A Hub-to-hub Network Revenue Management Model

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Abstract

The subject of this study is the revenue management problem in hub-to-hub airline networks. The network consists of two hubs and a connecting flight between them with spoke cities expanding outwards. The airline produces various itineraries within the network, and its flights compete with each other for limited flight capacities during a fixed booking period. Although stochastic dynamic network revenue management has been theoretically established, in reality its implementation is still heavily dependent on linear programming-based heuristics. Simpson (1989) and Williamson (1992) proposed bid price control, which is now widely adopted by major airlines. Bertsimas and de Boer (2003) proposed certainty equivalent control, which has been little studied by RM researchers. In this thesis, bid price control is first explained, and then the structural properties of the hub-to-hub network are investigated. Using the Lagrange dual-function and the primal-dual relationship, it is shown that the threshold values used in bid price control have some monotone properties in the network's capacity states. The certainty equivalent control is then applied to the hub-to-hub network. By linking the network revenue management problem with a maximum-weight circulation problem in network flow, the optimal value function is shown to be supermodular in certain capacity dimensions, and submodular in other dimensions. This leads to the monotonicity of CEC thresholds on some short-haul itineraries. The notion of $L^\natural$ concavity developed by Murota and Shioura (2005) is applied to this work, and it is shown that even the CEC thresholds on some two-leg or three-leg long-haul itineraries are monotonically increasing or decreasing in certain legs'

capacities. It is hoped that the new structural properties found in this thesis can lead to a reduction of the computational work in the implementation of both the bid price control and the certainty equivalent control in the hub-to-hub airline network.

keywords: Hub-to-hub network, bid-price control, certainty equivalent control, combinatorial optimization, structures, primal-dual, revenue management, airline network, monotone thresholds, supermodularity/submodularity, $L^\natural$ concavity, Lagrange dual.

摘要

在本文中我们研究一个从枢纽至枢纽的航空网络之收益管理问题。我们的焦点将放在它的结构性质上。此网络由两个枢纽城市和一条中间联线，以及周边的辐射城市和枢纽到它们之间的航班所组成。一个航空公司经营多条在各城市对之间的航线。在一个固定期限内，各航线之间为有限的航班资源而竞争座位分配额。虽然随机动态网络收益管理理论已被建立，但在现实中它的实行在很大程度上依赖于基于线性规划的启发式算法。于是我们将注意力放在两个启发式方法上：一个是由 Simpson （1989）和 Williamson（1992）提出的所谓 bid-price 控制 (BPC），另一个是由 Bertsimas 和 de Boer （2003）提出的 所谓 certainty equivalent 控制 (CEC)。我们先解释 什么叫 BPC，接着我们研究它在我们的枢纽至枢纽网络中的结构性质。通过使用 Lagrange 对偶函数与原有对偶关系，我们可以证明在 BPC 里使用的阈值有一些关于网络存量的单调性。然后我们应用 CEC 到这个网络。通过将网络收益管理问题和一个网络流中的最大加权环路问题相连接，我们揭示最优目标函数在一些存量维度上分别具有超模和次模性质。它们导致了一些短途航线上 CEC 阈值的单调性。接着我们将 Murota 和 Shioura (2005) 所发展的 L-natural 凹性概念引入到我们的工作。通过他们的一些结果，我们可以证明一些双联程和三联程航线上的 CEC 阈值也同样具有不同的单调性。希望通过这些新的结构性质本论文可以为有关行业带来一些系统上的新见解以及为航空公司管理人员提供一些直觉上的帮助。

关键词：枢纽至枢纽网络， bid price 控制，certainty equivalent 控制，组合优化，结构，原有对偶，收益管理，航空网络，单调阈，超模/次模，L-natural 凹性，Lagrange 对偶。

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Glossary

Bid price: A net value for a certain leg in an airline network.

Bid price control: A method used in controlling airline seat inventory in that it sums all the bid prices across a certain itinerary and use that as a threshold value for making accept/deny decisions on customer fare request. Only the higher value customers are accepted.

Booking limit: The maximum number of seats that can be sold to a particular booking class and the booking classes lower that it.

Cancellations: Events that customers cancel their air ticket bookings.

Circulation: In a graph, a flow is called a circulation if in every node the total amount of entering flows equals the that of leaving flows.

Code-sharing: A contract between two airline companies to form an alliance in that they can list the other airline's flight by using their own code.

Control algorithm: To employ a math program to approximately represent an MDP and use its optimal solution to heuristically instantiate the parameters of the decision rules of a given class of control policies.

Cutoff level: A threshold in the fare-class list. Any fare higher than this value is accepted a request. Also referred to as *threshold price*, *threshold value*.

Cycle: A closed (simple) path in a graph, with no other repeated vertices or edges other than the starting and ending vertices. This may also be called a *simple cycle.*

Displacement cost: The opportunity cost of consuming a consecutive set of flight legs' capacities by one customer.

DLP model: The deterministic linear programming model.

Dynamic models: Models in revenue management that take future booking process into account.

Expected marginal seat revenue (EMSR): The expected marginal revenue of a seat if held open.

Fare class: A class divided by airlines with corresponding restrictions.

Fleet assignment: Most airlines have a variety of aircraft types and sizes in their fleets. The fleet assignment process attempts to allocate aircraft to routes in the airline network to maximize contribution to profit. There are strong potential linkages between fleet assignment and revenue management processes because aircraft assignments determine leg capacities in the network.

Flight leg: A section of flight that involves a single takeoff and landing.

Hub-and-spoke network: Airline network with passengers transfer at major hub cities to arrive at their destinations.

Itinerary: A route that a passenger travels through via an airline network from his origin city to his destination city. Also called *OD*.

Leg: See flight leg.

$L^\natural$ concavity: A function $v : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is called *$L^\natural$-concave* if $v(x - x_0\mathbf{1})$ is supermodular in (x, x_0) for $\forall (x, x_0) \in \mathbf{R}^n \times \mathbf{R}$.

Load factor: The ratio of seats filled during a flight compared with the full capacity of the flight.

Low-before-high fares: The sequential arrival pattern of customers when low fare customers book before high fare ones.

Mark-up/markdown problems: Pricing problems that effect price changes in low-to-high or high-to-low order.

$M^\natural$ concavity: A function $v : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is called *$M^\natural$-concave* if it satisfies the following property: $\forall x, y \in \mathbf{R}^n, \forall i \in supp^+(x - y), \exists j \in supp^-(x - y) \cup \{0\}, \exists \alpha_0 > 0 : v(x) + v(y) \leq v(x - \alpha(\chi_i - \chi_j)) + v(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in [0, \alpha_0])$.

Nesting: A seat inventory control approach that makes sure that higher fare customers have higher priority in accessing to airline seats.

No-shows: The events that booked passengers don't show up when the scheduled flight departs.

OD: See itinerary.

ODF control: Origin-destination fare control. An approach to revenue management that accounts for all passenger bookings at origin destination level and differentiated by fare classes.

Opportunity cost: See displacement cost.

Overbooking: The practice of selling tickets beyond the airline's capacity in case no-shows occur.

Parallel arcs: In network flow, two arcs are said to be 'parallel' if every (undirected) simple cycle containing both of them orients them in the opposite direction.

Path: A sequence of consecutive arcs in a graph with no repetition of nodes.

Simple cycle: See *cycle.*

Protection levels: The seats protected for a certain fare class and higher level fare classes. Also referred to as *protection limits.*

Restrictions: Conditions that the discount-fare customers are required to meet.

Revenue management: The practice of controlling seat availability and pricing decisions for different booking classes during a ticket-selling process.

Seat inventory control: The practice of controlling seat availability to customer requests based on current inventory status.

Series arcs: In network flow, two arcs are said to be 'series' if every (undirected) simple cycle containing both of them orients them in the same direction.

Static models: Models that set seat protection levels without considering future adjustment. (Compare with dynamic models.)

Structural properties: Monotonicity of control thresholds and the sup/submodularity of the optimal value function in OR/MS field. Here we specifically indicate those properties in revenue management field.

Supermodularity: A function $f : \mathbf{X} \to \mathcal{R}$ is called *supermodular* if $f(\mathbf{x}) + f(\mathbf{y}) \leq f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

Submodularity: A function $f : \mathbf{X} \to \mathcal{R}$ is called *submodular* if $f(\mathbf{x}) + f(\mathbf{y}) \geq f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

Threshold price: (See *cutoff level*)

Yield management: The synonym of revenue management.

Notation

f_i	fare for fare class i.
$\mathbf{n}$	in dynamic network models means the capacities-state of the network; in other parts of this thesis means a counter.
$\mathbf{x}$	means the decision variable in static models as how many seats to allocate to all the airline network's itineraries, a vector.
k	the remaining time periods.
m	a certain itinerary.
$\mathbf{R}_t$	demand realization vector; detailed explanation see Talluri and van Ryzin (1998).
u_k^j	decision variable as whether or not to accept a request for itinerary j at time-to-go k.
$J_k(\mathbf{n})$	the maximum expected revenue (cost-to-go) for a given seat inventory state $\mathbf{n}$ at time k.
a_{ij}	the number of seats on leg i used by itinerary j.
A	the matrix $[a_{ij}]$.
A^m	the mth column vector in matrix A.
$\mathcal{D}^t$	the demand-to-come process.
$\bar{\mathcal{D}}^t$	the aggregate demand-to-come over the remaining periods after t.
F	the number of flight legs in a network.
odf	an arbitrary origin-destination-fare class combination.
c_l	the capacity of leg l.
f_{odf}	the fare for odf.

y_{odf} the decision variable of how many seats to allocate to odf.

d_{odf} the estimated demand-to-come in odf.

$\mathbf{C}$ the network's capacities vector space in the DLP model.

$\mathbf{\Lambda}$ the vector space of the network's dual variables for all legs in the DLP model.

$\mathbf{X}$ a lattice or the vector space of the network's allocation variables for all itineraries in the DLP model.

G a graph.

V the vertex set in graph G.

A the arc set in graph G.

P a parallel arc set.

S a series arc set.

w_P the weight vector of an arc set P.

c_P the upper bound vector of capacity constraint on an arc set P.

d_P the lower bound vector of capacity constraint on an arc set P.

w_S the weight vector of an arc set S.

c_S the upper bound vector of capacity constraint on an arc set S.

d_S the lower bound vector of capacity constraint on an arc set S.

χ_S takes value in $\{0,1\}^A$, the characteristic vector of $S \in A$.

Chapter 1

Introduction

Network revenue management (NRM) is a scientific subject that tries to control the availability and/or pricing of travel seats in different booking classes to maximize the expected revenues or profits of the network. In this thesis, only hub-to-hub airline networks are considered (see Figure 1.2 below).

1.1 Motivation

Hub-and-spoke networks are a mainstay of the global airline market. Hub cities are found across the world, such as Washington Dulles International Airport and Detroit Metropolitan Wayne County Airport in North America, Durban International Airport and Bole International Airport in Africa, Beijing Capital International Airport and Narita International Airport in Asia, Dublin Airport and

London Heathrow Airport in Europe, and so on. Around such hubs spoke cities further expand airline networks. Figure 1.1 illustrates a simple hub-and-spoke network.

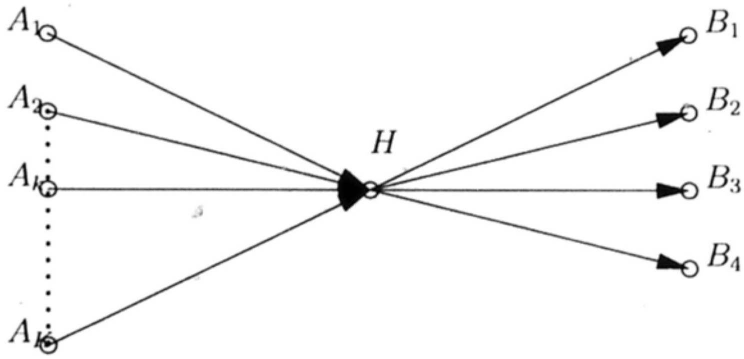

Figure 1.1: Early hub-and-spoke network

Scheduling, fleet assignment and revenue management are the three major operational aspects in this kind of airline network. Good revenue management decisions adds most value to the bottom line of an airline company, providing 4% - 10% increases in company revenues (Fuchs 1987). This thesis considers the network revenue management problem in which is a single company manages the whole network's revenue decisions.

Dynamic hub-and-spoke network revenue management is theoretically well understood, but the implementation of the dynamic model is problematic. The solution is to focus on a specific network structure and explore the structural properties of the model, which may lead to a reduction in the computation work.

Shifting attention to a specific hub-to-hub network structure, Figure 1.2 illustrates the two domestic hubs of Dallas/Fort Worth and Detroit Metropolitan airports in the U.S. operated by American Airlines.

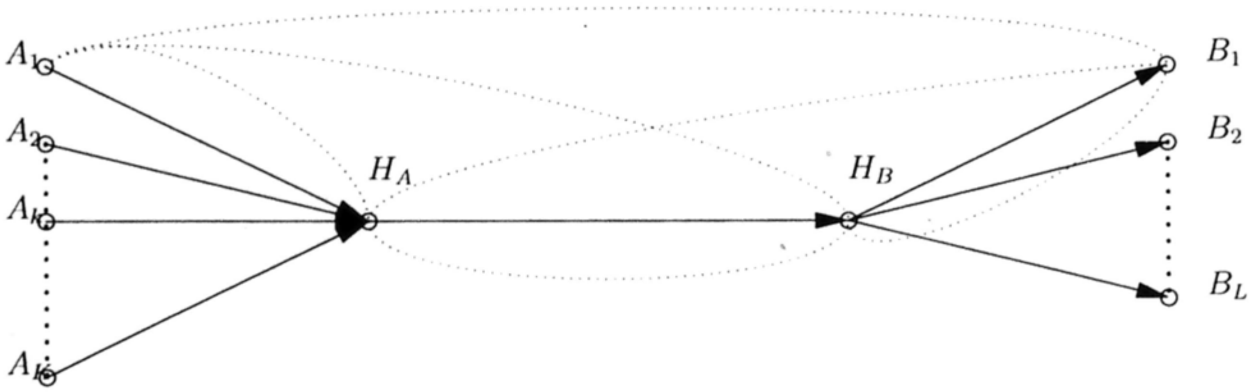

Figure 1.2: A hub-to-hub network

Similarly, the figure represents the KLM-Northwest alliance that connects the inter-continental hubs of Amsterdam Schiphol and London Heathrow airports. It's believed that through code-sharing or merger between two large airline companies such as the recent pairings of Air China with Cathy Pacific and British Airways with Iberia Airlines, ever more hub-to-hub networks will emerge in the global airline market. Efficient hub-to-hub network revenue management is essential in such a globalized context. Nevertheless, remarkably little research into hub-to-hub networks has been done in the revenue management field.

Although this thesis originally set out to study the structural properties of dynamic models, only the structural properties of static models are here considered, because of the time limitations of a PhD program. Future work will extend coverage to dynamic models.

1.2 Basic description of the model

The object of my study is the specific hub-to-hub network shown in Figure 1.2. The model tries to control the availability of seat inventory for various OD products across the whole network. In the network, customers depart from western cities and arrive at eastern cities, which the airline operates through so-called 'waves'. One wave of customers from different origins arrives at H_A in a short time interval. Some of them are transferred into another airplane (usually a large one), and arrive at destination hub H_B. Some passengers will then continue to fly to the B_l areas via another seamlessly scheduled departure plan. An airline may operate several such waves on the same network each day. The revenue management decision in the model assigns cut-off levels (also called threshold price) to the fares list for every origin-destination (OD) city pair. Any fares higher than this level are accepted, and any fares lower than this level are rejected. Many researchers have used the future expected value of a marginal seat as the cut-off value. A later development was to use the opportunity cost (OC) of the leg capacity as the cut-off value. The size of this practice becomes very large in a dynamic network situation, and is usually intractable using the optimal value differences as the opportunity cost. To solve this difficulty, two static heuristics are utilized in this thesis: the bid-price control (BP control) proposed by Simpson (1989) and Williamson (1992), and the certainty equivalent control (CEC) proposed by Betsimas and Popescu (2003). Emphasis is placed on the BP control.

1.3 Basic solution procedure

I adopt the bid-price control proposed by Simpson (1989) and Williamson (1992), and the certainty equivalent control (CEC) by Bertsimas and Popescu (2003). The former is used extensively in today's airline industry, though the latter reportedly offers better performance (Bertsimas and Popescu, 2003). The BP control heuristic works as follows. We first divide the booking horizon into multiple stages. For the beginning of each stage we then forecast the demand in that stage and do a mathematical programming-based optimization to optimize the capacity allocations over the entire airline's network. Then the *dual problem* is solved and we get a set of bid-prices for each leg in the airline network which is the shadow price of each leg's capacity constraint in the mathematical programming formulation. Bid-price control is a method that sums across a certain route all its component legs' bid prices as the cut-off value for making accept/reject decisions. Any fare request lower than this value is rejected, and higher fare requests are accepted. A shortcoming in bid-price control is that it needs to be frequently re-optimized in order to be asymptotically optimal. Williamson (1992) mainly discussed the usage of bid price control via illustrations on a three-leg network and a one-hub network with four spoke cities. In contrast, the model adopted here considers a more complex and realistic hub-to-hub network from which insights into the behavior of bid price controls in more general networks can be determined.

Certainty equivalent control (CEC) is another way to approximate the optimal thresholds and works as follows. Like the bid-price scheme, CEC aggregates the future demands and uses the LP value function to approximate the optimal value function. The value difference between two capacities is then computed as the threshold value. As in Williamson (1992), Bertsimas and Popescu (2003) confined their illustration to a three-leg and one-hub network, whereas the model considered here concentrates on a more general network, i.e. the hub-to-hub network. It is hoped that the theoretical analysis and numerical illustration offered here can readers can provide enhanced understanding into the behaviour of the two control schemes in more general network environments.

Both of the above methods need to be re-optimized frequently since they permit large group bookings even when the accepted fare class has very low value. Search-space reduction is thus essential in designing these algorithms. I therefore explore the structural properties of the basic optimization models embedded in the two control heuristics, e.g. some kind of monotonicity concerning the heuristical thresholds. It is hoped that such properties can reduce the search space when re-optimizing the model and re-solving the corresponding dual variables. A further contribution is that the results concerning the BP and the CEC heuristics may help better understand the properties concerning the optimal threshold values.

1.4 Our contributions

This thesis makes four contributions. Firstly, it offers a new object in network revenue management for the two-hub network which has not previously been explicitly studied by any RM researchers. Second, it is shown that the model is equivalent to a network flow problem, and thus has integer solutions. Third, I adopt the bid-price control method proposed by Simpson (1989) and Williamson (1992), and the certainty equivalent control method proposed by Betsimas and Popescu (2003) to solve the problem. Fourth, the structural properties within those controls are studied. In exploring those properties, new methodologies are devised, which may provide a new way for studying structural properties in dynamic models.

1.5 Structure of thesis

The following part of the thesis is divided into four chapters. Chapter 2 contains a literature review. I first review the historical development of airline yield management research, from the early low-before high single-leg models to the later dynamic network models. The study of structural properties of revenue management models is then reviewed.

Chapters 3 and 4 are the most substantive parts. Chapter 3 restricts attention to a network revenue management model and describes the BP control in detail. Chapter 4 then explores the structural properties of BP control and illustrates its potential usage. The focus here is on the monotone properties of the threshold values. Such monotonicity results may lead to algorithmic simplifications and can provide intuitive understanding into the controls of the network.

Chapter 5 first explains the certainty equivalent control developed by Bertsimas and Popescu (2003) and then adapts it to the hub-to-hub network revenue management problem. The structures of the certainty equivalent control are then studied. This is done by connecting the NRM problem with a maximum weight circulation formulation.

Chapter 6 serves as a numerical test component for the results in the previous chapters. This chapter applies real fare data drawn from airline web sites to extensively alter the capacity settings of the network. Tests are conducted with Matlab to gain more concrete understanding of the theoretical results obtained in the previous two chapters.

Chapter 7 concludes this dissertation, summarizing the research findings and contributions. The thesis ends by identifying some future research directions.

Chapter 2

Literature review

The purpose of this chapter is to provide a literature review for relevant works in the field of revenue management. I first review single-leg revenue management models. Then I review multi-leg and network revenue management models.

2.1 An introductory note

Airline RM can be classified as quantity-based and price-based. Talluri and van Ryzin (2004) offer a detailed account of price-based RM and show how it differs from quantity-based RM. Quantity-based RM include single-leg capacity control, network capacity control and overbooking, as classified in Talluri and van Ryzin (2004). This thesis focuses on the network-capacity control problem.

2.2 Single-leg revenue management models

Work in single-leg revenue management has utilized static or dynamic models.

2.2.1 Static models

Most static models in yield/revenue management assume that low-fare customers arrive before high-fare ones (see Littlewood (1972) for the first model of this kind), and must decide when to switch the opening to the higher fare classes. Littlewood (1972) proposed an EMSR approach which indicates that as long as the future expected marginal seat revenue exceeds the current low fare value, then one should stop accepting low-fare discount requests. This was applied by Buhr of Lufhansa in 1982 to the two-leg airline network seat allocation problem. Belobaba (1987, 1989) generalized the Littlewood model (1972) to a multiple fare-classes situation. Brumelle and Mcgill (1993) further established that the optimal EMSR approach is to determine a set of *protection levels.* These protection levels are determined via EMSRs, as done by Littlewood (1972) and Belobaba (1987).

2.2.2 Dynamic models

Feng and Gallego (1995), Feng (2000), Feng and Xiao (2000) extended the low-before-high model to incorporate mark-up/markdown and reversible price changes with a continuous-time dynamic-programming formulation. Hereafter, the assumptions that low fare customers arrive before high fare ones or vice versa are dropped. Instead, the decision to accept or deny a fare request is made in real-time by allowing the opening of multi-fares simultaneously. This is done by establishing a threshold value (cut-off level) on the fares list. The multi-fares refer to classes with restrictions such as refundability, upgradability, and transferability etc. This class of models is considered by Diamond and Stone (1991), Lee and Hersh (1993), Liang (1999), Robinson (1995), Lautenbacher and Stidham (1999), van Slyke and Young (2000), and Feng and Xiao (2001). For example, Lautenbacher and Stidham (1999) formulated the single-leg problem via Markov decision process (MDP), where they studied the structural properties of the optimal threshold value. After showing that the threshold value decreases in the current capacity, Lautenbacher and Stidham (1999) deduced that the optimal threshold policy is equivalent to the known booking-limit policy in the single-leg case. Other researchers obtained similar results. For example, Feng and Xiao (2001) obtained the same structural property results as Lautenbacher and Stidham (1999) under a continuous-time framework.

2.3 Two-leg network revenue management models

The two-leg network revenue management problem is that there are three itineraries competing for limited resources of two-leg flights. Each itinerary is further constituted of multi-fares. The decision in each period is whether or not to accept a customer request on each itinerary. You (1999) applied dynamic pricing to two-leg revenue management. Feng and Lin (2004) and Morton (2006) studied the structural properties of the two-leg revenue management problem by using continuous-time and discrete-time models. The former put emphasis on the monotonicity of the decision variable/control thresholds, while the latter focused on studying the second-order properties of the optimal-value function.

