1,\alpha}(\bar{E}_+)$ due to $\hat{W}_2^1 \in C^{2,\alpha}(\bar{E}_+)$ and $\hat{W}_5^1 \in C^{3,\alpha}[-\theta_0, \theta_0]$. In addition, the $C^{1,\alpha}$ estimate of $\beta_1 - \beta_2$ can be derived in terms of (3.5.5).

Indeed, it follows from (3.5.5) that

$$\begin{cases} \beta_1 - \beta_2 = \displaystyle\int_{z_1}^{0} \Big( O(\varepsilon)l(t) + O(1)\hat{Y}_2(t; z_2^2(t;\beta_2)) + O(\varepsilon)\big(\hat{Y}_5(z_2^2(t;\beta_2)), \hat{Y}_5'(z_2^2(t;\beta_2))\big) \Big) dt, \\ l(s) = \displaystyle\int_{z_1}^{s} \Big( O(\varepsilon)l(t) + O(1)\hat{Y}_2(t; z_2^2(t;\beta_2)) + O(\varepsilon)\big(\hat{Y}_5(z_2^2(t;\beta_2)), \hat{Y}_5'(z_2^2(t;\beta_2))\big) \Big) dt. \end{cases} \tag{3.5.6}$$

On the other hand, the estimate (3.3.9) implies

$$\|\partial_{z_1}(\beta_1, \beta_2)\|_{C^{1,\alpha}(\bar{E}_+)} \le C\varepsilon, \qquad \|\partial_{z_2}(\beta_1, \beta_2)\|_{C^{1,\alpha}(\bar{E}_+)} \le C.$$

This, together with (3.5.6), yields

$$\|\beta_1 - \beta_2\|_{C^{1,\alpha}(\bar{E}_+)} \le C\Big( \|\hat{Y}_2\|_{C^{1,\alpha}(\bar{E}_+)} + \varepsilon\|\hat{Y}_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]} \Big). \tag{3.5.7}$$

In addition, it follows from (3.3.11) that $\bar{Y}_4$ satisfies

$$\bar{Y}_4(z) = O(\varepsilon)\big(\beta_1 - \beta_2\big) + O(1)\bar{Y}_5(\beta_2) + \sum_{i=1}^{4} O(\varepsilon)\hat{Y}_i(0, \beta_2) + B_3 \int_{z_2}^{\beta_2(z)} \hat{Y}_5'(s)ds. \tag{3.5.8}$$

One should note that although $R_3$ contains the term $O(\Phi^- - \Phi_b^-)$, the coefficient of $\hat{Y}_5$ is $O(\varepsilon)$.

This, together with (3.3.10) and (3.5.4), shows that $Y_4$ admits the following estimate

$$
\begin{aligned}
&\|\bar{Y}_4\|_{C^{1,\alpha}(\bar{E}_+)}\\
&\leq C\bigg(\|\bar{Y}_5\|_{C^{1,\alpha}[-\theta_0,\theta_0]}+\varepsilon\|\beta_1-\beta_2\|_{C^{1,\alpha}(\bar{E}_+)}+\varepsilon\sum_{i=1}^{4}\|\hat{Y}_i\|_{C^{1,\alpha}(\bar{E}_+)}+\varepsilon\|\hat{Y}_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\bigg)\\
&\leq C\bigg(\|\bar{Y}_5\|_{C^{1,\alpha}[-\theta_0,\theta_0]}+\varepsilon\sum_{i=1}^{4}\|\hat{Y}_i\|_{C^{1,\alpha}(\bar{E}_+)}+\varepsilon\|\hat{Y}_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\bigg).
\end{aligned}
\tag{3.5.9}
$$

### 3.5.3 Estimate on $\bar{Y}_1(z)$.

It follows from (3.3.14) that

$$\bar{Y}_1=O(1)\bar{Y}_3+O(1)\bar{Y}_5+O(\varepsilon)(\hat{Y}_1,\hat{Y}_2,\hat{Y}_3,\hat{Y}_4,\hat{Y}_5). \tag{3.5.10}$$

