Global Robust Output Regulation for Nonlinear Output Feedback Systems and Its Applications

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The thesis is concerned with the global robust output regulation for nonlinear systems in the output feedback form by using output feedback control. For the nonlinear output feedback systems, we mainly study three typical output regulation problems. The first one is the output regulation problem with unknown control directions and input-to-state stable (ISS) inverse dynamics using a direct approach. The second one is the adaptive output regulation problem with an uncertain exosystem and ISS inverse dynamics. The third one is a case study on the solvability of the systems with integral input-to-state stable (iISS) inverse dynamics.

The nonlinear output regulation is a central control problem that involves nonlinear stabilization, tracking control and disturbance rejection as special cases. The control objective is to find a feedback controller to achieve asymptotic tracking and/or disturbance rejection while maintaining closed-loop stability. The output regulation study has experienced rapid developments in the past two decades or so. It is now well known that the problem can be systematically approached according to the general framework of tackling nonlinear output regulation that is composed of the following two steps. The first step is the problem conversion: from nonlinear output regulation to stabilization. The output regulation is generally more complicated than the stabilization problem. Therefore the problem conversion indeed reduces the complexity and makes it possible to be handled. In this step, the output regulation is converted into the stabilization of an augmented system consisting of the original plant and a suitable dynamic compensator called internal model. The second step is the stabilization of the augmented system whose solvability implies solvability of the output regulation problem.

In the past ten years or so, the output regulation of the strict output feedback systems has attracted a lot of attention. In contrast with the strict output feedback systems, the output feedback systems is more general since it not only involves the nonlinearity of the system output but also the unmodeled dynamics. Therefore, the usual design method is not applicable, which motivates us to develop some new methodology for the output

regulation design of the output feedback systems.

The main results of the thesis are outlined as follows.

i) A direct approach is proposed for the output regulation of the systems with ISS inverse dynamics and unknown control directions. The internal model is first designed for the control input. The output feedback control design is further achieved based on a type of partial state observer which is designed for the transformed augmented system. The Nussbaum function technique is successfully incorporated in the stabilization design to deal with the case of unknown control directions.

The result is applied to solve a tracking control problem associated with the well known Lorenz system and a class of generalized fourth-order Lorenz systems. By certain system decomposition, it is proved that the Lorenz system contains certain ISS inverse dynamics and the output feedback control is successfully realized.

ii) An adaptive output regulation design is proposed for the systems with ISS inverse dynamics and an uncertain exosystem. When the exosystem contains uncertain parameters, the direct approach can not be implemented any longer. To deal with this issue in the general case, by introducing an observer, we first derive an extended system composed of the plant and the observer. Then the output regulation problem of the extended system is solved. It is further shown that the unknown parameter vector of the exosystem can be exactly estimated if a controller containing a minimal internal model is employed.

The application of the result leads to the solution of several interesting control problems such as the global disturbance rejection of the FitzHugh-Nagumo (FHN) system and the robust output synchronization of the generalized third and fourth-order Lorenz system and the Harmonic system.

iii) A sufficient solvability condition of the global output regulation for the systems with iISS inverse dynamics is proposed. Since the concept of iISS is strictly weaker than the ISS one, the result allows us to handle a much larger class of nonlinear systems.

One of the motivations of the case study is to deal with the output regulation problem of a shunt-connected DC motor whose inverse dynamics is iISS but not ISS. As an illustration, a disturbance rejection problem of the shunt-connected DC motor is solved.

摘 要

本文针对非线性输出反馈控制系统, 研究基于输出反馈的全局鲁棒输出调节问题. 针对非线性输出反馈控制系统, 主要研究了三类典型的输出调节问题. 第一类设计问题是, 利用一种直接的方法, 在控制方向未知并且系统的逆动态具有输入到状态稳定(ISS)的性质的情况下的输出调节问题. 第二类设计问题是, 针对具有不确定外部系统和输出到状态稳定的逆动态的非线性系统的自适应输出调节问题. 第三类设计问题是, 是进一步考虑了具有积分输入到状态稳定(iISS)的逆动态的控制系统的一种输出调节问题.

非线性系统的输出调节问题是一个核心的控制问题, 涵盖了非线性系统的镇定控制, 跟踪控制和干扰抑制等特殊的控制问题. 输出调节问题的目标是寻找一种反馈控制器, 实现渐进跟踪或者渐进干扰抑制, 同时可以保持闭环系统的稳定性. 在过去大约二十年的时间里, 非线性输出调节问题的研究经历了快速的发展. 现在我们已经知道, 该问题可以利用输出调节问题的一般求解框架来系统的解决. 该框架包含了两个基本步骤, 第一步是问题的转化: 从输出调节问题到镇定控制问题. 一般情况下, 输出调节问题要比镇定控制问题复杂困难的多, 因此通过问题转化, 输出调节问题将大大简化, 并且更可能得到解决. 经过这一步转化, 输出调节问题转化成了关于增广系统的镇定问题, 该增广系统是由原被控系统和适当设计的动态补偿器构成, 这样的动态补偿器称为内模. 第二步是关于增广系统的镇定控制, 该镇定问题的可解性决定了输出调节问题的可解性.

在过去的十年左右的时间里, 相关严格输出反馈系统的输出调节问题吸引了研究人员的广泛注意. 相对于严格输出反馈系统, 输出反馈系统更具有一般性, 体现在该类系统不仅具有输出的非线性形式, 还具有未建模动态的非线性形式. 因此, 现有的严格输出反馈系统的输出调节求解方法将无法直接的运用于问题的求解, 我们将针对一般的输出反馈系统给出输出调节问题的设计方案.

本文的主要结论概括如下:

i) 针对具有未知控制方向以及输入到状态稳定的逆动态的输出反馈控制系统, 我们提出了一个直接方法. 首先对控制输入设计了内模, 然后通过设计适当的部分状态观测器, 实现了增广系统的输出反馈控制. Nussbaum函数方法成功的运用于镇定设计过程中, 解决了控制方向未知的问题.

该设计方法可以应用于解决广义的三阶和四阶Lorenz系统的跟踪控制问题. 经过分析证明, Lorenz控制系统具有输出到状态稳定的逆动态, 因此输出反馈控制得以实现.

ii) 针对具有不确定外部系统以及输入到状态稳定的逆动态的输出反馈控制系统, 我们提出了一个自适应输出调节设计方法. 当外部系统具有不确定参数的时候, 先前的直接方法将不再有效. 为了解决这个问题, 针对一般形式的输出反馈系统, 我们首先通过引入了某种类型的观测器, 得到一个新的扩展了的系统, 由原系统和该观测器构成. 然后针对扩展系统考虑了输出调节问题. 我们还进一步考虑的参数收敛性问题, 同时指出, 如果控制器包含了一个最低阶的内模, 那么外部系统的未知参数将可以得到精确的估计.

该设计方法可以应用于解决若干个有意义的控制问题, 比如FHN系统的全局干扰抑制问题, 广义Lorenz系统与谐振系统的同步控制问题等等.

iii) 针对具有积分输入到状态稳定的逆动态的输出反馈控制系统, 我们提出了一类输

出调节问题的可解性的充分条件. 相对于输入到状态稳定, 由于积分输入到状态稳定的概念更具有一般性, 该结论使得更广泛的一类输出反馈系统的输出调节问题得以解决.

该结果可以用来处理一种并励直流电机的干扰抑制问题, 可以证明这种并励直流电机具有积分输入到状态稳定的逆动态. 作为应用, 我们给出了该干扰抑制问题的仿真实验.

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Dabo XU
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Chapter 1

Introduction

1.1 Literature overview

The output regulation or the servomechanism problem is a central control problem in the area of automatic control. A general formulation of the problem is *design of a feedback control law for an uncertain control plant to achieve asymptotic tracking and/or disturbance rejection for a class of references and/or disturbances while maintaining closed-loop stability.* What makes the problem interesting and important is that it involves several basic control problems such as stabilization, asymptotic tracking, and disturbance rejection as special cases. In the problem setting, both the references and disturbances are supposed to be unavailable for feedback design, but they can be generated by an autonomous differential equation called the exosystem. In contrast with the similar problems, such as synchronization and trajectory tracking, where the desired trajectories can be fully used in the feedback design, a fundamental feature of the output regulation problem is that the reference inputs and disturbances are supposed to be unmeasurable but it can be generated by an exosystem.

The output regulation problem for linear systems was well studied during the 1970s [15] [20]. The problem for nonlinear systems has been a central control problem since the early 1990s. The problem aims at achieving, by feedback control, asymptotic tracking and disturbance rejection in an uncertain nonlinear plant where the reference inputs and disturbances are generated by an autonomous system called exosystem. The study of this problem for nonlinear systems was initiated in the early 1990s, name a few [24] [25] [27] [35] [38] [48]. In the past two decades or so, it has led to a powerful control design method called the internal model approach [4] [31] [32] [49] [77]. Various versions of this problem have been solved so far for several types of nonlinear systems as follows. The global state feedback output regulation for lower triangular systems was studied in [7] [9] [31] [32] [52]. The global state feedback output regulation for upper triangular systems was studied in [6]. The global output feedback control for strict output feedback systems with a known exosystem [8] [17] [18] [58] [61] [67] [76]. The semi-global output feedback output regulation for lower triangular systems can be found in [37] [47], and with an unknown exosystem [72] [77]. The practical output regulation problem was studied in [26] [66] [86]. Among them, applications of nonlinear output regulation can be found in [10] [31] [59] [60] [65].

A framework for handling the output regulation problem for a general nonlinear system has been given in [32]. According to this framework, the output regulation problem can be handled in two steps. In the first step, an appropriate dynamic compensator called internal model is designed. Attachment of the internal model to the given plant leads to an augmented system. The internal model has the property that the stabilization solution of the augmented system will lead to the output regulation solution of the given plant and exosystem. In the second step, the stabilization problem of the augmented system needs to be tackled. It is now quite clear what are the conditions under which the first step can be accomplished. Therefore, we will place the emphasis of the work on the second step. It should be noted that, in this general framework, the stabilization design for the augmented system is generally much more challenging than the original plant with the exogenous signals set to zero due to the introduction of internal model dynamics.

The class of systems studied in the thesis is more general than the strict output feedback systems in that the zero dynamics of the system is not linear. For this reason, the study of this class of systems will also be more interesting and meaningful. On one hand, this extension makes it possible to cover a larger class of nonlinear systems, and on the other hand, the existing design methodologies are usually not applicable to solve the new problems. This motivates us to investigate the solvability of the output regulation problems of nonlinear output feedback systems using output feedback control. The control of output feedback systems has gained a lot of attentions recently and most of the results are on the stabilization problem using output feedback control, see [41] [44] [45] [46] [71] for instances.

1.2 Contribution of the thesis

The contribution of the thesis is twofold. Firstly, based on the two-step general framework for tackling nonlinear output regulation, some new design methodology is developed for the output regulation of nonlinear output feedback systems by using output feedback control in the cases of unknown control direction, uncertain exosystem, and iISS inverse dynamics respectively. Secondly, the results in the present work are successfully applied to solve several interesting tracking control, synchronization, and disturbance rejection problems associated with the well known Lorenz system, FHN model and shunt-connected DC motor. Compared with the existing results on these applications, our design is output feedback control and the tracking or synchronization trajectory can be unmeasurable. In summary, there are three typical problems studied in the thesis.

i) We propose a direct approach to the output regulation problem of the systems with ISS inverse dynamics and an unknown control direction. In this case, the internal model is first designed for the control input. The resulting augmented system is not in the output feedback form. However, it can be transformed into a specific lower triangular form. The Nussbaum function technique is also successfully incorporated to deal with the case of unknown control directions.

The result is applied to solve a tracking control problem associated with the well known Lorenz system and a class of generalized fourth-order Lorenz systems. By certain system decomposition, it is proved that the Lorenz system exactly contains an ISS inverse dynamics and this makes the output feedback control design possible to be done.

ii) We propose an adaptive output regulation design methodology for the systems with ISS inverse dynamics and an uncertain exosystem. When the exosystem contains uncertain parameters, the direct approach can not be implemented any longer. To deal with this issue of the general case, by introducing a specific type of observer, we first derive an extended system composed of the plant and the observer. Then the output regulation design is performed for the extended system. We further show that this unknown parameter vector can be asymptotically estimated if a controller containing a minimal internal model is employed.

The main advantage of the design method is that it allows certain parameter uncertainty of the exosystem, and the parameter convergence issue is further discussed in this chapter. However, the sign of the control gain has to be known as a prerequisite.

The application of the result leads to the solution of several interesting control problems such as the global disturbance rejection of the FitzHugh-Nagumo (FHN) model and the robust output synchronization of the generalized Lorenz system and the Harmonic system.

iii) We present a case study on a sufficient solvability condition of the global output regulation for the systems with iISS inverse dynamics and unknown control directions. Since the iISS condition is strictly weaker than the ISS one, the extension allows us to handle a much larger class of nonlinear systems. We further point out that under certain conditions, the output regulation problem with an uncertain exosystem can be possibly solved by using a similar approach proposed in Chapter 4.

As an illustration, the result is applied to solve a disturbance rejection problem of the shunt-connected DC motor. It is shown that its inverse dynamics is iISS but not ISS.

1.3 Organization of the thesis

The remaining part of the thesis is organized as follows.

Chapter 2: Some fundamental background materials are collected and summarized, which is useful for deriving the results in the thesis. Some preliminary results as preparations for proving the results in the thesis are also placed in this part.

Chapter 3: The output regulation problem with unknown control directions and ISS inverse dynamics using output feedback control is solved in this chapter. Applications of the result to the tracking control of third and fourth-order Lorenz systems with unknown control directions are given in this chapter.

Chapter 4: The output regulation problem with an uncertain exosystem and ISS inverse dynamics using output feedback control is solved in this chapter. Applications of the result to the disturbance rejection of FHN model and the synchronization of Lorenz system and Harmonic oscillator are given in this chapter.

Chapter 5: For the systems with iISS inverse dynamics, the global output regulation problem is solved by using the developed changing supply function technique for iISS stable systems. It is pointed out that the design methods in Chapters 3 and 4 can be implemented under certain conditions.

An application of the result to a disturbance rejection problem of a shunt-connected DC motor is illustrated in this chapter.

Chapter 6: Some concluding remarks including the future work are given in this chapter.

The result presented in Chapter 3 is included in [87] [88], and the result in Chapter 4 is included in [89] [90].

The simulations throughout the thesis were performed by using MATLAB.

The thesis was typeset using LaTeX.

Notation and Acronym

The notations and acronyms used in the thesis are collected in Table 1.1.

□ **End of chapter.**

Table 1.1: Notations and Acronyms

Symbol	*Meaning*
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^+$	The set of all nonnegative real numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^{m\times n}$	The set of all real $m \times n$ matrices
$\|x\|$	The Euclidean norm of a vector x
$\|A\|$	The induced Euclidean norm of a real square matrix A
$A^\top$	The transpose of a matrix A
$\mathrm{col}(v_1, v_2)$	The compound column vector $[v_1^T, v_2^T]^T$ for any column vectors v_1 and v_2
$\lambda(A)$	The spectrum of a square matrix A
I_n	n-dimensional identity matrix
I	Identity matrix of an appropriate dimension
C^1	The class of continuously differentiable functions
p.d.	The class of continuous functions $f : \mathbb{R}^n \mapsto \mathbb{R}^+$ satisfying $f(0) = 0$ and $f(x) > 0, \forall\, x \neq 0$
$\mathcal{K}$	The class of strictly increasing p.d. functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$
$\mathcal{K}_\infty$	The class of unbounded class $\mathcal{K}$ functions
$\mathcal{L}$	The class of continuous, strictly decreasing, positive functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfying $f(s) > 0, \forall\, s \geq 0$
$\mathcal{KL}$	The class of continuous functions $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfying for each fixed $s \in \mathbb{R}^+$, $\beta(\cdot, s) \in \mathcal{K}$, and for each fixed $r \in \mathbb{R}^+$, $\beta(r, \cdot) \in \mathcal{L}$
BIBS	Bounded-input bounded-state
ISS	Input-to-state stability/stable
iISS	Integral input-to-state stability/stable
GAS	Globally asymptotically stable
PE	Persistence of excitation
CLF	Control Lyapunov function
FHN	The FitzHugh-Nagumo model

Chapter 2

Background and preliminaries

In this chapter, we review some fundamental definitions and results that will be extensively used in the thesis. These materials are well known and can be found in many books such as [21] [31] [36] [50] [51] [78]. Some preliminary results that will be referred to in the subsequent chapters are also presented in this chapter.

2.1 Fundamentals of nonlinear systems

2.1.1 Stability of nonlinear systems

We begin with the general nonlinear system described by

$$\dot{x} = f(x,t), \quad x(t_0) = x_0 \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the system state with initial state x_0 at initial time t_0, and f is locally Lipschitz in x and is piecewise continuous in t. Suppose system (2.1) has an equilibrium at $x = 0$, i.e.

$$f(0,t) = 0, \quad \forall\, t \geq t_0$$

Definition 2.1 The equilibrium $x = 0$ of system (2.1) is *stable* if for each $\varepsilon > 0$, there exists a constant $\delta(t_0, \varepsilon) > 0$ such that the trajectory of system (2.1) satisfies

$$\|x(t, t_0, x_0)\| < \varepsilon$$

for all $\|x_0\| < \delta$ and all $t \geq t_0 \geq 0$. If, in addition, the above $\delta > 0$ can be chosen independent of t_0, then $x = 0$ is *uniformly stable.*

The equilibrium $x = 0$ is *asymptotically stable* if it is stable and there exists a constant $c(t_0) > 0$ such that for all $\|x_0\| < c$

$$\lim_{t \to +\infty} x(t, t_0, x_0) = 0$$

The equilibrium $x = 0$ is *uniformly asymptotically stable* if there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and positive constant $c > 0$ which is independent of t_0, such that for all $\|x_0\| < c$

$$\|x(t, t_0, x_0)\| \leq \beta(\|x_0\|, t - t_0), \quad t \geq t_0 \geq 0 \tag{2.2}$$

If (2.2) holds for any initial state $x_0 \in \mathbb{R}^n$, then the equilibrium $x = 0$ is *globally uniformly asymptotically stable*. ∎

Theorem 2.1 [50] Let $x = 0$ be an equilibrium of (2.1) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. If there exists a C^1 positive definite (p.d.) function $V(t, x)$ be a C^1 function satisfying $W_1(x) \leq V(t, x) \leq W_2(x)$ for some p.d. functions $W_1(x)$ and $W_2(x)$ on D, such that along the trajectory of (2.1)

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x) \leq 0 \tag{2.3}$$

then $x = 0$ is uniform stable. ∎

Theorem 2.2 [50] Suppose that the assumptions of Theorem 2.1 are all satisfied with inequality (2.3) strengthened to

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x) \leq -W_3(x)$$

for some p.d. function $W_3(x)$ on D, then $x = 0$ is uniform asymptotically stable. If $D = \mathbb{R}^n$ and $V(t, x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable. ∎

Theorems 2.1 and 2.2 provide the sufficient conditions to determine the (uniform) stability or asymptotic stability of an equilibrium point of a (time-varying) uncontrolled nonlinear system. A function $V(t, x)$ is also called *positive definite* if $V(t, x) \geq W_1(x)$ with $W_1(x)$ being p.d., and $V(t, x)$ is *radially unbounded* if $W_1(x)$ radially unbounded.

In some cases, the function $V(t, x)$ in Theorems 2.1 and 2.2 can be chosen to be time-invariant for some time-varying nonlinear systems, that is $V = V(x)$ which is independent of t.

Particularly, consider the nonlinear system described by

$$\dot{x} = f(x, \mu(t)), \quad x(t_0) = x_0 \tag{2.4}$$

where $x \in \mathbb{R}^n$ is the system state with an initial state x_0 and $\mu(t) \in \Sigma$ with Σ being a compact subset of $\mathbb{R}^{n_\mu}$. Suppose system (2.4) has an equilibrium at $x = 0$, i.e. $f(0, \mu) = 0$ for all $\mu \in \Sigma$, the following theorem is sufficient to guarantee the global (uniform) asymptotic stability of $x = 0$.

Theorem 2.3 [50] Let $x = 0$ be an equilibrium point of (2.4) and f is locally Lipschitz in x and uniformly in $\mu \in \Sigma$. If there exists a C^1 function $V(x)$ satisfying

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|)$$

for some class $\mathcal{K}_\infty$ functions $\bar{\alpha}(\cdot)$ and $\underline{\alpha}(\cdot)$, such that along the trajectories of (2.4)

$$\frac{dV(x)}{dt} \leq -\alpha(x), \quad \forall\, x \in \mathbb{R}^n, \quad \forall\, \mu \in \Sigma \tag{2.5}$$

for some p.d. function $\alpha(x)$, then $x = 0$ is globally asymptotically stable. ∎

2.1.2 ISS and iISS

In this section, we review some basics of the iISS, iISS Lyapunov function and the issue of changing supply functions.

Consider a controlled nonlinear system described by

$$\dot{x} = f(x, u, \mu), \quad x(0) = x_0 \tag{2.6}$$

where $x \in \mathbb{R}^n$ is the system state with an initial state x_0, $u \in \mathbb{R}^m$ is the control input, $\mu(t) \in \Sigma$ is a piecewise continuous disturbance varying in a compact set $\Sigma \subset \mathbb{R}^{n_\mu}$, and f is locally Lipschitz in x and u.

Definition 2.2 [9] (ISS) The x system (2.6) is robust ISS stable with state x and input u with respect to Σ, if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function γ such that for any $x_0 \in \mathbb{R}^n$ and any trajectory $x(t) = x(t, x_0)$

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma\Big(\sup_{0<\tau<t} \|u(\tau)\|\Big) \tag{2.7}$$

where $\gamma(\cdot)$ is called the ISS gain function. ∎

Definition 2.3 [81] (iISS) The x system (2.6) is robust iISS with state x and input u with respect to Σ, if there exist a class $\mathcal{KL}$ function $\beta_1(\cdot,\cdot)$ and class $\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\gamma_1(\cdot)$, such that for any $x_0 \in \mathbb{R}^n$ and any trajectory $x(t) = x(t, x_0)$

$$\alpha_1(\|x(t)\|) \leq \beta_1(\|x_0\|, t) + \int_0^t \gamma_1(\|u(s)\|)ds \tag{2.8}$$

where $\gamma_1(\cdot)$ is called the iISS gain function. ∎

Remark 2.1 The relationships between ISS, iISS and BIBS are shown as follows.

$$\text{ISS} \Rightarrow \text{iISS}, \quad \text{ISS} \Rightarrow \text{BIBS}, \quad \text{iISS} \nRightarrow \text{BIBS}$$

Notice that for an iISS stable system, the bounded input may make the system state unbounded.

The concepts of ISS and iISS are also closely related to the concept of 0-GAS. System (2.6) is called 0-GAS if $x = 0$ of the system

$$\dot{x} = f(x, 0, \mu)$$

is globally asymptotically stable. Known from (2.7) and (2.8), we have the following

$$\text{ISS} \Rightarrow \text{0-GAS}, \quad \text{iISS} \Rightarrow \text{0-GAS}$$

That is to say, each ISS or iISS system is 0-GAS. ∎

Definition 2.4 [1] (iISS Lyapunov function) A C^1 function $V_o(x)$ is called a robust iISS Lyapunov function with respect to Σ for system (2.6), if it satisfies $\underline{\alpha}_o(\|x\|) \leq V_o(x) \leq \bar{\alpha}_o(\|x\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_o(\cdot)$ and $\bar{\alpha}_o(\cdot)$, and for any $\mu \in \Sigma$, along the trajectory of system (2.6)

$$\frac{\partial V_o}{\partial x}(x) \cdot F(x, u, \mu) \leq -\alpha_o(\|x\|) + \delta_o \gamma_o(u) \tag{2.9}$$

where both $\alpha_o(\cdot)$ and $\gamma_o(\cdot)$ are p.d. functions, and δ_o is an unknown positive constant. ∎

Proposition 2.1 System (2.6) is robust iISS stable with state x and input u w.r.t Σ, if and only if it has a robust iISS Lyapunov function $V_o(x)$ w.r.t Σ. ∎

The proof of Proposition 2.1 is omitted and it can be easily modified from [1].

Remark 2.2 In (2.9), $\gamma_o(\cdot)$ is also an iISS gain function for system (2.6) [1], and we call (α_o, γ_o) the iISS supply pair for system (2.6).

A specific property of system (2.6) having an iISS Lyapunov function satisfying (2.9) is that, if $\gamma_o(\cdot)$ is integrable over $[0, \infty)$, i.e. $\int_0^\infty \gamma_o(u(s))ds$ exists, then $x(t)$ will approach zero as $t \to +\infty$ [81]. ∎

Example 2.1 Consider the bilinear system described by

$$\dot{x} = -x + xu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \tag{2.10}$$

System (2.10) is not BIBS, so it is not ISS. But, it can be verified that (2.10) is iISS with state x and input u. In fact, choose $V(x) = \frac{1}{2}\ln(1 + x^\top x)$ which satisfies along the trajectory of (2.10)

$$\dot{V} = \frac{-x^\top x + x^\top xu}{1 + x^\top x} \leq -\frac{\|x\|^2}{1 + \|x\|^2} + u$$

Known from Proposition 2.1, it is iISS with state x and input u but clearly not BIBS. ∎

2.1.3 Stability of cascade-connected nonlinear systems

Consider the interconnected nonlinear system taking the following form

$$\begin{aligned} \dot{x}_1 &= \varphi_1(x_1, u, \mu) \\ \dot{x}_2 &= Ax_2 + \varphi_2(x_1, u, \mu) \end{aligned} \tag{2.11}$$

where $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is the system state, $u \in \mathbb{R}^m$ is the input, and $\mu \in \Sigma$ with $\Sigma \subset \mathbb{R}^m$ being a compact set. Both φ_1 and φ_2 are smooth functions satisfying $\varphi_1(0,0,\mu) = 0$, $\varphi_2(0,0,\mu) = 0$ for all $\mu \in \Sigma$.

Proposition 2.2 If x_1 subsystem of (2.11) is iISS stable with state x_1 and input u, then (x_1, x_2) system (2.11) is iISS stable with state (x_1, x_2) and input u. ∎

Proof: For x_1 subsystem, by using the definition of iISS and inequality (2.8), we have

$$\begin{aligned} \|x_1(t)\| &\leq \sup_{0\leq\tau\leq t} \|x_1(\tau)\| \\ &\leq \alpha_1^{-1}\Big(\beta_1(\|x_1(0)\|, 0) + \int_0^t \gamma_1(\|u(s)\|)ds\Big) \\ &\leq \alpha_1^{-1} \circ 2\beta_1(\|x_1(0)\|, 0) + \alpha_1^{-1}\Big(\int_0^t 2\gamma_1(\|u(s)\|)ds\Big) \end{aligned} \tag{2.12}$$

Since A is Hurwitz, we get that x_2 subsystem is ISS stable with state x_2 and input (x_1, u) and satisfies

$$\begin{aligned} \|x_2(t)\| \leq\; & \beta_2(\|x_2(0)\|, t) + \gamma_{21}\Big(\sup_{0\leq\tau\leq t} \|x_1(\tau)\|\Big) \\ & + \gamma_{22}\Big(\sup_{0\leq\tau\leq t} \|u(\tau)\|\Big) \end{aligned} \tag{2.13}$$

for some class $\mathcal{KL}$ function $\beta_2(\cdot,\cdot)$ and class $\mathcal{K}_\infty$ functions $\gamma_{21}(\cdot)$ and $\gamma_{22}(\cdot)$.

Substituting (2.12) into (2.13) gives

$$\begin{aligned} \|x_2(t)\| \leq\; & \beta_2(\|x_2(0)\|, 0) + \gamma_{22}\Big(\sup_{0\leq\tau\leq t} \|u(\tau)\|\Big) + \gamma_{21} \circ 2\alpha_1^{-1} \circ 2\beta_1(\|x_1(0)\|, 0) \\ & + \gamma_{21} \circ 2\alpha_1^{-1}\Big(\int_0^t 2\gamma_1(\|u(s)\|)ds\Big) \end{aligned} \tag{2.14}$$

From (2.12) and (2.14), we have

$$\begin{aligned} \|x_1(t)\| + \|x_2(t)\| \leq\; & \alpha_0(\|x_1(0), x_2(0)\|) + \gamma_{22}\Big(\sup_{0\leq\tau\leq t} \|u(\tau)\|\Big) \\ & + \gamma_{02}\Big(\int_0^t \gamma_1(\|u(s)\|)ds\Big) \end{aligned} \tag{2.15}$$

for some suitable functions $\alpha_0(\cdot)$ and $\gamma_{02}(\cdot)$

Known from estimates (2.12) and (2.13), (x_1, x_2) system is 0-GAS, i.e. the equilibrium of (x_1, x_2) system (2.11) with $u \equiv 0$ is globally asymptotically stable. This in conjunction with estimate (2.15) implies system (x_1, x_2) system (2.11) is iISS stable with state (x_1, x_2) and input u according to Proposition 3.2 of [2]. The proof is completed.

Remark 2.3 To our knowledge, it is now not clear about how to construct an iISS Lyapunov function and its supply pair for (x_1, x_2) system (2.11) by using the property shown in Proposition 2.2 directly. In order to obtain the iISS Lyapunov function of this case, we need one more condition. The result will be shown later by Lemma 5.1 on a changing iISS supply function technique. ∎

The following lemma is on a property of two interconnected ISS systems. It will be used several times in the subsequent chapters.

Lemma 2.1 Consider system (2.11). Assume, given any compact subset $\Sigma \subset \mathbb{R}^{n_\mu}$, there exists a C^1 function $V_{z_1}(z_1)$ satisfying $\underline{\alpha}_{z_1}(\|z_1\|) \leq V_{z_1}(z_1) \leq \bar{\alpha}_{z_1}(\|z_1\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{z_1}(\cdot)$ and $\bar{\alpha}_{z_1}(\cdot)$ such that, for any $\mu(t) \in \Sigma$, along the trajectory of system $\dot{z}_1 = \varphi_1(z_1, u, \mu)$

$$\dot{V}_{z_1} \leq -\alpha_{z_1}(\|z_1\|) + \delta_u \gamma_u(u) \tag{2.16}$$

for some class $\mathcal{K}_\infty$ function $\alpha_{z_1}(\cdot)$ satisfying $\lim_{s\to 0^+} \sup(\alpha_{z_1}^{-1}(s^2)/s) < \infty$, some positive number δ_u, and some known smooth p.d. function $\gamma_u(\cdot)$.

Then there exists a C^1 function $V_z(z_1, z_2)$ satisfying

$$\underline{\alpha}_z(\|z_1, z_2\|) \leq V_z(z_1, z_2) \leq \bar{\alpha}_z(\|z_1, z_2\|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_z(\cdot)$ and $\bar{\alpha}_z(\cdot)$ such that, for any $\mu(t) \in \Sigma$, along the trajectory of system (2.11),

$$\dot{V}_z \leq -\|z_1\|^2 - \|z_2\|^2 + \tilde{\delta}_u \tilde{\gamma}_u(u) \tag{2.17}$$

where $\tilde{\delta}_u$ is some positive number and $\tilde{\gamma}_u(\cdot)$ is some known smooth p.d. function. ∎

Remark 2.4 A class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ satisfying the condition $\lim_{s\to 0^+} \sup(\alpha^{-1}(s^2)/s) < \infty$ is said to be *locally quadratic*. The assumption (2.17) implies that the subsystem $\dot{z}_1 = \varphi_1(z_1, u, \mu(t))$ is ISS with state z_1 and input u, and the equilibrium $z_1 = 0$ of the system $\dot{z}_1 = \varphi_1(z_1, 0, \mu(t))$ is locally exponentially stable for any $\mu \in \Sigma$ if $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$ are also locally quadratic.

Under the condition (2.17), by applying the changing ISS supply functions technique in [80], for any smooth function $\Delta(z_1) > 0$, there exists a C^1 function $\bar{V}_{z_1}(z_1)$ satisfying

$$\underline{\alpha}_{1z_1}(\|z_1\|) \leq \bar{V}_{z_1}(z_1) \leq \bar{\alpha}_{1z_1}(\|z_1\|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{1z_1}(\cdot)$ and $\bar{\alpha}_{1z_1}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of system $\dot{z}_1 = \varphi_1(z_1, u, \mu)$

$$\dot{\bar{V}}_{z_1} \leq -\Delta(z_1)\|z_1\|^2 + \bar{\delta}_u \bar{\gamma}_u(u)u^2 \tag{2.18}$$

where constant $\bar{\delta}_u > 0$ and $\bar{\gamma}_u(\cdot)$ is some known smooth positive function. ∎

Proof of Lemma 2.1: Since $\varphi_2(z_1, u, \mu)$ is smooth and satisfies $\varphi_2(0, 0, \mu) = 0$, by Lemma 2.4 shown later, there exist some real constant $c > 0$, smooth positive functions $\psi_1(z_1)$ and $\psi_2(u)$ such that, for all $z_1 \in \mathbb{R}^{n_1}$, $u \in \mathbb{R}$, and $\mu \in \Sigma$,

$$\|\varphi_2(z_1, u, \mu)\| \leq c\Big(\psi_1(z_1)\|z_1\| + \psi_2(u)|u|\Big) \tag{2.19}$$

Next, by Remark 2.4, given any smooth function $\Delta(z_1) > 0$, there exists a C^1 function $\bar{V}_{z_1}(z_1)$ satisfying

$$\underline{\alpha}_{1z_1}(\|z_1\|) \leq \bar{V}_{z_1}(z_1) \leq \bar{\alpha}_{1z_1}(\|z_1\|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{1z_1}(\cdot)$ and $\bar{\alpha}_{1z_1}(\cdot)$ such that, for any $\mu(t) \in \Sigma$, along the trajectory of system $\dot{z}_1 = \varphi_1(z_1, u, \mu)$,

$$\dot{\bar{V}}_{z_1} \leq -\Delta(z_1)\|z_1\|^2 + \bar{\delta}_u \bar{\gamma}_u(u)u^2 \tag{2.20}$$

for some positive number $\bar{\delta}_u$ and some known smooth positive function $\bar{\gamma}_u(\cdot)$.

Let

$$V_z(z_1, z_2) = l\bar{V}_{z_1}(z_1) + 2z_2^\top P z_2$$

where l is some positive number to be specified later, and P is the positive definite solution of the Lyapunov equation

$$PA + A^\top P = -I_{n_2} \tag{2.21}$$

It can be seen that $V_z(z_1, z_2)$ is C^1 satisfying

$$\underline{\alpha}_z(\|z_1, z_2\|) \leq V_z(z_1, z_2) \leq \bar{\alpha}_z(\|z_1, z_2\|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_z(\cdot)$ and $\bar{\alpha}_z(\cdot)$. Using (2.20) and (2.21), $V_z(z_1, z_2)$ satisfies, for any $\mu(t) \in \Sigma$, along the trajectory of system (2.11)

$$\begin{aligned}\dot{V}_z &\leq -l\Delta(z_1)\|z_1\|^2 + l\bar{\delta}_u\bar{\gamma}_u(u)u^2 - 2\|z_2\|^2 + 4Pz_2\varphi_2(z_1, u, \mu)\\ &\leq -\left(l\Delta(z_1) - 8c^2\psi_1^2(z_1)\|P\|^2\right)\|z_1\|^2 - \|z_2\|^2\\ &\quad + \Big(l\bar{\delta}_u\bar{\gamma}_u(u) + 8c^2\psi_2^2(u)\|P\|^2\Big)u^2\end{aligned}$$

Letting

$$\begin{aligned}\Delta(z_1) &\geq 1 + \psi_1^2(z_1)\\ \tilde{\gamma}_u(u) &\geq \bar{\gamma}_u(u) + \psi_2^2(u)\end{aligned}$$

and

$$\begin{aligned} l &\geq \max\left\{1, 8c^2 \|P\|^2\right\} \\ \tilde{\delta}_u &\geq \max\left\{l\bar{\delta}_u, 8c^2 \|P\|^2\right\} \end{aligned}$$

yields (2.17). Thus the proof is completed.

2.2 Stabilization of nonlinear systems

2.2.1 Control Lyapunov function

Consider the controlled system described by

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{2.22}$$

where $f(0,0) = 0$, and $f(x, u)$ is a smooth function of x and u.

Definition 2.5 [21] (CLF) A C^1 p.d. function $V(x)$ is a control Lyapunov function (CLF) of (2.22) with respect to a controller $u = \phi(x)$ vanishing at $x = 0$ if

$$\dot{V}(x)\Big|_{u=\phi(x)} = \frac{\partial V(x)}{\partial x}(x) \cdot f(x, \phi(x)) < 0, \qquad \forall\, x \neq 0 \tag{2.23}$$

∎

The above standard CLF definition can be easily modified to robust CLF defined for uncertain nonlinear systems such as (2.11). The existence of CLF implies the equilibrium of the system can be made globally asymptotically stable by a static feedback controller. Roughly speaking, the stabilization design methods in the thesis is essentially based on the CLF method. It should be noted that the above defined CLF is with respect to a full state feedback controller, which is generally not necessary. As we will shown in the following chapters, for certain stabilization problem, the partial state feedback controller is sufficient to achieve the above condition (2.23).

2.2.2 Global stabilization by recursive design

By virtue of the above defined CLF function, we have a review of the basic backstepping design of nonlinear systems in triangular form.

Consider the nonlinear system described by

$$\begin{aligned}
\dot{z} &= f(z) + g(z)x_1 \\
\dot{x}_1 &= f_1(z, x_1) + g_1(z, x_1)x_2 \\
\dot{x}_2 &= f_2(z, x_1, x_2) + g_2(z, x_1, x_2)x_3 \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(z, x_1, ..., x_{n-1}) + g_{n-1}(z, x_1, ..., x_{n-1})x_n \\
\dot{x}_n &= f_n(z, x_1, ..., x_n) + g_n(z, x_1, ..., x_n)u
\end{aligned} \tag{2.24}$$

where $z \in \mathbb{R}^n$, $x_i \in \mathbb{R}$, $i = 1, \cdots, n$, and $u \in \mathbb{R}$. All the functions in (2.24) are supposed to be globally defined and smooth in their arguments, satisfying $f_i(0, \cdots, 0) = 0$ and $g_i(z, x_1, \cdots, x_i) \neq 0$ for any $(z, x_1, \cdots, x_i) \in \mathbb{R}^{n+i}$, $i = 1, \cdots, n$.

It can be seen that (2.24) can be put in the standard form

$$\dot{\chi} = f(\chi) + g(\chi)u \tag{2.25}$$

with $\chi = \text{col}(z, x_1, \cdots, x_n)$. The basic idea of backstepping is to derive a state feedback controller $u = \phi(\chi)$ such that the equilibrium $\chi = 0$ of (2.25) can be made globally asymptotically stable. By virtue of CLF of Definition 2.5, under certain conditions, we can obtained such a controller recursively.

We begin with a simple nonlinear system (2.26) and perform the control design procedure by using the CLF method.

Lemma 2.2 [51] Consider the following nonlinear system

$$\begin{aligned}
\dot{z} &= f(z) + g(z)x_1 \\
\dot{x}_1 &= u_1
\end{aligned} \tag{2.26}$$

where $z \in \mathbb{R}^n$, $x_1 \in \mathbb{R}$, $u_1 \in \mathbb{R}$, $f(0) = 0$, and the functions $f(z)$ and $g(z)$ are smooth.

If z subsystem (2.26) has a CLF $V_z(z)$ with respect to state feedback $x_1 = \phi(z)$, then the system described by (2.26) has a CLF

$$V_a(z, x_1) = V_z(z) + \frac{1}{2}(x_1 - \phi(z))^2$$

with respect to state feedback

$$\begin{aligned}
u_1 &= \phi_a(z, x_1) \\
&= \frac{\partial \phi(z)}{\partial z}[f(z) + g(z)x_1] - \frac{\partial V_z(z)}{\partial z}g(z) - k(x_1 - \phi(z))
\end{aligned} \tag{2.27}$$

where $k > 0$.

Thus the equilibrium point of the closed-loop system

$$\begin{aligned} \dot{z} &= f(z) + g(z)x_1 \\ \dot{x}_1 &= \phi_a(z, x_1) \end{aligned} \tag{2.28}$$

is AS. Moreover, if $V_z(z)$ is radially unbounded, so is $V_a(z, x_1)$. Thus, the equilibrium point of (2.28) is GAS. ∎

Using Lemma 2.2, the backstepping design for system (2.24) can be summarized as the following steps.

Initial Step: If $n = 1$ in (2.24), that is $u = x_2$

$$\begin{aligned} \dot{z} &= f(z) + g(z)x_1 \\ \dot{x}_1 &= f_1(z, x_1) + g_1(z, x_1)u \end{aligned} \tag{2.29}$$

by using Lemma 2.2, we have the following result.