2.4 Large network revenue management models

The study of large airline networks originated with Glover et al. (1982), who formulated the passenger mix problem in a complex airline network into network flows. Wollmer (1986) first proposed the linear programming model for large network revenue management.

Curry (1990) extended the low-before high model and the EMSR principle developed by Littlewood and Belobaba to the entire network. Curry first allocated the whole network's capacity into all the OD pairs. He then did the one-leg low-before-high allocation restriction in that specific OD pair based on the EMSR principle.

Williamson's (1992) doctoral dissertation proposed both the booking-limit and bid-price control to solve the *nesting* problem in network revenue management. de Boer et. al (2002) further compared the difference between booking-limit and bid-price control.

Bertsimas and Popescu (2003) compared the bid-price control with the certainty equivalent control. They used LP value function to approximate the optimal value function. And the marginal value $J_{t-1}^{LP}(n) - J_{t-1}^{LP}(n - A^m)$ is used to approximate the optimal marginal value $J_{t-1}(\mathbf{n}) - J_{t-1}(\mathbf{n} - A^m)$.

For randomized linear programming in computing bid-prices, see Talluri and van Ryzin (1999).

In theoretical aspect of bid-price control, Talluri and van Ryzin (1998) have shown that bid-price control is not optimal. They formulated the optimal control for the network revenue management problem by dynamic programming:

$$J_k(\mathbf{n}) = \max_{\mathbf{u}_k} E[R_k u_k(n, R_k) + J_{k-1}(\mathbf{n} - A u_k(n, R_k)))$$

Talluri and van Ryzin established that the structure of the optimal control is:

$$u_k^j(n, r^m) = \begin{cases} 1 & r^m \geq J_{k-1}(\mathbf{n}) - J_{k-1}(\mathbf{n} - A^m), \\ 0 & otherwise. \end{cases}$$

$J_{k-1}(\mathbf{n}) - J_{k-1}(\mathbf{n} - A^m)$ is called the *opportunity cost* (OC) on itinerary m. Here we also call it the optimal threshold. Williamson's BP control can be seen as a heuristic for approximating these optimal thresholds. Talluri and van Ryzin used a counter example to show that bid-price control is not actually optimal. However, they also showed that the BP control is asymptotically optimal. Thus in practice BP control still works rather well. For recent advances in computing bid prices, see Adelman (2007) and Topaloglu (2008, 2009).

For recent advances in computing booking-limits, see Bertsimas and de Boer (2005) and van Ryzin and Vulcano (2008).

In other aspects of network revenue management, see van Ryzin and Mcgill (2000) for protecting-limit updates with adaptive algorithms, and see Secomandi (2005) for applying the *control algorithm* approach to the NRM problem. Robust controls for network revenue management can be found in Perakis and Roels (2010).

Although many researchers have studied various mathematical programming-based network revenue management models, they generally do not explore the special structure of a specific airline network, as pointed out in [52] :"We use the now- standard term *network* RM–though the term is something of a misnomer because the theory and methodology do not require an explicit network structure as such." An exceptional case is Morton (2006). He studied substitutability and complementarity in network revenue management models and, particularly, in a network as shown in Figure 1.1, which he called a *bipartite* network. His model was similar to the model used here, i.e. the deterministic linear programming model (DLP model).

For the deterministic static network case, suppose there are K flight legs in the west and L flight legs in the east. Then the model is:

$$\begin{aligned} \max. \quad & \sum_{odf} f_{odf} y_{odf} \qquad (2.1) \\ s.t. \quad & \sum_{l \in odf} y_{odf} \leq c_l, l = 1, 2, \cdots, K + L \\ & y_{odf} \leq d_{odf}. \end{aligned}$$

Morton obtained results that resemble those presented in Chapter 5. Morton utilized certain results in network flows and proved that the optimal value function is supermodular in *series* arc pairs and submodular in *parallel* arc pairs.

Morton's approach was grounded in economics, i.e. the series arcs are economic complements while the parallel ones are economic substitutes. Morton did not, however, interpret these properties into the monotonicity of control thresholds. Furthermore, he did not get the $L^\natural$ concavity of the optimal value function in the capacities of a pair of series arcs, such as (A_1, H) and (H, B_1).

However, the concern here is more operational, in that by the super/submodularity and $L^\natural$ concavity it is shown that the CEC thresholds are monotone on some capacity parameters, and thus anticipate the reduction of computational work in the implementation process of the CEC. Furthermore, Morton did not obtain results on a multi-hub network such as shown in Figure 1.2. In comparison, based on this more complicated network, the model proposed here allows for long-route itinerary (three legs). To summarize, a new approach (different from Morton's (2006)) to studying network revenue management problems is here proposed which exploits the full structure lying behind a specific network. Therefore, it is hoped that it can both provide technical insights for algorithm designers and intuitive understanding of the hub-and-spoke systems for airline network managers.

Chapter 3

Network bid-price control

This chapter is a technical preparation for the next chapter. Chapters 3 and 4 contain the main structural properties for network BP control.

3.1 Introduction to network BP control

To reduce the computational effort to get the optimal thresholds $J_t(\mathbf{n}) - J_t(\mathbf{n} - A^m)$ as introduced in Chapter 2, one possible approximation is to associate each leg with a shadow price; then, for itinerary m, the sum of those shadow prices on its legs yields an approximation of the threshold. This is the essence of the so-called bid-price (BP) control.

The BP control heuristic works as follows: first let $\mathcal{D}^t$ denote the demand-to-come process and $\bar{\mathcal{D}}^t$ denote the aggregate demand-to-come over the remaining periods after t. Then we take its expectation (the aggregate demand-to-come) as $D^t = E[\bar{\mathcal{D}}^t]$. Then formulate

$$\begin{aligned} J_t^{LP}(\mathbf{n}) &= \max \sum_{odf} f_{odf} \min\{D_{odf}^t, y_{odf}\} \\ & \quad s.t. \sum_{l \in odf} y_{odf} \leq n_l, l = 1, 2 \cdots, F \end{aligned}$$

as a heuristic allocation model for the original problem. Then the dual LP of the above heuristic model is formulated. We then sum across a certain route with all its dual variables as its threshold price and use this dual value as an approximation to the optimal opportunity cost $J_{t-1}(\mathbf{n}) - J_{t-1}(\mathbf{n} - A^m)$; any fare class request on that route less than the route's threshold price will be rejected.

3.2 Our model

The model is restricted to the hub-to-hub airline network shown in Figure 1.2.

This kind of networks emerged in the airline industry during the past two decades due to the emergence of airline alliances or nationwide airline carriers such as American Airlines. Such networks are mostly used by international airlines that operate code-sharing agreements between two or more airlines that combine their respective networks together, such as the Northwest-KLM alliance.

There are three groups of cities in the network shown in Figure 1.2, two groups of spoke cities on the side and two hub cities in the middle. Between each city pair in $(A_i, H_A), (H_A, H_B)$ and (H_B, B_j) there is an aircraft running this route. A customer can travel from A_i to H_A, and then switch to another airplane to travel to H_B, which is his destination city or even further switch to another aircraft to the cities located in the B_j region.

Some notations are explained here. c_l denotes the capacity of leg l for $l = 1, \cdots, K+1+L$, which is a given constraint. y_{odf} denotes the capacity allocation into an ODF, which is a decision variable. Collectively, let $\mathbf{c} = (c_1, c_2, \cdots, c_{K+L+1})$. Although forecasting is very important in the implementation of NRM, the issue of how to do good forecasting is beyond the scope of this work to cover and I just simply denote the forecasted demand as d_{odf} for each odf combination. And we let $m \ni l$ to denote that itinerary m passes leg l and $l \in m$ to denote the vice versa. The integer programming version of the network revenue management problem is now addressed:

$$\begin{aligned} \max \quad & \sum_{odf} f_{odf} y_{odf} \\ s.t. \quad & \sum_{l \in odf} y_{odf} \leq c_l, l = 1, 2, \cdots, K + L + 1 \\ & y_{odf} \leq d_{odf} \\ & y_{odf} \quad integer \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} \max \quad & \sum_{odf} f_{odf} \min\{d_{odf}, y_{odf}\} \\ s.t. \quad & \sum_{l \in odf} y_{odf} \leq c_l, l = 1, 2, \cdots, K + L + 1 \\ & y_{odf} \quad integer. \end{aligned} \tag{3.2}$$

Before presenting the main results in Chapter 4 some preparatory work is presented.

Firstly, Model (3.1) is a network flow problem with the following form (see Subsection 5.2.3 for the graphical illustration) containing the artificial variables: $z_l, l = 1, 2, \cdots, K + L + 1$,

$$\begin{aligned} \max \quad & \sum_{odf} f_{odf} y_{odf} \\ s.t. \quad & \sum_{l \in odf} y_{odf} - z_l = 0, l = 1, 2, \cdots, K + L + 1 \\ & y_{odf} \leq d_{odf} \\ & z_l \leq c_l \\ & y_{odf}, z_l \quad integers. \end{aligned}$$

This immediately leads to:

Proposition 1 *We have an integer solution by solving the LP relaxation of Model* (3.1) *when all the parameters are integers.*

This will greatly cut down the computational effort in solving Model (3.1).

The LP relaxation of Model (3.2) is:

$$\begin{aligned} \max \quad & \sum_{odf} f_{odf} y_{odf} \\ s.t. \quad & \sum_{l \in odf} y_{odf} \leq c_l, l = 1, 2, \cdots, K + L + 1 \\ & y_{odf} \leq d_{odf}, \end{aligned} \tag{3.3}$$

which is called the DLP model for network revenue management. Proposition 1 indicates that it has the same optimal solution as Model (3.2).

The dual linear programming model for Model (3.3) is:

$$\begin{aligned} \min_{\lambda,\mu} \sum_{l} \lambda_l c_l + \sum_{odf} \mu_{odf} d_{odf} \\ s.t. \sum_{l \in odf} \lambda_l + \mu_{odf} \geq f_{odf}, \end{aligned} \tag{3.4}$$

where λ_l and μ_{odf} are the dual variables associated with the leg capacity constraints and the demand constraint in Model (3.3). λ_l is also called the bid-price for leg l. The bid-price control heuristic is obtained by summing across a certain route m all its bid prices: $\Lambda_m = \sum_{l \in m} \lambda_l$; this is then used as a threshold value; any fare class request on that route whose value is less than the route's threshold value is rejected, and any fare class request above that threshold value is accepted.

It is easy to see that the threshold value Λ_m defines a minimal acceptable fare on route m. This is denoted as f_k here.

The KKT conditions for the LP Model (3.3) and its dual (3.4) are:

$$\begin{aligned} \lambda_l(\sum_{l \in odf} y_{odf} - c_l) &= 0 \\ \mu_{odf}(y_{odf} - d_{odf}) &= 0 \\ y_{odf}(\sum_{l \in odf} \lambda_l + \mu_{odf} - f_{odf}) &= 0. \end{aligned} \tag{3.5}$$

3.3 Aggregate the revenues on an OD and smoothing

It is a little cumbersome for the above models to be expressed in terms of y_{odf} and d_{odf}. I therefore aggregate the total revenues on an OD route and now rewrite Model (3.3). Let us first order the fare classes on a specific OD pair m as $f_1, f_2, \cdots, f_J$, and let $d_1, d_2, \cdots, d_J$ be the corresponding demands. In addition, let $x_m = \sum_{odf \in I_m} y_{odf}$. Then the aggregate revenue function on the mth OD pair $r_m(x_m)$ is a piecewise linear concave function which is of the following form:

$$r_m(x) = \begin{cases} f_1 x & \text{for } 0 \leq x \leq d_1 \\ f_2 x + (f_1 - f_2) d_1 & \text{for } d_1 \leq x \leq d_1 + d_2 \\ f_3 x + (f_2 - f_3) d_2 + (f_1 - f_3) d_1 & \text{for } d_1 + d_2 \leq x \leq d_1 + d_2 + d_3 \\ \cdots & \\ f_J d_J + f_{J-1} d_{J-1} + \cdots + f_1 d_1 & \text{for } x \geq d_1 + d_2 + \cdots + d_J. \end{cases} \quad (3.6)$$

Given the value of x_m, there will be a specified threshold level k on the OD pair m, such that $f_1, \cdots, f_k$ are opened, but $f_{k+1}, \cdots, f_J$ are closed. Thus we can convert Problem (3.3) into the following form:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{1 \leq m \leq 2K+KL+2L+1} r_m(x_m) \\ s.t. \quad & \sum_{m \ni l} x_m \leq c_l, l = 1, 2, \cdots, K + L + 1. \end{aligned} \quad (3.7)$$

Let $\partial r_m(x_m)$ denote the subdifferential of $r_m(x_m)$. The next theorem addresses the optimality conditions for the reformulated problem (3.7):

Theorem 1 *The KKT conditions* (3.5) *in Model* (3.3) *are equivalent to the following conditions in Model* (3.7)*:*

$$\lambda_l(\sum_{l\in m} x_m - c_l) = 0 \tag{3.8}$$

$$x_m I(\sum_{m\ni l} \lambda_l \in \partial r_m(x_m)) = 0. \tag{3.9}$$

$I(\sum_{l\in m} \lambda_l \in \partial r_m(x_m))$ *is the indicator function with*

$$I = 0 \text{ if } \sum_{l\in m} \lambda_l \in \partial r_m(x_m)$$
$$I = 1 \text{ if else.}$$

Proof. We classify x_m^* into two cases:

- At (D_k, D_{k+1}), then $\partial r_m(x_m) = f_{k+1}^m$. From the KKT conditions (3.5),

$$y_{k+1}^m(\Lambda_m + \mu_{k+1}^m - f_{k+1}^m) = 0$$
$$\mu_{k+1}^m(y_{k+1}^m - d_{k+1}^m) = 0.$$

  Since $0 < y_{k+1}^m < d_{k+1}^m$ by assumption, we have $\mu_{k+1}^m = 0$ and $\Lambda_m + \mu_{k+1}^m - f_{k+1}^m = 0$. i.e. $\Lambda_m = f_{k+1}^m = \partial r_m(x_m)$, condition (3.9) is satisfied. Therefore the two conditions are equivalent.

- If $x_m^* = D_k$ for some k, then $\partial r_m(x_m) = [f_{k+1}, f_k]$. Obviously $\mu_{k+1}^m = 0$, from constraint in (3.4) $\Lambda_m \geq f_{k+1}^m$, and from condition (3.5) $\Lambda_m \leq f_k^m$. The condition (3.9) is satisfied. Conversely, if $f_{k+1}^m \leq \Lambda_m \leq f_k^m$, then we just let $\mu_i^m = f_i^m - \Lambda_m$ for $1 \leq i \leq k$. The KKT condition (3.5) is still satisfied for the original solution y_{odf}^*.

In conclusion, the KKT conditions are equivalent to (3.8)∼3.9 in the transformed problem. □

From the above discussion it can be seen that if $x_m^* \in (D_k, D_{k+1}]$ then $\partial r_m(x_m) = f_{k+1}^m$; if $x_m^* = D_k$ then $\partial r_m(x_m) = [f_{k+1}, f_k]$. Thus the bid-price Λ_m either equals f_{k+1}^m or takes value in $[f_k, f_{k+1}]$.

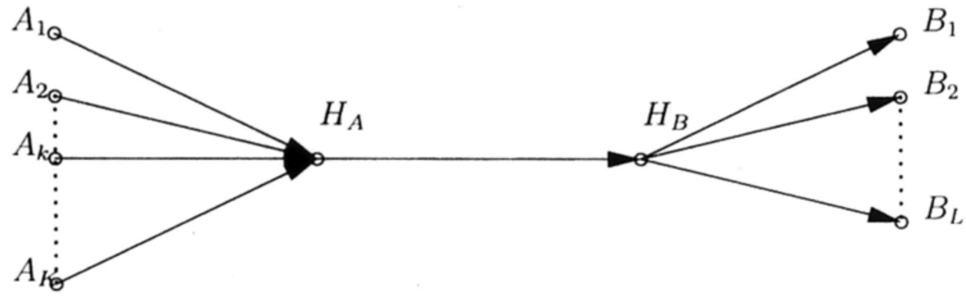

Figure 3.1: A hub-to-hub network

Because with those piecewise linear revenue functions it is hard to perform our subsequent analysis, we first prove that the piecewise linear revenue functions can be approximated by a sequence of differentiable, strictly concave functions. This is illustrated in the following *smoothing theorem*:

Theorem 2 *There exists a sequence of differentiable, strictly concave and strictly increasing functions $(h_m^n(x_m))$ that uniformly converges to the piecewise linear revenue function $r_m(x_m)$.*

Proof. By construction. Let $D_1 = d_1, D_2 = d_1 + d_2, \cdots, D_J = \sum_{j=1}^{J} d_j$. We first select a small interval $[D_k - D_k/n, D_k]$. Then we let

$$g(x) = ax^3 + bx^2 + cx + d \text{ for } x \in [D_k - D_k/n, D_k] \tag{3.10}$$

satisfying

$$\begin{aligned} g(D_k - D_k/n) &= r(D_k - D_k/n) \\ g(D_k) &= r(D_k) \\ g'(D_k - D_k/n) &= f_{k-1} \\ g'(D_k) &= f_k. \end{aligned} \tag{3.11}$$

Let $x = D_k - D_k/n + tD_k/n, t \in [0, 1]$, then let

$$g_1(t) = g(x) = a't^3 + b't^2 + c't + d'$$

and $\alpha = r(D_k - D_k/n), \beta = r(D_k)$. From Equation (3.11) we have

$$\begin{aligned} d' &= \alpha \\ a' + b' + c' + d' &= \beta \\ c' &= f_{k-1}D_k/n \\ 3a' + 2b' + c' &= f_kD_k/n. \end{aligned}$$

It reduces to

$$a' + b' + D_k/nf_{k-1} + \alpha = \beta$$

$$3a' + 2b' + c' = D_k/nf_k,$$

and so

$$\begin{aligned}
a' &= D_k/nf_k + D_k/nf_{k-1} \\
b' &= 3\beta - 3\alpha - 2D_k/nf_{k-1} - D_k/nf_k \\
c' &= D_k/nf_{k-1} \\
d' &= \alpha
\end{aligned}$$

and

$$\begin{aligned}
g''(x) &= \frac{n^2}{D_k^2} g_1''(t) \\
&= \frac{n^2}{D_k^2}((6D_k/nf_k + 6D_k/nf_{k-1} + 12\alpha - 12\beta)t + 6\beta - 6\alpha - 4D_k/nf_{k-1} - 2D_k/nf_k).
\end{aligned}$$

$$\begin{aligned}
g_1''(0) &= \frac{n^2}{D_k^2}(6\beta - 6\alpha - 4D_k/nf_{k-1} - 2D_k/nf_k) \\
g_1''(1) &= 4D_k/nf_k + 2D_k/nf_{k-1} + 6\alpha - 6\beta.
\end{aligned}$$

Since $\beta - \alpha = r(D_k) - r(D_k - D_k/n) = D_k/nf_{k-1}$, the last equations are transformed into

$$\begin{aligned}
g_1''(0) &= \frac{n^2}{D_k^2} 2D_k/n(f_{k-1} - f_k) = \frac{n}{D_k}(f_{k-1} - f_k) \\
g_1''(1) &= \frac{n^2}{D_k^2} 2D_k/n(f_k - f_{k-1}) = \frac{n}{D_k}(f_k - f_{k-1}).
\end{aligned}$$

To make it concave, let $h(x) = g_1(t) - Mt^2(1-t)^2$. It is easy to verify that $h(x)$ still satisfies Equation (3.11).

$$\begin{aligned}
h''(x) &= \frac{n^2}{D_k^2}(g_1''(t) - M(12t^2 - 12t + 2)) \\
&= \frac{n^2}{D_k^2}(2\frac{n}{D_k}(f_{k-1} - f_k)(1 - 3t) - M(12t^2 - 12t + 2)).
\end{aligned}$$

The above is maximum at $t = -n/(D_k M)(f_{k-1} - f_k) + 1/2$. Let $f_{k-1} - f_k = \Delta$, $n/D_k = \theta$; then the optimum value is expressed as $-\theta\Delta - 6\theta^2\Delta^2/M + M$. Because we can take M arbitrarily in $[2\theta\Delta, \infty]$, we just take it as $2\theta\Delta$ here. In this case:

$$-\theta\Delta - 6\theta^2\Delta^2/M + M = -\theta\Delta - 3\theta\Delta + 2\theta\Delta = -2\theta\Delta < 0.$$

Therefore by adopting M at $2\theta\Delta$ we can assure that $h(x)$ is strictly concave. Notice that this together with (3.11) indicate that $h(x)$ is also strictly increasing in $[D_k - D_k/n, D_k]$. We smooth For each k and then get a differentiable concave function that only takes different values at $(D_k - D_k/n, D_k)$ from $r_m(x_m)$. Suppose it is $h_m^n(x_m)$. Then

$$|h_m^n(x_m) - r_m(x_m)| < \max_{1 \le k \le J} \{r_m(D_k) - r_m(D_k - D_k/n)\}. \tag{3.12}$$

As $n \to \infty$, the above uniformly tends to zero. Therefore $(h_m^n(x_m))$ converges uniformly to $r_m(x_m)$. □

Now we have a sequence of strictly concave, differentiable and strictly increasing functions $(h_m^n(x_m))$. Let us take out a specific set of functions $h_m^n(x_m)$ for $m = 1, 2, \cdots, KL + 1 + K + L$ in the sequences here and temporarily omit the superscript n. Now let us substitute the revenue functions $r_m(x_m)$ in (3.7) with the strictly concave, differentiable and strictly increasing functions $h_m(x_m)$ for $m = 1, 2, \cdots, KL + 1 + K + L$. Then we get the following model:

$$\max_{\mathbf{x}} \sum_{1 \le m \le 2K + KL + 2L + 1} h_m(x_m) \tag{3.13}$$

$$s.t. \sum_{m \ni l} x_m \le c_l, l = 1, 2, \cdots, K + L + 1. \tag{3.14}$$

Notice that on each of the single leg routes $(A_k, H_A), (H_A, H_B), (H_B, B_t)$ there is a strictly increasing revenue function. Thus each leg's capacity must be used up. This reduces to the next formulation, i.e. Problem (3.13) can be rewritten as:

$$\max_{\mathbf{x}} \sum_{1 \le m \le 2K+KL+2L+1} h_m(x_m) \tag{3.15}$$

$$s.t. \sum_{m \ni l} x_m = c_l, l = 1, 2, \cdots, K+L+1, \tag{3.16}$$

which is called *the main model* in our thesis.

3.4 Concluding remarks

This chapter first examined the general bid price control scheme proposed by Simpson (1989) and Williamson (1992). The specific hub-to-hub NRM problem was then addressed. In investigating the specific structures in our hub-to-hub NRM problem, we found that the IP model of the problem can be recast into a network flow formulation. This assures that the LP relaxation of the IP model has integer solutions. This immediately leads on to restricting the focus on the DLP model for the hub-to-hub NRM problem.