Thus,

$$\|\bar{Y}_1\|_{C^{1,\alpha}(\bar{E}_+)}\leq C\bigg(\|\bar{Y}_3\|_{C^{1,\alpha}(\bar{E}_+)}+\|\bar{Y}_5\|_{C^{1,\alpha}[-\theta_0,\theta_0]}+\varepsilon\sum_{i=1}^{4}\|\hat{Y}_i\|_{C^{1,\alpha}(\bar{E}_+)}+\varepsilon\|Y_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\bigg). \tag{3.5.11}$$

### 3.5.4 Estimates on $\bar{Y}_2(z),\bar{Y}_3(z)$ and $\bar{Y}_5(-\theta_0)$

It is noted that $\bar{Y}_2,\bar{Y}_3$ and $\bar{Y}_5(-\theta_0)$ satisfy

$$
\begin{cases}
\partial_{z_1}\big(\lambda_1(z_1)\bar{Y}_2\big)+\partial_{z_2}\big(\lambda_2(z_1)\bar{Y}_3\big)-\lambda_3(z_1)\bar{Y}_2(0,z_2)=\sum\limits_{i=1}^{5}\big(O(\varepsilon)\hat{Y}_i+O(\varepsilon)\nabla\hat{Y}_i\big),\\
\partial_{z_1}\big(\lambda_4(z_1)\bar{Y}_3\big)-\partial_{z_2}\big(\lambda_5(z_1)\bar{Y}_2\big)+\lambda_6\big(\bar{Y}_5(-\theta_0)+\lambda_7\displaystyle\int_{-\theta_0}^{z_2}\bar{Y}_2(0,s)ds\big)\\
=\sum\limits_{i=1}^{5}\big(O(\varepsilon)\hat{Y}_i+O(\varepsilon)\nabla\hat{Y}_i\big)+O(\varepsilon)\big(\beta_1-\beta_2\big)+O(1)\displaystyle\int_{z_2}^{\beta_2(z)}\hat{Y}_5'(s)ds,\\
\bar{Y}_3(0,z_2)=B_1\big(\bar{Y}_5(-\theta_0)+\lambda_7\displaystyle\int_{-\theta_0}^{z_2}\bar{Y}_2(0,s)ds\big)+\sum\limits_{i=1}^{5}O(\varepsilon)\hat{Y}_i,\\
\bar{Y}_3(X_0+1-r_0,z_2)=0,\\
\bar{Y}_2(z_1,\pm\theta_0)=0.
\end{cases}
\tag{3.5.12}
$$

By proposition 4.4 in [74], one has the following estimates

$$\|\bar{Y}_2\|_{C^{1,\alpha}(\bar{E}_+)} + \|\bar{Y}_3\|_{C^{1,\alpha}(\bar{E}_+)} + |\bar{Y}_5(-\theta_0)| \leq C\varepsilon\Big(\sum_{i=1}^{4} \|\hat{Y}_i\|_{C^{1,\alpha}(\bar{E}_+)} + \|\hat{Y}_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\Big). \tag{3.5.13}$$

Collecting all the estimates in Step 1-Step 4 above shows that

$$\sum_{i=1}^{4} \|\bar{Y}_i\|_{C^{1,\alpha}(\bar{E}_+)} + \|\bar{Y}_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]} \leq C\varepsilon\Big(\sum_{j=1}^{4} \|\hat{Y}_j\|_{C^{1,\alpha}(\bar{E}_+)} + \|\hat{Y}_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\Big), \tag{3.5.14}$$

here the constant $C > 0$ depends only on $\alpha$ and the supersonic incoming flow. Thus, for suitably small $\varepsilon$, (3.5.14) implies that the mapping $T$ is contractible in $\big(C^{1,\alpha}(E_+)\big)^4 \times C^{2,\alpha}[-\theta_0, \theta_0]$. Therefore, there exists a unique solution $W = (W_1, W_2, W_3, W_4, W_5)$ in $\Xi_\delta$ which solves (3.2.28)-(3.2.29) and (3.2.30)-(3.2.31). Furthermore, by the definition of $\Xi_\delta$, we know that $W$ satisfies (3.2.33)-(3.2.34). Hence, we complete the proof of Theorem 3.2.1.

