A Study of Convertible Bond: Optimal Strategies and Pricing

WAN, Xiangwei

A Thesis Submitted in Partial Fulfillment

of the Requirements for the Degree of

Doctor of Philosophy

in

Systems Engineering and Engineering Management

The Chinese University of Hong Kong

June 2010

Thesis/Assessment Committee

Professor Man-Cho So (Chair)

Professor Nan Chen (Thesis Supervisor)

Professor Xunyu Zhou (Thesis Co-Supervisor)

Professor Duan Li (Committee Member)

Professor David Yao (Committee Member)

Professor Yue-Kuen Kwok (External Examiner)

ABSTRACT

This dissertation contains two parts: a non-zero-sum game approach of convertible bond and exotic options pricing under exponential-type jump-diffusion model.

In the first part, we propose a non-zero-sum stochastic game approach of pricing convertible bond under the framework that the capital structure of the firm involves tax rebate and endogenous default policy. Convertible bond is a hybrid security which embodies characteristics of both straight bond and equity. Beyond the bond provisions, it endows a conversion option for the bondholder to convert the bond for the equity of the issuing firm and a call option for the firm to buy the debt back. The conflict of interests between bondholder and shareholder affects the security prices significantly. In Chapter 2, we investigate how to use a non-zero-sum game framework to model their interaction and to evaluate the convertible bond accordingly. Mathematically, this problem can be reduced down to a system of variational inequalities. After we clarify the structure of the optimal exercise region of both parties, we manage to explicitly derive a unique Nash equilibrium to the constraint game and specify the associated optimal exercise strategies. Our model shows that tax benefit and credit risk can produce considerable impact on the optimal strategies of both parties. The firm may issue a call when the debt is out-of-the-money or in-the-money. This is consistent with the empirical findings of "late and early calls" (Ingersoll (1977), Mikkelson (1981), Cowan et al. (1993) and Ederington et al. (1997)) . In addition, the optimal call policy under our model offers an explanation to some

stylized patterns related to the returns of the company value as well.

In the second part, we use Laplace transform to study the pricing problems of various path-dependent exotic options with the underlying asset following an exponentially distributed jump diffusion process. These exotic options include double-barrier option and some occupation-time-related derivatives such as step options, corridor options, and quantile options. The result about double barrier options is presented in Chapter 3, where we prove non-singularity of a related high-dimensional matrix, which guarantees the existence and uniqueness of the solution. Chapter 4 is our work on occupation-time-related options, which presents an extension of the Black-Scholes setting to Kou's double-exponential jump diffusion model. We derive the closed-form Laplace transform of the joint distribution of the occupation time and the terminal value of the double-exponential jump diffusion process, and apply the result to price various occupation-time-related derivatives. This is done by solving the associated two correlated ordinary integro-differential equations, thanks to the special property of the exponential. All the Laplace transform-based analytical solutions can be inverted easily via Euler Laplace inversion algorithm, and the numerical results illustrate that our pricing methods are accurate and efficient.

Key words: Convertible Bond; Non-zero-sum Differential Game; Tax Benefit; Credit Risk; Early/Late Calls; Positive/Negative Stock Return; Double-barrier Options; Step Options; Corridor Options; Quantile Options; Occupation-Time; Jump-Diffusion Process.

摘要

本论文主要包含两个方面的内容: 可转换债券的非零和博弈模型和双边指数型跳跃扩散模型下的奇异期权定价.

在第一部分, 考虑到利息退税优惠和内在破产策略对公司资产结构的影响, 我们提出了可转换债券的一个非零和随机博弈模型. 可转换债券是一种混合证券, 它同时具有了一般债券和股票的特点. 除了债券的特征, 债券持有人具有在到期日前将其转换为可转换债券发行公司的股票的权利, 另一方面, 可转换债券发行公司具有在到期日前将其召回的权利. 债券持有人与发行公司之间的利益冲突对可转换债券价格有着显著的影响. 在第二章中, 我们探讨了如何使用非零和博弈去研究这种相互作用, 并以此来对可换股债券进行定价. 数学上, 这个问题可以转换到一组变分不等式. 我们首先利用变分不等式研究了债券持有人和股东各自的最优执行区域, 然后我们成功的解决了相关的博弈问题: 得到了解的存在唯一性, 给出了解的显式表达式, 并且指定了相关的最优执行策略. 我们的模型表明, 利息退税优惠和内在破产策略有可能对双方的最优策略产生相当大的影响. 公司的最优策略包括召回处于价内或者处于价外的可转换债券. 这与实证研究得出的公司"提前"或者"推迟"召回可转换债券的结论一致(参见Ingersoll (1977), Mikkelson (1981), Cowan et al. (1993) 和 Ederington et al. (1997)等). 此外，我们给出的最优召回策略也对公司股票回报率在公司召回可转换债券时的系统性偏差提供了一个可能的解释.

在第二部分，在假设标的资产的价格过程服从一个双边指数型跳跃扩散下, 我们使用Laplace变换来研究一些路径相关奇异期权的定价问题. 这些奇异期权包括双边障碍期权和一些与occupation-time相关的衍生产品,例如梯级期权,走廊期权和分位数期权等. 在第三章我们介绍双边障碍期权的结果，在那

里我们证明一个相关高维矩阵的非奇异性, 从而保证了解的存在唯一性. 第四章我们将occupation-time相关的衍生产品的研究从Black-Scholes模型扩展到了Kou的双指数跳跃扩散模型. 我们给出了occupation-time与终端标的资产联合分布Laplace变换的解析解, 并运用这个结果给出了一些occupation-time相关的衍生产品的定价. 得到此结果主要是用了指数分布的独特性质来解决了两个相互关联的常积分微分方程. 最后我们通过Laplace逆变换得到了期权价格, 数值试验结果表明我们的定价方法是准确和有效的.

关键字: 可转换债券; 非零和随机博弈; 利息退税; 信用风险; 提前/推迟召回; 正/负股票回报率; 双边障碍期权; 梯级期权; 走廊期权; 分位数期权; occupation-time; 跳跃扩散.

ACKNOWLEDGEMENTS

I would like to express my greatest gratitude to my supervisor, Professor Nan Chen, for his patient guidance in development of the idea in this research as well as in writing this thesis. Many thanks are given to Professor Xun-Yu Zhou, for his rigorous requirements when I first enter the area of stochastic calculus. I am also thankful to Professor Yue-Kuen Kwok for serving as the external examiner of my thesis and grateful to Professor Duan Li, Professor Man-Cho So and Professor David Yao for their expert serving as members of my thesis committee. Moreover, I would like to thank the coauthors of my papers, Professor Ning Cai and Professor Min Dai. I benefit a lot from the discussion with them. I also want to thank my fellow members and friends in Chinese University of Hong Kong. Finally, and most importantly, I would like to thank my family for their constant love and support.

To My Family

Contents

List of Tables

List of Figures

CHAPTER 1

OVERVIEW

This dissertation contains two parts: a non-zero-sum game approach of convertible bond and exotic options pricing under exponential-type jump-diffusion model.

The first part concerning convertible bond is the key point of this dissertation, the result of which is presented in Chapter 2. Convertible bond is a hybrid security which embodies characteristics of both straight bond and equity. Like straight bond, it distributes coupons continuously to the owner up to the maturity. However, unlike straight bond, it also entitles the owner the right to convert the security for a pre-specified portion of equity at her disposal. A typical convertible bond also contains a callable feature that the firm reserves a right to buy the debt back.

This mixed feature complicates the analysis of convertible bond. On one hand, the firm asset value is shared between bondholder and shareholder. The former should choose an optimal conversion strategy and the latter should set up optimal bankruptcy and call policies to maximize the values of their respective holdings. On the other hand, the firm can enjoy tax deductions from the government by serving interest payments to the bondholder. Hence, a non-zero-sum game approach should be a reasonable choice for handling the convertible bond problem.

Beyond the evaluation problem, a lot of studies try to explain two empirical puzzles regarding the convertible bond, that is, "early call and late call" puzzle and "positive and negative" stock return puzzle. The empirical studies show that the firm may issue a call when the conversion value- the equity value the convertible bond can exchange for- is significantly bigger or smaller than the call price, diversing the classic academic result that the firm should call the convertible bond back if and only if the conversion value equals the call price. This is called "early call and late call" puzzle to the out-of-the-money and in-the-money convertible bond, respectively. The "positive and negative" stock return puzzle is associated with the "early and late " call announcement. It is observed that there is a significantly positive stock return associated with the "early" call announcement and statistic negative stock return associated with the "late" call announcement.

To valuate the convertible bond and explain these puzzles, we propose a non-zero-sum stochastic game approach of pricing convertible bond under the framework that the capital structure of the firm involves tax rebate and endogenous default. We firstly transform the conventional non-zero-sum game formulation to a system of variational inequalities. An additional condition is imbedded to ensure our Nash equilibrium is non-trivial and more reasonable in a sense of Pareto optimality. Based on this variational inequalities, we derive the structure of optimal exercise region of each parties, which gives enough necessary conditions for us to construct the solution of the game. By simplifying the underlying firm value process to be a geometric Brownian motion and setting the bond with infinite time maturity, we derive the semi-analytic solutions for the game together with the specified optimal exercise strategies. Rigorous calculus show the existence and uniqueness of our candidate solution. Moreover, we derive diverse optimal call strategies, containing out-of-the-money "early" call, in-the-money "late" call and the classic call, according to different initial parameter setting. Especially, our optimal "early" call time for an out-of-the-money convertible bond is a first passage time of the underlying asset process to an upside flat barrier, and our optimal "late" call time for an in-the-money convertible bond is a first passage time of the underlying asset process to a downside flat barrier. These may also give an explanation of the "positive and negative" stock return puzzle.

It is worth mentioning that the candidate solution of the convertible bond problem is closely related to the first passage time of the underlying process to a double-side flat barrier. Hence in the second part of this dissertation, we use Lapalce transform technique to investigate the double-barrier option, which relates to the first passage time of double-barrier directly, and its extension, the occupation time related options, whose exercise payoff depending on the cumulative time spent by the underlying asset in a predetermined region. In this dissertation, we consider the exotic option pricing under Kou's double-exponential jump diffusion model. The model assumes the underlying asset return follows a jump diffusion process with Poisson jump intensity and double-exponentially distributed jump sizes. It is appealing in two respects. The associated asset returns have heavier tails than normal distributions and hence the model is capable of generating asymmetric leptokurtic feature for asset returns and volatility smiles for equity options, matching the empirical data better than the geometric Brownian motion model. The model also yields analytical solutions to many pricing problems, including both European and path-dependent derivatives, in terms of Laplace transforms. By applying numerical inversion algorithms we can easily obtain the prices. The result about double-barrier options is presented in Chapter 3. And the results about occupation-time-related options are delivered in Chapter 4. All the results can be applied to the hyper-exponentially jump diffusion model, an extension of Kou (2002)'s double-exponential jump diffusion model, proposed by Cai and Kou (2008), for the purpose of providing sufficient flexibility to capture the heaviness of the asset return tails.

Each chapter is organized self-contained with no reference to the others.

CHAPTER 2

A Non-Zero-Sum Game Approach to Convertible Bond: Tax Benefit, Bankrupt Cost and Early/Late Calls

2.1. Introduction

Convertible bonds are hybrid securities that have the characteristics of both straight bonds and equities. The bondholder receives coupons periodically and is entitled to a right to exchange the security at her discretion for part of the issuing company's equity. How many shares of common stock one bond can be converted for is pre-specified through a conversion ratio at its issuance. A typical convertible bond also contains a callable feature - the issuer retains the right to call the debt back. Upon calling, the company offers a price, which is also specified in the bond contract in advance, to the bondholder and forces her to either surrender the security for that price or to convert immediately.

Convertible bonds are quite popular as fund-raising tools among smaller and more speculative companies. Because they lack stable credit histories, the companies have to pay high interest to their debt holders if they choose to raise funds through straight bonds. Meanwhile, their stock are usually undervalued because the capital market is uncertain about the prospective of their business.

Convertible bonds may help to achieve financing with lower coupon, which is justified by the conversion right entitled to the bondholders. When the business turns out to be successful, the bondholders will opt to convert to equity voluntarily or compulsorily. This in turn will strengthen the company's capital base. However, the original shareholders of the company will suffer from a dilution after conversion. From the perspective of investors, convertible bonds are also attractive to some extent. They offer equity-like returns and put a "bond-floor" protection against the downside risk when the business of the issuing company turns sour.

In this paper, we investigate how to price convertible bonds. According to the preceding discussion, the interaction between bondholders and shareholders will affect the bond price significantly. If the bondholders convert earlier than the call announcement issued by the company, then the shareholders lose a chance to force the bondholders to surrender to their interest; if the company calls first, then the bondholders may have no way to act optimally. Hence, any rational pricing model should incorporate the interaction between the two parties. We use a game theoretic approach to tackle this problem.

2.1.1. Literature Review: a Tale of Two Puzzles

The pioneering work on convertible bond pricing dates back to Brennan and Schwartz (1977, 1980) and Ingersoll (1977a). These authors initiate a structural approach to analyzing the optimal call and conversion rules and evaluating convertibles. The key idea is to regard the bond as a contingent claim on the company's asset. They argue that a company should announce a call if and only if the conversion value – the equity value convertible bonds can be exchanged for — equals the call price.

However, later empirical studies do not support this conclusion. Ingersoll (1977b) finds that a majority of companies under examination (170 out of 179) significantly deviate from the theoretical "optimal" call policy. The median company does not issue a call until the conversion value is 43.9% in excess of the call price. This finding is also confirmed by a series of papers such as those of Constantinides and Grundy (1987), Asquith (1995) and so on. This phenomenon is well known in the literature as an "in-the-money call" or "late call" puzzle. More recent research, including those of Cowan et al. (1993) and Sarkar (2003), present empirical evidence which shows that a few convertibles are called when the conversion value is significantly smaller than the call price, which is known as an "out-of-the-money call" or "early call". The challenge lies in determining how to reconcile the discrepancy between the two puzzles in practice and the optimal policy in theory.

The second group of stylized facts we consider in this paper is related to returns of the stock and the total assets of the issuing company at the call announcement. Mikkelson (1981) reports that the average daily returns on the announcement day and one day before were around -1% for all 113 in-the-money calls tested, in contrast to the small returns of the market portfolio during the same period. This finding raises an interesting question: what motivates these companies to make a capital structure decision that reduces shareholders wealth? Cowan et al. (1993) document positive and statistically significant common stock price reactions to the announcement of out-of-the-money calls.

Extensive attempts have been made to explain these two puzzles. To name a few, Ingersoll (1977b), Asquith and Mullins (1991), Asquith (1995), Altintig and Butler (2005), and Dai and Kwok (2005) attribute the in-the-money call phenomenon to the call notice period, a 30-day window in which the issuing company allows the bondholders to ponder over their decision. Harris and Raviv (1985) and Kim and Kallberg (1998) suggest that the reason for in-the-money calls and negative security returns may be rooted in the asymmetric status of market participants and shareholders in their ability to access the company's asset information. Cowan et al. (1993) explain that the positive reaction on stock returns for out-of-the-money calling occurs because managers receive favorable private information about the value of the firm. Dunn and Eades (1984) think that the call delay is caused by passive investors and argue that an in-the-money call benefits the company if enough investors are expected to delay their voluntary conversions.

Other empirical evidence reveals that tax shields and credit risk play a role behind the scenes in the two puzzles (see, e.g., Mikkelson (1981), Asquith and Mullins (1991), Campbell et al. (1991), Jalan and Barone-Adesi (1995) and Sarkar (2003)). The interest payments of a company to its debt holders are tax-deductible expenses under the current tax codes. This may induce the company not to call the debt back even if the conversion value of the bond exceeds its call price. When the company calls, loss of the tax shield will decrease its after-tax value and yield negative return on the securities of the company, as suggested by Mikkelson (1981). In addition, Rosengren (1993) and Indro et al. (1999), among others, point out that credit risk significantly affects the pricing of convertible bonds in general. Impending danger of bankruptcy may prompt companies to call earlier.

2.1.2. Contribution of Our Paper

In this paper, we develop a two-person game model to incorporate the interaction between the shareholders and bondholders of an issuing company. We highlight a tradeoff of two major concerns, tax deduction on interest payments and the losses due to credit risk. On the one hand, the tax benefit entices companies to borrow from bondholders, which may explain why they make in-the-money calls. On the other hand, too much debt will give rise to the significant possibility of bankruptcy in the future. The costly reorganization procedure may prompt out-of-the-money calls to mitigate the impending credit risk facing the company. Encouraged by this intuition, we consider the effects of the combination of tax shield and bankrupt costs on the strategies and pricing of convertible bonds. Our model is capable of generating both in-the-money and out-of-the-money call phenomena. Furthermore, the special structure of the optimal call policy under the model yields a possible explanation for the above mentioned patterns on the security returns at calling.

Mathematically, the model can be formulated as a game involving two coupled optimal stopping problems. With the help of the theory of variational inequality systems, we explicitly solve the Nash equilibrium of the game and prove its uniqueness. Closed-form pricing formulae for both convertible bonds and common stocks are then obtained and the corresponding optimal call, bankrupt and conversion strategies are specified explicitly. The results provide a rigorous mathematical framework to accommodate the empirical evidence in Section 2.1.1.

The papers of Sîrbu et al. (2004) and Sîrbu and Shreve (2006) are closely related with ours. They discuss how to use a game model to price convertible bonds. However, due to the absence of tax effects, their setting is zero-sum: what the shareholders gain is what the bondholders lose. Thus, the two parties will try their best to minimize the size what the other party can acquire. The shareholders will never call in-the-money in their model. Bielecki et al. (2008) consider a general defaultable game-option formulation of convertible bonds under an abstract semimartingale market model. Kallsen and Kühn (2005) use a framework of game contingent claims to study convertible bonds, and introduce a mathematically rigorous concept of no arbitrage price for this kind of derivatives. However, both of the papers ignore tax effects and resembles a framework of game option discussed in Kifer (2002). In this paper we take tax effects into account, which leads to a non-zero-sum game described by a system of variational inequalities. In addition, we aim to obtain closed form price formulae and explain the empirical puzzles in convertible bonds.

We should acknowledge that there are many other factors which can influence the optimal strategies related to convertible bonds. The purpose of this paper is definitely not to claim that our model is complete. Instead, we intend to emphasize the impact of the tradeoff of tax and bankrupt costs and focus on the mathematical modeling of the problem, especially the application of game theory to convertible bond pricing. As the empirical literature in Section 1.1 points out, this tradeoff should not be the unique determinant, and introducing other factors may accentuate its effect. We leave this direction for future investigation.

2.1.3. Some Other Literatures: Reduced Form Approach

Most of the aforementioned literature can be classified under the structural approach, viewing convertible bonds as contingent claims on the company's asset value. The main criticism of this approach is that the company value is not directly observable. Practitioners would like to build up models that can be calibrated to liquid benchmark securities. Some studies thus suggest another approach: to decompose the security into fixed income and equity components and then to discount the associated cash flows in each component at different rates. Early papers in this area include McConnell and Schwartz (1986), Cheung and Nelken (1994), Ho and Pteffer (1996), Tsiveriotis and Fernandes (1998), Yigitbasioglu (2002). More recently, some researchers have introduced the effect of defaults on equity to this approach, stimulated by the progress of the intensity-based reduced-form modeling in the study of general credit risk. One can refer to the work of Takahashi et al. (2001), Davis and Lischka (2002), Ayache et al. (2003), Andersen and Buffum (2003) and Kovalov and Linetsky (2006) for further discussion.

This chapter is organized as follows. We specify our model in Section 2. Section 3 reduces the problem to a variational inequalities formulation and presents some preliminary results. A complete description on the Nash equilibrium is included in Section 4. The numerical experiments in Section 5 demonstrate sensitivity analysis on various parameters. We conclude this Chapter in section 6. All the proofs are deferred to the Appendix A.

2.2. Our Model

2.2.1. Asset Process, Debt Structure and Endogenous Default

Consider a company issuing two kinds of securities [1], common stock and perpetual convertible bond [2], at time 0. There are two players in the game: one bondholder and one shareholder. Assume the un-leveraged asset value of this company follows a geometric Brownian motion:

$$\frac{dV_t}{V_t} = (r - \delta)dt + \sigma dW_t, \quad V_0 = V, \tag{2.1}$$

under the risk neutral probability measure. Here r is the constant risk-free interest rate, W_t is a standard Brownian motion and σ is a positive constant. The company liquidates a portion of the total asset continuously to pay out to its bondholder and shareholder as interest payments and dividends, respectively. The liquidation rate is supposed to be $\delta V_t dt$ (after-tax) within $(t, t + dt)$ for all $t \geq 0$.

Denote P to be the total par value of the convertible bond issued at time 0. Assume that the company will not change its capital structure any more afterwards, until the moment of call, default or voluntary conversion. The bond pays out a stream of coupon flow to its holder continuously. Denote the coupon rate to be c. In every time interval $(t, t + dt)$, the bondholder will receive an amount of \$$cPdt$ coupon payments up to the first time when the bond is converted/called or the company is in default.

[1] Since the firm uses the convertible bond as an alternative method to raise capital instead of straight bond, usually there is no straight bond out-standing for a firm which issues convertible bond.

[2] Although the assumption of the infinite time horizon here is in purpose of simplifying the analysis and getting explicit solutions, it would be reasonable for practice since there is in general a very long time maturity for the issuing convertible and it is usually exercised long time before the maturity.

The bondholder is entitled a right to convert the security for some amount of common shares at her discretion. The conversion factor λ, $0 < \lambda < 1$, is defined as what percentage of the company asset value the bond can exchange for. For instance, if the company value is worth $\$V$ at conversion, then the bondholder will obtain λV after converting [3]. Meanwhile, the convertible bond is subject to redemption calls issued by the company at a preset strike price $\$K$. When calling, the bondholder must opt to surrender the security for $\$K$ or exercise the conversion immediately by force, that is we don't consider the call notice period here.

One important feature of our model is endogenous default, i.e., the stockholder can determine when to bankrupt. In the default event, the company will lose a portion ρ of the total asset due to its reorganization procedure. The bondholder will take over the rest part. We assume that $1 - \rho \geq \lambda$ [4].

Suppose that the corporate tax rate is κ. The company is assumed to enjoy tax exemption by serving its coupon payments. It can claim a tax credit of $\kappa cPdt$ from the government for the total due interest payment, $cPdt$, in $(t, t + dt)$. We incorporate this tax benefit in the model by simply assuming that the actual coupon payment for the company is $(1 - \kappa)cPdt$. Recall that an after-tax cash flow $\delta V_t dt$ is available to both bond and shareholder according to (2.1). The remaining cash flow after coupon obligation is then $(\delta V_t - (1 - \kappa)cP)dt$ and will be distributed to the shareholder as dividends. The quantity $\delta V_t - (1 - \kappa)cP$ may be negative. In this case, additional new equity is issued to finance the coupon payments. Such capital structure specification is quite standard. For instance,

[3] People call the conversion rate λ we define here as dulution ratio. Assume our convertible bond can be converted to $\bar{\lambda}$ share of stock. At conversion, the firm issues $\bar{\lambda}$ share of new stock to replace the convertible bond and the bondholder gets a proportion $\lambda = \frac{\bar{\lambda}}{1+\bar{\lambda}}$ of the firm value.

[4] Since the bondholder can still convert the bond for equity at the default, if the recovery rate $1 - \rho$ is less than the conversion rate λ, bondholder always has an incentive to convert the bond to equity before the default, which in consequence remove the debt obligation and the firm will never default.

Leland (1994) and Leland and Toft (1996) consider optimal leverage level and the pricing of straight bond under this framework; Hilberink and Rogers (2002) and Chen and Kou (2009) incorporate jump risk to the same capital structure to explain non-zero credit spreads of short-term straight bond.

2.2.2. A Non-Zero-Sum Game Between Bondholder and Shareholder

We follow a game-theoretic approach to model the conflict of interest between the bondholder and shareholder. According to the model description in the last subsection, the bondholder can choose when to convert and the shareholder have freedom to select both default and call times. Assume that both the two parties are risk neutral and hence they will behave to maximize the values of their own holdings at time 0. All the decisions are made at time 0. For the simplicity's sake, both parties are supposed to have equal access on the information regarding the company. We neglect information asymmetry in the model to concentrate our attention on the effects of tax benefits and bankrupt costs on the behavior of convertible bond.

Now, let us formulate the objective functions of the bondholder and shareholder respectively. Once the two players fix the conversion time τ_{con} and the bankruptcy and call time τ_b and τ_{cal}, the present value of the convertible bond can be decomposed into a sum of three components: coupon payments, conversion value and bankrupt recovery. The present value of coupon payments, up to the call/conversion or default, is equal to

$$E\left[\int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}} e^{-rt}cPdt\right].$$

When the default occurs, the bondholder will receive an amount of recovery payment by taking over the company's post-reorganization asset. Its present value should be given by

$$E[e^{-r\tau_b}(1-\rho)V_{\tau_b}\cdot\mathbf{1}_{\{\tau_b<\tau_{cal}\wedge\tau_{con}\}}].$$

When the call occurs first, the bondholder is forced to make a choice between K and $\lambda V_{\tau_{con}}$. Under the risk-neutral assumption, the bondholder surely prefers to the one which yields a better outcome in terms of the bond value. Thirdly, the bondholder can initiate a conversion voluntarily which brings her a payoff of $\lambda V_{\tau_{cal}}$. The present values of the cash flows associated with these two events is given by

$$E[e^{-r\tau_{con}}\lambda V_{\tau_{con}}\cdot \mathbf{1}_{\{\tau_{con}<\tau_b\wedge\tau_{cal}\}}] + E[e^{-r\tau_{cal}}\max\{K,\lambda V_{\tau_{cal}}\}\cdot \mathbf{1}_{\{\tau_{cal}<\tau_{con}\wedge\tau_b\}}].$$

In summary, the value of the convertible bond at time 0 is then given by

$$\begin{aligned} D(V;\tau_b,\tau_{cal};\tau_{con}) :=& E\Big[\int_0^{\tau_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-rt}cPdt \\ &+e^{-r(\tau_{con}\wedge\tau_b\wedge\tau_{cal})}\cdot(1-\rho)V_{\tau_b}\mathbf{1}_{\{\tau_b<\tau_{con}\wedge\tau_{cal}\}} \\ &+e^{\ r(\tau_{con}\wedge\tau_b\wedge\tau_{cal})}\cdot\max\{K,\lambda V_{\tau_{cal}}\}\mathbf{1}_{\{\tau_{cal}<\tau_{con}\wedge\tau_b\}} \\ &+e^{-r(\tau_{con}\wedge\tau_b\wedge\tau_{cal})}\cdot\lambda V_{\tau_{con}}\mathbf{1}_{\{\tau_{con}<\tau_b\wedge\tau_{cal}\}}\Big|V_0=V\Big]. \end{aligned} \quad (2.2)$$

Following the trade-off theory of capital structure (see, e.g., Brealey and Myers (2008), pp. 503 - 504), the market value of the equity of a company should be the difference between the market values of its total asset and outstanding debts. In the presence of corporate tax and bankruptcy cost, the former equals to the company's un-leveraged asset value plus the present value of tax shield minus the bankruptcy cost. The tax shield, which is defined as the present value of tax deductions, is given by

$$E\left[\int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}} e^{-rt}\kappa cPdt\right].$$

The default will force the shareholder out of the business and incur a loss of ρV_{τ_b} for the company. Thus, the present value of the bankrupt cost is

$$E[e^{-r\tau_b}\rho V_{\tau_b}\cdot\mathbf{1}_{\{\tau_b<\tau_{cal}\wedge\tau_{con}\}}].$$

Hence, the market value of the company at time 0 equals to

$$TF(V;\tau_b,\tau_{cal};\tau_{con}) = V + E\left[\int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}} e^{-rt}\kappa cPdt\right] - E[e^{-r\tau_b}\rho V_{\tau_b}\cdot\mathbf{1}_{\{\tau_b<\tau_{cal}\wedge\tau_{con}\}}]$$

and the equity market value is

$$E(V;\tau_b,\tau_{cal};\tau_{con}) = TF(V;\tau_b,\tau_{cal};\tau_{con}) - D(V;\tau_b,\tau_{cal};\tau_{con}).$$

By substituting D into the expression of function E, we may rewrite the equity value as follows:

$$\begin{aligned} E(V;\tau_b,\tau_{cal};\tau_{con}) = & E\Big[\int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}} e^{-rt}(\delta V_t - (1-\kappa)cP)dt \\ & + e^{-r(\tau_b\wedge\tau_{cal}\wedge\tau_{con})}\cdot 0\cdot \mathbf{1}_{\{\tau_b<\tau_{cal}\wedge\tau_{con}\}} \\ & + e^{-r(\tau_b\wedge\tau_{cal}\wedge\tau_{con})}\cdot (V_{\tau_{cal}} - \max\{K,\lambda V_{\tau_{cal}}\})\cdot \mathbf{1}_{\{\tau_{cal}<\tau_{con}\wedge\tau_b\}} \\ & + e^{-r(\tau_b\wedge\tau_{cal}\wedge\tau_{con})}\cdot (1-\lambda)V_{\tau_{con}}\mathbf{1}_{\{\tau_{con}<\tau_{cal}\wedge\tau_b\}}\Big| V_0 = V\Big]. \end{aligned} \quad (2.3)$$

The equation (2.3) has a clear interpretation too. The shareholder receives a random dividend flow, $(\delta V_t - (1-\kappa)cP)dt$, in every time interval $(t, t+dt)$ until one of the conversion, call and default events occurs. In the default event, the shareholder loses the total equity value. When the bondholder converts, the equity value will become $(1-\lambda)V_{\tau_{con}}$ after the conversion. When the company calls, the bondholder takes $\max\{K, \lambda V_{\tau_{cal}}\}$ and leaves the rest to the shareholder. Suppose that neither of the bondholder and shareholder is allowed to peer into the future. Then, all of the three times, τ_b, τ_{cal} and τ_{con}, must be stopping times with respect to the information filtration generated by $\{V_t, t \geq 0\}$. Denote $\mathcal{T}$ to be the set of all stopping times adaptive to the filtration of V. Then, we can formulate a game between the bondholder and shareholder as follows: both parties will take actions as a Nash equilibrium, in which τ^*_{con} and (τ^*_b, τ^*_{cal}) satisfy that all of them are in $\mathcal{T}$ and

$$\tau^*_{con} = \arg\max_{\tau_{con}\in\mathcal{T}} D(V;\tau^*_b,\tau^*_{cal};\tau_{con}) \quad (2.4)$$

and

$$(\tau^*_b,\tau^*_{cal}) = \arg\max_{\tau_b,\tau_{cal}\in\mathcal{T}} E(V;\tau_b,\tau_{cal};\tau^*_{con}). \quad (2.5)$$

For any fixed V such that $V \geq K/\lambda$, it is easy to see that $\overline{\tau}_b \wedge \overline{\tau}_{cal} = 0$ and $\overline{\tau}_{con} = 0$ is a Nash equilibrium of game (2.4-2.5). And the corresponding bond value $\overline{D}(V) = \lambda V$ and equity value $\overline{E}(V) = (1-\lambda)V$, that is, in this equilibrium, both players only get the minimum intrinsic value. It is clear that for any other Nash equilibrium of game (2.4-2.5) (if it exists), both players can get at least as much as they get in the equilibrium $\overline{\tau}_b \wedge \overline{\tau}_{cal} = 0$ and $\overline{\tau}_{con} = 0$. Hence in the following, we throw off this the trivial equilibrium ($\overline{\tau}_b \wedge \overline{\tau}_{cal} = 0$ and $\overline{\tau}_{con} = 0$), and aim to find the non-trivial equilibrium point which can advance both the bond value and the equity value [5]. From now on, for each V, we consider the following constraint game problem

$$\tau^*_{con} = \arg \max_{(\tau_{con}, \tau^*_b \wedge \tau^*_{cal}) \neq (0,0)} D(V; \tau^*_b, \tau^*_{cal}; \tau_{con}) \tag{2.6}$$

and

$$(\tau^*_b, \tau^*_{cal}) = \arg \max_{(\tau^*_{con}, \tau_b \wedge \tau_{cal}) \neq (0,0)} E(V; \tau_b, \tau_{cal}; \tau^*_{con}). \tag{2.7}$$

It is worth pointing out that the game (2.6-2.7) is of non-zero-sum feature. Given the un-leverage company value V at time 0, the sum of the market values of the equity and bond is

$$E + D = V + E\left[\int_0^{\tau_b \wedge \tau_{cal} \wedge \tau_{con}} e^{-rt} \kappa c P dt\right] - E[e^{-r\tau_b} \rho V_{\tau_b} \cdot \mathbf{1}_{\{\tau_b < \tau_{cal} \wedge \tau_{con}\}}], \tag{2.8}$$

which is not a constant. The right hand side of the above equality reflects two layers of concerns for the shareholder in determining his call and default policies. On one hand, keeping a proper level of debts may help to boost the market value of the company with the existence of the tax shield, the second term in right hand side of (2.8). On the other hand, too much debt amplifies the threat of default (cf. the third term in right hand side of (2.8)). In the later sections, we will see that this non-zero-sum feature plays a key role in the forming of optimal strategies, especially the optimal call and default strategies of the shareholder.

[5] The aimed equilibrium is actually called Pareto optimal Nash equilibrium, that is, it is Pareto optimal among all Nash equilibrium of game (2.4-2.5). And finally our uniqueness of the solution is also in the sense that there is a unique Pareto optimal Nash equilibrium.

At the end of this subsection, we emphasize the effect of the tax benefits. Let $\{E^*(V), D^*(V)\}$ be a pair of optimal equity value and bond value. If the tax rate $\kappa = 0$, then by (2.8),

$$E^*(V) + D^*(V) = V - E[e^{-r\tau_b}\rho V_{\tau_b} \cdot \mathbf{1}_{\{\tau_b < \tau_{cal} \wedge \tau_{con}\}}] \leq V.$$

On the other hand, if $V \geq K/\lambda$, it is easy to see that

$$E^*(V) \geq (1-\lambda)V \qquad \text{and} \qquad D^*(V) \geq \lambda V.$$

Hence on $\{V \geq K/\lambda\}$,

$$E^*(V) = (1-\lambda)V \qquad \text{and} \qquad D^*(V) = \lambda V.$$

Hence "late call" never happens in absence of tax benefit. This result is quite robust with the detail model setting, which is also the reason why there is no "late call" in Sirbu, Pikovsky and Shreve (2004), Gapeev and Kühn (2004), etc.

2.3. A Variational Inequalities Formulation

From now on, let us turn to solving (2.6-2.7) for a Nash equilibrium. Mathematically, the problem can be regarded as two optimal stopping problems coupled with each other. This observation leads us to reduce it down to a system of variational inequalities. We will present some preliminary results in this section on the structure of optimal policies by analyzing the inequalities. The work of Bensoussan and Friedman (1977) investigates non-zero-sum stochastic differential games defined by stopping times. We first rewrite the objective functions (2.2) and (2.3), following the general formulation provided by that work. This step assists to formulate the variational inequalities a lot.

Note that the company will not have sufficient funds to pay the bondholder off if the asset value is less than K when the shareholder calls. Thus, a rational shareholder never declares a call in that case. On the other hand, he will not issue a bankrupt announcement when the asset value is larger than K since this action leaves nothing to him. In light of these observations, we introduce two new functions as follows:

$$h(V) = \min((V-K)^+, (1-\lambda)V) = \begin{cases} 0, & V < K; \\ V-K, & K \le V < K/\lambda; \\ (1-\lambda)V, & V \ge K/\lambda. \end{cases}$$

and

$$g(V) = \begin{cases} (1-\rho)V, & V < K; \\ V - h(V), & V \ge K. \end{cases}$$

Functions h and g represent the respective payoffs of equity and bond securities upon call or default. With the help of these two notation, we can rewrite (2.2) and (2.3) to

$$\begin{aligned} D(V) &= E[\int_0^{\tau_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-rt}cPdt + e^{-r\tau_{con}} \cdot \lambda V_{\tau_{con}} \cdot \mathbf{1}_{\{\tau_{con}<\tau_b\wedge\tau_{cal}\}} \\ &\quad + e^{-r(\tau_b\wedge\tau_{cal})} g(V_{\tau_b\wedge\tau_{cal}}) \cdot \mathbf{1}_{\{\tau_{con}>\tau_b\wedge\tau_{cal}\}} | V_0 = V] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} E(V) &= E[\int_0^{\tau_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-rt}(\delta V_t - (1-\kappa)cP)dt \\ &\quad + e^{-r(\tau_b\wedge\tau_{cal})} h(V_{\tau_b\wedge\tau_{cal}}) \cdot \mathbf{1}_{\{\tau_{con}>\tau_b\wedge\tau_{cal}\}} \\ &\quad + e^{-r\tau_{con}} \cdot (1-\lambda)V_{\tau_{con}} \cdot \mathbf{1}_{\{\tau_{con}<\tau_b\wedge\tau_{cal}\}} \big| V_0 = V], \end{aligned} \tag{2.10}$$

respectively.

The objective functions (2.9) and (2.10) distinguish the payoffs due to voluntary and compulsory actions. For instance, in (2.9), g gives the payoff of the bondholder when the shareholder takes actions; $\lambda V_{\tau_{con}}$ is how much the bondholder can obtain when she converts voluntarily.

Under (2.9-2.10), it is straightforward to mimic the work of Bensoussan and Friedman (1977) to achieve a system of variational inequalities to formulate the game (2.6-2.7). Define an operator $\mathcal{L}$ as follows: it maps any function (with proper smooth conditions) f on $[0, +\infty)$ into

$$\mathcal{L}f(v) = -\frac{1}{2}\sigma^2 v^2 \frac{d^2 f}{dv^2}(v) - (r - \delta)v\frac{df}{dv}(v) + rf(v).$$

We consider the following variational inequality system: find two functions d and e such that

1. $d(V) \geq \lambda V$, $e(V) \geq h(V)$ for all $V \geq 0$.

2. If $d(V) = \lambda V$ for some V, then $e(V) = (1 - \lambda)V$.

3. If $e(V) = h(V)$ for some V, then $d(V) = g(V)$.

4. On the set $\{V \geq 0 : e(V) > h(V)\}$, the function d satisfies

   $$\min\{d(V) - \lambda V,\ \mathcal{L}d(V) - cP\} = 0.$$

5. On the set $\{V \geq 0 : d(V) > \lambda V\}$, the function e satisfies

   $$\min\{e(V) - h(V),\ \mathcal{L}e(V) - (\delta V - (1 - \kappa)cP)\} = 0.$$

6. On the set $\{V \geq 0 : e(V) = (1 - \lambda)V,\ d(V) = \lambda V\}$, either

   $$\mathcal{L}e(V) - (\delta V - (1 - \kappa)cP) \geq 0$$

   or

   $$\mathcal{L}d(V) - cP \geq 0,$$

   and they could not hold simultaneously.

Remark 2.1. *For the variational inequalities from* 1 *to* 6*, the value functions may not have the classic first-order and second-order derivatives at some points. Then we use the week derivatives. For example, f'' is defined as the second-order week derivative of f if*

$$\int_{-\infty}^{\infty} f''(x)g(x)dx = \int_{-\infty}^{\infty} f(x)g''(x)dx$$

for all second-order continuously differential function g, which have compact support.

Remark 2.2. *According to Bensoussan and Friedman (1977), Condition 1-5 is actually equivalent to the conventional game formulation (2.4-2.5). Condition 6 corresponds to our aim of finding non-trivial Nash equilibrium or a more reasonable optimal value functions, this is, the constraint game formulation (2.6-2.7). From Condition 6, we can intuitively see that, on the stopping region, either bondholder or shareholder voluntarily behavior optimal, and the situation that both parties behavior optimal is excluded.*

Heuristically, we can argue that the optimal bond and equity value functions D^* and E^* should satisfy the preceding system of variational inequalities. First, it is easy to see that the bond value equals to λV when the bondholder picks $\tau_{con} = 0$ as her conversion strategy. Due to the sub-optimality of this strategy, $D^*(V) \geq \lambda V$ for all V. On the shareholder side, the sub-optimality of $\tau_{cal} = 0$ and $\tau_b = 0$ will lead to $E^*(V) \geq h(V)$ for all V. These two observations implies Condition 1 in the inequality system. The second condition states that if the bondholder chooses to convert at time 0 when the company asset value is V, then this action will leave $(1 - \lambda)V$ to the shareholder. Condition 3 describes the payoff of the bondholder when the shareholder declares a default or call to stop the game at time 0.

Conditions 4 and 5 concern about the optimality of the respective strategies taken by both parties. In Condition 4, when the initial asset value satisfies $E^*(V) > h(V)$, the shareholder will not issue call and default announcements immediately at time 0; that is, the optimal strategy set up by the shareholder $\tau_b^* \wedge \tau_{cal}^* > 0$. Given the action of the counterpart, the bondholder faces to an optimal stopping problem to maximize the debt value by choosing a proper τ_{con}. It is well known that such optimal stopping problem can be described by a variational inequality. In particular, in our case D^* should be a solution to

$$\min\{D^*(V) - \lambda V,\ \mathcal{L}D^*(V) - cP\} = 0. \tag{2.11}$$

In a similar manner, we can obtain Condition 5.

Condition 6 means that a conversion in the game must be triggered by a proactive action from either of the two parties. For a V such that $E^*(V) = (1-\lambda)V$ and $D^*(V) = \lambda V$, the bondholder converts at time 0, either voluntarily or by force. Suppose that it is a voluntary conversion, i.e., $\tau^*_{con} = 0$. By the game formulation in (2.6), $\tau^*_b \wedge \tau^*_{cal} > 0$, which implies that the optimal call and default announcements will not happen at time 0. Similar as Condition 4, the optimal bond value $D^*(V)$ should satisfy the variational inequality (2.11) in the neighborhood of V. Specially $D^*(V)$ is smooth at V and $\mathcal{L}D^*(V) \geq cP$ in this case. On the other hand, if the conversion is compulsory, then $\tau^*_{cal} = 0$ and $\tau^*_{con} > 0$. The optimality of τ^*_{cal} in the interval $[0, \tau^*_{con}]$ will yield that $\mathcal{L}E^*(V) \geq \delta V - (1-\kappa)cP$.

Once we know the solutions to the system of variational inequalities 1-6, we can proceed to construct the Nash equilibrium to the game (2.6-2.7). According to the discussion in condition 6, the company asset value at a voluntary conversion is characterized by two properties: $D^*(V) = \lambda V$ and $\mathcal{L}D^*(V) \geq cP$. Hence, we may define

$$\tau^*_{con} = \inf\{t \geq 0 : D^*(V_t) = \lambda V_t, \mathcal{L}D^*(V_t) \geq cP\}.$$

As for the shareholder, natural candidates for the optimal bankruptcy and call times are $\tau^*_b = \inf\{t \geq 0 : E^*(V_t) = 0,\ \mathcal{L}E^*(V_t) \geq (\delta V - (1-\kappa)cP), V_t \leq K\}$ and

$$\tau^*_{cal} = \inf\{t \geq 0 : E^*(V_t) = h(V_t),\ \mathcal{L}E^*(V_t) \geq (\delta V - (1-\kappa)cP),\ V_t \geq K\},$$

respectively.

It is possible to present a clearer characterization for the structures of the aforementioned stopping times, even without solving the system 1-6 explicitly. Related results are summarized in the following proposition. They provide useful clues to how to find the optimal E^* and D^*, which is the main task in the next section.

Denote

$$\mathcal{S}_D := \{V \geq 0 : D^*(V) = \lambda V,\ \mathcal{L}D^*(V) \geq cP\},$$

$$\mathcal{S}_{EB} := \{V \geq 0 : E^*(V) = 0,\ \mathcal{L}E^*(V) \geq (\delta V - (1-\kappa)cP),\ V \leq K\},$$

and

$$\mathcal{S}_{EC} := \{V \geq 0 : E^*(V) = h(V),\ \mathcal{L}E^*(V) \geq (\delta V - (1-\kappa)cP),\ V \geq K\}.$$

Game formulation (2.6-2.7) indicates that $\overline{\mathcal{S}_D} \cap \overline{(\mathcal{S}_{EB} \cup \mathcal{S}_{EC})} = \emptyset$. We have

Proposition 2.3. *Suppose that two functions $E^*(V)$ and $D^*(V)$ solve the system of variational inequalities 1-6. Then, the following conclusions hold:*
(i). There exists a unique $V^*_{con} \in [(cP)/(\delta\lambda), +\infty)$ such that $\mathcal{S}_D = [V^*_{con}, +\infty)$ and $D^*(V)$ is smooth at V^*_{con}.
(ii). There exists a unique $V^*_b \in (0, \min\{K,\ (1-\kappa)cP/\delta\})$ such that $\mathcal{S}_{EB} = [0, V^*_b]$ and $E^*(V)$ is smooth at V^*_b.
(iii). $\mathcal{S}_{EC} \cap [K, K/\lambda) \neq \emptyset$ only if $K \leq (1-\kappa)cP/r$ and $\mathcal{S}_{EC} \cap (K/\lambda, +\infty) \neq \emptyset$ only if $K \leq (1-\kappa)cP/\delta$. Moreover, if $\mathcal{S}_{EC} \neq \emptyset$, then $K/\lambda \in \mathcal{S}_{EC}$ and there exist unique $V^*_{cal,1} \in (K, K/\lambda]$ and $V^*_{cal,2} \in [K/\lambda, (1-\kappa)cP/(\lambda\delta)]$ such that $\mathcal{S}_{EC} = [V^*_{cal,1}, V^*_{cal,2}]$. $E^*(V)$ is smooth at $V^*_{cal,1}$ if $V^*_{cal,1} < K/\lambda$; $E^*(V)$ is smooth at $V^*_{cal,2}$ if $V^*_{cal,2} > K/\lambda$.

We can interpret the meaning of Proposition 2.3 as follows. Conclusion (i) indicates that the bondholder should convert when the company asset value increases to a sufficiently large level. A default will occur if the company asset value is low enough, as shown in conclusion (ii). The call strategy of the shareholder depends on the magnitude of call price K. For a large K, conclusion (iii) indicates that $\mathcal{S}_{EC} = \emptyset$, i.e., the shareholder should not call at all during the life of the bond. This makes financial sense because he has to pay a high price in exchange for the bond security in this case. Conclusion (iii) also indicates that K/λ must be contained in $\mathcal{S}_{EC}$ if it is not empty.

2.4. Nash Equilibrium

In this section, we will present a complete description on the Nash equilibrium of the game (2.6-2.7). According to the guidance of Proposition 2.3, there are only two possibilities: $\mathcal{S}_{EC} = \emptyset$ when K is large and $\mathcal{S}_{EC} \neq \emptyset$ for a small K. How to form the equilibria in each scenario is specified in the subsections 2.4.1 and 2.4.2.

We need several notations to simplify the presentation. Introduce them here for later reference. For any three real numbers such that $0 < b \leq v \leq d$, let ς be the first passage time of V_t across double boundaries $V = b$ and $V = d$, i.e.,

$$\varsigma = \inf\{t \geq 0 : V_t \leq b \text{ or } V_t \geq d\}.$$

Define functions p and q to be the present values of two Arrow-Debreu securities, paying one dollar on the events of $V_\varsigma = b$ and $V_\varsigma = d$, respectively. In other words,

$$p(v; b, d) = E[e^{-r\varsigma}\mathbf{1}_{\{V_\varsigma = b\}}|V_0 = v] \quad \text{and} \quad q(v; b, d) = E[e^{-r\varsigma}\mathbf{1}_{\{V_\varsigma = d\}}|V_0 = v].$$

Under the specification of geometric Brownian motion (2.1), both of them admit closed-form solutions:

$$p(v; b, d) = \frac{d^{\beta+\gamma} - v^{\beta+\gamma}}{d^{\beta+\gamma} - b^{\beta+\gamma}}\left(\frac{b}{v}\right)^{\gamma} \quad \text{and} \quad q(v; b, d) = \frac{v^{\beta+\gamma} - b^{\beta+\gamma}}{d^{\beta+\gamma} - b^{\beta+\gamma}}\left(\frac{d}{v}\right)^{\gamma},$$

where two parameters β and γ are given by

$$\beta = \frac{-(r - \delta - \sigma^2/2) + \Delta}{\sigma^2}, \quad \gamma = \frac{(r - \delta - \sigma^2/2) + \Delta}{\sigma^2} \tag{2.12}$$

and $\Delta = \sqrt{(r - \delta - \sigma^2/2)^2 + 2r\sigma^2}$.

2.4.1. No Voluntary Calls

As illustrated in Proposition 2.3, $\mathcal{S}_{EC}$ will be an empty set if K is sufficiently large. In this case, the bondholder should choose to convert at the moment

$$\tau^*_{con} = \inf\{t \geq 0 : V_t \geq V^*_{con}\}$$

and the optimal default time for the shareholder should be in the form of

$$\tau_b^* = \inf\{t \geq 0 : V_t \leq V_b^*\}.$$

We can find the bond value function explicitly under the above stopping times. The bond value function satisfies that $D^*(V) = (1-\rho)V$ when $V \leq V_b^*$ and $D^*(V) = \lambda V$ when $V \geq V_{con}^*$. In the interval (V_b^*, V_{con}^*), it solves an ODE $\mathcal{L}D^*(V) = cP$. Appendix A.2 provides a general solution to this equation such that

$$D^*(V) = \frac{cP}{r} + c_1 V^\beta + c_2 V^{-\gamma},$$

where β and γ are given by (2.12). Constants c_1 and c_2 are determined by boundary conditions $D^*(V_{con}^*) = \lambda V_{con}^*$ and $D^*(V_b^*) = (1-\rho)(V_b^*)$. Some tedious calculation will yield that $D^*(V) = D_1(V; V_b^*, V_{con}^*)$, where $D_1(v; b, d)$ is a function defined as follows:

$$D_1(v;b,d) = \frac{cP}{r} + \left((1-\rho)b - \frac{cP}{r}\right) p(v;b,d) + \left(\lambda d - \frac{cP}{r}\right) q(v;b,d) \quad (2.13)$$

for $0 < b \leq v \leq d$.

The equity value function $E^*(V)$ is solvable in a similar manner. It equals to 0 when $V \leq V_b^*$ and $(1-\lambda)V$ when $V \geq V_{con}^*$. Meanwhile, $E^*(V), V \in [V_b^*, V_{con}^*]$ satisfies an ODE $\mathcal{L}E^*(V) = \delta V - (1-\kappa)cP$, whose general solution is given by

$$E^*(V) = V - \frac{(1-\kappa)cP}{r} + c_3 V^\beta + c_4 V^{-\gamma}$$

according to Appendix A.2. With the help of boundary conditions $E^*(V_{con}^*) = (1-\lambda)V_{con}^*$ and $E^*(V_b^*) = 0$, we can fix the values of constants c_3 and c_4 so that $E^*(V) = E_1(V; V_b^*, V_{con}^*)$, where E_1 is a function given by

$$\begin{aligned} E_1(v;b,d) &= v - \frac{(1-\kappa)cP}{r} \\ &+ \left(\frac{(1-\kappa)cP}{r} - b\right) p(v;b,d) + \left(\frac{(1-\kappa)cP}{r} - \lambda d\right) q(v;b,d). \end{aligned} \quad (2.14)$$

The optimal boundaries V_b^* and V_{con}^* can be determined through the principle of "smooth pasting". E^* is differentiable on the whole interval $[0, V_{con}^*]$, in particular, at $V = V_b^*$. Hence, V_b^* should satisfy

$$\frac{dE^*}{dV}(V)|_{V=V_b^*} = 0, \tag{2.15}$$

where 0 is the left derivative since $E^*(V) = 0$ for all $V \in [0, V_b^*)$. In the meantime, note that $D^*(V) = \lambda V$ for $V \geq V_{con}^*$. V_{con}^* should be a solution to another smooth pasting condition

$$\frac{dD^*}{dV}(V)|_{V=V_{con}^*} = \lambda. \tag{2.16}$$

Both (2.15) and (2.16) constitute a system of equations regarding V_b^* and V_{con}^*. We show in the following lemma that this system admits a unique solution.