To uncover the special structure of the hub-to-hub BP control, the total revenues on an OD were aggregated and a smoothing theorem was derived in which the piecewise revenue function for an OD can be approximated by a sequence of strictly increasing, strictly concave and infinitely differentiable revenue functions. Those functions are easier to deal with when doing subsequent analysis. It is hoped that by studying the structural properties in the reformulated problem with such revenue functions replacing the original ones we can also derive results from the original problem. This will be done in the next chapter.

Chapter 4

Structures in the main model

This chapter contains the main results of the thesis. In line with the previous chapter, we now consider the reformulated problem:

The main model

$$\max_{\mathbf{x}} \sum_{1 \le m \le 2K+KL+2L+1} h_m(x_m) \tag{4.1}$$

$$s.t. \sum_{m \ni l} x_m = c_l, l = 1, 2, \cdots, K + L + 1. \tag{4.2}$$

Section 4.3 of this chapter will show that properties obtained in this model can be extended to the same properties in Model (3.7) and thus give insights into the BP control.

We describe the Lagrangian function of Problem (4.1) as:

$$L(\mathbf{x}, \lambda) = \sum_{1 \leq m \leq 2K+KL+2L} h_m(x_m) - \sum_{l} \lambda_l \Big(\sum_{m \ni l)} x_m - c_l \Big). \tag{4.3}$$

To establish the dual function of Problem (4.1), let

$$l(\lambda) = \max_{\mathbf{x} \geq 0} L(\mathbf{x}, \lambda).$$

Then, solving the problem

$$\min_{\lambda \geq 0} l(\lambda)$$

gets the minimizer of the dual function. We denote an optimal primal-dual pair by $(\mathbf{x}^*, \lambda^*)$ or equivalently $(\mathbf{x}^*, \mathbf{\Lambda}^*)$. In later discussions we may neglect the asterisk sign and still use it to denote an optimal solution pair if confusion does not arise.

4.1 KKT conditions

The optimal shadow prices and optimal allocations for the replaced Problem (4.1) satisfy the following KKT conditions (necessary and sufficient conditions for optimality here):

$$\sum_{m \ni l} x_m = c_l, l = 1, 2, \cdots, K + L + 1, \tag{4.4}$$

$$x_m \Big(\sum_{l \in m} \lambda_l - h'_m(x_m) \Big) = 0, \tag{4.5}$$

$$h'_m(x_m) \leq \sum_{l \in m} \lambda_l, m = 1, 2, \cdots, 2K + KL + 2L. \tag{4.6}$$

We let $\Lambda_m = \sum_{l \in m} \lambda_l$ denote the shadow value for OD pair I_m (also called the threshold price for OD pair I_m). By adopting the notation of Λ_m we can rewrite Equation (4.5) as

$$x_m\Big(\Lambda_m - h'_m(x_m)\Big) = 0 \tag{4.7}$$

and rewrite (4.6) as

$$\Lambda_m - h'_m(x_m) \geq 0.$$

Then we study the monotone properties of the optimal solutions (allocation variables, bid prices, thresholds) to the transformed problems, with the original revenue functions replaced by differentiable, strictly concave functions.

4.2 Monotone thresholds

This section studies the monotonicity of the threshold values in the hub-to-hub network. Although structural properties (monotone thresholds) have been extensively studied in the literature, these kinds of structural properties in general networks have been overlooked. We thus focus on this issue and study the structural properties in a more general network model. The main results that were obtained are based on the deterministic LP model. Extension to dynamic models will be explored in future work. Apart from the considerations such as algorithmic simplification or to gain intuitive understanding, the monotonicity properties of those threshold values also have other applications. In his two-leg model, You (1999) summarized the optimal policy as critical booking capacities from the monotonicity of one leg's average seat value in another leg's capacity. This differs somewhat from virtue nesting (nested booking limit) since it depends on a set of threshold-curves in joint-variables. The optimality of the threshold curve policy relies on the sub/supermodularity of the optimal value function and the monotonicity of the average seat value in another leg's capacity. In the following by 'monotonically increasing' we mean 'nondecreasing'.

The main proof steps are summarized here: (1) We first project, by contradiction, that the dual value is not changed according to the assumed direction. (2) Then we show that the new allocation scheme will overflow the capacity constraints if any of them are different from the previous optimal allocation decision. (3) Then it must hold that all the allocations in the new system status are unchanged. (4) Then we change the dual variable under investigation to the opposite direction of the current change direction. (5) We then show that such a change will cause the dual Lagrange function $l(\lambda)$ value to be decreased, which contradicts the minimality of the function value at the optimal dual variables. (6) Thus the dual variable can only change in accordance with the pre-assumed directions. (7) By Lemma 1, the corresponding allocation decision's change direction can be determined. I am indebted to Simai He, a former fellow Ph.D. student and now an assistant professor at City University of Hong Kong, who sketched the above proof scheme for me. Our technique differs from that of Topkis (1978, 1998) and Granot and Veinott (1985), where the former did it in lattice programming and the latter did it in network flows. In a complicated environment, such as the NRM problem, neither method is entirely satisfactory (see Figure 1.2). In Topkis (1997), his prerequisite assumptions are too restrictive, requiring the objective function to be supermodular in all the state variables and the constraint region to be a lattice, conditions which are violated by the proposed network model. Although Granot and Veinott (1985) studied the monotone optimal solution in a network, they restricted their attention to the monotonicity of the optimal solution on parameters merely appearing in the objective function. But the proposed network model is in the constraint region. Although in the linear objective case the parameter in the constraint region can be transferred to the objective function by a dual-transform, the general objective is however a nonlinear one. Thus the vehicle adopted here is the Lagrange dual function, which stitches the objective function and the constraint region together.

Of specific interest is how a change in the capacity constraint of Model (3.3) will change the optimal allocations and bid-prices. Firstly, we found that the bid-price and allocation have the following inverse direction of change. This result is formally stated as:

Lemma 1 *In Problem* (3.13),

if $\tilde{\Lambda}_m > \Lambda^*_m$, *then* $\tilde{x}_m < x^*_m$ *or* $\tilde{x}_m = x^*_m = 0$.

if $\tilde{\Lambda}_m = \Lambda^*_m$, *then* $\tilde{x}_m = x^*_m$.

if $\tilde{\Lambda}_m < \Lambda^*_m$, *then* $\tilde{x}_m > x^*_m$ *or* $\tilde{x}_m = x^*_m = 0$.

Proof. In view of Equation (4.5) in the KKT conditions of the problem, $x^*_m(\Lambda^*_m - h'_m(x^*_m)) = 0$. If $\tilde{\Lambda}_m > \Lambda^*_m$ and $\tilde{x}_m \geq x^*_m$, then together considering the concavity

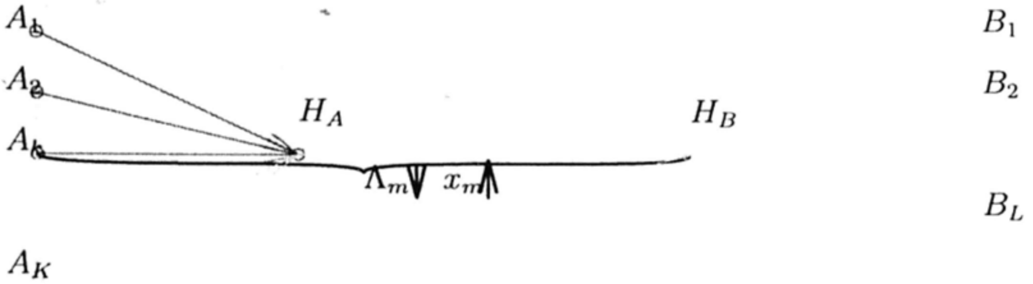

Figure 4.1: Illustration for Lemma 1

of $h_m(x_m)$, we get $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) > \Lambda^*_m - h'_m(x^*_m) \geq 0$. By virtue of the KKT condition, $\tilde{x}_m = 0$, and so $x^*_m = 0$. Therefore, when $\tilde{\Lambda}_m > \Lambda^*_m$, either $\tilde{x}_m < x^*_m$ or $\tilde{x}_m = x^*_m = 0$. Next, we assume that $\tilde{\Lambda}_m = \Lambda^*_m$. If $\tilde{x}_m > x^*_m \geq 0$, then, in view of the KKT conditions, $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) = 0$. Because $h_m(x_m)$ is concave, we have $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) > \Lambda^*_m - h'_m(x^*_m) \geq 0$, which is a contradiction. Thus, $\tilde{x}_m \leq x^*_m$. By symmetry, we also have $\tilde{x}_m \geq x^*_m$. It turns out that if $\tilde{\Lambda}_m = \Lambda^*_m$, $\tilde{x}_m = x^*_m$. By symmetry with the first case, the last assertion also holds. □

This can be understood intuitively as: the marginal value of a product decreases when its production quantity gets an increase.

Now we demonstrate that the bid price on the leg (H_A, H_B) is monotonically increasing in the capacities of $(A_k, H_A), k = 1, 2, \cdots, K$. Because of the combinatorial nature of the problem the proof here is also rather combinatorial.

Theorem 3 *The threshold Λ_m for $I_m = (H_A, H_B)$ is increasing in capacities c_l, $l = 1, \cdots, K$, or $l = K+2, \cdots, K+1+L$.*

This is illustrated in Figure 4.2.

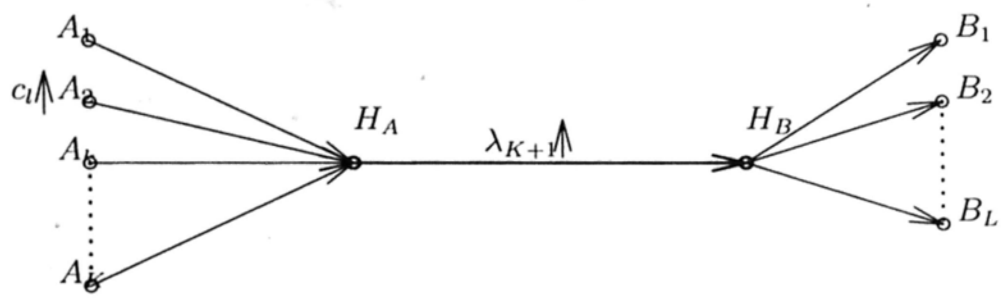

Figure 4.2: Illustration for Theorem 3

Proof. Because of symmetry, we only consider $l = 1$. We suppose c_1 is replaced by $\tilde{c}_1 > c_1$ and show that in the new set of optimal solution $\tilde{\mathbf{\Lambda}}$, $\tilde{\Lambda}_{k+1} \geq \Lambda_{k+1}$. To show this, we denote by U and V the cities in the origin and destination regions respectively. We then divide U into U_1 and U_2, where $U_1 = \{A_k : k \leq K, \tilde{\lambda}_k > \lambda_k^*\}$ ($\tilde{\lambda}_k$ and λ_k^* are the shadow prices of flight $F_k = (A_k, H_A)$) and $U_2 = U \backslash U_1$. We divide V into V_1 and V_2, where $V_1 = \{B_t : 1 \leq l \leq L, \tilde{\lambda}_{K+1+l} > \lambda^*_{K+1+l}\}$ ($\tilde{\lambda}_{K+1+l}$ and λ^*_{K+1+l} are the shadow prices of flight $F_n = (H_B, B_t)$), and $V_2 = V \backslash V_1$. Further, we denote by $V_1' = \{B_t : B_t \in V_1, \tilde{\Lambda}_m \geq \Lambda^*_m \text{ for } I_m = (H_A, B_t)\}$.

We suppose, contrarily, that $\tilde{\lambda}_{K+1} < \lambda^*_{K+1}$. Then we let $\lambda'_{K+1} = \tilde{\lambda}_{K+1} + \delta$, $\lambda'_n = \tilde{\lambda}_n - \delta$ for $F_n \in (U_1, H_A)$ and $\lambda'_n = \tilde{\lambda}_n - \delta$ for $F_n \in (H_B, V_1')$. In addition, we let $\lambda'_n = \tilde{\lambda}_n$ for any other flight F_n. It will be shown that λ' is still feasible with sufficiently small δ under $\tilde{\mathbf{x}}$ and that $(\tilde{\mathbf{x}}, \tilde{\lambda}')$ still satisfy the KKT conditions (4.4)∽ (4.5).

Then, we explore how the allocation variables of itineraries that pass through flight (H_A, H_B) are changed. We break down the detailed discussion in terms of three groups. Group 1 consists of itineraries included in (U_1, H_B) and (U_1, V). Group 2 consists of itineraries in (U_2, V_1) and (H_A, V_1), and Group 3 consists of itineraries in (U_2, H_B), (U_2, V_2), and (H_A, V_2).

1. In Group 1, consider $A_k \in U_1$ and $I_j = (A_k, H_A)$. By virtue of Lemma 1, $\tilde{x}_j < x_j^*$ or $\tilde{x}_j = x_j^* = 0$. By Equation (4.4), if $\tilde{x}_j < x_j^*$, then

$$\sum_{I_m \in (A_k, H_B) \cup (A_k, V)} \tilde{x}_m > \sum_{I_m \in (A_k, H_B) \cup (A_k, V)} x_m^* \tag{4.8}$$

   If $\tilde{x}_j = x_j^* = 0$ for $I_j = (A_k, H_A)$ then

$$\sum_{I_m \in (A_k, H_B) \cup (A_k, V)} \tilde{x}_m = \sum_{I_m \in (A_k, H_B) \cup (A_k, V)} x_m^*. \tag{4.9}$$

2. In Group 2, we divide our analysis into two subgroups

   (a) If $I_m \in (H_A, V_1 \backslash V_1') \cup (U_2, V_1 \backslash V_1')$, $\tilde{\Lambda}_m < \Lambda_m^*$, then by Lemma 1 $\tilde{x}_m > x_m^*$ or $\tilde{x}_m = x_m^* = 0$.

   (b) If $B_t \in V_1'$, then $\tilde{\Lambda}_m > \Lambda_m^*$ for $I_m \in (H_B, B_t) \cup (U_1, B_t)$ and thus, by Lemma 1 $\tilde{x}_m < x_m^*$ or $\tilde{x}_m = x_m^* = 0$. By applying Equation (4.4) to the flight (H_B, B_t), if $\tilde{x}_m < x_m^*$ for any $I_m \in (U_1, B_t) \cup (H_B, B_t)$, then,

$$\sum_{I_m \in (H_A, B_t) \cup (U_2, B_t)} \tilde{x}_m > \sum_{I_m \in (H_A, B_t) \cup (U_2, B_t)} x_m^*. \tag{4.10}$$

   If $\tilde{x}_m = x_m^* = 0$ for all $I_m \in (U_1, B_t) \cup (H_B, B_t)$, then,

$$\sum_{I_m \in (H_A, B_t) \cup (U_2, B_t)} \tilde{x}_m = \sum_{I_m \in (H_A, B_t) \cup (U_2, B_t)} x_m^*. \tag{4.11}$$

3. In Group 3, we consider $I_m \in (U_2, H_B) \cup (U_2, V_2) \cup (H_A, V_2)$. By definition, $\tilde{\Lambda}_m < \Lambda_m^*$, and so, by Lemma 1, $\tilde{x}_m > x_m^*$ or $\tilde{x}_m = x_m^* = 0$.

Aggregating the allocation variables of all the itineraries in the above groups, by noting that flight F_{K+1} is OD pair (H_A, H_B), yields

$$\sum_{(H_A,H_B)\in I_m, I_m\neq(H_A,H_B)} \tilde{x}_m \geq \sum_{(H_A,H_B)\in I_m, I_m\neq(H_A,H_B)} x^*_m. \tag{4.12}$$

For $I_m = (H_A, H_B)$, by Lemma 1

$$\tilde{x}_m > x^*_m \text{ or } \tilde{x}_m = x^*_m = 0. \tag{4.13}$$

If $\tilde{x}_m > x^*_m$, Equation (4.4) disproves (4.12). Thus $\tilde{x}_m = x^*_m = 0$. By Equation (4.4)

$$\sum_{(H_A,H_B)\in I_m, I_m\neq(H_A,H_B)} \tilde{x}_m = \sum_{(H_A,H_B)\in I_m, I_m\neq(H_A,H_B)} x^*_m,$$

and thus all the inequalities of optimal allocations in the three groups must be equations. Specifically, $\tilde{x}_m = x^*_m = 0$ for itineraries $(U_1, H_A)\cup(H_B, V_1')\cup(U_1, V_1')\cup (U_2, H_B) \cup (U_2, V_2) \cup (H_A, H_B) \cup (H_A, V_2)$.

Now consider the new set of solution λ'. Since $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) = \tilde{\Lambda}_m - h'_m(x^*_m) > \Lambda^*_m - h'_m(x^*_m) \geq 0$ for $I_m \in (U_1, H_A) \cup (H_B, V_1') \cup (U_1, V_1')$, when δ is sufficiently small $\tilde{\Lambda}_m - \delta - h'_m(\tilde{x}_m) > 0$ i.e. Equation (4.6) still holds for $(\lambda', \tilde{\mathbf{x}})$ on these itineraries. As for other itineraries passing (U_1, H_A) and (H_B, V_1'), $\Lambda'_m = \tilde{\Lambda}_m$. Thus Equation (4.6) obviously still holds. And from the above analysis $\tilde{x}_m = x^*_m = 0$ for itineraries $(U_1, H_A) \cup (H_B, V_1') \cup (U_1, V_1') \cup (U_2, H_B) \cup (U_2, V_2) \cup (H_A, H_B) \cup (H_A, V_2)$, Equation (4.5) also holds on these itineraries for the pair $(\tilde{\mathbf{x}}, \lambda')$. For other itineraries, it obviously remains true since nothing is changed. Thus λ' is also optimal in the new constraint state. This is a contradiction in that $\tilde{\lambda}$ is the unique optimal solution. Thus it must be that $\tilde{\lambda}_{K+1} \geq \lambda^*_{K+1}$. This completes the proof of the theorem. □

Since the shadow price measures the value of a component in a system, the above theorem reveals the complementary nature between the side leg and the value of the route (H_A, H_B). That is, the shadow value of the route (H_A, H_B) increases as a side leg's capacity increases.

As a direct consequence, we have

Corollary 1 *The optimal allocation x_m^* on route $I_m = (H_A, H_B)$ in Model (3.7) is monotonically decreasing in $c_l, l = 1, \cdots, K$ and the minimal acceptable fare level f_k on leg (H_A, H_B) is increasing in c_l.*

Proof. By virtue of Lemma 1, the above theorem implies that $\tilde{x}_m \leq x_m^*$ for $I_m = (H_A, H_B)$. From the definition of the piecewise linear revenue function, this means that the minimal acceptable fare level on (H_A, H_B) is increased. □

The following two lemmas are given to prove Theorem 4. The main result is that a large group of itineraries have unchanged capacity allocations under a certain capacities change.

Lemma 2 *Consider a capacity change in Problem (3.13) that changes $\mathbf{c}$ to $\tilde{\mathbf{c}}$. Then, suppose $\tilde{c}_n \geq c_n$ for the flights (U, H_A) and $\tilde{c}_n \leq c_n$ for the flights (H_B, V) while the capacity change in the middle flight is arbitrary. If $\tilde{\Lambda}_m \geq \Lambda_m^*$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$ simultaneously for some $k = 1, \cdots, K$, then, the capacity allocations to all the itineraries departing from the city A_s are unchanged, i.e. $\tilde{x}_m = x_m^*$ for $I_m \in (A_s, H_A) \cup (A_s, H_B) \cup (A_s, V)$.*

Proof. We prove this by a contradiction argument. First we define $\Delta_n = \tilde{\lambda}_n - \lambda^*_n, F_n \in \mathcal{F}$. If there exists $A_s \in U$ that $\tilde{\Lambda}_m \geq \Lambda^*_m$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$, and that an itinerary passing (A_s, H_A) has a different allocation from the original state, then we define $\mathcal{A}_1$ as the group of all such origin cities. We suppose that Δ_s for $F_s = (A_s, H_A)$ is the maximal among all such quantity for the flights departing from origin cities in $\mathcal{A}_1$ and arriving at hub H_A. Since $\tilde{\Lambda}_m \geq \Lambda^*_m$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$, then by Lemma 1 $\tilde{x}_m \leq x^*_m$ for these two itineraries.

- If it is the itinerary (A_s, H_A) that has a different allocation, then it must be that $\tilde{x}_j < x^*_j$ for $I_j = (A_s, H_A)$. By Equation (4.4)

$$\sum_{(A_s,H_A)\in I_m, I_m \neq I_j} \tilde{x}_m > \sum_{(A_s,H_A)\in I_m, I_m \neq I_j} x^*_m. \tag{4.14}$$

  But $\tilde{x}_m \leq x^*_m$ for $I_m = (A_s, H_B)$, thus there must exist $B_t \in V$ that $\tilde{x}_m > x^*_m$ for $I_m = (A_s, B_t)$.

- If it is not (A_s, H_A) but (A_s, H_B) has a different allocation, then $\tilde{x}_j = x^*_j$ for $I_j = (A_s, H_A)$ but $\tilde{x}_j < x^*_j$ for $I_j = (A_s, H_B)$. Again by applying Equation (4.4) to the flight (A_s, H_A) we have

$$\sum_{(A_s,H_A)\in I_m, I_m \neq (A_s,H_A)} \tilde{x}_m \geq \sum_{(A_s,H_A)\in I_m, I_m \neq (A_s,H_A)} x^*_m. \tag{4.15}$$

  But $\tilde{x}_j < x^*_j$ for $I_j = (A_s, H_B)$, we can deduce that there must also exist a $B_t \in V$ that $\tilde{x}_m > x^*_m$ for $I_m = (A_s, B_t)$.

- If not (A_s, H_A) or (A_s, H_B) but an $(A_s, B_s) \in (A_s, V)$ has a different allocation, i.e. $\tilde{x}_j = x^*_j$ for $I_j = (A_s, H_A)$ and $I_j = (A_s, H_B)$ but $\tilde{x}_j \neq x^*_j$ for

$I_j = (A_s, B_s)$, then either $\tilde{x}_j > x_j^*$ or $\tilde{x}_j < x_j^*$ for $I_j = (A_s, B_s)$. Again by applying Equation (4.4) to the flight (A_s, H_A), we have

$$\sum_{(A_s,H_A)\in I_m, I_m\neq(A_s,H_A)} \tilde{x}_m \geq \sum_{(A_s,H_A)\in I_m, I_m\neq(A_s,H_A)} x_m^*. \tag{4.16}$$

If $\tilde{x}_j < x_j^*$ for $I_j = (A_s, B_s)$, from the above inequality we can deduce that there must also exist a $B_t \in V$ that $\tilde{x}_m > x_m^*$ for $I_m = (A_s, B_t)$.

