Generalized Integral Transforms

related to the theory of Potential and Stokes flow

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GENERALIZED INTEGRAL TRANSFORMS RELATED TO THE THEORY OF POTENTIAL AND STOKES' FLOW

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Φαΐνε και σίγα

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Prologue

Many of the greatest, of the most important, aspects in science and mathematics were developed within the framework of physical science. These advances have been based on formulating the underlying mathematical equations for the process in question. Almost all physical, chemical and biological systems or processes obey mathematical laws that can be formulated by differential equations. This striking fact was first discovered by Isaac Newton (1642-1727) when he formulated the laws of mechanics and applied them to describe the motion of the planets.

Scientist, have extended over the centuries these type of connections to include a broad diversity of areas of science and technology, from which a field has emerged called mathematical modeling. A mathematical model, consisting of physical "reality" expressed in mathematical terms, is a partial differential equation, or more likely, a system of partial differential equations, whose solution describes the behavior of the physical system in view.

Partial differential equations were not consciously created as a subject, but emerged approximately in the beginning of the 18th century as ordinary differential equations "failed" to describe the physical principles studied. Partial differential equations are categorized into linear and nonlinear and the significance of the partition cannot be overstated. Linear equations enjoy an algebraic structure to their solution sets, i.e. their solutions superimpose. Nonlinear equations do not share this property. Nonlinear equations are harder to solve and their solutions are more difficult to analyze. There are in fact only a limited number of methods available to solve partial differential equations analytically without introducing approximate or numerical techniques.

Perhaps one of the oldest and most widely used technique for the solution of the partial differential equations of mathematical physics is the method of spectral decomposition, namely separation of variables. Introduced by d' Alembert, Daniel Bernoulli and Euler in the middle of the eighteenth century (for an exciting historical walkthrough see [Kli90]) it remains a method of great value today. Here, the unknown function is separated into a product of functions (*multiplicatively separable*) which depend solely on one of the variables. A set of such solutions is obtained which, due to the superposition principle, can be summed up to give a "general solution". The boundary conditions are applied to this solution and these restrict the summed functions to a subset, yielding the coefficients of the series. This method, in the form of additive separable solutions can also be applied to some nonlinear first-order equations. Moreover, a generalization of separation of variables exist [PZ03], i.e. obtain a solution in the general form

$$u(x_1, x_2) = \sum_{i=1}^{n} \phi_i(x_1)\, \psi_i(x_2)\,,$$

in order to solve PDE's with quadratic or power nonlinearities, viz

$$\sum_{j=1}^{m} f_j(x_1)\, g_j(x_2) \prod_j [u(x_1, x_2)] = 0$$

where $\prod_j [u]$ are differential forms of the products of nonnegative integer powers of the function u and its partial derivatives. It should be noted that often exact generalized separable solutions cannot be obtained by other well-known methods.

Another powerful methodology for solving PDE's are integral transforms. Integral transforms, in which the partial derivative are reduced to algebraic terms and ordinary derivatives, can be traced back to the pioneering work of Pierre Simon Laplace [Lap20] on probability theory in the late eighteenth century and of Jean Baptiste Joseph Fourier [Fou22] in his groundbreaking study *La Théorie analytique de la Chaleur*, published in 1822. The fundamental idea is to represent a function $f(\mathbf{x})$ in terms of a transform $\mathcal{F}(\mathbf{k})$, using an integral transform pair

$$\mathcal{F}(\mathbf{k}) = \int K(\mathbf{k}; \mathbf{x})\, f(\mathbf{x})\, \mathrm{d}\mathbf{x} \tag{1}$$

$$f(\mathbf{x}) = \int K'(\mathbf{x}; \mathbf{k})\, \mathcal{F}(\mathbf{k})\, \mathrm{d}\mathbf{k}\,. \tag{2}$$

A function $f(\mathbf{x})$ defined in terms of a function $\mathcal{F}(\mathbf{k})$ by means of an integral relation (1), is said to be the integral transform of the function $\mathcal{F}(\mathbf{k})$ by the kernel $K(\mathbf{k}; \mathbf{x})$. The application of the transform constitutes the transformed problem solvable and the original function space can be recovered by applying the inverse transform (2). Although Fourier is celebrated for his work on the conduction of heat, the mathematical methods involved, particularly trigonometric series, are very important and useful. He created a coherent mathematical method by which the different components of an equation and its solution in series were identified with the different aspects of the physical solution being analyzed. It is no exaggeration to say that the scientific achievements of Joseph Fourier provided the fundamental basis for modern developments of the theory and applications of partial differential equations.

Perhaps, the most important of partial differential equations in applied mathematics and mathematical physics is the one associated with the name of Pierre-Simon Laplace (1749-1827)(see the classic resource on the history of Mathematics [Bal60]). This equation was first discovered by Laplace while he threw himself into extensive research for seventeen years (1771-1787). Laplace developed the idea of the potential -a name first given by Green in 1828, appropriated from Lagrange who had used it in his memoirs of 1773, 1777 and 1780, a concept which is invaluable in a wide range, such as electromagnetism, hydrodynamics, etc.

Laplace's equation, which is time independent, arises in the study of a plethora of physical phenomena, including electrostatic or gravitational potential, the velocity potential of an incompressible fluid flow and the displacement field of a two- or three- dimensional elastic membrane. The relation with the physical world, however, dictates that certain conditions on the boundary of the region in which Laplace equation is to be solved, must be satisfied. The problem of finding solutions that takes on the given boundary values is known as the Dirichlet boundary-value problem, after Peter Gustav Lejeune Dirichlet

(1805-1859). If values of the normal derivative are prescribed on the boundary, the problem is known as Neumann boundary-value problem, in honor of Karl Gottfried Neumann (1832-1925).

Despite the efforts by many, great mathematicians including Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Simeon-Denis Poison (1781-1840), very little was known about the general properties of the solutions of Laplace's equation until 1828, when George Green (1793-1841) and Mikhail Ostrogradsky (1801-1861) independently investigated properties of a class of solutions know as harmonic functions.

In 1836-1837, Jasques Charles François Sturm (1803-1855) and Joseph Liouville (1809-1882) published a series of papers on second-order linear differential equations, originated from the study of a class of boundary-value problems. The influence of their work was such that this subject became known as Sturm-Liouville theory. This theory is a natural generalization of the theory of Fourier and extends the scope of the method of separation of variables.

Through the years, tremendous progress has been made on the general theory of ordinary and partial differential equations (for an excellent review see [BB98]). With the advent of new ideas and methods, new results and applications, both analytical and numerical studies are continually being added to this subject. Partial differential equations have been the subject of mathematical research for over three centuries and, owing to the increasing need in mathematics, science and engineering to solve more and more complicated real world problems, it seems quite likely that partial differential equations will remain a major area for many years to come.

The main concern of this dissertation is focused on the derivation of novel integral formulation for simple problems. These alternative integral representations display a rapid decay as the complex parameter involved tends to infinity and are therefore suitable for numerical computations and for the study of the asymptotic properties of those solutions. There is also another important advantage attached to the novel formulae presented. These integral representations are useful for solving changing-type boundary value problems (such as Dirichlet data on part of the boundary and Neumann data on the complementary of the boundary).

The dissertation is divided into a number of chapters as follows. The introductory chapter consist as the initiation of the reader to the generalized transform method, introduced by Prof. Fokas, which will then applied to a particular example, namely the Square, in Chapter 2.

Chapter 3 is devoted to the theory of Gegenbauer functions. The behavior of the Gegenbauer functions of the first and second kind of general complex degree ν and order λ on the cut $(-1, +1)$ are examined. Moreover, series representations together with asymptotic expansions, which to the authors knowledge are new, are presented. The *Gegenbauer Integral Operator* $\mathfrak{G}_{\nu}^{\lambda}$, which plays a crucial role in the derivation of novel integral representations associated with the new method, is here for the first time introduced. Last but not least, an alternative approach arriving at the Wronskian of an independent pair of solutions using recurrence relations is presented.

In Chapters 4 and 5 the Laplacian operator in the interior and exterior of a Sphere and the Stokes' operator concerning the irrotational flow of an incompressible, viscous fluid are analyzed. Technical calculations are left to the Appendices.

CHAPTER 1

The Generalized Transform Method

Mathematicians did not consciously created the subject of partial differential equations (PDE's). Their continuously exploration of physical problems† secured a better grasp of the physical principles underlying the phenomena and mathematical statements were formulated which are now comprised in partial differential equations.

A general approach for constructing a large class of solutions was invented almost with the advent of partial differential equations. This approach, based on the efforts of d' Alembert, Euler, D. Bernoulli and others, involved separating variables, an ingenious method that decomposes a partial differential equation into a set of ordinary differential equations (ODE's), and superimposing, namely building up complicated solutions from simple ones, the solutions of the resulting ordinary differential equations.

Separation of variables lies at the heart of the use of integral transform, and therefore led to solutions of PDE's by a transform pair. A great variety of integral transforms exist in the literature, such as the Fourier, Laplace, Mellin, Kontorovich-Lebedev, Mehler-Fock, naming only a few of them, as well as the finite analogs for certain of them. For a given boundary-value problem the appropriate transform is dictated by the differential operator, the fundamental domain and the boundary conditions prescribed. For simple boundary-value problems there exists an algorithmic procedure deriving associated transform pair [DB06, Sta97]. This procedure, based on the spectral decomposition of a single eigenvalue equation, has been remarkably successful for solving a variety of initial and boundary-value problems. However, for complicated problems the classical transform fails.

Within the last decade, a new approach for solving boundary value problems (BVP) has been developed by Fokas and it is presented in [Fok08]. The novelty of the Fokas method is based on the construction of a tailor-made transform, for each BVP, assimilating the geometrical and spectral characteristics of the problem. The key feature of this methodology

† One of the major problems of the eighteenth century consisted of the determination of the amount of gravitational attraction one mass exerts on another, which ultimately led to the theory of potential.

lies in the successful manipulation of the so-called global relation, a formula connecting the solution of the BVP with its derivatives on the boundary. Furthermore, one must underline two important notions in the new method presented. One is the concept of integrable nonlinear equations which is closely followed by Lax pairs. We will see these two concepts later one.

For linear PDE's with constant coefficients[‡] this new approach for solving boundary value problems involves the following steps.

1. Given a PDE, formulate the PDE as the compatibility condition of two linear equations, viz find a Lax pair for the given PDE.
2. Perform a simultaneous spectral analysis of both equations which yields a Riemann-Hilbert problem.
3. Given appropriate boundary conditions, analyze the global relation, satisfied by the boundary values of the solution and its derivatives.

Let us now proceed in more detail.

ELEMENTS OF THE GEOMETRY OF THE POLYGON. Consider a *convex* and *bounded* polygon $\Omega \subset \mathbb{C}$ in the complex $\mathbb{C}$-plane with vertices $z_1, \ldots, z_n, z_{n+1} = z_1$ as shown in Fig. 1.1. Furthermore, since the vertices z_j are finite in number, the sum of their angles equals

$$\sum_{j=1}^{n} \theta_j = (n-2)\pi\,.$$

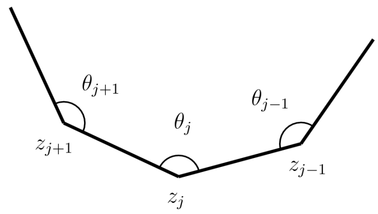

FIGURE 1.1: *Part of the convex and bounded polygon Ω with vertices z_j.*

The convexity of Ω implies $\theta_j \in (0,\pi)$ and thus from Fig. 1.2 it is easily deduced that $\theta_j = \varphi - \phi$. Since $\phi = \arg(z_{j-1} - z_j)$ and $\varphi = \arg(z_{j+1} - z_j)$ we find

$$\theta_j = \arg(z_{j-1} - z_j) - \arg(z_{j+1} - z_j)\,.$$

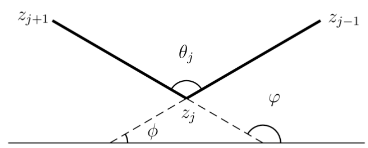

FIGURE 1.2: *The angles θ_j, ϕ and φ.*

Let $q(x,y)$ satisfy the Laplace equation in the interior $\Omega \subset \mathbb{C}$ of a *convex* and *bounded* polygon with vertices $z_1, \ldots, z_n, z_{n+1} = z_1$, $z = x + i\,y$ as shown in Fig. 1.1. In complex coordinates, using

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

the Laplace equation is written as

$$\frac{\partial^2}{\partial z\,\partial \bar{z}}\, q(z) = 0\,,$$

which can be reformulated in the form

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial q}{\partial z}\right) = 0\,, \tag{1.1}$$

where an overbar denotes complex conjugation.

From (1.1) it is obvious that $q(z)$ is harmonic if $\partial_z q(z)$ is an analytic function. From

[‡]In the case where the coefficients are functions of the variables, see [Fok04, TF07]

this observation we conclude that it is simpler to obtain an integral representation for an analytic function, namely for $\partial_z q(z)$, instead of $q(z)$.
Since $q(z)$ is a solution of the Laplace equation, the following differential form is closed

$$W(z,k) = e^{-ikz}\,\frac{\partial q}{\partial z}\,\mathrm{d}z, \quad k \in \mathbb{C}, \quad z \in \Omega \subset \mathbb{C}\,,$$

i.e.

$$\mathrm{d}W(z,k) = e^{-ikz}\,\frac{\partial^2 q}{\partial z\,\partial \bar{z}}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = 0, \quad k \in \mathbb{C}, \quad z \in \Omega \subset \mathbb{C}\,. \tag{1.2}$$

Then, Stoke's theorem

$$\oint\limits_{\partial\Omega} W = \iint\limits_{\Omega} \mathrm{d}W\,,$$

implies that

$$\int\limits_{\partial\Omega} e^{-ikz}\,\frac{\partial q}{\partial z}\,\mathrm{d}z = 0, \quad k \in \mathbb{C}, \quad z \in \Omega \subset \mathbb{C}\,. \tag{1.3}$$

We will refer to this equation as the global relation. Rewrite the foregoing expression as

$$\sum_{j=1}^{n} \rho_j(k) = 0, \quad k \in \mathbb{C}, \tag{1.4}$$

where (j) corresponds to the side (z_{j+1}, z_j) of the polygon and the spectral functions $\rho_j(k)$ are defined as

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\,\frac{\partial q^{(j)}}{\partial z}\,\mathrm{d}z, \quad k \in \mathbb{C}, \quad j = 1, 2, \ldots, n, \quad z_{n+1} = z_1\,. \tag{1.5}$$

Proposition 1.0.1 ([FK03, FFX04]) *Let Ω be a bounded convex polygon in the complex $\mathbb{C}$-plane with vertices $z_1, \ldots, z_n$. Let f be a smooth complex-valued function defined on the boundary $\partial\Omega$ of the polygon Ω and consider f_j the restriction of f on the side (z_{j+1}, z_j). Assume that there exist a function f such that the spectral functions* (1.5) *with $\partial_z\, q^{(j)} = f_j$ satisfy* (1.4). *If, the analytic function $\partial_z\, q$ is defined as the representation*

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi} \sum_{j=1}^{n} \int\limits_{\ell_j} e^{ikz} \rho_j(k)\,\mathrm{d}k\,, \tag{1.6}$$

where the rays ℓ_j in the complex k-plane are defined as

$$\ell_j = \Big\{ k \in \mathbb{C} \,\Big|\, \arg k = -\arg(z_j - z_{j+1}) \Big\}\,, \tag{1.7}$$

then $q(z)$ satisfies the Laplace equation in Ω and on the jth side, $\partial_z\, q^{(j)} = f_j$.

Indeed, consider an auxiliary function $\mu(z,k)$, $k \in \mathbb{C}$, $z \in \Omega \subset \mathbb{C}$ which satisfies the system

$$(\partial_z - i\,k)\,\mu(z,k) = \partial_z\, q(z), \tag{1.8}$$

$$\partial_{z\bar{z}}\,\mu(z,k) = 0\,. \tag{1.9}$$

Introducing an exponential Euler factor, equation (1.8) becomes

$$\mathrm{d}\Big(e^{-i\,k\,z}\,\mu(z,k)\Big) = e^{-ikz}\,\frac{\partial q}{\partial z}\,\mathrm{d}z\,.$$

Integrating the above equation along any curve $\Gamma(z_j, z)$ connecting the vertice z_j with any point z inside Ω, we find

$$\mu_j(z,k) = \int\limits_{\Gamma(z_j,z)} e^{ik(z-\zeta)}\,\frac{\partial q}{\partial \zeta}\,\mathrm{d}\zeta\,.$$

Moreover, since $\partial_z\, q$ is analytic, the contour $\Gamma(z_j, z) \subset \Omega$ can be deformed in any convenient way. Given the convexity of the polygon Ω let us consider the line segment which connect the vertice z_j with any point z inside the given domain. Then the latter equation reads

$$\mu_j(z,k) = \int_{z_j}^{z} e^{ik(z-\zeta)}\,\frac{\partial q}{\partial \zeta}\,\mathrm{d}\zeta\,. \tag{1.10}$$

From the above relation it is obvious that $\mu_j(z, k)$ is an entire function with respect to $k \in \mathbb{C}$, with $k = \infty$ as the only possible singularity. Therefore, in order for μ_j to be bounded as k tends to ∞, the following inequality must hold

$$\mathrm{Re}\Big(ik(z-\zeta)\Big) \leq 0, \quad z \in \Omega \subset \mathbb{C}, \quad \zeta \in \Gamma(z_j, z), \quad k \in \mathbb{C}\,. \tag{1.11}$$

The above inequality implies the restriction of the complex variable k to the sector S_j associated with the vertice z_j

$$S_j = \Big\{k \in \mathbb{C} : \arg k \in \big[-\arg(z_{j-1} - z_j), \pi - \arg(z_{j+1} - z_j)\big]\Big\}\,. \tag{1.12}$$

Indeed, for the straight line (z_j, z) we have

$$\arg(z-\zeta) \in \big[\arg(z_{j-1} - z_j), \arg(z_{j+1} - z_j)\big]$$

which together with (1.12) implies

$$\arg k + \arg(z-\zeta) \in [0, \pi]\,,$$

namely, inequality (1.11). The angle of the sector S_j, denoted by ψ_j, depicted in Fig. 1.3, equals

$$\psi_j = \pi - \arg(z_{j+1} - z_j) + \arg(z_{j-1} - z_j) = \pi - \theta_j\,,$$

from which

$$\sum_{j=1}^{n} \psi_j = \sum_{j=1}^{n} (\pi - \theta_j) = n\pi - (n-2)\pi = 2\pi\,.$$

The sectors S_j and S_{j+1} share the ray ℓ_j as common boundary and therefore

$$\ell_j = S_j \cap S_{j+1} = \Big\{k \in \mathbb{C}\,\Big|\,\arg k = -\arg(z_j - z_{j+1})\Big\}\,.$$

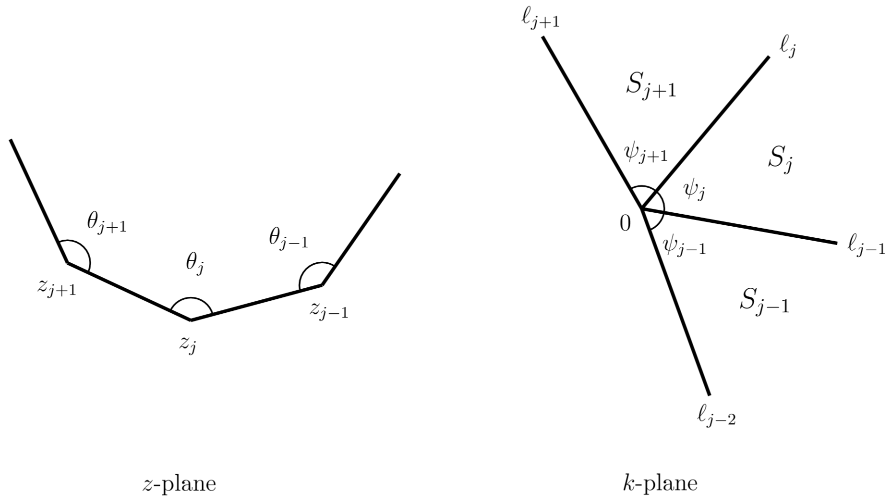

FIGURE 1.3: *The rays ℓ_j in the k-plane associated with the polygon Ω in the z-plane.*

Subtracting (1.10) with the equation resulting from (1.10) with j replaced by $j+1$, we derive the so-called jump condition

$$\mu_{j+1}(z,k) - \mu_j(z,k) = e^{i\,kz}\,\rho_j(k), \quad k \in \mathbb{C}\,. \tag{1.13}$$

Moreover, integrating (1.10) by-parts the asymptotic behavior for the auxiliary function $\mu_j(z,k)$ is obtained

$$\mu_j(z,k) = \mathcal{O}\Big(\frac{1}{k}\Big), \quad k \to \infty, \quad z \in \Omega \subset \mathbb{C}\,. \tag{1.14}$$

Equations (1.13) and (1.14) refer to a so-called Riemann-Hilbert problem* the solution to which is [AF03]

$$\mu(z,k) = \frac{1}{2i\pi}\sum_{j=1}^{n}\int_{\ell_j}\frac{e^{i\kappa z}\,\rho_j(\kappa)}{\kappa - k}\,\mathrm{d}\kappa, \tag{1.15}$$

where the rays $\{\ell_j\}_{j=1}^n$ are defined by (1.7). Substituting (1.15) into the first equation (1.8) of the Lax pair we immediately obtain $\partial_z\, q(z)$. Note that the operator $(\partial_z - i\,k)$ is such that it annihilates the k-dependency.

Remark 1.0.1 *Equation*

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\,\frac{\partial q^{(j)}}{\partial z}\,\mathrm{d}z\,,$$

can be seen as the (finite) Fourier transform along the straight line segment (z_{j+1}, z_j). Then, the inversion formula implies

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{\ell_j} e^{ikz}\rho_j(k)\,\mathrm{d}k\,.$$

*The notion of a Riemann-Hilbert problem will be explained in more depth in section 1.2

The solution is then obtained by integrating the expression for $\partial_z\, q(z)$ and by the use of the reality condition

$$q(x, y) = 2\mathrm{Re} \int_{z_0}^{z} \frac{\partial q}{\partial z}\, \mathrm{d}z + \text{const.} \tag{1.16}$$

The main differences between the two methods solving PDE's i.e. the classical transform (separation of variables) and the new (generalized transform) method, can be described briefly as: (1) Applying the classical transform, we assume that the solution to a given boundary-value problem can be expanded in a series of eigenfunctions. The generalized transform method on the other hand, constructs the solutions without the need of using eigenfunction expansions, arriving at separable solutions without actually assuming separation [DF05]. (2) In contrast with the method of separation of variables, which is strongly based on the geometry of the fundamental domain, the new method does not depend on the geometry of the domain at hand, but on the linearity of the PDE. An overview is provided in [Das03].

1.1 Integrability and Lax pairs

Since their exist different definitions of integrability, the question "What is Integrability?" results in a synthesis of many answers usually depending on how one chooses to attack the problem. For example, one type of attack involves perturbative or asymptotic methods. A second approach is algebraic, involving the classification of symmetries. Yet another method is based on the analytic behavior of solutions in the complex domain, the technique of Painlevé analysis where one examines if a given nonlinear PDE has Painlevé property, i.e. the only movable singularities are poles (see [Mus99] and the references given there). Another form of tackling involves the Lax pair formulation. Given a nonlinear PDE it is very difficult to find a Lax pair associated with the PDE, so it is actually simpler to postulate a Lax pair and determine to which PDE the pair correspond [IR00].

In what follows, we will call an equation integrable if it admits a Lax pair formulation, i.e. it can be written as the compatibility condition of two linear eigenvalue equations. The importances that the Lax pair consist of two *eigenvalue* equations must be emphasized, since if the pair does not contain a spectral parameter, then it cannot be used to solve the associated PDE. Moreover, for *linear* PDE's, the existence of a Lax pair is usually related to a closed 1-form [Ash08, FZ02].

Peter D. Lax, in his fundamental mathematical paper [Lax68], showed that it is possible to solve nonlinear equations introducing linear techniques. Lax proved that if it possible to find two linear PDE's (the so-called Lax pair), such that the compatibility of these two equations is equivalent to the initial PDE, then the equation accepts an analytic solution. As mentioned, not every nonlinear PDE possess a Lax pair, but those that do are identified as integrable. His approach in brief.

Consider the PDE

$$\partial_t\, u(\mathbf{x}, t) = \mathcal{L}(\mathbf{x}, t)\, u(\mathbf{x}, t)\,, \tag{1.17}$$

where $\mathcal{L}(\mathbf{x}, t)$ a nonlinear operator. Lax separated the *nonlinear* operator $\mathcal{L}(\mathbf{x}, t)$ into two *linear* $A(\mathbf{x}, t)$ and $B(\mathbf{x}, t)$ who actually "absorve" the nonlinearity of $\mathcal{L}(\mathbf{x}, t)$ through the coefficients of A and B which are polynomials of $u, \nabla u$, etc. Thus, given an auxiliary function $\mu(\mathbf{x},t)$ and assuming that

$$A(\mathbf{x},t)\,\mu(\mathbf{x},t) = \lambda\,\mu(\mathbf{x},t) \tag{1.18}$$
$$\partial_t\,\mu(\mathbf{x},t) = B(\mathbf{x},t)\,\mu(\mathbf{x},t)\,, \tag{1.19}$$

where λ is a time-independent parameter, one can easily show, by cross-differentiation, that

$$\partial_t\,A(\mathbf{x},t) + \big[A(\mathbf{x},t),\ B(\mathbf{x},t)\big] = 0\,, \tag{1.20}$$

where

$$\big[A(\mathbf{x},t),\ B(\mathbf{x},t)\big] = A(\mathbf{x},t)\,B(\mathbf{x},t) - B(\mathbf{x},t)\,A(\mathbf{x},t)$$

is the commutator.
Equation (1.20) is called the Lax representation, where else equations (1.18),(1.19) constitute the Lax pair. The difficulty with this method, as Lax points out, is that one must "guess" a suitable A and then find an B in order to satisfy equations (1.18),(1.19). As an alternative, Ablowitz, Kaup, Newell and Segur [AKNS74] proposed a technique which, very generally, can be formulated as follows.

Consider two *linear* equations

$$\partial_{x_1}\,\mu(x_1,x_2) = X_1\,\mu(x_1,x_2) \tag{1.21}$$
$$\partial_{x_2}\,\mu(x_1,x_2) = X_2\,\mu(x_1,x_2)\,. \tag{1.22}$$

Cross differentiation yields

$$\partial_{x_2}\,X_1 - \partial_{x_1}\,X_2 + \big[X_1,\ X_2\big] = 0\,. \tag{1.23}$$

This is, in essence, the equivalent of (1.20). Given X_1, it turns out there is a simple procedure to find X_2 such that (1.23) contains a nonlinear equation. However, in order for (1.23) to be effective, the operator X_1 should include a (time-independent) parameter which plays the role of an eigenvalue.

Remark 1.1.1 *In the case where the vector field* $\mathbf{X} = (X_1, X_2)$ *is irrotational, i.e.* $\nabla\times\mathbf{X} = 0$, *equation* (1.23) *simplifies as*

$$\big[X_1,\ X_2\big] = 0\,,$$

viz. the operators X_1 *and* X_2 *commute.*

However, the ingenious method introduced by Lax was adopted a few decades later by Fokas and Gelfand [FG94] who proved that every linear equation has at least one Lax pair. The importance is that the Lax pair technique in contradiction for nonlinear equations can *always* be applied to linear PDE's and provides a new point of view in dealing with linearity and separability [Fok09].

1.2 The Riemann-Hilbert formulation

In what follows, a short survey on the Riemann-Hilbert problems is presented. Missing details are found in the standard references of F.D. Gakhov [Gak90], N.I. Muskhelishvili [Mus53] and N.P. Vekua [Vek67].

1.2.1 Historical notes

In his 1857 paper "Theorie der Abel'schen Functionen" Bernhard Riemann [Rie57] first posed the problem of finding an analytic function, given a relation of the boundary values, in a certain domain. This problem became known as the Riemann problem (sometimes also referred as the linear conjugation problem). A few years later, David Hilbert studied the problem in more details, which in modern days is known as the Riemann-Hilbert problem. A particular aspect of the Riemann-Hilbert problem, namely the existence of a Fuchsian system with given singularities and a given monodromy group, was addressed by Hilbert, among other problems, as the twenty-first problem at the Paris conference of the International Congress for Mathematicians in 1900 [Hil02]* (see also [Gak90, p.137]). Riemann-Hilbert problems are, moreover, associated with the notion of integrability of a system [Its03].

1.2.2 The Riemann-Hilbert problem

The Riemann-Hilbert problem can be stated, in a simplified form, in the following way: Find a sectionally analytic function $\Phi(z)$, which takes the values $\Phi^{\pm}(z)$ for $z \in \Omega^{\pm}$, vanishes at infinity and undergoes a jump $\varphi(t)$, viz satisfies the condition

$$\Phi^{+}(\tau) - \Phi^{-}(\tau) = \varphi(\tau)\,,$$

passing through an oriented simple contour $\mathcal{C}$ in the complex plane. The solution to this problem is closely related to the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\varphi(\tau)}{\tau - z}\,\mathrm{d}\tau\,. \tag{1.24}$$

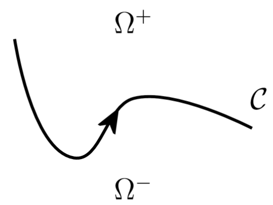

FIGURE 1.4: *The regions Ω^+ and Ω^- on either side of the contour $\mathcal{C}$*

Dropping the additional condition $\Phi(z) \to 0$ as z tends to ∞, the solution of the problem is given by the formula

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\varphi(\tau)}{\tau - z}\,\mathrm{d}\tau + \text{const.}$$

The above formula indicates that an appropriate limit (e.g. at infinity) reduces the number of solutions.

When z approaches $\mathcal{C}$ along a path entirely in Ω^+, $\Phi(z)$ has a limit $\Phi^+(\tau)$. Similarly, $\Phi(z)$ has a limit $\Phi^-(\tau)$ in the case where z approaches $\mathcal{C}$ along a path entirely in Ω^-. These limits are given by the Sokhotski formulae†

$$\Phi^{\pm}(\tau) = \pm\frac{1}{2}\varphi(\tau) + \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\varphi(t)}{t - \tau}\,\mathrm{d}t\,. \tag{1.25}$$

If the contour $\mathcal{C}$ displays a corner point, depicted in Figure 1.5, the above Sokhotski

*The original speech in German, "Mathematische Probleme", can be found at `http://www.mathematik.uni-bielefeld.de/~kersten/hilbert/rede.html`. There also exist a radio speech of Hilbert recorded in Königsberg in 1930.

†also known as the Sokhotski-Plemelj formulae.

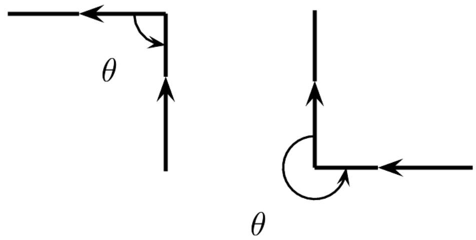

FIGURE 1.5: *Contour with corner point*

formulae are reformulated as follows

$$\Phi^{+}(\tau) = \left(1 - \frac{\theta}{2\pi}\right)\varphi(\tau) + \frac{1}{2\pi i}\int\limits_{\mathcal{C}} \frac{\varphi(t)}{t-\tau}\,\mathrm{d}t, \tag{1.26}$$

$$\Phi^{-}(\tau) = -\frac{\theta}{2\pi}\varphi(\tau) + \frac{1}{2\pi i}\int\limits_{\mathcal{C}} \frac{\varphi(t)}{t-\tau}\,\mathrm{d}t\,. \tag{1.27}$$

CHAPTER 2

Harmonic functions in rectangular domains*

2.1 Introduction

In most cases, a given, well-posed, boundary-value problem can be solved by means of separation of variables, if there exist a coordinate system that fits the boundary of the fundamental domain and at the same time it separates the partial differential equation (PDE). Furthermore, separation of variables leads to the solution of PDE's by a transform pair. The "prototype" of such a pair is the Fourier transform. However, for complicated problems the classical transform method fails. For example, there do not exist proper transforms for solving many boundary-value problems for elliptic equations of second order and in simple domains.

In 1997, A.S. Fokas [Fok97, Fok01, FK03] proposed a general method for solving boundary-value problems for two-dimensional linear and integrable nonlinear PDE's. An equation in two dimensions (x_1, x_2) is called integrable if it can be expressed as the condition that a certain differential 1-form $W(x_1, x_2; k)$, $k \in \mathbb{C}$, is closed, e.g. linear PDE's with constant coefficients. This novel approach can be seen as a generalization of the separation of variables method, but more effectively (for a review see [Das07b]). It is based on the *simultaneous* spectral analysis of the two compatible equations of the Lax pair associated with the PDE, i.e. construct two scalar linear equations whose compatibility condition is the given PDE. In general, one of this equations defines an eigenvalue problem and the other is an evolution equation. The method expresses the solution in terms of the solution of a matrix Riemann-Hilbert problem in the complex plane of the spectral parameter k. The spectral functions $\rho(k)$ determining the Riemann-Hilbert problem are given in terms of the boundary values of the solution. Since for a well posed boundary-value problem only one boundary condition is prescribed, some of the boundary values appearing in $\rho(k)$ are unknown. The fact that these boundary values are in general related can be expressed

*This work has been published as [Dosa]

in a simple way in terms of a global relation, which plays a crucial role in the analysis of boundary-value problems, satisfied by the corresponding spectral functions.

More recently, Fokas and Novokshenov [FN] have shown that by algebraic manipulation of the so-called global relation in specific subdomains it is possible to rederive the classical transform for certain elliptic PDE's. Moreover, the new method provides an alternative approach deriving this transform and also yields integral representations useful for solving changing-type BVP's and, since this integral representations involve a strong decay as k tends to ∞, they are suitable for numerical computations and for the study of the asymptotic properties of the solutions.

A question at hand concerns the kind of domain one should choose for this comparison. Obviously, it depends on the coordinate system one is interested in. Let us focus on the Cartesian coordinate system. Bearing this in mind, the domains in which one can implement *both* techniques, are the rectangles. The simplest rectangle is the Square.

The present chapter is organized as follows. In section 2.2, a brief introduction of the Fokas method applied to the case of a Square is given. In the sequence, in order to fix notation and terminology, the classical transform is presented, which is then rederived in section 2.6, by means of the analysis of the global relation. In the second part, consisting of sections 2.7 and 2.8, the new method is implemented to derive alternative formulae for the solution in terms of an integral instead of a series. This is realized by algebraic manipulation of the global relation in appropriate subdomains of the Square. Moreover, the machinery introduced is utilized in section 2.9 to solve a changing-type boundary value problem. In the latter case, one must combine the new method with the Riemann-Hilbert formulation.

2.1.1 Formulation of the Problem

The two dimensional Laplace equation in Cartesian coordinates, namely

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)q(x,y)=0,\quad (x,y)\in\Omega\,, \tag{2.1}$$

in the interior Ω of a Square defined by

$$\Omega=\Big\{-L\leq x\leq L,\ -L\leq y\leq L\Big\} \tag{2.2}$$

and depicted in Figure 2.1, where $q(x,y)$ is a real valued function, is investigated.

We analyze the general Dirichlet problem

$$q(L,y)=f_D^{(1)}(y),\quad q(x,-L)=f_D^{(2)}(x),\quad q(-L,y)=f_D^{(3)}(y),\quad q(x,L)=f_D^{(4)}(x) \tag{2.3}$$

which, after a suitable parametrization, becomes

$$q^{(j)}(s)=f_D^{(j)}(s)\quad s\in[-L,L],\quad j=1,2,3,4\,, \tag{2.4}$$

where (j) corresponds to the jth side of the Square.
We assume that the functions $f_D^{(j)}$ are smooth and compatible at the corners of the Square.

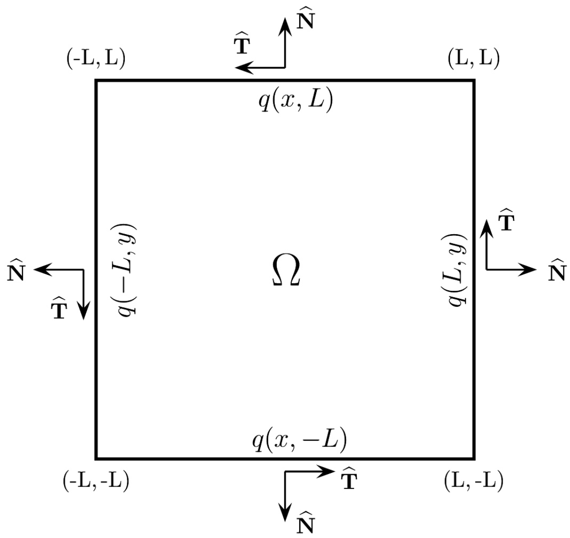

FIGURE 2.1: *The domain* $\Omega = \big\{ -L \leq x \leq L,\, -L \leq y \leq L \big\}$

The general Neumann problem can be treated in the same manner, where, furthermore, the Neumann data have to satisfy the compatibility condition

$$\oint_{\partial\Omega} \left(-\frac{\partial q}{\partial y}\,\mathrm{d}x + \frac{\partial q}{\partial x}\,\mathrm{d}y \right) = 0\,,$$

and $\partial\Omega$ is the boundary of the domain.
Throughout the analysis presented, emanating from the linearity of the Laplacian operator, the fact that the solution $q(x, y)$ can be written as a linear combination of "partial solutions" $q_j(x, y)$, corresponding to specific subproblems, namely particular boundary conditions, is applied.

2.2 The Generalized Transform Method. A Brief Introduction

For the Square, equation (1.3) deduces to

$$\int_{z_2}^{z_1} e^{-ikz}\,\partial_z\, q^{(1)}(z)\,\mathrm{d}z + \int_{z_3}^{z_2} e^{-ikz}\,\partial_z\, q^{(2)}(z)\,\mathrm{d}z + \int_{z_4}^{z_3} e^{-ikz}\,\partial_z\, q^{(3)}(z)\,\mathrm{d}z \\ + \int_{z_1}^{z_4} e^{-ikz}\,\partial_z\, q^{(4)}(z)\,\mathrm{d}z = 0, \quad k \in \mathbb{C}, \quad z \in \Omega \subset \mathbb{C}\,,$$

where the complex numbers $z_1 = L + iL$, $z_2 = L - iL$, $z_3 = -L - iL$, $z_4 = -L + iL$ denote the vertices of the Square and (j) corresponds to the side (z_{j+1}, z_j), $j = 1, 2, 3, 4$, $z_5 = z_1$.
Rewrite the foregoing expression as

$$\sum_{j=1}^{4} \rho_j(k) = 0, \quad k \in \mathbb{C} \tag{2.5}$$

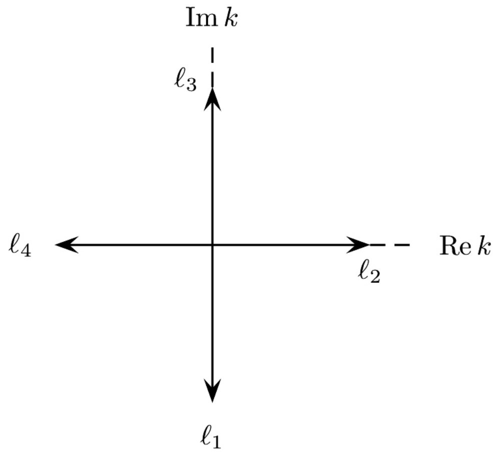

FIGURE 2.2: *The rays ℓ_j associated with the Square*

where

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\,\frac{\partial q^{(j)}}{\partial z}\,\mathrm{d}z, \quad z_5 = z_1,\ j = 1,2,3,4\,. \tag{2.6}$$

Equation (2.5) is the so-called global relation for the particular case and the functions $\{\rho_j(k)\}_{j=1}^4$ are called the spectral functions.
Introducing the local coordinate system $(\hat{\mathbf{T}}, \hat{\mathbf{N}})$ on each side of the Square, as shown in Figure 2.1, we obtain

$$\frac{\partial q^{(j)}}{\partial z}\,\mathrm{d}z = \frac{1}{2}\Big(\partial_T\, q^{(j)}(s) + i\,\partial_N\, q^{(j)}(s)\Big)\mathrm{d}s, \quad s \in [-L, L], \quad j = 1,2,3,4$$

where $\partial_T q^{(j)}(s)$ is the derivative of the solution along the boundary and $\partial_N q^{(j)}(s)$ is the derivative of the solution normal to the boundary of the jth side of the Square.
Substituting the latter into (2.6) yields

$$\rho_j(k) = \frac{1}{2}\Big(G^{(j)}(-ik) + i\,\Psi^{(j)}(-ik)\Big)\,,$$

where

$$G^{(j)}(-ik) = \int_{-L}^{L} e^{-ikz^{(j)}(s)}\,\partial_T\, q^{(j)}(s)\,\mathrm{d}s, \quad \Psi^{(j)}(-ik) = \int_{-L}^{L} e^{-ikz^{(j)}(s)}\,\partial_N\, q^{(j)}(s)\,\mathrm{d}s$$

and $z^{(j)}(s)$ a suitable parametrization for each side (j) of the Square.
Following the analysis, the solution is obtained from the reality condition

$$q(x,y) = 2\mathrm{Re}\int_{z_0}^{z}\frac{\partial q}{\partial z}\,\mathrm{d}z + \text{const.} \tag{2.7}$$

where

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi} \sum_{j=1}^{4} \int_{\ell_j} e^{ikz} \rho_j(k)\, \mathrm{d}k \tag{2.8}$$

and the rays ℓ_j are defined by (1.7) and depicted in Figure 2.2.

2.2.1 The Global Relation

The so-called global relation, i.e. an integral relation connecting the boundary values of the solution (Dirichlet data) with the normal derivative of the solution on the boundary (Neumann data), for the particular case of the Square becomes,

$$e^{-ikL}\, \Psi^{(1)}(k) + e^{-kL}\, \Psi^{(2)}(-ik) + e^{ikL}\, \Psi^{(3)}(-k) + e^{kL}\, \Psi^{(4)}(ik) = i\, \mathcal{G}(k)\,, \tag{2.9}$$

where

$$\mathcal{G}(k) = e^{-ikL}\, G^{(1)}(k) + e^{-kL}\, G^{(2)}(-ik) + e^{ikL}\, G^{(3)}(-k) + e^{kL}\, G^{(4)}(ik)$$

and $\Psi^{(j)}(k)$, $G^{(j)}(k)$ are the following transforms of the Neumann and Dirichlet boundary data

$$\Psi^{(j)}(k) = \int_{-L}^{L} e^{ks}\, \partial_N\, q^{(j)}(s)\, \mathrm{d}s, \quad G^{(j)}(k) = \int_{-L}^{L} e^{ks}\, \partial_T\, q^{(j)}(s)\, \mathrm{d}s, \quad j = 1, 2, 3, 4, \quad k \in \mathbb{C}$$

respectively.

2.3 The Classical Transform

When we apply the classical transform we assume the solution expanded in a series of eigenfunctions of one of the variables, with the coefficient depending upon the other variable. Separation of variables relies upon the completeness of the eigenfunctions corresponding to one of the variables. The solution will depend on functions which enter into the boundary conditions, and since the spatial domain Ω is rectangular, the relative eigenfunctions are trigonometric.

Furthermore, every function can be written uniquely as the sum of an even and an odd function, or in terms of a Fourier expansion, every function, satisfying Dirichlet's conditions, which enters into the boundary conditions can be written as

$$f(s) \sim \sum_n \left[\alpha_n \sin\left(\tfrac{n\pi}{L} s\right) + \beta_n \cos\left(\left(n + \tfrac{1}{2}\right) \tfrac{\pi}{L} s\right)\right], \quad s \in [-L, L] \tag{2.10}$$

where the set $\mathcal{S} = \{1\} \cup \{\sin \frac{n\pi}{L} s,\, n \in \mathbb{N} - \{0\}\} \cup \{\cos\left(n + \frac{1}{2}\right) \frac{\pi}{L} s,\, n \in \mathbb{Z}\}$ form a complete orthogonal basis of $L_2[-L, L]$.

Proposition 2.3.1 *Let the real valued function $q(x, y)$ satisfy the Laplace equation* (2.1) *in the domain Ω defined in* (2.2)*, with boundary conditions* (2.4)*, where the given functions $f_D^{(j)}(s)$, $j = 1, 2, 3, 4$ have sufficient smoothness and are continuous at the vertices. Then the* *classical representation for the solution is given by*

$$
\begin{aligned}
q(x,y) = \sum_{n=1}^{\infty} & \left[a_n \sinh\left(\tfrac{n\pi}{L}(x+L)\right) + c_n \sinh\left(\tfrac{n\pi}{L}(x-L)\right) \right] \sin\left(\tfrac{n\pi}{L} y\right) \\
+ \sum_{n=0}^{\infty} & \left[b_n \sinh\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}(x+L)\right) + d_n \sinh\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}(x-L)\right) \right] \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L} y\right) \\
+ \sum_{n=1}^{\infty} & \left[e_n \sinh\left(\tfrac{n\pi}{L}(y-L)\right) + g_n \sinh\left(\tfrac{n\pi}{L}(y+L)\right) \right] \sin\left(\tfrac{n\pi}{L} x\right) \\
+ \sum_{n=0}^{\infty} & \left[f_n \sinh\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}(y-L)\right) + h_n \sinh\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}(y+L)\right) \right] \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L} x\right),
\end{aligned}
\tag{2.11}
$$

where, by introducing a intrinsic coordinate system $(\hat{\mathbf{T}},\, \hat{\mathbf{N}})$ *on each side of the Square, the Fourier coefficients* a_n, b_n, c_n, d_n, e_n, f_n, g_n *and* h_n *can be expressed as follows*

$$a_n = \frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(1)}(s) \sin\left(\tfrac{n\pi}{L} s\right) \,\mathrm{d}s \tag{2.12}$$

$$b_n = \frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(1)}(s) \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L} s\right) \mathrm{d}s \tag{2.13}$$

$$c_n = \frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(3)}(-s) \sin\left(\tfrac{n\pi}{L} s\right) \,\mathrm{d}s \tag{2.14}$$

$$d_n = -\frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(3)}(-s) \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L} s\right) \mathrm{d}s \tag{2.15}$$

$$e_n = -\frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(2)}(s) \sin\left(\tfrac{n\pi}{L} s\right) \,\mathrm{d}s \tag{2.16}$$

$$f_n = -\frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(2)}(s) \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L} s\right) \mathrm{d}s \tag{2.17}$$

$$g_n = -\frac{1}{L\sinh(2n\pi)} \int_{-L}^{L} f_D^{(4)}(-s) \sin\left(\tfrac{n\pi}{L} s\right) \,\mathrm{d}s \tag{2.18}$$

$$h_n = \frac{1}{L\sinh(2n+1)\pi} \int_{-L}^{L} f_D^{(4)}(-s) \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L} s\right) \mathrm{d}s\,. \tag{2.19}$$

2.4 Analysis of the Global Relation

Replacing k with $-k$, the global relation (2.9)

$$\sum_{j=1}^{4} e^{(-i)^j\, kL}\, \Psi^{(j)}\left((-i)^{j-1}\, k\right) = i\, \mathcal{G}(k), \quad k \in \mathbb{C} \tag{2.20}$$

together with its Schwarz conjugate

$$\sum_{j=1}^{4} e^{i^j\, kL}\, \Psi^{(j)}\left(i^{j-1}\, k\right) = -i\, \overline{\mathcal{G}(\overline{k})}, \quad k \in \mathbb{C} \tag{2.21}$$

and

$$\mathcal{G}(k) = \sum_{j=1}^{4} e^{(-i)^j\, kL}\, G^{(j)}\Big((-i)^{j-1}\, k\Big), \quad k \in \mathbb{C}\,,$$

become

$$\sum_{j=1}^{4} e^{-(-i)^j\, kL}\, \Psi^{(j)}\Big(-(-i)^{j-1}\, k\Big) \;=\; i\,\mathcal{G}(-k), \quad k \in \mathbb{C}\,, \tag{2.22}$$

$$\sum_{j=1}^{4} e^{-i^j\, kL}\, \Psi^{(j)}\Big(-\,i^{j-1}\, k\Big) \;=\; -i\,\overline{\mathcal{G}(-\bar{k})}, \quad k \in \mathbb{C}\,, \tag{2.23}$$

where $\overline{\mathcal{G}(\bar{k})}$ denotes the Schwarz conjugate of the function $\mathcal{G}(k)$.
By simple algebraic manipulations the above expressions can be combined to give,

$$\begin{aligned} &-i\sin kL\left[\Big(\Psi^{(1)}(k)+\Psi^{(1)}(-k)\Big)-\Big(\Psi^{(3)}(k)+\Psi^{(3)}(-k)\Big)\right]\\ &-\cosh kL\Big[\Big(\Psi^{(2)}(ik)-\Psi^{(2)}(-ik)\Big)-\Big(\Psi^{(4)}(ik)-\Psi^{(4)}(-ik)\Big)\Big]=\frac{i}{2}\,\Gamma_1(k)\,, \end{aligned} \tag{2.24}$$

$$\begin{aligned} &-i\sin kL\Big[\Big(\Psi^{(1)}(k)-\Psi^{(1)}(-k)\Big)+\Big(\Psi^{(3)}(k)-\Psi^{(3)}(-k)\Big)\Big]\\ &+\sinh kL\Big[\Big(\Psi^{(2)}(ik)-\Psi^{(2)}(-ik)\Big)+\Big(\Psi^{(4)}(ik)-\Psi^{(4)}(-ik)\Big)\Big]=\frac{i}{2}\,\Gamma_2(k)\,, \end{aligned} \tag{2.25}$$

$$\begin{aligned} &\cos kL\Big[\Big(\Psi^{(1)}(k)-\Psi^{(1)}(-k)\Big)-\Big(\Psi^{(3)}(k)-\Psi^{(3)}(-k)\Big)\Big]\\ &-\sinh kL\Big[\Big(\Psi^{(2)}(ik)+\Psi^{(2)}(-ik)\Big)-\Big(\Psi^{(4)}(ik)+\Psi^{(4)}(-ik)\Big)\Big]=\frac{i}{2}\,\Gamma_3(k)\,, \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\cos kL\Big[\Big(\Psi^{(1)}(k)+\Psi^{(1)}(-k)\Big)+\Big(\Psi^{(3)}(k)+\Psi^{(3)}(-k)\Big)\Big]\\ &+\cosh kL\Big[\Big(\Psi^{(2)}(ik)+\Psi^{(2)}(-ik)\Big)+\Big(\Psi^{(4)}(ik)+\Psi^{(4)}(-ik)\Big)\Big]=\frac{i}{2}\,\Gamma_4(k)\,, \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} \Gamma_1(k) &= \Big(\mathcal{G}(k)+\overline{\mathcal{G}(\bar{k})}\Big)-\Big(\mathcal{G}(-k)+\overline{\mathcal{G}(-\bar{k})}\Big)\,,\\ \Gamma_2(k) &= \Big(\mathcal{G}(k)+\overline{\mathcal{G}(\bar{k})}\Big)+\Big(\mathcal{G}(-k)+\overline{\mathcal{G}(-\bar{k})}\Big)\,,\\ \Gamma_3(k) &= \Big(\mathcal{G}(k)-\overline{\mathcal{G}(\bar{k})}\Big)-\Big(\mathcal{G}(-k)-\overline{\mathcal{G}(-\bar{k})}\Big)\,,\\ \Gamma_4(k) &= \Big(\mathcal{G}(k)-\overline{\mathcal{G}(\bar{k})}\Big)+\Big(\mathcal{G}(-k)-\overline{\mathcal{G}(-\bar{k})}\Big)\,. \end{aligned}$$

The Dirichlet and Neumann problems can be solved by evaluating expressions (2.24)-(2.27) at discrete values of k. This yields the unknown boundary values in terms of infinite series. In particular, evaluating equations (2.24)-(2.27) at those values of k for which the coefficients of $\Psi^{(j)}(k) \pm \Psi^{(j)}(-k)\; j = 1, 3$ and $\Psi^{(j)}(ik) \pm \Psi^{(j)}(-ik)$, $j = 2, 4$ vanishes, we find

$$\int_{-L}^{L} \sin\left(\tfrac{n\pi}{L}s\right)\,\partial_{N}\,q^{(1)}(s)\,\mathrm{d}s = \frac{\Gamma_2(i\frac{n\pi}{L})}{8\sinh n\pi} + \frac{\Gamma_3(i\frac{n\pi}{L})}{8\cosh n\pi}\,, \tag{2.28}$$

$$\int_{-L}^{L} \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}s\right)\partial_{N}\,q^{(1)}(s)\,\mathrm{d}s = i\frac{\Gamma_1(i(n+\frac{1}{2})\frac{\pi}{L})}{8\sinh(n+\frac{1}{2})\pi} + i\frac{\Gamma_4(i(n+\frac{1}{2})\frac{\pi}{L})}{8\cosh(n+\frac{1}{2})\pi}\,, \tag{2.29}$$

$$\int_{-L}^{L} \sin\left(\tfrac{n\pi}{L}s\right)\,\partial_{N}\,q^{(2)}(s)\,\mathrm{d}s = -\frac{\Gamma_1(\frac{n\pi}{L})}{8\cosh n\pi} + \frac{\Gamma_2(\frac{n\pi}{L})}{8\sinh n\pi}\,, \tag{2.30}$$

$$\int_{-L}^{L} \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}s\right)\partial_{N}\,q^{(2)}(s)\,\mathrm{d}s = i\frac{\Gamma_4((n+\frac{1}{2})\frac{\pi}{L})}{8\cosh(n+\frac{1}{2})\pi} - i\frac{\Gamma_3((n+\frac{1}{2})\frac{\pi}{L})}{8\sinh(n+\frac{1}{2})\pi}\,, \tag{2.31}$$

$$\int_{-L}^{L} \sin\left(\tfrac{n\pi}{L}s\right)\,\partial_{N}\,q^{(3)}(s)\,\mathrm{d}s = \frac{\Gamma_2(i\frac{n\pi}{L})}{8\sinh n\pi} - \frac{\Gamma_3(i\frac{n\pi}{L})}{8\cosh n\pi}\,, \tag{2.32}$$

$$\int_{-L}^{L} \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}s\right)\partial_{N}\,q^{(3)}(s)\,\mathrm{d}s = -i\frac{\Gamma_1(i(n+\frac{1}{2})\frac{\pi}{L})}{8\sinh(n+\frac{1}{2})\pi} + i\frac{\Gamma_4(i(n+\frac{1}{2})\frac{\pi}{L})}{8\cosh(n+\frac{1}{2})\pi}\,, \tag{2.33}$$

$$\int_{-L}^{L} \sin\left(\tfrac{n\pi}{L}s\right)\,\partial_{N}\,q^{(4)}(s)\,\mathrm{d}s = \frac{\Gamma_1(\frac{n\pi}{L})}{8\cosh n\pi} + \frac{\Gamma_2(\frac{n\pi}{L})}{8\sinh n\pi}\,, \tag{2.34}$$

$$\int_{-L}^{L} \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}s\right)\partial_{N}\,q^{(4)}(s)\,\mathrm{d}s = i\frac{\Gamma_4((n+\frac{1}{2})\frac{\pi}{L})}{8\cosh(n+\frac{1}{2})\pi} + i\frac{\Gamma_3((n+\frac{1}{2})\frac{\pi}{L})}{8\sinh(n+\frac{1}{2}\pi)}\,. \tag{2.35}$$

Proposition 2.4.1 *Let the real valued function* $q(x,y)$ *satisfy the Laplace equation* (2.1) *in the domain* (2.2)*, with boundary conditions* (2.4)*, where the given functions* $f_{D}^{(j)}(s)$ *have sufficient smoothness and are continuous at the vertices. Then the Neumann data* $\partial_{N}q^{(j)}(s)$*,* $j=1,2,3,4$ *can be expressed in terms of the given Dirichlet data by the Fourier series*

$$\partial_{N}\,q^{(j)}(s) = \sum_{n=1}^{\infty}\left[\mathfrak{A}_n^{(j)}\sin\left(\tfrac{n\pi}{L}s\right) + \mathfrak{B}_n^{(j)}\cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}s\right)\right],\; j=1,2,3,4 \tag{2.36}$$

where the coefficients $\mathfrak{A}_n^{(j)}$ *and* $\mathfrak{B}_n^{(j)}$ *are given by equations* (2.28)*-*(2.35)*.*

The coefficients $\mathfrak{A}_n^{(j)}$ and $\mathfrak{B}_n^{(j)}$ can be correlated with the Fourier coefficients (2.12)-(2.19) through the known functions $\Gamma_j(k)$, e.g. for $j=1$ and $k_n = i(n+\frac{1}{2})\frac{\pi}{L}$ we obtain

$$\begin{aligned}\Gamma_1\left(i(n+\tfrac{1}{2})\tfrac{\pi}{L}\right) &= -4i\pi(n+\tfrac{1}{2})\,\cosh\left[(n+\tfrac{1}{2})\pi\right]\,\sinh\left[(2n+1)\pi\right](b_n+d_n)\\ &+ 8i(n+\tfrac{1}{2})(-1)^n\,\sinh\left[(n+\tfrac{1}{2})\pi\right]\sum_{m=1}^{\infty}(-1)^m\frac{m}{m^2+(n+\frac{1}{2})^2}\,\sinh(2m\pi)\,(e_m-g_m)\,.\end{aligned}$$

Finally, the solution is then given by the expression

$$q(x,y) = \mathrm{Re}\left\{\frac{1}{2i\pi}\sum_{j=1}^{4}\int_{0}^{\infty}\frac{\exp\left[\left(-i^{j+1}\,z-L\right)\,k\right]}{k}\left[G^{(j)}\left(-ik\right)+i\Psi^{(j)}\left(-ik\right)\right]\mathrm{d}k + \mathrm{const.}\right\}. \tag{2.37}$$

After long and tedious calculations (see for details Appendix A), (2.37) yields (2.11).