Using the input transformation

$$u = \frac{1}{g_1(z, x_1)}[u_a - f_1(z, x_1)]$$

gives

$$\begin{aligned} \dot{z} &= f(z) + g(z)x_1 \\ \dot{x}_1 &= u_a \end{aligned} \tag{2.30}$$

Therefore, by Lemma 2.2, if z subsystem of (2.30) has a CLF $V(z)$ with respect to state feedback $x_1 = \phi(z)$, then (2.30) has a CLF

$$V_a(z, x_1) = V(z) + \frac{1}{2}(x_1 - \phi(z))^2$$

with respect to

$$\phi_a(z, x_1) = \frac{\partial \phi(z)}{\partial z}[f(z) + g(z)x_1] - \frac{\partial V(z)}{\partial z}g(z) - k(x_1 - \phi(z))$$

where $k > 0$. Therefore, (2.30) has the same CLF with respect to

$$u = \frac{1}{g_1(z, x_1)}[\phi_a(z, x_1) - f_1(z, x_1)] \tag{2.31}$$

Mathematical Induction: Suppose when $n = i$, the following system

$$\begin{aligned} \dot{z} &= f(z) + g(z)x_1 \\ \dot{x}_1 &= f_1(z, x_1) + g_1(z, x_1)x_2 \\ &\vdots \\ \dot{x}_i &= f_i(z, x_1, ..., x_i) + g_i(z, x_1, ..., x_i)x_{i+1} \end{aligned} \tag{2.32}$$

has a CLF $V_i(z, x_1, \cdots, x_i)$ with respect to a state feedback controller $u_i = \phi_i(z, x_1, \cdots, x_i)$.

Consider the system described by

$$\begin{aligned}
\dot{z} &= f(z) + g(z)x_1 \\
\dot{x}_1 &= f_1(z, x_1) + g_1(z, x_1)x_2 \\
&\vdots \\
\dot{x}_i &= f_i(z, x_1, ..., x_i) + g_i(z, x_1, ..., x_i)x_{i+1} \\
\dot{x}_{i+1} &= f_{i+1}(z, x_1, ..., x_{i+1}) + g_{i+1}(z, x_1, ..., x_{i+1})x_{i+2}
\end{aligned} \tag{2.33}$$

which can be put into the following form

$$\begin{aligned}
\dot{\zeta} &= \bar{f}(\zeta) + \bar{g}(\zeta)x_{i+1} \\
\dot{x}_{i+1} &= \bar{f}_{i+1}(\zeta, x_{i+1}) + \bar{g}_{i+1}(\zeta, x_{i+1})x_{i+2}
\end{aligned} \tag{2.34}$$

Using Lemma 2.2 again, similar with the method in the initial step, we conclude that there exists a CLF with respect to a controller $u_{i+1} = \phi_i(\zeta, x_{i+1})$.

Therefore, as a result of mathematical induction, the global stabilization problem of (2.24) can be solved by a recursive design procedure.

Remark 2.5 The above global stabilization design is for nonlinear systems without uncertainties and based on a static full-state feedback controller. In general, the nonlinear systems may involve both parameter uncertainty and unmodeled dynamic uncertainty. So the design method can not be applied directly in many cases. The stabilization problems in the thesis are involved with both static uncertainty and dynamics uncertainty. Therefore, we need to find some robust, dynamic partial state/output feedback controller to solve them. ∎

2.3 Output regulation of nonlinear systems

In this section, we review some basics of the nonlinear output regulation. For a specific class of nonlinear systems, we first formulate the problem of nonlinear output regulation and review the general framework for handling nonlinear output regulation [31]. The plant is an output feedback nonlinear system in simple case of relative degree one described by (2.35).

2.3.1 Problem description

Consider the single-input single-output nonlinear system described by

$$\begin{aligned}
\dot{z} &= f(z, y, v, w) \\
\dot{y} &= g(z, y, v, w) + u \\
e &= y - q(v, w)
\end{aligned} \tag{2.35}$$

with an exosystem described by

$$\dot{v} = A_1(\sigma)v, \quad v(0) = v_0 \tag{2.36}$$

where $(z, y) \in \mathbb{R}^n \times \mathbb{R}$ is the system state, $e(t) \in \mathbb{R}$ is the tracking error to be regulated, and $u(t) \in \mathbb{R}$ is the control input to be designed. w, σ are the parameter vectors of the plant and exosystem which may be uncertain and for each $\sigma \in \mathbb{R}^{n_\sigma}$, all the eigenvalues of $A_1(\sigma)$ are distinct with zero real parts. All the functions in (2.35) and (2.36) are supposed to be sufficiently smooth and satisfying $f(0,0,0,w) = 0$, $g(0,0,0,w) = 0$ and $q(0,w) = 0$ for all $w \in \mathbb{R}^{n_w}$.

The output regulation problem can be formulated as follows.

Problem 2.1 (Global Output Regulation) The global output regulation problem is to find a control law in the form

$$\begin{aligned} u &= u_K(\xi, e) \\ \dot{\xi} &= g_K(\xi, e) \end{aligned} \tag{2.37}$$

where $\xi \in \mathbb{R}^{n_\xi}$ is the controller state and u_K, g_K are smooth functions, such that the closed-loop system composed of (2.35), (2.36) and (2.37) has the properties that, for any initial state and any system uncertainty

i) the trajectory of the closed-loop system exists and is bounded over $[0, +\infty)$, and

ii) the tracking error $e(t)$ asymptotically decays to zero as time t tends to infinity.

∎

The controller (2.37) solving Problem 2.1 is called a dynamic error output regulator. It has been known that the solvability of the regulator equations of the plant (2.35) and exosystem (2.36) is a necessary condition [31] [38]. To fulfill this condition, we need the following standard assumption.

Assumption 2.1 (Solvability of Regulator Equations) There exists a solution composed of $\mathbf{z}(v,w,\sigma)$, $q(v,w)$ and $\mathbf{u}(v,w,\sigma)$ satisfying the following regulator equations

$$\begin{aligned} \frac{d\mathbf{z}(v,w,\sigma)}{dt} &= f(\mathbf{z}(v,w,\sigma), q(v,w), v, w) \\ 0 &= g(\mathbf{z}(v,w,\sigma), q(v,w), v, w) + \mathbf{u}(v,w,\sigma) \end{aligned}$$

for all $(v, w, \sigma) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$. ∎

Existence of the output zeroing manifolds [31]

$$\{\mathbf{z}(v,w,\sigma), \quad q(v,w), \quad \mathbf{u}(v,w,\sigma)\}$$

is necessary to achieve the solvability of the output regulation problem. In particular, the zeroing output manifold $\mathbf{u}(v, w, \sigma)$ provides the necessary feedforward control information to the controller. Nevertheless, $\mathbf{u}(v, w, \sigma)$ cannot be directly used for designing feedback control law as it depends on the exogenous signal v and unknown parameter (w, σ). We need to design a dynamic compensator called internal model which can asymptotically provide the information of $\mathbf{u}(v, w, \sigma)$.

2.3.2 Tackling output regulation via problem conversion

To achieve the problem conversion, we need the following definitions of steady-state generator and internal model.

Definition 2.6 (Steady-State Generator)[32] Under Assumption 2.1, the plant (2.35) and exosystem (2.36) is said to have a steady-state generator with output u if there exists a triple $\{\theta, \alpha, \beta\}$ with $\theta : \mathbb{R}^{n_v+n_w+n_\sigma} \mapsto \mathbb{R}^{n_s}$, $\alpha : \mathbb{R}^{n_s} \mapsto \mathbb{R}^{n_s}$ and $\beta : \mathbb{R}^{n_s} \mapsto \mathbb{R}$, all vanishing at the origin, satisfying

$$\begin{aligned} \frac{d\theta(v, w, \sigma)}{dt} &= \alpha(\theta(v, w, \sigma)) \\ \mathbf{u}(v, w, \sigma) &= \beta(\theta(v, w, \sigma)) \end{aligned}$$

for all $(v, w, \sigma) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$. ∎

Definition 2.7 (Internal Model)[32] Suppose Assumption 2.1 is satisfied and there exists a steady-state generator with output u for the plant (2.35) and exosystem (2.36). The following dynamics is called internal model

$$\dot{\eta} = \gamma(\eta, u, e) \tag{2.38}$$

if function γ vanishes at the origin and satisfies

$$\gamma(\theta(v, w, \sigma), \mathbf{u}(v, w, \sigma), 0) = \alpha(\theta(v, w, \sigma))$$

for all $(v, w, \sigma) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$. ∎

It has been shown in [32] that the steady-state generator will lead to some well defined internal model. Note that the internal model is not unique. Therefore, design of the internal model in some suitable form is a key problem to achieve the solvability of the output regulation problem.

Using the internal model in the general form (2.38), we next illustrate the procedure of problem conversion. The *augmented system* is described by

$$\begin{aligned} \dot{z} &= f(z, y, v, w) \\ \dot{\eta} &= \gamma(\eta, u, e) \\ \dot{y} &= g(z, y, v, w) + u \end{aligned} \tag{2.39}$$

To formulate our stabilization problem, by performing the following coordinate transformations

$$\begin{aligned}\bar{z} &= z - \mathbf{z}(v, w, \sigma) \\ \bar{\eta} &= \eta - \theta(v, w, \sigma)\end{aligned}$$

we get the transformed augmented system described by

$$\begin{aligned}\dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\ \dot{\bar{\eta}} &= \bar{\gamma}(\bar{\eta}, e, u, v, w, \sigma) \\ \dot{e} &= \bar{g}(\bar{z}, e, v, w) - \beta(\bar{\eta} + \theta) + u \end{aligned} \tag{2.40}$$

where

$$\begin{aligned}\bar{f}(\bar{z}, e, v, w, \sigma) &= f(\bar{z} + \mathbf{z}(v, w, \sigma), e + q(v, w), v, w) - f(\mathbf{z}(v, w, \sigma), q(v, w), v, w) \\ \bar{\gamma}(\bar{\eta}, e, u, v, w, \sigma) &= \gamma(\bar{\eta} + \theta(v, w, \sigma), u, e) \\ \bar{g}(\bar{z}, e, v, w, \sigma) &= g(\bar{z} + \mathbf{z}(v, w, \sigma), e + q(v, w), v, w) - g(\mathbf{z}(v, w, \sigma), q(v, w), v, w)\end{aligned}$$

Notice that the state $\bar{\eta}$ is not available for feedback design any longer. The stabilization problem of the augmented system (2.40) can be formulated as follows.

Problem 2.2 (Global Robust Stabilization) Find a control law of the following form

$$\begin{aligned}u &= u_k(\zeta, e, \eta) \\ \dot{\zeta} &= g_k(\zeta, e, \eta)\end{aligned} \tag{2.41}$$

where u_k and g_k are globally defined and continuous such that, for any initial state and all $(v, w, \sigma) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$, the solution of the closed-loop system composed of (2.40) and (2.41) exists and is bounded over $[0, \infty)$, and

$$\lim_{t \to \infty} \|\bar{z}\| + \|\bar{\eta}\| + |e| = 0$$

That is to say, the control objective is to make the point $(\bar{z}, \bar{\eta}, e) = 0$ globally asymptotically stable. ∎

It is obvious that if Problem 2.2 is solvable, we can get a solution of Problem 2.1. Thus, the problem conversion is achieved.

2.4 Some useful lemmas

In this section, we summarize some useful lemmas that will be referred to frequently throughout Chapters 3 to 5.

Lemma 2.3 (Barbalat' Lemma)[51] Suppose $f(t)$ is continuously differentiable for $t \geq t_0$ for some t_0, $f(t)$ has a finite limit as $t \to \infty$, and $\dot{f}(t)$ is uniformly continuous. Then $\dot{f}(t) \to 0$ as $t \to \infty$. ∎

Corollary 2.1 If the differentiate function $f(t)$ has a finite limit as $t \to \infty$, and is such that $\ddot{f}$ exists and is bounded, then $\dot{f}(t) \to 0$ as $t \to \infty$. ∎

Corollary 2.2 If a function $x : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is uniformly continuous, and there exists a p.d. quadratic function $V(x)$ such that

$$\int_{t_0}^{\infty} V(x(t))dt < \infty$$

then $x(t)$ tends to zero as $t \to \infty$. ∎

The following theorem can be proved by using Lemma 2.3.

Theorem 2.4 (LaSalle-Yoshizawa)[51] Consider system (2.1) and suppose f is locally Lipschitz in x uniformly in t. Let $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+$ be a C^1 function such that, for some class $\mathcal{K}_\infty$ functions $\bar{\alpha}(\cdot)$, $\underline{\alpha}(\cdot)$, and a continuous nonnegative function $\alpha(x) \geq 0$

i) $\bar{\alpha}(|x|) \leq V(x,t) \leq \underline{\alpha}(|x|)$

ii) $\dot{V}(x,t) \overset{def}{=} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -\alpha(x), \quad \forall\, x \in \mathbb{R}^n, \quad \forall\, t \geq t_0$

Then all solutions of (2.1) are globally uniformly bounded and satisfies

$$\lim_{t \to \infty} \alpha(x(t)) = 0$$

Moreover, if in addition $\alpha(x)$ is p.d., then the equilibrium $x = 0$ is globally uniformly asymptotically stable. ∎

The following lemma is adapted from [31] which will be used several times to derive the dominating functions for an uncertain function.

Lemma 2.4 (Dominating Functions' Inequality) For each $i = 1, \cdots, n$, let $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_\mu} \mapsto \mathbb{R}$ be a C^1 function satisfying $f_i(0,0,\mu) = 0$ for all $\mu \in \Sigma$ with Σ being a compact set and

$$f(x_1, x_2, \mu) = \begin{bmatrix} f_1(x_1, x_2, \mu) \\ \vdots \\ f_n(x_1, x_2, \mu) \end{bmatrix}$$

Then there exist smooth p.d. functions $\phi_1 : \mathbb{R}^{n_1} \mapsto \mathbb{R}^+$ and $\phi_2 : \mathbb{R}^{n_2} \mapsto \mathbb{R}^+$ such that

$$\|f(x_1, x_2, \mu)\| \leq \phi_1(x_1) + \phi_2(x_2)$$

for all $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and $\mu \in \Sigma$. ∎

The Nussbaum function, for instances, $\mathcal{N}(s) = \exp(s^2)\cos(\pi s/2)$ or $\exp(s^2)\cos(s)$, was first introduced in [70]. We will use the Nussbaum function to deal with the stabilization problems in the case of unknown control directions. The following lemma collected from [58] will be used in Chapters 3 and 5.

Lemma 2.5 [58] Let $U(t) \geq 0$ and $k(t)$ be smooth functions defined on $[0, T)$, $\mathcal{N}(\cdot)$ be an even smooth Nussbaum-type function, and b be a nonzero constant. If it holds that

$$U(t) \leq \int_0^t \Big(b\mathcal{N}(k(\tau)) + 1\Big)\dot{k}(\tau)d\tau + p, \quad t \in [0, T)$$

where p is some constant, then $U(t)$, $k(t)$ and $\int_0^t \Big(b\mathcal{N}(k(\tau)) + 1\Big)\dot{k}(\tau)d\tau$ are all bounded over $[0, T)$. ∎

The following result will used to analyze the parameter convergence issue of the adaptive output regulation problem studied in Chapter 4.

Definition 2.8 A bounded piecewise continuous function $f : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is said to be persistence exciting (PE) if there exist positive constants ϵ, t_0, T_0 such that, for any unit row vector $\mathbf{c}$ of dimension n, and any $t \geq t_0$,

$$\frac{1}{T_0}\int_t^{t+T_0} |\mathbf{c}f(s)|ds \geq \epsilon$$

Or equivalently,

$$\frac{1}{T_0}\int_t^{t+T_0} f(s)f^{\mathsf{T}}(s)ds \geq \epsilon I$$

∎

Lemma 2.6 [61] Let $g : \mathbb{R}^+ \mapsto \mathbb{R}^n$ be a C^1 function and $f : \mathbb{R}^+ \mapsto \mathbb{R}^n$ is PE. Then

$$\lim_{t\to\infty} g(t) = 0 \tag{2.42}$$

if

$$\lim_{t\to\infty} \dot{g}(t) = 0 \tag{2.43}$$

and

$$\lim_{t\to\infty} g^{\mathsf{T}}(t)f(t) = 0 \tag{2.44}$$

∎

Remark 2.6 This lemma gives the convergence condition of the function $g(t)$ to the origin based on two asymptotic properties (2.43) and (2.44) of $g(t)$ and the PE condition of $f(t)$. The result is of interest in that it does not assume that $g(t)$ has to be governed by some linear differential equation as assumed in the literature of adaptive control of linear systems. Thus, it also applies to adaptive control of nonlinear systems. It should be noted that the fact that $\lim_{t\to\infty} \dot{g}(t) = 0$ alone plus the PE condition of f does not necessarily imply the existence of $\lim_{t\to\infty} g(t)$. ∎

□ **End of chapter.**

Chapter 3

Global output regulation using output feedback control

In this chapter, we address the global output regulation problems for output feedback systems (3.1) and (3.29) with unknown control directions and ISS inverse dynamics using output feedback control. The class of output feedback systems described by (3.1) is in a simple form and has a relative degree equaling one. We first present the solution for this special case in Section 3.1. In the general case, the class of output feedback systems described by (3.29) has a relative degree larger than one and the problem is then considered in Section 3.2. For the general case, to achieve the output feedback control, we need to introduce some type of partial state observer.

3.1 Special case

In this section, we study the global robust output regulation problem for an output feedback systems described by (3.1) and an exosystem described by (3.2). This problem aims at designing a feedback controller for an uncertain nonlinear plant such that the output of the closed-loop system will asymptotically track a class of reference inputs and rejects a class of disturbances. Here both the reference inputs and disturbances are generated by a linear neutrally stable system described by (3.2). Various versions of the problem related to (3.1) have been studied for ten years or so, see [5] [16] [17] [18] [37] [58] for instances.

Some special cases of the system of the form (3.1) have been studied in [16], [58]. Compared with the result in [16], [58], the main feature of system (3.1) is that we allow the inverse dynamics $\dot{z} = f(z, y, 0, w)$ to be nonlinear both in z and y so that our result applies to a larger class of systems. It will be seen later that the Lorenz system does not belong to the class of systems studied in [16], [58]. Technically, the approach in [16], [58] cannot handle the nonlinear zero-dynamics, and we need to apply Lemma 2.1 to deal with the nonlinear zero-dynamics. On the other hand, the output regulation problem of system (3.1) with unstable zero dynamics is studied in [5]. However, our result is global as opposed to the semi-global result in [5]. Moreover, we will not assume the knowledge of the sign of the high frequency gain b.

In Section 3.1.1, we formulate the problem and introduce a set of basic assumptions on system (3.1) so that the robust output regulation problem of system (3.1) can be converted into a global robust stabilization problem of an augmented system based on the general framework as shown in Chapter 2. Section 3.1.2 presents the main result. A design example is illustrated in Section 3.1.3.

3.1.1 Problem formulation and preliminaries

Consider the class of uncertain nonlinear systems described by

$$\begin{aligned} \dot{z} &= f(z,y,v,w) \\ \dot{y} &= g(z,y,v,w) + bu \\ e &= y - q(v,w) \end{aligned} \tag{3.1}$$

where $(z,y) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $e \in \mathbb{R}$ is the error output and $u \in \mathbb{R}$ is the control input. $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$ with $\mathcal{W}$ nonempty is a constant uncertain parameter vector, and $v(t) \in \mathbb{R}^{n_v}$ represents the time-varying reference and/or disturbance. The functions f, g and q are assumed to be sufficiently smooth of their arguments satisfying

$$f(0,0,0,w) = 0, \quad g(0,0,0,w) = 0, \quad q(0,w) = 0, \quad \forall\, w \in \mathcal{W}$$

and the uncertain gain b is nonzero with an unknown sign.

It is assumed that $v(t)$ is generated by a linear exosystem

$$\dot{v} = A_1 v, \quad v(0) = v_0 \tag{3.2}$$

where A_1 is a known constant matrix with all its eigenvalues distinct with zero real parts. As a result, the general solution of the exosystem is a sum of finite sinusoidal functions with the amplitudes and phase angles depending on the initial condition v_0.

The problem of the global robust output regulation problem is precisely formulated as follows: For any given $\mathcal{W}$, design an output feedback control law of the form:

$$\begin{aligned} u &= u_K(\zeta, e) \\ \dot{\zeta} &= g_K(\zeta, e) \end{aligned} \tag{3.3}$$

where both u_K and g_K are sufficiently smooth vanishing at the origin such that, for any initial condition $(z(0), y(0), v(0), \zeta(0))$, and any constant parameter $w \in \mathcal{W}$, the solution of the closed-loop system composed of (3.1) to (3.3) exists and is bounded over $[0, +\infty)$ and the error output $e(t)$ asymptotically approaches zero as $t \to +\infty$.

The subsystem $\dot{z} = f(z, 0, 0, w)$ is called the zero dynamics of system (3.1) with v set to zero. For the special case where the zero dynamics is a linear stable system, i.e., $f(z, 0, 0, w) = H(w)z$ with $H(w)$ a Hurwitz matrix for all w, the output regulation problem of system (3.1) with relative degree greater than or equal to 1 has been studied in several papers, e.g., [16], [58]. More recently, the semi-global robust output regulation problem of system (3.1) is further studied which allows the origin $z = 0$ of the system $\dot{z} = f(z, 0, 0, w)$ to be unstable [5].

Let us first list two standard assumptions to achieve the problem conversion.

Assumption 3.1 There exists a globally defined smooth function $\mathbf{z} : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^n$ with $\mathbf{z}(0, w) = 0$ such that

$$\frac{\partial \mathbf{z}(v, w)}{\partial v} A_1 v = f(\mathbf{z}(v, w), q(v, w), v, w) \tag{3.4}$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathcal{W}$. ∎

Under Assumption 3.1, let $\mathbf{y}(v, w) = q(v, w)$ and

$$\mathbf{u}(v, w) = b^{-1}\big((\partial q(v, w)/\partial v) A_1 v - g(\mathbf{z}(v, w), q(v, w), v, w)\big) \tag{3.5}$$

Then, $\{\mathbf{z}(v, w),\ \mathbf{y}(v, w),\ \mathbf{u}(v, w)\}$ is the solution of the regulator equations associated with (3.1) and (3.2). For the existence of an suitable internal model, we make another assumption.

Assumption 3.2 $\mathbf{u}(v, w)$ is a polynomial in v with coefficients depending on w. ∎

Remark 3.1 Under Assumption 3.2, there exists an integer s such that $\mathbf{u}(v, w)$ satisfies, for all trajectories $v(t)$ of the exosystem and all $w \in \mathcal{W}$

$$\frac{d^s \mathbf{u}(v, w)}{dt^s} = a_1 \mathbf{u}(v, w) + a_2 \frac{d\mathbf{u}(v, w)}{dt} + \cdots + a_s \frac{d^{s-1}\mathbf{u}(v, w)}{dt^{(s-1)}} \tag{3.6}$$

where $a_1, a_2, \cdots, a_s$ are real scalars such that all the roots of the polynomial

$$P(\lambda) = \lambda^s - a_1 - a_2\lambda - \cdots - a_s\lambda^{s-1}$$

are distinct with zero real parts [31].

Let $\tau(v, w) = \mathrm{col}\big(\mathbf{u}, \dot{\mathbf{u}}, \cdots, \mathbf{u}^{(s-1)}\big)$, $\Phi = \left[\begin{array}{c|c} 0 & I_{s-1} \\ \hline a_1 & a_2, \cdots, a_s \end{array}\right]$ and $\Gamma = [1, 0, \cdots, 0]_{1\times s}$. Then $\tau(v, w)$, Φ and Γ satisfy the following equations:

$$\begin{aligned} \frac{\partial \tau(v, w)}{\partial v} A_1 v &= \Phi \tau(v, w) \\ \mathbf{u}(v, w) &= \Gamma \tau(v, w) \end{aligned} \tag{3.7}$$

∎

Remark 3.2 System (3.7) can be used to generate the steady-state input $\mathbf{u}(v,w)$, and thus called a steady-state generator with output u in the sense of Definition 2.6. Since (Γ, Φ) is observable and the eigenvalues of Φ have zero real parts, for any controllable pair (M, N) with $M \in \mathbb{R}^{s\times s}$ a Hurwitz matrix and $N \in \mathbb{R}^{s\times 1}$ a column vector, there is a unique nonsingular matrix T satisfying the Sylvester equation

$$T\Phi - MT = N\Gamma$$

Let $\theta(v,w) = T\tau(v,w)$ which satisfies

$$\begin{aligned} \dot{\theta}(v,w) &= (M + N\Psi)\theta(v,w) \\ \mathbf{u}(v,w) &= \Psi\theta(v,w) \end{aligned}$$

with $\Psi = \Gamma T^{-1}$. Then we can define a dynamic compensator as follows:

$$\dot{\eta} = M\eta + Nu \tag{3.8}$$

which is an internal model with output u in the sense of Definition 2.7. ∎

Attaching the internal model (3.8) to (3.1) and performing the following coordinate and input transformation:

$$\bar{z} = z - \mathbf{z}(v,w), \quad \tilde{\eta} = \eta - \theta(v,w) - Nb^{-1}e, \quad e = y - q(v,w), \quad \bar{u} = u - \Psi\eta \tag{3.9}$$

gives a system as follows:

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, \mu) \\ \dot{\tilde{\eta}} &= M\tilde{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z}, e, \mu) \\ \dot{e} &= \bar{g}(\bar{z}, e, \mu) + b\Psi\tilde{\eta} + \Psi Ne + b\bar{u} \end{aligned} \tag{3.10}$$

where $\mu = (v,w)$, $\bar{f}(\bar{z},e,\mu) = f(\bar{z} + \mathbf{z}, e + q, v, w) - f(\mathbf{z}, q, v, w)$, and $\bar{g}(\bar{z},e,\mu) = g(\bar{z} + \mathbf{z}, e + q, v, w) - g(\mathbf{z}, q, v, w)$. It can be verified that $\bar{f}(0,0,\mu) = 0$ and $\bar{g}(0,0,\mu) = 0$ for any $\mu \in \mathbb{R}^{n_v} \times \mathcal{W}$.

Let $z = \mathrm{col}(\bar{z}, \tilde{\eta})$ and $F(z,e,\mu) = \mathrm{col}\big(\bar{f}(\bar{z},e,\mu), M\tilde{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z},e,\mu)\big)$. Then system (3.10) takes the following form:

$$\begin{aligned} \dot{z} &= F(z, e, \mu) \\ \dot{e} &= \bar{g}(\bar{z}, e, \mu) + b\Psi\tilde{\eta} + \Psi Ne + b\bar{u} \end{aligned} \tag{3.11}$$

Remark 3.3 System (3.11) is the desired augmented system. The quantity $\mu(t)$ in the augmented system (3.11) can be viewed as an unknown time-varying disturbance. It can be seen that if there exists a control law of the form

$$\begin{aligned} \bar{u} &= k_\zeta(\zeta, e) \\ \dot{\zeta} &= g_\zeta(\zeta, e) \end{aligned}$$

that solves the global robust stabilization problem of system (3.11) in the sense that, for any initial condition of the closed-loop system and the exosystem, and any fixed unknown parameter $w \in \mathcal{W}$, the solution of the closed-loop system is bounded for all $t \geq 0$, and the state of the augmented system (3.11) tends to zero as $t \to +\infty$, then the following control law:

$$\begin{aligned} u &= k_\zeta(\zeta, e) + \Psi\eta \\ \dot{\eta} &= M\eta + Nu \\ \dot{\zeta} &= g_\zeta(\zeta, e) \end{aligned}$$

solves the global robust output regulation for the original system (3.1). ∎

3.1.2 Main result

In this section, we will consider the global robust stabilization problem of system (3.11) without the knowledge of the control direction, that is the controller can be independent of the sigh of parameter b. For this purpose, we need one more assumption as follows.

Assumption 3.3 For any compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W}$, there exists a C^1 function $V_{\bar{z}}$ satisfying $\underline{\alpha}(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \bar{\alpha}(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of the subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$

$$\dot{V}_{\bar{z}} \leq -\alpha(\|\bar{z}\|) + \delta\gamma(e) \tag{3.12}$$

where δ is some unknown positive constant, $\alpha(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying $\lim_{s\to 0^+} \sup(\alpha^{-1}(s^2)/s) < \infty$ and $\gamma(\cdot)$ is a known smooth p.d. function. ∎

Remark 3.4 Assumption 3.3 is modified from what was used in [42]. This assumption is slightly stronger than ISS condition of the subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$ with state $\bar{z}$ and input e.

When $v = 0$, the subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$ reduces to $\dot{z} = f(z, 0, 0, w)$ which can be viewed as the zero dynamics of system (3.1) with y as the output and v set to be zero. Thus Assumption 3.3 implies that system (3.1) is minimum-phase. ∎

Lemma 3.1 Consider system (3.11). Under Assumption 3.3, there exist a smooth positive function $\rho(e) \geq 1$ and a controller of the form

$$\begin{aligned} \bar{u} &= \mathcal{N}(k)\rho(e)e \\ \dot{k} &= \rho(e)e^2 \end{aligned} \tag{3.13}$$

such that the closed-loop system composed of system (3.11) and controller (3.13) has the property that, for any $v(t)$ generated by the exosystem (3.2), and any $w \in \mathcal{W}$, there exists a C^1 function $V(z,e)$ satisfying $\underline{\alpha}(\| z,e \|) \leq V \leq \bar{\alpha}(\| z,e \|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$, such that, along the trajectory of the closed-loop system

$$\dot{V} \leq \Big(b\mathcal{N}(k) + p\Big)\dot{k} \tag{3.14}$$

where p is a positive constant. ∎

Proof: Consider the subsystem $\dot{z} = F(z,e,\mu)$ which can be decomposed into the form (2.11) with $z_1 = \bar{z}$, $z_2 = \tilde{\eta}$, and $u = e$. Let $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W}$ be a compact subset such that $\mu(t) = (v(t), w) \in \Sigma$ for all $t \geq 0$. Recall that M is Hurwitz, and $\bar{f}(0,0,v,w) = 0$ and $\bar{g}(0,0,v,w) = 0$ for all $(v,w) \in \mathbb{R}^{n_w} \times \mathcal{W}$.

Thus, by Lemma 2.1, under Assumption 3.3, there exists a C^1 function $V_z(z)$ satisfying $\underline{\alpha}_{1z}(\|z\|) \leq V_z(z) \leq \bar{\alpha}_{1z}(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{1z}(\cdot)$ and $\bar{\alpha}_{1z}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of system $\dot{z} = F(z,e,\mu)$

$$\dot{V}_z \leq -\|z\|^2 + \delta_e \gamma_e(e) \tag{3.15}$$

for some positive number δ_e and smooth p.d. function $\gamma_e(\cdot)$.

Further, by Remark 2.4, given any smooth function $\Delta(z) > 0$, there exists a C^1 function $U(z)$ satisfying $\underline{\alpha}_{2z}(\|z\|) \leq U(z) \leq \bar{\alpha}_{2z}(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{2z}(\cdot)$ and $\bar{\alpha}_{2z}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of system $\dot{z} = F(z,e,\mu)$

$$\dot{U} \leq -\Delta(z)\|z\|^2 + \bar{\delta}_e \bar{\gamma}_e(e) e^2 \tag{3.16}$$

for some positive number $\bar{\delta}_e$ and some known smooth positive function $\bar{\gamma}_e(\cdot)$. Let

$$V(z,e) = \frac{1}{2}e^2 + U(z)$$

which satisfies, for any $\mu \in \Sigma$, along the trajectory of the closed-loop system composed of (3.11) and (3.13)

$$\dot{V} \leq e\big(\bar{g}(\bar{z},e,\mu) + b\Psi\tilde{\eta} + \Psi N e + b\mathcal{N}(k)\rho(e)e\big) - \Delta(z)\|z\|^2 + \bar{\delta}_e\bar{\gamma}_e(e)e^2. \tag{3.17}$$

In (3.17), denote $\tilde{g}(z,e,\mu) := \bar{g}(\bar{z},e,\mu) + b\Psi\tilde{\eta} + \Psi N e$ which satisfies $\tilde{g}(0,0,\mu) = 0$ for all $\mu \in \Sigma$. Using Lemma 2.4 again yields

$$|\tilde{g}(z,e,\mu)| \leq p_1\Big(\Delta_1(z)\|z\| + \rho_1(e)e\Big)$$

for some positive smooth functions $\Delta_1(z)$ and $\rho_1(e)$, and some constant $p_1 > 0$. Thus, by completing the squares, we have

$$\begin{aligned} |e\tilde{g}(z,e,\mu)| &\leq \frac{1}{2}e^2 + \frac{1}{2}|\tilde{g}(z,e,\mu)|^2 \leq \frac{1}{2}e^2 + p_1^2\Big(\Delta_1^2(z)\|z\|^2 + \rho_1^2(e)e^2\Big) \\ &\leq p_1^2\Big(\Delta_1^2(z)\|z\|^2 + \Big(\rho_1^2(e) + \frac{1}{2p_1^2}\Big)e^2\Big) \\ &\leq p_2\Big(\Delta_2(z)\|z\|^2 + \rho_2(e)e^2\Big) \end{aligned} \tag{3.18}$$

where $p_2 \geq p_1^2$, $\Delta_2(z) \geq \Delta_1^2(z)$ and $\rho_2(e) \geq \rho_1^2(e) + \frac{1}{2p_1^2}$. Substituting (3.18) into (3.17) gives

$$\dot{V} \leq b\mathcal{N}(k)\rho(e)e^2 + \left(p_2\Delta_2(z) - \Delta(z)\right)\|z\|^2 + p_2\rho_2(e)e^2 + \bar{\delta}_e\bar{\gamma}_e(e)e^2 \tag{3.19}$$

Choosing functions $\Delta(z)$, $\rho(e)$ such that

$$\Delta(z) \geq p_2\Delta_2(z) + 1, \quad \rho(e) \geq \max\left\{\rho_2(e),\ \bar{\gamma}_e(e),\ 1\right\}$$

and constant $p \geq p_1 + \bar{\delta}_e$ gives

$$\dot{V} = \left(b\mathcal{N}(k) + p\right)\rho(e)e^2 - \|z\|^2 \tag{3.20}$$

Hence, we have (3.14). The proof is completed.

Theorem 3.1 Under Assumptions 3.1 to 3.3, let $\rho(e)$ be what is defined in Lemma 3.1. Then the following controller

$$\begin{aligned} u &= \mathcal{N}(k)\rho(e)e + \Psi\eta \\ \dot{\eta} &= M\eta + Nu \\ \dot{k} &= \rho(e)e^2 \end{aligned} \tag{3.21}$$

solves the global robust stabilization problem of system (3.11). ∎

Proof: For any given $v_0 \in \mathbb{R}^{n_v}$ and $w \in \mathcal{W}$, there exists a compact subset Σ such that $\mu(t) = (v(t), w) \in \Sigma$ for all $t \geq 0$. Thus, inequality (3.20) holds for this pair of $v(t)$ and w. Integrating both sides of (3.14) over $[0, t)$, $\forall\, t \geq 0$, gives

$$V(t) \leq \int_0^t \left(b\mathcal{N}(k(\tau)) + p\right)\dot{k}(\tau)d\tau + V(0) \tag{3.22}$$

By Lemma 2.5, the above inequality shows that $V(t)$ and $k(t)$ are bounded over each time interval $[0, T)$ with $0 < T \leq +\infty$. So the solution of the closed-loop system composed of system (3.11) and controller law (3.13) is defined and bounded over $[0, +\infty)$.

We now show $e(t)$ will approach the origin as $t \to +\infty$. Since $k(t)$ is bounded over $[0, +\infty)$ and $\dot{k}(t) = \rho(e)e^2$ with $\rho(e) \geq 1$, e is square integrable over $[0, +\infty)$. Furthermore, both $e(t)$ and $\dot{e}(t)$ are bounded over $[0, +\infty)$. By using Lemma 2.3, we conclude that $e(t)$ tends to zero as $t \to +\infty$. This completes the proof.

Remark 3.5 The control law (3.21) utilizes the well known universal adaptive high gain employed in, say, [34] [73]. This type of controller works well in a noise-free environment. However, in the presence of noises or repeated disturbances, the gain k might drift to infinity since the derivative of k is always nonnegative. This phenomenon is called "bursting phenomenon" in some literatures. The authors of [34] [73] have addressed this issue by employing the *deadzone* technique. Nevertheless, employment of such technique can only guarantee that the output of the closed-loop system will asymptotically approach, instead of the origin, some sufficiently small neighborhood of the origin. ∎

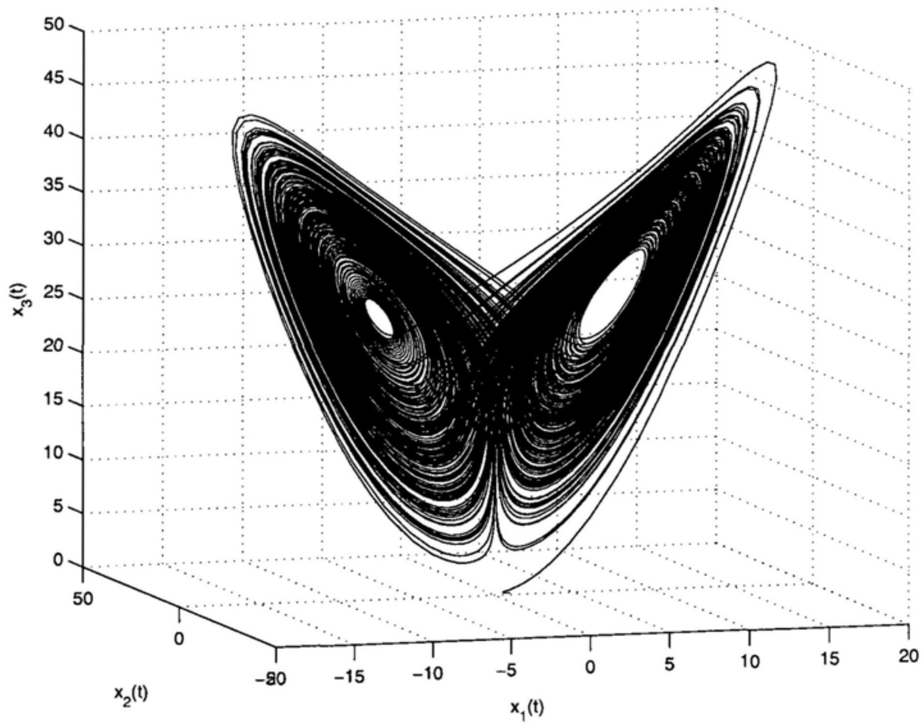

Figure 3.1: 3-D plot of Lorenz system.

3.1.3 Application

Consider the controlled Lorenz system [13] described by

$$\begin{aligned} \dot{x}_1 &= -L_1x_1 + L_1x_2 \\ \dot{x}_2 &= L_3x_1 - x_2 - x_1x_3 + bu \\ \dot{x}_3 &= L_2x_3 + x_1x_2 \end{aligned}$$

and an error output $e = x_2 - F(t)$ where (L_1, L_2, L_3, b) is a constant parameter vector satisfying $L_1 > 0$, $L_2 < 0$ and $b \neq 0$. $F(t) = A_m \sin(\omega t + \phi)$ is the reference input with known frequency ω and unknown amplitude $A_m > 0$ and initial phase ϕ. When $u \equiv 0$, system (3.23) is the well-known Lorenz system [62] which exhibits a chaotic behavior with $L_1 = 10$, $L_2 = -8/3$, and $L_3 = 28$ as shown in Figure 3.1. We consider the problem of the global asymptotic tracking by output feedback with e as the output for any sinusoidal reference input $F(t)$ without knowing the sign and value of b.

First note that $F(t) = v_1(t)$ which can be generated by

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = A_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \ v_0 = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} = \begin{bmatrix} A_m \sin\phi \\ A_m \cos\phi \end{bmatrix} \tag{3.23}$$

By letting $(z_1, z_2, y) = (x_1, x_3, x_2)$, we can put system (3.23) in the standard form (3.1) as follows

$$\begin{aligned}\dot{z}_1 &= -L_1 z_1 + L_1 y\\ \dot{z}_2 &= L_2 z_2 + z_1 y\\ \dot{y} &= bu + L_3 z_1 - y - z_1 z_2\end{aligned}\tag{3.24}$$

To make our problem more interesting, we allow the parameter (L_1, L_2, L_3) to undergo some perturbation. To be more specific, let

$$L = (\bar{L}_1, \bar{L}_2, \bar{L}_3) + (w_1, w_2, w_3)$$

where $(\bar{L}_1, \bar{L}_2, \bar{L}_3)$ represents the nominal value of L and (w_1, w_2, w_3) the uncertainty of L. To guarantee $L_1 > 0$ and $L_2 < 0$, we define

$$\mathcal{W} = \{w \mid w^{\mathsf{T}} \in \mathbb{R}^3, \bar{L}_1 + w_1 > 0, \bar{L}_2 + w_2 < 0\}$$

Now it is clear that if we can solve the global robust output regulation problem for system (3.24) with exosystem (3.23) and $\mathcal{W}$ as described above, we can solve the global robust asymptotic tracking problem for system (3.23) by output feedback control for any sinusoidal reference input $F(t)$ in the presence of the parameter variation $w \in \mathcal{W}$. Moreover, since the inverse dynamics of system (3.24) contains a nonlinear term $z_1 y$, the result in [58] does not apply to this example.