Thus in all the above cases there is always a $B_t \in V$ that $\tilde{x}_m > x_m^*$ for $I_m = (A_s, B_t)$ and hence by Lemma 1 $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m = (A_s, B_t)$. Since $\tilde{\Lambda}_m \geq \Lambda_m^*$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$, we immediately deduce that $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m = (H_A, B_t)$ and $I_m = (H_B, B_t)$. Thus $\tilde{x}_m > x_m^*$ or $\tilde{x}_m = x_m^* = 0$ for $I_m = (H_A, B_t)$ and $I_m = (H_B, B_t)$. Applying Equation (4.4) to the flight (H_B, B_t) we have

$$\sum_{(H_B,B_t)\in I_m, I_m\neq(H_B,B_t), I_m\neq(H_A,B_t)} \tilde{x}_m \leq \sum_{(H_B,B_t)\in I_m, I_m\neq(H_B,B_t), I_m\neq(H_A,B_t)} x_m^*. \tag{4.17}$$

But $\tilde{x}_m > x_m^*$ for $I_m = (A_s, B_t)$. We conclude from the above inequality that there must exist $I_m = (A_{s'}, B_t)$ that $\tilde{x}_m < x_m^*$. Because $\tilde{x}_m > x_m^*$ for $I_m = (A_s, B_t)$, we get $k' \neq k$. Notice also that $\tilde{\Lambda}_m > \Lambda_m^*$ for $I_m = (A_{s'}, B_t)$. Because $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m = (A_s, B_t)$, we have $\tilde{\Lambda}_m - \Lambda_m^*|_{I_m=(A_{s'},B_t)} > \tilde{\Lambda}_m - \Lambda_m^*|_{I_m=(A_k,B_t)}$, i.e.

$$\tilde{\lambda}_n - \lambda_n^*|_{F_n=(A_{s'},H_A)} > \tilde{\lambda}_n - \lambda_n^*|_{F_n=(A_k,H_A)}. \tag{4.18}$$

Note that from $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m = (H_A, B_t)$ and $I_m = (H_B, B_t)$ we deduce $\tilde{\Lambda}_m > \Lambda_m^*$ for $I_m = (A_{s'}, H_A)$ and $I_m = (A_{s'}, H_B)$ and there is an itinerary $I_m = (A_{s'}, B_t)$ passing $(A_{s'}, H_A)$ that has a different allocation as in the original state ($\tilde{x}_m < x_m^*$ for $I_m = (A_{s'}, B_t)$). Thus $A_{s'}$ also belongs to the group of cities

A_1. But $\Delta_{s'} > \Delta_s$. This is in contradiction with the maximality of Δ_s for $F_s = (A_s, H_A)$ in all such side flights. This in turn shows that all the itineraries departing from the city A_s in the new capacity state must have the same allocation as in the original state. □

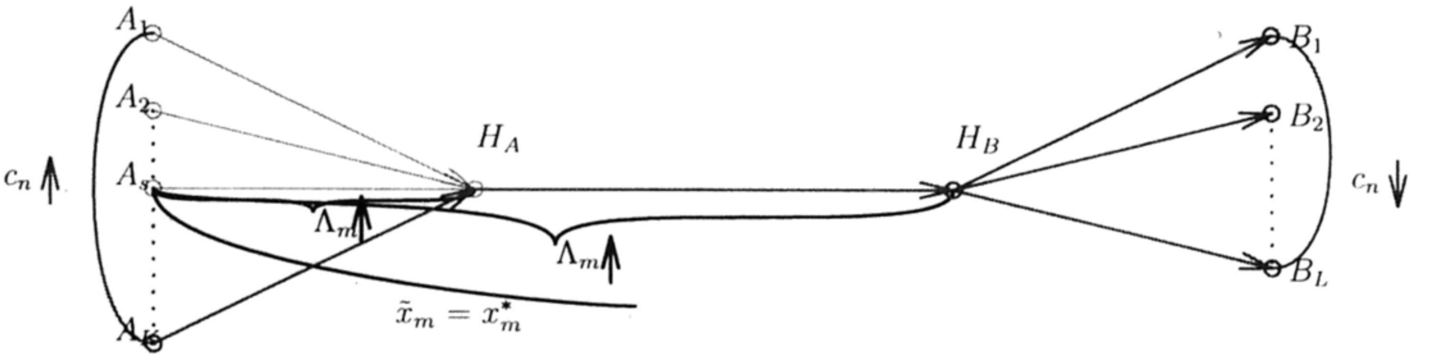

Figure 4.3: Illustration for Lemma 2

Following the similar line of reasoning, we can draw the following conclusion which is symmetric to the previous one except for a minor change.

Lemma 3 *Consider a capacity change in Problem* (3.13) *that changes* $\mathbf{c}$ *to* $\tilde{\mathbf{c}}$. *Suppose that* $\tilde{c}_1 > c_1$, *and* $\tilde{c}_n = c_n$ *for* $n > 1$. *If* $\tilde{\Lambda}_m \leq \Lambda^*_m$ *for* $I_m = (A_s, H_A)$ *and* $I_m = (A_s, H_B)$ *for some* $k = 2, \cdots, K$ *and* $\tilde{\Lambda}_m > \Lambda^*_m$ *for* $I_m = (A_1, H_B)$, *then, the capacity allocations to all the itineraries departing from the city* A_s *are unchanged, i.e.* $\tilde{x}_m = x^*_m$ *for* $I_m \in (A_s, H_A) \cup (A_s, H_B) \cup (A_s, V)$.

Proof. We prove this by contradiction. First we define $\Delta_n = \tilde{\lambda}_n - \lambda^*_n, F_n \in \mathcal{F}$. If there exists $A_s \in U$ that $\tilde{\Lambda}_m \leq \Lambda^*_m$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$, and that an itinerary passing (A_s, H_A) has a different allocation from the original state, then we define $\mathcal{A}_2$ as the group of all such origin cities. We suppose that Δ_s for $F_s = (A_s, H_A)$ is minimal among all such quantities for the flights departing from origin cities $\mathcal{A}_2$ and arriving at hub H_A. Because $\Delta_1 > 0$ for $F_1 = (A_1, H_A)$, we have $\Delta_s < \Delta_1$. Since $\tilde{\Lambda}_m \leq \Lambda^*_m$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$, then by Lemma 1 $\tilde{x}_m \geq x^*_m$ for these two itineraries.

- If the itinerary (A_s, H_A) has a different allocation, then it must be that $\tilde{x}_j > x^*_j$ for $I_j = (A_s, H_A)$. By applying Equation (4.4) to the flight (A_s, H_A) whose capacity is unchanged we get

$$\sum_{(A_s,H_A)\in I_m, I_m \neq I_j} \tilde{x}_m < \sum_{(A_s,H_A)\in I_m, I_m \neq I_j} x^*_m \tag{4.19}$$

  But $\tilde{x}_m \geq x^*_m$ for $I_m = (A_s, H_B)$; thus, there must exist $B_t \in V$ that $\tilde{x}_m < x^*_m$ for $I_m = (A_s, B_t)$.

- If not (A_s, H_A) but (A_s, H_B) has a different allocation, i.e. $\tilde{x}_j = x^*_j$ for $I_j = (A_s, H_A)$ but $\tilde{x}_j > x^*_j$ for $I_j = (A_s, H_B)$, then again by (4.4)

$$\sum_{(A_s,H_A)\in I_m, I_m \neq (A_s,H_A)} \tilde{x}_m < \sum_{(A_s,H_A)\in I_m, I_m \neq (A_s,H_A)} x^*_m \tag{4.20}$$

  can deduce that there must also exist a $B_t \in V$ that $\tilde{x}_m < x^*_m$ for $I_m = (A_s, B_t)$.

- If not (A_s, H_A) or (A_s, H_B) but an $(A_s, B_s) \in (A_s, V)$ has a different allocation, i.e. $\tilde{x}_j = x^*_j$ for $I_j = (A_s, H_A)$ and $I_j = (A_s, H_B)$ but $\tilde{x}_j \neq x^*_j$ for $I_j = (A_s, B_s)$, then either $\tilde{x}_j > x^*_j$ or $\tilde{x}_j < x^*_j$ for $I_j = (A_s, B_s)$. If $\tilde{x}_j > x^*_j$ for $I_j = (A_s, B_s)$ again by applying (4.4) to the flight (A_s, H_A) we have

$$\sum_{(A_s,H_A)\in I_m, I_m \neq (A_s,H_A)} \tilde{x}_m < \sum_{(A_s,H_A)\in I_m, I_m \neq (A_s,H_A)} x^*_m. \tag{4.21}$$

We can deduce that there must also exist a $B_t \in V$ that $\tilde{x}_m < x^*_m$ for $I_m = (A_s, B_t)$.

Thus in all the above cases, there is always a $B_t \in V$ that $\tilde{x}_m < x^*_m$ for $I_m = (A_s, B_t)$ and hence by Lemma 1 $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (A_s, B_t)$. Since $\tilde{\Lambda}_m \leq \Lambda^*_m$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$, we instantly deduce that $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (H_A, B_t)$ and $I_m = (H_B, B_t)$. Thus $\tilde{x}_m < x^*_m$ or $\tilde{x}_m = x^*_m = 0$ for $I_m = (H_A, B_t)$ and $I_m = (H_B, B_t)$. By applying (4.4) to the flight (H_B, B_t) we have

$$\sum_{(H_B,B_t)\in I_m, I_m \neq (H_B,B_t)} \tilde{x}_m = \sum_{(H_B,B_t)\in I_m, I_m \neq (H_B,B_t)} x^*_m \tag{4.22}$$

But $\tilde{x}_m < x^*_m$ for $I_m = (A_s, B_t)$ and $\tilde{x}_m \leq x^*_m$ for $I_m = (H_A, B_t)$ and $I_m = (H_B, B_t)$. We conclude from the above inequality that there must exist $I_m = (A_{s'}, B_t)$ that $\tilde{x}_m > x^*_m$. Because $\tilde{x}_m < x^*_m$ for $I_m = (A_s, B_t)$, we get $k' \neq k$. Because $\tilde{x}_m > x^*_m$ for $I_m = (A_{s'}, B_t)$, we have $\tilde{\Lambda}_m < \lambda^*_m$ for $I_m = (A_{s'}, B_t)$. Because $\tilde{\Lambda}_m > \lambda^*_m$ for $I_m = (A_s, B_t)$ we have $\tilde{\Lambda}_m - \lambda^*_m|_{I_m=(A_{s'},B_t)} < \tilde{\Lambda}_m - \lambda^*_m|_{I_m=(A_k,B_t)}$, i.e.

$$\tilde{\lambda}_n - \lambda^*_n|_{F_n=(A_{s'},H_A)} < \tilde{\lambda}_n - \lambda^*_n|_{F_n=(A_k,H_A)} \tag{4.23}$$

Note that from $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (H_A, B_t)$ and $I_m = (H_B, B_t)$, we deduce $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m = (A_{s'}, H_A)$ and $I_m = (A_{s'}, H_B)$, and there is an itinerary $I_m = (A_{s'}, B_t)$ passing $(A_{s'}, H_A)$ that has a different allocation as in the original state($\tilde{x}_m > x^*_m$ for $I_m = (A_{s'}, B_t)$); thus, $A_{s'} \in \mathcal{A}_2$. But $\Delta_{s'} < \Delta_s$. This contradicts the minimality of Δ_s for $F_s = (A_s, H_A)$ in all such side flights. Thus all the itineraries passing the flight (A_s, H_A) in the new capacity state must have the same allocation as in the original state. □

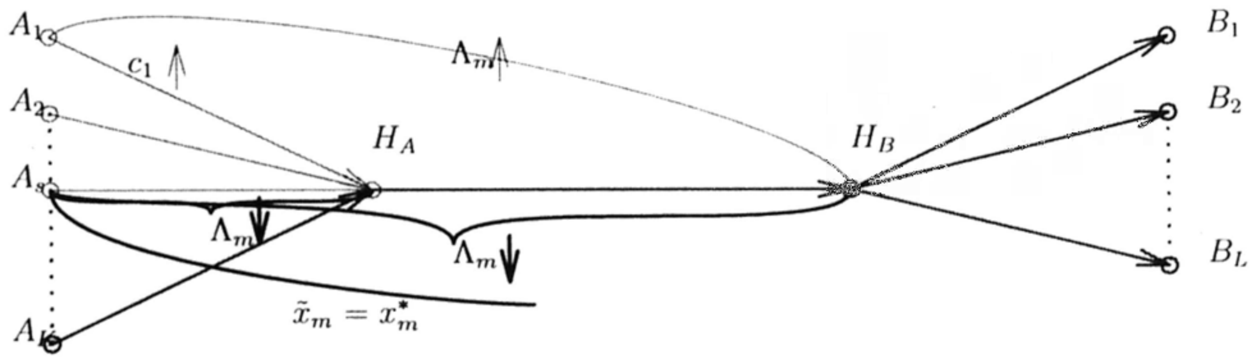

Figure 4.4: Illustration for Lemma 3

Note that it is possible that when $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m = (A_1, H_A)$ and $I_m = (A_1, H_B)$ the itineraries passing the flight (A_1, H_A) have different allocations, since its capacity is increased, and $\tilde{x}_m > x^*_m$ for $I_m = (A_1, H_A)$ or $I_m = (A_1, H_B)$ will possibly not violate the capacity constraint. Thus in the assumption of Lemma 3 we singled out the case where $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m = (A_1, H_A)$ and $I_m = (A_1, H_B)$ from the general side legs.

The above two lemmas are very important since they comprise a major block in proving Theorem 4 and Theorem 5.

When the first side arc's capacity increases, as Theorem 3 states, the middle arc would be of higher marginal value. Thus for the combining routes of left-side flight and middle-flight, they would have less accessibility to the middle-flight leg's capacity. Therefore the OD markets that just using a side leg on the left should have increased allocations on themselves. And thus they have decreased marginal value (or shadow price). Below is a theorem to address this fact.

Theorem 4 *The bid-price* Λ_m *for any* $I_m = (A_s, H_A), s = 1, \cdots, K$ *is monoton-*

ically decreasing in any $c_l, 1 \leq l \leq K$.

Proof. Without loss of generality, we suppose $l = 1$. And we pick up two capacities $\tilde{c}_1 > c_1$, but $\tilde{c}_n = c_n$ for other capacities $n \neq 1$. We suppose, by contradiction, that there is $F_n \in (U, H_A)$ that $\tilde{\lambda}_n - \lambda^*_n > 0$. Let $\Delta = \max\{\tilde{\lambda}_n - \lambda^*_n, F_n \in (U, H_A)\}$. Then obviously $\Delta > 0$. We let $\lambda'_n = \tilde{\lambda}_n - \delta$ for $F_n \in \mathcal{F}_1$, $\lambda'_n = \tilde{\lambda}_n + \delta$ for $F_n \in \mathcal{F}_2$ and $\lambda'_n = \tilde{\lambda}_n$ for $F_n \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. Note that the monotone relation stated in Theorem 3, i.e. $\lambda'_n \leq \lambda^*_n$ for $F_n = (H_A, H_B)$ is preserved under such adjustment. It shall be shown that λ' is still feasible with sufficiently small δ under $\tilde{\mathbf{x}}$ and $(\tilde{\mathbf{x}}, \tilde{\lambda}')$ still satisfies the KKT conditions (4.4)∽ (4.5).

Let $\mathcal{F}_1 = \{F_n | F_n \in (U, H_A), \tilde{\lambda}_n - \lambda^*_n = \Delta\}$. If $(A_s, H_A) \in \mathcal{F}_1$, then $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (A_s, H_A)$ and by Theorem 3 for $I_m = (A_s, H_B)$. By Lemma 3 $\tilde{x}_m = x^*_m = 0$ for $I_m = (A_s, H_A)$ and $I_m = (A_s, H_B)$, $\tilde{x}_m = x^*_m$ for all $I_m \in (A_s, V)$. Further if $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) = 0$ for $I_m = (A_s, B_t)$, then by Equation (4.6) $\tilde{\Lambda}_m \leq \Lambda^*_m$, which implies $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m \in (H_B, B_t) \cup (H_A, B_t) \bigcup_{(A_s, H_A) \subsetneq \mathcal{F}_1} (A_s, B_t)$. By Lemma 3 and Lemma 1 $\tilde{x}_m = x^*_m = 0$ for these itineraries. Thus we can conclude that $\Lambda^*_m > h'_m(x^*_m)$ for these itineraries.

To characterize such B_t's, let $\mathcal{F}_2 = \{F_n | F_n = (H_B, B_t) \in (H_B, V)$, there exists $(A_s, H_A) \in \mathcal{F}_1$ that $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) = 0$ for $I_m = (A_s, B_t)\}$.

Since $\Lambda'_m = \tilde{\Lambda}_m$ for $I_m \in \bigcup_{(A_s, H_A) \subsetneq \mathcal{F}_1} ((A_s, H_A) \cup (A_s, H_B)) \bigcup_{(H_B, B_t) \subsetneq \mathcal{F}_2} ((A_s, B_t) \cup (H_A, B_t) \cup (H_B, B_t)) \cup (H_A, H_B)$, $(\Lambda'_m, \tilde{x}_m)$ still satisfy Equation (4.5) for these itineraries. Since $\Lambda'_m = \tilde{\Lambda}_m$ for $I_m = (A_s, B_t)$ where $(A_s, H_A) \in \mathcal{F}_1, (H_B, B_t) \in \mathcal{F}_2$, $(\Lambda'_m, \tilde{x}_m)$ still satisfy Equation (4.5) for these itineraries. Since $\tilde{x}_m = 0$ for

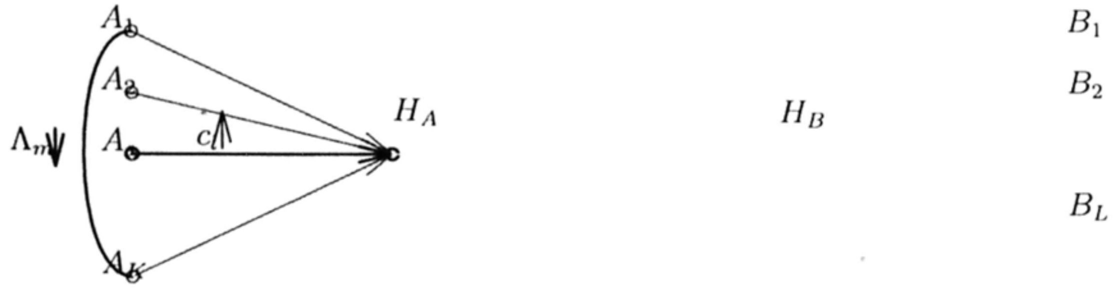

Figure 4.5: Illustration for Theorem 4

$I_m \in \bigcup_{(A_s,H_A)\in\mathcal{F}_1} ((A_s, H_A) \cup (A_s, H_B) \bigcup_{(H_B,B_t)\subsetneq\mathcal{F}_2} (A_s, B_t))$ (By the definition of $\mathcal{F}_2$ $\tilde{x}_m = 0$ for $I_m \in (A_s, B_t)$ where $(H_B, B_t) \subsetneq \mathcal{F}_2$),$(\Lambda'_m, \tilde{x}_m)$ still satisfy Equation (4.5) for these itineraries. Since

$$\tilde{x}_m = x^*_m = 0 \text{ for } I_m \in \mathcal{F}_2 \bigcup_{(H_B,B_t)\in\mathcal{F}_2} ((H_A, B_t) \bigcup_{(A_s,H_A)\subsetneq\mathcal{F}_1} (A_s, B_t)),$$

$(\Lambda'_m, \tilde{x}_m)$ still satisfy Equation (4.5) for these itineraries as long as δ is sufficiently small. Thus λ' is also optimal in the new constraint state. This is in contradiction with the uniqueness of the optimal dual solution. Thus it must hold that $\tilde{\lambda}_n \leq \lambda^*_n$ for $F_n \in (U, H_A)$. By Lemma 1 $\tilde{x}_m \geq x^*_m$ for $I_m \in (U, H_A)$. □

This is illustrated in Figure 4.5.

The thresholds of the left side legs are decreased. The economic interpretation of this theorem is that the shadow value of those flights is decreased and they would become less important to the whole network.

Corollary 2 *In Model* (3.7) , *the optimal allocation* x^*_m *to the route* $I_m = (A_l, H_A)$ *is monotonically increasing in any* $c_s, 1 \leq s \leq K$ *and the minimal acceptable fare level* f_k *on route* (A_l, H_A) *is decreasing in* c_s.

Proof. Since Theorem 4 says Λ_m^* for any $I_m \in (U, H_A)$ is decreasing in $c_s, 1 \leq s \leq K$ in Model (3.7), by Lemma 1 it indicates that x_m^* is increasing in c_s. By the construction of $r_m(x_m)$ we know that the minimal acceptable fare level f_k is decreasing in c_s. □

Now we study how a side flight's capacity change would affect a long-haul market travelling that side flight.

Theorem 5 *The threshold Λ_m for $I_m = (A_k, H_B)$ is monotonically decreasing in c_k for any $k = 1, \cdots, K$.*

Proof. Without loss of generality, we take $k = 1$. We suppose $\tilde{\Lambda}$ already satisfy the monotone relation stated in Theorems 3, 4. If $\tilde{\Lambda}_m > \Lambda_m^*$ for $I_m = (A_1, H_B)$, then it is obvious that $\tilde{\Lambda}_m > \Lambda_m^*$ for $I_m = (H_A, H_B)$.

Now in the new constraint state, we let $\lambda_n' = \tilde{\lambda}_n - \delta$ for $F_n = (H_A, H_B)$; let $\mathcal{F}_3 = \{F_n | F_n = (A_s, H_A)$ or $(H_B, B_t), \tilde{\Lambda}_m - h_m'(\tilde{x}_m) = 0$ for $I_m = (A_s, H_B)$ and $I_m = (H_A, B_t)\}$; let $\lambda_n' = \tilde{\lambda}_n + \delta$ for $F_n \in \mathcal{F}_3$. Note that the monotone relations stated in Theorems 3 , 4 7 are preserved under such adjustment as long as δ is sufficiently small. It shall be shown that λ' is still feasible under $\tilde{x}$ and $(\tilde{\mathbf{x}}, \tilde{\lambda}')$ still satisfy the KKT conditions (4.4)∽ (4.5).

Consider other itineraries in (U, H_B). If for $I_m = (A_s, H_B) \in (U, H_B), \tilde{\Lambda}_m < \Lambda_m^*$, then by Lemma 3 $\tilde{x}_m = x_m^* = 0$ for $I_m = (A_s, H_B)$ and $I_m = (A_s, H_A)$. If for an $I_m \in (U, H_B), \tilde{\Lambda}_m \geq \Lambda_m^*$, then by Lemma 1 $\tilde{x}_m \leq x_m^*$ for I_m.