2.5 The Global Relation Revisited

Let $q(x,y)$ and $\overline{q}(x,y)$ satisfy the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)q(x,y)=0\,, \tag{2.38}$$

and the formal adjoint of the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\overline{q}(x,y)=0\,. \tag{2.39}$$

Multiplying (2.38) by $\overline{q}(x,y)$ and (2.39) by $q(x,y)$ and subtracting them, we obtain the divergence form

$$\frac{\partial}{\partial x}\left(\overline{q}\,\frac{\partial q}{\partial x}-q\,\frac{\partial \overline{q}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\overline{q}\,\frac{\partial q}{\partial y}-q\,\frac{\partial \overline{q}}{\partial y}\right)=0\,. \tag{2.40}$$

Equation (2.40) holds true everywhere in $\mathbb{R}^2$ and applying Green's theorem to a closed subdomain of $\mathbb{R}^2$, yields

$$\int_C\left[\left(\overline{q}\,\frac{\partial q}{\partial x}-q\,\frac{\partial \overline{q}}{\partial x}\right)\mathrm{d}y+\left(q\,\frac{\partial \overline{q}}{\partial y}-\overline{q}\,\frac{\partial q}{\partial y}\right)\mathrm{d}x\right]=0\,, \tag{2.41}$$

where C is the boundary of the subdomain.
Equation (2.41) provides the *global relation*, since it relates the boundary values of the solution with the values of the normal derivatives of the solution on the boundary.
Letting $\overline{q}(x,y;k)=\overline{X}(x;k)\overline{Y}(y;k)$ where k is the complex separation constant, it follows that $\overline{X}(x;k)$ and $\overline{Y}(y;k)$ satisfy the ODE's

$$\left.\begin{aligned}\overline{X}''+k^2\,\overline{X}&=0\\ \overline{Y}''-k^2\,\overline{Y}&=0\end{aligned}\right\},\quad k\in\mathbb{C}\,,$$

where the prime denotes differentiation with respect to the argument.
Solving the above ODE's yields $\overline{q}(x,y)=e^{\pm ikx}\,e^{\sigma ky}$, where $\sigma=\pm1$. Then, equations (2.40) and (2.41) become

$$\frac{\partial}{\partial x}\left[e^{\pm ikx}\,e^{\sigma ky}\left(\pm ikq-\frac{\partial q}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[e^{\pm ikx}\,e^{\sigma ky}\left(\sigma kq-\frac{\partial q}{\partial y}\right)\right]=0 \tag{2.42}$$

and

$$\int_C e^{\pm ikx}\,e^{\sigma ky}\left[\left(\pm ikq-\frac{\partial q}{\partial x}\right)\mathrm{d}y-\left(\sigma kq-\frac{\partial q}{\partial y}\right)\mathrm{d}x\right]=0,\quad k\in\mathbb{C}\,, \tag{2.43}$$

respectively. Equations (2.42) imply two items. First, applying Green's theorem we obtain *immediately* the global relation, and second it yields a Lax pair formulation.
Indeed, if q(x,y) is the solution of the Laplace equation in a closed subdomain $\Omega\subset\mathbb{R}^2$, then (2.42) implies the existences of a function $\Xi(x,y;k)$, such that

$$\left.\begin{aligned}\tfrac{\partial}{\partial y}\,\Xi&=e^{\pm ikx}\,e^{\sigma ky}\left(\pm ikq-\tfrac{\partial q}{\partial x}\right)\\ \tfrac{\partial}{\partial x}\,\Xi&=-e^{\pm ikx}\,e^{\sigma ky}\left(\sigma kq-\tfrac{\partial q}{\partial y}\right)\end{aligned}\right\},\quad k\in\mathbb{C}\,.$$

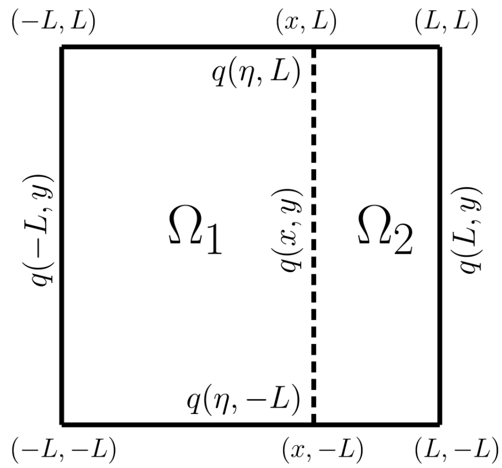

FIGURE 2.3: *The subdomains* $\Omega_1 \subset \Omega$ *and* $\Omega_2 \subset \Omega$ *defined as* $\Omega_1 = \{-L \le \eta \le x,\, |y| \le L\}$, $\Omega_2 = \{x \le \eta \le L,\, |y| \le L\}$, *respectively.*

The assumption $\Xi(x, y; k) = e^{\pm ikx}\, e^{\sigma ky}\, \mu(x, y; k)$ where $\mu(x, y; k)$ an auxiliary function, leads right away to the Lax pairs

$$\begin{aligned} \left(\frac{\partial}{\partial y} + \sigma k\right) \mu &= \pm ikq - \frac{\partial q}{\partial x}\,, \\ \left(\frac{\partial}{\partial x} \pm ik\right) \mu &= \frac{\partial q}{\partial y} - \sigma kq\,. \end{aligned}$$

Furthermore, (2.42) implies that if the differential form

$$W(x, y; k) = e^{\pm ikx}\, e^{\sigma ky} \left\{ \left(\pm ikq - \frac{\partial q}{\partial x}\right) \mathrm{d}y - \left(\sigma kq - \frac{\partial q}{\partial y}\right) \mathrm{d}x \right\}$$

is closed, viz

$$\mathrm{d}W(x, y; k) = e^{\pm ikx}\, e^{\sigma ky} \left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right) \mathrm{d}x \wedge \mathrm{d}y = 0\,,$$

then Stoke's theorem (1.2) provides (2.43).

2.6 The Classical Representation

To rederive the classical transform (2.11), apply the global relation (2.43) in the subdomains Ω_1 and Ω_2 defined by

$$\Omega_1 = \Big\{ -L \le \eta \le x,\, |y| \le L \Big\}, \quad \Omega_2 = \Big\{ x \le \eta \le L,\, |y| \le L \Big\}$$

and depicted in Figure 2.3, with the following boundary conditions

$$\left.\begin{array}{ll} q(L, y) = f_{D}^{(1)}(y), & q(-L, y) = f_{D}^{(3)}(y) \\ q(\eta, L) = q(\eta, -L) = 0, & \partial_y\, q(\eta, L) = \partial_y\, q(\eta, -L) = 0 \end{array}\right\}, \tag{2.44}$$

where the functions $f_D^{(1)}(y)$ and $f_D^{(3)}(y)$ have sufficient smoothness and the sum of the integrals of $f_N^{(1)}(y)$ and $f_N^{(3)}(y)$ vanishes.
Thus we derive the following equations

$$\begin{aligned}&\int_{-L}^{L} e^{\sigma k y}\Big(\pm ik\, q_1(x,y) - \partial_x\, q_1(x,y)\Big)\,\mathrm{d}y\\ &= e^{\mp ik(x+L)}\int_{-L}^{L} e^{\sigma k y}\Big(\pm ik\, q(-L,y) - \partial_x\, q(-L,y)\Big)\,\mathrm{d}y, \quad k\in\mathbb{C}, \quad (x,y)\in\Omega_1\,, \end{aligned} \tag{2.45}$$

$$\begin{aligned}&\int_{-L}^{L} e^{\sigma k y}\Big(\pm ik q_1(x,y) - \partial_x\, q_1(x,y)\Big)\,\mathrm{d}y\\ &= e^{\mp ik(x-L)}\int_{-L}^{L} e^{\sigma k y}\Big(\pm ik\, q(L,y) - \partial_x\, q(L,y)\Big)\,\mathrm{d}y, \quad k\in\mathbb{C}, \quad (x,y)\in\Omega_2\,, \end{aligned} \tag{2.46}$$

where $q_1(x,y)$ the solution corresponding to the specific boundary conditions (2.44). To eliminate the unknown function $\partial_x q_1(x,y)$, subtract equations $(2.45)^+$ and $(2.46)^-$

$$\begin{aligned}\int_{-L}^{L} e^{\sigma k y} q_1(x,y)\,\mathrm{d}y = \frac{1}{2ik}\Bigg[&e^{ik(x-L)}\int_{-L}^{L} e^{\sigma k y}\Big(ik\, q(L,y) + \partial_x\, q(L,y)\Big)\,\mathrm{d}y\\ &+ e^{-ik(x+L)}\int_{-L}^{L} e^{\sigma k y}\Big(ikq(-L,y) - \partial_x\, q(-L,y)\Big)\,\mathrm{d}y\Bigg], \; k\in\mathbb{C}-\{0\}\,. \end{aligned} \tag{2.47}$$

Using boundary conditions (2.44) and denoting

$$\mathfrak{D}^{(j)}(\sigma k) = \int_{-L}^{L} e^{\sigma k y} f_D^{(j)}(y)\mathrm{d}y, \qquad \mathfrak{N}^{(j)}(\sigma k) = \int_{-L}^{L} e^{\sigma k y} f_N^{(j)}(y)\mathrm{d}y, \quad j=1,3\,, \tag{2.48}$$

where the unknown Neumann boundary values are defined as

$$\frac{\partial q}{\partial n}\Big|_{x=x_{\max},\, x_{\min}} = f_N^{(j)}(y), \quad j=1,3$$

and $\hat{\mathbf{n}}$ is the outgoing normal to the boundary, equation (2.47) rewrites

$$\begin{aligned}\int_{-L}^{L} e^{\sigma k y} q_1(x,y)\,\mathrm{d}y = \frac{1}{2ik}\Bigg[&e^{ik(x-L)}\Big(ik\,\mathfrak{D}^{(1)}(\sigma k) + \mathfrak{N}^{(1)}(\sigma k)\Big)\\ &+ e^{-ik(x+L)}\Big(ik\,\mathfrak{D}^{(3)}(\sigma k) + \mathfrak{N}^{(3)}(\sigma k)\Big)\Bigg], \; k\in\mathbb{C}-\{0\}\,. \end{aligned} \tag{2.49}$$

In order to compute the two unknowns $\mathfrak{N}^{(1)}(\sigma k)$ and $\mathfrak{N}^{(3)}(\sigma k)$, apply the global relation (2.43) in the domain Ω depicted in Figure 2.1, with boundary conditions (2.44) to derive the Dirichlet-to-Neumann correspondence,

$$e^{\pm ikL}\Big(\pm ik\,\mathfrak{D}^{(1)}(\sigma k) - \mathfrak{N}^{(1)}(\sigma k)\Big) - e^{\mp ikL}\Big(\pm ik\,\mathfrak{D}^{(3)}(\sigma k) + \mathfrak{N}^{(3)}(\sigma k)\Big) = 0, \quad k\in\mathbb{C}\,.$$

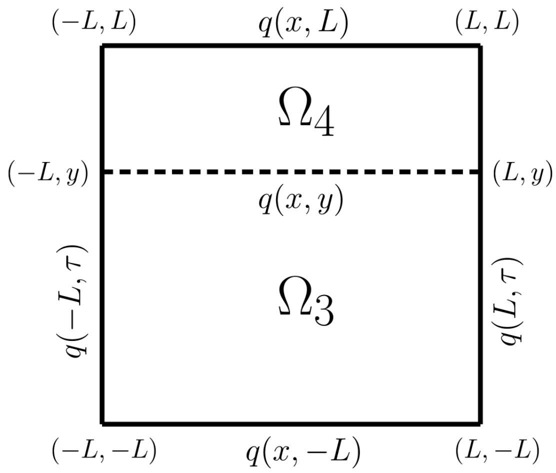

FIGURE 2.4: *The subdomains* Ω_3 *and* Ω_4 *defined as* $\Omega_3 = \{|x| \leq L \ \ -L \leq \tau \leq y\}$ *and* $\Omega_4 = \{|x| \leq L,\ y \leq \tau \leq L\}$, *respectively.*

Solving the above system with respect to the unknown Neumann data and substituting the resulting expressions into (2.49) we obtain

$$\int_{-L}^{L} e^{\sigma k y} q_1(x,y)\,\mathrm{d}y = \frac{1}{e^{i2kL} - e^{-i2kL}} \Bigg[\Big(e^{ik(x+L)} - e^{-ik(x+L)} \Big) \mathfrak{D}^{(1)}(\sigma k) - \Big(e^{ik(x-L)} - e^{-ik(x-L)} \Big) \mathfrak{D}^{(3)}(\sigma k) \Bigg],\ k \in \mathbb{C} - \left\{ \frac{n\pi}{2L} \right\},\ n \in \mathbb{Z}\,. \tag{2.50}$$

Replacing $\sigma = 1$ and $\sigma = -1$ in the above equation respectively, and performing simple algebraic manipulations of the resulting two equations, we derive the relations

$$\int_{-L}^{L} \begin{matrix}\cosh\\ \sinh\end{matrix}\,(ky)\, q_1(x,y)\,\mathrm{d}y = \frac{1}{\sin(2kL)} \Bigg[\sin\Big(k(x+L)\Big) \int_{-L}^{L} \begin{matrix}\cosh\\ \sinh\end{matrix}\,(ky)\, f_D^{(1)}(y)\,\mathrm{d}y - \sin\Big(k(x-L)\Big) \int_{-L}^{L} \begin{matrix}\cosh\\ \sinh\end{matrix}\,(ky)\, f_D^{(3)}(y)\,\mathrm{d}y \Bigg],\ k \in \mathbb{C} - \left\{ \frac{n\pi}{2L} \right\},\ n \in \mathbb{Z}\,. \tag{2.51}$$

Evaluating equations (2.51) at $k = i(n+\frac{1}{2})\frac{\pi}{L}$ and at $k = i\frac{n\pi}{L}$, yields the cosine and sine Fourier transform of $q_1(x,y)$, respectively. The inversion formulae then gives

$$q_1^c(x,y) = \sum_{n=0}^{\infty} \Bigg[b_n \sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(x+L) \Big) + d_n \sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(x-L) \Big) \Bigg] \cos\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L} y \Big) \tag{2.52}$$

and

$$q_1^s(x,y) = \sum_{n=1}^{\infty} \Bigg[a_n \sinh\Big(\tfrac{n\pi}{L}(x+L) \Big) + c_n \sinh\Big(\tfrac{n\pi}{L}(x-L) \Big) \Bigg] \sin\Big(\tfrac{n\pi}{L} y \Big), \tag{2.53}$$

where the Fourier constants a_n, b_n, c_n, d_n are given by equations (2.12)-(2.15). Analogous, applying the global relation (2.43)$^-$ in the subdomains

$$\Omega_3 = \Big\{|x| \le L,\ -L \le \tau \le y\Big\}$$

and

$$\Omega_4 = \Big\{|x| \le L,\ y \le \tau \le L\Big\},$$

depicted in Figure 2.4, with the following boundary conditions

$$\left.\begin{array}{cc} q(x,-L) = f_D^{(2)}(x), & q(x,L) = f_D^{(4)}(x) \\ q(L,\tau) = q(-L,\tau) = 0, & \partial_x\, q(L,\tau) = \partial_x\, q(-L,\tau) = 0 \end{array}\right\}, \tag{2.54}$$

we find the following equations

$$\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q_2(x,y) - \partial_y\, q_2(x,y)\Big)\mathrm{d}x$$
$$= e^{-\sigma k(y+L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,-L) - \partial_y\, q(x,-L)\Big)\mathrm{d}x\,,\quad k\in\mathbb{C},\quad (x,y)\in\Omega_3\,, \tag{2.55}$$

$$\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q_2(x,y) - \partial_y\, q_2(x,y)\Big)\mathrm{d}x$$
$$= e^{-\sigma k(y-L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,L) - \partial_y\, q(x,L)\Big)\mathrm{d}x\,,\quad k\in\mathbb{C},\quad (x,y)\in\Omega_4\,, \tag{2.56}$$

where $q_2(x,y)$ is the solution corresponding to the boundary conditions (2.54). In order to eliminate the unknown function $\partial_y q_2(x,y)$, subtract (2.55) evaluated for $\sigma = 1$ and (2.56) evaluated for $\sigma = -1$

$$\int_{-L}^{L} e^{-ikx}\, q_2(x,y)\,\mathrm{d}x = \frac{1}{2k}\Bigg[e^{-k(y+L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,-L) - \partial_y\, q(x,-L)\Big)\mathrm{d}x$$
$$+ e^{k(y-L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,L) - \partial_y\, q(x,L)\Big)\mathrm{d}x\Bigg],\quad k\in\mathbb{C}-\{0\}\,. \tag{2.57}$$

Using boundary conditions (2.54) and denoting

$$\mathfrak{D}^{(j)}(-ik) = \int_{-L}^{L} e^{-ikx} f_D^{(j)}(x)\mathrm{d}x,\quad \mathfrak{N}^{(j)}(-ik) = \int_{-L}^{L} e^{-ikx} f_N^{(j)}(x)\mathrm{d}x,\quad j=2,4\,, \tag{2.58}$$

where the unknown Neumann boundary values are defined as

$$\left.\frac{\partial q}{\partial n}\right|_{y=y_{\min},\, y_{\max}} = f_N^{(j)}(x),\quad j=2,4$$

and $\hat{\mathbf{n}}$ is the outgoing normal to the boundary, equation (2.57) rewrites

$$\int_{-L}^{L} e^{-ikx} q_2(x,y)\mathrm{d}x = \frac{1}{2k}\Bigg[e^{-k(y+L)}\Big(k\mathfrak{D}^{(2)}(-ik) + \mathfrak{N}^{(2)}(-ik)\Big)$$
$$+ e^{k(y-L)}\Big(k\mathfrak{D}^{(4)}(-ik) + \mathfrak{N}^{(4)}(-ik)\Big)\Bigg],\ k\in\mathbb{C}-\{0\}\,. \tag{2.59}$$

To compute the unknowns $\mathfrak{N}^{(2)}(-ik)$ and $\mathfrak{N}^{(4)}(-ik)$, apply (2.43)$^{-}$ in Ω with boundary conditions (2.54) to obtain

$$e^{-\sigma kL}\Big(\sigma k\mathfrak{D}^{(2)}(-ik)+\mathfrak{N}^{(2)}(-ik)\Big)-e^{\sigma kL}\Big(\sigma k\mathfrak{D}^{(4)}(-ik)-\mathfrak{N}^{(4)}(-ik)\Big)=0,\quad k\in\mathbb{C}\,. \tag{2.60}$$

Replacing $\sigma=1$ and $\sigma=-1$ in (2.60) respectively, we obtain two equations with unknowns the Fourier transforms of the Neumann data $\mathfrak{N}^{(j)}(-ik)$, $j=2,4$. Solving this system and substituting into (2.59) yields

$$\int_{-L}^{L}e^{-ikx}q_2(x,y)\mathrm{d}x=\frac{1}{e^{2kL}-e^{-2kL}}\Bigg[-\Big(e^{k(y-L)}-e^{-k(y-L)}\Big)\mathfrak{D}^{(2)}(-ik)$$
$$+\Big(e^{k(y+L)}-e^{-k(y+L)}\Big)\mathfrak{D}^{(4)}(-ik)\Bigg],\quad k\in\mathbb{C}-\Big\{i\frac{n\pi}{2L}\Big\},\quad n\in\mathbb{Z}\,. \tag{2.61}$$

Simple algebraic manipulations of the latter equation together with (2.61) with k replaced by $-k$, leads to

$$\int_{-L}^{L}\begin{matrix}\cos\\ \sin\end{matrix}\,(kx)\,q_2(x,y)\,\mathrm{d}x=\frac{1}{\sinh(2kL)}\Bigg[-\sinh\Big(k(y-L)\Big)\int_{-L}^{L}\begin{matrix}\cos\\ \sin\end{matrix}\,(kx)\,f_D^{(2)}(x)\,\mathrm{d}x$$
$$+\sinh\Big(k(y+L)\Big)\int_{-L}^{L}\begin{matrix}\cos\\ \sin\end{matrix}\,(kx)\,f_D^{(4)}(x)\,\mathrm{d}x\Bigg],\quad k\in\mathbb{C}-\Big\{i\frac{n\pi}{2L}\Big\},\quad n\in\mathbb{Z}\,. \tag{2.62}$$

Evaluating equations (2.62) at $k=(n+\frac{1}{2})\frac{\pi}{L}$ and at $k=\frac{n\pi}{L}$ yields the cosine and sine Fourier transform of $q_2(x,y)$ respectively. The inversion formulae then implies

$$q_2^c(x,y)=\sum_{n=0}^{\infty}\Bigg[f_n\sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(y-L)\Big)+h_n\sinh\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}(y+L)\Big)\Bigg]\cos\Big(\big(n+\tfrac{1}{2}\big)\tfrac{\pi}{L}x\Big) \tag{2.63}$$

and

$$q_2^s(x,y)=\sum_{n=1}^{\infty}\Bigg[e_n\sinh\Big(\tfrac{n\pi}{L}(y-L)\Big)+g_n\sinh\Big(\tfrac{n\pi}{L}(y+L)\Big)\Bigg]\sin\big(\tfrac{n\pi}{L}x\big)\,, \tag{2.64}$$

where the Fourier constants e_n, f_n, g_n, h_n are given by equations (2.16)-(2.19).
Adding equations (2.52),(2.53),(2.63) and (2.64) yields the classical transform (2.11).

2.7 Novel Integral Formulae

Proposition 2.7.1 *Let $q(x,y)$ satisfy the Laplace equation* (2.1) *in the interior Ω of the Square defined by*

$$\Omega=\Big\{|x|\le L,\ |y|\le L\Big\}$$

and with boundary conditions specified in (2.3). *Then* $q(x, y)$ *admits the following integral representation*

$$\begin{aligned} q(x,y) = &- \frac{1}{2i\pi} \int_{\mathcal{L}} e^{(y+L)k} \, \frac{\sin k(x-L)\, \mathcal{N}^{(3)}(k) - \sin k(x+L)\, \mathcal{N}^{(1)}(k)}{\sin 2kL} \, \mathrm{d}k \\ &+ \frac{1}{2i\pi} \int_{\mathcal{R}} e^{(y-L)k} \, \frac{\sin k(x+L)\, \mathcal{M}^{(1)}(k) - \sin k(x-L)\, \mathcal{M}^{(3)}(k)}{\sin 2kL} \, \mathrm{d}k \\ &- \frac{1}{2\pi} \int_{\mathcal{U}} e^{i(x+L)k} \, \frac{\sinh k(y+L)\, \mathcal{N}^{(4)}(k) - \sinh k(y-L)\, \mathcal{N}^{(2)}(k)}{\sinh 2kL} \, \mathrm{d}k \\ &+ \frac{1}{2\pi} \int_{\mathcal{D}} e^{i(x-L)k} \, \frac{\sinh k(y+L)\, \mathcal{M}^{(4)}(k) - \sinh k(y-L)\, \mathcal{M}^{(2)}(k)}{\sinh 2kL} \, \mathrm{d}k \,, \end{aligned} \tag{2.65}$$

where the functions $\mathcal{N}^{(j)}(k), \mathcal{M}^{(j)}(k), j = 1, 2, 3, 4$ *are defined as*

$$\mathcal{N}^{(j)}(k) = \sum_n (-1)^n \left(\alpha_n^{(j)} \, \frac{\frac{n\pi}{L}}{k^2 + \frac{n^2\pi^2}{L^2}} + \beta_n^{(j)} \, \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 + (n+\frac{1}{2})^2 \frac{\pi^2}{L^2}} \right) \tag{2.66}$$

$$\mathcal{M}^{(j)}(k) = \sum_n (-1)^n \left(\beta_n^{(j)} \, \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 + (n+\frac{1}{2})^2 \frac{\pi^2}{L^2}} - \alpha_n^{(j)} \, \frac{\frac{n\pi}{L}}{k^2 + \frac{n^2\pi^2}{L^2}} \right) , \tag{2.67}$$

for every $k \in \mathbb{C} - \left\{ \pm i\frac{n\pi}{L}, \pm i(n+\frac{1}{2})\frac{\pi}{L} \right\}$, *if* $j = 1, 3$, *and*

$$\mathcal{N}^{(j)}(k) = \sum_n (-1)^n \left(\alpha_n^{(j)} \, \frac{\frac{n\pi}{L}}{k^2 - \frac{n^2\pi^2}{L^2}} + \beta_n^{(j)} \, \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 - (n+\frac{1}{2})^2 \frac{\pi^2}{L^2}} \right) \tag{2.68}$$

$$\mathcal{M}^{(j)}(k) = \sum_n (-1)^n \left(\alpha_n^{(j)} \, \frac{\frac{n\pi}{L}}{k^2 - \frac{n^2\pi^2}{L^2}} - \beta_n^{(j)} \, \frac{(n+\frac{1}{2})\frac{\pi}{L}}{k^2 - (n+\frac{1}{2})^2 \frac{\pi^2}{L^2}} \right) , \tag{2.69}$$

for every $k \in \mathbb{C} - \left\{ \pm\frac{n\pi}{L}, \pm(n+\frac{1}{2})\frac{\pi}{L} \right\}$, *if* $j = 2, 4$. *The Fourier coefficients* $\alpha_n^{(j)}$ *and* $\beta_n^{(j)}$ *correlate with the coefficients* (2.12)-(2.19) *as* $\alpha_n^{(1)} = \sinh 2n\pi \, a_n$, $\alpha_n^{(2)} = -\sinh 2n\pi \, e_n$, $\alpha_n^{(3)} = -\sinh 2n\pi \, c_n$, $\alpha_n^{(4)} = \sinh 2n\pi \, g_n$ *and* $\beta_n^{(1)} = \sinh(2n+1)\pi \, b_n$, $\beta_n^{(2)} = -\sinh(2n+1)\pi \, f_n$, $\beta_n^{(3)} = -\sinh(2n+1)\pi \, d_n$, $\beta_n^{(4)} = \sinh(2n+1)\pi \, h_n$.
The contours $\mathcal{L}$, $\mathcal{R}$, $\mathcal{U}$ *and* $\mathcal{D}$ *are obtained by deformation processes described in the sequence and depicted in Figure 2.5.*

Equation (2.47), with σ replaced by -1, can be thought as the bilateral Laplace transform of $q_1(x, y)$, provided that the function $q_1(x, y)$ is such that the integral is convergent for some values of k. The inversion formula then implies

$$\begin{aligned} q_1(x,y) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{e^{ky}}{2ik} \Bigg[& e^{ik(x-L)} \int_{-L}^{L} e^{\sigma k y} \Big(ik\, q(L,y) + \partial_x\, q(L,y) \Big) \, \mathrm{d}y \\ & + e^{-ik(x+L)} \int_{-L}^{L} e^{\sigma k y} \Big(ikq(-L,y) - \partial_x\, q(-L,y) \Big) \, \mathrm{d}y \Bigg] \mathrm{d}k \,, \end{aligned} \tag{2.70}$$

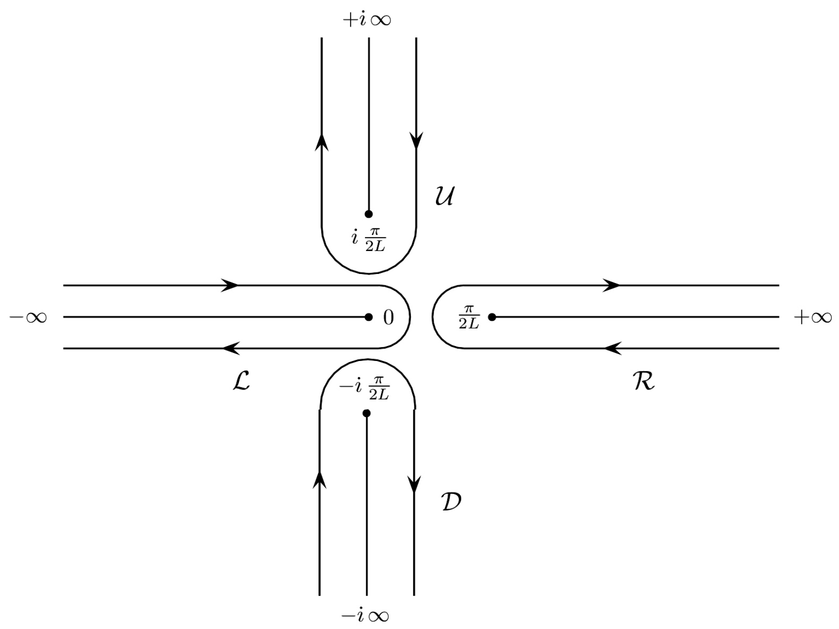

FIGURE 2.5: *The contours $\mathcal{L}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{D}$.*

a formula useful for changing-type boundary value problems, as we will see in section 9. But since we are primarily concerned with Dirichlet data prescribed on the boundary, the inversion of (2.50) implies

$$q_1(x,y) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{e^{ky}}{e^{i2kL}-e^{-i2kL}}\Bigg[\left(e^{ik(x+L)}-e^{-ik(x+L)}\right)\mathfrak{D}^{(1)}(-k) - \left(e^{ik(x-L)}-e^{-ik(x-L)}\right)\mathfrak{D}^{(3)}(-k)\Bigg]\,\mathrm{d}k\,, \tag{2.71}$$

where the Dirichlet transforms $\mathfrak{D}^{(j)}$ are given by equations (2.48).
Expanding the Dirichlet data f^j_D in a series of the form (2.10) yields $\mathfrak{D}^{(j)}(-k) = e^{kL}\,\mathcal{N}^{(j)}(k) + e^{-kL}\,\mathcal{M}^{(j)}(-k)$, where we note that

$$\mathcal{N}^{(j)}(k),\, \mathcal{M}^{(j)}(k) = \mathcal{O}\left(\frac{1}{k^2}\right)\,.$$

Plugging the latter expression into eq. (2.71) we find

$$q_1(x,y) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(y+L)k}\,\frac{\sin k(x+L)\,\mathcal{N}^{(1)}(k) - \sin k(x-L)\,\mathcal{N}^{(3)}(k)}{\sin 2kL}\,\mathrm{d}k + \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(y-L)k}\,\frac{\sin k(x+L)\,\mathcal{M}^{(1)}(k) - \sin k(x-L)\,\mathcal{M}^{(3)}(k)}{\sin 2kL}\,\mathrm{d}k\,. \tag{2.72}$$

The Laplace transform of the function $q_1(x, y)$ displays a rapid decay as k approaches large values. Indeed, as $k \to \infty$ the denominator $e^{i\,2kL} - e^{-i\,2kL}$ is dominated by $e^{-i\,2kL}$ for $\mathrm{Im}\,k < 0$ and by $-e^{i\,2kL}$ for $\mathrm{Im}\,k > 0$. On the other hand, the nominator e^{ky} is bounded in the left ($\mathrm{Re}\,k \leq 0$) complex k-plane if $y \in [0, L]$ and in the right ($\mathrm{Re}\,k > 0$) complex k-plane if $y \in [-L, 0]$. Hence as $k \to \infty$,

$$\frac{e^{i\,k(x+L)} - e^{-i\,k(x+L)}}{e^{i\,2kL} - e^{-i\,2kL}} \sim \begin{cases} e^{i\,k(x-L)} - e^{-i\,k(x+3L)} & ,\mathrm{Im}\,k < 0 \\ -e^{i\,k(x+3L)} + e^{-i\,k(x-L)} & ,\mathrm{Im}\,k > 0 \end{cases}, \quad k \to \infty\,,$$

$$\frac{e^{i\,k(x-L)} - e^{-i\,k(x-L)}}{e^{i\,2kL} - e^{-i\,2kL}} \sim \begin{cases} e^{i\,k(x-3L)} - e^{-i\,k(x+L)} & ,\mathrm{Im}\,k < 0 \\ -e^{i\,k(x+L)} + e^{-i\,k(x-3L)} & ,\mathrm{Im}\,k > 0 \end{cases}, \quad k \to \infty\,.$$

Furthermore, the exponentials $e^{(y+L)k}$ and $e^{(y-L)k}$ are bounded in the left (Re < 0) or the right (Re > 0) complex k-plane, respectively.
The aforementioned analysis implies that the Bromwich contour in (2.72) can be replaced either by the contour $\mathcal{L}$ or by the contour $\mathcal{R}$, depicted in Figure 2.6. Equation (2.72) then becomes

$$\begin{aligned} q_1(x,y) = &-\frac{1}{2i\pi}\int_{\mathcal{L}} e^{(y+L)k}\,\frac{\sin k(x-L)\,\mathcal{N}^{(3)}(k) - \sin k(x+L)\,\mathcal{N}^{(1)}(k)}{\sin 2kL}\,\mathrm{d}k \\ &+\frac{1}{2i\pi}\int_{\mathcal{R}} e^{(y-L)k}\,\frac{\sin k(x+L)\,\mathcal{M}^{(1)}(k) - \sin k(x-L)\,\mathcal{M}^{(3)}(k)}{\sin 2kL}\,\mathrm{d}k\,. \end{aligned} \tag{2.73}$$

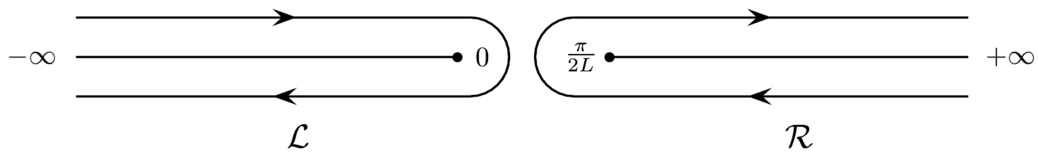

FIGURE 2.6: *The contours $\mathcal{L}$ and $\mathcal{R}$.*

The contour $\mathcal{L}$ begins and ends in the left ($\mathrm{Re}\,k < 0$) complex $k-$plane, such that $\mathrm{Re}\,k$ tends to $-\infty$ at each end, a technique known as Talbot's method [Tal79]. In Talbot's method the initial contour is deformed to the region of the complex $k-$plane in which the factor $e^{f(k)}$ reduces in magnitude as much as possible. Analogous, the contour $\mathcal{R}$ begins and ends in the right ($\mathrm{Re}\,k > 0$) complex $k-$plane, such that $\mathrm{Re}\,k \to \infty$ at each end.
Similarly, equation (2.57) can be seen as the Fourier transform of $q_2(x, y)$. Thus, the inversion formula implies

$$\begin{aligned} q_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ikx}}{2k}\Bigg[& e^{-k(y+L)}\int_{-L}^{L} e^{-ikx}\Big(\sigma k\, q(x,-L) - \partial_y\, q(x,-L)\Big)\mathrm{d}x \\ &+ e^{k(y-L)}\int_{-L}^{L} e^{-i\,kx}\Big(\sigma k\, q(x,L) - \partial_y\, q(x,L)\Big)\mathrm{d}x\Bigg]\mathrm{d}k\,, \end{aligned} \tag{2.74}$$

is a relation which will prove valuable for changing-type boundary value problems. For

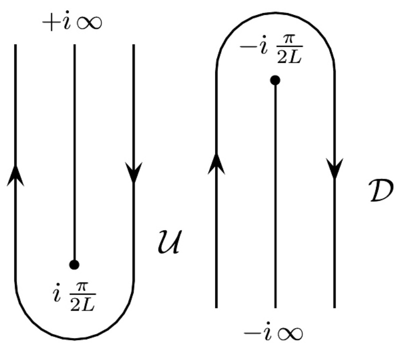

FIGURE 2.7: *The contours $\mathcal{U}$ and $\mathcal{D}$.*

Dirichlet data the inversion of (2.61) yields

$$q_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{e^{2kL}-e^{-2kL}}\left[-\left(e^{k(y-L)}-e^{-k(y-L)}\right)\mathfrak{D}^{(2)}(-ik) + \left(e^{k(y+L)}-e^{-k(y+L)}\right)\mathfrak{D}^{(4)}(-ik)\right]\mathrm{d}k\,.$$

Applying the previous analysis, the above equations yields

$$\begin{aligned} q_2(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x+L)k}\,\frac{\sinh k(y-L)\,\mathcal{N}^{(2)}(k)-\sinh k(y+L)\,\mathcal{N}^{(4)}(k)}{\sinh 2kL}\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x-L)k}\,\frac{\sinh k(y+L)\,\mathcal{M}^{(4)}(k)-\sinh k(y-L)\,\mathcal{M}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k\,. \end{aligned} \tag{2.75}$$

From (2.75) it is evident that the Fourier transform of the function $q_2(x,y)$ displays a rapid decay as k approaches large values. Indeed, as $k\to\infty$ the denominator $e^{2kL}-e^{-2kL}$ is dominated by e^{-2kl} for $\mathrm{Re}\,k>0$ and by $-e^{2kl}$ for $\mathrm{Re}\,k<0$. The nominator $e^{i\,kx}$ on the other hand is bounded in the lower $(\mathrm{Im}\,k<0)$ complex k-plane for every $x\in[-L,0]$ and in the upper $(\mathrm{Im}\,k>0)$ complex k-plane for every $x\in[0,L]$. Hence as $k\to\infty$,

$$\frac{e^{k(y-L)}-e^{-k(y-L)}}{e^{2kL}-e^{-2kL}} \sim \begin{cases} -e^{k(y+L)}+e^{-k(y-3L)} & ,\mathrm{Re}\,k<0 \\ e^{k(y-3L)}-e^{-k(y+L)} & ,\mathrm{Re}\,k>0 \end{cases}, \quad k\to\infty\,,$$

$$\frac{e^{k(y+L)}-e^{-k(y+L)}}{e^{2kL}-e^{-2kL}} \sim \begin{cases} -e^{k(y+3L)}+e^{-k(y-L)} & ,\mathrm{Re}\,k<0 \\ e^{k(y-L)}-e^{-k(y+3L)} & ,\mathrm{Re}\,k>0 \end{cases}, \quad k\to\infty\,.$$

Moreover, the exponentials $e^{i(x+L)k}$ and $e^{i(x-L)k}$ are bounded in the upper (Im >0) or the lower (Im <0) complex k-plane, respectively.
Thus, the line with endpoints $-\infty$ and $+\infty$ present in (2.75), can be replaced by either the

contour $\mathcal{U}$ or by the contour $\mathcal{D}$ depicted in Figure 2.7. Hence (2.75) can be rewritten as

$$\begin{aligned} q_2(x,y) = & -\frac{1}{2\pi}\int_{\mathcal{U}} e^{i\,(x+L)k}\,\frac{\sinh k(y+L)\,\mathcal{N}^{(4)}(k) - \sinh k(y-L)\,\mathcal{N}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k \\ & + \frac{1}{2\pi}\int_{\mathcal{D}} e^{i\,(x-L)k}\,\frac{\sinh k(y+L)\,\mathcal{M}^{(4)}(k) - \sinh k(y-L)\,\mathcal{M}^{(2)}(k)}{\sinh 2kL}\,\mathrm{d}k\,. \end{aligned} \tag{2.76}$$

Adding equations (2.73) and (2.76) yields (2.65).

2.7.1 Existence of the Integral transforms and the Inversion formulae

The aforementioned operations are justified introducing the functional space $L_1(\mathbb{R})$ for every function $q:\mathbb{R}\to\mathbb{C}$ exhibiting exponential growth, i.e. equipped with the property

$$|q(\mathbf{x})| \le C\,e^{\sigma\,x}.$$

Then [Sne72, GPS06],

Theorem 2.7.2 (Existence of the Bilateral Laplace Transform) *Let $q \in L_1(\epsilon, E)$, $-\infty < \epsilon < E < +\infty$, belonging to both $L_1(\mathbb{R}; e^{-\sigma_1\,x_i})$ and $L_1(\mathbb{R}; e^{-\sigma_2\,x_i})$. Then the bilateral Laplace transform $Q(x_2;k) = \mathcal{BL}\{q(x_1,x_2);k\}$ exist and the integral*

$$Q(x_2;k) = \int_{-\infty}^{\infty} e^{-k\,x_1}\,q(x_1,x_2)\,\mathrm{d}x_1$$

is absolutely and uniformly convergent in the strip $\sigma_1 < c < \sigma_2$

Theorem 2.7.3 (Inversion formula) *Let $q(x_1,x_2), e^{-k\,x_i}\,q(x_1,x_2) \in C[\epsilon,E] \cap L_1(\mathbb{R})$, $\sigma_1 < c = Re\,k < \sigma_2$. Then the following inversion formula for the bilateral Laplace transformation*

$$q(x_1,x_2) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{c-iR}^{c+iR} e^{k\,x_1}\,Q(x_2;k)\,\mathrm{d}k\,,$$

is valid for every interval $[\epsilon, E] \subset \mathbb{R}$.

Similar conclusions, due do the connection with the (bilateral) Laplace transform, are valid for the Fourier transform.

2.8 A Novel Integral Representation

Proposition 2.8.1 *Suppose that there exist a function $q(x,y)$ with sufficient smoothness all the way to the boundary, satisfying the Laplace equation* (2.1) *in the interior of the Square Ω defined by*

$$\Omega = \Big\{|x| \le L,\ |y| \le L\Big\},$$

with Dirichlet boundary conditions prescribed by equations (2.3)*. Then the solution* $q(x,y)$ *admits the following integral representation*

$$\begin{aligned} q(x,y) = {} & \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\Big(\mathcal{J}(y;k)\, f_{D}^{(1)}(\tau)\Big)\,\mathrm{d}k \\ & + \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y+L)}\Big(\mathcal{I}(x;k)\, f_{D}^{(2)}(\eta)\Big)\,\mathrm{d}k \\ & - \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x+L)}\Big(\mathcal{J}(y;k)\, f_{D}^{(3)}(\tau)\Big)\,\mathrm{d}k \\ & - \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} e^{k(y-L)}\Big(\mathcal{I}(x;k)\, f_{D}^{(4)}(\eta)\Big)\,\mathrm{d}k, \quad k\in\mathbb{C}\,, \end{aligned} \tag{2.77}$$

where the integral operators $\mathcal{I}(x;k)$ *and* $\mathcal{J}(y;k)$ *are defined as*

$$\mathcal{I}(x;k) = \int_{-L}^{x} d\eta\, e^{ik(\eta-x)} + \int_{x}^{L} d\eta\, e^{-ik(\eta-x)}, \quad k\in\mathbb{C}\,, \tag{2.78}$$

and

$$\mathcal{J}(y;k) = \int_{-L}^{y} d\tau\, e^{k(\tau-y)} + \int_{y}^{L} d\tau\, e^{-k(\tau-y)}, \quad k\in\mathbb{C}\,, \tag{2.79}$$

respectively.

Employing the global relation (2.43) in the subdomains Ω_3 and Ω_4 depicted in Figure 2.4, with boundary conditions

$$q(L,\tau) = f_{D}^{(1)}(\tau), \qquad \left.\begin{aligned} q(x,-L) = q(-L,\tau) = q(x,L) = 0 \\ \partial_y\, q(x,-L) = \partial_x\, q(-L,\tau) = \partial_y\, q(x,L) = 0 \end{aligned}\right\}\,, \tag{2.80}$$

we derive the following equations

$$\begin{aligned} & \int_{-L}^{L} e^{\pm i\,kx}\Big(\sigma k\, q_1(x,y) - \partial_y\ q_1(x,y)\Big)\mathrm{d}x \\ & = -e^{\pm i\,kL}\int_{-L}^{y} e^{\sigma k(\tau-y)}\Big(\pm\, i\,k f_{D}^{(1)}(\tau) - f_{N}^{(1)}(\tau)\Big)d\tau,\ k\in\mathbb{C},\ (x,y)\in\Omega_3\,, \end{aligned} \tag{2.81}$$

$$\begin{aligned} & \int_{-L}^{L} e^{\pm i\,kx}\Big(\sigma k\, q_1(x,y) - \partial_y\ q_1(x,y)\Big)\mathrm{d}x \\ & = e^{\pm i\,kL}\int_{y}^{L} e^{\sigma k(\tau-y)}\Big(\pm\, i\,k f_{D}^{(1)}(\tau) - f_{N}^{(1)}(\tau)\Big)d\tau,\ k\in\mathbb{C},\ (x,y)\in\Omega_4\,, \end{aligned} \tag{2.82}$$

where the solution $q_1(x,y)$ corresponds to the boundary conditions (2.80). Replace in the former $\sigma = 1$ and in the latter $\sigma = -1$. Subtracting the resulting equations, not only eliminates the unknown function $\partial_y\, q_1(x,y)$, but also provides the Fourier transform for the solution $q_1(x,y)$,

$$\int_{-L}^{L} e^{\pm ikx}\ q_1(x,y)\,\mathrm{d}x = -\frac{e^{\pm ikL}}{2k}\,\mathcal{J}(y;k)\Big(\pm\, ik f_{D}^{(1)}(\tau) - f_{N}^{(1)}(\tau)\Big), \quad k\in\mathbb{C}-\{0\}, \tag{2.83}$$

where the integral operator $\mathcal{J}(y;k)$ is defined by eq. (2.79).
The inverse of (2.83)$^-$ gives

$$q_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\frac{1}{2k}\mathcal{J}(y;k)\Big(ikf_D^{(1)}(\tau) + f_N^{(1)}(\tau)\Big)\mathrm{d}k\,. \tag{2.84}$$

Eliminating the unknown Neumann boundary data $f_N^{(1)}(\tau)$ in (2.84), with the aid of (2.83)$^+$, we find

$$\begin{aligned} q_1(x,y) =& \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\Big(\mathcal{J}(y;k)f_D^{(1)}(\tau)\Big)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\Bigg\{\int_{-L}^{L} e^{ik(x-L)}q_1(x,y)\mathrm{d}x\Bigg\}\mathrm{d}k\,. \end{aligned} \tag{2.85}$$

As k tends to infinity, both $e^{k(\tau-y)}$ and $e^{-k(\tau-y)}$ tend to zero since $\tau - y \leq 0$ for $\tau \in [-L,y]$ and $\tau - y \geq 0$ for $\tau \in [y,L]$, respectively. Thus, the integral operator $\mathcal{J}(y;k)$ is bounded as a function of k in the right ($\mathrm{Re}\,k \geq 0$) complex k-plane. Furthermore, since $x - L \leq 0$, the exponential $e^{i\,k(x-L)}$ is bounded in the lower ($\mathrm{Im}\,k \leq 0$) complex k-plane.
Assuming the change of the order of integration being permitted, the second integral appearing on the right-hand side of eq. (2.85) takes the form

$$\int_{-\infty}^{+\infty} e^{ik2(x-L)}\,\mathrm{d}k\,. \tag{2.86}$$

By deforming the line with endpoints $-\infty$ and $+\infty$ into a contour that begins and ends in the lower ($\mathrm{Im}\,k \leq 0$) complex k-plane, such that $\mathrm{Im}\,k \to -\infty$ at each end, the integral (2.86) yields a zero contribution since $e^{i\,k(x-L)}$ is *analytic* and *bounded* in $\mathrm{Im}\,k \leq 0$.
Hence, (2.85) becomes

$$q_1(x,y) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-L)}\Big(\mathcal{J}(y;k)f_D^{(1)}(\tau)\Big)\,\mathrm{d}k\,. \tag{2.87}$$

Repeating the above procedure in the subdomains Ω_3 and Ω_4 with boundary conditions

$$\begin{aligned} &q(-L,\tau) = f_D^{(3)}(\tau), \quad q(L,\tau) = q(x,-L) = q(x,L) = 0 \\ &\partial_x\,q(L,\tau) = \partial_y\,q(x,-L) = \partial_y\,q(x,L) = 0\,, \end{aligned}$$

we derive the relation

$$q_3(x,y) = -\frac{i}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x+L)}\Big(\mathcal{J}(y;k)f_D^{(3)}(\tau)\Big)\,\mathrm{d}k\,, \tag{2.88}$$

where the solution $q_3(x,y)$ corresponds to the specific boundary conditions described above.
Similar, by applying the global relation (2.43) in the subdomains Ω_1 and Ω_2, depicted in Figure 2.3, with boundary conditions

$$\left.\begin{aligned} q(\eta,-L) = f_D^{(2)}(\eta), \qquad & q(L,y) = q(-L,y) = q(\eta,L) = 0 \\ & \partial_x\,q(L,y) = \partial_x\,q(-L,y) = \partial_y\,q(\eta,L) = 0 \end{aligned}\right\}, \tag{2.89}$$

we derive the following equations

$$\int_{-L}^{L} e^{\sigma k y}\Big(\pm i\,k\,q_2(x,y) - \partial_x\; q_2(x,y)\Big)\mathrm{d}y$$
$$= e^{-\sigma k L}\int_{-L}^{x} e^{\pm i\,k(\eta - x)}\Big(\sigma k f_D^{(2)}(\eta) + f_N^{(2)}(\eta)\Big)d\eta,\; k\in\mathbb{C},\; (x,y)\in\Omega_1\,, \tag{2.90}$$

$$\int_{-L}^{L} e^{\sigma k y}\Big(\pm i\,k\,q_2(x,y) - \partial_x\; q_2(x,y)\Big)\mathrm{d}y$$
$$= -e^{-\sigma k L}\int_{x}^{L} e^{\pm i\,k(\eta - x)}\Big(\sigma k f_D^{(2)}(\eta) + f_N^{(2)}(\eta)\Big)d\eta,\; k\in\mathbb{C},\; (x,y)\in\Omega_2\,, \tag{2.91}$$

where the solution $q_2(x,y)$ corresponds to the boundary conditions (2.89). The unknown function $\partial_x\, q_2(x,y)$, is eliminated by adding equations (2.90)$^+$ and (2.91)$^-$

$$\int_{-L}^{L} e^{\sigma k y}\, q_2(x,y)\,\mathrm{d}y = \frac{e^{-\sigma k L}}{2i\,k}\,\mathcal{I}(x;k)\Big(\sigma\,k\,f_D^{(2)}(\eta) + f_N^{(2)}(\eta)\Big),\quad k\in\mathbb{C}-\{0\}\,, \tag{2.92}$$

where the integral operator $\mathcal{I}(x;k)$ is defined by eq. (2.78).
Evaluate equation (2.92) for $\sigma = -1$ to retrieve the bilateral Laplace transform for the solution $q_2(x,y)$, provided that $q_2(x,y)$ is such that the integral is convergent for some values of k. Then inversion implies the representation

$$q_2(x,y) = -\frac{1}{2i\,\pi}\int_{c-i\,\infty}^{c+i\,\infty} e^{k(y+L)}\,\frac{1}{2i\,k}\,\mathcal{I}(x;k)\Big(k f_D^{(2)}(\eta) - f_N^{(2)}(\eta)\Big)\,\mathrm{d}k\,. \tag{2.93}$$

The unknown Neumann boundary values $f_N^{(2)}(\eta)$ are eliminated with the aid of (2.92) evaluated at $\sigma = 1$.
Eq. (2.93) then becomes

$$\begin{aligned} q_2(x,y) =& \frac{1}{2\pi}\int_{c-i\,\infty}^{c+i\,\infty} e^{k(y+L)}\,\Big(\mathcal{I}(x;k) f_D^{(2)}(\eta)\Big)\,\mathrm{d}k \\ &+\frac{1}{2i\,\pi}\int_{c-i\,\infty}^{c+i\,\infty} e^{k(y+L)}\left\{\int_{-L}^{L} e^{k(y+L)} q_2(x,y)\mathrm{d}y\right\}\mathrm{d}k\,. \end{aligned} \tag{2.94}$$

The exponentials appearing in equation (2.78) are bounded in the lower ($\mathrm{Im}\,k \le 0$) complex k-plane. Hence, as $k\to\infty$, the integral operator $\mathcal{I}(x;k)$ is bounded as a function of k in the lower ($\mathrm{Im}\,k\le 0$) complex k-plane. Moreover, as $k\to\infty$ the exponential $e^{k(y+L)}$ tends to zero in the left ($\mathrm{Re}\,k\le 0$) complex k-plane.
Interchanging the order of integration in the second integral appearing on the right-hand side of eq. (2.94) we find

$$\int_{c-i\,\infty}^{c+i\,\infty} e^{2k(y+L)}\,\mathrm{d}k\,. \tag{2.95}$$

By deforming the Bromwich line into a contour that begins and ends in the left ($\mathrm{Re}\,k\le 0$) complex k-plane, such that $\mathrm{Re}\,k\to -\infty$ at both ends, the integral (2.95) yields a zero

contribution.
Hence, (2.94) yields

$$q_2(x,y) = \frac{1}{2\pi}\int_{c-i\,\infty}^{c+i\,\infty} e^{k(y+L)}\Big(\mathcal{I}(x;k)f_{D}^{(2)}(\tau)\Big)\,\mathrm{d}k\,, \tag{2.96}$$

An analysis similar to the one described previously, applied in the subdomains Ω_1 and Ω_2, with boundary conditions

$$\left.\begin{array}{ll} q(\eta,L) = f_{D}^{(4)}(\eta), & q(L,y) = q(-L,y) = q(\eta,-L) = 0 \\ & \partial_x\, q(L,y) = \partial_x\, q(-L,y) = \partial_y\, q(\eta,-L) = 0 \end{array}\right\}, \tag{2.97}$$

reveals that

$$q_4(x,y) = -\frac{1}{2\pi}\int_{c-i\,\infty}^{c+i\,\infty} e^{k(y-L)}\Big(\mathcal{I}(x;k)f_{D}^{(4)}(\eta)\Big)\,\mathrm{d}k\,, \tag{2.98}$$

where $q_4(x,y)$ is the solution corresponding to the boundary conditions (2.97).
Finally, adding equations (2.87), (2.88), (2.96) and (2.98) we obtain (2.77).