Finally, we prove Theorem 3.1.3.

**Proof of Theorem 3.1.3.** By Theorem 3.2.1, there exists a unique solution

$$(U_1(z), \omega(z), P(z), S(z); \xi(z_2))$$

to the problem (3.2.19)-(3.2.21), and further

$$(U_1(r,\theta), \omega(r,\theta), P(r,\theta), S(r,\theta); \xi(\theta))$$

solves the problem (3.2.12)-(3.2.15). Moreover, in terms of the transformations (3.2.1)-(3.2.2), one obtains a solution

$$(u_1(x), u_2(x), P(x), S(x); \eta(x_2))$$

to the problem (3.1.1) with (3.1.2)-(3.1.5) and admits the following estimates

$$\|\eta(x_2) - \sqrt{r_0^2 - x_2^2}\|_{C^{3,\alpha}[x_2^1, x_2^2]} \leq C\|\xi - r_0\|_{C^{3,\alpha}[-\theta_0,\theta_0]} \leq C\varepsilon \tag{3.5.15}$$

with $(x_1^i, x_2^i)(i = 1, 2)$ standing for the intersection points of $x_1 = \eta(x_2)$ with $x_2 = (-1)^i x_1 tan\theta_0$ for $i = 1, 2$, and

$$
\begin{aligned}
&\|(u_1(x), u_2(x), P(x), S(x)) - (\hat{U}_0^+(r)\frac{x_1}{r}, \hat{U}_0^+(r)\frac{x_2}{r}, \hat{P}_0^+(r), S_0^+)\|_{C^{1,\alpha}(\bar{\Omega}_+)} \\
&\leq C\|(U_1(r,\theta), \omega(r,\theta), P(r,\theta), S(r,\theta)) - (\hat{U}_0^+(r), 0, \hat{P}_0^+(r), S_0^+)\|_{C^{1,\alpha}(\bar{R}_+)} \\
&\leq C\Big(\|(U_1(z), \omega(z), P(z), S(z)) - (\tilde{U}_0^+(z_1), 0, \tilde{P}_0^+(z_1), S_0^+)\|_{C^{1,\alpha}(\bar{E}_+)} \\
&\qquad\qquad + \|\xi - r_0\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\Big) \\
&\leq C\varepsilon.
\end{aligned}
$$

Thus, we complete the proof of Theorem 3.1.3.

At the end of this chapter, we want to discuss some properties of the transonic shock solution obtained before.

To this end, we assume that the system (3.2.28),(3.2.30)-(3.2.32) has two solutions $(W_{11}, W_{21}, W_{31}, W_{41}, W_{51})$ and $(W_{12}, W_{22}, W_{32}, W_{42}, W_{52})$ when the end pressure is replaced by

$$
P(\theta) = P_e + \varepsilon \tilde{P}_1(\theta), \qquad on \qquad r = X_0 + 1, \tag{3.5.16}
$$

and

$$
P(\theta) = P_e + \varepsilon \tilde{P}_2(\theta), on \qquad r = X_0 + 1. \tag{3.5.17}
$$

Denote $Y_i = W_{i1} - W_{i2}, i = 1, 2, 3, 4, 5$.

We have the following properties:

$$
\sum_{i=1}^{4} \|Y_i\|_{C^{1,\alpha}(E_+)} + \|Y_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]} \leq C\varepsilon \|P_1(z_2) - P_2(z_2)\|_{C^{1,\alpha}(E_+)}. \tag{3.5.18}
$$

This implies the continuous dependence of the transonic shock solution with respect to the exit pressure.