We now verify that the composite system (3.23) and (3.24) satisfies Assumptions 3.1 to 3.3. From the last equation of (3.24), we have $\mathbf{y}(v, w) = v_1$. Substituting $\mathbf{y}(v, w)$ into the first equation of (3.24) yields

$$\mathbf{z}_1(v, w) = r_{11} v_1 + r_{12} v_2$$

where

$$r_{11}(w) = \frac{L_1^2}{\omega^2 + L_1^2}, \quad r_{12}(w) = -\frac{L_1 \omega}{\omega^2 + L_1^2}$$

Substituting $\mathbf{y}(v, w)$ and $\mathbf{z}_1(v, w)$ into the second equation of (3.24) gives

$$\mathbf{z}_2(v, w) = r_{21} v_1^2 + r_{22} v_2^2 + r_{23} v_1 v_2$$

where

$$\begin{aligned}r_{21}(w) &= -\frac{\omega}{L_2} r_{23} - \frac{r_{11}}{L_2}\\ r_{22}(w) &= \frac{\omega}{L_2} r_{23}\\ r_{23}(w) &= \frac{-r_{12} L_2 - 2\omega^2 - 2\omega r_{11}}{L_2^2 + 2\omega^2}\end{aligned}$$

Finally, substituting $\mathbf{y}(v,w)$ to $\mathbf{z}_2(v,w)$ into the third equation of (3.24) gives

$$\begin{aligned}\mathbf{u}(v,w) &= b^{-1}(\omega v_2 + v_1 - L_3\mathbf{z}_1 + \mathbf{z}_1\mathbf{z}_2)\\ &= r_{31}v_1 + r_{32}v_2 + r_{33}v_1^3 + r_{34}v_2^3 + r_{35}v_1^2v_2 + r_{36}v_1v_2^2\end{aligned}$$

where

$$\begin{aligned}r_{31}(w) &= -b^{-1}(-1 + L_3r_{11})\\ r_{32}(w) &= b^{-1}(\omega - L_3r_{12})\\ r_{33}(w) &= b^{-1}r_{11}r_{21}, \quad r_{34}(w) = b^{-1}r_{12}r_{22}\\ r_{35}(w) &= b^{-1}(r_{12}r_{21} + r_{11}r_{23})\\ r_{36}(w) &= b^{-1}(r_{11}r_{22} + r_{12}r_{23})\end{aligned}$$

Thus both Assumptions 3.1 and 3.2 are satisfied. It can be further verified that

$$\frac{d^4\mathbf{u}(v,w)}{dt^4} + 9\omega^4\mathbf{u}(v,w) + 10\omega^2\frac{d^2\mathbf{u}(v,w)}{dt^2} = 0$$

By Remark 3.1, the steady-state generator of the form (3.7) is given by

$$\begin{aligned}\tau(v,w) &= \operatorname{col}\big(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}\big)\\ \Phi &= \left[\begin{array}{c|c} 0 & I_3 \\ \hline -9\omega^4 & 0, -10\omega^2, 0\end{array}\right], \quad \Gamma = [1, 0, 0, 0]\end{aligned}$$

which leads to the internal model described by (3.8) with

$$M = \left[\begin{array}{c|c} 0 & I_3 \\ \hline -m_1 & -m_2, -m_3, -m_4\end{array}\right], \quad N = \operatorname{col}(0, 0, 0, 1)$$

and the parameter (m_1, m_2, m_3, m_4) being such that M is Hurwitz. By solving the Sylvester equation $T\Phi - MT = N$ with $(m_1, m_2, m_3, m_4) = (4, 12, 13, 6)$, we have

$$\Psi = \Gamma T^{-1} = \big[4 - 9\omega^4, 12, 13 - 10\omega^2, 6\big]$$

Now performing the coordinate and input transformation (3.9) for the augmented system composed of (3.24) and (3.8) gives

$$\begin{aligned}\dot{\bar{z}}_1 &= -L_1\bar{z}_1 + L_1e\\ \dot{\bar{z}}_2 &= L_2\bar{z}_2 + (\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1v_1\\ \dot{\bar{\eta}} &= M\bar{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z}_1, \bar{z}_2, e, \mu)\\ \dot{e} &= -e + L_3\bar{z}_1 - (\bar{z}_1 + \mathbf{z}_1)(\bar{z}_2 + \mathbf{z}_2) + \mathbf{z}_1\mathbf{z}_2 + b\Psi\bar{\eta} + \Psi Ne + b\bar{u}\end{aligned} \tag{3.25}$$

We now verify that system (3.25) satisfies Assumption 3.3. In fact, for any fixed compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W}$, let

$$V_{\bar{z}} = \frac{\hbar}{2}\bar{z}_1^2 + \frac{\hbar}{4}\bar{z}_1^4 + \frac{1}{2}\bar{z}_2^2$$

for some $\hbar > 0$ which satisfies along the trajectory of $(\bar{z}_1, \bar{z}_2)$ subsystem

$$\begin{aligned}\dot{V}_{\bar{z}} &= -\hbar L_1\bar{z}_1^2 + \hbar L_1\bar{z}_1 e - \hbar L_1\bar{z}_1^4 + \hbar L_1\bar{z}_1^3 e + L_2\bar{z}_2^2 + \bar{z}_2\big((\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1\big)\\ &= -\hbar L_1\bar{z}_1^2 + \hbar L_1\bar{z}_1 e - \hbar L_1\bar{z}_1^4 + \hbar L_1\bar{z}_1^3 e + L_2\bar{z}_2^2 + \bar{z}_2\bar{z}_1 e + v_1\bar{z}_2\bar{z}_1 + \mathbf{z}_1\bar{z}_2 e \qquad (3.26)\end{aligned}$$

In (3.26), using Young's inequality gives, for any $\varepsilon > 0$

$$\begin{aligned}\hbar L_1\bar{z}_1 e &\le \frac{1}{2}\bar{z}_1^2 + \frac{\hbar^2 L_1^2}{2}e^2\\ \hbar L_1\bar{z}_1^3 e &\le \frac{3}{4}\bar{z}_1^4 + \frac{\hbar^4 L_1^4}{4}e^4\\ v_1\bar{z}_2\bar{z}_1 &\le \frac{1}{2\varepsilon}\bar{z}_1^2 + \frac{\varepsilon v_1^2}{2}\bar{z}_2^2\\ \mathbf{z}_1\bar{z}_2 e &\le \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{\mathbf{z}_1^2}{2\varepsilon}e^2\\ \bar{z}_2\bar{z}_1 e &\le \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{1}{2\varepsilon}\bar{z}_1^2 e^2 \le \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{1}{4}\bar{z}_1^4 + \frac{1}{4\varepsilon^2}e^4 \qquad (3.27)\end{aligned}$$

Substituting (3.27) into (3.26) gives

$$\begin{aligned}\dot{V}_{\bar{z}} \le\ & \Big(-\hbar L_1 + \frac{1}{2} + \frac{1}{2\varepsilon}\Big)\bar{z}_1^2 + \Big(-\hbar L_1 + 1\Big)\bar{z}_1^4 + \Big(L_2 + \varepsilon + \frac{\varepsilon v_1^2}{2}\Big)\bar{z}_2^2\\ & + \Big(\frac{\hbar^2 L_1^2}{2} + \frac{\mathbf{z}_1^2}{2\varepsilon}\Big)e^2 + \Big(\frac{\hbar^4 L_1^4}{4} + \frac{1}{4\varepsilon^2}\Big)e^4 \qquad (3.28)\end{aligned}$$

Since Σ is compact, for all $(v, w) \in \Sigma$, there exist a sufficiently small $\varepsilon > 0$, a sufficiently large $\hbar > 0$, and constants $\ell_1, \cdots, \ell_5 > 0$ such that

$$\dot{V}_{\bar{z}} \le -\ell_1\bar{z}_1^2 - \ell_2\bar{z}_1^4 - \ell_3\bar{z}_2^2 + \ell_4 e^2 + \ell_5 e^4$$

As a result, by Theorem 3.1, the global robust output regulation problem for the composite system (3.23) and (3.24) is solvable by an output feedback controller. In fact, following the design method detailed in Section 3.1.2, we can obtain a controller of the form (3.21) with $\rho(e) = 5(e^6 + 1)$.

Simulation is performed for the closed-loop system composed of (3.2), (3.11) and (3.12) with $\omega = 0.8$ and $L_1 = 10, L_2 = -8/3, L_3 = 28$, and initial conditions $v_0 = \mathrm{col}(1, 0)$, $x(0) = (-1, 1, 2)$, $\eta(0) = 0$ and $k(0) = 1$. Some results are shown in Figure 3.2 and 3.3 with $b = 1$ and $b = -1$, respectively. It shows that the controller is independent of the control direction and the control objective is achieved.

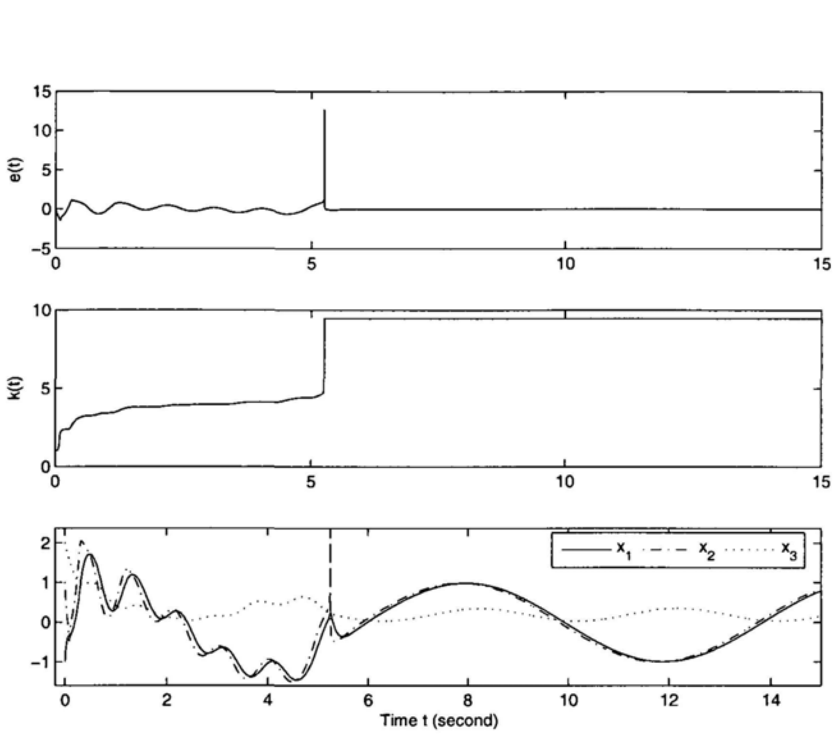

Figure 3.2: Responses of $e(t)$, $k(t)$, and $\left(x_1(t), x_2(t), x_3(t)\right)$ when $b = 1$.

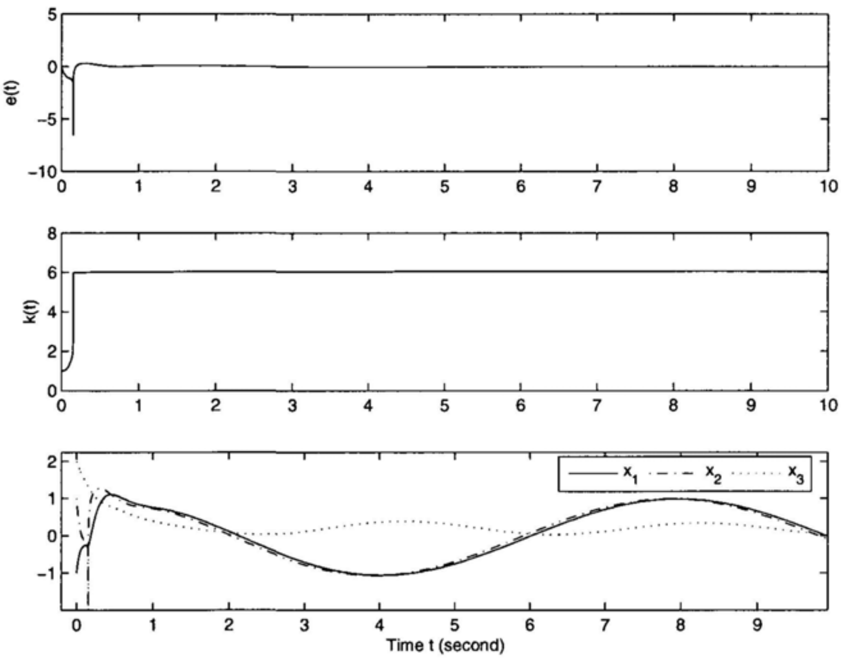

Figure 3.3: Responses of $e(t)$, $k(t)$, and $\left(x_1(t), x_2(t), x_3(t)\right)$ when $b = -1$.

3.2 General case

In this section, we consider the global robust output regulation problem of the nonlinear output feedback systems in the following general form

$$\begin{aligned}\dot{z} &= f(z,y,v,w)\\ \dot{x}_i &= x_{i+1}+g_i(z,y,v,w),\ i=1,\cdots,r-1\\ \dot{x}_r &= bu+g_r(z,y,v,w)\\ y &= x_1\\ e &= x_1-q(v,w)\end{aligned} \tag{3.29}$$

where $(z,x)\in\mathbb{R}^n\times\mathbb{R}^r$ with $r\geq 2$ is the state, $u\in\mathbb{R}$ is the input, $y\in\mathbb{R}$ is the output, b is a nonzero constant with an unknown sign, $w\in\mathbb{W}\subset\mathbb{R}^{n_w}$ is an uncertain parameter vector with $\mathbb{W}$ an arbitrarily prescribed subset of $\mathbb{R}^{n_w}$, and $v(t)\in\mathbb{R}^{n_v}$ is an exogenous signal representing both reference input and disturbance.

It is assumed that $v(t)$ is generated by a linear system of the form (3.2) where all the eigenvalues of matrix A_1 are simple with zero real parts. All functions in (3.29) are supposed to be globally defined, sufficiently smooth, and satisfy $f(0,0,0,w)=0$, $g_i(0,0,0,w)=0$, and $q(0,w)=0$ for all $w\in\mathcal{W}$.

The quantity e represents the tracking error. The precise statement of our problem is given as follows.

Problem 3.1 Design a dynamic output feedback control law of the form

$$u=u_K(\zeta,e),\ \dot{\zeta}=g_K(\zeta,e) \tag{3.30}$$

where $\zeta\in\mathbb{R}^{n_\zeta}$ for some integer $n_\zeta>0$. u_K and g_K are globally defined sufficiently smooth functions vanishing at the origin such that, for all initial conditions, and all $w\in\mathbb{W}$, the trajectory of the closed-loop system composed of (3.29) to (3.30) exists and is bounded over $[0,+\infty)$, and the error output $e(t)$ asymptotically approaches zero as $t\to+\infty$. ∎

The global robust stabilization problem for the output feedback systems with $v_0=0$ has been studied in [42][44]. A subclass of (3.29) is given as follows

$$\begin{aligned}\dot{z} &= H(w)z+g_0(y,w)\\ \dot{x}_i &= x_{i+1}+g_i(y,w),\ i=1,\cdots,r-1\\ \dot{x}_r &= bu+Q(w)z+g_r(y,w)\\ y &= x_1\end{aligned} \tag{3.31}$$

where $H(w)$ and $Q(w)$ are matrices of appropriate dimensions, and $H(w)$ is Hurwitz for each constant uncertainty w [58] [67]. A disturbance rejection problem for system (3.31) has been studied in [16] which can be viewed as a special case of the output regulation. A special feature of (3.31) is that its zero dynamics $\dot{z} = H(w)z$ is a linear stable system. This special feature lends itself to an effective approach to designing an output feedback control law. In contrast, system (3.29) does not possess this feature and hence the approach in [16] is not applicable here. Therefore, we need to employ a different technique to tackle our problem which involves the use of some type of observer. Moreover, unlike [16], we will not assume the knowledge of the sign of the high frequency gain b.

In Section 3.2.1, we introduce a set of basic assumptions on system (3.29) and convert the problem into a global robust stabilization problem of an augmented system. Section 3.2.2 will give the main result of this section, and a design example will be illustrated in Section 3.2.3.

3.2.1 Assumptions and problem conversion

To achieve the problem conversion as shown in [32], we will first list some standard assumptions as follows.

Assumption 3.4 There exists a smooth function $\mathbf{z}(v,w) : \mathbb{R}^{n_v+n_w} \mapsto \mathbb{R}^n$ with $\mathbf{z}(0,0) = 0$ such that

$$\frac{\partial \mathbf{z}(v,w)}{\partial v} A_1 v = f\big(\mathbf{z}(v,w), q(v,w), v, w\big) \tag{3.32}$$

for all $(v,w) \in \mathbb{R}^{n_v} \times \mathcal{W}$. ∎

Let

$$\mathbf{x}(v,w) = \mathrm{col}(\mathbf{x}_1(v,w), \cdots, \mathbf{x}_r(v,w))$$

with $\mathbf{x}_1(v,w) = q(v,w)$ and for $i = 2, \cdots, r$

$$\begin{aligned} \mathbf{x}_i(v,w) &= L_{A_1 v}\mathbf{x}_{i-1}(v,w) - g_{i-1}(\mathbf{z}(v,w), q(v,w), v, w) \\ \mathbf{u}(v,w) &= b^{-1}\big[L_{A_1 v}\mathbf{x}_r(v,w) - g_r(\mathbf{z}(v,w), q(v,w), v, w)\big] \end{aligned}$$

where $L_{A_1 v} q(v,w) = \frac{\partial q(v,w)}{\partial v} A_1 v$. Then, under Assumption 3.4, the solution of the regulator equations associated with system (3.29) and exosystem (3.2) is provided with $\mathbf{z}(v,w)$, $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$.

Assumption 3.5 There exist an integer n_s, a sufficiently smooth function $\tau : \mathbb{R}^{n_v+n_w} \mapsto \mathbb{R}^{n_s}$ vanishing at the origin, and a pair of matrices $\Phi \in \mathbb{R}^{n_s \times n_s}$ and $\Psi \in \mathbb{R}^{1 \times n_s}$, such that

$$\frac{d\tau(v,w)}{dt} = \Phi\tau(v,w), \;\; \mathbf{u}(v,w) = \Psi\tau(v,w)$$

for all $(v,w) \in \mathbb{R}^{n_v} \times \mathcal{W}$. Moreover, the pair (Ψ, Φ) is observable and all the eigenvalues of Φ are simple with zero real parts. ∎

Assumption 3.5 guarantees the existence of the internal model. In fact, under Assumption 3.5, the Sylvester equation

$$T\Phi - MT = N\Psi$$

has a unique nonsingular solution T for a given controllable pair (M, N) with M a Hurwitz matrix and N a vector of appropriate dimensions. Let $\theta(v,w) = T\tau(v,w)$. Then, we have

$$\begin{aligned} \dot{\theta}(v,w) &= (M + N\Psi T^{-1})\theta(v,w) \\ \mathbf{u}(v,w) &= \Psi T^{-1}\theta(v,w) = \Psi_o\theta(v,w) \end{aligned} \tag{3.33}$$

Therefore, we can define the following dynamics

$$\dot{\eta} = M\eta + Nu \tag{3.34}$$

as an internal model with output u.

Attaching the internal model (3.34) to system (3.29) and performing the following coordinate and input transformation

$$\bar{z} = z - \mathbf{z}(v,w), \quad \bar{x} = x - \mathbf{x}(v,w), \quad \bar{\eta} = \eta - \theta(v,w), \quad \bar{u} = u - \Psi_o\eta \tag{3.35}$$

yields

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\ \dot{\bar{\eta}} &= (M + N\Psi_o)\bar{\eta} + N\bar{u} \\ \dot{\bar{x}}_i &= \bar{x}_{i+1} + \bar{g}_i(\bar{z}, e, v, w) \\ & \quad i = 1, \cdots, r-1 \\ \dot{\bar{x}}_r &= b\bar{u} + b\Psi_o\bar{\eta} + \bar{g}_r(\bar{z}, e, v, w) \end{aligned} \tag{3.36}$$

where $\bar{x} = \text{col}(\bar{x}_1, \cdots, \bar{x}_r)$ and

$$\begin{aligned} \bar{f}(\bar{z}, e, v, w) &= f(\bar{z} + \mathbf{z}(v,w), e + q(v,w), v, w) \\ &\quad - f(\mathbf{z}(v,w), q(v,w), v, w) \\ \bar{g}_i(\bar{z}, e, v, w) &= g_i(\bar{z} + \mathbf{z}(v,w), e + q(v,w), v, w) \\ &\quad - g_i(\mathbf{z}(v,w), q(v,w), v, w) \end{aligned} \tag{3.37}$$

for $i = 1, \cdots, r$.

System (3.36) is called augmented system and it has the following property:

$$\bar{f}(0, 0, v, w) = 0, \quad \bar{g}_i(0, 0, v, w) = 0, \quad i = 1, \cdots, r$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathcal{W}$. Therefore, the global robust output regulation problem of system (3.29) as described in Problem 3.1 will be solved if the following global robust stabilization problem is solvable.

Problem 3.2 Design a dynamic output feedback control law of the form

$$\bar{u} = \bar{u}_K(\bar{\zeta}, e), \ \dot{\bar{\zeta}} = \bar{g}_K(\bar{\zeta}, e) \tag{3.38}$$

where $\bar{\zeta} \in \mathbb{R}^{n_{\bar{\zeta}}}$ for some integer $n_{\bar{\zeta}} > 0$. $\bar{u}_K$ and $\bar{g}_K$ are globally defined sufficiently smooth functions vanishing at the origin such that, for any fixed $w \in \mathbb{W}$ and any $v(t)$ generated by (3.2), the solution of the closed-loop system composed of (3.36) and (3.38) is bounded and $\bar{x}_1(t)(= e(t))$ approaches zero asymptotically. ∎

3.2.2 Main result

A specific difficulty with the global robust stabilization problem of system (3.36) is that it is not in the output feedback form as displayed in (3.29) due to the presence of the internal model. Moreover, like $\bar{z}$, the state $\bar{\eta}$ is not available for feedback. Nevertheless, performing, as in [43], the following coordinate transformation on (3.36)

$$\tilde{\eta} = \bar{\eta} - c_r\bar{x}_r - \cdots - c_1\bar{x}_1 \tag{3.39}$$

where $c_r = b^{-1}N$, $c_{i-1} = Mc_i$ for $i = 2, \cdots, r$, gives

$$\begin{aligned} \dot{\tilde{\eta}} &= M\tilde{\eta} + Mc_1e - \sum_{i=1}^{r} c_i\bar{g}_i(\bar{z}, e, v, w) \\ \dot{\bar{x}} &= A_s\bar{x} + bB\Psi_o\tilde{\eta} + \bar{g}(\bar{z}, e, v, w) + bB\bar{u} \end{aligned} \tag{3.40}$$

where $\bar{g}(\bar{z}, e, v, w) = \text{col}(\bar{g}_1(\bar{z}, e, v, w), \cdots, \bar{g}_r(\bar{z}, e, v, w))$

$$A_s = \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline s_r & s_{r-1}, \cdots, s_1 \end{array}\right], \ B = \text{col}(\underbrace{0, \cdots, 0}_{r-1}, 1)$$

and real scalars $s_i = b\Psi_o c_{r+1-i}$ for $i = 1, \cdots, r$. Further performing another coordinate transformation on $\bar{x}$-system

$$\xi = b^{-1}U_s \cdot \bar{x} \tag{3.41}$$

where

$$U_s = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -s_1 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ -s_{r-2} & -s_{r-3} & \cdots & 1 & 0 \\ -s_{r-1} & -s_{r-2} & \cdots & -s_1 & 1 \end{bmatrix}$$

gives

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\ \dot{\tilde{\eta}} &= M\tilde{\eta} + Mc_1e - \sum_{i=1}^{r} c_i\bar{g}_i(\bar{z}, e, v, w) \\ \dot{\xi} &= A_c\xi + B\Psi_o\tilde{\eta} + G(\bar{z}, e, v, w) + B\bar{u} \end{aligned} \tag{3.42}$$

where $A_c = \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline 0 & 0 \end{array}\right]$ and

$$\begin{aligned} G(\bar{z},e,v,w) &= \text{col}\big(G_1(\bar{z},e,v,w),\cdots,G_r(\bar{z},e,v,w)\big) \\ G_1(\bar{z},e,v,w) &= s_1 e + b^{-1}\bar{g}_1(\bar{z},e,v,w) \\ G_i(\bar{z},e,v,w) &= s_i e - b^{-1}\sum_{j=1}^{i-1} s_j \bar{g}_j(\bar{z},e,v,w) + b^{-1}\bar{g}_i(\bar{z},e,v,w) \end{aligned} \tag{3.43}$$

for $i = 2,\cdots,r$. It is noted that U_s is such that

$$U_s A_s U_s^{-1} = \left[\begin{array}{c|c} s_{[r-1]} & I_{r-1} \\ \hline s_r & 0 \end{array}\right]$$

where $s_{[r-1]} = \text{col}(s_1,\cdots,s_{r-1})$, and $e = \bar{x}_1 = b\xi_1$.

As our purpose is to design an output feedback control law that only relies on $e(t)$, we need to introduce some sort of observer to estimate the state $\xi(t)$. We will adopt a standard observer such as what can be found in [42] as follows:

$$\dot{\hat{\xi}} = A_c\hat{\xi} + \lambda(e - \hat{\xi}_1) + B\bar{u} \tag{3.44}$$

where $\lambda = \text{col}(\lambda_1,\cdots,\lambda_r)$ is chosen such that the matrix

$$A_o = \left[\begin{array}{c|c} -\lambda_{[r-1]} & I_{r-1} \\ \hline -\lambda_r & 0\cdots 0 \end{array}\right]$$

is Hurwitz. The observation error $\tilde{\xi} = \xi - \hat{\xi}$ satisfies

$$\dot{\tilde{\xi}} = A_o\tilde{\xi} - \lambda(1-b^{-1})e + B\Psi_o\tilde{\eta} + G(\bar{z},e,v,w) \tag{3.45}$$

Attaching (3.45) to (3.42) and replacing the state variable vector ξ by $(e,\hat{\xi}_2,\cdots,\hat{\xi}_r)$ gives the following system

$$\begin{aligned} \dot{z} &= F(z,e,\mu) \\ \dot{e} &= b\hat{\xi}_2 + b\tilde{\xi}_2 + bG_1(\bar{z},e,\mu) \\ \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \lambda_i(e-\hat{\xi}_1),\ i = 2,\cdots,r-1 \\ \dot{\hat{\xi}}_r &= \bar{u} + \lambda_r(e-\hat{\xi}_1) \end{aligned} \tag{3.46}$$

where $z = \text{col}(\bar{z},\tilde{\eta},\tilde{\xi})$, $\mu = (v,w)$, and

$$F(z,e,\mu) = \begin{bmatrix} \bar{f}(\bar{z},e,v,w) \\ M\tilde{\eta} + Mc_1 e - \sum_{i=1}^{r} c_i\bar{g}_i(\bar{z},e,v,w) \\ A_o\tilde{\xi} - \lambda(1-b^{-1})e + B\Psi_o\tilde{\eta} + G(\bar{z},e,v,w) \end{bmatrix}$$

It can be seen that system (3.46) is in a standard lower triangular form. The global stabilization problem for such a system is solvable if the subsystem $\dot{z} = F(z, e, \mu)$ has certain ISS property. To apply Lemma 2.1 to system (3.46), we need one more assumption as follows.

Assumption 3.6 For any compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W}$, there exists a C^1 function $V_{\bar{z}}(\bar{z})$ satisfying $\underline{\alpha}_{\bar{z}}(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \bar{\alpha}_{\bar{z}}(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\bar{z}}(\cdot)$ and $\bar{\alpha}_{\bar{z}}(\cdot)$ such that, for any $(v, w) \in \Sigma$, along the trajectory of $\bar{z}$ subsystem

$$\frac{\partial V_{\bar{z}}}{\partial \bar{z}}(\bar{z}) \cdot \bar{f}(\bar{z}, e, v, w) \leq -\alpha_{\bar{z}}(\|\bar{z}\|) + \delta_e \gamma_e(e)$$

where δ_e is some unknown constant, $\alpha_{\bar{z}}(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying

$$\limsup_{s \to 0^+} (\alpha_{\bar{z}}^{-1}(s^2)/s) < \infty$$

and $\gamma_e(\cdot)$ is a known smooth p.d. function. ∎

Remark 3.6 Assumption 3.6 implies that the subsystem

$$\dot{\bar{z}} = \bar{f}(\bar{z}, e, v, w) \tag{3.47}$$

is ISS with state $\bar{z}$ and input e and the equilibrium $\bar{z} = 0$ of $\dot{\bar{z}} = \bar{f}(\bar{z}, 0, v, w)$ is locally exponentially stable if the functions $\underline{\alpha}$ and $\bar{\alpha}$ are also locally quadratic. Under Assumption 3.6, by Remark 2.4, for any smooth function $\Delta(\bar{z}) > 0$, there exists a C^1 function $\bar{V}_{\bar{z}}(\bar{z})$ satisfying $\underline{\alpha}_{1\bar{z}}(\|\bar{z}\|) \leq \bar{V}_{\bar{z}}(\bar{z}) \leq \bar{\alpha}_{1\bar{z}}(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{1\bar{z}}(\cdot)$ and $\bar{\alpha}_{1\bar{z}}(\cdot)$ such that, for any $(v, w) \in \Sigma$, along the trajectory of system (3.47),

$$\dot{\bar{V}}_{\bar{z}} \leq -\Delta(\bar{z}) \|\bar{z}\|^2 + \bar{\delta}_e \bar{\gamma}_e(e) e^2$$

where $\bar{\delta}_e$ is some unknown positive constant and $\bar{\gamma}_e(\cdot)$ is some known smooth positive function. ∎

We are now ready to construct our control law using a recursive method modified from the tuning function approach described in [51]. For this purpose, we introduce the following notation.

$$\begin{aligned}
\kappa_1(e, k) &= \mathcal{N}(k)\rho(e)e \\
\kappa_2(e, k, \hat{b}, \hat{\xi}_1, \hat{\xi}_2) &= -2\omega_1 - \lambda_2(e - \hat{\xi}_1) - \hat{b}E_1\hat{\xi}_2 - \omega_1 E_1^2 - K_1 \\
\phi_2(e, k, \hat{b}, \hat{\xi}_2) &= \omega_1 E_1 \hat{\xi}_2 \\
\kappa_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i) &= -\omega_{i-2} - \omega_{i-1} - \lambda_i(e - \hat{\xi}_1) + \sum_{j=1}^{i-1} \frac{\partial \kappa_{i-1}}{\partial \hat{\xi}_j} \dot{\hat{\xi}}_j \\
&\quad + \frac{\partial \kappa_{i-1}}{\partial \hat{b}} \dot{\hat{b}} - \hat{b}E_{i-1}\hat{\xi}_2 - \omega_{i-1}E_{i-1}^2 - K_{i-1} \\
\phi_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i) &= \phi_{i-1} + \omega_{i-1}E_{i-1}\hat{\xi}_2, \ i = 3, \cdots, r
\end{aligned} \tag{3.48}$$

where, for $i = 1, \cdots, r$, $E_i = -\frac{\partial \kappa_i}{\partial e}$ and $K_i = -\frac{\partial \kappa_i}{\partial k}\dot{k}$, k is a variable governed by the second equation of (3.50), $\mathcal{N}(k)$ is a Nussbaum-type function, $\hat{b}$ is an estimate for b and governed by the third equation of (3.50), and $\rho(e)$ is a positive continuous function to be specified later by (3.62). Also, for $i = 1, \cdots, r-1$

$$\omega_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i) = \hat{\xi}_{i+1} - \kappa_i(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_i)$$

For convenience, we let $\omega_r = 0$ and $\hat{\xi}_{r+1} = \bar{u}$. The derivative of ω_i satisfies

$$\begin{aligned} \dot{\omega}_1 &= \dot{\hat{\xi}}_2 - \dot{\kappa}_1 = \omega_2 + \kappa_2 + \lambda_2(e - \hat{\xi}_1) + E_1(b\hat{\xi}_2 + b\tilde{\xi}_2 + bG_1) + K_1 \\ \dot{\omega}_i &= \dot{\hat{\xi}}_{i+1} - \dot{\kappa}_i = \omega_{i+1} + \kappa_{i+1} + \lambda_{i+1}(e - \hat{\xi}_1) + E_i\dot{e} \\ &\quad - \sum_{j=1}^{i} \frac{\partial \kappa_i}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j - \frac{\partial \kappa_i}{\partial \hat{b}}\dot{\hat{b}} + K_i \end{aligned} \tag{3.49}$$

for $i = 2, \cdots, r-1$.

Lemma 3.2 Under Assumptions 3.4 to 3.6, there exist a sufficiently smooth function $\rho(e) \geq 1$, a control law of the form

$$\begin{aligned} \bar{u} &= \kappa_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r) \\ \dot{k} &= \rho(e)e^2 \\ \dot{\hat{b}} &= \phi_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r) \end{aligned} \tag{3.50}$$

and a C^1 function $V(z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1})$, where $\tilde{b}(t) = b - \hat{b}(t)$, satisfying

$$\underline{\alpha}(\| z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1} \|) \leq V \leq \bar{\alpha}(\| z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1} \|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$, such that, for any $(v, w) \in \Sigma$ where Σ is as specified in Assumption 3.6, along the trajectory of the closed-loop system composed of system (3.46) and control law (3.50),

$$\dot{V} \leq \Big(b\mathcal{N}(k) + p\Big)\dot{k} \tag{3.51}$$

with p a positive constant. ∎

Proof: Consider the subsystem $\dot{z} = F(z, e, \mu)$ which can be decomposed into the form (2.11) with $z_1 = \bar{z}$, $z_2 = \text{col}(\tilde{\eta}, \tilde{\xi})$, and $u = e$. Recall that both M and A_o are Hurwitz, and the functions $\bar{f}(0, 0, v, w) = 0$, $\bar{g}_i(0, 0, v, w) = 0$, and $G_i(0, 0, v, w) = 0$ for $i = 1, \cdots, r$, and for all $(v, w) \in \mathbb{R}^{n_w} \times \mathcal{W}$.

Thus, by Lemma 2.1, under Assumption 3.6, there exists a C^1 function $V_z(z)$ satisfying $\underline{\alpha}_{1z}(\|z\|) \leq V_z(z) \leq \bar{\alpha}_{1z}(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{1z}(\cdot)$ and $\bar{\alpha}_{1z}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of system $\dot{z} = F(z, e, \mu)$

$$\dot{V}_z \leq -\|z\|^2 + \delta_e \gamma_e(e)$$

for some positive number δ_e and smooth p.d. function $\gamma_e(\cdot)$.

Further, again by Remark 2.4, given any smooth function $\Delta(z) > 0$, there exists a C^1 function $U(z)$ satisfying $\underline{\alpha}_{2z}(\|z\|) \leq U(z) \leq \bar{\alpha}_{2z}(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{2z}(\cdot)$ and $\bar{\alpha}_{2z}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of system $\dot{z} = F(z, e, \mu)$

$$\dot{U} \leq -\Delta(z)\,\|z\|^2 + \bar{\delta}_e \bar{\gamma}_e(e) e^2 \tag{3.52}$$

for some positive number $\bar{\delta}_e$ and some known smooth positive function $\bar{\gamma}_e(\cdot)$. The function U will be used later in constructing the function V in (3.60).

Next, we will construct a Lyapunov-like function for the subsystem of (3.46) governing the state variables $(e, \hat{\xi}_2, \cdots, \hat{\xi}_r)$. The spirit of the approach is similar with that in [44].

Step 1 Let $V_1(e) = \frac{1}{2}e^2$. Then, by completing the squares, the derivative of V_1 along the trajectory of the e subsystem is

$$\begin{aligned} \dot{V}_1 &= e\big[b\hat{\xi}_2 + b\tilde{\xi}_2 + bG_1(\bar{z}, e, v, w)\big] \\ &= b\big[\omega_1 + \mathcal{N}(k)\rho(e)e\big]e + b\tilde{\xi}_2 e + bG_1 e \\ &\leq b\mathcal{N}(k)\rho(e)e^2 + \omega_1^2 + \Pi_1(z, e, \mu) \end{aligned} \tag{3.53}$$

where $\Pi_1(z, e, \mu)$ is defined as follows:

$$\Pi_1(z, e, \mu) = b\big(\tilde{\xi}_2 + G_1 + \frac{b}{4}e\big)e$$

Step 2 Let $V_2(e, \tilde{b}, \omega_1) = V_1(e) + \frac{1}{2}\tilde{b}^2 + \frac{1}{2}\omega_1^2$. Then, by completing the squares, the derivative of V_2 along the trajectory of the (e, ω_1) subsystem is

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + \omega_1\dot{\omega}_1 - \tilde{b}\cdot\dot{\hat{b}} \\ &= \dot{V}_1 + \omega_1(\omega_2 + \kappa_2) + \omega_1\lambda_2(e - \hat{\xi}_1) + \omega_1 K_1 + \omega_1 E_1(b\hat{\xi}_2 + b\tilde{\xi}_2 \\ &\quad + bG_1) - \tilde{b}\cdot\dot{\hat{b}} \\ &\leq b\mathcal{N}(k)\rho(e)e^2 + \omega_1^2 + \Pi_1(z, e, \mu) + \omega_1\omega_2 + \omega_1\kappa_2 + \omega_1\lambda_2(e - \hat{\xi}_1) \\ &\quad + \omega_1 K_1 + \hat{b}\omega_1 E_1\hat{\xi}_2 + \tilde{b}\omega_1 E_1\hat{\xi}_2 + \frac{1}{2}(\omega_1 E_1)^2 + \frac{1}{2}b^2\tilde{\xi}_2^2 \\ &\quad + \frac{1}{2}(\omega_1 E_1)^2 + \frac{1}{2}b^2 G_1^2 - \tilde{b}\cdot\dot{\hat{b}} \\ &\leq b\mathcal{N}(k)\rho(e)e^2 + \omega_1^2 + \omega_1\omega_2 + \omega_1\kappa_2 + \omega_1\lambda_2(e - \hat{\xi}_1) + \omega_1 K_1 \\ &\quad + \hat{b}\omega_1 E_1\hat{\xi}_2 + (\omega_1 E_1)^2 - \tilde{b}(\dot{\hat{b}} - \omega_1 E_1\hat{\xi}_2) + \Pi_2(z, e, \mu) \end{aligned}$$

where $\Pi_2(z, e, \mu)$ is used to denote the following function

$$\begin{aligned} \Pi_2(z, e, \mu) &= \Pi_1(z, e, \mu) + \Pi_0(z, e, \mu) \\ \Pi_0(z, e, \mu) &= \frac{1}{2}b^2(\tilde{\xi}_2^2 + G_1^2) \end{aligned}$$

Further using the expressions of $\kappa_2(e,k,\hat{b},\hat{\xi}_1,\hat{\xi}_2)$ and $\phi_2(e,k,\hat{b},\hat{\xi}_2)$ as given in (3.48) yields

$$\dot{V}_2 \le b\mathcal{N}(k)\rho(e)e^2 + \omega_1\omega_2 - \omega_1^2 - \tilde{b}(\dot{\hat{b}} - \phi_2) + \Pi_2(z,e,\mu)$$

If $r = 2$, we have $\hat{\xi}_3 = \bar{u}$. The proof is completed by letting $\omega_2 = 0$. Otherwise, continue the design as follows.