Now consider a flight $F_n = (H_B, B_t) \in (H_B, V)$. If $\tilde{\lambda}_n > \lambda_n^*$, then $\tilde{\Lambda}_m > \Lambda_m^*$

for $I_m \in (A_1, B_t) \cup (H_A, B_t) \cup (H_B, B_t)$, by the same procedure as in Lemma 2 we can show that $\tilde{x}_m = x^*_m = 0$ for all these itineraries. If $\tilde{\lambda}_n \leq \lambda^*_n$, then $\tilde{x}_m \geq x^*_m$ for $I_m = (H_B, B_t)$. Since $\tilde{x}_m \leq x^*_m$ for all $I_m \in (U, H_B)$ and $\tilde{x}_m < x^*_m$ or $\tilde{x}_m = x^*_m = 0$ for $I_m = (H_A, H_B)$, by applying Equation (4.4) to the flight (H_A, H_B) we have $\sum_{I_m \in (U,V) \cup (H_A,V)} \tilde{x}_m \geq \sum_{I_m \in (U,V) \cup (H_A,V)} x^*_m$. But for $I_m = (H_B, B_t)$ with $\tilde{\lambda}_m > \lambda^*_m$ all $I_m \in (A_s, B_t) \cup (H_A, B_t)$ we have $\tilde{x}_m = x^*_m$. Therefore $\sum_{I_m \in (U,V_2) \cup (H_A,V_2)} \tilde{x}_m \geq \sum_{I_m \in (U,V_2) \cup (H_A,V_2)} x^*_m$.Thus once $\tilde{x}_m > x^*_m$ for any itinerary $\in (H_B, V_2)$ the total allocation on the right side flights overflows the total capacity of these flights, which contradicts Equation (4.4). Therefore $\tilde{x}_m = x^*_m$ for all itineraries $\in (H_B, V)$.

If $\tilde{\Lambda}_m = \Lambda^*_m$ for $I_m = (A_s, H_B)$, then it is obvious that $\tilde{\lambda}_n < \lambda^*_n$ for $F_n = (A_s, H_A)$. By Lemma 3 $\tilde{x}_m = x^*_m = 0$ for $I_m = (A_s, H_A)$.

If $\tilde{\lambda}_n \geq \lambda^*_n$ for $F_n = (H_B, B_t)$ then by Lemma 3 $\tilde{x}_m = x^*_m = 0$ for $I_m = (H_A, B_t)$. Thus $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) > 0$ for $I_m = (H_A, B_t)$. Therefore if $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) = 0$ for $I_m = (H_A, B_t)$ then it must hold that $\tilde{\lambda}_n < \lambda^*_n$ for $F_n = (H_B, B_t)$. Since the above analysis says that it must be $\tilde{x}_m = x^*_m$, it must hold that $\tilde{x}_m = x^*_m = 0$ for $I_m = (H_B, B_t)$. In addition, if $\tilde{\Lambda}_m - h'_m(\tilde{x}_m) = 0$ for $I_m = (A_s, H_B)$, then $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m = (A_s, B_t)$. By Lemma 3 $\tilde{x}_m = x^*_m = 0$ for $I_m = (A_s, B_t)$.

Therefore, by the above analysis (4.5) still holds for $(\lambda', \tilde{\mathbf{x}})$ as long as δ is sufficiently small.

This means λ' is also optimal in the new constraint state. This is a contradiction with the uniqueness of $\tilde{\lambda}$. Thus it must hold that $\tilde{\Lambda}_m \leq \Lambda^*_m$ for $I_m = (A_1, H_B)$.

□

This is illustrated in Figure 4.6.

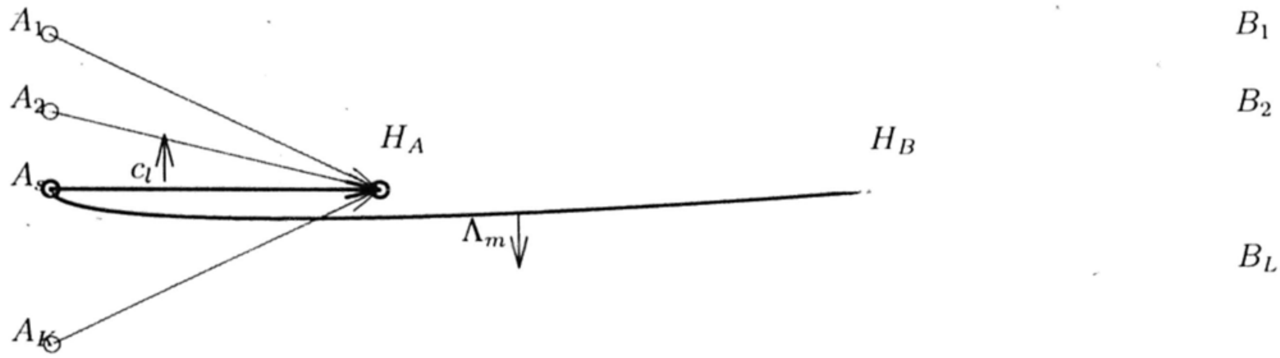

Figure 4.6: Illustration for Theorem 5

Corollary 3 *In Model* (3.7) , *the optimal allocation* x^*_m *to the route* $I_m = (A_l, H_B)$ *is monotonically increasing in any* c_l *for any* $1 \leq l \leq K$ *and the minimal acceptable fare level* f_k *on route* (A_l, H_B) *is decreasing in* c_l.

Proof. By Lemma 1 x^*_m for $I_m \in (A_l, H_B)$ is increasing in $c_l, 1 \leq l \leq K$. And by the construction of $r_m(x_m)$ we know that the minimal acceptable fare level f_k on the route (A_l, H_B) is decreasing in c_l. □

The following theorem addresses how a side flight's capacity change would affect a through market that travels that side flight to a destination city in the B_l $(l = 1, 2, \cdots, L)$ area.

Theorem 6 *The threshold* Λ_m *for* $I_m = (A_k, B_t)$ *is monotonically decreasing in* c_k *for any* $k = 1, \cdots, K$, $t = 1, \cdots, L$.

Proof. Without loss of generality, we take $k = 1$ and let $\tilde{c}_1 > c_1$ to replace c_1. Suppose the corresponding optimal solution $(\tilde{\lambda}, \tilde{x})$ already satisfy the monotone relation stated in Theorems 3, 4 , 5. If there is $I_m = (A_1, B_t)$ that $\tilde{\Lambda}_m - \Lambda^*_m > 0$, by Theorem 6 this means $\tilde{\lambda}_{K+1+t} - \lambda^*_{K+1+t} > 0$. Then we let $\Delta = \max\{\tilde{\lambda}_n - \lambda^*_n, F_n \in (H_B, V)\}$ and $\mathcal{F}_4 = \{F_n | F_n \in (H_B, V), \tilde{\lambda}_n - \lambda^*_n = \Delta\}$, $\mathcal{F}_5 = \{F_n | F_n = (A_s, H_A), x^*_m > 0 \text{ for any } I_m \in (A_s, B_t), (H_B, B_t) \in \mathcal{F}_4\}$. Now in the new constraint state, we let $\lambda'_n = \tilde{\lambda}_n - \delta$ for $F_n \in \mathcal{F}_4$, $\lambda'_n = \tilde{\lambda}_n + \delta$ for $F_n \in \mathcal{F}_5$. Note that the monotonicity relations stated in Theorems 3, 4 , 5 are preserved under such adjustment. We shall show that λ' is still feasible with sufficiently small δ under $\tilde{x}$ and $(\tilde{\mathbf{x}}, \tilde{\lambda}')$ still satisfy the KKT conditions (4.4)∽ (4.5).

If $(H_B, B_t) \in \mathcal{F}_4$, then $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (A_1, B_t)$ and by Theorem 5 for $I_m = (H_B, B_t)$, by Theorem 4 for $I_m = (H_A, B_t)$. By the same line as in Lemma 2 $\tilde{x}_m = x^*_m = 0$ for these itineraries and $\tilde{x}_m = x^*_m$ for all $I_m \in (U, B_t)$. If $\tilde{x}_m = x^*_m > 0$ for $I_m = (A_s, B_t)$, i.e. $(A_s, H_A) \in \mathcal{F}_5$, then by Equation (4.5) $\tilde{\Lambda}_m = \Lambda^*_m$, which implies $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m \in (A_s, H_A) \cup (A_s, H_B) \bigcup_{(H_B, B_t) \subsetneq \mathcal{F}_4} (A_s, B_t)$. By Lemma 2 $\tilde{x}_m = x^*_m = 0$ for these itineraries.

From the above analysis, $(\tilde{\mathbf{x}}, \lambda')$ still satisfy the KKT conditions (4.4)∽ (4.5). Thus λ' is also optimal in the new constraint state. This is a contradiction with the uniqueness of $\tilde{\lambda}$ as the optimal dual solution. Therefore it must hold that $\tilde{\Lambda}_m \leq \Lambda^*_m$ for $I_m \in (A_1, V)$. □

This is illustrated in Figure 4.7.

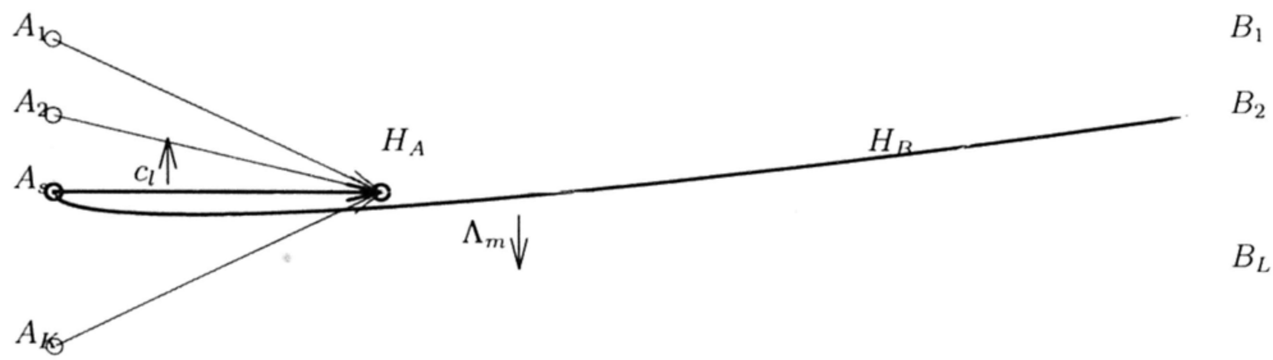

Figure 4.7: Illustration for Theorem 6

Corollary 4 *In Model* (3.7) , *the optimal allocation* x_m^* *to the route* $I_m = (A_l, B_t)$ *is monotonically increasing in any* c_l *for any* $1 \le l \le K, 1 \le t \le L$ *and the minimal acceptable fare level* f_k *on route* (A_l, B_t) *is decreasing in* c_l.

Proof. By Lemma 1 x_m^* for $I_m = (A_l, B_t)$ is increasing in $c_l, 1 \le l \le K$. And by the construction of $r_m(x_m)$ we know that the minimal acceptable fare level level f_k on the route (A_l, B_t) is decreasing in c_l. □

Theorem 7 *The threshold* Λ_m *for* $I_m = (H_A, B_t)$, $t = 1, \cdots, L$ *is monotonically increasing in* c_l *for any* $l = 1, \cdots, K$.

Proof. Suppose $\tilde{\Lambda}$ already satisfies the monotone relation stated in Theorems 3, 4. If $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m = (H_A, B_t)$, then because $\tilde{\Lambda}_m \ge \Lambda_m^*$ for $I_m = (H_A, H_B)$ it must be that $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m = (H_B, B_t)$. Since $\tilde{\Lambda}_m \le \Lambda_m^*$ for $I_m \in (U, H_A)$, we also get $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m \in (U, B_t)$. By applying Lemma 3 to the flight (H_B, B_t) we know that $\tilde{x}_m = x_m^* = 0$ for $I_m = (H_B, B_t), (H_A, B_t)$ and for itineraries $I_m \in (U, B_t)$. This is impossible as all the itineraries passing the flight (H_B, B_t)

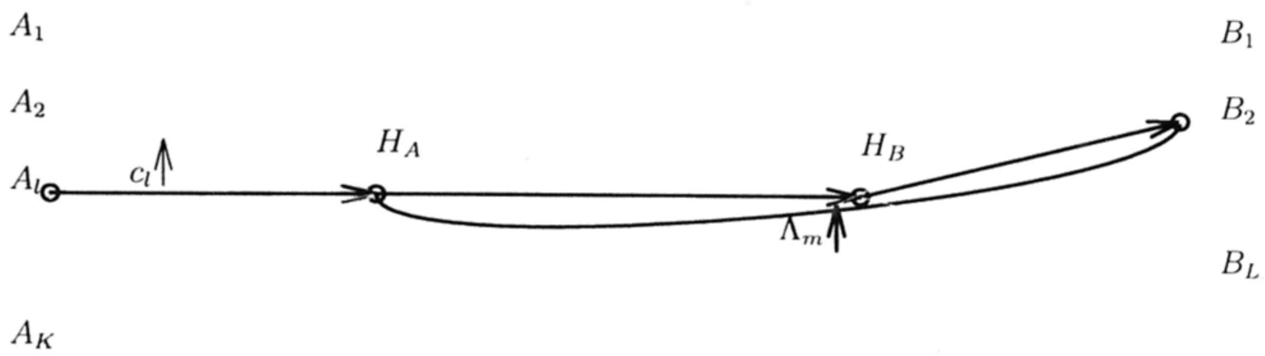

Figure 4.8: Illustration for Theorem 7

have a null allocation. The capacity of it is not used at all, which is contradictory with Equation (4.4). Therefore it must be that $\tilde{\Lambda}_m \geq \Lambda^*_m$ for $I_m = (H_A, B_t)$. By Lemma 1 $\tilde{x}_m \leq x^*_m$ for $I_m = (H_A, B_t)$. □

Figure 4.8 illustrates this.

Corollary 5 *In Model* (3.7), *the optimal allocation* x^*_m *to the route* $I_m = (H_A, B_t)$ *is monotonically decreasing in any* c_l *for any* $1 \leq l \leq K, 1 \leq t \leq L$ *and the minimal acceptable fare level* f_k *on route* (H_A, B_t) *is increasing in* c_l.

Proof. By Lemma 1 x^*_m for $I_m = (H_A, B_t)$ is decreasing in $c_l, 1 \leq l \leq K$. And by the construction of $r_m(x_m)$, we know that the minimal acceptable fare level level f_k on the route (H_A, B_t) is increasing in c_l. □

Likewise, the monotonicity property of the thresholds on the middle leg's capacity are as follows. Preparatory work is first done. From Lemma 2, we take out the special case where $\tilde{c}_n = c_n$ for the flights (U, H_A) and (H_B, V), while c_n for the flight (H_A, H_B) is changed and can immediately draw out the following corollary:

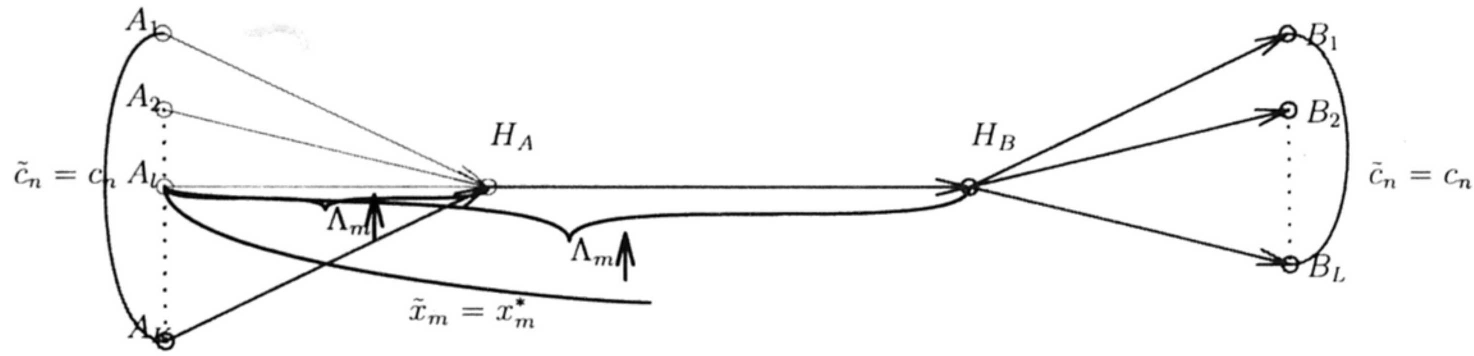

Figure 4.9: Illustration for Corollary 6

Corollary 6 *If $\tilde{c}_n = c_n$ for the flights (U, H_A) and (H_B, V), while c_n for the flight (H_A, H_B) is changed. Then, if $\tilde{\Lambda}_m \geq \Lambda_m^*$ for $I_m = (A_l, H_A)$ and $I_m = (A_l, H_B)$ simultaneously for some $l = 1, \cdots, K$, then, the capacity allocations to all the itineraries passing the flight (A_l, H_A) are unchanged, i.e. $\tilde{x}_m = x_m^*$ for $I_m = (A_l, H_A), (A_l, H_B)$, and for all $I_m \in (A_l, V)$.*

See the illustration in Figure 4.9.

By symmetry we can draw other conclusions as well. Now we are ready to prove the theorems below, which are illustrated consecutively and through a set of graphs. The first is Theorem 8, which addresses the monotonicity of the threshold on the route (H_A, H_B).

Theorem 8 *The threshold Λ_m for $I_m = (H_A, H_B)$ is monotonically decreasing in c_{K+1}.*

Proof. Similar to the proof of Theorem 3, we let U_3 and U_4 divide the cities in the origin region by $U_3 = \{A_s | A_s \in U, \tilde{\lambda}_n < \lambda_n^* \text{ for } F_n = (A_s, H_A)\}$ and $U_4 = U/U_1$

respectively. Meanwhile, let $V_3 = \{B_t | B_t \in V, \tilde{\lambda}_n \geq \lambda^*_n \text{ for } F_n = (H_B, B_t)\}$, and $V_4 = V/V_1$ divide those cities in the destination region, respectively as well. And $V'_4 = \{B_t | B_t \in V_4, \tilde{\Lambda}_m \leq \Lambda^*_m \text{ for } I_m = (H_A, B_t)\}$. We suppose, by contradiction, that $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (H_A, H_B)$. We classify the allocation of itineraries that pass flight (H_A, H_B) into three groups: 1. The aggregate allocation of itineraries (U_3, H_B) and (U_3, V); 2. That of itineraries (U_4, V_4) and (H_A, V_4); 3. That of itineraries (U_4, H_B), (U_4, V_3), (H_A, H_B) and (H_A, V_3).

1. For $A_s \in U_3$, $I_m = (A_s, H_A)$, by Lemma 1, $\tilde{x}_m > x^*_m$ or $\tilde{x}_m = x^*_m = 0$. By Equation (4.4), if $\tilde{x}_m > x^*_m$ for $I_m = (A_s, H_A)$ then

$$\sum_{I_m \in (A_s, H_B) \cup (A_s, V)} \tilde{x}_m < \sum_{I_m \in (A_s, H_B) \cup (A_s, V)} x^*_m. \tag{4.24}$$

If $\tilde{x}_m = x^*_m = 0$ for $I_m = (A_s, H_A)$ then

$$\sum_{I_m \in (A_s, H_B) \cup (A_s, V)} \tilde{x}_m = \sum_{I_m \in (A_s, H_B) \cup (A_s, V)} x^*_m. \tag{4.25}$$

2. (a) If $B_t \in V'_4$, then $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m = (H_B, B_t)$ and $\tilde{\Lambda}_m < \Lambda^*_m$ for $I_m \in (U_3, B_t)$. Thus, by Lemma 1 $\tilde{x}_m > x^*_m$ or $\tilde{x}_m = x^*_m = 0$ for $I_m \in (U_3, B_t) \cup (H_B, B_t)$. By Equation (4.4) if $\tilde{x}_m < x^*_m$ for any $I_m \in (U_3, B_t) \cup (H_B, B_t)$ then

$$\sum_{I_m \in (H_A, B_t) \cup (U_4, B_t)} \tilde{x}_m > \sum_{I_m \in (H_A, B_t) \cup (U_4, B_t)} x^*_m. \tag{4.26}$$

If $\tilde{x}_m = x^*_m = 0$ for all $I_m \in (U_3, B_t) \cup (H_B, B_t)$ then

$$\sum_{I_m \in (H_A, B_t) \cup (U_4, B_t)} \tilde{x}_m = \sum_{I_m \in (H_A, B_t) \cup (U_4, B_t)} x^*_m. \tag{4.27}$$

(b) If $I_m \in (H_A, V_4/V_4') \cup (U_4, V_4/V_4')$, $\tilde{\Lambda}_m > \Lambda^*_m$, then by Lemma 1 $\tilde{x}_m < x^*_m$ or $\tilde{x}_m = x^*_m = 0$.

3. For $I_m \in (U_4, H_B) \cup (U_4, V_3) \cup (H_A, H_B) \cup (H_A, V_3)$, $\tilde{\Lambda}_m > \Lambda^*_m$, by Lemma 1, $\tilde{x}_m < x^*_m$ or $\tilde{x}_m = x^*_m = 0$.

To sum up all the above groups,

$$\sum_{(H_A,H_B)\in I_m, I_m \neq (H_A,H_B)} \tilde{x}_m \leq \sum_{(H_A,H_B)\in I_m, I_m \neq (H_A,H_B)} x^*_m \tag{4.28}$$

and

$$\tilde{x}_m < x^*_m \text{ or } \tilde{x}_m = x^*_m = 0 \text{ for } I_m = (H_A, H_B). \tag{4.29}$$

If $\tilde{x}_m < x^*_m$ for $I_m = (H_A, H_B)$, then by Equation (4.4)

$$\sum_{(H_A,H_B)\in I_m, I_m \neq (H_A,H_B)} \tilde{x}_m > \sum_{(H_A,H_B)\in I_m, I_m \neq (H_A,H_B)} x^*_m. \tag{4.30}$$

This contradicts inequality (4.28). If $\tilde{x}_m = x^*_m = 0$ for $I_m = (H_A, H_B)$, then by (4.4)

$$\sum_{(H_A,H_B)\in I_m, I_m \neq (H_A,H_B)} \tilde{x}_m = \tilde{c}_{K+1} > c_{K+1}. \tag{4.31}$$

This also contradicts (4.28). Therefore, it must hold that $\tilde{\Lambda}_m \leq \Lambda^*_m$ for $I_m = (H_A, H_B)$. □

This is illustrated in Figure 4.10.

As a direct consequence,

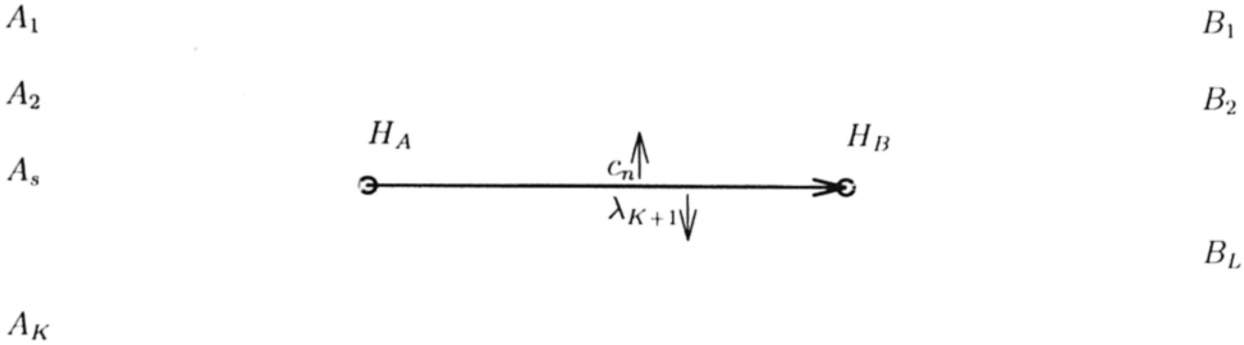

Figure 4.10: Illustration for Theorem 8

Corollary 7 *In Model* (3.7), *the optimal allocation* x_m^* *to the route* $I_m = (H_A, H_B)$ *is monotonically increasing in any* c_{K+1} *and the minimal acceptable fare level* f_k *on route* (H_A, H_B) *is decreasing in* c_{K+1}.