Lemma 2.4. *There exist unique V_b^* and V_{con}^* satisfying (2.15) and (2.16) simultaneously. $V_b^* \leq (1-\kappa)cP/\delta$ and $V_{con}^* > cP/(\delta\lambda)$. Substitute them into (2.13) and (2.14) and consider an equation*

$$E_1^*(V; V_b^*, V_{con}^*) = (1-\lambda)V.$$

It has at most a unique solution, $V = k_1$, within the interval (V_b^*, V_{con}^*). Define $K_1 = \lambda k_1$ if such solution exists or $K_1 = \lambda V_{con}^*$ otherwise. Then, when $K \geq K_1$, we have

$$E^*(V; V_b^*, V_{con}^*) \geq h(V) \quad \text{and} \quad D^*(V; V_b^*, V_{con}^*) \geq \lambda V$$

for any $V \in [V_b^*, V_{con}^*]$.

The following theorem is the main result of this subsection. It shows that for $K \geq K_1$, the stopping times discussed at the beginning of the subsection constitute the Nash equilibrium and the explicit forms of the bond and equity values are obtainable through (2.13) and (2.14).

Theorem 2.5. *Suppose that $K \geq K_1$ and V_b^* and V_{con}^* are solutions to the equations (2.15) and (2.16). Then, in an equilibrium the bondholder should convert at*

$$\tau_{con}^* = \inf\{t \geq 0 : V_t \geq V_{con}^*\}$$

and the shareholder never calls and should announce a default at the moment

$$\tau_b^* = \inf\{t \geq 0 : V_t \leq V_b^*\}.$$

Furthermore, the optimal equity and bond value at time 0 *are given by*

$$(E^*(V), D^*(V)) = \begin{cases} (0, (1-\rho)V) & \text{if } V \leq V_b^*; \\ (E_1(V; V_b^*, V_{con}^*), D_1(V; V_b^*, V_{con}^*)) & \text{if } V_b^* < V \leq V_{con}^*; \\ ((1-\lambda)V, \lambda V), & \text{if } V \geq V_{con}^*. \end{cases}$$

In Theorem 2.5, the optimal default and call boundaries are specified through the smooth pasting conditions (2.15) and (2.16). Computing the anticipated appreciation in equity value around the bankrupt trigger V_b^* casts more financial insight on the optimality of the bankruptcy policy. Applying Ito's lemma to the equity value function E^* with respect to V_t, we have

$$dE^*(V_t) = \frac{\partial E^*}{\partial V} dV_t + \frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 E^*}{\partial V^2} dt = \frac{\partial E_1}{\partial V} dV_t + \frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 E_1}{\partial V^2} dt$$

when $V_t > V_b^*$. As $V_t \downarrow V_b^*$, $\partial E_1/\partial V$ converges to 0 according to (2.15). After substituting (2.14), the expression of E_1, into $\partial^2 E_1/\partial^2 V$, we can easily show that

$$\frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 E_1}{\partial V^2}(V_t) \to (1-\kappa)cP - \delta V_b^*.$$

Consequently, the expectation of $dE^*(V_t)$ should satisfy

$$\lim_{V_t \downarrow V_b^*} E[dE^*(V_t)] = [(1-\kappa)cP - \delta V_b^*]dt.$$

The left hand side of the above equality may be interpreted as the expected capital gain for the shareholder at the default boundary $V_t = V_b^*$ if he puts off the default to a moment later. The right hand side is the additional cash flow required from him to keep the company solvent for this moment. It is the difference between the after-tax coupon payment and the cash flow available for paying out by liquidating a portion of the company's asset. From this equality, we can see that the smooth pasting condition (2.15) implies that at $V = V_b^*$, the equity capital gain just equals to the amount of cash flow which must be provided by the shareholder to meet the debt obligation. Hence, he should choose to announce a default when $V_t = V_b^*$ because it will not be an attractive option any longer to continue contributing new capital to make the company run.

Following similar calculation, we can show as well that under the smooth pasting condition (2.16),

$$\lim_{V_t \uparrow V_{con}^*} E[dD^*(V_t) + cPdt] = r\lambda V_{con}^* dt.$$

Its left hand side means the expected sum of the capital gain in bond value and received coupon payments, if the current company value is V_{con}^* and the bondholder opts to postpone the conversion decision until dt. The right hand side is the total value appreciation under the risk neutral probability if the bondholder chooses to convert at V_{con}^* and carries the post-conversion value over $(0, dt)$. Both ways offer her the same payoffs and she should convert immediately at V_{con}^* consequently.

The financial explanation of the pricing formulas in Theorem 2.5 is very clear too. If the default and conversion never occurred during the whole life of the company, the present value of the total debt obligation for the company at time 0 would be

$$\int_0^{+\infty} (1-\kappa)cPe^{-rt}dt = \frac{(1-\kappa)cP}{r}$$

in the presence of the corporate tax exemption. Accordingly, the equity value would be $V - (1-\kappa)cP/r$ at time 0. Upon the moment the conversion happens, the company's capital structure changes and it is released from a continuous debt payment flow whose value is worth $(1-\kappa)cP/r$. At the same time, the shareholder has to give up λV_{con}^* to the bondholder. The net equity value change for the shareholder when converting is then

$$\frac{(1-\kappa)cP}{r} - \lambda V_{con}.$$

When the default occurs, the shareholder loses the total asset value due to the bankruptcy, albeit he does not need to serve the debt obligation any longer.

Hence, the net equity value change at that time will be

$$\frac{(1-\kappa)cP}{r} - V_b.$$

Recall the probabilistic meaning of p and q. The last two terms in the expression of $E_1(V; V_b^*, V_{con}^*)$ reflect the present values of these two changes. Similar observation applies to the bond value as well. We leave detailed discussion to the readers.

2.4.2. Early and Late Calls

In this subsection, we consider the cases with cheap strike price, more specifically, $K < K_1$. For such K, the shareholder will keep calling the debt back as an option, i.e., $\mathcal{S}_{EC} \neq \emptyset$. According to Proposition 2.3, there exist two critical points $K < V_{cal,1}^* \leq K/\lambda \leq V_{cal,2}^*$ so that $\mathcal{S}_{EC} = [V_{cal,1}^*, V_{cal,2}^*]$. Meanwhile, he will announce default on the company's debt obligation if the company value is lower than $V_b^* \leq K$. On the other hand, the bondholder's conversion region is specified by $[V_{con}^*, +\infty)$, which does not have any overlapping with $\mathcal{S}_{EC}$, $V_{con}^* > V_{cal,2}^*$.

In two disjoint intervals $(V_b^*, V_{cal,1}^*)$ and $(V_{cal,2}^*, V_{con}^*)$, both parties of the game do not take actions to stop the running of the company. Therefore, the bond and equity value functions should follow the ODEs

$$\mathcal{L}D^*(V) = cP$$

and

$$\mathcal{L}E^*(V) = \delta V - (1-\kappa)cP$$

respectively. Some boundary conditions are needed to fix their solutions. Take the interval $(V_b^*, V_{cal,1}^*)$ for instance. Since $D^*(V) = (1-\rho)V$, $E^*(V) = 0$ for $V \leq V_b^*$ and $D^*(V) = K$, $E^*(V) = V - K$ for $V \in [V_{con,1}^*, K/\lambda]$, the continuous property of D^* and E^* requires that

$$\begin{cases} D^*(V_b^*) = (1-\rho)V_b^* \\ E^*(V_b^*) = 0 \end{cases} \quad \text{and} \quad \begin{cases} D^*(V_{cal,1}^*) = K \\ E^*(V_{cal,1}^*) = V_{cal,1}^* - K. \end{cases}$$

It is straightforward to verify that $D^*(V) = D_2(V; V_b^*, V_{cal,1}^*)$ and $E^*(V) = E_2(V; V_b^*, V_{cal,1}^*)$, where

$$D_2(v; b, d) = \frac{cP}{r} + \left((1-\rho)b - \frac{cP}{r}\right) p(v; b, d) + \left(K - \frac{cP}{r}\right) q(v; b, d)$$

and

$$\begin{aligned} E_2(v; b, d) =& v - \frac{(1-\kappa)cP}{r} \\ &+ \left(\frac{(1-\kappa)cP}{r} - b\right) p(v; b, d) + \left(\frac{(1-\kappa)cP}{r} - K\right) q(v; b, d) \end{aligned}$$

for all $0 \le b \le v \le d$.

The financial interpretation of D_2 and E_2 is achievable as well, following similar analysis as what we did for D_1 and E_1. The only difference is that at $V_{cal,1}^*$, the bondholder prefers to settling the call with cash. She receives K and terminates the coupon payment whose present value is given by cP/r. The shareholder saves $(1-\kappa)cP/r$ for the company but he needs to pay K to the bondholder.

For the interval $(V_{cal,2}^*, V_{con}^*)$, introduce two functions

$$D_3(v; b, d) \;=\; \frac{cP}{r} + \left(\lambda b - \frac{cP}{r}\right) p(v; b, d) + \left(\lambda d - \frac{cP}{r}\right) q(v; b, d)$$

and

$$\begin{aligned} E_3(v; b, d) =& v - \frac{(1-\kappa)cP}{r} \\ &+ \left(\frac{(1-\kappa)cP}{r} - \lambda b\right) p(v; b, d) + \left(\frac{(1-\kappa)cP}{r} - \lambda d\right) q(v; b, d) \end{aligned}$$

for all $0 \le b \le v \le d$. The boundary conditions of D^* and E^* at $V_{cal,2}^*$ and V_{con}^*

$$\begin{cases} D^*(V_{cal,2}^*) = \lambda V_{cal,2}^* \\ E^*(V_{cal,2}^*) = (1-\lambda) V_{cal,2}^* \end{cases} \quad \text{and} \quad \begin{cases} D^*(V_{con}^*) = \lambda V_{con}^* \\ E^*(V_{con}^*) = (1-\lambda) V_{con}^* \end{cases}$$

can help us to determine bond and equity value functions in the interval $(V_{cal,2}^*, V_{con}^*)$ as $D^*(V) = D_3(V; V_{cal,2}^*, V_{con}^*)$ and $E^*(V) = E_3(V; V_{cal,2}^*, V_{con}^*)$. One may figure out the corresponding interpretation to E_3 and D_3 by itself.

Invoke the principle of smooth pasting once again to look for the optimal boundaries $V_b^*, V_{cal,1}^*, V_{cal,2}^*$ and V_{con}^*. Note that $E^*(V) = 0$ for $V \leq V_b^*$ and $E^*(V) = V - K$ for $V \in [V_{cal,1}^*, K/\lambda]$. We can use

$$\frac{dE^*}{dV}(V)|_{V=V_b^*} = 0 \quad \text{and} \quad \frac{dE^*}{dV}(V)|_{V=V_{cal,1}^*} = 1 \tag{2.17}$$

to determine V_b^* and $V_{cal,1}^*$. As shown in the next lemma, substituting the expression of E_2 into (2.17) will generate the solutions. In the meantime, since $E^*(V) = (1-\lambda)V$ for $V \in [K/\lambda, V_{cal,2}^*]$ and $D^*(V) = \lambda V$ for $V \in [V_{con}^*, +\infty)$, $V_{cal,2}^*$ and V_{con}^* should satisfy

$$\frac{dE^*}{dV}(V)|_{V=V_{cal,2}^*} = 1 - \lambda \quad \text{and} \quad \frac{dD^*}{dV}(V)|_{V=V_{con}^*} = \lambda, \tag{2.18}$$

where E^* and D^* are given by E_3 and D_3 respectively.

However, the $V_{cal,1}^*$ and $V_{cal,2}^*$ obtained through equations (2.17) and (2.18) may not satisfy the requirement that $V_{cal,1}^* < K/\lambda < V_{cal,2}^*$, which guarantee that the smooth-pasting condition holds at $V_{cal,1}^*$ and $V_{cal,2}^*$. In the following lemma, we show that this is true only for small strike price K.

Lemma 2.6. *(i). For each $K < (1-\kappa)cP/r$, the equation (2.17) has unique solutions V_b^* and $V_{cal,1}^*$. $V_b^* < \min\{K, (1-\kappa)cP/\delta\}$ and $V_{cal,1}^* > K$. Notice that the definition of E_2 involves K. View such obtained $V_{cal,1}^*$ as a function of K and consider an equation*

$$V_{cal,1}^*(K) = K/\lambda.$$

There exists a unique root $K_2 < (1-\kappa)cP/r$ to this equation. $K_2 < K_1$. Furthermore, when $K < K_2$, we have $V_{cal,1}^* < K/\lambda$,

$$E_2(V; V_b^*, V_{cal,1}^*) \geq (V-K)^+ \quad \textit{and} \quad D_2(V; V_b^*, V_{cal,1}^*) \geq \lambda V$$

for all $V \in [V_b^*, V_{cal,1}^*]$.

(ii). The equation (2.18) yields unique solutions $V_{cal,2}^*$ and V_{con}^*. $V_{cal,2}^* < (1-\kappa)cP/(\lambda\delta)$ and $V_{con}^* > cP/(\lambda\delta)$. For all $V \in (V_{cal,2}^*, V_{con}^*)$,

$$E_3(V; V_{cal,2}^*, V_{con}^*) \geq (1-\lambda)V \quad \textit{and} \quad D_3(V; V_{cal,2}^*, V_{con}^*) \geq \lambda V.$$

Notice that both of the definitions of D_3 and E_3 are independent of K. Such obtained $V^*_{cal,2}$ and V^*_{con} does not depend on K either. Let $K_3 = \lambda V^*_{cal,2}$. For any $K < K_3$, we have $K/\lambda < V^*_{cal,2}$.

Proposition 2.3 states that K/λ is an interior point of $\mathcal{S}_{EC} = [V^*_{cal,1}, V^*_{cal,2}]$ when K is small. For $K \geq K_2$, the left boundary of $\mathcal{S}_{EC}$ for the shareholder degenerates to K/λ. In this case, we use the continuity of the value function at the boundary instead of the smooth-pasting principle. The boundary conditions for E^* and D^* will change to

$$D^*(K/\lambda) = K \quad \text{and} \quad E^*(K/\lambda) = K/\lambda - K.$$

Combining with the boundary conditions of E^* and D^* at $V = V_b^*$, we have

$$D^*(V) = D_2(V; V_b^*, K/\lambda) \quad \text{and} \quad E^*(V) = E_2(V; V_b^*, K/\lambda)$$

for $V_b^* \leq V \leq K/\lambda$. The optimal default boundary V_b^* is once again determined by a smooth pasting condition

$$\frac{dE^*}{dV}(V)|_{V=V_b^*} = \frac{dE_2}{dV}(V_b^*; V_b^*, K/\lambda) = 0. \tag{2.19}$$

In a similar way, one can establish the corresponding results under the scenario $K_3 < K < K_1$ if $K_3 < K_1$. The right boundary of $\mathcal{S}_{EC}$, $V^*_{cal,2}$, degenerates to K/λ. The bond and equity value functions should be given by

$$D^*(V) = D_3(V; K/\lambda, V^*_{con}) \quad \text{and} \quad E^*(V) = E_3(V; K/\lambda, V^*_{con})$$

for all $K/\lambda \leq V \leq V^*_{con}$ respectively. The optimal conversion boundary V^*_{con} is a solution to

$$\frac{dD^*}{dV}(V)|_{V=V^*_{con}} = \frac{dD_3}{dV}(V^*_{con}; K/\lambda, V^*_{con}) = \lambda. \tag{2.20}$$

We summarize some related properties of functions E^* and D^* for such intermediate sized K in the following lemma:

Lemma 2.7. *(i). When $K \in [K_2, K_1)$, we can find a unique $V_b^* < \min\{K, (1-\kappa)cP/\delta\}$ satisfying (2.19). Furthermore, once we substitute such V_b^* in the functions E_2 and D_2,*

$$E_2(V; V_b^*, K/\lambda) \geq (v - K)^+ \quad and \quad D_2(V; V_b^*, K/\lambda) \geq \lambda V$$

for $V \in [V_b^*, K/\lambda]$.

(ii). Suppose that $K_3 < K_1$. For any $K \in [K_3, K_1)$ and $K < cP/\delta$, equation (2.20) yields a unique solution V_{con}^*. $V_{con}^* > cP/(\delta\lambda)$. In addition,

$$E_3(V; K/\lambda, V_{con}^*) > (1-\lambda)V \quad and \quad D_3(V; K/\lambda, V_{con}^*) > \lambda V$$

for any $V \in (K/\lambda, V_{con}^*)$.

Now, we are ready to present the main conclusions in this subsection. The following theorem builds up a Nash equilibrium in the case of $K \leq K_1$, using the aforementioned stopping regions and the critical points in call prices.

Theorem 2.8. *When $K < K_1$, a Nash equilibrium to the game (2.6-2.7) is formed if the bondholder converts her security at the moment*

$$\tau_{con}^* = \inf\{t \geq 0 : V_t \geq V_{con}^*\}$$

and the shareholder declares bankruptcy and call at

$$\tau_b^* = \inf\{t \geq 0 : V_t \leq V_b^*\} \quad and \quad \tau_{cal}^* = \inf\{t \geq 0 : V_t \in [V_{cal,1}^*, V_{cal,2}^*]\}$$

respectively. Under such equilibrium, the equity and bond value functions should be given by

$$(E^*(V), D^*(V)) = \begin{cases} (0, (1-\rho)V), & if\ V \leq V_b^*; \\ (E_2(V; V_b^*, V_{cal,1}^*), D_2(V; V_b^*, V_{cal,1}^*)), & if\ V_b^* < V \leq V_{cal,1}^*; \\ (V - K, K), & if\ V_{cal,1}^* < V \leq K/\lambda; \\ ((1-\lambda)V, \lambda V), & if\ K/\lambda < V \leq V_{cal,2}^*; \\ (E_3(V; V_{cal,2}^*, V_{con}^*), D_3(V; V_{cal,2}^*, V_{con}^*)), & if\ V_{cal,2}^* < V \leq V_{con}^*; \\ ((1-\lambda)V, \lambda V), & if\ V > V_{con}^*. \end{cases}$$

In addition, when $K < K_2$, V_b^* *and* $V_{cal,1}^*$ *are determined by equation* (2.17) *and the endpoint of the call region* $V_{cal,1}^* < K/\lambda$; *when* $K_2 \leq K < K_1$, $V_{cal,1}^*$ *degenerates to* K/λ *and* V_b^* *is determined by equation* (2.19); *when* $K < K_3$, $V_{cal,2}^*$ *and* V_{con}^* *are determined by equation* (2.18) *and the endpoint of the call region* $V_{cal,2}^* > K/\lambda$; *when* $K_3 \leq K < K_1$, $V_{cal,2}^*$ *degenerates to* K/λ *and* V_{con}^* *is determined by equation* (2.20).

Finally, we show that the Nash equilibrium in both scenarios is unique.

Theorem 2.9. *For all* $K > 0$, *the bond and equity value functions in any Nash equilibrium of the game (2.6-2.7) should be identical with the functions given in Theorems 2.5 and 2.8.*

Some observations on Theorem 2.8 are of special interest to us. First, K/λ is the optimal call boundary only in some cases. Note that if the shareholder opts to call the debt back when the company value $V_t = K/\lambda$, the bond's conversion value equals to the call price. This is exactly corresponding to the optimal conversion policy established by Brennan and Schwartz (1977, 1980) and Ingersoll (1977a) under an ideal market assumption. However, our model reveals that this policy may not be necessarily optimal if we take the tax benefit and credit risk into account.

Second, our model is capable of generating out-of-the-money calls. When $K < K_2$, the shareholder should make a call announcement at the first time when the asset value surges up to $V_{cal,1}$, a level less than K/λ. When he calls, the conversion value of the bond is less than K. The return rate of the company asset at this out-of-the-money calling must be positive because the call is triggered by an up-crossing of the asset process. This is consistent with the empirical finding we mentioned in the introduction.

Third, sometimes the shareholder may also call the debt back when it is in-the-money. For $K < K_3$, if the company starts with an initial value larger than K/λ, the shareholder will issue a call at $V_{con,2}$. The conversion value of the bond at the call is then $\lambda V_{con,2}$, exceeding its call price K. The asset return of the company on in-the-money calls in our model is negative, since such calls are triggered by a down-crossing of the asset process. Consequently, our model can reproduce late calls and associated negative returns, consistent with the pattern found by Mikkelson (1981) and so on.

Finally, we find that there will not be a consistent pattern for the asset return when the convertible bond is called at the moment when the company asset value first reaches K/λ. The return could be positive if the company starts from an initial value less than K/λ and could be negative if the company starts from an initial value more than K/λ.

2.5. Numerical Results

In this section, we will use some numerical experiments to demonstrate the impacts of various parameters on the equity and bond values and the optimal call policy. Table 1 summarizes the parameters we use in the base case. In addition,

Macroeconomic parameters: $r = 8\%$, $\kappa = 35\%$.
Company-specific parameters: $\delta = 6\%$, $\sigma = 22\%$, $\rho = 50\%$.
Bond Contract parameters: $c = 7\%$, $\lambda = 20\%$, $P = 100$.

Table 2.1: Basic parameters for numerical illustration. The risk-free rate $r = 8\%$ is close to the average historical treasury rate during 1973-1998, and the corporate tax rate $\kappa = 35\%$ is chosen according to Leland and Toft (1996). We set the paying-out ratio at $\delta = 6\%$, which is consistent with the average coupon and dividend payments in the US during 1973-1998 (Huang and Huang (2003)). The diffusion volatility $\sigma = 0.22$, which is reported as the average asset volatility for companies with credit rating A to Baa by Schaefer and Strebulaev (2007). The recovery ratio after the default is assumed to be 50%, i.e., $\rho = 50\%$. The coupon rate $c = 7\%$. Note that this is slightly lower than the risk free interest rate. We choose it to reflect low coupon payment for the convertible bond. The conversion ratio and the bond face value are 20% and \$100 respectively.

we assume that one year is equal to 252 trading days.

2.5.1. No Call and In-the-Money, Out-of-the-Money Calls: The Impact of K

In the base case given by Table 1, we can calculate that there is no voluntary call if and only if the strike price K is larger than \$87.51. Figure 2.1 shows the convertible bond value function with respect to the company value V when we take $K = 100$. The shareholder will announce a default at the first time

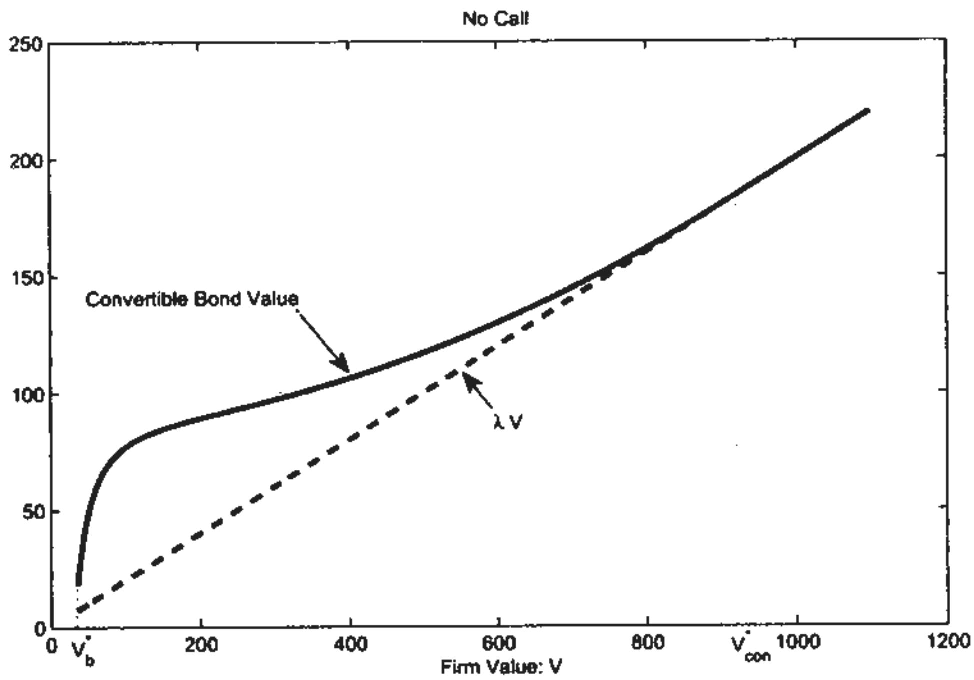

Figure 2.1: The convertible bond value in a case with large call price. The default barrier $V_b^* = 36.43$ and the conversion barrier $V_{con}^* = 914.62$. The shareholder will never call the debt voluntarily.

when the company value drops down to $V_b^* = 36.43$ and the bondholder opts to convert at $V_{con}^* = 914.62$. From this figure, we can see that the bond value will converge to its conversion value as V is large, because the bondholder has more incentive to convert when the company value increases. This will lead that the bond behaves more like an equity security. On the other hand, when V is close to V_b, the convertible bond is similar as a regular defaultable bond because of

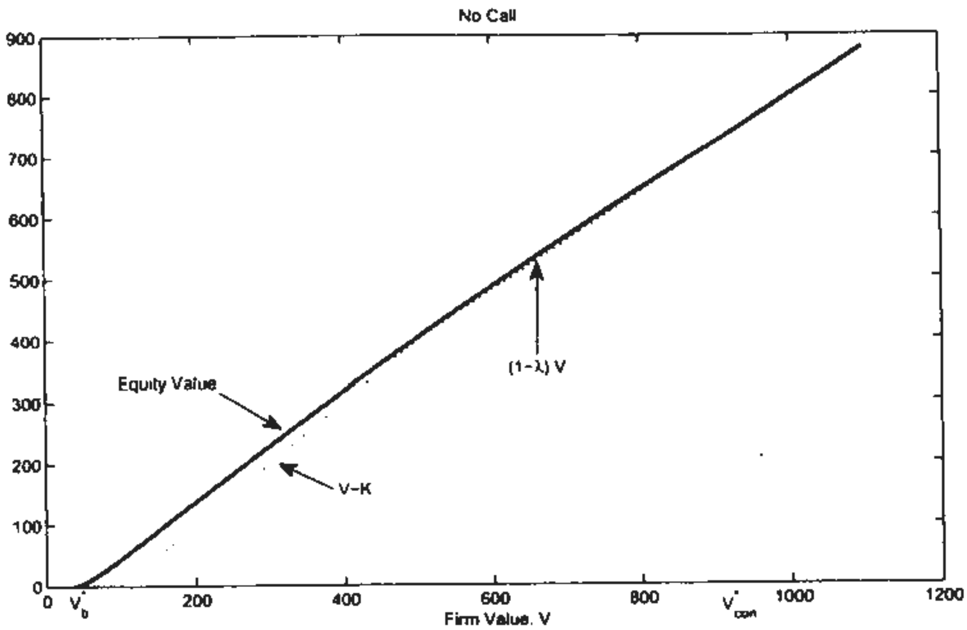

Figure 2.2: The equity value in a case with large call price. The default barrier $V_b^* = 36.43$ and the conversion barrier $V_{con}^* = 914.62$. The shareholder will never call the debt voluntarily.

the influence of credit risk. Figure 2.2 illustrates the equity value function in this case that firm never calls.

Figure 2.3 illustrates the bond value function in a case with smaller call price. We choose $K = 50$, which is less than $K_2 = 55.73$ and $K_3 = 53.90$. The default barrier V_b^*, forcing surrender barrier $V_{cal,1}^*$, forcing conversion barrier $V_{cal,2}^*$, and conversion barrier V_{con}^* divide the whole range of the company value V into five segments. If the initial company value falls between V_b^* and $V_{cal,1}^*$, the shareholder will call the debt back when V_t crosses $V_{cal,1}^*$ for the first time. Note that $V_{cal,1}^* < K/\lambda = 250$. Such call must occur out of the money. However, if the company starts somewhere between $V_{cal,2}^*$ and V_{con}^*, then the debt-calling will be in-the-money since it occurs at $V_{cal,2}^*$ and $V_{cal,2}^* > K/\lambda$. Figure 2.4 illustrates the equity value function in this case with smaller call price.

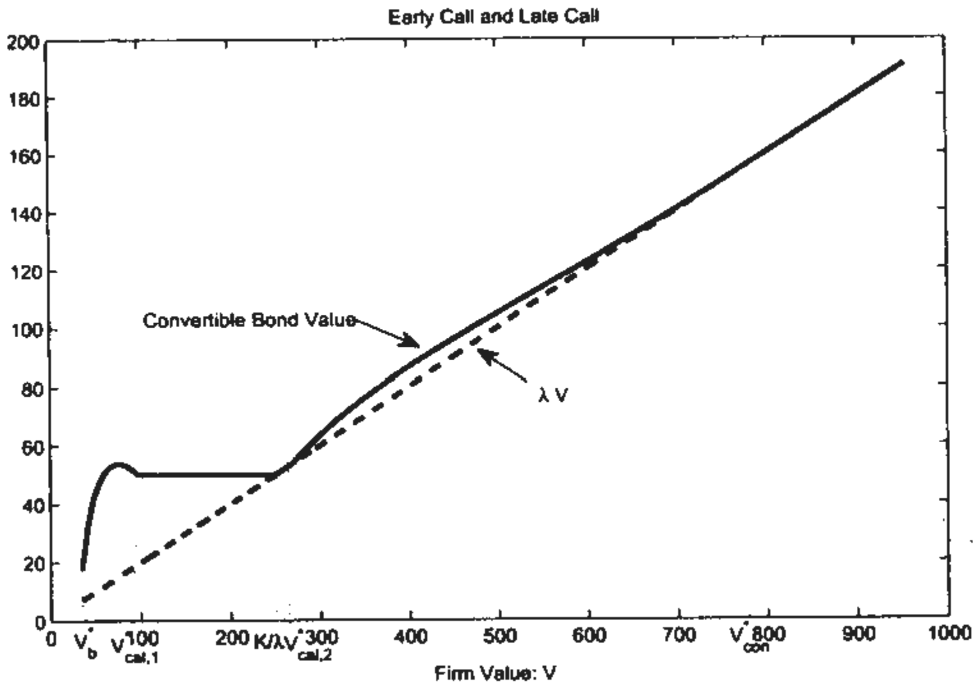

Figure 2.3: The convertible bond value in a case with smaller call price. The default barrier is $V_b^* = 35.44$. The forcing surrender and conversion barriers are $V_{cal,1}^* = 97.90$ and $V_{cal,2}^* = 269.51$, respectively. The conversion barrier is $V_{con}^* = 782.00$. The horizontal straight line between $V_{cal,1}^*$ and K/λ indicates that the bond value equals to \$50. This is because the shareholder will call the debt once the company value falls in this interval and the bondholder responds to this call by a forced surrender. The bond value function coincides with λV in the interval $(K/\lambda, V_{cal,2})$. The shareholder will issue a call as well in this interval but the bondholder opts to convert in response.

2.5.2. Comparative Statics

This subsection reports the effects of variation in selected parameters on the optimal strategies of both parties and the convertible bond value. The parameters are the risk free interest rate r, the bond coupon rate c, the paying-out rate δ, and the corporate tax rate κ.

Table 2.2 displays the changes of default, conversion and call barriers in response to changes of the parameters. To clarify the interpretation on the results in the table, we consider two companies in the following discussion. Both of them are identical except their initial asset values. Company A starts with $V_0 = 300$ and B starts from $V_0 = 70$.

Effect of risk-free interest rate. When r increases, we can see that the optimal call region $[V^*_{cal,1}, V^*_{cal,2}]$ shrinks in its size, converging to $K/\lambda = 250$, the call barrier predicted by the classical literature. Under all r, Company A falls in a region in which only in-the-money calls are possible. For larger r, the call barrier $V^*_{cal,2}$ is farther away from V_0. It will take longer for V_t to hit the barrier. Therefore, the company tends to delay the call decision when r is high. Meanwhile, this observation applies for Company B too. The call for this company will be out-of-the-money. As we raise r, $V^*_{cal,1}$ increases. Thus, given all other parameters unchanged, Company B will wait longer until it issues a call announcement under a higher r. The economic intuition of this conclusion is fairly apparent: for a given coupon rate, a higher interest rate environment means that the company is paying the bondholder a relatively lower coupon. This makes the convertible bond more attractive to the company and leads to a delayed call.

In addition, a higher r also implies a lower default barrier V^*_b and a smaller conversion barrier V^*_{con}, as shown in Table 2.2. This is also what we can expect. Relatively low coupon payments in the settings of high r encourage the bondholder to convert for the equity sooner, because staying in bond to receive coupons is not attractive in that case. From the perspective of the shareholder, less coupon payments means less debt obligation. Thus, the shareholder postpones the default by pushing the barrier down.

Effect of coupon rate. The bond coupon rate c affects the optimal strategies in a way totally reverse to the risk free interest rate. When c increases, the optimal call region $[V^*_{cal,1}, V^*_{cal,2}]$ is enlarged and both companies tend to call in a shorter period of time after time 0. Accordingly, high coupon payments prompts a call decision because the convertible bond becomes an expensive fund-raising tool for the company for a higher c. Moreover, if c is large, the shareholder will also adopt a higher V^*_b to interrupt the cash flow of coupons to the bondholder, while the bondholder will be attracted to holding the bond for a longer time,

	V_b^*	$V_{cal,1}^*$	$V_{cal,2}^*$	V_{con}^*
r				
0.07	36.49	80.49	271.20	795.59
0.08	35.44	97.90	269.51	782.00
0.09	33.98	250.00	268.14	768.97
c				
0.06	30.96	250.00	250.00	656.96
0.07	35.44	97.90	269.51	782.00
0.08	38.49	74.11	308.01	893.72
δ				
0.05	36.61	92.71	318.83	928.80
0.06	35.44	97.90	269.51	782.00
0.07	34.17	104.33	250.00	664.61
κ				
0.15	40.84	65.28	423.86	675.69
0.25	38.68	73.22	339.04	733.19
0.35	35.44	97.90	269.51	782.00
0.45	30.58	250.00	250.00	795.60
ρ				
0.2	35.44	97.90	269.51	782.00
0.5	35.44	97.90	269.51	782.00
0.8	35.44	97.90	269.51	782.00

Table 2.2: Effects of various parameters on the optimal strategies. The defaulting parameter used is $K = 50$. We vary the parameter of interest each time and keep all the other parameters the same as those in Table 1.

which implies a higher V^*_{con}.

Effect of paying-out rate. Given the coupon rate unchanged, the effect of higher paying-out rate is to augment the dividends paid to the shareholder and in turn, to reduce the value of the bond. Under a high δ setting, the shareholder will have less incentive to eliminate the bondholder from the game since the bond value does not shift too much wealth away. This intuition is consistent with the observation on Table 2.2. No matter $V_0 = 300$ or 70, the distance between the call barriers and V_0 tends to be larger as δ rises. In other words, the call will be delayed if δ is high. The effect of δ on the default and conversion policies is similar as those of r. A high δ tempts the bondholder to convert sooner and the shareholder to announce a bankruptcy later.

Effect of tax rate. In our model, the tax shield is an important factor for the shareholder to borrow. Therefore, we expect that a high corporate tax will encourage the company to put off the call announcement. Table 2.2 illustrates that $V^*_{cal,1}$ and $V^*_{cal,2}$ are increasing and decreasing functions of κ, respectively. Hence, the convertible bond should be called in an early stage if κ is small.

Effect of default cost rate. For the small strike price, the default cost rate doesn't affect the optimal exercise strategies. For any default cost rate, shareholder get nothing at default. And the default and forcing surrendering polices are determined by the shareholder, hence they are not affected by the default cost rate.

In summary, the above numerical experiments project that delayed calls should be associated with low coupon rate, high corporate tax, high paying-out ratio and high risk free interest rate. These implications are supported by some empirical tests done by Sarkar (2003).

Table 2.3 provides a sensitivity analysis of the value of an in-the-money convertible bond at time 0 with respect to risk-free interest rate, coupon rate, paying-out rate and corporate tax rate. It shows that the bond value is positively related to the coupon rate, tax rate and conversion ratio, and

Interest rate r	0.06	0.07	0.08	0.09
Bond value	105.60	105.25	104.88	104.49
Equity Value	405.70	405.48	405.22	404.93
Coupon rate c	0.06	0.07	0.08	0.09
Bond value	101.88	104.88	107.12	108.57
Equity Value	404.46	405.22	404.52	403.37
Paying out rate δ	0.05	0.06	0.07	0.08
Bond value	106.62	104.88	102.39	100.45
Equity Value	403.55	405.22	405.43	403.26
Tax rate κ	0.15	0.25	0.35	0.45
Bond value	100.67	102.42	104.88	105.74
Equity Value	400.33	402.04	405.22	408.67
Default Cost ρ	0.10	0.30	0.50	0.70
Bond value	104.88	104.88	104.88	104.88
Equity Value	405.22	405.22	405.22	405.22

Table 2.3: Effects of various parameters on the convertible bond value. The defaulting parameters used are $K = 50$ and $V_0 = 500$. We vary the parameter of interest each time and keep all the other parameters the same as those in Table 1.

negatively related to the interest rate and payout rate. The former factors determine the cash inflows for the bondholder. Thus, higher values in those factors would boost the security value. The latter two factors push down the bond value as they rise. High risk free interest will discount the cash flow of the bond more, which generates a lower present value. High paying-out ratio implies a high dividend payment to the shareholder, which will shift the wealth away from the bondholder. Equity Value is positively related to the tax rate.

2.5.3. Negative and Positive Stock Returns

This subsection illustrates that our model can generate negative stock returns at an in-the-money call and positive stock returns at an out-of-the-money call.

In our investigation, we use Monte Carlo method to simulate the stock price changes for a specific company around the debt-calling date. More precisely, consider the default parameters in Table 2.5 and Company A starting with $V_0 = 300$. Simulate daily sample paths of V_t, following the geometric Brownian motion (2.1). The equity value for each day is obtained if substitute V_t in the equity function E. According to our calculation, such V_0 falls in the interval between $V^*_{cal,2} = 269.51$ and $V^*_{con} = 782$. When a call occurs, the call must be in-the-money for Company A. We choose the discrete time unit to be 1 trading day (i.e., 1/252 year) to simulate the call and conversion time. Record the sample path of the stock daily values, in which a call occurs. Figure 2.5 shows a typical realization of such path in a time window from 60 days before the call to 60 days after. We can see that the daily returns of the company's stock is not significant at all (less than 0.5%) except for the day in which a call announcement is issued. The daily return on calling drops down almost 2%. Note that the call is in-the-money because it happens when $V_t = V^*_{cal,2} = 269.51$, which is larger than $K/\lambda = 250$.

Figure 2.6 shows the daily stock returns in a 121-day time window centering on the day in which an out-of-the-money call is issued. In this figure, we consider Company B with $V_0 = 90$. There is a significant positive stock return at the calling day, which is larger than 3.5%. However, the returns of the rest days are less than 1%.

2.6. Conclusion and Future Work

We have established a non-zero-sum game framework to study the pricing problem of callable convertible bond. The impact of a trade-off, tax shield and bankrupt costs, is highlighted in the dissertation. Taking this trade-off into account will significantly change the strategies of the bondholder and shareholder, compared with the zero-sum setting in Sîrbu et al. (2004) and Sîrbu and Shreve (2006). In the presence of tax benefits and credit risk, the shareholder may call the debt in-the-money or out-of-the-money. The corresponding stock returns on the calling day exhibit some patterns consistent with the well-documented empirical results.

These results show that we should be in the right direction to study the convertible bonds. Then we have the motivation to extend the underlying process to be the general diffusion process or consider the convertible bonds with finite time maturity.

For the more general diffusion process with infinite maturity, we can use Dayanik and Karatzas (2003)'s method on the optimal stopping problem for one-dimensional diffusions. The extra efforts should be paid to the interaction of the two optimal stopping problems. Explicit formula may be possible for some cases and it can be expected that the effects of tax and default will still diverse the optimal call strategies.

Other extension within infinite time horizon is the case of Sirbu, Pikovsky and Shreve (2004), in which it assumes that the dividend is proportional to the equity value, such that the underlying firm value process involves the unknown value function of convertible bonds. This involves a nonlinear ODE, or equivalently an invariant solution of a parameterized linear ODE.

The optimal stopping problem with finite horizon is more challenging. The explicit solution may be impossible. But it's worth working on characterizing the optimal stopping boundaries of the bondholders and shareholders. According to our results in infinite time horizon, the optimal call boundaries may include double curves in finite time horizon, which differ from the standard American option problem.

We have to stress again that the corporate tax and credit risk are among many factors that have influence over the decisions related with convertible bond. Introducing other factors may accentuate the effect of the aforementioned trade-off and this leaves several possible directions for future investigation. For instance, the indentures of many convertible bond prohibit the issuers from calling for a certain period of time. Our model can be extended to cover such prohibition by viewing the problem as a two-stage sequential game. The first stage is the call protection period, in which the two parties interact with each other choosing optimal conversion and default policies. The analysis in this chapter constitutes the second stage. Another possible extension is to incorporate the asymmetric status in information access for the bondholder and shareholder. In reality, bond investors cannot observe the company's asset directly and suffer from imperfect accounting information (see, e.g., Duffie and Lando (2001)). A game framework with imperfect information would be an appropriate model under this setting.

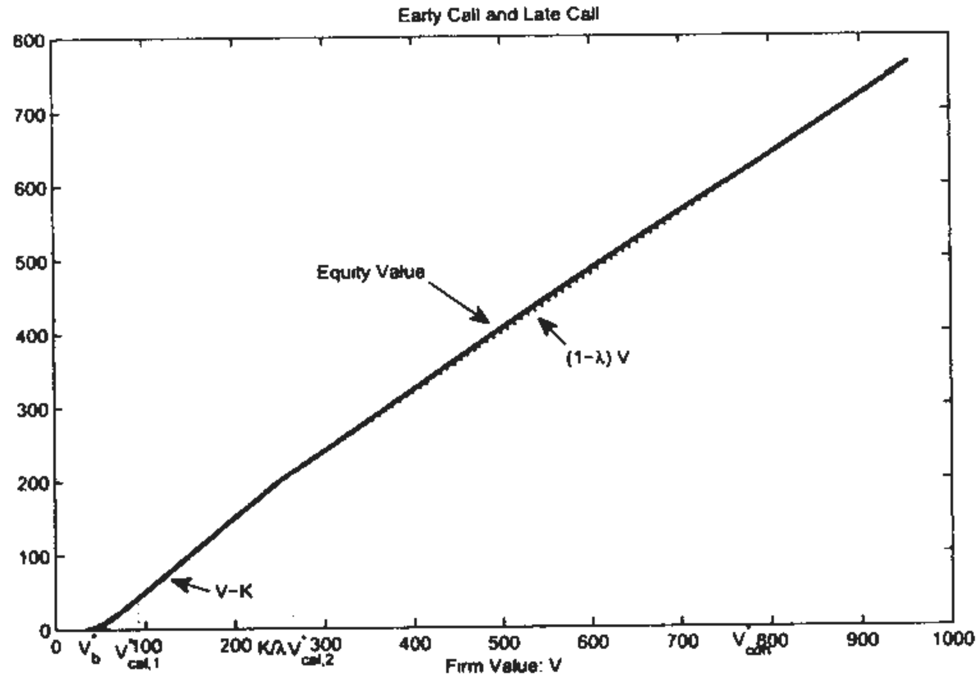

Figure 2.4: The equity value in a case with smaller call price. The default barrier is $V_b^* = 35.44$. The forcing surrender and conversion barriers are $V_{cal,1}^* = 97.90$ and $V_{cal,2}^* = 269.51$, respectively. The conversion barrier is $V_{con}^* = 782.00$. The horizontal straight line between $V_{cal,1}^*$ and K/λ indicates that the bond value equals to \$50. This is because the shareholder will call the debt once the company value falls in this interval and the bondholder responds to this call by a forced surrender. The bond value function coincides with λV in the interval $(K/\lambda, V_{cal,2})$. The shareholder will issue a call as well in this interval but the bondholder opts to convert in response.

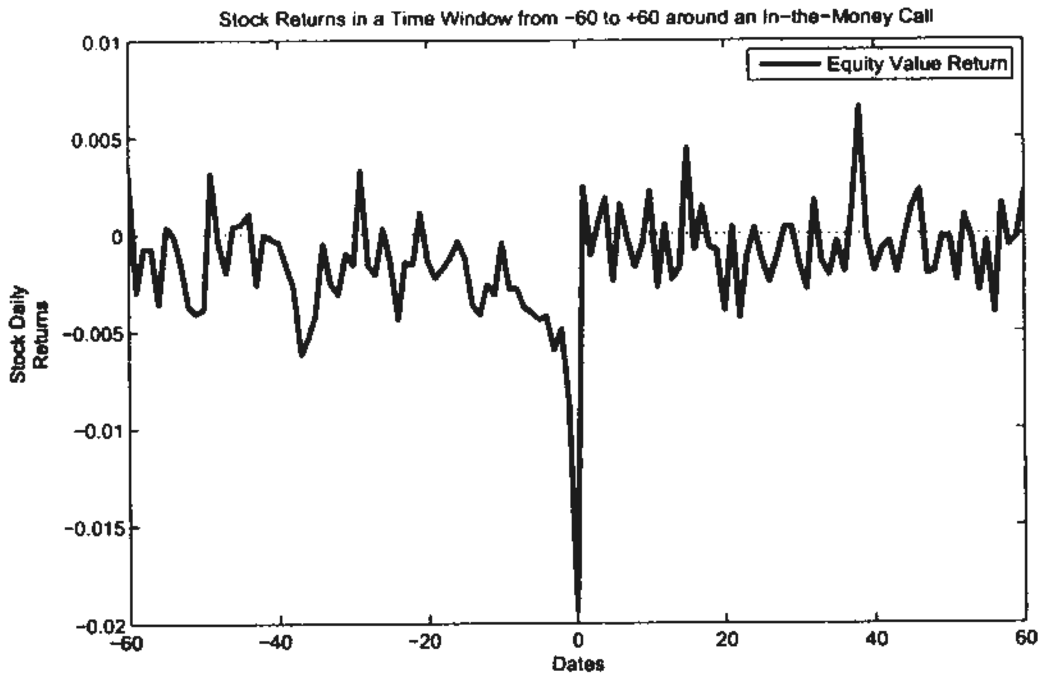

Figure 2.5: Daily returns of equity in a time window from 60 days before the call to 60 days after. We use the default parameters in Table 2.5 and let $V_0 = 300$. The call price is 50. The call boundary is $V^*_{cal,2} = 269.51$ and the conversion boundary is $V^*_{con} = 782$. We simulate 100 sample paths and draw the average.

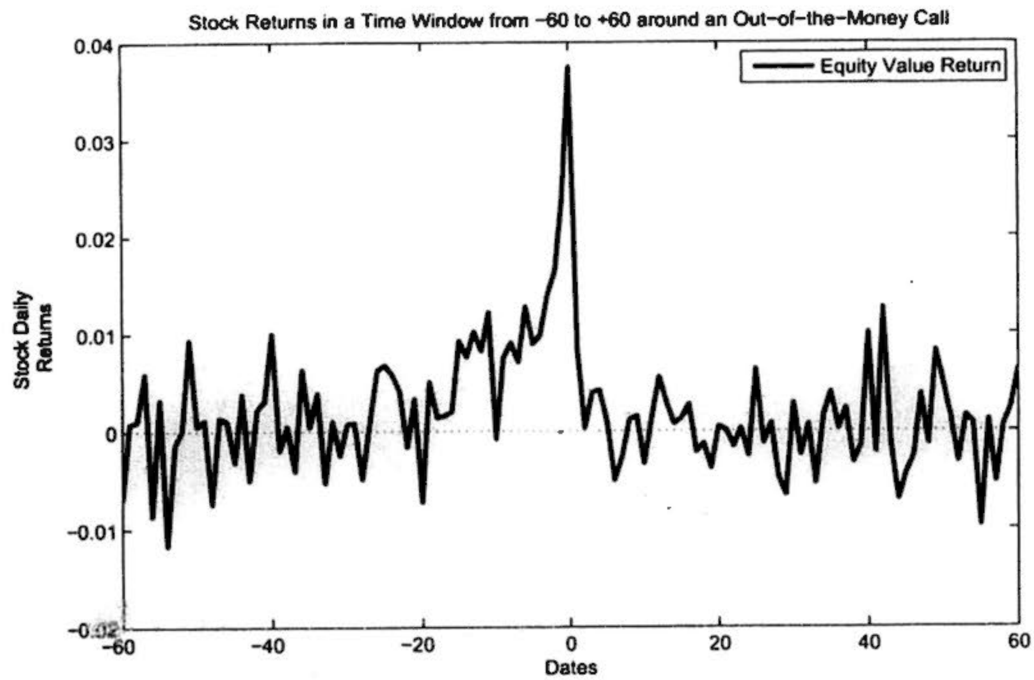

Figure 2.6: Daily returns of equity in a time window from 60 days before the call to 60 days after. We use the default parameters in Table 2.5 and let $V_0 = 90$. The call price is 50. The call boundary is $V^*_{cal,1} = 97.90$ and the default boundary is $V^*_b = 35.44$. We simulate 100 sample paths and draw the average.

CHAPTER 3

Pricing Double-Barrier Options under a Hyper-Exponential Jump Diffusion Model

3.1. Introduction

Barrier options are among the most popular exotic options traded in financial markets. A barrier option offers the holder a payoff like that of a vanilla option, contingent on whether or not the underlying asset price process crosses some level(s) called the barrier(s) before or at the maturity date. In this chapter we are going to study the pricing problem of double-barrier options under a flexible jump diffusion process for the underlying asset price.

The research of barrier options has been attracting a lot of attention in computational finance. It is motivated by both practical and theoretical reasons. In practice, barrier options are actively traded in the markets, especially in the Over-the-Counter markets (See Das (2004) and Zhang (1998)). In comparison with vanilla options, they have at least two advantages as argued by Derman and Kani (1996,1997). First, they may more closely match investor beliefs about the future behavior of the asset price. Second, they are always cheaper than vanilla options and hence are more attractive for investors. Meanwhile, barrier options also provide a useful tool to some theoretical studies outside the context of literal options. For example, Sircar and Xiong (2000) used a double-barrier-option framework to model executive stock options; the work of Goldstein, Ju and Leland (2001) on optimal dynamic capital structure was based on a double barrier structure, one barrier for firm bankruptcy and the other for capital readjustments.

Most studies on barrier option pricing are conducted under the Black-Scholes model (BSM). Closed form pricing formulae for double-barrier options can be easily derived under this setting. One may refer to Kunitomo and Ikeda (1992), Geman and Yor (1994), Pelsser (2000), and Schroder (2000). Despite its simplicity, the BSM has obvious shortcomings to be a good description for the movements of the underlying asset prices. It assumes the asset returns are normally distributed and their variances remain constant. Empirical studies invalidate such assumptions by suggesting two observations for asset returns: the asymmetric leptokurtic feature, i.e., the actual return has much heavier tails than normal, and the volatility smile, i.e., the volatility implied from equity option prices is not a constant but presents a curve resembling a "smile". To overcome the difficulties encountered by the BSM, many alternative models have been proposed in the literature to incorporate both of the empirical phenomena and correspondingly, the pricing problem of barrier options is needed to be re-investigated.