Step i ($3 \le i \le r$) Let

$$V_{i-1}(z,e,\tilde{b},\omega_1,\cdots,\omega_{i-2}) = V_1(e) + \frac{1}{2}\tilde{b}^2 + \frac{1}{2}\sum_{j=1}^{i-2}\omega_j^2$$

Assume the derivative of V_{i-1} along the trajectory of the $(e,\omega_1,\cdots,\omega_{i-2})$ subsystem satisfies

$$\dot{V}_{i-1} \le b\mathcal{N}(k)\rho(e)e^2 + \omega_{i-2}\omega_{i-1} - \sum_{j=1}^{i-2}\omega_j^2 - \tilde{b}(\dot{\hat{b}} - \phi_{i-1}) + \Pi_{i-1}(z,e,\mu) \tag{3.54}$$

where

$$\Pi_{i-1}(z,e,\mu) = \Pi_1(z,e,v,w) + (i-2)\Pi_0(z,e,\mu)$$

Next, define

$$V_i(z,e,\tilde{b},\omega_1,\cdots,\omega_{i-1}) = V_{i-1} + \frac{1}{2}\omega_{i-1}^2$$

Then, the derivative of V_i along the trajectory of the $(e,\omega_1,\cdots,\omega_{i-1})$ subsystem satisfies

$$\begin{aligned}\dot{V}_i &= \dot{V}_{i-1} + \omega_{i-1}\left[\omega_i + \kappa_i + \lambda_i(e - \hat{\xi}_1)\right] + \omega_{i-1}E_{i-1}\dot{e} - \omega_{i-1}\sum_{j=1}^{i-1}\frac{\partial\kappa_{i-1}}{\partial\hat{\xi}_j}\dot{\hat{\xi}}_j \\ &\quad -\omega_{i-1}\frac{\partial\kappa_{i-1}}{\partial\hat{b}}\dot{\hat{b}} + \omega_{i-1}K_{i-1}\end{aligned} \tag{3.55}$$

By completing the squares, we have

$$\begin{aligned}\omega_{i-1}E_{i-1}\dot{e} &= b\omega_{i-1}E_{i-1}(\hat{\xi}_2 + \tilde{\xi}_2 + G_1) \\ &\le \hat{b}\omega_{i-1}E_{i-1}\hat{\xi}_2 + \tilde{b}\omega_{i-1}E_{i-1}\hat{\xi}_2 + \omega_{i-1}^2E_{i-1}^2 + \Pi_0(z,e,\mu)\end{aligned}$$

This inequality together with (3.55) and (3.54) yields

$$\begin{aligned}
\dot{V}_i \leq\ & \dot{V}_{i-1} + \omega_{i-1}\big[\omega_i + \kappa_i + \lambda_i(e - \hat{\xi}_1)\big] + \hat{b}\omega_{i-1}E_{i-1}\hat{\xi}_2 + \tilde{b}\omega_{i-1}E_{i-1}\hat{\xi}_2 \\
& + \omega_{i-1}^2 E_{i-1}^2 - \omega_{i-1}\sum_{j=1}^{i-1}\frac{\partial \kappa_{i-1}}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j - \omega_{i-1}\frac{\partial \kappa_{i-1}}{\partial \hat{b}}\dot{\hat{b}} \\
& + \omega_{i-1}K_{i-1} + \Pi_0(z, e, \mu) \\
\leq\ & b\mathcal{N}(k)\rho(e)e^2 + \omega_{i-2}\omega_{i-1} - \sum_{j=1}^{i-2}\omega_j^2 + \omega_{i-1}\omega_i + \omega_{i-1}\Big[\kappa_i + \lambda_i(e - \hat{\xi}_1) \\
& + \hat{b}E_{i-1}\hat{\xi}_2 + \omega_{i-1}E_{i-1}^2 - \sum_{j=1}^{i-1}\frac{\partial \kappa_{i-1}}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j - \frac{\partial \kappa_{i-1}}{\partial \hat{b}}\dot{\hat{b}} + K_{i-1}\Big] \\
& - \tilde{b}(\dot{\hat{b}} - \phi_{i-1} - \omega_{i-1}E_{i-1}\hat{\xi}_2) + \Pi_{i-1}(z, e, \mu) + \Pi_0(z, e, \mu)
\end{aligned}$$

Further using the expressions of $\kappa_i(e, k, \hat{b}, \hat{\xi}_1, \hat{\xi}_2)$ and $\phi_i = (e, k, \hat{b}, \hat{\xi}_2)$ as given in (3.48) yields

$$\dot{V}_i \leq b\mathcal{N}(k)\rho(e)e^2 + \omega_{i-1}\omega_i - \sum_{j=1}^{i-1}\omega_j^2 + \Pi_i(z, e, \mu)$$

where $\Pi_i(z, e, \mu) = \Pi_1(z, e, v, w) + (i-1)\Pi_0(z, e, \mu)$.

Noting $\hat{\xi}_{r+1} = \bar{u}$ and $\omega_r = 0$, we have

$$\dot{V}_r \leq b\mathcal{N}(k)\rho(e)e^2 - \sum_{j=1}^{r-1}\omega_j^2 + \Pi_r(z, e, \mu) \tag{3.56}$$

We will now obtain an upper bound for function $\Pi_r(z, e, \mu)$. To this end, again by Lemma 2.4, there exist some number $p_1 > 0$ and known smooth positive functions $\varphi_{\bar{z}}(\bar{z})$ and $\varphi_e(e)$ such that, for any $(v, w) \in \Sigma$

$$|G_1(\bar{z}, e, v, w)|^2 \leq p_1\big[\varphi_{\bar{z}}(\bar{z})\,\|\bar{z}\|^2 + \varphi_e(e)e^2\big] \tag{3.57}$$

Using the inequality (3.57), an upper bound for the function $\Pi_r(z, e, \mu)$ can be given as follows

$$\begin{aligned}
& \Pi_r(z, e, \mu) \\
=\ & \Pi_1(z, e, \mu) + (r-1)\Pi_0(z, e, \mu) \\
=\ & b\Big(\tilde{\xi}_2 + G_1 + \frac{b}{4}e\Big)e + (r-1)\frac{1}{2}b^2(\tilde{\xi}_2^2 + G_1^2) \\
\leq\ & \Big(|b| + \frac{r-1}{2}b^2\Big)\tilde{\xi}_2^2 + \frac{b^2 + 2|b|}{4}e^2 + (r - 1 + |b|) \\
& \times\Big(p_1[\varphi_{\bar{z}}(\bar{z})\,\|\bar{z}\|^2 + \varphi_e(e)e^2]\Big) \\
\leq\ & p_\pi\big(\varphi_Z(z)\,\|z\|^2 + \varphi_e(e)e^2\big)
\end{aligned} \tag{3.58}$$

for any $\mu \in \Sigma$, where p_π is some sufficiently large number and $\varphi_Z(\cdot)$ is some known smooth positive function.

Using (3.58) in (3.56) gives

$$\dot{V}_r \leq b\mathcal{N}(k)\rho(e)e^2 - \sum_{j=1}^{r-1} \omega_j^2 + p_\pi\big(\varphi_Z(z)\,\|z\|^2 + \varphi_e(e)e^2\big) \tag{3.59}$$

for any $\mu \in \Sigma$. Finally, let

$$V(z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1}) = V_r(z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1}) + U(z) \tag{3.60}$$

Clearly, V satisfies

$$\underline{\alpha}(\| z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1} \|) \leq V \leq \bar{\alpha}(\| z, e, \tilde{b}, \omega_1, \cdots, \omega_{r-1} \|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$. Furthermore, from (3.52) and (3.59), the derivative of V satisfies, for any $\mu \in \Sigma$

$$\begin{aligned}
\dot{V} &\leq b\mathcal{N}(k)\rho(e)e^2 - \sum_{j=1}^{r-1} \omega_j^2 + p_\pi\big(\varphi_Z(z)\,\|z\|^2 + \varphi_e(e)e^2\big) \\
&\quad -\big(\Delta(z)\,\|z\|^2 - \bar{\delta}_e\bar{\gamma}_e(e)e^2\big) \\
&\leq b\mathcal{N}(k)\rho(e)e^2 - \sum_{j=1}^{r-1} \omega_j^2 - \big(\Delta(z) - p_\pi\varphi_Z(z)\big)\,\|z\|^2 \\
&\quad +\big(\bar{\delta}_e\bar{\gamma}_e(e) + p_\pi\varphi_e(e)\big)e^2
\end{aligned} \tag{3.61}$$

Letting $\Delta(z) \geq p_\pi\varphi_Z(\|z\|) + 1$, $p \geq p_\pi + \bar{\delta}_e$ and

$$\rho(e) \geq \max\left\{\bar{\gamma}_e(e),\ \varphi_e(e)\right\} \tag{3.62}$$

gives, for any $\mu \in \Sigma$

$$\begin{aligned}
\dot{V} &\leq b\mathcal{N}(k)\rho(e)e^2 + p\cdot\rho(e)e^2 - \sum_{j=1}^{r-1} \omega_j^2 - \|z\|^2 \\
&= \Big(b\mathcal{N}(k) + p\Big)\dot{k} - \sum_{j=1}^{r-1} \omega_j^2 - \|z\|^2
\end{aligned} \tag{3.63}$$

Hence, (3.51) is obtained. The proof is completed.

Theorem 3.2 Under Assumption 3.4 through 3.6, Problem 3.2 is solvable. ∎

Proof: For any given $v_0 \in \mathbb{R}^{n_v}$ and $w \in \mathcal{W}$, there exists a compact set Σ such that $\mu(t) = (v(t), w) \in \Sigma$ for all $t \geq 0$. Thus, inequality (3.63) holds for this pair of $v(t)$ and w. Integrating both sides of (3.51) over $[0, t)$, $\forall\, t \geq 0$ gives

$$V(t) \leq \int_0^t \Big(b\mathcal{N}(k(\tau)) + p\Big)\dot{k}(\tau)d\tau + V(0)$$

By Lemma 2.5, the above inequality shows that $V(t)$ and $k(t)$ are bounded over each time interval $[0, T)$ with $0 < T \leq +\infty$. So the solution of the closed-loop system composed of system (3.46) and control law (3.50) is defined on $[0, +\infty)$ and bounded over $[0, +\infty)$.

We now show $e(t)$ will approach the origin as $t \longrightarrow +\infty$. Since $k(t)$ is bounded over $[0, +\infty)$ and $\dot{k}(t) = \rho(e)e^2$, the function $\rho(e)e^2$ is integrable over $[0, +\infty)$. Furthermore, $\rho(e)e^2$ is uniformly continuous since both $e(t)$ and $\dot{e}(t)$ are bounded over $[0, +\infty)$. In fact, the boundedness of $\dot{e}(t)$ can be induced from the boundedness of the right-hand side of the e subsystem equation. Using Lemma 2.3 concludes that $\dot{k}(t)$ tends to zero, that is, $e(t)$ tends to zero as $t \longrightarrow +\infty$. This completes the proof.

Remark 3.7 As a result of the above theorem, the following control law

$$\begin{aligned} u &= \kappa_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r) + \Psi_o \eta \\ \dot{k} &= \rho(e)e^2 \\ \dot{\hat{b}} &= \phi_r(e, k, \hat{b}, \hat{\xi}_1, \cdots, \hat{\xi}_r) \\ \dot{\eta} &= M\eta + Nu \\ \dot{\hat{\xi}} &= A_c\hat{\xi} + \lambda(e - \hat{\xi}_1) + Bu - B\Psi_o\eta \end{aligned} \tag{3.64}$$

which is in the form of (3.30) solves the global robust output regulation problem for system (3.29). ∎

3.2.3 Application

The controlled single-input single-output hyperchaotic Lorenz system [39] is described by the following equations:

$$\begin{aligned} \dot{z}_1 &= a_{11}z_1 + a_{12}x_1 \\ \dot{z}_2 &= a_3 z_2 + z_1 x_1 \\ \dot{x}_1 &= x_2 + a_{21}z_1 + a_{22}x_1 - z_1 z_2 \\ \dot{x}_2 &= bu + a_4 z_1 \end{aligned} \tag{3.65}$$

where $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4)$ is a constant parameter vector satisfying a_{11}, $a_3 < 0$ and b is some unknown nonzero constant. A detailed analysis of this system with $u = 0$ has been given and various types of chaotic behaviors for different values of parameter $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4)$ are exhibited. Also, a full state feedback stabilization of this system is studied in [39]. Here, by designating an output $y = x_1$ and defining a tracking error $e = y - F(t)$ where $F(t) = A_m \sin(\omega t + \phi)$, we will consider a more challenging control problem of designing an error output feedback control law such that all the states of the closed-loop system are bounded and the tracking error e asymptotically approaches zero.

To make the problem more interesting, we allow the amplitude A_m to be an arbitrary positive number and initial phase ϕ an arbitrary real number. We will show that the above problem can be formulated as the global robust output regulation problem.

Let $v = \mathrm{col}(v_1, v_2)$ and a linear autonomous system in the form (3.2) has been shown by (3.23) for some $\omega > 0$ with $v_1(t) = F(t)$. Also, we allow the parameter vector $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4)$ to undergo some perturbation. To be more specific, let

$$a = (\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3, \bar{a}_4) + (w_1, \cdots, w_6)$$

where $(\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3, \bar{a}_4)$ represents the nominal value of a and $(w_1, \cdots, w_6)$ the uncertainty of a. To guarantee $a_{11}, a_3 < 0$, we define $\mathbb{W}$ as $\mathbb{W} = \{w | w \in \mathbb{R}^6, \bar{a}_{11} + w_1 < 0, \bar{a}_3 + w_5 < 0\}$.

System (3.65) is in the form (3.29) with $r = 2$ and it cannot be transformed into the form (3.31). Therefore, none of existing results, e.g., the design method in [16], can solve Problem 3.1 for system (3.65).

It can be easily verified that the regulator equations associated with (3.65) and (3.23) are solvable. In fact, from the error equation $e = x_1 - v_1$, we have

$$\mathbf{x}_1(v, w) = v_1 \tag{3.66}$$

Substituting (3.66) into the first equation of (3.65) yields

$$\mathbf{z}_1(v, w) = r_{11} v_1 + r_{12} v_2 \tag{3.67}$$

where

$$r_{11} = -\frac{a_{11} a_{12}}{\omega^2 + a_{11}^2}, \quad r_{12} = -\frac{a_{12}\omega}{\omega^2 + a_{11}^2}$$

Substituting (3.67) and (3.66) into the second equation of (3.65) gives

$$\mathbf{z}_2(v, w) = r_{21} v_1^2 + r_{22} v_1 v_2 + r_{23} v_2^2 \tag{3.68}$$

where

$$r_{21} = -\frac{\omega}{a_3} r_{22} - \frac{r_{11}}{a_3}, \quad r_{22} = \frac{-r_{12} a_3 - 2\omega^2 - 2\omega r_{11}}{a_3^2 + 2\omega^2}, \quad r_{23} = \frac{\omega}{a_3} r_{22}$$

Substituting (3.66) and (3.68) into the third equation of (3.65) gives

$$\begin{aligned} \mathbf{x}_2(v, w) &= \omega v_2 - a_{22} v_1 - a_{21} \mathbf{z}_1 + \mathbf{z}_1 \mathbf{z}_2 \\ &= r_{31} v_1 + r_{32} v_2 + r_{33} v_1^3 + r_{34} v_1^2 v_2 + r_{35} v_1 v_2^2 + r_{36} v_2^3 \end{aligned}$$

where

$$\begin{aligned} r_{31} &= -a_{22} - a_{21} r_{11}, \ r_{32} = \omega - a_{21} r_{12}, \ r_{33} = r_{11} r_{21} \\ r_{34} &= r_{12} r_{21} + r_{11} r_{22}, \ r_{35} = r_{11} r_{23} + r_{12} r_{22}, \ r_{36} = r_{12} r_{23} \end{aligned}$$

Thus $\mathbf{x}_2(v,w)$ can be put into the following form

$$\mathbf{x}_2(v,w) = \mathcal{X}_{21}(w)v^{[1]} + \mathcal{X}_{23}(w)v^{[3]} \tag{3.69}$$

where $v^{[1]} = \mathrm{col}(v_1, v_2)$, $v^{[3]} = \mathrm{col}(v_1^3, v_1^2 v_2, v_1 v_2^2, v_2^3)$, and $\mathcal{X}_{21}(w), \mathcal{X}_{23}(w)$ are appropriate row vectors. Finally, substituting (3.69) into the fourth equation of (3.65) gives

$$\begin{aligned}\mathbf{u}(v,w) &= \sum_{l=1,3} \mathcal{X}_{2l}(w)A^{[l]}(\omega)v^{[l]} + a_4\mathbf{z}_1(v,w)\\ &= r_{41}v_1 + r_{42}v_2 + r_{43}v_1^3 + r_{44}v_1^2v_2 + r_{45}v_1v_2^2 + r_{46}v_2^3\end{aligned}$$

where

$$A^{[1]}(\omega) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad A^{[3]}(\omega) = \begin{bmatrix} 0 & 3\omega & 0 & 0 \\ -\omega & 0 & 2\omega & 0 \\ 0 & -2\omega & 0 & \omega \\ 0 & 0 & -3\omega & 0 \end{bmatrix}$$

and

$$\begin{aligned} r_{41} &= -\omega r_{32} + a_4 r_{11}, \; r_{42} = \omega r_{31} + a_4 r_{12}, \; r_{43} = 3\omega r_{34} \\ r_{44} &= -\omega r_{33} + 2\omega r_{35}, \; r_{45} = -2\omega r_{34} + \omega r_{36}, \; r_{46} = -3\omega r_{35} \end{aligned}$$

Therefore, the steady-state generator described by (3.33) can be constructed as follows

$$\begin{aligned} \tau(v,w) &= \mathrm{col}\big(\mathbf{u}, L_{A_1v}\mathbf{u}, L_{A_1v}^2\mathbf{u}, L_{A_1v}^3\mathbf{u}\big) \\ \Phi &= \left[\begin{array}{c|c} 0 & I_3 \\ \hline -9\omega^4 & 0, -10\omega^2, 0 \end{array}\right] \\ \Psi &= [1,0,0,0] \end{aligned} \tag{3.70}$$

Hence, Assumption 3.4 and Assumption 3.5 are satisfied. So we can define the following internal model

$$\dot{\eta} = M\eta + Nu$$

where

$$M = \left[\begin{array}{c|c} 0 & I_3 \\ \hline -m_1 & -m_2, -m_3, -m_4 \end{array}\right], \quad N = \mathrm{col}(0,0,0,1)$$

and parameters $m_i > 0$ are such that M is Hurwitz.

Performing transformation (3.35) gives

$$\begin{aligned} \dot{\bar{z}}_1 &= a_{11}\bar{z}_1 + a_{12}e \\ \dot{\bar{z}}_2 &= a_3\bar{z}_2 + (\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1 \\ \dot{\bar{\eta}} &= (M + N\Psi_o)\bar{\eta} + N\bar{u} \\ \dot{\bar{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\bar{x} + \begin{bmatrix} \bar{g}_1 \\ \Psi_o\bar{\eta} + \bar{u} + \bar{g}_2 \end{bmatrix} \end{aligned}$$

where

$$\bar{g}_1 = a_{21}\bar{z}_1 + a_{22}e - (\bar{z}_1 + \mathbf{z}_1)(\bar{z}_2 + \mathbf{z}_2) + \mathbf{z}_1\mathbf{z}_2, \quad \bar{g}_2 = a_4\bar{z}_1$$

It can be seen that Assumption 3.6 is satisfied for the $(\bar{z}_1, \bar{z}_2)$ subsystem. By Theorem 3.1, the global robust output regulation problem for this system is solvable. In fact, by choosing

$$c_1 = b^{-1}MN, \ c_2 = b^{-1}N, \ s_1 = \Psi_o N, \ s_2 = \Psi_o MN$$

performing the transformations (3.39) and (3.41), and incorporating the observer (3.44), we can obtain system (3.46) as follows

$$\begin{aligned}
\dot{\bar{z}}_1 &= a_{11}\bar{z}_1 + a_{12}e \\
\dot{\bar{z}}_2 &= a_3\bar{z}_2 + (\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1 \\
\dot{\tilde{\eta}} &= M\tilde{\eta} + (Mc_1 e - c_1\bar{g}_1 - c_2\bar{g}_2) \\
\dot{\tilde{\xi}} &= \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 0 \end{bmatrix} \tilde{\xi} + \begin{bmatrix} -\lambda_1(1 - b^{-1})e + G_1 \\ -\lambda_2(1 - b^{-1})e + \Psi_o\tilde{\eta} + G_2 \end{bmatrix} \\
\dot{e} &= b\hat{\xi}_2 + b\tilde{\xi}_2 + bG_1 \\
\dot{\hat{\xi}}_2 &= u - \Psi_o\eta + \lambda_2(e - \hat{\xi}_1)
\end{aligned}$$

where $G_1 = s_1 e + b^{-1}\bar{g}_1$ and $G_2 = s_2 e - b^{-1}s_1\bar{g}_1 + b^{-1}\bar{g}_1$. According to the design procedure detailed in Section 3, we can obtain a specific control law in the form of (3.64) with various design functions as follows

$$\begin{aligned}
\omega_1 &= \hat{\xi}_2 - \mathcal{N}(k)\rho(e)e \\
\mathcal{N}(k) &= k^2\cos(k), \ \rho(e) = 5(e^6 + 1) \\
\kappa_2 &= -2\omega_1 - \lambda_2(e - \hat{\xi}_1) - \hat{b}E_1\hat{\xi}_2 - \omega_1 E_1^2 - K_1 \\
\phi_2 &= \omega_1 E_1\hat{\xi}_2, \ E_1 = -5\mathcal{N}(k)(7e^6 + 1) \\
K_1 &= \left[k^2\sin(k) - 2k\cos(k)\right]\dot{k}\rho(e)e
\end{aligned} \tag{3.71}$$

Simulations are performed for the closed-loop system composed of system (3.65) and a controller in the form (3.64). Various parameters are chosen as follows.

$$\lambda = \mathrm{col}(2,3), \ (m_1, m_2, m_3, m_4) = (4, 12, 13, 6), \ \omega = 1, \ b = 1$$
$$(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_4) = (-10, 10, 28, -1, -8/3, -1)$$

The initial conditions are $(z_1(0), z_2(0), x_1(0), x_2(0)) = (-2, 1, 2, 1)$, $v_0 = \mathrm{col}(1, 0)$, $\eta(0) = 0$, $\hat{\xi}(0) = 0$, $\hat{b}(0) = 0$, and $k(0) = 1$. The responses of the tracking error, control input, and the plant state variables are shown in Figures 3.4 and 3.5.

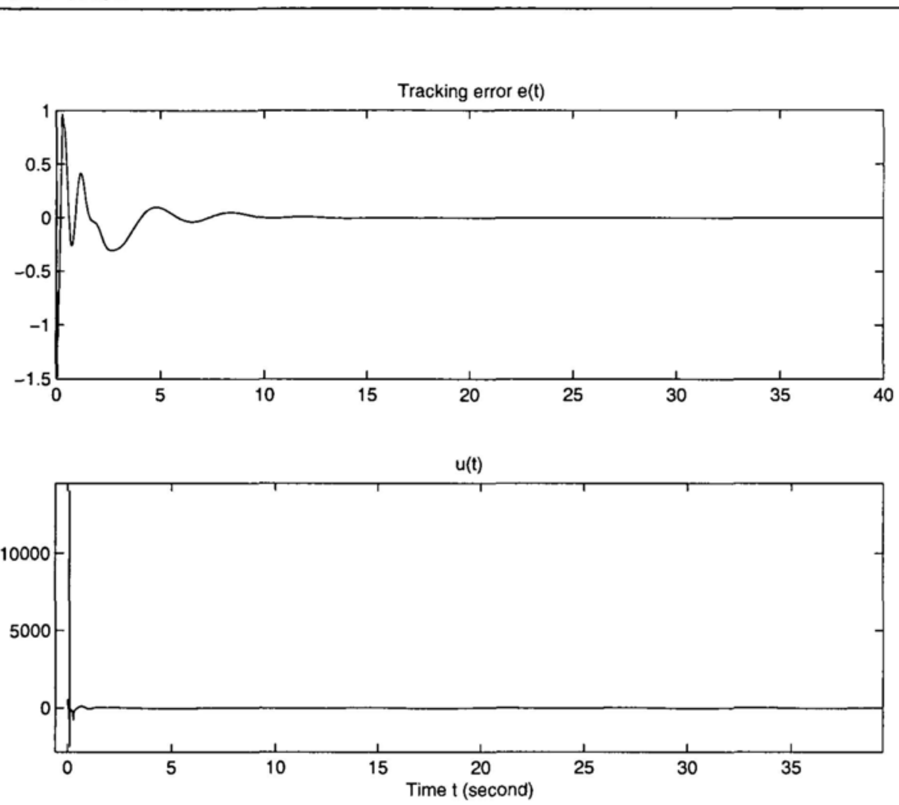

Figure 3.4: Profiles of tracking error $e(t)$ and control input $u(t)$.

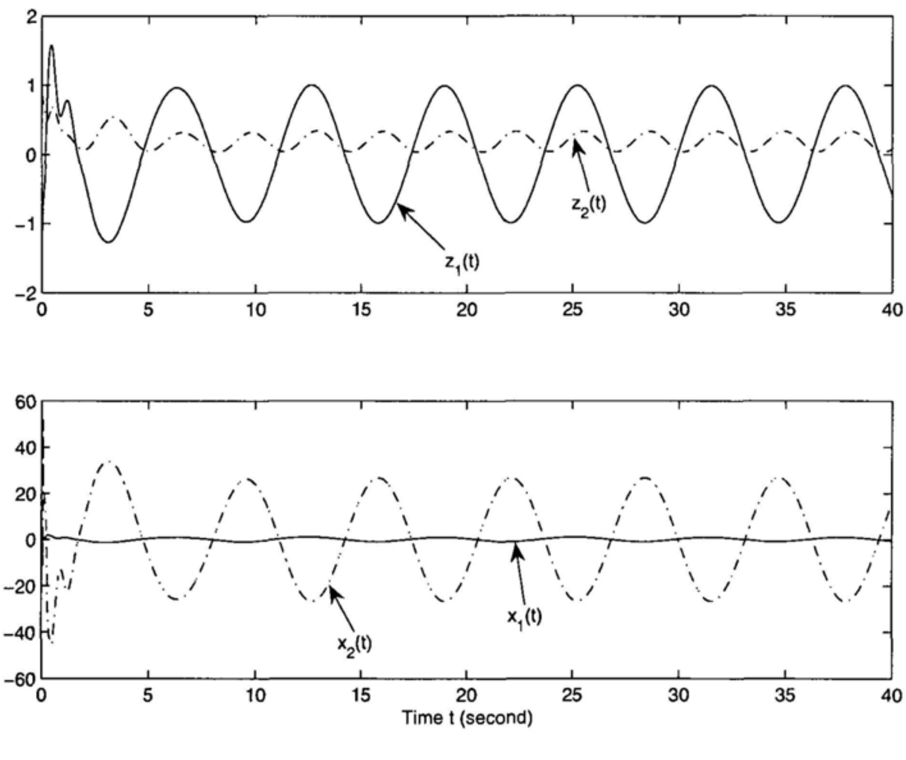

Figure 3.5: State responses of (z_1, z_2, x_1, x_2).

3.3 Conclusion

In this chapter, we have presented the solvability conditions for the global robust output regulation problem for nonlinear systems (3.1) and (3.29) by output feedback control. Since the zero dynamics of our system is not linear, the existing approach as used in [16] is not applicable here. Moreover, our approach does not assume the sign of high frequency gain is known. To illustrate the effectiveness of our approach, we have applied our approach to the global robust asymptotic tracking problem of the well known third-order and fourth-order Lorenz systems.

□ **End of chapter.**

Chapter 4

Global adaptive output regulation using output feedback control

In this chapter, we address the global adaptive output regulation problem of output feedback systems. As the exosystem described by equation (4.2) contains some uncertain parameter, the approach presented in Chapter 3 is not applicable here. However, the problem can still be solved by a new approach. We present the solution with two cases. In Section 4.1, we derive the solution for the systems in a special form. Then in Section 4.2, the solution for the general case is given.

4.1 Special case

In this section, we consider the output feedback system in the following simple form

$$\begin{aligned} \dot{z} &= f(z,y,v,w) \\ \dot{y} &= g(z,y,v,w) + b(w)u \\ e &= y - q(v,w) \end{aligned} \tag{4.1}$$

where $(z,y) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $e \in \mathbb{R}$ is the error output and $u \in \mathbb{R}$ is the control input. $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$ with $\mathcal{W}$ nonempty is a constant uncertain parameter vector, and $v(t) \in \mathbb{R}^{n_v}$ represents the time-varying reference and/or disturbance. The functions f, g and q, and b are supposed to be sufficiently smooth in their arguments satisfying $f(0,0,0,w) = 0$, $g(0,0,0,w) = 0$, $q(0,w) = 0$, and $b(w) > 0$, respectively, for any $w \in \mathcal{W}$. It is also assumed that $v(t)$ is generated by a linear exosystem

$$\dot{v} = A_1(\sigma)v, \quad v(0) = v_0 \tag{4.2}$$

where $\sigma \in \mathbb{S} \subset \mathbb{R}^{n_\sigma}$ represents the uncertainty in the exosystem. To have our problem well posed, we also assume that all the eigenvalues of $A_1(\sigma)$ are distinct with zero real parts for all $\sigma \in \mathbb{S}$.

As a result, the general solution of the exosystem is a sum of finitely many sinusoidal functions with their frequencies depending on the eigenvalues of $A_1(\sigma)$ and amplitudes and phase angles on the initial condition v_0.

Briefly, the problem can be stated as follows: Given $\mathcal{W}$ and $\mathbb{S}$, design an output feedback control law of the form:

$$u = u_K(\zeta, e), \quad \dot{\zeta} = g_K(\zeta, e) \tag{4.3}$$

where both u_K and g_K are sufficiently smooth vanishing at the origin such that, for any initial condition $(z(0), y(0), v_0, \zeta(0))$, and any constant parameter $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$, the solution of the closed-loop system composed of (4.1) to (4.3) exists and is bounded over $[0, +\infty)$ and the error output $e(t)$ asymptotically approaches zero.

As the exosystem can generate a trigonometric polynomial of arbitrary amplitudes, phase angles, and frequencies, the problem includes the asymptotic tracking of an unknown sinusoidal signal, or asymptotic disturbance rejection of an unknown sinusoidal signal as special cases. It also includes the global stabilization problem as a special case if the initial condition v_0 is set to zero. Moreover, if one considers the exosystem (4.2) as a master system and the plant (4.1) as a slave system, then the problem can also be interpreted as the global robust output synchronization of these two systems.

This class of systems is general enough to include some well known systems such as the controlled FHN model, Lorenz system, and Chua's circuit as special cases [14] [19] [57] [59] [74] [85] [93]. On the other hand, the formulation of our problem is also general enough to include several interesting control problems such as global robust stabilization, global robust output synchronization/asymptotic tracking, global disturbance rejection, etc., as special problems. Thus, the solvability of the problem described in this section will automatically lead to the solution of several interesting control problems involving such systems as the controlled FHN model and the controlled Lorenz system.

Section 4.1.1 lists some assumptions and some preliminaries. In Section 4.1.2, we will present the solution of the problem formulated in Section 4.1.1. The problem is solved by combining the internal model approach and some adaptive control technique. In Section 4.1.3, as applications of the main result obtained in Section 4.1.2, we solve two typical control problems associated with two well known systems, i.e., the FHN model and the Lorenz system.

4.1.1 Assumptions and preliminaries

To accomplish the problem conversion, we need two standard assumptions in the following.

Assumption 4.1 There exists a globally defined smooth function $\mathbf{z} : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma} \mapsto$

$\mathbb{R}^n$ with $\mathbf{z}(0, w, \sigma) = 0$ such that

$$\frac{\partial \mathbf{z}(v, w, \sigma)}{\partial v} A_1(\sigma) v = f(\mathbf{z}(v, w, \sigma), q(v, w), v, w) \tag{4.4}$$

for all $(v, w, \sigma) \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$. ∎

Under Assumption 4.1, let $\mathbf{y}(v, w, \sigma) = q(v, w)$, and

$$\mathbf{u}(v, w, \sigma) = b^{-1}\big[\mathcal{L}_{A_1(\sigma)v} q(v, w) - g(\mathbf{z}(v, w, \sigma), q(v, w), v, w)\big] \tag{4.5}$$

where $\mathcal{L}_{A_1(\sigma)v} q(v, w) = \big(\partial q(v, w)/\partial v\big) A_1(\sigma) v$. Then, $\mathbf{z}(v, w, \sigma)$, $\mathbf{y}(v, w, \sigma)$ and $\mathbf{u}(v, w, \sigma)$ is the solution of the regulator equations associated with (4.1) and (4.2). For the existence of the internal model, we need one more assumption.

Assumption 4.2 The function $\mathbf{u}(v, w, \sigma)$ is a polynomial in v with coefficients depending on w and σ. ∎

Remark 4.1 Neither Assumption 4.1 nor Assumption 4.2 is restrictive as it might appear to be. As will be seen later, all the systems considered in this section satisfy these two assumptions. ∎

Remark 4.2 Under Assumption 4.2, there exists an integer s such that $\mathbf{u}(v, w, \sigma)$ satisfies, for all trajectories $v(t)$ of the exosystem, all $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$,

$$\begin{aligned} \frac{d^s \mathbf{u}(v, w, \sigma)}{dt^s} &= a_1(\sigma) \mathbf{u}(v, w, \sigma) + a_2(\sigma) \frac{d\mathbf{u}(v, w, \sigma)}{dt} \\ &\quad + \cdots + a_s(\sigma) \frac{d^{s-1} \mathbf{u}(v, w, \sigma)}{dt^{(s-1)}} \end{aligned} \tag{4.6}$$

where $a_1(\sigma), a_2(\sigma), \cdots, a_s(\sigma)$ are real scalars such that all the roots of the polynomial

$$P^\sigma(\lambda) = \lambda^s - a_1(\sigma) - a_2(\sigma)\lambda - \cdots - a_s(\sigma)\lambda^{s-1} \tag{4.7}$$

are distinct with zero real part for all $\sigma \in \mathbb{S}$ [30].

Let $\tau(v, w, \sigma) = \mathrm{col}\big(\mathbf{u}, \mathcal{L}_{A_1(\sigma)v}\mathbf{u}, \cdots, \mathcal{L}^{s-1}_{A_1(\sigma)v}\mathbf{u}\big)$ where $\mathcal{L}^{i+1}_{A_1(\sigma)v}\mathbf{u} = \big(\partial \mathcal{L}^{i}_{A_1(\sigma)v}\mathbf{u}/\partial v\big) A_1(\sigma) v$ for any integer $i \geq 1$, and

$$\begin{aligned} \Phi(\sigma) &= \left[\begin{array}{c|c} 0 & I_{s-1} \\ \hline a_1(\sigma) & a_2(\sigma), \cdots, a_s(\sigma) \end{array}\right] \\ \Gamma &= [1, 0, \cdots, 0]_{1 \times s} \end{aligned} \tag{4.8}$$

Then $\tau(v, w, \sigma)$ satisfies

$$\begin{aligned} \frac{\partial \tau(v, w, \sigma)}{\partial v} A_1(\sigma) v &= \Phi(\sigma) \tau(v, w, \sigma) \\ \mathbf{u}(v, w, \sigma) &= \Gamma \tau(v, w, \sigma) \end{aligned} \tag{4.9}$$

∎

Remark 4.3 System (4.9) is used to generate the steady-state input $\mathbf{u}(v, w, \sigma)$, and thus it is a steady-state generator with output u according Definition 2.6. Since, for each $\sigma \in \mathbb{S}$, $(\Gamma, \Phi(\sigma))$ is observable and the eigenvalues of $\Phi(\sigma)$ have zero real part, for any controllable pair (M, N) with $M \in \mathbb{R}^{s\times s}$ a Hurwitz matrix and $N \in \mathbb{R}^{s\times 1}$ a column vector, there is a unique nonsingular matrix $T(\sigma)$ satisfying the following Sylvester equation:

$$T(\sigma)\Phi(\sigma) - MT(\sigma) = N\Gamma \tag{4.10}$$

Let $\theta(v, w, \sigma) = T(\sigma)\tau(v, w, \sigma)$ which satisfies

$$\dot{\theta} = (M + N\Psi^{\sigma})\theta, \quad \mathbf{u} = \Psi^{\sigma}\theta$$

with $\Psi^{\sigma} = \Gamma T^{-1}(\sigma)$. Then we can define the internal model as follows:

$$\dot{\eta} = M\eta + Nu \tag{4.11}$$

∎

Attaching the internal model (4.11) to (4.1) and performing the following coordinate transformation:

$$\bar{z} = z - \mathbf{z}(v, w, \sigma), \ \tilde{\eta} = \eta - \theta(v, w, \sigma) - Nb^{-1}e \tag{4.12}$$

gives a system as follows:

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, \mu) \\ \dot{\tilde{\eta}} &= M\tilde{\eta} + \bar{f}_2(\bar{z}, e, \mu) \\ \dot{e} &= \bar{g}_e(\bar{z}, \tilde{\eta}, e, \mu) + b(u - \Psi^{\sigma}\eta) \end{aligned} \tag{4.13}$$

where $\mu = (v, w, \sigma)$ and

$$\begin{aligned} \bar{f}(\bar{z}, e, \mu) &= f(\bar{z} + \mathbf{z}(v, w, \sigma), e + q(v, w), v, w) \\ &\quad - f(\mathbf{z}(v, w, \sigma), q(v, w), v, w) \\ \bar{f}_2(\bar{z}, e, \mu) &= MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z}, e, \mu) \\ \bar{g}_e(\bar{z}, \tilde{\eta}, e, \mu) &= \bar{g}(\bar{z}, e, \mu) + b\Psi^{\sigma}\tilde{\eta} + \Psi^{\sigma}Ne \\ \bar{g}(\bar{z}, e, \mu) &= g(\bar{z} + \mathbf{z}(v, w, \sigma), e + q(v, w), v, w) \\ &\quad - g(\mathbf{z}(v, w, \sigma), q(v, w), v, w) \end{aligned}$$

It can be verified that, for any $\mu \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$

$$\bar{f}(0, 0, \mu) = 0, \quad \bar{f}_2(0, 0, \mu) = 0, \quad \bar{g}_e(0, 0, 0, \mu) = 0$$

Remark 4.4 The quantity $\mu(t)$ in the augmented system (4.13) can be viewed as an unknown time-varying disturbance. It can be seen that if there exists a control law of the form

$$\begin{aligned} u &= k_\zeta(\zeta, \eta, e) \\ \dot{\zeta} &= g_\zeta(\zeta, \eta, e) \end{aligned} \tag{4.14}$$

that solves the global robust stabilization problem of system (4.13) in the sense that, for any initial condition of the closed-loop system and the exosystem, and any fixed unknown parameter $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$, the solution of the closed-loop system is bounded for all $t \geq 0$, and the state of the augmented system (4.13) tends to zero as t tends to infinity, then the following control law

$$\begin{aligned} u &= k_\zeta(\zeta, \eta, e) \\ \dot{\eta} &= M\eta + Nu \\ \dot{\zeta} &= g_\zeta(\zeta, \eta, e) \end{aligned}$$

solves the global robust output regulation of the original system (4.1). ∎

4.1.2 Main result

We will consider the global robust stabilization problem of system (4.13). For this purpose, we need one more assumption as follows.

Assumption 4.3 For any compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$, there exists a C^1 function $V_{\bar{z}}$ satisfying $\underline{\alpha}(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \bar{\alpha}(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of the subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$

$$\dot{V}_{\bar{z}} \leq -\alpha(\|\bar{z}\|) + \delta\gamma(e)$$

where δ is some unknown positive constant, $\alpha(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying $\lim_{s \to 0^+} \sup(\alpha^{-1}(s^2)/s) < \infty$ and $\gamma(\cdot)$ is a known smooth p.d. function. ∎

Next, we will derive a stabilizability property for the following auxiliary system

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, \mu) \\ \dot{\tilde{\eta}} &= M\tilde{\eta} + \bar{f}_2(\bar{z}, e, \mu) \\ \dot{e} &= \bar{g}_e(\bar{z}, \tilde{\eta}, e, \mu) + b\tilde{u} \end{aligned} \tag{4.15}$$

which is obtained by letting $\tilde{u} = u - \Psi^\sigma \eta$ in (4.13).

Lemma 4.1 Consider system (4.15). Under Assumption 4.3, there exist a smooth positive function $\rho(e) \geq 1$, a controller of the form

$$\tilde{u} = -k\rho(e)e, \quad \dot{k} = \rho(e)e^2 \tag{4.16}$$

such that the closed-loop system composed of (4.15) and (4.16) has the property that, for any given $(v_0, w, \sigma) \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$, there exists a C^1 Lyapunov function candidate $V(\bar{z}, \tilde{\eta}, e, k)$ such that

$$\dot{V} \leq -\|\bar{z}\|^2 - \|\tilde{\eta}\|^2 \tag{4.17}$$

∎

Proof: For the given $(v_0, w, \sigma) \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$, let Σ be a compact subset of $\mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$ containing $(v(t), w, \sigma)$, $t \geq 0$, with $v(t)$ being generated by (4.2) with initial state v_0.