Indeed, same analysis as in the proof of Theorem 3.1.3, we have the following estimates:

$$\|Y_5'\|_{C^{1,\alpha}[-\theta_0,\theta_0]} \leq C\|Y_2\|_{C^{1,\alpha}(\bar{E}_+)} + C\varepsilon\left(\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}(\bar{E}_+)} + \|Y_5\|_{C^{1,\alpha}[-\theta_0,\theta_0]}\right),$$

$$\|\bar{Y}_4\|_{C^{1,\alpha}(\bar{E}_+)} \leq C\left(\|Y_5\|_{C^{1,\alpha}[-\theta_0,\theta_0]} + \varepsilon\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}(\bar{E}_+)} + \varepsilon\|Y_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\right),$$

$$\|Y_1\|_{C^{1,\alpha}(\bar{E}_+)} \leq C\Bigg(\|Y_3\|_{C^{1,\alpha}(\bar{E}_+)} + \|Y_5\|_{C^{2,\alpha}([-\theta_0,\theta_0])} + \varepsilon\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}(\bar{E}_+)} + \varepsilon\|Y_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\Bigg),$$

$$\|Y_2\|_{C^{1,\alpha}(\bar{E}_+)} + \|Y_3\|_{C^{1,\alpha}(\bar{E}_+)} + |Y_5(-\theta_0)| \leq C\varepsilon\left(\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}(\bar{E}_+)} + \|Y_5\|_{C^{2,\alpha}[-\theta_0,\theta_0]}\right) + C\varepsilon\|P_1(z_2 - P_2(z_2))\|_{C^{1,\alpha}(E_+)}.$$

Hence one can easily obtain the estimates (3.5.18).

One may try to apply a similar approach in [74] to establish the monotonicity of shock location with respect to the exit pressure. But we did not succeed. Actually, due to the porous medium boundary condition $W_2$ is a main term in our solvability condition (3.4.10), which is quite different from the solvability condition in [74].

# Chapter 4

# Summary and discussion on future work

In this chapter, we will briefly summarize previous work and discuss some open problems which are closely related to the results obtained in this thesis.

In Chapter 2, we consider 3-D axially symmetric Euler flows through infinitely long nozzles without assuming irrotational condition. Global existence and uniqueness of subsonic solution are proved for a general nozzle, when the variation of Bernoulli's function in the upstream is sufficiently small and mass flux has an upper critical value. We use a stream function formulation, by which, 3-D Euler equations are equivalent to a quasilinear second order equation for a stream function. A key point here is to have the gradient estimate near the axis. Then, the existence of solution to the BVP and asymptotic behavior for the stream function are obtained. Finally, the uniqueness of the solutions will be a consequence of the asymptotic behavior.

In addition, we showed only the uniqueness of uniformly subsonic flows in class of axially symmetric flows. Is the flow unique among all 3-dimensional axially symmetric nozzles? Moreover, note that subsonic flow obtained is under assumption that the mass flux is less than the critical mass, there may be rich phenomena in the nozzle if the mass flux is beyond the critical value and transonic shocks appear. More generally, can we show existence and uniqueness for subsonic flows through 3-dimensional infinitely long nozzles?

In Chapter 3, we establish the existence and uniqueness of a transonic shock solution to the full steady compressible Euler system in a class of de Laval nozzles with porous medium. For this class of nozzles, we have solved the transonic shock problem posed by Courant-Friedrichs: Given a appropriately large receiver pressure $p_e$, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes $p_e$.

Furthermore, we would like to consider similar boundary conditions for flat nozzles, since Z. P. Xin, W. Yan and H. C. Yin in [119] proved that there is no such transonic shock solution for flat nozzles with some perturbed pressure given at exit. We expect such problem is the well-posed if porous medium boundary condition is concerned.