It is inappropriate to give a comprehensive overview of all models in such a chapter and here we shall focus on the double-barrier option pricing under a hyper-exponential jump diffusion model (HEM) proposed by Cai and Kou (2008) recently. Their model assumes the asset return follows a jump diffusion process with Poisson jump intensity and hyper-exponentially distributed jump sizes. As a result, the empirical asset returns have heavier tails than normal distributions. But, as shown in Heyde and Kou (2006), it may be very difficult to distinguish empirically the exponential-type tails from power-type tails even given a long period of daily data. So, a sensible asset model should be with more flexibility about the heaviness of the asset return tails. The HEM is appealing in this sense, thanks to the property of hypo-exponential distribution that it can approximate various distributions ranging from power tails to exponential tails (see, e.g., Feldmann and Whitt (1998)).

Mathematically, the contributions of our work are two-fold. First, we obtain analytical solutions to the prices of the standard double-barrier options in terms of Laplace transforms and then are able to invert them numerically via some efficient and accurate algorithms such as the Euler inversion algorithm proposed by Abate and Whitt (1992) and Choudhury, Lucantoni and Whitt (1994). Second, we show the existence and uniqueness of the solutions. More precisely, our analytical pricing formulae involve solutions of some high-dimensional linear systems and thus their existence and uniqueness are reduced down to the non-singularity of the associated high-dimensional matrix. We manage to prove the matrix is invertible in this chapter.

It is worth pointing out that similar technical issues also arise in some related work such as Cai and Kou (2008) and Sepp (2004). Cai and Kou (2008) considered the single-barrier option pricing. They also showed the existence and uniqueness of their solution through non-singularity of a simpler matrix, which turns out to be a sub-matrix of ours in the double-barrier case. As a by-product of our work, we can duplicate their conclusion with a new proof. Sepp (2004) priced standard double barrier options under the Kou's double exponential jump diffusion model (Kou (2002), Kou and Wang (2004)). The Kou's model assumes a double exponential distribution for jumps and therefore it is a special case of the HEM. In addition, Sepp (2004) did not prove the existence and uniqueness of his solution.

Beyond the jump diffusion model and Laplace transforms, there is a bulk of research on pricing barrier options under different models or with different methodologies. For instance, Davydov and Linetsky (2001) derived analytical solutions for both single- and double-barrier options under the CEV model; Broadie, Glasserman, and Kou (1997), Broadie and Yamamoto (2005), Feng and Linetsky (2008), Howison and Steinberg (2005), Petrella and Kou (2004), etc. developed many numerical methods to pricing discrete barrier options.

The rest of the chapter is organized as follows. In Section 2, we introduce the hyper-exponential jump diffusion model. Section 3 concentrates on deriving a general analytical formula relating to the joint distribution of the first passage time of the HEM to two flat barriers and the value of the HEM at the first passage time. Section 4 presents the analytical solution to the pricing problem of standard double-barrier options. Meanwhile, numerical results are also provided via the Euler inversion algorithm. Section 5 concludes this chapter. The main proof about the non-singularity of a high-dimensional matrix is given in the Appendix B.

3.2. The Model

Under the HEM, the asset price process $\{S_t : t \geq 0\}$ under the risk-neutral probability measure $\mathbb{P}$ is defined as $S_t := S_0 e^{X_t}$ and the log-return process $\{X_t : t \geq 0\}$ follows

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{3.1}$$

where $\sigma > 0$, $\mu := r - \sigma^2/2 - \lambda\zeta$ with risk-free rate $r > 0$, $\zeta = E[e^{Y_1}]$, $\{W_t : t \geq 0\}$ is a standard Brownian motion, $\{N_t : t \geq 0\}$ is a Poisson process with intensity λ, and $\{Y_i : i = 1, 2, \ldots\}$ is a sequence of independent identically distributed hyper-exponential random variables with a probability density function given by

$$f_Y(y) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} \mathbf{1}_{\{y \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j y} \mathbf{1}_{\{y < 0\}}, \tag{3.2}$$

where $p_i \geq 0$, $\eta_i > 1$ for all $i = 1, \ldots, m$, $q_j \geq 0$, $\theta_j > 0$ for all $j = 1, \ldots, n$, and $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$. From (3.2), we can see that there are m up-jumps and n down-jumps, among which the i^{th} up-jump occurs with probability p_i and then has an exponentially distributed size with mean $1/\eta_i$ for any $i = 1, 2, \cdots, m$, and the j^{th} down-jump occurs with probability q_j and then has an exponentially distributed size with mean $1/\theta_j$ for any $j = 1, 2, \cdots, n$. We also assume $\{W_t\}, \{N_t\}$ and $\{Y_i\}$ are independent.

Due to the jumps, the risk-neutral measure is not unique. Here we assume the risk-neutral measure $\mathbb{P}$ is chosen within a rational expectations equilibrium setting such that the equilibrium price of an option is given by the expectation under $\mathbb{P}$ of the discounted option payoff. For details, we refer to Lucas (1978), Naik and Lee (1990), and Kou (2002).

Thus, it is easy to see that the infinitesimal generator of $\{X_t\}$ is given by

$$\begin{aligned}(Lu)(x) =& \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) \\ &+ \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] f_Y(y) dy,\end{aligned}$$

for any twice continuously differentiable function $u(x)$ and the the Lévy exponent of $\{X_t\}$ is given by

$$\begin{aligned}G(x) :=& \frac{1}{t} \log \mathbf{E}[\exp(xX_t)] \\ =& x\mu + \frac{1}{2}x^2\sigma^2 + \lambda \left(\sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right)\end{aligned}$$

for any $x \in (-\theta_1, \eta_1)$. By some elementary calculus, we can show for any given $a > 0$, the equation $G(x) = a$ has exactly $m + n + 2$ real roots $\beta_1, \ldots, \beta_{m+1}, -\gamma_1, \ldots, -\gamma_{n+1}$ satisfying

$$0 < \beta_1 < \eta_1 < \beta_2 < \cdots < \eta_m < \beta_{m+1} < \infty, \tag{3.3}$$

$$0 < \gamma_1 < \theta_1 < \gamma_2 < \cdots < \theta_n < \gamma_{n+1} < \infty. \tag{3.4}$$

We record this result for later references.

3.3. Distribution of the First Passage Time to Two Flat Barriers

Define the first passage time τ of a general HEM $\{X_t := X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\}$ to two flat barriers h and H $(h < H)$ as follows

$$\tau := \inf\{t \geq 0 : X_t \geq H \text{ or } X_t \leq h\}. \tag{3.5}$$

Here $\{X_t\}$ slightly differs from that given in (3.1) in that it starts from the point X_0 rather than 0. From now on, we use $\mathbf{E}^x$ and $\mathbf{P}^x$ to represent the expectation and the probability, respectively, when $\{X_t\}$ starts from $X_0 \equiv x$.

The joint distribution of τ and X_τ plays a crucial role when pricing double-barrier options. Our idea is to get it via the Laplace transform

$$\mathbf{E}^x[e^{-\alpha\tau+\theta X_\tau}].$$

The following theorem reaches a more general result for any expectations in the form of $\mathbf{E}^x[e^{-\alpha\tau}f(X_\tau)]$, where f could be any nonnegative measurable function. The Laplace transform then becomes a direct corollary.

Theorem 3.1. *Consider any nonnegative measurable function f such that $\int_0^{+\infty} f(y+H)e^{-\eta_i y}dy$ and $\int_{-\infty}^0 f(y+h)e^{\theta_j y}dy$ are integrable for all $1 \le i \le m$ and $1 \le j \le n$. For any $\alpha > 0$ and $x \in (h, H)$, we have*

$$\mathbf{E}^x[e^{-\alpha\tau}f(X_\tau)] = \boldsymbol{\varpi}(x)\mathbf{N}^{-1}\mathbf{f}, \tag{3.6}$$

where $\boldsymbol{\varpi}(x)$ is a row vector defined as

$$\begin{aligned}\boldsymbol{\varpi}(x) = \big(&e^{\beta_1(x-H)}, \ldots, e^{\beta_{m+1}(x-H)}, \\ &e^{-\gamma_1(x-h)}, \ldots, e^{-\gamma_{n+1}(x-h)}\big),\end{aligned} \tag{3.7}$$

$\mathbf{f}$ is a column vector such that $\mathbf{f} = (f_0^u, \ldots, f_m^u, f_0^d, \ldots, f_n^d)^T$,

$$\begin{aligned} f_0^u &= f(H),\ f_i^u = \int_0^{+\infty} f(y+H)e^{-\eta_i y}dy,\ 1 \le i \le m, \\ f_0^d &= f(h),\ f_j^d = \int_{-\infty}^0 f(y+h)e^{\theta_j y}dy, \quad 1 \le j \le n; \end{aligned} \tag{3.8}$$

and $\mathbf{N}$ *is a* $(m+n+2)\times(m+n+2)$ *non-singular matrix given by*

$$
\begin{bmatrix}
1 & \cdots & 1 & \bar{x}^{\gamma_1} & \cdots & \bar{x}^{\gamma_{n+1}} \\
\frac{1}{\eta_1-\beta_1} & \cdots & \frac{1}{\eta_1-\beta_{m+1}} & \frac{\bar{x}^{\gamma_1}}{\eta_1+\gamma_1} & \cdots & \frac{\bar{x}^{\gamma_{n+1}}}{\eta_1+\gamma_{n+1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\eta_m-\beta_1} & \cdots & \frac{1}{\eta_m-\beta_{m+1}} & \frac{\bar{x}^{\gamma_1}}{\eta_m+\gamma_1} & \cdots & \frac{\bar{x}^{\gamma_{n+1}}}{\eta_m+\gamma_{n+1}} \\
\bar{x}^{\beta_1} & \cdots & \bar{x}^{\beta_{m+1}} & 1 & \cdots & 1 \\
\frac{\bar{x}^{\beta_1}}{\theta_1+\beta_1} & \cdots & \frac{\bar{x}^{\beta_{m+1}}}{\theta_1+\beta_{m+1}} & \frac{1}{\theta_1-\gamma_1} & \cdots & \frac{1}{\theta_1-\gamma_{n+1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\bar{x}^{\beta_1}}{\theta_n+\beta_1} & \cdots & \frac{\bar{x}^{\beta_{m+1}}}{\theta_n+\beta_{m+1}} & \frac{1}{\theta_n-\gamma_1} & \cdots & \frac{1}{\theta_n-\gamma_{n+1}}
\end{bmatrix}
$$

with $\bar{x} := e^{h-H}$.

To prove Theorem 3.1, the most difficult part is to show the non-singularity of the matrix $\mathbf{N}$. We summarize the conclusion in the following proposition and defer its proof to the Appendix B.1.

Proposition 3.2. *For any* $\{\beta_i\}_{i=1}^{m+1}$ *and* $\{\gamma_j\}_{j=1}^{n+1}$ *satisfying* (3.3) *and* (3.4), *the matrix* $\mathbf{N}$ *is non-singular.*

With the help of Proposition 3.2, we can show Theorem 3.1 now.

Proof of Theorem 3.1. Notice that τ is the first time the process X exits the band (h, H). It may leave the band at the boundaries, i.e., $X_\tau = H$ or $X_\tau = h$; or it may jump across the boundaries when leaving. Therefore, we introduce a sequence of events: $F_0 := \{\omega : X_\tau = H\}$, $G_0 := \{\omega : X_\tau = h\}$, indicating two possibilities that X leaves the band at the boundaries; $F_i := \{\omega : X_\tau - H > 0, Y_{N_\tau} \sim \text{Exp}(\eta_i)\}$ for $i = 1, 2, \cdots, m$ and $G_j := \{\omega : X_\tau - h < 0, -Y_{N_\tau} \sim \text{Exp}(\theta_j)\}$ for $j = 1, 2, \cdots, n$, indicating with which type of jump the process jumps across the boundaries when leaving. By the law of total probability, we have

$$\mathbf{E}^x[e^{-\alpha\tau} f(X_\tau)] = \sum_{i=0}^{m} \mathbf{E}^x[e^{-\alpha\tau} f(X_\tau)\mathbf{1}_{F_i}] + \sum_{j=0}^{n} \mathbf{E}^x[e^{-\alpha\tau} f(X_\tau)\mathbf{1}_{G_j}]. \tag{3.9}$$

Emulating the proofs of Proposition 2.1 in Kou and Wang (2003) and Theorem 3.1 in Cai (2008a), we can easily show that conditional on F_i, τ and X_τ are independent and moreover the overshoot $X_\tau - H$ is still exponentially distributed with mean $1/\eta_i$, thanks to the memoryless property of exponential distribution. Thus, for any $i = 1, 2, \cdots, m$,

$$\begin{aligned}\mathbf{E}^x[e^{-a\tau} f(X_\tau)\mathbf{1}_{F_i}] &= \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_i}]\mathbf{E}^x[f(X_\tau - H + H)|F_i] \\ &= \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_i}] \cdot \eta_i f_i^u. \end{aligned} \tag{3.10}$$

Similarly, for any $j = 1, 2, \cdots, n$, we have

$$\mathbf{E}^x[e^{-a\tau} f(X_\tau)\mathbf{1}_{G_j}] = \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_j}] \cdot \theta_j f_j^d. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11),

$$\mathbf{E}^x[e^{-a\tau} f(X_\tau)] = \sum_{i=0}^{m} \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_i}] \cdot \eta_i f_i^u + \sum_{j=0}^{n} \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_j}] \cdot \theta_j f_j^d, \tag{3.12}$$

with $\eta_0 = \theta_0 = 1$.

On the other hand, we are also able to obtain closed-form expressions for $\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_i}]$ and $\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_j}]$. Note that for any $a > 0$ and imaginary number b with the real part being 0,

$$M_t := \exp(-at + bX_t) - \exp(bX_0) - (G(b) - a)\int_0^t \exp(-as + bX_s)ds$$

is a zero-mean martingale. By the optional sampling theorem, we know $\mathbf{E}^x[M_\tau] = 0$, i.e.,

$$0 = \mathbf{E}^x[\exp(-a\tau + bX_\tau)] - e^{bx} - (G(b) - a)\mathbf{E}^x[\int_0^\tau \exp(-as + bX_s)ds].$$

Applying (3.12) in the first term on the right hand side of the above equality,

$$\begin{aligned} 0 =& \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_0}]e^{bH} + \sum_{i=1}^{m} \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_i}]e^{bH}\frac{\eta_i}{\eta_i - b} \\ &+ \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_0}]e^{bh} + \sum_{j=1}^{n} \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_j}]e^{bh}\frac{\theta_j}{\theta_j + b} \\ &- e^{bx} - (G(b) - a)\mathbf{E}^x[\int_0^\tau \exp(-as + bX_s)ds]. \end{aligned} \tag{3.13}$$

Denote the right hand side of (3.13) by $h(b)$ and define $H(b) := \prod_{i=1}^{m}(\eta_i - b) \cdot \prod_{j=1}^{n}(\theta_j + b) \cdot h(b)$. Then $H(b)$ is well defined and analytic in the whole complex domain $\mathbb{C}$. By (3.13), $H(b)$ equals zero when b is a pure imaginary number. By the identity theorem of analytic functions in the complex domain (Theorem 10.18, Rudin (1987)), we get $H(b) = 0$ for all $b \in \mathbb{C}$. Accordingly, $h(b) = 0$ for all $b \in \mathbb{C} \setminus \{-\theta_n, \ldots, -\theta_1, \eta_1, \ldots, \eta_m\}$.

Replace b by β_i and $-\gamma_j$ in $h(b) = 0$, respectively. Note that β_i and $-\gamma_j$ are all the roots to $G(x) = a$. We have the following linear equations with respect to $\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_i}]$ and $\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_j}]$:

$$\begin{aligned} e^{\beta_i x} =& \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_0}]e^{\beta_i H} + \sum_{l=1}^{m}\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_l}]e^{\beta_i H}\frac{\eta_l}{\eta_l - \beta_i} \\ &+ \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_0}]e^{\beta_i h} + \sum_{j=1}^{n}\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_j}]e^{\beta_i h}\frac{\theta_j}{\theta_j + \beta_i} \end{aligned}$$

and

$$\begin{aligned} e^{-\gamma_i x} =& \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_0}]e^{-\gamma_i H} + \sum_{l=1}^{m}\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_l}]e^{-\gamma_i H}\frac{\eta_l}{\eta_l + \gamma_i} \\ &+ \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_0}]e^{-\gamma_i h} + \sum_{j=1}^{n}\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_j}]e^{-\gamma_i h}\frac{\theta_j}{\theta_j - \gamma_i}. \end{aligned}$$

Proposition 3.2 shows the non-singularity of $\mathbf{N}$. It follows that the vector

$$\begin{aligned} &\Big(\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_0}], \ldots, \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{F_m}], \\ &\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_0}], \ldots, \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{G_n}]\Big) \\ =&\boldsymbol{\varpi}(x)\mathbf{N}^{-1}\mathrm{Diag}\left\{\frac{1}{\eta_0}, \ldots, \frac{1}{\eta_m}, \frac{1}{\theta_0}, \ldots, \frac{1}{\theta_n}\right\}, \end{aligned} \tag{3.14}$$

where $\mathrm{Diag}\left\{\frac{1}{\eta_0}, \ldots, \frac{1}{\eta_m}, \frac{1}{\theta_0}, \ldots, \frac{1}{\theta_n}\right\}$ is a diagonal matrix. Plugging (3.14) into (3.12) yields (3.6) immediately. □

From Theorem 3.1, we can obtain a variety of closed-form expressions for expectations of some functions with respect to τ and X_τ. For instance, choosing $f(x)$ to be $e^{\theta x}$ with $\theta \in (-\theta_1, \eta_1)$ in the above theorem, we are able to derive the Laplace transform $\mathbf{E}^x[e^{-a\tau + \theta X_\tau}]$, which is presented in Corollary ??. Fixing

$f(x)$ to be $\mathbf{1}_{\{x \geq H\}}$ and $\mathbf{1}_{\{x \leq h\}}$, respectively, we then have $\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{\{X_\tau \geq H\}}]$ and $\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{\{X_\tau \leq h\}}]$, which reflect the present values of \$1 received when the underlying asset price crosses the upper boundary H and the lower boundary h, respectively. This result is given in Corollary 3.4.

Corollary 3.3. *For any $\theta \in (-\theta_1, \eta_1)$, we have*

$$\mathbf{E}^x[e^{-a\tau+\theta X_\tau}] = e^{\theta H} \cdot (\sum_{i=1}^{m+1} \omega_i e^{\beta_i (x-H)} + \sum_{j=1}^{n+1} \nu_j e^{-\gamma_j (x-h)}) \tag{3.15}$$

where

$$(\omega_1, \dots, \omega_{m+1}, \nu_1, \dots, \nu_{n+1})^T = \mathbf{N}^{-1}\mathbf{J}(\theta)$$

and

$$\mathbf{J}(\theta) = (1, \frac{1}{\eta_1 - \theta}, \dots, \frac{1}{\eta_m - \theta}, \overline{x}^\theta, \frac{\overline{x}^\theta}{\theta_1 + \theta}, \dots, \frac{\overline{x}^\theta}{\theta_n + \theta})^T.$$

Corollary 3.4.

$$\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{\{X_\tau \geq H\}}] = \sum_{i=1}^{m+1} \omega_i^{(1)} e^{\beta_i (x-H)} + \sum_{j=1}^{n+1} \nu_j^{(1)} e^{-\gamma_j (x-h)} \tag{3.16}$$

$$\mathbf{E}^x[e^{-a\tau}\mathbf{1}_{\{X_\tau \leq h\}}] = \sum_{i=1}^{m+1} \omega_i^{(2)} e^{\beta_i (x-H)} + \sum_{j=1}^{n+1} \nu_j^{(2)} e^{-\gamma_j (x-h)} \tag{3.17}$$

where

$$(\omega_1^{(1)}, \dots, \omega_{m+1}^{(1)}, \nu_1^{(1)}, \dots, \nu_{n+1}^{(1)})^T = \mathbf{N}^{-1}\mathbf{J_1} \text{ with } \mathbf{J_1} = (1, \frac{1}{\eta_1}, \dots, \frac{1}{\eta_m}, 0, \dots, 0)^T,$$

and

$$(\omega_1^{(2)}, \dots, \omega_{m+1}^{(2)}, \nu_1^{(2)}, \dots, \nu_{n+1}^{(2)})^T = \mathbf{N}^{-1}\mathbf{J_2} \text{ with } \mathbf{J_2} = (0, \dots, 0, 1, \frac{1}{\theta_1}, \dots, \frac{1}{\theta_m})^T.$$

Remark 3.5. *We can also show Corollary 3.1 and Corollary 3.2 through another route. Actually, consider the following two ordinary integro-differential equations (OIDEs)*

$$\begin{cases} Lu(x) = au(x), & h < x < H; \\ u(x) = e^{\theta x}, & x \leq h \text{ or } x \geq H \end{cases} \tag{3.18}$$

,

and

$$\begin{cases} Lu(x) = au(x), & h < x < H; \\ u(x) = 1, & x \geq H; \\ u(x) = 0, & x \leq h. \end{cases} \tag{3.19}$$

First, it can be proved that under the condition that $u(h) = u(h+)$ *and* $u(H) = u(H-)$, *the right hand sides of* (3.15) *and* (3.16), *denoted by* $v_0(x)$ *and* $v_1(x)$, *are unique solutions to* (3.18) *and* (3.19), *respectively. Second, applying the martingale method, we can show that* $v_0(x) \equiv \mathbf{E}^x[e^{-a\tau+\theta X_\tau}]$ *and* $v_1(x) \equiv \mathbf{E}^x[e^{-a\tau}\mathbf{1}_{\{X_\tau \geq H\}}]$. *Similarly, we can obtain (3.17). Thus we complete the proofs for Corollary 3.1 and Corollary 3.2 in a different way. Meanwhile, the non-singularity of* $\mathbf{N}$ *guarantees the uniqueness of such solutions. For details of this route of argument, we refer to Cai and Kou (2008).*

3.4. Pricing Double-Barrier Options

In this section, we are going to derive pricing formulae for standard double-barrier options, based on the theoretical results obtained in the last section.

3.4.1. Standard Double-Barrier Options

The payoff of a standard double-barrier option is activated (knocked in) or extinguished (knocked out) when the price of the underlying asset crosses barriers. For example, a knock-out put option will not give the holder the payoff of a European put option unless the underlying price remains within some pre-specified range before the option matures. More precisely, consider an interval (L, U) and the initial asset price S_0 is in it. The holder will receive $(K - S_T)^+\mathbf{1}_{\{\tau > T\}}$ at the maturity T, where $\tau = \inf\{t \geq 0 : S_t \leq L \text{ or } S_t \geq U\}$. Under the risk-neutral measure $\mathbb{P}$ and the assumption that the underlying asset follows the HEM, the price of such an option is given by

$$P(K, T) = e^{-rT}\mathbf{E}[(S_T - K)^+\mathbf{1}_{\{\tau > T\}}|S_0]. \tag{3.20}$$

We may use Corollary 3.3 to obtain a double Laplace transform for the expectation in $P(K,T)$. For this purpose, change some variables in (3.20) first. Let $h := \log(L/S_0)$, $H := \log(U/S_0)$ and $\kappa := -\log K$. Then, the expectation in $P(K,T)$ can be represented as

$$C(\kappa, T) := \mathbf{E}^x\left[(S_0 e^{X_T} - e^{-\kappa})\mathbf{1}_{\{\tau > T, S_0 e^{X_T} > e^{-\kappa}\}}\right],$$

where $\tau = \inf\{t \geq 0 : X_t \leq h \text{ or } X_t \geq H\}$. Conduct a double Laplace transform on the new function $C(\kappa, T)$ with respect to κ and T. Note that the definition domains for κ and T are $(-\infty, \infty)$ and $(0, \infty)$, respectively. We have the following theorem:

Theorem 3.6. *For any* $0 < \varphi < \eta_1 - 1$ *and* $a > \max\{G(\varphi+1), 0\}$, *let*

$$g(\varphi, a) = \int_0^\infty \int_{-\infty}^\infty e^{-\varphi\kappa - aT} C(\kappa, T) d\kappa dT. \tag{3.21}$$

Then,

$$g(\varphi, a) = \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \frac{1}{a - G(\varphi+1)} \left(1 - e^{(\varphi+1)H}\left(\sum_{i=1}^{m+1} \omega_i e^{-\beta_i H} + \sum_{j=1}^{n+1} \nu_j e^{\gamma_j h}\right)\right), \tag{3.22}$$

where

$$(\omega_1, \omega_2, \ldots, \omega_{m+1}, \nu_1, \nu_2, \ldots, \nu_{n+1})^T = \mathbf{N}^{-1}\mathbf{J}(\varphi+1).$$

Proof. For any fixed T, by the Fubini theorem,

$$\begin{aligned}\int_{-\infty}^\infty e^{-\varphi\kappa} C(\kappa, T) d\kappa =& \mathbf{E}^x\Big[\int_{-\log(S_0 e^{X_T})}^\infty (S_0 e^{X_T} e^{-\varphi\kappa} - e^{-(\varphi+1)\kappa}) d\kappa \cdot \mathbf{1}_{\{\tau > T\}}\Big] \\ =& \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \mathbf{E}^x[e^{(\varphi+1)X_T}(1 - \mathbf{1}_{\{\tau \leq T\}})].\end{aligned}$$

From the definition of the Lévy exponent, we know $\mathbf{E}^x[e^{(\varphi+1)X_T}] = \exp(G(\varphi+1)T)$. Then, using the Fubini's theorem again,

$$\begin{aligned} g(\varphi, a) =& \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \int_0^\infty e^{-aT} \mathbf{E}^x[e^{(\varphi+1)X_T}(1 - \mathbf{1}_{\{\tau \leq T\}})] dT \\ =& \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \cdot \frac{1}{a - G(\varphi+1)} - \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \cdot \int_0^\infty \mathbf{E}^x[e^{-aT+(\varphi+1)X_T} \mathbf{1}_{\{\tau < T\}}] dT. \end{aligned}$$

Conditional on the filtration up to τ, the expectation in the above equality should be the same as

$$\mathbf{E}^x[e^{-aT}\cdot\mathbf{E}^x[e^{(\varphi+1)X_T}|\mathcal{F}_\tau]\mathbf{1}_{\{\tau<T\}}]=\mathbf{E}^x[e^{-aT+G(\varphi+1)(T-\tau)+(\varphi+1)X_\tau}\mathbf{1}_{\{\tau\leq T\}}],$$

where the equality holds due to the Markovian property of $\{X_t\}$, the fact $X_T - X_\tau \stackrel{d}{=} X_{T-\tau}$, and the definition of the Lévy exponent. In summary, $g(\varphi, a)$ is then equal to

$$\frac{S_0^{\varphi+1}}{\varphi(\varphi+1)}\cdot\frac{1}{a-G(\varphi+1)}(1-\mathbf{E}^x[e^{-a\tau+(\varphi+1)X_\tau}]).$$

Applying Corollary 3.3 here, we can immediately obtain the conclusion. □

Once we have the double Laplace transform, we apply some numerical inversion algorithm to recover the value of the function $C(\kappa, T)$ at some specific κ and T we want to price. There are several other double-barrier options such as knock-out put, knock-in call or put traded in the market. The pricing formulae for them can be obtained through similar derivations and we leave all the details for interested readers.

Remark 3.7.

$$g(\varphi,a)=\frac{1}{\varphi(\varphi+1)}\frac{1}{a-G(\varphi+1)}\left(S_0^{\varphi+1}-U^{\varphi+1}\left(\sum_{i=1}^{m+1}\omega_i\left(\frac{S_0}{U}\right)^{\beta_i}+\sum_{j=1}^{n+1}\nu_j\left(\frac{L}{S_0}\right)^{\gamma_j}\right)\right), \tag{3.23}$$

then

$$\frac{\partial g(\varphi,a)}{\partial S_0}=\frac{1}{\varphi(\varphi+1)}\frac{1}{a-G(\varphi+1)}\left((\varphi+1)S_0^{\varphi}-\frac{U^{\varphi+1}}{S_0}\left(\sum_{i=1}^{m+1}\omega_i\beta_i\left(\frac{S_0}{U}\right)^{\beta_i}-\sum_{j=1}^{n+1}\nu_j\gamma_j\left(\frac{L}{S_0}\right)^{\gamma_j}\right)\right). \tag{3.24}$$

3.4.2. Numerical Examples

In this section, we intend to price the above standard knock-out call options by inverting the associated Laplace transforms (3.21) numerically via the Euler inversion algorithm. This algorithm was introduced by Abate and Whitt (1992) and Choudhury, Lucantoni and Whitt (1994) and a few new developments are accumulated in the literature based on their work. Since we need to invert a two-sided Laplace transform with respect to κ, we suggest to use Petrella (2004), which is faster and more stable numerically than the original Euler inversion when dealing with two-sided transforms, due to the introduction of a scaling factor.

In our numerical example, m and n are both 2 in the hyper-exponential distribution (3.2). The numerical results for the standard double-barrier options (denoted by EI Price) are given in Table 3.1, where we also show the Monte Carlo simulation result (denoted by MC Value) as a benchmark together with the associated 95% confidence interval (denoted by 95% CI). We can see that all the EI Prices stay within the 95% confidence intervals of the associated MC Values. Besides, based on a PC with Pentium(R) 4 CPU 2.80GHz, 1 GB of RAM, the CPU time to produce one numerical result via Euler inversion algorithm is only around 6 seconds, while it takes about 20 minutes to generate one MC Value. Consequently, we draw the conclusion that the pricing method based on our analytical pricing formulae as well as the Euler inversion algorithm is accurate and efficient. It is worth mentioning that in Table 3.1, MC Values tend to be greater than EI Prices partly because the Monte Carlo simulation method overestimates the option prices due to the systematic discretization bias. Since our main purpose is to study the analytical solution rather than the Monte Carlo simulation method. We refer the interested readers to Metwally and Atiya (2002) for more detailed discussions on the systematic discretization error reduction.

From the table, we can also see that the option price descreases as the strike K increases. This is intuitive because the payoff is a decreasing function in K.

Meanwhile, when either σ or λ increases, the option price depreciates. That is because the option tends to be more likely knocked out when the underlying is more volatile.

3.5. Conclusion

In this Chapter, we investigate the pricing problem of double-barrier options under a flexible, hyper-exponential jump diffusion model. Specifically, we derive the closed form expression for the double-Laplace transform of the standard double-barrier option by studying the joint distribution of the first passage time of a hyper-exponential jump diffusion process to two flat barriers and the value of this process at the first passage time. Moreover, this closed form double-Laplace transform can be inverted numerically via a two-sided Laplace inversion algorithm. Numerical examples indicate that the pricing algorithm is accurate, efficient, and easy to implement. One of our theoretical contribution is that we show the non-singularity of a complicated, high-dimensional matrix, therefore guaranteeing the existence and uniqueness of our analytical pricing formula.

Pricing Knock-Out Call Options				
K	λ	EI Price	MC Value	95% CI
	5	0.1052	0.1063	(0.1019, 0.1107)
105	3	0.1156	0.1189	(0.1142, 0.1236)
	1	0.1270	0.1300	(0.1252, 0.1348)
	5	0.3456	0.3471	(0.3375, 0.3567)
100	3	0.3804	0.3847	(0.3746, 0.3948)
	1	0.4191	0.4210	(0.4105, 0.4315)
	5	0.7812	0.7831	(0.7666, 0.7996)
95	3	0.8606	0.8676	(0.8499, 0.8847)
	1	0.9487	0.9478	(0.9298, 0.9658)

Table 3.1: The Laplace inversion (EI Price) vs. the Monte Carlo simulation (MC Value). For unvarying parameters, the default choices are $r = 0.05$, $\sigma = 0.2$, $m = n = 2$, $\eta_1 = 30$, $\eta_2 = 50$, $\theta_1 = 30$, $\theta_2 = 40$, $p_1 = p_2 = q_1 = q_2 = 0.25$, $S_0 = 100$, $U = 115$, $L = 80$, $T = 1$, and $\rho = 1$. Parameters for the Laplace inversion method are $A_1 = A_2 = 28.3$, $(n_1, n_2) = (11, 38)$, and the scaling factor $X = 1000$; while the MC values along with the associated 95% confidence intervals are obtained by using 60,000 time steps and simulating 100,000 sample paths. To generate one numerical result, the CPU time is about 6 seconds for the Laplace inversion method and is about 20 minutes for Monte Carlo simulation method. Moreover, we can see that all the EI prices stay within the 95% confidence intervals of the associated MC values.

CHAPTER 4

Occupation Times of Jump-Diffusion Processes with Double Exponential Jumps and the Pricing of Options

4.1. Introduction

Occupation-time-related derivatives are recently introduced products that have been attracting much attention from investors and researchers. A defining characteristic of these contracts is an exercise payoff that depends on the time spent by the underlying asset in a predetermined region(s). Typically, the specification of the occupation regions involves flat barrier(s). In that sense, these contracts can be viewed as a generalized type of barrier option.

The payoffs of barrier options are activated or extinguished as soon as the underlying asset prices cross barriers. This discontinuity at the barriers poses an obstacle to the risk management of both option writers and buyers. Take the knock-out barrier option as an illustration. Even if the buyer has a correct view on the overall market trend, an accidental price jump across the barrier can easily wipe out his or her entire investment in the options. Furthermore, as Chesney, Jeanblanc-Picqué and Yor (1997) and Linetsky (1998) argued, market manipulators also like to take advantage of the fact that the payoffs are associated with barrier crossing, driving the underlying price to trigger a crossing and profiting from the massive losses of the other party to the transaction.

Several scholars have proposed a series of occupation-time-related options to alleviate the risk management difficulties inherent in barrier options caused by the discontinuity around the barriers. The payoffs now depend not only on the barrier crossing; but also on how long the underlying price spends above/below the barrier. Thus, option buyers can receive or lose value more gradually. One of the most popular examples is the step option suggested by Linetsky (1998,1999). This derivative's payoff is discounted at a rate defined by the occupation time. Under the geometric Brownian motion (GBM) model, Linetsky (1999) derived closed-form pricing formulae for various single-barrier step options, while Davydov and Linetsky (2002) investigated the pricing of double-barrier step options via Laplace inversion. A second example is the corridor option traded in the foreign exchange and interest rate markets. This option pays an amount at maturity that is associated with the time spent by a reference index, usually an exchange or interest rate, below a given level or inside a band. Fusai (2000) priced this derivative under the GBM model by studying the distribution of the time spent by a Brownian motion with drift inside a band. Another important type of occupation-time-related option is the quantile option, which Miura (1992) suggested as an alternative to the standard barrier option. A quantile is the minimum barrier to ensure that the fraction of the occupation time during the lifetime of the option exceeds a given level. Dassios (1995) provided a formula for the quantile distribution of a Brownian motion with drift, as did Embrechts, Rogers and Yor (1995) and Yor (1995). Akahori (1995) and Dassios (1995) calculated the prices of α-quantile options for the GBM model. Kwok and Lau (2001) developed a pricing algorithm for quantile options based on the forward shooting grid method under the GBM model. Leung and Kwok (2006) derived the distribution functions of occupation times under the constant elasticity of variance (CEV) process. Using an identity on quantiles of the processes with stationary and independent increments developed by Dassios (1996), Cai

(2008b) priced both the fixed- and floating-strike quantile options numerically by applying Laplace inversion twice under a hyper-exponential jump diffusion model.

In reality, many occupation-time-related options are based on a discrete time monitoring. In other words, such derivatives specify a series of reference dates. The occupation time is defined through the portion of monitoring dates in which the underlying price is below/above some level or between two levels. Some research is devoted to the study of such kind of options. However, the common feature of such research is that the underlying asset price is assumed to follow a GBM model. For instance, Atkinson and Fusai (2007) studied discrete quantile options using the Spitzer identity of Brownian motions; Fusai and Tagliani (2001) applied some numerical methods of PDEs to price discrete corridor options; and Davydov and Linetsky (2002) considered step options under the discrete monitoring scheme.

In this article, we investigate the pricing and hedging problems of occupation-time-related options under Kou's double exponential jump diffusion model Kou (2002). The model assumes the underlying asset return follows a jump diffusion process with Poisson jump intensity and double-exponentially distributed jump sizes. It is appealing in two respects. The associated asset returns have heavier tails than normal distributions and hence the model is capable of generating asymmetric leptokurtic feature for asset returns and volatility smiles for equity options, matching the empirical data better than the GBM model. The model also yields analytical solutions to many pricing problems, including both European and path-dependent derivatives, in terms of Laplace transforms. By applying numerical inversion algorithms we can easily obtain the prices.

The main result of this article is to derive the Laplace transform of the distribution of occupation times regarding one barrier under Kou's model, which enables us to calculate the prices of various related options such as step options, corridor options, and quantile options. It turns out that the Laplace transform solves a partial integro-differential equation (PIDE). We manage to reduce the equation to an ordinary integro-differential equation (OIDE) using an integral transform. Note that derivatives of exponential functions are still exponential. Then we can transform the OIDE into an ODE and rigorously show the existence and uniqueness of the solution to the OIDE. This article contributes to the literature of occupation-time-related options by generalizing the formulae for the GBM model to a model with discontinuous sample paths. It is simple to recover all of the classical results obtained with the GBM model from ours by letting the jump intensity be zero. The closed-form expressions of the Laplace transforms of the option prices also facilitate the calculation of price sensitivities in relation with market variables and model parameters. As shown in Section 4, not much extra effort is needed to obtain deltas, the price sensitivity with respect to the change of the underlying price. Such sensitivities play a vital role in risk management of derivatives, and traders can use it to rebalance the portfolio accordingly to achieve a desired exposure. In addition, our PIDE-OIDE approach can easily be extended to derive a close form solution for the Laplace transform of the distribution of occupation times spent within two barriers (a corridor).

Beyond financial settings, we should emphasize that the mathematical results about occupation times of a jump diffusion process may find potential applications in other branches of applied probability more generally. One candidate case we can think of is in queuing theory. When service times or interarrival times have heavy-tailed distributions, heavy-traffic limits for the queue-length process usually are given by jump diffusions (see Whitt (2002), Chapter 6). The results presented in this chapter may be of interest to those who want to study the occupation time above/below single level or between two levels for a heavy-traffic queue. The literature accumulates some progress in this direction. For instance, Cohen and Hooghiemstra (1981) discussed occupation times of Brownian excursions, a special kind of diffusions, and their link with the M/M/1 queue. We hope that our results may stimulate further investigation in jump-diffusion setting.

The organization of this article is as follows. Section 2 introduces Kou's model and some of its elementary properties. Section 3 demonstrates how to solve the PIDE to obtain the Laplace transform of the distribution of the occupation times. Section 4 applies the results of Section 3 to pricing various derivatives including step options, corridor options, and quantile options. Numerical results are given in Section 5. Appendices C.1-C.3 are included to deal with some technical issues arise in the body text and Appendix C.4 discusses the extension of our approach to the occupation times in a corridor.

4.2. Kou's Model and Its Basic Properties

Consider a market consisting of three securities only: a risk-free bond, a stock, and an occupation-time-related option contingent upon the stock. The bond offers investors risk free interest rate r. In Kou's double-exponential jump diffusion model (DEM), the stock price under the physical probability measure is governed by the following dynamic,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t}(V_i - 1)\right),$$

where μ and σ are constants, $\{W_t : t \geq 0\}$ is a standard Brownian motion, $\{N_t : t \geq 0\}$ is a Poisson process with arrival rate λ, and $\{V_i : i = 1, 2, \ldots\}$ is a sequence of independent identically distributed (i.i.d.) random variables. According to the model, the instantaneous asset return rate is subject to the effects of three factors: a deterministic trend μ, small fluctuations described by the Brownian motion, and large market shocks captured by the Poisson-arrival jump part. To make the model more mathematically tractable, we further assume that $Y_i := \log(V_i)$ follows a double exponential distribution, the probability density function (pdf) of which is

$$f_Y(y) = p\eta e^{-\eta y}\mathbf{1}_{\{y>0\}} + q\theta e^{\theta y}\mathbf{1}_{\{y<0\}},$$

where $\eta > 1$, $\theta > 0$, $p \geq 0$, $q \geq 0$, and $p + q = 1$. In other words, there are two types of jumps in the process: upward jumps with occurrence probability p

and average jump size $1/\eta$, and downward jumps with occurrence probability q and average jump size $1/\theta$. Both types of jumps are exponentially distributed. We also assume that $\{W_t : t \geq 0\}$, $\{N_t : t \geq 0\}$, and $\{Y_i : i = 1, 2, \ldots\}$ are independent. This model, proposed by Kou (2002) and Kou and Wang (2003,2004), is known as the double-exponential jump diffusion model in the financial engineering literature.

We need to work on a risk-neutral probability measure to calculate the option price. However, that measure is not unique because of the jump diffusion assumption. Following Lucas (1978) and Naik and Lee (1990), Kou (2002) showed that there is a particular probability measure P^* so that the equilibrium price of an option is given by the expectation under this measure of the discounted option payoff if we consider a representative agent economy with a HARA-type utility function. We point out that our argument will work under any equivalent martingale measure that preserves the model structure, particularly the exponential type of the jumps. Under this risk-neutral probability measure P^*, S_t follows another double-exponential jump diffusion model. More specifically, S_t obeys

$$\frac{dS_t}{S_t} = rdt + \sigma^* dW_t^* + d\left(\sum_{i=1}^{N_t^*}(V_i^* - 1)\right).$$

Under P^*, $\{W_t^* : t \geq 0\}$ is a standard Brownian motion, $\{N_t^* : t \geq 0\}$ is a Poisson process with arrival rate λ^*, and $\{Y_i^* := \log(V_i^*) : i = 1, 2, \cdots\}$ is also a sequence of i.i.d. double-exponentially distributed random variables, but with different parameters. The distribution of Y_i^* is given by

$$f_{Y^*}(y) = p^*\eta^* e^{-\eta^* y} 1_{\{y \geq 0\}} + q^*\theta^* e^{\theta^* y} 1_{\{y<0\}},$$

where the new set of parameters satisfy $\eta^* > 1$, $\theta^* > 0$, $p^* \geq 0$, $q^* \geq 0$, and $p^* + q^* = 1$. Moreover, $\{W_t : t \geq 0\}$, $\{N_t^* : t \geq 0\}$, and $\{Y_i^* : i = 1, 2, \ldots\}$ are also independent under P^*. As we are only interested in option pricing, the difference between the physical and risk-neutral probability measures plays no role. From now on we drop the superscript $*$, with the understanding that all of the processes and parameters in the subsequent discussions are under P^*.

Let X_t be the log-return of the asset, i.e., $X_t := \log(S_t/S_0)$. By Itô's formula (cf. Protter (2005), Theorem II. 32, p. 78), one can easily obtain

$$X_t = X_0 + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad X_0 = 0, \tag{4.1}$$

where ζ is the mean percentage jump size

$$\zeta = E[e^Y - 1] = \frac{p\eta}{\eta - 1} + \frac{q\theta}{\theta + 1} - 1.$$

An additional requirement $\eta > 1$ is needed to ensure that $E[V_1] = E[e^{Y_1}] < \infty$ and $E[e^{X_t}] < \infty$; this essentially means that the average upward jump cannot exceed 100%, which is quite reasonable in the reality of stock markets. For notational simplicity, denote $\mu := r - \frac{1}{2}\sigma^2 - \lambda\zeta$.

Mathematically, the double-exponential jump diffusion process (4.1) is a special Lévy processes because it has stationary and independent increments. Its Lévy exponent is defined as

$$G(x) := \frac{1}{t}\log E[\exp(xX_t)|X_0 = 0] = \frac{\sigma^2}{2}x^2 + \mu x + \lambda\left(\frac{p\eta}{\eta - x} + \frac{q\theta}{\theta + x} - 1\right). \tag{4.2}$$

Consider an algebraic equation $G(x) = r + a$ for any given $a > -r$. It is easy to show that all four roots of the equation are real numbers (cf. Lemma 2.1, Kou and Wang (2003)). Denote them by $\beta_{1,a}, \beta_{2,a}, -\gamma_{1,a}, -\gamma_{2,a}$. These roots satisfy

$$0 < \beta_{1,a} < \eta < \beta_{2,a} < \infty, \quad 0 < \gamma_{1,a} < \theta < \gamma_{2,a} < \infty.$$

We will use these roots frequently when we derive the distributions of occupation times of (4.1) in Section 3. Explicit formulae for the four roots are also presented in Appendix C.1 for reference.

Another important tool to establish the key results of the article is the infinitesimal generator of X_t. Note that X_t is a Markovian process and its infinitesimal generator is given by

$$\begin{aligned}(\mathcal{L}u)(x) &:= \lim_{t \downarrow 0} \frac{E[u(X_t)|X_0 = x] - u(x)}{t} \\ &= \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] f_Y(y)dy \end{aligned} \tag{4.3}$$

for any twice continuously differentiable function u.

4.3. Distribution of the Occupation Times

In the section, we will present the main results of the article the Laplace transforms of the distributions of occupation times of the double-exponential jump diffusion process $\{X_t\}$ given by (4.1). Once it is known, in principle we are able to calculate any option prices related with occupation times. Consider a constant barrier h and let τ_t denote the occupation time the log-return process $\{X_t\}$ spends below h until t, i.e.,

$$\tau_t \equiv \tau_t(h) := \int_0^t \mathbf{1}_{\{X_u<h\}}du. \tag{4.4}$$

An occupation time related option with maturity T usually has a payoff associated with τ_T and X_T. Suppose it is given by $f(\tau_T, X_T)$ for a general function f. Then pricing the option is equivalent to evaluating the following discounted expected payoff

$$e^{-rT}E[f(\tau_T, X_T)|X_0 = x] \tag{4.5}$$

under the risk neutral probability. This section is devoted to the calculation of the expectation.

Before jumping into mathematical details, we would like to motivate readers by the intuition behind the scenes first. If the joint probability distribution of (τ_t, X_t) is available explicitly for all t, the expectation in (4.5) is obtainable by numerically integrating

$$E[f(\tau_T, X_T)|X_0 = x] = \int_0^T \int_{-\infty}^{\infty} f(s,y)F(ds,dy;T,x),$$

where $F(ds,dy;T,x) = P\{\tau_T \in ds, X_T \in dy|X_0 = x\}$. So our pricing strategy starts from finding a closed-form expression for the distribution $F(ds,dy;T,x)$. The Laplace transform is a powerful tool in characterizing probability distributions. We can invert the transforms to recover distributions easily, either using transform tables when possible or resorting to other numerical methods. For any $\rho > 0$ and $\gamma \in \mathbb{R}$, define $V(\rho, \gamma; t, x)$ as the Laplace transform of $F(ds, dy; t, x)$

with respect to s and y, i.e.,

$$V(\rho,\gamma;t,x)=\int_0^t\int_{-\infty}^{\infty}e^{-\rho s+\gamma y}F(ds,dy;t,x)=E[e^{-\rho\tau_t+\gamma X_t}|X_0=x].$$

As mentioned in the introduction section, V can be determined by the solution of a PIDE for any fixed pair of ρ and γ. A heuristic approach is now presented to obtain the equation and a much more rigorous treatment is deferred to Theorem 4.2 below. Choose a short time duration δ. τ_t can be decomposed into two parts, the contribution of $\mathbf{1}_{\{X_u\le h\}}$ prior to δ and the contribution after δ:

$$\tau_t=\int_0^t\mathbf{1}_{\{X_u\le h\}}du=\int_0^\delta\mathbf{1}_{\{X_u\le h\}}du+\int_\delta^t\mathbf{1}_{\{X_u\le h\}}du.$$

By the Markovian property and the Lévy properties of $\{X_t\}$ we have,

$$E[e^{-\rho\int_\delta^t\mathbf{1}_{\{X_u\le h\}}du+\gamma X_t}|X_\delta=x]=E[e^{-\rho\int_0^{t-\delta}\mathbf{1}_{\{X_u\le h\}}du+\gamma X_{t-\delta}}|X_0=x]=V(t-\delta,x).$$

If applying the Taylor expansion on $V(t-\delta,X_\delta)$,

$$\begin{aligned}V(t,x)&=E[e^{-\rho\tau_t+\gamma X_t}|X_0=x]\\&=E[e^{-\rho\int_0^\delta\mathbf{1}_{\{X_u\le h\}}du}\cdot V(t-\delta,X_\delta)|X_0=x]\\&\approx E[e^{-\rho\int_0^\delta\mathbf{1}_{\{X_u\le h\}}du}\cdot(V(t,X_\delta)-\delta\frac{\partial V}{\partial t}(t,X_\delta))|X_0=x]+o(\delta).\end{aligned}\quad(4.6)$$

Note that $e^x\approx 1+x+o(x)$. Hence, (4.6) can be rewritten approximately as

$$\begin{aligned}V(t,x)-E[V(t,X_\delta)|X_0=x]=&-\delta E\left[\frac{\partial V}{\partial t}(t,X_\delta)|X_0=x\right]\\&-\rho E\left[\int_0^\delta\mathbf{1}_{\{X_u\le h\}}du\cdot V(t,X_\delta)\right]+o(\delta).\end{aligned}\quad(4.7)$$

Divide both sides of (4.7) by δ and take it to 0. The left-hand side converges to $-\mathcal{L}V(t,x)$, thanks to (4.3), the definition of the infinitesimal generator $\mathcal{L}$. The right-hand side converges to

$$-\frac{\partial V}{\partial t}(t,x)-\rho\mathbf{1}_{\{x\le h\}}V(t,x).$$

In addition, we also know one boundary condition for the function V such that $V(0,x)=e^{\gamma x}$.

In summary, V should solve the following PIDE with Cauchy boundary condition

$$\begin{cases} \frac{\partial V}{\partial t} + \rho \mathbf{1}_{\{x\leq h\}} V = \mathcal{L}V, & \text{for } t \in (0,T] \text{ and } x \in \mathbb{R}; \\ V(0,x) = e^{\gamma x}, & \text{for } x \in \mathbb{R}. \end{cases} \tag{4.8}$$

Theorem 4.1 rigorously establishes the relationship between the Laplace transform V and the solution to PIDE (4.8) via the martingale problem formulation.

Theorem 4.1. *Assume that $V : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a solution to PIDE (4.8), which is of class $C^{1,1}$ on $[0,T] \times \mathbb{R}$ and $C^{1,2}$ on $[0,T] \times \mathbb{R} \setminus \{h\}$. Moreover, the left and right second derivatives at h, $\partial^2 V(t,h-)/\partial x^2$ and $\partial^2 V(t,h+)/\partial x^2$, exist and V is bounded by*

$$\max_{0\leq t\leq T} |V(t,x)| \leq C_1 e^{C_2|x|}, \qquad x \in \mathbb{R}, \tag{4.9}$$

for constants $C_1 > 0$ and $0 < C_2 < \min\{\eta, \theta\}$. Then V admits the following stochastic representation:

$$V(t,x) = E[e^{-\rho\int_0^t \mathbf{1}_{\{X_s\leq h\}}ds} e^{\gamma X_t} | X_0 = x], \qquad 0 \leq t \leq T, \quad x \in \mathbb{R}. \tag{4.10}$$

And such a solution is unique.