Consider the $(\bar{z}, \tilde{\eta})$ subsystem of the system (4.15) which is in the form of (2.11) with $z_1 = \bar{z}$, $z_2 = \tilde{\eta}$, $u = e$ and $\mu(t) = \mathrm{col}(v(t), w, \sigma)$. Since M is Hurwitz, $\bar{f}(0, 0, \mu) = 0$ and $\bar{g}(0, 0, \mu) = 0$, by Lemma 2.1, there exists a C^1 function $V_1(\bar{z}, \tilde{\eta})$ satisfying $\underline{\alpha}_1(\|\bar{z}, \tilde{\eta}\|) \leq V_1(\bar{z}, \tilde{\eta}) \leq \bar{\alpha}_1(\|\bar{z}, \tilde{\eta}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1(\cdot)$ and $\bar{\alpha}_1(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of $(\bar{z}, \tilde{\eta})$ subsystem

$$\dot{V}_1 \leq -\|\bar{z}\|^2 - \|\tilde{\eta}\|^2 + \delta_e \gamma_e(e)$$

for some positive constant δ_e and smooth p.d. function $\gamma_e(\cdot)$.

Let $z = \mathrm{col}(\bar{z}, \tilde{\eta})$. By Remark 2.4, for any smooth function $\Delta(z) > 0$, there exists a C^1 function $V_z(z)$ satisfying $\underline{\alpha}_2(\|z\|) \leq V_z(z) \leq \bar{\alpha}_2(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_2(\cdot)$ and $\bar{\alpha}_2(\cdot)$ such that

$$\dot{V}_z \leq -\Delta(z)\|z\|^2 + \bar{\delta}_e \bar{\gamma}_e(e) e^2 \tag{4.18}$$

for some positive constant $\bar{\delta}_e$ and smooth positive function $\bar{\gamma}_e(\cdot)$.

Let

$$V(\bar{z}, \tilde{\eta}, e, k) = V_z(z) + \frac{1}{2}e^2 + \frac{1}{2}b(k - \bar{k})^2 \tag{4.19}$$

for some real constant $\bar{k} > 0$. Then by completing the squares, the derivative of V along the trajectory of system (4.15) under the control law (4.16) satisfies

$$\begin{aligned} \dot{V} &= \dot{V}_z + e\big[\bar{g}_e(\bar{z}, \tilde{\eta}, e, \mu) - bk\rho(e)e\big] + b\dot{k}(k - \bar{k}) \\ &\leq \dot{V}_z + \frac{1}{2}\bar{g}_e^2 + \frac{1}{2}e^2 - bk\rho(e)e^2 + b\dot{k}(k - \bar{k}) \end{aligned} \tag{4.20}$$

By Lemma 2.4, we have, for all $\mu \in \Sigma$

$$|\bar{g}_e(\bar{z}, \tilde{\eta}, e, \mu)|^2 \leq p_1\big(\phi_1(z)\|z\|^2 + \phi_2(e)e^2\big) \tag{4.21}$$

for some positive constant p_1 and smooth positive functions $\phi_1(z)$ and $\phi_2(e)$.

Using inequalities (4.18) and (4.21) in (4.20) gives

$$\begin{aligned} \dot{V} &\leq -\Delta(z)\|z\|^2 - \bar{\delta}_e\bar{\gamma}_e(|e|)e^2 + \frac{p_1}{2}\big(\phi_1(z)\|z\|^2 \\ &\quad + \phi_2(e)e^2\big) + \frac{1}{2}e^2 - bk\rho(e)e^2 + b\dot{k}(k - \bar{k}) \\ &\leq -\Big(\Delta(z) - \frac{p_1}{2}\phi_1(z)\Big)\|z\|^2 + \Big(\bar{\delta}_e\bar{\gamma}_e(|e|) \\ &\quad + \frac{p_1}{2}\phi_2(e) + \frac{1}{2}\Big)e^2 - bk\rho(e)e^2 + b\dot{k}(k - \bar{k}) \end{aligned}$$

Letting $\Delta(z)$ and $\rho(e)$ be smooth functions satisfying

$$\begin{aligned}\Delta(z) &\geq \frac{1}{2}p_1\phi_1(z) + \frac{b}{2}\|\Psi^\sigma\|^2 + 1 \\ \rho(e) &\geq \max\{\bar{\gamma}_e(|e|),\ \phi_2(e),\ 1\}\end{aligned}$$

and $\bar{k}$ be such that

$$\bar{k} \geq b^{-1}\Big(\bar{\delta}_e + \frac{1}{2}p_1 + \frac{1+b}{2}\Big) \tag{4.22}$$

gives

$$\dot{V} \leq -\|z\|^2 + b(\dot{k} - \rho(e)e^2)(k - \bar{k})$$

Hence, the function $V(\bar{z}, \tilde{\eta}, e, k)$ defined by (4.19) satisfies (4.17) for system (4.15) under the controller (4.16).

Remark 4.5 Note that Lemma 4.1 implies that the trajectory of the closed-loop system composed of (4.15) and (4.16) is bounded and $\lim_{t\to\infty}(\|\bar{z}(t)\| + \|\tilde{\eta}(t)\|) = 0$. The boundedness of $e(t)$ and $\dot{e}(t)$ implies $\dot{k}(t)$ is bounded and uniformly continuous. Thus by Lemma 2.3, $\dot{k}(t) = \rho(e)e^2$ tends to zero as $t \to +\infty$, that is, $e(t)$ tends to zero as $t \to +\infty$. Therefore we have the following result. ∎

Corollary 4.1 Under Assumptions 4.1 to 4.3, if the parameter σ is known, then the controller

$$u = -k\rho(e)e + \Psi^\sigma\eta, \quad \dot{k} = \rho(e)e^2 \tag{4.23}$$

solves the global robust stabilization problem of system (4.13) in the sense described in Remark 4.4. ∎

Remark 4.6 The global robust stabilization of system (4.1) with v set to zero can be viewed as a special case of the global robust output regulation. In fact, when v is set to zero, there is no need to introduce an internal model. We can set the dimension of $\tilde{\eta}$ in (4.13) to zero. Then system (4.13) reduces to system (4.1) with v set to zero. The control law (4.23) reduces to the following simpler control law

$$u = -k\rho(e)e, \quad \dot{k} = \rho(e)e^2 \tag{4.24}$$

∎

Remark 4.7 In control law (4.23), k is called dynamic gain and is introduced to generate a high gain independent of $v(t)$, w, and σ. This control technique is called self tuning regulator in literature. When $(v(t), w, \sigma)$ is contained in some known compact subset, k can be taken to be some sufficiently large number determined by the boundaries of the compact subset. In this case, control law (4.23) can be further reduced to $u = -k\rho(e)e + \Psi^\sigma\eta$ where k is a known positive number. ∎

In the case where σ is unknown, the control law (4.23) is not implementable. Nevertheless, we can still solve the problem by further incorporating the adaptive control technique into our design method.

Theorem 4.1 Under Assumptions 4.1 to 4.3, let $\rho(e)$ be the same as in (4.23). Then the following controller

$$\begin{aligned} u &= -k\rho(e)e + \hat{\Psi}\eta \\ \dot{\hat{\Psi}} &= -\eta^{\top}e, \quad \dot{k} = \rho(e)e^2 \end{aligned} \tag{4.25}$$

solves the global robust stabilization problem of system (4.13) in the sense described in Remark 4.4. As a result, the following controller

$$\begin{aligned} u &= -k\rho(e)e + \hat{\Psi}\eta \\ \dot{\eta} &= M\eta + Nu, \quad \dot{\hat{\Psi}} = -\eta^{\top}e, \quad \dot{k} = \rho(e)e^2 \end{aligned} \tag{4.26}$$

solves the global robust output regulation for system (4.1). ∎

Proof: Let $V(\bar{z}, \tilde{\eta}, e, k)$ be defined as in (4.19), and $\tilde{\Psi}(t) = \Psi^{\sigma} - \hat{\Psi}(t)$ where $\hat{\Psi}$ is viewed as the estimation of Ψ^{σ}. Let U be defined as follows:

$$U(\bar{z}, \tilde{\eta}, e, k) = V(\bar{z}, \tilde{\eta}, e, k) + \frac{1}{2}b\tilde{\Psi}\tilde{\Psi}^{\top}$$

Then the derivative of U along the trajectory of system (4.13) and (4.26) satisfies

$$\begin{aligned} \dot{U}|_{(4.13)+(4.26)} &= \dot{V}|_{(4.13)+(4.26)} - b\dot{\tilde{\Psi}}\tilde{\Psi}^{\top} \\ &= \dot{V}|_{(4.15)+(4.16)} + \frac{\partial V}{\partial e}\cdot(-\tilde{\Psi}\eta) - b\dot{\tilde{\Psi}}\tilde{\Psi}^{\top} \\ &\leq -\|z\|^2 - b(\dot{\hat{\Psi}} + \eta^T e)\tilde{\Psi}^{\top} \\ &= -\|z\|^2 \end{aligned}$$

Now using the same argument as Remark 4.5 completes the proof.

Remark 4.8 We will now consider the convergence issue of the parameter $\hat{\Psi}$ which is related to the dimension of the internal model used. Recall from [61] that a monic polynomial (4.7) is called a global *zeroing polynomial* of $\mathbf{u}(v, w, \sigma)$ on $\mathbb{S}$, if, along all trajectories $v(t)$ of the exosystem (4.2) and all $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$, $\mathbf{u}(v, w, \sigma)$ satisfies a differential equation of the form (4.6). A monic polynomial $P^{\sigma}(\lambda)$ is called a minimal zeroing polynomial of $\mathbf{u}(v, w, \sigma)$ on $\mathbb{S}$ if $P^{\sigma}(\lambda)$ is a zeroing polynomial of $\mathbf{u}(v, w, \sigma)$ on $\mathbb{S}$ of least degree. An internal model whose dimension is equal to the degree of the minimal zeroing polynomial of $\mathbf{u}(v, w, \sigma)$ is called the minimal internal model. To determine the minimal zeroing polynomial, assume the nonzero eigenvalues of $A_1(\sigma)$ be $\pm \jmath\omega_1, \cdots, \pm \jmath\omega_{n_k}$[1] with $\omega_i > 0$, $i = 1, \cdots, n_k$ where $n_k = n_v/2$ if n_v is even and $n_k = (n_v - 1)/2$ if n_v is odd. Then, it can be deduced from the result of [30] that, there exist a set

$$\Omega = \left\{ l_1\omega_1 + \cdots + l_{n_k}\omega_{n_k}, l_1, \cdots, l_{n_k} = 0, \pm 1, \cdots, \pm\infty \right\}$$

an integer r, and r distinct members of Ω denoted by $\hat{\omega}_l$, $l = 1, \cdots, r$, such that, along all trajectories $v(t)$ of the exosystem, for any $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$,

$$\mathbf{u}(v(t), w, \sigma) = \sum_{l=1}^{r} C_l(v_0, w, \sigma) e^{\jmath\hat{\omega}_l t} \tag{4.27}$$

where $C_l(v_0, w, \sigma) \in \mathbb{C}$ are not identically zero for all (v_0, w, σ). Thus, the minimal zeroing polynomial of $\mathbf{u}(v, w, \sigma)$ is $P^\sigma(\lambda) = \prod_{l=1}^{r}(\lambda + \jmath\hat{\omega}_l)$. By Lemma 2.6, under Assumptions 4.1 to 4.2, if the internal model is of minimal dimension, and v_0, w and σ are such that none of $C_l(v_0, w, \sigma)$ is zero, then the feedback controller (4.26) is such that $\lim_{t\to\infty}(\hat{\Psi} - \Psi^\sigma) = 0$. ∎

Remark 4.9 Clearly, the dimension of the minimal internal model cannot be less than the dimension of the matrix $A_1(\sigma)$. Thus, any internal model whose dimension is equal to the dimension of the matrix $A_1(\sigma)$ is a minimal internal model. As will be seen in the next section, both of the two examples will employ minimal internal model. ∎

4.1.3 Applications

Global disturbance rejection of FHN model

Consider the controlled FHN model described by the following equations:

$$\begin{aligned} \dot{x}_1 &= x_1 - \frac{1}{3}x_1^3 - x_2 + x_3 + F(t) + F_u \\ \dot{x}_2 &= \varepsilon_1(x_1 + a_1 - a_2 x_2) \\ \dot{x}_3 &= \varepsilon_2(-x_1 + a_3 - a_4 x_3) \end{aligned} \tag{4.28}$$

where $F(t) = A_m \sin(\omega t + \phi)$ is the external disturbance, F_u is the control input, and $a_1, \cdots, a_4$, ε_1 and ε_2 are given positive constants. The system with $F_u = 0$ is taken from [74] which studies the bursting mechanism in excitable systems. Without the control F_u, the system may exhibit chaotic behavior as shown in Figure 4.1 where $\varepsilon_1 = \varepsilon_2 = 0.1$, $a_1 = \cdots = a_4 = 1$, and $(A_m, \omega, \phi) = (0.5, 0.2, -2\pi/3)$

Here by introducing a control input F_u, we aim at achieving global disturbance rejection in the sense that the control can guarantee that the trajectory of the closed-loop system starting from any initial condition is globally bounded and all states (x_1, x_2, x_3) of

[1] Throughout this thesis, let $\jmath := \sqrt{-1}$ be the imaginary unit.

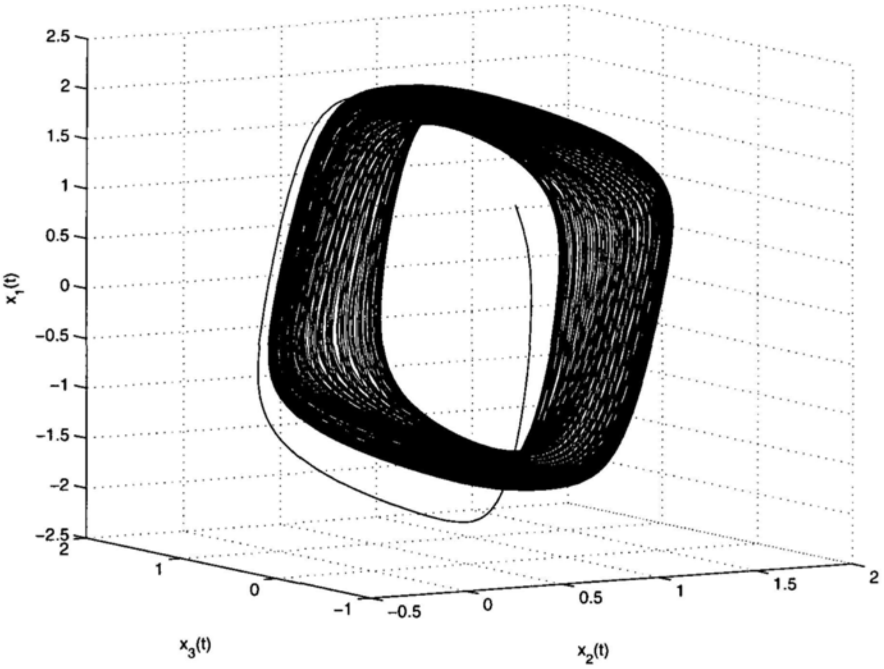

Figure 4.1: 3-D plot of FHN model with initial state $x(0) = (1.2, 0.5, -0.5)$.

the plant (4.28) converge to the point $(0, a_1/a_2, a_3/a_4)$ as $t \to +\infty$. To make the problem more interesting, we allow the amplitude A_m and frequency ω to be arbitrary positive numbers and phase angle ϕ an arbitrary real number. Also, we allow the system parameters ε_1 and ε_2 to be arbitrary positive numbers. We will show that the above problem can be formulated as the global robust output regulation problem described in Section 4.1.1.

Let $v = \text{col}(v_1, v_2)$ and define a linear autonomous uncertain system in the form (4.2) as follows:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = A_1(\omega) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad v_0 := \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} = \begin{bmatrix} A_m \sin \phi \\ A_m \cos \phi \end{bmatrix} \tag{4.29}$$

where

$$A_1(\omega) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

which has a pair of distinct eigenvalues $\pm \jmath\omega$ for all $\omega \in \mathbb{S} = \{\omega | \omega > 0\}$. It can be seen that $v_1(t) = F(t)$. Also, let $\varepsilon_1 = \bar{\varepsilon}_1 + w_1$ and $\varepsilon_2 = \bar{\varepsilon}_2 + w_2$ with $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ the nominal values of ε_1 and ε_2, respectively, and with w_1 and w_2 being uncertainties.

To put system (4.28) in the form (4.1), letting

$$\begin{aligned} z_1 &= x_2 - a_1/a_2 \\ z_2 &= x_3 - a_3/a_4 \\ e &= y = x_1 \\ u &= F_u - a_1/a_2 + a_3/a_4 \end{aligned}$$

gives

$$\begin{aligned} \dot{z}_1 &= -\varepsilon_1 a_2 z_1 + \varepsilon_1 e \\ \dot{z}_2 &= -\varepsilon_2 a_4 z_2 - \varepsilon_2 e \\ \dot{y} &= y - \frac{1}{3}y^3 - z_1 + z_2 + v_1 + u \\ e &= y \end{aligned} \tag{4.30}$$

Thus, system (4.30) is in the form (4.1) with $\sigma = \omega$ and $\mathcal{W} = \{(w_1, w_2) | w_1 \geq 0, w_2 \geq 0\}$. Clearly, if we can solve the global robust output regulation problem for system (4.30) described in Section 4.1.1 with $q(v, w) = 0$, $\mathcal{W}$ and $\mathbb{S}$ defined above, we can also solve the global disturbance rejection problem described above.

We now verify that system (4.30) satisfies Assumptions 4.1 to 4.3. From the last equation of system (4.30), we have $\mathbf{y}(v, w, \omega) = 0$, and from the first two equations of (4.30), we have

$$\mathbf{z}_1(v, w, \omega) = \mathbf{z}_2(v, w, \omega) = 0$$

It follows from the third equation of (4.30) that $\mathbf{u}(v, w, \omega) = -v_1$. Thus Assumptions 4.1 and 4.2 are satisfied. Also it can be easily verified that

$$\frac{d^2\mathbf{u}(v, w, \omega)}{dt^2} + \omega^2 \mathbf{u}(v, w, \omega) = 0 \tag{4.31}$$

Thus, by Remark 4.2, the steady-state generator can be obtained with

$$\tau(v, w, \omega) = \begin{bmatrix} -v_1 \\ -\omega v_2 \end{bmatrix}, \ \Phi = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \ \Gamma = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4.32}$$

Given any controllable pair (M, N) of the form

$$M = \begin{bmatrix} 0 & 1 \\ -m_1 & -m_2 \end{bmatrix}, \ N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $m_1, m_2 > 0$, we can obtain an internal model as follows:

$$\dot{\eta} = M\eta + Nu \tag{4.33}$$

Solving the Sylvester equation (4.10) gives

$$\begin{aligned} T^{-1}(\omega) &= \begin{bmatrix} m_1 - \omega^2 & m_2 \\ -m_2\omega^2 & m_1 - \omega^2 \end{bmatrix} \\ \Psi^{\omega} &= \begin{bmatrix} m_1 - \omega^2 & m_2 \end{bmatrix} \end{aligned} \tag{4.34}$$

Attaching the internal model (4.33) to (4.30) and performing the coordinate transformation (4.12) gives the augmented system as follows:

$$\begin{aligned} \dot{\bar{z}}_1 &= -\varepsilon_1 a_2 \bar{z}_1 + \varepsilon_1 e \\ \dot{\bar{z}}_2 &= -\varepsilon_2 a_4 \bar{z}_2 - \varepsilon_2 e \\ \dot{\tilde{\eta}} &= M\tilde{\eta} + MNe - N\bar{g}(\bar{z}, e, \mu) \\ \dot{e} &= \bar{g}(\bar{z}, e, \mu) + \Psi^{\omega}(\tilde{\eta} + Ne) + (u - \Psi^{\omega}\eta) \end{aligned} \tag{4.35}$$

where $\bar{g}(\bar{z}, e, \mu) = e - \frac{1}{3}e^3 - \bar{z}_1 + \bar{z}_2$.

Using the Lyapunov function candidate $V(\bar{z}_1, \bar{z}_1) = 0.5\bar{z}_1^2 + 0.5\bar{z}_2^2$, it can be shown that the first two equations of (4.35) satisfy Assumption 4.3. Hence, all the conditions for Theorem 4.1 are satisfied. A control law of the form (4.26) can be constructed with $\rho(e) = e^4 + 1$.

Simulation is performed for the case where $(m_1, m_2) = (2, 3)$, $\omega = 1$, $(w_1, w_2) = (0.4, 0.3)$, $\bar{\varepsilon}_1 = 0.1$, $\bar{\varepsilon}_2 = 0.2$, $a_2 = a_4 = 1$. The initial values are randomly produced as follows: $z(0) = [0.6589; -1.3279]$, $e(0) = 0.2439$, $v_0 = [2.1579; -0.8240]$, $\eta(0) = [0.7777; -0.5405]$, $\hat{\Psi}(0) = [0.6366, 0.9954]$, $k(0) = -0.5141$. Figures 4.2 and 4.3 show the profiles of various states of the closed-loop system.

Next, we show that $\hat{\Psi}(t)$ will converge to its real value Ψ^{ω}. For this purpose, we need to prove that (4.33) is the minimal internal model. In fact

$$\begin{aligned} \mathbf{u}(v, w, \omega) &= -v_1 \\ &= -\Big(\frac{v_{10}}{2} + \frac{v_{20}}{2\jmath}\Big)e^{\jmath\omega t} - \Big(\frac{v_{10}}{2} - \frac{v_{20}}{2\jmath}\Big)e^{-\jmath\omega t} \\ &= \sum_{l=1}^{2} C_l(v_0, w, \omega)e^{\jmath\hat{\omega}_l t} \end{aligned}$$

with $\hat{\omega}_{1,2} = \pm\omega$. By Remark 4.8, the minimal zeroing polynomial is

$$P^{\omega}(\lambda) = \prod_{i=1}^{2}(\lambda + \hat{\omega}_i) = \lambda^2 + \omega^2$$

Consequently, the internal model (4.33) is the minimal one. Moreover, for any $v_0 \neq 0$, any $w \in \mathcal{W}$, and any $\omega > 0$,

$$\begin{aligned} C_1(v_0, w, \omega) &= -\Big(\frac{v_{10}}{2} + \frac{v_{20}}{2\jmath}\Big) \neq 0 \\ C_2(v_0, w, \omega) &= -\Big(\frac{v_{10}}{2} - \frac{v_{20}}{2\jmath}\Big) \neq 0 \end{aligned}$$

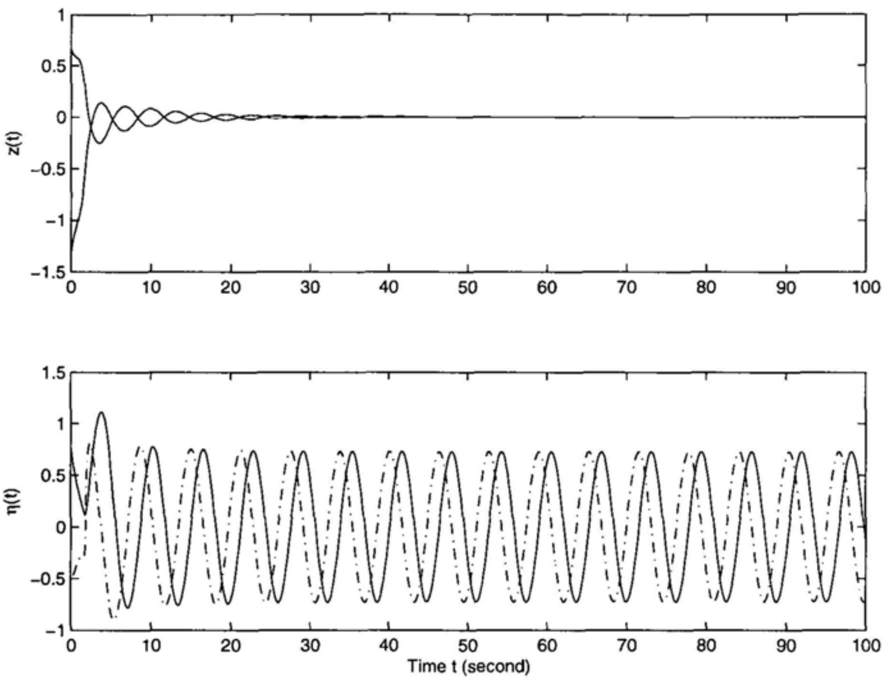

Figure 4.2: Responses of states $z(t)$ and $\eta(t)$.

As a result, for any $v_0 \neq 0$, any $w \in \mathcal{W}$, and any $\omega > 0$

$$\lim_{t \to +\infty} \hat{\Psi}(t) = \Psi^{\omega}$$

The simulation shown in Figure 4.4 validates this conclusion.

Using (4.34), the estimated frequency $\hat{\omega}$ can be determined by

$$\hat{\omega} = \sqrt{m_1 - \hat{\Psi}_1}$$

with $\hat{\Psi}_1$ the first component of $\hat{\Psi}$. Thus the convergence of $\hat{\Psi}$ to its true value also implies the convergence of the estimated frequency $\hat{\omega}$ to the true frequency ω. Figure 4.5 illustrates the convergence property of the frequency ω.

Tracking control of the generalized Lorenz system

Consider the following system [55] described by

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 - x_1x_3 + bu \\ \dot{x}_3 &= a_3x_3 + x_1x_2 \\ e &= x_2 - F(t) \end{aligned} \tag{4.36}$$

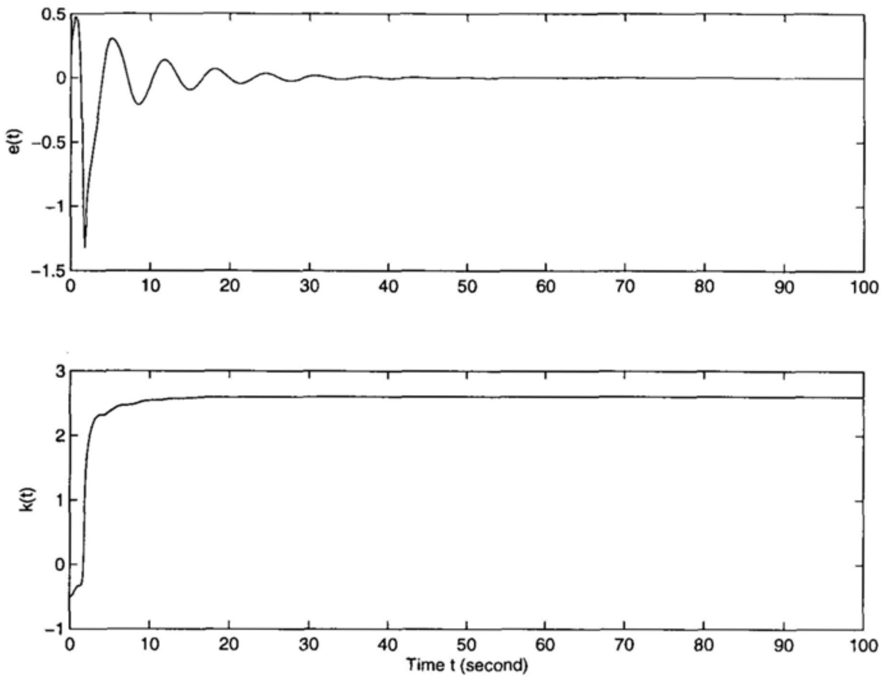

Figure 4.3: Responses of tracking error $e(t)$ and state $k(t)$.

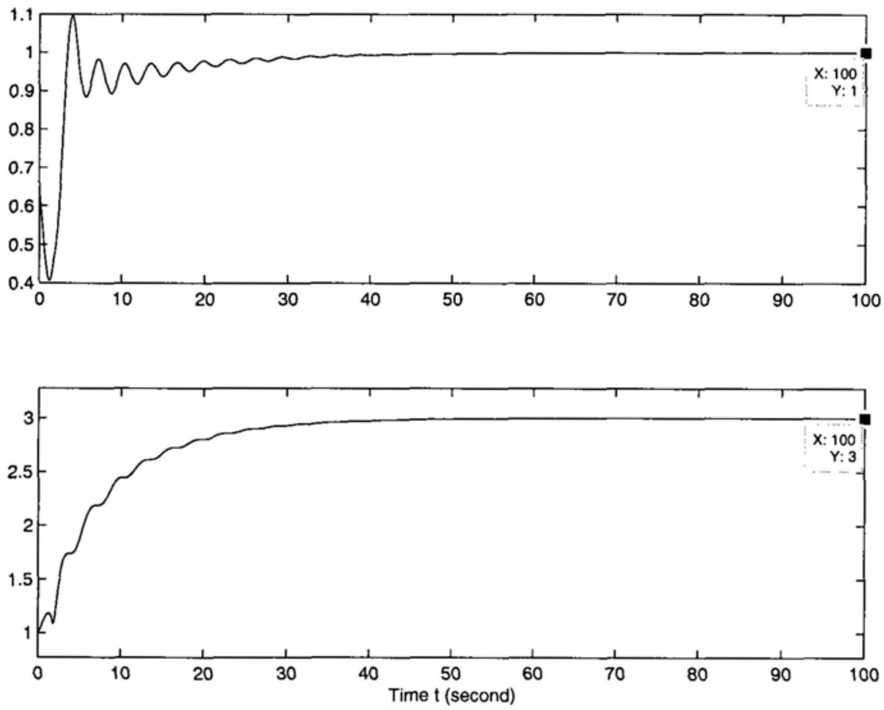

Figure 4.4: Response of $\hat{\Psi}(t)$.

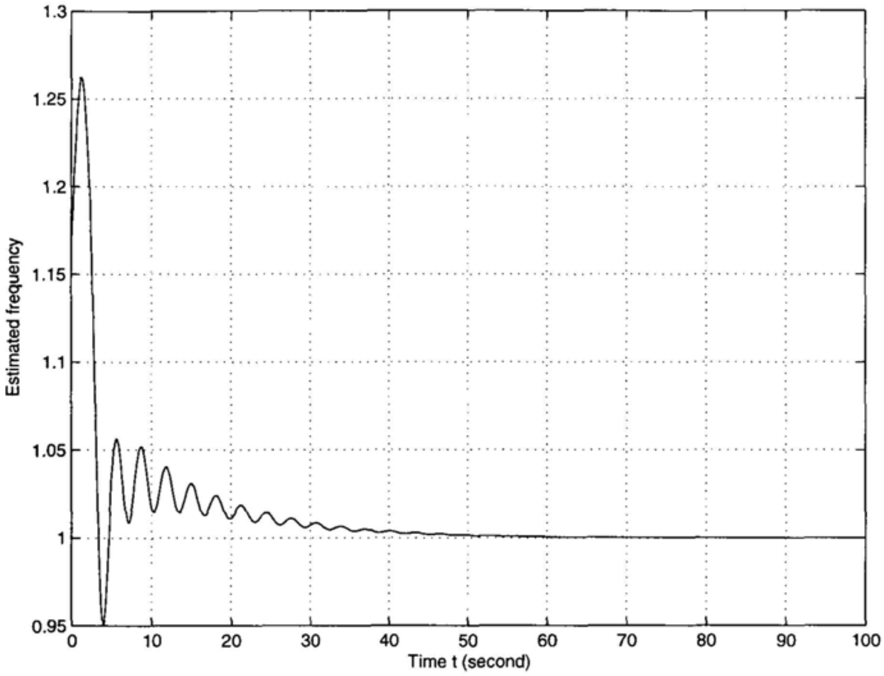

Figure 4.5: Response of estimated frequency $\hat{\omega}$.

where $(a_{11}, a_{12}, a_{21}, a_{22}, a_3, b)$ is a constant parameter vector satisfying $a_{11}, a_3 < 0, b > 0$, $e \in \mathbb{R}$ is the error output, and $F(t) = A_m \sin(\omega t + \phi)$ is the reference input. When $a_{11} = -a_{12} = -L_1 < 0$, $a_{21} = L_3 > 0$, $a_{22} = -1$, $a_3 = -L_2 < 0$, and $u = 0$, system (4.36) reduces to the following form

$$\begin{aligned} \dot{x}_1 &= L_1(x_2 - x_1) \\ \dot{x}_2 &= L_3 x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - L_2 x_3 \end{aligned} \qquad (4.37)$$

which is the well known Lorenz system discovered by Edward N. Lorenz [62] where $L_1 > 0$ is called the Prandtl number, $L_2 > 0$ is a geometric factor and L_3 is called the Rayleigh number. This system is known to exhibit chaotic behavior as we have shown in Figure 3.1. In the past decades, the Lorenz system has become a popular model for testing various control design problems [13] [33] [53] [56] [64] [84] [92].

For example, in [53] [92], a single control input is added to the second equation of (4.37) leading to a controlled Lorenz system.

System (4.36) is called a controlled generalized Lorenz system. Let us first point out that the problem of designing a control law to asymptotically track the reference input $F(t)$ for system (4.36) is an output regulation problem. In fact, by performing the coordinate transformation $(z_1, z_2, y) = (x_1, x_3, x_2)$ and noting that $F(t) = v_1$ where v_1 is the first state of the same exosystem (4.29), we can put system (4.36) in the standard form (4.1) as follows:

$$\begin{aligned}\dot{z}_1 &= a_{11}z_1 + a_{12}y \\ \dot{z}_2 &= a_3z_2 + z_1y \\ \dot{y} &= bu + a_{21}z_1 + a_{22}y - z_1z_2 \\ e &= y - v_1 \end{aligned} \tag{4.38}$$

Again, we allow the parameter $(a_{11}, a_{12}, a_{21}, a_{22}, a_3)$ to undergo some perturbation. To be more specific, let

$$a = (\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3) + (w_1, \cdots, w_5)$$

where $(\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3)$ represents the nominal value of a and $(w_1, \cdots, w_5)$ the uncertainty of a. To guarantee $a_{11}, a_3 < 0$, we define $\mathcal{W} = \{w | w^{\mathsf{T}} \in \mathbb{R}^5, \bar{a}_{11} + w_1 < 0, \bar{a}_3 + w_5 < 0\}$. Also let $\mathbb{S} = \{\omega | \omega > 0\}$.

Now it is clear that if we can solve the global robust output regulation problem for system (4.38) with the exosystem (4.29) and $\mathcal{W}$ and $\mathbb{S}$ as described above, we can solve the global robust asymptotic tracking problem for system (4.36) with x_2 as the output for any sinusoidal reference input $F(t)$ in the presence of the parameter variation $w \in \mathcal{W}$.

We now verify that the composite system (4.38) and (4.29) satisfies Assumptions 4.1 to 4.3. From the last equation of (4.38), we have

$$\mathbf{y}(v, w, \omega) = v_1 \tag{4.39}$$

Substituting (4.39) into the first equation of (4.38) yields

$$\mathbf{z}_1(v, w, \omega) = r_{11}v_1 + r_{12}v_2 \tag{4.40}$$

where

$$r_{11}(w, \omega) = -\frac{a_{11}a_{12}}{\omega^2 + a_{11}^2}, \quad r_{12}(w, \omega) = -\frac{a_{12}\omega}{\omega^2 + a_{11}^2}$$

Substituting (4.40) and (4.39) into the second equation of (4.38) gives

$$\mathbf{z}_2(v, w, \omega) = r_{21}v_1^2 + r_{22}v_2^2 + r_{23}v_1v_2 \tag{4.41}$$

where

$$\begin{aligned} r_{21}(w, \omega) &= -\frac{a_3^2 r_{11} - a_3\omega r_{12} + 2\omega^2 r_{11}}{a_3(a_3^2 + 4\omega^2)} \\ r_{22}(w, \omega) &= \frac{\omega}{a_3} r_{23}, \; r_{23}(w, \omega) \;=\; -\frac{r_{12}a_3 + 2\omega r_{11}}{a_3^2 + 4\omega^2} \end{aligned}$$

Finally, substituting (4.39) to (4.41) into the third equation of (4.38) gives

$$\begin{aligned} \mathbf{u}(v, w, \omega) &= b^{-1}(\omega v_2 - a_{22}v_1 - a_{21}\mathbf{z}_1 + \mathbf{z}_1\mathbf{z}_2) \\ &= r_{31}v_1 + r_{32}v_2 + r_{33}v_1^3 + r_{34}v_2^3 \\ &\quad + r_{35}v_1^2v_2 + r_{36}v_1v_2^2 \end{aligned} \tag{4.42}$$

where

$$\begin{aligned} r_{31}(w,\omega) &= -b^{-1}(a_{22}+a_{21}r_{11}) \\ r_{32}(w,\omega) &= b^{-1}(\omega - a_{21}r_{12}) \\ r_{33}(w,\omega) &= b^{-1}r_{11}r_{21}, \quad r_{34}(w,\omega) \;=\; b^{-1}r_{12}r_{22} \\ r_{35}(w,\omega) &= b^{-1}(r_{12}r_{21}+r_{11}r_{23}) \\ r_{36}(w,\omega) &= b^{-1}(r_{11}r_{22}+r_{12}r_{23}) \end{aligned}$$

Thus both Assumptions 4.1 and 4.2 are satisfied. It can be further verified that

$$\frac{d^4\mathbf{u}(v,w,\omega)}{dt^4} + 9\omega^4\mathbf{u}(v,w,\omega) + 10\omega^2\frac{d^2\mathbf{u}(v,w,\omega)}{dt^2} = 0$$

By Remark 4.2, the steady-state generator of the form (4.9) is given by

$$\begin{aligned} \tau(v,w,\omega) &= \mathrm{col}\big(\mathbf{u}, \mathcal{L}_{A_1(\omega)v}\mathbf{u}, \mathcal{L}^2_{A_1(\omega)v}\mathbf{u}, \mathcal{L}^3_{A_1(\omega)v}\mathbf{u}\big) \\ \Phi(\omega) &= \left[\begin{array}{c|c} 0 & I_3 \\ \hline -9\omega^4 & 0, -10\omega^2, 0 \end{array}\right], \;\Gamma \;=\; [1,0,0,0] \end{aligned}$$

which leads to the internal model as follows:

$$\dot{\eta} = M\eta + Nu \tag{4.43}$$

where (M, N) is any controllable pair of the form

$$M = \left[\begin{array}{c|c} 0 & I_3 \\ \hline -m_1 & -m_2, -m_3, -m_4 \end{array}\right], \; N = \mathrm{col}(0,0,0,1)$$

and the parameter (m_1, m_2, m_3, m_4) is such that M is Hurwitz. Solving the Sylvester equation (4.10) with $(m_1, m_2, m_3, m_4) = (4, 12, 13, 6)$ yields

$$T^{-1}(\omega) = \begin{bmatrix} 4-9\omega^4 & 12 & 13-10\omega^2 & 6 \\ -54\omega^4 & 4-9\omega^4 & 12-60\omega^2 & 13-10\omega^2 \\ \star_{(1)} & -54\omega^4 & \star_{(2)} & 12-60\omega^2 \\ \star_{(3)} & \star_{(1)} & \star_{(4)} & \star_{(2)} \end{bmatrix}$$

where

$$\begin{aligned} \star_{(1)} &:= 9\omega^4(10\omega^2-13), \; \star_{(2)} \;:=\; 91\omega^4 - 130\omega^2 + 4 \\ \star_{(3)} &:= 108\omega^4(5\omega^2-1), \; \star_{(4)} \;:=\; 6\omega^2(91\omega^2-20) \end{aligned}$$

Hence, we have

$$\Psi^\omega = \Gamma T^{-1}(\omega) = \begin{bmatrix} 4-9\omega^4 & 12 & 13-10\omega^2 & 6 \end{bmatrix} \tag{4.44}$$

To verify Assumption 4.3, performing the coordinate transformation (4.12)

$$\bar{z}_1 = z_1 - \mathbf{z}_1,\ \bar{z}_2 = z_2 - \mathbf{z}_2,\ \tilde{\eta} = \eta - \theta - Nb^{-1}e$$

for the system composed of (4.38) and (4.43) gives the following augmented system

$$\begin{aligned}
\dot{\bar{z}}_1 &= a_{11}\bar{z}_1 + a_{12}e \\
\dot{\bar{z}}_2 &= a_3\bar{z}_2 + (\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1 \\
\dot{\tilde{\eta}} &= M\tilde{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z}_1, \bar{z}_2, e, \mu) \\
\dot{e} &= \bar{g}(\bar{z}_1, \bar{z}_2, e, \mu) + \Psi^{\omega}(b\tilde{\eta} + Ne) + b(u - \Psi^{\omega}\eta)
\end{aligned} \tag{4.45}$$

where $\bar{g} = a_{22}e + a_{21}\bar{z}_1 - (\bar{z}_1 + \mathbf{z}_1)(\bar{z}_2 + \mathbf{z}_2) + \mathbf{z}_1\mathbf{z}_2$.