# Chapter 5

# Appendix

In this appendix, we will give a description on the transonic solution of the problem (3.1.1) with (3.1.2)-(3.1.5) when the exit pressure is a suitable constant $P_e$ under the assumptions on the nozzle walls and the uniform supersonic incoming flow in Chapter 3. Such a solution is called a background solution and can be obtained by solving the related ordinary differential equations. In fact, the related analysis has been given in Section 147 of [41] and the details can be seen in [120]. In this appendix, we give a detailed illustration.

**Theorem** (Existence of a transonic shock for the constant end pressure) For the 2-D nozzle and the uniform supersonic incoming flow given in §1 of Chapter 1, then there exist two constant pressures $P_1$ and $P_2$ with $P_1 \le P_2$ such that if the exit pressure $P_e \in (P_1, P_2)$, then the system (3.1.1) has a symmetric transonic shock solution,

$$(u_1, u_2, P, S) = \begin{cases} (u_{1,0}^-, u_{2,0}^-, P_0^-(r), S_0^-), & for \quad r \le r_0, \\ (u_{1,0}^+, u_{2,0}^+, P_0^+(r), S_0^+), & for \quad r > r_0. \end{cases}$$

here $u_{i,0}^+(x) = U_0^+(r)\frac{x_i}{r}, X_0 < r < X_0 + 1$, $S_0^+$ is a constant, and $(P_0^+(r), U_0^+(r))$ is $C^3$-smooth.

**Remark 1.** By the assumption (3.1.6), one has for $r_0 \le r \le X_0 + 1$

$$|\frac{d^k U_0^+(r)}{dr^k}| + |\frac{d^k P_0^+(r)}{dr^k}| \le \frac{C_k}{X_0^k}, \qquad k = 1, 2, 3.$$

**Remark 2.** One can obtain an extension $(\hat{\rho}_0^+(r), \hat{U}_0^+(r))$ of $(\rho_0^+(r), U_0^+(r))$ for $r \in (X_0, X_0+1)$ by solving the Euler system.

For notational conveniences, the superscripts "-" will be neglected.

As in chapter 3, it is convenient to use the polar coordinate and set $D_0 = \frac{1}{y_1}, D_1 = \partial_{y_1}, D_2 = \frac{1}{y_1}\partial_{y_2}$, then the system (3.2.5) can be rewritten into the following non-divergence form

$$AD_1U + BD_2U = C. \tag{B.1}$$

where

$$A = \begin{bmatrix} \rho & 0 & 0 & U_1 \\ \rho U_1 & 0 & 1 & 0 \\ 0 & \rho U_1 & 0 & 0 \\ 1 & 0 & \frac{U_1}{\gamma P} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \rho & 0 & U_2 \\ \rho U_2 & 0 & 0 & 0 \\ 0 & \rho U_2 & 1 & 0 \\ 0 & 1 & \frac{U_2}{\gamma P} & 0 \end{bmatrix}.$$

and

$$C = \begin{bmatrix} -D_0\rho U_1 \\ D_0 U_2^2 \\ -D_0U_1U_2 \\ -\gamma D_0 P U_1 \end{bmatrix}, U = \begin{bmatrix} U_1 \\ U_2 \\ P \\ \rho \end{bmatrix}.$$

The background solution satisfies the following problem:

$$A_0D_1U_0 + B_0D_2U_0 = C_0. \tag{B.2}$$

where

$$A_0 = \begin{bmatrix} \rho_0 & 0 & 0 & U_{1,0} \\ \rho_0 U_{1,0} & 0 & 1 & 0 \\ 0 & \rho_0 U_{1,0} & 0 & 0 \\ 1 & 0 & \frac{U_{1,0}}{\gamma P_0} & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & \rho_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C_0 = \begin{bmatrix} -D_0 \rho_0 U_{1,0} \\ 0 \\ 0 \\ -\gamma D_0 P_0 U_{1,0} \end{bmatrix}.$$

Set $W = U - U_0$, then $W$ satisfies the following system:

$$AD_1 W + BD_2 W = E(W). \tag{B.3}$$

where $A = A(W + U_0)$, $B = B(W + U_0)$ and $E(W) = C - C_0 - (A - A_0)D_1 U_0 - (B - B_0)D_2 U_0$.