2.9 Changing-type Boundary Value Problems

The Dirichlet-to-Neumann correspondence, i.e. the global relation implemented at the boundary of the fundamental domain, can be used for the analysis of problems with changing-type boundary conditions. For example, consider the following problem

$$q(L,y) = f_{D}^{(1)}(y), \quad y\in[-L,0] \quad ; \quad \partial_x\, q(L,y) = f_{N}^{(1)}(y), \quad y\in[0,L]\,, \tag{2.99}$$

$$q(x,-L) = f_{D}^{(2)}(x), \quad x\in[-L,0] \quad ; \quad -\,\partial_y\, q(x,-L) = f_{N}^{(2)}(x), \quad x\in[0,L]\,, \tag{2.100}$$

$$q(-L,y) = f_{D}^{(3)}(y), \quad y\in[-L,0] \quad ; \quad -\,\partial_x\, q(-L,y) = f_{N}^{(3)}(y), \quad y\in[0,L]\,, \tag{2.101}$$

$$q(x,L) = f_{D}^{(4)}(x), \quad x\in[-L,0] \quad ; \quad \partial_y\, q(x,L) = f_{N}^{(4)}(x), \quad x\in[0,L]\,, \tag{2.102}$$

where we assume that the functions $f_{D}^{(j)}$ and $f_{N}^{(j)}$ are smooth and continuous at the corners of the Square and also at the points $(0,L)$, $(0,-L)$, $(L,0)$ and $(-L,0)$.
It is a well known fact that, due to the linearity of the Laplacian operator, the solution $q(x,y)$ can be written as a linear combination of "partial solutions" which correspond to specific boundary conditions. Therefore, implementing the global relation (2.43)$^+$, with σ replaced by -1, in the domain Ω depicted in Figure 2.1, we obtain the following relation

$$\int_{-L}^{L} e^{-ky}\Big(i\,k\,q_1(L,y) - \partial_x\, q_1(L,y)\Big)\,\mathrm{d}y = 0\,, \tag{2.103}$$

where $q_1(x,y)$ is a "partial solution" corresponding to given boundary conditions prescribed on side 1 of the Square and zero boundary conditions on the remaining sides.
Splitting the above integral into one part valid in the interval $-L \le y \le 0$ and a second part valid in the remaining interval and using boundary conditions (2.99) we find

$$\begin{aligned} &ik\int_0^L e^{-ky}\,q_1(L,y)\,\mathrm{d}y - \int_{-L}^0 e^{-ky}\,\partial_x\, q_1(L,y)\,\mathrm{d}y \\ &= \int_0^L e^{-ky}\,f_{N}^{(1)}(y)\,\mathrm{d}y - ik\int_{-L}^0 e^{-ky}\,f_{D}^{(1)}(y)\,\mathrm{d}y\,. \end{aligned} \tag{2.104}$$

Introducing the variable $z = e^{-kL}$, eq. (2.104) becomes the Riemann-Hilbert problem

$$\Phi_1^+(z) - \Phi_1^-(z) = \varphi_1(z), \quad z \in \mathbb{C}\,, \tag{2.105}$$

where

$$\Phi_1^+(z) = i\,k \int_0^L e^{-ky}\, q_1(L,y)\,\mathrm{d}y, \qquad \Phi_1^-(z) = \int_{-L}^0 e^{-ky}\, \partial_x\, q_1(L,y)\,\mathrm{d}y\,, \tag{2.106}$$

and $\varphi_1(z)$ is the known function

$$\varphi_1(z) = \int_0^L e^{-ky}\, f_N^{(1)}(y)\,\mathrm{d}y - ik \int_{-L}^0 e^{-ky}\, f_D^{(1)}(y)\,\mathrm{d}y\,. \tag{2.107}$$

Note that $\Phi_1^+(z)$ is analytic as z tends to zero, where else $\Phi_1^-(z)$ is analytic as $z \to \infty$. Moreover, $\Phi_1^-(z) \to 0$ as $z \to \infty$.
Employing the global relation (2.43)$^-$, with σ replaced by 1, in the domain Ω depicted in Figure 2.1, for the "partial solution" $q_2(x,y)$ corresponding to given boundary conditions prescribed on side 2 of the Square and zero boundary conditions on the remaining sides, we obtain

$$\int_{-L}^L e^{-i\,kx} \Big(k\, q_2(x,-L) - \partial_y\, q_2(x,-L) \Big)\,\mathrm{d}x = 0\,. \tag{2.108}$$

Splitting the above integral into two parts and using boundary conditions (2.100) we find

$$\begin{aligned} & k \int_0^L e^{-ikx}\, q_2(x,-L)\,\mathrm{d}x - \int_{-L}^0 e^{-ikx}\, \partial_y\, q_2(x,-L)\,\mathrm{d}x = \\ & - \int_0^L e^{-ikx}\, f_N^{(2)}(x)\,\mathrm{d}x - k \int_{-L}^0 e^{-ikx}\, f_D^{(2)}(x)\,\mathrm{d}x\,. \end{aligned} \tag{2.109}$$

Introducing the variable $z' = e^{-ikL}$, eq. (2.109) becomes the Riemann-Hilbert problem

$$\Phi_2^+(z') - \Phi_2^-(z') = \varphi_2(z'), \quad z' \in \mathbb{C}\,, \tag{2.110}$$

where

$$\Phi_2^+(z') = k \int_0^L e^{-ikx}\, q_2(x,-L)\,\mathrm{d}x \qquad \Phi_2^-(z') = \int_{-L}^0 e^{-ikx}\, \partial_y\, q_2(x,-L)\,\mathrm{d}x\,, \tag{2.111}$$

and $\varphi_2(z')$ is the known function

$$\varphi_2(z') = -\int_0^L e^{-ikx}\, f_N^{(2)}(x)\,\mathrm{d}x - k \int_{-L}^0 e^{-ikx}\, f_D^{(2)}(x)\,\mathrm{d}x\,. \tag{2.112}$$

Note that $\Phi_2^+(z')$ is analytic as z' tends to zero, where else $\Phi_2^-(z')$ is analytic as $z' \to \infty$. Moreover, $\Phi_2^-(z') \to 0$ as $z' \to \infty$.
Repeating the above procedures for the sides 3 and 4, one is led to the Riemann-Hilbert problems

$$\Phi_3^+(z) - \Phi_3^-(z) = \varphi_3(z), \quad z = e^{-kL} \tag{2.113}$$

and

$$\Phi_4^+(z') - \Phi_4^-(z') = \varphi_4(z'), \quad z' = e^{-i\,kL}\,, \tag{2.114}$$

where

$$\Phi_3^+(z) = i\,k\int_0^L e^{-ky}\, q_3(L,y)\,\mathrm{d}y, \qquad \Phi_3^-(z) = \int_{-L}^0 e^{-ky}\,\partial_x\, q_3(L,y)\,\mathrm{d}y\,, \tag{2.115}$$

$$\Phi_4^+(z') = k\int_0^L e^{-ikx}\, q_4(x,-L)\,\mathrm{d}x, \qquad \Phi_4^-(z') = \int_{-L}^0 e^{-ikx}\,\partial_y\, q_4(x,-L)\,\mathrm{d}x\,, \tag{2.116}$$

and $\varphi_3(z)$, $\varphi_4(z')$ are the known functions

$$\varphi_3(z) = -\int_0^L e^{-ky}\, f_N^{(3)}(y)\,\mathrm{d}y - i\,k\int_{-L}^0 e^{-ky}\, f_D^{(3)}(y)\,\mathrm{d}y\,, \tag{2.117}$$

$$\varphi_4(z') = \int_0^L e^{-ikx}\, f_N^{(4)}(x)\,\mathrm{d}x - k\int_{-L}^0 e^{-ikx}\, f_D^{(4)}(x)\,\mathrm{d}x\,. \tag{2.118}$$

The *scalar* Riemann-Hilbert problems (2.105), (2.110), (2.113) and (2.114) can be solved in closed form (see [Mus53] and specially Appendix 2 of the reference given, since the boundary of the fundamental domain Ω is a *piecewise* smooth contour).
The solution $q(x,y)$ is given by adding equations (2.70) and (2.74). Splitting the integrals on the right-hand side of the resulting equation into two parts and given boundary conditions (2.99)-(2.102), the unknown boundary conditions are obtained by solving the Riemann-Hilbert problems derived in this section, and hence the solution $q(x,y)$ is completely determined.

Remark 2.9.1 *It is possible to obtain the solution $q(x,y)$ in terms of a series instead of an integral by using equation* (2.11) *together with the Fourier coefficients* (2.12)-(2.19).

CHAPTER 3

Gegenbauer functions*

3.1 Some identities satisfied by the Gamma functions

In this section, the main properties of the $\Gamma-$functions are introduced, which will be frequently used in the sequence, without proof. Guided by the duplication formula

$$2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\tfrac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2z)\;, \tag{3.1}$$

it is straightforward to show, by replacing z with $-z$, that

$$\Gamma\left(z+\tfrac{1}{2}\right)\,\Gamma\left(\tfrac{1}{2}-z\right)=4\pi\,\frac{\Gamma(2z)\,\Gamma(-2z)}{\Gamma(z)\,\Gamma(-z)}\;. \tag{3.2}$$

Furthermore, utilizing the well known properties

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin z\pi}\;, \tag{3.3}$$

or, replacing z with $-z$,

$$\Gamma(-z)\,\Gamma(1+z)=-\frac{\pi}{\sin z\pi}\;, \tag{3.4}$$

and

$$\Gamma(1+z)=z\,\Gamma(z)\;, \tag{3.5}$$

it is easily shown that

$$\Gamma(z)\,\Gamma(-z)=-\frac{\pi}{z\,\sin z\pi}\;, \tag{3.6}$$

from which, replacing z with $2z$

$$\Gamma(2z)\,\Gamma(-2z)=-\frac{\pi}{2z\,\sin 2z\pi}\;. \tag{3.7}$$

Replacing everything into (3.2) we find

$$\Gamma\left(z+\tfrac{1}{2}\right)\,\Gamma\left(\tfrac{1}{2}-z\right)=\frac{\pi}{\cos z\pi}\;. \tag{3.8}$$

*This work has been published as [Dosb]

3.2 The Gegenbauer Differential Equation

The second-order ordinary differential equation

$$\left(1-z^2\right)\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2}-(2\lambda+1)\,z\,\frac{\mathrm{d}w(z)}{\mathrm{d}z}+\nu\,(\nu+2\lambda)\,w(z)=0\,, \tag{3.9}$$

where ν, λ and z may be any complex numbers, introduced by Leopold Gegenbauer in 1875 [Geg75], is known as the Gegenbauer or ultraspherical differential equation and can be seen as a particular case ($\mu=\nu$) of the generalized Gegenbauer differential equation

$$\left(1-z^2\right)\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2}+(\nu-2\lambda-\mu-1)\,z\,\frac{\mathrm{d}w(z)}{\mathrm{d}z}+\left[\mu\,(2\nu-\mu)+2\lambda\,\frac{\mu-(2\nu-\mu)z^2}{1-z^2}\right]w(z)=0\,. \tag{3.10}$$

The solutions of (3.9) are

$$w(z)=c_1\,C_\nu^\lambda(z)+c_2\,D_\nu^\lambda(z) \tag{3.11}$$

where $C_\nu^\lambda(z)$ and $D_\nu^\lambda(z)$ are known as the Gegenbauer or Ultraspherical functions of the first and second kind of degree ν and order λ, respectively. They are defined as [Erd53, p. 175,179, plus errata]

$$C_\nu^\lambda(z)=\frac{\Gamma\,(\nu+2\lambda)}{\Gamma\,(\nu+1)\;\Gamma\,(2\lambda)}\,F\left(-\nu,\nu+2\lambda,\lambda+\frac{1}{2};\frac{1-z}{2}\right),\quad |1-z|<2\,, \tag{3.12}$$

and

$$D_\nu^\lambda(z)=2^{2\lambda-1}\,\frac{\Gamma\,(\nu+2\lambda)\;\Gamma\,(\lambda)}{\Gamma\,(\nu+\lambda+1)}\,(2z)^{-\nu-2\lambda}\,F\left(\frac{\nu+2\lambda+1}{2},\frac{\nu+2\lambda}{2},\nu+\lambda+1;\frac{1}{z^2}\right),\ |z|>1\,, \tag{3.13}$$

applicable if $\nu+2\lambda\neq 0,-1,-2,\ldots$, and λ is not zero or a negative integer. Slightly modified definitions as well as expansion formulas and addition theorems for the Gegenbauer functions can be found in [DFS76]. If ν is replaced by a positive integer n, the Gegenbauer functions of the first kind (3.12) degenerate to the well known Gegenbauer polynomials $C_n^\lambda(z)$.

3.3 Asymptotic Expressions for the Gegenbauer functions

Kummer's formula

$$F\,(\alpha,\beta,\gamma;z)=(1-z)^{-\alpha}\,F\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right),\quad |\arg(-z)|<\pi\,, \tag{3.14}$$

together with the duplication theorem

$$F\left(2\alpha,2\beta,\alpha+\beta+\frac{1}{2};z\right)=F\left(\alpha,\beta,\alpha+\beta+\frac{1}{2};4z-4z^2\right)\,, \tag{3.15}$$

lead to the useful relation

$$F\left(\alpha,\beta,\frac{\alpha+\beta+1}{2};\frac{1-z}{2}\right)=z^{-\alpha}\,F\left(\frac{\alpha}{2},\frac{\alpha+1}{2},\frac{\alpha+\beta+1}{2};1-\frac{1}{z^2}\right)\,. \tag{3.16}$$

Indeed, applying (3.14) to the right-hand side of (3.15) we find

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; z\right) = F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; 4z-4z^2\right)$$
$$= (2z-1)^{-2\alpha}\, F\left(\alpha, \alpha+\frac{1}{2}, \alpha+\beta+\frac{1}{2}; \frac{4z^2-4z}{(2z-1)^2}\right) .$$

Put $2z-1=-u$ to find

$$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-u}{2}\right) = u^{-2\alpha}\, F\left(\frac{2\alpha}{2}, \frac{2\alpha+1}{2}, \frac{2\alpha+2\beta+1}{2}; 1-\frac{1}{u^2}\right) ,$$

which, by replacing 2α, 2β and u with α, β and z respectively, becomes (3.16).
Kummer's solutions of the hypergeometric equation together with the fact that any three of them can be connected by a linear relation with constant coefficients produces [Erd53, WG89]

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\, F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z)$$
$$+ \frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\,(1-z)^{\gamma-\alpha-\beta}\, F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; 1-z) .$$

Employing the above formula to the right-hand side of (3.16), we find

$$F\left(\alpha, \beta, \frac{\alpha+\beta+1}{2}; \frac{1-z}{2}\right) = z^{-\alpha}\, F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\alpha+\beta+1}{2}; 1-\frac{1}{z^2}\right)$$
$$= \frac{\Gamma(\frac{\alpha+\beta+1}{2})\,\Gamma(\frac{\beta-\alpha}{2})}{\Gamma(\frac{\beta+1}{2})\,\Gamma(\frac{\beta}{2})}\,\frac{1}{z^\alpha}\, F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\alpha-\beta}{2}+1; \frac{1}{z^2}\right) + \frac{\Gamma(\frac{\alpha+\beta+1}{2})\,\Gamma(\frac{\alpha-\beta}{2})}{\Gamma(\frac{\alpha+1}{2})\,\Gamma(\frac{\alpha}{2})}\,\frac{1}{z^\beta}$$
$$\times F\left(\frac{\beta+1}{2}, \frac{\beta}{2}, \frac{\beta-\alpha}{2}+1; \frac{1}{z^2}\right) . \tag{3.17}$$

Applying (3.17) to the definition of the Gegenbauer functions of the first kind (3.12), we obtain

$$C_\nu^\lambda(z) = \frac{\Gamma(\nu+\lambda)}{\Gamma(\nu+1)\,\Gamma(\lambda)}\,(2z)^\nu\, F\left(-\frac{\nu}{2}, \frac{1-\nu}{2}, -\nu-\lambda+1; \frac{1}{z^2}\right)$$
$$+ \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1)\,\Gamma(\lambda)}\,\frac{\sin\nu\pi}{\sin(\nu+\lambda)\pi}\,\frac{1}{(2z)^{\nu+2\lambda}}\, F\left(\frac{\nu+2\lambda+1}{2}, \frac{\nu+2\lambda}{2}, \nu+\lambda+1; \frac{1}{z^2}\right) , \tag{3.18}$$

where equations (3.1), (3.3) and (3.5) are used. For very large values of z, equations (3.18) and (3.13) read as

$$C_\nu^\lambda(z) \sim \frac{\Gamma(\nu+\lambda)}{\Gamma(\nu+1)\,\Gamma(\lambda)}\,(2z)^\nu, \quad z\to\infty \tag{3.19}$$

$$D_\nu^\lambda(z) \sim 2^{2\lambda-1}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+\lambda+1)}\,\frac{1}{(2z)^{\nu+2\lambda}}, \quad z\to\infty , \tag{3.20}$$

as long as $\mathrm{Re}\,\nu$ is positive.

3.4 The Wronskian for the Gegenbauer functions

In order to evaluate the Wronskian of the independent pair $C_\nu^\lambda(z)$, $D_\nu^\lambda(z)$, write the Gegenbauer differential equation (3.9) for both the first and second kind Gegenbauer functions, namely

$$(z^2-1)\frac{d^2C_\nu^\lambda(z)}{\mathrm{d}z^2}+(2\lambda+1)\,z\,\frac{\mathrm{d}C_\nu^\lambda(z)}{\mathrm{d}z}-\nu\,(\nu+2\lambda)\,C_\nu^\lambda(z)=0\,,$$
$$(z^2-1)\frac{\mathrm{d}^2D_\nu^\lambda(z)}{\mathrm{d}z^2}+(2\lambda+1)\,z\,\frac{\mathrm{d}D_\nu^\lambda(z)}{\mathrm{d}z}-\nu\,(\nu+2\lambda)\,D_\nu^\lambda(z)=0\,.$$

Multiplying the former by $(z^2-1)^{\lambda-\frac{1}{2}}\,D_\nu^\lambda(z)$ and the latter by $(z^2-1)^{\lambda-\frac{1}{2}}\,C_\nu^\lambda(z)$ and subtracting the resulting equations side-by-side, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(z^2-1)^{\lambda+\frac{1}{2}}\left(C_\nu^\lambda(z)\,\frac{\mathrm{d}D_\nu^\lambda(z)}{\mathrm{d}z}-D_\nu^\lambda(z)\,\frac{\mathrm{d}C_\nu^\lambda(z)}{\mathrm{d}z}\right)\right]=0\,,$$

which integrated once becomes

$$C_\nu^\lambda(z)\,\frac{\mathrm{d}D_\nu^\lambda(z)}{\mathrm{d}z}-D_\nu^\lambda(z)\,\frac{\mathrm{d}C_\nu^\lambda(z)}{\mathrm{d}z}=\frac{c(\nu;\lambda)}{(z^2-1)^{\lambda+\frac{1}{2}}} \tag{3.21}$$

where the function $c(\nu;\lambda)$ can be obtained by calculating the above expression for some specific value of z.
Choosing the point at infinity where we can use equations (3.19) and (3.20), it is straightforward to show that

$$c(\nu;\lambda)=-\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}$$

which replaced in (3.21) provides the final result,

$$C_\nu^\lambda(z)\,\frac{\mathrm{d}D_\nu^\lambda(z)}{\mathrm{d}z}-D_\nu^\lambda(z)\,\frac{\mathrm{d}C_\nu^\lambda(z)}{\mathrm{d}z}=e^{\mp i(\lambda-\frac{1}{2})\pi}\,\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}\,\frac{1}{(1-z^2)^{\lambda+\frac{1}{2}}} \tag{3.22}$$

where the upper sign corresponds to $\mathrm{Im}\,z>0$ and the lower sign to $\mathrm{Im}\,z<0$.

A different approach to evaluate the Wronskian of the independent pair $\Big(C_n^\lambda(z),\,D_n^\lambda(z)\Big)$, where now $C_n^\lambda(z)$ the Gegenbauer *polynomial* of the first kind of degree n and order λ and $D_n^\lambda(z)$ the Gegenbauer function of the second kind of degree n and order λ, is described as follows.

Changing n into $n-1$ in the recurrence relation [WG89, p. 274]

$$(n+1)C_{n+1}^\lambda(z)-2(\lambda+n)\,z\,C_n^\lambda(z)+(2\lambda+n-1)C_{n-1}^\lambda(z)=0,$$

we obtain

$$nC_n^\lambda(z)+(2\lambda+n-2)C_{n-2}^\lambda(z)=2(\lambda+n-1)\,z\,C_{n-1}^\lambda(z)\;. \tag{3.23}$$

The Gegenbauer functions of the second kind $D_n^\lambda(z)$ satisfy the same recurrence relations as the Gegenbauer polynomials $C_n^\lambda(z)$ [Erd53, p. 179].
Thus

$$nD_n^\lambda(z)+(2\lambda+n-2)D_{n-2}^\lambda(z)=2(\lambda+n-1)\,z\,D_{n-1}^\lambda(z)\;. \tag{3.24}$$

Multiplying (3.23) by $D^\lambda_{n-1}(z)$ and (3.24) by $C^\lambda_{n-1}(z)$ and subtracting the resulting equations, we obtain

$$n\Big(D^\lambda_n(z)\, C^\lambda_{n-1}(z) - D^\lambda_{n-1}(z)\, C^\lambda_n(z)\Big) = (2\lambda + n - 2)\Big(D^\lambda_{n-1}(z)\, C^\lambda_{n-2}(z) - D^\lambda_{n-2}(z)\, C^\lambda_{n-1}(z)\Big)\,. \tag{3.25}$$

Evaluating the above equation for increasing n*

$$\begin{aligned}
n = 2 \quad & 2\Big(D^\lambda_2(z)\, C^\lambda_1(z) - D^\lambda_1(z)\, C^\lambda_2(z)\Big) && = 2\lambda\Big(D^\lambda_1(z)\, C^\lambda_0(z) - D^\lambda_0(z)\, C^\lambda_1(z)\Big)\\
n = 3 \quad & 3\Big(D^\lambda_3(z)\, C^\lambda_2(z) - D^\lambda_2(z)\, C^\lambda_3(z)\Big) && = (2\lambda+1)\Big(D^\lambda_2(z)\, C^\lambda_1(z) - D^\lambda_1(z)\, C^\lambda_2(z)\Big)\\
& && = \tfrac{2\lambda+1}{2}\, 2\lambda\Big(D^\lambda_1(z)\, C^\lambda_0(z) - D^\lambda_0(z)\, C^\lambda_1(z)\Big)\\
n = 4 \quad & 4\Big(D^\lambda_4(z)\, C^\lambda_3(z) - D^\lambda_3(z)\, C^\lambda_4(z)\Big) && = (2\lambda+2)\Big(D^\lambda_3(z)\, C^\lambda_2(z) - D^\lambda_2(z)\, C^\lambda_3(z)\Big)\\
& && = \tfrac{2\lambda+2}{3}\,\tfrac{2\lambda+1}{2}\, 2\lambda\Big(D^\lambda_1(z)\, C^\lambda_0(z) - D^\lambda_0(z)\, C^\lambda_1(z)\Big)\\
& \qquad\vdots && \qquad\vdots\\
& n\Big(D^\lambda_n(z)\, C^\lambda_{n-1}(z) - D^\lambda_{n-1}(z)\, C^\lambda_n(z)\Big) && = \tfrac{(2\lambda)_{n-1}}{(2)_{n-2}}\Big(D^\lambda_1(z)\, C^\lambda_0(z) - D^\lambda_0(z)\, C^\lambda_1(z)\Big)\,,
\end{aligned}$$

where $(a)_n$ the Pochhammer symbol defined as $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ and

$$\frac{(2\lambda)_{n-1}}{(2)_{n-2}} = \frac{\Gamma(2\lambda+n-1)}{\Gamma(n)\Gamma(2\lambda)}\,,$$

so that (3.26) rewrites

$$n\Big(D^\lambda_n(z)\, C^\lambda_{n-1}(z) - D^\lambda_{n-1}(z)\, C^\lambda_n(z)\Big) = \frac{\Gamma(2\lambda+n-1)}{\Gamma(n)\Gamma(2\lambda)}\Big(D^\lambda_1(z)\, C^\lambda_0(z) - D^\lambda_0(z)\, C^\lambda_1(z)\Big)\,. \tag{3.26}$$

The Wronskian of the independent pair $\Big(C^\lambda_n(z),\, D^\lambda_n(z)\Big)$ is

$$W_n\Big(C^\lambda_n(z),\, D^\lambda_n(z)\Big) = C^\lambda_n(z)\,\frac{\mathrm{d}D^\lambda_n(z)}{\mathrm{d}z} - D^\lambda_n(z)\,\frac{\mathrm{d}C^\lambda_n(z)}{\mathrm{d}z}\,,$$

which multiplied by $(1-z^2)$ and using the recurrence relation (3.170)

$$(1-z^2)\,\frac{\mathrm{d}}{\mathrm{d}z}\,\mathcal{G}^\lambda_n(z) = (n+2\lambda-1)\,\mathcal{G}^\lambda_{n-1}(z) - nz\,\mathcal{G}^\lambda_n(z),$$

where $\mathcal{G}^\lambda_n(z)$ any solution of the Gegenbauer equation, becomes

$$(1-z^2)W_n = -(n+2\lambda-1)\Big(D^\lambda_n(z)\, C^\lambda_{n-1}(z) - D^\lambda_{n-1}(z)\, C^\lambda_n(z)\Big)\,.$$

Substituting (3.26) into the above equation we finally obtain

$$(1-z^2)\Big(C^\lambda_n(z)\,\frac{\mathrm{d}D^\lambda_n(z)}{\mathrm{d}z} - D^\lambda_n(z)\,\frac{\mathrm{d}C^\lambda_n(z)}{\mathrm{d}z}\Big) = -\frac{\Gamma(n+2\lambda)}{\Gamma(n+1)\Gamma(2\lambda)}\Big(D^\lambda_1(z)\, C^\lambda_0(z) - D^\lambda_0(z)\, C^\lambda_1(z)\Big) \tag{3.27}$$

*For $n = 0$ and $n = 1$ we obtain $0 = 0$ and $C^\lambda_1(z) = 2\lambda\, z$ respectively.

where we used (3.5).
A formula connecting $D_n^\lambda(z)$ and $C_n^\lambda(z)$ similar to Christoffel's relation between $Q_n(z)$ and $P_n(z)$ has been given by Watson [Wat38]

$$\begin{aligned} D_n^\lambda(z) =& C_n^\lambda(z)\, D_0^\lambda(z) \\ & - \frac{\Gamma(2\lambda)}{(z^2-1)^{\lambda-\frac{1}{2}}} \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor} (\lambda+n-2m-1) \frac{(1-\lambda)_m\,(2\lambda+n-m)_m}{(n-m)_{m+1}\,(\lambda)_{m+1}} C_{n-2m-1}^\lambda(z)\,, \end{aligned} \tag{3.28}$$

provided that $2\lambda-1$ is not a negative integer and $\lfloor\frac{n-1}{2}\rfloor$ is the greatest integer which is less then or equals to $\frac{n-1}{2}$.
For $n=1$ equation (3.28) becomes

$$D_1^\lambda(z) = C_1^\lambda(z)\, D_0^\lambda(z) - \frac{\Gamma(2\lambda)}{(z^2-1)^{\lambda-\frac{1}{2}}}\,.$$

Substituting the above expression into (3.27) we finally find

$$C_n^\lambda(z)\,\frac{\mathrm{d}D_n^\lambda(z)}{\mathrm{d}z} - D_n^\lambda(z)\,\frac{\mathrm{d}C_n^\lambda(z)}{\mathrm{d}z} = -\frac{\Gamma(n+2\lambda)}{\Gamma(n+1)}\,\frac{1}{(z^2-1)^{\lambda+\frac{1}{2}}}\,. \tag{3.29}$$

Replacing in (3.29) specific values of λ, namely $\lambda = 0,\ \frac{1}{2},\ \frac{3}{2}$ we obtain the Chebyshev, Legendre or Gegenbauer functions of order $\frac{3}{2}$ respectively.
Hence for,

$$\begin{array}{lll} \lambda=0: & C_n^0(z)\equiv T_n(z),\ D_n^0(z)\equiv U_n(z) & ,T_n(z)\,\dot{U}_n(z)-U_n(z)\,\dot{T}_n(z) = -\frac{1}{n}\,\frac{1}{\sqrt{z^2-1}} \\ \lambda=\frac{1}{2}: & C_n^{\frac{1}{2}}(z)\equiv P_n(z),\ D_n^{\frac{1}{2}}(z)\equiv Q_n(z) & ,P_n(z)\,\dot{Q}_n(z)-Q_n(z)\,\dot{P}_n(z) = -\frac{1}{z^2-1} \\ \lambda=\frac{3}{2}: & & ,C_n^{\frac{3}{2}}(z)\,\dot{D}_n^{\frac{3}{2}}(z)-D_n^{\frac{3}{2}}(z)\,\dot{C}_n^{\frac{3}{2}}(z) = -\frac{(n+1)(n+2)}{(z^2-1)^2}\,. \end{array}$$

3.5 The Gegenbauer functions of the first kind $C_\nu^\lambda(x)$ on the cut $-1<x<+1$

Consider equation (3.12) which is the fundamental representation of $C_\nu^\lambda(z)$

$$C_\nu^\lambda(z) = \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,\Gamma(2\lambda)}\,F\left(-\nu,\nu+2\lambda,\lambda+\frac{1}{2};\frac{1-z}{2}\right),\quad |1-z|<2\,, \tag{3.30}$$

applicable for general degree ν and order λ. Following [Hob31], denote $z-1=\rho\,e^{i\phi}$, $|z|<1$, so that $1-z=-\rho\,e^{i\phi}$. In the complex $z-$plane, as depicted in Figure 3.1, if z lies just above the real axis, namely $z=x+i\varepsilon$, $\varepsilon>0$, $|\varepsilon|<<1$, then $\phi\simeq\pi$. On the other hand, if z lies just below the real axis, namely $z=x+i\varepsilon$, $\varepsilon<0$, $|\varepsilon|<<1$, then $\phi\simeq-\pi$. Hence

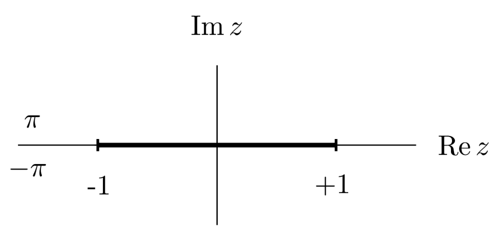

FIGURE 3.1: *The complex $z-$plane*

$$\begin{aligned} 1-z &= -\rho\,e^{i\phi} \simeq -\sqrt{(1-x)^2+\varepsilon^2}\,e^{\pm i\pi} \\ &= \sqrt{(1-x)^2+\varepsilon^2} = \sqrt{(1-x)^2+\varepsilon^2} \end{aligned}$$

As ε tends to zero, (3.12) becomes

$$C_\nu^\lambda(x) = \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,\Gamma(2\lambda)} F\left(-\nu, \nu+2\lambda, \lambda+\frac{1}{2}; \frac{1-x}{2}\right), \quad -1 < x < +1 . \tag{3.31}$$

The above expression is suitable to examine the behavior of $C_\nu^\lambda(x)$ as $x \to 1^-$. Indeed, in the limit $x \to 1^-$ (3.31) reduces to

$$C_\nu^\lambda(1) = \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,\Gamma(2\lambda)} . \tag{3.32}$$

Equation (3.31) becomes cumbersome to work with as $x \to -1^+$. Therefore we rewrite the hypergeometric function on the right-hand side of (3.32) so that the argument becomes $\frac{1+x}{2}$. This is achieved using the transformation [WG89, p. 160, eq. (4)]

$$\begin{aligned} F(\alpha,\beta,\gamma;z) = {} & \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z) \\ & + \frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z) , \end{aligned} \tag{3.33}$$

applicable if $|\arg(1-z)| < \pi$, γ is not zero or a negative integer and also $\gamma-\alpha-\beta$ must also not be an integer. Thus, (3.31) with the aid of (3.8), becomes

$$\begin{aligned} C_\nu^\lambda(x) = {} & \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,\Gamma(2\lambda)} \left[\frac{\cos(\nu+\lambda)\pi}{\cos\lambda\pi} F\left(-\nu,\nu+2\lambda,\lambda+\frac{1}{2};\frac{1+x}{2}\right)\right. \\ & \left. + \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(-\nu)\,\Gamma(\nu+2\lambda)} \left(\frac{1+x}{2}\right)^{-\lambda+\frac{1}{2}} F\left(\nu+\lambda+\frac{1}{2},-\nu-\lambda+\frac{1}{2},-\lambda+\frac{3}{2};\frac{1+x}{2}\right)\right] . \end{aligned} \tag{3.34}$$

As x tends to -1^+ (3.34) reads

$$\lim_{x\to-1^+} C_\nu^\lambda(x) = \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,\Gamma(2\lambda)} \left[\frac{\cos(\nu+\lambda)\pi}{\cos\lambda\pi} + \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(-\nu)\,\Gamma(\nu+2\lambda)} \lim_{x\to-1^+}\left(\frac{1+x}{2}\right)^{-\lambda+\frac{1}{2}}\right] \tag{3.35}$$

or, more conclusive

$$\lim_{x\to-1^+} C_\nu^\lambda(x) = \begin{cases} \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,\Gamma(2\lambda)} \frac{\cos(\nu+\lambda)\pi}{\cos\lambda\pi} & , -\frac{1}{2} < \lambda < \frac{1}{2} \\ \infty & , \lambda > \frac{1}{2} \end{cases} , \quad \lambda \neq n+\frac{1}{2}, n = 0,1,2,\dots . \tag{3.36}$$

From (3.35) it is seen that as x tends to -1^+, $C_\nu^\lambda(x) \sim (1+x)^{-\lambda+\frac{1}{2}}$, $\lambda \neq n+\frac{1}{2}$, $n = 0,1,2,\dots$.

As mentioned, when $\lambda \neq n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$ (3.33) is not applicable and (3.35) also does not hold. However, implementing [WG89, p. 167, eq. (8)]

$$\begin{aligned} F(\alpha, \beta, \gamma; z) = & \frac{\Gamma(n)\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} (1-z)^{-n} \sum_{k=0}^{n-1} \frac{(\alpha-n)_k\,(\beta-n)_k}{k!\,(1-n)_k} (1-z)^k \\ & + (-1)^{n+1} \frac{\Gamma(\gamma)}{\Gamma(\alpha-n)\,\Gamma(\beta-n)} \sum_{k=0}^{\infty} \frac{(\alpha)_k\,(\beta)_k}{k!\,(n+k)!} (1-z)^k \\ & \times \Big(\psi(\alpha+k) + \psi(\beta+k) - \psi(1+n+k) - \psi(1+k) + \ln(1-z) \Big) , \end{aligned} \tag{3.37}$$

for every $\gamma - \alpha - \beta = -n$, $n = 0, 1, 2, \ldots$ (in the case where $n = 0$ the finite sum is to be neglected) and $\alpha, \beta \neq 0, -1, -2 \ldots$, we obtain from (3.31)

$$\begin{aligned} C_\nu^{n+\frac{1}{2}}(x) = & -\frac{\sin\nu\pi}{\pi} \frac{\Gamma(n)\,\Gamma(n+1)}{\Gamma(2n+1)} \left(\frac{1+x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(-\nu-n)_k\,(\nu+n+1)_k}{k!\,(1-n)_k} \left(\frac{1+x}{2}\right)^k \\ & + (-1)^n \frac{\Gamma(\nu+2n+1)\,\Gamma(n+1)}{\Gamma(\nu+1)\,\Gamma(2n+1)\,\Gamma(-\nu-n)\,\Gamma(\nu+n+1)} \sum_{k=0}^{\infty} \frac{(-\nu)_k\,(\nu+2n+1)_k}{k!\,(n+k)!} \left(\frac{1+x}{2}\right)^k \\ & \times \left(\psi(-\nu+k) + \psi(\nu+2n+1+k) - \psi(1+n+k) - \psi(1+k) + \ln\frac{1+x}{2} \right) . \end{aligned} \tag{3.38}$$

From formula (3.38) we see that, as x tends to -1^+ and ν is not an integer, then, if $n = 0$, $C_\nu^{\frac{1}{2}}(x) \equiv P_\nu(x) \sim \ln(1+x)$, where else $C_\nu^{n+\frac{1}{2}}(x) \sim (1+x)^{-n}$ if $n > 0$, all becoming infinite.

3.6 The Gegenbauer functions of the second kind $D_\nu^\lambda(x)$ on the cut $-1 < x < +1$

Consider (3.13) which is the fundamental expression of $D_\nu^\lambda(x)$

$$D_\nu^\lambda(z) = 2^{2\lambda-1} \frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+\lambda+1)} \left(\frac{1}{2z}\right)^{\nu+2\lambda} F\left(\frac{\nu+2\lambda}{2}, \frac{\nu+2\lambda+1}{2}, \nu+\lambda+1; \frac{1}{z^2}\right), \; |z| > 1 \, . \tag{3.39}$$

Expression (3.39) may be used to deal with the segments $z \in (-\infty, -1] \cup [+1, +\infty)$, but is not suitable for the segment $-1 < z < +1$. This is true, since in the open interval $z \in (-1, +1)$ it is not possible to pin down a value for the argument $\arg z$ independent of z, i.e. if $z \in (-1, 0)$, then if z lies just the above the real axis (see Figure 3.1), $z = x + i\varepsilon, \varepsilon > 0$ and $|\varepsilon| << 1$, the argument of z is $\arg z \simeq \pi$. On the other hand, if z lies just below the real axis, namely $z = x + i\varepsilon, \varepsilon < 0$ and $|\varepsilon| << 1$, then $\arg z \simeq -\pi$. Therefore, as ε tends to $0^\pm$, we find

$$z = x + i\varepsilon = \sqrt{x^2+\varepsilon^2}\, e^{\pm i\pi} \Rightarrow z = x\, e^{\pm i\pi}$$

and

$$z^{-\nu-2\lambda} = x^{-\nu-2\lambda}\, e^{\mp i\pi(\nu+2\lambda)} \, .$$

If $z \in (0, +1)$, then, as $\varepsilon \to 0^\pm$

$$z^{-\nu-2\lambda} = x^{-\nu-2\lambda}\, e^0 \, .$$

This is why (3.13) is unsuitable for the segment $-1 < x < +1$ and some transformations are needed. Using the transformation

$$F\left(\alpha,\beta,\gamma;z\right) = (1-z)^{-\alpha}\, F\left(\alpha, \gamma-\beta, \gamma; \frac{z}{z-1}\right), \quad |\arg(1-z)| < \pi\,, \tag{3.40}$$

on (3.13) we obtain

$$D_\nu^\lambda(z) = 2^{-\nu-1}\frac{\Gamma\left(\nu+2\lambda\right)\Gamma\left(\lambda\right)}{\Gamma\left(\nu+\lambda+1\right)}\left(z^2-1\right)^{-\frac{\nu+2\lambda}{2}} F\left(\frac{\nu+2\lambda}{2}, \frac{\nu+1}{2}, \nu+\lambda+1; \frac{1}{1-z^2}\right), \tag{3.41}$$

for every $\mathrm{Re}\, z > 0$.
The restriction $\mathrm{Re}\, z > 0$ is needed because of the part of the branch cut of $\left(z^2-1\right)^{-\frac{\nu+2\lambda}{2}}$ associated with $z^2 \le 0$. However, the restriction can be dropped adopting the convention that $\left(z^2-1\right)^{-\frac{\nu+2\lambda}{2}}$ is defined as $(z-1)^{-\frac{\nu+2\lambda}{2}}\, (z+1)^{-\frac{\nu+2\lambda}{2}}$, so that the only branch cut is $(-\infty, 1]$.
Applying then consecutively the transformations [WG89, p. 161, eq.(8)]

$$\begin{aligned} F\left(\alpha,\beta,\gamma;z\right) =& \frac{\Gamma\left(\gamma\right)\Gamma\left(\beta-\alpha\right)}{\Gamma\left(\gamma-\alpha\right)\Gamma\left(\beta\right)}(-z)^{-\alpha}\, F\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1; z^{-1}\right) \\ &+ \frac{\Gamma\left(\gamma\right)\Gamma\left(\alpha-\beta\right)}{\Gamma\left(\gamma-\beta\right)\Gamma\left(\alpha\right)}(-z)^{-\beta}\, F\left(\beta, \beta-\gamma+1, \beta-\alpha+1; z^{-1}\right), \, |\arg(-z)| < \pi\,, \end{aligned} \tag{3.42}$$

which holds only if $\alpha-\beta$ is not an integer, and [WG89, p. 179, eq.(9) substituting $t = \frac{1-z}{2}$]

$$F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}; 1-z^2\right) = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-z}{2}\right), \tag{3.43}$$

to (3.41), gives

$$\begin{aligned} D_\nu^\lambda(z) =& \frac{\Gamma\left(\nu+2\lambda\right)\Gamma\left(\lambda\right)\Gamma\left(\frac{1}{2}-\lambda\right)}{2\sqrt{\pi}\,\Gamma\left(\nu+1\right)}\, F\left(-\nu, \nu+2\lambda, \lambda+\frac{1}{2}; \frac{1-z}{2}\right) \\ &+ 2^{2\lambda-2}\,\frac{\Gamma\left(\lambda\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\sqrt{\pi}}\left(z^2-1\right)^{\frac{1}{2}-\lambda} F\left(\nu+1, -\nu-2\lambda+1, \frac{3}{2}-\lambda; \frac{1-z}{2}\right), \end{aligned} \tag{3.44}$$

valid only if $\lambda-\frac{1}{2}$ is not an integer, i.e. $\lambda \neq n+\frac{1}{2}$, $n = 0, 1, 2, \ldots$.
In view of the definition for the Gegenbauer functions of the first kind (3.12), equation (3.44) rewrites

$$D_\nu^\lambda(z) = \frac{1}{\sin 2\lambda\pi}\left\{2^{2\lambda-1}\sin\lambda\pi\left(\Gamma\left(\lambda\right)\right)^2\, C_\nu^\lambda(z) - \pi\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+1\right)}(z^2-1)^{\frac{1}{2}-\lambda}\, C_{\nu+2\lambda-1}^{1-\lambda}(z)\right\}. \tag{3.45}$$

The latter holds for every $\lambda \neq \frac{n}{2}$. Solving (3.45) with respect to $C_\nu^\lambda(z)$ we obtain

$$C_\nu^\lambda(z) = \frac{2^{1-2\lambda}}{\sin\lambda\pi\,\left(\Gamma\left(\lambda\right)\right)^2}\left\{\sin 2\lambda\pi\, D_\nu^\lambda(z) + \pi\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+1\right)}(z^2-1)^{\frac{1}{2}-\lambda}\, C_{\nu+2\lambda-1}^{1-\lambda}(z)\right\}, \tag{3.46}$$

which shows clearly that $C_\nu^\lambda(z)$ and $C_{\nu+2\lambda-1}^{1-\lambda}(z)$ are linearly independent, except if λ is a half odd integer. Furthermore, replacing in (3.13) λ with $1-\lambda$ and ν with $\nu+2\lambda-1$ and applying the transformation formula

$$F(\alpha,\beta,\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma;z)\,, \tag{3.47}$$

it is straightforward to show that

$$D_{\nu+2\lambda-1}^{1-\lambda}(z) = 2^{1-2\lambda}\frac{\pi}{\sin\lambda\pi}\frac{\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)\,(\Gamma(\lambda))^2}\,(z^2-1)^{\lambda-\frac{1}{2}} D_\nu^\lambda(z)\,. \tag{3.48}$$

Consider $z\pm1 = \rho_\pm\, e^{i\phi_\pm}$ so that

$$z^2-1 = (z+1)(z-1) = \rho_+\,\rho_-\, e^{i(\phi_+ + \phi_-)}\,,$$

where $\rho_\pm = \sqrt{(x\pm1)^2+y^2}$ and $\phi_\pm = \arg(z\pm1)$. If z lies just above the real axis, then $\phi_- \simeq \pi$ and $\phi_+ \simeq 0$. Thus $z^2-1 = (x^2-1)\,e^{i\pi}$. On the other hand, if z lies just below the real axis, then $\phi_- \simeq -\pi$, $\phi_+ \simeq 0$ and $z^2-1 = (x^2-1)\,e^{-i\pi}$. Hence the values of $D_\nu^\lambda(z)$ on the cut from -1 to +1, are

$$\begin{aligned} e^{i\lambda\pi}\, D_\nu^\lambda(x+i0) =& \frac{1}{\sin 2\lambda\pi}\Bigg\{2^{2\lambda-1}\,\sin\lambda\pi\,\left(\Gamma(\lambda)\right)^2\, e^{i\lambda\pi}\, C_\nu^\lambda(x) \\ & + \pi\,\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}(1-x^2)^{\frac{1}{2}-\lambda}\, e^{-i\lambda\pi}\, C_{\nu+2\lambda-1}^{1-\lambda}(x)\Bigg\}\,, \end{aligned} \tag{3.49}$$

and

$$\begin{aligned} e^{-i\lambda\pi}\, D_\nu^\lambda(x-i0) =& \frac{1}{\sin 2\lambda\pi}\Bigg\{2^{2\lambda-1}\,\sin\lambda\pi\,\left(\Gamma(\lambda)\right)^2\, e^{-i\lambda\pi}\, C_\nu^\lambda(x) \\ & - \pi\,\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}(1-x^2)^{\frac{1}{2}-\lambda}\, e^{-i\lambda\pi}\, C_{\nu+2\lambda-1}^{1-\lambda}(x)\Bigg\}\,. \end{aligned} \tag{3.50}$$

Eliminating $C_{\nu+2\lambda-1}^{1-\lambda}(x)$ from (3.49) and (3.50) implies

$$e^{i\lambda\pi}\, D_\nu^\lambda(x+i0) + e^{-i\lambda\pi}\, D_\nu^\lambda(x-i0) = 2^{2\lambda-1}\,\left(\Gamma(\lambda)\right)^2\, C_\nu^\lambda(x)\,, \tag{3.51}$$

if $-1 < x < +1$, and shows why the cut must be extended to the point $z=1$ in the case of the Gegenbauer function of the second kind. Equation (3.51) can be rewritten as

$$e^{i\lambda\pi}\, D_\nu^\lambda(x+i0) - e^{\pm i\pi}\, e^{-i\lambda\pi}\, D_\nu^\lambda(x-i0) = 2^{2\lambda-1}\,\left(\Gamma(\lambda)\right)^2\, C_\nu^\lambda(x)\,. \tag{3.52}$$

Note that for $\lambda = \frac{1}{2}$, eq. (3.51) reduces to the well known relation

$$Q_\nu(x+i0) - Q_\nu(x-i0) = -i\pi\, P_\nu(x)\,.$$

Furthermore, if we stipulate that in $-1 < x < +1$*

$$D_\nu^\lambda(x) = -\frac{i}{2}\left(e^{i\lambda\pi}\, D_\nu^\lambda(x+i0) - e^{-i\lambda\pi}\, D_\nu^\lambda(x-i0)\right)\,, \tag{3.53}$$

*An alternative definition can be given as

$$D_\nu^\lambda(x) = \frac{i}{2}\left(e^{-i\lambda\pi}\, D_\nu^\lambda(x+i0) + e^{i\lambda\pi}\, D_\nu^\lambda(x-i0)\right)\,.$$

then $D_\nu^\lambda(x)$ clearly satisfies the Gegenbauer differential equation (3.9). Moreover, $D_\nu^\lambda(x)$ and $C_\nu^\lambda(x)$ are linearly independent of each other. The values of the Gegenbauer functions of the second kind on the cut $-1 < x < +1$, are obtained by substituting (3.49) and (3.50) into (3.53)

$$D_\nu^\lambda(x) = \frac{1}{\sin 2\lambda\pi}\Bigg\{2^{2\lambda-1}\big(\sin\lambda\pi\,\Gamma(\lambda)\big)^2\,C_\nu^\lambda(x) - i\pi\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}(1-x^2)^{\frac{1}{2}-\lambda}\,e^{-i\lambda\pi}\,C_{\nu+2\lambda-1}^{1-\lambda}(x)\Bigg\}. \tag{3.54}$$

To obtain the value of $D_\nu^\lambda(x)$ as x tends to 1^- we may use (3.31) of the preceding section together with the transformation (3.47) applied to the second hypergeometric function on the right-hand side of (3.54) and thus the foregoing expression for $D_\nu^\lambda(x)$ rewrites

$$\begin{aligned} D_\nu^\lambda(x) &= 2^{2\lambda-2}\frac{\sin\lambda\pi}{\cos\lambda\pi}\frac{\Gamma(\nu+2\lambda)\ (\Gamma(\lambda))^2}{\Gamma(\nu+1)\ \Gamma(2\lambda)}\,F\left(-\nu,\nu+2\lambda,\lambda+\frac{1}{2};\frac{1-x}{2}\right)\\ &+ i\,\frac{e^{-i\lambda\pi}}{\sqrt{\pi}}\,2^{\lambda-\frac{3}{2}}\,\Gamma(\lambda)\,\Gamma\left(\lambda-\tfrac{1}{2}\right)\left(\frac{1}{1-x}\right)^{\lambda-\frac{1}{2}} F\left(\nu+\lambda+\frac{1}{2},-\nu-\lambda+\frac{1}{2},\frac{3}{2}-\lambda;\frac{1-x}{2}\right). \end{aligned} \tag{3.55}$$

From (3.55) we see that as $x \to 1^-$

$$\begin{aligned} \lim_{x\to 1^-} D_\nu^\lambda(x) =& 2^{2\lambda-2}\frac{\sin\lambda\pi}{\cos\lambda\pi}\frac{\Gamma(\nu+2\lambda)\ (\Gamma(\lambda))^2}{\Gamma(\nu+1)\ \Gamma(2\lambda)}\\ &+ i\frac{e^{-i\lambda\pi}}{\sqrt{\pi}}\,2^{\lambda-\frac{3}{2}}\,\Gamma(\lambda)\,\Gamma\left(\lambda-\tfrac{1}{2}\right)\lim_{x\to 1^-}\left(\frac{1}{1-x}\right)^{\lambda-\frac{1}{2}}, \end{aligned} \tag{3.56}$$

which tends in general to infinity if λ is an (half odd or not) integer greater than $\frac{1}{2}$. In particular, if λ is a positive integer and $\nu + 2\lambda$ is not a negative integer then the first term vanishes and $D_\nu^n(x) \sim (1-x)^{\frac{1}{2}-n}$. If $\lambda = 0$, (3.55) becomes an indeterminate form which can be evaluated with the aid of Weierstrass' infinite product for the $\Gamma-$functions. For values of λ in the open interval from $-\frac{1}{2}$ to $\frac{1}{2}$, excluding zero,

$$\lim_{x\to 1^-} D_\nu^\lambda(x) = \frac{\sqrt{\pi}}{2}\frac{\sin\lambda\pi}{\cos\lambda\pi}\frac{\Gamma(\nu+2\lambda)\ \Gamma(\lambda)}{\Gamma(\nu+1)\ \Gamma\left(\lambda+\frac{1}{2}\right)}.$$

In order to study the value of $D_\nu^\lambda(x)$ as x tends to -1^+, we apply (3.33) to (3.55) to find

$$\begin{aligned} D_\nu^\lambda(x) &= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\nu+2\lambda)\ \Gamma(\lambda)}{\Gamma(\nu+1)\ \Gamma\left(\lambda+\frac{1}{2}\right)}\,\frac{1}{\cos^2\lambda\pi}\left(\sin\lambda\pi\ \cos(\nu+\lambda)\pi - i\,e^{-i\lambda\pi}\ \sin(\nu+2\lambda)\pi\right)\\ &\times F\left(-\nu,\nu+2\lambda,\lambda+\frac{1}{2};\frac{1+x}{2}\right) - \frac{2^{2\lambda-2}}{\sqrt{\pi}}\,\Gamma(\lambda)\,\Gamma\left(\lambda-\tfrac{1}{2}\right)\frac{1}{\cos\lambda\pi}\Big(\sin\lambda\pi\ \sin\nu\pi\\ &+ i\,e^{-i\lambda\pi}\ \cos(\nu+\lambda)\pi\Big)\,(1-x^2)^{\frac{1}{2}-\lambda}\,F\left(\nu+1,-\nu-2\lambda+1,\frac{3}{2}-\lambda;\frac{1+x}{2}\right), \end{aligned} \tag{3.57}$$

where (3.47) is also used. As x tends to -1^+ the above equation becomes

$$\lim_{x\to-1^+} D_\nu^\lambda(x) = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+1)\,\Gamma\left(\lambda+\frac{1}{2}\right)}\,\frac{1}{\cos^2\lambda\pi}\left(\sin\lambda\pi\,\cos(\nu+\lambda)\pi - i\,e^{-i\lambda\pi}\,\sin(\nu+2\lambda)\pi\right)$$
$$-\frac{2^{\lambda-\frac{3}{2}}}{\sqrt{\pi}}\,\Gamma(\lambda)\,\Gamma\left(\lambda-\tfrac{1}{2}\right)\frac{1}{\cos\lambda\pi}\left(\sin\lambda\pi\,\sin\nu\pi + i\,e^{-i\lambda\pi}\,\cos(\nu+\lambda)\pi\right)\lim_{x\to-1^+}(1+x)^{\frac{1}{2}-\lambda}\,, \tag{3.58}$$

which tends in general to infinity if λ is an (half odd or not) integer greater than $\frac{1}{2}$. In particular, if λ is a positive integer and $\nu+2n$ is not a negative integer then the first term equals

$$-i(-1)^n\,\frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\nu+2n)}{n\,\Gamma\left(n+\frac{1}{2}\right)}\sin\nu\pi,$$

where else the second term tends to infinity and therefore $D_\nu^n(x)\sim(1+x)^{\frac{1}{2}-n}$. If $\lambda=0$, (3.57) becomes an indeterminate form. For values of λ in the open interval from $-\frac{1}{2}$ to $\frac{1}{2}$, excluding zero,

$$\lim_{x\to-1^+} D_\nu^\lambda(x) = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+1)\,\Gamma\left(\lambda+\frac{1}{2}\right)}\,\frac{1}{\cos^2\lambda\pi}\left(\sin\lambda\pi\,\cos(\nu+\lambda)\pi - i\,e^{-i\lambda\pi}\,\sin(\nu+2\lambda)\pi\right).$$