We are now ready to verify that the $(\bar{z}_1, \bar{z}_2)$ subsystem satisfies Assumption 4.3. In fact, for any fixed compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$, let $V_{\bar{z}} = \frac{\hbar}{2}\bar{z}_1^2 + \frac{\hbar}{4}\bar{z}_1^4 + \frac{1}{2}\bar{z}_2^2$ for some $\hbar > 0$ which satisfies, along the trajectory of $(\bar{z}_1, \bar{z}_2)$ subsystem

$$\begin{aligned}
\dot{V}_{\bar{z}} &= \hbar a_{11}\bar{z}_1^2 + \hbar a_{12}\bar{z}_1 e + \hbar a_{11}\bar{z}_1^4 + \hbar a_{12}\bar{z}_1^3 e \\
&\quad + a_3\bar{z}_2^2 + \bar{z}_2\bar{z}_1 e + v_1\bar{z}_2\bar{z}_1 + \mathbf{z}_1\bar{z}_2 e
\end{aligned} \tag{4.46}$$

In (4.46), using Young's inequality gives

$$\begin{aligned}
\hbar a_{12}\bar{z}_1 e &\leq \frac{1}{2}\bar{z}_1^2 + \frac{\hbar^2 a_{12}^2}{2}e^2 \\
\hbar a_{12}\bar{z}_1^3 e &\leq \frac{3}{4}\bar{z}_1^4 + \frac{\hbar^4 a_{12}^4}{4}e^4 \\
\bar{z}_2\bar{z}_1 e &\leq \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{1}{2\varepsilon}\bar{z}_1^2 e^2 \leq \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{1}{4}\bar{z}_1^4 + \frac{1}{4\varepsilon^2}e^4 \\
v_1\bar{z}_2\bar{z}_1 &\leq \frac{1}{2\varepsilon}\bar{z}_1^2 + \frac{\varepsilon v_1^2}{2}\bar{z}_2^2 \\
\mathbf{z}_1\bar{z}_2 e &\leq \frac{\varepsilon}{2}\bar{z}_2^2 + \frac{\mathbf{z}_1^2}{2\varepsilon}e^2
\end{aligned} \tag{4.47}$$

for any $\varepsilon > 0$. Substituting (4.47) into (4.46) gives

$$\begin{aligned}
\dot{V}_{\bar{z}} &\leq \Big(\hbar a_{11} + \frac{1}{2} + \frac{1}{2\varepsilon}\Big)\bar{z}_1^2 + \Big(\hbar a_{11} + 1\Big)\bar{z}_1^4 + \Big(a_3 + \varepsilon + \frac{\varepsilon v_1^2}{2}\Big)\bar{z}_2^2 \\
&\quad + \Big(\frac{\hbar^2 a_{12}^2}{2} + \frac{\mathbf{z}_1^2}{2\varepsilon}\Big)e^2 + \Big(\frac{\hbar^4 a_{12}^4}{4} + \frac{1}{4\varepsilon^2}\Big)e^4
\end{aligned} \tag{4.48}$$

Since Σ is compact, it can be seen that there exist constants $\ell_i > 0$, $i = 1, \cdots, 5$, such that, for all $(v, w, \sigma) \in \Sigma$

$$\dot{V}_{\bar{z}} \leq -\ell_2\bar{z}_1^2 - \ell_3\bar{z}_1^4 - \ell_1\bar{z}_2^2 + \ell_4 e^2 + \ell_5 e^4 \tag{4.49}$$

Thus, by Theorem 4.1, the global robust output regulation problem for the composite system (4.38) and (4.29) is solvable by an output feedback control law. In fact, following

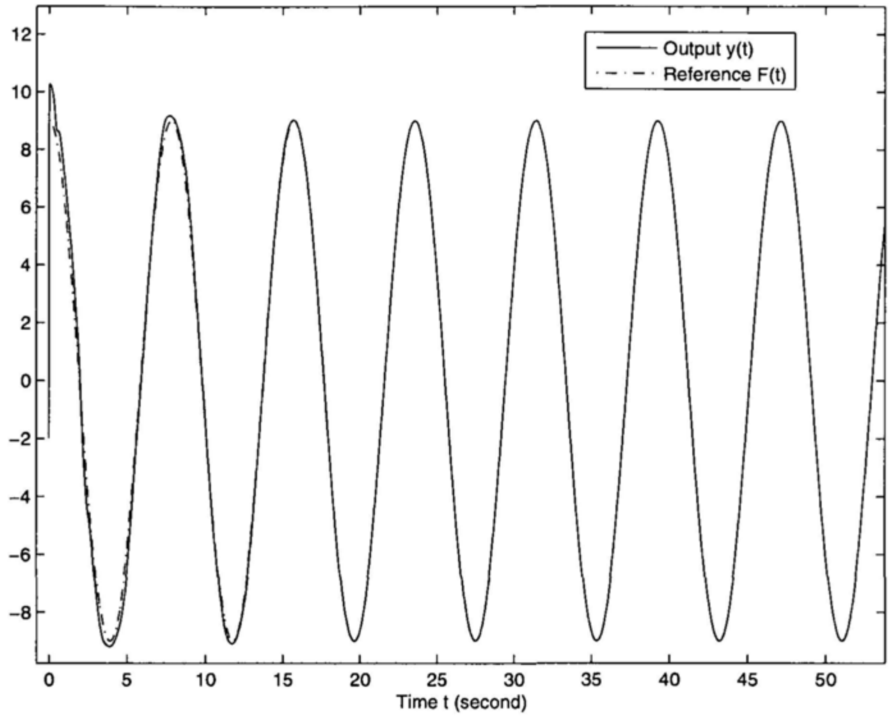

Figure 4.6: Tracking performance.

the design method detailed in Section 4.1.2, we can obtain a controller of the form (4.26) with the design function $\rho(e) = 2(e^6 + 1)$.

The simulation is performed with $\omega = 0.8$, $a = (-10, 10, 28, -1, -8/3)$ and $b = 1$. Various initial values are randomly chosen to be $(z_1(0), z_2(0), y(0)) = (3, -1, -2)$, $v_0 = \text{col}(9, 0)$, $\eta(0) = \hat{\Psi}(0) = 0$ and $k(0) = 1$. Figures 4.6 and 4.7 show the performance of the closed-loop system.

Remark 4.10 The global stabilization problem associated with (4.37) was studied in [53] [92] by using state feedback. Similar results on the control of the Lorenz system can also be found in [13] [33] [64] [63] [84]. The stabilization problem in [53] [92] can be viewed as a special case of global robust output regulation of this section by having $F(t) \equiv 0$. In such a case, as pointed out in Remark 4.6, the stabilizing controller can be given by (4.24) with $\rho(y) = 5(y^6 + 1)$. Simulation results are shown in Figures 4.8 and 4.9 with initial values $(x_1(0), y(0), x_3(0), k(0)) = (2, -1.5, -1.8, 1.2)$. It can be seen that all the state variable of the plant converge to the origin while the dynamic gain $k(t)$ tends to a constant gain as the time goes to infinity. Also, the tracking control problem studied in [13] is also a special case of our problem where A_m and σ are known nonzero numbers. ∎

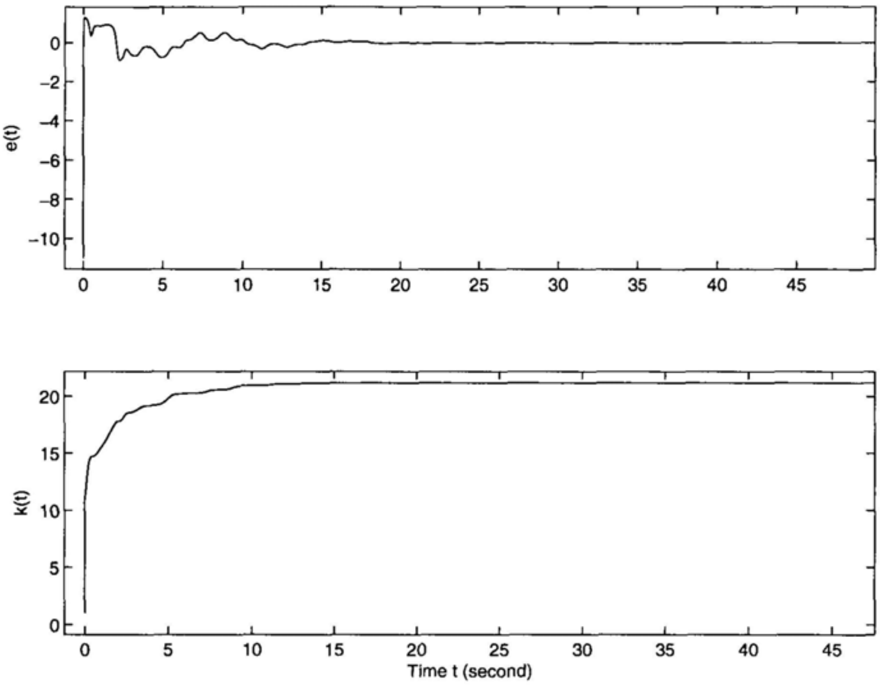

Figure 4.7: Responses of tracking error $e(t)$ and state $k(t)$.

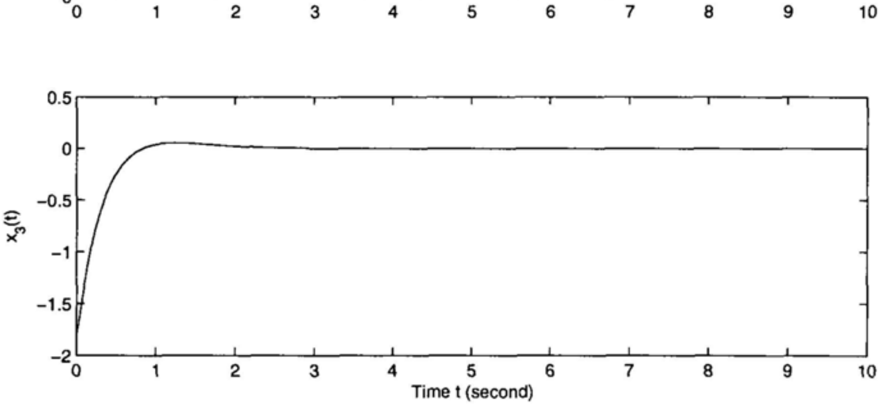

Figure 4.8: Responses of $x_1(t)$ and $x_3(t)$.

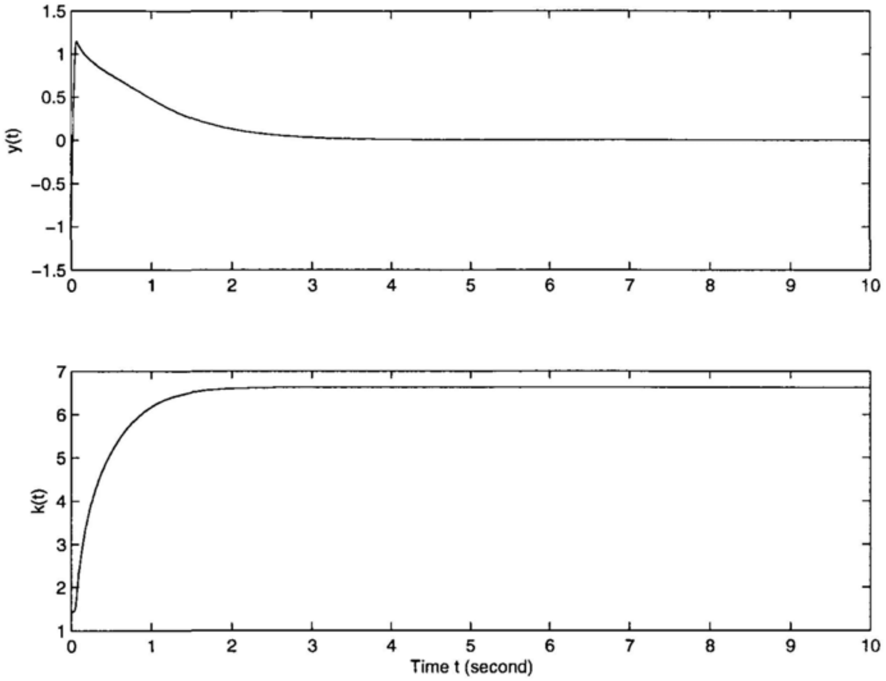

Figure 4.9: Responses of $y(t)$ and $k(t)$.

Next, we show the convergence of $\hat{\Psi}(t)$ to Ψ^{ω}, i.e.,

$$\lim_{t \to +\infty} \hat{\Psi}(t) = \Psi^{\omega} \tag{4.50}$$

To this end, we will show that (4.43) is the minimal internal model. In fact, from

$$\begin{aligned} v_1(t) &= \left(\frac{v_{10}}{2} + \frac{v_{20}}{2\jmath}\right)e^{\jmath\omega t} + \left(\frac{v_{10}}{2} - \frac{v_{20}}{2\jmath}\right)e^{-\jmath\omega t} \\ v_2(t) &= \left(\frac{v_{10}}{2} + \frac{v_{20}}{2\jmath}\right)\jmath e^{\jmath\omega t} - \left(\frac{v_{10}}{2} - \frac{v_{20}}{2\jmath}\right)\jmath e^{-\jmath\omega t} \end{aligned}$$

we can put $\mathbf{u}(v, w, \omega)$ in the form (4.42) with $r = 4$, $\hat{\omega}_{1,2} = \pm\omega$ and $\hat{\omega}_{3,4} = \pm 3\omega$ where none of the coefficients $C_l(v_0, w, \omega), l = 1, \cdots, 4$ is identically zero for all $(v_0, w, \omega) \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$. In particular, none of the coefficients $C_l(v_0, w, \omega), l = 1, \cdots, 4$ is zero for the given v_0, w and ω. By Remark 4.8, the control law guarantees the parameter convergence property (4.50). Figures 4.10 and 4.11 confirm this property for $\omega = 0.8$ and $\Psi^{\omega} = \left[0.3136, 12, 6.6, 6\right]$.

From (4.44), the estimated frequency $\hat{\omega}$ can be related to the third component $\hat{\Psi}_3$ of $\hat{\Psi}$ as follows:

$$\hat{\omega} = \sqrt{0.1(13 - \hat{\Psi}_3)}$$

Thus, the estimated frequency $\hat{\omega}$ will also approach the true frequency as shown in Figure 4.12.

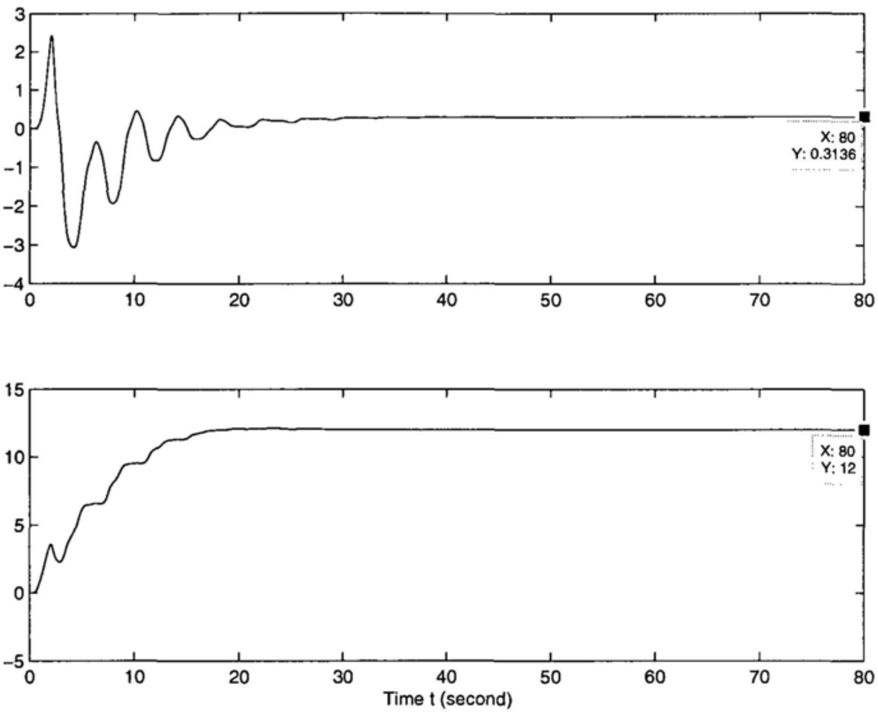

Figure 4.10: Response of estimation $\hat{\Psi}(t)$ (first two components).

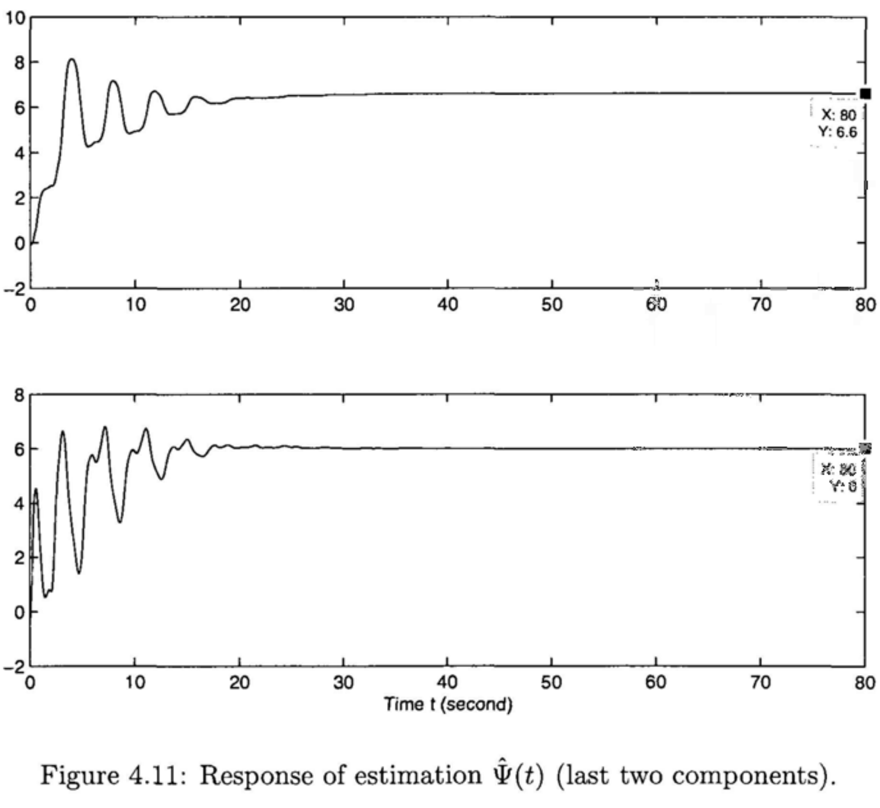

Figure 4.11: Response of estimation $\hat{\Psi}(t)$ (last two components).

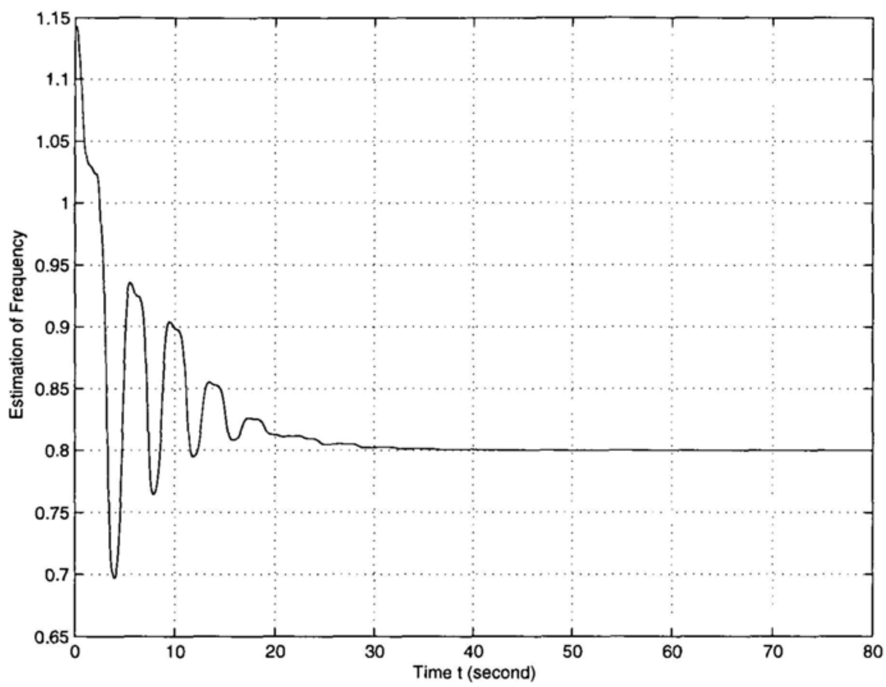

Figure 4.12: Response of estimated frequency $\hat{\omega}$.

4.2 General case

In this section, we consider the system in the general form with relative degree $n \geq 2$

$$\begin{aligned} \dot{z} &= f(z, y, v, w) \\ \dot{x}_i &= x_{i+1} + g_i(z, y, v, w), \ i = 1, \cdots, n-1 \\ \dot{x}_n &= bu + g_n(z, y, v, w), \ n \geq 2 \\ y &= x_1 \\ e &= x_1 - q(v, w) \end{aligned} \tag{4.51}$$

where $(z, x) \in \mathbb{R}^{n_z} \times \mathbb{R}^n$ is the state with $x = \mathrm{col}(x_1, \cdots, x_n)$, $u \in \mathbb{R}$ is the input, and $e \in \mathbb{R}$ is the error output. $b > 0$ is an uncertain constant, $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is the parameter uncertainty, and $v(t) \in \mathbb{R}^{n_v}$ represents the time-varying reference and/or disturbance. The functions f, g_i and q are sufficiently smooth in their arguments satisfying $f(0, 0, 0, w) = 0$, $g_i(0, 0, 0, w) = 0$ and $q(0, w) = 0$ for all $w \in \mathcal{W}$.

It is also assumed that $v(t)$ is generated by a linear exosystem described by (4.2) where $\sigma \in \mathbb{S} \subset \mathbb{R}^{n_\sigma}$ represents the uncertainty in the exosystem. To have our problem well posed, we assume all the eigenvalues of $A_1(\sigma)$ are distinct with zero real parts for all $\sigma \in \mathbb{S}$.

We will describe the control problem for system (4.51) and (4.2) as follows.

Problem 4.1 Given $\mathcal{W}$ and $\mathbb{S}$, design an output feedback control law of the form:

$$u = u_K(\zeta, e), \ \dot{\zeta} = g_K(\zeta, e) \tag{4.52}$$

where both u_K and g_K are sufficiently smooth vanishing at the origin such that, for any initial state $(z(0), x(0), v_0, \zeta(0))$ and any constant parameter $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$, the solution of the closed-loop system composed of (4.51) to (4.52) exists and is bounded over $[0, +\infty)$, and the error output $e(t)$ approaches zero as t tends to infinity. ∎

Compared with the relative degree one case, the current case involves some specific technical difficulties. First, an observer has to be introduced to estimate the state variables $x_1, \cdots, x_n$. Thus, we actually need to deal with the output regulation problem for the extended system (4.56) composed of (4.51) and the observer (4.54) instead of system (4.51). Even though the extended system is more complicated than the original system (4.51), we have managed to show that, under some standard assumptions on the *original* system (4.51), the output regulation problem of the extended system can still be converted into a stabilization problem of an augmented system consisting of the extended system and a canonical linear internal model. Second, the augmented system can also be put in a lower triangular form with the relative degree n . The augmented system contains a linear parameterized uncertainty incurred by the unknown parameter in the exosystem. We need to employ a recursive adaptive control design method to handle the adaptive stabilization problem of the augmented system and to obtain the estimation of the unknown parameter in the exosystem.

Section 4.2.1 introduces some standard assumptions and shows that the output regulation problem for system (4.51) and (4.2) can be converted into an adaptive stabilization problem of an augmented system composed of the original system, a partial state observer, and the internal model. Section 4.2.2 further considers the stabilization problem of the augmented system. An adaptive control law will be derived via a recursive design method. In Section 4.2.3, we will apply the design method to solve a tracking control problem related to a generalized fourth order Lorenz system.

4.2.1 Problem conversion

To derive the augmented system associated with the plant (4.51), we repeat the following two standard assumptions.

Assumption 4.4 There exists a globally defined sufficiently smooth function $\mathbf{z} : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma} \mapsto \mathbb{R}^n$ with $\mathbf{z}(0, w, \sigma) = 0$ such that

$$\frac{\partial \mathbf{z}(v, w, \sigma)}{\partial v} A_1(\sigma) v = f(\mathbf{z}(v, w, \sigma), q(v, w), v, w) \tag{4.53}$$

for all $(v, w, \sigma) \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$. ∎

Under Assumption 4.4, let $\mathbf{x}_1(v,w,\sigma) = q(v,w)$, and

$$\begin{aligned}\mathbf{x}_i(v,w,\sigma) &= \mathcal{L}_{A_1(\sigma)v}\mathbf{x}_{i-1}(v,w) - g_{i-1}(\mathbf{z}(v,w,\sigma), q(v,w), v, w), \quad i = 2,\cdots,n \\ \mathbf{u}(v,w,\sigma) &= b^{-1}\big[\mathcal{L}_{A_1(\sigma)v}\mathbf{x}_n(v,w,\sigma) - g_n(\mathbf{z}(v,w,\sigma), q(v,w), v, w)\big]\end{aligned}$$

where $\mathcal{L}_{A_1(\sigma)v}q(v,w) = \big(\partial q(v,w)/\partial v\big)A_1(\sigma)v$. Then, $\mathbf{z}(v,w,\sigma)$, $\mathbf{x}(v,w,\sigma) := \text{col}\big(\mathbf{x}_1(v,w,\sigma), \cdots, \mathbf{x}_n(v,w,\sigma)\big)$ and $\mathbf{u}(v,w,\sigma)$ constitute the solution of the regulator equations associated with (4.51) and (4.2).

For the existence of the internal model, we need one more assumption.

Assumption 4.5 The function $\mathbf{u}(v,w,\sigma)$ is a polynomial in v with coefficients depending possibly on w and σ. ∎

Remark 4.11 Assumption 4.5 holds if the solution $\mathbf{z}(v,w)$ of (4.53), the function $q(v,w)$ are polynomials in v, and the functions $g_i(z,y,v,w)$ are polynomials in (z,y,v). ∎

As we need to design an output feedback control law, as in [44], we will first introduce an input driven filter to system (4.51) as follows:

$$\dot{\xi} = A_c\xi + Bu, \;\; \xi \in \mathbb{R}^n, u \in \mathbb{R} \tag{4.54}$$

where

$$A_c = \left[\begin{array}{c|c} -\lambda_{[n-1]} & I_{n-1} \\ \hline -\lambda_n & 0 \end{array}\right], \quad B = \big[0,\cdots,0,1\big]^\top$$

$\lambda_{[n-1]} := \text{col}(\lambda_1,\cdots,\lambda_{n-1})$, and $\lambda_1,\cdots,\lambda_n$ are given positive constants such that A_c is Hurwitz. This filter can also be viewed as an observer for the subsystem governing the state variables x_i, $i = 1,\cdots,n$.

Next, by performing the following coordinate transformation

$$\epsilon_i = b^{-1}x_i - \xi_i, \quad i = 1,\cdots,n \tag{4.55}$$

we get the following extended system

$$\begin{aligned}\dot{z} &= f(z,y,v,w) \\ \dot{\epsilon} &= A_c\epsilon + h(z,y,v,w) \\ \dot{y} &= b\xi_2 + b\epsilon_2 + g_1(z,y,v,w) \\ \dot{\xi}_i &= \xi_{i+1} - \lambda_i\xi_1, \; i = 2,\cdots,n-1 \\ \dot{\xi}_n &= u - \lambda_n\xi_1\end{aligned} \tag{4.56}$$

where $\epsilon = \text{col}(\epsilon_1,\cdots,\epsilon_n)$, $h_i(z,y,v,w) = \lambda_i b^{-1}y + b^{-1}g_i(z,y,v,w)$, $i = 1,\cdots,n$, and

$$h(z,y,v,w) = \begin{bmatrix} h_1(z,y,v,w) \\ \vdots \\ h_n(z,y,v,w) \end{bmatrix}$$

It can be seen that if the global robust output regulation problem for the extended system (4.56) is solvable by a control law depending on the partial states e and ξ only, then this control law together with (4.54) is in the form (4.52) and solves the global robust output regulation problem of system (4.51). Thus, in what follows, we only need to consider the global robust output regulation problem for the extended system (4.56).

Remark 4.12 It is necessary to verify the solvability of the regulator equations associated with (4.56) and (4.2). Under Assumptions 4.4 and 4.5, $\mathbf{u}(v, w, \sigma)$ can be uniquely expressed as follows:

$$\mathbf{u}(v, w, \sigma) = \sum_{l=1}^{K} \mathcal{U}_l(w, \sigma) v^{[l]} \tag{4.57}$$

where K is some positive integer, $v^{[1]} = v = \left[v_1, \cdots, v_{n_v}\right]^\top$, and for each $l \geq 2$,

$$v^{[l]} = \left[v_1^l, v_1^{l-1}v_2, \cdots, v_1^{l-1}v_{n_v}, v_1^{l-2}v_2^2, v_1^{l-2}v_2v_3, \cdots, v_1^{l-2}v_2v_{n_v}, \cdots, v_{n_v}^l\right]^\top$$

and $\mathcal{U}_l(w, \sigma)$ is a suitable constant coefficient vector. Moreover, from Chapter 4 of [31], for each $l \geq 1$, there is a matrix $\mathcal{M}_l(\sigma)$ whose eigenvalues are of zero real part such that

$$\frac{\partial v^{[l]}}{\partial v} A_1(\sigma) v = \mathcal{M}_l(\sigma) v^{[l]}.$$

Thus, for each $l = 1, 2, \cdots, K$, the following Sylvester equation

$$\mathcal{X}_l(w, \sigma)\mathcal{M}_l(\sigma) = A_c \mathcal{X}_l(w, \sigma) + B\mathcal{U}_l(w, \sigma)$$

has a unique solution $\mathcal{X}_l(w, \sigma)$ since A_c is Hurwitz. Let

$$\begin{aligned} \Xi(v, w, \sigma) &= \sum_{l=1}^{K} \mathcal{X}_l(w, \sigma) v^{[l]} \\ \mathbf{E}(v, w, \sigma) &= b^{-1}\mathbf{x}(v, w, \sigma) - \Xi(v, w, \sigma) \end{aligned} \tag{4.58}$$

and $\mathbf{y}(v, w, \sigma) = q(v, w)$. It can be verified that $\{\mathbf{z}, \mathbf{E}, \mathbf{y}, \Xi, \mathbf{u}\}$ is the solution of the regulator equations associated with the extended system (4.56) and exosystem (4.2). ∎

Remark 4.13 Under Assumption 4.5, each component of $\Xi(v, w, \sigma)$ is also a polynomial. Thus, there exists an integer s such that $\Xi_2(v, w, \sigma)$, the second component of $\Xi(v, w, \sigma)$, satisfies, for all trajectories $v(t)$ of the exosystem (4.2), all $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$

$$\begin{aligned} \frac{d^s \Xi_2(v, w, \sigma)}{dt^s} &= a_1(\sigma)\Xi_2(v, w, \sigma) + a_2(\sigma)\frac{d\Xi_2(v, w, \sigma)}{dt} \\ &\quad + \cdots + a_s(\sigma)\frac{d^{s-1}\Xi_2(v, w, \sigma)}{dt^{(s-1)}} \end{aligned} \tag{4.59}$$

where $a_1(\sigma), a_2(\sigma), \cdots, a_s(\sigma)$ are real scalars such that all the roots of the polynomial

$$P^\sigma(\lambda) = \lambda^s - a_1(\sigma) - a_2(\sigma)\lambda - \cdots - a_s(\sigma)\lambda^{s-1} \tag{4.60}$$

are distinct with zero real part for all $\sigma \in \mathbb{S}$ [30].

Let $\tau(v,w,\sigma) = \operatorname{col}(\Xi_2, \mathcal{L}_{A_1(\sigma)v}\Xi_2, \cdots, \mathcal{L}^{s-1}_{A_1(\sigma)v}\Xi_2)$ where $\mathcal{L}^{i+1}_{A_1(\sigma)v}\Xi_2 = \frac{\partial \mathcal{L}^{i}_{A_1(\sigma)v}\Xi_2}{\partial v}A_1(\sigma)v$ for any integer $i \geq 1$, and

$$\Phi(\sigma) = \left[\begin{array}{c|c} 0 & I_{s-1} \\ \hline a_1(\sigma) & a_2(\sigma), \cdots, a_s(\sigma) \end{array}\right], \quad \Gamma = [1, 0, \cdots, 0]_{1\times s} \tag{4.61}$$

Then $\tau(v,w,\sigma)$ satisfies

$$\frac{\partial \tau(v,w,\sigma)}{\partial v}A_1(\sigma)v = \Phi(\sigma)\tau(v,w,\sigma), \quad \Xi_2(v,w,\sigma) = \Gamma\tau(v,w,\sigma) \tag{4.62}$$

∎

Remark 4.14 Since, for each $\sigma \in \mathbb{S}$, $(\Gamma, \Phi(\sigma))$ is observable and the eigenvalues of $\Phi(\sigma)$ have zero real part, for any controllable pair (M, N) with $M \in \mathbb{R}^{s\times s}$ a Hurwitz matrix and $N \in \mathbb{R}^{s\times 1}$ a column vector, there is a unique nonsingular matrix $T(\sigma)$ satisfying the following Sylvester equation [69]

$$T(\sigma)\Phi(\sigma) - MT(\sigma) = N\Gamma \tag{4.63}$$

Let $\theta(v,w,\sigma) = T(\sigma)\tau(v,w,\sigma)$ which satisfies

$$\dot{\theta} = (M + N\Psi^{\sigma})\theta, \quad \Xi_2 = \Psi^{\sigma}\theta$$

with $\Psi^{\sigma} = \Gamma T^{-1}(\sigma)$. Then we can define the internal model as follows

$$\dot{\eta} = M\eta + N\xi_2 \tag{4.64}$$

∎

Therefore, attaching the internal model (4.64) to (4.56) and performing the following coordinate transformations

$$\bar{z} = z - \mathbf{z}(v,w,\sigma), \quad \bar{\epsilon} = \epsilon - \mathbf{E}(v,w,\sigma), \quad \tilde{\eta} = \eta - \theta(v,w,\sigma) - Nb^{-1}e, \quad e = y - q(v,w) \tag{4.65}$$

yields a system described by

$$\begin{aligned} \dot{z} &= F(z,e,\mu) \\ \dot{e} &= b\xi_2 - b\Psi^{\sigma}\eta + \bar{g}_e(z,e,\mu) \\ \dot{\xi}_i &= \xi_{i+1} - \lambda_i\xi_1, \ i = 2, \cdots, n-1 \\ \dot{\xi}_n &= u - \lambda_n\xi_1 \end{aligned} \tag{4.66}$$

where $z(t) = \operatorname{col}(\bar{z}, \bar{\epsilon}, \tilde{\eta})$, $\mu(t) = \operatorname{col}(v,w,\sigma)$,

$$F(z,e,\mu) = \begin{bmatrix} \bar{f}(\bar{z},e,\mu) \\ A_c\bar{\epsilon} + \bar{h}(\bar{z},e,\mu) \\ M\tilde{\eta} + MNb^{-1}e - N\bar{\epsilon}_2 - Nb^{-1}\bar{g}_1(\bar{z},e,\mu) \end{bmatrix}$$

and

$$\begin{aligned}
\bar{f}(\bar{z},e,\mu) &= f(\bar{z}+\mathbf{z}(v,w,\omega),e+q(v,w),v,w) - f(\mathbf{z}(v,w,\omega),q(v,w),v,w) \\
\bar{g}_1(\bar{z},e,\mu) &= g_1(\bar{z}+\mathbf{z}(v,w,\omega),e+q(v,w),v,w) - g_1(\mathbf{z}(v,w,\omega),q(v,w),v,w) \\
\bar{h}(\bar{z},e,\mu) &= h(\bar{z}+\mathbf{z}(v,w,\omega),e+q(v,w),v,w) - h(\mathbf{z}(v,w,\omega),q(v,w),v,w) \\
\bar{g}_e(z,e,\mu) &= b\Psi^\sigma\tilde{\eta} + \Psi^\sigma Ne + b\bar{\epsilon}_2 + \bar{g}_1(\bar{z},e,\mu)
\end{aligned}$$

Remark 4.15 System (4.66) is called the semi-translated augmented system. The quantity $\mu(t)$ in (4.66) can be viewed as an unknown time-varying disturbance. It can be easily verified that for any $\mu \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$, $F(0,0,\mu)=0$, $\bar{f}(0,0,\mu)=0$, $\bar{g}_1(0,0,\mu)=0$, $\bar{h}(0,0,\mu)=0$ and $\bar{g}_e(0,0,\mu)=0$. ∎

For system (4.66), we can define a stabilization problem as follows.

Problem 4.2 Given (4.66) with η any bounded piecewise continuous function of time t, find a control law in the form

$$u = u_\zeta(\zeta,\eta,\xi,e), \ \dot{\zeta} = g_\zeta(\zeta,\eta,\xi,e) \tag{4.67}$$

where both u_ζ and g_ζ are sufficiently smooth vanishing at the origin such that for any initial state and any $(w,\sigma) \in \mathcal{W}\times\mathbb{S}$, the solution of the closed-loop system composed of (4.2), (4.66) and (4.67) exists and is bounded over $[0,+\infty)$. Moreover, $\lim_{t\to\infty}(\|z(t)\| + |e(t)|) = 0$. ∎

Remark 4.16 It can be readily seen that if Problem 2.1 for system (4.66) is solvable, then the following control law

$$\begin{aligned}
u &= u_\zeta(\zeta,\eta,\xi,e) \\
\dot{\zeta} &= g_\zeta(\zeta,\eta,\xi,e) \\
\dot{\xi} &= A_c\xi + Bu_\zeta(\zeta,\eta,\xi,e), \quad \dot{\eta} = M\eta + N\xi_2
\end{aligned} \tag{4.68}$$

which is in the form (4.52) solves Problem 1.1, for the plant (4.51) and exosystem (4.2). Thus, we have completed the conversion of the output regulation problem of the given plant into the stabilization problem of the system (4.66). ∎

4.2.2 The stabilization problem

System (4.66) is a lower triangular system with relative degree n. What makes this system special is that it contains dynamic uncertainty z, static uncertainty Ψ^σ. The dynamic uncertainty z can be handled in the same way as what has been done in the last section with following condition.

Assumption 4.6 For any compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$, there exists a C^1 function $V_{\bar{z}}$ satisfying $\underline{\alpha}(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \bar{\alpha}(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of the subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$

$$\dot{V}_{\bar{z}} \leq -\alpha(\|\bar{z}\|) + \delta\gamma(e) \tag{4.69}$$

where δ is some unknown positive constant, $\alpha(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying $\lim_{s\to 0^+} \sup(\alpha^{-1}(s^2)/s) < \infty$ and $\gamma(\cdot)$ is a known smooth p.d. function. ∎

We are now ready to carry on the recursive design. For this purpose, we introduce the following notations:

$$\begin{aligned}
\tilde{\xi}_1(\xi_2, k, \eta, \hat{\Psi}, e) &= \xi_2 - \kappa_1(k, \eta, \hat{\Psi}, e) \\
\kappa_1(k, \eta, \hat{\Psi}, e) &= -k\rho(e)e + \hat{\Psi}\eta \\
\chi_1(\xi_{[2]}, k, \eta, \hat{\Psi}, e) &= -\lambda_2\xi_1 + \frac{\partial\tilde{\xi}_1}{\partial k}\dot{k} + \frac{\partial\tilde{\xi}_1}{\partial \eta}\dot{\eta} + \frac{\partial\tilde{\xi}_1}{\partial \hat{\Psi}}\dot{\hat{\Psi}} \\
\phi_1(\eta, e) &= \eta^\top e \\
\Delta_1(z, e, \mu) &= \bar{g}_e(z, e, \mu)e
\end{aligned} \tag{4.70}$$

and for $i = 2, \cdots, n$

$$\begin{aligned}
\tilde{\xi}_i(\xi_{[i+1]}, k, \eta, \hat{\Psi}, \hat{b}, e) &= \xi_{i+1} - \kappa_i(\xi_{[i]}, k, \eta, \hat{\Psi}, \hat{b}, e) \\
\kappa_i(\xi_{[i]}, k, \eta, \hat{\Psi}, \hat{b}, e) &= -\chi_{i-1} - \tilde{\xi}_{i-1} - \tilde{\xi}_{i-2} - \hat{b}E_{i-1}(\tilde{\xi}_1 - k\rho(e)e) - \frac{1}{2}\tilde{\xi}_{i-1}E_{i-1}^2 \\
\chi_{i-1}(\xi_{[i]}, k, \eta, \hat{\Psi}, \hat{b}, e) &= -\lambda_i\xi_1 + \sum_{j=1}^{i-1}\frac{\partial\tilde{\xi}_{i-1}}{\partial\xi_j}\dot{\xi}_j + \frac{\partial\tilde{\xi}_{i-1}}{\partial k}\dot{k} + \frac{\partial\tilde{\xi}_{i-1}}{\partial\eta}\dot{\eta} + \frac{\partial\tilde{\xi}_{i-1}}{\partial\hat{\Psi}}\dot{\hat{\Psi}} + \frac{\partial\tilde{\xi}_{i-1}}{\partial\hat{b}}\dot{\hat{b}} \\
\phi_i(\xi_{[i]}, k, \eta, \hat{\Psi}, \hat{b}, e) &= \eta^\top\Big(e + \sum_{j=1}^{i-1}\tilde{\xi}_j E_j\Big) \\
\phi_{1i}(\xi_{[i]}, k, \eta, \hat{\Psi}, \hat{b}, e) &= \tilde{\xi}_1 e + (\tilde{\xi}_1 - k\rho(e)e)\sum_{j=1}^{i-1}\tilde{\xi}_j E_j \\
\Delta_i(z, e, \mu) &= \Delta_{i-1} + \frac{1}{2}\bar{g}_e^2
\end{aligned} \tag{4.71}$$

where for convenience, we let $\tilde{\xi}_0 := \hat{b}e$, $\tilde{\xi}_n := 0$, $\xi_{r+1} := u$, and $E_i := \partial\tilde{\xi}_i/\partial e$, $i = 1, \cdots, n-1$. k is a variable governed by (4.72), and $\hat{\Psi}$, $\hat{b}$ are estimations of Ψ^σ, b with their estimation errors denoted by $\tilde{\Psi} = \hat{\Psi} - \Psi^\sigma$ and $\tilde{b} = \hat{b} - b$, respectively. $\rho(e)$ is a continuous positive design function which is determined by (4.81). Additionally, we will introduce a variable $\tilde{k} := k - \bar{k}$ with $\bar{k}$ being some real positive constant to be specified by (4.81).