By solving $\det(B - \lambda A) = 0$, we find the eigenvalues of (B.3):

$$\lambda_1 = \lambda_2 = \frac{U_2}{U_1}, \lambda_{3,4} = \frac{U_1 U_2 \pm c\sqrt{U_1^2 + U_2^2 - c^2}}{U_1^2 - c^2}.$$

The corresponding left eigenvectors are

$$\ell_1 = (1, 0, 0, -\rho), \ell_2 = (0, U_1, U_2, 0), \ell_{3,4} = (0, \lambda_{3,4}, -1, \rho(U_2 - U_1 \lambda_{3,4})).$$

Multiplication of (B.3) by $\ell_i$ yields

$$\ell_i A(D_1 U + \lambda_i D_2 U) = \ell_i E(W). \tag{B.4}$$

Simple calculations show that

$$\ell_1 A = (0, 0, -\frac{\rho U_1}{\gamma P}, U_1), \qquad \ell_2 A = (\rho U_1^2, \rho U_1 U_2, U_1, 0),$$

$$\ell_{3,4} A = (\rho U_2, -\rho U_1, \mp \frac{\sqrt{U_1^2 + U_2^2 - c^2}}{c}, 0).$$

The system (B.4) can be rewritten as

$$\sum_{j=1}^{4} \zeta_{kj}(D_1 W_j + \lambda_k D_2 W_j) = \ell_k E(W) =: \mu_k, k = 1, 2, 3, 4. \tag{B.5}$$

where $(\zeta_{k1}, \zeta_{k2}, \zeta_{k3}, \zeta_{k4}) = \ell_k A$.

Denote $R_\delta = (X_0, X_0 + \delta) \times (-1, 1)$, where $\delta$ is to be determined later. Set the iteration space

$$\Xi_\varepsilon = \{W \in C^{2,\alpha}(R_\delta) : \|W\|_{C^{2,\alpha}(R_\delta)} < \varepsilon\}.$$

Take $V \in \Xi_\varepsilon$, we set $\bar{\zeta}_{kj} = \zeta_{kj}(V + U_0), \bar{\mu}_k = \mu_k(V + U_0)$, hence we obtain the linearized equation:

$$\sum_{j=1}^{4} \bar{\zeta}_{kj}(D_1 W_j + \lambda_k D_2 W_j) = \bar{\mu}_k, k = 1, 2, 3, 4. \tag{B.6}$$

Next we consider the boundary condition. We already know $W_2 = 0$ on the boundary $y_2 = \pm 1$. We need three more conditions on the boundary to prescribe $W_1, W_3$ and $W_4$. We use the characteristic method to determine the data on the boundary. Let us only consider the data on the lower boundary $y_2 = -1$.

Define the i-th characteristics $f_i$ passing through $(y_1, -1)$ by

$$\begin{cases} \dfrac{df_i(\tau; y_1, -1)}{d\tau} = \dfrac{\lambda_i((V + U_0)(\tau, f_i))}{\tau}, \\ f_i(y_1; y_1, -1) = -1. \end{cases} \tag{B.7}$$

Using the equations, along the i-th characteristic, we have

$$\frac{d}{d\tau}(\ell_i A W) = \frac{d}{d\tau}(\ell_i A) W + \mu_i(W). \tag{B.8}$$