Solving (3.54) with respect to the Gegenbauer functions of the first kind $C_\nu^\lambda(x)$, we obtain

$$C_\nu^\lambda(x) = 2^{2-2\lambda}\,\frac{\cos\lambda\pi}{\sin\lambda\pi}\,\frac{1}{(\Gamma(\lambda))^2}\,D_\nu^\lambda(x) + i\pi\,2^{1-2\lambda}e^{-i\lambda\pi}\,\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,(\Gamma(\lambda))^2}\,\frac{1}{\sin^2\lambda\pi}\,(1-x^2)^{\frac{1}{2}-\lambda}\,C_{\nu+2\lambda-1}^{1-\lambda}(x) \tag{3.59}$$

from which it can be seen that $C_\nu^\lambda(x)$ and $C_{\nu+2\lambda-1}^{1-\lambda}(x)$ are linearly independent on the cut. For positive half integer values of λ, namely $\lambda=n+\frac{1}{2}$, formula (3.59) reduces to

$$C_\nu^{n+\frac{1}{2}}(x) = (-1)^n\,\frac{\pi}{2^{2n}}\,\frac{\Gamma(\nu+2n+1)}{\Gamma(\nu+1)\,\left(\Gamma\left(n+\frac{1}{2}\right)\right)^2}\,(1-x^2)^{-n}\,C_{\nu+2n}^{-n+\frac{1}{2}}(x) \tag{3.60}$$

and for this values of λ, $C_\nu^{n+\frac{1}{2}}(x)$ and $C_{\nu+2n}^{-n+\frac{1}{2}}(x)$ become linearly dependent.
In order to study the behavior of $D_\nu^\lambda(x)$ for $\lambda=n+\frac{1}{2}$ as x tends to $\pm1^\mp$, we first observe that (3.54), in view of (3.60), becomes an indeterminate form $\frac{0}{0}$.
Hence, (3.54) for $\lambda=n+\frac{1}{2}$ reads

$$D_\nu^{n+\frac{1}{2}}(x) = \lim_{\lambda\to n+\frac{1}{2}}\frac{1}{\sin 2\lambda\pi}\left\{2^{2\lambda-1}\big(\sin\lambda\pi\,\Gamma(\lambda)\big)^2\,C_\nu^\lambda(x) - i\pi\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}\,(1-x^2)^{\frac{1}{2}-\lambda}\,e^{-i\lambda\pi}\,C_{\nu+2\lambda-1}^{1-\lambda}(x)\right\}. \tag{3.61}$$

Applying L' Hospital's rule and utilizing the property

$$\frac{\mathrm{d}}{\mathrm{d}z}\Gamma(z) = \Gamma(z)\,\psi(z)\,,$$

where $\psi(z)$ is the logarithmic derivative of the Gamma function, we find

$$\frac{\mathrm{d}}{d\lambda}\left(2^{2\lambda-1}\big(\sin\lambda\pi\,\Gamma\left(\lambda\right)\big)^2 C_\nu^\lambda(x)\right)\Bigg|_{\lambda=n+\frac{1}{2}} = 2^{2n}\left(\Gamma\left(n+\tfrac{1}{2}\right)\right)^2\Bigg(2\ln 2+2\psi(n+\tfrac{1}{2})$$
$$+\psi(n+1)-2\psi(2n+1)\Bigg)C_\nu^{n+\frac{1}{2}}(x)+2^{2n}\frac{\Gamma\left(\nu+2n+1\right)\left(\Gamma\left(n+\frac{1}{2}\right)\right)^2}{\Gamma\left(\nu+1\right)\,\Gamma\left(2n+1\right)}$$
$$\times\sum_{k=0}^{\infty}\Bigg(2\psi(\nu+2n+1+k)-\psi(n+1+k)\Bigg)\frac{(-\nu)_k\,(\nu+2n+1)_k}{k!\,(n+1)_k}\left(\frac{1-x}{2}\right)^k \tag{3.62}$$

and

$$\frac{\mathrm{d}}{d\lambda}\left(i\pi\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+1\right)}\left(1-x^2\right)^{\frac{1}{2}-\lambda}e^{-i\lambda\pi}\,C_{\nu+2\lambda-1}^{1-\lambda}(x)\right)=i\pi\,e^{-i\lambda\pi}\,\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+1\right)}\Bigg(-\ln(1-x^2)$$
$$-i\pi+2\ln 2+\psi(\lambda)+\psi(\lambda-\tfrac{1}{2})+2\psi(-\nu-2\lambda+1)-\psi(\tfrac{3}{2}-\lambda)+2\pi\frac{\cos 2\lambda\pi}{\sin 2\lambda\pi}\Bigg)\left(1-x^2\right)^{\frac{1}{2}-\lambda}$$
$$\times\,C_{\nu+2\lambda-1}^{1-\lambda}(x)-i\,e^{-i\lambda\pi}\,2^{2\lambda-2}\,\frac{\sin 2\lambda\pi}{\sqrt{\pi}}\Gamma\left(\lambda\right)\,\Gamma\left(\lambda-\tfrac{1}{2}\right)\,\left(1-x^2\right)^{\frac{1}{2}-\lambda}$$
$$\times\sum_{k=0}^{\infty}\Bigg(\psi(\tfrac{3}{2}-\lambda+k)-2\psi(-\nu-2\lambda+1+k)\Bigg)\frac{(\nu+1)_k\,(-\nu-2\lambda+1)_k}{k!\,(\frac{3}{2}-\lambda)_k}\left(\frac{1-x}{2}\right)^k. \tag{3.63}$$

From the theory of the $\Gamma-$functions (and therefore also for the $\psi-$functions) it is known that $\psi(-n)\to\infty$ for $n=0,1,2,\ldots$. Hence, utilizing the property

$$\psi(1-z)=\psi(z)+\pi\cot z\pi\,, \tag{3.64}$$

the term $2\psi(-\nu-2\lambda+1)-\psi(\frac{3}{2}-\lambda)$ becomes $2\psi(\nu+2\lambda)-\psi(\lambda-\frac{1}{2})+2\pi\cot(\nu+2\lambda)\pi+\pi\frac{\sin\lambda\pi}{\cos\lambda\pi}$ and

$$\frac{\mathrm{d}}{d\lambda}\left(i\pi\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+1\right)}\left(1-x^2\right)^{\frac{1}{2}-\lambda}e^{-i\lambda\pi}\,C_{\nu+2\lambda-1}^{1-\lambda}(x)\right)=i\pi\,e^{-i\lambda\pi}\,\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+1\right)}\Bigg(-\ln(1-x^2)$$
$$-i\pi+2\ln 2+\psi(\lambda)+2\psi(\nu+2\lambda)+2\pi\cot(\nu+2\lambda)\pi+\pi\cot\lambda\pi\Bigg)\left(1-x^2\right)^{\frac{1}{2}-\lambda}C_{\nu+2\lambda-1}^{1-\lambda}(x)$$
$$+\,i\,e^{-i\lambda\pi}\,\frac{2^{2\lambda-1}}{\sqrt{\pi}}\,\sin\lambda\pi\,\sin(\nu+2\lambda)\pi\,\Gamma\left(\lambda\right)\,\Gamma\left(\nu+2\lambda\right)\,\left(1-x^2\right)^{\frac{1}{2}-\lambda}$$
$$\times\sum_{k=0}^{\infty}\Bigg(\psi(\tfrac{3}{2}-\lambda+k)-2\psi(-\nu-2\lambda+1+k)\Bigg)\frac{(\nu+1)_k\,\Gamma\left(-\nu-2\lambda+1+k\right)}{k!\,\Gamma\left(\frac{3}{2}-\lambda+k\right)}\left(\frac{1-x}{2}\right)^k. \tag{3.65}$$

Putting everything together we find

$$\begin{aligned}
D_\nu^{n+\frac{1}{2}}(x) &= \frac{2^{2n-1}}{\pi}\left(\Gamma\left(n+\tfrac{1}{2}\right)\right)^2 \Big(-\ln(1-x^2) - i\pi + 2\psi(2n+1) - \psi(n+\tfrac{1}{2}) - \psi(n+1) \\
&+ 2\psi(\nu+2n+1) + 2\pi\cot\nu\pi\Big) C_\nu^{n+\frac{1}{2}}(x) - \frac{2^{2n-1}}{\pi}\frac{\Gamma(\nu+2n+1)\left(\Gamma\left(n+\frac{1}{2}\right)\right)^2}{\Gamma(\nu+1)\,\Gamma(2n+1)} \\
&\times \sum_{k=0}^{\infty}\Big(2\psi(\nu+2n+1+k) - \psi(n+1+k)\Big)\frac{(-\nu)_k\,(\nu+2n+1)_k}{k!\,(n+1)_k}\left(\frac{1-x}{2}\right)^k \\
&- \frac{2^{2n-1}}{\pi\sqrt{\pi}}\sin\nu\pi\,\Gamma(\nu+2n+1)\,\Gamma\left(n+\tfrac{1}{2}\right)(1-x^2)^{-n}\lim_{\lambda\to n+\frac{1}{2}}\sum_{k=0}^{\infty}\Big(\psi(\tfrac{3}{2}-\lambda+k) \\
&- 2\psi(-\nu-2\lambda+1+k)\Big)\frac{(\nu+1)_k\,\Gamma(-\nu-2\lambda+1+k)}{k!\,\Gamma\left(\frac{3}{2}-\lambda+k\right)}\left(\frac{1-x}{2}\right)^k . \qquad (3.66)
\end{aligned}$$

As mentioned, $\psi(-n)\to\infty$ for $n=0,1,2,\ldots$, and $\lim_{\lambda\to n+\frac{1}{2}}\psi(\frac{3}{2}-\lambda+k)$ makes sense only if $k>n-1$. Thus, splitting the last of the series into two parts, one with index counting from zero to $n-1$ and the other from n to infinity, i.e.

$$\sum_{k=0}^{\infty} = \sum_{k=0}^{n-1} + \sum_{k=n}^{\infty},$$

we merely notice that the second series behaves properly, where else the first one needs further manipulation. Employing (3.64) we find

$$\begin{aligned}
&\psi(\tfrac{3}{2}-\lambda+k) - 2\psi(-\nu-2\lambda+1+k) \\
&= \psi(\lambda-\tfrac{1}{2}-k) - 2\psi(\nu+2\lambda-k) + \pi\cot(\lambda-\tfrac{1}{2}-k)\pi - 2\pi\cot(\nu+2\lambda-k)\pi
\end{aligned}$$

and as λ tends to $n+\frac{1}{2}$

$$\begin{aligned}
&\lim_{\lambda\to n+\frac{1}{2}}\psi(\tfrac{3}{2}-\lambda+k) - 2\psi(-\nu-2\lambda+1+k) \\
&= \psi(n-k) - 2\psi(\nu+2n+1-k) + \pi\cot(n-k)\pi - 2\pi\cot(\nu+2n+1-k)\pi\,,
\end{aligned}$$

valid if $k<n$. Furthermore, making use of (3.3) it can be shown that

$$\lim_{\lambda\to n+\frac{1}{2}}\frac{1}{\Gamma\left(\frac{3}{2}-\lambda+k\right)} = \frac{\sin(n-k)\pi}{\pi}\,\Gamma(n-k)\ .$$

Finally

$$\begin{aligned}
D_\nu^{n+\frac{1}{2}}(x) = {} & \frac{2^{2n-1}}{\pi}\left(\Gamma\left(n+\tfrac{1}{2}\right)\right)^2 \Big(-\ln(1-x^2) - i\pi + 2\psi(2n+1) - \psi(n+\tfrac{1}{2}) - \psi(n+1) \\
& + 2\psi(\nu+2n+1) + 2\pi\cot\nu\pi \Big) C_\nu^{n+\frac{1}{2}}(x) - \frac{2^{2n-1}}{\pi}\frac{\Gamma(\nu+2n+1)\left(\Gamma\left(n+\frac{1}{2}\right)\right)^2}{\Gamma(\nu+1)\,\Gamma(2n+1)} \\
& \times \sum_{k=0}^{\infty}\Big(2\psi(\nu+2n+1+k) - \psi(n+1+k)\Big)\frac{(-\nu)_k\,(\nu+2n+1)_k}{k!\,(n+1)_k}\left(\frac{1-x}{2}\right)^k \\
& - (-1)^n\frac{2^{2n-1}}{\pi\sqrt{\pi}}\sin\nu\pi\,\Gamma(\nu+2n+1)\,\Gamma\left(n+\tfrac{1}{2}\right)(1-x^2)^{-n}\sum_{k=0}^{n-1}(-1)^k\frac{(\nu+1)_k}{k!\,\Gamma(n-k)} \\
& \times \Gamma(-\nu-2n+k)\left(\frac{1-x}{2}\right)^k - \frac{2^{n-1}}{\pi\sqrt{\pi}}\sin\nu\pi\,\Gamma(\nu+2n+1)\,\Gamma\left(n+\tfrac{1}{2}\right)(1+x)^{-n} \\
& \times \sum_{k=0}^{\infty}\Big(\psi(k+1) - 2\psi(-\nu-n+k)\Big)\frac{(\nu+1+n)_k\,(\nu+1)_n\,\Gamma(-\nu-n+k)}{k!\,(n+k)!}\left(\frac{1-x}{2}\right)^k
\end{aligned} \tag{3.67}$$

or, applying the relations in section 3.1 the latter can be simplified as

$$\begin{aligned}
D_\nu^{n+\frac{1}{2}}(x) = {} & \frac{2^{2n-1}}{\pi}\left(\Gamma\left(n+\tfrac{1}{2}\right)\right)^2 \Big(-\ln(1-x^2) - i\pi + 2\psi(2n+1) - \psi(n+\tfrac{1}{2}) - \psi(n+1) \\
& + 2\psi(\nu+2n+1) + 2\pi\cot\nu\pi \Big) C_\nu^{n+\frac{1}{2}}(x) - \frac{2^{2n-1}}{\pi}\frac{n!}{(2n)!}\left(\Gamma\left(n+\tfrac{1}{2}\right)\right)^2 \\
& \times \sum_{k=0}^{\infty}(-1)^k\Big(2\psi(\nu+2n+1+k) - \psi(n+1+k)\Big)\frac{\Gamma(\nu+2n+1+k)}{k!\,\Gamma(\nu+1-k)}\left(\frac{1-x}{2}\right)^k \\
& + (-1)^n\frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(\nu+2n+1)\,\Gamma\left(n+\tfrac{1}{2}\right)(1-x^2)^{-n}\sum_{k=0}^{n-1}\frac{(\nu+1)_k\,(n-k-1)!}{k!\,\Gamma(\nu+2n+1-k)}\left(\frac{1-x}{2}\right)^k \\
& + \frac{2^{n-1}}{\sqrt{\pi}}\Gamma(\nu+2n+1)\,\Gamma\left(n+\tfrac{1}{2}\right)(1+x)^{-n}\sum_{k=0}^{\infty}\Big(\psi(k+1) - 2\psi(-\nu-n+k)\Big) \\
& \times \frac{(\nu+1+n)_k\,(\nu+1)_n}{k!\,(n+k)!\,\Gamma(\nu+1+n-k)}\left(\frac{1-x}{2}\right)^k,
\end{aligned} \tag{3.68}$$

where the finite series is to be neglected if $n = 0$.
From (3.67) we see that if $x \to 1^-$, $D_\nu^{n+\frac{1}{2}}(x)$ tends to infinity in general. In particular, if $n = 0$ and by the fact that $C_\nu^{\frac{1}{2}}(1) \equiv P_\nu(1) = 1$, $D_\nu^{\frac{1}{2}}(x) \equiv Q_\nu(x) \sim \ln(1-x)$ and for $n > 0$, $D_\nu^{n+\frac{1}{2}}(x) \sim (1-x)^{-n}$. Repeating the aforementioned procedure in the case where x tends to -1^+, one obtains $D_\nu^{\frac{1}{2}}(x) \equiv Q_\nu(x) \sim \ln(1+x)$ if $n = 0$ and $D_\nu^{n+\frac{1}{2}}(x) \sim (1+x)^{-n}$ if $n > 0$.

3.7 Some Relations satisfied by the Gegenbauer functions [†]

The Gegenbauer differential equation (3.9) does not change if we replace ν by $-\nu-2\lambda$ or z by $-z$, and therefore it has solutions $C^{\lambda}_{-\nu-2\lambda}(z)$, $D^{\lambda}_{-\nu-2\lambda}(z)$, $C^{\lambda}_{\nu}(-z)$ and $D^{\lambda}_{\nu}(-z)$, as well as $C^{\lambda}_{\nu}(z)$ and $D^{\lambda}_{\nu}(z)$. Since every three solutions of a second-order linear differential equation are linearly dependent, there exist certain functional relations between the solutions just enumerated. The simplest such relation is the formula

$$C^{\lambda}_{-\nu-2\lambda}(z) = -\frac{\sin(\nu+2\lambda)\pi}{\sin\nu\pi}\, C^{\lambda}_{\nu}(z)\,, \tag{3.69}$$

which is easily proven by the definition of the Gegenbauer function of the first kind (3.12) and the symmetry property of the hypergeometric function

$$F\left(\alpha,\beta,\gamma;z\right) = F\left(\beta,\alpha,\gamma;z\right)\,.$$

To obtain a relation connecting $C^{\lambda}_{\nu}(z)$, $D^{\lambda}_{\nu}(z)$ and $D^{\lambda}_{-\nu-2\lambda}(z)$ we assume temporarily that $z>1$. It follows then that

$$C^{\lambda}_{\nu}(z) = 2^{1-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma\left(\lambda\right)\right)^{2}\,\sin(\nu+\lambda)\pi}\left(D^{\lambda}_{\nu}(z) - D^{\lambda}_{-\nu-2\lambda}(z)\right). \tag{3.70}$$

Indeed, utilizing formula (3.18)

$$\begin{aligned} C^{\lambda}_{\nu}(z) &= \frac{\Gamma\left(\nu+\lambda\right)}{\Gamma\left(\nu+1\right)\,\Gamma\left(\lambda\right)}\,(2z)^{\nu}\,F\left(-\frac{\nu}{2},\frac{1-\nu}{2},-\nu-\lambda+1;\frac{1}{z^{2}}\right)\\ &+\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+\lambda+1\right)\,\Gamma\left(\lambda\right)}\,\frac{\sin\nu\pi}{\sin(\nu+\lambda)\pi}\,(2z)^{-\nu-2\lambda}\,F\left(\frac{\nu+2\lambda+1}{2},\frac{\nu+2\lambda}{2},\nu+\lambda+1;\frac{1}{z^{2}}\right) \end{aligned}$$

we notice that the second term on the right-hand side of the above equation can be rewritten as

$$\begin{aligned} &\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+\lambda+1\right)\,\Gamma\left(\lambda\right)}\,\frac{\sin\nu\pi}{\sin(\nu+\lambda)\pi}\,(2z)^{-\nu-2\lambda}\,F\left(\frac{\nu+2\lambda+1}{2},\frac{\nu+2\lambda}{2},\nu+\lambda+1;\frac{1}{z^{2}}\right)\\ &=\frac{1}{2^{2\lambda-1}}\,\frac{1}{\left(\Gamma\left(\lambda\right)\right)^{2}}\,\frac{\sin\nu\pi}{\sin(\nu+\lambda)\pi}\,D^{\lambda}_{\nu}(z)\,, \end{aligned} \tag{3.71}$$

where we used relation (3.13)

$$D^{\lambda}_{\nu}(z) = 2^{2\lambda-1}\,\frac{\Gamma\left(\nu+2\lambda\right)\,\Gamma\left(\lambda\right)}{\Gamma\left(\nu+\lambda+1\right)}\,(2z)^{-\nu-2\lambda}\,F\left(\frac{\nu+2\lambda+1}{2},\frac{\nu+2\lambda}{2},\nu+\lambda+1;\frac{1}{z^{2}}\right)\,. \tag{3.72}$$

Replace in the latter expression ν with $-\nu-2\lambda$ to obtain

$$D^{\lambda}_{-\nu-2\lambda}(z) = 2^{2\lambda-1}\,\frac{\Gamma\left(-\nu\right)\,\Gamma\left(\lambda\right)}{\Gamma\left(-\nu-\lambda+1\right)}\,(2z)^{\nu}\,F\left(-\frac{\nu}{2},\frac{1-\nu}{2},-\nu-\lambda+1;\frac{1}{z^{2}}\right)\,. \tag{3.73}$$

Employing the very useful relation (3.4) satisfied by the $\Gamma-$function we obtain at once

$$\frac{\Gamma\left(-\nu\right)\,\Gamma\left(\lambda\right)}{\Gamma\left(-\nu-\lambda+1\right)} = -\frac{\Gamma\left(\lambda\right)\,\Gamma\left(\nu+\lambda\right)\,\sin(\nu+\lambda)\pi}{\Gamma\left(\nu+1\right)\,\sin\nu\pi}\,,$$

[†] Based on section 7.5 of [Leb72]

and the foregoing expression for $D^{\lambda}_{-\nu-2\lambda}(z)$ reads

$$D^{\lambda}_{-\nu-2\lambda}(z) = -2^{2\lambda-1}\,\frac{\Gamma(\nu+\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+1)}\,\frac{\sin(\nu+\lambda)\pi}{\sin\nu\pi}\,(2z)^{\nu}\,F\left(-\frac{\nu}{2},\frac{1-\nu}{2},-\nu-\lambda+1;\frac{1}{z^2}\right). \tag{3.74}$$

The first term on the right-hand side of (3.18) becomes now

$$\frac{\Gamma(\nu+\lambda)}{\Gamma(\nu+1)\,\Gamma(\lambda)}\,(2z)^{\nu}\,F\left(-\frac{\nu}{2},\frac{1-\nu}{2},-\nu-\lambda+1;\frac{1}{z^2}\right)$$
$$= -\frac{1}{2^{2\lambda-1}}\,\frac{1}{\left(\Gamma(\lambda)\right)^2}\,\frac{\sin\nu\pi}{\sin(\nu+\lambda)\pi}\,D^{\lambda}_{-\nu-2\lambda}(z)\,,$$

and (3.18) becomes (3.70).
Note that for $\lambda=\frac{1}{2}$ we obtain the known expression

$$P_{\nu}(z) = \frac{1}{\pi}\,\tan\nu\pi\,\Big(Q_{\nu}(z)-Q_{-\nu-1}(z)\Big).$$

Formula (3.70) remains valid for all z cut along $(-\infty,+1]$, since in this region both sides are analytic functions of z.
Another relation between the solutions of (3.9) enumerated above, can be derived assuming temporarily that $|z|>1$ and $|\arg z|<\pi$. Then formula (3.13) gives

$$D^{\lambda}_{\nu}(-z) = e^{\pm i2\lambda\pi}\,e^{\pm i\nu\pi}\,D^{\lambda}_{\nu}(z), \quad \nu+2\lambda\neq 0,-1,-2,\dots\,, \tag{3.75}$$

where the upper sign corresponds to $\operatorname{Im} z>0$ and the lower sign to $\operatorname{Im} z<0$.
Using the principle of analytic continuation, we can drop the condition $|z|>1$, thereby establishing the validity of (3.75) for arbitrary z in the plane cut along $(-\infty,+1]$ and arbitrary $\nu+2\lambda\neq 0,-1,-2,\dots$.
Replacing z by $-z$ in (3.70) which combined with (3.75) yields

$$C^{\lambda}_{\nu}(-z) = 2^{1-2\lambda}\,\frac{1}{\left(\Gamma(\lambda)\right)^2}\,\frac{\sin\nu\pi}{\sin(\nu+\lambda)\pi}\left(e^{\pm i2\lambda\pi}\,e^{\pm i\nu\pi}\,D^{\lambda}_{\nu}(z) - e^{\mp i\nu\pi}\,D^{\lambda}_{-\nu-2\lambda}(z)\right). \tag{3.76}$$

Eliminating $D^{\lambda}_{-\nu-2\lambda}(z)$ using (3.70) gives

$$C^{\lambda}_{\nu}(-z) = \pm i2^{2-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma(\lambda)\right)^2}\,e^{\pm i\lambda\pi}\,D^{\lambda}_{\nu}(z) + e^{\mp i\nu\pi}\,C^{\lambda}_{\nu}(z)\,. \tag{3.77}$$

The upper sign is chosen if $\operatorname{Im} z>0$ and the lower sign if $\operatorname{Im} z<0$. Equation (3.77) shows the nature of the singularity of $C^{\lambda}_{\nu}(z)$. Unless ν is an integer or zero, $C^{\lambda}_{\nu}(z)$ has a logarithmic singularity at $z=-1$. In the case where ν is an integer n, then (3.77) becomes

$$C^{\lambda}_{n}(-z) = (-1)^n\,C^{\lambda}_{n}(z)\,. \tag{3.78}$$

As z approaches the real line it follows from (3.77) that

$$i2^{2-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma(\lambda)\right)^2}\,e^{i\lambda\pi}\,D^{\lambda}_{\nu}(x+i0) = C^{\lambda}_{\nu}(-x) - e^{-i\nu\pi}\,C^{\lambda}_{\nu}(x)\,, \tag{3.79}$$

$$-i2^{2-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma\left(\lambda\right)\right)^2}\,e^{-i\lambda\pi}\,D^\lambda_\nu(x-i0)=C^\lambda_\nu(-x)-e^{i\nu\pi}\,C^\lambda_\nu(x)\,. \tag{3.80}$$

Eliminating $C^\lambda_\nu(-x)$ from (3.79) and (3.80) recovers (3.51).
In order to obtain a relation which connects the Gegenbauer functions C^λ_ν, D^λ_ν and $D^\lambda_{-\nu-2\lambda}$ on the real line, similar to (3.70), we work as follows.
As seen, (3.70) is valid in the whole complex $z-$plane. Assume that z approaches the real axis from the upper bank. Then (3.70) becomes

$$C^\lambda_\nu(x+i\,\varepsilon)=2^{1-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma\left(\lambda\right)\right)^2\,\sin(\nu+\lambda)\pi}\Big(D^\lambda_\nu(x+i\,\varepsilon)-D^\lambda_{-\nu-2\lambda}(x+i\,\varepsilon)\Big)\,. \tag{3.81}$$

As ε tends to zero and since $C^\lambda_\nu(x+i0)=C^\lambda_\nu(x)$, the above relation rewrites

$$C^\lambda_\nu(x)=2^{1-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma\left(\lambda\right)\right)^2\,\sin(\nu+\lambda)\pi}\Big(D^\lambda_\nu(x+i0)-D^\lambda_{-\nu-2\lambda}(x+i0)\Big)\,. \tag{3.82}$$

Similarly, let z approach the real axis from the lower bank. Then as $\varepsilon\to 0$, (3.70) becomes

$$C^\lambda_\nu(x)=2^{1-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma\left(\lambda\right)\right)^2\,\sin(\nu+\lambda)\pi}\Big(D^\lambda_\nu(x-i0)-D^\lambda_{-\nu-2\lambda}(x-i0)\Big)\,. \tag{3.83}$$

Multiply equation (3.82) throughout $-\frac{i}{2}\,e^{i\lambda\pi}$ and (3.83) by $\frac{i}{2}\,e^{-i\lambda\pi}$. Adding the resulting equations and taking into account (3.53) yields

$$\sin\lambda\pi\,C^\lambda_\nu(x)=2^{1-2\lambda}\,\frac{\sin\nu\pi}{\left(\Gamma\left(\lambda\right)\right)^2\,\sin(\nu+\lambda)\pi}\Big(D^\lambda_\nu(x)-D^\lambda_{-\nu-2\lambda}(x)\Big)\,. \tag{3.84}$$

3.8 Series representation of the Gegenbauer functions‡

To derive expansions of the Gegenbauer functions which hold in the part of the cut plane where $|z|<1$, we first note that the substitution $t=z^2$ transforms the Gegenbauer differential equation (3.9) into

$$t\,(1-t)\,\frac{d^2w(t)}{\mathrm{d}t^2}+\left(\frac{1}{2}-(\lambda+1)\,t\right)\frac{\mathrm{d}\,w(t)}{\mathrm{d}t}-\left(-\frac{\nu}{2}\right)\left(\frac{\nu}{2}+\lambda\right)w(t)=0\,, \tag{3.85}$$

which is a special case of the hypergeometric equation

$$t\,(1-t)\,\frac{d^2u}{\mathrm{d}t^2}+\big(\gamma-(\alpha+\beta+1)\,t\big)\,\frac{du}{\mathrm{d}t}-\alpha\,\beta\,u=0\,, \tag{3.86}$$

corresponding to the values

$$\alpha=-\frac{\nu}{2},\quad \beta=\frac{\nu}{2}+\lambda,\quad \gamma=\frac{1}{2}\,.$$

The general solution of (3.85) for $|z|<1$ can be written in the form [Leb72, p.163, eq.(7.2.6)]

$$w(z)=A(\nu;\lambda)\,F\left(-\frac{\nu}{2},\frac{\nu+2\lambda}{2},\frac{1}{2};z^2\right)+B(\nu;\lambda)\,z\,F\left(\frac{1-\nu}{2},\frac{\nu+2\lambda+1}{2},\frac{3}{2};z^2\right)\,, \tag{3.87}$$

‡Based on section 7.6 of [Leb72]

where A and B are arbitrary functions of the variables ν and λ. In particular, if the values of these functions are chosen to be $A = C_\nu^\lambda(0)$, $B = \frac{\mathrm{d}}{\mathrm{d}z} C_\nu^\lambda(z)\Big|_{z=0}$, then $w \equiv C_\nu^\lambda(z)$, and to obtain the desired expansion, we need only calculate the values of the Gegenbauer functions $C_\nu^\lambda(z)$ and its derivative at the point $z = 0$.
With this aim, we set $z = 0$ in the series (3.12), obtaining

$$
\begin{aligned}
C_\nu^\lambda(0) &= \frac{\Gamma(\nu + 2\lambda)}{\Gamma(\nu + 1)\, \Gamma(2\lambda)}\, F\left(-\nu, \nu + 2\lambda, \lambda + \frac{1}{2}; \frac{1}{2}\right) \\
&= \frac{\Gamma(\nu + 2\lambda)}{\Gamma(\nu + 1)\, \Gamma(2\lambda)} \sum_{n=0}^{+\infty} \frac{(-\nu)_n\, (\nu + 2\lambda)_n}{n!\, \left(\lambda + \frac{1}{2}\right)_n} \frac{1}{2^n}\,,
\end{aligned}
\tag{3.88}
$$

where $(\alpha)_n$ is the Pochhammer symbol defined as $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(n)}$.
Thus,

$$
C_\nu^\lambda(0) = -\frac{\sin \nu\pi}{\pi} \frac{\sqrt{\pi}}{2^{2\lambda-1}\, \Gamma(\lambda)} \sum_{n=0}^{+\infty} \frac{\Gamma(n - \nu)\, \Gamma(n + \nu + 2\lambda)}{n!\, \Gamma\left(n + \lambda + \frac{1}{2}\right)} \frac{1}{2^n}\,,
\tag{3.89}
$$

where we employed formula (3.3) together with the duplication theorem (3.1) from the theory of the Γ−functions.
In the sequence, consider the Beta function $B(x, y)$ defined as

$$
B(x, y) = \int_0^1 t^{x-1}\, (1 - t)^{y-1}\, \mathrm{d}t, \quad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0
\tag{3.90}
$$

which can be, more conveniently, expressed as the ratio of Γ−functions

$$
B(x, y) = \frac{\Gamma(x)\, \Gamma(y)}{\Gamma(x + y)}\,.
\tag{3.91}
$$

Let $x + y$ equal to $n + \lambda + \frac{1}{2}$, so that (3.91) becomes

$$
B\left(n - \nu, \nu + \lambda + \tfrac{1}{2}\right) = \frac{\Gamma(n - \nu)\, \Gamma\left(\nu + \lambda + \frac{1}{2}\right)}{\Gamma\left(n + \lambda + \frac{1}{2}\right)}, \quad n = 0, 1, 2, \ldots\,.
\tag{3.92}
$$

In view of equations (3.90) and (3.92), (3.89) yields

$$
\begin{aligned}
C_\nu^\lambda(0) &= -\frac{\sin \nu\pi}{\pi} \frac{\sqrt{\pi}}{2^{2\lambda-1}\, \Gamma(\lambda)} \sum_{n=0}^{+\infty} \frac{\Gamma(n + \nu + 2\lambda)}{2^n\, n!\, \Gamma\left(\nu + \lambda + \frac{1}{2}\right)} \int_0^1 t^{n-\nu-1}\, (1 - t)^{\nu+\lambda-\frac{1}{2}}\, \mathrm{d}t \\
&= -\frac{\sin \nu\pi}{\pi} \frac{\Gamma(\nu + 2\lambda)}{\Gamma\left(\nu + \lambda + \frac{1}{2}\right)} \frac{\sqrt{\pi}}{2^{2\lambda-1}\, \Gamma(\lambda)} \int_0^1 t^{-\nu-1}\, (1 - t)^{\nu+\lambda-\frac{1}{2}} \left(1 - \frac{t}{2}\right)^{-\nu-2\lambda} \mathrm{d}t\,,
\end{aligned}
\tag{3.93}
$$

where the reversal of the order of summation and integration is justified by an absolute convergence argument.
The last line of the above expression is obtained employing the binomial series

$$
(1 - z)^{-r} = \sum_{n=0}^{+\infty} \frac{(r)_n}{n!}\, z^n = F(r, 1, 1; z)\,, \quad |z| < 1\,,
$$

where r is an arbitrary variable (complex or real) and

$$\binom{r}{n} = \frac{(r)_n}{n!},$$

are the binomial coefficients.
With this in mind, equation (3.93) becomes

$$C_\nu^\lambda(0) = -\frac{\sin\nu\pi}{\pi}\frac{\Gamma(\nu+2\lambda)}{\Gamma\left(\nu+\lambda+\frac{1}{2}\right)}\frac{\sqrt{\pi}}{2^{2\lambda-1}\,\Gamma(\lambda)}\int_0^1 t^{-\nu-1}\,(1-t)^{\nu+\lambda-\frac{1}{2}}\,F\left(\nu+2\lambda,1,1;\frac{t}{2}\right)\,\mathrm{d}t\,,$$

which is handled by means of the Beta function (see, e.g. [GR00, pp 806-807]).

A more convenient way is to employ the following summation formula ([WG89, eq.(3), p. 185]) based on Kummer's formula

$$F\left(\alpha,\beta,\frac{1+\alpha+\beta}{2};\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1+\alpha+\beta}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)\,\Gamma\left(\frac{1+\beta}{2}\right)}\,.$$

Hence, (3.88) rewrites

$$C_\nu^\lambda(0) = \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\,\Gamma(2\lambda)}\,\frac{\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)\,\Gamma\left(\frac{1+\nu+2\lambda}{2}\right)}\,. \tag{3.94}$$

Since both sides of (3.94) are entire functions of ν, our result holds for arbitrary values of ν. Using (3.3) we can write (3.94) also in the form

$$C_\nu^\lambda(0) = \frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{1+\nu}{2}\right)\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)\,\Gamma(2\lambda)\,\Gamma\left(\frac{1+\nu+2\lambda}{2}\right)}\,\cos\frac{\nu\pi}{2}\,. \tag{3.95}$$

Once we have found $C_\nu^\lambda(0)$, we can easily deduce $\frac{\mathrm{d}}{\mathrm{d}z}\,C_\nu^\lambda(0)$ by using the recurrence formula

$$(1-z^2)\,\frac{\mathrm{d}}{\mathrm{d}z}\,C_\nu^\lambda(z) + \nu\,z\,C_\nu^\lambda(z) - (\nu+2\lambda-1)\,C_{\nu-1}^\lambda(z) = 0\,,$$

obtained, relating the Gegenbauer functions of the first kind $C_\nu^\lambda(z)$ with the associated Legendre functions of the first kind $P_\nu^\mu(z)$ via

$$P_\nu^\mu(z) = 2^\mu\,\frac{\Gamma(\nu+\mu+1)\,\Gamma(1-2\mu)}{\Gamma(\nu-\mu+1)\,\Gamma(1-\mu)}\,(z^2-1)^{-\frac{\mu}{2}}\,C_{\nu+\mu}^{-\mu+\frac{1}{2}}(z)\,,$$

and employing the recurrence relation

$$(z^2-1)\,\frac{\mathrm{d}}{\mathrm{d}z}\,P_\nu^\mu(z) = \nu\,z\,P_\nu^\mu(z) - (\nu+\mu)\,P_{\nu-1}^\mu(z)\,.$$

This gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\,C_\nu^\lambda(0) &= (\nu+2\lambda-1)\,C_{\nu-1}^\lambda(0)\\ &= \frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{\nu}{2}\right)\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu)\,\Gamma(2\lambda)\,\Gamma\left(\frac{\nu+2\lambda}{2}\right)}\,\sin\frac{\nu\pi}{2}\,,\end{aligned} \tag{3.96}$$

where we take account of formula (3.5).
Combining (3.87), (3.95) and (3.96), we obtain the following series expansion of the Gegenbauer function of the first kind, valid for $|z| < 1$ and arbitrary ν

$$\begin{aligned} C_\nu^\lambda(z) =& \frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{1+\nu}{2}\right)\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)\,\Gamma(2\lambda)\,\Gamma\left(\frac{1+\nu+2\lambda}{2}\right)} \cos\frac{\nu\pi}{2}\, F\left(-\frac{\nu}{2}, \frac{\nu+2\lambda}{2}, \frac{1}{2}; z^2\right) \\ &+ \frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{\nu}{2}\right)\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu)\,\Gamma(2\lambda)\,\Gamma\left(\frac{\nu+2\lambda}{2}\right)} \sin\frac{\nu\pi}{2}\, z\, F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; z^2\right), \end{aligned} \tag{3.97}$$

which, making use of the duplication formula (3.1) reads

$$\begin{aligned} C_\nu^\lambda(z) =& 2^{1-2\lambda}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\lambda)\,\Gamma(\nu+1)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)} \cos\frac{\nu\pi}{2}\, F\left(-\frac{\nu}{2}, \frac{\nu+2\lambda}{2}, \frac{1}{2}; z^2\right) \\ &+ 2^{1-2\lambda}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{\nu}{2}\right)}{\Gamma(\lambda)\,\Gamma(\nu)\,\Gamma\left(\frac{\nu+2\lambda}{2}\right)} \sin\frac{\nu\pi}{2}\, z\, F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; z^2\right). \end{aligned} \tag{3.98}$$

The corresponding expansion for the Gegenbauer function of the second kind is obtained from (3.77) with the aid of (3.98). After simple manipulations we arrive at

$$\begin{aligned} e^{\pm i\lambda\pi}\, D_\nu^\lambda(z) =& e^{\mp i\frac{\nu\pi}{2}}\,\frac{\Gamma(\lambda)}{2}\left[\frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)}\, F\left(-\frac{\nu}{2}, \frac{\nu+2\lambda}{2}, \frac{1}{2}; z^2\right)\right. \\ &\left.\pm i\,\frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\frac{\nu}{2}\right)}{\Gamma(\nu)\,\Gamma\left(\frac{\nu+2\lambda}{2}\right)}\, z\, F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; z^2\right)\right], \end{aligned} \tag{3.99}$$

where $|z| < 1$, and the upper sign is chosen if $\operatorname{Im} z > 0$, and the lower sign if $\operatorname{Im} z < 0$.
A formula of practical interest is the series expansion of $D_\nu^\lambda(x)$, obtained from (3.53) and (3.99)

$$\begin{aligned} D_\nu^\lambda(x) =& \frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)\,\Gamma\left(\frac{\nu}{2}\right)}{2\,\Gamma(\nu)\,\Gamma\left(\frac{\nu+2\lambda}{2}\right)} \cos\frac{\nu\pi}{2}\, x\, F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; x^2\right) \\ &- \frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)\,\Gamma\left(\frac{\nu+1}{2}\right)}{2\,\Gamma(\nu+1)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)} \sin\frac{\nu\pi}{2}\, F\left(-\frac{\nu}{2}, \frac{\nu+2\lambda}{2}, \frac{1}{2}; x^2\right), \end{aligned} \tag{3.100}$$

for every $-1 < x < +1$.* From equations (3.98) and (3.100) it is straightforward to show

*Utilizing the alternative definition for the Gegenbauer functions of the second kind

$$D_\nu^\lambda(x) = \frac{i}{2}\left(e^{-i\lambda\pi}\, D_\nu^\lambda(x+i0) + e^{i\lambda\pi}\, D_\nu^\lambda(x-i0)\right),$$

introduced in section 3.6, the series expansion of $D_\nu^\lambda(x)$ yields

$$\begin{aligned} D_\nu^\lambda(x) =& i\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)\,\Gamma\left(\frac{\nu+1}{2}\right)}{2\,\Gamma(\nu+1)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)} \cos\left(\frac{\nu}{2}+2\lambda\right)\pi\, F\left(-\frac{\nu}{2}, \frac{\nu+2\lambda}{2}, \frac{1}{2}; x^2\right) \\ &+ i\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)\,\Gamma\left(\frac{\nu}{2}\right)}{2\,\Gamma(\nu)\,\Gamma\left(\frac{\nu+2\lambda}{2}\right)} \sin\left(\frac{\nu}{2}+2\lambda\right)\pi\, x\, F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; x^2\right). \end{aligned}$$

that

$$\left[C_\nu^\lambda(x)\,\frac{dD_\nu^\lambda(x)}{\mathrm{d}\,x} - D_\nu^\lambda(x)\,\frac{dC_\nu^\lambda(x)}{\mathrm{d}\,x}\right]_{x=0} = \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}\,. \tag{3.101}$$

Furthermore, we have from section 3.4 that the Wronskian of the independent pair $\left(C_\nu^\lambda(x), D_\nu^\lambda(x)\right)$ is

$$W\left(C_\nu^\lambda(x), D_\nu^\lambda(x)\right) = \frac{\mathfrak{C}(\nu;\lambda)}{(1-x^2)^{\lambda+\frac{1}{2}}}\,,$$

where the unknown function $\mathfrak{C}(\nu;\lambda)$ may be determined putting $x = 0$ and using (3.101). Thus, we obtain

$$C_\nu^\lambda(x)\,\frac{dD_\nu^\lambda(x)}{\mathrm{d}\,x} - D_\nu^\lambda(x)\,\frac{dC_\nu^\lambda(x)}{\mathrm{d}\,x} = \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)}\,\frac{1}{(1-x^2)^{\lambda+\frac{1}{2}}}\,. \tag{3.102}$$

3.9 Asymptotic expansions for the Gegenbauer functions

To obtain the asymptotic expansion of the Gegenbauer functions, the clue lies in rewriting the hypergeometric function in such a way, so that ν occurs only in the third parameter γ. Starting from the Gegenbauer function of the second kind

$$D_\nu^\lambda(z) = 2^{2\lambda-1}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+\lambda+1)}\,(2z)^{-\nu-2\lambda}\,F\left(\frac{\nu+2\lambda+1}{2}, \frac{\nu+2\lambda}{2}, \nu+\lambda+1; \frac{1}{z^2}\right)\,,$$

and applying successively the transformations (3.47) and [WG89, p. 183, eq.(15)]

$$F\left(\alpha, \alpha+\frac{1}{2}, \gamma; z\right) = (1-z)^{-\alpha}\,F\left(2\alpha, 2\gamma-2\alpha-1, \gamma; \frac{\sqrt{1-z}-1}{2\sqrt{1-z}}\right)\,,$$

we arrive at

$$\begin{aligned} D_\nu^\lambda(z) =& 2^{\lambda-1}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+\lambda+1)}\,\left(z^2-1\right)^{-\frac{\lambda}{2}}\,\left(\sqrt{z^2-1}+z\right)^{-\nu-\lambda} \\ & \times F\left(1-\lambda, \lambda, \nu+\lambda+1; \frac{\sqrt{z^2-1}-z}{2\sqrt{z^2-1}}\right)\,. \end{aligned} \tag{3.103}$$

Let $z = \cos\theta \pm i\,0$ to find

$$D_\nu^\lambda(\cos\theta \pm i\,0) = 2^{\lambda-1}\,\frac{\Gamma(\lambda)}{(\sin\theta)^\lambda}\,\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1)}\,e^{\mp i\left[(\nu+\lambda)\theta+\frac{\pi\lambda}{2}\right]}\,F\left(1-\lambda, \lambda, \nu+\lambda+1; \frac{\pm i\,e^{\mp i\theta}}{2\sin\theta}\right) \tag{3.104}$$

where the upper sign corresponds to $\operatorname{Im} z > 0$ and the lower sign to $\operatorname{Im} z < 0$.
Substituting (3.104) into (3.51) we immediately obtain the asymptotic expansion for the Gegenbauer function of the first kind $C_\nu^\lambda(\cos\theta)$

$$\begin{aligned} C_\nu^\lambda(\cos\theta) =& \frac{1}{2^\lambda\,\Gamma(\lambda)\,(\sin\theta)^\lambda}\,\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1)}\left[e^{-i\left[(\nu+\lambda)\theta-\frac{\pi\lambda}{2}\right]}\,F\left(1-\lambda, \lambda, \nu+\lambda+1; \frac{i\,e^{-i\theta}}{2\sin\theta}\right)\right. \\ & \left. + e^{i\left[(\nu+\lambda)\theta-\frac{\pi\lambda}{2}\right]}\,F\left(1-\lambda, \lambda, \nu+\lambda+1; \frac{-i\,e^{i\theta}}{2\sin\theta}\right)\right]\,. \end{aligned}$$

Expressing the above result in terms of the hypergeometric series, we find

$$C_\nu^\lambda(\cos\theta) = \frac{2^{1-\lambda}}{\Gamma(\lambda)\ (\sin\theta)^\lambda} \frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1)} \sum_{n=0}^{+\infty} \frac{(1-\lambda)_n\,(\lambda)_n}{n!\,(\nu+\lambda+1)_n\,(2\sin\theta)^n}$$
$$\times\cos\left[(\nu+\lambda+n)\theta - \frac{\lambda+n}{2}\pi\right], \quad \varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0)\ . \tag{3.105}$$

When $\mathrm{Re}\,\nu\to\infty$, the series on the right-hand side of (3.105) is the asymptotic expansion of $C_\nu^\lambda(\cos\theta)$ for fixed λ.
Rewrite (3.105) as

$$C_\nu^\lambda(\cos\theta) = 2^{1-\lambda}\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1)}\frac{1}{\Gamma(\lambda)\ (\sin\theta)^\lambda}\left[\cos\left[(\nu+\lambda)\theta-\frac{\pi\lambda}{2}\right]+\mathcal{O}\left(\frac{1}{\nu}\right)\right],$$
$$\varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0)\ . \tag{3.106}$$

Furthermore, from the asymptotic expansion of the $\Gamma-$function

$$\frac{\Gamma(\nu+\alpha)}{\Gamma(\nu+\beta)} = \nu^{\alpha-\beta}\left[1+\mathcal{O}\left(\frac{1}{\nu}\right)\right], \quad |\arg\nu|<\pi\ ,$$

we have

$$\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1)} = \nu^{\lambda-1}\left[1+\mathcal{O}\left(\frac{1}{\nu}\right)\right], \quad |\arg\nu|<\pi\,, \tag{3.107}$$

and therefore

$$C_\nu^\lambda(\cos\theta) = \frac{1}{\Gamma(\lambda)\ (\sin\theta)^\lambda}\left(\frac{\nu}{2}\right)^{\lambda-1}\cos\left[(\nu+\lambda)\theta-\frac{\pi\lambda}{2}\right]+\mathcal{O}\left(\frac{1}{\nu}\right),$$
$$\varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0), \quad |\arg\nu|<\pi\ . \tag{3.108}$$

Also, substituting (3.104) into (3.53) we find the corresponding asymptotic expansion for the Gegenbauer function of the second kind.

$$D_\nu^\lambda(\cos\theta) = -\,2^{\lambda-1}\frac{\Gamma(\lambda)}{(\sin\theta)^\lambda}\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1)}\sum_{n=0}^{+\infty}\frac{(1-\lambda)_n\,(\lambda)_n}{n!\,(\nu+\lambda+1)_n\,(2\sin\theta)^n}$$
$$\sin\left[(\nu+\lambda+n)\theta-\frac{n+\lambda}{2}\pi\right], \varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0)\ . \tag{3.109}$$

Employing (3.107) the above equation reduces to

$$D_\nu^\lambda(\cos\theta) = -\frac{\Gamma(\lambda)}{(\sin\theta)^\lambda}(2\nu)^{\lambda-1}\sin\left[(\nu+\lambda)\theta-\frac{\pi\lambda}{2}\right]+\mathcal{O}\left(\frac{1}{\nu}\right),$$
$$\varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0), \quad |\arg\nu|<\pi\ . \tag{3.110}$$

The corresponding expressions for $\mathrm{Re}\,\nu\to-\infty$ can be determined with the aid of the symmetry relations (3.69) and (3.70).

More specific, substituting (3.105) into (3.69) we easily obtain

$$C^{\lambda}_{-\nu-2\lambda}(\cos\theta) = -2^{1-\lambda}\,\frac{\sin(\nu+2\lambda)\pi}{\sin\nu\pi}\,\frac{\Gamma\left(\nu+2\lambda\right)}{\Gamma\left(\nu+\lambda+1\right)}\,\frac{1}{\Gamma\left(\lambda\right)\,\left(\sin\theta\right)^{\lambda}}$$
$$\times\sum_{n=0}^{+\infty}\frac{(1-\lambda)_n\,(\lambda)_n}{n!\,(\nu+\lambda+1)_n\,(2\sin\theta)^n}\cos\left[(\nu+\lambda+n)\theta-\frac{\lambda+n}{2}\pi\right],\quad \varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0)\,. \tag{3.111}$$

or, in a more compact form

$$C^{\lambda}_{-\nu-2\lambda}(\cos\theta) = -\frac{1}{\Gamma\left(\lambda\right)\,\left(\sin\theta\right)^{\lambda}}\,\left(\frac{\nu}{2}\right)^{\lambda-1}\frac{\sin(\nu+2\lambda)\pi}{\sin\nu\pi}\cos\left[(\nu+\lambda)\theta-\frac{\pi\lambda}{2}\right]+\mathcal{O}\left(\frac{1}{\nu}\right),$$
$$\varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0),\quad |\arg\nu|<\pi\,. \tag{3.112}$$

In order to obtain a expression for the Gegenbauer function of the second kind, replace ν by $-\nu-2\lambda$ in (3.103)

$$D^{\lambda}_{-\nu-2\lambda}(z) = -\,2^{\lambda-1}\,\frac{\sin(\nu+\lambda)\pi}{\sin\nu\pi}\,\frac{\Gamma\left(\nu+\lambda\right)\,\Gamma\left(\lambda\right)}{\Gamma\left(\nu+1\right)}\,\left(z^2-1\right)^{-\frac{\lambda}{2}}\,\left(\sqrt{z^2-1}+z\right)^{\nu+\lambda}$$
$$\times F\left(-\lambda+1,\lambda,-\nu-\lambda+1;\frac{\sqrt{z^2-1}-z}{2\sqrt{z^2-1}}\right)\,. \tag{3.113}$$

Let $z=\cos\theta\pm i0$ in the foregoing equation to find

$$D^{\lambda}_{-\nu-2\lambda}(\cos\theta\pm i0) = -\,2^{\lambda-1}\,\frac{\sin(\nu+\lambda)\pi}{\sin\nu\pi}\,\frac{\Gamma\left(\nu+\lambda\right)\,\Gamma\left(\lambda\right)}{\Gamma\left(\nu+1\right)\,\left(\sin\theta\right)^{\lambda}}\,e^{\pm i\left[(\nu+\lambda)\theta+\frac{\pi\lambda}{2}\right]}$$
$$\times F\left(-\lambda+1,\lambda,-\nu-\lambda+1;\frac{\pm i\,e^{\mp i\theta}}{2\sin\theta}\right)\,. \tag{3.114}$$

Substituting (3.114) into (3.53) with ν replaced by $-\nu-2\lambda$

$$D^{\lambda}_{-\nu-2\lambda}(\cos\theta) = -\frac{i}{2}\left(e^{i\lambda\pi}\,D^{\lambda}_{-\nu-2\lambda}(\cos\theta+i0)-e^{-i\lambda\pi}\,D^{\lambda}_{-\nu-2\lambda}(\cos\theta-i0)\right)$$
$$= -2^{\lambda-1}\,\frac{\sin(\nu+\lambda)\pi}{\sin\nu\pi}\,\frac{\Gamma\left(\nu+\lambda\right)\,\Gamma\left(\lambda\right)}{\Gamma\left(\nu+1\right)\,\left(\sin\theta\right)^{\lambda}}\sum_{n=0}^{+\infty}\frac{(1-\lambda)_n\,(\lambda)_n}{n!\,(-\nu-\lambda+1)_n\,(2\sin\theta)^n}$$
$$\times\sin\left[(\nu+\lambda-n)\theta+\frac{n+3\lambda}{2}\pi\right],\quad \varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0)\,, \tag{3.115}$$

or

$$D^{\lambda}_{-\nu-2\lambda}(\cos\theta) = -\frac{\sin(\nu+\lambda)\pi}{\sin\nu\pi}\,\frac{\Gamma\left(\lambda\right)}{\left(\sin\theta\right)^{\lambda}}\,(2\nu)^{\lambda-1}\sin\left[(\nu+\lambda)\theta+\frac{3\pi\lambda}{2}\right]+\mathcal{O}\left(\frac{1}{\nu}\right),$$
$$\varepsilon<\theta<\pi-\varepsilon,\ (\varepsilon>0),\quad |\arg\nu|<\pi\,. \tag{3.116}$$

Note that, due do the factor $\frac{1}{\sin^{\lambda}\theta}$, equations (3.105), (3.109), (3.111) and (3.115) are unsuitable in the vicinity of $\theta\sim 0,\pi$ for every $\operatorname{Re}\lambda>0$. To overcome this obstacle, an analysis seen, for example in [Mac99, Mac14], is in order.