Theorem 4.2 Under Assumptions 4.4 to 4.6, there exist a continuous function $\rho(e) \geq 1$ and a control law of the form

$$\begin{aligned} u &= \kappa_n(\xi_{[n]}, k, \eta, \hat{\Psi}, \hat{b}, e) \\ \dot{\hat{\Psi}} &= -\phi_n(\xi_{[n]}, k, \eta, \hat{\Psi}, \hat{b}, e) \\ \dot{\hat{b}} &= \phi_{1n}(\xi_{[n]}, k, \eta, \hat{\Psi}, \hat{b}, e), \quad \dot{k} = \rho(e)e^2 \end{aligned} \tag{4.72}$$

that solves Problem 4.2. ∎

Proof: First, notice that for any given $(v_0, w, \sigma) \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$ and $v(t)$ being generated by (4.2) with initial state v_0, there exists a compact subset Σ of $\mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$ containing $(v(t), w, \sigma)$ for all $t \geq 0$. Let $z_1 = \bar{z}$, $z_2 = \text{col}(\bar{\epsilon}, \bar{\eta})$. Then the z subsystem of (4.66) can be put in the following form (2.11) with $A = \begin{bmatrix} A_c & 0 \\ \bar{N} & M \end{bmatrix}$, $\bar{N}$ satisfying $\bar{N}\bar{\epsilon} = -N\bar{\epsilon}_2$, and

$$\begin{aligned} \varphi_1(z_1, e, \mu) &= \bar{f}(z_1, e, \mu(t)) \\ \varphi_2(z_1, e, \mu) &= \begin{bmatrix} \bar{h}(\bar{z}, e, \mu) \\ MNb^{-1}e - Nb^{-1}\bar{g}_1(\bar{z}, e, \mu) \end{bmatrix} \end{aligned}$$

Since both A_c and M are Hurwitz, and $\varphi_1(0,0,\mu) = 0$ and $\varphi_2(0,0,\mu) = 0$, by Lemma 2.1, there exists a C^1 function $V_1(z)$ satisfying $\underline{\alpha}_1(\|z\|) \leq V_1(z) \leq \bar{\alpha}_1(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1(\cdot)$ and $\bar{\alpha}_1(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of z subsystem

$$\dot{V}_z \leq -\|z\|^2 + \delta_e \gamma_e(e) \tag{4.73}$$

for some positive constant δ_e and smooth p.d. function $\gamma_e(\cdot)$.

Further, by Remark 2.4, given any smooth function $\Delta(z) > 0$, there exists a C^1 function $U_z(z)$ satisfying $\underline{\alpha}_{2z}(\|z\|) \leq U_z(z) \leq \bar{\alpha}_{2z}(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{2z}(\cdot)$ and $\bar{\alpha}_{2z}(\cdot)$ such that, along the trajectory of system $\dot{z} = F(z, e, \mu)$

$$\dot{U}_z \leq -\Delta(z)\|z\|^2 + \bar{\delta}_e \bar{\gamma}_e(e) e^2 \tag{4.74}$$

for some positive number $\bar{\delta}_e$ and some known smooth function $\bar{\gamma}_e(\cdot) \geq 1$. The function U_z will be used later in constructing the function U in (4.77).

Next, define

$$\begin{aligned} V_1 &= \frac{1}{2}(e^2 + b\tilde{\Psi}\tilde{\Psi}^\top + \tilde{b}^2) \\ V_i &= V_{i-1} + \frac{1}{2}\tilde{\xi}_{i-1}^2, \quad 2 \leq i \leq n \end{aligned}$$

Then, using (4.70), V_1 satisfies along the trajectory of e subsystem

$$\dot{V}_1 \leq b\tilde{\xi}_1 e - bk\rho(e)e^2 + b(\dot{\hat{\Psi}} + \phi_1)\tilde{\Psi}^\top + \Delta_1 + \dot{\hat{b}}\tilde{b} \tag{4.75}$$

and for $2 \leq i \leq n$, V_i satisfies along the trajectory of $(e, \xi_{[i]})$ subsystem

$$\dot{V}_i \leq \tilde{\xi}_{i-1}\tilde{\xi}_i - \sum_{j=1}^{i-1} \tilde{\xi}_j^2 - bk\rho(e)e^2 + b(\dot{\hat{\Psi}} + \phi_i)\tilde{\Psi}^\top + (\dot{\hat{b}} - \phi_{1i})\tilde{b} + \Delta_i$$

At $i = n$, we have, along the trajectory of system (4.66)

$$\dot{V}_n \leq -\sum_{j=1}^{n-1} \tilde{\xi}_j^2 - bk\rho(e)e^2 + \Delta_n(z, e, \mu) \tag{4.76}$$

Now let

$$U = U_z + V_n + \frac{1}{2}b\tilde{k}^2 \tag{4.77}$$

which satisfies along the trajectory of the closed-loop system composed of (4.66) and (4.72)

$$\begin{aligned} \dot{U} &= \dot{U}_z + \dot{V}_n + b\dot{k}(k - \bar{k}) \\ &\leq -\Delta(z)\,\|z\|^2 + \bar{\delta}_e\bar{\gamma}_e(e)e^2 - \sum_{j=1}^{n-1} \tilde{\xi}_j^2 - bk\rho(e)e^2 + \Delta_n(z, e, \mu) + b\dot{k}(k - \bar{k}). \end{aligned} \tag{4.78}$$

It is easy to verify that the above defined U satisfies

$$\underline{\alpha}(\| z, e, \tilde{\Psi}, \tilde{b}, \tilde{k}, \tilde{\xi}_{[n-1]} \|) \leq U \leq \bar{\alpha}(\| z, e, \tilde{\Psi}, \tilde{b}, \tilde{k}, \tilde{\xi}_{[n-1]} \|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$.

Since $\Delta_n(z, e, \mu)$ defined by the last equation of (4.71) satisfies $\Delta_n(0, 0, \mu) = 0$ for any $\mu \in \Sigma$, using Lemma 2.4, one can show that

$$\Delta_n(z, e, \mu) \leq \Delta_o(z)\,\|z\|^2 + \hbar\Delta_e(e)e^2 \tag{4.79}$$

for some positive continuous function $\Delta_o(z)$, some real positive constant $\hbar$, and some known positive continuous function $\Delta_e(e)$. Using (4.79) in (4.78) gives

$$\dot{U} \leq -\big(\Delta(z) - \Delta_o(z)\big)\,\|z\|^2 - \sum_{j=1}^{n-1} \tilde{\xi}_j^2 + b\dot{k}(k - \bar{k}) - bk\rho(e)e^2 + \bar{\delta}_e\bar{\gamma}_e(e)e^2 + \hbar\Delta_e(e)e^2 \tag{4.80}$$

Choosing

$$\begin{aligned} \rho(e) &\geq \max\{\bar{\gamma}_e(e), \Delta_e(e), 1\} \\ \Delta(z) &\geq \Delta_o(z) + 1 \\ \bar{k} &\geq \bar{\delta}_e + \hbar \end{aligned} \tag{4.81}$$

in (4.80) and noting $\dot{k} = \rho(e)e^2$ gives

$$\dot{U} \leq -\|z\|^2 - \sum_{j=1}^{n-1} \tilde{\xi}_j^2 \tag{4.82}$$

By (4.82), the solution of the closed-loop system composed of (4.66) and (4.72) is defined on $[0, +\infty)$ and is bounded over $[0, +\infty)$, and $z(t)$, $\tilde{\xi}_i$ converge to zero as $t \to +\infty$.

We now show $e(t)$ will approach the origin as $t \to +\infty$. Since $k(t)$ is bounded over $[0, +\infty)$ and $\dot{k}(t) = \rho(e)e^2$ with $\rho(e) \geq 1$, e is square integrable over $[0, +\infty)$. Furthermore, both $e(t)$ and $\dot{e}(t)$ are bounded over $[0, +\infty)$. Using Lemma 2.3 concludes that $e(t)$ tends to zero as $t \to +\infty$. This completes the proof.

Remark 4.17 The controller (4.72) is in the form (4.67). By Remark 4.16, this controller will lead to the controller (4.68) which solves Problem 4.1 of systems (4.51) and (4.2). In what follows, we summarize the main design procedure for Problem 4.1 by the following steps:

Step 1 Solve the regulator equations associated with systems (4.51) and (4.2) to get $\mathbf{u}(v, w, \sigma)$;

Step 2 Introduce the filter (4.54) and rewrite $\mathbf{u}(v, w, \sigma)$ in the form (4.57);

Step 3 According to Remark 4.12, get $\Xi(v, w, \sigma)$ described by (4.58);

Step 4 Calculate the polynomial (4.60) and construct the internal model (4.64);

Step 5 Choose the design function $\rho(e)$ according to (4.81);

Step 6 Determine the design functions κ_n, ϕ_n and ϕ_{1n} according to (4.70) or (4.71).

Note that Assumptions 4.4 and 4.5 can be verified at *Step 1*. If these two conditions are fulfilled, after *Step 1*, it is ready to test condition (4.69) for subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$ of the augmented system (4.66). ∎

We will now consider the convergence issue of the parameter $\hat{\Psi}$ which will in turn determine the convergence of the unknown parameter σ in the exosystem. For this purpose, let us establish a lemma.

Lemma 4.2 Under Assumptions 4.4 to 4.6, the closed-loop system composed of (4.66) and (4.72) has the following properties

i)

$$\lim_{t \to \infty} \dot{\hat{\Psi}} = 0 \tag{4.83}$$

ii)

$$\lim_{t \to \infty} (\hat{\Psi} - \Psi^{\sigma}) T(\sigma) \tau(v, w, \sigma) = 0 \tag{4.84}$$

∎

Proof: From the third equation of (4.72), $\dot{\hat{\Psi}}$ is a linear function of e and $\tilde{\xi}_i$, $i = 1, \cdots, n-1$. Thus $\lim_{t\to\infty} \dot{\hat{\Psi}}(t) = 0$ since $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} \tilde{\xi}_i(t) = 0$ for $i = 1, \cdots, n-1$.

To show (4.84), note that from the convergence of $\tilde{\xi}_1$ and e, i.e.

$$\lim_{t\to+\infty} \tilde{\xi}_1 = \lim_{t\to+\infty} \left(\xi_2 + k\rho(e)e - \hat{\Psi}\eta\right) = 0$$

we have

$$\lim_{t\to+\infty} \left(\xi_2 - \hat{\Psi}\eta\right) = 0 \tag{4.85}$$

Since $\lim_{t\to+\infty} \bar{g}_e(z, e, \mu) = 0$ and both e and $\ddot{e}$ are bounded over $[0, +\infty)$, using Lemma 2.3 gives

$$\lim_{t\to+\infty} \dot{e} = \lim_{t\to+\infty} \left(b\xi_2 - b\Psi^\sigma \eta + \bar{g}_e\right) = 0$$

which implies

$$\lim_{t\to+\infty} \left(\xi_2 - \Psi^\sigma \eta\right) = 0 \tag{4.86}$$

Therefore, (4.84) can be derived from (4.85), (4.86) and the fact that $\hat{\Psi}$ is bounded for all $t \geq 0$ and

$$\lim_{t\to+\infty} (\eta - \theta) = \lim_{t\to+\infty} \bar{\eta} = 0$$

Remark 4.18 Since $\hat{\Psi}$ satisfies conditions (4.83) and (4.84), by Lemma 4.1 of [61], $\hat{\Psi}(t)$ will converge to Ψ^σ asymptotically provided that $\tau(v(t), w, \sigma)$ is PE. To give conditions under which $\tau(v(t), w, \sigma)$ is PE, recall from [61] that a monic polynomial (4.60) is called a *zeroing polynomial* of $\Xi_2(v, w, \sigma)$ on $\mathbb{S}$, if, along all trajectories $v(t)$ of the exosystem (4.2) and all $(w, \sigma) \in \mathcal{W} \times \mathbb{S}$, $\Xi_2(v, w, \sigma)$ satisfies a differential equation of the form (4.59). A monic polynomial $P^\sigma(\lambda)$ is called the minimal zeroing polynomial of $\Xi_2(v, w, \sigma)$ on $\mathbb{S}$ if $P^\sigma(\lambda)$ is a zeroing polynomial of $\Xi_2(v, w, \sigma)$ on $\mathbb{S}$ of least degree. An internal model whose dimension is equal to the degree of the minimal zeroing polynomial of $\Xi_2(v, w, \sigma)$ is called the minimal internal model. As $\Xi_2(v, w, \sigma)$ is a polynomial in v, it is known from [61] that there exists an integer r such that, along all trajectories $v(t)$ of the exosystem, for any $(w, \sigma) \in W \times \mathbb{S}$

$$\Xi_2(v(t), w, \sigma) = \sum_{l=1}^{r} C_l(v_0, w, \sigma) e^{\jmath \hat{\omega}_l t} \tag{4.87}$$

where $\jmath = \sqrt{-1}$, $\hat{\omega}_1, \cdots, \hat{\omega}_l$ are determined by the eigenvalues of A_1 and are distinct, and $C_l(v_0, w, \sigma) \in \mathbb{C}$ are not identically zero for all (v_0, w, σ). Thus, the minimal zeroing polynomial of $\Xi_2(v, w, \sigma)$ is $P^\sigma(\lambda) = \prod_{l=1}^{r}(\lambda + \jmath\hat{\omega}_l)$. ∎

Combining Lemma 4.2 and Remark 4.18 gives the following result.

Theorem 4.3 Under Assumptions 4.4 to 4.6, if the internal model is of minimal dimension, and v_0, w and σ are such that none of $C_l(v_0, w, \sigma)$ is zero, then the feedback controller (4.72) is such that $\lim_{t\to\infty}(\hat{\Psi} - \Psi^\sigma) = 0$. ∎

4.2.3 Application

Consider a class of nonlinear systems in the form (4.51) described by the following equations:

$$\begin{aligned} \dot{z}_1 &= a_{11}z_1 + a_{12}x_1 \\ \dot{z}_2 &= a_3 z_2 + z_1 x_1 \\ \dot{x}_1 &= x_2 + a_{21}z_1 + a_{22}x_1 - z_1 z_2 \\ \dot{x}_2 &= u + a_{41}z_1 + a_{42}z_2 x_1 \\ y &= x_1 \end{aligned} \tag{4.88}$$

where $a = \text{col}(a_{11}, a_{12}, a_{21}, a_{22}, a_3, a_{41}, a_{42})$ is a parameter vector. To make system (4.88) general enough, we choose $\bar{a} = \text{col}(-10, 10, 28, 1, -8/3, 0, -0.2)$ as the nominal value of parameter a and assume $a = \bar{a} + w$ where $w \in \mathcal{W}$ is some uncertainty and the set $\mathcal{W} \subset \mathbb{R}^7$ is defined as follows:

$$\mathcal{W} = \{w \mid w \in \mathbb{R}^7,\ -10 + w_1 < 0,\ -8/3 + w_5 < 0\}$$

In particular, when $u \equiv 0$ and $w = 0$, system (4.88) in terms of $(z_1, z_2, x_1, -x_2)$ is the hyperchaotic Lorenz system in [22].

Our objective is to design an output feedback controller such that for any constants $A_m, \omega > 0, w \in \mathcal{W}$, for any initial conditions, the solution of the closed-loop system exists and is bounded over $[0, +\infty)$ and the output y asymptotically tracks the class of reference signals $F(t) = A_m \sin(\omega t + \phi)$.

Notice that the class of reference signals $F(t)$ can be generated by a linear autonomous system in the form (4.2) described by (4.29). Let $\mathbb{S} = \{\omega \mid \omega > 0\}$ and

$$e = x_1 - v_1 \tag{4.89}$$

This tracking problem has been formulated as Problem 4.1.

We now verify that systems (4.88) to (4.89) satisfy Assumptions 4.4 to 4.6. From equation (4.89), we have

$$\mathbf{x}_1(v, w) = v_1 \tag{4.90}$$

Substituting (4.90) into the first equation of (4.88) yields

$$\mathbf{z}_1(v, w, \omega) = r_{11}v_1 + r_{12}v_2 \tag{4.91}$$

where

$$r_{11}(w, \omega) = -\frac{a_{11}a_{12}}{\omega^2 + a_{11}^2},\ r_{12}(w, \omega) = -\frac{a_{12}\omega}{\omega^2 + a_{11}^2}$$

Thus, Assumption 4.4 is verified.

Substituting (4.91) and (4.90) into the second equation of (4.88) gives

$$\mathbf{z}_2(v,w,\omega) = r_{21}v_1^2 + r_{22}v_1v_2 + r_{23}v_2^2 \tag{4.92}$$

where

$$\begin{aligned} r_{21}(w,\omega) &= -\frac{a_3^2 r_{11} - a_3\omega r_{12} + 2\omega^2 r_{11}}{a_3(a_3^2+4\omega^2)} \\ r_{22}(w,\omega) &= -\frac{r_{12}a_3 + 2\omega r_{11}}{a_3^2+4\omega^2}, \quad r_{23}(w,\omega) = \frac{\omega}{a_3}r_{22} \end{aligned}$$

Substituting (4.90) to (4.92) to the third equation of (4.88) gives

$$\begin{aligned} \mathbf{x}_2(v,w,\omega) &= \omega v_2 - a_{22}v_1 - a_{21}\mathbf{z}_1 + \mathbf{z}_1\mathbf{z}_2 \\ &= r_{31}v_1 + r_{32}v_2 + r_{33}v_1^3 + r_{34}v_1^2v_2 + r_{35}v_1v_2^2 + r_{36}v_2^3 \end{aligned} \tag{4.93}$$

where

$$\begin{aligned} r_{31}(w,\omega) &= -a_{22} - a_{21}r_{11} \\ r_{32}(w,\omega) &= \omega - a_{21}r_{12}, \quad r_{33}(w,\omega) = r_{11}r_{21} \\ r_{34}(w,\omega) &= r_{12}r_{21} + r_{11}r_{22} \\ r_{35}(w,\omega) &= r_{11}r_{23} + r_{12}r_{22}, \quad r_{36}(w,\omega) = r_{12}r_{23} \end{aligned}$$

So $\mathbf{x}_2(v,w,\omega)$ can be put into the form (4.57) as follows:

$$\mathbf{x}_2(v,w,\omega) = \mathcal{X}_{21}(w,\omega)v^{[1]} + \mathcal{X}_{23}(w,\omega)v^{[3]} \tag{4.94}$$

where $v^{[1]} = \left[v_1, v_2\right]^\top$, $v^{[3]} = \left[v_1^3, v_1^2v_2, v_1v_2^2, v_2^3\right]^\top$, and

$$A^{[1]}(\omega) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad A^{[3]}(\omega) = \begin{bmatrix} 0 & 3\omega & 0 & 0 \\ -\omega & 0 & 2\omega & 0 \\ 0 & -2\omega & 0 & \omega \\ 0 & 0 & -3\omega & 0 \end{bmatrix}$$

Finally, substituting (4.94) into the fourth equation of (4.88) gives

$$\begin{aligned} \mathbf{u}(v,w,\omega) &= \sum_{l=1,3} \mathcal{X}_{2l}(w,\omega)A^{[l]}(\omega)v^{[l]} - a_{41}\mathbf{z}_1(v,w,\omega) - a_{42}\mathbf{z}_2(v,w,\omega)\mathbf{x}_1(v,w) \\ &= r_{41}v_1 + r_{42}v_2 + r_{43}v_1^3 + r_{44}v_1^2v_2 + r_{45}v_1v_2^2 + r_{46}v_2^3 \end{aligned} \tag{4.95}$$

where

$$\begin{aligned} r_{41}(w,\omega) &= -(\omega r_{32} + a_{41}r_{11}), \quad r_{42}(w,\omega) = \omega r_{31} - a_{41}r_{12} \\ r_{43}(w,\omega) &= -\omega r_{34} - a_{42}r_{21}, \quad r_{44}(w,\omega) = \omega(3r_{33} - 2r_{35}) - a_{42}r_{22} \\ r_{45}(w,\omega) &= \omega(2r_{34} - 3r_{36}) - a_{42}r_{23}, \quad r_{46}(w,\omega) = \omega r_{35} \end{aligned}$$

Thus, Assumption 4.5 is also verified.

For system (4.88), we introduce the input driven filter described by

$$\dot{\xi} = A_c \xi + Bu, \quad \xi \in \mathbb{R}^2 \tag{4.96}$$

where

$$A_c = \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $\lambda_1, \lambda_2 > 0$, and the $\epsilon = \mathrm{col}(x_1 - \xi_1, x_2 - \xi_2)$ dynamics as described in (4.56) is

$$\dot{\epsilon} = A_c \epsilon + h(z, x_1, v, w), \ \epsilon \in \mathbb{R}^2 \tag{4.97}$$

where

$$h(z, x_1, v, w) = \begin{bmatrix} \lambda_1 x_1 + a_{21} z_1 + a_{22} x_1 - z_1 z_2 \\ \lambda_2 x_1 + a_{41} z_1 + a_{42} z_2 x_1 \end{bmatrix}$$

To calculate the solution $\Xi(v, w, \omega)$ of the regulator equations associated with (4.88), (4.29) and (4.96), put $\mathbf{u}(v, w, \omega)$ into the form:

$$\mathbf{u}(v, w, \omega) = \mathcal{U}_1(w, \omega) v^{[1]} + \mathcal{U}_3(w, \omega) v^{[3]}$$

where $\mathcal{U}_1 = \begin{bmatrix} r_{41}, r_{42} \end{bmatrix}$ and $\mathcal{U}_3 = \begin{bmatrix} r_{43}, r_{44}, r_{45}, r_{46} \end{bmatrix}$. Then,

$$\Xi(v, w, \omega) = \mathcal{S}_1(w, \omega) v^{[1]} + \mathcal{S}_3(w, \omega) v^{[3]}$$

where $\mathcal{S}_l(w, \omega)$, $l = 1, 3$, are governed by the Sylvester equation:

$$\mathcal{S}_l(w, \omega) A^{[l]}(\omega) = A_c \mathcal{S}_l(w, \omega) + B \mathcal{U}_l(w, \omega)$$

Put

$$\mathcal{S}_1(w, \omega) = \begin{bmatrix} \mathcal{S}_{11}(w, \omega) \\ \mathcal{S}_{21}(w, \omega) \end{bmatrix}, \quad \mathcal{S}_3(w, \omega) = \begin{bmatrix} \mathcal{S}_{13}(w, \omega) \\ \mathcal{S}_{23}(w, \omega) \end{bmatrix}$$

Then

$$\Xi_2(v, w, \omega) = \mathcal{S}_{21}(w, \omega) v^{[1]} + \mathcal{S}_{23}(w, \omega) v^{[3]}$$

As $\Xi_2(v, w, \omega)$ is also a polynomial in v, it can be verified that

$$\frac{d^4 \Xi_2(v, w, \omega)}{dt^4} = -9\omega^4 \Xi_2(v, w, \omega) - 10\omega^2 \frac{d^2 \Xi_2(v, w, \omega)}{dt^2}$$

By Remark 4.13, the steady-state generator in the form (4.62) is given with

$$\begin{aligned} \tau(v, w, \omega) &= \left[\Xi_2, \mathcal{L}_{A_1(\omega)v} \Xi_2, \mathcal{L}^2_{A_1(\omega)v} \Xi_2, \mathcal{L}^3_{A_1(\omega)v} \Xi_2 \right]^{\top} \\ \Phi(\omega) &= \left[\begin{array}{c|c} 0 & I_3 \\ \hline -9\omega^4 & 0, -10\omega^2, 0 \end{array} \right], \quad \Gamma = [1, 0, 0, 0] \end{aligned} \tag{4.98}$$

which leads to an internal model as follows:

$$\dot{\eta} = M\eta + N\xi_2 \tag{4.99}$$

where (M, N) is any controllable pair of the form

$$M = \left[\begin{array}{c|c} 0 & I_3 \\ \hline -m_1 & -m_2, -m_3, -m_4 \end{array}\right], \quad N = \left[0, 0, 0, 1\right]^\top$$

and the parameter $(m_1, m_2, m_3, m_4) = (4, 12, 13, 6)$ is such that M is Hurwitz. Solving the Sylvester equation (4.63) yields

$$T^{-1}(\omega) = \begin{bmatrix} 4 - 9\omega^4 & 12 & 13 - 10\omega^2 & 6 \\ -54\omega^4 & 4 - 9\omega^4 & 12 - 60\omega^2 & 13 - 10\omega^2 \\ 9\omega^4(10\omega^2 - 13) & -54\omega^4 & 91\omega^4 - 130\omega^2 + 4 & 12 - 60\omega^2 \\ 108\omega^4(5\omega^2 - 1) & 9\omega^4(10\omega^2 - 13) & 6\omega^2(91\omega^2 - 20) & 91\omega^4 - 130\omega^2 + 4 \end{bmatrix}$$

Hence, we have

$$\begin{aligned} \Psi^\omega &= \Gamma T^{-1}(\omega) \\ &= \left[4 - 9\omega^4, \;\; 12, \;\; 13 - 10\omega^2, \;\; 6\right] \end{aligned} \tag{4.100}$$

Performing the coordinate transformation (4.65) gives the transformed augmented system as follows:

$$\begin{aligned} \dot{\bar{z}}_1 &= a_{11}\bar{z}_1 + a_{12}e \\ \dot{\bar{z}}_2 &= a_3\bar{z}_2 + (\bar{z}_1 + \mathbf{z}_1)(e + v_1) - \mathbf{z}_1 v_1 \\ \dot{\bar{\epsilon}}_1 &= -\lambda_1\bar{\epsilon}_1 + \bar{\epsilon}_2 + \bar{h}_1(\bar{z}, e, \mu) \\ \dot{\bar{\epsilon}}_2 &= -\lambda_2\bar{\epsilon}_1 + \bar{h}_2(\bar{z}, e, \mu) \\ \dot{\bar{\eta}} &= M\bar{\eta} + MNe - N\bar{\epsilon}_2 - N\bar{g}_1(\bar{z}, e, \mu) \\ \dot{e} &= \xi_2 - \Psi^\omega\eta + \bar{g}_e(z, e, \mu) \\ \dot{\xi}_2 &= u - \lambda_2\xi_1 \end{aligned}$$

where

$$\begin{aligned} \bar{g}_1(\bar{z}, e, \mu) &= a_{21}\bar{z}_1 + a_{22}e - (\bar{z}_1 + \mathbf{z}_1)(\bar{z}_2 + \mathbf{z}_2) + \bar{z}_1\bar{z}_2 \\ \bar{g}_2(\bar{z}, e, \mu) &= a_{41}\bar{z}_1 + a_{42}(\bar{z}_2 + \mathbf{z}_2)(e + v_1) - a_{42}\mathbf{z}_2 v_1 \\ \bar{g}_e(z, e, \mu) &= \Psi^\omega\bar{\eta} + \Psi^\omega Ne + \bar{\epsilon}_2 + \bar{g}_1(\bar{z}, e, \mu) \\ \bar{h}_i(\bar{z}, e, \mu) &= \lambda_i e + \bar{g}_i(\bar{z}, e, \mu), \;\; i = 1, 2 \end{aligned}$$

It remains to verify Assumption 4.6. In fact, for any fixed compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W}$, let

$$V_{\bar{z}} = \frac{\hbar}{2}\bar{z}_1^2 + \frac{\hbar}{4}\bar{z}_1^4 + \frac{1}{2}\bar{z}_2^2$$

for some sufficiently large $\hbar > 0$. It can be shown that, for all $(v, w) \in \Sigma$, along the trajectory of $(\bar{z}_1, \bar{z}_2)$ system

$$\dot{V}_{\bar{z}} \leq -\ell_1 \bar{z}_1^2 - \ell_2 \bar{z}_1^4 - \ell_3 \bar{z}_2^2 + \ell_4 e^2 + \ell_5 e^4 \tag{4.101}$$

for some constants $\ell_i > 0$, $i = 1, \cdots, 5$.

As a result of Theorem 4.2, we conclude the problem is solvable. Using the design method detailed in Section 4.2.2 gives the controller as follows.

$$\begin{aligned} u &= \kappa_2(\xi_{[2]}, k, \eta, \hat{\Psi}, e) \\ \dot{\hat{\Psi}} &= -\eta^{\mathsf{T}} e - \eta^{\mathsf{T}} \tilde{\xi}_1 E_1, \quad \dot{k} = \rho(e) e^2 \end{aligned} \tag{4.102}$$

where $\rho(e) = 9(e^6 + 1)$.

Next, we show the convergence of $\hat{\Psi}(t)$ to Ψ^ω, i.e.

$$\lim_{t \to +\infty} \hat{\Psi}(t) = \Psi^\omega \tag{4.103}$$

To this end, we will show that (4.99) is the minimal internal model. In fact, from

$$\begin{aligned} v_1(t) &= \Big(\frac{v_{10}}{2} + \frac{v_{20}}{2\jmath}\Big) e^{\jmath\omega t} + \Big(\frac{v_{10}}{2} - \frac{v_{20}}{2\jmath}\Big) e^{-\jmath\omega t} \\ v_2(t) &= \Big(\frac{v_{10}}{2} + \frac{v_{20}}{2\jmath}\Big) \jmath e^{\jmath\omega t} - \Big(\frac{v_{10}}{2} - \frac{v_{20}}{2\jmath}\Big) \jmath e^{-\jmath\omega t} \end{aligned}$$

we can put $\Xi_2(v, w, \omega)$ in the form (4.95) with $r = 4$, $\hat{\omega}_{1,2} = \pm\omega$ and $\hat{\omega}_{3,4} = \pm 3\omega$ where none of the coefficients $C_l(v_0, w, \omega), l = 1, \cdots, 4$ is identically zero for all $(v_0, w, \omega) \in \mathbb{R}^{n_v} \times \mathcal{W} \times \mathbb{S}$. By Theorem 4.3, for any given v_0, w and ω, the control law guarantees the parameter convergence property (4.103) as long as none of the coefficients $C_l(v_0, w, \omega), l = 1, \cdots, 4$, is zero.

From (4.100), the estimated frequency $\hat{\omega}$ can be related to the third component $\hat{\Psi}_3$ of $\hat{\Psi}$ as follows:

$$\hat{\omega} = \sqrt{0.1(13 - \hat{\Psi}_3)}$$

Thus, the estimated frequency $\hat{\omega}$ will also approach the true frequency. For $\omega = 1$, $\lambda_{[2]} = \text{col}(2, 3)$, $a = \text{col}(-10, 10, 3, 5, -4, 1, 1)$, and initial value $(z_1(0), z_2(0), x_1(0), x_2(0)) = (1, -2, 2, -1)$, $v_0 = \text{col}(5, 0)$, $k(0) = 1$ and $\eta(0) = \hat{\Psi}(0) = 0$, the simulation is performed and some results are shown in Figures 4.13 and 4.14. The real value of Ψ^ω is $[-5, 12, 3, 6]$. The control goal is achieved.

4.3 Conclusion

This chapter has presented the solvability conditions for the global robust output regulation problem of the output feedback systems by using output feedback control. An

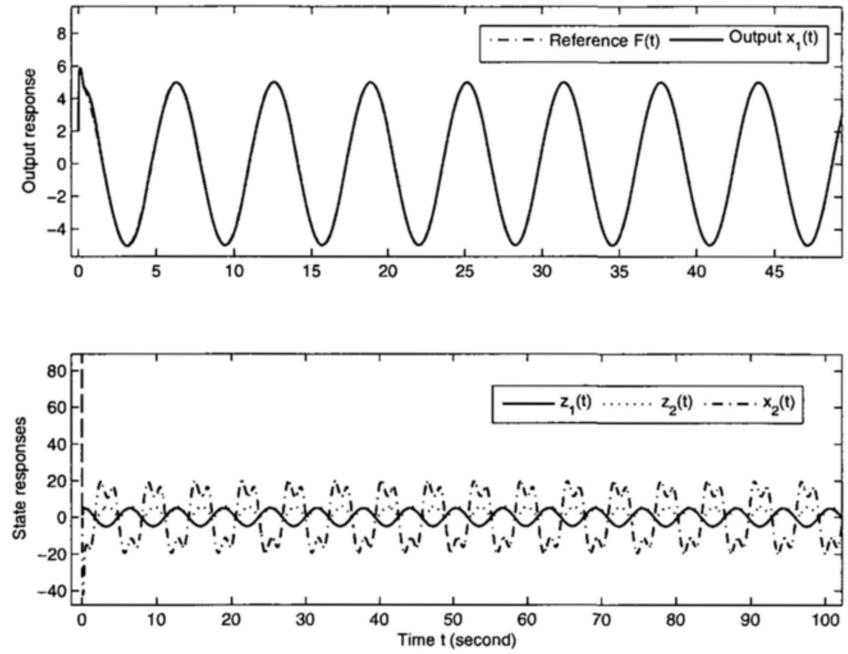

Figure 4.13: Responses of system output $x_1(t)$ and states $(z_1(t), z_2(t), x_2(t))$.

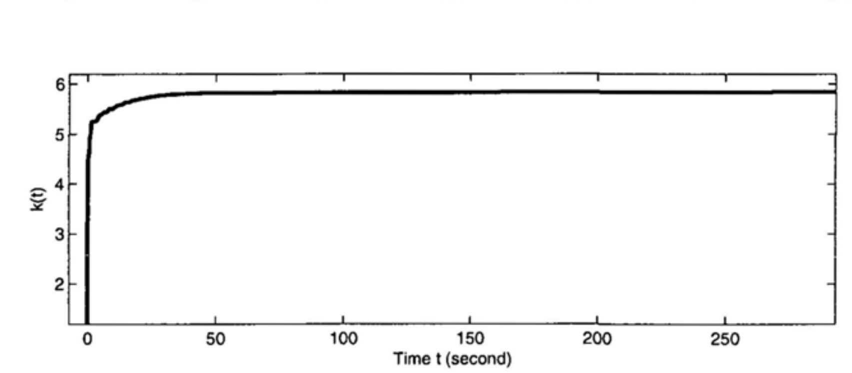

Figure 4.14: Responses of estimation vector $\hat{\Psi}(t)$ and variable $k(t)$.

adaptive control technique is used to handle the unknown parameter vector in the exosystem. It has been shown that this unknown parameter vector can be exactly estimated asymptotically if a controller containing a minimal internal model is employed. Applications of our result to the FHN model and the generalized third and fourth-order Lorenz system have been illustrated.

It should be noted that the design methods in this chapter rely on the ISS property of the inverse dynamics. This implies the system is minimum-phase, that is the equilibrium of the zero dynamics is asymptotically stable. In some case, this condition will not be satisfied. For instance, consider the controlled Chua's circuit in [59] described by the following equations:

$$\begin{aligned} C_1\frac{dV_{C_1}}{dt} &= \frac{1}{R}(V_{C_2} - V_{C_1}) - f_d(V_{C_1}) \\ C_2\frac{dV_{C_2}}{dt} &= \frac{1}{R}(V_{C_1} - V_{C_2}) + I_L \\ L\frac{dI_L}{dt} &= -V_{C_2} - R_0 I_L + u \end{aligned} \tag{4.104}$$

where the cubic function

$$f_d(V_{C_1}) = a_1 V_{C_1} + a_3 V_{C_1}^3$$

with $a_1 < 0$ and $a_3 > 0$ and the nominal values of the various parameters are $R = 1,\ C_1 = 1/9.5,\ C_2 = 1,\ R_0 = 0,\ a_1 = -8/7$. If $y = I_L$ is taken as the output, the system is non-minimum phase. In fact, letting

$$z_1 = V_{C_1}, \quad z_2 = V_{C_2}$$

gives

$$\begin{aligned} \dot{z}_1 &= \frac{-Ra_1 - 1}{RC_1} z_1 + \frac{1}{RC_1} z_2 - \frac{a_3}{C_1} z_1^3 \\ \dot{z}_2 &= \frac{1}{RC_2} z_1 - \frac{1}{RC_2} z_2 + \frac{1}{C_2} y \\ \dot{y} &= \frac{1}{L} u - \frac{1}{L} z_2 - \frac{R_0}{L} y \end{aligned} \tag{4.105}$$

The Jacobian matrix at the origin of the first two equations with $y = 0$ is

$$J_0 = \begin{bmatrix} \frac{-Ra_1-1}{RC_1} & \frac{1}{RC_1} \\ \frac{1}{RC_2} & -\frac{1}{RC_2} \end{bmatrix} = \begin{bmatrix} 9.5/7 & 9.5 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues of the matrix J_0 are

$$\lambda_1 = 3.4784, \quad \lambda_2 = -3.1213$$

Thus, the system is non-minimum phase.

□ **End of chapter.**

Chapter 5

Global output regulation for systems with iISS inverse dynamics

This chapter presents a case study of the systems with iISS inverse dynamics. We will further study the problem in Chapter 3.1 of the output feedback systems with iISS inverse dynamics. As we have shown in Chapter 2, iISS concept is strictly weaker than ISS. Thus, the result of this chapter allows us to handle a much larger class of nonlinear systems. In fact, one of the motivations is to handle the disturbance rejection problem of a shunt-connected DC motor to be described in Section 5.4, where it will be seen that the inverse dynamics of this system is iISS and the problem can be solved.

5.1 Introduction

As a case study, we consider the class of output feedback systems in the special case follows

$$\begin{aligned} \dot{z} &= f(z, y, v, w) \\ \dot{y} &= g(z, y, v, w) + bu \\ e &= y - q(v, w) \end{aligned} \tag{5.1}$$

where $(z, y) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $e \in \mathbb{R}$ is the error output and $u \in \mathbb{R}$ is the control input. $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$ with $\mathcal{W}$ nonempty is a constant uncertain parameter vector, and $v(t) \in \mathbb{R}^{n_v}$ represents the time-varying reference and/or disturbance. The functions f, g and q are supposed to be sufficiently smooth in their arguments satisfying for any $w \in \mathcal{W}$, $f(0,0,0,w) = 0$, $g(0,0,0,w) = 0$, $q(0,w) = 0$, and the control gain $b > 0$ is uncertain.

It is assumed that $v(t)$ is generated by a linear exosystem

$$\dot{v} = A_1 v, \quad v(0) = v_0 \tag{5.2}$$

and all the eigenvalues of the system matrix $A_1 \in \mathbb{R}^{n_v \times n_v}$ are distinct with zero real parts.

The problem concerned in this chapter is precisely formulated as follows: *For any given $\mathcal{W}$, design an output feedback control law of the form:*

$$u = u_K(\zeta, e), \quad \dot{\zeta} = g_K(\zeta, e) \tag{5.3}$$

where both u_K and g_K are sufficiently smooth vanishing at the origin such that, for any initial state $(z(0), x(0), v_0, \zeta(0))$, and any $w \in \mathcal{W}$, the solution of the closed-loop system composed of (5.1) to (5.3) exists and is bounded over $[0, +\infty)$ and the error output $e(t)$ asymptotically approaches zero as t tends to infinity.

Technically, the current problem is most relevant to [44] where the global stabilization problem of the output feedback systems, described by (5.1) with $v_0 = 0$ and an iISS inverse dynamics is studied. The problem here is more challenging than the problem in [44] in that we need to handle asymptotic tracking and disturbance rejection problem for (5.1) with the disturbance and reference input being generated by the exosystem (5.2).