Since $\lambda_1 = \lambda_2 = 0$ and $\lambda_4 < 0$ on $y_2 = -1$, the first, second and fourth characteristics can travel to the left and reach the initial boundary on $y_1 = X_0$. Let $\xi_i(y_1) = f_i(X_0; y_1, -1)$, integrate along the above equation to obtain

$$\ell_i A W(y_1, -1) = \ell_i A \Phi_0(\xi_i(y_1)) + \int_{X_0}^{y_1} \frac{d}{d\tau}(\ell_i A) U d\tau =: \chi_i.$$

Now we will linearize these conditions as in [23]: replace $\ell_i A$ by $\bar{\ell}_i \bar{A}$ and $\chi_i$ by $\bar{\chi}_i$, where

$$\bar{\chi}_i = \bar{\ell}_i \bar{A} \Phi_0(\xi_i(y_1)) + \int_{X_0}^{y_1} \frac{d}{d\tau}(\ell_i A)(V + U_0) d\tau.$$

Together with the boundary condition $W_2(y_1, -1) = 0$, we have

$$\begin{cases} -\frac{U_{1,0}+V_1}{\bar{c}^2}W_3 + (U_{1,0}+V_1)W_4 = \bar{\chi}_1, \\ (V_3+\rho)(U_{1,0}+V_1)^2W_1 + (U_{1,0}+V_1)W_3 = \bar{\chi}_2, \\ -\bar{\lambda}_4(1-\frac{(U_{1,0}+V_1)^2}{\bar{c}^2})W_3 = \bar{\chi}_4. \end{cases} \tag{B.9}$$

After these preparations, we are able to use the characteristic method to solve the problem.

Let $l_3^+$ be the 3-th characteristic going through $(X_0, -1)$ and the $l_4^-$ be the 4-th characteristic going through $(X_0, 1)$. Set $R_1 = R_2 = R_\delta$ and $l_3^+, l_4^-$ will separate the domain $R_\delta$ into two parts, which will be denoted by $R_3, R_3^+$ and $R_4, R_4^-$ respectively.

By the characteristic method, we have

$$\begin{cases} \sum_{j=1}^{4}\bar{\zeta}_{lj}W_j(y_1,y_2) = \sum_{j=1}^{4}\bar{\zeta}_{lj}(X_0,y)W_j(X_0,y_2) + \int_{X_0}^{y_1}(\frac{d}{d\tau}(\bar{\ell}_l\bar{A})W + \mu_l(\bar{W}))d\tau \\ \qquad\qquad =: I_l(W), (y_1,y_2)\in R_l, l=1,2,3,4, \\ \sum_{j=1}^{4}\bar{\zeta}_{3j}W_j(y_1,y_2) = \sum_{j=1}^{4}\bar{\zeta}_{3j}(\xi_3(y_1,y_2),-1)W_j(\xi_3(y_1,y_2),-1) \\ \qquad + \int_{\xi_3(y_1,y_2)}^{y_1}(\frac{d}{d\tau}(\bar{\ell}_3\bar{A})W + \mu_3(\bar{W}))d\tau =: I_3(W), (y_1,y_2)\in R_3^+, \\ \sum_{j=1}^{4}\bar{\zeta}_{4j}W_j(y_1,y_2) = \sum_{j=1}^{4}\bar{\zeta}_{4j}(\xi_4(y_1,y_2),1)W_j(\xi_4(y_1,y_2),1) \\ \qquad + \int_{\xi_4(y_1,y_2)}^{y_1}(\frac{d}{d\tau}(\bar{\ell}_4\bar{A})W + \mu_l(\bar{W}))d\tau =: I_4(W), (y_1,y_2)\in R_4^-, \end{cases} \tag{B.10}$$

where $\xi_3(y_1,y_2), \xi_4(y_1,y_2)$ are defined as follows:

$$\begin{cases} \frac{d}{d\tau}f_3(\tau;y_1,y_2) = \frac{\lambda_3(\tau, f_3(\tau;y_1,y_2))}{y_1}, \\ f_3(y_1;y_1,y_2) = y_2. \end{cases}$$

and

$$\begin{cases} \frac{d}{d\tau}f_4(\tau;y_1,y_2) = \frac{\lambda_4(\tau, f_4(\tau;y_1,y_2))}{y_1}, \\ f_4(y_1;y_1,y_2) = y_2. \end{cases}$$

and $\xi_3(y_1, y_2), \xi_4(y_1, y_2)$ satisfy $f_3(\xi_3(y_1, y_2); y_1, y_2) = -1$ and $f_4(\xi_4(y_1, y_2); y_1, y_2) = 1$.