3.10 Evaluation of the limit $(1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}\,x} C_\nu^\lambda(x)$, $m \in \mathbb{R}$ as x tends to $\pm 1^\mp$

From the definition of the Gegenbauer functions of the first kind (3.12) on the cut, we obtain, by differentiation with respect to the argument

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,x} C_\nu^\lambda(x) &= \frac{1}{2\lambda+1}\,\frac{\Gamma(\nu+2\lambda+1)}{\Gamma(\nu)\,\Gamma(2\lambda)}\,F\left(-\nu+1, \nu+2\lambda+1, \lambda+\frac{3}{2}; \frac{1-x}{2}\right), \\ &= 2\lambda C_{\nu-1}^{\lambda+1}(x) \end{aligned} \tag{3.117}$$

where we used the property

$$\frac{\mathrm{d}}{\mathrm{d}\,x} F\left(\alpha, \beta, \gamma; x\right) = \frac{\alpha\,\beta}{\gamma}\,F\left(\alpha+1, \beta+1, \gamma+1; x\right),$$

together with formula (3.6).
As x tends to 1^-, the latter expression, multiplied by a factor $(1-x^2)^m$, $m \in \mathbb{R}$, becomes

$$\lim_{x\to 1^-}(1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}\,x} C_\nu^\lambda(x) = \frac{1}{2\lambda+1}\,\frac{\Gamma(\nu+2\lambda+1)}{\Gamma(\nu)\,\Gamma(2\lambda)} \lim_{x\to 1^-}(1-x^2)^m, \tag{3.118}$$

from which, if $\nu+2\lambda$ is not a negative integer

$$\lim_{x\to 1^-}(1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}\,x} C_\nu^\lambda(x) = \begin{cases} 0 & \text{, if } \; m > 0 \\ \frac{1}{2\lambda+1}\,\frac{\Gamma(\nu+2\lambda+1)}{\Gamma(\nu)\,\Gamma(2\lambda)} & \text{, if } \; m = 0 \\ \infty & \text{, if } \; m < 0 \end{cases}. \tag{3.119}$$

Employing the transformation formula (3.47), equation (3.117) yields

$$\frac{\mathrm{d}}{\mathrm{d}\,x} C_\nu^\lambda(x) = \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1}\,\frac{\Gamma(\nu+2\lambda+1)}{\Gamma(\nu)\,\Gamma(2\lambda)}\,(1+x)^{-\lambda-\frac{1}{2}}\,F\left(\nu+\lambda+\frac{1}{2}, -\nu-\lambda+\frac{1}{2}, \lambda+\frac{3}{2}; \frac{1-x}{2}\right).$$

As x tends to -1^+ and bearing in mind that

$$F\left(\alpha, \beta, \gamma; 1\right) = \frac{\Gamma\left(\gamma\right)\,\Gamma\left(\gamma-\alpha-\beta\right)}{\Gamma\left(\gamma-\alpha\right)\,\Gamma\left(\gamma-\beta\right)}, \quad \mathrm{Re}(\gamma-\alpha-\beta) > 0\,,$$

the above equation, multiplied by $(1-x^2)^m$, becomes

$$\lim_{x\to -1^+}(1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}\,x} C_\nu^\lambda(x) = 2^{m+\lambda-\frac{1}{2}}\,\frac{\sin\nu\pi}{\pi}\,\frac{\left(\Gamma(\lambda+\frac{1}{2})\right)^2}{\Gamma(2\lambda)} \lim_{x\to -1^+}(1+x)^{m-\lambda-\frac{1}{2}}. \tag{3.120}$$

For fixed, real values of λ greater then $-\frac{1}{2}$ it is straightforward to show that

$$\lim_{x\to -1^+}(1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}\,x} C_\nu^\lambda(x) = \begin{cases} 0 & \text{, if } \; m-\lambda-\frac{1}{2} > 0 \\ 2^{2m-1}\,\frac{\sin\nu\pi}{\pi}\,\frac{(\Gamma(m))^2}{\Gamma(2m-1)} & \text{, if } \; m-\lambda-\frac{1}{2} = 0 \\ \infty & \text{, if } \; m-\lambda-\frac{1}{2} < 0 \end{cases}. \tag{3.121}$$

As an example, consider $\lambda = \frac{1}{2}$. For this particular choice $C_\nu^{\frac{1}{2}}(x) \equiv P_\nu(x)$ and (3.121) gives

$$\lim_{x\to -1^+}(1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}\,x} P_\nu(x) = \begin{cases} 0 & \text{, if } \; m > 1 \\ \frac{2}{\pi}\sin\nu\pi & \text{, if } \; m = 1 \\ \infty & \text{, if } \; m < 1 \end{cases}. \tag{3.122}$$

The machinery introduced can be generalized bearing in mind that

$$\frac{d^k}{\mathrm{d}x^k} F\left(\alpha,\beta,\gamma;x\right) = \frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k}\, F\left(\alpha+k,\beta+k,\gamma+k;x\right)\,.$$

3.11 Evaluation of the limit $(1-x^2)^m\,\frac{\mathrm{d}}{\mathrm{d}x}\,D_\nu^\lambda(x),\ m\in\mathbb{R}$ as x tends to $\pm 1^{\mp}$

The Gegenbauer functions of the second kind admits the series expansion (3.100), which differentiated with respect to the argument and bearing in mind the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x} = 2x\,\frac{\mathrm{d}}{\mathrm{d}x^2}\,,$$

gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} D_\nu^\lambda(x) =& 2^{2\lambda}\,\frac{\Gamma\left(\nu-1\right)\,\Gamma\left(\lambda\right)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)\,\Gamma\left(\nu\right)}\,\cos\frac{\nu\pi}{2}\,F\left(\frac{1-\nu}{2},\frac{\nu+2\lambda+1}{2},\frac{3}{2};x^2\right)\\ &-\frac{2^{2\lambda+1}}{3}\,\frac{\Gamma\left(\lambda\right)\,\Gamma\left(\frac{\nu+2\lambda+3}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)}\,\cos\frac{\nu\pi}{2}\,x^2\,F\left(\frac{3-\nu}{2},\frac{\nu+2\lambda+3}{2},\frac{5}{2};x^2\right)\\ &+2^{2\lambda}\,\frac{\Gamma\left(\lambda\right)\,\Gamma\left(\frac{\nu+2\lambda+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\,\sin\frac{\nu\pi}{2}\,x\,F\left(\frac{2-\nu}{2},\frac{\nu+2\lambda+2}{2},\frac{3}{2};x^2\right)\,. \end{aligned} \tag{3.123}$$

Moreover, utilizing formula (3.47) on the last two hypergeometric functions of the right-hand side of (3.123), we obtain, multiplying by a factor $(1-x^2)^m$

$$\begin{aligned}&(1-x^2)^m\,\frac{\mathrm{d}}{\mathrm{d}x} D_\nu^\lambda(x) = 2^{2\lambda}\,\frac{\Gamma\left(\nu-1\right)\,\Gamma\left(\lambda\right)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)\,\Gamma\left(\nu\right)}\,\cos\frac{\nu\pi}{2}\,(1-x^2)^m\\ &\times F\left(\frac{1-\nu}{2},\frac{\nu+2\lambda+1}{2},\frac{3}{2};x^2\right)-\frac{2^{2\lambda+1}}{3}\,\frac{\Gamma\left(\lambda\right)\,\Gamma\left(\frac{\nu+2\lambda+3}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)}\,\cos\frac{\nu\pi}{2}\,x^2\,(1-x^2)^{m-\lambda-\frac{1}{2}}\\ &\times F\left(\frac{\nu+2}{2},\frac{-\nu-2\lambda+2}{2},\frac{5}{2};x^2\right)+2^{2\lambda}\,\frac{\Gamma\left(\lambda\right)\,\Gamma\left(\frac{\nu+2\lambda+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\,x\,(1-x^2)^{m-\lambda-\frac{1}{2}}\\ &\times F\left(\frac{\nu+1}{2},\frac{-\nu-2\lambda+1}{2},\frac{3}{2};x^2\right)\,, \end{aligned} \tag{3.124}$$

where we notice that for both resulting expressions $\mathrm{Re}(\gamma-\alpha-\beta)>0$, if $\mathrm{Re}\lambda>-\frac{1}{2}$.
As x tends to $\pm 1^{\mp}$, and bearing in mind that

$$\lim_{x\to\pm 1^{\mp}} F\left(\alpha,\beta,\gamma;x^2\right) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)},\quad \mathrm{Re}(\gamma-\alpha-\beta)>0\,, \tag{3.125}$$

we find

$$\begin{aligned}&\lim_{x\to\pm 1^{\mp}}(1-x^2)^m\,\frac{\mathrm{d}}{\mathrm{d}x} D_\nu^\lambda(x) = 2^{2\lambda}\,\frac{\Gamma\left(\nu-1\right)\,\Gamma\left(\lambda\right)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)\,\Gamma\left(\nu\right)}\,\cos\frac{\nu\pi}{2}\,\lim_{x\to\pm 1^{\mp}}(1-x^2)^m\\ &\times F\left(\frac{1-\nu}{2},\frac{\nu+2\lambda+1}{2},\frac{3}{2};x^2\right)+\Gamma\left(2\lambda\right)\,\left(\cos^2\frac{\nu\pi}{2}\pm\sin^2\frac{\nu\pi}{2}\right)\lim_{x\to\pm 1^{\mp}}(1-x^2)^{m-\lambda-\frac{1}{2}}\,. \end{aligned} \tag{3.126}$$

In the sequence, consider the hypergeometric function

$$F\left(\frac{1-\nu}{2}-1, \frac{\nu+2\lambda+1}{2}-1, \frac{3}{2}-1; x^2\right) = F\left(\frac{-\nu-1}{2}, \frac{\nu+2\lambda-1}{2}, \frac{1}{2}; x^2\right).$$

Differentiating the latter with respect to the argument

$$\frac{\mathrm{d}}{d(x^2)} F\left(\frac{-\nu-1}{2}, \frac{\nu+2\lambda-1}{2}, \frac{1}{2}; x^2\right) = -\frac{(\nu+1)(\nu+2\lambda-1)}{2} F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; x^2\right)$$

and integrating the resulting expression over the interval $[0, x^2]$ with respect to the argument, we find

$$F\left(\frac{-\nu-1}{2}, \frac{\nu+2\lambda-1}{2}, \frac{1}{2}; x^2\right) = 1 - \frac{(\nu+1)(\nu+2\lambda-1)}{2} \times \int_0^{x^2} F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; x^2\right) d(x^2). \tag{3.127}$$

As x tends to $\pm 1^{\mp}$, the left-hand side of the above formula remains bounded if $\operatorname{Re}\lambda < \frac{3}{2}$, and so must the right-hand side. This implies that as $x \to \pm 1^{\mp}$, $F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; x^2\right)$ enjoys the asymptotic behavior

$$F\left(\frac{1-\nu}{2}, \frac{\nu+2\lambda+1}{2}, \frac{3}{2}; x^2\right) \sim \frac{1}{(1-x^2)^p}, \quad 0 < p < 1, \tag{3.128}$$

and thus (3.126) rewrites

$$\lim_{x\to\pm 1^{\mp}} (1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}x} D_\nu^\lambda(x) \sim 2^{2\lambda} \frac{\Gamma(\nu-1)\,\Gamma(\lambda)\,\Gamma\left(\frac{\nu+2\lambda+1}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)\Gamma(\nu)} \cos\frac{\nu\pi}{2} \lim_{x\to\pm 1^{\mp}} (1-x^2)^{m-p} + \Gamma(2\lambda)\left(\cos^2\frac{\nu\pi}{2} \pm \sin^2\frac{\nu\pi}{2}\right) \lim_{x\to\pm 1^{\mp}} (1-x^2)^{m-\lambda-\frac{1}{2}}, \tag{3.129}$$

which holds for fixed, real values of λ in the open interval from $-\frac{1}{2}$ to $\frac{3}{2}$. If $m < p$ or $\lambda = 0$, the above expression becomes infinite. For values of m equal or greater then unity, we obtain

$$\lim_{x\to\pm 1^{\mp}} (1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}x} D_\nu^\lambda(x) = \begin{cases} 0 & \text{, if } m-\lambda-\frac{1}{2} > 0 \\ \Gamma(2\lambda)\left(\cos^2\frac{\nu\pi}{2} \pm \sin^2\frac{\nu\pi}{2}\right) & \text{, if } m-\lambda-\frac{1}{2} = 0 \\ \infty & \text{, if } m-\lambda-\frac{1}{2} < 0 \end{cases}. \tag{3.130}$$

As an example, consider $\lambda = \frac{1}{2}$ and therefore $D_\nu^{\frac{1}{2}}(x) \equiv Q_\nu(x)$. Hence, from (3.130) one can deduce that

$$\lim_{x\to\pm 1^{\mp}} (1-x^2)^m \frac{\mathrm{d}}{\mathrm{d}x} Q_\nu(x) = \begin{cases} 0 & \text{, if } m > 1 \\ \cos^2\frac{\nu\pi}{2} \pm \sin^2\frac{\nu\pi}{2} & \text{, if } m = 1 \\ \infty & \text{, if } m < 1 \end{cases}. \tag{3.131}$$

A formula valid for the evaluation of the limit in consideration for values of λ greater then $\frac{1}{2}$, is obtained transforming the first of the hypergeometric functions on the right-hand side of (3.126) via (3.47)

$$\begin{aligned}\lim_{x\to\pm 1^{\mp}}(1-x^2)^m\,\frac{\mathrm{d}}{\mathrm{d}x}D_\nu^\lambda(x) =&\frac{2^{2\lambda-2}}{\sqrt{\pi}}\,\Gamma(\lambda)\,\Gamma\left(\lambda-\tfrac{1}{2}\right)\,\cos^2\frac{\nu\pi}{2}\lim_{x\to\pm 1^{\mp}}(1-x^2)^{m-\lambda+\frac{1}{2}}\\ &+\Gamma(2\lambda)\,\left(\cos^2\frac{\nu\pi}{2}\pm\sin^2\frac{\nu\pi}{2}\right)\lim_{x\to\pm 1^{\mp}}(1-x^2)^{m-\lambda-\frac{1}{2}}\,,\end{aligned} \tag{3.132}$$

from which we see that as x tends to $\pm 1^{\mp}$, $(1-x^2)^m\,\frac{\mathrm{d}}{\mathrm{d}x}D_\nu^\lambda(x)\sim(1\mp x)^{m-\lambda-\frac{1}{2}}$. As an example, consider $\lambda=\frac{3}{2}$ and the above formula implies

$$\lim_{x\to\pm 1^{\mp}}(1-x^2)^m\,\frac{\mathrm{d}}{\mathrm{d}x}D_\nu^{\frac{3}{2}}(x)=\begin{cases}\infty & ,\text{if}\quad m<2\\ 2\left(\cos^2\frac{\nu\pi}{2}\pm\sin^2\frac{\nu\pi}{2}\right) & ,\text{if}\quad m=2\\ 0 & ,\text{if}\quad m>2\end{cases}\,. \tag{3.133}$$

3.12 Connecting formulae

In this section the formulae relating the Gegenbauer functions with the associated Legendre functions and the Legendre functions are formulated. The associated Legendre functions of the first and second kind are defined as follows [Erd53, p.122, equations (7) and (8)]

$$P_\nu^\mu(z)=\frac{2^\mu}{\Gamma(1-\mu)}\,(z^2-1)^{-\frac{\mu}{2}}\,F\left(1+\nu-\mu,-\nu-\mu,1-\mu;\frac{1-z}{2}\right),\quad |1-z|<2\,, \tag{3.134}$$

$$\begin{aligned}Q_\nu^\mu(z)=&\frac{\sqrt{\pi}\,e^{i\mu\pi}}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)}\,\Gamma(\nu+\mu+1)\,z^{\mu-\nu-1}\,(z^2-1)^{-\frac{\mu}{2}}\\ &\times F\left(\frac{1+\nu-\mu}{2},1+\frac{\nu-\mu}{2},\nu+\frac{3}{2};\frac{1}{z^2}\right),\quad |z|>1,\quad \nu+\mu+1\neq 0,-1,-2,\dots\,.\end{aligned} \tag{3.135}$$

On the other hand, the Gegenbauer functions of the first and second kind are defined as follows [Erd53, p.175, equation (3) and p.179, equation (32) plus errata]

$$C_{\nu'}^\lambda(z)=\frac{\Gamma(\nu'+2\lambda)}{\Gamma(\nu'+1)\,\Gamma(2\lambda)}\,F\left(-\nu',\nu'+2\lambda,\lambda+\frac{1}{2};\frac{1-z}{2}\right),\quad |1-z|<2,\quad \lambda>-\frac{1}{2} \tag{3.136}$$

and

$$\begin{aligned}D_{\nu'}^\lambda(z)=2^{2\lambda-1}\,\frac{\Gamma(\nu'+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu'+\lambda+1)}\,(2z)^{-\nu'-2\lambda}\,F\left(\frac{\nu'+2\lambda}{2},\frac{\nu'+2\lambda+1}{2},\nu'+\lambda+1;\frac{1}{z^2}\right),\\ |z|>1,\quad \nu'+2\lambda\neq 0,-1,-2,\dots\,,\quad \lambda\neq 0,-1,-2,\dots\,,\end{aligned} \tag{3.137}$$

where ν, μ, ν', λ arbitrary complex numbers.
Replacing in equations (3.136) and (3.137) ν' with $\nu+\mu$ and λ by $-\mu+\frac{1}{2}$ it is easy to show that

$$C_{\nu+\mu}^{-\mu+\frac{1}{2}}(z) = 2^{-\mu}\,\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,\frac{\Gamma(1-\mu)}{\Gamma(1-2\mu)}\,(z^2-1)^{\frac{\mu}{2}}\,P_\nu^\mu(z), \quad |z|<1\,, \tag{3.138}$$

and

$$D_{\nu+\mu}^{-\mu+\frac{1}{2}}(z) = \frac{e^{-i\mu\pi}}{2^\mu\sqrt{\pi}}\,\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,\Gamma\left(-\mu+\tfrac{1}{2}\right)\,(z^2-1)^{\frac{\mu}{2}}\,Q_\nu^\mu(z), \quad |z|>1\,, \tag{3.139}$$

or

$$C_\nu^\lambda(z) = 2^{\lambda-\frac{1}{2}}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\nu+1)\,\Gamma(2\lambda)}\,(z^2-1)^{\frac{1}{4}-\frac{\lambda}{2}}\,P_{\nu+\lambda-\frac{1}{2}}^{-\lambda+\frac{1}{2}}(z), \quad |z|<1\,, \tag{3.140}$$

$$D_\nu^\lambda(z) = 2^{\lambda-\frac{1}{2}}\,\frac{e^{-i\lambda\pi}}{\sqrt{\pi}}\,\frac{\Gamma(\nu+2\lambda)\,\Gamma(\lambda)}{\Gamma(\nu+1)}\,(z^2-1)^{\frac{1}{4}-\frac{\lambda}{2}}\,Q_{\nu+\lambda-\frac{1}{2}}^{-\lambda+\frac{1}{2}}(z), \quad |z|>1\,. \tag{3.141}$$

Replacing μ by $-\mu$, equations (3.138) and (3.139) yield

$$C_{\nu-\mu}^{\mu+\frac{1}{2}}(z) = 2^\mu\,\frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)}\,(z^2-1)^{-\frac{\mu}{2}}\,\left(P_\nu^\mu(z) - \frac{2}{\pi}\,e^{-i\mu\pi}\,\sin\mu\pi\,Q_\nu^\mu(z)\right)\,, \tag{3.142}$$

$$D_{\nu-\mu}^{\mu+\frac{1}{2}}(z) = 2^\mu\,\frac{e^{-i\mu\pi}}{\sqrt{\pi}}\,\Gamma\left(\mu+\tfrac{1}{2}\right)\,(z^2-1)^{-\frac{\mu}{2}}\,Q_\nu^\mu(z)\,, \tag{3.143}$$

where we used the relations [Erd53, p. 140, equations (2) and (5)]

$$Q_\nu^{-\mu}(z) = e^{-i2\mu\pi}\,\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,Q_\nu^\mu(z)\,,$$
$$P_\nu^{-\mu}(z) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,\left(P_\nu^\mu(z) - \frac{2}{\pi}\,e^{-i\mu\pi}\,\sin\mu\pi\,Q_\nu^\mu(z)\right)\,.$$

When $\mu = m$, $m = 0, 1, 2, \dots$ equations (3.142) and (3.143) rewrite at once as

$$C_{\nu-m}^{m+\frac{1}{2}}(z) = 2^m\,\frac{m!}{(2m)!}\,\frac{d^m}{\mathrm{d}z^m}P_\nu(z)\,, \tag{3.144}$$

$$D_{\nu-m}^{m+\frac{1}{2}}(z) = (-1)^m\,\frac{2^m}{\sqrt{\pi}}\,\Gamma\left(m+\tfrac{1}{2}\right)\,\frac{d^m}{\mathrm{d}z^m}Q_\nu(z)\,, \tag{3.145}$$

since [Erd53, p. 148, equations (4) and (5)]

$$P_\nu^m(z) = (z^2-1)^{\frac{m}{2}}\,\frac{d^m}{\mathrm{d}z^m}P_\nu(z)\,,$$
$$Q_\nu^m(z) = (z^2-1)^{\frac{m}{2}}\,\frac{d^m}{\mathrm{d}z^m}Q_\nu(z)\,.$$

In particular, for $m = 1$ we find $C_{\nu-1}^{\frac{3}{2}}(z) = \frac{\mathrm{d}}{\mathrm{d}z}P_n(z)$. Replacing ν with $\nu+1$ we obtain the more convenient form

$$C_\nu^{\frac{3}{2}}(z) = \frac{\mathrm{d}}{\mathrm{d}z}P_{\nu+1}(z)\,. \tag{3.146}$$

The latter follows also up by differentiating the expression

$$\int C_\nu^\lambda(z)\mathrm{d}z = \frac{1}{2(\lambda-1)} C_{\nu+1}^{\lambda-1}(z)$$

for $\lambda = \frac{3}{2}$.
Letting $\nu = n$, $n = 0, 1, 2, \ldots$ and $z = x$ the latter becomes

$$C_n^{\frac{3}{2}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} P_{n+1}(x)\,. \tag{3.147}$$

The derivative of a Legendre polynomial is given by Christoffel's formula [Chr58] as

$$\frac{\mathrm{d}}{\mathrm{d}x} P_{n+1}(x) = \sum_{m=0}^{\lfloor \frac{n}{2},\, \frac{n-1}{2} \rfloor} (2n - 4m + 1) P_{n-2m}(x)\,.$$

Substituting the above equation into (3.147) we obtain

$$C_n^{\frac{3}{2}}(x) = \sum_{m=0}^{\lfloor \frac{n}{2},\, \frac{n-1}{2} \rfloor} (2n - 4m + 1) P_{n-2m}(x)\,. \tag{3.148}$$

Based on (3.148), consider a second solution of the form

$$D_n^{\frac{3}{2}}(x) = \sum_{m=0}^{\lfloor \frac{n}{2},\, \frac{n-1}{2} \rfloor} (2n - 4m + 1) Q_{n-2m}(x)\,. \tag{3.149}$$

Also, based on (3.146) we write

$$D_n^{\frac{3}{2}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} Q_{n+1}(x)\,. \tag{3.150}$$

where we omitted a constant multiplier.

3.13 The Gegenbauer Integral Operator

Define the Gegenbauer Integral Operator of degree ν and order λ as

$$\mathfrak{G}_\nu^\lambda(z) = C_\nu^\lambda(z) \int_{-1}^{z} d\tau\, D_\nu^\lambda(\tau) + D_\nu^\lambda(z) \int_{z}^{+1} d\tau\, C_\nu^\lambda(\tau)\,. \tag{3.151}$$

Replacing in the latter ν with $-\nu - 2\lambda$ and employing the symmetry relations (3.69)

$$C_{-\nu-2\lambda}^\lambda(z) = -\frac{\sin(\nu+2\lambda)\pi}{\sin \nu\pi}\, C_\nu^\lambda(z)$$

and (3.70)

$$D_{-\nu-2\lambda}^\lambda(z) = D_\nu^\lambda(z) - 2^{2\lambda-1} \frac{\big(\Gamma(\lambda)\big)^2 \sin(\nu+\lambda)\pi}{\sin \nu\pi}\, C_\nu^\lambda(z),$$

equation (3.151) rewrites

$$\mathfrak{G}_\nu^\lambda(z) + \frac{\sin \nu\pi}{\sin(\nu+2\lambda)\pi}\, \mathfrak{G}_{-\nu-2\lambda}^\lambda(z) = 2^{2\lambda-1} \frac{(\Gamma(\lambda))^2 \sin(\nu+\lambda)\pi}{\sin \nu\pi}\, C_\nu^\lambda(z) \int_{-1}^{+1} \mathrm{d}z\, C_\nu^\lambda(z)\,. \tag{3.152}$$

3.13.1 Specific Examples

For $\lambda = 0$ the Gegenbauer functions $C_\nu^\lambda(z)$ and $D_\nu^\lambda(z)$ become the Chebyshev functions of the first $T_\nu(z)$ and second kind $U_\nu(z)$ respectively.
Thus, (3.151) becomes the Gegenbauer Integral Operator of order $\lambda = 0$ or the *Chebyshev Integral Operator*

$$\mathfrak{T}_\nu(z) = T_\nu(z) \int_{-1}^{z} d\tau\, U_\nu(\tau) + U_\nu(z) \int_{z}^{+1} d\tau\, T_\nu(\tau)\,, \tag{3.153}$$

and (3.152) reads

$$\mathfrak{T}_\nu(z) + \mathfrak{T}_{-\nu}(z) = \frac{1}{2}\, T_\nu(z) \int_{-1}^{+1} \mathrm{d}z\, T_\nu(z)\,. \tag{3.154}$$

For $\lambda = \frac{1}{2}$ the Gegenbauer functions $C_\nu^\lambda(z)$ and $D_\nu^\lambda(z)$ become the Legendre functions of the first $P_\nu(z)$ and second kind $Q_\nu(z)$ respectively.
Thus, (3.151) becomes the Gegenbauer Integral Operator of order $\lambda = \frac{1}{2}$ or the *Legendre Integral Operator*

$$\mathfrak{P}_\nu(z) = P_\nu(z) \int_{-1}^{z} d\tau\, Q_\nu(\tau) + Q_\nu(z) \int_{z}^{+1} d\tau\, P_\nu(\tau), \tag{3.155}$$

which appears solving the Laplace equation in a spherical domain. Also (3.152) becomes

$$\mathfrak{P}_\nu(z) - \mathfrak{P}_{-\nu-1}(z) = \pi\, \cot \nu\pi\, P_\nu(z) \int_{-1}^{+1} \mathrm{d}z\, P_\nu(z)\,. \tag{3.156}$$

For $\lambda = \frac{3}{2}$ (3.151) becomes the Gegenbauer Integral Operator of order $\lambda = \frac{3}{2}$

$$\mathfrak{G}_\nu^{\frac{3}{2}}(z) = C_\nu^{\frac{3}{2}}(z) \int_{-1}^{z} d\tau\, D_\nu^{\frac{3}{2}}(\tau) + D_\nu^{\frac{3}{2}}(z) \int_{z}^{+1} d\tau\, C_\nu^{\frac{3}{2}}(\tau)\,, \tag{3.157}$$

which appears solving the irrotational Stoke's flow in a spherical domain.
Also (3.152) becomes

$$\mathfrak{G}_\nu^{\frac{3}{2}}(z) - \mathfrak{G}_{-\nu-3}^{\frac{3}{2}}(z) = -\pi\, \cot \nu\pi\, C_\nu^{\frac{3}{2}}(z) \int_{-1}^{+1} \mathrm{d}z\, C_\nu^{\frac{3}{2}}(z)\,. \tag{3.158}$$

3.14 Recurrence relations for the Gegenbauer polynomials

Gegenbauer's polynomial $C_n^\lambda(x)$ for positive integral values of n is defined to be the coefficient of t in the expansion of $(1-2xt+t^2)^{-\lambda}$ in powers of t

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x)\, t^n, \quad \lambda > -\frac{1}{2}\,. \tag{3.159}$$

In the case where n is a negative integer, we define $C_n^\lambda(x) = 0$.
Differentiating the generating function (3.159) with respect to t,

$$-2\lambda\,(t-x)\,(1-2xt+t^2)^{-\lambda-1} = \sum_{n=0}^{\infty} n\, C_n^\lambda(x)\, t^{n-1}\,, \tag{3.160}$$

which with the aid of (3.159) yields

$$(n+1)\, C^{\lambda}_{n+1}(x) - 2(n+\lambda)\, x\, C^{\lambda}_{n}(x) + (n+2\lambda-1)\, C^{\lambda}_{n-1}(x) = 0\,. \tag{3.161}$$

Repeating the above procedure, namely differentiating the generating function (3.159) but this time with respect to x, it is straightforward to show that

$$2\lambda\, t\, (1-2xt+t^2)^{-\lambda-1} = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n}(x)\, t^n\,. \tag{3.162}$$

Multiplying the latter throughout $t-x$ and using (3.160), we find

$$x\, \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n}(x) - \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n-1}(x) = n\, C^{\lambda}_{n}(x)\,. \tag{3.163}$$

Continuing from (3.162) and in view of (3.159) we obtain

$$2\lambda\, C^{\lambda}_{n-1}(x) = \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n}(x) - 2x\, \frac{\mathrm{d}}{\mathrm{d}x}\, C^{\lambda}_{n-1}(x) + \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n-2}(x)\,, \tag{3.164}$$

which, replacing n with $n+1$, reads

$$2\lambda\, C^{\lambda}_{n}(x) = \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n+1}(x) - 2x\, \frac{\mathrm{d}}{\mathrm{d}x}\, C^{\lambda}_{n}(x) + \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n-1}(x)\,. \tag{3.165}$$

Differentiating (3.161) with respect to x and using (3.165) implies

$$(1-\lambda)\, \frac{\mathrm{d}}{\mathrm{d}x}\, C^{\lambda}_{n+1}(x) + (n+2\lambda-1) \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n-1}(x) = 2(n+\lambda)(1-\lambda) C^{\lambda}_{n}(x)\,, \tag{3.166}$$

which then, with the help of (3.163), rewrites

$$(1-\lambda)\, \frac{\mathrm{d}}{\mathrm{d}x}\, C^{\lambda}_{n+1}(x) + (n+2\lambda-1) x\, \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n}(x) = \left(n(n+1) - 2\lambda(\lambda-1)\right)\, C^{\lambda}_{n}(x)\,. \tag{3.167}$$

Differentiate (3.161) with respect to x to find

$$\frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n+1}(x) - \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n-1}(x) = 2(n+\lambda)\, C^{\lambda}_{n}(x)\,. \tag{3.168}$$

Eliminating from equations (3.163) and (3.168) $C^{\lambda}_{n-1}(x)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n+1}(x) - x\, \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n}(x) = (n+2\lambda)\, C^{\lambda}_{n}(x)\,. \tag{3.169}$$

Multiplying (3.163) by x, replacing in (3.169) n with $n-1$ and subtracting the resulting expressions, yields

$$(1-x^2) \frac{\mathrm{d}}{\mathrm{d}x} C^{\lambda}_{n}(x) = (n+2\lambda-1)\, C^{\lambda}_{n-1}(x) - n\, x\, C^{\lambda}_{n}(x)\,. \tag{3.170}$$

3.15 Recurrence relations for the Gegenbauer polynomials of order $-\frac{1}{2}$

The generating function for the Gegenbauer polynomials of integral degree n and order $-\frac{1}{2}$ is given by Sampson [Sam91] as follows

$$\sqrt{1 - 2x\,t + t^2} = -\sum_{n=0}^{\infty} C_n^{-\frac{1}{2}}(x)\,t^n\,. \tag{3.171}$$

Following a similar analysis as seen in the previous section, it is easy to show that the following recurrence relations are valid

$$(2n-1)\,x\,C_n^{-\frac{1}{2}}(x) = (n+1)\,C_{n+1}^{-\frac{1}{2}}(x) + (n-2)\,C_{n-1}^{-\frac{1}{2}}(x)\,, \tag{3.172}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}C_n^{-\frac{1}{2}}(x) - 2x\,\frac{\mathrm{d}}{\mathrm{d}x}\,C_{n-1}^{-\frac{1}{2}}(x) + \frac{\mathrm{d}}{\mathrm{d}x}C_{n-2}^{-\frac{1}{2}}(x) = -C_{n-1}^{-\frac{1}{2}}(x)\,, \tag{3.173}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}C_{n-1}^{-\frac{1}{2}}(x) - x\,\frac{\mathrm{d}}{\mathrm{d}x}\,C_n^{-\frac{1}{2}}(x) = -n\,C_n^{-\frac{1}{2}}(x)\,. \tag{3.174}$$

A generating function for the Gegenbauer polynomials if the order is in general a negative integer, is given by De Duffahel [Duf35] as

$$(1 - 2x\,t + t^2)^{\lambda-1}\,\ln(1 - 2x\,t + t^2) = \sum_{n=0}^{\infty} C_n^{1-\lambda}(x)\,t^n\,. \tag{3.175}$$

CHAPTER 4

On the Global Relation and the Dirichlet-to-Neumann Correspondence for harmonic functions *

4.1 Introduction

Within the last decade a generalized transform has been developed by Fokas and his collaborators [Fok08]. The novelty of this transformation is focused on the fact that it is a transform that meets the particular analytical and geometrical characteristics of the problem at hand. In fact, the integral kernel of the transformation carries the analytical properties of the partial differential operator and the geometry of the fundamental domain specifies the appropriate contour of integration [Das07b]. A crucial part of the theory concerns the manipulation of the so-called *global relation*, which is an integral relation connecting the boundary values of the solution (Dirichlet data) with the normal derivative of the solution on the boundary (Neumann data). In many cases, it is possible to obtain the missing boundary data directly from the given ones exactly in the form that they appear in the integral representation of the solution. As far as elliptic boundary value problems in two dimensions are concerned, an important contribution of this theory is the integral representation of the solutions in the interior of a convex polygon [Fok01, FK03, DF05].
To the authors knowledge no successful extension of the method of generalized transform to three dimensions has been achieved yet. The present work aims in this direction. We actually use the method of Fokas to solve the Laplace equation inside and outside a sphere, under the assumption that the boundary data, and therefore the solution as well, is independent of the azimuthal angle. As we demonstrate, this problem, although two-

*This work has been published as [DDa]

dimensional in its basic nature, exhibits fundamental differences from the corresponding problem for the Laplace equation inside a disc [FN]. The solution obtained by the Fokas method, has the advantage of being uniformly convergent on the boundary. Furthermore, it is useful in numerical applications and also for studying asymptotic properties of the solutions, since the integral representations converge much faster than the series. Moreover, with the novel integral representations introduced in section 6 it is possible to solve problems with mixed boundary conditions, such as, Dirichlet data on a part of the boundary and Neumann data on the complementary part. This kind of problems involve the solution of a Riemann-Hilbert problem.
The analysis of the global relation has the following advantages: (i) It provides the most effective approach for constructing the Dirichlet-to-Neumann correspondence. (ii) Formulating the global relation in specific subdomains of Ω, it is possible to re-derive the classical representations or, depending on the operator, to yield alternative series representations. This approach will be presented in a forthcoming paper.
The chapter is organized as follows.

A brief review of the classical solutions for the interior and exterior Dirichlet and Neumann problems is given in section 4.2 in order to fix notation and terminology. In section 4.3 the general Global Relation is derived, which is further used in section 4.4 to establish the Dirichlet-to-Neumann correspondence. Section 4.5 is devoted to the steps that one has to follow in order to recover the classical solutions from the Global Relation. The novel integral representations on which the present work is focused is developed in section 4.6.

4.2 The Classical Representation

Let S be a sphere with center at the origin and radius a. We denote by Ω^i the interior and by Ω^e the exterior of S. Our goal is to find harmonic functions $q_D^i, q_D^e, q_N^i, q_N^e$ that solve the interior Dirichlet, the exterior Dirichlet, the interior Neumann and the exterior Neumann problems, respectively. We denote the Dirichlet data on the boundary by g_D, the Neumann data on the boundary by g_N and we assume azimuthal independence, that is

$$\frac{\partial}{\partial \phi}\, g_D(\mathbf{r}) = \frac{\partial}{\partial \phi}\, g_N(\mathbf{r}) = 0, \quad r = a, \tag{4.1}$$

where r denotes the radial spherical coordinate. Furthermore for the well-posedness of the exterior problems we demand that the solution of the Laplace equation should satisfy the asymptotic condition

$$q^e(\mathbf{r}) = \mathcal{O}\Big(\frac{1}{r}\Big), \quad r \to \infty, \tag{4.2}$$

where q^e stands for both q_D^e and q_N^e.
The asymptotic condition (4.2) secures the uniqueness of the exterior problem, corresponding to the normalization

$$\lim_{r\to\infty} q^e(\mathbf{r}) = 0\ , \tag{4.3}$$

which also eliminates the arbitrary additive constant that the solutions of Neumann problem involve. Similarly, for the interior Neumann problem we can preassign a value to the solution at the origin. In addition, the Neumann data have to satisfies the compatibility relation

$$\oint_S g_N(\mathbf{r})\, ds(\mathbf{r}) = 0. \tag{4.4}$$

Using the spherical coordinates (r, θ, ϕ), utilizing the fact that g_D and g_N are $\phi-$independent, and introducing the variable

$$\zeta = \cos\theta, \quad \theta \in (0, \pi)$$

we write Laplace's equation in the form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial \zeta^2} - \frac{2\zeta}{r^2}\frac{\partial}{\partial \zeta} \right) q(r,\zeta) = 0 \ . \tag{4.5}$$

Separating variables in the above equation we obtain the following two ordinary differential equations connected by the complex separation constant κ

$$(1-\zeta^2)Z''(\zeta) - 2\zeta Z'(\zeta) + \kappa Z(\zeta) = 0, \quad \kappa \in \mathbb{C} \ , \tag{4.6}$$

$$r^2 R''(r) + 2rR'(r) - \kappa R(r) = 0, \quad \kappa \in \mathbb{C} \ , \tag{4.7}$$

where the prime denotes differentiation with respect to the argument. The solution of (4.6), after a suitable transformation, is expressed in terms of the hypergeometric function [Hil97]. On the other hand, the solution of (4.7) takes one of the following forms

$$R_1(r;\kappa) = r^{\frac{-1+\sqrt{1+4\kappa}}{2}}, \qquad R_2(r;\kappa) = r^{\frac{-1-\sqrt{1+4\kappa}}{2}}, \quad \kappa \in \mathbb{C}. \tag{4.8}$$

Replacing the separation constant κ by $\nu(\nu+1), \ \nu \in \mathbb{C}$, we identify equation (4.6) with the Legendre equation while the solutions (4.8) simplify to

$$R_1(r;\nu) = r^{\nu}, \qquad R_2(r;\nu) = r^{-\nu-1}, \quad \nu \in \mathbb{C} \ . \tag{4.9}$$

Therefore, the general solution of equation (4.5) is represented as a linear combination of functions of the form

$$\left.\begin{array}{rcl} \Theta^{(1)}(r,\zeta;\nu) &=& r^{\nu}\, P_\nu(\zeta) \\ \Theta^{(2)}(r,\zeta;\nu) &=& r^{\nu}\, Q_\nu(\zeta) \\ \Theta^{(3)}(r,\zeta;\nu) &=& r^{-\nu-1}\, P_\nu(\zeta) \\ \Theta^{(4)}(r,\zeta;\nu) &=& r^{-\nu-1}\, Q_\nu(\zeta) \end{array}\right\}, \quad \nu \in \mathbb{C} \ , \tag{4.10}$$

where P_ν and Q_ν are the Legendre functions of the first and the second kind respectively. For $\nu = n = 0, 1, 2, \ldots$ the eigensolutions (4.10) recover the well known zonal harmonics

$$\left.\begin{array}{rcl} \Theta_n^{(1)} &=& r^{n}\, P_n(\zeta) \\ \Theta_n^{(2)} &=& r^{n}\, Q_n(\zeta) \\ \Theta_n^{(3)} &=& r^{-(n+1)}\, P_n(\zeta) \\ \Theta_n^{(4)} &=& r^{-(n+1)}\, Q_n(\zeta) \end{array}\right\}, \tag{4.11}$$

and a complete representation of a harmonic function is written as

$$q(r,\zeta) = \sum_{n=0}^{+\infty}\sum_{j=1}^{4} A_n^{(j)}\, \Theta_n^{(j)} \ . \tag{4.12}$$

Furthermore, since in most applications of interest no singularities are present along the polar axis, we disregard the Legendre functions of the second kind Q_n, which are singular for $\zeta = \pm 1$.
For interior problems, the coefficients $A_n^{(j)}$, $j = 3, 4$ have to vanish and (4.12) is written as

$$q^i(r,\zeta) = \sum_{n=0}^{+\infty} A_n^{(1)} r^n P_n(\zeta). \tag{4.13}$$

In particular, for the Dirichlet problem with data g_D we find

$$q_D^i(r,\zeta) = \frac{1}{2}\sum_{n=0}^{+\infty}(2n+1)\left(\frac{r}{a}\right)^n \mathfrak{D}_n\, P_n(\zeta)\ , \tag{4.14}$$

where

$$\mathfrak{D}_n = \int_{-1}^{+1} g_D(\zeta) P_n(\zeta) d\zeta, \quad n = 0, 1, 2, \ldots \tag{4.15}$$

Similarly, for the Neumann problem with data g_N prescribed on the boundary, the solution assumes the form

$$q_N^i(r,\zeta) = \frac{a}{2}\sum_{n=1}^{+\infty}\frac{2n+1}{n}\left(\frac{r}{a}\right)^n \mathfrak{N}_n\, P_n(\zeta)\ , \tag{4.16}$$

where

$$\mathfrak{N}_n = \int_{-1}^{+1} g_N(\zeta) P_n(\zeta) d\zeta, \quad n = 1, 2, 3, \ldots\ . \tag{4.17}$$

The corresponding solutions for the exterior problems are

$$q_D^e(r,\zeta) = \frac{1}{2}\sum_{n=0}^{+\infty}(2n+1)\left(\frac{a}{r}\right)^{n+1} \mathfrak{D}_n\, P_n(\zeta)\ , \tag{4.18}$$

for the Dirichlet, and

$$q_N^e(r,\zeta) = -\frac{a}{2}\sum_{n=1}^{+\infty}\frac{2n+1}{n+1}\left(\frac{a}{r}\right)^{n+1} \mathfrak{N}_n\, P_n(\zeta)\ , \tag{4.19}$$

for the Neumann problem, where the coefficients $\mathfrak{D}_n$ and $\mathfrak{N}_n$ are given by equations (4.15) and (4.17), respectively and $\mathfrak{N}_0 = 0$.

4.3 The Global Relation

Let $q(r,\zeta)$ satisfy the Laplace equation (4.5) and $\overline{q}(r,\zeta)$ satisfies the formal adjoint of equation (4.5) which reads as

$$\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{2}{r^2} + \frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial \zeta^2} - \frac{2\zeta}{r^2}\frac{\partial}{\partial \zeta}\right)\overline{q}(r,\zeta) = 0. \tag{4.20}$$

Multiply (4.5) by $\overline{q}(r,\zeta)$, (4.20) by $q(r,\zeta)$ and subtracting the resulting equations we obtain, after some algebraic manipulations, the divergence form

$$\frac{\partial}{\partial r}\left(\overline{q}\,\frac{\partial q}{\partial r}+\Big(\frac{2}{r}\,\overline{q}-\frac{\partial \overline{q}}{\partial r}\Big)q\right)+\frac{\partial}{\partial \zeta}\left(\frac{1-\zeta^2}{r^2}\,\Big(\overline{q}\,\frac{\partial q}{\partial \zeta}-q\,\frac{\partial \overline{q}}{\partial \zeta}\Big)\right)=0\,. \tag{4.21}$$

Consider an arbitrary function $\Xi(r,\zeta;\nu)$, such that

$$\frac{\partial}{\partial \zeta}\,\Xi=\overline{q}\,\frac{\partial q}{\partial r}+\Big(\frac{2}{r}\,\overline{q}-\frac{\partial \overline{q}}{\partial r}\Big)q, \tag{4.22}$$

$$\frac{\partial}{\partial r}\,\Xi=-\frac{1-\zeta^2}{r^2}\,\Big(\overline{q}\,\frac{\partial q}{\partial \zeta}-q\,\frac{\partial \overline{q}}{\partial \zeta}\Big), \tag{4.23}$$

then (4.21) implies

$$\left[\frac{\partial}{\partial r},\frac{\partial}{\partial \zeta}\right]\,\Xi(r,\zeta;\nu)=0,\quad \nu\in\mathbb{C}\,,$$

and therefore equations (4.22) and (4.23) constitute a Lax Pair for the Laplace equation (4.5).

Equation (4.21) holds true everywhere in any meridian disc of radius a and applying Green's theorem to a closed subdomain of the meridian disc, we obtain the *global relation*

$$\oint_C\left[\left(\overline{q}\,\frac{\partial q}{\partial r}+\Big(\frac{2}{r}\,\overline{q}-\frac{\partial \overline{q}}{\partial r}\Big)q\right)d\zeta-\frac{1-\zeta^2}{r^2}\left(\overline{q}\,\frac{\partial q}{\partial \zeta}-q\,\frac{\partial \overline{q}}{\partial \zeta}\right)\mathrm{d}r\right]=0\,, \tag{4.24}$$

where C is the boundary of the subdomain.

4.4 The Dirichlet-To-Neumann Correspondence

In this section we are going to utilize the global relation to construct the Dirichlet-to-Neumann correspondence. Taking advantage of the separability, we can replace in (4.20) $\overline{q}(r,\zeta)$ by $\overline{R}(r)\overline{Z}(\zeta)$, and obtain the equation

$$r^2\frac{d^2\overline{R}}{\mathrm{d}r^2}-2r\frac{d\overline{R}}{\mathrm{d}r}-(\nu-1)(\nu+2)\overline{R}=0,\quad \nu\in\mathbb{C}\,, \tag{4.25}$$

for the $\overline{R}(r;\nu)$ function, and the equation

$$(1-\zeta^2)\frac{d^2\overline{Z}}{d\zeta^2}-2\zeta\frac{d\overline{Z}}{d\zeta}+\nu(\nu+1)\overline{Z}=0,\quad \nu\in\mathbb{C}\,, \tag{4.26}$$

for the $\overline{Z}(\zeta;\nu)$ function. Hence, the $\zeta-$dependence remain the same, as in the Laplace's equation, while the $r-$dependence is replaced by

$$\overline{R}_1(r;\nu)=r^{\nu+2},\qquad \overline{R}_2(r;\nu)=r^{-\nu+1},\quad \nu\in\mathbb{C}\,. \tag{4.27}$$

Equation (4.20) accepts solutions of the form

$$\overline{q}(r,\zeta;\nu)=\overline{R}(r;\nu)X_\nu(\zeta)\,, \tag{4.28}$$

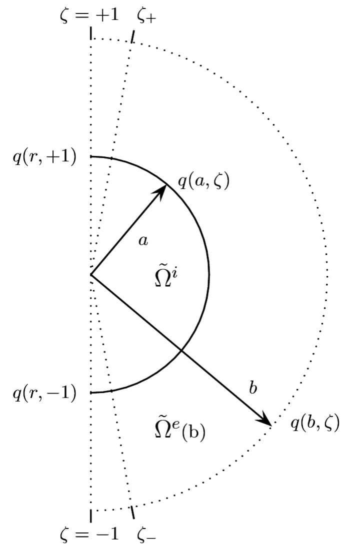

FIGURE 4.1: *The interior subdomain* $\tilde{\Omega}^i = \{0 \le r \le a,\, \zeta_- < \zeta < \zeta_+\}$ *which tend to the interior domain* $\Omega^i = \{0 \le r \le a,\, -1 < \zeta < +1\}$ *as* $\zeta_\pm \to \pm 1$, *and the exterior shell subdomain* $\tilde{\Omega}^e(b) = \{a \le r < b,\, \zeta_- < \zeta < \zeta_+\}$ *which tend to the exterior domain* $\Omega^e = \{a \le r < +\infty,\, -1 < \zeta < +1\}$ *as* $\zeta_\pm \to \pm 1$ *and* $b \to \infty$.

where $\overline{R}(r;\nu)$ are given by (4.27) and $X_\nu(\zeta)$ stands for any Legendre function.
Let

$$\Xi(r,\zeta;\nu) = \overline{q}(r,\zeta;\nu)\,\mu(r,\zeta;\nu), \quad \nu \in \mathbb{C}\,, \tag{4.29}$$

where $\mu(r,\zeta;\nu)$ an auxiliary function. Replacing (4.29) into equations (4.22) and (4.23) it is straightforward to show that the Lax pair (4.22), (4.23) assumes the form

$$\left(\frac{\partial}{\partial\zeta} + \frac{d\ln X_\nu(\zeta)}{d\zeta}\right)\mu(r,\zeta;\nu) = \left[\frac{\partial}{\partial r} + \left(\frac{2}{r} - \frac{d\ln\overline{R}(r)}{\mathrm{d}\,r}\right)\right]q(r,\zeta), \quad \nu \in \mathbb{C}\,, \tag{4.30}$$

$$r^2\left(\frac{\partial}{\partial r} + \frac{d\ln\overline{R}(r)}{\mathrm{d}\,r}\right)\mu(r,\zeta;\nu) = -(1-\zeta^2)\left(\frac{\partial}{\partial\zeta} - \frac{d\ln X_\nu(\zeta)}{d\zeta}\right)q(r,\zeta), \quad \nu \in \mathbb{C}\,. \tag{4.31}$$

The solution $\overline{q}(r,\zeta;\nu)$ remains bounded in the neighborhood of $r=0$ for $\mathrm{Re}\,\nu \in [-2,+\infty)$ when the functions $\overline{R}_1(r;\nu)$ are chosen and for $\mathrm{Re}\,\nu \in (-\infty,+1]$ when the functions $\overline{R}_2(r;\nu)$ are chosen. This regions characterize the *interior* solutions of (4.20). Thus, applying (4.24) in the domain $\tilde{\Omega}^i$, as shown in Figure 4.1 it is straightforward to show that

$$
\begin{aligned}
&-\int_0^a (1-\zeta_-^2)\left(X_\nu(\zeta_-)\,\frac{\partial q(r,\zeta_-)}{\partial\zeta} - \frac{dX_\nu(\zeta_-)}{d\zeta}\,q(r,\zeta_-)\right)\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2}\\
&+\int_{\zeta_-}^{\zeta_+}\left[\overline{R}(a;\nu)\,\frac{\partial q(a,\zeta)}{\partial r} + \left(\frac{2}{a}\,\overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{\mathrm{d}\,r}\right)\,q(a,\zeta)\right]X_\nu(\zeta)\,d\zeta\\
&+\int_0^a (1-\zeta_+^2)\left(X_\nu(\zeta_+)\,\frac{\partial q(r,\zeta_+)}{\partial\zeta} - \frac{dX_\nu(\zeta_+)}{d\zeta}\,q(r,\zeta_+)\right)\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2} = 0
\end{aligned}
\tag{4.32}
$$

The Legendre functions of the first kind are regular at $\zeta = +1$ with value $P_\nu(+1) = 1$, and exhibit the singular behavior $P_\nu(\zeta) \sim \ln\frac{1+\zeta}{2}$ as $\zeta \to -1^+$. On the other hand, the Legendre functions of the second kind are irregular along the ζ-axis. In particular, they exhibit the singular behavior $Q_\nu(\zeta) \sim \ln(1 \mp \zeta)$ as ζ tends to $\pm 1^\mp$ [WG89, pp. 255-261]. From this observation it follows that $(1-\zeta^2)\,X_\nu(\zeta)$ tends to zero as ζ tends to $\pm 1^\mp$, where else the following limits hold (see Appendix)

$$
\lim_{\zeta\to\pm 1^\mp}(1-\zeta^2)\frac{d}{d\zeta}\,Q_\nu(\zeta) = \begin{cases}1\\ \cos\nu\pi\end{cases}, \quad \nu\in\mathbb{C}, \quad \nu \neq -1,-2,\dots\,, \tag{4.33}
$$

$$
\lim_{\zeta\to\pm 1^\mp}(1-\zeta^2)\frac{d}{d\zeta}\,P_\nu(\zeta) = \begin{cases}0\\ \frac{2}{\pi}\sin\pi\nu\end{cases}, \quad \nu\in\mathbb{C}\,, \tag{4.34}
$$

where for the penultimate expression we used the recurrence relation

$$
(1-\zeta^2)\frac{d}{d\zeta}\,P_\nu(\zeta) = \nu\Big(P_{\nu-1}(\zeta) - \zeta\,P_\nu(\zeta)\Big)\,.
$$

Thus, (4.32) is evaluated as

$$
\begin{aligned}
&\overline{R}(a;\nu)\,\mathfrak{N}(\nu|X_\nu) + \left(\frac{2}{a}\,\overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{\mathrm{d}\,r}\right)\mathfrak{D}(\nu|X_\nu)\\
&= \begin{cases} -\frac{2}{\pi}\sin\pi\nu\int_0^a q(r,-1)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2}, & \text{if } X_\nu(\zeta) = P_\nu(\zeta)\\ \int_0^a \big(q(r,+1) - \cos\nu\pi\,q(r,-1)\big)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2}, & \text{if } X_\nu(\zeta) = Q_\nu(\zeta)\end{cases}, \ \nu \neq -1,-2,\dots\,,
\end{aligned}
\tag{4.35}
$$

where the Legendre transforms of the boundary data $\mathfrak{D}(\nu|X_\nu)$ and $\mathfrak{N}(\nu|X_\nu)$ are given by

$$
\mathfrak{D}(\nu|X_\nu) = \int_{-1}^{+1} g_D(\zeta)X_\nu(\zeta)d\zeta\,, \tag{4.36}
$$

$$
\mathfrak{N}(\nu|X_\nu) = \int_{-1}^{+1} g_N(\zeta)X_\nu(\zeta)d\zeta\,. \tag{4.37}
$$

Both integrals in (4.36) and (4.37) exist due do the logarithmic singularities of the Legendre functions.