Proof: Introduce $v(t,x) = V(T-t,x)$ for any $t \in [0,T]$. Following the arguments leading to the Feynman-Kac formula (cf. e.g. Theorem 4.4.2, Karatzas and Shreve (1991)), we attempt to apply Itô's formula on $v(t,X_t)\exp(-\rho\int_0^t \mathbf{1}_{\{X_s\leq h\}}ds)$ to calculate its expectation. However the irregularity of $v(t,\cdot)$ at barrier h forbids us from doing so directly. From Lemma C.1 in Appendix C.2 we know that there exist a series of functions $\{v_n(t,x) : n = 1,2,\cdots\}$ such that: (1). $v_n(t,x)$ converges to $v(t,x)$ as $n \to \infty$ for any $(t,x) \in [0,T] \times \mathbb{R}$; (2). $v_n(t,x)$ is of class $C^{1,2}$ in $[0,T) \times \mathbb{R}$ for any n; (3). $v_n(t,x) \equiv v(t,x)$ for any $(t,x) \in [0,T] \times (-\infty,h] \cup [h+1/n,\infty)$; and (4) for any $(t,x) \in [0,T] \times (h,h+1/n)$ and any $n \in \mathbb{N}$, $\max\{|v_n(t,x)|, |\partial v_n(t,x)/\partial t|, |\partial v_n(t,x)/\partial x|, |\partial^2 v_n(t,x)/\partial x^2|\} \leq M$, where M is a positive constant independent of t, x, and n.

Define

$$e_n(t,x) := \frac{\partial v_n}{\partial t}(t,x) - \rho 1_{\{x<h\}} v_n(t,x) + \mathcal{L} v_n(t,x).$$

According to the construction of $\{v_n(t,x)\}$ and (4.9), some routine algebra manipulation will yield that there exist positive constants M_1 and M_2, independent of n, t, and x, such that

$$|e_n(t,x)| \le \frac{M_1}{n} < +\infty, \quad \text{for } (t,x) \in [0,T] \times (-\infty, h] \cup [h + \frac{1}{n}, \infty) \tag{4.11}$$

and

$$|e_n(t,x)| \le M_2 < +\infty, \quad \text{for } (t,x) \in [0,T] \times (h, h + \frac{1}{n}). \tag{4.12}$$

Now we are able to apply Itô's formula to $v_n(t+\alpha, X_\alpha)e^{-\rho\int_0^\alpha 1_{\{X_s \le h\}}ds}$ because v_n is twice differentiable on the whole real line with respect to x. Let $T_m := \inf\{t \in [0,T] : |X_t| \ge m\}$ for any $m \in \mathbb{N}$. Itô's formula for jump diffusions (cf. Protter (2005), Theorem II. 32, p. 78) implies that

$$\begin{aligned}\mathrm{MG}^{(n,m)}(\alpha) :=& v_n(t + \alpha \wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m} 1_{\{X_s \le h\}}ds} \\ &- \int_{0+}^{\alpha\wedge T_m} e^{-\rho\int_0^s 1_{\{X_\xi \le h\}}d\xi} e_n(t+s-, X_{s-})ds\end{aligned}$$

is a local martingale for any fixed $t \in [0,T]$, m, $n \in \mathbb{N}$, and $0 \le \alpha \le T - t$. In other words, there should be a nondecreasing sequence of stopping times $\{\pi_k, k = 1, 2, \cdots\}$ such that $\mathrm{P}(\lim_{k\to+\infty} \pi_k = +\infty) = 1$ and $\{\mathrm{MG}^{(n,m)}(\alpha \wedge \pi_k) : \alpha \in [0, T-t]\}$ is a true martingale. It follows that for any $0 \le s < \alpha \le T - t$, we have

$$\mathrm{E}[\mathrm{MG}^{(n,m)}(\alpha \wedge \pi_k)|\mathcal{F}_s] = \mathrm{MG}^{(n,m)}(s \wedge \pi_k). \tag{4.13}$$

Fix $n \in \mathbb{N}$ and a sufficiently large m such that $m > |h| + 1$. We intend to show that $\{\mathrm{MG}^{(n,m)}(\alpha) : \alpha \in [0, T-t]\}$ is actually a true martingale. It suffices to show that $\sup_{\alpha\in[0,T-t]} |\mathrm{MG}^{(n,m)}(\alpha)|$ is integrable. Indeed, if this is true, we can apply the dominated convergence theorem on (4.13). Letting $k \to +\infty$ will yield

$$\mathrm{E}[\mathrm{MG}^{(n,m)}(\alpha)|\mathcal{F}_s] = \mathrm{MG}^{(n,m)}(s)$$

for any $0 \le s < \alpha \le T - t$; i.e., $\mathrm{MG}^{(n,m)}(\alpha)$ is a martingale.

Fortunately, the integrability of $\sup_{\alpha\in[0,T-t]} |\mathrm{MG}^{(n,m)}(\alpha)|$ is implied by the observation that the two terms in the expression of $\mathrm{MG}^{(n,m)}(\alpha)$ can be bounded as follows. For the second term, we can show that

$$\begin{aligned}
&\left|\int_{0+}^{\alpha\wedge T_m-} e^{-\rho\int_0^s 1_{\{X_\xi\le h\}}d\xi} e_n(t+s-, X_{s-})ds\right| \\
\le & \int_{0+}^{\alpha\wedge T_m} |e_n(t+s-, X_{s-})|\, I_{\{X_{s-}\in[h,h+\frac{1}{n}]\}}ds \\
&+ \int_{0+}^{\alpha\wedge T_m} |e_n(t+s-, X_{s-})|\, I_{\{X_{s-}\in[-m,h]\cup[h+\frac{1}{n},m]\}}ds \\
\le & M_2\int_{0+}^{\alpha\wedge T_m} I_{\{X_{s-}\in[h,h+\frac{1}{n}]\}}ds + \frac{M_1}{n}(\alpha\wedge T_m-) \le (M_2+\frac{M_1}{n})T, \qquad (4.14)
\end{aligned}$$

where the second inequality holds due to (4.11) and (4.12). For the first term, it is easy to see that $\exp(-\rho\int_0^{\alpha\wedge T_m} 1_{\{X_s\le h\}}ds)$ is always bounded by 1. Thus,

$$|v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m} 1_{\{X_s\le h\}}ds}| \le |v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})|. \qquad (4.15)$$

When $\alpha < T_m$, $v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m}) = v_n(t+\alpha, X_\alpha)$, which is bounded by $\max_{s\in[0,T],x\in[-m,m]} |v_n(s,x)|$ because $|X_\alpha| \le m$ by the definition of T_m. When $\alpha > T_m$, $v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m}) = v_n(t+T_m, X_{T_m})$. By (4.9), its absolute value is bounded by

$$|v(t+T_m, X_{T_m})| \le C_1 e^{C_2\max_{0\le s\le T}|X_s|} \le C_1 e^{C_2|x|+C_2|\mu|T} e^{C_2\sigma\max_{0\le s\le T}|W_s|} e^{C_2\sum_{i=1}^{N_T}|Y_i|}.$$

Now we intend to show $E[|v(t+T_m, X_{T_m})|] < +\infty$. On the one hand, some calculation illustrates that

$$E[e^{C_2\sum_{i=1}^{N_T}|Y_i|}] = \exp\left\{\lambda T\left(\frac{p\eta}{\eta - C_2} + \frac{q\theta}{\theta - C_2} - 1\right)\right\} < +\infty, \qquad (4.16)$$

thanks to $0 < C_2 < \min\{\eta, \theta\}$. On the other hand, we also have

$$E[e^{C_2\sigma\max_{0\le s\le T}|W_s|}] < +\infty. \qquad (4.17)$$

Actually, notice that $e^{C_2\sigma\max_{0\le s\le T}|W_s|} \le Z_+ Z_-$, where

$$Z_+ := e^{C_2\sigma\max_{0\le s\le T} W_s} \quad \text{and} \quad Z_- := e^{C_2\sigma\max_{0\le s\le T}(-W_s)}.$$

Since both $\max_{0\le s\le T} W_s$ and $\max_{0\le s\le T}(-W_s)$ have the same distribution as $|W_T|$, it follows that

$$\mathrm{E}Z_+^2 = \mathrm{E}Z_-^2 = \mathrm{E}e^{2C_2\sigma|W_T|} = 2e^{2C_2^2\sigma^2 T}\Phi(2C_2\sigma\sqrt{T}),$$

where $\Phi(x)$ is the cumulative normal distribution function. According to Minkowski's integral inequality, we can obtain that

$$\begin{aligned}\left[\mathrm{E}e^{C_2\sigma\max_{0\le s\le T}|W_s|}\right]^{1/2} &\le [\mathrm{E}(Z_+Z_-)]^{1/2} \le \frac{1}{2}\left[\mathrm{E}((Z_+ + Z_-)^2)\right]^{1/2}\\ &\le \frac{1}{2}\left[\mathrm{E}(Z_+^2)\right]^{1/2} + \frac{1}{2}\left[\mathrm{E}(Z_-^2)\right]^{1/2}\\ &= \left[\mathrm{E}(Z_+^2)\right]^{1/2} = \left[2e^{2C_2^2\sigma^2 T}\Phi(2C_2\sigma\sqrt{T})\right]^{1/2} < +\infty.\end{aligned}$$

Then (4.17) is proved.

From (4.16) and (4.17), we then have $E[|v(t+T_m, X_{T_m})|] < +\infty$. Therefore, the right-hand side of (4.15) will be bounded by

$$\begin{aligned}|v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})| &\le |v_n(t+\alpha, X_\alpha)1_{\{\alpha<T_m\}}|\\ &\quad + E[|v_n(t+T_m, X_{T_m})1_{\{\alpha\ge T_m\}}|\\ &\le \max_{s\in[0,T],x\in[-m,m]}|v_n(s,x)| + |v(t+T_m, X_{T_m})|.\end{aligned}$$

Note that the right-hand side of this inequality has nothing to do with α. It follows that $\sup_{\alpha\in[0,T-t]}|v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})|$ is integrable. Combining with (4.14), we have already shown that $\sup_{\alpha\in[0,T-t]}|\mathrm{MG}^{(n,m)}(\alpha)|$ is integrable. Consequently, $\{\mathrm{MG}^{(n,m)}(\alpha) : \alpha\in[0,T-t]\}$ is a true martingale.

The martingale property of $\mathrm{MG}^{(n,m)}(\alpha)$ implies that

$$E[\mathrm{MG}^{(n,m)}(\alpha)|X_0 = x] = E[\mathrm{MG}^{(n,m)}(0)|X_0 = x] = v_n(t,x).$$

In other words,

$$\begin{aligned}v_n(t,x) &= E\left[v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}1_{(X_s\le h)}ds}|X_0 = x\right]\\ &\quad - E\left[\int_{0+}^{\alpha\wedge T_m} e^{-\rho\int_0^s 1_{(X_\xi\le h)}d\xi}e_n(t+s-, X_{s-})ds|X_0 = x\right]. \quad (4.18)\end{aligned}$$

Let n go to $+\infty$ in (4.18). The left-hand side converges to v. Meanwhile, (4.14) and (4.15) allow us to apply the dominated convergence theorem on the right-hand side. Note that the second term on the right-hand side of (4.18) goes to zero. After taking the limit, (4.18) becomes

$$v(t,x) = E\left[v(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}1_{\{X_s<h\}}ds}|X_0=x\right]. \quad (4.19)$$

Note that the term inside the expectation of (4.19) is bounded by

$$\begin{aligned} & |v(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}1_{\{X_s<h\}}ds}| \\ \leq\ & |v(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})| \\ \leq\ & C_1 e^{C_2|x|+C_2|\mu|T}e^{C_2\sigma\max_{0\leq s\leq T}|W_s|}e^{C_2\sum_{i=1}^{N_T}|Y_i|} \end{aligned}$$

and the right-hand side can be shown to be integrable. We may be able to apply the dominated convergence theorem again on (4.19) to get the limit as m goes to $+\infty$. It follows that

$$v(t,x) = E[v(t+\alpha, X_\alpha)e^{-\rho\int_0^\alpha 1_{\{X_s\leq h\}}ds}].$$

Let $\alpha = T - t$ in the last equation and recall the definition of v. We have

$$\begin{aligned} V(T-t,x) = v(t,x) =& E[v(T, X_{T-t})e^{-\rho\int_0^{T-t}1_{\{X_s\leq h\}}ds}] \\ =& E[V(0, X_{T-t})e^{-\rho\int_0^{T-t}1_{\{X_s<h\}}ds}]. \end{aligned}$$

The right-hand side is equal to $E[e^{\gamma X_{T-t}-\rho\int_0^{T-t}1_{\{X_s\leq h\}}ds}]$. As t is arbitrary, the proof is completed. □

Equation (4.8) is a PIDE with a Cauchy boundary, noting that $\mathcal{L}$ involves both differential and integral operators. We intend to use the Laplace transform once again to convert it into an OIDE, which is much easier to solve. Consider the first equation in (4.8). Introduce the following Laplace transform on the (discounted) value of V:

$$u(x;a) = \int_0^{+\infty} e^{-at}\cdot e^{-rt}V(t,x)dt$$

for a sufficiently large positive a. Routine calculation shows that u must satisfy

$$
\begin{aligned}
(\rho \mathbf{1}_{\{x<h\}} + r + a)u(x;a) - e^{\gamma x} =& \mathcal{L}u(x;a) = \frac{1}{2}\sigma^2 u''(x;a) + \mu u'(x;a) \\
&+\lambda \int_{-\infty}^{\infty} [u(x+y;a) - u(x;a)] f_Y(y)dy. \qquad (4.20)
\end{aligned}
$$

Thus, we have successfully removed the partial derivative in (4.8). For a general jump density f_Y, it could still be very difficult to solve (4.20) for a closed-form solution. However, when f_Y is a double exponential density, (4.20) is solvable explicitly. We summarize the solution in the following theorem.

Theorem 4.2. *For any* $0 \le \gamma < \min\{\eta, \theta\}$, $\rho > 0$ *and*

$$
a + r > |\mu|\gamma + 2\sigma^2\gamma^2 + \lambda\left(\frac{p\eta}{\eta-\gamma} + \frac{q\theta}{\theta-\gamma} - 1\right), \qquad (4.21)
$$

the Laplace transform

$$
\begin{aligned}
u(x;\rho,\gamma,a,h) &= \int_0^\infty e^{-(a+r)t} E[e^{-\rho\tau_t + \gamma X_t} | X_0 = x] dt \\
&= \begin{cases} \omega_1 e^{\beta_{1,a+\rho}(x-h)} + \omega_2 e^{\beta_{2,a+\rho}(x-h)} - c_1 e^{\gamma(x-h)}, & x \le h; \\ -\nu_1 e^{-\gamma_{1,a}(x-h)} - \nu_2 e^{-\gamma_{2,a}(x-h)} - c_2 e^{\gamma(x-h)}, & x > h, \end{cases}
\end{aligned}
$$

where

$$
c_1 = \frac{e^{\gamma h}}{G(\gamma) - a - r - \rho}, \qquad c_2 = \frac{e^{\gamma h}}{G(\gamma) - a - r},
$$

and

$$
\omega_1 = \frac{(\beta_{2,a+\rho}-\gamma)(-\gamma_{1,a}-\gamma)(-\gamma_{2,a}-\gamma)(\eta-\beta_{1,a+\rho})(\theta+\beta_{1,a+\rho})}{(\beta_{2,a+\rho}-\beta_{1,a+\rho})(-\gamma_{1,a}-\beta_{1,a+\rho})(-\gamma_{2,a}-\beta_{1,a+\rho})(\eta-\gamma)(\theta+\gamma)} c_{12}, \qquad (4.22)
$$

$$
\omega_2 = \frac{(\beta_{1,a+\rho}-\gamma)(-\gamma_{1,a}-\gamma)(-\gamma_{2,a}-\gamma)(\eta-\beta_{2,a+\rho})(\theta+\beta_{2,a+\rho})}{(\beta_{1,a+\rho}-\beta_{2,a+\rho})(-\gamma_{1,a}-\beta_{2,a+\rho})(-\gamma_{2,a}-\beta_{2,a+\rho})(\eta-\gamma)(\theta+\gamma)} c_{12}, \qquad (4.23)
$$

$$
\nu_1 = \frac{(\beta_{1,a+\rho}-\gamma)(\beta_{2,a+\rho}-\gamma)(-\gamma_{2,a}-\gamma)(\eta+\gamma_{1,a})(\theta-\gamma_{1,a})}{(\beta_{1,a+\rho}+\gamma_{1,a})(\beta_{2,a+\rho}+\gamma_{1,a})(-\gamma_{2,a}+\gamma_{1,a})(\eta-\gamma)(\theta+\gamma)} c_{12}, \qquad (4.24)
$$

$$
\nu_2 = \frac{(\beta_{1,a+\rho}-\gamma)(\beta_{2,a+\rho}-\gamma)(-\gamma_{1,a}-\gamma)(\eta+\gamma_{2,a})(\theta-\gamma_{2,a})}{(\beta_{1,a+\rho}+\gamma_{2,a})(\beta_{2,a+\rho}+\gamma_{2,a})(-\gamma_{1,a}+\gamma_{2,a})(\eta-\gamma)(\theta+\gamma)} c_{12}. \qquad (4.25)
$$

with

$$
c_{12} = \frac{\rho e^{\gamma h}}{(G(\gamma) - a - r - \rho)(G(\gamma) - a - r)}. \qquad (4.26)
$$

Proof: Fix constants ρ, γ, a, and h. Define

$$u(x) = \begin{cases} u_1(x), & x < h \\ u_2(x), & x > h \end{cases} \tag{4.27}$$

Then the non-homogenous OIDE (4.20) can be rewritten as two separate equations in the regions $(-\infty, h)$ and $(h, +\infty)$. For $x < h$,

$$\frac{\sigma^2}{2}u_1''(x) + \mu u_1'(x) - (\lambda + a + r + \rho)u_1(x) + \lambda\Big[\int_{-\infty}^{0} u_1(x+y)q\theta e^{\theta y}dy + \int_0^{h-x} u_1(x+y)p\eta e^{-\eta y}dy + \int_{h-x}^{+\infty} u_2(x+y)p\eta e^{-\eta y}dy\Big] = -e^{\gamma x}. \tag{4.28}$$

and for $x > h$,

$$\frac{\sigma^2}{2}u_2''(x) + \mu u_2'(x) - (\lambda + a + r)u_2(x) + \lambda\Big[\int_{-\infty}^{h-x} u_1(x+y)q\theta e^{\theta y}dy + \int_{h-x}^{0} u_2(x+y)q\theta e^{\theta y}dy + \int_0^{+\infty} u_2(x+y)p\eta e^{-\eta y}dy\Big] = -e^{\gamma x}. \tag{4.29}$$

We claim that the solution $u_1(x)$ and $u_2(x)$ must be of the following form

$$\begin{cases} u_1(x) = \omega_1 e^{\beta_{1,a+\rho}(x-h)} + \omega_2 e^{\beta_{2,a+\rho}(x-h)} - c_1 e^{\gamma(x-h)}, & x \leq h; \\ u_2(x) = -\nu_1 e^{-\gamma_{1,a}(x-h)} - \nu_2 e^{-\gamma_{2,a}(x-h)} - c_2 e^{\gamma(x-h)}, & x > h, \end{cases} \tag{4.30}$$

where ω_1, ω_2, ν_1, ν_2, c_1, and c_2 are constants to be determined. Indeed, for equation (4.29), under a change of variable $z = x + y$, it is transformed further to

$$\frac{\sigma^2}{2}u_2''(x) = -\mu u_2'(x) + (\lambda + a + r)u_2(x) - \lambda e^{-\theta x}\int_{-\infty}^{h} u_1(z)q\theta e^{\theta z}dz - \lambda e^{-\theta x}\int_h^x u_2(z)q\theta e^{\theta z}dz - \lambda e^{\eta x}\int_x^{+\infty} u_2(z)p\eta e^{-\eta z}dz + e^{\gamma x}. \tag{4.31}$$

Our purpose is to remove the three integrals in (4.31), one by one, to reduce the OIDE to an ODE in order to make use of the theory of ODEs to solve the equation completely and to show the uniqueness of the solution at the same time. First, any solution to (4.31) must have the third-order derivative. This point is easily seen from the right-hand side of (4.31) because all terms are differentiable and so is $u''(x)$. Multiplying both sides of (4.31) by $e^{\theta x}$,

$$\frac{\sigma^2}{2}e^{\theta x}u_2''(x) = -\mu e^{\theta x}u_2'(x) + (\lambda + a + r)e^{\theta x}u_2(x) - \lambda\int_{-\infty}^{h} u_1(z)q\theta e^{\theta z}dz$$
$$- \lambda\int_{h}^{x} u_2(z)q\theta e^{\theta z}dz - \lambda e^{(\theta+\eta)x}\int_{x}^{+\infty} u_2(z)p\eta e^{-\eta z}dz + e^{(\theta+\gamma)x}.$$

Take differentiation on both sides of this equation to remove the first integral. Dividing the resulting OIDE by $e^{-\theta x}$ yields

$$\frac{\sigma^2}{2}u_2'''(x) = -(\frac{\sigma^2}{2}\theta + \mu)u_2''(x) - (\mu\theta - \lambda - a - r)u_2'(x)$$
$$+ [(\lambda + a)\theta - \lambda q\theta + \lambda p\eta]u_2(x) + \lambda(\eta + \theta)e^{\eta x}\int_{x}^{+\infty} u_2(z)p\eta e^{-\eta z}dz + (\theta + \gamma)e^{\gamma x}. \tag{4.32}$$

From (4.32), u should also be fourth-order differentiable. Hence, we can take a similar step to remove the integral in (4.32) to obtain a non-homogeneous ODE with constant coefficients as follows:

$$\frac{\sigma^2}{2}u_2^{(4)}(x) + [-\frac{\sigma^2}{2}(\eta - \theta) + \mu]u_2'''(x) + [-\frac{\sigma^2}{2}\eta\theta - \mu(\eta - \theta) - \lambda - a - r]u_2''(x)$$
$$+ [(\eta - \theta)(\lambda + a + r) - \mu\eta\theta + \lambda q\theta - \lambda p\eta]u_2'(x) + a\eta\theta u_2(x) = (\eta - \gamma)(\theta + \gamma)e^{\gamma x}. \tag{4.33}$$

On one hand, it is easy to see that $c_2e^{\gamma x}$ is a particular solution to the ODE (4.33) for a constant c_2. On the other hand, the characteristic equation of the corresponding homogeneous ODE turns to be

$$(G(y) - a - r)(y + \theta)(y - \eta) = 0,$$

which has four real roots as mentioned in Section 2. Therefore, any solution to (4.33) can be expressed as

$$u_2(x) = \bar{\nu}_1e^{\beta_{1,a}(x-h)} + \bar{\nu}_2e^{\beta_{2,a}(x-h)} - \nu_1e^{-\gamma_{1,a}(x-h)} - \nu_2e^{-\gamma_{2,a}(x-h)} - c_2e^{\gamma(x-h)},$$

for any $x > h$,with $\bar{\nu}_1$, $\bar{\nu}_2$, ν_1, ν_2, and c_2 undetermined. Furthermore, we can argue that the first two coefficients $\bar{\nu}_1$ and $\bar{\nu}_2$ should be 0. In fact, we know that

$$\frac{u_2(x)}{e^{\gamma x}} = \int_0^{\infty} e^{-(a+r)t}E[e^{-\rho\tau_t+\gamma(X_t-x)}|X_0 = x]dt$$
$$= \int_0^{\infty} e^{-(a+r)t}E[e^{-\rho\tau_t+\gamma X_t}|X_0 = 0]dt,$$

where the last equality is because of the Lévy property of X. The right-hand side of the above equality is less than

$$\int_0^\infty e^{-(a+r-G(\gamma))t}dt = \frac{1}{a+r-G(\gamma)} < +\infty,$$

because $E[\exp(-\rho\tau_t + \gamma X_t)|X_0 = 0] < E[\exp(\gamma X_t)|X_0 = 0] = \exp(G(\gamma))$. Thus, $\lim_{x\to+\infty} u_2(x)/e^{\gamma x} < +\infty$. Note $\beta_{2,a} > \beta_{1,a} > \gamma$, which implies $\tilde{\nu}_1$ and $\tilde{\nu}_2$ must be 0. Consequently, any solution to the OIDE in (4.29) can be expressed as

$$u_2(x) = -\nu_1 e^{-\gamma_{1,a}(x-h)} - \nu_2 e^{-\gamma_{2,a}(x-h)} - c_2 e^{\gamma(x-h)}, \quad \text{for } x > h,$$

with c_2, ν_1, and ν_2 to be determined. Similarly, we also can show any solution to the first OIDE in (4.28) is expressed as

$$u_1(x) = \omega_1 e^{\beta_{1,a+\rho}(x-h)} + \omega_2 e^{\beta_{2,a+\rho}(x-h)} - c_1 e^{\gamma(x-h)}, \quad \text{for } x < h,$$

with c_1, ω_1, and ω_2 to be determined.

Now we need six equations to determine these coefficients. Substituting $u_1(x), u_2(x)$ into (4.28-4.29) yields that for any $x < h$,

$$\begin{aligned}&\left[c_1 e^{-\gamma h}(G(\gamma) - a - r - \rho) - 1\right] e^{\gamma x}\\&+ \lambda p \eta \left[\frac{\omega_1}{\eta - \beta_{1,a+\rho}} + \frac{\omega_2}{\eta - \beta_{2,a+\rho}} + \frac{\nu_1}{\eta + \gamma_{1,a}} + \frac{\nu_2}{\eta + \gamma_{2,a}} - \frac{c_1 - c_2}{\eta - \gamma}\right] e^{\eta(x-h)} = 0.\end{aligned}$$

and for any $x > h$,

$$\begin{aligned}&-\left[c_2 e^{-\gamma h}(G(\gamma) - a - r) - 1\right] e^{\gamma x}\\&+ \lambda q \theta \left[\frac{\omega_1}{\theta + \beta_{1,a+\rho}} + \frac{\omega_2}{\theta + \beta_{2,a+\rho}} + \frac{\nu_1}{\theta - \gamma_{1,a}} + \frac{\nu_2}{\theta - \gamma_{2,a}} - \frac{c_1 - c_2}{\theta + \gamma}\right] e^{-\theta(x-h)} = 0.\end{aligned}$$

Therefore, u is a solution if and only if the coefficients ω_1, ω_2, ν_1, ν_2, c_1, and c_2 satisfy the following four equations:

$$\begin{aligned}&c_1(G(\gamma) - a - r - \rho) = e^{\gamma h}\\&\frac{\omega_1}{\eta - \beta_{1,a+\rho}} + \frac{\omega_2}{\eta - \beta_{2,a+\rho}} + \frac{\nu_1}{\eta + \gamma_{1,a}} + \frac{\nu_2}{\eta + \gamma_{2,a}} = \frac{c_1 - c_2}{\eta - \gamma},\\&c_2(G(\gamma) - a - r) = e^{\gamma h},\\&\frac{\omega_1}{\theta + \beta_{1,a+\rho}} + \frac{\omega_2}{\theta + \beta_{2,a+\rho}} + \frac{\nu_1}{\theta - \gamma_{1,a}} + \frac{\nu_2}{\theta - \gamma_{2,a}} = \frac{c_1 - c_2}{\theta + \gamma}.\end{aligned}$$

In addition, we can also obtain another two equations from the fact that $u(x)$ is continuously differentiable at barrier h:

$$\omega_1 + \omega_2 - c_1 = -\nu_1 - \nu_2 - c_2,$$
$$\beta_{1,a+\rho}\omega_1 + \beta_{2,a+\rho}\omega_2 - c_1\gamma = \gamma_{1,a}\nu_1 + \gamma_{2,a}\nu_2 - c_2\gamma.$$

All of these equations are linear with respect to the undetermined parameters. To solve them, first we can easily obtain that

$$c_1 = \frac{e^{\gamma h}}{G(\gamma) - a - r - \rho} \quad \text{and} \quad c_2 = \frac{e^{\gamma h}}{G(\gamma) - a - r}.$$

Substituting these two into the above linear system will reduce it further to

$$\mathbf{A}(\rho)\mathbf{c}(\rho, \gamma) = \mathbf{J}(\rho, \gamma), \tag{4.34}$$

with $\mathbf{c}(\rho, \gamma) = (\omega_1, \omega_2, \nu_1, \nu_2)^T$, $\mathbf{J}(\rho, \gamma) = c_{12}\left(1, \gamma, \frac{1}{\eta - \gamma}, \frac{1}{\theta + \gamma}\right)^T$, and

$$\mathbf{A}(\rho) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_{1,a+\rho} & \beta_{2,a+\rho} & -\gamma_{1,a} & -\gamma_{2,a} \\ \frac{1}{\eta - \beta_{1,a+\rho}} & \frac{1}{\eta - \beta_{2,a+\rho}} & \frac{1}{\eta + \gamma_{1,a}} & \frac{1}{\eta + \gamma_{2,a}} \\ \frac{1}{\theta + \beta_{1,a+\rho}} & \frac{1}{\theta + \beta_{2,a+\rho}} & \frac{1}{\theta - \gamma_{1,a}} & \frac{1}{\theta - \gamma_{2,a}} \end{bmatrix}$$

where $c_{12} = c_1 - c_2 = \frac{\rho e^{\gamma h}}{(G(\gamma) - a - r - \rho)(G(\gamma) - a - r)}$. Appendix C.3 shows that the matrix $\mathbf{A}(\rho)$ is non-singular and the coefficients defined by (4.22)-(4.25) solve the linear equations (4.34). □

We also can extend the above approach to derive the distribution of occupation times the process spends within two barriers. A minor technical gap remains. All detailed discussion is included in Appendix C.4.

Remark 4.3. *The key step in the whole proof lies in (4.31). The assumption of exponential-type jump distributions in Kou's model allows us to differentiate the OIDE in order to transform the OIDE to an ODE. It seems that our method does not apply for any jump distribution other than exponential-type distributions. For instance, this transformation will not be workable for Merton's jump diffusion model.*

Remark 4.4. *Cai and Kou (2008) studied a similar OIDE under a more general hyper-exponential jump diffusion model as follows.*

$$\begin{cases} (\mathcal{L}u)(x) - (a + r)u(x) = 0, & x < x_0 \\ \bar{u}(x) = g(x), & x \geq x_0, \end{cases} \tag{4.35}$$

where $a > 0$ and $g(x)$ is a known function. By transforming (4.35) into a homogeneous linear ODE with constant coefficients, Cai and Kou managed to show that the solution to (4.35) must be of the form

$$u(x) = \bar{\omega}_1 e^{\beta_{1,a}(x-x_0)} + \omega_2 e^{\beta_{2,a}(x-x_0)} + \tilde{\omega}_1 e^{-\gamma_{1,a}(x-x_0)} + \tilde{\omega}_2 e^{-\gamma_{2,a}(x-x_0)}.$$

Despite the similarity, (4.8) *is much more complicated because it is "non-homogeneous" and furthermore it contains two OIDEs in two disjoint regions that are intertwined together due to the integral parts. We are still able to reduce it down to a linear ODE, applying the same technique as in Cai and Kou (2008) after some modification.*

Remark 4.5. *Note that several structured products issued on the real financial market have a payoff written on the occupation time, but with an interest rate or a spread of swap rates with different maturities as underlying. These underlying processes are usually of mean reversion structure. However, our approach would be hard to be extended to the mean reversion jump diffusion cases. The primary technical barrier lies in the fact that the corresponding OIDE, in which the coefficient of the first derivative is not a constant but a linear function of state variable, is difficult to solve explicitly.*

4.4. Pricing Occupation-Time-Related Options

In this section, several examples of occupation-time-related options accumulated in the literature are considered, including the step options suggested by Linetsky (1998), the corridor options studied by Fusai (2000), and the quantile options proposed by Miura (1992). Thanks to Theorem 4.2 and the special structures of these options, we can obtain closed-form expressions for the option prices in terms of their Laplace transforms and then make it possible to suggest hedging strategies accordingly. Furthermore, we are also able to calculate the price sensitivities very easily from the Laplace transforms, which is convenient for risk management on the options. This section uses delta as an example. The calculation of other greeks is similar and thus omitted due to the space limitation.

¿From now on, we assume that L is the constant barrier to define the occupation times. Define $h = \log(L/S_0)$ as the associated barrier for the log-return process $\{X_t\}$.

4.4.1. Pricing Step Options

As mentioned in the introduction, Linetsky (1999) introduced the step option to overcome the hedging problem inherent in standard barrier option around the barrier. For down-and-out step call options, the payoff at maturity is defined as the payoff of a standard European call option discounted at a rate that depends on the amount of time spent by the underlying asset below a pre-specified barrier. We can classify these options into proportional step options, simple step options, and delayed barrier options according to different discounting schemes used.

Proportional (Geometric) Step Options

In this section, we focus on pricing a proportional step call option, which has the payoff

$$e^{-\rho\tau_T(h)}(S_0 e^{X_T} - K)^+,$$

where ρ is the non-negative knock-out rate, S_0 is the initial underlying asset price, X_T is the log-return value of the underlying asset price at maturity T, and $\tau_T(h)$ is the occupation time as defined in (4.4). The pricing method also applies to proportional step put options.

In some sense the proportional step option can be regarded as an extension of the standard barrier option and the vanilla European option. With a finite positive knock-out rate ρ, it is obvious that

$$\mathbf{1}_{\{\varsigma_h > T\}}(S_0 e^{X_T} - K)^+ \leq e^{-\rho\tau_T(h)}(S_0 e^{X_T} - K)^+ \leq (S_0 e^{X_T} - K)^+, \qquad (4.36)$$

where ς_h is defined as the first passage time of $\{X_t\}$ to the barrier h, i.e., $\varsigma_h = \inf\{t \geq 0 : X_t \leq h\}$. The payoff of the proportional step call option is sandwiched by the payoff of the vanilla European call on the right- hand side of (4.36) and the payoff of the down-and-out barrier call on the left-hand side of (4.36). When $\rho = 0$, the payoff of the step option coincides with that of the vanilla call. As ρ approaches $+\infty$, it tends to the payoff of the down-and-out barrier call.

Additionally, (4.36) also reveals one advantage of the step option over the standard barrier option. The down-and-out barrier call eliminates the payoff to the investor immediately if the underlying process $\{X_t\}$ touches the barrier h at or before T, i.e., $\mathbf{1}_{\{\varsigma_h > T\}} = 0$. However, the payoff of the step option does not disappear when X crosses the boundary. Investors still receive a portion of the original payoff, discounted depending upon the length of the period that $\{X_t\}$ spends below h. This mollifies the discontinuity of the barrier options around h, which eases the difficulty of risk management on barrier options to some degree. We have discussed it briefly in the introduction section and Linetsky (1999) has offered more details.

Under the risk-neutral probability measure, the proportional step call option price is

$$C_1(K,T) = e^{-rT}E[e^{-\rho\tau_T(h)}(S_0 e^{X_T} - K)^+ | S_0].$$

Make a change of variable $\kappa = -\log K$ for the convenience of later applying Laplace transforms. Then, we have

$$C_1(\kappa,T) = e^{-rT}E[e^{-\rho\tau_T(h)}(S_0 e^{X_T} - e^{-\kappa})^+ | S_0].$$

Taking double Laplace transforms on the price function $C_1(\kappa, T)$ with respect to κ and T, respectively, and applying the Fubini theorem to interchange the order of the expectation and the integral with respect to κ, we obtain

$$
\begin{aligned}
g_1(\varphi, a) &:= \int_0^\infty dT \int_{-\infty}^\infty e^{-\varphi\kappa - aT} C_1(\kappa, T) d\kappa \\
&= \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \int_0^\infty e^{-aT} E[e^{-\int_0^T k(X_s)ds + (\varphi+1)X_T}] dT.
\end{aligned}
\tag{4.37}
$$

Using Theorem 4.2, we can derive an explicit closed-form expression for the double Laplace transform above.

Theorem 4.6. *With the initial underlying asset price S_0 and barrier L, assuming that (4.21) is satisfied, then for any $a > 0$ and $0 < \varphi < \min\{\eta, \theta\} - 1$, the double Laplace transform of the proportional step call option price $C_1(\kappa, T)$ is*

$$
g_1(\varphi, a) = \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} u(0; \rho, \varphi + 1, a, \log(L/S_0)),
$$

where $u(x; \rho, \gamma, a, h)$ is given by Theorem 4.2.

The delta of an option is defined as the derivative of the option price with respect to the current underlying price S_0. Taking differentiation under the integral (4.37), we can easily see that

$$
\frac{\partial}{\partial S_0} g_1(\varphi, a) = \int_0^\infty dT \int_{-\infty}^\infty e^{-\varphi\kappa - aT} \frac{\partial}{\partial S_0} C_1(\kappa, T) d\kappa.
$$

Accordingly, the transform of the delta is just the derivative of the transform of the price function with respect to S_0. Hence, the delta of the step option is also obtainable through the Laplace transform.

Simple (Arithmetic) Step Options and Delayed Barrier Options

In addition to the proportional step options, Linetsky (1998) also discussed two other kinds of step options, simple (arithmetic) step options and delayed barrier options. Laplace transform techniques can also lead to analytical solutions to pricing problems of these two step options.

The simple step option uses a discounting scheme that is different from what is used for the proportional step option. The payoff of a simple step call option is defined as

$$
(1 - \tau_T(h)/\vartheta)^+ \cdot (S_T - K)^+.
$$

With a positive knock-out rate $1/\vartheta$, investors will lose the option payoff gradually until the occupation time accumulates up to ϑ, when they will lose all of the value. This is a major difference from the proportional step option, where investors will never lose the entire option value.

It is simple to convert the pricing problem of simple step options into that of the proportional step options we discussed in Section 4.1.1 via Laplace transform. Note that for any $\rho > 0$,

$$\begin{aligned}
&\int_0^\infty \vartheta C_2(K,T,\vartheta)e^{-\rho\vartheta}d\vartheta \\
= \; & e^{-rT}E[\int_0^\infty \vartheta(1-\tau_T(h)/\vartheta)^+ e^{-\rho\vartheta}d\vartheta \cdot (S_0e^{X_T}-K)^+|S_0] \\
= \; & \frac{e^{-rT}}{\rho^2}E[e^{-\rho\tau_T(h)}(S_0e^{X_T}-K)^+] = \frac{1}{\rho^2}C_1(\rho;K,T).
\end{aligned}$$

The right-hand side of the formula above is calculable via double Laplace inversion. Thus, we can essentially apply triple Laplace inversion to obtain C_2. The numerical experiment in Section 5 indicates that the computation is still very efficient.

The delayed barrier option poses an alternative discount factor $\mathbf{1}_{\{\tau_T(h)<\vartheta\}}$ on the payoff of the vanilla European call. Hence, the option value is wiped out completely if and only if $\tau_T(h) > \vartheta$. We can also convert the associated pricing problem into that of a proportional option formulation by taking a Laplace transform with respect to ϑ.

$$\begin{aligned}
\int_0^\infty e^{-\rho\vartheta}C_3(K,T,\vartheta)d\vartheta &= e^{-rT}E[\int_0^\infty \mathbf{1}_{\{t<\vartheta\}}e^{-\rho\vartheta}d\vartheta \cdot (S_0e^{X_T}-K)^+|S_0] \\
&= \frac{1}{\rho}C_1(\rho;K,T).
\end{aligned}$$

Hence, triple Laplace inversion can also be applied to price delayed barrier options numerically.

4.4.2. Pricing Corridor Options

The corridor option is another example of occupation-time-related options. It pays an amount at the maturity, dependent upon the time spent by a reference market variable below (or above) a given barrier or inside an interval. The former option, i.e., the corridor option with single barrier, is usually referred to as the hurdle option. In this subsection, we will concentrate on hurdle options only. Corridor options with double barriers can be priced similarly. For details, see Appendix C.4. It is worth mentioning that Fusai (2000) studied the pricing of corridor options with double barriers under the GBM model. His approach relied on the special properties of Brownian motion.

A corridor option with single barrier has the payoff $\max\{\tau_T(h) - K, 0\}$ for a given strike $K < T$, and its price at time 0 is thus given by

$$Cor(K,T) = e^{-rT} E[\max\{\tau_T(h) - K, 0\}].$$

We need the expectation of $\tau_T(h)$ to proceed the price calculation. A nice property of the Laplace transform of a probability distribution is that we can obtain any order moments of the distribution through the derivatives of its Laplace transform at zero. Keeping this property in mind and using the notations in Theorem 4.2, we have

$$\begin{aligned}\int_0^\infty e^{-(a+r)T} E[\tau_T(h)]dT &= \int_0^\infty e^{-(a+r)T} \frac{\partial}{\partial\rho}\bigg|_{\rho=0} E[e^{-\rho\tau_T(h)+\gamma X_t}|X_0 = x]dT \\ &= \frac{\partial u}{\partial\rho}(x; 0, \gamma, a, h). \end{aligned} \tag{4.38}$$

Then, taking a double Laplace transform of $Cor(K,T)$ with respect to K and T, i.e.,

$$g_{cor}(\varphi, a) = \int_0^\infty \int_0^\infty e^{-\varphi K - aT} Cor(K,T)dKdT,$$

we can obtain Theorem 4.7 as follows:

Theorem 4.7. *For any φ and $a > 0$, we have*

$$\begin{aligned} g_{cor}(\varphi, a) &= -\frac{1}{\varphi}\frac{\partial u}{\partial\rho}(0; 0, 0, a, \log(L/S_0)) \\ &\quad + \frac{1}{\varphi^2} u(0; \varphi, 0, a, \log(L/S_0)) - \frac{1}{(a+r)\varphi^2}. \end{aligned} \tag{4.39}$$

Proof: Applying the Fubini theorem to interchange the order of expectation and integrals in g_{cor}, we have

$$\begin{aligned} g_{cor}(\varphi, a) &= \int_0^\infty e^{-(a+r)T} dT E[\int_0^\infty e^{-\varphi K} \max\{\tau_T(h) - K, 0\} dK] \\ &= -\frac{1}{\varphi}\int_0^\infty e^{-(a+r)T} E[\tau_T(h)] dT \\ &\quad +\frac{1}{\varphi^2}\int_0^\infty e^{-(a+r)T} E[e^{-\varphi\tau_T(h)}] dT - \frac{1}{(a+r)\varphi^2}. \end{aligned}$$

The integral in the second term on the right-hand side above can be represented by $u(0; \varphi, 0, a, h)$. In addition, we know from (4.38) that the integral in the first term is $\partial u(0; \rho, 0, a, h)/\partial\rho$. The theorem is proved. □

What is interesting here is that we can also obtain a closed-form expression for $\partial u/\partial\rho$, which is convenient when calculating g_{cor}.

Proposition 4.8. *For any $a > 0$, we have*

$$\frac{\partial u}{\partial\rho}(0; 0, 0, a, \log(L/S_0)) = \begin{cases} \widetilde{\omega}_1 \left(\frac{S_0}{L}\right)^{\beta_{1,a}} + \widetilde{\omega}_2 \left(\frac{S_0}{L}\right)^{\beta_{2,a}} - \frac{1}{(a+r)^2}, & S_0 \le L; \\ -\widetilde{\nu}_1 \left(\frac{L}{S_0}\right)^{\gamma_{1,a}} - \widetilde{\nu}_2 \left(\frac{L}{S_0}\right)^{\gamma_{2,a}}, & S_0 > L, \end{cases} \quad (4.40)$$

where

$$\begin{aligned} \widetilde{\omega}_1 &= \frac{\beta_{2,a}\gamma_{1,a}\gamma_{2,a}}{\eta\theta(a+r)^2} \frac{(\beta_{1,a} - \eta)(\beta_{1,a} + \theta)}{(\beta_{1,a} - \beta_{2,a})(\beta_{1,a} + \gamma_{1,a})(\beta_{1,a} + \gamma_{2,a})}, \\ \widetilde{\omega}_2 &= \frac{\beta_{1,a}\gamma_{1,a}\gamma_{2,a}}{\eta\theta(a+r)^2} \frac{(\beta_{2,a} - \eta)(\beta_{2,a} + \theta)}{(\beta_{2,a} - \beta_{1,a})(\beta_{2,a} + \gamma_{1,a})(\beta_{2,a} + \gamma_{2,a})}, \\ \widetilde{\nu}_1 &= \frac{\beta_{1,a}\beta_{2,a}\gamma_{2,a}}{\eta\theta(a+r)^2} \frac{(\gamma_{1,a} + \eta)(\gamma_{1,a} - \theta)}{(\gamma_{1,a} + \beta_{1,a})(\gamma_{1,a} + \beta_{2,a})(\gamma_{1,a} - \gamma_{2,a})}, \\ \widetilde{\nu}_2 &= \frac{\beta_{1,a}\beta_{2,a}\gamma_{1,a}}{\eta\theta(a+r)^2} \frac{(\gamma_{2,a} + \eta)(\gamma_{2,a} - \theta)}{(\gamma_{2,a} + \beta_{1,a})(\gamma_{2,a} + \beta_{2,a})(\gamma_{2,a} - \gamma_{1,a})}. \end{aligned}$$

Proof: According to Theorem 4.2, u is a piecewise-defined function. To emphasize their dependence on ρ and γ, we rewrite c_1, c_2, ω_1, ω_2, ν_1, and ν_2 as $c_1(\rho,\gamma)$, $c_2(\gamma)$, $\omega_1(\rho,\gamma)$, $\omega_2(\rho,\gamma)$, $\nu_1(\rho,\gamma)$, and $\nu_2(\rho,\gamma)$, respectively. When $x \le h \equiv \log(L/S_0)$, by the product rule of function derivative, we have

$$\begin{aligned} \frac{\partial u}{\partial\rho} &= e^{\beta_{1,a+\rho}(x-h)} \left(\frac{\partial\omega_1(\rho,\gamma)}{\partial\rho} + \omega_1(\rho,\gamma)\frac{\partial\beta_{1,a+\rho}}{\partial\rho}\right) \\ &\quad + e^{\beta_{2,a+\rho}(x-h)} \left(\frac{\partial\omega_2(\rho,\gamma)}{\partial\rho} + \omega_2(\rho,\gamma)\frac{\partial\beta_{2,a+\rho}}{\partial\rho}\right) - \frac{\partial c_1(\rho,\gamma)}{\partial\rho} e^{\gamma(x-h)} \quad (4.41) \end{aligned}$$

Letting $x = \rho = \gamma = 0$ and noting that $\omega_1(0,\gamma) = \omega_2(0,\gamma) = 0$ (cf. (4.22), (4.23), and (4.26)), we obtain that when $S_0 \le L$,

$$\frac{\partial u}{\partial \rho}\Big|_{\rho=0} = \left(\frac{S_0}{L}\right)^{\beta_{1,a}} \frac{\partial \omega_1(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)} + \left(\frac{S_0}{L}\right)^{\beta_{2,a}} \frac{\partial \omega_2(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)} - \frac{\partial c_1(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)}. \tag{4.42}$$

Similarly, when when $S_0 > L$, we have

$$\frac{\partial u}{\partial \rho}\Big|_{\rho=0} = -\left(\frac{L}{S_0}\right)^{\gamma_{1,a}} \frac{\partial \nu_1(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)} - \left(\frac{L}{S_0}\right)^{\gamma_{1,a}} \frac{\partial \nu_2(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)}. \tag{4.43}$$

Note $\mathbf{c}(\rho,\gamma) = (\omega_1(\rho,\gamma), \omega_2(\rho,\gamma), \nu_1(\rho,\gamma), \nu_2(\rho,\gamma))$ is the solution of the linear system (4.34), i.e., $A(\rho)c(\rho,\gamma) = J(\rho,\gamma)$. Then,

$$\frac{\partial \mathbf{A}(\rho)}{\partial \rho}\Big|_{\rho=0} \mathbf{c}(0,0) + \mathbf{A}(0)\frac{\partial \mathbf{c}(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)} = \frac{\partial \mathbf{J}(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)}.$$

The fact that $\mathbf{c}(0,0) = \mathbf{0}$ implies

$$\mathbf{A}(0)\frac{\partial \mathbf{c}(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)} = \frac{\partial \mathbf{J}(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)}.$$

In other words, we can obtain $\frac{\partial \mathbf{c}(\rho,\gamma)}{\partial \rho}\Big|_{(\rho,\gamma)=(0,0)}$, i.e., the partial derivatives of $(\omega_1(\rho,\gamma), \omega_2(\rho,\gamma), \nu_1(\rho,\gamma), \nu_2(\rho,\gamma))$ at $(0,0)$ by solving the above equations. Then substituting the result back into (4.42) and (4.43) yields (4.40) immediately, which completes the proof. □

4.4.3. Pricing Quantile Options

Miura (1992) introduced α-quantile options as an extension of lookback options. Its payoff depends on the α-quantile of the underlying asset price process, which is defined as

$$q(\alpha, T) = \inf\{h : \tau_T(h) > \alpha T\}, \qquad \text{for any } \alpha \in [0,1].$$

Following Dassios (1995), we will investigate the pricing of the fixed-strike α-quantile call option with payoff $(S_0 e^{\gamma q(\alpha,T)} - K)^+$. It is worth mentioning that when $\alpha = 0$ and $\gamma = 1$, $q(\alpha, T)$ is the running maximum of $\{X_t\}$ over $[0, T]$ so that the quantile option is reduced to the lookback option.

For any $0 \le v \le T$, let

$$Qua(v, T) = e^{-rT} E[(S_0 e^{\gamma q(v/T, T)} - K)^+]$$

be the (v/T)-quantile option price. A key observation that

$$\{\tau_T(h) < v\} \equiv \{q(v/T, T) > h\} \tag{4.44}$$

links the quantile options with occupation times. The Laplace transform of $\tau_T(h)$ helps us again to establish a theorem as follows on the closed-form double Laplace transform of the quantile option price. Inverting the transform can then produce numerical prices.

Theorem 4.9. *Assume that $0 < \gamma < \min\{\eta, \theta\}$. For any $a > 0$ and $\rho > 0$ such that $G(\gamma) < a + \rho + r$, the double Laplace transform of $Qua(v, T)$ with respect to v and T is given by*

$$\begin{aligned} &g_{Qua}(\rho, a) \\ &= \int_0^\infty \int_0^\infty e^{-\rho v} e^{-aT} Qua(v, T) \mathbf{1}_{\{v<T\}} dT dv \\ &= \begin{cases} \frac{\gamma K}{\rho} \frac{\omega_1}{\beta_{1,a+\rho} - \gamma} (S_0/K)^{\frac{\beta_{1,a+\rho}}{\gamma}} + \frac{\gamma K}{\rho} \frac{\omega_2}{\beta_{2,a+\rho} - \gamma} (S_0/K)^{\frac{\beta_{2,a+\rho}}{\gamma}}, & \textit{if } K \ge S_0; \\ \\ \frac{\gamma S_0}{\rho} \frac{\omega_1}{\beta_{1,a+\rho} - \gamma} + \frac{\gamma S_0}{\rho} \frac{\omega_2}{\beta_{2,a+\rho} - \gamma} - \frac{\gamma S_0}{\rho} \frac{\nu_1}{\gamma_{1,a} + \gamma} (1 - (K/S_0)^{\frac{\gamma_{1,a} + \gamma}{\gamma}}) \\ \quad - \frac{\gamma S_0}{\rho} \frac{\nu_2}{\gamma_{2,a} + \gamma} (1 - (K/S_0)^{\frac{\gamma_{2,a} + \gamma}{\gamma}}) + \frac{S_0 - K}{(a + r)(a + r + \rho)}, & \textit{if } K < S_0, \end{cases} \end{aligned}$$

where ω_1, ω_2, ν_1 and ν_2 are given by Theorem 4.2 with both γ and h replaced by 0.