Next we will use the contraction principle to solve (B.10). Take any $W \in \Xi_\varepsilon$, solving the linear system

$$\sum_{j=1}^{4} \zeta_{lj}(y_1, y_2) Z_j(y_1, y_2) = I_l(W)$$

Hence we can define a mapping $S : Z = S(W)$. In order to get the contraction of $S$, we introduce the following norm in $\Xi_\delta$:

$$\begin{aligned} \|W\|_* &= sup_{R_\delta} |e^{-ay_1} W(y_1, y_2)| \\ &\quad + \frac{1}{b}(sup_{R_\delta} |e^{-ay_1} \partial_{y_1} W(y_1, y_2)| + sup_{R_\delta} |e^{-ay_1} \partial_{y_2} W(y_1, y_2)|). \end{aligned}$$

where $a, b$ are positive constants. A direct computation shows that $S$ maps $\Xi_\delta$ to itself.

One can show that for sufficiently large $a$ and $b$, $S$ is a contraction operator. Hence there exists a unique fixed point $W$ such that $W = S(W)$. Hence we have solved the linearized equation. This enable us to define another mapping $T : V \longrightarrow W$. Using the integral equations and Gronwall's inequality, one can derive the estimate of $W$. Differentiating the equations with respect to $y_2$, one can derive the equation of $\partial_{y_2} Z$, which takes the same form as $Z$, one can derive the estimate of $\partial_{y_2} Z$ as for $Z$. Finally using the equation itself, one can derive the estimate of $\partial_{y_1} Z$. In a word, one can prove that the mapping $T$ maps $\Xi$ to itself if $\delta$ is small enough. For the details, one may refer to [83].

One can do the same estimates to show that $T$ is a contraction operator with respect to $C^0$ norm. Since $\Xi_\varepsilon$ is compact in $C^1(R_\delta)$ and closed in the $C^0$ norm, then $T$ has a unique fixed point in $\Xi_\delta$.

Hence we have proved that for the initial-boundary value problem (3.1.1),(3.1.2)-(3.1.9) with $\|\Phi_0 - U_0\|_{C^{2,\alpha}} < \varepsilon_0$ for $0 < \varepsilon_0 < 1$, there exits a small $\delta_0 > 0$ depending only on $\varepsilon_0, U_0$ such that there exists a unique solution $U$ on $R_{\delta_0}$ satisfying the following estimate:

$$\|U - U_0\|_{C^{2,\alpha}(R_\delta)} \leq C\|\Phi_0 - U_0\|_{C^{2,\alpha}}.$$

where $C_1$ depends only on $U_0, \delta_0$.

Now take $\varepsilon = \dfrac{\varepsilon_0}{C_1^{\frac{2}{\delta}+1}}$, then we have

$$\|U - U_0\|_{C^{2,\alpha}(R_{\delta_0})} \leq C_1\|\Phi_0 - U_0\|_{C^{2,\alpha}(-1.1)} < \frac{\varepsilon}{C_1^{\frac{2}{\delta_0}}}.$$

Hence $\|U(\delta_0, \cdot) - U_0\|_{C^{2,\alpha}} < \varepsilon$. Then one can apply the above result again, continue this process (up to $\dfrac{2}{\delta_0} + 1$ times), we extend the local solution to the whole region. The corresponding estimate holds also. (3.1.15) can be checked directly. Hence we have completed the proof.

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