The parameter ν lives in appropriate subdomains of $\mathbb{C}$, specified by the regularity of the radial factors of the solution of the formal adjoint (4.27) at the origin, and therefore (4.35) rewrites as

$$a\,\mathfrak{N}(\nu|X_\nu) - \nu\,\mathfrak{D}(\nu|X_\nu) = \begin{cases} -\frac{2}{\pi}\sin\pi\nu\int_0^a q(r,-1)\,\left(\frac{r}{a}\right)^{\nu+1}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta) \\ \int_0^a\left[q(r,+1)-\cos\nu\pi\,q(r,-1)\right]\,\left(\frac{r}{a}\right)^{\nu+1}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta) \end{cases}, \quad \mathrm{Re}\,\nu\geq 0 \quad , \tag{4.38}$$

$$a\,\mathfrak{N}(\nu|X_\nu) + (\nu+1)\,\mathfrak{D}(\nu|X_\nu) = \begin{cases} -\frac{2}{\pi}\sin\pi\nu\int_0^a q(r,-1)\,\left(\frac{a}{r}\right)^{\nu}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta) \\ \int_0^a\left[q(r,+1)-\cos\nu\pi\,q(r,-1)\right]\,\left(\frac{a}{r}\right)^{\nu}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta) \end{cases}, \mathrm{Re}\,\nu<-1,\ \nu\neq-2,\dots \tag{4.39}$$

where (4.38) is derived with the use of $\overline{R}_1$ and (4.39) with the use of $\overline{R}_2$. Evaluating (4.38a) for $\nu=0$, one obtains the compatibility condition (4.4).
Similarly, $\overline{q}(r,\zeta;\nu)$ stays bounded as r tends to infinity for every ν in the half plane $\mathrm{Re}\,\nu\in(-\infty,-2]$ for the solutions $r^{\nu+2}$, and for $\mathrm{Re}\,\nu\in[1,+\infty)$ for the solutions $r^{-\nu+1}$. Hence, the global relation (4.24) in the domain $\tilde{\Omega}^e(b)$, depicted in Figure 4.1 takes the form

$$\begin{aligned} &-\int_a^b(1-\zeta_-^2)\left(X_\nu(\zeta_-)\,\frac{\partial q(r,\zeta_-)}{\partial\zeta}-\frac{dX_\nu(\zeta_-)}{d\zeta}\,q(r,\zeta_-)\right)\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2} \\ &+\int_{\zeta_-}^{\zeta_+}\left[\overline{R}(b;\nu)\,\frac{\partial q(b,\zeta)}{\partial r}+\left(\frac{2}{b}\,\overline{R}(b;\nu)-\frac{d\overline{R}(b;\nu)}{\mathrm{d}\,r}\right)\,q(b,\zeta)\right]X_\nu(\zeta)\,d\zeta \\ &+\int_a^b(1-\zeta_+^2)\left(X_\nu(\zeta_+)\,\frac{\partial q(r,\zeta_+)}{\partial\zeta}-\frac{dX_\nu(\zeta_+)}{d\zeta}\,q(r,\zeta_+)\right)\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2} \\ &-\int_{\zeta_-}^{\zeta_+}\left[\overline{R}(a;\nu)\,\frac{\partial q(a,\zeta)}{\partial r}+\left(\frac{2}{a}\,\overline{R}(a;\nu)-\frac{d\overline{R}(a;\nu)}{\mathrm{d}\,r}\right)\,q(a,\zeta)\right]X_\nu(\zeta)\,d\zeta=0\,. \end{aligned} \tag{4.40}$$

As $b\to\infty$ the second integral vanishes for both $\overline{R}_1$ and $\overline{R}_2$. Then, in analogy to the relations (4.38), (4.39) we obtain

$$a\,\mathfrak{N}(\nu|X_\nu) - \nu\,\mathfrak{D}(\nu|X_\nu) = \begin{cases} \frac{2}{\pi}\sin\pi\nu\int_a^{+\infty} q(r,-1)\,\left(\frac{r}{a}\right)^{\nu+1}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta) \\ -\int_a^{+\infty}\left[q(r,+1)-\cos\nu\pi\,q(r,-1)\right]\,\left(\frac{r}{a}\right)^{\nu+1}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta) \end{cases}, \tag{4.41}$$

for every $\mathrm{Re}\,\nu\leq 0$, $\nu\neq-1,-2,\dots$, and

$$a\,\mathfrak{N}(\nu|X_\nu) + (\nu+1)\,\mathfrak{D}(\nu|X_\nu) = \begin{cases} \frac{2}{\pi}\sin\pi\nu\int_a^{+\infty} q(r,-1)\,\left(\frac{a}{r}\right)^{\nu}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta) \\ -\int_a^{+\infty}\left[q(r,+1)-\cos\nu\pi\,q(r,-1)\right]\,\left(\frac{a}{r}\right)^{\nu}\,\frac{\mathrm{d}\,r}{r}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta) \end{cases}, \quad \mathrm{Re}\,\nu>-1\,, \tag{4.42}$$

which are the appropriate relations for the exterior solutions.
The connection formulae (4.38), (4.39) and (4.41), (4.42) provide in a global form the coefficients of the Legendre expansion of the Dirichlet data in terms of the Neumann data and vice versa.

Remark 4.4.1 *We note here that in spherical coordinates with axial symmetry, the $\zeta-$dependence of the Laplace equation, of the formal adjoint of the Laplace equation and of the biharmonic equation comes via the Legendre functions. The corresponding $r-$dependence for the Laplace equation is given by r^{ν} and $r^{-\nu-1}$, for the formal adjoint Laplace is given by $r^{\nu+2}$ and $r^{-\nu+1}$, and for the biharmonic as r^{ν}, $r^{\nu+2}$, $r^{-\nu-1}$, and $r^{-\nu+1}$. Hence, the solutions of the formal adjoint Laplace equation are the additional two independent solutions introduced by the second application of the Laplace operator on itself.*

4.5 From Complex to Real: Recovering Classical Solutions

In certain cases, depending on the form of the operator acting on $q(\xi_1, \xi_2)$, it is possible to re-derive the classical representations via the global relation. This procedure will be analytically described in the sequence.

4.5.1 Part 1: The Interior Problem

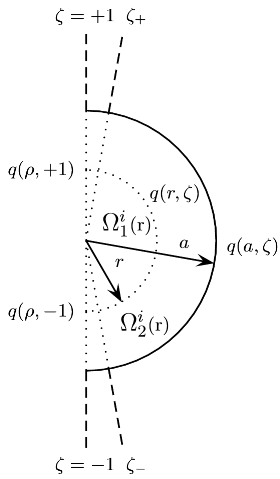

FIGURE 4.2: *The interior subdomains* $\Omega_1^i(r) = \{0 \le \rho \le r,\ -1 < \zeta < +1\}$ *and* $\Omega_2^i(r) = \{r \le \rho \le a,\ -1 < \zeta < +1\}$.

Applying the global relation (4.24) to the domain

$$\Omega_1^i(r) = \Big\{0 \le \rho \le r,\ -1 < \zeta < +1\Big\} \tag{4.43}$$

depicted in Figure 4.2, we obtain

$$\int_{-1}^{+1}\left[\overline{R}(r;\nu)\frac{\partial q(r,\zeta)}{\partial r}+\left(\frac{2}{r}\,\overline{R}(r;\nu)-\frac{d\overline{R}(r;\nu)}{\mathrm{d}\,r}\right)q(r,\zeta)\right]X_\nu(\zeta)d\zeta$$
$$=\begin{cases}-\frac{2}{\pi}\sin\pi\nu\int_0^r q(\rho,-1)\,\overline{R}(\rho;\nu)\,\frac{d\rho}{\rho^2} & ,\text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ \int_0^r\left[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\right]\overline{R}(\rho;\nu)\,\frac{d\rho}{\rho^2} & ,\text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases}\quad,\quad \nu\neq-1,-2,\dots\,, \tag{4.44}$$

where $\overline{R}(r;\nu)$ and $\frac{d\overline{R}(r;\nu)}{\mathrm{d}r}$ have to be bounded as r tends to zero, that is $\mathrm{Re}\,\nu\geq-2$ if $\overline{R}(r;\nu)=r^{\nu+2}$ and $\mathrm{Re}\,\nu\leq 1$ if $\overline{R}(r;\nu)=r^{-\nu+1}$. Similarly, for the domain

$$\Omega_2^i(r)=\left\{r\leq\rho\leq a,\ -1<\zeta<+1\right\} \tag{4.45}$$

depicted in Figure 4.2, we find

$$\int_{-1}^{+1}\left[\overline{R}(r;\nu)\,\frac{\partial q(r,\zeta)}{\partial r}+\left(\frac{2}{r}\,\overline{R}(r;\nu)-\frac{d\overline{R}(r;\nu)}{\mathrm{d}\,r}\right)q(r,\zeta)\right]X_\nu(\zeta)d\zeta$$
$$=\left[\overline{R}(a;\nu)\mathfrak{N}(\nu|X_\nu)+\left(\frac{2}{a}\,\overline{R}(a;\nu)-\frac{d\overline{R}(a;\nu)}{\mathrm{d}\,r}\right)\mathfrak{D}(\nu|X_\nu)\right]$$
$$+\begin{cases}\frac{2}{\pi}\sin\pi\nu\int_r^a q(\rho,-1)\,\overline{R}(\rho;\nu)\,\frac{d\rho}{\rho^2} & ,\text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ -\int_r^a\left[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\right]\overline{R}(\rho;\nu)\,\frac{d\rho}{\rho^2} & ,\text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},\ \nu\neq-1,-2,\dots\,. \tag{4.46}$$

Note that the domain $\Omega_2^i(r)$ does not include either the singularity at zero or the singularity at infinity, thus there are no restrictions on ν in (4.46). However, in order to eliminate the unknown function $\frac{\partial q(r,\zeta)}{\partial r}$ one has to combine equations (4.44) and (4.46) in either one of the domains $\mathrm{Re}\,\nu\in[-2,+\infty)$ or $\mathrm{Re}\,\nu\in(-\infty,1]$.

The half plane Re $\nu\in[-2,+\infty)$

Introducing in eq. (4.44) $\overline{R}(r;\nu)=r^{\nu+2}$ we obtain

$$\int_{-1}^{+1}\left(r\,\frac{\partial q(r,\zeta)}{\partial r}-\nu\,q(r,\zeta)\right)X_\nu(\zeta)d\zeta$$
$$=\begin{cases}-\frac{2}{\pi}\sin\pi\nu\int_0^r q(\rho,-1)\,\left(\frac{\rho}{r}\right)^{\nu+1}\,\frac{d\rho}{\rho} & ,\text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ \int_0^r\left[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\right]\,\left(\frac{\rho}{r}\right)^{\nu+1}\,\frac{d\rho}{\rho} & ,\text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases}\quad,\quad \mathrm{Re}\,\nu>-1 \tag{4.47}$$

where the restriction on ν is due to the fact that the ratios $\left(\frac{r}{a}\right)^{\nu}$, $\left(\frac{r}{\rho}\right)^{\nu}$ must remain bounded as r tends to zero. Also, introducing $\overline{R}(r;\nu) = r^{-\nu+1}$ in (4.46) we obtain

$$\int_{-1}^{+1}\left(r\,\frac{\partial q(r,\zeta)}{\partial r} + (\nu+1)\,q(r,\zeta)\right)X_\nu(\zeta)\,d\zeta = \left(\frac{r}{a}\right)^{\nu}\Big(a\,\mathfrak{N}(\nu|X_\nu) + (\nu+1)\,\mathfrak{D}(\nu|X_\nu)\Big)$$
$$+\begin{cases}\frac{2}{\pi}\sin\pi\nu\int_r^a q(\rho,-1)\,\left(\frac{r}{\rho}\right)^{\nu}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta) = P_\nu(\zeta)\\[2ex] -\int_r^a\big[q(\rho,+1) - \cos\nu\pi\,q(\rho,-1)\big]\,\left(\frac{r}{\rho}\right)^{\nu}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta) = Q_\nu(\zeta)\end{cases},\ \nu\neq -1,-2,\dots \tag{4.48}$$

Subtracting the above two equations eliminates the function $\frac{\partial q(r,\zeta)}{\partial r}$

$$\int_{-1}^{+1} q(r,\zeta)X_\nu(\zeta)d\zeta = \frac{1}{2k+1}\left(\frac{r}{a}\right)^{\nu}\Big(a\,\mathfrak{N}(\nu|X_\nu) + (\nu+1)\,\mathfrak{D}(\nu|X_\nu)\Big)$$
$$+\frac{1}{2k+1}\times\begin{cases}\frac{2}{\pi}\sin\pi\nu\left(\mathbb{P}_0^a(r;\nu)q(\rho,-1)\right) & \text{, if } X_\nu(\zeta) = P_\nu(\zeta)\\[2ex] -\mathbb{P}_0^a(r;\nu)\big[q(\rho,+1) - \cos\nu\pi\,q(\rho,-1)\big] & \text{, if } X_\nu(\zeta) = Q_\nu(\zeta)\end{cases},\ \mathrm{Re}\,\nu\geq 0, \tag{4.49}$$

which is the $\nu-$Legendre coefficient of the solution $q(r,\zeta)$ valid in the domain $r\in[0,a]$. The integral operator $\mathbb{P}_0^a(r;\nu)$ is defined as

$$\mathbb{P}_0^a(r;\nu) := \int_0^r \frac{d\rho}{\rho}\left(\frac{\rho}{r}\right)^{\nu+1} + \int_r^a \frac{d\rho}{\rho}\left(\frac{r}{\rho}\right)^{\nu}. \tag{4.50}$$

The inversion of (4.49) leads to an integral representation for $q(r,\zeta)$. However, in order to recover the classical representations (4.14) and (4.16) we have to use the orthogonality relation of the Legendre polynomials. Thus, letting $\nu = n = 0,1,2,\dots$ in (4.49a) we obtain

$$\int_{-1}^{+1} q(r,\zeta)P_n(\zeta)d\zeta = \frac{1}{2n+1}\left(\frac{r}{a}\right)^{n}\Big(a\,\mathfrak{N}_n + (n+1)\,\mathfrak{D}_n\Big),\quad n=0,1,2,\dots, \tag{4.51}$$

where $\mathfrak{D}_n$ and $\mathfrak{N}_n$ are given by (4.15) and (4.17) respectively.
Using the Dirichlet-to-Neumann correspondence given by (4.38a) with ν replaced by n as well as the expansion of q as Legendre polynomials, we recover trivially the expansions (4.14) or (4.16), respectively.

The half plane Re $\nu\in(-\infty,+1]$

Replacing, in (4.44) and (4.46), $\overline{R}(r;\nu)$ by $r^{-\nu+1}$ and $r^{\nu+2}$ respectively, we find

$$\int_{-1}^{+1}\left(r\,\frac{\partial q(r,\zeta)}{\partial r} + (\nu+1)\,q(r,\zeta)\right)X_\nu(\zeta)d\zeta$$
$$=\begin{cases}-\frac{2}{\pi}\sin\pi\nu\int_0^r q(\rho,-1)\,\left(\frac{r}{\rho}\right)^{\nu}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta) = P_\nu(\zeta)\\[2ex] \int_0^r\big[q(\rho,+1) - \cos\nu\pi\,q(\rho,-1)\big]\,\left(\frac{r}{\rho}\right)^{\nu}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta) = Q_\nu(\zeta)\end{cases},$$
$$\mathrm{Re}\,\nu\leq 0,\quad \nu\neq -1,-2,\dots$$

$$\int_{-1}^{+1}\left(r\,\frac{\partial q(r,\zeta)}{\partial r}-\nu\,q(r,\zeta)\right)X_\nu(\zeta)d\zeta=\left(\frac{a}{r}\right)^{\nu+1}\Big(a\,\mathfrak{N}(\nu|X_\nu)-\nu\,\mathfrak{D}(\nu|X_\nu)\Big)$$
$$+\begin{cases}\frac{2}{\pi}\sin\pi\nu\int_r^a q(\rho,-1)\,\left(\frac{\rho}{r}\right)^{\nu+1}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ -\int_r^a\big[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\big]\,\left(\frac{\rho}{r}\right)^{\nu+1}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},$$
$$\nu\in\mathbb{C},\quad \nu\neq-1,-2,\ldots.$$

Subtracting the above equations to eliminate the function $\frac{\partial q(r,\zeta)}{\partial r}$, we obtain

$$\int_{-1}^{+1}q(r,\zeta)X_\nu(\zeta)d\zeta=-\frac{1}{2k+1}\left(\frac{a}{r}\right)^{\nu+1}\Big(a\,\mathfrak{N}(\nu|X_\nu)-\nu\,\mathfrak{D}(\nu|X_\nu)\Big)$$
$$-\frac{1}{2k+1}\times\begin{cases}\frac{2}{\pi}\sin\pi\nu\Big(\mathbb{P}^{\dagger}{}_0^a(r;\nu)\,q(\rho,-1)\Big), & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ -\mathbb{P}^{\dagger}{}_0^a(r;\nu)\big[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\big], & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases}, \tag{4.52}$$
$$\operatorname{Re}\nu<-1,\quad \nu\neq-2,-3,\ldots \tag{4.53}$$

which is the ν-Legendre coefficient of the function $q(r,\zeta)$ valid in the domain $r\in[0,a]$. The integral operator $\mathbb{P}^{\dagger}{}_0^a(r;\nu)$ is defined as

$$\mathbb{P}^{\dagger}{}_0^a(r;\nu):=\int_0^r\frac{d\rho}{\rho}\,\left(\frac{r}{\rho}\right)^{\nu}+\int_r^a\frac{d\rho}{\rho}\,\left(\frac{\rho}{r}\right)^{\nu+1},\quad \mathbb{P}_0^a(r;-\nu-1)=\mathbb{P}^{\dagger}{}_0^a(r;\nu) \tag{4.54}$$

4.5.2 Part 2: The Exterior Problem

Following the same procedure as in the previous section, namely applying (4.24) in the domains $\Omega_1^e(r)$ and $\Omega_2^e(r,b)$ defined by

$$\Omega_1^e(r)=\Big\{a\le\rho\le r,\ -1<\zeta<+1\Big\} \tag{4.55}$$
$$\Omega_2^e(r,b)=\Big\{r\le\rho\le b,\ -1<\zeta<+1\Big\} \tag{4.56}$$

and depicted in Figure 4.3, we arrive at the following equations

$$\int_{-1}^{+1}\left[\overline{R}(r;\nu)\,\frac{\partial q(r,\zeta)}{\partial r}+\left(\frac{2}{r}\,\overline{R}(r;\nu)-\frac{d\overline{R}(r;\nu)}{\mathrm{d}r}\right)q(r,\zeta)\right]X_\nu(\zeta)d\zeta$$
$$=\left[\overline{R}(a;\nu)\mathfrak{N}(\nu|X_\nu)+\left(\frac{2}{a}\,\overline{R}(a;\nu)-\frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right)\mathfrak{D}(\nu|X_\nu)\right]$$
$$+\begin{cases}-\frac{2}{\pi}\sin\pi\nu\int_a^r q(\rho,-1)\,\overline{R}(\rho;\nu)\,\frac{d\rho}{\rho^2}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ \int_a^r\big[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\big]\,\overline{R}(\rho;\nu)\,\frac{d\rho}{\rho^2}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},\ \nu\neq-1,-2,\ldots \tag{4.57}$$

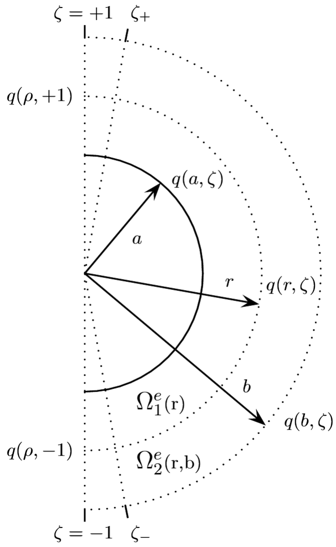

FIGURE 4.3: *The exterior subdomains* $\Omega_1^e(r) = \{a \le \rho \le r,\ -1 < \zeta < +1\}$ *and* $\Omega_2^e(r, b) = \{r \le \rho \le b,\ -1 < \zeta < +1\}$.

which holds for every $\nu \in \mathbb{C}$ since $\Omega_1^e(r)$ is isolated both from the origin and from infinity, and

$$\int_{-1}^{+1} \left[\overline{R}(r;\nu)\, \frac{\partial q(r,\zeta)}{\partial r} + \left(\frac{2}{r}\, \overline{R}(r;\nu) - \frac{d\overline{R}(r;\nu)}{\mathrm{d}r} \right) q(r,\zeta) \right] X_\nu(\zeta) d\zeta$$
$$= \begin{cases} \frac{2}{\pi} \sin \pi\nu \int_r^\infty q(\rho,-1)\, \overline{R}(\rho;\nu)\, \frac{d\rho}{\rho^2}, & \text{if } X_\nu(\zeta) = P_\nu(\zeta) \\ -\int_r^\infty \left[q(\rho,+1) - \cos\nu\pi\, q(\rho,-1) \right] \overline{R}(\rho;\nu)\, \frac{d\rho}{\rho^2}, & \text{if } X_\nu(\zeta) = Q_\nu(\zeta) \end{cases}, \ \nu \ne -1,-2,\dots \tag{4.58}$$

which holds for those complex values of ν for which the limit of the corresponding integral as $b \to \infty$ vanishes, i.e. for $\mathrm{Re}\,\nu \le -2$, when we pick $\overline{R}_1 = r^{\nu+2}$, and for $\mathrm{Re}\,\nu \ge 1$, when the solutions $\overline{R}_2 = r^{-\nu+1}$ are chosen.

The Half plane Re $\nu \in [+1, +\infty)$

Setting $\overline{R}_1(r;\nu) = r^{\nu+2}$ in (4.57), and $\overline{R}_2(r;\nu) = r^{-\nu+1}$ in (4.58), we obtain

$$\int_{-1}^{+1} \left(r\, \frac{\partial q(r,\zeta)}{\partial r} - \nu\, q(r,\zeta) \right) X_\nu(\zeta) d\zeta = \left(\frac{a}{r}\right)^{\nu+1} \left(a\, \mathfrak{N}(\nu|X_\nu) - \nu\, \mathfrak{D}(\nu|X_\nu) \right)$$
$$+ \begin{cases} -\frac{2}{\pi} \sin\pi\nu \int_a^r q(\rho,-1)\, \left(\frac{\rho}{r}\right)^{\nu+1} \frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta) = P_\nu(\zeta) \\ \int_a^r \left[q(\rho,+1) - \cos\nu\pi\, q(\rho,-1) \right] \left(\frac{\rho}{r}\right)^{\nu+1} \frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta) = Q_\nu(\zeta) \end{cases}, \ \nu \in \mathbb{C},\ \nu \ne -1,-2,\dots$$

$$\int_{-1}^{+1}\left(r\,\frac{\partial q(r,\zeta)}{\partial r}+(\nu+1)\,q(r,\zeta)\right)X_\nu(\zeta)d\zeta$$
$$=\begin{cases}\frac{2}{\pi}\sin\pi\nu\int_r^\infty q(\rho,-1)\,\left(\frac{r}{\rho}\right)^\nu\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ -\int_r^\infty\left[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\right]\,\left(\frac{r}{\rho}\right)^\nu\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},\quad \mathrm{Re}\,\nu\geq 1\quad .$$

Subtracting the above equations eliminates the function $\frac{\partial q(r,\zeta)}{\partial r}$

$$\int_{-1}^{+1}q(r,\zeta)X_\nu(\zeta)d\zeta=-\frac{1}{2k+1}\left(\frac{a}{r}\right)^{\nu+1}\left(a\,\mathfrak{N}(\nu|X_\nu)-\nu\,\mathfrak{D}(\nu|X_\nu)\right)$$
$$-\frac{1}{2k+1}\times\begin{cases}-\frac{2}{\pi}\sin\pi\nu\left(\mathbb{P}_a^\infty(r;\nu)q(\rho,-1)\right), & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ \mathbb{P}_a^\infty(r;\nu)\left[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\right], & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},\quad \mathrm{Re}\,\nu\geq 1\quad , \tag{4.59}$$

which is the $\nu-$Legendre coefficient of the solution $q(r,\zeta)$, for $r\geq a$, and the integral operator $\mathbb{P}_a^\infty(r;\nu)$ is defined as

$$\mathbb{P}_a^\infty(r;\nu):=\int_a^r\frac{d\rho}{\rho}\,\left(\frac{\rho}{r}\right)^{\nu+1}+\int_r^\infty\frac{d\rho}{\rho}\,\left(\frac{r}{\rho}\right)^\nu\,. \tag{4.60}$$

Inverting the latter relation we obtain an integral representation for $q(r,\zeta)$ valid in the domain $r\in[a,+\infty)$. However, the classical representations (4.18) and (4.19) are recovered utilizing the orthogonality relation of the Legendre polynomials.
Thus, letting $\nu=n=1,2,\ldots$ (4.59a) reads

$$\int_{-1}^{+1}q(r,\zeta)P_n(\zeta)\,d\zeta=-\frac{1}{2n+1}\left(\frac{a}{r}\right)^{n+1}\left(a\,\mathfrak{N}_n-n\,\mathfrak{D}_n\right), \tag{4.61}$$

and using the Dirichlet-to-Neumann correspondence (4.42a) with ν replaced by n, we recover equations (4.18) and (4.19) respectively.

The half plane Re $\nu\in(-\infty,-2]$

Replacing $\overline{R}(r;\nu)$ with $r^{-\nu+1}$ and $r^{\nu+2}$ in (4.57) and (4.58), respectively, we find

$$\int_{-1}^{+1}\left(r\,\frac{\partial q(r,\zeta)}{\partial r}+(\nu+1)\,q(r,\zeta)\right)X_\nu(\zeta)d\zeta=\left(\frac{r}{a}\right)^\nu\left(a\,\mathfrak{N}(\nu|X_\nu)+(\nu+1)\,\mathfrak{D}(\nu|X_\nu)\right)$$
$$+\begin{cases}-\frac{2}{\pi}\sin\pi\nu\int_a^r q(\rho,-1)\,\left(\frac{r}{\rho}\right)^\nu\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ \int_a^r\left[q(\rho,+1)-\cos\nu\pi\,q(\rho,-1)\right]\,\left(\frac{r}{\rho}\right)^\nu\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},\nu\in\mathbb{C},\,\nu\neq-1,-2,\ldots\,, \tag{4.62}$$

$$\int_{-1}^{+1}\left(r\,\frac{\partial q(r,\zeta)}{\partial r}-\nu\, q(r,\zeta)\right)X_\nu(\zeta)d\zeta$$
$$=\begin{cases}\frac{2}{\pi}\sin\pi\nu\int_r^\infty q(\rho,-1)\,\left(\frac{\rho}{r}\right)^{\nu+1}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ -\int_r^\infty\left[q(\rho,+1)-\cos\nu\pi\, q(\rho,-1)\right]\,\left(\frac{\rho}{r}\right)^{\nu+1}\,\frac{d\rho}{\rho}, & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},$$
$$\mathrm{Re}\,\nu<-2,\ \nu\neq-3,-4,\ldots. \tag{4.63}$$

Eliminating the function $\frac{\partial q(r,\zeta)}{\partial r}$ by subtracting the above equations, leads to the expression

$$\int_{-1}^{+1} q(r,\zeta)\,X_\nu(\zeta)\,d\zeta=\frac{1}{2k+1}\left(\frac{r}{a}\right)^\nu\left(a\,\mathfrak{N}(\nu|X_\nu)+(\nu+1)\,\mathfrak{D}(\nu|X_\nu)\right)$$
$$+\frac{1}{2k+1}\times\begin{cases}-\frac{2}{\pi}\sin\pi\nu\left(\mathbb{P}{\dagger}_a^\infty(r;\nu)\,q(\rho,-1)\right), & \text{if } X_\nu(\zeta)=P_\nu(\zeta)\\ \mathbb{P}{\dagger}_a^\infty(r;\nu)\left[q(\rho,+1)-\cos\nu\pi\, q(\rho,-1)\right], & \text{if } X_\nu(\zeta)=Q_\nu(\zeta)\end{cases},$$
$$\mathrm{Re}\,\nu<-2,\ \nu\neq-3,-4,\ldots \tag{4.64}$$

where the integral operator $\mathbb{P}{\dagger}_a^\infty(r;\nu)$ is defined as

$$\mathbb{P}{\dagger}_a^\infty(r;\nu):=\int_a^r\frac{d\rho}{\rho}\,\left(\frac{r}{\rho}\right)^\nu+\int_r^\infty\frac{d\rho}{\rho}\,\left(\frac{\rho}{r}\right)^{\nu+1}. \tag{4.65}$$

Remark 4.5.1 *Relation (4.52a) is obtained from (4.49a) if we replace* ν *by* $-(\nu+1)$ *and observe that Legendre's equation* (4.6) *remains invariant under this transformation. That is* $P_{-(\nu+1)}=P_k$ *and the half-plane Re*$\,\nu\geq 0$ *is mapped to Re*$\,\nu\leq-1$*. Similarly, this transformation maps (4.64a) to (4.59a) and the half plane Re*$\,\nu\leq-2$ *to the half plane Re*$\,\nu\geq 1$.

4.6 A Novel Integral Representation

Novel integral representations are obtained by applying the global relation to particular subdomains. This is realized in the sequel.

4.6.1 Part 1: Solutions valid in the interior

Applying the global relation (4.24) in the subdomain

$$\Omega_3^i(\zeta)=\left\{0\leq r\leq a,\ \zeta\leq t<+1\right\} \tag{4.66}$$

depicted in Figure 4.4, with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\,P_\nu(\zeta)$, we obtain

$$\int_0^a(1-\zeta^2)\left(P_\nu(\zeta)\,\frac{\partial q(r,\zeta)}{\partial\zeta}-\frac{dP_\nu(\zeta)}{d\zeta}\,q(r,\zeta)\right)\overline{R}(r;\nu)\,\frac{\mathrm{d}r}{r^2}$$
$$=\int_\zeta^{+1}\left[\overline{R}(a;\nu)\,g_N(t)+\left(\frac{2}{a}\overline{R}(a;\nu)-\frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right)g_D(t)\right]P_\nu(t)\mathrm{d}\,t,\quad (r,\zeta)\in\partial\Omega_3^i(\zeta)\ , \tag{4.67}$$

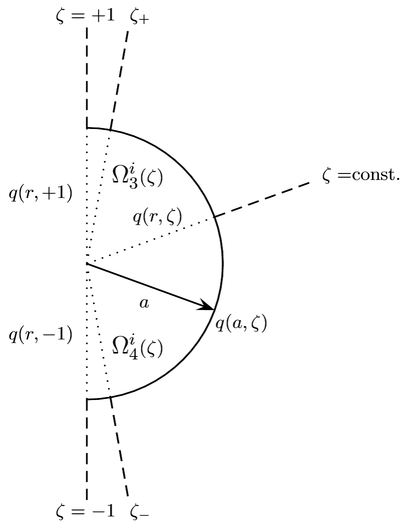

FIGURE 4.4: *The interior subdomains* $\Omega_3^i(\zeta) = \{0 \le r \le a,\ \zeta \le t < +1\}$ *and* $\Omega_4^i(\zeta) = \{0 \le r \le a,\ -1 < t \le \zeta\}$.

where the regularity of the Legendre functions at $\zeta = +1$ has been used. Similarly, applying the global relation (4.24) in the subdomain

$$\Omega_4^i(\zeta) = \Big\{0 \le r \le a,\ -1 < t \le \zeta\Big\} \tag{4.68}$$

depicted in Figure 4.4, with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\,Q_\nu(\zeta)$ we obtain

$$\begin{aligned}
&\int_0^a (1-\zeta^2)\left(Q_\nu(\zeta)\,\frac{\partial q(r,\zeta)}{\partial \zeta} - \frac{dQ_\nu(\zeta)}{d\zeta}\,q(r,\zeta)\right)\overline{R}(r;\nu)\frac{\mathrm{d}r}{r^2}\\
&= -\int_{-1}^{\zeta}\left[\overline{R}(a;\nu)\,g_N(t) + \left(\frac{2}{a}\,\overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right)g_D(t)\right]Q_\nu(t)\mathrm{d}\,t\\
&\quad - \cos\nu\pi\int_0^a q(\tau,-1)\,\overline{R}(\tau;\nu)\frac{\mathrm{d}\tau}{\tau^2}, \quad \nu \ne -1,-2,\dots, \quad (r,\zeta) \in \partial\Omega_4^i(\zeta)\,.
\end{aligned} \tag{4.69}$$

Multiply (4.67) by $Q_\nu(\zeta)$ and (4.69) by $P_\nu(\zeta)$ and subtracting them side by side in order to eliminate the unknown function $\frac{\partial q(r,\zeta)}{\partial \zeta}$ we obtain

$$\begin{aligned}
&(1-\zeta^2)\Big(P_\nu(\zeta)\,\frac{dQ_\nu(\zeta)}{d\zeta} - Q_\nu(\zeta)\,\frac{dP_\nu(\zeta)}{d\zeta}\Big)\int_0^a q(r,\zeta)\,\overline{R}(r;\nu)\frac{\mathrm{d}r}{r^2}\\
&=\mathfrak{P}_\nu(\zeta)\left[\overline{R}(a;\nu)\,g_N(t) + \left(\frac{2}{a}\,\overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right)g_D(t)\right]\\
&\quad + \cos\nu\pi\,P_\nu(\zeta)\int_0^a q(\tau,-1)\,\overline{R}(\tau;\nu)\frac{\mathrm{d}\tau}{\tau^2}, \quad \nu \ne -1,-2,\dots\,,
\end{aligned} \tag{4.70}$$

where

$$\mathfrak{G}_\nu^{\frac{1}{2}}(\zeta)\, f \equiv \mathfrak{P}_\nu(\zeta)\, f = P_\nu(\zeta)\int_{-1}^{\zeta} Q_\nu(t)\, f(t)\,\mathrm{d}\,t + Q_\nu(\zeta)\int_{\zeta}^{1} P_\nu(t)\, f(t)\,\mathrm{d}\,t, \quad \nu \neq -1,-2,\dots \,, \tag{4.71}$$

is an integral operator, which we will refer to as the *Gegenbauer Integral Operator* of order $\frac{1}{2}$ or simple *Legendre Integral Operator* $\mathfrak{P}_\nu(\zeta)$ acting on the function

$$f : [-1,+1] \to \mathbb{C}\ .$$

The first integral in (4.71) exist since $Q_\nu(\zeta)$ exhibits a logarithmic singularity at $\zeta = -1$. Utilizing the Wronskian relation

$$P_\nu(\zeta)\,\frac{dQ_\nu(\zeta)}{d\zeta} - Q_\nu(\zeta)\,\frac{dP_\nu(\zeta)}{d\zeta} = \frac{1}{1-\zeta^2}\,,$$

we finally write equation (4.70) as follows

$$\int_0^a q(r,\zeta)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}r}{r^2} = \mathfrak{P}_\nu(\zeta)\left[\overline{R}(a;\nu)\, g_N(t) + \left(\frac{2}{a}\,\overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right) g_D(t)\right] + \cos\nu\pi\, P_\nu(\zeta)\int_0^a q(\tau,-1)\,\overline{R}(\tau;\nu)\frac{\mathrm{d}\,\tau}{\tau^2}, \quad \nu \neq -1,-2,\dots\,. \tag{4.72}$$

Relation (4.72) provides a global connection between the values of the interior solution along the radii ζ =constant and $\zeta = -1$, with its Dirichlet and Neumann values on the boundary. The parameter ν lives in appropriate subdomains of $\mathbb{C}$, specified by the regularity of the radial factors of the solution of the formal adjoint at the origin.

The Half-plane Re $\nu \in [-2,+\infty)$

In the half-plane $\mathrm{Re}\,\nu \in [-2,+\infty)$, $\overline{q}(r,\zeta;\nu)$ remains bounded in the vicinity of $r = 0$ whenever $\overline{R}(r;\nu)$ is replaced by $r^{\nu+2}$. Thus, introducing $\overline{R}(r;\nu) = r^{\nu+2}$ in (4.72) we obtain

$$\int_0^a \left(\frac{r}{a}\,q(r,\zeta)\right)\left(\frac{r}{a}\right)^{\nu-1} \mathrm{d}\left(\frac{r}{a}\right) = \mathfrak{P}_\nu(\zeta)\left(a\,g_N(t) - \nu\,g_D(t)\right) + \frac{1}{a}\cos\nu\pi\,P_\nu(\zeta)\int_0^a q(\tau,-1)\left(\frac{\tau}{a}\right)^{\nu}\mathrm{d}\,\tau\,. \tag{4.73}$$

The inverse Mellin transform then implies

$$q(r,\zeta) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu - i\infty}^{\mathrm{Re}\,\nu + i\infty}\left(\frac{a}{r}\right)^{\nu+1}\mathfrak{P}_\nu(\zeta)\Big(a\,g_N(t) - \nu\,g_D(t)\Big)\mathrm{d}\nu + \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu - i\infty}^{\mathrm{Re}\,\nu + i\infty} r^{-\nu-1}\,\cos\pi\nu\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\,\tau\right]\mathrm{d}\nu,\ \nu \neq -1,-2,\dots\,, \tag{4.74}$$

where the integral is taken over any open contour Γ, connecting the points $\mathrm{Re}\,\nu - iR$ and $\mathrm{Re}\,\nu + iR$ in the complex $\nu-$plane as $R \to \infty$.

The Legendre functions of the first kind are in general defined as

$$P_\nu(\zeta) = F\left(-\nu, \nu+1, 1; \frac{1-\zeta}{2}\right),$$

thus, because of the symmetry of F, we obtain

$$P_\nu(\zeta) = P_{-\nu-1}(\zeta) \tag{4.75}$$

for all values of ν and

$$Q_{-\nu-1}(\zeta) = Q_\nu(\zeta) - \pi \cot \pi\nu \, P_\nu(\zeta) \tag{4.76}$$

for every $\nu \in \mathbb{C}$ except for integral values [MF53, p.599].
Hence, replacing ν with $-\nu - 1$ in (4.73), we obtain the formula

$$\begin{aligned}\int_0^a q(r,\zeta)\, r^{-\nu-1}\,\mathrm{d}r &= a^{-\nu}\,\mathfrak{P}_{-\nu-1}(\zeta)\left(a\,g_N(t) + (\nu+1)\,g_D(t)\right)\\ &- \cos\nu\pi\, P_{-\nu-1}(\zeta)\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\,\tau, \ \mathrm{Re}\,\nu < -1\,, \quad \nu \neq -2,-3,\dots\,.\end{aligned} \tag{4.77}$$

Combining (4.71) together with (4.75) and (4.76) it can be easily shown that

$$\mathfrak{P}_{-\nu-1}(\zeta) = \mathfrak{P}_\nu(\zeta) - \pi\cot\pi\nu\, P_\nu(\zeta)\int_{-1}^{+1}\mathrm{d}\,t\, P_\nu(t), \quad \nu\in\mathbb{C}, \quad \nu\neq n\in\mathbb{Z}\,, \tag{4.78}$$

and (4.77), with the aid of (4.39), rewrites

$$\begin{aligned}a^\nu \int_0^a q(r,\zeta)\, r^{-\nu-1}\,\mathrm{d}r &= \mathfrak{P}_\nu(\zeta)\left(a\,g_N(t) + (\nu+1)\,g_D(t)\right)\\ &+ a^\nu\ \cos\nu\pi\, P_\nu(\zeta)\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\,\tau, \quad \mathrm{Re}\,\nu < -1\ , \quad \nu \neq -2,-3,\dots\,.\end{aligned} \tag{4.79}$$

Remark 4.6.1 *Equations* (4.73)

$$\begin{aligned}&a\left(\mathfrak{P}_\nu(\zeta)\,g_N(t)\right) - \nu\left(\mathfrak{P}_\nu(\zeta)\,g_D(t)\right)\\ &= \int_0^a \left[q(r,\zeta) - \cos\nu\pi\, P_\nu(\zeta)\, q(r,-1)\right]\left(\frac{r}{a}\right)^{\nu+1}\frac{\mathrm{d}\,r}{r}, \quad \mathit{Re}\,\nu \geq 0 \quad ,\end{aligned}$$

and (4.79)

$$\begin{aligned}&a\left(\mathfrak{P}_\nu(\zeta)\,g_N(t)\right) + (\nu+1)\left(\mathfrak{P}_\nu(\zeta)\,g_D(t)\right)\\ &= \int_0^a \left[q(r,\zeta) - \cos\nu\pi\, P_\nu(\zeta)\, q(r,-1)\right]\left(\frac{a}{r}\right)^{\nu}\frac{\mathrm{d}\,r}{r}, \quad \mathit{Re}\,\nu < -1, \quad \nu\neq -2,-3,\dots\,,\end{aligned}$$

constitute the generalized Dirichlet-to-Neumann correspondence for the interior of the sphere. Evaluating the above relations at $\zeta = +1$, equations (4.38b) and (4.39b) are retrieved. In order to obtain the correspondences (4.38a) and (4.39a) one must replace $\overline{q}(r,\zeta;\nu)$ by $\overline{R}(r;\nu)\,Q_\nu(\zeta)$ in the subdomain $\Omega_3^i(\zeta)$ and by $\overline{R}(r;\nu)\,P_\nu(\zeta)$ in $\Omega_4^i(\zeta)$.

Utilizing (4.79) to eliminate the unknown boundary data from (4.74), we obtain the following equations

$$
\begin{aligned}
q(r,\zeta) =& -\frac{1}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}(2\nu+1)\Big(\mathfrak{P}_\nu(\zeta)\,g_D(t)\Big)\,\mathrm{d}\nu \\
&+\frac{1}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\left[\int_0^a q(\tau,\zeta)\,\left(\frac{a}{\tau}\right)^{\nu}\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}\nu \\
&-\frac{1}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\cos\pi\nu\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\left(\frac{a}{\tau}\right)^{\nu}\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}\nu \\
&+\frac{1}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}r^{-\nu-1}\cos\pi\nu\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\quad \nu\neq-1,-2,\dots,
\end{aligned}
\tag{4.80}
$$

if Dirichlet boundary values are described, or

$$
\begin{aligned}
q(r,\zeta) =& \frac{a}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{2\nu+1}{\nu+1}\Big(\mathfrak{P}_\nu(\zeta)\,g_N(t)\Big)\,\mathrm{d}\nu \\
&-\frac{1}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{\nu}{\nu+1}\left[\int_0^a q(\tau,\zeta)\,\left(\frac{a}{\tau}\right)^{\nu}\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}\nu \\
&+\frac{1}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{\nu}{\nu+1}\cos\pi\nu\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\left(\frac{a}{\tau}\right)^{\nu}\frac{\mathrm{d}\tau}{\tau}\right]\mathrm{d}\nu \\
&+\frac{1}{2\pi i}\int_{\operatorname{Re}\nu-i\infty}^{\operatorname{Re}\nu+i\infty}r^{-\nu-1}\cos\pi\nu\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\quad \nu\neq-1,-2,\dots,
\end{aligned}
\tag{4.81}
$$

for Neumann data.

As $\nu \to \infty$ the Legendre functions of the first and second kind assume the asymptotic forms [Erd53, p. 162]

$$P_\nu(\cos\theta) = \sqrt{\frac{2}{\nu\pi\sin\theta}}\Big(\alpha(\theta)\,\cos(\nu\theta)+\beta(\theta)\,\sin(\nu\theta)\Big)+\mathcal{O}\Big(\frac{1}{\nu}\Big),\quad |\arg\,\nu|<\pi\,, \tag{4.82}$$

and

$$Q_\nu(\cos\theta) = \sqrt{\frac{\pi}{2\nu\sin\theta}}\Big(\beta(\theta)\,\cos(\nu\theta)-\alpha(\theta)\,\sin(\nu\theta)\Big)+\mathcal{O}\Big(\frac{1}{\nu}\Big),\quad |\arg\,\nu|<\pi\,, \tag{4.83}$$

for every $0<\theta<\pi$, where we used the asymptotic behavior of the ratio of Gamma functions [Mar83, p. 49]

$$\frac{\Gamma(\nu+c)}{\Gamma(\nu+d)} = \nu^{c-d}\left[1+\mathcal{O}\left(\frac{1}{\nu}\right)\right],\quad |\arg\,\nu|<\pi\,.$$

The corresponding expressions for $\operatorname{Re}\nu\to-\infty$ can be determined by employing the symmetry relations (4.75) and (4.76) respectively. More specific, as $\operatorname{Re}\nu\to-\infty$, (4.82) holds, where else (4.83) must be replaced by the expression

$$Q_{-\nu-1}(\cos\theta) = -\sqrt{\frac{\pi}{2\nu\sin\theta}}\,\cot\pi\nu\,\Big(\beta(\theta)\,\cos\nu\theta - \alpha(\theta)\,\sin\nu\theta\Big) + \mathcal{O}\left(\frac{1}{\nu}\right), \quad |\arg\,\nu| < \pi\,, \tag{4.84}$$

for every $0 < \theta < \pi$, where $\alpha(\theta) = \frac{\sqrt{2}}{2}\Big(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\Big)$, $\beta(\theta) = \frac{\sqrt{2}}{2}\Big(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\Big)$ and $\cot\nu\pi$ remains bounded as $\nu \to \infty$. Note that for θ sufficiently close to 0 or π the asymptotic formulas (4.82), (4.83) and (4.84) become unsuitable [MO48].
Moreover, the fact that the trigonometric functions $\cos\nu$ and $\sin\nu$ are unbounded as ν approaches large values in $\mathbb{C}$, implies that the Legendre integral operator $\mathfrak{P}_\nu(\zeta)$ is unbounded as a function of ν.
Nevertheless, the functions

$$\begin{aligned} &\left(\frac{a}{r}\right)^{\nu+1}(2\nu+1)\Big(\mathfrak{P}_\nu(\zeta)g_D(t)\Big), && \left(\frac{a}{r}\right)^{\nu+1}\left(\frac{a}{\tau}\right)^{\nu+1} \\ &\left(\frac{a}{r}\right)^{\nu+1}\left(\frac{a}{\tau}\right)^{\nu+1}\cos\nu\pi\,P_\nu(\zeta), && \frac{1}{r}\left(\frac{\tau}{r}\right)^{\nu}\cos\pi\nu\,P_\nu(\zeta)\,, \end{aligned} \tag{4.85}$$

for Dirichlet problems, and

$$\begin{aligned} &\left(\frac{a}{r}\right)^{\nu+1}\frac{2\nu+1}{\nu+1}\Big(\mathfrak{P}_\nu(\zeta)g_N(t)\Big), && \left(\frac{a}{r}\right)^{\nu+1}\left(\frac{a}{\tau}\right)^{\nu+1}\frac{\nu}{\nu+1}, \\ &\left(\frac{a}{r}\right)^{\nu+1}\left(\frac{a}{\tau}\right)^{\nu+1}\frac{\nu}{\nu+1}\cos\nu\pi\,P_\nu(\zeta), && \frac{1}{r}\left(\frac{\tau}{r}\right)^{\nu}\cos\nu\pi\,P_\nu(\zeta), \end{aligned} \tag{4.86}$$

for Neumann problems, must remain bounded in order for the improper integrals on the right-hand side of (4.80) and (4.81) to make sense.
More specific, the function $\frac{a\nu+b}{\nu+1}$, $a, b \in \mathbb{R}$ is bounded in the whole complex $\nu-$plane, as the ratio of equal degree polynomials, while the exponentials $\left(\frac{a}{\chi}\right)^{\nu+1}$, where χ stands for r or τ respectively, and $\left(\frac{\tau}{r}\right)^{\nu+1}$, $r \le \tau$ are bounded for every ν in the half-plane $\mathrm{Re}\,\nu < -1$.
Furthermore, it can be shown that

$$\begin{aligned} \left|\left(\frac{a}{r}\right)^{\nu+1}\left(\frac{a}{\tau}\right)^{\nu+1}\frac{\nu}{\nu+1}\cos\nu\pi\,P_\nu(\zeta)\right| &\le A(\theta)\,\frac{\mathrm{Re}\,\nu}{\sqrt{\mathrm{Re}\,\nu}\,(\mathrm{Re}\,\nu+1)} \\ \times\exp\left[\left(\ln\left(\frac{a}{r}\right)+\ln\left(\frac{a}{\tau}\right)\right)(\mathrm{Re}\,\nu+1)\right]&\exp\left[B_\pm(\theta)\,\mathrm{Im}\,\nu\right]\,, \end{aligned} \tag{4.87}$$

and

$$\begin{aligned} \left|\frac{1}{r}\left(\frac{\tau}{r}\right)^{\nu}\cos\nu\pi\,P_\nu(\zeta)\right| &\le A(\theta)\,\frac{\mathrm{Re}\,\nu}{\sqrt{\mathrm{Re}\,\nu}\,(\mathrm{Re}\,\nu+1)} \\ \times\exp\left[\ln\left(\frac{\tau}{r}\right)\,\mathrm{Re}\,\nu\right]&\exp\left[B_\pm(\theta)\,\mathrm{Im}\,\nu\right]\,, \end{aligned} \tag{4.88}$$

where $A(\theta) = \sqrt{\frac{1}{2\sin\theta}}\left(\alpha^2(\theta)+\beta^2(\theta)\right)$, $B_\pm(\theta) = \pi \pm \theta$.
Deforming the contour Γ in the left ($\mathrm{Re}\,\nu < -1$) complex $\nu-$plane in such a way so that

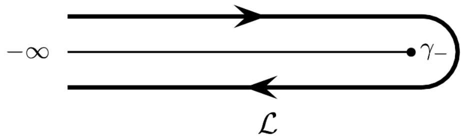

FIGURE 4.5: *The contour $\mathcal{L}$. The point γ_- can be chosen such that $\gamma_- < \mathrm{Re}\,\nu$. Note that the imaginary part of ν is held finite along the path.*

the imaginary part of ν is held finite, the expressions (4.85) and (4.86) remain absolutely convergent, and the interchange of the order of integration in (4.80) and (4.81) is permitted. Thus, we rewrite the second, third and fourth integral of the right-hand side of (4.80) and (4.81) as

$$\frac{1}{a}\int_0^a q(\tau,\zeta)\left[\int_{\mathcal{L}}\left(\frac{a}{r}\right)^{\nu+1}\left(\frac{a}{\tau}\right)^{\nu+1}\frac{\nu}{\nu+1}\,\mathrm{d}\nu\right]\mathrm{d}\tau\ , \tag{4.89}$$

$$\frac{1}{a}\int_0^a q(\tau,-1)\left[\int_{\mathcal{L}}\left(\frac{a}{r}\right)^{\nu+1}\left(\frac{a}{\tau}\right)^{\nu+1}\frac{\nu}{\nu+1}\cos\nu\pi\,P_\nu(\zeta)\,\mathrm{d}\nu\right]\mathrm{d}\tau\ , \tag{4.90}$$

$$\int_0^a q(\tau,-1)\left[\int_{\mathcal{L}}\left(\frac{\tau}{r}\right)^{\nu}\cos\pi\nu\,P_\nu(\zeta)\,\mathrm{d}\nu\right]\mathrm{d}\tau\ , \tag{4.91}$$

where for the Dirichlet data the term $\frac{\nu}{\nu+1}$ in the first two expressions must be omitted.

The deformed contour $\mathcal{L}$, depicted in Figure 4.5, begins and ends in the left complex $\nu-$plane, such that $\mathrm{Re}\,\nu\to-\infty$ at each end (a technique known as Talbot's method [Tal79]).
Thus, the integrals (4.89), (4.90) and (4.91) yield a zero contribution and equations (4.80), (4.81) take their final form

$$q^i_D(r,\zeta)=\frac{1}{2\pi i}\int_{\mathcal{L}}\left(\frac{a}{r}\right)^{\nu+1}(2\nu+1)\Big(\mathfrak{P}_\nu(\zeta)\,g_D(t)\Big)\,\mathrm{d}\nu,\quad \nu\neq-1,-2,\dots\ , \tag{4.92}$$

if Dirichlet boundary values are described, or

$$q^i_N(r,\zeta)=-\frac{a}{2\pi i}\int_{\mathcal{L}}\left(\frac{a}{r}\right)^{\nu+1}\frac{2\nu+1}{\nu+1}\Big(\mathfrak{P}_\nu(\zeta)\,g_N(t)\Big)\,\mathrm{d}\nu,\quad \nu\neq-1,-2,\dots\ , \tag{4.93}$$

if Neumann data are given.

The half-plane Re $\nu\in(-\infty,+1]$

Replace $\overline{R}(r;\nu)$ with $r^{-\nu+1}$ in order that the solution remains bounded in the neighborhood of $r=0$. Equation (4.72) then reads

$$\begin{aligned}\int_0^a q(r,\zeta)\left(\frac{r}{a}\right)^{-\nu-1}\mathrm{d}\left(\frac{r}{a}\right) &= \mathfrak{P}_\nu(\zeta)\Big(a\,g_N(t)+(\nu+1)\,g_D(t)\Big)\\ &+\frac{1}{a}\cos\nu\pi\,P_\nu(\zeta)\int_0^a q(\tau,-1)\left(\frac{\tau}{a}\right)^{-\nu-1}\mathrm{d}\tau,\ \mathrm{Re}\,\nu<-1,\ \nu\neq-2,-3,\dots\end{aligned} \tag{4.94}$$

Replacing in the above equation $-\nu - 1$ with ν and bearing in mind equations (4.78) and (4.38), we obtain the Mellin transform for the function $\frac{r}{a}\, q(r,\zeta)$

$$\int_0^a \left(\frac{r}{a}\, q(r,\zeta)\right) \left(\frac{r}{a}\right)^{\nu-1} \mathrm{d}\left(\frac{r}{a}\right) = \mathfrak{P}_\nu(\zeta)\left(a\, g_N(t) - \nu\, g_D(t)\right) + \frac{1}{a}\cos\nu\pi\, P_\nu(\zeta) \int_0^a q(\tau,-1) \left(\frac{\tau}{a}\right)^{\nu} \mathrm{d}\,\tau, \quad \mathrm{Re}\,\nu \geq 0\,. \tag{4.95}$$

Inverting (4.95) we find

$$q(r,\zeta) = \frac{1}{2\pi i} \int_{\mathrm{Re}\,\nu - i\infty}^{\mathrm{Re}\,\nu + i\infty} \left(\frac{a}{r}\right)^{\nu+1} \mathfrak{P}_\nu(\zeta)\Big(a\, g_N(t) - \nu\, g_D(t)\Big)\mathrm{d}\nu + \frac{1}{2\pi i} \int_{\mathrm{Re}\,\nu - i\infty}^{\mathrm{Re}\,\nu + i\infty} r^{-\nu-1} \cos\nu\pi\, P_\nu(\zeta) \left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu, \quad \nu \neq -1,-2,\dots, \tag{4.96}$$

which coincides with equation (4.74).