Proof: With the change of variable $s = T - v$, we have

$$g_{Qua}(\rho, a) = \int_0^\infty \int_0^\infty e^{-(a+\rho)v} e^{-as} Qua(v, v + s) ds dv. \tag{4.45}$$

Note that for any random variable Y,

$$E[(Y - K)^+] = \int_K^{+\infty} P(Y > u) du.$$

In particular,

$$Qua(v, v+s) = e^{-r(v+s)} \int_K^{+\infty} P[q(\frac{v}{v+s}, v+s) > \frac{1}{\gamma}\log(u/S_0)]du.$$

Introduce another change-of-variable such that $h = \log(u/S_0)/\gamma$. Then,

$$Qua(v, v+s) = e^{-r(v+s)}\gamma S_0 e^{\gamma h} \int_k^{\infty} P[q(\frac{v}{v+s}, v+s) > h]dh,$$

where $k = \log(K/S_0)/\gamma$. The equivalence (4.44) implies

$$Qua(v, v+s) = e^{-r(v+s)}\gamma S_0 e^{\gamma h} \int_k^{\infty} P[\tau_{v+s}(h) < v]dh. \tag{4.46}$$

Substituting (4.46) back into (4.45) leads to

$$g_{Qua}(\rho, a) = \int_k^{\infty} \gamma S_0 e^{\gamma h} \left(\int_0^{\infty} \int_0^{\infty} e^{-(a+r)(v+s)-\rho v} P[\tau_{v+s}(h) < v] ds dv \right) dh. \tag{4.47}$$

The double integral (4.47) becomes

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} e^{-(a+r)(v+s)-\rho v} P[\tau_{v+s}(h) < v] ds dv \\ = & \int_0^{\infty} e^{-(a+r)t} dt \int_0^t e^{-\rho v} P[\tau_t(h) < v] dv \\ = & \frac{1}{\rho} \int_0^{\infty} e^{-(a+r)t} E[e^{-\rho \tau_t(h)}] dt - \frac{1}{\rho(a+r+\rho)} \end{aligned}$$

under a change of variable $t = v + s$. The integral on the right-hand side of this equality is equal to $u(0, \rho, 0, a, h)$ by Theorem 4.2. Hence,

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} e^{-(a+r)(v+s)-\rho v} P[\tau_{v+s}(h) < v] ds dv \\ = & \begin{cases} \frac{1}{\rho}\left(\omega_1 e^{-\beta_{1,a+\rho} h} + \omega_2 e^{-\beta_{2,a+\rho} h}\right), & h \geq 0; \\ -\frac{1}{\rho}\left(\nu_1 e^{\gamma_{1,a} h} + \nu_2 e^{\gamma_{2,a} h}\right) + \frac{1}{(a+r)(a+r+\rho)}, & h < 0. \end{cases} \end{aligned}$$

Plugging this into (4.47), routine calculation will complete the proof. □

Remark 4.10. *Cai (2008b) developed a method to price both the fixed- and floating-strike quantile options numerically using Laplace inversion twice under a more general hyper-exponential jump diffusion model. Our method improves the efficiency because it requires inversion once only. His method can also be used to price floating-strike quantile options under a more general jump diffusion model.*

4.5. Numerical Results

In this section we present numerical results of the options prices and hedging parameters. For numerical pricing and hedging of options via Laplace inversion, we use the analytical formulae in Section 4.4 and the multi-dimensional Euler inversion algorithm, which was introduced by Choudhury, Lucantoni, and Whitt (1994) and was extended to the two-sided case by Petrella (2004).

4.5.1. Proportional Step Options

We use the modified two-sided Euler inversion algorithm of Petrella (2004) to invert the two-sided Laplace transform with respect to κ for the proportional step option. This algorithm is faster and more stable numerically than the original Euler inversion when dealing with two-sided transforms, due to the introduction of a scaling factor. The numerical results for the proportional step option prices (denoted by EI Price) are given in Table 4.1, where we also show the Monte Carlo simulation results (denoted by MC Value) as a benchmark together with the associated 95% confidence intervals (denoted by 95% CI). The numerical prices are given at the top and the delta values are given at the bottom. We can see that all the EI Prices stay within the 95% confidence intervals of the associated MC Values. The pricing method based on our analytical pricing formulae as well as the Euler inversion algorithm is accurate and efficient.

As λ approaches 0, the double exponential jump diffusion model will converge to a geometric Brownian motion. Therefore, we can expect both the price and delta of occupation-time-related options under the DEM should also converge to those under the GBM. Table 4.2 verifies this intuition. Furthermore, it shows that our numerical method works for GBM as well because it replicates Linesky's result when we take $\lambda = 0$.

Prices of Proportional Step Options under the DEM				
S_0	K	EI price	MC value	Std Err
	90	13.81882988	13.84674076	0.01999824
100	100	9.42438004	9.45073300	0.02077236
	110	5.97929056	6.00093565	0.02087176
	90	19.04025239	19.06951901	0.01933582
105	100	13.45926395	13.48746393	0.02121837
	110	8.90133738	8.93024825	0.02272951

Deltas of Proportional Step Options under the DEM				
S_0	K	EI price	MC value	Std Err
	90	0.96243741	0.96296267	0.00149024
100	100	0.73048507	0.73064749	0.00128122
	110	0.51700296	0.51785311	0.00122150
	90	1.07858913	1.07768208	0.00156524
102	100	0.82650108	0.82629754	0.00133499
	110	0.59299438	0.59407950	0.00126350

Table 4.1: The double Laplace inversion (EI price) vs. Monte Carlo simulation (MC value) under the DEM. The default choices are $\lambda = 3$, $r = 0.05$, $\sigma = 0.2$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\rho = 1$, and $t = 1$. The CPU time for the Laplace inversion method is around 3.5 seconds. MC values along with the associated standard errors (denoted by Std Err) are obtained by using 50,000 time steps and simulating 100,000 sample paths, and the CPU time is around 10 minutes. This table shows that all of the EI prices stay within the 95% confidence intervals of the associated MC values.

4.5.2. Simple Step, Delayed Barrier, Corridor, and Quantile Options

The numerical prices and delta values of other occupation-time-related options, including simple step, delayed barrier, corridor, and quantile options, are given

Numerical Results When the Jump Intensity is Small		
	Prices	Deltas
λ	P	Δ
0.1	6.802016390247875	0.616344715465678
0.01	6.782466159399428	0.616240150248286
0.001	6.780507945848759	0.616229613080276
0.0001	6.780312092511664	0.616228558551593
0.00001	6.780292506857593	0.616228453089914
0	6.780290330669454	0.616228441372041

Table 4.2: How the prices and deltas of a proportional step option change as λ goes to 0. When $\lambda \to 0$, both of the prices and deltas converge to those under the GBM model. The parameters we use are the same as the setting in TABLE 5.3 of Linetsky (1999): $r = 0.05$, $\sigma = 0.6$, $L = 95$, $S_0 = 100$, $K = 100$, and $t = 0.5$. The jump parameters are $\eta = 30$, $\theta = 20$, and $p = q = 0.5$. When $\lambda = 0$, our results are the same as Linetsky's.

in Table 4.3-4.6.

For the pricing and hedging of the simple step and the delayed barrier options, we need to do triple Laplace inversions. First, we use a two-dimensional Euler inversion formula for the complex-valued function (Formula (2.7) in Choudhury, Lucantoni, and Whitt (1994) with $l_1 = l_2 = 1$) and then we do an extra one-dimensional Euler inversion (Formula (4.6) in Abate and Whitt (1992)). Our results show that the average time spent by one triple Laplace inversion is around 2 minutes, which is still very efficient compared to the Monte Carlo simulation. For the numerical results of corridor and quantile option prices, it suffices to use a two-dimensional Euler inversion algorithm. Our method is more efficient than Cai's method Cai (2008b).

Prices of Simple Step Options under the DEM				
S_0	K	EI price	MC value	Std Err
	90	9.67457995	9.70774495	0.02985213
100	100	7.07669587	7.10395013	0.02529039
	110	4.75390837	4.77502124	0.02191003
	90	12.16683520	12.20418981	0.03073929
102	100	8.92866361	8.95838153	0.02642060
	110	6.03645208	6.05956925	0.02348342

Prices of Delayed Barrier Options under the DEM				
S_0	K	EI price	MC value	Std Err
	90	14.25719729	14.28598897	0.03006500
100	100	10.08003700	10.10164481	0.02591103
	110	6.52095740	6.53852537	0.02366194
	90	16.39440581	16.43625061	0.02796657
102	100	11.63440011	11.66789910	0.02483299
	110	7.59164287	7.61545252	0.02366583

Table 4.3: The Laplace inversion (EI price) vs. Monte Carlo simulation (MC value). For the simple step and delayed barrier options, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\sigma = 0.2$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\vartheta = 0.5$, and $t = 1$. All Monte Carlo values (denoted by MC value) along with the associated standard errors (denoted by Std Err) are obtained using 50,000 time steps and simulating 100,000 sample paths. The CPU time of our numerical methods for generating one price of simple step or delayed barrier options is around 2 minutes. The CPU time for Monte Carlo simulation is around 10 minutes for the two type of options. The table indicates that all the EI prices stay within the 95% confidence intervals of the associated MC values.

4.5.3. Discretization Frequency Effect

Our EI price is given under an assumption that the underlying price is continuously monitored. However, in reality a sizable portion of contracts specify fixed reference times for monitoring and the occupation time is defined according to the number of the monitoring dates in which the underlying price is above/below some level or within a band. This may introduce substantial differences between the two monitoring schemes. Some scholars have already studied the effect of discretization frequency on the pricing of occupation-time-related options under GBM models. The main literature includes Atkinson and Fusai (2007), Davydov and Linetsky (2002) and Fusai and Tagliani (2001).

In this subsection, we aim to investigate how the discretization frequency will affect the pricing results under the double exponential jump diffusions. Table 4.7 and Figure 4.1 compare our continuous-time outcomes in one proportional step option example with the prices under discrete time monitoring, which are obtained through Monte Carlo simulation, for various initial underlying prices. The monitoring frequencies we use are monthly, biweekly, weekly and daily. That is, the time horizon, 1 year, is divided into 12, 26, 52 and 252 subintervals, respectively. For discrete monitoring contracts, define the occupation time as follows:

$$\tau_L = \sum_{i=1}^{N} (t_i - t_{i-1}) \mathbf{1}_{\{S_{t_i} < L\}},$$

where $0 = t_0 < \cdots < t_N = T$ are the reference dates.

It is clear to see that the relative differences between the two schemes reduce significantly when the discretization becomes more frequent. Therefore, the continuous results should be a good approximation to those contracts under high-frequent monitoring (say, daily or weekly). However, we should admit that significant differences exist (e.g., more than 9% for $S_0 = 105$ in the case of monthly monitoring) between the continuous-time scheme and the less-frequent discrete monitoring. It will then be important to distinguish these two under this scenario.

A similar convergence can be observed for the delta too. As the discretization becomes finer and finer, the deltas under discrete monitoring will converge to the delta under continuous monitoring. Table 4.8 and Figure 2 demonstrate the related numerical experiments.

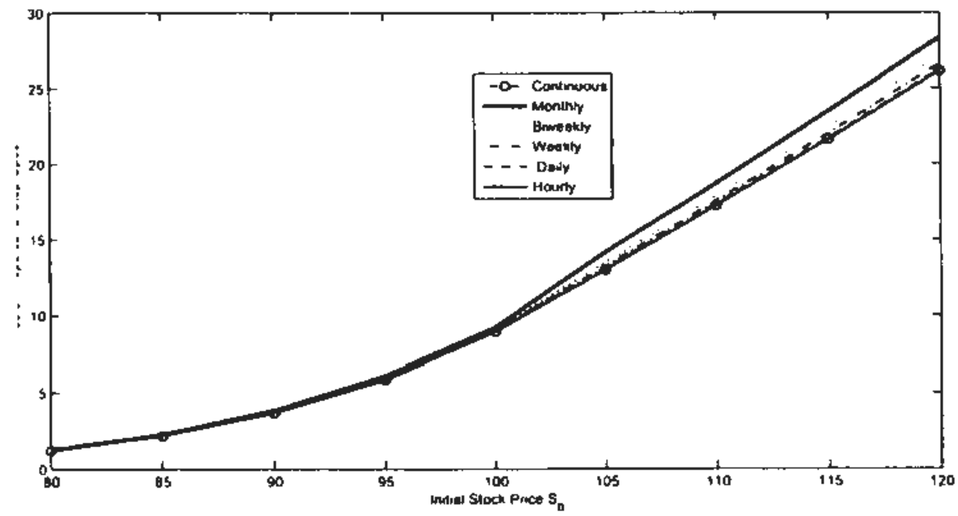

Figure 4.1: Comparison of continuous and discrete monitoring results under the DEM model. As the discretization becomes finer, the discrete-time monitoring option prices converge to the continuous-time option prices under all initial stock prices. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$ and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete prices.

4.5.4. Robustness of Our Pricing Algorithm

We point out that our Laplace inversion based pricing algorithm is robust. As illustrated in Figure 4.3, our pricing algorithm retains its accuracy when some model parameters vary within realistic ranges. More precisely, when η (θ and p, respectively) changes in $[15, 100]$ ($[15, 100]$ and $[0, 1]$, respectively), the relative errors between our numerical prices and MC prices are all less than 0.3%. These ranges cover most cases in reality. For example, $\eta \in [15, 100]$ and $\theta \in [15, 100]$ mean that the expected upward and downward jump sizes of return are between 1% and 6.67%. Note that the minimum and maximum daily returns of S&P 500 from Aug 1, 2007 to Oct 26, 2009 (during the ongoing financial crisis) are -4.76% and 4.11%, respective. Absolute values of them are both smaller than 6.67%. Consequently, we draw the conclusion that our pricing algorithm is robust and thus reliable.

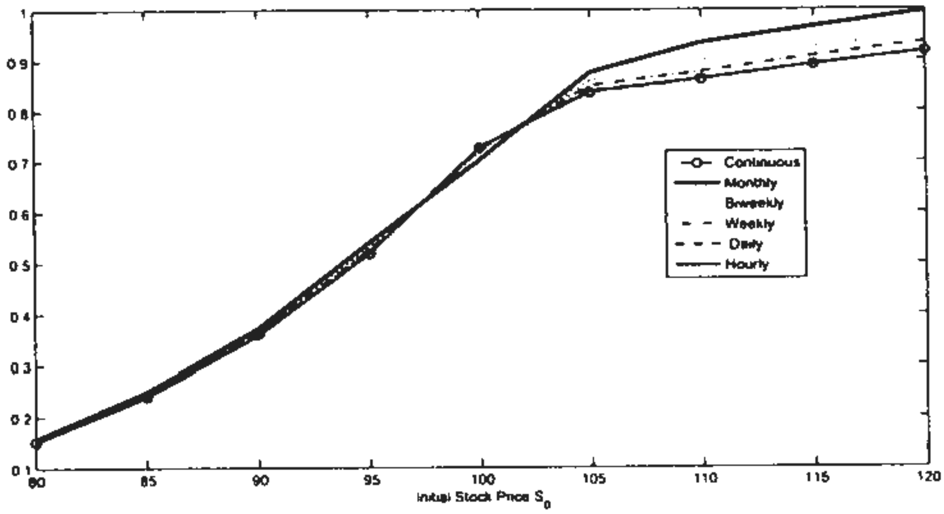

Figure 4.2: Comparison of continuous and discrete monitoring deltas under the DEM model. As the discretization becomes finer, the deltas of discrete monitoring converge to those of continuous monitoring under all initial stock prices. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$ and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete deltas.

4.6. Conclusion

In this Chapter, we investigate pricing and hedging problems of occupation-time-related options such as step options, corridor options, and quantile options under Kou's double-exponential jump diffusion model. By studying the occupation-time distribution, we derive the Laplace transform-based analytical solutions to these pricing problems, which can be inverted numerically via the Euler Laplace inversion algorithm. The numerical results indicate that our pricing formulae are both accurate and efficient.

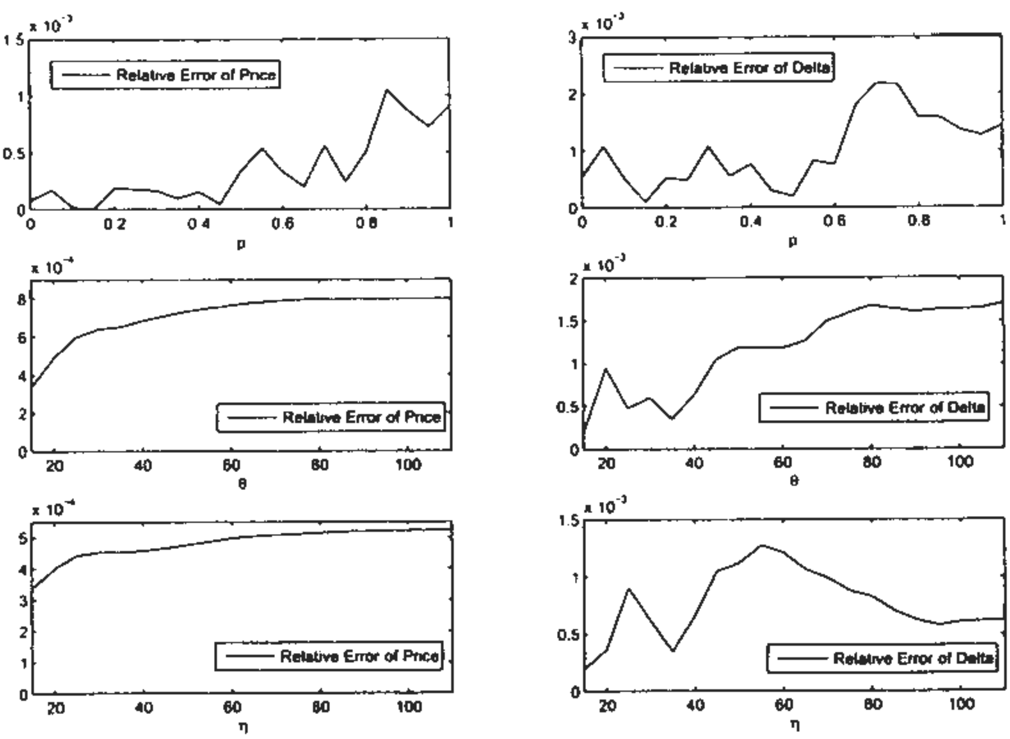

Figure 4.3: The relative errors between the Euler Inversion and MC Simulation for varying p, θ and η. We test the robustness of our method using the proportional step option. The default parameters of the jump diffusion processes are $r = 0.05$, $\sigma = 0.2$, $\lambda = 1$, $\eta = \theta = 15$ and $p = q = 0.5$. The current underlying asset price is $S_0 = 105$. The option contract parameters are $\rho = 1$, $K = 100$ and $L = 90$. The occupation time is accumulated when the underlying price is less than 90.

Prices of Corridor Options with single barrier under the DEM				
K	S_0	EI price	MC value	Std Err
	95	0.46627793	0.46580529	0.00060334
0.2	100	0.34654861	0.34620580	0.00064820
	105	0.22446654	0.22460171	0.00061260
	95	0.31194613	0.31159386	0.00050566
0.4	100	0.22032156	0.22018846	0.00052382
	105	0.13161829	0.13177739	0.00046635

Prices of Quantile Options under the DEM				
α	K	EI price	MC value	Std Err
	90	6.98491715	7.00339911	0.01605925
0.2	100	2.08465538	2.09972912	0.01122946
	110	0.37724012	0.38423388	0.00552578
	90	12.59539246	12.61168267	0.02098495
0.5	100	5.90331831	5.92048348	0.01876866
	110	2.29109044	2.30873387	0.01459738

Table 4.4: The Laplace inversion (EI price) vs. Monte Carlo simulation (MC value). For the corridor options with single barrier, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\sigma = 0.2$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, and $t = 1$. For the quantile options, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\sigma = 0.2$, $\eta = 34$, $\theta = 34$, $p = 0.6$, $q = 0.4$, $S_0 = 100$, $\gamma = 1$, and $t = 1$. All Monte Carlo values (denoted by MC value) along with the associated standard errors (denoted by Std Err) are obtained using 50,000 time steps and simulating 100,000 sample paths. The CPU time of our numerical methods for generating one price of corridor options, and quantile options is around 3 seconds and 3 seconds, respectively. The CPU time for Monte Carlo simulation is around 22 minutes for the quantile options and around 10 minutes for the corridor options. The table indicates that all the EI prices stay within the 95% confidence intervals of the associated MC values.

Delta of Simple Step Options under the DEM				
S_0	K	EI value	MC value	Std Err
100	90	1.13763436	1.13898389	0.00377164
	100	0.84343613	0.84598812	0.00282432
	110	0.58164514	0.58407656	0.00219618
102	90	1.35962149	1.35725685	0.00391806
	100	1.01249653	1.01289927	0.00291876
	110	0.70396190	0.70573895	0.00226845

Delta of Delayed Barrier Options under the DEM with $\lambda = 3$				
S_0	K	EI value	MC value	Std Err
100	90	1.04040458	1.00570473	0.02406215
	100	0.75353143	0.73751457	0.01574238
	110	0.51485647	0.50789855	0.00969742
102	90	1.09523707	1.07269986	0.02342268
	100	0.79990286	0.78274507	0.01498936
	110	0.55545444	0.54777543	0.00905981

Table 4.5: The Laplace inversion (EI value) vs. Monte Carlo simulation (MC value). For the simple step and delayed barrier options, the default parameter choices are $\lambda = 3$, $\sigma = 0.2$, $r = 0.05$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\vartheta = 0.5$, $\Delta S_0 = 0.1$, and $t = 1$. Monte Carlo values for simple step and delayed barrier options along with the associated standard errors (denoted by Std Err) are obtained by using 100,000 time steps and simulating 100,000 sample paths. The CPU time of our numerical methods for generating one price of simple step or delayed barrier options is around 100 seconds. The CPU time for Monte Carlo simulation is around 25 minutes for the simple step or delayed barrier options. The table indicates that all the EI values stay within the 95% confidence intervals of the associated MC values.

Delta of Corridor Options with single barrier under the DEM				
K	S_0	EI value	MC value	Std Err
0.2	100	-0.02563184	-0.02556980	0.00008900
	102	-0.02669485	-0.02658852	0.00008981
	104	-0.02209558	-0.02199631	0.00008277
0.4	100	-0.01912417	-0.01908412	0.00008163
	102	-0.01957286	-0.01956383	0.00008143
	104	-0.01569789	-0.01561718	0.00007313

Delta of Quantile Options under the DEM				
α	S_0	EI value	MC value	Std Err
0.2	90	0.06855937	0.06908873	0.00077633
	100	0.33498655	0.33507900	0.00118391
	110	0.62926410	0.62827708	0.00108407
0.5	90	0.25471549	0.25497213	0.00115890
	100	0.57434958	0.57383915	0.00121216
	110	0.82522134	0.82466107	0.00096239

Table 4.6: The Laplace inversion (EI value) vs. Monte Carlo simulation (MC value). For the corridor options with single barrier, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\sigma = 0.2$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\Delta S_0 = 0.1$, and $t = 1$. For the quantile options, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\sigma = 0.2$, $\eta = 34$, $\theta = 34$, $p = 0.6$, $q = 0.4$, $S_0 = 100$, $\gamma = 1$, $\Delta S_0 = 0.1$, and $t = 1$. Monte Carlo values for corridor and quantile options along with the associated standard errors (denoted by Std Err) are obtained by using 20,000 time steps and simulating 100,000 sample paths. The CPU time of our numerical methods for generating one price of corridor options and quantile options is around 3 seconds. The CPU time for Monte Carlo simulation is around 4.3, and 9 minutes for the corridor, and quantile options, respectively. The table indicates that all the EI values stay within the 95% confidence intervals of the associated MC values.

	Monitoring frequency				
	Relative differences				
S_0	Monthly	Biweekly	Weekly	Daily	Continuous Prices
95	4.698%	1.974%	1.011%	0.922%	7.22634078
100	3.489%	1.583%	0.998%	0.783%	10.35784700
105	9.458%	4.049%	2.027%	0.828%	14.37387610
110	8.940%	3.918%	2.003%	0.720%	18.50956926
115	8.866%	3.947%	1.982%	0.616%	22.75134627
120	8.841%	3.957%	1.975%	0.534%	27.12165429

Table 4.7: Comparison of continuous and discrete step option pricing. The relative difference is defined as (discrete price − continuous price)/continuous price. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$ and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete prices.

	Monitoring frequency				
	Relative differences				
S_0	Monthly	Biweekly	Weekly	Daily	Continuous Delta
95	3.990%	2.366%	1.368%	0.361%	0.53553845
100	-3.411%	-1.907%	-0.643%	0.275%	0.72754990
105	4.342%	2.670%	1.752%	0.466%	0.82098985
110	8.048%	4.428%	1.915%	0.256%	0.83620619
115	8.850%	3.936%	1.824%	0.195%	0.86111472
120	8.935%	3.986%	2.108%	0.138%	0.88675445

Table 4.8: Comparison of the deltas of the continuous and discrete step options. The relative difference is defined as (discrete delta – continuous delta)/continuous delta. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$ and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete deltas.

APPENDIX A

APPENDIX FOR CHAPTER 2

A.1. Proof of Proposition 2.3

To begin with, let us prove some preliminary properties of $\mathcal{S}_D$, $\mathcal{S}_{EB}$ and $\mathcal{S}_{EB}$:

$$\mathcal{S}_D \subset [cP/(\delta\lambda), +\infty), \tag{A.1}$$

$$\mathcal{S}_{EB} \subset (0, \min(K, (1-\kappa)cP/\delta)], \tag{A.2}$$

$$\mathcal{S}_{EC} \cap [K, K/\lambda) \subset [K, (1-\kappa)cP/(\lambda r)) \text{ if } \mathcal{S}_{EC} \cap [K, K/\lambda) \neq \emptyset, \tag{A.3}$$

$$\mathcal{S}_{EC} \cap (K/\lambda, +\infty) \subset [K/\lambda, (1-\kappa)cP/(\lambda\delta)] \text{ if } \mathcal{S}_{EC} \cap (K/\lambda, +\infty) \neq \emptyset. \tag{A.4}$$

Consider the set $\mathcal{S}_D$ first. For any $V \in \mathcal{S}_D$, it must be a local minimum of the function $D^*(v) - \lambda v$ because $D^*(v) \geq \lambda v$ for all $v \geq 0$. This implies that [1]

$$\frac{d}{dv}D^*(v)|_{v=V} = \lambda \quad \text{and} \quad \frac{d^2}{dv^2}D^*(v)|_{v=V} \geq 0.$$

Hence,

$$cP \leq \mathcal{L}D^*(V) = -\frac{1}{2}\sigma^2 V^2 \frac{d^2}{dv^2}D^*(V) - (r-\delta)V\frac{d}{dv}D^*(V) + rD^*(V) \leq \delta\lambda V,$$

that is, $V \geq cP/(\delta\lambda)$. This implies (A.1).

[1]The classic first order derivative of D^* at V exist and the second order derivative of D^* at V may refer to week derivative. However the second order derivative of D^* at V always exist. This is because by Condition 4 and 6, D^* satisfies a variational inequality on the neighborhood of V. The similar results hold for E^* in the following argument.

The proof of (A.2)-(A.4) is similar. For any $V \in \mathcal{S}_{EB}$, Using Condition 5, we then have

$$\mathcal{L}E^*(V) \geq \delta V - (1-\kappa)cP \tag{A.5}$$

for such V. On the other hand, E^* achieves its local minimum at V, which implies that

$$\frac{d}{dv}E^*(v)|_{v=V} = 0 \quad \text{and} \quad \frac{d^2}{dv^2}E^*(v)|_{v=V} \geq 0. \tag{A.6}$$

(A.6) implies that

$$\begin{aligned}\mathcal{L}E^*(V) &= -\frac{1}{2}\sigma^2V^2\frac{d^2}{dv^2}E^*(V) - (r-\delta)V\frac{d}{dv}E^*(V) + rE^*(V)\\ &= -\frac{1}{2}\sigma^2V^2\frac{d^2}{dv^2}E^*(V) \leq 0.\end{aligned}$$

Combining it with (A.5) will lead to a conclusion that $V \in [0, (1-\kappa)cP/\delta)$. It is clear that $V \leq K$. These leads to (A.2). For any $V \in \mathcal{S}_{EC} \cap [K, K/\lambda)$, $E^*(V) = V - K$. Consider a function $E^*(v) - (v - K)$. $v = V$ is a local minimum of the function and therefore,

$$\frac{d}{dv}E^*(v)|_{v=V} = 1 \quad \text{and} \quad \frac{d^2}{dv^2}E^*(v)|_{v=V} \geq 0.$$

Then, we have

$$\begin{aligned}\mathcal{L}E^*(V) &= -\frac{1}{2}\sigma^2V^2\frac{d^2}{dv^2}E^*(V) - (r-\delta)V\frac{d}{dv}E^*(V) + rE^*(V)\\ &\leq -(r-\delta)V + r(V-K) = \delta V - rK.\end{aligned}$$

Note that $\mathcal{L}E^*(V) \geq \delta V - (1-\kappa)cP$, which implies $\mathcal{S}_{EC} \cap [K, K/\lambda) \neq \emptyset$ only if $K \leq (1-\kappa)cP/r$. That is, (A.3) follows. For any $V \in \mathcal{S}_{EC} \cap (K/\lambda, +\infty)$, it is a local minimum of $E^*(v) - (1-\lambda)v$ on the interval $[K/\lambda, +\infty)$. From this, we may derive that

$$\mathcal{L}E^*(V) \leq -(r-\delta)(1-\lambda)V + r(1-\lambda)V = \delta(1-\lambda)V.$$

Combining it with $\mathcal{L}E^*(V) \geq \delta V - (1-\kappa)cP$, we have $V \leq (1-\kappa)cP/(\lambda\delta)$. Note that $V \geq K/\lambda$ in this case. The strike price K should satisfy that $K \leq$

$(1-\kappa)cP/\delta$. Consequently, a necessary condition for $\mathcal{S}_{EC} \cap (K/\lambda, +\infty) \neq \emptyset$ is that $K \leq (1-\kappa)cP/\delta$. Furthermore, (A.4) holds. It is worth pointing out that the first sentence of part (iii) has been proved meanwhile.

We now prove part (i). First we claim that $\mathcal{S}_D \neq \emptyset$. Suppose that it is not true, i.e., $\mathcal{S}_D = \emptyset$. Consider a sufficiently large number C such that $C > \max\{K/\lambda, (1-\kappa)cP/(\lambda\delta), (1-\kappa)cP/(\lambda r)\}$. For any $V \geq C$, it cannot be an element in $\mathcal{S}_{EC}$ according to (A.3) and (A.4). Hence, either

$$E^*(V) = (1-\lambda)V, \quad \mathcal{L}E^*(V) < \delta V - (1-\kappa)cP$$

or $E^*(V) > (1-\lambda)V$ holds for such V. If the first case is true, $D^*(V) = \lambda V$. By condition 6, $\mathcal{L}D^*(V) \geq cP$, which means $V \in \mathcal{S}_D$. However, this contradicts to the assumption that $\mathcal{S}_D = \emptyset$. Hence, the second case holds for all $V \geq C$. Using condition 4, it is easy to see that $\mathcal{L}D^*(V) \geq cP$. The assumption of $\mathcal{S}_D = \emptyset$ implies $D^*(V) > \lambda V$. We may reach that $\mathcal{L}D^*(V) = cP$ for all $V \geq C$ with the help of condition 4 again. According to Appendix A.2, the ODE $\mathcal{L}D^*(V) = cP$ admits a general solution in the form of

$$D^*(V) = \frac{cP}{r} + c_1 V^{\beta} + c_2 V^{-\gamma},$$

where $\beta > 1$ and $\gamma > 0$. On the other hand, by (2.2), we have that

$$\begin{aligned} D^*(V) = \sup_{\tau_b, \tau_{cal}, \tau_{con}} D(V; \tau_b, \tau_{cal}; \tau_{con}) &\leq E\left[\int_0^{+\infty} e^{-rt} cP dt + \sup_{0 \leq t < +\infty} e^{-rt}(V_t + K)\right] \\ &\leq \frac{cP}{r} + K + VE\left[\sup_{0 \leq t \leq \infty} e^{-(\delta + \frac{1}{2}\sigma^2)t + \sigma W_t}\right]. \end{aligned}$$

It is straightforward to argue the finiteness of the expectation on the right hand side of the above inequality. This implies that D^* grows at most linearly and c_1 should be 0. As V tends to $+\infty$, D^* converges to cP/r. Contradicting to the condition that $D^*(V) \geq \lambda V$. Therefore, $\mathcal{S}_D \neq \emptyset$.

Second, we show that if some $V_1 \in \mathcal{S}_D$, then $[V_1, +\infty) \subset \mathcal{S}_D$. Following the arguments leading to the conclusion $\mathcal{S}_D \neq \emptyset$, we can see that there is a unbounded, monotonically increasing sequence $\left\{\tilde{V}_N\right\}$ such that $D^*(\tilde{V}_N) = \lambda \tilde{V}_N$

for all N. It suffices to prove that $(V_1, \tilde{V}_N) \subset \mathcal{S}_D$ for any N. We claim that $D^*(V)$ satisfies the variational inequality problem

$$\begin{cases} \min\{\mathcal{L}D^*(V) - cP, D^*(V) - \lambda V\} = 0 \text{ in } (V_1, \tilde{V}_N) \\ D^*(V_1) = \lambda V_1, \ D^*(\tilde{V}_N) = \lambda \tilde{V}_N. \end{cases} \tag{A.7}$$

Actually for any $V \in (V_1, \tilde{V}_N)$, $V \notin \mathcal{S}_{EC}$. Hence, either

$$E^*(V) = (1-\lambda)V, \quad \mathcal{L}E^*(V) < \delta V - (1-\kappa)cP$$

or $E^*(V) > (1-\lambda)V$ holds for such V. If the late case is true, (A.7) follows from condition 4. If it is the first case, $E^*(V) = (1-\lambda)V$ implies $D^*(V) = \lambda V$, and together with condition 6, (A.7) holds.

By (A.1), we have $V_1 \geq cP/(\delta\lambda)$, thus $V \geq cP/(\delta\lambda)$ for all $V \in (V_1, \tilde{V}_N)$, which indicates

$$\mathcal{L}(\lambda V) - cP = \delta\lambda V - cP \geq 0 \text{ in } (V_1, \tilde{V}_N).$$

As a result, λV is a supersolution to problem (A.7), i.e., $D^*(V) \leq \lambda V$ in $(V_1, \tilde{V}_N)$, which leads to the desired results $D^*(V) = \lambda V$ and $\mathcal{L}D^*(V) - cP \geq 0$ in $(V_1, \tilde{V}_N)$. Let $V^*_{con} = \inf\{V : V \in \mathcal{S}_D\}$, then $\mathcal{S}_D = [V^*_{con}, +\infty)$ and $V^*_{con} \geq cP/(\delta\lambda)$ because of (A.1).

We now move to the proof of part (ii). The nonemptyness of $\mathcal{S}_{EB}$ can be proved in a similar way as we did for $\mathcal{S}_D$. Indeed, if $\mathcal{S}_{EB} = \emptyset$, consider a sufficiently small number $\widetilde{C}$ such that $\widetilde{C} < \min\{K, cP/(\lambda\delta)\}$. For any $V \leq \widetilde{C}$, it cannot be an element in $\mathcal{S}_D$ according to (A.1). Hence, either

$$D^*(V) = \lambda V, \quad \mathcal{L}D^*(V) < cP$$

or $D^*(V) > \lambda V$ holds for such V. If the first case is true, $E^*(V) = (1-\lambda)V > h(V)$. By condition 4, $\mathcal{L}D^*(V) \geq cP$. Contradiction. Hence, the second case holds for all $V \leq \widetilde{C}$. Using condition 5, we have $\mathcal{L}E^*(V) \geq \delta V - (1-\kappa)cP$ and $E^*(V) \geq h(V)$. Then $E^*(V) > h(V)$ since the assumption of $\mathcal{S}_{EB} = \emptyset$ and $V < K$. We the get that $\mathcal{L}E^*(V) = \delta V - (1-\kappa)cP$ for all $V \leq \widetilde{C}$ with the help of condition 5 again. According to Appendix A.2, the ODE $\mathcal{L}E^*(V) = \delta V - (1-\kappa)cP$ admits a general solution in the form of

$$E^*(V) = V - \frac{(1-\kappa)cP}{r} + c_3 V^\beta \to -\frac{(1-\kappa)cP}{r} \text{ as } V \to 0,$$

a contradiction! Second, we will show that if some $V \in \mathcal{S}_{EB}$, then $v \in \mathcal{S}_{EB}$ for all $v \le V$. For this purpose, consider the interval $[0, V]$. Suppose that $\max_x E^*(x) > 0$ and denote the maximum point by x^*. Thus,

$$\frac{d}{dx}E^*(x^*) = 0 \quad \text{and} \quad \frac{d^2}{dx^2}E^*(x^*) \le 0.$$

They imply that $\mathcal{L}E^*(x^*) > 0$. Furthermore, note that $x^* \le V$ and $V \in \mathcal{S}_{EB} \subset [0, (1-\kappa)cP/\delta]$. Therefore, $\delta x^* - (1-\kappa)cP < 0 < \mathcal{L}E^*(x^*)$.

On the other hand, $E^*(x^*) > 0 = h(x^*)$ implies that $\mathcal{L}D^*(x^*) \ge cP$ according to condition 4. Since $x^* \notin \mathcal{S}_D$, we know that $D^*(x^*) > \lambda x^*$. By condition 5, $\mathcal{L}E^*(x^*) = \delta x^* - (1-\kappa)cP$ because of the assumption $E^*(x^*) > 0$. Contradiction. Consequently, $E^*(x) \equiv 0$ for all $x \le V$. Define $V_b^* = \sup \mathcal{S}_D$. We have $\mathcal{S}_D = [0, V_b^*]$. Due to (A.2), $V_b^* \le \min(K, (1-\kappa)cP/\delta)$.

It remains to prove part (iii) when $\mathcal{S}_{EC} \ne \emptyset$. In this case, either $\mathcal{S}_{EC} \cap (K, K/\lambda]$ or $\mathcal{S}_{EC} \cap (K/\lambda, +\infty)$ is not empty. For the first case $\mathcal{S}_{EC} \cap (K, K/\lambda] \ne \emptyset$, suppose $V_1 \in \mathcal{S}_{EC} \cap (K, K/\lambda]$. We might as well assume $V_1 \ne K/\lambda$. It suffices to prove $E^*(x) = x - K$ for all $x \in (V_1, K/\lambda]$. Owing to part (i) and part (iii), we have $V_{con}^* > K/\lambda$ and $E^*(V_{con}^*) = (1-\lambda)V_{con}^* < V_{con}^* - K$. Noticing $E^*(x) \ge x - K$ in $x \in (V_1, K/\lambda]$, we then infer that there exists a point $V_2 \in [K/\lambda, V_{con}^*)$ such that $E^*(V_2) = V_2 - K$. Consider the interval (V_1, V_2) in which $E^*(V)$ is governed by the variational inequality problem

$$\begin{cases} \min\{\mathcal{L}E^*(V) - \delta V + (1-\kappa)cP, E^*(V) - h(V)\} = 0, \\ E^*(V_1) = V_1 - K, \; E^*(V_2) = V_2 - K \end{cases} \tag{A.8}$$

Thanks to part (iii)we infer $K \le (1-\kappa)cP/r$, thus

$$\mathcal{L}(V - K) - \delta V + (1-\kappa)cP = -rK + (1-\kappa)cP \ge 0$$

which, combined with $V - K \geq h(V)$ in (V_1, V_2), implies that the function $V - K$ is a supersolution to the problem (A.8) in (V_1, V_2), i.e., $E^*(V) \leq V - K$ for all $V \in (V_1, V_2)$. We then deduce $E^*(V) = V - K$ for all $V \in (V_1, K/\lambda]$.

For the second case $\mathcal{S}_{EC} \cap (K/\lambda, +\infty) \neq \emptyset$, suppose $V_1 \in \mathcal{S}_{EC} \cap (K/\lambda, +\infty)$. It suffices to prove $E^*(x) = (1-\lambda)x$ for all $x \in [K/\lambda, V_1]$. Owing to part (ii), we have $V_b^* < K/\lambda$ and $E^*(V_b^*) = 0 < (1-\lambda)V_b^*$. Noticing $E^*(x) \geq (1-\lambda)x$ in $x \in [K/\lambda, V_1]$, we then infer that there exists a point $V_2 \in [V_b^*, K/\lambda]$ such that $E^*(V_2) = (1-\lambda)V_2$. Consider the interval (V_2, V_1) in which $E^*(V)$ is governed by the variational inequality problem

$$\begin{cases} \min\{\mathcal{L}E^*(V) - \delta V + (1-\kappa)cP, E^*(V) - h(V)\} = 0, \\ E^*(V_2) = (1-\lambda)V_2, \ E^*(V_1) = (1-\lambda)V_1 \end{cases} \tag{A.9}$$

Thanks to (A.4), we have $V < V_1 \leq (1-\kappa)cP/(\lambda\delta)$, thus

$$\mathcal{L}((1-\lambda)V) - \delta V + (1-\kappa)cP = -\lambda\delta V + (1-\kappa)cP \geq 0$$

which, combined with $(1-\lambda)V \geq h(V)$ in (V_2, V_1), implies that the function $(1-\lambda)V$ is a supersolution to the problem (A.9) in (V_2, V_1), i.e., $E^*(V) \leq (1-\lambda)V$ for all $V \in (V_2, V_1)$. We then deduce $E^*(V) = (1-\lambda)V$ for all $V \in [K/\lambda, V_1]$. The proof is complete. □

A.2. The Euler-Cauchy ODE

Consider two second-order non-homogeneous ODEs such as

$$\mathcal{L}D(v) = -\frac{1}{2}\sigma^2 v^2 \frac{d^2}{dv^2} D(v) - (r-\delta)v\frac{d}{dv}D(v) + rD(v) = cP$$

and

$$\mathcal{L}E(v) = -\frac{1}{2}\sigma^2 v^2 \frac{d^2}{dv^2} E(v) - (r-\delta)v\frac{d}{dv}E(v) + rE(v) = \delta v - (1-\kappa)cP.$$

Explicit general solutions to both equations are known (Zwillinger (1997), p. 120). The general solution to the former equation is given by

$$D(v) = \frac{cP}{r} + c_1 v^\beta + c_2 v^{-\gamma};$$

and the solution to the latter is

$$E(v) = v - \frac{(1-\kappa)cP}{r} + c_3 v^{\beta} + c_4 v^{-\gamma}$$

where $c_i, 1 \le i \le 4$ are constants to be determined by the boundary conditions we introduce. It is also easy to see that $\beta > 1$ and $\gamma > 0$.

A.3. Properties of Some Elementary Functions

To simplify the proof of some technique Lemmas, we summarize the properties of some elementary function here. The functions are defined by

$$\begin{aligned}
f_1(x) &= \beta + \gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma}, \\
f_2(x) &= \gamma + \beta x^{\beta+\gamma} - (\beta+\gamma)x^{\beta}, \\
f_3(x;\lambda) &= \lambda\left((\beta-1) + (\gamma+1)x^{\beta+\gamma}\right) - (\beta+\gamma)x^{\gamma+1}, \\
f_4(x;\lambda) &= (\gamma+1) + (\beta-1)x^{\beta+\gamma} - \lambda(\beta+\gamma)x^{\beta-1} \\
f_5(x) &= \beta(\beta-1) - (\beta-1)(\gamma+1)(\beta+\gamma)x^{\gamma} \\
&\quad + \beta\gamma(\beta+\gamma)x^{\gamma+1} - \gamma(\gamma+1)x^{\beta+\gamma}, \\
f_6(x;\lambda) &= \gamma(\gamma+1) - \lambda\beta\gamma(\beta+\gamma)x^{\beta-1} \\
&\quad + (\beta-1)(\gamma+1)(\beta+\gamma)x^{\beta} - \beta(\beta-1)x^{\beta+\gamma}, \\
g_1(x;a,\epsilon) &= (\beta-1)(\gamma+1)xf_1(x) - a\beta\gamma f_3(x;1) + b\beta\gamma(\beta+\gamma)x^{\gamma+1}, \\
g_2(x;\lambda,b) &= b\beta\gamma x f_4(x;\lambda) - (\beta-1)(\gamma+1)f_2(x), \\
g_3(x;\lambda) &= x\left(\beta(\gamma+1) - \gamma(\beta-1)x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma}\right) \\
&\quad - \lambda\left(\gamma(\beta-1) - \beta(\gamma+1)x^{\beta+\gamma} + (\beta+\gamma)x^{\beta}\right), \\
g_4(x;a) &= (1-a)\left((\beta-1)\gamma x^{\beta+\gamma} - \beta(\gamma+1)\right) + (\beta+\gamma)x^{\gamma}, \\
g_5(x;a) &= \beta\gamma x f_4(x;1) - a(\beta-1)(\gamma+1)f_2(x), \\
g_6(x;\lambda,\kappa,\rho) &= xf_1(x)f_4(x;\lambda) - (1-\kappa)f_2(x)(f_3(x;\lambda) + \rho(\beta+\gamma)x^{\gamma+1}) \\
g_7(x;\kappa) &= xf_1(x)f_4(x;1) - (1-\kappa)f_2(x)f_3(x;1).
\end{aligned}$$

Lemma A.1. *For parameters $\beta > 1, \gamma > 0, \lambda \in (0,1), a \in (0,1), \kappa \in (0,1), 1-\rho \geq \lambda, b > 0, \epsilon \geq 0$, considering x as variable, we have that 1). $f_1, f_2, f_4, f_5, f_6, f_7$ and $f_3(x;1), f_6(x;1), -g_3(x;1)$ are all positive on $(0,1)$; 2) $-f_3, g_i (1 \leq i \leq 6)$ all have a unique zero point on $(0,1)$, negative on the left and positive on the right.*

Proof. Fixed $\beta > 1, \gamma > 0$.

(1.1) For fixed $\lambda \in (0,1)$, $f_1(x), f_2(x), f_3(x;1), f_4(x;1), f_4(x;\lambda)$ are positive for any $x \in (0,1)$. The derivative of $f_1(x)$ is given by

$$f_1'(x) = (\beta+\gamma)\gamma x^{\gamma-1}(x^\beta - 1) < 0, \forall x \in (0,1),$$

That is $f_1(x)$ is strictly decreasing on $(0,1)$. Then $f_1(x) > f_1(1) = 0$ for all $x \in (0,1)$.

By similar process, we have $f_2(x), f_3(x;1), f_4(x;1)$ are positive on $(0,1)$. And then $f_4(x;\lambda)$ is positive on $(0,1)$ since $f_4(x;\lambda) > f_4(x;1)$ for all $x \in (0,1)$.

(1.2) For fixed $\lambda \in (0,1)$, $f_5(x), f_6(x;1), f_6(x;\lambda)$ are positive for any $x \in (0,1)$. The derivative of $f_5(x)$ is given by

$$f_5'(x) = -(\beta+\gamma)\gamma(\gamma+1)(x^\beta + (\beta-1) - \beta x) < 0, \forall x \in (0,1),$$

since it is easy to verify that $x^\beta + (\beta-1) - \beta x > 0, \forall x \in (0,1)$. Then $f_5(x) > f_5(1) = 0$ for all $x \in (0,1)$.

By similar process, we have $f_6(x;1)$ is positive on $(0,1)$. And then $f_6(x;\lambda)$ is positive on $(0,1)$ since $f_6(x;\lambda) > f_6(x;1)$ for all $x \in (0,1)$.

(1.3) $-g_3(x;1) > 0$ for all $x \in (0,1)$. Denote $w(x) = -g_3(x;1)$. $w(0) = \gamma(\beta-1) > 0$, $w(1) = 0$ and

$$w'(x) = \gamma(\beta-1)(\beta+\gamma+1)x^{\beta+\gamma} - \beta(\gamma+1)(\beta+\gamma)x^{\beta+\gamma-1}$$
$$+ (\beta+\gamma)(\beta x^{\beta-1} + (\gamma+1)x^\gamma) - \beta(\gamma+1).$$

$w'(0) = -\beta(\gamma+1) < 0$, $w'(1) = 0$ and $w''(x) = (\beta+\gamma)x^{-1}w^{(1)}(x)$, where

$$w^{(1)}(x) = \gamma(\beta-1)(\beta+\gamma+1)x^{\beta+\gamma} - \beta(\gamma+1)(\beta+\gamma-1)x^{\beta+\gamma-1}$$
$$+ \beta(\beta-1)x^{\beta-1} + \gamma(\gamma+1)x^{\gamma}.$$

If $\beta < \gamma+1$, let $w^{(2)}(x) = w^{(1)}(x)/x^{\beta-1}$, that is

$$w^{(2)}(x) = \gamma(\beta-1)(\beta+\gamma+1)x^{\gamma+1} - \beta(\gamma+1)(\beta+\gamma-1)x^{\gamma}$$
$$+ \beta(\beta-1) + \gamma(\gamma+1)x^{\gamma+1-\beta}.$$

$w^{(2)}(0) = \beta(\beta-1) > 0$, $w^{(2)}(1) = 0$ and $(w^{(2)})'(x) = \gamma(\gamma+1)x^{\gamma-\beta}w^{(3)}(x)$, where

$$w^{(3)}(x) = (\beta-1)(\beta+\gamma+1)x^{\beta} - \beta(\beta+\gamma-1)x^{\beta-1} + (\gamma+1-\beta).$$

$w^{(3)}(0) = \gamma+1-\beta > 0$, $w^{(3)}(1) = 0$ and $(w^{(3)})'(x) = (\beta-1)\beta x^{\beta-2}((\beta+\gamma+1)x - (\beta+\gamma-1))$. Hence $w^{(3)}(x)$ is firstly strictly decreasing and later strictly increasing on $(0,1)$, with $w^{(3)}(0) > 0$ and $w^{(3)}(1) = 0$, $w^{(3)}(x) = 0$ has only one root on $(0,1)$. Hence $w^{(2)}(x)$ is firstly strictly increasing and later strictly decreasing. With $w^{(2)}(0) > 0$, $w^{(2)}(1) = 0$, we have $w^{(2)}(x) > 0$ on $(0,1)$, and so is $w''(x)$. Then $w'(x)$ is strictly increasing from $w'(0) < 0$ to $w'(1) = 0$. That is $w'(x) < 0$ for all $(0,1)$. Then $w(0)$ is strictly decreasing from $w(0) > 0$ to $w(1) = 0$, which implies that $w(x) > 0$ for all $(0,1)$.

If $\beta \geq \gamma+1$, by the similar process as above, we also get that $w(x) > 0$ for all $(0,1)$.

(2.1) *For fixed $\lambda \in (0,1)$, $f_3(x;\lambda) = 0$ has a unique solution $x^* \in (0,1)$ and $f_3(x;\lambda) > 0$ for $x \in (0,x^*)$, $f_3(x;\lambda) < 0$ for $x \in (x^*,1)$.* It is true since that $f_3(0;\lambda) = \lambda(\beta-1) > 0, f_3(1;\lambda) = -(1-\lambda)(\beta+\gamma) < 0$ and $f_3'(x;\lambda) = (\gamma+1)(\beta+\gamma)x^{\gamma}(\lambda x^{\beta-1} - 1) < 0$.