For convenience, we use the following notation. For a pair of p.d. functions $\kappa_1, \kappa_2 : \mathbb{R}^+ \mapsto \mathbb{R}^+$, $\kappa_1(s) \in \mathcal{O}(\kappa_2(s))$ means that $\kappa_1(s) \leq c\kappa_2(s)$ for some constant $c > 0$ and all s in a neighborhood of the origin, and if $\kappa_2(s)$ is bounded over $[0, +\infty)$, so is $\kappa_1(s)$.

5.2 Problem conversion

As in Chapter 3, the first step of our approach is to form the desired augmented system composed of the original plant and a suitable internal model. For completeness, we repeat the following two standard assumptions and the problem conversion procedure similar with what has been done in Chapter 3.

Assumption 5.1 There exists a globally defined sufficiently smooth function $\mathbf{z} : \mathbb{R}^{n_v} \times \mathcal{W} \mapsto \mathbb{R}^n$ with $\mathbf{z}(0, w) = 0$ such that

$$\frac{\partial \mathbf{z}(v, w)}{\partial v} A_1 v = f(\mathbf{z}(v, w), q(v, w), v, w) \tag{5.4}$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathcal{W}$. ∎

Under Assumption 5.1, let $\mathbf{y}(v, w) = q(v, w)$ and

$$\mathbf{u}(v, w) = b^{-1}\big[\big(\partial q(v, w)/\partial v\big) A_1 v - g(\mathbf{z}(v, w), q(v, w), v, w)\big]$$

Then, $\{\mathbf{z}(v, w),\ \mathbf{y}(v, w),\ \mathbf{u}(v, w)\}$ is the solution of the regulator equations associated with (5.1) and (5.2).

Assumption 5.2 The function $\mathbf{u}(v, w)$ is a polynomial in v with coefficients possibly depending on w. ∎

Remark 5.1 Under Assumption 5.2, by Remark 3.2 in Chapter 3, we can define the following dynamic compensator

$$\dot{\eta} = M\eta + Nu \tag{5.5}$$

as in internal model with output u in the sense of Definition 2.7. ∎

Attaching the internal model (5.5) to (5.1) and performing the following coordinate and input transformations

$$\bar{z} = z - \mathbf{z}(v, w),\ \tilde{\eta} = \eta - \theta(v, w) - Nb^{-1}e,\ \bar{u} = u - \Psi\eta \tag{5.6}$$

yields a system described by

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, \mu) \\ \dot{\tilde{\eta}} &= M\tilde{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z}, e, \mu) \\ \dot{e} &= \bar{g}(\bar{z}, e, \mu) + b\Psi\tilde{\eta} + \Psi Ne + b\bar{u} \end{aligned} \tag{5.7}$$

where $\mu = (v, w)$ and

$$\begin{aligned} \bar{f}(\bar{z}, e, \mu) &= f(\bar{z} + \mathbf{z}(v, w), e + q(v, w), v, w) \\ &\quad - f(\mathbf{z}(v, w), q(v, w), v, w) \\ \bar{g}(\bar{z}, e, \mu) &= g(\bar{z} + \mathbf{z}(v, w), e + q(v, w), v, w) \\ &\quad - g(\mathbf{z}(v, w), q(v, w), v, w). \end{aligned} \tag{5.8}$$

In (5.8), it can be verified that, $\bar{f}(0, 0, \mu) = 0$ and $\bar{g}(0, 0, \mu) = 0$ for any $\mu \in \mathbb{R}^{n_v} \times \mathcal{W}$.

Denote $z = \text{col}(\bar{z}, \tilde{\eta})$ and

$$F(z, e, \mu) = \begin{bmatrix} \bar{f}(\bar{z}, e, \mu) \\ M\tilde{\eta} + MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z}, e, \mu) \end{bmatrix}$$

Then, system (5.7) takes the following form

$$\begin{aligned} \dot{z} &= F(z, e, \mu) \\ \dot{e} &= \bar{g}(\bar{z}, e, \mu) + b\Psi\tilde{\eta} + \Psi Ne + b\bar{u} \end{aligned} \tag{5.9}$$

Remark 5.2 We have obtained the augmented system (5.9). It can be seen that if there exists a control law of the form

$$\bar{u} = k_\zeta(\zeta, e),\quad \dot{\zeta} = g_\zeta(\zeta, e) \tag{5.10}$$

that solves the global robust stabilization problem of system (5.9) in the sense that, for any initial state of the closed-loop system and the exosystem, and any fixed unknown parameter $w \in \mathcal{W}$, the solution of the closed-loop system is bounded for all $t \geq 0$, and the state of the augmented system (5.9) approaches zero as t tends to infinity, then the following control law

$$u = k_\zeta(\zeta, e) + \Psi\eta, \ \dot{\eta} = M\eta + Nu, \ \dot{\zeta} = g_\zeta(\zeta, e) \tag{5.11}$$

solves the global robust output regulation problem of the original plant (5.1) and exosystem (5.2). ∎

5.3 Main result

We have shown that system (5.9) is a time-varying output feedback system with relative degree one by viewing $\mu(t)$ an unknown external signal. The existing approaches such as that in Chapter 3 require that the inverse dynamics $\dot{z} = F(z, e, \mu)$ be ISS. In this section, we will further show that the ISS condition on $\dot{z} = F(z, e, \mu)$ can be weakened to iISS condition. Moreover, what makes our problem more interesting is that we actually can prove that the inverse dynamics $\dot{z} = F(z, e, \mu)$ is iISS provided that the subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$ is iISS as it can be shown by Proposition 2.2 and later by Lemma 5.2.

For this purpose, let us first make an iISS-like assumption as follows.

Assumption 5.3 The subsystem $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$ is (robust) iISS with state $\bar{z}$ and input e in the sense that, for any compact subset $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W}$, there exists a C^1 function $V_o(\bar{z})$ satisfying $\underline{\alpha}_o(\|\bar{z}\|) \leq V_o(\bar{z}) \leq \bar{\alpha}_o(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_o(\cdot)$ and $\bar{\alpha}_o(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of system $\dot{\bar{z}} = \bar{f}(\bar{z}, e, \mu)$

$$\frac{\partial V_o}{\partial \bar{z}}(\bar{z}) \cdot \bar{f}(\bar{z}, e, \mu) \leq -\alpha_o(\|\bar{z}\|) + \delta_o \gamma_o(e) e^2 \tag{5.12}$$

where $\alpha_o(\cdot)$ is p.d., $\gamma_o(\cdot) \geq 1$ is continuous and positive, and δ_o is an unknown positive constant. ∎

Remark 5.3 In order to consider a system subject to a time-varying uncertainty function $\mu(t)$, we have slightly modified the iISS concept by introducing an unknown constant δ_o in (5.12) to account for the unknown boundary of Σ, as it has been shown in Chapter 2.

The function $\alpha_o(\cdot)$ in (5.12) is called a supply function. As the supply function $\alpha_o(\cdot)$ in (5.12) is generally bounded, the inverse dynamics $\dot{z} = F(z, e, \mu)$ of (5.9) is not ISS. Therefore, in this case, the design methods in Chapters 3 and 4 are not applicable. Nevertheless, we can show, under Assumption 5.3, that $\dot{z} = F(z, e, \mu)$ is iISS. For this purpose, we will first generalize the changing supply function result of [80] from ISS system to iISS system as follows. ∎

For convenience, we rewrite the augmented system (5.9) as follows

$$\begin{aligned} \dot{z}_1 &= \varphi_1(z_1, e, \mu) \\ \dot{z}_2 &= Mz_2 + \varphi_2(z_1, e, \mu) \\ \dot{e} &= \varphi_e(z_1, z_2, e, \mu) + b\bar{u} \end{aligned} \tag{5.13}$$

with $z_1 = \bar{z}$, $z_2 = \tilde{\eta}$, $\varphi_1 = \bar{f}(\bar{z}, e, \mu)$, $\varphi_2 = MNb^{-1}e - Nb^{-1}\bar{g}(\bar{z}, e, \mu)$ and $\varphi_e = \bar{g}(\bar{z}, e, \mu) + b\Psi\tilde{\eta} + \Psi Ne$.

Before stating the main theorem, we first give the following key lemma of a changing iISS supply function technique.

Lemma 5.1 (Changing iISS Supply Function) Assume, z_1 subsystem of (5.13) is iISS stable with state z_1 and input e and has an iISS Lyapunov function $V_o(z_1)$ satisfying (5.12) with an iISS supply function $\alpha_o(\cdot)$.

For any p.d. function $\psi(\|z_1\|) \in \mathcal{O}(\alpha_o(\|z_1\|))$, there exists a C^1 function $V_{z_1}(z_1)$ satisfying $\underline{\alpha}_{z_1}(\|z_1\|) \leq V_{z_1}(z_1) \leq \bar{\alpha}_{z_1}(\|z_1\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{z_1}(\cdot)$ and $\bar{\alpha}_{z_1}(\cdot)$ such that for any $\mu \in \Sigma$, along the trajectory of z_1 system

$$\dot{V}_{z_1} \leq -\psi(\|z_1\|) + \bar{\delta}_o \bar{\gamma}_o(e)e^2 \tag{5.14}$$

for a continuous function $\bar{\gamma}_o(\cdot) \geq 1$ and a positive constant $\bar{\delta}_o > 0$. ▮

Remark 5.4 When $\alpha_o(\cdot)$ in (5.12) is unbounded, the z_1 subsystem of (5.13) is actually ISS. Thus, this lemma can be viewed as a generalization of the changing supply functions technique of [80] from ISS to iISS. It should be noted that a result similar to this lemma has been given in [44] without a detailed proof. Moreover, the result here is more specific in that the iISS gain function can still be chosen with $\bar{\gamma}_o(\cdot) = \gamma_o(\cdot)$ given by (5.12). For completeness, we give a proof for the case that $\alpha_o(\cdot)$ is bounded over $[0, +\infty)$. ▮

Proof: Suppose $\alpha_o(\cdot)$ is p.d. and bounded over $[0, +\infty)$. As in [80], with the given $V_o(z_1)$ satisfying (5.12), define

$$V_{z_1}(z_1) = \kappa \circ V_o(z_1), \quad \kappa(s) \triangleq \int_0^s \varsigma(\tau)d\tau, \quad s \geq 0$$

where $\varsigma(s) > 0$ for all $s > 0$ is some bounded, continuous and nondecreasing function to be determined. Clearly, the above defined function V_{z_1} satisfies $\underline{\alpha}_{2z_1}(\|z_1\|) \leq V_{z_1}(z_1) \leq \bar{\alpha}_{2z_1}(\|z_1\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{2z_1}(\cdot)$ and $\bar{\alpha}_{2z_1}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of z_1 system

$$\dot{V}_{z_1}(z_1) \leq -\varsigma(V_o(z_1))\alpha_o(\|z_1\|) + \delta_o \varsigma(V_o(z_1))\gamma_o(e)e^2 \tag{5.15}$$

In (5.15), since $\varsigma(\cdot)$ is bounded and $\varsigma(V_{z_1}(z_1)) \geq \varsigma \circ \underline{\alpha}_{z_1}(\|z_1\|)$, we have

$$\dot{V}_{z_1}(z_1) \leq -\varsigma \circ \underline{\alpha}_{z_1}(\|z_1\|)\alpha_o(\|z_1\|) + \bar{\delta}_o \gamma_o(e)e^2 \tag{5.16}$$

for some positive constant $\bar{\delta}_o > 0$.

By virtue of $\psi(\|z_1\|) \in \mathcal{O}(\alpha_o(\|z_1\|))$, we can define the following function

$$\tilde{\varsigma}(s) = \begin{cases} \limsup_{s \to 0+} \frac{\psi(s)}{\alpha_o(s)}, & s = 0; \\ \sup_{\tau \in (0,s]} \frac{\psi(\tau)}{\alpha_o(\tau)}, & s > 0. \end{cases}$$

which is a well defined, positive, continuous and increasing function and satisfies

$$\tilde{\varsigma}(\|z_1\|)\alpha_o(\|z_1\|) \geq \psi(\|z_1\|) \tag{5.17}$$

Using the above constructed function $\tilde{\varsigma}(\cdot)$, further define

$$\varsigma(s) = \tilde{\varsigma} \circ \underline{\alpha}_{z_1}^{-1}(s), \quad s \in [0, +\infty) \tag{5.18}$$

Substituting (5.18) into (5.16) and using (5.17) yields (5.14) with $\bar{\gamma}_o(\cdot) = \gamma_o(\cdot)$. The proof is completed.

Next we will establish a result on the iISS property of the interconnected (z_1, z_2) subsystem of (5.13). For this purpose, we recall that, by Lemma 2.4, there exist a class $\mathcal{K}$ functions $\phi_1(\cdot)$, a continuous function $\phi_2(\cdot) \geq 1$ and some (unknown) positive constants $p_{21}, p_{22} > 0$ such that, for any $\mu(t) \in \Sigma$

$$\|\varphi_2(z_1, e, \mu(t))\| \leq p_{21}\phi_1(\|z_1\|) + p_{22}\phi_2(e)|e| \tag{5.19}$$

Lemma 5.2 For (z_1, z_2) subsystem of (5.13), assume z_1 subsystem satisfies Assumption 5.3 with $z_1 = \bar{z}$, and $\phi_1(\|z_1\|) \in \mathcal{O}(\alpha_o(\|z_1\|))$. Then there exists a C^1 function $V_z(z_1, z_2)$ satisfying $\underline{\alpha}_z(\|z_1, z_2\|) \leq V_z(z_1, z_2) \leq \bar{\alpha}_z(\|z_1, z_2\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_z(\cdot)$ and $\bar{\alpha}_z(\cdot)$ such that, for any $\mu(t) \in \Sigma$, along the trajectory of system (5.13)

$$\dot{V}_z \leq -\phi_1^2(\|z_1\|) - \|z_2\|^2 + \tilde{\delta}_e\tilde{\gamma}_e(e)e^2 \tag{5.20}$$

where $\tilde{\gamma}_e(\cdot) \geq 1$ is a positive and continuous function and $\tilde{\delta}_e$ is an unknown positive constant. ∎

Proof: First, suppose the conditions in Lemma 5.1 for z_1 subsystem are all fulfilled and $\phi_1^2(\|z_1\|) \in \mathcal{O}(\alpha_o(\|z_1\|))$, by applying Lemma 5.1, we immediately have $\bar{V}_{z_1}$ satisfying

$$\dot{\bar{V}}_{z_1} \leq -\phi_1^2(\|z_1\|) + \bar{\delta}_e\gamma_e(e)e^2 \tag{5.21}$$

with $\psi(\cdot) = \phi_1^2(\cdot)$. Next, define

$$V_z(z_1, z_2) = l\bar{V}_{z_1}(z_1) + 2z_2^\top Pz_2$$

where l is some positive number to be specified and P is the positive definite solution of the Lyapunov equation

$$PM + M^\top P = -I_{n_2} \tag{5.22}$$

It can be seen that the above defined function $V_z(z_1, z_2)$ is C^1 satisfying $\underline{\alpha}_z(\|z_1, z_2\|) \leq V_z(z_1, z_2) \leq \bar{\alpha}_z(\|z_1, z_2\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_z(\cdot)$ and $\bar{\alpha}_z(\cdot)$. Moreover, using (5.21), $V_z(z_1, z_2)$ satisfies, along the trajectory of system (5.13)

$$\dot{V}_z \leq -l\phi_1^2(\|z_1\|) + l\bar{\delta}_e\gamma_e(e)e^2 - 2\|z_2\|^2 + 4Pz_2\varphi_2(z_1, e, \mu)$$

In the above inequality, using (5.19) and completing the squares gives

$$\begin{aligned}\dot{V}_z &\leq -l\phi_1^2(\|z_1\|) + l\bar{\delta}_e\gamma_e(e)e^2 - 2\|z_2\|^2 \\ &\quad + \|z_2\|^2 + 4\|P\|^2\|\varphi_2(z_1, e, \mu)\|^2 \\ &\leq -l\phi_1^2(\|z_1\|) + l\bar{\delta}_e\gamma_e(e)e^2 - \|z_2\|^2 \\ &\quad + 8\|P\|^2 p_{21}^2\phi_1^2(\|z_1\|) + 8\|P\|^2 p_{21}^2\phi_2^2(e)e^2\end{aligned}$$

Finally, choosing

$$\begin{aligned} l &\geq 8\|P\|^2 p_{21}^2 + 1 \\ \tilde{\delta}_e &\geq \max\{l\bar{\delta}_e, 8\|P\|^2 p_{21}^2\} \\ \tilde{\gamma}_e(e) &\geq \max\{\gamma_e(e), \phi_2^2(e)\}\end{aligned}$$

yields (5.20). This completes the proof.

Remark 5.5 Lemma 5.2 suggests that, when z_1 subsystem is iISS in the sense of Assumption 5.3, and the condition $\phi_1^2(\|z_1\|) \in \mathcal{O}(\alpha_o(\|z_1\|))$ holds, the (z_1, z_2) subsystem is also iISS with state (z_1, z_2), input e, and an iISS gain function $\tilde{\gamma}_e(e)e^2$. ∎

We are now ready to consider the stabilization problem for the augmented system (5.13).

Theorem 5.1 Consider the augmented system (5.13). Under Assumption 5.3, if $\phi_1^2(\|z\|) \in \mathcal{O}(\alpha_o(\|z\|))$, there exist a positive and continuous function $\rho(e) \geq 1$ and a controller of the form

$$\bar{u} = -k\rho(e)e, \quad \dot{k} = \rho(e)e^2 \tag{5.23}$$

such that the closed-loop system composed of the augmented system (5.13) and control law (5.23) has a property that, for any given $(v_0, w) \in \mathbb{R}^{n_v} \times \mathcal{W}$, there exists a C^1 function $V(z, e, \tilde{k})$ satisfying $\underline{\alpha}_c(\|z, e, \tilde{k}\|) \leq V \leq \bar{\alpha}_c(\|z, e, \tilde{k}\|)$ such that

$$\dot{V}\big|_{(5.13)+(5.23)} \leq -\phi_1^2(\|z_1\|) - \|z_2\|^2 \tag{5.24}$$

where $\tilde{k} = p - k$ with p being some positive constant. ∎

Proof: Consider the augmented system (5.13). For any $(v_0, w) \in \mathbb{R}^{n_v} \times \mathcal{W}$, let $\Sigma \subset \mathbb{R}^{n_v} \times \mathcal{W}$ be a compact subset such that $\mu(t) = (v(t), w) \in \Sigma$ for all $t \geq 0$. Recall that M is Hurwitz, and the functions $\varphi_i(0, 0, \mu) = 0$, $i = 1, 2$, and $\varphi_e(0, 0, 0, \mu) = 0$ for all $\mu \in \mathbb{R}^{n_v} \times \mathcal{W}$.

Thus, by Lemma 5.2, under Assumption 5.3 and condition $\phi_1^2(\|z_1\|) \in \mathcal{O}(\alpha_o(\|z_1\|))$, there exists a C^1 iISS Lyapunov function $V_z(z)$ satisfying $\underline{\alpha}_{1z}(\|z\|) \leq V_z(z) \leq \bar{\alpha}_{1z}(\|z\|)$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{1z}(\cdot)$ and $\bar{\alpha}_{1z}(\cdot)$ such that, for any $\mu \in \Sigma$, along the trajectory of system $\dot{z} = F(z, e, \mu)$

$$\dot{V}_z \leq -\phi_1^2(\|z_1\|) - \|z_2\|^2 + \delta_e \gamma_e(e) e^2 \tag{5.25}$$

for some positive constant δ_e and a known positive and continuous function $\gamma_e(\cdot) \geq 1$.

Using the above obtained V_z, we further define

$$V(z, e, \tilde{k}) = \hbar V_z + \frac{1}{2}e^2 + \frac{1}{2}b\tilde{k}^2 \tag{5.26}$$

for some $\hbar > 0$ to be specified. Then for any $\mu \in \Sigma$, the above defined V satisfies

$$\begin{aligned} \dot{V}\Big|_{(5.13)+(5.23)} \leq & -\hbar\phi_1^2(\|z\|) - \hbar\|z_2\|^2 + \hbar\delta_e\gamma_e(e)e^2 \\ & + e\big(\varphi_e(z_1, z_2, e, \mu) - bk\rho(e)e\big) - b\tilde{k}\dot{k} \end{aligned} \tag{5.27}$$

In (5.27), by using Lemma 2.4 again, we have

$$|\varphi_e(z_1, z_2, e, \mu)| \leq p_1\phi_1(\|z_1\|) + p_2\|z_2\| + p_e\phi_e(e)|e| \tag{5.28}$$

for a p.d. function $\phi_1(\cdot)$, a continuous function $\phi_e(\cdot) \geq 1$ and some positive constants $p_1, p_2, p_e > 0$. Note that we have chosen the same function $\phi_1(\cdot)$ as that in (5.19) to simplify our derivation.

Using (5.28), by completing the squares, we have

$$\begin{aligned} & e\varphi_e(z_1, z_2, e, \mu) \\ \leq & |e| \cdot \big(p_1\phi_1(\|z_1\|) + p_2\|z_2\| + p_e\phi_e(e)|e|\big) \\ \leq & \frac{1}{4}e^2 + p_1^2\phi_1^2(\|z_1\|) + \frac{1}{4}e^2 + p_2^2\|z_2\|^2 + p_e\phi_e(e)e^2 \\ \leq & p_1^2\phi_1^2(\|z_1\|) + p_2^2\|z_2\|^2 + \frac{1}{2}e^2 + p_e\phi_e(e)e^2 \end{aligned} \tag{5.29}$$

Substituting (5.29) into (5.27) gives

$$\begin{aligned} \dot{V} \leq & -(\hbar - p_1^2)\phi_1^2(\|z_1\|) - (\hbar - p_2^2)\|z_2\|^2 + \Big(\hbar\delta_e\gamma_e(e) \\ & + p_e^2\phi_e^2(e) + \frac{1}{2}\Big)e^2 - bk\rho(e)e^2 - b\tilde{k}\dot{k} \end{aligned} \tag{5.30}$$

In (5.30), choosing function $\rho(e)$ such that

$$\rho(e) \geq \max\left\{\phi_e^2(e), \gamma_e(e)\right\} \tag{5.31}$$

and constants

$$\hbar \geq \max\left\{p_1^2, p_2^2\right\} + 1, \quad p \geq b^{-1}\Big(\hbar\delta_e + p_e^2 + \frac{1}{2}\Big)$$

yields

$$\dot{V} \leq -\phi_1^2(\|z_1\|) - \|z_2\|^2 - b\tilde{k}(\dot{k} - \rho(e)e^2) \tag{5.32}$$

This completes the proof.

Remark 5.6 By Theorem 5.1 and Theorem 2.4, $k(t), z(t), e(t)$ are globally uniformly bounded over $[0, +\infty)$ and $z(t)$ approaches zero as $t \to +\infty$. Since $\dot{k} = \rho(e)e^2 \geq e^2$, e^2 is integrable over $[0, +\infty)$. Also, the boundedness of e and $\dot{e}$ implies that e^2 is uniformly continuous. By Lemma 2.3, $e(t)$ tends to zero as $t \to +\infty$. Thus, the global robust output regulation for the plant (5.1) and the exosystem (5.2) has been solved as a result of Remark 5.2. ∎

Remark 5.7 Our result can also be extended to the case where the sign of the control gain b is unknown by utilizing the Nussbaum function technique, similar with the result in Chapter 3. In fact, if the sign of the control gain b is unknown, modify the control law (5.23) to the following

$$\bar{u} = \mathcal{N}(k)\rho(e)e, \quad \dot{k} = \rho(e)e^2 \tag{5.33}$$

where $\mathcal{N}(k) = \exp(k^2)\cos(k)$ is a Nussbaum-type function. It can be seen that the same Lyapunov function $\mathcal{V}(z, e, k)$ satisfies

$$\dot{V}\big|_{(5.13)+(5.33)} \leq \mathcal{N}(k)\rho(e)e^2 \tag{5.34}$$

Using Lemma 2.5 shows that the control law (5.33) solves the stabilization problem of the augmented (5.13). ∎

5.4 Application

A shunt-connected DC motor as shown in Figure 5.1 is described by the following equations [50]

$$\begin{aligned} v_f &= R_f i_f + L_f \frac{di_f}{dt} \\ v_a &= c_1 i_f \omega + L_a \frac{di_a}{dt} + R_a i_a \\ J\frac{d\omega}{dt} &= c_2 i_f i_a - \tau_L \end{aligned} \tag{5.35}$$

where in the first equation v_f, i_f, R_f, and L_f are the voltage, current, resistance, and inductance of the field circuit, and in the second equation, v_a, i_a, R_a, and L_a are the corresponding variables for the armature circuit. The third equation is a torque equation for the shaft, with J as the rotor inertia, $\tau_L = c_3\omega$ as the load torque and c_3 as a damping coefficient [50].

As shown in [50], the field and armature windings of the shunt-connected DC motor are connected in parallel and an external resistance R_x is connected in series with the field winding to limit the field flux, that is

$$v_a = v_f + R_x i_f$$

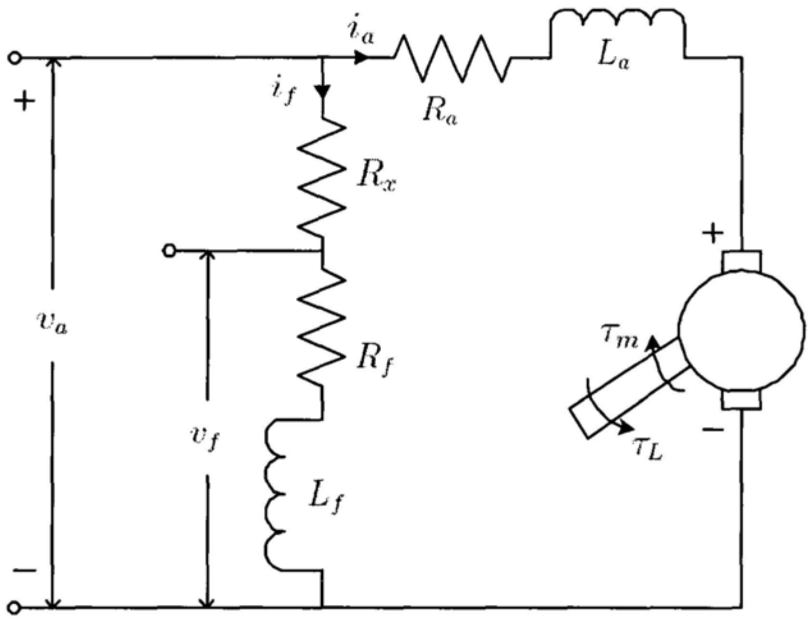

Figure 5.1: A shunt-connected DC motor.

where v_f is seen as the system input.

In [11] [12], the feedback and input-output linearization methods are applied to control a shunt DC motor. We consider the control problem for system (5.35) in the presence of certain input disturbances by using output feedback control.

Suppose $v_f = u_f + d(t)$ where u_f is the actual control input and $d(t) = A_m \sin(\sigma t + \varrho)$ with $A_m, \sigma > 0$ is the input disturbance with an unknown amplitude and initial phase. Notice that the class of disturbances $d(t)$ can be generated by

$$\dot{v} = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix} v, \; v_0 := \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} = \begin{bmatrix} A_m \sin \varrho \\ A_m \cos \varrho \end{bmatrix} \tag{5.36}$$

with $v = \text{col}(v_1, v_2)$ and $v_1 = d(t)$. In particular, when $d(t) \equiv 0$, i.e. $v_0 = 0$, the disturbance rejection problem reduces to certain stabilization problem as in [11] [12].

It is assumed that $J, L_a, L_f, R_a, c_3, \sigma > 0$ and all the other parameters in (5.38) and (5.39) are arbitrary. In addition, the parameter vector $(J, L_a, R_a, c_3, c_1, c_2, R_f, L_f, R_x)$ is defined by

$$(\bar{J}, \bar{L}_a, \bar{R}_a, \bar{c}_3, \bar{c}_1, \bar{c}_2, \bar{R}_f, \bar{L}_f, \bar{R}_x) + w^\top$$

with $w \in \mathcal{W}$ and $\mathcal{W} = \{w \in \mathbb{R}^9 | J, L_a, L_f, R_a, c_3 > 0\}$ where $(\bar{J}, \bar{L}_a, \bar{R}_a, \bar{c}_3, \bar{c}_1, \bar{c}_2, \bar{R}_f, \bar{L}_f, \bar{R}_x)$ denotes its nominal value, say $\bar{J} = 0.0007046$, $\bar{R}_f = 1$, $\bar{L}_f = 0.1236$, $\bar{L}_a = 0.0917$, $\bar{R}_a = 2.5$, $\bar{c}_1 = 1$, $\bar{c}_2 = 1$, $\bar{c}_3 = 0.0004$, $\bar{R}_x = 7.2$. Here all the (nominal) parameter values are borrowed from [68].

Letting $x_1 = i_f$, $x_2 = i_a$, $x_3 = \omega$ and $u = u_f$ gives

$$\begin{aligned}
\dot{x}_1 &= -L_f^{-1} R_f x_1 + L_f^{-1} u + L_f^{-1} d \\
\dot{x}_2 &= -L_a^{-1} c_1 x_1 x_3 - L_a^{-1} R_a x_2 + L_a^{-1} R_x x_1 \\
&\quad + L_a^{-1} u + L_a^{-1} d \\
\dot{x}_3 &= J^{-1} c_2 x_1 x_2 - J^{-1} c_3 x_3
\end{aligned} \tag{5.37}$$

To transform system (5.37) into the form (5.1), further letting

$$z_1 = x_3, \quad z_2 = x_2 - L_a^{-1} L_f x_1, \quad y = x_1$$

gives

$$\begin{aligned}
\dot{z}_1 &= -J^{-1} c_3 z_1 + J^{-1} c_2 (z_2 + L_a^{-1} L_f y) y \\
\dot{z}_2 &= -L_a^{-1} R_a z_2 - L_a^{-1} c_1 z_1 y \\
&\quad + L_a^{-1} (R_f + R_x - R_a L_a^{-1} L_f) y \\
\dot{y} &= -L_f^{-1} R_f y + L_f^{-1} d + L_f^{-1} u
\end{aligned}$$

or in a compact form

$$\begin{aligned}
\dot{z} &= A_z z + H_z z y + f_z(y) \\
\dot{y} &= g(y, v) + b u
\end{aligned} \tag{5.38}$$

where $b = L_f^{-1}$ and

$$\begin{aligned}
A_z &= \begin{bmatrix} -J^{-1} c_3 & 0 \\ 0 & -L_a^{-1} R_a \end{bmatrix} \\
H_z &= \begin{bmatrix} 0 & J^{-1} c_2 \\ -L_a^{-1} c_1 & 0 \end{bmatrix} \\
f_z(y) &= \begin{bmatrix} -J^{-1} L_a^{-1} L_f c_2 y^2 \\ L_a^{-1} (R_x + R_f - R_a L_a^{-1} L_f) y \end{bmatrix} \\
g(y, v) &= -L_f^{-1} R_f y + L_f^{-1} v_1
\end{aligned}$$

The conditions in Theorem 5.1 are all satisfied. In fact, Assumptions 5.1 and 5.2 are satisfied with $\{\mathbf{z}(v,w) = \mathbf{y}(v,w) = 0, \mathbf{u}(v,w) = -v_1\}$ which is the solution of the regulator equations associated with (5.38) and (5.39). Since $\ddot{\mathbf{u}} = -\sigma^2 \mathbf{u}$, we have the steady-state generator matrices as follows

$$\Phi = \begin{bmatrix} 0 & 1 \\ -\sigma^2 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

For any controllable pair (M, N) of the form

$$M = \begin{bmatrix} 0 & 1 \\ -m_1 & -m_2 \end{bmatrix}, \ N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $m_1, m_2 > 0$, we obtain the internal model (5.5) and further calculation shows that $\Psi = \begin{bmatrix} m_1 - \sigma^2 & m_2 \end{bmatrix}$.

To verify Assumption 5.3, consider the inverse dynamics described by

$$\dot{z} = A_z z + H_z z y + f_z(y) \tag{5.39}$$

We state that system (5.39) is iISS with state z and input y. To show this fact, define $V_z = \ln(1 + z^T P z)$ where matrix $P > 0$ is such that $PA_z + A_z^T P = -I$ which satisfies along the trajectory of system (5.39)

$$\dot{V}_z = -\frac{\|z\|^2}{1 + z^T P z} + \frac{2z^T P\big(H_z z y + f_z(y)\big)}{1 + z^T P z} \tag{5.40}$$

In (5.40), completing the squares gives

$$\begin{aligned} &\frac{2z^T P\big(H_z z y + f_z(y)\big)}{1 + z^T P z} \\ &\leq \frac{\epsilon_1 \|z\|^2 + \epsilon_1^{-1} \|P H_z z y + P f_z(y)\|^2}{1 + z^T P z} \\ &\leq \frac{\epsilon_1 \|z\|^2 + 2\epsilon_1^{-1} (\|P H_z z\|^2 y^2 + \|P f_z(y)\|^2)}{1 + z^T P z} \end{aligned}$$

for any real number $0 < \epsilon_1 < 1$. Moreover, since there exist real numbers $\epsilon_2, \epsilon_3 > 0$ such that

$$\frac{2\epsilon_1^{-1} (\|P H_z z\|^2 y^2 + \|P f_z(y)\|^2)}{1 + z^T P z} \leq \epsilon_2 y^2 + \epsilon_3 \|f_z(y)\|^2$$

we have

$$\dot{V}_z \leq -\frac{(1 - \epsilon_1) \|z\|^2}{1 + z^T P z} + \epsilon_2 y^2 + \epsilon_3 \|f_z(y)\|^2 \tag{5.41}$$

Therefore, $\alpha_o(\|z\|) = \frac{(1-\epsilon_1)\|z\|^2}{1+z^T P z}$ in (5.41) corresponding to (5.12) is bounded and p.d. and Assumption 5.3 is verified. Consider function $\bar{g}(\bar{z}, e, \mu) = -L_f^{-1} R_f y$ corresponding to (5.8). The condition $\phi_1^2(\|z\|) \in \mathcal{O}(\alpha_o(\|z\|))$ in Theorem 5.1 is certainly satisfied. As a result, the problem is solvable.

The simulation is performed for system (5.38) and (5.39) under a controller in the form (5.11) with $\rho(y) = y^2 + 1$, $(m_1, m_2) = (2, 3)$, $\sigma = 1.2$, and $\Psi = [0.56, 3]$. The initial value is $(z_1(0), z_2(0), y(0)) = (2, 3, -1)$, $v_0 = \text{col}(2, 0)$, $k(0) = \eta(0) = 0$. Some results are shown in Figure 5.2 with $w = \text{col}(0.01, \cdots, 0.01)$.

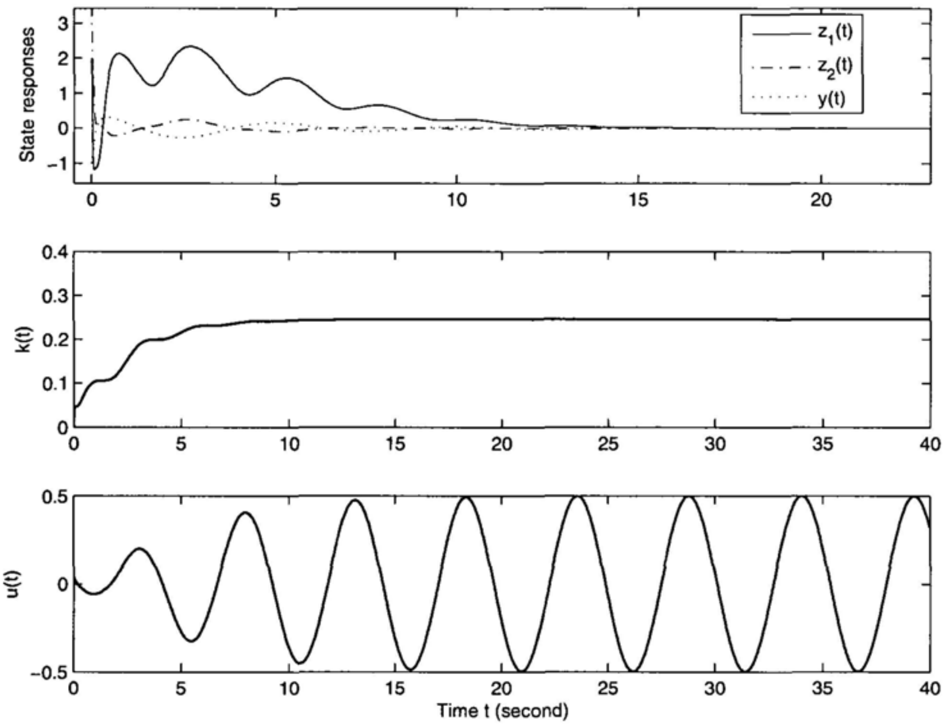

Figure 5.2: Profiles of $(z(t), y(t))$, $k(t)$ and $u(t)$.

5.5 Conclusion

We have presented the solution to the global robust output regulation problem for output feedback systems in the special case with iISS inverse dynamics by using output feedback control. Based on the iISS changing supply function technique, a sufficient condition has been obtained to achieve the output regulation design. As an illustration, a disturbance rejection problem of a shunt-connected DC motor was solved. For output feedback systems in general form and the similar problems studied in Chapter 4, it is possible to modify the current approaches to achieve the output regulation design under certain conditions. They are left for future study. It should be noted that the main result of this chapter, together with those of Chapters 3 and 4, can be seen as a design paradigm that contains many other control results in literature as special cases.

□ **End of chapter.**

Chapter 6

Conclusions

The global robust output regulation for nonlinear output feedback systems has been studied and the results have been applied to solve several tracking and disturbance rejection problems related to well known Lorenz system, FHN model and DC motor. The proposed design methodology strictly extends the results on the output regulation for strict output feedback systems. The result has been presented to cover a larger class of output feedback systems with iISS inverse dynamics, which are strictly larger than the systems with ISS inverse dynamics.

We first studied the output regulation problem for the output feedback systems with ISS inverse dynamics and presented a direct design approach. By designing an internal model in a suitable form, we then obtained an augmented system. Although the augmented system is not in the output feedback form, we have shown that the system can be transformed into a system in a special lower triangular form. Without the knowledge of the sign of the uncertain control gain, the stabilization problem has been solved by using a dynamic output feedback controller, based on a type of observer which is constructed for the transformed augmented system. The result has been applied to solve a tracking control problem of the Lorenz system. In contrast with the previous results on the control of Lorenz systems, the developed design is output feedback control and independent of the control direction. Moreover, the tracking trajectory can be unmeasurable.

Then we studied the output regulation problem with an uncertain exosystem. The design procedure for the general output feedback systems has been presented involving three steps. In the first step, by introducing an input driven filter, an extended system was obtained. Then the output regulation design was equivalently performed for the extended system. In the second step, the internal model was designed and in the last step, an adaptive backstepping design was successfully performed for the transformed augmented systems. We have shown that the parameter convergence is guaranteed if a controller containing a minimal internal model is employed. Three applications of the result to the disturbance rejection of FHN model, the synchronization of the Lorenz system and Harmonic oscillator, and the tracking control of generalized Lorenz system has been illustrated.

Finally, we have presented the solution of the global output regulation problem of the output feedback systems with iISS inverse dynamics using output feedback control. A changing iISS supply function technique was shown as a key lemma to achieve the iISS for the augmented system, which would be useful for further study of the output regulation problem or stabilization problem of the iISS nonlinear systems. As an application, using the current result, we have obtained a solution of a disturbance rejection problem associated with a shunt-connected DC motor.

We close the thesis with some prospects of the future research related to the current work. As it has been pointed out that the presented results are provided with the ISS or iISS property of the inverse dynamics. This means the system is minimum-phase, that is the equilibrium of the zero dynamics is asymptotically stable. The minimum-phase condition is necessary for the current design. However, this condition may not be satisfied in some case, such as the example (4.104) that we have shown in Chapter 4. It is, therefore, interesting to further develop a design method that does not rely on the minimum phase assumption. Another interesting work is to extend the presented results of single-input single-output output feedback systems to multi-input multi-output output feedback systems.

□ **End of chapter.**

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1. D. Xu and J. Huang. Global output regulation for output feedback systems with an uncertain exosystem and its application. *International Journal of Robust and Nonlinear Control*, 2009, DOI: 10.1002/rnc.1539.

2. D. Xu and J. Huang. Output regulation for a class of nonlinear systems using the observer based output feedback control. *Submitted to Journal of DCDIS.* (*Under review*)

3. D. Xu and J. Huang. Robust adaptive control of a class of nonlinear systems and its applications. *IEEE Transactions on Circuits and Systems I*, 2010, (*to appear*)

4. D. Xu and J. Huang. Global robust output regulation for a class of nonlinear systems with unknown control directions. *Journal of Dynamic Systems, Measurement and Control-Transactions of the ASME*, vol. 132, no. 1, 014503, 2010.

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