Remark 4.6.2 *In order to obtain solutions valid in the right complex ν-plane, consider* (4.94) *as the Mellin transform of the solution $q(r,\zeta)$. The inverse formula then implies*

$$q(r,\zeta) = \frac{1}{2\pi i} \int_{-Re\,\nu - i\infty}^{-Re\,\nu + i\infty} \left(\frac{r}{a}\right)^{\nu} \mathfrak{P}_\nu(\zeta)\Big(a\, g_N(t) + (\nu+1)\, g_D(t)\Big)\mathrm{d}\nu + \frac{1}{2\pi i} \int_{-Re\,\nu - i\infty}^{-Re\,\nu + i\infty} r^{\nu} \cos\nu\pi\, P_\nu(\zeta) \left[\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu, \quad \nu \neq -1,-2\dots, \tag{4.97}$$

where the integral is to be taken over any contour Γ' which joins two points $-Re\,\nu - iR$ and $-Re\,\nu + iR$ in the complex $\nu-$plane as $R \to \infty$.
Eliminating the unknown boundary data in (4.97), *with the aid of* (4.95), *we derive the following equations*

$$\begin{aligned} q(r,\zeta) = &\frac{1}{2\pi i} \int_{-Re\,\nu - i\infty}^{-Re\,\nu + i\infty} \left(\frac{r}{a}\right)^{\nu} (2\nu+1) \Big(\mathfrak{P}_\nu(\zeta)\, g_D(t)\Big)\, \mathrm{d}\nu \\ &+ \frac{1}{2\pi i} \int_{-Re\,\nu - i\infty}^{-Re\,\nu + i\infty} \left(\frac{r}{a}\right)^{\nu} a^{-\nu-1} \left[\int_0^a q(\tau,\zeta)\,\tau^{\nu}\,\mathrm{d}\tau\right] \mathrm{d}\nu \\ &+ \frac{1}{2\pi i} \int_{-Re\,\nu - i\infty}^{-Re\,\nu + i\infty} \left(\frac{r}{a}\right)^{\nu} a^{-\nu-1} \cos\nu\pi\, P_\nu(\zeta) \left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right] \mathrm{d}\nu \\ &+ \frac{1}{2\pi i} \int_{-Re\,\nu - i\infty}^{-Re\,\nu + i\infty} r^{\nu} \cos\nu\pi\, P_\nu(\zeta) \left[\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right] \mathrm{d}\nu, \quad \nu \neq -1,-2\dots, \end{aligned} \tag{4.98}$$

in the case where Dirichlet boundary values are prescribed, or

$$
\begin{aligned}
q(r,\zeta) =& \frac{a}{2\pi i}\int_{-Re\,\nu-i\infty}^{-Re\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{2\nu+1}{\nu}\left(\mathfrak{P}_\nu(\zeta)\,g_{N}(t)\right)\mathrm{d}\nu\\
&-\frac{1}{2\pi i}\int_{-Re\,\nu-i\infty}^{-Re\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,a^{-\nu-1}\left[\int_0^a q(\tau,\zeta)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu\\
&-\frac{1}{2\pi i}\int_{-Re\,\nu-i\infty}^{-Re\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,a^{-\nu-1}\,\cos\nu\pi\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu\\
&+\frac{1}{2\pi i}\int_{-Re\,\nu-i\infty}^{-Re\,\nu+i\infty}r^{\nu}\,\cos\nu\pi\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\quad \nu\neq-1,-2\ldots,
\end{aligned}
\tag{4.99}
$$

if Neumann data are prescribed.

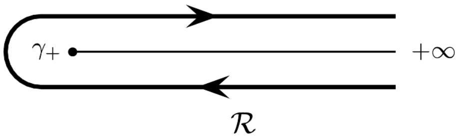

FIGURE 4.6: *The contour $\mathcal{R}$. The point γ_+ can be chosen such that $\gamma_+ > Re\,\nu$.*

The analysis as seen in section 4.6.1 implies that the contour Γ' can be replaced with the contour $\mathcal{R}$ depicted in Figure 4.6, and thus, the interchange of the order of integration is applied to rewrite the second, third and fourth integral of the right-hand side of the foregoing equations as

$$\frac{1}{a}\int_0^a q(\tau,\zeta)\left[\int_{\mathcal{R}}\left(\frac{r}{a}\right)^{\nu}\left(\frac{\tau}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,\mathrm{d}\nu\right]\mathrm{d}\tau \tag{4.100}$$

$$\frac{1}{a}\int_0^a q(\tau,-1)\left[\int_{\mathcal{R}}\left(\frac{r}{a}\right)^{\nu}\left(\frac{\tau}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,\cos\nu\pi\,P_\nu(\zeta)\,\mathrm{d}\nu\right]\mathrm{d}\tau \tag{4.101}$$

$$\int_0^a q(\tau,-1)\left[\int_{\mathcal{R}}\left(\frac{r}{\tau}\right)^{\nu}\cos\nu\pi\,P_\nu(\zeta)\,\mathrm{d}\nu\right]\frac{\mathrm{d}\tau}{\tau}\,, \tag{4.102}$$

respectively, where for the Dirichlet boundary values the term $\frac{\nu+1}{\nu}$ in the first two expressions has to be omitted.
The deformed contour $\mathcal{R}$ begins and ends in the right complex $\nu-$plane, such that $Re\,\nu\to+\infty$ at each end. Thus, the integrals (4.100),(4.101) *and* (4.102) *yield a zero contribution and equations* (4.98), (4.99) *become*

$$q_{D}^{i}(r,\zeta)=\frac{1}{2\pi i}\int_{\mathcal{R}}\left(\frac{r}{a}\right)^{\nu}(2\nu+1)\left(\mathfrak{P}_\nu(\zeta)\,g_{D}(t)\right)\mathrm{d}\nu,\quad \nu\neq-1,-2\ldots, \tag{4.103}$$

if Dirichlet boundary values are described, or

$$q^i_{N}(r,\zeta) = \frac{a}{2\pi i}\int_{\mathcal{R}}\left(\frac{r}{a}\right)^{\nu}\frac{2\nu+1}{\nu}\Big(\mathfrak{P}_\nu(\zeta)\,g_{N}(t)\Big)\,\mathrm{d}\nu, \quad \nu\neq -1,-2\ldots\,, \tag{4.104}$$

if Neumann data are given.

4.6.2 Part 2: Solutions valid in the exterior

Utilizing the global relation (4.24) in the subdomain

$$\Omega^e_3(b,\zeta) = \Big\{a\le r\le b\,,\zeta\le t<+1\Big\}, \tag{4.105}$$

depicted in Figure 4.7, with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\,P_\nu(\zeta)$ we obtain

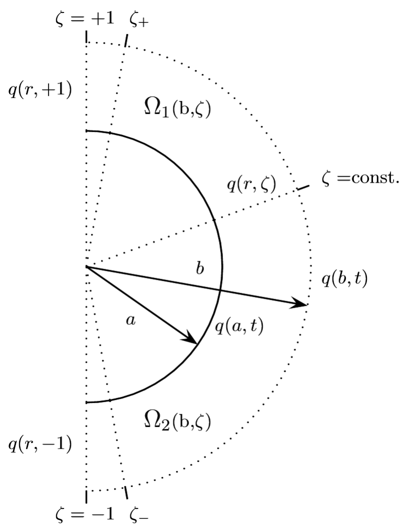

FIGURE 4.7: *The exterior subdomains* $\Omega^e_3(b,\zeta) = \{a\le r\le b\,,\zeta\le t<+1\}$ *and* $\Omega^e_4(b,\zeta) = \{a\le r\le b\,,-1<t\le\zeta\}$.

$$\begin{aligned}&\int_a^{+\infty}(1-\zeta^2)\left(P_\nu(\zeta)\,\frac{\partial q(r,\zeta)}{\partial\zeta}-\frac{dP_\nu(\zeta)}{\mathrm{d}\,\zeta}\,q(r,\zeta)\right)\overline{R}(r;\nu)\,\frac{\mathrm{d}r}{r^2}\\ &= -\int_\zeta^{+1}\left[\overline{R}(a;\nu)\,g_{N}(\zeta)+\left(\frac{2}{a}\,\overline{R}(a;\nu)-\frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right)g_{D}(\zeta)\right]P_\nu(t)\,\mathrm{d}\,t,\ (r,\zeta)\in\partial\Omega^e_3(b,\zeta)\,,\end{aligned} \tag{4.106}$$

where we used the regularity of the Legendre functions at $\zeta=+1$. Employing the global relation (4.24) in the subdomain

$$\Omega^e_4(b,\zeta) = \Big\{a\le r\le b\,,-1<t\le\zeta\Big\}, \tag{4.107}$$

depicted in Figure 4.7, with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\,Q_\nu(\zeta)$ we obtain

$$\begin{aligned}&\int_a^{+\infty}(1-\zeta^2)\left(Q_\nu(\zeta)\,\frac{\partial q(r,\zeta)}{\partial\zeta}-\frac{dQ_\nu(\zeta)}{\mathrm{d}\,\zeta}\,q(r,\zeta)\right)\overline{R}(r;\nu)\,\frac{\mathrm{d}r}{r^2}\\&=\int_{-1}^{\zeta}\left[\overline{R}(a;\nu)\,g_N(t)+\left(\frac{2}{a}\,\overline{R}(a;\nu)-\frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right)g_D(t)\right]Q_\nu(t)\,\mathrm{d}\,t\\&-\cos\nu\pi\int_a^{+\infty}q(r,-1)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}r}{r^2},\quad \nu\neq-1,-2,\dots,\quad (r,\zeta)\in\partial\Omega_4^e(b,\zeta)\,.\end{aligned}\tag{4.108}$$

In order to eliminate the unknown function $\frac{\partial q(r,\zeta)}{\partial\zeta}$ we subtract (4.106) multiplied by $Q_\nu(\zeta)$ and (4.108) multiplied by $P_\nu(\zeta)$. Then

$$\begin{aligned}\int_a^{+\infty}q(r,\zeta)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}r}{r^2}&=-\,\mathfrak{P}_\nu(\zeta)\left[\overline{R}(a;\nu)\,g_N(t)+\left(\frac{2}{a}\,\overline{R}(a;\nu)-\frac{d\overline{R}(a;\nu)}{\mathrm{d}r}\right)g_D(t)\right]\\&+\cos\nu\pi\,P_\nu(\zeta)\int_a^{+\infty}q(r,-1)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}r}{r^2},\quad \nu\neq-1,-2,\dots\,,\end{aligned}\tag{4.109}$$

where $\mathfrak{P}_\nu(\zeta)$ is the Legendre integral operator defined in (4.71) and the parameter ν lives in appropriate subdomains of $\mathbb{C}$, specified by the regularity of the radial factors of the solution of the formal adjoint at infinity.

The half-plane Re $\nu\in(-\infty,-2]$

In the half-plane $\mathrm{Re}\,\nu\in(-\infty,-2]$, $\overline{q}(r,\zeta;\nu)$ remains bounded as r tends to infinity, whenever $\overline{R}(r;\nu)$ is replaced by $r^{\nu+2}$. Thus, introducing $\overline{R}(r;\nu)=r^{\nu+2}$ in (4.109) we find

$$\begin{aligned}&\int_a^{+\infty}\left(\frac{r}{a}\,q(r,\zeta)\right)\left(\frac{r}{a}\right)^{\nu-1}\mathrm{d}\left(\frac{r}{a}\right)=-\,\mathfrak{P}_\nu(\zeta)\Big(a\,g_N(t)-\nu\,g_D(t)\Big)\\&+\frac{1}{a}\cos\nu\pi\,P_\nu(\zeta)\int_a^{+\infty}q(\tau,-1)\left(\frac{\tau}{a}\right)^{\nu}\mathrm{d}\tau,\quad \mathrm{Re}\,\nu\le0,\ \nu\neq-1,-2,\dots\,,\end{aligned}\tag{4.110}$$

which is recognized as the Mellin transform for the function $\frac{r}{a}\,q(r,\zeta)$. The restriction on ν is due the fact that r^ν must remain bounded as r tends to infinity. The Mellin inversion then implies

$$\begin{aligned}q(r,\zeta)=&-\frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\mathfrak{P}_\nu(\zeta)\Big(a\,g_N(t)-\nu\,g_D(t)\Big)\mathrm{d}\nu\\&+\frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}r^{-\nu-1}\cos\nu\pi\,P_\nu(\zeta)\left[\int_a^{+\infty}q(\tau,-1)\,\tau^\nu\,\mathrm{d}\tau\right]\mathrm{d}\nu,\ \nu\neq-1,-2\dots\,.\end{aligned}\tag{4.111}$$

Setting ν with $-\nu-1$ in (4.110) one finds, with the aid of (4.78) and (4.42)

$$\begin{aligned}-\int_a^{+\infty}q(r,\zeta)\left(\frac{a}{r}\right)^{\nu}\frac{\mathrm{d}r}{r}=&\,\mathfrak{P}_\nu(\zeta)\Big(a\,g_N(t)+(\nu+1)\,g_D(t)\Big)\\&-\cos\nu\pi\,P_\nu(\zeta)\int_a^{+\infty}q(\tau,-1)\left(\frac{a}{\tau}\right)^{\nu}\frac{\mathrm{d}\tau}{\tau},\quad \mathrm{Re}\,\nu>1\,.\end{aligned}\tag{4.112}$$

Remark 4.6.3 *Equations* (4.111)

$$a\left(\mathfrak{P}_\nu(\zeta)\, g_N(t)\right) - \nu\left(\mathfrak{P}_\nu(\zeta)\, g_D(t)\right)$$
$$= -\int_a^\infty \left[q(r,\zeta) - \cos\nu\pi\, P_\nu(\zeta)\, q(r,-1)\right]\left(\frac{r}{a}\right)^{\nu+1}\frac{\mathrm{d}\,r}{r}, \quad Re\,\nu \le 0, \quad \nu \neq -1,-2,\dots,$$

and (4.112)

$$a\left(\mathfrak{P}_\nu(\zeta)\, g_N(t)\right) + (\nu+1)\left(\mathfrak{P}_\nu(\zeta)\, g_D(t)\right)$$
$$= -\int_a^\infty \left[q(r,\zeta) - \cos\nu\pi\, P_\nu(\zeta)\, q(r,-1)\right]\left(\frac{a}{r}\right)^{\nu}\frac{\mathrm{d}\,r}{r}, \quad Re\,\nu > -1,$$

constitute the generalized Dirichlet-to-Neumann correspondence for the exterior of the sphere. Evaluating the above relations at $\zeta = +1$, we derive equations (4.41b) and (4.42b). On the other hand, in order to obtain the correspondences (4.41a) and (4.42a), one must replace $\overline{q}(r,\zeta;\nu)$ by $\overline{R}(r;\nu)\, Q_\nu(\zeta)$ in the subdomain $\Omega_3^i(\zeta)$ and by $\overline{R}(r;\nu)\, P_\nu(\zeta)$ in $\Omega_4^i(\zeta)$.

Eliminating the unknown boundary values with the help of (4.112) and substituting into (4.111) we derive the following equations,

$$\begin{aligned} q(r,\zeta) =& \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}(2\nu+1)\left(\mathfrak{P}_\nu(\zeta)\, g_D(t)\right)\mathrm{d}\nu \\ &+ \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1} a^\nu\left[\int_a^{+\infty} q(\tau,\zeta)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &- \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1} a^\nu\,\cos\pi\nu\, P_\nu(\zeta)\left[\int_a^{+\infty} q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &+ \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty} r^{-\nu-1}\,\cos\pi\nu\, P_\nu(\zeta)\left[\int_a^{+\infty} q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\ \nu \neq -1,-2\dots. \end{aligned} \tag{4.113}$$

if Dirichlet data are given, or

$$\begin{aligned} q(r,\zeta) =& -\frac{a}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{2\nu+1}{\nu+1}\left(\mathfrak{P}_\nu(\zeta)\, g_N(t)\right)\mathrm{d}\nu \\ &- \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{\nu}{\nu+1} a^\nu\left[\int_a^{+\infty} q(\tau,\zeta)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &+ \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{\nu}{\nu+1} a^\nu\,\cos\pi\nu\, P_\nu(\zeta)\left[\int_a^{+\infty} q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &+ \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty} r^{-\nu-1}\,\cos\pi\nu\, P_\nu(\zeta)\left[\int_a^{+\infty} q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\ \nu \neq -1,-2\dots. \end{aligned} \tag{4.114}$$

for Neumann boundary values.
Following the analysis presented in section 4.6.1, in order that the three last integrals on the right-hand side vanish, the contour Γ' must be replaced by the contour $\mathcal{R}$, depicted in Figure 4.6. Hence equations (4.113) and (4.114) rewrite

$$q^e_{D}(r,\zeta) = -\frac{1}{2\pi i}\int_{\mathcal{R}}\left(\frac{a}{r}\right)^{\nu+1}(2\nu+1)\Big(\mathfrak{P}_\nu(\zeta)\,g_{D}(t)\Big)\,\mathrm{d}\nu,\ \nu \neq -1,-2\dots, \tag{4.115}$$

for the Dirichlet, or

$$q^e_{N}(r,\zeta) = \frac{a}{2\pi i}\int_{\mathcal{R}}\left(\frac{a}{r}\right)^{\nu+1}\frac{2\nu+1}{\nu+1}\Big(\mathfrak{P}_\nu(\zeta)\,g_{N}(t)\Big)\,\mathrm{d}\nu,\ \nu \neq -1,-2\dots, \tag{4.116}$$

for the Neumann case.

The half-plane Re $\nu \in [1,+\infty)$

In the half-plane $\mathrm{Re}\,\nu \in [1,+\infty)$, $\overline{q}(r,\zeta;\nu)$ remains bounded as r tends to infinity, for every $\overline{R}(r;\nu)$ replaced by $r^{-\nu+1}$. Eq. (4.109) then becomes the Mellin transform of the solution $q(r,\zeta)$

$$\int_a^{+\infty} q(r,\zeta)\left(\frac{r}{a}\right)^{-\nu-1}\mathrm{d}\left(\frac{r}{a}\right) = -\,\mathfrak{P}_\nu(\zeta)\Big(a\,g_{N}(t) + (\nu+1)\,g_{D}(t)\Big) + \frac{1}{a}\cos\nu\pi\,P_\nu(\zeta)\int_a^{+\infty} q(\tau,-1)\left(\frac{\tau}{a}\right)^{-\nu-1}\mathrm{d}\tau,\ \mathrm{Re}\,\nu > -1\,. \tag{4.117}$$

The inversion formula then gives the representation

$$q(r,\zeta) = -\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}\mathfrak{P}_\nu(\zeta)\Big(a\,g_{N}(t)+(\nu+1)\,g_{D}(t)\Big)\mathrm{d}\nu + \frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty} r^{\nu}\cos\nu\pi\,P_\nu(\zeta)\left[\int_a^{+\infty} q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\ \nu\neq -1,-2\dots. \tag{4.118}$$

Replacing in (4.117) ν with $-\nu-1$, and using (4.78) and (4.41) we arrive at

$$-\int_a^{+\infty} q(r,\zeta)\left(\frac{r}{a}\right)^{\nu+1}\frac{\mathrm{d}r}{r} = \mathfrak{P}_\nu(\zeta)\Big(a\,g_{N}(t) - \nu\,g_{D}(t)\Big) - \cos\nu\pi\,P_\nu(\zeta)\int_a^{+\infty} q(\tau,-1)\left(\frac{\tau}{a}\right)^{\nu+1}\frac{\mathrm{d}\tau}{\tau},\quad \mathrm{Re}\,\nu < -2,\ \nu\neq -3,-4,\dots, \tag{4.119}$$

Equation (4.119) must be used in order to eliminate the unknown boundary values in (4.118), which then reads as

$$\begin{aligned} q(r,\zeta) = &-\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}(2\nu+1)\left(\mathfrak{P}_\nu(\zeta)\,g_D(t)\right)\mathrm{d}\nu \\ &+\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}a^{-\nu-1}\left[\int_a^{+\infty}q(\tau,\zeta)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &-\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}a^{-\nu-1}\cos\nu\pi\,P_\nu(\zeta)\left[\int_a^{+\infty}q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &+\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}r^{\nu}\cos\pi\nu\,P_\nu(\zeta)\left[\int_a^{+\infty}q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\ \nu\neq-1,-2\ldots, \end{aligned} \tag{4.120}$$

if Dirichlet data are prescribed, or

$$\begin{aligned} q(r,\zeta) = &-\frac{a}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{2\nu+1}{\nu}\left(\mathfrak{P}_\nu(\zeta)\,g_N(t)\right)\mathrm{d}\nu \\ &-\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,a^{-\nu-1}\left[\int_a^{+\infty}q(\tau,\zeta)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &+\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,a^{-\nu-1}\cos\nu\pi\,P_\nu(\zeta)\left[\int_a^{+\infty}q(\tau,-1)\,\tau^{\nu}\,\mathrm{d}\tau\right]\mathrm{d}\nu \\ &+\frac{1}{2\pi i}\int_{-\mathrm{Re}\,\nu-i\infty}^{-\mathrm{Re}\,\nu+i\infty}r^{\nu}\cos\nu\pi\,P_\nu(\zeta)\left[\int_a^{+\infty}q(\tau,-1)\,\tau^{-\nu-1}\,\mathrm{d}\tau\right]\mathrm{d}\nu,\ \nu\neq-1,-2\ldots, \end{aligned} \tag{4.121}$$

if Neumann data are prescribed.
Replacing the contour Γ with the contour $\mathcal{R}$ depicted in Figure 4.5, the aforementioned equations reduce to

$$q_D^e(r,\zeta) = -\frac{1}{2\pi i}\int_{\mathcal{L}}\left(\frac{r}{a}\right)^{\nu}(2\nu+1)\left(\mathfrak{P}_\nu(\zeta)\,g_D(t)\right)\mathrm{d}\nu, \quad \nu\neq-1,-2\ldots, \tag{4.122}$$

for the Dirichlet and

$$q_N^e(r,\zeta) = -\frac{a}{2\pi i}\int_{\mathcal{L}}\left(\frac{r}{a}\right)^{\nu}\frac{2\nu+1}{\nu}\left(\mathfrak{P}_\nu(\zeta)\,g_N(t)\right)\mathrm{d}\nu, \quad \nu\neq-1,-2\ldots, \tag{4.123}$$

for the Neumann problem.

4.7 Existence of the Integral transforms and the Inversion formulae

The aforementioned operations in chapters are justified introducing the set $L_1(0,\infty)$ for every real or complex-valued function $F(\mathbf{r})$ of the real variable r. Then [Mar83, GPS06],

Theorem 4.7.1 (Existence of the Mellin transform) *Consider* $F(\mathbf{r}) \in L_1(\epsilon, E)$, $0 < \epsilon < E < \infty$ *with the estimate*

$$|F(\mathbf{r})| \leq C \begin{cases} r^{-a} & r \in (0, \epsilon) \\ r^{-b} & r > E \end{cases} , \tag{4.124}$$

where C *is a constant. If* $a < b$*, then the Mellin transform* $F^*(\mathbf{r}; \nu) = \mathcal{M}\{F(\mathbf{r}); \nu\}$ *exist and the integral*

$$\int_0^\infty F(\mathbf{r})\, r^{\nu-1} \,\mathrm{d}\, r \,,$$

converges uniformly.

Theorem 4.7.2 (Inversion of the Mellin transform) *Let* $F(\mathbf{r}), r^{Re\,\nu-1} F(\mathbf{r}) \in C(0, \infty) \cap L_1(0, \infty)$, $a < Re\,\nu < b$. *Then the following inversion formula for the Mellin transformation*

$$F(\mathbf{r}) = \frac{1}{2\pi i} \lim_{Im\,\Gamma \to \infty} \int_\Gamma r^{-\nu} F^*(\mathbf{r}; \nu) \,\mathrm{d}\, \nu \,,$$

is valid for every $r \in (0, \infty)$.

Similar conclusions hold replacing the Mellin kernel $r^{\nu-1}$ with $r^{-\nu-1}$.

4.8 The “*mirrored*” Mellin transform

In what follows, a brief sketch of the proof based on [Sne72] is given.
In the definition of the Fourier transform and the the inversion formula

$$G(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t)\, e^{i\xi t} \,\mathrm{d}t \tag{4.125}$$

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G(\xi)\, e^{-i\xi t} \,\mathrm{d}\xi \,. \tag{4.126}$$

Replacing in the above relations x with e^t and s with $-c - i\xi$, it is straightforward to show that the following formulas hold

$$F(s) = \int_0^{+\infty} f(x)\, x^{-s-1} \,\mathrm{d}x \tag{4.127}$$

$$f(x) = \frac{1}{2i\pi} \int_{-c-i\infty}^{-c+i\infty} F(s)\, x^s \,\mathrm{d}s \,, \tag{4.128}$$

where $F(s) = G(is + ic)$ and $f(x) = \frac{1}{\sqrt{2\pi}}\, x^c\, g(\ln x)$.

CHAPTER 5

Irrotational Stokes' Flow in a Spherical Shell *

5.1 Stokes' Flow. A brief Introduction†

For a Newtonian viscous fluid in the absence of body forces, characterized by constant density ρ and viscosity μ, the Navier-Stokes equations, valid for incompressible flow, are given as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \mu\,\Delta\mathbf{u} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0\,, \tag{5.1}$$

which can be re-formulated in *dimensionless* form as

$$Re\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \Delta\mathbf{u} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0\,, \tag{5.2}$$

where Re is a dimensionless number introduced, in concept, by Sir G. G. Stokes in 1851 but named after Osborne Reynolds (1842-1912) [Rey83, Rot90].

A very interesting and important flow regime results from the assumption that the Reynolds number is very small compared to unity, $Re << 1$, but nonzero. Since $Re = \frac{\rho\, u\, L}{\mu}$, the low Reynolds number limit can be achieved by dealing with very large viscosities μ, or considering very small length-scales L of the flow, or by treating flows where the fluid velocities u are very slow, so-called *creeping flow* or *Stokes' flow*.

Following this scenario‡, the system (5.1) reduces to the linear system

$$\nabla p - \mu\Delta\mathbf{u} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0\,. \tag{5.3}$$

*This work has been published as [DDb]

†Based on section 4-7 of [HB86]

‡Details can be found in essentially any book dealing with fluid mechanics.

Stokes equations represent a considerable simplification of the Navier-Stokes equations, especially in the incompressible Newtonian case and carry some important properties such as

- INSTANTANEITY. The only dependency on time is through time-dependent boundary conditions.
- REVERSIBILITY. If $\mathbf{u}$ and p satisfy (5.3) then $-\mathbf{u}$ and $-p$ also satisfy the same equation.
- TIME REVERSIBILITY. In the sense that a time-reversed Stokes flow solves the same equations as the original Stokes flow and comes as an immediate consequence of instantaneity.

In most applications, the motion of a fluid represented by a streaming flow past a body of revolution, parallel to its symmetry axis, are very important. Such motions are called axisymmetric. Denoting with ϕ the azimuthal angle, the axisymmetrical flow is then one for which

I. The velocity $\mathbf{u}$ is independent of ϕ, viz

$$\frac{\partial \mathbf{u}}{\partial \phi} = 0\,. \tag{5.4}$$

II. The azimuthal component of $\mathbf{u}$ is anywhere zero, i.e.

$$\hat{\boldsymbol{\phi}} \cdot \mathbf{u} = 0\,. \tag{5.5}$$

If one is interested in the case where the fluid is considered *incompressible*, i.e. ρ =const. (see for details section 5.8 of [Tri88]), the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0\,,$$

yields

$$\nabla \cdot \mathbf{u} = 0\,,$$

which in orthogonal curvilinear coordinates (ξ_1, ξ_2, ξ_3) expands as

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi_1}(h_2 h_3 u_1) + \frac{\partial}{\partial \xi_2}(h_1 h_3 u_2) + \frac{\partial}{\partial \xi_3}(h_1 h_2 u_3)\right] = 0 \tag{5.6}$$

Let $\xi_3 = \phi$ and stipulate that the scale factors $h_i, i = 1, 2, 3$ are independent of the azimuthal angle ϕ. Then, bearing in mind (5.5), relation (5.6) simplifies as

$$\frac{\partial}{\partial \xi_1}(h_2 h_\phi u_1) + \frac{\partial}{\partial \xi_2}(h_1 h_\phi u_2) = 0 \tag{5.7}$$

and this can always be satisfied by introducing a *scalar* function $\Psi(\xi_1, \xi_2)$ such that

$$u_1 = -\frac{1}{h_2 h_\phi}\frac{\partial \Psi}{\partial \xi_2}\,, \quad u_2 = \frac{1}{h_1 h_\phi}\frac{\partial \Psi}{\partial \xi_1}\,. \tag{5.8}$$

The function $\Psi(\xi_1, \xi_2)$ is known as the stream function, relating to the fact that it remains constant along the streamlines. The stream function has been defined in such a way as to vanish everywhere on the axis of revolution. A stream function exists in *all cases* of incompressible fluid motion in two dimensions and also in the three-dimensional case only when the latter are axisymmetric (for more details see [HB86, pp.102-103]).

The Navier-Stokes equations (5.1) for the case of an incompressible fluid, encountered at the beginning of this section, rewrite, introducing the vector identities

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\,\nabla\mathbf{u}^2 - \mathbf{u}\times(\nabla\times\mathbf{u})$$
$$\Delta\mathbf{u} = \nabla\,(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})\ ,$$

as

$$\frac{\partial\mathbf{u}}{\partial t} + \frac{1}{2}\,\nabla\mathbf{u}^2 - \mathbf{u}\times(\nabla\times\mathbf{u}) + \frac{1}{\rho}\,\nabla p + \eta\,\nabla\times(\nabla\times\mathbf{u}) = \mathbf{0}\,, \tag{5.9}$$

where $\eta = \frac{\mu}{\rho}$ is the kinematic viscosity. The term $\nabla\cdot\mathbf{u}$ vanishes on account of the assumption of incompressibility. Eliminating the pressure by taking the curl on both sides yields

$$\frac{\partial\boldsymbol{\omega}}{\partial t} - \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) + \eta\,\nabla\times(\nabla\times\boldsymbol{\omega}) = \mathbf{0}\,, \tag{5.10}$$

where the vorticity $\boldsymbol{\omega}$ is defined as follows

$$\boldsymbol{\omega} = \nabla\times\mathbf{u}\,.$$

Employing the relation

$$\nabla\times\mathbf{u} = \frac{1}{2}\,\frac{\hat{\boldsymbol{\xi}}_i\times\hat{\boldsymbol{\xi}}_j}{h_i\,h_j}\left(\frac{\partial}{\partial\xi_i}\,(h_j\,u_j) - \frac{\partial}{\partial\xi_j}\,(h_i\,u_i)\right), \quad \text{(Einstein summation assumed)}$$

and with the aid of (5.4), (5.5) and (5.8), we find

$$\boldsymbol{\omega} = \frac{\hat{\boldsymbol{\phi}}}{h_\phi}\,E^2\,\Psi \tag{5.11}$$

where the differential operator E^2 is defined as

$$E^2 \equiv \frac{h_\phi}{h_1 h_2}\left\{\frac{\partial}{\partial\xi_1}\left(\frac{h_2}{h_1 h_\phi}\,\frac{\partial}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_1}{h_2 h_\phi}\,\frac{\partial}{\partial\xi_2}\right)\right\}. \tag{5.12}$$

Moreover, repeated application of the operator $\nabla\times$ to (5.11) reveals that

$$\nabla\times\boldsymbol{\omega} = \left(\frac{\hat{\boldsymbol{\xi}}_1}{h_2 h_\phi}\,\frac{\partial}{\partial\xi_2} - \frac{\hat{\boldsymbol{\xi}}_2}{h_1 h_\phi}\,\frac{\partial}{\partial\xi_1}\right)E^2\,\Psi\,, \tag{5.13}$$

and

$$\nabla\times(\nabla\times\boldsymbol{\omega}) = -\frac{\hat{\boldsymbol{\phi}}}{h_\phi}\,E^4\,\Psi\,, \tag{5.14}$$

where the operator E^4 comes from a successive application of E^2 on itself. Furthermore, it is possible to express the velocity $\mathbf{u}$ in terms of the stream function as [HB86, p.99]

$$\mathbf{u} = \frac{\hat{\boldsymbol{\phi}}}{h_\phi} \times \nabla\Psi \,,$$

and therefore, using the vector identity

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}\,(\mathbf{C}\cdot\mathbf{A}) - \mathbf{A}\,(\mathbf{C}\cdot\mathbf{B}) \,,$$

it can be shown that

$$\mathbf{u} \times \boldsymbol{\omega} = \frac{E^2\,\Psi}{h_\phi^2}\,\nabla\Psi \,. \tag{5.15}$$

Replacing equations (5.11), (5.14) and (5.15) into (5.10), we obtain the nonlinear differential equation satisfied by the stream function Ψ

$$E^4\,\Psi + \hat{\boldsymbol{\phi}}\cdot\frac{h_\phi}{\eta}\,\nabla\left(\frac{E^2\,\Psi}{h_\phi^2}\right)\times\nabla\Psi - \frac{1}{\eta}\frac{\partial}{\partial t}\,(E^2\,\Psi) = 0\,, \tag{5.16}$$

or

$$E^4\,\Psi + \frac{1}{\eta}\,\frac{1}{h_1\,h_2\,h_\phi}\left(\frac{\partial\Psi}{\partial\xi_2}\,\frac{\partial}{\partial\xi_1} - \frac{\partial\Psi}{\partial\xi_1}\,\frac{\partial}{\partial\xi_2}\right)(E^2\,\Psi) - \frac{1}{\eta}\frac{\partial}{\partial t}\,(E^2\,\Psi) = 0\,. \tag{5.17}$$

where the middle term

$$\nabla \times \left(\frac{E^2\,\Psi}{h_\phi^2}\,\nabla\Psi\right)$$

has been altered using the vector identity

$$\nabla \times (f\,\mathbf{F}) = \nabla f \times \mathbf{F} + f\,\nabla \times \mathbf{F}$$

for (5.16), or through the equality

$$\nabla \times \mathbf{F} = \frac{1}{h_1\,h_2\,h_\phi}\begin{vmatrix} h_1\,\hat{\boldsymbol{\xi}}_\mathbf{1} & h_2\,\hat{\boldsymbol{\xi}}_\mathbf{3} & h_\phi\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\phi} \\ h_1\,F_1 & h_2\,F_2 & h_3\,F_3 \end{vmatrix}$$

for (5.17).

In the low-Reynolds number limit, equation (5.16) (or (5.17)) simplify as

$$E^4\,\Psi - \frac{1}{\eta}\frac{\partial}{\partial t}\,(E^2\,\Psi) = 0\,, \tag{5.18}$$

that is, the nonlinear central term is omitted and hence, (5.18) constitutes the equation of motion for creeping flows. If, furthermore, the motion of the fluid is assumed to be steady, equation (5.18) becomes

$$E^4\,\Psi = 0$$

as the equation of motion.
Of particular interest is the case of irrotational motion of the fluid, namely the case where $\boldsymbol{\omega} = \mathbf{0}$, then, the equation of motion derived from (5.11) is

$$E^2\,\Psi = 0\,.$$

The present chapter is organized as follows. In section 5.2 a brief review of classical representations, namely solutions in form of series expansions are given, followed by the formulation of the problem in section 5.3. In section 5.4 the general global relation is derived, on which section 5.5 is based, in order to establish the Dirichlet-to-Neumann correspondence together with a Lax pair formulation. Section 5.6 is devoted to the steps that one has to follow in order to recover the classical solutions from the global relation. Moreover, alternative formulae for the solutions in terms of integrals instead of a series can be derived. The novel integral representations on which the present work is focused on, is developed in section 5.7.

5.2 The Classical Representation

The stream function Ψ for irrotational axisymmetric Stokes' flow in spherical coordinates satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right)\Psi(r,\theta) = 0\,.$$

Introducing the variable

$$\zeta = \cos\theta, \quad \theta\in(0,\pi)\,,$$

the latter equation takes the form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial\zeta^2}\right)\Psi(r,\zeta) = 0\,. \tag{5.19}$$

Since the separability of the irrotational Stokes' operator E^2 is closely related to the separability of the Laplacian operator, putting $\Psi(r,\zeta) = R(r)\,Z(\zeta)$ we obtain the two ordinary differential equations

$$r^2\frac{d^2R(r)}{dr^2} - \alpha R(r) = 0\,, \tag{5.20}$$

$$(1-\zeta^2)\frac{d^2Z(\zeta)}{\mathrm{d}\,\zeta^2} + \alpha Z(\zeta) = 0\,, \tag{5.21}$$

where α a complex parameter introduced during the process of separation of variables. The latter of the ODEs has three regular singular points at $\zeta = \pm1$ and ∞. Thus by replacing ζ with $1-2t$, eq. (5.21) reduces to hypergeometric form

$$t(1-t)\frac{d^2Z(t)}{dt^2} + \alpha Z(t) = 0\,,$$

with parameters

$$\begin{aligned} a &= b = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\alpha} \\ c &= 0 \end{aligned}$$

Choose b to be the larger of the two and hence the first solution is [WG89, p. 149],

$$Z_1(t) = t\,F\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\alpha}, \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha}, 2; t\right) ,$$

or, for the original equation,

$$Z_1(\zeta) = \frac{1-\zeta}{2}\,F\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\alpha}, \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha}, 2; \frac{1-\zeta}{2}\right) .$$

Following [WG89, p.149], the second solution is given by the expression

$$\begin{aligned} Z_2(z) =& z\,\ln z\,F\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\alpha}, \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha}, 2; z\right) \\ &+ \sum_{n=0}^{\infty} \frac{(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\alpha})_n\,(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha})_n}{n!\,(2)_n}\, z^{n+1} \Big[\psi\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\alpha} + n\right) \\ &+ \psi\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha} + n\right) - \psi(n+2) - \psi(n+1) - \psi\left(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\alpha}\right) \\ &- \psi\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha}\right) - 2\gamma\Big] , \end{aligned}$$

where $z = \zeta \pm i0$, is to be interpreted as the limiting value of the complex value z approaching the real axis from above $(+)$ or from below $(-)$, respectively, $\psi(z)$ is the logarithmic derivative of the $\Gamma-$function (or digamma function)

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) ,$$

and $\gamma = -\psi(1) = 0.57721566$ is Euler's constant.
Since above relation is cumbersome to work with, we note that since $1 + a + b - c = 0, -1, -2, \ldots$ a second solution is [Erd53, p. 75, eq.(8)]

$$Z(z) = (1-z)\,F\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\alpha}, \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\alpha}, 2; 1-z\right) .$$

In order to simplify calculations, note that (5.21) is a particular case of the Gegenbauer differential equation

$$(1-z^2)\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} - (2\lambda+1)z\,\frac{\mathrm{d}w(z)}{\mathrm{d}z} + \nu(\nu+2\lambda)\,w(z) = 0\,, \tag{5.22}$$

if $\lambda = -\frac{1}{2}$ and $\alpha = \nu(\nu-1)$, so that two independent solutions are,

$$\begin{aligned} Z_1(\zeta) &= \frac{1-\zeta}{2}\,F\left(-\nu+1, \nu, 2; \frac{1-\zeta}{2}\right), \quad \left|\frac{1-\zeta}{2}\right| < 1\,, \\ Z_2(z) &= (1-z)\,F\left(-\nu+1, \nu, 2; 1-z\right), \quad |1-z| < 1\,. \end{aligned}$$

Following current literature [HB86, KK91] we adopt the definition for the Gegenbauer functions of order $\lambda = -\frac{1}{2}$ from [Sam91]. There are obtained by putting $Z(\zeta) = \zeta^{\nu}\, u(\zeta)$, where ν is a complex parameter, and $\zeta^2 = \frac{1}{t}$ into (5.21), which then rewrites

$$t(1-t)\frac{\mathrm{d}^2 u(t)}{\mathrm{d}\,t^2} + \left[\left(\tfrac{3}{2}-\nu\right) - \left(\tfrac{3}{2}-\nu\right)\, t\right]\frac{\mathrm{d}u(t)}{\mathrm{d}t} - \left[\frac{\alpha - \nu(\nu-1)}{4t} + \frac{\nu(\nu-1)}{4}\right] u(t) = 0\,,$$

and is of hypergeometric form only if

$$\alpha = \nu(\nu-1)\,.$$

After some manipulations we obtain

$$Z(\zeta) = \zeta^{\nu}\, F\left(-\frac{\nu}{2}, -\frac{\nu-1}{2}, -\nu+\frac{3}{2}; \zeta^{-2}\right), \quad |\zeta^2| > 1\,.$$

Denote with $C_\nu^{-\frac{1}{2}}(\zeta)$ the Gegenbauer functions of the first kind and order $\lambda = -\frac{1}{2}$

$$C_\nu^{-\frac{1}{2}}(\zeta) = A(\nu)\, \zeta^{\nu}\, F\left(-\frac{\nu}{2}, -\frac{\nu-1}{2}, -\nu+\frac{3}{2}; \zeta^{-2}\right), \quad |\zeta^2| > 1\,,$$

where $A(\nu)$ is an arbitrary function of the complex parameter ν.
Sampson [Sam91] showed that a second solution of (5.21) can be derived by simple changing ν into $-\nu+1$ and multiplying by a constant factor. Thus, denoting $D_\nu^{-\frac{1}{2}}(\zeta)$ as the Gegenbauer functions of the second kind and order $-\frac{1}{2}$, we find

$$D_\nu^{-\frac{1}{2}}(\zeta) = B(\nu)\, \zeta^{-\nu+1}\, F\left(\frac{\nu-1}{2}, \frac{\nu}{2}, \nu+\frac{1}{2}; \zeta^{-2}\right)\,,$$

where again, $B(\nu)$ is an arbitrary function of ν.
In order to be compatible with [Hei78, Hei81, Sam91] set the functions $A(\nu)$ and $B(\nu)$ equal to

$$A(\nu) = 2^{\nu-1}\frac{\Gamma(\nu-\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu+1)}\,,$$

and

$$B(\nu) = -\frac{\sqrt{\pi}\,\Gamma(\nu-1)}{2^{\nu}\,\Gamma(\nu+\frac{1}{2})}\,.$$

The corresponding expressions for the Gegenbauer functions of the first and second kind of order $\lambda = -\frac{1}{2}$, valid in the interval $|\zeta| < 1$, are given by [Sam91, pp. 455-456 and p. 473] in the case where ν is an positive integer, including zero, as follows.

$$\left.\begin{aligned} C_n^{-\frac{1}{2}}(\zeta) &= (-1)^{\frac{n}{2}}\,\frac{\Gamma(\frac{n-1}{2})}{2\sqrt{\pi}\,\Gamma(\frac{n+2}{2})}\,F\left(-\tfrac{n}{2}, \tfrac{n}{2}-\tfrac{1}{2}, \tfrac{1}{2}; \zeta^2\right) \\ &= (-1)^{\frac{n}{2}}\,\frac{\Gamma(\frac{n-1}{2})}{2\sqrt{\pi}\,\Gamma(\frac{n+2}{2})}\,(1-\zeta^2)\,F\left(\tfrac{n}{2}+\tfrac{1}{2}, -\tfrac{n}{2}+1, \tfrac{1}{2}; \zeta^2\right) \end{aligned}\right\}, n \text{ is even}\,,$$

$$\left.\begin{aligned} &= (-1)^{\frac{n-1}{2}}\,\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}\,\zeta\, F\left(-\tfrac{n}{2}+\tfrac{1}{2}, \tfrac{n}{2}, \tfrac{3}{2}; \zeta^2\right) \\ &= (-1)^{\frac{n-1}{2}}\,\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}\,\zeta\,(1-\zeta^2)\,F\left(\tfrac{n}{2}+1, -\tfrac{n}{2}+\tfrac{3}{2}, \tfrac{3}{2}; \zeta^2\right) \end{aligned}\right\}, n \text{ is odd}\,,$$

$$\begin{aligned} D_n^{-\frac{1}{2}}(\zeta) &= (-1)^{\frac{n}{2}} \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2})}{2\Gamma(\frac{n+1}{2})}\,\zeta\, F\left(-\tfrac{n}{2}+\tfrac{1}{2}, \tfrac{n}{2}, \tfrac{3}{2}; \zeta^2\right), \quad n \text{ is even} \\ &= (-1)^{\frac{n+1}{2}} \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2})}{2\Gamma(\frac{n+1}{2})}\, F\left(-\tfrac{n}{2}, \tfrac{n}{2}-\tfrac{1}{2}, \tfrac{1}{2}; \zeta^2\right), \quad n \text{ is odd}. \end{aligned}$$

Equation (5.20) with $\alpha = \nu(\nu-1)$, $\nu \in \mathbb{C}$ accepts functions of the form

$$R_1(r;\nu) = r^\nu, \qquad R_2(r;\nu) = r^{-\nu+1}$$

as solutions. The general solution for the stream function is given by the real part of $\Psi(r,\zeta;\nu)$, i.e.

$$\Psi(r,\zeta) = \text{Re}\,\Psi(r,\zeta;\nu),$$

where

$$\Psi(r,\zeta;\nu) = \sum_{i=1}^{4} \mathcal{A}^{(i)}(\nu)\,\Theta^{(i)}(r,\zeta;\nu),$$

and

$$\left.\begin{aligned} \Theta^{(1)}(r,\zeta;\nu) &= r^\nu\, C_\nu^{-\frac{1}{2}}(\zeta) \\ \Theta^{(2)}(r,\zeta;\nu) &= r^\nu\, D_\nu^{-\frac{1}{2}}(\zeta) \\ \Theta^{(3)}(r,\zeta;\nu) &= r^{-\nu+1}\, C_\nu^{-\frac{1}{2}}(\zeta) \\ \Theta^{(4)}(r,\zeta;\nu) &= r^{-\nu+1}\, D_\nu^{-\frac{1}{2}}(\zeta) \end{aligned}\right\}, \quad \nu \in \mathbb{C}, \tag{5.23}$$

and $\mathcal{A}^{(i)}(\nu)$ arbitrary functions.
However, the procedure can be significant simplified by letting $\nu = n \in \mathbb{Z}^+$. The general solution is then given as

$$\Psi(r,\zeta) = \sum_{n\in\mathbb{Z}^+} \sum_{i=1}^{4} A_n^{(i)} \Theta_n^{(i)}, \tag{5.24}$$

where now the irrotational eigensolutions (5.23) are

$$\left.\begin{aligned} \Theta_n^{(1)} &= r^n\, C_n^{-\frac{1}{2}}(\zeta) \\ \Theta_n^{(2)} &= r^n\, D_n^{-\frac{1}{2}}(\zeta) \\ \Theta_n^{(3)} &= r^{-n+1}\, C_n^{-\frac{1}{2}}(\zeta) \\ \Theta_n^{(4)} &= r^{-n+1}\, D_n^{-\frac{1}{2}}(\zeta) \end{aligned}\right\}, \quad \nu \in \mathbb{C}. \tag{5.25}$$

The eigenfunctions of the second kind with respect to the variable ζ are defined as [Sam91, p. 470]

$$D_n^{-\frac{1}{2}}(\zeta) = C_n^{-\frac{1}{2}}(\zeta)\, Q_0(\zeta) - K_{n-1}(\zeta) \tag{5.26}$$

where

$$Q_0(\zeta) = \begin{cases} \frac{1}{2} \ln \frac{1+\zeta}{1-\zeta}, & |\zeta| < 1, \\ \frac{1}{2} \ln \frac{\zeta+1}{\zeta-1}, & |\zeta| > 1, \end{cases} \tag{5.27}$$

is the Legendre function of the second kind and $K_{n-1}(\zeta)$ a polynomial of degree $n-1$ defined as

$$K_{n-1}(\zeta) = \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{2(2n-4m+1)}{(2m-1)(n-m)} \left[1 - \frac{(2m-1)(n-m)}{n(n-1)}\right] C_{n-2m+1}^{-\frac{1}{2}}(\zeta), \tag{5.28}$$

where $\lfloor \frac{n-1}{2} \rfloor$ is the greatest integer which is less then or equals to $\frac{n-1}{2}$.
The Gegenbauer functions of order $-\frac{1}{2}$ are in general related with the Legendre functions of the first and second kind, viz

$$\begin{array}{ll}
\frac{dC_n^{-\frac{1}{2}}(\zeta)}{\mathrm{d}\,\zeta} = P_{n-1}(\zeta) & \frac{dD_n^{-\frac{1}{2}}(\zeta)}{\mathrm{d}\,\zeta} = Q_{n-1}(\zeta) \\
C_n^{-\frac{1}{2}}(\zeta) = \frac{P_n(\zeta)-P_{n-2}(\zeta)}{2n-1} & D_n^{-\frac{1}{2}}(\zeta) = \frac{Q_n(\zeta)-Q_{n-2}(\zeta)}{2n-1} \\
C_n^{-\frac{1}{2}}(\zeta) = -P_n(\zeta) + 2\zeta\, P_{n-1}(\zeta) - P_{n-2}(\zeta) & D_n^{-\frac{1}{2}}(\zeta) = -Q_n(\zeta) + 2\zeta\, Q_{n-1}(\zeta) - Q_{n-2}(\zeta)
\end{array}$$

For $n = 0, 1$ the Gegenbauer functions $C_n^{-\frac{1}{2}}(\zeta)$ and $D_n^{-\frac{1}{2}}(\zeta)$ are defined as

$$\begin{aligned}
C_0^{-\frac{1}{2}}(\zeta) &= -D_1^{-\frac{1}{2}}(\zeta) = -1 \\
C_1^{-\frac{1}{2}}(\zeta) &= D_0^{-\frac{1}{2}}(\zeta) = \zeta\,,
\end{aligned}$$

where else the first few Gegenbauer polynomials $C_n^{-\frac{1}{2}}(\zeta)$ for $n \geq 2$ are given below

$$\left.\begin{array}{rcc}
C_2^{-\frac{1}{2}}(\zeta) & = & \frac{1}{2}(\zeta^2-1) \\
C_3^{-\frac{1}{2}}(\zeta) & = & \frac{1}{2}\zeta\,(\zeta^2-1) \\
C_4^{-\frac{1}{2}}(\zeta) & = & \frac{1}{8}(\zeta^2-1)(5\zeta^2-1) \\
C_5^{-\frac{1}{2}}(\zeta) & = & \frac{1}{8}\zeta\,(\zeta^2-1)(7\zeta^2-3) \\
C_6^{-\frac{1}{2}}(\zeta) & = & \frac{1}{16}(\zeta^2-1)(21\zeta^4-14\zeta^2+1) \\
C_7^{-\frac{1}{2}}(\zeta) & = & \frac{1}{16}\zeta\,(\zeta^2-1)(33\zeta^4-30\zeta^2+5) \\
C_8^{-\frac{1}{2}}(\zeta) & = & \frac{1}{128}(\zeta^2-1)(429\zeta^6-495\zeta^4+135\zeta^2-5)
\end{array}\right\} . \tag{5.29}$$

Note, that since in most applications the variable ζ lives on the interval $(-1, +1)$, standard references [HB86, KK91] define the Gegenbauer functions $C_n^{-\frac{1}{2}}(\zeta)$ and $D_n^{-\frac{1}{2}}(\zeta)$ as

$$C_n^{-\frac{1}{2}}(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1} \tag{5.30}$$

$$D_n^{-\frac{1}{2}}(\zeta) = \frac{Q_{n-2}(\zeta) - Q_n(\zeta)}{2n-1}\,, \tag{5.31}$$

and the factors $\zeta^2 - 1$ in equations (5.29) must be replaced with $1 - \zeta^2$.
Employing the theory of hypergeometric functions, it can be shown that $D_\nu^{-\frac{1}{2}}(\zeta)$, and subsequently $D_n^{-\frac{1}{2}}(\zeta)$, is unbounded at both poles $\zeta = \pm 1$, and therefore the second solution is for the time being, disregarded. Hence, the irrotational eigensolutions (5.24) simplify as

$$\Psi(r,\zeta) = \sum_{n\in\mathbb{Z}^+} \left(a_n\, r^n + b_n\, r^{-n+1}\right) C_n^{-\frac{1}{2}}(\zeta)\,. \tag{5.32}$$

It can be shown [Das07a] that every Gegenbauer polynomial of the first kind and order $\lambda = -\frac{1}{2}$ enjoys the structure

$$C_n^{-\frac{1}{2}}(\zeta) = (1-\zeta^2)\, M_{n-2}(\zeta), \quad n \geq 2\,, \tag{5.33}$$

where $M_{n-2}(\zeta)$ is a polynomial of degree $(n-2)$. This implies that

$$C_n^{-\frac{1}{2}}(\pm 1) = 0, \quad n \geq 2$$

and secures that the stream function vanishes along the axis of revolution.
Therefore, (5.32) reads

$$\Psi(r,\zeta) = \sum_{n=2}^{+\infty} \Big(a_n\, r^n + b_n\, r^{-n+1} \Big)\, C_n^{-\frac{1}{2}}(\zeta)\,. \tag{5.34}$$

Once the stream function $\Psi(r,\zeta)$ is obtained the axisymmetric velocity field

$$\mathbf{u}(r,\zeta) = u_r(r,\zeta)\widehat{\mathbf{r}} + u_\zeta(r,\zeta)\widehat{\boldsymbol{\zeta}}$$

is given by

$$u_r(r,\zeta) = -\frac{1}{r^2}\,\frac{\partial\Psi(r,\zeta)}{\partial\zeta}\,,$$
$$u_\zeta(r,\zeta) = \frac{1}{r\sqrt{1-\zeta^2}}\,\frac{\partial\Psi(r,\zeta)}{\partial r}\,.$$

5.2.1 Alternative Solutions

The expressions of the solutions as far as it concerns the $\zeta-$direction, motivates to consider expressions for the stream function of the form

$$\Psi(r,\zeta) = \sum_{n,m} \Big(a_{n,m}\, r^n + b_{n,m}\, r^{-n+1} \Big)\, (1-\zeta^2)^m\, Z(\zeta)\,, \tag{5.35}$$

where $a_{n,m}$ and $b_{n,m}$ are arbitrary constants.
Replacing (5.35) into (5.19) it is straightforward to shown that $\Psi(r,\zeta)$ satisfies (5.19) only if $Z(\zeta)$ satisfies the differential equation

$$(1-\zeta^2)\, Z''(\zeta) - 4m\,\zeta\, Z'(\zeta) + \left[\Big(n(n-1) - 2m \Big) + 4m(m-1)\,\frac{\zeta^2}{1-\zeta^2} \right] Z(\zeta) = 0\,, \tag{5.36}$$

where the prime denotes differentiation with respect to the argument.
Putting $Z(\zeta) = (1-\zeta^2)^\mu\, u(\zeta)$, then $u(\zeta)$ satisfies the equation*

$$(1-\zeta^2)\, u'' - 4(\mu+m)\,\zeta\, u' + \Big[n(n-1) - 2(\mu+m) \Big] u = 0\,, \tag{5.37}$$

where μ assumes the values $1-m$ or $-m$.
If $\mu = -m$, equation (5.37) is recognized as the Gegenbauer equation of order $\lambda = -\frac{1}{2}$ and

*Letting $m = \frac{k}{2}\,(k+1)$ equation (5.36) becomes

$$(1-\zeta^2)\, Z''(\zeta) - 2k(k+1)\,\zeta\, Z'(\zeta) + \left[(n+k)(n-k-1) + (k-1)k(k+1)(k+2)\,\frac{\zeta^2}{1-\zeta^2} \right] Z(\zeta) = 0\,,$$

which remains invariant under the transformations $k \to -k-1$ and $n \to -n+k+1$.

degree n. On the other hand, replacing μ by $1-m$, equation (5.37) is recognized as the Gegenbauer equation of order $\lambda = \frac{3}{2}$ and degree $n-2$. Hence the solutions to (5.36) are

$$Z_n(\zeta) = (1-\zeta^2)^{-m}\Big(a_n\, C_n^{-\frac{1}{2}}(\zeta) + b_n\, D_n^{-\frac{1}{2}}(\zeta)\Big)\,,$$

and

$$Z_n(\zeta) = (1-\zeta^2)^{-m}\Big(c_n\,(1-\zeta^2)\, C_{n-2}^{\frac{3}{2}}(\zeta) + d_n\,(1-\zeta^2)\, D_{n-2}^{\frac{3}{2}}(\zeta)\Big)\,,$$

where a_n, b_n, c_n and d_n are arbitrary constants.
From the above relations it is easily deduced that

$$\left.\begin{aligned} C_n^{-\frac{1}{2}}(\zeta) &= \alpha_n\,(1-\zeta^2)\, C_{n-2}^{\frac{3}{2}}(\zeta) \\ D_n^{-\frac{1}{2}}(\zeta) &= \beta_n\,(1-\zeta^2)\, D_{n-2}^{\frac{3}{2}}(\zeta) \end{aligned}\right\}, \quad n \geq 2\,. \tag{5.38}$$

Taking a step ahead, the constants α_n are easily evaluated with the aid of the orthogonality relations (5.44) and (5.86) and equals $\alpha_n = \frac{1}{n\,(n-1)}$. Hence

$$C_n^{-\frac{1}{2}}(\zeta) = \frac{1}{n\,(n-1)}\,(1-\zeta^2)\, C_{n-2}^{\frac{3}{2}}(\zeta), \quad n \geq 2\,. \tag{5.39}$$

The constant β_n on the other hand, can be evaluated employing equations (5.26)-(5.28) and a formula connecting $D_n^{\lambda}(z)$ and $C_n^{\lambda}(z)$ similar to Christoffel's relation [Chr58] between $Q_n(z)$ and $P_n(z)$ given by Watson [Wat38].