(2.2) *For fixed $a \in (0,1)$ and $b \geq 0$, $g_1(x;a,b) = 0$ has a unique solution $x^* \in (0,1)$ and $g_1(x;a,b) < 0$ for $x \in (0,x^*)$, $g_1(x;a,b) > 0$ for $x \in (x^*,1)$.*

Define $u(x) = (\beta - 1)(\gamma + 1)xf_1(x) - a\beta\gamma f_3(x;1)$ and then $g_1(x;a,b) = u(x) + b\beta\gamma(\beta+\gamma)x^{\gamma+1}$. $u(0) = -\beta\gamma(\beta-1) < 0, u(1) = 0$ and

$$\begin{aligned} u'(x) =& (\beta-1)(\gamma+1)\left(\beta + \gamma(\beta+\gamma+1)x^{\beta+\gamma} - (\beta+\gamma)(\gamma+1)x^{\gamma}\right) \\ &- a\beta\gamma(\beta+\gamma)(\gamma+1)(x^{\beta+\gamma-1} - x^{\gamma}), \end{aligned}$$

which gives $u'(0) = (\beta-1)(\gamma+1)\beta > 0$ and $u'(1) = 0$. $u''(x) = (\beta+\gamma)\gamma(\gamma+1)x^{\gamma-1}u^{(2)}(x)$, with

$$u^{(2)}(x) = (\beta-1)\left((\beta+\gamma+1)x^{\beta} - (\gamma+1)\right) - a\beta\left((\beta+\gamma-1)x^{\beta-1} - \gamma\right),$$

$u^{(2)}(0) = a\beta\gamma - (\beta-1)(\gamma+1)$, $u^{(2)}(1) = \beta(\beta-1)(1-a) > 0$ and

$$(u^{(2)})'(x) = \beta(\beta-1)x^{\beta-2}\left((\beta+\gamma+1)x - a(\beta+\gamma-1)\right).$$

In $(0,1)$, $(u^{(2)})'(x)$ is firstly negative and later positive, which implies that $u^{(2)}(x)$ is firstly strictly decreasing and later strictly increasing to $u^{(2)}(1) > 0$. Then $u^{(2)}(x) = 0(u''(x) = 0)$ has at most two roots in $(0,1)$. If there is no root or one root (local minimum of $u^{(2)}(x)$), then $u^{(2)}(x) > 0(u''(x) > 0)$ in $(0,1)$, except at most one point, which implies that $u'(x)$ is strictly increasing. That contradicts with the fact that $u'(0) > 0 = u'(1)$; If there is exactly one root x_1 (not local minimum), then $u^{(2)}(x)(u''(x)) < 0$ for $x \in (0, x_1)$ and $u^{(2)}(x) > 0(u''(x) > 0)$ for $x \in (x_1, 1)$, which implies that $u'(x)$ is firstly strictly decreasing and later strictly increasing. With $u'(0) > 0, u'(1) = 0$, we have $u'(x) = 0$ has a unique solution x_2 in $(0,1)$ and $u'(x) > 0$ for $x \in (0, x_2)$, $u'(x) < 0$ for $x \in (x_2, 1)$; If there is exactly two roots $0 < x_1 < x_2 < 1$, then $u^{(2)}(x) > 0(u''(x) > 0)$ for $x \in (0, x_1) \cup (x_2, 1)$ and $u^{(2)}(x) < 0(u''(x) < 0)$ for $x \in (x_1, x_2)$, which implies that $u'(x)$ is firstly strictly increasing in $(0, x_1)$, then strictly deceasing in (x_1, x_2) and later strictly increasing in $(x_2, 1)$. With $u'(0) > 0, u'(1) = 0$, we have $u'(x) = 0$ has a unique solution x_3 in $(0,1)$ and $u'(x) > 0$ for $x \in (0, x_3)$, $u'(x) < 0$ for $x \in (x_3, 1)$. In all, $u'(x)$ is firstly positive and later negative. With $u(0) < 0, u(1) = 0$, we have $u(x) = 0$ has a unique solution x_4 in $(0,1)$ and $u(x) < 0$ for $x \in (0, x_4)$, $u(x) > 0$ for $x \in (x_4, 1)$; $u(x)$ is strictly increasing on $[0, x_4]$.

If $b = 0$, $f_1(x;a,b) = u(x)$ has a unique solution on $(0,1)$ and is negative on the left and positive on the right. If $b > 0$ then the function $b\beta\gamma(\beta+\gamma)x^{\gamma+1}$ is strictly positive in the interval $(0,1)$. Since $u(x)$ is positive in $(x_4, 1)$, $f_1(x;a,b)$ should also be strictly positive in this interval. In other words, $f_1(x;a,b) = 0$ has no roots in $(x_4, 1)$. However, $f_1(x;a,b)$ is increasing in $(0, x_4)$ because both of $u_0(x)$ and $b\beta\gamma(\beta+\gamma)x^{\gamma+1}$ are increasing on this region. Note that $f_1(0;a,b) = u_0(0) < 0$ and $f_1(x_4;a,b) > 0$. Thus, the equation $f_1(x;a,b) = 0$ has a unique solution in $(0, x_4)$. Denote it by x^*. We know from the preceding discussion that $f_1(x;a,b) < 0$ for $x \in (0, x^*)$ and $f_1(x;a,b) > 0$ for $x \in (x^*, 1)$.

(2.3) For fixed $\lambda \in (0,1)$ and $b > 0$, $g_2(x;\lambda,b) = 0$ has a unique solution $x^* \in (0,1)$ and $g_2(x;\lambda,b) < 0$ for $x \in (0,x^*)$, $g_2(x;\lambda,b) > 0$ for $x \in (x^*,1)$. Denote $g_2(x) = g_2(x;\lambda,b)$. Since $g_2(0) = -(\beta-1)(\gamma+1)\gamma < 0$ and $g_2(1) = b(1-\lambda)\beta\gamma(\beta+\gamma) > 0$, $g_2(x) = 0$ has solutions in $(0,1)$. Next we show that the solution is unique.

$$\begin{aligned} g_2'(x) =& b\beta\gamma\left((\gamma+1) + (\beta-1)(\beta+\gamma+1)x^{\beta+\gamma} - \lambda\beta(\beta+\gamma)x^{\beta-1}\right) \\ &- (\beta-1)(\gamma+1)\beta(\beta+\gamma)\left(x^{\beta+\gamma-1} - x^{\beta-1}\right). \end{aligned}$$

$g_2'(0) = b\beta\gamma(\gamma+1) > 0$, $g_2'(1) = b(1-\lambda)\beta^2\gamma(\beta+\gamma) > 0$ and $g_2''(x) = \beta(\beta-1)(\beta+\gamma)x^{\beta-2}g_2^{(2)}(x)$, where

$$g_2^{(2)}(x) = b\gamma\left((\beta+\gamma+1)x^{\gamma+1} - \lambda\beta\right) - (\gamma+1)\left((\beta+\gamma-1)x^{\gamma} - (\beta-1)\right).$$

$$(g_2^{(2)})'(x) = \gamma(\gamma+1)x^{\gamma-1}\left(b(\beta+\gamma+1)x - (\beta+\gamma-1)\right).$$

Then in $(0,1)$, $(g_2^{(2)})'(x)$ is firstly negative and later change its sign at most once, which implies that $g_2^{(2)}(x)$ is firstly strictly decreasing and later may be strictly increasing. Then $g_2^{(2)}(x) = 0 (g_2''(x) = 0)$ has at most two roots in $(0,1)$. And then $g_2'(x)$ has at most three monotonic interval. With $g_2'(0) > 0, g_2'(1) > 0$, we have that $g_2'(x) = 0$ has at most two roots in $(0,1)$. If there is no root or one root (local minimum of $g_2'(x)$), then $g_2'(x) > 0$ on $(0,1)$, except at most one point, which implies that $g_2(x)$ is strictly increasing and only has one zero point on $(0,1)$; If there is one root x_1(not local minimum of $g_2^{(2)}(x)$) , it contradicts to $g_2'(0) > 0, g_2'(1) > 0$; If there is exactly two roots $0 < x_1 < x_2 < 1$. Then $g_2'(x)$ is positive at the intervals $(0, x_1)$ and $(x_2, 1)$ and negative at the interval (x_1, x_2). Then $g_2(x)$ is strictly increasing at the intervals $(0, x_1)$ and $(x_2, 1)$ and strictly decreasing at the interval (x_1, x_2). Since $g_2(0) < 0$ and $g_2(1) > 0$, to show $g_2(x) = 0$ has a unique solution between 0 and 1, we only need to show that $g_2(x_2) > 0$. Actually $g_2'(x_2) = b\beta\gamma(x_2 f_4(x_2;\lambda))' - (\beta-1)(\gamma+1)f_2'(x_2) = 0$, then

$$\begin{aligned}
g_2(x_2) &= b\beta\gamma x_2 f_4(x_2;\lambda) - (\beta-1)(\gamma+1)f_2(x_2)\\
&= b\beta\gamma x_2 f_4(x_2;\lambda) - \frac{b\beta\gamma(x_2 f_4(x_2;\lambda))'}{f_2'(x_2)} f_2(x)\\
&= \frac{b\beta\gamma}{f_2'(x_2)}(f_2'(x_2)(x_2 f_4(x_2;\lambda)) - f_2(x_2)(x_2 f_4(x_2;\lambda))')\\
&= \frac{b\beta\gamma}{\beta(\beta+\gamma)x_2^{\beta-1}(1-x_2^{\gamma})}(1-x_2^{\beta+\gamma})f_6(x_2;\lambda) > 0
\end{aligned}$$

(2.4) *For fixed* $\lambda \in (0,1)$, $g_3(x;\lambda) = 0$ *has a unique solution* $x^* \in (0,1)$ *and* $g_3(x;\lambda) < 0$ *for* $x \in (0, x^*)$, $g_3(x;\lambda) > 0$ *for* $x \in (x^*, 1)$. Denote $v(x) = -g_3(x;\lambda)$. $v(0) = \lambda\gamma(\beta-1) > 0$, $v(1) = 0$ and

$$\begin{aligned}
v'(x) = {} & \gamma(\beta-1)(\beta+\gamma+1)x^{\beta+\gamma} - \lambda\beta(\gamma+1)(\beta+\gamma)x^{\beta+\gamma-1}\\
& + (\beta+\gamma)(\lambda\beta x^{\beta-1} + (\gamma+1)x^{\gamma}) - \beta(\gamma+1).
\end{aligned}$$

$v'(0) = -\beta(\gamma+1) < 0$, $v'(1) = (1-\lambda)\beta\gamma(\beta+\gamma) > 0$ and $v''(x) = (\beta+\gamma)x^{-1}v^{(1)}(x)$, where

$$\begin{aligned}
v^{(1)}(x) = {} & \gamma(\beta-1)(\beta+\gamma+1)x^{\beta+\gamma} - \lambda\beta(\gamma+1)(\beta+\gamma-1)x^{\beta+\gamma-1}\\
& + \lambda\beta(\beta-1)x^{\beta-1} + \gamma(\gamma+1)x^{\gamma}.
\end{aligned}$$

If $\beta < \gamma+1$, let $v^{(2)}(x) = v^{(1)}(x)/x^{\beta-1}$, that is

$$\begin{aligned}
v^{(2)}(x) = {} & \gamma(\beta-1)(\beta+\gamma+1)x^{\gamma+1} - \lambda\beta(\gamma+1)(\beta+\gamma-1)x^{\gamma}\\
& + \lambda\beta(\beta-1) + \gamma(\gamma+1)x^{\gamma+1-\beta}.
\end{aligned}$$

$v^{(2)}(0) = \lambda\beta(\beta-1) > 0$, $v^{(2)}(1) = (1-\lambda)\beta\gamma(\beta+\gamma) > 0$ and $(v^{(2)})'(x) = \gamma(\gamma+1)x^{\gamma-\beta}v^{(3)}(x)$, where

$$v^{(3)}(x) = (\beta-1)(\beta+\gamma+1)x^{\beta} - \lambda\beta(\beta+\gamma-1)x^{\beta-1} + (\gamma+1-\beta).$$

$v^{(3)}(0) = \gamma+1-\beta > 0$, $v^{(3)}(1) = (1-\lambda)\beta(\beta+\gamma-1) > 0$ and $(v^{(3)})'(x) = (\beta-1)\beta x^{\beta-2}((\beta+\gamma+1)x - \lambda(\beta+\gamma-1))$. Hence $v^{(3)}(x)$ is firstly strictly decreasing and later strictly increasing on $(0,1)$, and $v^{(3)}(x) = 0$ has at most two roots on $(0,1)$. With $v^{(3)}(0) > 0$, $v^{(3)}(1) > 0$, if there are no root or one root, then $v^{(3)}(x) > 0$ on $(0,1)$, except at most one point. Hence $v^{(2)}(x)$ is strictly increasing, with $v^{(2)}(0) > 0$, $v^{(2)}(1) > 0$, we have $v^{(2)}(x) > 0$ on $(0,1)$, and so is $v''(x)$. Then $v'(0)$ is strictly increasing from $v'(0) < 0$ to $v'(1) > 0$. Then $v(0)$ is firstly decreasing from $v(0) > 0$, and later increasing to $v(1) = 0$, which implies that $v(x) = 0$ has a unique root on $(0,1)$. If $v^{(3)}(x) = 0$ has exactly two roots $x_1 < x_2$ on $(0,1)$, then $v^{(2)}(x)$ is strictly increasing on $(0,x_1) \cup (x_2,1)$ and strictly decreasing on (x_1,x_2), that is, $v^{(2)}(x)$ has a local minimum at $x = x_2$. If $v^{(2)}(x_2) \geq 0$, then it refers to the previous case. If $v^{(2)}(x_2) < 0$, then $v^{(2)}(x) = 0$ has exactly two roots $x_3 < x_4$ on $(0,1)$, and so is $v''(x)$. Hence $v'(x)$ has a local maximum at $x = x_3$. Since $v''(x_3) = 0$, we have

$$\gamma(\beta-1)(\beta+\gamma+1)x_3^{\beta+\gamma} = \lambda\beta(\gamma+1)(\beta+\gamma-1)x_3^{\beta+\gamma-1} - \lambda\beta(\beta-1)x_3^{\beta-1} - \gamma(\gamma+1)x_3^{\gamma}.$$

Then

$$\begin{aligned}
v'(x_3) =& \gamma(\beta-1)(\beta+\gamma+1)x_3^{\beta+\gamma} \\
&- \lambda\beta(\gamma+1)(\beta+\gamma)x_3^{\beta+\gamma-1} + (\beta+\gamma)(\lambda\beta x_3^{\beta-1} + (\gamma+1)x_3^{\gamma}) - \beta(\gamma+1) \\
=& \lambda\beta(\gamma+1)(\beta+\gamma-1)x_3^{\beta+\gamma-1} - \lambda\beta(\beta-1)x_3^{\beta-1} - \gamma(\gamma+1)x_3^{\gamma} \\
&- \lambda\beta(\gamma+1)(\beta+\gamma)x_3^{\beta+\gamma-1} + (\beta+\gamma)(\lambda\beta x_3^{\beta-1} + (\gamma+1)x_3^{\gamma}) - \beta(\gamma+1) \\
=& -\beta(\gamma+1)\left(\lambda x_3^{\beta+\gamma-1} - \lambda x_3^{\beta-1} - x_3^{\gamma} + 1\right) \\
=& -\beta(\gamma+1)(1-\lambda x_3^{\beta-1})(1-x_3^{\gamma}) < 0.
\end{aligned}$$

Hence $v(x)$ is firstly decreasing from $v(0) > 0$, and later increasing to $v(1) = 0$, which implies that $v(x) = 0$ has a unique root on $(0,1)$ and positive on the left and negative on the right. Consequently $g_3(x;\lambda) = 0$ has a unique root on $(0,1)$ and negative on the left and positive on the right.

If $\beta \geq \gamma + 1$, by the similar process as above, we also get that $g_3(x;\lambda) = 0$ has a unique root on $(0,1)$ and negative on the left and positive on the right.

(2.5) *For fixed* $a \in (0,1)$, $g_4(x;a) = 0$ *has a unique solution* $x^* \in (0,1)$ *and* $g_4(x;a) < 0$ *for* $x \in (0,x^*)$, $g_4(x;a) > 0$ *for* $x \in (x^*,1)$. $g_4(0;a) = -(1-a)\beta(\gamma+1) < 0$, $g_4(1;a) = a(\beta+\gamma) > 0$ and $\partial g_4(x;a)/\partial x = \gamma(\beta+\gamma)x^{\gamma-1}\left((1-a)(\beta-1)x^{\beta}+1\right) > 0$, we get that $g_4(x;a) = 0$ has a unique root on $(0,1)$ and negative on the left and positive on the right.

(2.6) *For fixed* $a \in (0,1)$, $g_5(x;a) = 0$ *has a unique solution* $x^* \in (0,1)$ *and* $g_5(x;a) < 0$ *for* $x \in (0,x^*)$, $g_5(x;a) > 0$ *for* $x \in (x^*,1)$. Denote $g_5(x) = g_5(x;a)$. Since $g_5(0) = -a(\beta-1)(\gamma+1)\gamma < 0$ and $g_5(1) = 0$.

$$\begin{aligned} g_5'(x) =& \beta\gamma\left((\gamma+1) + (\beta-1)(\beta+\gamma+1)x^{\beta+\gamma} - \beta(\beta+\gamma)x^{\beta-1}\right) \\ & - a(\beta-1)(\gamma+1)\beta(\beta+\gamma)\left(x^{\beta+\gamma-1} - x^{\beta-1}\right). \end{aligned}$$

$g_5'(0) = \beta\gamma(\gamma+1) > 0$, $g_5'(1) = 0$ and $g_5''(x) = \beta(\beta-1)(\beta+\gamma)x^{\beta-2}g_5^{(2)}(x)$, where

$$g_5^{(2)}(x) = \gamma\left((\beta+\gamma+1)x^{\gamma+1} - \beta\right) - a(\gamma+1)\left((\beta+\gamma-1)x^{\gamma} - (\beta-1)\right).$$

$g_5^{(2)}(1) = (1-a)\gamma(\gamma+1) > 0$, and

$$(g_5^{(2)})'(x) = \gamma(\gamma+1)x^{\gamma-1}\left((\beta+\gamma+1)x - a(\beta+\gamma-1)\right).$$

Then in $(0,1)$, $(g_5^{(2)})'(x)$ is firstly negative and later positive, which implies that $g_5^{(2)}(x)$ is firstly strictly decreasing and later strictly increasing. Then $g_5^{(2)}(x) = 0(g_5''(x) = 0)$ has at most two roots in $(0,1)$. If there is no root or one root(local minimum), then $g_5^{(2)}(x) \geq 0(g_5''(x) \geq 0)$ since $g_5^{(2)}(1) > 0$, which implies that $g_5'(x)$ is increasing, contradicting to the fact that $g_5'(0) > g_5'(1)$; If there is one root x_1(not local minimum), then $g_5^{(2)}(x) < 0(g_5''(x) < 0)$ for $x \in (0,x_1)$ and $g_5^{(2)}(x) > 0(g_5''(x) > 0)$ on $(x_1,1)$, which implies that $g_5'(x)$ is firstly strictly decreasing and later strictly increasing. With $g_5'(0) > 0, g_5'(1) = 0$, we have $g_5'(x)$ is firstly positive and later negative on $(0, 1)$. Then $g_5(x)$ is firstly strictly increasing from $g_5(0) < 0$ and later strictly decreasing to $g_5(1) = 0$. Hence $g_5(x) = 0$ has a unique solution on $(0, 1)$; If there is exactly two roots $0 < x_1 < x_2 < 1$, then $g_5^{(2)}(x) > 0(g_5''(x) > 0)$ for $x \in (0, x_1) \cup (x_2, 1)$ and $g_5^{(2)}(x) < 0(g_5''(x) < 0)$ for $x \in (x_1, x_2)$, which implies that $g_5'(x)$ is firstly strictly increasing in $(0, x_1)$, then strictly decreasing in (x_1, x_2) and later strictly increasing in $(x_2, 1)$. With $g_5'(0) > 0, v_3'(1) = 0$, we have $g_5'(x)$ is firstly positive and later negative on $(0, 1)$. Then $g_5(x)$ is firstly strictly increasing from $g_5(0) < 0$ and later strictly decreasing to $g_5(1) = 0$. Hence $g_5(x) = 0$ has a unique solution on $(0, 1)$.

(2.7) For fixed $\lambda \in (0, 1), \kappa \in [0, 1), \rho \in [0, 1 - \lambda]$, $g_6(x; \lambda, \kappa, \rho) = 0$ has a unique solution $x^* \in (0, 1)$ and $g_6(x; \lambda, \kappa, \rho) < 0$ for $x \in (0, x^*)$, $g_6(x; \lambda, \kappa, \rho) > 0$ for $x \in (x^*, 1)$. Note that

$$(1 - x^{\beta+\gamma})g_3(x; \lambda) = xf_1(x)f_4(x; \lambda) - f_2(x)f_3(x; \lambda); \tag{A.10}$$

$$f_3(x; \lambda) + \rho(\beta + \gamma)x^{\gamma+1} = (1 - \rho)f_3(x; \frac{\lambda}{1 - \rho}). \tag{A.11}$$

It is easy to verify that

$$\begin{aligned} g_6(x; \lambda, \kappa, \rho) =& (1 - x^{\beta+\gamma})g_3(x; \lambda) - \rho(\beta + \gamma)x^{\gamma+1}f_2(x) \\ & + \kappa f_2(x)(f_3(x; \lambda) + \rho(\beta + \gamma)x^{\gamma+1}) && \text{(A.12)} \\ =& xf_1(x)f_4(x; \lambda) - (1 - \kappa)f_2(x)(f_3(x; \lambda) + \rho(\beta + \gamma)x^{\gamma+1}) && \text{(A.13)} \\ =& \kappa xf_1(x)f_4(x; \lambda) + (1 - \kappa)\rho xf_1(x)f_4(x; \lambda) \\ & + (1 - \kappa)(1 - \rho)(1 - x^{\beta+\gamma})g_3(x; \frac{\lambda}{1 - \rho}). && \text{(A.14)} \end{aligned}$$

Denote the roots of $g_3(x; \lambda) = 0$, $g_3(x; \frac{\lambda}{1-\rho}) = 0$, $f_3(x; \frac{\lambda}{1-\rho}) = 0$ as $x_1, x_2, x_3 \in (0, 1)$, respectively. Firstly we show that $x_1 \leq x_2 < x_3$. Actually by equation (A.10), $g_3(x_2; \frac{\lambda}{1-\rho}) = 0$ implies that $f_3(x_2; \frac{\lambda}{1-\rho}) = x_2f_1(x_2)f_4(x_2; \frac{\lambda}{1-\rho})/f_2(x_2) > 0$.

By the property of $f_3(x; \frac{\lambda}{1-\rho})$, we have $x_2 < x_3$. Again, by $g_3(x_2; \frac{\lambda}{1-\rho}) = 0$, i.e.

$$\begin{aligned}&\frac{\lambda}{1-\rho}\left(\gamma(\beta-1) - \beta(\gamma+1)x_2^{\beta+\gamma} + (\beta+\gamma)x_2^{\beta}\right)\\ =&x_2\left(\beta(\gamma+1) - \gamma(\beta-1)x_2^{\beta+\gamma} - (\beta+\gamma)x_2^{\gamma}\right),\end{aligned}$$

we have

$$\begin{aligned}g_3(x_2;\lambda) =&x_2\left(\beta(\gamma+1) - \gamma(\beta-1)x_2^{\beta+\gamma} - (\beta+\gamma)x_2^{\gamma}\right)\\ &- \lambda\left(\gamma(\beta-1) - \beta(\gamma+1)x_2^{\beta+\gamma} + (\beta+\gamma)x_2^{\beta}\right)\\ =&x_2\left(\beta(\gamma+1) - \gamma(\beta-1)x_2^{\beta+\gamma} - (\beta+\gamma)x_2^{\gamma}\right)\\ &- (1-\rho)x_2\left(\beta(\gamma+1) - \gamma(\beta-1)x_2^{\beta+\gamma} - (\beta+\gamma)x_2^{\gamma}\right)\\ =&\rho x_2\left(\beta(\gamma+1) - \gamma(\beta-1)x_2^{\beta+\gamma} - (\beta+\gamma)x_2^{\gamma}\right) \geq 0,\end{aligned}$$

where the last inequality holds since for any $x \in (0,1)$, we have

$$\beta(\gamma+1) - \gamma(\beta-1)x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma} > 0.$$

By the property of $g_3(x;\lambda)$, we have $x_1 \leq x_2$.

By (A.14) and the property $g_3(x; \frac{\lambda}{1-\rho})$, for any $x \in (x_2, 1)$,

$$\begin{aligned}g_6(x;\lambda,\kappa,\rho) = &\kappa x f_1(x) f_4(x;\lambda) + (1-\kappa)\rho x f_1(x) f_4(x;\lambda)\\ &+ (1-\kappa)(1-\rho)(1-x^{\beta+\gamma})g_3(x;\frac{\lambda}{1-\rho}) > 0.\end{aligned}$$

With $g_6(0;\lambda,\kappa,\rho) = -\lambda(1-\kappa)\gamma(\beta-1) < 0$, we have that $g_6(x;\lambda,\kappa,\rho) = 0$ has roots on $(0,1)$. Fixed $\lambda \in (0,1)$ and $\rho \in [0, 1-\lambda]$, denote the largest root of $g_6(x;\lambda,\kappa,\rho) = 0$ as $x^* = x^*(\kappa)$, that is, a function of κ, then $x^*(\kappa) \in (0, x_2]$, for any $\kappa \in [0,1)$. Since $g_6(x;\lambda,\kappa,\rho)$ can only have zero points on $(0, x_2]$, from now on we will restrict $x \in (0, x_2]$

For $\kappa = 0$, reminding that $g_3(x_1;\lambda) = 0$, by (A.12) we have

$$g_6(x_1;\lambda,0,\rho) = -\rho(\beta+\gamma)x_1^{\gamma+1} f_2(x) \leq 0.$$

Then $x_1 \leq x^*(0) \leq x_2$. For $\kappa \to 1$, by (A.13) we have $g_6(x_1;\lambda,1,\rho) \to x f_1(x) f_4(x;\lambda)$. Then $x^*(1-) = 0$. Next we show that $\kappa \mapsto x^*(\kappa) : [0,1) \mapsto$

$(0, x^*(0)]$ is a strictly decreasing bijective function, that is, $\partial x^*(\kappa)/\partial\kappa < 0$, for any $\kappa \in [0, 1)$. Actually for $x \in (0, x_2]$, $0 \le \kappa_1 < \kappa_2 < 1$, by equation (A.12), we have

$$\begin{aligned} g_6(x; \lambda, \kappa_2, \rho) - g_6(x; \lambda, \kappa_1, \rho) =& (\kappa_2 - \kappa_1) f_2(x)(f_3(x; \lambda) + \rho(\beta+\gamma)x^{\gamma+1}) \\ =& (\kappa_2 - \kappa_1)(1-\rho) f_2(x) f_3(x; \frac{\lambda}{1-\rho}) > 0, \end{aligned}$$

where $f_3(x; \frac{\lambda}{1-\rho}) > 0$ since $x \le x_2 < x_3$. By the definition of $x^*(\kappa)$, we have that $x^*(\kappa_2) < x^*(\kappa_1)$. Then $\partial x^*(\kappa)/\partial\kappa < 0$, for any $\kappa \in [0, 1)$. It follows that $g_6(x; \lambda, \kappa, \rho)$ has a unique zero point on $(0, 1)$, for any $\kappa \in [0, 1)$. Otherwise, assume for some $\kappa_3 \in [0, 1)$, $g_6(x; \lambda, \kappa_3, \rho)$ has other zero point $y^* \in (0, 1)$ and $y^* < x^*(\kappa_3)$. Then there exists a $\kappa_4 \in (\kappa_3, 1)$, such that $x^*(\kappa_4) = y^*$. And

$$0 = g_6(y^*; \lambda, \kappa_4, \rho) - g_6(y^*; \lambda, \kappa_3, \rho) = (\kappa_4 - \kappa_3)(1-\rho) f_2(y^*) f_3(y^*; \frac{\lambda}{1-\rho}) > 0.$$

Contradiction.

(2.8) *For fixed* $\kappa \in (0, 1)$, $g_7(x; \kappa) = 0$ *has a unique solution* $x^* \in (0, 1)$ *and* $g_7(x; \kappa) < 0$ *for* $x \in (0, x^*)$, $g_7(x; \kappa) > 0$ *for* $x \in (x^*, 1)$. Note that

$$(1 - x^{\beta+\gamma}) g_3(x; 1) = x f_1(x) f_4(x; 1) - f_2(x) f_3(x; 1). \tag{A.15}$$

Then

$$g_7(x; \kappa) = (1 - x^{\beta+\gamma}) g_3(x; 1) + \kappa f_2(x) f_3(x; 1) \tag{A.16}$$

Note that $g_7(0; \kappa) = -(1-\kappa)\gamma(\beta - 1) < 0$. And $g_7^{(i)}(1; \kappa) = 0, i = 0, 1, 2, 3$, $g_7^{(4)}(1; \kappa) = \kappa\beta\gamma(\beta-1)(\gamma+1)(\beta+\gamma)^2 > 0$. We have that $g_7(x; \kappa) = 0$ has roots on $(0, 1)$. Denote the largest root of $g_7(x; \kappa) = 0$ as $x^* = x^*(\kappa)$, that is, a function of κ, then $x^*(\kappa) \in (0, 1)$, for any $\kappa \in (0, 1)$.

For $\kappa \to 0$, reminding that $g_3(x; 1) < 0$, for any $x \in (0, 1)$, we have

$$g_7(x; \kappa) = (1 - x^{\beta+\gamma}) g_3(x; 1) < 0, \text{ for any } x \in (0, 1).$$

Hence $x^*(0+) = 1$. For $\kappa \to 1$,

$$g_7(x; \kappa) = x f_1(x) f_4(x; 1) > 0, \text{ for any } x \in (0, 1).$$

Then $x^*(1-) = 0$. Next we show that $\kappa \mapsto x^*(\kappa) : (0,1) \mapsto (0,1)$ is a strictly decreasing bijective function, that is, $\partial x^*(\kappa)/\partial\kappa < 0$, for any $\kappa \in (0,1)$. Actually for $x \in (0,1)$, $0 \le \kappa_1 < \kappa_2 < 1$, by equation (A.12), we have

$$g_7(x;\kappa_2) - g_7(x;\kappa_1) = (\kappa_2 - \kappa_1) f_2(x) f_3(x;1) > 0.$$

By the definition of $x^*(\kappa)$, we have that $x^*(\kappa_2) < x^*(\kappa_1)$. Then $\partial x^*(\kappa)/\partial\kappa < 0$, for any $\kappa \in (0,1)$. It follows that $g_7(x;\kappa)$ has a unique zero point on $(0,1)$, for any $\kappa \in (0,1)$. Otherwise, assume for some $\kappa_3 \in (0,1)$, $g_7(x;\kappa)$ has other zero point $y^* \in (0,1)$ and $y^* < x^*(\kappa_3)$. Then there exists a $\kappa_4 \in (\kappa_3, 1)$, such that $x^*(\kappa_4) = y^*$. And

$$0 = g_7(y^*;\kappa_4) - g_7(y^*;\kappa_3) = (\kappa_4 - \kappa_3)\kappa f_2(y^*) f_3(y^*;1) > 0.$$

Contradiction.

A.4. Properties of the Candidate Value Functions

Lemma A.2. *When $\lambda b < Pc/\delta$, the smooth pasting condition*

$$\frac{\partial D_1}{\partial v}(v;b,d)|_{v=d} = \lambda \tag{A.17}$$

determines a unique finite number $d^* = d(b) > b$, satisfying $\lambda d^* > Pc/\delta$, such that $D_1(v,b,d) > \lambda v$, for any $b < v < d \le d^*$. Specially if $1-\rho = \lambda$, let $x(b) = b/d(b)$, then $\partial x(b)/\partial b \ge 0$, for $b < Pc/(\delta\lambda)$.

Proof. Note that

$$\beta\gamma = \frac{2r}{\sigma^2}, \quad (\beta-1)(\gamma+1) = \frac{2\delta}{\sigma^2}.$$

It is easy to verify that

$$\frac{\partial D_1}{\partial v}(v;b,d)|_{v=d} - \lambda = -\frac{\sigma^2 cP}{2\delta r b(1 - x^{\beta+\gamma})} \cdot g_1(x;\omega,\epsilon),$$

where $x = b/d, \omega = \delta\lambda b/cP \in (0,1), \epsilon = (1-\rho-\lambda)\omega/\lambda \geq 0$ and $g_1(x;\omega,\epsilon)$ is defined in Appendix A.3. Furthermore, $g_1(x;\omega,\epsilon)$ has a unique solution $x^* \in (0,1)$. Hence, for any given $b < Pc/\delta\lambda$, if we define $d^* := b/x^* > b$, it is a unique solution to (A.17).

To prove that $\lambda d^* > cP/\delta$, that is $b/d^* < \delta\lambda b/cP$ or $x^* < \omega$, it is sufficient to show that

$$g_1(\omega;\omega,\epsilon) > 0 \text{ for any } 0 < \omega < 1, \epsilon \geq 0, \tag{A.18}$$

due to the fact that on $(0,1)$, $g_1(x;\omega,\epsilon)$ is strictly negative on the left side of its zero point x^* and strictly positive on the right side of its zero point x^*. And (A.18) is true, since $g_1(\omega;\omega,\epsilon) \geq \omega f_5(\omega) > 0$ for all $\omega \in (0,1)$, where $f_5(x)$ is defined in Appendix A.3.

By the definition, $D_1(v;b,v) = \lambda v$ for any $v > b$. If we show that, for fixed b, v and $b < v$, $D_1(v;b,d)$ is strictly increasing on $[v, d^*]$ with respect to d, then $D_1(v;b,d) > D_1(v;b,v) = \lambda v$ for all $d \in (v, d^*]$. Actually when taking a partial derivative on the function $D_1(v;b,d)$ with respect to d, we get

$$\frac{\partial D_1}{\partial d}(v;b,d) = \left(\frac{d}{v}\right)^{\gamma} \cdot \frac{v^{\beta+\gamma} - b^{\beta+\gamma}}{d^{\beta+\gamma} - b^{\beta+\gamma}} \cdot \frac{\sigma^2 cP}{2\delta r b(1 - x^{\beta+\gamma})} \cdot g_1(x;\omega,\epsilon).$$

For any $v < d < d^*$, $x = b/d > b/d^* = x^*$. By the property of $g_1(x;\omega,\epsilon)$, we have $g_1(x;\omega,\epsilon) > 0$ for any $x \in (x^*, 1)$ and so does $\partial D_1/\partial d$ for any $b < v < d \leq d^*$.

If $1 - \rho = \lambda$, let $x(b) = b/d(b)$, then $g_1(x(b);\omega,0) = 0$. Solving b in term of $x(b)$, we have $b = \frac{cP}{r\lambda}\frac{x(b)f_1(x(b))}{f_3(x(b);1)}$. Then

$$\frac{\partial x(b)}{\partial b} = \frac{1}{\frac{\partial b}{\partial x}|_{x=x(b)}} = \frac{r\lambda}{cP}\frac{(f_3(x(b);1))^2}{(1-(x(b))^{\beta+\gamma})f_5(x(b))} > 0.$$

□

Lemma A.3. *(i). For any given $d > 0$, the smooth pasting condition*

$$\frac{\partial E_1}{\partial v}(v;b,d)|_{v=b} = 0 \tag{A.19}$$

determines a unique finite number $b^* = b(d) < d$, satisfying $b^* < (1-\kappa)Pc/\delta$, such that $E_1(V, b^*, d) > 0$ for any $v \in (b^*, d)$; (ii). For any given $d > K$, the smooth pasting condition

$$\frac{\partial E_2}{\partial v}(v; b, d)|_{v=b} = 0 \tag{A.20}$$

determines a unique finite number $b^* = b(d) < d$, satisfying $b^* < (1-\kappa)Pc/\delta$, such that $E_2(V, b^*, d) > 0$ for any $v \in (b^*, d)$.

Proof. (i). We can verify that,

$$\frac{\partial E_1}{\partial v}(v; b, d)\Big|_{v=b} = \frac{\sigma^2(1-\kappa)cP}{2r\delta x(1-x^{\beta+\gamma})d} \cdot g_2(x; \lambda, \nu),$$

where $x = b/d, \nu = \delta d/((1-\kappa)cP) > 0$ and $g_2(x; \lambda, \nu)$ is defined in Appendix A.3. Furthermore, $g_2(x; \lambda, \nu)$ has a unique solution $x^* \in (0,1)$. Hence $b^* := d * x^*$ is the unique solution to (A.19).

When $d \leq (1-\kappa)cP/\delta$, $b^* = dx^* < (1-\kappa)cP/\delta$. When $d > (1-\kappa)cP/\delta$, to prove that $b^* < (1-\kappa)cP/\delta$, that is $b^*/d < (1-\kappa)cP/\delta d$ or $x^* < 1/\nu$, it is sufficient to show that

$$g_2(1/\nu; \lambda, \nu) > 0 \text{ for any } \ \lambda \in (0,1), \nu > 1, \tag{A.21}$$

due to the fact that on $(0,1)$, $g_2(x; \lambda, \nu)$ is strictly negative on the left side of its zero point x^* and strictly positive on the right side of its zero point x^*. And (A.21) is true, since $g_2(1/\nu; \lambda, \nu) = f_6(1/\nu; \lambda) > 0$ for all $\lambda \in (0,1), \nu > 1$, where $f_6(x; \lambda)$ is defined in Appendix A.3.

To show $E_1(v; b^*, d) > 0$ for any $b^* < v < d$, we take a partial derivative on $E_1(v; b, d)$ with respect to b,

$$\frac{\partial E_1}{\partial b}(v; b, d) = -\left(\frac{b}{v}\right)^{\gamma} \cdot \frac{d^{\beta+\gamma} - v^{\beta+\gamma}}{d^{\beta+\gamma} - b^{\beta+\gamma}} \cdot \frac{\sigma^2(1-\kappa)cP}{2r\delta x(1-x^{\beta+\gamma})d} \cdot g_2(x; \lambda, \nu),$$

where $x = b/d, \nu = \delta d/((1-\kappa)cP) > 0$. For $b^* < b < d$, $x = b/d > b^*/d = x_*$. Hence $g_2(x; \lambda, \nu) > 0$. Then $\frac{\partial E_1}{\partial b} < 0$. Consequently, $E_1(v; b, d) > E_1(v; v, d) = 0$ for any $v \in (b^*, d)$.

(ii).The proof is the same as the (i). We only need to replace λ by K/d. □

Lemma A.4. *Let $b(d)$ be given as in Lemma A.3 for each d. (i). The smooth pasting condition*

$$\frac{\partial E_1}{\partial v}(v; b(d), d)\Big|_{v=d} = 1 - \lambda, \tag{A.22}$$

determines a unique finite number $d_3^* > 0$, such that for fixed $v > b(d)$, $E_1(v, b(d), d)$ is strictly decreasing on $[0, d_3^*)$ and strictly increasing on (d_3^*, ∞). For $d > d_3^*$, there exists a unique $k_1 \in (b(d), d_3^*)$, such that $E_1(k_1; b(d), d) = (1-\lambda)k_1$ and $E_1(v; b(d), d) > (1-\lambda)v$ for any $v \in (k_1, d)$. Let $x(d) = b(d)/d$, then $\partial x(d)/\partial d < 0$ for $d < d_3^*$.

(ii).If $K \geq \frac{(1-\kappa)cP}{r}$, $E_2(v; b(d), d)$ is strictly increasing with respect to $d > K$ and then $E_2(v; b(d), d) > v - K$ for any $v \in (K, K/\lambda)$. If $K < \frac{(1-\kappa)cP}{r}$, the smooth pasting condition

$$\frac{\partial E_2}{\partial v}(v; b(d), d)\Big|_{v=d} = 1, \tag{A.23}$$

determines a unique finite number $d_2^* > K$, such that for fixed $v \in (b(d), d)$, $E_2(v; b(d), d)$ is strictly increasing on $[K, d_2^*)$ and strictly decreasing on (d_2^*, ∞), following that $E_2(v; b(d_2^*), d_2^*) > v - K$ for any $v \in (K, d_2^*)$.

Proof. (i). It is easy to verify that

$$\frac{\partial E_1}{\partial v}(v; b(d), d)\Big|_{v=d} - (1-\lambda) = \frac{1}{x(1-x^{\beta+\gamma})d} \cdot \left(\frac{(1-\kappa)cP}{r} x f_1(x) - b(d) f_3(x; \lambda)\right),$$

where $x = b(d)/d$. By the proof of Lemma A.3 (i), x also satisfies $g_2(x; \lambda, \nu) = 0$, where $\nu = \delta d/((1-\kappa)cP)$. Then

$$\frac{\partial E_1}{\partial v}(v; b(d), d)\Big|_{v=d} - (1-\lambda) = \frac{(1-\kappa)cP}{r x f_4(x; \lambda) d} \cdot g_3(x; \lambda),$$

where $f_4(x; \lambda), g_3(x; \lambda)$ are defined in Appendix A.3 , $f_4(x; \lambda) > 0$ for all $x \in (0, 1)$ and $g_3(x; \lambda)$ has a unique solution $x_3^* \in (0, 1)$. Plugging into $g_2(x; \lambda, \nu) = 0$, we get $d_3^* = \frac{(1-\kappa)Pcf_2(x_3^*)}{r x_3^* f_4(x_3^*; \lambda)}$ is the unique solution to (A.22).

Letting $x_3 = x_3(d) = b(d)/d$, take a partial derivative on $E_1(v; b(d), d)$ with respect to d,

$$\begin{aligned}\frac{\partial E_1(v; b(d), d)}{\partial d} &= \frac{\partial E_1(v; b, d)}{\partial b}\Big|_{b=b(d)} \frac{\partial b(d)}{\partial d} + \frac{\partial E_1(v; b, d)}{\partial d}\Big|_{b=b(d)} \\ &= -\left(\frac{d}{v}\right)^{\gamma} \frac{v^{\beta+\gamma} - b(d)^{\beta+\gamma}}{d^{\beta+\gamma} - b(d)^{\beta+\gamma}} \frac{g_3(x_3(d); \lambda)}{f_2(x_3(d))},\end{aligned}$$

where we have used the fact that $\frac{\partial E_1(v;b,d)}{\partial b}|_{b=b(d)} = 0$ since $b = b(d)$ is a local maximum of $E_1(v; b, d)$ with respect to b. For $\nu_1 < \nu_2$, $g_2(x; \lambda, \nu_1) < g_2(x; \lambda, \nu_2)$ for any $x \in (0, 1)$. Then by the property of $g_2(x; \lambda, \nu)$ and the fact that $g_2(x_3(d); \lambda, \delta d/((1 - \kappa)cP)) = 0$, we have that $\frac{\partial x_3(d)}{\partial d} < 0$ for all $d > 0$. When $0 < d < d_3^*$, $x_3 > x_3^*$. Then $g_3(x_3; \lambda) > 0$ and then $\frac{\partial E_1(v;b(d),d)}{\partial d} < 0$. Hence $E_1(v; b(d), d)$ is strictly decreasing on $[0, d_3^*)$. When $d > d_3^*$, $x_3 < x_3^*$. Then $g_3(x_3; \lambda) < 0$ and then $\frac{\partial E_1(v;b(d),d)}{\partial d} > 0$. Hence $E_1(v; b(d), d)$ is strictly increasing on (d_3^*, ∞).

By the monotonicity, if $d > d_3^*$, for any $v \in [d_3^*, d)$, $E_1(v; b(d), d) > E_1(v; b(v), v) = (1 - \lambda)v$. On the other hand, $E_1(b(d); b(d), d) = 0 < (1 - \lambda)b(d)$. Hence $E_1(v; b(d), d)$ will intersect with $(1 - \lambda)v$ on $(b(d), d_3^*)$. The uniqueness of the intersection follows from the monotonicity of $E_1(v; b(d), d)$ on $(0, d_3^*)$ with respect to d. Denoting the unique intersection as k_1, then $E_1(v; b(d), d) > (1 - \lambda)v$, for any $v \in (k_1, d)$.

(ii). It is easy to verify that

$$\begin{aligned}\frac{\partial E_2}{\partial v}(v; b(d), d)\Big|_{v=d} - 1 =& \frac{1}{(1 - x^{\beta+\gamma})d} \cdot \Big(\frac{(1 - \kappa)cP}{r} f_1(x) \\ & - K(\beta + \gamma x^{\beta+\gamma}) + d(\beta + \gamma)x^{\gamma+1}\Big),\end{aligned}$$

where $x = b(d)/d$. By the proof of Lemma A.3 (ii), x also satisfies $g_2(x; K/d, \nu) = 0$, where $\nu = \delta d/((1 - \kappa)cP)$. That is,

$$d\, x((\gamma + 1) + (\beta - 1)x^{\beta+\gamma}) - K(\beta + \gamma)x^{\beta} - \frac{(1 - \kappa)cP}{r} f_2(x) = 0. \tag{A.24}$$

Then

$$\frac{\partial E_2}{\partial v}(d; b(d), d)\Big|_{v=d} - 1 = -\frac{(1 - \kappa)cP}{r((\gamma + 1) + (\beta - 1)x^{\beta+\gamma})d} \cdot g_4(x; \omega),$$

where $\omega = Kr/((1-\kappa)cP)$ and $g_4(x;\omega)$ is defined in Appendix A.3.

If $K < (1-\kappa)cP/r$, i.e. $\omega \in (0,1)$, $g_4(x;\omega)$ has a unique solution $x_2^* \in (0,1)$. That is

$$(1-\kappa)cP/r = K/v(x_2^*),$$

where

$$v(x) \triangleq \frac{\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma}}{\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma}}. \tag{A.25}$$

Plugging into (A.24), we get $d_2^* = K/u(x_2^*)$ is the unique solution to (A.23), where

$$u(x) \triangleq \frac{x(\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma})}{\beta\gamma(1-x^{\beta+\gamma})}. \tag{A.26}$$

is a strictly increasing function from $(0,1)$ to $(0,1)$, since $u(0) = 0, u(1) = 1$ and

$$u(x)' = \frac{\beta\gamma((\gamma+1) + (\beta-1)(x)^{\beta+\gamma})f_1(x)}{(\beta\gamma(1-(x)^{\beta+\gamma}))^2} > 0.$$

Hence $d_2^* > K$.

Letting $x_2 = x_2(d) = b(d)/d$, take a partial derivative on $E_2(v;b(d),d)$ with respect to d,

$$\frac{\partial E_2(v;b(d),d)}{\partial d} = \left(\frac{d}{v}\right)^{\gamma} \frac{v^{\beta+\gamma} - b(d)^{\beta+\gamma}}{d^{\beta+\gamma} - b(d)^{\beta+\gamma}} \frac{g_4(x_2;\omega)}{(\gamma+1) + (\beta-1)x_2^{\beta+\gamma}}. \tag{A.27}$$

Similarly as the first part proof, we have $\frac{\partial x_2(d)}{\partial d} < 0$ for all $d > K$. Further more we get that $E_2(v;b(d),d)$ is strictly increasing on $[K, d_2^*)$ and is strictly decreasing on (d_2^*, ∞). And then $E_2(v;b(d_2^*),d_2^*) > E_2(v;b(v),v) = v - K$ for any $v \in (K, d_2^*)$.

If $K \geq \frac{(1-\kappa)cP}{r}$, that is $\omega \geq 0$, then $g_4(x;\omega)$ is strictly positive on $(0,1)$. By equation (A.27), $\frac{\partial E_2(v;b(d),d)}{\partial d} > 0$. Hence $E_2(v;b(d),d)$ is strictly increasing with respect to $d > K$. And then $E_2(v;b(d),d) > E_2(v;b(v),v) = v - K$ for any $v \in (K,d)$. □

Lemma A.5. *Let $d_2^* = d_2^*(K)$ be given by Lemma A.4, which is a function of strick price K for $K \in (0,(1-\kappa)cP/r)$. There exists a unique $K_2 \in (0,(1-\kappa)cP/r)$, satisfying $d_2^*(K_2) = K_2/\lambda$ and, $d_2^*(K) < K/\lambda$, for $K \in (0,K_2)$; $d_2^*(K) > K/\lambda$, for $K \in (K_2,(1-\kappa)cP/r)$.*

Proof. By the proof of Lemma A.4, $d_2^* = K/u(x_2^*)$. Then $d_2^* = K/\lambda$ implies that $\lambda = u(x_2^*)$, where $u(x)$ defined by equation (A.26) is strictly increasing bijective function from $(0,1)$ to $(0,1)$. Hence for any $\lambda \in (0,1)$, there is a unique $x_2^* \in (0,1)$ satisfying $\lambda = u(\bar{x}_2^*)$.

For any $K \in (0,(1-\kappa)cP/r)$. Let $x_2^* = x_2^*(K)$ be the unique solution of $g_4(x; Kr/((1-\kappa)cP))$=0, where g_4 is defined in Appendix A.3. Then $K = \frac{(1-\kappa)Pc}{r}v(x_2^*)$, where $v(x)$, given by (A.25), is a strictly decreasing bijective function from $(0,1)$ to $(0,1)$ since $v(0) = 1, v(1) = 0$ and

$$v'(x) = \frac{-\beta\gamma(\beta+\gamma)x^{\gamma-1}((\gamma+1)+(\beta-1)x^{\beta+\gamma})}{(\beta(\gamma+1)-(\beta-1)\gamma x^{\beta+\gamma})^2} < 0.$$

Hence $x_2^*(K) = \frac{r}{(1-\kappa)Pc}v^{-1}(K)$ is a strictly decreasing bijective function from $(0,(1-\kappa)cP/r)$ to $(0,1)$. Following that there exists a unique $K_2 \in (0,(1-\kappa)cP/r)$, such that $\bar{x}_2^*$ is the unique solution to $f_5(x; K_2r/((1-\kappa)cP)) = 0$ on $(0,1)$, and $d_2^*(K_2) = K_2/\lambda$.

Recalling that $K/d_2^*(K) = u(x_2^*(K))$, taking derivative with respect to K on both side, we get that

$$\frac{\partial(K/d_2^*(K))}{\partial K} = \frac{\partial u(x)}{\partial x}|_{x=x_2^*}\frac{\partial x_2^*(K)}{\partial K} = \frac{r}{(1-\kappa)Pc}u'(x_2^*)(v^{-1})'(K) < 0.$$

Hence for any fixed λ, when $K < K_2$, $K/d_2^*(K) > K_2/d_2^*(K_2) = \lambda$, which implies $d_2^*(K) < K/\lambda$. Similarly, $\frac{(1-\kappa)Pc}{r} > K > K_2 = K(\lambda)$, we have $d_2^*(K) > K/\lambda$. □

Lemma A.6. *When $\lambda d > (1-\kappa)Pc/\delta$, the smooth pasting condition*

$$\frac{\partial E_3}{\partial v}(v; b(d), d)\,|_{v=d} = 1-\lambda, \tag{A.28}$$

determines a unique finite number $b^* = b(d) < d$, satisfying $\lambda b^* < (1-\kappa)Pc/\delta$, such that $E_3(v; b, d) > \lambda v$, for any $b^* \le b < v < d$. Furthermore, fixed $v > b(d)$, $E_3(v; b(d), d)$ is strictly increasing with respect to d for $d > (1-\kappa)Pc/\delta\lambda$.