Proof. The same as the proof of Corollary 3. □

The theorem below addresses another result about the monotonicity of the thresholds in the left side routes (A_l, H_A).

Theorem 9 *The threshold* Λ_m *for* $I_m = (A_l, H_A)$ *is monotonically increasing in* c_{K+1}.

Proof. Suppose the optimal solutions $(\tilde{\lambda}, \tilde{\mathbf{x}})$ already satisfy the monotone relation stated in Theorem 8. If there is $F_n \in (U, H_A)$ that $\tilde{\lambda}_n - \lambda_n^* < 0$, then let $\Delta = \min\{\tilde{\lambda}_n - \lambda_n^*, F_n \in (U, H_A)\}$. Let $\mathcal{F}_6 = \{F_n | F_n \in (U, H_A), \tilde{\lambda}_n - \lambda_n^* = \Delta\}$. If $(A_s, H_A) \in \mathcal{F}_6$, then $\tilde{\Lambda}_m < \Lambda_m^*$ for $I_m = (A_s, H_A)$ and by Theorem 3 for $I_m = (A_s, H_B)$. By the same logic as in Corollary 6, $\tilde{x}_m = x_m^* = 0$ for $I_m =$

(A_s, H_A) and $I_m = (A_s, H_B)$, $\tilde{x}_m = x^*_m$ for all $I_m \in (A_s, V)$. If $\tilde{x}_m = x^*_m > 0$ for $I_m = (A_s, B_t)$, then by (4.5), $\tilde{\Lambda}_m = \Lambda^*_m$, which implies $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m \in (H_B, B_t) \cup (H_A, B_t) \bigcup_{(A_i, H_A) \subsetneq \mathcal{F}_6} (A_i, B_t)$. By the same logic as in Corollary 6 and Lemma 1, $\tilde{x}_m = x^*_m = 0$ for $I_m \in (H_B, B_t) \cup (H_A, B_t) \bigcup_{(A_i, H_A) \subsetneq \mathcal{F}_6} (A_i, B_t)$.

To explicitly characterize the above type of B_t's, let $\mathcal{F}_7 = \{F_n | F_n = (H_B, B_t)$, there exists $I_m = (A_s, B_t)$ where $(A_s, H_A) \in \mathcal{F}_6$ that $x^*_m > 0\}$.

Let $\lambda'_n = \tilde{\lambda}_n + \delta$ for all $F_n \in \mathcal{F}_6$, $\lambda'_n = \tilde{\lambda}_n - \delta$ for all $F_n \in \mathcal{F}_7$. Note that the monotone relation stated in Theorem 8 is preserved under such adjustment.
Since $x^*_m = 0$ for $I_m \in \mathcal{F}_7 \bigcup_{(A_s, H_A) \in \mathcal{F}_6, H_A)} ((A_s, H_A) \cup (A_s, H_B) \bigcup_{(H_B, B_t) \subsetneq \mathcal{F}_7} (A_s, B_t))$ and $\tilde{\Lambda}_m > \Lambda^*_m$ for these itineraries, (4.5) and (4.6) still hold for (Λ'_m, x^*_m) on them as long as δ is sufficiently small. Since $\Lambda'_m = \Lambda^*_m$ for $I_m = (A_s, B_t)$ where $(A_s, H_A) \in \mathcal{F}_6, (H_B, B_t) \in \mathcal{F}_7$, (4.5) also holds for (Λ'_m, x^*_m) on this kind of itinerary. Since $\tilde{x}_m = x^*_m = 0$ for $I_m \in \mathcal{F}_7 \cup \bigcup_{(H_B, B_t) \in \mathcal{F}_7} (H_A, B_t) \bigcup_{(A_s, H_A) \subsetneq \mathcal{F}_6} (A_s, B_t)$, (4.5) also holds for (Λ'_m, x^*_m) as long as δ is sufficiently small.

Thus λ' is also optimal in the new constraint state. But from the construction of λ' we know that $\Delta' > \Delta$. This is a contradiction with the uniqueness of $\tilde{\lambda}$ as the optimal dual solution. Then from the above analysis, it must hold that $\tilde{\lambda}_n \geq \lambda^*_n$ for $F_n \in (U, H_A)$. □

This is illustrated in Figure 4.11.

As a direct consequence,

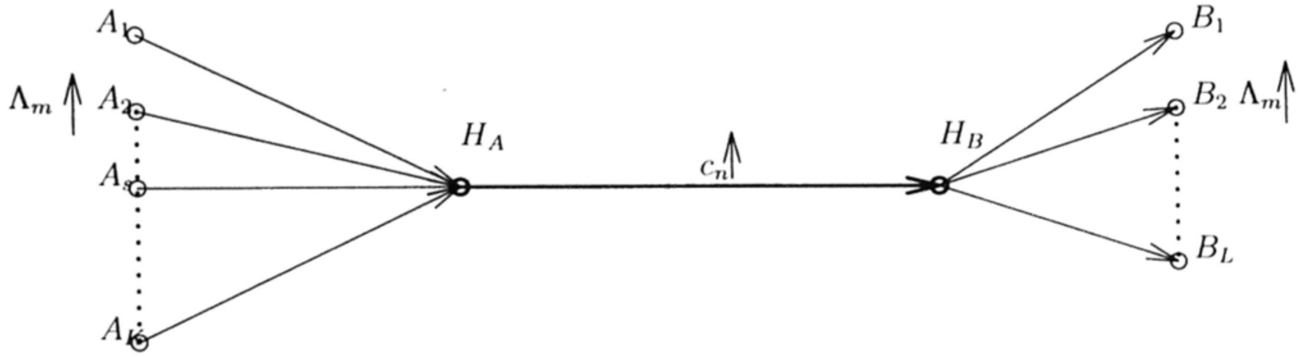

Figure 4.11: Illustration for Theorem 9

Corollary 8 *In Model* (3.7), *the optimal allocation* x_m^* *to the route* $I_m = (A_l, H_A)$ *is monotonically decreasing in* c_{K+1} *and the minimal acceptable fare level* f_k *on route* (A_l, H_A) *is increasing in* c_{K+1}.

Proof. The same as the proof of Corollary 3. □

By symmetry we get:

Theorem 10 *The threshold* Λ_m *for* $I_m = (H_B, B_t)$ *is monotonically increasing in* c_{K+1}.

Corollary 9 *In Model* (3.7), *the optimal allocation* x_m^* *to the route* $I_m = (H_B, B_t)$ *is monotonically increasing in any* c_{K+1} *and the minimal acceptable fare level* f_k *on route* (H_B, B_t) *is decreasing in* c_{K+1}.

Now the optimal allocations on the two set of itineraries (U, H_A) and (H_B, V) are all decreased. The next theorem addresses the monotonicity of the threshold on route (H_A, B_t).

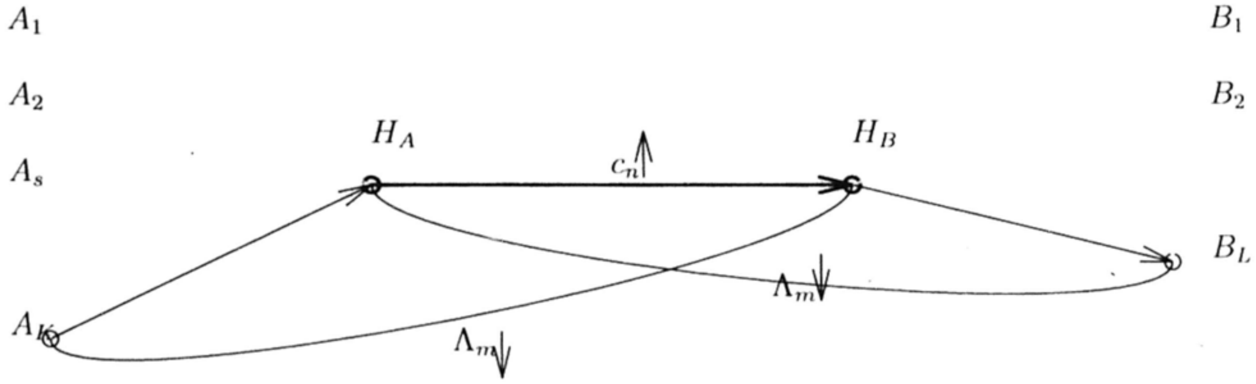

Figure 4.12: Illustration for Theorem 11

Theorem 11 *The threshold Λ_m for $I_m = (H_A, B_t)$ is monotonically decreasing in c_{K+1}.*

Proof. Suppose $\tilde{\Lambda}$ already satisfy the monotone relation stated in Theorems 8, 9. If $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (H_A, B_t)$, then from Theorem 9 $\tilde{\Lambda}_m > \Lambda^*_m$ for all $I_m \in (U, B_t)$. And from Theorem 8, $\tilde{\Lambda}_m > \Lambda^*_m$ for $I_m = (H_B, B_t)$ as well. Thus by the same logic as in Corollary 6, $\tilde{x}_m = x^*_m = 0$ for all the itineraries passing (H_B, B_t). This is impossible by applying (4.4) to the flight (H_B, B_t). Therefore it must be that $\tilde{\Lambda}_m \leq \Lambda^*_m$ for $I_m = (H_A, B_t)$. □

This is illustrated in Figure 4.12.

As a direct consequence,

Corollary 10 *In Model (3.7), the optimal allocation x^*_m to the route $I_m = (H_A, B_t)$ is monotonically increasing in c_{K+1} and the minimal acceptable fare level f_k on route (H_A, B_t) is decreasing in c_{K+1}.*

Proof. The same as the proof of Corollary 3. □

By symmetry we get:

Theorem 12 *The threshold* Λ_m *for* $I_m = (A_l, H_B)$ *is monotonically decreasing in* c_{K+1}.

Corollary 11 *In Model* (3.7) , *the optimal allocation* x_m^* *to the route* $I_m = (A_l, H_B)$ *is monotonically increasing in any* c_{K+1} *and the minimal acceptable fare level* f_k *on route* (A_l, H_B) *is decreasing in* c_{K+1}.

4.3 Extension back to the original model

Now we get back to the original Problem (3.7). We first prove that the optimal solutions to a sequence of problem instances (4.1) with the smoothed revenue functions converge to an optimal solution of Problem (3.7)(with piecewise linear revenue functions). This is illustrated in the following *optimal solution convergence theorem.*

Theorem 13 *Any sequence* $(\mathbf{x}^{*n}, \lambda^{*n})$ *obtained in Problem* (4.1) *has a subsequence that converges to the optimal solution pair of Problem* (3.7) *as* $h_m^n(x_m)$ *tends to* $r_m(x_m)$ *by our prescribed construction.*

Proof. There must be a subsequence that converges. Suppose the limit is $(\dot{x}, \dot{\lambda})$. Then if $\dot{x}$ has no component in the points D_k, it is obvious that they still satisfy the optimality conditions (3.8)$\sim$ (3.9). If a component of $\dot{x}$, such as $\dot{x}_m$, equals D_k^m, then obviously from (4.6) that $\dot{\Lambda}_m \geq f_{k+1}^m$. In the sequence, after some large enough N, for each (Λ_m^*, x_m^*), they satisfy that $\Lambda_m = h_m'(x_m)$. Therefore it should be that $\dot{\Lambda}_m \leq f_k^m$ since $h_m'(x_m) \leq f_k^m$ by the construction of $h_m(x_m)$. Therefore the optimality conditions (3.8)$\sim$ (3.9) are still satisfied, i.e. $(\dot{x}, \dot{\lambda})$ is the optimal solution to the original Problem (3.7). □

Then we reduce these results back to the original problem by utilizing the following *monotone property preserving theorem*:

Theorem 14 *If in the optimal solution* $(\mathbf{\Lambda^n}, \mathbf{x^n})$ *of Model* (3.13), *some components* $(\Lambda_{m_1}^*, x_{m_1}^*), (\Lambda_{m_2}^*, x_{m_2}^*), \cdots, (\Lambda_{m_t}^*, x_{m_t}^*)$ *are monotonically increasing (or decreasing) in some components of* $\mathbf{c}$, *then there exists a projection* $\mathbf{C} \rightarrow \mathbf{\Lambda} \times \mathbf{X}$ *that* $(\lambda(\mathbf{c}), \mathbf{x}(\mathbf{c}))$ *is an optimal solution pair to Model* (3.7) *with capacities state* $\mathbf{c}$ *and that* $(\Lambda_{m_1}^*, x_{m_1}^*), (\Lambda_{m_2}^*, x_{m_2}^*), \cdots, (\Lambda_{m_t}^*, x_{m_t}^*)$ *preserve those monotone properties in the corresponding components of* $\mathbf{c}$.

Proof. For each capacities state $\mathbf{c}$ we take out a limit point of $(\lambda^{\mathbf{n}}, \mathbf{x^n})$. Suppose it is $(\lambda, \mathbf{x}; \mathbf{c})$. By Theorem 13 it is the optimal solution pair of Problem (3.7) under constraint $\mathbf{c}$. We now prove that $(\lambda, \mathbf{x}; \mathbf{c})$ still preserves those monotone relations as $(\lambda^{\mathbf{n}}, \mathbf{x^n})$ do. To show this, consider two capacities state, $\tilde{\mathbf{c}}$ and $\mathbf{c}$. We suppose the optimal solutions are denoted by $(\tilde{\lambda}, \tilde{\mathbf{x}})$ and $(\lambda, \mathbf{x})$ respectively. Without loss of generality, we just assume $\tilde{c}_l > c_l$ and others in the capacities are unchanged, and there is an arbitrary I_m that holds a monotone relation. Without loss of generality, we suppose that $\tilde{\Lambda}_m^n \geq \Lambda_m^n, \tilde{x}_m^n \leq x_m^n$ for each n. Since

$\lim_n \tilde{\Lambda}^n_m \geq \lim_n \Lambda^n_m$, we have $\tilde{\Lambda}_m \geq \Lambda_m$. In the same way $\tilde{x}_m \leq x_m$. □

This assures that the monotonicity results we obtained within Model (3.13) still hold for an optimal selection of the original Problem (3.7).

4.4 Concluding remarks

This chapter focused attention on the main model for the hub-to-hub network RM problem. After explicitly presenting the Lagrange function of the main model, the primal-dual argument was applied by utilizing the KKT conditions to derive certain structural properties on the BP control thresholds for this model.

The main observation is that the BP threshold on a long-haul route are decreasing in the individual legs' capacity of that route. Interestingly, the BP threshold in the hub-to-hub route is increasing in two side legs' capacities. Furthermore, the BP threshold on the side single-leg route is decreasing in the same side leg's capacities. Even more interesting is that a long haul route's threshold, such as on (H_A, B_l), is also increasing in side leg (A_k, H_A)'s capacity. It is hoped that all these structural properties can render management insight for the RM department in airline companies.

On the technical side, the primal-dual argument employed here is new in exploring structural properties in revenue management problems. It will hopefully further widen the toolkit for doing monotone properties/comparative statics analysis in OM/MS field. The technique is briefly summarized here. In investigating the change in directions of the decision variables after a constraint status change, we first disclose that the change direction of the optimal allocation is inherently related to that of the dual optimal solution, as illustrated in Lemma 1. By studying the change in direction of the dual solution, we can draw the conclusions for the primal solution. Since the problem is a nonlinear one, the Lagrange dual function was employed to incorporate both the primal variables and the dual variables.

Beyond this, most literature on bid-price control has addressed the *re-optimization* issue. It is said that only when the bid-prices are frequently re-optimized (updated in near real-time) can the BP control scheme work nearly optimally. See Williamson (1992), Talluri and van Ryzin (1998, 2004), Bertsimas and de Boer (2003) for detailed account of this issue. The problem is how these bid-prices can be re-calculated in near real-time? This is a challenging question given the current capability of major airlines' computation systems. The structural properties provided here give hint for deeper understanding into the patterns that bid-prices produce in a dynamic situation. And by such properties it is hopeful that the difficulty in the real-time updating of bid-prices may be overcome.

Chapter 5

On network CE control

This chapter focuses on the CE control. Since the emphasis of this thesis is on BP control, the materials in this chapter are necessarily shorter and are not intended to be comparable to those in the previous two chapters. This is because, rather than doing analysis from scratch as in Chapter 3 and Chapter 4, this chapter attempts to bridge **existing results** in network flows with the structures in network CE control thresholds. This chapter on the CE control is not, however, unimportant as it may point to future research directions which are even more important than its BP control counterpart.

5.1 Introduction to network CE control

One way to approximate the optimal marginal value threshold $J_{t-1}(\mathbf{n})-J_{t-1}(\mathbf{n}-A_j)$ is to use the so-called certainty equivalent control. This was proposed by Bertsimas and Popescu (2003) and works as follows. As with the bid-price scheme, one still aggregates the future demands and uses the LP Equation (3.1) to approximate the optimal value function. The marginal value $J_{t-1}^{LP}(n) - J_{t-1}^{LP}(n - A_j)$ is used to approximate the optimal marginal value $J_{t-1}(\mathbf{n}) - J_{t-1}(\mathbf{n} - A_j)$. This is the so-called certainty equivalent control.

Having examined the structure of network BP control in the previous chapter, it is natural to enquire if the thresholds in network CE control exhibit the same patterns as the thresholds in network BP control? The following sections aim to address. The key is to establish the relationship between super-/sub-modularity and $L^\natural$ concavity of the optimal value function with the monotone property of those CEC thresholds.

5.2 Supermodularity, $L^\natural$ concavity and the CEC thresholds

In exploring the super-/sub-modularity and other second order properties of the optimal value function of Model 3.3, we link the analysis with results in network flows. Other researchers in this field are Glover et al. (1982) and Morton (2006).

Section 5.2 elaborates on the difference between the present study and Morton's (2006).

5.2.1 Related literature in network flows

In network flows, $G = (V, A)$ represents a graph where V is the nodes set and A is the arcs set.

The maximum-weight circulation problem requires one find a circulation that maximizes the weighted sum of arc flows subject to capacities constraints on the arcs. Readers are referred to Murota and Shioura (2005) and the Appendix for its standard formulation.

Gale and Politof (1981) showed the super-/sub-modularity of the optimal value function in its series/parallel arcs capacities. Murota and Shioura (2005) also showed that the optimal value function is $L^\natural$ concave in its series capacities dimensions, and $M^\natural$ concave in its parallel capacities dimensions.

Theorem 15 *(Gale and Politof (1981)). In a maximum weight circulation problem,*

- *the optimal value function is submodular in the upper and lower bounds of parallel arcs' capacity constraints;*
- *the optimal value function is supermodular in upper and lower bounds of series arcs' capacity constraints.*

Theorem 16 *(Murota and Shioura (2005)). In a maximum weight circulation problem,*

- *the optimal value function is $L^\natural$ concave in the upper and lower bounds of series arcs' capacity constraints;*
- *the optimal value function is $M^\natural$ concave in the upper and lower bounds of parallel arcs' capacity constraints.*

In addition to these, there is a large body of literature on discrete convex analysis addressing the supermodularity/$L^\natural$ concavity issues. Interested reader can find a full account in Murota (2003, 2005).

For convenience, in this section the DLP Model (3.3) is restated to study its behavior:

$$\begin{aligned} \max \quad & \sum_{odf} f_{odf} x_{odf}, \\ s.t. \quad & \sum_{l \in odf} x_{odf} \leq c_l, l = 1, 2, \cdots, K + L + 1, \\ & x_{odf} \leq d_{odf}, \\ & x_{odf} \quad integer. \end{aligned} \tag{5.1}$$

We represent the optimal value of the above model as $V(\mathbf{c})$, where $\mathbf{c}$ is the capacities vector.

5.2.2 For a two-leg network

The first task in this part is to reformulate the DLP model for network revenue management into a graphical illustration of the network flow problem and identify parallel/series arcs. We write the DLP model for two-leg network revenue management as:

$$\begin{aligned} V(c_1, c_2) = \quad & \max \sum_{odf} f_{odf} x_{odf}, \\ & s.t. \sum_{l \in odf} x_{odf} \leq c_l, l = 1, 2 \\ & x_{odf} \leq d_{odf} \\ & x_{odf} \quad integer. \end{aligned} \tag{5.2}$$

It can be formulated into a weighted circulation problem, by adding artificial variables $y_l, l = 1, 2$:

$$\begin{aligned} V(c_1, c_2) = \quad & \max \sum_{odf} f_{odf} x_{odf}, \\ & s.t. \sum_{l \in odf} x_{odf} - y_l = 0 \\ & y_l \leq c_l, l = 1, 2 \\ & x_{odf} \leq d_{odf} \\ & x_{odf} \quad integer. \end{aligned} \tag{5.3}$$

(5.4)

See Figure 5.1 for illustration. All the passengers in each original itinerary are considered a *flow* in Figure 5.1. Furthermore, because the LP relaxation of network flow problems have integer solutions, the integer constraints (5.4) can be dropped.

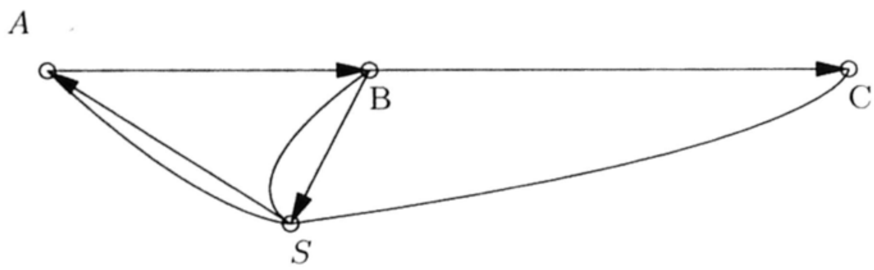

Figure 5.1: A circulation problem

The two arcs $S \to A$ represent the original flows in (A, B) and (A, C) respectively; arc (A, B) is the capacity arc. Thus in node A the flow conservation law is satisfied. The arc $S \to B$ represents the original flow in (B, C); the back arc $B \to S$ represents an artificial arc to carry back the flow (A, B) with infinite capacity. Thus in node B, the flow conservation law is satisfied. The arc $C \to S$ is to carry back the original flow in itinerary (B, C) and (A, C) with infinite capacity. Thus in node C the flow conservation law is also satisfied. As for node S, the $S \to A$ arc carrying the original flow in (A, B) trades off with the arc (B, S); the $S \to A$ arc carrying the original flow in (A, C) plus arc $S \to B$ trades off with the arc (C, S). Therefore the flow conservation law is again preserved at the auxiliary node S.