Proof. It is easy to verify that

$$\frac{\partial E_3}{\partial v}(v; b, d)|_{v=b} - (1-\lambda) = \frac{\lambda\sigma^2}{2rx(1-x^{\beta+\gamma})}\cdot g_5(x;\omega),$$

where $x = b/d, \omega = (1-\kappa)cP/\delta\lambda d \in (0,1)$ and $g_5(x;\omega)$ is defined in Appendix A.3. Furthermore, $g_5(x;\omega)$ has a unique solution $x^* \in (0,1)$. Hence, for any given d, if we define $b^* := d\,x^*$, it is a unique solution to (A.28).

To prove that $\lambda b^* < (1-\kappa)Pc/\delta$, that is $b^*/d < (1-\kappa)Pc/\delta\lambda d$ or $x^* < \omega$, it is sufficient to show that

$$g_5(\omega;\omega) > 0 \text{ for any } 0 < \omega < 1, \tag{A.29}$$

due to the fact that on $(0,1)$, $g_5(x;\omega)$ is strictly negative on the left side of its zero point x^* and strictly positive on the right side of its zero point x^*. And (A.29) is true, since $g_5(\omega;\omega) = \omega f_6(\omega;1) > 0$ for all $\omega \in (0,1)$, where $f_6(x;1)$ is defined in Appendix A.3.

Fix $v < d$, taking a partial derivative on the function $E_3(v;b,d)$ with respect to b,

$$\frac{\partial E_3}{\partial b}(v;b,d) = -\left(\frac{b}{v}\right)^{\gamma} \cdot \frac{d^{\beta+\gamma} - v^{\beta+\gamma}}{d^{\beta+\gamma} - b^{\beta+\gamma}} \cdot \frac{\lambda\sigma^2}{2rx(1-x^{\beta+\gamma})} \cdot g_5(x;\omega).$$

For any $b > b^*$, $x = b/d > b^*/d = x^*$. By the property of $g_5(x;\omega)$, we have $g_5(x;\omega) > 0$ for any $x \in (x^*,1)$ and then $\partial E_3/\partial b < 0$ for any $b^* < b < d$.

Consequently, if fix $v < d$ and regard E_3 as a function of b, it should be strictly decreasing on $[b^*, v]$. We then have $E_3(v;b,d) > E_3(v;v,d) = \lambda v$ for all $b \in (b^*, v]$.

Take a partial derivative on $E_3(v;b(d),d)$ with respect to d, the last statement follows from,

$$\begin{aligned}\frac{\partial E_3(v;b(d),d)}{\partial d} &= \frac{\partial E_3(v;b,d)}{\partial b}\Big|_{b=b(d)} \frac{\partial b(d)}{\partial d} + \frac{\partial E_3(v;b,d)}{\partial d}\Big|_{b=b(d)} \\ &= \lambda\left(\frac{d}{v}\right)^{\gamma} \frac{v^{\beta+\gamma} - b(d)^{\beta+\gamma}}{d^{\beta+\gamma} - b(d)^{\beta+\gamma}} \frac{(-g_3(x_4;1))}{f_2(x_4)}\Big|_{x_4=b(d)/d} > 0,\end{aligned}$$

where $-g_3(x;1)$ and $f_2(x)$ are defined in Appendix A.3, which are both positive for $x \in (0,1)$. □

Lemma A.7. *For $K < K_1$, we have*

$$E^*(K/\lambda-) \geq E^*(K/\lambda+),$$

where E^* is the equity value function defined by Theorem 2.8.

Proof. In Theorem 2.8, there are four cases. Let $\varepsilon > 0$ be positive and small enough.

Case 1: $K \le K_2$ and $K \le K_3$.

$E^*(V) = V - K$ on $(K/\lambda - \varepsilon, K/\lambda)$ and $E^*(V) = (1-\lambda)V$ on $(K/\lambda, K/\lambda + \varepsilon)$.

Hence

$$(E^*)'(K/\lambda-) = 1 > 1 - \lambda = (E^*)'(K/\lambda+).$$

Case 2: $K \le K_2$ and $K > K_3$.

$$E^*(V) = V - K \text{ on } (K/\lambda - \varepsilon, K/\lambda)$$
$$E^*(V) = E_3(V; K/\lambda, d(K/\lambda)) \text{ on } (K/\lambda, K/\lambda + \varepsilon).$$

Hence

$$(E^*)'(K/\lambda-) = 1 > \partial E_3/\partial V(K/\lambda+; K/\lambda, d(K/\lambda)) = (E^*)'(K/\lambda+),$$

where $\partial E_3/\partial V(K/\lambda+; K/\lambda, d(K/\lambda)) < 1$ follows from (A.35) with the condition that $K < K_2 < (1-\kappa)cP/r$.

Case 3: $K > K_2$ and $K \le K_3$.

$$E^*(V) = E_1(V; b(K/\lambda), K/\lambda) \text{ on } (K/\lambda - \varepsilon, K/\lambda)$$
$$E^*(V) = (1-\lambda)V \text{ on } (K/\lambda, K/\lambda + \varepsilon).$$

By Lemma A.4, $E_1(v; b(d), d)$ is strictly decreasing on $(0, d_3^*)$, with respect to d. Since $K/\lambda < K_1/\lambda < d_3^*$, we get that $E_1(v; b(K/\lambda), K/\lambda) < E_1(v; b(v), v) = (1-\lambda)v$ for $v \in (K/\lambda - \varepsilon, K/\lambda)$. Together with the fact that $E_1(K/\lambda; b(K/\lambda), K/\lambda) = (1-\lambda)K/\lambda$, we get that

$$(E^*)'(K/\lambda-) = \partial E_1/\partial V(K/\lambda-; b(K/\lambda), K/\lambda) \ge 1 - \lambda = (E^*)'(K/\lambda+).$$

Case 4: $K > K_2$ and $K > K_3$.

$$E^*(V) = E_1(V; b(K/\lambda), K/\lambda) \text{ on } (K/\lambda - \varepsilon, K/\lambda)$$
$$E^*(V) = E_3(V; K/\lambda, d(K/\lambda)) \text{ on } (K/\lambda, K/\lambda + \varepsilon).$$

We need to show

$$\partial E_1(K/\lambda-, b(K/\lambda), K/\lambda)/\partial V \geq \partial E_3(K/\lambda+, K/\lambda, d(K/\lambda))/\partial V.$$

Similar as Lemma A.4,

$$\frac{\partial E_1}{\partial V}(V; b(K/\lambda), K/\lambda)\Big|_{V=K/\lambda} - (1-\lambda) = \frac{(1-\kappa)cP}{r x_5^* f_4(x_5^*;\lambda)(K/\lambda)} \cdot g_3(x_5^*;\lambda), \quad \text{(A.30)}$$

where $x_5^* = b(K/\lambda)/(K/\lambda)$, satisfying $g_2(x_5^*; \lambda, \delta(K/\lambda)/((1-\kappa)cP)) = 0$. And

$$\frac{\partial E_3(V, K/\lambda, d(K/\lambda))}{\partial V}\Big|_{V=K/\lambda} - (1-\lambda) = \frac{cP}{r f_3(x_6^*;1)(K/\lambda)} g_7(x_6^*;\kappa), \quad \text{(A.31)}$$

where $x_6^* = (K/\lambda)/d(K/\lambda)$, satisfying $g_1(x_6^*; \delta K/(cP), 0) = 0$. Numerical test shows that the right hand side of (A.30) is always bigger than the right hand side of (A.31). □

A.5. Proof of Lemma 2.4, 2.6 and 2.7

Proof of Lemma 2.4. Please note that the definition and properties of elementary functions $f_i (1 \leq i \leq 6), g_j (1 \leq j \leq 7)$ ar all given in Appendix A.3.

Let $x = V_b^*/V_{con}^*$. Substituting the expressions of D_1 and E_1 into (2.15) and (2.16). The smooth pasting condition (2.16) gives

$$g_1(x; \omega, (1-\rho-\lambda)\omega/\lambda) = 0,$$

where $\omega = \delta\lambda V_b^*/cP$; The smooth pasting condition (2.15) gives

$$g_2(x; \lambda, \nu) = 0,$$

where $\nu = \delta V_{con}^*/((1-\kappa)cP)$. Note that $\omega = \lambda(1-\kappa)x\nu = \lambda(1-\kappa)(\beta-1)(\gamma+1)f_2(x)/(\beta\gamma f_4(x;\lambda))$. Plugging into g_1, we have

$$0 = g_1(x; \omega, (1-\rho-\lambda)\omega/\lambda) = \frac{(\beta-1)(\gamma+1)}{f_4(x;\lambda)} g_6(x; \lambda, \kappa, \rho).$$

Furthermore, $g_6(x; \lambda, \kappa, \rho) = 0$ has a unique solution $x_1^* \in (0,1)$. Plugging x_1^* into g_1 and g_2, we get

$$\begin{cases} V_b^* = \frac{(1-\kappa)cP}{r} \frac{\gamma+\beta(x_1^*)^{\beta+\gamma}-(\beta+\gamma)(x_1^*)^{\beta}}{(\gamma+1)+(\beta-1)(x_1^*)^{\beta+\gamma}-\lambda(\beta+\gamma)(x_1^*)^{\beta-1}}, \\ V_{con}^* = \frac{cP}{r} \frac{\beta+\gamma(x_1^*)^{\beta+\gamma}-(\beta+\gamma)(x_1^*)^{\gamma}}{\lambda(\beta-1)+\lambda(\gamma+1)(x_1^*)^{\beta+\gamma}-(1-\rho)(\beta+\gamma)(x_1^*)^{\gamma+1}}, \end{cases} \quad \text{(A.32)}$$

with $x_1^* = V_b^*/V_{con}^*$. Besides, $V_b^* = V_b(V_{con}^*) < (1-\kappa)cP/\delta$ and $E_1(V; V_b^*, V_{con}^*) > 0$, for any $V_b^* < V < V_{con}^*$, according to Lemma A.3. Furthermore, $\lambda V_b^* < V_b^* < (1-\kappa)cP/\delta < cP/\delta$, by Lemma A.2, $V_{con}^* = V_{con}(V_b^*) > cP/\delta\lambda$ and $D_1(V; V_b^*, V_{con}^*) > \lambda V$, for any $V_b^* < V < V_{con}^*$.

For the given V_{con}^*, k_1, which satisfies $E_1(k_1; V_b^*, V_{con}^*) = (1-\lambda)k_1$, is given by Lemma A.4 (If $V_{con}^* \leq d_3^*$, let $k_1 = V_{con}^*$). For $K > \lambda k_1$, to show that $E_1(V; V_b^*, V_{con}^*) > h(v) = \min\{(v-K)^+, (1-\lambda)V\}$, for any $V_b^* < V < V_{con}^*$, we still need to show that

$$E_1(V; V_b^*, V_{con}^*) \geq V - K \text{ for } K_1 < K < V < k_1, \tag{A.33}$$

$$E_1(V; V_b^*, V_{con}^*) \geq (1-\lambda)V \text{ for } k_1 < V < V_{con}^*.. \tag{A.34}$$

The equation (A.34) follows from the definition of k_1 and Lemma A.4. Define $K_1 = \lambda k_1$. If $K_1 \geq (1-\kappa)cP/r$, by Lemma A.4 (ii),

$$E_1(V; V_b^*, V_{con}^*) > V - K_1 \geq V - K, \text{ for any } K_1 \leq K < V < k_1.$$

If $K_2 < K_1 < (1-\kappa)cP/r$ (K_2 is given by Lemma A.5), by Lemma A.5, $V_{cal,1}^*(K_1) > K_1/\lambda$. By Lemma A.4 (ii), we also have

$$E_1(V; V_b^*, V_{con}^*) > V - K_1 \geq V - K, \text{ for any } K_1 \leq K < V < k_1.$$

Hence if $K_1 > K_2$, the equation (A.33) holds. Inversely, consider $K_1 \leq K_2$.

$$E_1(V; V_b^*, V_{con}^*) = E_2(V; V_b(k_1), k_1)\mathbf{1}_{\{V \leq k_1\}}|_{K=K_1} + E_3(V; k_1, V_{con}^*)\mathbf{1}_{\{k_1 < V \leq V_{con}^*\}}.$$

If $K_1 = K_2$, then the left hand side derivative of $E_1(V; V_b^*, V_{con}^*)$ at k_1 equals 1; If $K_1 < K_2$, $V_{cal,1}^*(K_1) < K_1/\lambda = k_1$. By Lemma A.4, $E_2(V; V_b(k_1), k_1)|_{K=K_1} < E_2(V; V_b(V), V)|_{K=K_1} = V - k_1$ for any $V \in (V_{cal,1}^*(K_1), K_1/\lambda)$. Following that the left hand side derivative of $E_1(V; V_b^*, V_{con}^*)$ at k_1 is no less than 1. On the other hand, consider the right hand side derivative of $E_1(V; V_b^*, V_{con}^*)$ at k_1. With $x = k_1/V_{con}^*$,

$$\begin{aligned} \frac{\partial E_3}{\partial v}(k_1+; k_1, V_{con}^*) - (1-\lambda) &= \frac{K_1 f_4(x;1) - (1-\kappa)cP/r f_2(x)}{(1-x^{\beta+\gamma})V_m} \\ &< \lambda - \lambda\frac{(\beta+\gamma)x^{\beta-1}(1-x)}{(1-x^{\beta+\gamma})} < \lambda, \end{aligned} \tag{A.35}$$

where we have used $K_1 \leq K_2 < (1-\kappa)cP/r$. Then the right hand side derivative of $E_3(v; k_1, V^*_{con})$ at k_1 is strictly less than 1, and so does $E_1(V; V^*_b, V^*_{con})$. This is a contradiction since $E_1(V; V^*_b, V^*_{con})$ should be smooth at k_1. Hence $K_1 > K_2$ always holds. □

Proof of Lemma 2.6. (i). Most results follow from Lemma A.3 (ii), Lemma A.4 (ii) and Lemma A.5 in Appendix A.4. We only need to show that, when $K < K_2$,

$$D_2(V; V^*_b, V^*_{cal,1}) \geq \lambda V \text{ for all } V \in [V^*_b, V^*_{cal,1}]. \tag{A.36}$$

Before we show (A.36), we firstly give some formulas for computing the critical references. The critical early call reference K_2 is given by

$$K_2 = \frac{(1-\kappa)Pc}{r}\frac{\beta(\gamma+1) - (\beta-1)\gamma(x^*_3)^{\beta+\gamma} - (\beta+\gamma)(x^*_3)^{\gamma}}{\beta(\gamma+1) - (\beta-1)\gamma(x^*_3)^{\beta+\gamma}}. \tag{A.37}$$

where x^*_3 is the unique solution on $(0,1)$ of the following equation

$$\frac{x(\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma})}{\beta\gamma(1-x^{\beta+\gamma})} = \lambda.$$

If $K < (1-\kappa)cP/r$, the equation,

$$K = \frac{(1-\kappa)cP}{r}\frac{\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma}}{\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma}},$$

has a unique solution $x^*_2 \in (0,1)$, $x^*_2 = V^*_b/V^*_{cal,1}$ and

$$\begin{cases} V^*_b = K\frac{\beta\gamma(1-(x^*_2)^{\beta+\gamma})}{\beta(\gamma+1)-(\beta-1)\gamma(x^*_2)^{\beta+\gamma}-(\beta+\gamma)(x^*_2)^{\gamma}}, \\ V^*_{cal,1} = K\frac{\beta\gamma(1-(x^*_2)^{\beta+\gamma})}{(x^*_2)(\beta(\gamma+1)-(\beta-1)\gamma(x^*_2)^{\beta+\gamma}-(\beta+\gamma)(x^*_2)^{\gamma})}. \end{cases} \tag{A.38}$$

Now let us return to prove (A.36). When $K < K_2$, by Lemma A.4 (ii), $E_2(V; V^*_b, V^*_{cal,1}) > E_2(V; V_b(K/\lambda), K/\lambda)$ for $V > V_b(K/\lambda)$. Then $E_2(V; V^*_b, V^*_{cal,1})$ will intersect $(1-\lambda)V$ at some $\widehat{V} \in (V^*_{cal,1}, K/\lambda)$. Then $\widehat{V} < K/\lambda < K_2/\lambda < k_1$. Denoting $\overline{V}_{con} = V_{con}(b^*)$, if $\overline{V}_{con} > \widehat{V}$, by Lemma A.2, we have $D_2(V; V^*_b, V^*_{cal,1}) \geq \lambda V$ for all $V \in [V^*_b, V^*_{cal,1}]$.

Now we show that $\overline{V}_{con} > \widehat{V}$. Note that $V^*_b = V_b(\widehat{V})$. Letting $\widehat{x} = V^*_b/\widehat{V}$, by Lemma A.3(i), $g_2(\widehat{x}; \lambda, \nu) = 0$, where $\nu = \delta\widehat{V}/((1-\kappa)cP)$. To show $\overline{V}_{con} > \widehat{V}$, that is $V_b^*/\overline{V}_{con} \le V_b^*/\widehat{V} = \widehat{x}$. By the definition of $\overline{V}_{con}$ and Lemma A.2, $V_b^*/\overline{V}_{con}$ satisfies $g_1(V_b^*/\overline{V}_{con};\omega,\epsilon) = 0$, where $\omega = \delta\lambda V_b^*/cP \in (0,1), \epsilon = (1-\rho-\lambda)\omega/\lambda \ge 0$. By Lemma A.1 in Appendix A.3, g_1 has a unique solution $V_b^*/\overline{V}_{con} \in (0,1)$, and is negative on the left, positive on the right. Hence $V_b^*/\overline{V}_{con} \le \widehat{x}$ is equivalent to that $g_1(\widehat{x};\omega,\epsilon) \ge 0$.

Note that $\omega/\nu = \lambda(1-\kappa)\widehat{x}$. Plugging $g_2(\widehat{x};\lambda,\nu) = 0$ into $g_1(\widehat{x};\omega,\epsilon)$, we have

$$g_1(\widehat{x};\omega,\epsilon) = \frac{(\beta-1)(\gamma+1)}{f_4(\widehat{x};\lambda)} g_6(\widehat{x}).$$

Since $\widehat{V} \le k_1 < d_3^*$, where d_3^* is given by Lemma A.4, $\widehat{x} = V_b^*/\widehat{V} \ge V_b(k_1)/k_1 \ge x_1^*$ due to $\partial x(V_c)/\partial V_c < 0$ in Lemma A.4, where x_1^* is the root of $g_6(x) = 0$ on $(0,1)$. Similarly by Lemma A.1 in Appendix A.3, g_6 has a unique solution $x_1^* \in (0,1)$, and is negative on the left, positive on the right. $\widehat{x} \ge x_1^*$ implies $g_6(\widehat{x}) \ge 0$, and so does $g_1(\widehat{x};\omega,\epsilon)$.

(ii). Let $x = V_{cal,2}^*/V_{con}^*$. The smooth pasting conditions (2.18) give

$$g_1(x;\omega,0) = 0, \text{ where } \omega = \delta\lambda V_{cal,2}^*/cP$$

$$g_5(x;\nu) = 0, \text{ where } \nu = (1-\kappa)cP/(\delta\lambda V_{con}^*)$$

Note that $\omega = (1-\kappa)x/\nu = (1-\kappa)(\beta-1)(\gamma+1)f_2(x)/(\beta\gamma f_4(x;1))$. Plugging in to g_1, we have

$$\begin{aligned} 0 =& g_1(x;\omega,0) = \frac{(\beta-1)(\gamma+1)}{f_4(x;1)}(xf_1(x)f_4(x;1) - (1-\kappa)f_2(x)f_3(x;1)) \\ =& \frac{(\beta-1)(\gamma+1)}{f_4(x;1)} g_7(x;\kappa). \end{aligned}$$

where $g_7(x;\kappa)$ is defined in Appendix A.3. Further more, $g_7(x;\kappa) = 0$ has a unique solution $x_4^* \in (0,1)$. Plugging x_4^* into g_1 and g_5, we get $x_4^* = V_{cal,2}^*/V_{con}^*$ and

$$\begin{cases} V_{cal,2}^* = \frac{(1-\kappa)cP}{\lambda r}\frac{\gamma+\beta(x_4^*)^{\beta+\gamma}-(\beta+\gamma)(x_4^*)^{\beta}}{(\gamma+1)+(\beta-1)(x_4^*)^{\beta+\gamma}-(\beta+\gamma)(x_4^*)^{\beta-1}}, \\ V_{con}^* = \frac{cP}{\lambda r}\frac{\beta+\gamma(x_4^*)^{\beta+\gamma}-(\beta+\gamma)(x_4^*)^{\gamma}}{(\beta-1)+(\gamma+1)(x_4^*)^{\beta+\gamma}-(\beta+\gamma)(x_4^*)^{\gamma+1}}. \end{cases} \tag{A.39}$$

The other statements follow from Lemma A.2 and Lemma A.6. □

Proof of Lemma 2.7. (i). Most results follow from Lemma A.4 (ii) in Appendix A.4. We only need to show that, when $K_2 \leq K < K_1$,

$$D_2(V; V_b^*, V_{cal,1}^*) \geq \lambda V \text{ for all } V \in [V_b^*, V_{cal,1}^*]. \tag{A.40}$$

Again before we show (A.40), we firstly give some formulas for computing the critical references. The optimal default barrier V_b^* in this case is given by

$$V_b^* = \frac{(1-\kappa)cP}{r} \frac{\gamma + \beta(x_5^*)^{\beta+\gamma} - (\beta+\gamma)(x_5^*)^{\beta}}{(\gamma+1) + (\beta-1)(x_5^*)^{\beta+\gamma} - \lambda(\beta+\gamma)(x_5^*)^{\beta-1}}. \tag{A.41}$$

where x_5^* is the unique solution on $(0,1)$ of $g_2(x; \lambda, \nu)$, with $\nu = \delta(K/\lambda)/((1-\kappa)cP)$.

Now let us return to prove (A.40). When $K \in (K_2, K_1)$, let $\widehat{V} = K/\lambda$, then $\widehat{V} < K_1/\lambda = k_1$. Denoting $\overline{V}_{con} = V_{con}(b^*)$, if $\overline{V}_{con} > \widehat{V}$, by Lemma A.2, we get (A.40). The proof for $\overline{V}_{con} > \widehat{V}$ is the same as that in the proof of Lemma 2.6.

(ii). Most results follow from Lemma A.6 (ii) in Appendix A.4. We only need to show that, when $K_3 < K < K_1$,

$$E_3(V; K/\lambda, V_{con}^*) \geq (1-\lambda)V \text{ for all } V \in [K/\lambda, V_{con}^*]. \tag{A.42}$$

Again before we show (A.42), we firstly give the optimal exercise barriers $\{K/\lambda, V_{con}^*\}$ in this case:

$$V_{con}^* = V_{con}^* = \frac{cP}{\lambda r} \frac{\beta + \gamma(x_6^*)^{\beta+\gamma} - (\beta+\gamma)(x_6^*)^{\gamma}}{(\beta-1) + (\gamma+1)(x_6^*)^{\beta+\gamma} - (\beta+\gamma)(x_6^*)^{\gamma+1}}. \tag{A.43}$$

where x_6^* is the unique solution on $(0,1)$ of $g_1(x; \omega, 0)$, with $\omega = \delta K/(cP)$.

Now let us return to prove (A.42). When $K_3 \leq K < cP/\delta)$, denote $\overline{V}_{cal,2} = V_{cal,2}(V_{con}^*)$. To show (A.42), by Lemma A.6, we only need to show that $\overline{V}_{cal,2} \leq K/\lambda$. By definition and Lemma A.6, $\overline{V}_{cal,2}/V_{con}^*$ is the unique solution of $g_5(x; \nu) = 0$, where $\nu = (1-\kappa)cP/(\delta\lambda V_{con}^*)$. Since g_5 is positive if and only if it is on the right hand side its unique solution, to show $\overline{V}_{cal,2} \leq K/\lambda$, we only need to show that $g_5(\widehat{x}; \nu) \geq 0$ where $\widehat{x} = (K/\lambda)/V_{con}^*$.

Consider $1-\rho = \lambda$ in Lemma A.2. Note that $V^*_{con} = V_{con}(K/\lambda)$. $\widehat{x} = (K/\lambda)/V^*_{con}$ satisfies $g_1(\widehat{x};\omega,0)=0$, where $\omega = \delta K/(cP)$. Note that $\nu = (1-\kappa)\widehat{x}/\omega = (1-\kappa)\beta\gamma f_3(\widehat{x};1)/((\beta-1)(\gamma+1)f_1(\widehat{x}))$. Plugging it into $g_5(\widehat{x};\nu)$, we have

$$g_5(\widehat{x};\nu) = \frac{\beta\gamma}{f_1(\widehat{x})} g_7(\widehat{x};\kappa).$$

Since $V^*_{cal,2} \le K/\lambda < cP/(\lambda\delta)$, $x^*_4 \le \widehat{x}$ due to $\partial x(V_b)/\partial V_b > 0$, where $V^*_{cal,2}$ is given in (A.39) and x^*_4 is the unique root of $g_7(x;\kappa)=0$. Since g_7 is positive if and only if it is on the right hand side its unique solution, we get $g_7(\widehat{x}) \ge 0$ and so does $g_5(\widehat{x};\nu)$. □

A.6. Proof of Theorem 2.5, 2.8 and 2.9

Proof of Theorem 2.5. We first prove that τ^*_{con} solves the optimization problem (2.6), given τ^*_b and τ^*_{cal} defined in the theorem. Note that the function D^* defined in the theorem is differentiable on the interval $(V^*_b, +\infty)$ and has a second-order derivative except at $V = V^*_{con}$. Thus, it must be a difference of two convex functions. Applying Ito's formula for linear combination of convex functions (see, e.g., Karatzas and Shreve (1991), Problem 3.6.24 and Corollary 3.7.2), we have

$$\begin{aligned} e^{-r(\tau^*_b\wedge\tau)}D^*(V_{\tau^*_b\wedge\tau}) =& D^*(V) + \int_0^{\tau^*_b\wedge\tau} e^{-ru}\sigma V_u \frac{dD^*}{dV}(V_u)dW_u \\ &- \int_0^{\tau^*_b\wedge\tau} e^{-ru}\mathcal{L}D^*(V_u)\mathbf{1}_{\{V_u\neq V^*_{con}\}}du \end{aligned}$$

for any stopping time $\tau \ge 0$. Taking expectations on both sides and rearranging the order of the terms in the last equality,

$$\begin{aligned} D^*(V) &= E[e^{-r(\tau^*_b\wedge\tau)}D^*(V_{\tau^*_b\wedge\tau}) + \int_0^{\tau^*_b\wedge\tau} e^{-ru}\mathcal{L}D^*(V_u)\mathbf{1}_{\{V_u\neq V^*_{con}\}}du|V_0=V] \\ &-E[\int_0^{\tau^*_b\wedge\tau} e^{-ru}\sigma V_u\frac{dD^*}{dV}(V_u)dW_u|V_0=V]. \end{aligned} \tag{A.44}$$

$D^*(V)$ satisfies $\mathcal{L}D^*(V) = cP$ in the interval (V_b^*, V_{con}^*). When $V > V_{con}^*$, $D^*(V) = \lambda V$ and hence

$$\mathcal{L}D^*(V) = \mathcal{L}[\lambda V] = \delta\lambda V > \delta\lambda V_{con}^* \geq cP,$$

where the last inequality is due to $\lambda V_{con}^* \geq cP/\delta$ by Lemma 2.4. Consequently, $\mathcal{L}D^*(V) \geq cP$ for all $V \in [V_b^*, \infty)$. Furthermore, the boundedness of the function dD^*/dV implies that

$$E\left[\int_0^{\tau_b^* \wedge \tau} e^{-ru}\sigma V_u \frac{dD^*}{dV}(V_u)dW_u \middle| V_0 = V\right] = 0.$$

From all the above observation and (A.44), we obtain that

$$D^*(V) \geq E\left[e^{-r(\tau_b^* \wedge \tau)}D^*(V_{\tau_b^* \wedge \tau}) + \int_0^{\tau_b^* \wedge \tau} e^{-ru}cPdu \middle| V_0 = V\right]$$

for any stopping time τ. Note that when $\tau_b^* < \tau$, $D^*(V_{\tau_b^* \wedge \tau}) = D^*(V_b^*) = (1-\rho)V_{\tau_b^*}$; when $\tau < \tau_b^*$, we have $V_{\tau_b^* \wedge \tau} > V_b^*$ and hence $D^*(V_{\tau_b^* \wedge \tau}) = D^*(V_\tau) > \lambda V_\tau$ according to Lemma 2.4. Therefore,

$$\begin{aligned} D^*(V) \geq & E\Big[e^{-r\tau_b^*}(1-\rho)V_{\tau_b^*}\mathbf{1}_{\{\tau_b^* < \tau\}} + e^{-r\tau}\lambda V_\tau \cdot \mathbf{1}_{\{\tau_b^* > \tau\}} \\ & + \int_0^{\tau_b^* \wedge \tau} e^{-ru}cPdu \Big| V_0 = V\Big]. \end{aligned} \tag{A.45}$$

The inequality (A.45) becomes an equality when we take τ to be τ_{con}^*. Indeed, the right hand side of (A.45) equals to, once we let $\tau = \tau_{con}^*$,

$$\begin{aligned} & E\Big[\left\{e^{-r\tau_b^*}(1-\rho)V_b^* + \frac{cP}{r}[1 - e^{-r\tau_b^*}]\right\} \cdot \mathbf{1}_{\{\tau_b^* < \tau_{con}^*\}} \\ & + \left\{e^{-r\tau_{con}^*}\lambda V_{con}^* + \frac{cP}{r}[1 - e^{-r\tau_{con}^*}]\right\} \cdot \mathbf{1}_{\{\tau_b^* > \tau_{con}^*\}}\Big]. \end{aligned}$$

Note that τ_b^* and τ_{con}^* are respectively the first passage times of V_t hitting a lower boundary V_b^* and an upper boundary V_{con}^*. With the help of notations p and q, it is easy to verify that the right hand side equals to $D^*(V)$. In summary, we have already shown that τ_{con}^* solves the following optimization problem

$$\sup_{\tau \in \mathcal{T}} E\left[e^{-r\tau_b^*}(1-\rho)V_{\tau_b^*}\mathbf{1}_{\{\tau_b^* < \tau\}} + e^{-r\tau}\lambda V_\tau \cdot \mathbf{1}_{\{\tau_b^* > \tau\}} + \int_0^{\tau_b^* \wedge \tau} e^{-ru}cPdu \middle| V_0 = V\right]$$

and D^* is the corresponding optimal value function. In other words, τ^*_{con} is the best response of the bondholder, given that the shareholder chooses τ^*_b and τ^*_{cal} defined as in the theorem statement.

Fix $\tau^*_{con} = \inf\{t \geq 0 : V_t \geq V^*_{con}\}$. The optimality of τ^*_b and τ^*_{cal} can be argued in a similar fashion. The function E^* defined in Theorem 2.5 is twice differentiable on $(0, V^*_{con})$, except at $V = V^*_b$. Invoking the generalized Ito's formula again, for any stopping times τ_b and τ_{cal}, we have

$$\begin{aligned} E^*(V) =& E\Big[e^{-r(\tau^*_{con}\wedge\tau_b\wedge\tau_{cal})}E^*(V_{\tau^*_{con}\wedge\tau_b\wedge\tau_{cal}}) \\ &+ \int_0^{\tau^*_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-ru}\mathcal{L}E^*(V_u)\mathbf{1}_{\{V_u\neq V^*_b\}}du|V_0 = V\Big]. \end{aligned} \tag{A.46}$$

From Lemma 2.4, we know that $V^*_b \leq (1-\kappa)cP/\delta$. For all $V \in (0, V^*_b)$, $E^*(V) = 0$ and thus

$$\mathcal{L}E^*(V) = \mathcal{L}[0] = 0 \geq \delta V - (1-\kappa)cP.$$

For $V \in (V^*_b, V^*_{con})$,

$$\mathcal{L}E^*(V) = \mathcal{L}E_1(V) = \delta V - (1-\kappa)cP.$$

Combining these two facts with (A.46) yields that

$$\begin{aligned} E^*(V) \geq \sup_{\tau_b,\tau_{cal}\in\mathcal{T}} & E\Big[e^{-r(\tau^*_{con}\wedge\tau_b\wedge\tau_{cal})}E^*(V_{\tau^*_{con}\wedge\tau_b\wedge\tau_{cal}}) \\ &+ \int_0^{\tau^*_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-ru}(\delta V_u - (1-\kappa)cP)du\big|V_0 = V\Big]. \end{aligned}$$

In addition, we have

$$E^*(V_{\tau^*_{con}\wedge\tau_b\wedge\tau_{cal}}) \geq (1-\lambda)V^*_{con}\mathbf{1}_{\{\tau^*_{con}<\tau_b\wedge\tau_{cal}\}} + h(V_{\tau_b\wedge\tau_{cal}})\cdot\mathbf{1}_{\{\tau_{con}>\tau_b\wedge\tau_{cal}\}}$$

because $E^*(V) \geq h(V)$ for all $V \leq V^*_{con}$ according to Lemma 2.4. Therefore,

$$\begin{aligned} E^*(V) \quad \geq \quad \sup_{\tau_b,\tau_{cal}\in\mathcal{T}} & E\Big[e^{-r\tau^*_{con}}(1-\lambda)V^*_{con}\mathbf{1}_{\{\tau^*_{con}<\tau_b\wedge\tau_{cal}\}} \\ &+e^{-r(\tau_b\wedge\tau_{cal})}h(V_{\tau_b\wedge\tau_{cal}})\cdot\mathbf{1}_{\{\tau^*_{con}>\tau_b\wedge\tau_{cal}\}} \\ &+\int_0^{\tau^*_{con}\wedge\tau} e^{-ru}(\delta V_u - (1-\kappa)cP)du\big|V_0 = V\Big]. \end{aligned}$$

On the other hand, it is straightforward to verify that the equality holds in the preceding inequality if we choose $\tau_b = \tau_b^*$ and $\tau_{cal} = +\infty$. Consequently, we have established the optimality of τ_b^* and τ_{cal}^* under the given τ_{con}^*. Theorem 2.5 is proved. □

Proof of Theorem 2.8. The proof of this theorem is highly similar as the last one. We provide some sketch only. Suppose that the shareholder has already set up his policies such as

$$\tau_b^* = \inf\{t \geq 0 : V_t \leq V_b^*\} \quad \text{and} \quad \tau_{cal}^* = \inf\{t \geq 0 : V_t \in [V_{cal,1}^*, V_{cal,2}^*]\}.$$

When the company asset value V falls in $[0, V_b^*]$ or $[V_{cal,1}^*, V_{cal,2}^*]$, the game is stopped immediately and the bond and equity values should be given by the theorem statement.

Consider a case when $V \in (V_b^*, V_{cal,1}^*)$. The function D^* defined in the theorem is twice differentiable in this interval. Suppose that V_t starts from $V_0 = V$. Applying Ito's formula, we have

$$\begin{aligned} e^{-r(\tau\wedge\tau_b^*\wedge\tau_{cal}^*)}D^*(V_{\tau\wedge\tau_b^*\wedge\tau_{cal}^*}) =& D^*(V) + \int_0^{\tau\wedge\tau_b^*\wedge\tau_{cal}^*} e^{-ru}\sigma V_u \frac{dD^*}{dV}(V_u)dW_u \\ &- \int_0^{\tau\wedge\tau_b^*\wedge\tau_{cal}^*} e^{-ru}\mathcal{L}D^*(V_u)du \end{aligned} \tag{A.47}$$

for any stopping time τ.

When $V_t \in (V_b^*, V_{cal,1}^*)$, $D^*(V_t) = D_3(V_t; V_b^*, V_{cal,1}^*)$, which satisfies $\mathcal{L}D^*(V_t) = cP$. Therefore,

$$\int_0^{\tau\wedge\tau_b^*\wedge\tau_{cal}^*} e^{-ru}\mathcal{L}D^*(V_u)du = \int_0^{\tau\wedge\tau_b^*\wedge\tau_{cal}^*} e^{-ru}cPdu. \tag{A.48}$$

Take expectations on both sides of (A.47). Combining with (A.48), we have

$$D^*(V) = E\left[\int_0^{\tau\wedge\tau_b^*\wedge\tau_{cal}^*} e^{-ru}cPdu + e^{-r(\tau\wedge\tau_b^*\wedge\tau_{cal}^*)}D^*(V_{\tau\wedge\tau_b^*\wedge\tau_{cal}^*}) \mid V_0 = V\right].$$

In addition, Lemma 2.6 and Lemma 2.7 shows that $D^*(V) = D_3(V; V_b^*, V_{cal,1}^*) \geq \lambda V$ when $V \in (V_b^*, V_{cal,1}^*)$, no matter whether or not $V_{cal,1}^* = K/\lambda$. If $\tau < \tau_b^* \wedge \tau_{cal}^*$, we have $V_{\tau\wedge\tau_b^*\wedge\tau_{cal}^*} \in (V_b^*, V_{cal,1}^*)$ and hence

$$D^*(V_{\tau\wedge\tau_b^*\wedge\tau_{cal}^*}) \geq \lambda V_{\tau\wedge\tau_b^*\wedge\tau_{cal}^*}.$$

On the other hand, if $\tau \geq \tau_b^* \wedge \tau_{cal}^*$, $D^*(V_{\tau\wedge\tau_b^*\wedge\tau_{cal}^*})$ equals to either $D^*(V_{\tau_b^*})$ or $D^*(V_{\tau_{cal}^*})$. By the definitions of τ_b^* and τ_{cal}^*, we know $D^*(V_{\tau_b^*}) = (1-\rho)V_b^*$ and $D^*(V_{\tau_{cal}^*}) = g(V_{cal,1}^*)$. In summary,

$$D^*(V) \geq E\Big[e^{-r(\tau_b^*\wedge\tau_{cal}^*)} g(V_{\tau_b^*\wedge\tau_{cal}^*}) \mathbf{1}_{\{\tau_b^*\wedge\tau_{cal}^*\leq\tau\}} + e^{-r\tau}\lambda V_\tau \cdot \mathbf{1}_{\{\tau_b^*\wedge\tau_{cal}^*>\tau\}} + \int_0^{\tau\wedge\tau_b^*\wedge\tau_{cal}^*} e^{-ru} cP du \Big| V_0 = V\Big].$$

It is straightforward to verify the equality is achieved when we take $\tau = \tau_{con}^*$. So far we have proven that the optimal action for the bondholder is to convert at τ_{con}^*. Emulating the above arguments, we also can establish the optimality of τ_{con}^* when the initial company asset value is in $(V_{cal,2}^*, V_{con}^*)$.

Now turn to investigate the optimal behavior for the shareholder, given that the bondholder converts her security at V_{con}^*. Note that the equity function E^* defined in the theorem is differentiable on $(0, V_{con}^*)$, except for at the point K/λ. Apply the generalized Ito's formula for convex functions (see, e.g., Problem 3.6.24, p. 215, Karatzas and Shreve (1991)),

$$\begin{aligned} & e^{-r(\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*)} E^*(V_{\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*}) \\ = \; & E^*(V) + \int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*} e^{-ru}\sigma V_u \frac{dE^*}{dV}(V_u) dW_u - \int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*} e^{-ru}\mathcal{L}E^*(V_u) du \\ & + e^{-r(\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*)} \Lambda_{\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*} \cdot \left[\lim_{V\downarrow K/\lambda} \frac{dE^*}{dV}(V) - \lim_{V\uparrow K/\lambda} \frac{dE^*}{dV}(V)\right], \end{aligned} \tag{A.49}$$

for any stopping time τ_b and τ_{cal}, where $\{\Lambda_t, t \geq 0\}$ is the local time process of $\{V_t, t \geq 0\}$ at K/λ. According to Lemma A.7, we have that

$$\lim_{V\downarrow K/\lambda} \frac{dE^*}{dV}(V) - \lim_{V\uparrow K/\lambda} \frac{dE^*}{dV}(V) \leq 0,$$

If taking expectations on both sides of (A.49), term rearrangement will lead to

$$E^*(V) \geq E\Big[\int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*} e^{-ru}\mathcal{L}E^*(V_u) du + e^{-r(\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*)} E^*(V_{\tau_b\wedge\tau_{cal}\wedge\tau_{con}^*}) \mid V_0 = V\Big]. \tag{A.50}$$

When $V \in (0, V_b^*)$,

$$\delta V - (1-\kappa)cP < \delta V_b^* - (1-\kappa)cP \leq 0$$

according to Lemma 2.6 and Lemma 2.7. On that interval, $E^*(V) = 0$ and hence $\mathcal{L}E^*(V) = 0$. Accordingly, $\mathcal{L}E^*(V) > \delta V - (1-\kappa)cP$. When $V \in (V_b^*, V_{cal,1}^*)$ or $V \in (V_{cal,2}^*, V_{con}^*)$, E^* satisfies

$$\mathcal{L}E^*(V) = \delta V - (1-\kappa)cP.$$

When $V \in [V_{cal,1}^*, V_{cal,2}^*]$, there are two possibilities for the value of $E^*(V)$: it equals to either $V - K$ or $(1-\lambda)V$. No matter which possibility it is, E^* satisfies $\mathcal{L}E^*(V) \geq \delta V - (1-\kappa)cP$ on the interval $[V_{cal,1}^*, V_{cal,2}^*]$. Indeed, by the discussion in Appendix A.1, $E^*(V) = V - K$ only in the interval $(V_{cal,1}^*, K/\lambda)$ and this interval is not degenerate only if $K < K_2$. When $K < K_2$, $\mathcal{L}E^*(V) = \delta V - rK > \delta V - (1-\kappa)cP$ because $K < K_2 < (1-\kappa)Pc/r$ according to Lemma 2.6. On the other hand, the equity value $E^*(V) = (1-\lambda)V$ in the interval $(K/\lambda, V_{cal,2}^*)$. Therefore,

$$\mathcal{L}E^*(V) = \mathcal{L}[(1-\lambda)V] = \delta V - \lambda\delta V > \delta V - \lambda\delta V_{cal,2}^* \geq \delta V - (1-\kappa)cP$$

since $V_{cal,2}^* \leq (1-\kappa)cP/(\delta\lambda)$ from Lemma 2.6.

So far we have already established

$$\mathcal{L}E^*(V) \geq \delta V - (1-\kappa)cP$$

for all $V \in (0, V_{con}^*)$. In addition, Lemma 2.6 and Lemma 2.7 prove that $E^*(V) \geq h(V)$ for all $V \in (0, V_{con}^*)$. Consequently, (A.50) implies

$$\begin{aligned} E^*(V) \geq & E\Big[\int_0^{\tau_b \wedge \tau_{cal} \wedge \tau_{con}^*} e^{-ru}(\delta V_u - (1 - \kappa cP))du \\ & + e^{-r(\tau_b \wedge \tau_{cal})} h(V_{\tau_b \wedge \tau_{cal}}) \cdot \mathbf{1}_{\{V_{\tau_b \wedge \tau_{cal}} < \tau_{con}^*\}} \\ & + e^{-r\tau_{con}^*} \cdot (1-\lambda) V_{\tau_{con}^*} \cdot \mathbf{1}_{\{\tau_{con}^* < \tau_b \wedge \tau_{cal}\}} \Big| V_0 = V\Big]. \end{aligned}$$

Furthermore, it is easy to verify that the equality holds in the above inequality if substituting τ_b^* and τ_{cal}^* specified in the theorem to its right hand side.

Thus, both of them are respectively optimal default and call times under the assumption that the bondholder converts at τ^*_{con}. We have shown that a Nash equilibrium should be given by what the theorem states. □

Proof of Theorem 2.9. The uniqueness of the value functions in equilibrium is self-evident. Proposition 2.3 proves necessary conditions for a function being the bond or equity value function. Lemmas 2.4, 2.6 and 2.7 further confirms that these necessary conditions can determine the value function uniquely. Theorems 2.5 and 2.8 verify the optimality of the equity and bond functions we find from the necessary conditions in Proposition 2.3. Consequently, the equilibrium value functions should be unique. □

APPENDIX B

APPENDIX FOR CHAPTER 3

B.1. The Non-Singularity of the Matrix N.

Note that $\mathbf{N}$ can be divided into four blocks

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{X}_\gamma \\ \mathbf{C}\mathbf{X}_\beta & \mathbf{D} \end{bmatrix},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are given by

$$\begin{bmatrix} 1 & \cdots & 1 \\ \frac{1}{\eta_1-\beta_1} & \cdots & \frac{1}{\eta_1-\beta_{m+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{\eta_m-\beta_1} & \cdots & \frac{1}{\eta_m-\beta_{m+1}} \end{bmatrix}, \begin{bmatrix} 1 & \cdots & 1 \\ \frac{1}{\eta_1+\gamma_1} & \cdots & \frac{1}{\eta_1+\gamma_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{\eta_m+\gamma_1} & \cdots & \frac{1}{\eta_m+\gamma_{n+1}} \end{bmatrix},$$

$$\begin{bmatrix} 1 & \cdots & 1 \\ \frac{1}{\theta_1+\beta_1} & \cdots & \frac{1}{\theta_1+\beta_{m+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{\theta_n+\beta_1} & \cdots & \frac{1}{\theta_n+\beta_{m+1}} \end{bmatrix}, \begin{bmatrix} 1 & \cdots & 1 \\ \frac{1}{\theta_1-\gamma_1} & \cdots & \frac{1}{\theta_1-\gamma_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{\theta_n-\gamma_1} & \cdots & \frac{1}{\theta_n-\gamma_{n+1}} \end{bmatrix},$$

respectively, $\mathbf{X}_\beta \equiv \text{Diag}\{\overline{x}^{\beta_1}, \ldots, \overline{x}^{\beta_{m+1}}\}$, and $\mathbf{X}_\gamma \equiv \text{Diag}\{\overline{x}^{\gamma_1}, \ldots, \overline{x}^{\gamma_{n+1}}\}$.

To facilitate showing Proposition 3.2, we first notice a famous result about the diagonal dominance matrices in Lemma B.1 and then prove the other two lemmas: Lemma B.2 and Lemma B.4.

Lemma B.1. *(Lévy-Desplanques, Theorem 6.1.11, Roger and Johnson (1985)) If a complex matrix* $\mathbf{P} = (p_{ij})_{n*n}$ *satisfies* $|p_{ii}| > \sum_{j=1,j\neq i}^{n} |p_{ij}|$, *for* $i = 1, \ldots, n$, *then* $\mathbf{P}$ *is non-singular.*

Lemma B.2. *For any* $m \geq 1$, $n \geq 1$, *and* $\{\beta_i\}_{i=1}^{m+1}$ *and* $\{\gamma_j\}_{j=1}^{n+1}$ *satisfying* (3.3) *and* (3.4), *the determinants of the two matrices* $\mathbf{A}$ *and* $\mathbf{D}$ *are given by*

$$det(\mathbf{A}) = \frac{\Pi_{1\leq i<j\leq m+1}(\beta_j - \beta_i)\Pi_{1\leq i<j\leq m}(\eta_i - \eta_j)}{\Pi_{1\leq i\leq m, 1\leq j\leq m+1}(\eta_i - \beta_j)} \neq 0,$$

and

$$det(\mathbf{D}) = \frac{\Pi_{1\leq i<j\leq n+1}(\gamma_j - \gamma_i)\Pi_{1\leq i<j\leq n}(\theta_i - \theta_j)}{\Pi_{1\leq i\leq n, 1\leq j\leq n+1}(\theta_i - \gamma_j)} \neq 0.$$

Proof. Due to the analogy between $\mathbf{A}$ and $\mathbf{D}$, we only consider the matrix $\mathbf{A}$. To explicitly describing the dependence of the matrix $\mathbf{A}$ on the parameters, we rewrite $\mathbf{A}$ as $\mathbf{A}_{m+1}(\eta_1, \cdots, \eta_m; \beta_1, \cdots, \beta_{m+1})$ for any $m = 0, 1\cdots$, where $(m+1)$ in the subscript denotes the dimension. When $m = 1$, it is trivial that $\det(\mathbf{A}_1(;\beta_1)) = 1$. When $m \geq 1$, to calculate its determinant, a natural idea is to perform elementary operations such that the first m elements in the last column of $\mathbf{A}$ becomes zero. Subtract the $(m+1)^{th}$ linear equation times $(\eta_m - \beta_{m+1})$ from the first linear equation, and subtract the $(m+1)^{th}$ linear equation times $(\eta_m - \beta_{m+1})$ from the $(i+1)^{th}$ linear equation times $(\eta_i - \beta_{m+1})$ for $i = 1, 2, \cdots, m-1$. Then after eliminating the last row and the last column, we obtain an $m \times m$ matrix as follows.

$$\begin{pmatrix} \frac{\beta_{m+1}-\beta_1}{\eta_m-\beta_1} & \cdots & \frac{\beta_{m+1}-\beta_m}{\eta_m-\beta_m} \\ \frac{\beta_{m+1}-\beta_1}{\eta_m-\beta_1}\frac{\eta_1-\eta_m}{\eta_1-\beta_1} & \cdots & \frac{\beta_{m+1}-\beta_m}{\eta_m-\beta_m}\frac{\eta_1-\eta_m}{\eta_1-\beta_m} \\ \frac{\beta_{m+1}-\beta_1}{\eta_m-\beta_1}\frac{\eta_2-\eta_m}{\eta_2-\beta_1} & \cdots & \frac{\beta_{m+1}-\beta_m}{\eta_m-\beta_m}\frac{\eta_2-\eta_m}{\eta_2-\beta_m} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{m+1}-\beta_1}{\eta_m-\beta_1}\frac{\eta_{m-1}-\eta_m}{\eta_{m-1}-\beta_1} & \cdots & \frac{\beta_{m+1}-\beta_m}{\eta_m-\beta_m}\frac{\eta_{m-1}-\eta_m}{\eta_{m-1}-\beta_m} \end{pmatrix}$$

where for any $j = 1, 2, \cdots, m+1$, the (i,j) element equals $\frac{\beta_{m+1}-\beta_j}{\eta_{m+1}-\beta_j}$ if $i = 1$ and equals $\frac{\beta_{m+1}-\beta_j}{\eta_m-\beta_j}\frac{\eta_{i-1}-\eta_m}{\eta_{i-1}-\beta_j}$ if $i = 2, 3, \cdots, m+1$. Taking the common factor $\frac{\beta_{m+1}-\beta_j}{\eta_m-\beta_j}$ out of the j^{th} column for all $j = 1, 2, \cdots, m$ and taking the common factor $(\eta_{i-1} - \eta_m)$ out of the i^{th} row for all $i = 2, 3, \cdots, m$, we can obtain the following recursion for any $m = 1, 2 \cdots$.

$$\det(\mathbf{A}_{m+1}(\eta_1, \cdots, \eta_m; \beta_1, \cdots, \beta_{m+1})) = \frac{\prod_{j=1}^{m}(\beta_{m+1} - \beta_j) \prod_{i=1}^{m-1}(\eta_i - \eta_m)}{\prod_{j=1}^{m}(\eta_m - \beta_j)} \times \det(\mathbf{A}_m(\eta_1, \cdots, \eta_{m-1}; \beta_1, \cdots, \beta_m)).$$

Noting that $\det(\mathbf{A}_1(;\beta_1)) = 1$ when $m = 1$ and applying the above recursion repeatedly for m times, the argument is completed immediately. □

Remark B.3. *It is easy to establish a simple relationship between the matrix* $\mathbf{A}$ *in Lemma B.2 and the matrix* $\mathbf{A}$ *in Cai and Kou (2008). Then the non-singularities of these two matrices are equivalent. Accordingly, here we actually provide a different approach to show the non-singularity of the matrix* $\mathbf{A}$ *in Cai and Kou (2008). Cai and Kou (2008) achieved this objective by constructing a polynomial function and analyzing its roots.*

Lemma B.4. *Let* $\mathbf{Q} := (\mathbf{D}^{-1}\mathbf{C}\mathbf{X}_\beta\mathbf{A}^{-1}\mathbf{B}\mathbf{X}_\gamma)^T \equiv (q_{ij})_{(n+1)*(n+1)}$. *We have* $q_{kl} > 0$ *and* $\sum_{l=1}^{n+1} q_{kl} < 1$.

Proof. Consider the matrix $\mathbf{A}^{-1}\mathbf{B}$. Let z_{kl} be the k^{th} row, l^{th} column element of the matrix $\mathbf{A}^{-1}\mathbf{B}$, then by Cramer's rule,

$$z_{kl} = \frac{\det(\mathbf{A}_{kl})}{\det(\mathbf{A})}, \tag{B.1}$$

where $\mathbf{A}_{kl}$ is the matrix formed by replacing the k^{th} column of matrix $\mathbf{A}$ by the l^{th} column of matrix $\mathbf{B}$, that is

$$\mathbf{A}_{kl} = \begin{bmatrix} 1 & \cdots & 1_{(k^{th}\text{ column})} & \cdots & 1 \\ \frac{1}{\eta_1 - \beta_1} & \cdots & \frac{1}{\eta_1 + \gamma_l} & \cdots & \frac{1}{\eta_1 - \beta_{m+1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{1}{\eta_m - \beta_1} & \cdots & \frac{1}{\eta_m + \gamma_l} & \cdots & \frac{1}{\eta_m - \beta_{m+1}} \end{bmatrix}.$$

Since $\mathbf{A}_{kl}$ has the same structure as matrix $\mathbf{A}$, similar as Lemma B.2, we have,

$$\det(\mathbf{A}_{kl}) = \frac{\prod_{1\le i<j\le m+1, i\ne k, j\ne k} (\beta_j - \beta_i) \prod_{1\le i<k} (-\gamma_l - \beta_i)}{\prod_{1\le i\le m, 1\le j\le m+1, j\ne k} (\eta_i - \beta_j)} \cdot \frac{\prod_{k<j\le m+1} (\beta_j + \gamma_l) \prod_{1\le i<j\le m} (\eta_i - \eta_j)}{\prod_{1\le i\le m} (\eta_i + \gamma_l)}$$

Then :

$$z_{kl} = \frac{\prod_{i=1}^{k-1}(-\gamma_l - \beta_i) \prod_{j=k+1}^{m+1} (\beta_j + \gamma_l) \prod_{i=1}^{m}(\eta_i - \beta_k)}{\prod_{i=1}^{k-1}(\beta_k - \beta_i) \prod_{j=k+1}^{m+1} (\beta_j - \beta_k) \prod_{i=1}^{m}(\eta_i + \gamma_l)} > 0 \qquad \text{(B.2)}$$

Furthermore, since the elements of the first row of matrices $\mathbf{A}$ and $\mathbf{B}$ are all 1, by matrix multiplication, we have

$$\sum_{k=1}^{m+1} z_{kl} = 1, 1 \le l \le n+1. \qquad \text{(B.3)}$$

Let y_{ij} be the i^{th} row, j^{th} column element of the matrix $\mathbf{D}^{-1}\mathbf{C}$, since matrices $\mathbf{D}$ and $\mathbf{C}$ have the same structure as matrices $\mathbf{A}$ and $\mathbf{B}$, by the similar process as above, we have

$$y_{ij} > 0, \sum_{k=1}^{n+1} y_{kj} = 1, 1 \le i \le n+1, 1 \le j \le m+1. \qquad \text{(B.4)}$$

By direct computation, the i^{th} row, j^{th} column element of the matrix $\mathbf{X}_\beta \mathbf{A}^{-1}\mathbf{B}\mathbf{X}_\gamma$ is $z_{ij}\overline{x}^{\beta_i+\gamma_j}$. Hence

$$q_{kl} = \left(\mathbf{D}^{-1}\mathbf{C}\mathbf{X}_\beta \mathbf{A}^{-1}\mathbf{B}\mathbf{X}_\gamma\right)_{lk} = \sum_{i=1}^{m+1} y_{li} z_{ik} \overline{x}^{\beta_i+\gamma_k} > 0,$$

and

$$\sum_{l=1}^{n+1} q_{kl} = \sum_{l=1}^{n+1}\sum_{i=1}^{m+1} y_{li} z_{ik} \overline{x}^{\beta_i+\gamma_k} < \sum_{i=1}^{m+1} z_{ik}(\sum_{l=1}^{n+1} y_{li}) = \sum_{i=1}^{m+1} z_{ik} = 1,$$

due to equations (B.2), (B.3), (B.4) and the fact that $0 < \overline{x} < 1$. Lemma $B.4$ is proved. □

Based on the three lemmas above, we start to show Proposition 3.2.

Proof of Proposition 3.2. Since both the matrix $\mathbf{A}$ and $\mathbf{D}$ are non-singular, we can perform elementary operations on the matrix $\mathbf{N}$ as follows.

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{C}\mathbf{X}_\beta\mathbf{A}^{-1} & \mathbf{I} \end{bmatrix} \mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{X}_\gamma \\ \mathbf{0} & \mathbf{D}(\mathbf{I} - \mathbf{D}^{-1}\mathbf{C}\mathbf{X}_\beta\mathbf{A}^{-1}\mathbf{B}\mathbf{X}_\gamma) \end{bmatrix},$$

where $\mathbf{I}$ denotes the identity matrix. It follows that

$$\det(\mathbf{N}) = \det(\mathbf{A})\det(\mathbf{D})\det(\mathbf{I} - \mathbf{D}^{-1}\mathbf{C}\mathbf{X}_\beta\mathbf{A}^{-1}\mathbf{B}\mathbf{X}_\gamma).$$

From Lemma $B.1$ and Lemma $B.4$, we can easily see that $\det(\mathbf{I} - \mathbf{D}^{-1}\mathbf{C}\mathbf{X}_\beta\mathbf{A}^{-1}\mathbf{B}\mathbf{X}_\gamma) \neq 0$. Therefore, $\det(\mathbf{N}) \neq 0$, which completes the proof. □

APPENDIX C

APPENDIX FOR CHAPTER 4

C.1. Roots of the Equation $G(x) = r + a$.

The equation $G(x) = r + a$, with $G(x)$ defined as (4.2), can be reduced down to

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0,$$

where

$$a_4 = \sigma^2, \quad a_3 = 2\mu - \sigma^2(\eta - \theta), \quad a_2 = -\sigma^2\eta\theta - 2\mu(\eta - \theta) - 2\lambda - 2(r + a),$$
$$a_1 = -2\mu\eta\theta - 2\lambda p(\eta + \theta) + 2\lambda\eta + 2(r + a)(\eta - \theta), \quad a_0 = 2(r + a)\eta\theta.$$

It has four roots given by

$$\beta_{1,a} = -\frac{a_3}{4a_4} + \frac{p_1 - p_3}{2}, \quad \beta_{2,a} = -\frac{a_3}{4a_4} + \frac{p_1 + p_3}{2},$$
$$\gamma_{1,a} = \frac{a_3}{4a_4} + \frac{p_1 - p_2}{2}, \quad \gamma_{2,a} = \frac{a_3}{4a_4} + \frac{p_1 + p_2}{2},$$

where

$$p_1 = \sqrt{B_3 + C_0 + C_1}, \quad p_2 = \sqrt{B_4 - C_0 - C_1 - \frac{B_5}{4p_1}}, \quad p_3 = \sqrt{B_4 - C_0 - C_1 + \frac{B_5}{4p_1}},$$

$$B_0 = a_2^2 - 3a_1a_3 + 12a_0a_4, \quad B_1 = 2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4 + 27a_0a_3^2 - 72a_0a_2a_4,$$
$$B_2 = \sqrt{B_1^2 - 4B_0^3}, \quad B_3 = \frac{a_3^2}{4a_4^2} - \frac{2a_2}{3a_4}, \quad B_4 = \frac{a_3^2}{2a_4^2} - \frac{4a_2}{3a_4}, \quad B_5 = \frac{4a_2a_3}{a_4^2} - \frac{8a_1}{a_4} - \frac{a_3^3}{a_4^3},$$
$$B_6 = \sqrt[3]{B_1 + B_2}, \quad C_0 = \frac{\sqrt[3]{2}B_0}{3a_4B_6}, \quad C_1 = \frac{B_6}{3\sqrt[3]{2}a_4}.$$

C.2. Lemma C.1.

Lemma C.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function of class $C^{1,1}$ on $[0,T] \times \mathbb{R}$ and $C^{1,2}$ on $[0,T] \times \mathbb{R} \setminus \{h\}$. The left and right second derivatives $\frac{\partial^2 f}{\partial x^2}(t,h-)$, $\frac{\partial^2 f}{\partial x^2}(t,h+)$ exist. Then, we can find a sequence of $\{f_n\} \in C^{1,2}([0,T] \times \mathbb{R})$ and a positive constant M, independent of t, x, and n, such that (1) $f_n(t,x)$ converges to $f(t,x)$ as $n \to \infty$ for any $(t,x) \in [0,T] \times \mathbb{R}$; (2) $f_n(t,x) \equiv f(t,x)$ for any $(t,x) \in [0,T] \times (-\infty,h] \cup [h+\frac{1}{n},\infty)$; and (3) $\max\{|f_n|,|\frac{\partial f_n}{\partial t}|,|\frac{\partial f_n}{\partial x}|,|\frac{\partial^2 f_n}{\partial x^2}|\} \le M$ for any $(t,x) \in [0,T] \times (h,h+\frac{1}{n})$.*

Proof: Introduce a polynomial to smooth the irregular point at $x = h$ for the function f. Let $f_n(t,x) = f(t,x)$ for $(t,x) \in [0,T] \times (-\infty,h] \cup [h+\frac{1}{n},\infty)$ and $f_n(t,x) = P_n(t,n(x-h))$ for $(t,x) \in [0,T] \times (h,h+\frac{1}{n})$, where P_n is a fifth order polynomial given by

$$P_n(t,x) = \frac{a}{n^2}x^5 + \frac{b}{n^2}x^4 + \frac{c}{n^2}x^3 + \frac{\frac{\partial^2 f}{\partial x^2}(t,h-)}{2n^2}x^2 + \frac{\frac{\partial f}{\partial x}(t,h)}{n}x + V(t,h).$$

f_n must be twice differentiable at $x = h$ and $x = h+1/n$. It is easy to check that f_n has second order derivative at $x = h$ and its differentiability at $x = h+1/n$ is equivalent to requiring a, b, c to satisfy $P_n(t,1) = f(t,h+1/n)$,

$$\frac{\partial P_n(t,1)}{\partial x} = \frac{\partial f(t,h+\frac{1}{n})}{\partial x} \text{ and } \frac{\partial^2 P_n(t,1)}{\partial x^2} = \frac{\partial^2 f(t,h+\frac{1}{n})}{\partial x^2}.$$

That is, $\{a,b,c\}$ is a set of roots of the following linear equations:

$$a+b+c = n(n(f(t,h+\frac{1}{n}) - f(t,h)) - \frac{\partial f(t,h)}{\partial x}) - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,h-); \quad \text{(C.1)}$$

$$5a+4b+3c = n(\frac{\partial f(t,h+\frac{1}{n})}{\partial x} - \frac{\partial f}{\partial x}(t,h)) - \frac{\partial^2 f}{\partial x^2}(t,h-); \quad \text{(C.2)}$$

$$20a+12b+6c = \frac{\partial^2 f(t,h+\frac{1}{n})}{\partial x^2} - \frac{\partial^2 f}{\partial x^2}(t,h-). \quad \text{(C.3)}$$

Note that the foregoing linear equations are solvable for any t and n. Using the conditions of f, we can show that the right hand sides of (C.1-C.3) are in the order of $o(1)$ as $n \to +\infty$. Thus, the coefficients a, b, and c are also in the order of $o(1)$, which yields the property (3). From our construction it is also easy to see that such f_n satisfies (1) and (2). □

C.3. The Property of the Matrix A.

By Gauss elimination of elementary column operation, we can show that the determinant of the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ \frac{1}{\eta - a_1} & \frac{1}{\eta - a_2} & \frac{1}{\eta - a_3} & \frac{1}{\eta - a_4} \\ \frac{1}{\theta + a_1} & \frac{1}{\theta + a_2} & \frac{1}{\theta + a_3} & \frac{1}{\theta + a_4} \end{bmatrix},$$

is given by

$$\det(\mathbf{A}) = -\frac{(\eta+\theta)\Pi_{1\le i<j\le 4}(a_i - a_j)}{\Pi_{1\le i\le 4, 1\le j\le 4}(\eta - a_i)(\theta + a_j)} \neq 0.$$

$\mathbf{A}$ is thus non-singular. Let $\mathbf{b} = (1, b, \frac{1}{\eta - b}, \frac{1}{\theta + b})^T$, then the linear equations

$$\mathbf{Ax} = \mathbf{b}$$

have a unique solution $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)^T$, with

$$x_i^* = \frac{\Pi_{j\neq i}(a_j - b)(\eta - a_i)(\theta + a_i)}{\Pi_{j\neq i}(a_j - a_i)(\eta - b)(\theta + b)}, \qquad i = 1, 2, 3, 4.$$

C.4. Occupation Times with Double Barriers

Our Euler-inversion-based approach can be extended to cover the occupation time that the underlying process spends inside two flat barriers, i.e., a corridor with double barriers. There is one minor technical difficulty remaining: we cannot show non-singularity of an 8×8 matrix rigorously, which we believe is true. However we can choose the Laplace transform parameter 'a' big enough to ensure the non-singular property of the matrix $\mathbf{B}$ (please refer to Remark C.2 below). Note that numerical experiments demonstrate that the matrix should be invertible. Moreover, it turns out that this does not affect the validity of our numerical methods for pricing occupation-time-related options.

In this subsection we first presents the closed-form Laplace transform of the joint distribution of the occupation time with double barriers and the log-return of the underlying at the maturity. Then this result is applied to price corridor options with double barriers, and numerical results are provided in Table C.1. To price other options related to occupation times with two barriers, readers may mimic the arguments in Section 4.

Consider two barriers h and H with $h < H$ and let $\tau_{(h,H)}$ denote the occupation times spent between the lower barrier h and the upper barrier H until the maturity T, that is,

$$\tau_{(h,H)} := \int_0^T \mathbf{1}_{\{h<X_t<H\}} dt.$$

Given any $0 \leq \gamma < \min\{\eta, \theta\}$ and $\rho > 0$, our objective is to compute the following Laplace transform of $\tau_{(h,H)}$ and X_T:

$$V(T, x; \rho, \gamma; h, H) := e^{-rT} \cdot E[e^{-\rho\tau_{(h,H)}+\gamma X_T} | X_0 = x]. \tag{C.4}$$

Following similar derivation as in Theorem 3.1, we can show that such V uniquely solves the following PIDE system:

$$\begin{cases} \frac{\partial V}{\partial t} + \rho \mathbf{1}_{\{h<x<H\}} V = \mathcal{L}V, & \text{for } t \in (0,T] \text{ and } x \in \mathbb{R} \setminus \{h, H\}; \\ V(0,x) = e^{\gamma x}, & \text{for } x \in \mathbb{R}. \end{cases} \tag{C.5}$$

For $a > 0$ satisfying (4.21), consider the Laplace transform of $V(T, x; \rho, \gamma)$ with respect to the maturity T

$$\tilde{u}(x; \rho, \gamma, a; h, H) \triangleq \int_0^\infty e^{-aT} V(T, x; \rho, \gamma) dT.$$

Similarly as in the case of occupation times with single barrier, we can transform the PIDE (C.5) into an OIDE. Some algebra can yield the closed-form solution for $\tilde{u}$ as follows

$$\tilde{u}(x; \rho, \gamma, a; h, H)$$
$$= \begin{cases} \omega_1^L e^{\beta_{1,a}(x-h)} + \omega_2^L e^{\beta_{2,a}(x-h)} - c_L e^{\gamma(x-h)}, & x \leq h; \\ -\omega_1^0 e^{\beta_{1,a+\rho}(x-H)} - \omega_2^0 e^{\beta_{2,a+\rho}(x-H)} - \nu_1^0 e^{-\gamma_{1,a+\rho}(x-h)} & \\ \qquad -\nu_2^0 e^{-\gamma_{2,a+\rho}(x-h)} - c_0 e^{\gamma(x-H)}, & h < x < H; \\ \nu_1^U e^{-\gamma_{1,a}(x-H)} + \nu_2^U e^{-\gamma_{2,a}(x-H)} - c_U e^{\gamma(x-H)}, & x \geq H, \end{cases}$$

where

$$c_L = \frac{e^{\gamma h}}{G(\gamma) - a - r}, \quad c_0 = \frac{e^{\gamma H}}{G(\gamma) - a - r - \rho}, \quad \text{and} \quad c_U = \frac{e^{\gamma H}}{G(\gamma) - a - r}.$$

In other words, the solution $\tilde{u}$ is a linear combination of exponential functions. The coefficients vector

$$\boldsymbol{d} = (\omega_1^L, \omega_2^L, \nu_1^0, \nu_2^0, \omega_1^0, \omega_2^0, \nu_1^U, \nu_2^U)^T$$

satisfies a linear system

$$\boldsymbol{Bd} = \boldsymbol{R}. \tag{C.6}$$

Here $\boldsymbol{R}$ is an 8-dimensional vector

$$\boldsymbol{R} = (c_U - c_0) \cdot \left(x^\gamma, \gamma \bar{x}^\gamma, \frac{\bar{x}^\gamma}{\eta - \gamma}, \frac{x^\gamma}{\theta + \gamma}, 1, \gamma, \frac{1}{\eta - \gamma}, \frac{1}{\theta + \gamma} \right)^T,$$

where $x := e^{h-H}$. $\mathbf{B}$ is an 8×8 matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{N}\mathbf{Z}_{\boldsymbol{\beta}} \\ \mathbf{M}\mathbf{Z}_{\boldsymbol{\gamma}} & \mathbf{N} \end{bmatrix},$$

where $\mathbf{Z}_{\boldsymbol{\beta}}$ and $\mathbf{Z}_{\boldsymbol{\gamma}}$ are two 4×4 diagonal matrices with the diagonal elements being $\{\bar{x}^{\beta_{1,a+\rho}}, \bar{x}^{\beta_{2,a+\rho}}, 0, 0\}$ and $\{0, 0, \bar{x}^{\gamma_{1,a+\rho}}, \bar{x}^{\gamma_{2,a+\rho}}\}$, respectively, and $\mathbf{M}$ and $\mathbf{N}$ are given by

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_{1,a} & \beta_{2,a} & -\gamma_{1,a+\rho} & -\gamma_{2,a+\rho} \\ \frac{1}{\eta - \beta_{1,a}} & \frac{1}{\eta - \beta_{2,a}} & \frac{1}{\eta + \gamma_{1,a+\rho}} & \frac{1}{\eta + \gamma_{2,a+\rho}} \\ \frac{1}{\theta + \beta_{1,a}} & \frac{1}{\theta + \beta_{2,a}} & \frac{1}{\theta - \gamma_{1,a+\rho}} & \frac{1}{\theta - \gamma_{2,a+\rho}} \end{bmatrix},$$

and

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_{1,a+\rho} & \beta_{2,a+\rho} & -\gamma_{1,a} & -\gamma_{2,a} \\ \frac{1}{\eta - \beta_{1,a+\rho}} & \frac{1}{\eta - \beta_{2,a+\rho}} & \frac{1}{\eta + \gamma_{1,a}} & \frac{1}{\eta + \gamma_{2,a}} \\ \frac{1}{\theta + \beta_{1,a+\rho}} & \frac{1}{\theta + \beta_{2,a+\rho}} & \frac{1}{\theta - \gamma_{1,a}} & \frac{1}{\theta - \gamma_{2,a}} \end{bmatrix}.$$

Remark C.2. *To guarantee that the linear system* (C.6) *has a unique solution, we need the condition that the Matrix* $\mathbf{B}$ *is non-singular. Actually this appears to* *be the most different part when generalizing to the double barriers. We haven't proven it rigorously right now. However we can choose the Laplace transform parameter 'a' big enough to ensure the non-singular property of the matrix* **B**.

Note that we can compute the determinant of matrix **M** *and* **N**, *which are non-zero. And* $\beta_{i,a+\rho}, -\gamma_{i,a+\rho}, i = 1, 2$ *are four solutions to*

$$G(x) = a + \rho,$$

and $\beta_{i,a}, -\gamma_{i,a}, i = 1, 2$ *are four solutions to*

$$G(x) = a.$$

When $a \gg \rho$, $\beta_{i,a+\rho} \approx \beta_{i,a}, \gamma_{i,a+\rho} \approx \gamma_{i,a}, i = 1, 2$, *then*

$$\begin{aligned} det(\mathbf{B}) &= det \begin{bmatrix} \mathbf{M} & \mathbf{N}\mathbf{Z}_\beta \\ \mathbf{M}\mathbf{Z}_\gamma & \mathbf{N} \end{bmatrix} = det \left\{ \begin{bmatrix} \mathbf{M} & \mathbf{N}\mathbf{Z}_\beta \\ \mathbf{M}\mathbf{Z}_\gamma & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{Z}_\beta \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \right\} \\ &= det \begin{bmatrix} \mathbf{M} & (\mathbf{N}-\mathbf{M})\mathbf{Z}_\beta \\ \mathbf{M}\mathbf{Z}_\gamma & \mathbf{N}-\mathbf{M}\mathbf{Z}_\gamma\mathbf{Z}_\beta \end{bmatrix} = det \begin{bmatrix} \mathbf{M} & (\mathbf{N}-\mathbf{M})\mathbf{Z}_\beta \\ \mathbf{M}\mathbf{Z}_\gamma & \mathbf{N} \end{bmatrix} \\ &\approx det(\mathbf{M}) * det(\mathbf{N}) \end{aligned}$$

is non-zero, where we have used the facts that $\mathbf{Z}_\gamma \mathbf{Z}_\beta = \mathbf{0}$ *and all the components of matrix* $\mathbf{N} - \mathbf{M}$ *are quite close to zero as* $a \gg \rho$. *With numerical verification, further more, we conjecture that when a satisfies the inequality* (4.21) *in Theorem 4.2, the matrix* **B** *is non-singular.*

The conclusion that **B** is invertible for large a is good enough for our objective of option pricing. Taking inversion on the Laplace transform $\tilde{u}(x; \rho, \gamma, a; h, H)$,

$$V(T, x; \rho, \gamma; h, H) = \frac{1}{2\pi i} \lim_{M \to +\infty} \int_{a-iM}^{a+iM} e^{sT} \tilde{u}(x; \rho, \gamma, s; h, H) ds,$$

The integration in the right hand side is done along any contour path $Re(s) = a$ on the complex plane as long as a is greater than the real part of all singularities of u. Therefore, we can choose such a large a to complete the inversion. Numerical experiments indicate that it works quite well for the Euler inversion. Moreover, it turns out that our pricing methods for occupation-time-related options based on the Laplace transform result of the joint distribution of X_T and $\tau_{(h,H)}$ should be valid. Next we apply the result to price corridor options with double barriers to illustrate the effectiveness of our pricing method. Due to similarities, pricing of other options related to occupation times with two barriers is omitted. Consider a corridor call option with double barriers, whose price is given by

$$Cor(K,T) = e^{-rT}E[\max\{\tau_{(\log(l/S_0),\log(L/S_0))} - K, 0\}],$$

where l and L ($l < L$) are two barriers of the underlying asset price process S_t that starts from S_0. Mimicking the proofs of Theorem 4.7 and Proposition 4.8, the double Laplace transform of $Cor(K,T)$ with respect to K and T

$$\tilde{g}_{cor}(\varphi, a) = \int_0^\infty \int_0^\infty e^{-\varphi K - aT} Cor(K,T) dK dT \tag{C.7}$$

should be equal to

$$\begin{aligned}\tilde{g}_{cor}(\varphi, a) &= -\frac{1}{\varphi}\frac{\partial \tilde{u}}{\partial \rho}(0;0,0,a;\log(l/S_0),\log(L/S_0)) \\ &\quad + \frac{1}{\varphi^2}\tilde{u}(0;\varphi,0,a;\log(l/S_0),\log(L/S_0)) - \frac{1}{(a+r)\varphi^2},\end{aligned}$$

where

$$\begin{aligned}&\frac{\partial \tilde{u}}{\partial \rho}(0;0,0,a;\log(l/S_0),\log(L/S_0)) \\ &= \begin{cases} \tilde{\omega}_1^L \cdot (S_0/l)^{\beta_{1,a}} + \tilde{\omega}_2^L \cdot (S_0/l)^{\beta_{2,a}}, & S_0 \le l; \\ -\tilde{\omega}_1^0 \cdot (S_0/L)^{\beta_{1,a}} - \tilde{\omega}_2^0 \cdot (S_0/L)^{\beta_{2,a}} - \tilde{\nu}_1^0 \cdot (l/S_0)^{\gamma_{1,a}} & \\ \qquad -\tilde{\nu}_2^0 \cdot (l/S_0)^{\gamma_{2,a}} - \frac{1}{(a+r)^2}, & l < S_0 < L; \\ \tilde{\nu}_1^U (L/S_0)^{\gamma_{1,a}} + \tilde{\nu}_2^U (L/S_0)^{\gamma_{2,a}}, & S_0 \ge L \end{cases}\end{aligned}$$

and $\widetilde{\mathbf{d}} = (\tilde{\omega}_1^L, \tilde{\omega}_2^L, \tilde{\nu}_1^0, \tilde{\nu}_2^0, \tilde{\omega}_1^0, \tilde{\omega}_2^0, \tilde{\nu}_1^U, \tilde{\nu}_2^U)^T$ satisfies the following linear system:

$$\mathbf{B}(0)\widetilde{\mathbf{d}} = -\frac{1}{(a+r)^2} \cdot \left(1, 0, \frac{1}{\eta}, \frac{1}{\theta}, 1, 0, \frac{1}{\eta}, \frac{1}{\theta}\right)^T.$$

Her B(0) is B with $\rho = 0$. Inverting the Laplace transform (C.7) via the Euler inversion algorithm, we can price corridor options with double barriers numerically. Numerical results are given in Table C.1, where we can see that all the numerical prices obtained using our pricing method (denoted by EI value) stay within the 95% confidence intervals of the associated MC simulation estimates (denoted by MC value). This demonstrates that our pricing method is also accurate for pricing corridor option with double barriers. In addition, similarly as in the case of corridor options with single barrier, we can also calculate deltas for corridor options with double barriers numerically. Numerical results are also given in Table C.1, which also indicate the effectiveness of our numerical method.

Prices of Corridor Options with Double Barriers under the DEM				
K	S_0	EI value	MC value	Std Err
	95	0.49444505	0.49360862	0.00075583
0.2	100	0.45098018	0.45017582	0.00073051
	105	0.37305021	0.37252713	0.00070721
	95	0.32304472	0.32235313	0.00065997
0.4	100	0.28990787	0.28934707	0.00062511
	105	0.23235612	0.23187688	0.00058409

Deltas of Corridor Options with Double Barriers under the DEM				
K	S_0	EI value	MC value	Std Err
	100	-0.01853381	-0.01852545	0.00007402
0.2	102	-0.02008015	-0.02014496	0.00007719
	104	-0.02149747	-0.02153651	0.00007988
	100	-0.01499858	-0.01496464	0.00007158
0.4	102	-0.01588576	-0.01590179	0.00007396
	104	-0.01664286	-0.01667591	0.00007538

Table C.1: Prices and deltas of corridor options with double barriers (denoted by EI value). The default parameter choices are $\lambda = 3$, $r = 0.05$, $\sigma = 0.2$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $l = 80$ for pricing part or $l = 50$ for delta part, $L = 110$, and $t = 1$. The Monte Carlo simulation estimates (denoted by MC value) along with the associated standard errors (denoted by Std Err) are obtained by using 50,000 time steps for pricing part or 20,000 time steps for delta part and by simulating 100,000 sample paths. The CPU time of our numerical method for generating one corridor option price or delta is around 3 seconds. The CPU times for producing one MC value of corridor option price and one MC value of delta are around 10 minutes and 4.3 minutes, respectively. The table indicates that all the EI values stay within the 95% confidence intervals of the associated MC values.

BIBLIOGRAPHY

[1] Abate, J. and Whitt, W. (1992): The Fourier-series method for inverting transforms of probability distributions. *Queueing Systems*, **10**, pp. 5-88.

[2] Akahori, J. (1995). Some formulae for a new type of path-dependent option. *Annals of Applied Probability*, **5**, pp. 383-388.

[3] Altintig, Z. A., and A. W. Butler, 2005, Are they still called late? The effect of notice period on calls of convertible bonds. *Journal of Corporate Finance* **11**, 337 350.

[4] Andersen, L., and D. Buffum, 2003-2004, Calibration and implementation of convertible bond models. *Journal of Computational Finance* **2**, 1 34.

[5] Asquith, P., 1995. Convertible bonds are not called late. *Journal of Finance* **50**, 1275 1289.

[6] Asquith, P., and D. W. Mullins, Jr., 1991, Convertible debt: corporate call policy and voluntary conversion. *Journal of Finance*, **46**, 1273 1289.

[7] Atkinson, C. and Fusai, G. (2007). Discrete extrema of the Brownian motion and pricing of lookback options. *Journal of Computational Finance*, **10**, pp. 1-43.

[8] Ayache, E., P.A . Forsyth and Kenneth R . Vetzal (2003). Valuation of Convertible Bonds With Credit Risk. *The Journal of Derivatives*, **11**(1), pp. 9-29.

[9] Bensoussan, A. and A., Friedman, 1974. Non-linear variational inequalities and differential games with stopping times. *Journal of Functional Analysis* **16**, 305 352.

[10] Bensoussan, A. and A., Friedman, 1977. Nonzero-sum stochastic differential games with stopping times and free boundary problems. *Transactions of the American Mathematical Society* . **231**, 275 327.

[11] Bielecki, T.R., and M. Rutkowski (2002). *Credit Risk: Modeling, Valuation, and Hedging.* Springer, Berlin.

[12] Bielecki, T. R., Crépey, S., Jeanblanc, M. and M. Rutkowski, 2008, Arbitrage pricing of defaultable game options with applications to convertible bonds. *Quantitative Finance* **8**, 795 810.

[13] Billingsley, R. S., and D.S., Smith ,1996, Why do firms issue convertible bond? *Financial Management*, **25** (2), 93-99.

[14] Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, **81**, 637-654.

[15] Brealey, R. A., and S. C. Myers, 2008, *Principles of Corporate Finance.* 9th edition. McGraw-Hill, New York.

[16] Brennan, M., and E. Schwartz, 1977, Convertible bonds: valuation and optimal strategies for call and conversion, *Journal of Finance* **32**, 1699 1715.

[17] Brennan, M., and E. Schwartz, 1980, Analyzing convertible bonds, *Journal of Financial and Quantitative Analysis* **15**, 907–929.

[18] Brennan, M., and E. Schwartz, 1988, The case for convertibles, *Journal of Applied Corporate Finance* **1**(2), 55-64.

[19] Brigham E.F., 1966, An Analysis of Convertible Debentures: Theory and Some Empirical Evidence. *The Journal of Finance*, **21**(1), 35 54.

[20] Broadie, M., Glasserman, P. and Kou, S. G. (1997): A continuity correction for discrete barrier options. *Mathematical Finance*, **7**, pp. 325-348.

[21] Broadie, M. and Yamamoto, Y. (2005): A double-exponential fast Gauss transform algorithm for pricing discrete path-dependent options. *Operations Research*, **53**, pp. 764-779.

[22] Buckdahn,R., P., Cardaliaguet and C., Rainer, 2004, Nash Equilibrium Payoffs for Nonzero-Sum Stochastic Differential Games *SIAM Journal of Control and Optimization*. **43**(2), 624 642.

[23] Byrd, A. K., and W., Moore, 1996, On the information content of calls of convertible securities. *Journal of Business* **68** (January), 89-101.

[24] Cai, N. (2008a): On first passage times of a hyper-exponential jump diffusion process. Working Paper of the Hong Kong University of Science and Technology.

[25] Cai, N. (2008b). Pricing quantile options in a flexible jump diffusion model. Preprint.

[26] Cai, N. and Kou, S. G. (2008). Option pricing under a hyper-exponential jump diffusion model. Preprint.

[27] Campbell, C. J., Ederington, L. H., and P. Vankudre, 1991, Tax shields, sample-selection bias, and the information content of conversion-forcing bond calls. *Journal of Finance* **46**, 1291-1324.

[28] Carr, P. (1995): Two extensions to barrier option valuation. *Applied Mathematical Finance*, **2**, pp. 173-209.

[29] Cattiaux, P. and J. P. Lepeltier,1990, Existence of an quasi-Markov Nash equilibrium for nonzero sun Markov stopping games, *Stochastics and Stochastics Reports*, **30**, 85-103.

[30] Chen, N., and S. G. Kou, 2009, Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk. *Mathematical Finance* **19**, 343 378.

[31] Chesney, M., Jeanblanc-Picqué, M. and Yor, M. (1997). Brownian excursions and Parisian barrier options. *Advances in Applied Probability*. **29**, pp. 165-184.

[32] Cheung, W., and I. Nelken, 1994, Costing converts. *Risk* **7**, 47 49.

[33] Choudhury, G. L., Lucantoni, D. M. and Whitt, W. (1994): Multidimensional transform inversion with applications to the transient M/G/1 queue. *Annals of Applied Probability*, **4**, pp. 719-740.

[34] Constantinides, G. M. and B. D. Grundy, 1987, Call and conversion of convertible corporate bonds: theory and evidence. Graduate School of Business, University of Chicago.

[35] Cohen, J. W. and Hooghiemstra, G. (1981). Brownian Excursion, the M/M/1 Queue and Their Occupation Times. *Mathematics of Operations Research*, **6**, pp. 608-629.

[36] Cowan, A. R., Nayar, N., and A. K. Singh, 1990, Stock Returns before and After Calls of Convertible Bonds. *The Journal of Financial and Quantitative Analysis*, **25**(4), 549-554.

[37] Cowan, A. R., Nayar, N., and A. K. Singh, 1993, Calls of out-of-themoney convertible bonds. *Financial Management* **22**, 106 116.

[38] Dann L.Y. and W.H. Mikkelson, 1984, Convertible debt issuance, capital structure change and financing-related information: Some new evidence. *Journal of Financial Economics*, **13**, 157-186.

[39] Das, S. (2004): *Swaps/Financial Derivatives: Products, Pricing, Applications and Risk Management*. Volume 3. Third Edition. John Wiley & Sons.

[40] Dassios, A. (1995). The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options. *Annals of Applied Probability*, **5**, pp. 389-398.

[41] Dassios, A. (1996). Sample Quantiles of stochastic processes with stationary and independent increments. *Annals of Applied Probability*, **6**, pp. 1041-1043.

[42] Davis, M. H. A., and F. Lischka, 2002, Convertible bonds with market risk and credit risk. In *Applied Probability*, R. Chan, Y.-K. Kwok, D. Yao, and Q. Zhang, eds., AMS/IP Studies in Advanced Mathematics 26, American Mathematical Society, Providence, RI. pp. 45-58.

[43] Davydov, A. and Linetsky, V. (2001): Pricing and hedging path-dependent options under the CEV process. *Management Science*, **47**, pp. 949-965.

[44] Davydov, A. and Linetsky, V. (2002). Structuring, pricing and hedging double-barrier step options. *Journal of Computational Finance*, **5**, pp. 55-87.

[45] Derman, E and Kani, I. (1996): The ins and outs of barrier options: part 1. *Derivatives Quarterly*, Winter 1996, pp. 55-67.

[46] Derman, E and Kani, I. (1997): The ins and outs of barrier options: part 2. *Derivatives Quarterly*, Spring 1997, pp. 73-80.

[47] Duffie, D. and D. Lando. (2001). Term structures of credit spreads with incomplete accounting information. *Econometrica*, **69**, 599-632.

[48] Duffie, D. and K. Singleton (1999). Modeling term structures of defaultable bonds. *Review of Financial Studies*. **12**, 687-720.

[49] Duffie, D. and K. J. Singleton (2003). *Credit Risk: Pricing, Measurement, and Management.* Princeton University Press, Princeton, N.J.

[50] Dunn K.B. and K.M., Eades, 1989, Voluntary conversion of convertible securities and the optimal call strategy. *Journal of Financial Economics*, **23**(2), 273 301.

[51] Dynkin, E.B., 1969, Game variant of a problem on optimal stopping. *Soviet mathematics Doklady* . **10**, 270 274.

[52] Ederington, L. H., Caton, G. L., and C. J. Campbell, 1997, To call or not to call convertible debt. *Financial Management* **26**, 22 31.

[53] Ederington L. H. and J. C. Goh, 2001, Is a Convertible Bond Call Really Bad News? *Journal of Business*, **74**(3), 459-476.

[54] Embrechts, P., Rogers, L. C. G., and Yor, M. (1995). A proof of Dassios' representation of the α-quantile of Brownian motion with drift. *Annals of Applied Probability*, **5**, pp. 757-767.

[55] Elliott, C. M. and J. R. Ockendon, 1982, *Weak and Variational Methods for Moving and Boundary Problems*, Pitman Advanced Publishing Program, Boston.

[56] Feldmann, A. and Whitt, W. (1998): Fitting mixtures of exponentials to long-tail distributions to analyze network performance models. *Performance Evaluation*, 31, pp. 245-279.

[57] Feng, L. and Linetsky, V., (2008): Pricing discretely monitored barrier options and defaultable bonds in Lévy process models: A fast Hilbert transform approach. *Mathematical Finance*, **18**, pp. 337-384.

[58] Friedman, A., 1988, *Variational Principles and Free-Boundary Problems.* Robert E. Krieger Publishing Company, Malabar, Florida.

[59] Fusai, G. (2000). Corridor options and arc-sine law. *Annals of Applied Probability*, **10**, pp. 634-663.

[60] Fusai, G. and Tagliani, A. (2001). Pricing of occupation time derivatives: continuous and discrete monitoring. *Journal of Computational Finance*, **5**, pp. 1-37.

[61] Gapeev P. and C., Kühn , 2004, Perpetual convertible bonds in jump-diffusion models. *Statistics & Decisions*, **23**(1), 15 31.

[62] Gaunt, J., 2008, Investors warming up to convertibles, *New York Times*, May, 14.

[63] Geman, H. and Yor, M. (1996): Pricing and hedging double barrier options: a probabilistic approach. *Mathematical Finance*, **6**, pp. 365-378.

[64] Glasserman P. and Z. Wang 2009. Valuing the Treasury' s Capital Assistance Program. Working paper.

[65] Goldstein, R., Ju, N. and Leland, H. (2001): An EBIT-based model of optimal capital structure. *Journal of Business*, **74**, pp. 483-512.

[66] Greenhalgh, H., 2009, A positive year for convertible bonds, *Financail Times*, Feb. 09.

[67] Hamadène, S., and J., Zhang, 2010, The Continuous Time Nonzero-Sum Dynkin Game Problem and Application in Game Options, *SIAM Journal of Control and Optimization*, **48**(5), 3659 3669.

[68] Harris, M. and A. Raviv, 1985, A sequential signaling model of convertible debt call policy. *Journal of Finance* **5**, 1263 1281.

[69] Hennessy C.A. and Y., Tserlukevich, 2008, Taxation, agency conflicts, and the choice between callable and convertible debt. *Journal of Economic Theory*, 143(1), 374 404.

[70] Heyde, C. C. and Kou, S. G. (2006): On the controversy over tailweight of distributions. *Operations Research Letters*, **32**, pp. 399-408.

[71] Hilberink, B., and L.C.G. Rogers, 2002, Optimal capital structure and endogenous default. *Finance and Stochastics* **6**, 237 263.

[72] Ho, T. and M. Pfeffer, 1996, Convertible bonds: Model, value, attribution and analytics. *Financial Analysts Journal* **52**, 35 44.

[73] Howison, S. and Steinberg, M. (2005): A matched asympotic expansions approach to continuity corrections for discretely sampled options. Part 1: Barrier options. *Applied Mathematical Finance*, **14**, pp. 63 - 89.

[74] Ingersoll, J. E., 1977a, A contingent-claims valuation of convertible securities, *Journal of Financial Economics* **4**, 289 322.

[75] Ingersoll, J. E., 1977b, An examination of corporate call policies on convertible securities, *Journal of Finance* **32**, 463 478.

[76] Jaffee, D., and A., Shleifer, 1990, Costs of financial distress, delayed calls of convertible bonds, and the role of investment banks, *Journal of Business*, **63**(2), 107 124.

[77] Jalan, P., and G. Barone-Adesi, 1995, Equity financing and corporate convertible bond policy. *Journal of Banking and Finance* **19**, 187 206.

[78] Jarrow, R. and P. Protter (2004). Structural versus reduced-form models: A new information based perspective. *Journal of Investment Management*, **2**, 34 43.

[79] Jarrow, R. A. and S. Turnbull (1995). Pricing derivartives on financial securities subject to credit risk. *Journal of Finance*, **50**, 53 86.

[80] Kallsen, J., and C. Kühn, 2005, Convertible bonds: financial derivatives of game type. In *Exotic Option Pricing and Advanced Lévy Models*, A. Kyprianov, W. Schoutems, and P. Willmott, eds., Wiley, New York. pp. 277 292.

[81] Karatzas, I., and S. Shreve, 1991, *Brownian Motion and Stochastic Calculus*. 2nd Edition. Springer-Verlag, New York.

[82] Karatzas, I., and S. Shreve, 1998, *Methods of Mathematical Finance*. 2nd Edition. Springer-Verlag, New York.

[83] Kifer, Y., 1971, Optimal stopping in games with continuous time. *Theory of Probability and Its Applications* **16**, 545–550

[84] Kifer, Y., 2000, Game options. *Finance and Stochastics* **4**, 443–463.

[85] Kim, Y. O., and J. Kallberg, 1998, Convertible calls and corporate taxes under asymmetric information. *Journal of Banking and Finance*, **22**, 19–40.

[86] Kinderlehrer, D., and G. Stampacchia, 1980, *An Introduction to Variational Inequalities and Their Applications*. Academic Press, New York.

[87] Klarnet, K., and E., Stiefel, 2009, Revisiting the Case for Convertibles. Special Report, Palisade Capital Management, LLC.

[88] Kou, S. G. (2002): A jump-jiffusion model for option pricing. *Management Science*, **48**, pp. 1086-1101.

[89] Kou, S. G., Petrella, G. and Wang, H. (2005): Pricing path-dependent options with jump risk via Laplace transforms. *Kyoto Economic Review*, **74**, pp. 1-23.

[90] Kou. S. G. and Wang, H. (2003): First passage times of a jump diffusion processes, *Advances in Applied Probability*, **35**, pp. 504-531

[91] Kou, S. G. and Wang, H. (2004): Option pricing under a double exponential jump diffusion model. *Management Science*. **50**, pp. 1178-1192.

[92] Kovalov, P., and V. Linetsky, 2008, Pricing convertible bonds with stock price, volatility, interest rate, and credit risks. Working Paper.

[93] Kreps D. M. and R. Wilson, 1982, Sequential Equilibria, *Econometrica*, **50**(4), 863–894.

[94] Kunitomo, N. and Ikeda, M. (1992): Pricing options with curved boundaries. *Mathematical Finance*, **2**, pp. 275-298.

[95] Kwok, Y. K. and Lau, K. W. (2001). Pricing algorithms for options with exotic path-dependence. *Journal of Defivatives*, **9**, pp. 23-38.

[96] Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives Research*. **2**, 99 120.

[97] Lando, D. (2004). *Credit Risk Modelling: Theory and Applications*. Princeton University Press, Princeton, N.J.

[98] Leland, H. E., 1994, Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance* **49**, 1213 1252.

[99] Leland, H. E., and K. B. Toft, 1996, Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads, *Journal of Finance* **51**, 987 1019.

[100] Lewis, C. M, R.J., Rogalski and J.K., Seward, 1998, Understanding the Design of Convertible Debt. *Journal of Applied Corporate Finance*, **11**(1), 45 53.

[101] Lewis, C. M, R.J., Rogalski and J.K., Seward, 1999, Is Convertible Debt a Substitute for Straight Debt or for Common Equity? *Financial Management*, **28**(3), 5 27.

[102] Leung, K. S. and Kwok, Y. K. (2006). Distribution of occupation times for CEV diffusions and pricing of a-quantile options. *Quantitative Finance*, **7**, pp. 87-94.

[103] Linetsky, V. (1998). Steps to the barrier. *RISK*, April, pp. 62-65.

[104] Linetsky, V. (1999). Step options. *Mathematical Finance*, **9**, pp. 55-96.

[105] Longstaff, F. and E. Schwartz (1995). Valuing risky debt: A new approach. *Journal of Finance*, **50**, 789 821.

[106] Lucas, R. E. (1978). Asset prices in an exchange economy. *Econometrica*, **46**. pp. 1429-1445.

[107] Mayer, D., (1998). Why firms issue convertible bonds: The matching of financial and real investment options. *Journal of Financial Economics* **47**(1),83-102

[108] Mazzeo, M. A., and W. T., Moore. 1992, Liquidity costs and stock price response to convertible security calls. *Journal of Business*, **65** (July), 353 69.

[109] Merton, R.C. (1974). On the pricing of corporate debt: The risky structure of interest rates. *Journal of Finance*, **29**, 449-470.

[110] Metwally, S. and Atiya, A. (2002): Using the Brownian bridge for fast simulation of jump-diffusion processes and barrier options. *Journal of Derivatives*, (Fall), 43-54.

[111] Miura, R. (1992). A note on look-back options based on order statistics. *Hitotsubashi Journal of Commerce Management*, **27**, pp. 15-28.

[112] McConnell, J. J., and E. S. Schwartz, 1986, LYON taming. *Journal of Finance* **41**, 561 577.

[113] Mikkelson, W. H., 1981, Convertible calls and security returns. *Journal of Financial Economics* **9**, 237 264.

[114] Mikkelson, W. H., 1985, Capital structure change and decreases in stockholders' wealth: a cross-sectional study of convertible security calls, in *Corporate Capital Structures in the United States* B. M. Friedman ed., The University of Chicago Press, 265 296.

[115] Nagai, H. 1987, Non-zero-sum stopping games of symmetric Markov processes, *Probababılıty Theory Related Fields*, **75** , 487 497.

[116] Naik, V. and Lee, M. (1990): General equilibrium pricing of options on the market portfolio with discontinuous returns. *Review of Financial Studies*, 3, pp. 493-521.

[117] Nowak, A.S., andK., Szajowski ,1999, Nonzero-sum stochastic games. In *Stochastic and Differential Games* (M. Bardi, T. E. S. Raghavan and T. Parthasarathy, eds.) 297 342. Birkhäuser, Boston.

[118] Ofer, A. R., and A., Natarajan, 1987, Convertible call policies: An empirical analysis of an information signaling hypothesis. *Journal of Financial Economics*, **19** (September), 91 108.

[119] Ohtsubo, Y., 1986, Optimal stopping in sequential games with or without a constraint of always terminating. *Mathematics of Operations Research*. **11**, 591 607.

[120] Ohtsubo, Y., 1987, A nonzero-sum extension of Dynkins stopping problem, *Mathematics of Operations Research*, **12**, 277 296.

[121] Pelsser, A. (2000): Pricing double barrier options using Laplace transforms. *Finance and Stochastics*, **4**, pp. 95-104.

[122] Peskir. G., and A. Shiryaev, 2006, *Optimal Stopping and Free-boundary Problems*. Springer-Verlag.

[123] Petrella, G. (2004). An extension of the Euler Laplace transform inversion algorithm with applications in option pricing. *Operations Research Letters*, **32**, pp. 380-389.

[124] Petrella, G., and Kou, S. G. (2004): Numerical pricing of discrete barrier and lookback options via Laplace transforms. *Journal of Computational Finance*. **8**, pp. 1-37.

[125] Protter, P. (2005). *Stochastic Integration and Differntial Equations. A New Approach*, 2nd Edition. Springer, Berlin.

[126] Roger, A. H. and Johnson, C. (1985): *Matrix Analysis*. Cambridge University Press.

[127] Rudin, W. (1987): *Real and Complex Analysis*. Third Edition, McGraw-Hill, New York.

[128] Sarkar, S., 2003, Early and late calls of convertible bonds: Theory and evidence. *Journal of Banking and Finance* **27**, 1349 1374.

[129] Schroder, M. (2000): On the valuation of double-barrier options: computational aspects. *Journal of Computational Finance*, **3**, pp. 5-33.

[130] Sepp, A. (2004): Analytical pricing of the double-barrier options under a double-exponential jump diffusion processes: applications of Laplace transform. *International Journal of Theoretical and Applied Finance*, **7**, pp. 151-175.

[131] Shivers, M.A, 2003, Convertibel Bond Valuation and Pricing: Theory and Evidence. PhD dissertation, University of California, Berkeley.

[132] Shamaya, E., and E., Solan, 2004, Two-player Nonzero-sum Stopping Games in Discrete Time. *The Annals of Probability* **32**(3), 2733 2764.

[133] Singh, A. K., A. R., Cowan and N. , Nayar, 1991. Underwritten calls of convertible bonds. *Journal of Financial Economics*, **29** (March), 173 96.

[134] Sîrbu, M., I. Pikovsky, and S. Shreve, 2004, Perpetual convertible bonds. *SIAM Journal of Control and Optimization* **43**, 58 85.

[135] Sîrbu, M., and S. Shreve, 2006, A two-person game for pricing convertible bonds. *SIAM Journal of Control and Optimization* **45**, 1508 1639.

[136] Sircar, R. and Xiong, W. (2000): A general framework for evaluating executive stock options. *Journal of Economic Dynamics and Control*, **31**, pp. 2317-2349

[137] Stein, J., 1992, Convertible bond as backdoor equity financing, *Journal of Financial Economics*, **32**(1), 3–21.

[138] Takahashi, A., Kobayashi, T., and N. Nakagawa, 2001, Pricing convertible bonds with default risk. *Journal of Fixed Income* **11**, 20–29.

[139] Tsiveriotis, K., and C. Fernandes, 1998, Valuing convertible bonds with credit risk. *Journal of Fixed Income* **8**, 95–102.

[140] Whitt, W. (2002). *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Applications to Queues*. Springer-Verlag, New York.

[141] Yigitbasioglu, A. B., 2002, Pricing convertible bonds with interest rate, equity, credit and FX risk. EFMA London Meeting Paper, ISMA Center, University of Reading, Reading, UK.

[142] Yor, M. (1995). The distribution of Brownian quantiles. *Journal of Applied Probability*, **32**, pp. 405-416.

[143] Zhang, P. G. (1998): *Exotic Options: A Guide to Second Generation Options*. Second Edition. World Scientific.

[144] Zhou, C. (2001). The term structure of credit spreads with jump risk. *Journal of Banking and Finance*, **25**, 2015–2040.

[145] Zwillinger, D., 1997, *Handbook of Differential Equations*, 3rd ed. Academic Press, Boston, MA.