Therefore by such transformation on every node, the flow conservation balance is satisfied. This ensures that Model (5.3) fits into the maximum weight circulation problem framework.

Lemma 4 *In the maximum weight circulation Model* (5.3) ,(A, B) *and* (B, C) *are series arcs.*

Proof. Since every simple cycle containing both of the two arcs (A, B) and (B, C) orients them in the same direction, by definition, they are series arcs. □

From the above theorem, the following theorem directly follows.

Theorem 17 *The value function $V(c_1, c_2)$ of Model (5.2) is concave in c_1, c_2 separately and supermodular in (c_1, c_2) jointly.*

Proof. The former part follows from the well known result in parametric linear programming that the optimal value of a linear program is concave in its capacities parameters. By Lemma 4, (A, B) and (B, C) are series arcs. By Theorem 15, the optimal value function of a maximum weight circulation problem is supermodular in the capacity upper-bounds of a set of series arcs. Here c_1 and c_2 are the two upper bounds. Therefore the theorem follows. □

Corollary 12 *The CEC threshold value $V(c_1, c_2) - V(c_1 - 1, c_2)$ to control the fare requests on itinerary (A, B) is decreasing in c_1.*

Proof. From the first part of Theorem 17 it follows that

$$V(c_1, c_2) - V(c_1 - 1, c_2) \leq V(c_1 - 1, c_2) - V(c_1 - 2, c_2).$$

This completes the proof. □

Corollary 13 *The CEC threshold value $V(c_1, c_2) - V(c_1 - 1, c_2)$ to control fare requests on itinerary (A, B) is increasing in c_2.*

Proof. From the definition of supermodularity Theorem 17 indicates

$$V(c_1, c_2+1) - V(c_1-1, c_2+1) \geq V(c_1, c_2) - V(c_1-1, c_2).$$

This completes the proof. □

Taking this further, because (A, B) and (B, C) are series arcs, we have

Theorem 18 *The value function $V(c_1, c_2)$ of Model (5.2) is $L^\natural$ concave in its dimensions.*

Proof. From Theorem 16 this theorem follows. □

Corollary 14 *The threshold value $V(c_1, c_2) - V(c_1 - 1, c_2 - 1)$ to control fare requests on itinerary (A, C) is decreasing in either of the capacities c_1 and c_2.*

Proof. From the definition of $L^\natural$ concavity Theorem 18 implies:

$$V(c_1, c_2) - V(c_1-1, c_2-1) \geq V(c_1+1, c_2) - V(c_1, c_2-1)$$

and

$$V(c_1, c_2) - V(c_1-1, c_2-1) \geq V(c_1, c_2+1) - V(c_1, c_2).$$

This completes the proof. □

Theorem 17 and Theorem 18 are the same properties as obtained in Morton (2006). Corollaries 12, 13 and 14 are the same results on monotone thresholds as obtained in Feng and Lin (2004). They differ in that Morton (2006) did not draw conclusions concerning the monotonicity of the threshold values. However, they both adopted dynamic models and neglected to prove those properties in a static model. This research gap is here satisfied.

5.2.3 CEC thresholds for the hub-to-hub network

The hub-to-hub network in Figure 5.2 is now studied:

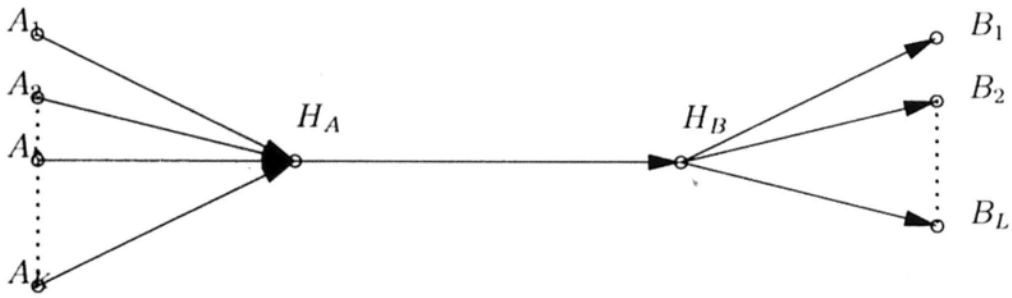

Figure 5.2: A hub-to-hub network

As illustrated in previous parts of this thesis, we can represent Model (3.3) with a maximum weight circulation formulation:

$$\begin{aligned} \max \quad & \sum_{odf} f_{odf} x_{odf}, \\ s.t. \quad & \sum_{l \in odf} x_{odf} - y_l = 0, l = 1, 2, \cdots, K + L + 1 \\ & x_{odf} \leq d_{odf} \\ & y_l \leq c_l. \end{aligned}$$

The graphical illustration is as follows:

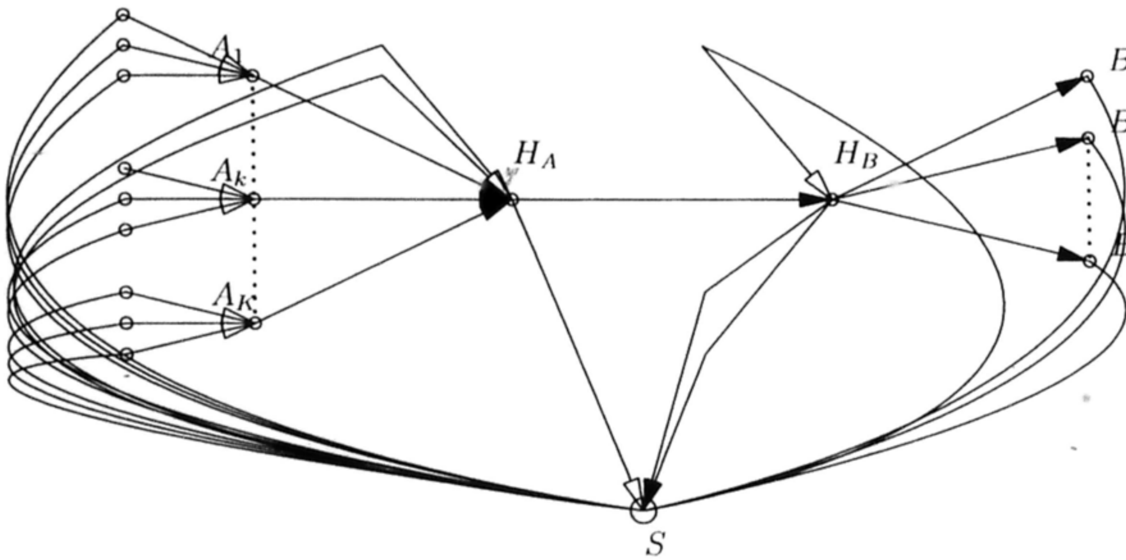

Figure 5.3: The network flow (circulation) representation of the hub-to-hub RM network

Let all the artificial nodes and B_l's be combined into one node. Then the problem is transformed into a maximum weight circulation problem. The three arcs entering A_i represent the three kind of flows in (A_i, H_A): $(A_i, H_A), (A_i, H_B)$ and (A_i, B_j); the two upper arcs entering H_A represents the flow (H_A, H_B) and the flows (H_A, B_j) with infinite capacities; the lower arc exiting H_A represents the flows (A_i, H_A) with infinite capacity; the upper arc entering H_B represents the flows (H_B, B_j) with infinite capacity; the lower arcs exiting H_B represent the flows $(H_A, H_B), (A_i, H_B)$ respectively with infinite capacities .

Below is the main theorem in this chapter.

Theorem 19 *By the above graphical construction, Model* (5.5) *fits into a maximum weight circulation framework.*

Proof. Similarly as in the two-leg case, one can easily verify that by such transformation the flow conservation balance on each node is satisfied. Therefore, this indeed fits into the maximum weight circulation problem framework. □

For the same reason as in the proof of Lemma 4, we have the following lemma:

Lemma 5 *In Model* (5.5), *the arcs* $(A_k, H_A), (H_A, H_B)$ *and* $(H_B, B_l), k = 1, 2, \cdots, K, l = 1, 2, \cdots, L$ *are pairwise series arcs.*

Proof. Obviously, every simple cycle would orient arcs (A_k, H_A) and (H_A, H_B) in the same direction. Thus they are series arcs. For the same reason (H_A, H_B) and (H_B, B_l) are also series arcs. Now we show that (A_k, H_A) and (H_B, B_l) are series arcs also. In the above graph, every path containing both (A_k, H_A) and (H_B, B_l) would be either going through $A_k \to H_A \to H_B \to B_l$ or going through $A_k \to H_A \to S \to H_B \to B_l$. However, since it is a cycle, the latter case cannot occur. Thus it can only go through $A_k \to H_A \to H_B \to B_l$. Therefore (A_k, H_A) and (H_B, B_l) are series arcs. □

Lemma 6 *In Model* (5.5), *the arcs* $(i, j), 1 \leq i, j \leq K$ *and* $(i_1, j_1), K + 2 \leq i_1, j_1 \leq K + 1 + L$ *are parallel arcs.*

Proof. This is a direct result from Granot and Veinott (1985). □

As a consequence of Lemma 5 and Lemma 6, we have the following theorem:

Theorem 20 *The value function* $V(\mathbf{c})$ *of Model (3.3) for the network in Figure 5.2 is supermodular in dimensions* $(c_i, c_{K+1}), i \leq L$ *and dimensions* $(c_{K+1}, c_j), K+2 \leq j \leq K+1+L$*; submodular in dimensions* $(c_i, c_j), 1 \leq i, j \leq K$ *and submodular in dimensions* $(c_i, c_j), K+2 \leq i, j \leq K+1+L$.

Proof. Because in the network flow formulation, $(A_i, H_A), (H_A, H_B)$ are series arcs, $(A_i, H_A), (A_j, H_A)$ are parallel arcs and $(H_B, B_i), (H_B, B_j)$ are parallel arcs, it directly follows from Gale and Politof (1981) that this theorem holds. □

This result is similar to the result on a bipartite network obtained in Morton (2006). His approach was similar in that he also represented the DLP model (5.1) with a maximum weight circulation formulation (his representation is a little different) and applied the results developed by Gale and Politof (1981).

And therefore we have

Corollary 15 *The threshold value* $V(c_1, c_2, \cdots, c_k, \cdots, c_{K+1}, c_{K+1+L}) - V(c_1 - 1, c_2, \cdots, c_k, \cdots, c_{K+1}, \cdots, c_{K+1+L})$ *to control the itinerary* (A_1, H_A) *is increasing in* c_{K+1} *while decreasing in* $c_k, 1 \leq k \leq K$.

Proof. From the definition of super/submodularity Theorem 20 indicates:

$$\begin{aligned} &V(c_1, c_2, \cdots, c_k, \cdots, c_{K+1}, \cdots, c_{K+1+L}) \\ &-V(c_1 - 1, c_2, \cdots, c_k, \cdots, c_{K+1}, \cdots, c_{K+1+L}) \geq \\ &V(c_1, c_2, \cdots, c_k, \cdots, c_{K+1} - 1, \cdots, c_{K+1+L}) \\ &-V(c_1 - 1, c_2, \cdots, c_k, \cdots, c_{K+1} - 1, \cdots, c_{K+1+L}). \qquad (5.5) \end{aligned}$$

This completes the proof. □

See the same illustration as in the bid price control part in Figure 4.11 and Figure 4.5.

Further,

Corollary 16 *The threshold value* $V(c_1, \cdots, c_k, \cdots, c_{K+1}, \cdots, c_{K+1+L}) - V(c_1, \cdots, c_k, \cdots, c_{K+1}-1, \cdots, c_{K+1+L})$ *to control the itinerary* (H_A, H_B) *is decreasing in* c_{K+1} *while increasing in* $c_k, 1 \leq k \leq K$ *and increasing in* $c_{K+1+l}, 1 \leq l \leq L$.

Proof. The same as the above corollary. □

Since $(A_k, H_A), (H_A, H_B)$ and (H_B, B_l) are series arcs, we have

Theorem 21 *The value function* $V(\mathbf{c})$ *for Model* (5.1) *is* $L^\natural$ *concave in* $(c_k, c_{K+1}, c_{K+1+l})$ $, k \leq K, l \leq L$.

Proof. The same as in Theorem 18 (This is from Murota and Shioura (2005)). □

Consequently, we have:

Corollary 17 *The threshold value* $V(c_1, \cdots, c_k, \cdots, c_{K+1}, c_{K+2}, \cdots, c_{K+1+L}) - V(c_1, \cdots, c_k - 1, \cdots, c_{K+1} - 1, c_{K+2}, \cdots, c_{K+1+L})$ *to control the fare requests on itinerary* (A_k, H_B) *is decreasing in* $c_k, 1 \leq k \leq K$ *and* c_{K+1}.

Proof. The same as the proof in Corollary 15. □

Corollary 18 *The CEC threshold value* $V(c_1, \cdots, c_k, \cdots, c_{K+1}, \cdots, c_{K+1+l}, \cdots, c_{K+1+L}) - V(c_1, \cdots, c_k - 1, \cdots, c_{K+1} - 1, \cdots, c_{K+1+l} - 1, \cdots, c_{K+1+L})$ *to control the fare requests on itinerary* (A_k, B_l) *is decreasing in* $c_k, 1 \leq k \leq K, c_{K+1}$ *and* c_{K+1+l}

Proof. The same as the proof in Corollary 15. □

Corollary 19 *The CEC threshold value* $V(c_1, \cdots, c_k, \cdots, c_{K+1}, c_{K+2}, c_{K+3}, \cdots, c_{K+1+L}) - V(c_1, \cdots, c_k, \cdots, c_{K+1} - 1, c_{K+2} - 1, c_{K+3}, \cdots, c_{K+1+L})$ *to control the itinerary* (H_A, B_1) *is increasing in* $c_k, 1 \leq k \leq K$.

Proof. The same as the proof in Corollary 15. □

These patterns are the same as in bid price control (Chapter 4). See the illustrations in Figure 4.6, Figure 4.7 and Figure 4.8 in that chapter.

However, the answer for other monotone results remains unknown, such as whether CEC threshold on (H_A, B_1) is increasing in capacity c_1 (leg (A_1, H_A)'s capacity). This requires other second order properties beyond the super/submodularity and $L^\natural$ concavity developed by Gale and Politof (1981), Murota and Shioura (2005). This remains an important question for future research.

5.3 Chapter summary

The main contribution of this chapter is that the notions of super/submodulatiry and $L^\natural$ concavity were bridged with the monotonicity of CEC thresholds. This was done by first reformulating the DLP model into a network flow representation. Secondly, the parallel and series arcs were identified. Thirdly, the results developed by Gale and Politof (1981) and Murota and Shioura (2003) were applied to show the value function's supermodularity in series arcs' capacities, its submodularity in parallel arcs' capacities, its $L^\natural$ concavity in series arcs' capacities. Finally, those properties were translated into the CEC thresholds' monotonicities.

Network CE control reportedly has better performance than bid price control (see Bertsimas and de Boer (2003)). Here the structures of the CEC threshold values, which have the similar pattern as those of bid price thresholds, were analyzed. For example, the threshold on itinerary (H_A, H_B) is increasing in leg 1's capacity, and the threshold on (A_1, H_B) is decreasing in leg $1, (A_1, H_A)$ and leg $K+1, (H_A, H_B)$'s capacities. In obtaining these results, the methodology and results developed by Gale and Politof (1981) and Murota and Shioura (2005) were utilized. These properties will add intuitive understanding to CEC and provide technical insights.

Appendix

Formulation of the maximum weight circulation problem Let $G = (V, A)$

be a directed graph with vertex set V and arc set A. We add some additional notation here:

a an arc.

v a node.

w the weight vector.

c the upper bound vector of capacity constraints.

d the lower bound vector of capacity constraints.

ξ a feasible flow.

Murota and Shioura (2005) formulated the *maximum weight circulation problem* in a network as:

$$\max\{w'\xi \mid \sum\{\xi(a)|a \text{ leaves } v\} - \sum\{\xi(a)|a \text{ enters } v\} = 0 \quad (v \in V), \quad d \leq \xi \leq c\}.$$

Chapter 6

Numerical examples

In this chapter mathematical programs are run using Matlab to determine how the thresholds mentioned in the previous chapters behave in real data-driven systems.

6.1 Test for allocation variables

This section considers a simple network as shown in Figure 6.1.

This network contains just two cities, A_1, A_2, on the left side, and two cities, B_1, B_2, on the right side, with H_A, H_B still being the two connecting hub-cities. For simplicity, we assume that there is just one fare class on each OD pair market. We set $\mathbf{c}$ as the capacities of the network, and $\mathbf{b} = (1, 3, 5, 5, 1, 3, 5, 5, 1, 3, 3, 1, 1)$

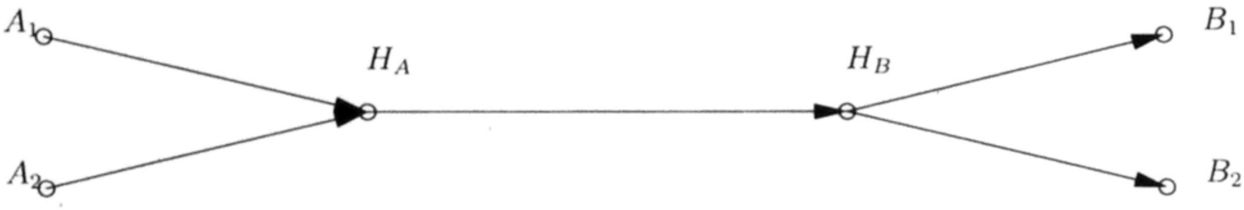

Figure 6.1: A simple network

as the unit revenues for the 13 itineraries in the network.

For simplicity, we first verify the allocations variables' change as specified from Corollary 1 to Corollary 10. By changing the values of the capacities $\mathbf{c}$, we can see the change directions of the allocation variables $\mathbf{x}$, as illustrated in Table 6.1 and Table 6.2. f means the objective function value in the table.

It can be seen from Table 6.1 that as leg 1's capacity increases from $(5, 3, 11, 5, 6)$ to $(8, 3, 11, 5, 6)$, the route allocations to itineraries $1, 2, 3$ and 4 are non-decreasing, as stated in corollaries 2, 3 and 4. Flow on itinerary 5,(A_2, H_A), is also nondecreasing (remaining at 0) as stated in Corollary 2; on itinerary 9, (H_A, H_B) stays 0; on itinerary 10, (H_A, B_1) decreases from 0.8372 to 0. These results equate to the theoretical results obtained in Chapter 3. It is also interestingly to note that, as stated in Corollary 2, the allocation to itinerary 5, (A_2, H_A), is increasing in leg 1's capacity, as the capacity increases from $(7, 7, 11, 5, 7)$ to $(10, 7, 11, 5, 7)$.

Intuitively, as leg 1's capacity increases, its usage on itineraries $1, 2, 3, 4$ would naturally increase. Consequently, leg (H_A, H_B)'s capacity is less used on other itineraries. This therefore results in a decrease in the sum of allocations on itineraries $(A_2, H_B), (A_2, B_1), (A_2, B_2), \cdots, (A_2, B_L)$. This causes an increase in the allocation of leg (A_2, H_A)'s capacity on its own leg route, itinerary (A_2, H_A).

In other cases (not proven in these theorems or corollaries), the thresholds still behave in patterns. For example, the allocation to itinerary 6, (A_2, H_B), increases from 2.6592 to 2.6888 as the capacity increases from $(5, 13, 15, 7)$ to $(8, 13, 15, 7)$, while at the same time allocation to itinerary 7 decreases; the allocation to itinerary 12, (H_B, B_1), increases as the capacity increases from $(7, 7, 11, 5, 7)$ to $(10, 7, 11, 5, 7)$, while the allocation on itinerary 13, (H_B, B_2), decreases at the same time. It will be noted that these patterns might be violated in other cases since we observe that on itinerary 8, (A_2, B_2), from capacity $(5, 13, 15, 7)$ to $(8, 13, 15, 7)$, the allocation first increases from 6.1944 to 6.1981, and then drops to 6.1930.

Now in case of degeneracy (multiple solutions), let $\mathbf{b} = (1, 5, 8, 9, 2, 9, 10, 11, 5, 10, 7, 5, 4)$. As we change the middle leg's capacity, we can see the change directions of the allocation variables as illustrated in Table 6.2.

Let $\mathbf{b} = (1, 3, 5, 5, 1, 3, 5, 5, 1, 3, 3, 1, 1)$. By altering $\mathbf{c}$ incrementally, we can see the change directions of the five dual variables for the five flight legs, as illustrated in Table 6.3.

This table shows that as the middle leg's capacity increases from $(2, 3, 4, 5, 6)$ to $(2, 3, 100, 5, 6)$, and the dual price on the middle leg decreases from $3, 2.4, \cdots$ to 1. At the same time, the dual variables on legs $1, 2, 4, 5$ all increase. Furthermore, as leg 1's capacity increases from $(2, 3, 11, 5, 6), (3, 3, 11, 5, 6)$ to $(8, 3, 11, 5, 6)$, we see that the middle leg's dual variable increases from 1.4385, 1.4385 to 1.8811. At the same time, the dual variables on legs 1 and 2 decrease from 2.0000 to 1.3920. These results agree with the theoretical assertions in Theorems 8, 9, 3 and 4.

6.2 Numerical test for BP control

Now the BP control is consider in a larger network. These data have been extracted from web-sites. Fare data is given in Table 6.4, and demand data in Table 6.5.

In the run, let

$$\mathbf{c} = (50 \quad 142 \quad 189 \quad 350 \quad 189 \quad 142 \quad 142).$$

We input all the parameters into Model 3.3 and obtain the following dual variable solution shown in Figure 6.2. The numbers on the legs represent the optimal dual variables of DLP Model 3.3.

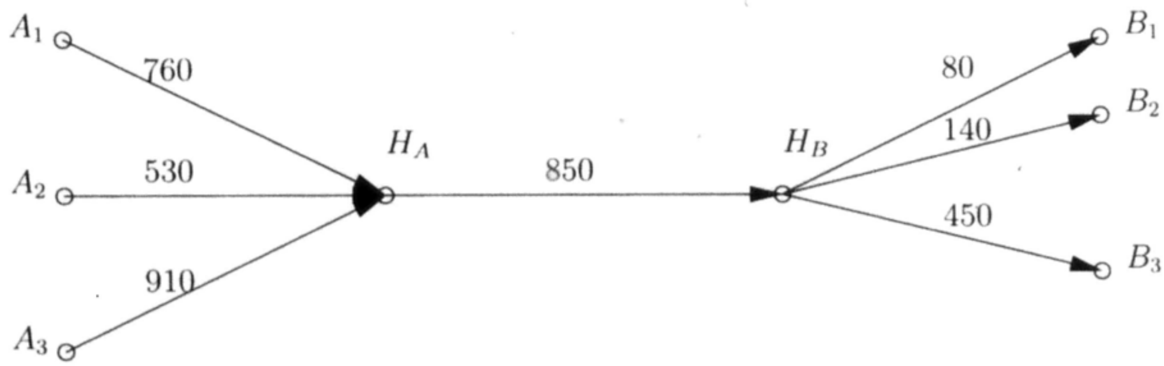

Figure 6.2: Example 1 of dual values

We then get the values given in Threshold Price Table 6.6.

This threshold price shows which ODF is open and which is closed. For example, the Chang Chun to Shang Hai (itinerary 2) threshold price is 1610; therefore, all three classes on this route are closed. In addition, the Chang Chun to Hang

Zhou (itinerary 4) threshold price is 1750; therefore, only the higher class Y is opened for booking. Since the focus is on the structural properties of these threshold values rather than the control mechanisms, readers interested in how these controls work may wish to pursue this elsewhere. Table 6.6∽Table 6.9 are given for implementation purposes.

In the second run, let

$$\mathbf{c} = (100 \quad 142 \quad 189 \quad 350 \quad 189 \quad 142 \quad 142).$$

We get the following optimal dual variables shown in Figure 6.3:

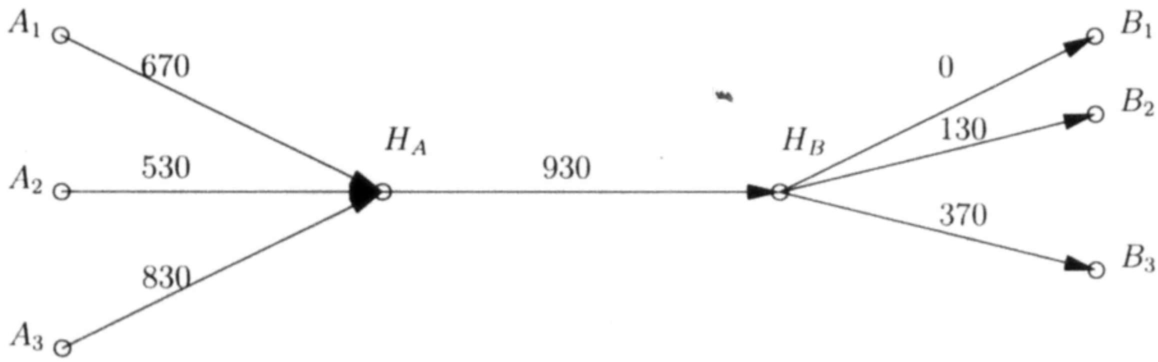

Figure 6.3: Example 2 of dual values

We get the values given in the second Threshold Price Table 6.7. A comparison of Table 6.6 and Table 6.7 shows that itineraries $1, 2, 3, 4$ and 5 all have decreased threshold prices that comply with theorems 4, 5 and 6; itinerary 11 also has a decreased threshold price, from 910 to 830; itinerary 6 stays unchanged at 530, complying with Theorem 4; itineraries $16, 17, 18$ and 19 all have increased threshold prices, in accordance with theorems 3 and 7.

In the run, let

$$\mathbf{c} = (189 \quad 142 \quad 189 \quad 350 \quad 189 \quad 142 \quad 142).$$

We get λ in Figure 6.4,

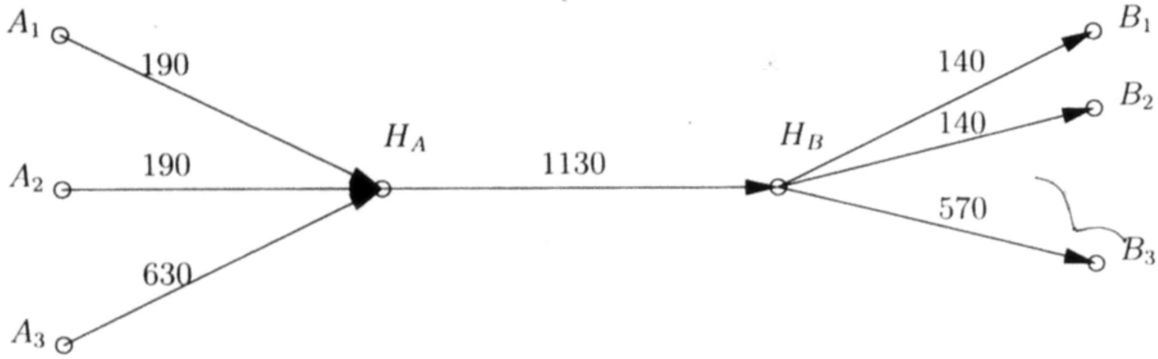

Figure 6.4: Example 3 of dual values

and we get Threshold Price Table 6.8. We observe that the threshold prices still behave in a manner as prescribed in the monotonicity results of theorems 3~7. In the run, we let

$$\mathbf{c} = (189 \quad 142 \quad 189 \quad 550 \quad 189 \quad 142 \quad 142).$$

We get λ as shown in Figure 6.5 and threshold Price Table 6.9.

Comparing Table 6.8 with Table 6.9 reveals that the bid prices on $(A_k, H_A), k = 1, 2, 3$ are considerably improved, while those on (H_A, H_B) are greatly reduced. Threshold prices on itineraries $2, 7, 12, 17, 18$ and 19 all decrease, as stated in Theorem 12 and Theorem 11. For three-leg itineraries, there is no fixed pattern since the threshold price on (A_1, B_1) remains unchanged, while it grows larger on (A_2, B_2) and smaller on (A_3, B_3).

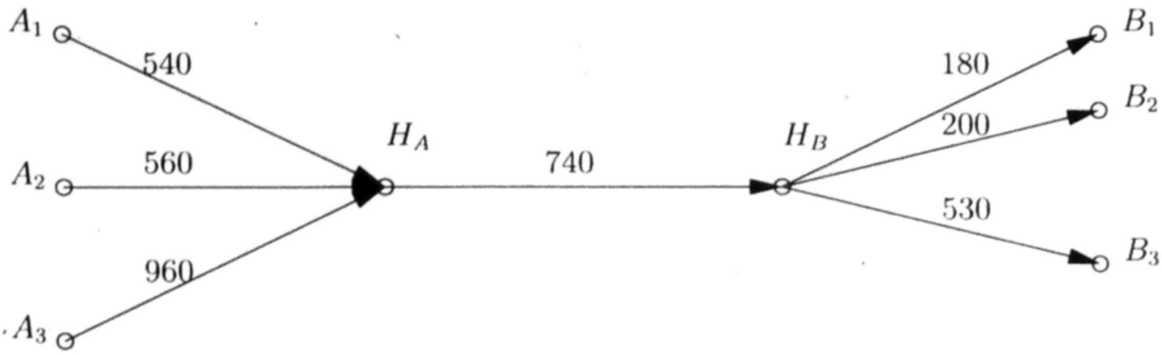

Figure 6.5: Example 4 of dual values

6.3 Numerical test for CE control

As Chapter 5 on network CE control supplements chapters 3 and 4, a numerical test is done here for the structures in network CE control as a supplementary part to the numerical test part of network BP control (Section 6.2).

As Corollaries 15 ∽ 19 indicate, the monotonicity of many CEC thresholds reduces to the super/submodularity and $L^\natural$ concavity of the optimal value function of Model (3.3). It is therefore necessariy to verify these second-order properties. We first verify that it is submodular in (c_1, c_2). Taking four different capacities states, we get Table 6.10. We calculate $f_1 + f_3 - f_2 - f_4 = 0$. The submodularity of value function in capacities (c_1, c_2) is satisfied. We then calculate $f_2 - f_1 = 190$, which means, the threshold value in controlling fare requests on itinerary (A_2, H_A) when flight leg 1's capacity is 188, is 190. We then calculate $f_3 - f_4 = 190$, which means, the threshold value in controlling fare requests on itinerary (A_2, H_A) when flight leg 1's capacity is 189, is again 190. Thus in this scenario the threshold value is not changed.

Taking another four different capacities states gives Table 6.11. We now calculate $f_5 + f_6 - f_7 - f_8 = -1.1850e + 004$. The submodularity of the value function in (c_1, c_2) is obvious. The threshold values calculation is neglected hereafter since the emphasis is on the second-order properties of the optimal value function.

Then we verify that the optimal value function is supermodular in (c_2, c_4), as stated by Theorem 20. This is shown in Table 6.12

We now calculate $f_9 + f_{12} - f_{10} - f_{11} = 2.4880e + 004 > 0$. Thus the supermodularity of the optimal value function in (c_2, c_4) is verified.

Finally, we verify that the optimal value function is $L^\natural$ concave in (c_1, c_4). Note that $L^\natural$ concavity implies the threshold in itinerary (A_1, H_B) by using CEC that is decreasing in c_1 or c_4. Now we calculate $f_{13} - f_{14} = 1320$, $f_{15} - f_{16} = 1610$. Because $f_{13} - f_{14} < f_{15} - f_{16}$, $L^\natural$ concavity of the optimal value function in (c_1, c_4) is verified, as addressed in Theorem 21. By Corollary 17, it means that the threshold on (A_1, H_B) is decreasing in c_1 and c_4.

6.4 Chapter summary

This chapter began with a simple test to check that all the allocations variables behave in the ordered way prescribed by corollaries in Chapter 3, and that all the dual variables also behave in a proper way as prescribed in the main theorems of Chapter 3. The numerical test was then extended to a more realistic network, with fare data drawn from websites and demand data based on realistic estimations. The LP associated with this network RM model was then calculated through Matlab to sum out all the threshold prices on the network's itineraries in four capacities scenarios. By comparing different scenarios, all the threshold prices were shown to behave in accordance with the main theorems stated in Chapter 3. Such a pattern may long have been observed by airline yield managers in real NRM applications, yet this thesis establishes the theoretical foundation for such an ordered phenomenon in real life.

For the certainty equivalent control, the supermodularity/submodularity of the optimal value function has been verified as has the $L^\natural$ concavity in consecutive flight legs' capacities. These properties lead to the assertion that some threshold values in CEC have monotonicity with respect to certain legs' capacities.

In summary, these numerical tests have provided more concrete understanding of the theoretical properties obtained in the previous chapters.

Table 6.1: Allocations change as side leg's capacity changes

o	**x**	f
(5,3,11,5,6)	(0.0000,0.0000,2.6853,2.3147,0.0000,0.0000,1.4775, 1.5225,0.0000,0.8372,2.1628,0.0000,0.0000)	49
(6,3,11,5,6)	(0.0000,0.0000,3.1521,2.8479,0.0000,0.0000,1.4423, 1.5577,0.0000,0.4056,1.5944,0.0000,0.0000)	51
(7,3,11,5,6)	(0.0000,0.0000,3.4874,3.5126,0.0000,0.0000,1.2407, 1.7593,0.0000,0.2719,0.7281,0.0000,0.0000)	53
(8,3,11,5,6)	(0.0000,0.0000,4.0749,3.9251,0.0000,0.0000,0.9251, 2.0749,0.0000, 0.0000,0.0000,0.0000,0.0000)	55
(7,7,11,5,7)	(1.5000,0.0000,2.4225,3.0775,1.5000,0.0000,2.4225, 3.0775,0.0000, 0.0000,0.0000,0.1551,0.8449)	59
(8,7,11,5,7)	(2.4163,0.0000,2.4947,3.0891,1.5837,0.0000,2.2764, 3.1399,0.0000, 0.0000,0.0000,0.2289,0.7711)	60
(9,7,11,5,7)	(3.0434,0.0000,2.5713,3.3853,1.9566,0.0000,2.1886, 2.8548,0.0000,0.0000,0.0000,0.2401,0.7599)	61
(10,7,11,5,7)	(3.6503,0.0000,2.6841,3.6656,2.3497,0.0000,2.0725, 2.5778,0.0000,0.0000,0.0000,0.2434,0.7566)	62
(5,13,15,5,7)	(3.0000,0.3408,0.8537,0.8056,0.0000,2.6592,4.1463, 6.1944,0.0000,0.0000,0.0000,0.0000,0.0000)	98
(6,13,15,5,7)	(4.0000,0.3359,0.8622,0.8019,0.0000,2.6641,4.1378, 6.1981,0.0000,0.0000,0.0000,0.0000,0.0000)	99
(7,13,15,5,7)	(5.0000,0.3232,0.8699,0.8070,0.0000,2.6768,4.1301, 6.1930,0.0000,0.0000,0.0000,0.0000,0.0000)	100
(8,13,15,5,7)	(6.0000,0.3112,0.8766,0.8121,0.0000,2.6888,4.1234, 6.1879,0.0000,0.0000,0.0000,0.0000,0.0000)	101
(···)	(···)	(···)

Table 6.2: Allocations change as middle leg's capacity changes

c	**x**	f
(8,3,12,5,6)	(8.0000,0.0000,0.0000,0.0000,0.0000,3.0000,0.0000, 0.0000,7.8222,1.1778,0.0000,3.8222,6.0000)	129
(8,3,13,5,6)	(8.0000,0.0000,0.0000,0.0000,0.0000,3.0000,0.0000, 0.0000,8.8079,1.1921,0.0000,3.8079,6.0000)	134
(8,3,14,5,6)	(8.0000,0.0000,0.0000,0.0000,0.0000,3.0000,0.0000, 0.0000,9.7806,1.2194,0.0000,3.7806,6.0000)	139
(8,3,15,5,6)	(8.0000,0.0000,0.0000,0.0000,0.0000,3.0000,0.0000, 0.0000,10.7459,1.2541,0.0000,3.7459,6.0000)	144
(8,3,16,5,6)	(8.0000,0.0000,0.0000,0.0000,0.0000,3.0000,0.0000, 0.0000,11.7065,1.2935,0.0000,3.7065,6.0000)	149
(8,3,17,5,6)	(8.0000,0.0000,0.0000,0.0000,0.0000,3.0000,0.0000, 0.0000,12.6938,1.3062,0.0000,3.6938,6.0000)	154
(···)	(···)	(···)

Table 6.3: Change of dual variables

c	λ
(2,3,4,5,6)	(1.0000,1.0000,3.0000,1.0000,1.0000)
(2,3,5,5,6)	(1.5968,1.5968,2.4032,1.0000,1.0000)
(2,3,6,5,6)	(2.0000,2.0000,2.0000,1.0000,1.0000)
(2,3,7,5,6)	(2.0000,2.0000,2.0000,1.0000,1.0000)
(2,3,10,5,6)	(2.0000,2.0000,2.0000,1.0000,1.0000)
(2,3,30,5,6)	(2.0000,2.0000,1.0000,2.0000,2.0000)
(2,3,50,5,6)	(2.0000,2.0000,1.0000,2.0000,2.0000)
(2,3,15,5,6)	(2.0000,2.0000,1.0000,2.0000,2.0000)
(2,3,12,5,6)	(2.0000,2.0000,1.0000,2.0000,2.0000)
(2,3,11,5,6)	(2.0000,2.0000,1.4385,1.5615,1.5615)
(3,3,11,5,6)	(2.0000,2.0000,1.4385,1.5615,1.5615)
(4,3,11,5,6)	(2.0000,2.0000,1.4385,1.5615,1.5615)
(5,3,11,5,6)	(2.0000,2.0000,1.4409,1.5591,1.5591)
(6,3,11,5,6)	(2.0000,2.0000,1.4473,1.5527,1.5527)
(7,3,11,5,6)	(2.0000,2.0000,1.4655,1.5345,1.5345)
(8,3,11,5,6)	(1.3920,1.3920,1.8811,1.7269,1.7269)
$(\cdots)$	$(\cdots)$

Table 6.4: Fare data

	Bei Jing	Shang Hai	Nan Jing	Hang Zhou	Fu Zhou
Chang Chun	Y 960 B 860 K 770	Y 1600 R 960 W 640	Y 1460 S 950 R 880	Y 1750 S 1140 Q 1050	Y 1810 K 1450 M 1270
Shen Yang	Y 700 K 600 S 460	Y 1300 B 1170 M 910	Y 1460 T 1310 G 1020	Y 1590 T 1430 K 1270	Y 1830 S 1190 Q 1100
Ha Erbin	Y 960 T 860 L 770	Y 1760 L 1320 V 880	Y 1650 M 1160 Q 990	Y 1900 R 760 M	Y 1990 K 1590
Bei Jing		Y 1130 B 1020 L 850	Y 930 K 750 Q 650	K 850 Q 750 G 630	U 570
Shang Hai	K	Y R W	Y 800 W 320 R	Y 200 S Q	Y 780 N 510 R 310

Table 6.5: Demand data

	Bei Jing	Shang Hai	Nan Jing	Hang Zhou	Fu Zhou
Chang Chun	Y 30 B 40 K 50	Y 60 R 20 W 70	Y 80 S 90 R 10	Y 30 S 20 Q 10	Y 50 K 40 M 60
Shen Yang	Y 70 K 80 S 10	Y 30 B 40 M 60	Y 80 T 90 G 12	Y 33 T 45 K 66	Y 100 S 200 Q 34
Ha Erbin	Y 85 T 67 L 29	Y 110 L 23 V 24	Y 37 M 38 Q 39	Y 49 R 41 M	Y 43 K 44
Bei Jing		Y 46 B 47 L 48	Y 49 K 50 Q 51	K 52 Q 53 G 54	U 55
Shang Hai	K	Y R W	Y 58 W 59 R	Y 61 S Q	Y 64 N 65 R 66

Table 6.6: Example 1 of threshold price

Itinerary	Threshold Price
1	760
2	1610
3	1690
4	1750
5	2060
6	530
7	1380
8	1460
9	1520
10	1830
11	910
12	1760
13	1840
14	1900
15	2210
16	850
17	930
18	990
19	1300
20	80
21	140
22	450

Table 6.7: Example 2 of threshold price

Itinerary	Threshold Price
1	670
2	1600
3	1600
4	1730
5	1970
6	530
7	1460
8	1460
9	1590
10	1830
11	830
12	1760
13	1760
14	1890
15	2130
16	930
17	930
18	1060
19	1300
20	0
21	130
22	370

Table 6.8: Example 3 of threshold price

Itinerary	Threshold Price
1	190
2	1320
3	1460
4	1460
5	1890
6	190
7	1320
8	1460
9	1460
10	1890
11	630
12	1760
13	1900
14	1900
15	2330
16	1130
17	1270
18	1270
19	1700
20	140
21	140
22	570

Table 6.9: Example 4 of threshold price

Itinerary	Threshold Price
1	540
2	1280
3	1460
4	1480
5	1810
6	560
7	1300
8	1480
9	1680
10	2010
11	960
12	1700
13	1880
14	1900
15	2230
16	740
17	920
18	940
19	1270
20	180
21	200
22	530

Table 6.10: Illustration 1 for the submodularity of value function

Number	c	f
1	[188 141 189 350 189 142 142]	882320
2	[188 142 189 350 189 142 142]	882510
3	[189 142 189 350 189 142 142]	882700
4	[189 141 189 350 189 142 142]	882510

Table 6.11: Illustration 2 for the submodularity of value function

Number	c	f
5	[50 50 189 350 189 142 142]	751310
6	[149 149 189 350 189 142 142]	874910
7	[50 149 189 350 189 142 142]	815290
8	[149 50 189 350 189 142 142]	822779

Table 6.12: Illustration for the supermodularity of value function

Number	c	f
9	[189 50 189 150 189 142 142]	577979
10	[189 50 189 350 189 142 142]	843439
11	[189 142 189 150 189 142 142]	591979
12	[189 142 189 350 189 142 142]	882699

Table 6.13: Illustration for the $L^\natural$ concavity of value function

Number	c	f
13	[189 142 189 350 189 142 142]	882700
14	[188 142 189 349 189 142 142]	881380
15	[50 142 189 350 189 142 142]	811580
16	[49 142 189 349 189 142 142]	809970

Chapter 7

Conclusion

Chapter 2 of this thesis began by introducing the optimal control of the network revenue management formulated by Talluri and van Ryzin (1998). Subsequent chapters then focused on two heuristic control methods, the bid price control and the certainty equivalent control, and applied them to the so-called hub-to-hub airline network.

7.1 Summary of research findings

This study of the bid price control first aggregated the revenues on each OD and disclosed an optimality condition in terms of the sub-differentials of the aggregated piecewise-linear revenue functions. Attention was then shifted to the monotone structure of the bid price control thresholds. By applying a primal-dual argument, proof was demonstrated that the BP thresholds are monotone in some legs' capacities.

The study of the certainty equivalent control investigated the problem from a combinatorial optimization perspective, bridging the DLP model for network revenue management with network flows. By applying the notion of $L^\natural$ concavity developed by Murota (2003, 2005), it is demonstrated that the CEC thresholds exhibit the same monotone patterns as those of the BP thresholds.

The numerical test part studied the intricate interrelationship between the different flights in the multi-hub network via numerical exploration. More specifically, it demonstrated how a change in the network configuration affects the allocation balance, and how the dual price of the itineraries changes, given that the economic features (fare structure, estimated demand, etc.) of the various markets are fixed. These numerical tests make two valuable contributions: 1) To gain more concrete management insight into the relationships between the network flights; 2) To suggest possibilities for the reduction of computational work of the decision making process. Moreover, the numerical comparisons were shown to agree with the structural properties obtained in Chapter 4 and Chapter 5.

7.2 Contributions

The principle contribution of this thesis is that it offers the first explicit model and study of the structure of a specific type of multi-hub airline network, the hub-to-hub network. Another is that it adopts the use of a primal-dual technique to analyze monotone structural properties. This novel approach may benefit researchers in the mathematical programming field since many traditional primal-dual techniques are applied in algorithm design or in merely proving the optimality of some solutions. This thesis also fully exploited the use of Lagrange dual variables and the primal-dual relationship to derive the monotone properties of the bid price control thresholds. This can be viewed as a good example of merging analytical intricacy with geometric simplicity.

Chapter 5 clarified the relationship between the DLP model in a hub-to-hub network RM context with the maximum weight circulation problem in network flows. This method was first proposed in a simple form by Glover et al. (1982) and later extended by Morton (2006) to a more general 'bipartite' network. This thesis has extended the method to a more complex hub-to-hub network. It is hoped that this will reveal new ways of handling large-scale revenue management problems.

The notion of $L^\natural$ concavity in network flows was developed by Murota and Shioura (2003,2005) in the discrete convex analysis field, and has been used by Zipkin (2008) to study a lost-sales inventory model. This thesis provides another concrete example demonstrating its power to explore structural properties in the OM/MS field.

7.3 Future work

The emphasis of this thesis is on the theoretical side of revenue management models. Future work will focus on its implementation. One promising direction is the extension of the structural properties obtained here to dynamic models. This may face difficulty since stochastic dynamic models differ considerably from static ones. However, we will try to adopt the spirit in this thesis and may merge it with another kind of methodology.

Another possibility would be to try to exploit such properties to enhance the computational efficiency in the heuristic dynamic-programming process. These properties may also characterize the optimality of certain control policies as *threshold curves*, as is done by You (1999). The model could also be further developed to incorporate cancellations, no-shows and overbooking.

Regarding the relationship between NRM problems and network flows investigated in Chapter 5, strengthening the relationship between RM network and network flows opens up a rich research direction in revenue management field. An interesting question to ask in future work is what kind of network RM problems can be recast into network flow representations and whether these can lead to the promotion of higher computational efficiency.

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