Remark 5.2.1 *Observe that at the axis of revolution* ($\zeta = \pm 1$) *in the case where* $C_n^{-\frac{1}{2}}(\zeta)$ *is replaced by* (5.39)*, the stream function vanishes due do the term* $(1-\zeta^2)$ *without imposing any restrictions on the Gegenbauer polynomials of order* $\frac{3}{2}$*. Moreover, since the properties of the Gegenbauer polynomials of order greater then* $-\frac{1}{2}$*, are well known [Erd53, Hei78, Hei81, GR00, AS65], the use of* (5.39) *is suggested.*

5.3 Formulation of the problem

Consider a spherical shell $\mathcal{S}$ centered at the origin with inner radius r_1 and outer radius r_2 as depicted in Figure 5.1. The motion of the fluid is restraint in the interior domain Ω^i defined as

$$\Omega^i = \Big\{(r,\zeta)\,\Big|\, r_1 \leq r \leq r_2,\ -1 < \zeta < +1\Big\}.$$

Scope of the article is to obtain expressions for the stream function $\Psi_D(r,\zeta)$ and $\Psi_N(r,\zeta)$ valid for Dirichlet and Neumann problem, respectively, in the interior of the spherical shell $\mathcal{S}$. Moreover, by a limiting procedure, the corresponding streamfunctions $\Psi_D^i(r,\zeta)$, $\Psi_D^e(r,\zeta)$, $\Psi_N^i(r,\zeta)$, and $\Psi_N^e(r,\zeta)$, which solve the interior Dirichlet, exterior Dirichlet, the interior Neumann and the exterior Neumann problems, respectively, are found. The Dirichlet boundary values are denoted by $g_D^{(j)}$, $j=1,2$ where else we denote the Neumann data on the boundaries by $g_N^{(j)}$, $j=1,2$. In order to secure the uniqueness of the exterior problem a asymptotic condition must be applied, e.g. if the fluid is at rest at infinity, we demand that the solution of (5.19) should satisfy the asymptotic condition

$$\frac{\Psi^e}{r^2} \to 0, \quad \text{as} \quad r \to \infty\,, \tag{5.40}$$

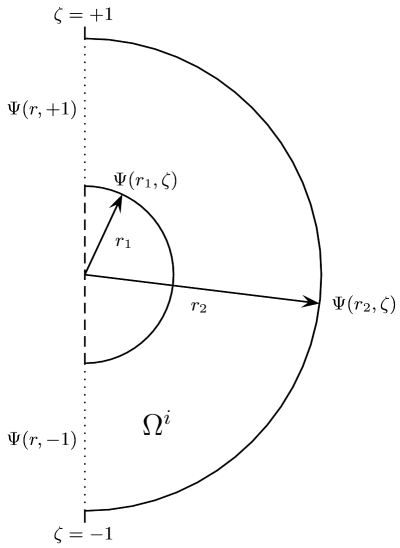

FIGURE 5.1: *The interior domain* Ω^i *defined as* $\Omega^i = \left\{(r,\zeta)\,\middle|\, r_1 \le r \le r_2,\ -1 < \zeta < +1\right\}$.

in order that the velocity vanishes. On the other hand, if the body is considered at rest with the fluid streaming past it, the appropriate asymptotic condition is

$$\Psi^e \to f(r,\zeta;U), \quad \text{as} \quad r \to \infty\,, \tag{5.41}$$

where U is the uniform velocity of the fluid and Ψ^e stands for both Ψ^e_D and Ψ^e_N. In addition, the Neumann boundary values have to satisfy the compatibility condition (see Appendix D for details)

$$\int_{\partial\Omega(r_1)\cup\partial\Omega(r_2)} \sum_{j=1}^{2} g_N^{(j)}(\mathbf{r})\, dS(\mathbf{r}) = \begin{cases} \frac{8}{3}\pi\left(r_2^3 - r_1^3\right), & n = 2 \\ 0, & n > 2 \end{cases}. \tag{5.42}$$

In the case where Dirichlet boundary values $g_D^{(j)}$, $j = 1, 2$ are prescribed, equation (5.34) implies

$$\begin{aligned} \Psi_D(r,\zeta) = {} & \frac{1}{2}\sum_{n=2}^{+\infty} n(n-1)(2n-1)\left[r_2\left(\frac{r_1}{r_2}\right)^n - r_1\left(\frac{r_2}{r_1}\right)^n\right]^{-1} \\ & \times \left[\left(r_2^{-n+1}\,\widetilde{\mathfrak{D}}_n^{(1)} - r_1^{-n+1}\,\widetilde{\mathfrak{D}}_n^{(2)}\right) r^n + \left(r_1^n\,\widetilde{\mathfrak{D}}_n^{(2)} - r_2^n\,\widetilde{\mathfrak{D}}_n^{(1)}\right) r^{-n+1}\right] C_n^{-\frac{1}{2}}(\zeta)\,, \end{aligned} \tag{5.43}$$

where we used the fact that the Gegenbauer polynomials of the first kind and order $-\frac{1}{2}$ satisfy the orthogonality relations

$$\int_{-1}^{+1} \frac{C_n^{-\frac{1}{2}}(\zeta) C_m^{-\frac{1}{2}}(\zeta)}{1-\zeta^2}\mathrm{d}\,\zeta = \frac{2}{n(n-1)(2n-1)}\delta_{nm}, \quad n \ge 2,\ m \ge 2\,, \tag{5.44}$$

and the Gegenbauer transforms of order $-\frac{1}{2}$ of the Dirichlet data are given as

$$\widetilde{\mathfrak{D}}_n^{(j)} = \int_{-1}^{+1} \frac{g_D^{(j)} C_n^{-\frac{1}{2}}(\zeta)}{1-\zeta^2} \mathrm{d}\,\zeta, \quad n \geq 2, \quad j = 1, 2\,. \tag{5.45}$$

Similarly, for the Neumann problem with data $g_N^{(j)}$, $j = 1, 2$ the solution assumes the form

$$\Psi_N(r,\zeta) = \frac{1}{2}\sum_{n=2}^{+\infty}(2n-1)\left[\frac{1}{r_2}\left(\frac{r_2}{r_1}\right)^n - \frac{1}{r_1}\left(\frac{r_1}{r_2}\right)^n\right]^{-1}$$
$$\times\left[(n-1)\left(r_1^{-n}\,\widetilde{\mathfrak{N}}_n^{(2)} - r_2^{-n}\,\widetilde{\mathfrak{N}}_n^{(1)}\right) r^n + n\left(r_1^{n-1}\,\widetilde{\mathfrak{N}}_n^{(2)} - r_2^{n-1}\,\widetilde{\mathfrak{N}}_n^{(1)}\right) r^{-n+1}\right] C_n^{-\frac{1}{2}}(\zeta)\,, \tag{5.46}$$

where

$$\widetilde{\mathfrak{N}}_n^{(j)} = \int_{-1}^{+1} \frac{g_N^{(j)} C_n^{-\frac{1}{2}}(\zeta)}{1-\zeta^2} \mathrm{d}\,\zeta, \quad n \geq 2, \quad j = 1, 2\,. \tag{5.47}$$

As r_1 tends to zero, equation (5.34) implies

$$\lim_{r_1 \to 0} \Psi_X(r,\zeta) = \Psi_X^i(r,\zeta)\,, \qquad X = D \,\text{or}\, N\,,$$

where

$$\Psi_D^i(r,\zeta) = \frac{1}{2}\sum_{n=2}^{+\infty} n(n-1)(2n-1)\,\widetilde{\mathfrak{D}}_n^{(2)}\left(\frac{r}{r_2}\right)^n C_n^{-\frac{1}{2}}(\zeta)\,, \tag{5.48}$$

and

$$\Psi_N^i(r,\zeta) = \frac{r_2}{2}\sum_{n=2}^{+\infty}(n-1)(2n-1)\,\widetilde{\mathfrak{N}}_n^{(2)}\left(\frac{r}{r_2}\right)^n C_n^{-\frac{1}{2}}(\zeta)\,, \tag{5.49}$$

corresponding to the irrotational flow of a viscous fluid in the interior of a sphere of radius r_2.
In a similar manner, as $r_2 \to \infty$, we have $\lim_{r_2 \to \infty} \Psi_X(r,\zeta) = \Psi_X^e(r,\zeta)$, $X = D$ or N, where

$$\Psi_D^e(r,\zeta) = \frac{1}{2}\sum_{n=2}^{+\infty} n(n-1)(2n-1)\,\widetilde{\mathfrak{D}}_n^{(1)}\left(\frac{r_1}{r}\right)^{n-1} C_n^{-\frac{1}{2}}(\zeta)\,, \tag{5.50}$$

and

$$\Psi_N^e(r,\zeta) = -\frac{r_1}{2}\sum_{n=2}^{+\infty} n(2n-1)\,\widetilde{\mathfrak{N}}_n^{(1)}\left(\frac{r_1}{r}\right)^{n-1} C_n^{-\frac{1}{2}}(\zeta)\,, \tag{5.51}$$

describing the irrotational flow of a viscous fluid in the exterior of a sphere of radius r_1.
The coefficients $\widetilde{\mathfrak{D}}_n^{(j)}$ and $\widetilde{\mathfrak{N}}_n^{(j)}$ are given by equations (5.45) and (5.47) respectively.

5.4 The Global Relation

The *global relation*, namely a expression explicitly connecting the Dirichlet with the Neumann boundary values (and vice versa) can be derived by the algorithmic steps analytically described bellow.
Let $\Psi(r,\zeta)$ satisfy the differential equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial\zeta^2}\right)\Psi(r,\zeta)=0 \tag{5.52}$$

and suppose $\overline{q}(r,\zeta)$ any solution of the formal adjoint† $\left(E^2\right)^*$,

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial\zeta^2}\left(\frac{1-\zeta^2}{r^2}\right)\right)\overline{q}(r,\zeta)=0\,. \tag{5.53}$$

Multiply (5.52) by $\overline{q}(r,\zeta)$ and (5.53) by $\Psi(r,\zeta)$ and subtracting them yields, after some algebraic manipulations, the divergence form

$$\frac{\partial}{\partial r}\left(\overline{q}\,\frac{\partial\Psi}{\partial r}-\frac{\partial\overline{q}}{\partial r}\,\Psi\right)+\frac{\partial}{\partial\zeta}\left[\frac{1-\zeta^2}{r^2}\left(\overline{q}\,\frac{\partial\Psi}{\partial\zeta}-\frac{\partial\overline{q}}{\partial\zeta}\,\Psi\right)+\frac{2\zeta}{r^2}\,\overline{q}\,\Psi\right]=0\,. \tag{5.54}$$

In the sequence, consider an arbitrary function $\Xi(r,\zeta;\nu)$, such that

$$\frac{\partial\Xi}{\partial\zeta}=\overline{q}\,\frac{\partial\Psi}{\partial r}-\frac{\partial\overline{q}}{\partial r}\,\Psi\,, \tag{5.55}$$

$$\frac{\partial\Xi}{\partial r}=-\left[\frac{1-\zeta^2}{r^2}\left(\overline{q}\,\frac{\partial\Psi}{\partial\zeta}-\frac{\partial\overline{q}}{\partial\zeta}\,\Psi\right)+\frac{2\zeta}{r^2}\,\overline{q}\,\Psi\right], \tag{5.56}$$

then the above relations imply

$$\left[\frac{\partial}{\partial r},\frac{\partial}{\partial\zeta}\right]\Xi(r,\zeta;\nu)=0,\quad \nu\in\mathbb{C}\,,$$

where $[\cdot,\cdot]$ denotes the commutator, i.e.

$$\left[\frac{\partial}{\partial r},\frac{\partial}{\partial\zeta}\right]=\frac{\partial}{\partial r}\frac{\partial}{\partial\zeta}-\frac{\partial}{\partial\zeta}\frac{\partial}{\partial r}$$

and therefore equations (5.55) and (5.56) constitute a Lax Pair for (5.19).
Equation (5.54) holds true in any meridian plane of $\mathbb{R}^3$. Applying Green's second identity to a closed subdomain of the meridian plane, one obtains immediately the global relation

$$\int_{\partial\Omega}\left\{\left(\overline{q}\,\frac{\partial\Psi}{\partial r}-\frac{\partial\overline{q}}{\partial r}\,\Psi\right)\mathrm{d}\,\zeta-\left[(1-\zeta^2)\left(\overline{q}\,\frac{\partial\Psi}{\partial\zeta}-\frac{\partial\overline{q}}{\partial\zeta}\,\Psi\right)+2\zeta\,\overline{q}\,\Psi\right]\frac{\mathrm{d}\,r}{r^2}\right\}=0\,, \tag{5.57}$$

where $\partial\Omega$ is the boundary of the subdomain.

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†Note that the self-adjoint assumes the form

$$\left(\frac{1}{1-\zeta^2}\,E^2\right)=\left(\frac{1}{1-\zeta^2}\,E^2\right)^*$$

5.5 The Dirichlet-To-Neumann Correspondence and a Lax pair formulation

Utilizing the global relation for constructing the Dirichlet-to-Neumann correspondence is the most effective approach. Observing (5.57) we notice that the algorithmic part starts as soon as a solution to (5.53) is found. Taking advantage of the fact that the domain in question is separable, i.e. we can replace $\overline{q}(r,\zeta)$ with $\overline{R}(r)\overline{Z}(\zeta)$, it follows that $\overline{R}(r)$ satisfies the ODE

$$r^2\,\frac{d^2\overline{R}(r)}{\mathrm{d}\,r^2}-\beta\,\overline{R}(r)=0,\quad \beta\in\mathbb{C}$$

whereas $\overline{Z}(\zeta)$ is a solution to the equation

$$\frac{d^2}{\mathrm{d}\,\zeta^2}\Big((1-\zeta^2)\overline{Z}(\zeta)\Big)+(\beta-2)\,\overline{Z}(\zeta)=0,\quad \beta\in\mathbb{C}\,,$$

where β is the separation constant. The latter is a particular case of the Gegenbauer differential equation (5.22) with $\lambda=\frac{3}{2}$ and $\beta=\nu(\nu+3)+2$.
Hence

$$\overline{q}(r,\zeta;\nu)=\overline{R}(r;\nu)\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,, \tag{5.58}$$

where

$$\overline{R}_1(r;\nu)=r^{\nu+2},\quad \overline{R}_2(r;\nu)=r^{-\nu-1}\,, \tag{5.59}$$

and $\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)$ any solution of the Gegenbauer equation of order $\lambda=\frac{3}{2}$.
Concluding, for the Lax pair introduced in the previous section, consider

$$\Xi(r,\zeta;\nu)=\overline{q}(r,\zeta;\nu)\,\mu(r,\zeta;\nu),\quad \nu\in\mathbb{C}\,, \tag{5.60}$$

where $\mu(r,\zeta;\nu)$ an auxiliary function. Replacing (5.60) into equations (5.55) and (5.56) it is straightforward to show that the Lax pair assumes the form

$$\left(\frac{\partial}{\partial\zeta}+\frac{d\ln\mathcal{G}_\nu^{\frac{3}{2}}}{\mathrm{d}\,\zeta}\right)\mu(r,\zeta;\nu)=\left(\frac{\partial}{\partial r}-\frac{d\ln\overline{R}}{\mathrm{d}\,r}\right)\Psi(r,\zeta),\quad \nu\in\mathbb{C}\,,$$

$$\left(r^2\frac{\partial}{\partial r}+r^2\,\frac{d\ln\overline{R}}{\mathrm{d}\,r}\right)\mu(r,\zeta;\nu)=-\left[(1-\zeta^2)\left(\frac{\partial}{\partial\zeta}-\frac{d\ln\mathcal{G}_\nu^{\frac{3}{2}}}{\mathrm{d}\,\zeta}\right)+2\zeta\right]\Psi,\quad \nu\in\mathbb{C}\,.$$

In the sequence, apply the global relation (5.57) in the domain $\widetilde{\Omega}$ defined as

$$\widetilde{\Omega}=\{r_1\le r\le r_2,\ \zeta_-\le\zeta\le\zeta_+\}$$

and depicted in Figure 5.2, to find

$$\begin{aligned}
&-\int_{r_1}^{r_2}\left[(1-\zeta_-^2)\left(\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_-)\,\frac{\partial\Psi(r,\zeta_-)}{\partial\zeta}-\frac{d\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_-)}{\mathrm{d}\,\zeta}\,\Psi(r,\zeta_-)\right)+2\zeta_-\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_-)\,\Psi(r,\zeta_-)\right]\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2}\\
&+\int_{\zeta_-}^{\zeta_+}\left(\overline{R}(r_2;\nu)\,\frac{\partial\Psi(r_2,\zeta)}{\partial r}-\frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\Psi(r_2,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta\\
&+\int_{r_1}^{r_2}\left[(1-\zeta_+^2)\left(\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_+)\,\frac{\partial\Psi(r,\zeta_+)}{\partial\zeta}-\frac{d\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_+)}{\mathrm{d}\,\zeta}\,\Psi(r,\zeta_+)\right)+2\zeta_+\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_+)\,\Psi(r,\zeta_+)\right]\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2}\\
&-\int_{\zeta_-}^{\zeta_+}\left(\overline{R}(r_1;\nu)\,\frac{\partial\Psi(r_1,\zeta)}{\partial r}-\frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\Psi(r_1,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta=0\,,
\end{aligned} \tag{5.61}$$

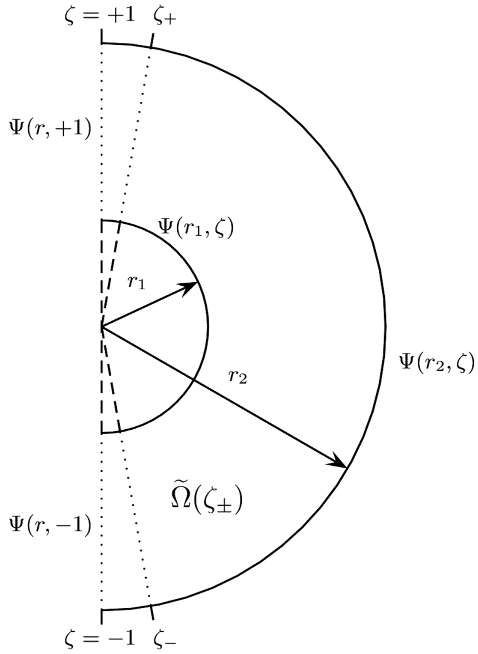

FIGURE 5.2: *The interior subdomain* $\widetilde{\Omega}$.

where $\overline{q}(r,\zeta;\nu)$ is replaced by (5.58).
Stipulate that

$$\Psi(r,\zeta) = (1-\zeta^2)\,\widetilde{\Psi}(r,\zeta)\,, \tag{5.62}$$

then the term

$$(1-\zeta^2)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\frac{\partial\Psi(r,\zeta)}{\partial\zeta} + 2\zeta\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\Psi(r,\zeta)$$

equals

$$(1-\zeta^2)^2\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\frac{\partial\widetilde{\Psi}(r,\zeta)}{\partial\zeta}\,,$$

and (5.61) rewrites

$$\begin{aligned}
&-\int_{r_1}^{r_2}\left[(1-\zeta_-^2)^2\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_-)\,\frac{\partial\widetilde{\Psi}(r,\zeta_-)}{\partial\zeta} - (1-\zeta_-^2)^2\frac{d\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_-)}{\mathrm{d}\,\zeta}\,\widetilde{\Psi}(r,\zeta_-)\right]\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2}\\
&+\int_{\zeta_-}^{\zeta_+}\left(\overline{R}(r_2;\nu)\,\frac{\partial\Psi(r_2,\zeta)}{\partial r} - \frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\Psi(r_2,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta\\
&+\int_{r_1}^{r_2}\left[(1-\zeta_+^2)^2\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_+)\,\frac{\partial\widetilde{\Psi}(r,\zeta_+)}{\partial\zeta} - (1-\zeta_+^2)^2\frac{d\mathcal{G}_\nu^{\frac{3}{2}}(\zeta_+)}{\mathrm{d}\,\zeta}\,\widetilde{\Psi}(r,\zeta_+)\right]\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2}\\
&-\int_{\zeta_-}^{\zeta_+}\left(\overline{R}(r_1;\nu)\,\frac{\partial\Psi(r_1,\zeta)}{\partial r} - \frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\Psi(r_1,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta = 0\,.
\end{aligned} \tag{5.63}$$

The Gegenbauer functions of the first kind of general degree ν and order $\frac{3}{2}$ are regular at $\zeta = 1$, where else the behave as $(1+\zeta)^{-1}$ near the singular point $\zeta = -1$. On the other hand, the Gegenbauer functions of the second kind behave as $(1 \mp \zeta)^{-1}$ as ζ tends to $\pm 1^{\mp}$. This implies that the factors $(1-\zeta_{\pm}^2)^2\, \mathcal{G}_{\nu}^{\frac{3}{2}}(\zeta_{\pm})$ vanish as $\zeta_{\pm} \to \pm 1^{\mp}$. Moreover, it can be shown that (see chapter 3, sections 3.10 and 3.11)

$$\lim_{\zeta \to 1^-} (1-\zeta^2)^2 \frac{dC_{\nu}^{\frac{3}{2}}(\zeta)}{\mathrm{d}\,\zeta} = 0\,, \tag{5.64}$$

$$\lim_{\zeta \to -1^+} (1-\zeta^2)^2 \frac{dC_{\nu}^{\frac{3}{2}}(\zeta)}{\mathrm{d}\,\zeta} = \frac{4}{\pi}\sin \nu\pi\,, \tag{5.65}$$

and

$$\lim_{\zeta \to \pm 1^{\mp}} (1-\zeta^2)^2 \frac{dD_{\nu}^{\frac{3}{2}}(\zeta)}{\mathrm{d}\,\zeta} = 2\left(\cos^2 \frac{\nu\pi}{2} \pm \sin^2 \frac{\nu\pi}{2}\right)\,. \tag{5.66}$$

Thus, as $\zeta_{\pm}$ tends to $\pm 1^{\mp}$ respectively, (5.63) becomes

$$\begin{aligned}
&\left(\overline{R}(r_2;\nu)\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}}) - \frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}})\right) \\
&- \left(\overline{R}(r_1;\nu)\,\mathfrak{N}^{(1)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}}) - \frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}})\right) \\
&= \begin{cases} -\frac{4}{\pi}\sin\nu\pi \int_{r_1}^{r_2} \widetilde{\Psi}(r,-1)\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2} & ,\text{if} \quad \mathcal{G}_{\nu}^{\frac{3}{2}}(\zeta) = C_{\nu}^{\frac{3}{2}}(\zeta) \\ 2\int_{r_1}^{r_2}\left(\widetilde{\Psi}(r,1) - \cos\nu\pi\,\widetilde{\Psi}(r,-1)\right)\overline{R}(r;\nu)\frac{\mathrm{d}\,r}{r^2} & ,\text{if} \quad \mathcal{G}_{\nu}^{\frac{3}{2}}(\zeta) = D_{\nu}^{\frac{3}{2}}(\zeta) \end{cases}, \\
&\qquad \nu \in \mathbb{C}, \quad \nu \neq -3,-4,\ldots
\end{aligned} \tag{5.67}$$

where $\mathfrak{D}^{(j)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}})$ and $\mathfrak{N}^{(j)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}}), j = 1,2$ denote the *weightless* Gegenbauer transforms of the first and second kind of order $\frac{3}{2}$ for the functions $\Psi(r_j,\zeta)$ and $\frac{\partial \Psi(r_j,\zeta)}{\partial r}$ respectively, i.e.

$$\mathfrak{D}^{(j)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}}) = \int_{-1}^{+1} \Psi(r_j,\zeta)\,\mathcal{G}_{\nu}^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta, \quad j = 1,2\,, \tag{5.68}$$

$$\mathfrak{N}^{(j)}(\nu|\mathcal{G}_{\nu}^{\frac{3}{2}}) = \int_{-1}^{+1} \frac{\partial \Psi(r_j,\zeta)}{\partial r}\,\mathcal{G}_{\nu}^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta, \quad j = 1,2\,. \tag{5.69}$$

Replacing $\overline{R}(r;\nu)$ in (5.67) with a linear combination of $r^{\nu+2}$ and $r^{-\nu-1}$ leads to a relation valid in the interior Ω^i of the spherical shell $\mathcal{S}$ which is valid in the entire complex $\nu-$plane, except for the points $\nu \neq -3,-4,\ldots$ As $r_1 \to 0$, the resulting expression is valid only in interval $\mathrm{Re}\,\nu \in (-2,-1)$. On the contrary, none expression can be derived in the case where

r_2 tends to infinity. Therefore, in order to keep things simple, rewrite (5.67) as

$$r_2^{\nu+1}\left(r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) - (\nu+2)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right) - r_1^{\nu+1}\left(r_1\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) - (\nu+2)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$= \begin{cases} -\frac{4}{\pi}\sin\nu\pi\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\,r^{\nu}\,\mathrm{d}\,r & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = C_\nu^{\frac{3}{2}}(\zeta) \\ 2\int_{r_1}^{r_2}\left(\widetilde{\Psi}(r,1) - \cos\nu\pi\,\widetilde{\Psi}(r,-1)\right) r^{\nu}\,\mathrm{d}\,r & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = D_\nu^{\frac{3}{2}}(\zeta) \end{cases}, \tag{5.70}$$

$$r_2^{-\nu-2}\left(r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) + (\nu+1)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right) - r_1^{-\nu-2}\left(r_1\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) + (\nu+1)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$= \begin{cases} -\frac{4}{\pi}\sin\nu\pi\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\,r^{-\nu-3}\,\mathrm{d}\,r & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = C_\nu^{\frac{3}{2}}(\zeta) \\ 2\int_{r_1}^{r_2}\left(\widetilde{\Psi}(r,1) - \cos\nu\pi\,\widetilde{\Psi}(r,-1)\right) r^{-\nu-3}\,\mathrm{d}\,r & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = D_\nu^{\frac{3}{2}}(\zeta) \end{cases}, \tag{5.71}$$

for every $\nu \in \mathbb{C}$, $\nu \neq -3, -4, \dots$, where (5.70) is derived with the use of $\overline{R}_1$ and (5.71) is derived with the use of $\overline{R}_2$. As r_1 tends to zero, a singularity at $r = 0$ is introduced and thus the function $\overline{R}(r;\nu)$ needs to be bounded and equations (5.70) and (5.71) rewrite

$$r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) - (\nu+2)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})$$
$$= \begin{cases} -\frac{4}{\pi}\sin\nu\pi\int_0^{r_2}\widetilde{\Psi}(r,-1)\left(\frac{r}{r_2}\right)^{\nu+1}\frac{\mathrm{d}\,r}{r} & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = C_\nu^{\frac{3}{2}}(\zeta) \\ 2\int_0^{r_2}\left(\widetilde{\Psi}(r,1) - \cos\nu\pi\,\widetilde{\Psi}(r,-1)\right)\left(\frac{r}{r_2}\right)^{\nu+1}\frac{\mathrm{d}\,r}{r} & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = D_\nu^{\frac{3}{2}}(\zeta) \end{cases}, \tag{5.72}$$

valid for every $\operatorname{Re}\nu > -1$, and

$$r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) + (\nu+1)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})$$
$$= \begin{cases} -\frac{4}{\pi}\sin\nu\pi\int_0^{r_2}\widetilde{\Psi}(r,-1)\left(\frac{r_2}{r}\right)^{\nu+2}\frac{\mathrm{d}\,r}{r} & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = C_\nu^{\frac{3}{2}}(\zeta) \\ 2\int_0^{r_2}\left(\widetilde{\Psi}(r,1) - \cos\nu\pi\,\widetilde{\Psi}(r,-1)\right)\left(\frac{r_2}{r}\right)^{\nu+2}\frac{\mathrm{d}\,r}{r} & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = D_\nu^{\frac{3}{2}}(\zeta) \end{cases}, \nu \neq -3, -4, \dots \tag{5.73}$$

valid for every $\operatorname{Re}\nu \leq -2$.

On the other hand, sending r_2 to infinity, the exterior problem for a sphere of radius r_1 is recovered. The singularity at infinity is handled introducing $\overline{R}_1(r;\nu) = r^{\nu+2}$ for every $\nu \in \mathbb{C}$ less then -1 or introducing $\overline{R}_2(r;\nu) = r^{-\nu-1}$ for every $\nu \in \mathbb{C}$ greater then -2. Hence, equations (5.70) and (5.71) rewrite as

$$r_1\,\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) - (\nu+2)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})$$
$$= \begin{cases} \frac{4}{\pi}\sin\nu\pi\int_{r_1}^{\infty}\widetilde{\Psi}(r,-1)\left(\frac{r}{r_1}\right)^{\nu+1}\frac{\mathrm{d}\,r}{r} & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = C_\nu^{\frac{3}{2}}(\zeta) \\ -2\int_{r_1}^{\infty}\left(\widetilde{\Psi}(r,1) - \cos\nu\pi\,\widetilde{\Psi}(r,-1)\right)\left(\frac{r}{r_1}\right)^{\nu+1}\frac{\mathrm{d}\,r}{r} & \text{, if } \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = D_\nu^{\frac{3}{2}}(\zeta) \end{cases},$$
$$\operatorname{Re}\nu \leq -1, \quad \nu \neq -3, -4, \dots, \tag{5.74}$$

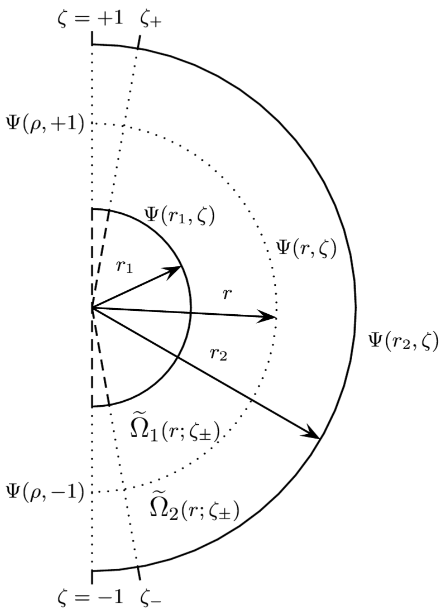

FIGURE 5.3: *The shell subdomains* $\widetilde{\Omega}_1(r,\zeta)=\left\{r_1\le\rho\le r,\ \zeta_-\le\zeta\le\zeta_+\right\}$ *and* $\widetilde{\Omega}_2(r,\zeta)=\left\{r\le\rho\le r_2,\ \zeta_-\le\zeta\le\zeta_+\right\}$ *which tend to the shell subdomains* $\Omega_1(r)=\left\{r_1\le\rho\le r,\ -1<\zeta<+1\right\}$ *and* $\Omega_2(r)=\left\{r\le\rho\le r_2,\ -1<\zeta<+1\right\}$ *as* $\zeta_\pm\to\pm1^\mp$.

and

$$r_1\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})+(\nu+1)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})$$
$$=\begin{cases}\frac{4}{\pi}\sin\nu\pi\int_{r_1}^\infty\widetilde{\Psi}(r,-1)\,\left(\frac{r_1}{r}\right)^{\nu+2}\,\frac{\mathrm{d}\,r}{r} & ,\ \text{if}\quad \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ -2\int_{r_1}^\infty\left(\widetilde{\Psi}(r,1)-\cos\nu\pi\,\widetilde{\Psi}(r,-1)\right)\,\left(\frac{r_1}{r}\right)^{\nu+2}\,\frac{\mathrm{d}\,r}{r} & ,\ \text{if}\quad \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},\ \operatorname{Re}\nu\ge-2\,. \tag{5.75}$$

5.6 From Complex to Real: Classical Solutions Recovered

Applying the global relation (5.57) in the shell subdomains $\Omega_1(r,\zeta)$ defined as

$$\Omega_1(r,\zeta)=\left\{(r,\zeta)\middle|r_1\le\rho\le r,\ -1<\zeta<+1\right\},$$

and $\Omega_2(r,\zeta)$ defined as

$$\Omega_2(r,\zeta)=\left\{(r,\zeta)\middle|r\le\rho\le r_2,\ -1<\zeta<+1\right\},$$

depicted in Figure 5.3, with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)$ yields,

$$\int_{-1}^{+1}\left(\overline{R}(r;\nu)\,\frac{\partial\Psi(r,\zeta)}{\partial r}-\frac{d\overline{R}(r;\nu)}{\mathrm{d}\,r}\,\Psi(r,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta$$
$$=\left(\overline{R}(r_1;\nu)\,\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})-\frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$+\begin{cases}-\frac{4}{\pi}\sin\nu\pi\int_{r_1}^{r}\widetilde{\Psi}(\rho,-1)\,\overline{R}(\rho;\nu)\,\frac{\mathrm{d}\rho}{\rho^2} & \text{, if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ -2\int_{r_1}^{r}\left(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1)-\widetilde{\Psi}(\rho,1)\right)\overline{R}(\rho;\nu)\,\frac{\mathrm{d}\rho}{\rho^2} & \text{, if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},$$
$$,\nu\in\mathbb{C},\ \nu\neq-3,-4,\dots,\tag{5.76}$$

and

$$\int_{-1}^{+1}\left(\overline{R}(r;\nu)\,\frac{\partial\Psi(r,\zeta)}{\partial r}-\frac{d\overline{R}(r;\nu)}{\mathrm{d}\,r}\,\Psi(r,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta$$
$$=\left(\overline{R}(r_2;\nu)\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})-\frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$+\begin{cases}\frac{4}{\pi}\sin\nu\pi\int_{r}^{r_2}\widetilde{\Psi}(\rho,-1)\,\overline{R}(\rho;\nu)\,\frac{\mathrm{d}\rho}{\rho^2} & \text{, if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ 2\int_{r}^{r_2}\left(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1)-\widetilde{\Psi}(\rho,1)\right)\overline{R}(\rho;\nu)\,\frac{\mathrm{d}\rho}{\rho^2} & \text{, if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},$$
$$,\nu\in\mathbb{C},\ \nu\neq-3,-4,\dots,\tag{5.77}$$

where $\mathfrak{D}^{(j)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})$ and $\mathfrak{N}^{(j)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}),\ j=1,2$ are given by equations (5.68). Note that the subdomains Ω_1 and Ω_2 do not include the singularities at $r=0$ or $r=\infty$, and therefore no restrictions on $\overline{R}(r;\nu)$ in (5.76) or (5.77) have to imposed.
Replacing in equations (5.76) and (5.77) $\overline{R}$ with (5.59), the following 4 equations valid for every $\nu\in\mathbb{C}$, $\nu\neq-3,-4,\dots$, are the result

$$\int_{-1}^{+1}\left(r\,\frac{\partial\Psi(r,\zeta)}{\partial r}-(\nu+2)\,\Psi(r,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta$$
$$=\left(\frac{r_1}{r}\right)^{\nu+1}\left(r_1\,\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})-(\nu+2)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$+\begin{cases}-\frac{4}{\pi}\sin\nu\pi\int_{r_1}^{r}\widetilde{\Psi}(\rho,-1)\left(\frac{\rho}{r}\right)^{\nu+1}\frac{\mathrm{d}\rho}{\rho} & \text{, if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ -2\int_{r_1}^{r}\left(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1)-\widetilde{\Psi}(\rho,1)\right)\left(\frac{\rho}{r}\right)^{\nu+1}\frac{\mathrm{d}\rho}{\rho} & \text{, if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},\tag{5.78}$$

$$\int_{-1}^{+1}\left(r\,\frac{\partial\Psi(r,\zeta)}{\partial r}+(\nu+1)\,\Psi(r,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta$$
$$=\left(\frac{r}{r_1}\right)^{\nu+2}\left(r_1\,\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})+(\nu+1)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$+\begin{cases}-\frac{4}{\pi}\sin\nu\pi\int_{r_1}^{r}\widetilde{\Psi}(\rho,-1)\left(\frac{r}{\rho}\right)^{\nu+2}\frac{\mathrm{d}\rho}{\rho}, & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ -2\int_{r_1}^{r}\left(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1)-\widetilde{\Psi}(\rho,1)\right)\left(\frac{r}{\rho}\right)^{\nu+2}\frac{\mathrm{d}\rho}{\rho} & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},\tag{5.79}$$

$$\int_{-1}^{+1}\left(r\,\frac{\partial\Psi(r,\zeta)}{\partial r}-(\nu+2)\,\Psi(r,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta$$
$$=\left(\frac{r_2}{r}\right)^{\nu+1}\left(r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})-(\nu+2)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$+\begin{cases}\frac{4}{\pi}\sin\nu\pi\int_{r}^{r_2}\widetilde{\Psi}(\rho,-1)\left(\frac{\rho}{r}\right)^{\nu+1}\frac{\mathrm{d}\rho}{\rho} & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ 2\int_{r}^{r_2}\left(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1)-\widetilde{\Psi}(\rho,1)\right)\left(\frac{\rho}{r}\right)^{\nu+1}\frac{\mathrm{d}\rho}{\rho} & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},\tag{5.80}$$

and

$$\int_{-1}^{+1}\left(r\,\frac{\partial\Psi(r,\zeta)}{\partial r}+(\nu+1)\,\Psi(r,\zeta)\right)\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta$$
$$=\left(\frac{r}{r_2}\right)^{\nu+2}\left(r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})+(\nu+1)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$+\begin{cases}\frac{4}{\pi}\sin\nu\pi\int_{r}^{r_2}\widetilde{\Psi}(\rho,-1)\left(\frac{r}{\rho}\right)^{\nu+2}\frac{\mathrm{d}\rho}{\rho} & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ 2\int_{r}^{r_2}\left(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1)-\widetilde{\Psi}(\rho,1)\right)\left(\frac{r}{\rho}\right)^{\nu+2}\frac{\mathrm{d}\rho}{\rho} & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},\tag{5.81}$$

Subtracting equation (5.78) from (5.79) and (5.80) from (5.81), equations (5.70) and (5.71) are recovered.

In order to eliminate the unknown function $\frac{\partial\Psi(r,\zeta)}{\partial r}$, subtract equations (5.78) and (5.81) side-by-side, to find

$$\int_{-1}^{+1}\Psi(r,\zeta)\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta=\frac{1}{2\nu+3}\left[\left(\frac{r}{r_2}\right)^{\nu+2}\left(r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})+(\nu+1)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)\right.$$
$$\left.-\left(\frac{r_1}{r}\right)^{\nu+1}\left(r_1\,\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})-(\nu+2)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)\right]+\frac{1}{2\nu+3}$$
$$\times\begin{cases}\frac{4}{\pi}\sin\nu\pi\,\mathbb{R}_{r_1}^{r_2}(r;\nu)\widetilde{\Psi}(\rho,-1) & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=C_\nu^{\frac{3}{2}}(\zeta)\\ 2\mathbb{R}_{r_1}^{r_2}(r;\nu)\Big(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1)-\widetilde{\Psi}(\rho,1)\Big) & ,\text{ if }\ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta)=D_\nu^{\frac{3}{2}}(\zeta)\end{cases},$$
$$,\nu\in\mathbb{C},\quad \nu\neq-3,-4,\ldots,\tag{5.82}$$

where

$$\mathbb{R}_{r_1}^{r_2}(r;\nu) := \int_{r_1}^{r} \frac{d\rho}{\rho}\left(\frac{\rho}{r}\right)^{\nu+1} + \int_{r}^{r_2} \frac{d\rho}{\rho}\left(\frac{r}{\rho}\right)^{\nu+2},$$

is an integral operator, which we will refer to as the *Radial Integral Operator*.
Similarly, subtracting equations (5.79) and (5.80) side-by-side, yields

$$\int_{-1}^{+1} \Psi(r,\zeta)\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta = \frac{1}{2\nu+3}\Bigg[\left(\frac{r}{r_1}\right)^{\nu+2}\left(r_1\,\mathfrak{N}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) + (\nu+1)\,\mathfrak{D}^{(1)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)$$
$$-\left(\frac{r_2}{r}\right)^{\nu+1}\left(r_2\,\mathfrak{N}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}}) - (\nu+2)\,\mathfrak{D}^{(2)}(\nu|\mathcal{G}_\nu^{\frac{3}{2}})\right)\Bigg] - \frac{1}{2\nu+3}$$
$$\times\begin{cases} \frac{4}{\pi}\sin\nu\pi\,\mathbb{R}^{\dagger}{}_{r_1}^{r_2}(r;\nu)\widetilde{\Psi}(\rho,-1) & ,\text{ if } \ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = C_\nu^{\frac{3}{2}}(\zeta) \\ 2\mathbb{R}^{\dagger}{}_{r_1}^{r_2}(r;\nu)\Big(\cos\nu\pi\,\widetilde{\Psi}(\rho,-1) - \widetilde{\Psi}(\rho,1)\Big), & ,\text{ if } \ \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = D_\nu^{\frac{3}{2}}(\zeta) \end{cases},$$
$$,\nu\in\mathbb{C},\quad \nu\neq -3,-4,\dots, \tag{5.83}$$

where

$$\mathbb{R}^{\dagger}{}_{r_1}^{r_2}(r;\nu) := \int_{r_1}^{r} \frac{d\rho}{\rho}\left(\frac{r}{\rho}\right)^{\nu+2} + \int_{r}^{r_2} \frac{\mathrm{d}\rho}{\rho}\left(\frac{\rho}{r}\right)^{\nu+1}.$$

Note that

$$\mathbb{R}^{\dagger}{}_{r_1}^{r_2}(r;-\nu-3) = \mathbb{R}_{r_1}^{r_2}(r;\nu), \quad \nu\in\mathbb{C},$$

which implies that (5.82a) and (5.83a) differ only by a transformation based on the symmetry of the Gegenbauer functions of the first kind and order $\frac{3}{2}$. A similar conclusion does not hold for the Gegenbauer functions of the second kind and order $\frac{3}{2}$, since there doesn't exist a linear relation connecting them 3.
The inversion of (5.82), or (5.83), leads to an integral representation for the stream function $\Psi(r,\zeta)$ in the case of the irrotational Stokes' flow valid in a spherical shell with inner radius r_1 and outer radius r_2. However, in order to arrive at classical representations, i.e. solutions in form of a series expansion, we must take advantage of the orthogonality relation for the Gegenbauer polynomials. Thus, letting $\nu = n = 0,1,2,\dots$ in (5.82) yields

$$\int_{-1}^{+1} \Psi(r,\zeta)\,C_n^{\frac{3}{2}}(\zeta)\,\mathrm{d}\,\zeta = \frac{1}{2n+3}\Bigg[\left(\frac{r}{r_2}\right)^{n+2}\left(r_2\,\mathfrak{N}_n^{(2)} + (n+1)\,\mathfrak{D}_n^{(2)}\right)$$
$$-\left(\frac{r_1}{r}\right)^{n+1}\left(r_1\,\mathfrak{N}_n^{(1)} - (n+2)\,\mathfrak{D}_n^{(1)}\right)\Bigg], \tag{5.84}$$

where now

$$\mathfrak{D}_n^{(j)} = \int_{-1}^{+1} g_D^{(j)}(\zeta)\,C_n^{\frac{3}{2}}(\zeta), \qquad \mathfrak{N}_n^{(j)} = \int_{-1}^{+1} g_N^{(j)}(\zeta)\,C_n^{\frac{3}{2}}(\zeta). \tag{5.85}$$

Proposition 5.6.1 *The set of functions $C_n^\lambda(x)$ form a complete, orthogonal system with weight*

$$\frac{\Gamma(n+1)\Gamma^2(\lambda)\,(n+\lambda)}{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}(1-x^2)^{\lambda-\frac{1}{2}}, \quad \lambda > -\frac{1}{2} \tag{5.86}$$

on the interval $(-1,+1)$.
Consider a class of functions of the form

$$f(x) = \frac{g(x)}{(1-x^2)^{\lambda-\frac{1}{2}}} \tag{5.87}$$

expanded in a series of Gegenbauer polynomials, namely

$$f(x) = \sum_{n=0}^{+\infty} \alpha_n C_n^{\lambda}(x)\,, \tag{5.88}$$

then in view of (5.87)

$$g(x) = (1-x^2)^{\lambda-\frac{1}{2}} \sum_{n=0}^{+\infty} \alpha_n\, C_n^{\lambda}(x)\,, \tag{5.89}$$

where the coefficients α_n *are evaluated as follows*

$$\alpha_n = 2^{2\lambda-1}\frac{\Gamma(n+1)\left(\Gamma(\lambda)\right)^2(n+\lambda)}{\pi\Gamma(n+2\lambda)}\int_{-1}^{+1} g(x)\, C_n^{\lambda}(x)\, dx\,. \tag{5.90}$$

Thus, the inversion formula for (5.84) implies

$$\Psi(r,\zeta) = \frac{1-\zeta^2}{2}\sum_{n=0}^{+\infty}\frac{1}{(n+1)(n+2)}\left\{\left(\frac{r}{r_2}\right)^{n+2}\left(r_2\,\mathfrak{N}_n^{(2)} + (n+1)\,\mathfrak{D}_n^{(2)}\right)\right.$$
$$\left. -\left(\frac{r_1}{r}\right)^{n+1}\left(r_1\,\mathfrak{N}_n^{(1)} - (n+2)\,\mathfrak{D}_n^{(1)}\right)\right\} C_n^{\frac{3}{2}}(\zeta)\,. \tag{5.91}$$

The unknown boundary values $\mathfrak{D}_n^{(j)}$ or $\mathfrak{N}_n^{(j)}$, $j=1,2$ depending on the given boundary value problem, are eliminated by the use of equations (5.70) and (5.71) with ν replaced by $n\in\mathbb{N}$. Substituting the resulting formulae into (5.91) the solutions $\Psi_D(r,\zeta)$, and $\Psi_N(r,\zeta)$ respectively, are obtained as follows

$$\Psi_D(r,\zeta) = \frac{1-\zeta^2}{2}\sum_{n=0}^{+\infty}\frac{2n+3}{(n+1)(n+2)}\left[r_2\left(\frac{r_1}{r_2}\right)^{n+2} - r_1\left(\frac{r_2}{r_1}\right)^{n+2}\right]^{-1}$$
$$\times\left\{\left(r_2^{-n-1}\,\mathfrak{D}_n^{(1)} - r_1^{-n-1}\,\mathfrak{D}_n^{(2)}\right) r^{n+2} - \left(r_2^{n+2}\,\mathfrak{D}_n^{(1)} - r_1^{n+2}\,\mathfrak{D}_n^{(2)}\right) r^{-n-1}\right\} C_n^{\frac{3}{2}}(\zeta) \tag{5.92}$$

$$\Psi_N(r,\zeta) = \frac{1-\zeta^2}{2}\sum_{n=0}^{+\infty}\frac{2n+3}{(n+1)(n+2)}\left[\frac{1}{r_2}\left(\frac{r_2}{r_1}\right)^{n+2} - \frac{1}{r_1}\left(\frac{r_1}{r_2}\right)^{n+2}\right]^{-1}$$
$$\times\left\{\frac{r^{n+2}}{n+2}\left(r_1^{-n-2}\,\mathfrak{N}_n^{(2)} - r_2^{-n-2}\,\mathfrak{N}_n^{(1)}\right) - \frac{r^{-n-1}}{n+1}\left(r_2^{n+1}\,\mathfrak{N}_n^{(1)} - r_1^{n+1}\,\mathfrak{N}_n^{(2)}\right)\right\} C_n^{\frac{3}{2}}(\zeta) \tag{5.93}$$

As r_1 tends to zero, the above relations simplify as

$$\Psi_D^i(r,\zeta) = \frac{1-\zeta^2}{2}\sum_{n=0}^{+\infty}\frac{2n+3}{(n+1)(n+2)}\left(\frac{r}{r_2}\right)^{n+2}\mathfrak{D}_n^{(2)}\,C_n^{\frac{3}{2}}(\zeta)\,, \tag{5.94}$$

and

$$\Psi^i_{N}(r,\zeta) = \frac{1-\zeta^2}{2}\, r_2 \sum_{n=0}^{+\infty} \frac{2n+3}{(n+1)(n+2)^2} \left(\frac{r}{r_2}\right)^{n+2} \mathfrak{N}^{(2)}_n\, C^{\frac{3}{2}}_n(\zeta)\,, \tag{5.95}$$

describing the irrotational Stokes' flow of a fluid in the interior of a sphere of radius r_2. Similarly, as $r_2 \to \infty$, equations (5.92) and (5.93) yield

$$\Psi^e_{D}(r,\zeta) = \frac{1-\zeta^2}{2} \sum_{n=0}^{+\infty} \frac{2n+3}{(n+1)(n+2)} \left(\frac{r_1}{r}\right)^{n+1} \mathfrak{D}^{(1)}_n\, C^{\frac{3}{2}}_n(\zeta)\,, \tag{5.96}$$

and

$$\Psi^e_{N}(r,\zeta) = -\frac{1-\zeta^2}{2}\, r_1 \sum_{n=0}^{+\infty} \frac{2n+3}{(n+1)^2(n+2)} \left(\frac{r_1}{r}\right)^{n+1} \mathfrak{N}^{(1)}_n\, C^{\frac{3}{2}}_n(\zeta)\,, \tag{5.97}$$

which describe the irrotational flow of a viscous flow in the exterior of a sphere of radius r_1.
After simple manipulations utilizing (5.39) and therefore $\widetilde{\mathfrak{X}}^{(j)}_n = \frac{1}{n\,(n-1)}\, \mathfrak{X}^{(j)}_{n-2}$, where $\mathfrak{X}$ stands for $\mathfrak{D}$ or $\mathfrak{N}$ respectively, equations (5.92) - (5.97) recover equations (5.43), (5.46) and (5.48) - (5.51).

5.7 A Novel Integral Representation

Utilizing the global relation (5.57) in the subdomains $\widetilde{\Omega}_3$ and $\widetilde{\Omega}_4$ defined as

$$\begin{aligned} \widetilde{\Omega}_3(\zeta;\zeta_+) &= \Big\{(r,\zeta)\,\Big|\, r_1 \le r \le r_2,\ \zeta \le t \le \zeta_+\Big\},\\ \widetilde{\Omega}_4(\zeta;\zeta_-) &= \Big\{(r,\zeta)\,\Big|\, r_1 \le r \le r_2,\ \zeta_- \le t \le \zeta\Big\}, \end{aligned}$$

depicted in Figure 5.4, with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\,\mathcal{G}^{\frac{3}{2}}_\nu(\zeta)$, where $\mathcal{G}^{\frac{3}{2}}_\nu(\zeta)$ is any solution of the Gegenbauer equation (5.22) of order $\lambda = \frac{3}{2}$, we obtain

$$\begin{aligned} &\int_{r_1}^{r_2} \left[(1-\zeta^2)\left(\mathcal{G}^{\frac{3}{2}}_\nu(\zeta)\,\frac{\partial\Psi(r,\zeta)}{\partial\zeta} - \frac{\mathrm{d}\mathcal{G}^{\frac{3}{2}}_\nu(\zeta)}{\mathrm{d}\zeta}\,\Psi(r,\zeta)\right) + 2\,\zeta\,\mathcal{G}^{\frac{3}{2}}_\nu(\zeta)\,\Psi(r,\zeta)\right]\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2}\\ &= \int_{\zeta}^{+1}\left\{\left(\overline{R}(r_2;\nu)\,\frac{\partial\Psi(r_2,t)}{\partial r} - \frac{\mathrm{d}\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\Psi(r_2,t)\right)\right.\\ &\qquad\left. - \left(\overline{R}(r_1;\nu)\,\frac{\partial\Psi(r_1,t)}{\partial r} - \frac{\mathrm{d}\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\Psi(r_1,t)\right)\right\}\mathcal{G}^{\frac{3}{2}}_\nu(t)\,\mathrm{d}t\\ &+\begin{cases} 0 & ,\text{if}\quad \mathcal{G}^{\frac{3}{2}}_\nu(\zeta) = C^{\frac{3}{2}}_\nu(\zeta)\\ -2\int_{r_1}^{r_2}\widetilde{\Psi}(r,+1)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2} & ,\text{if}\quad \mathcal{G}^{\frac{3}{2}}_\nu(\zeta) = D^{\frac{3}{2}}_\nu(\zeta), \end{cases} ,\ \nu\in\mathbb{C},\quad \nu\neq -3,-4,\dots\,, \end{aligned} \tag{5.98}$$

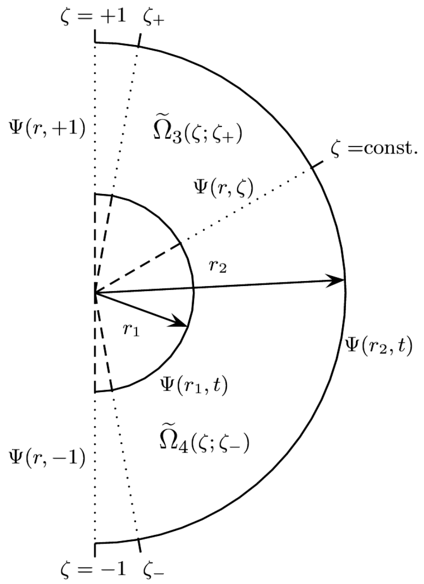

FIGURE 5.4: *The subdomains* $\widetilde{\Omega}_3(\zeta;\zeta_+)$ *and* $\widetilde{\Omega}_4(\zeta;\zeta_-)$ *defined as* $\widetilde{\Omega}_3(\zeta;\zeta_+) = \left\{(r,\zeta)\,\middle|\, r_1 \le r \le r_2,\ \zeta \le t \le \zeta_+\right\}$ *and* $\widetilde{\Omega}_4(\zeta;\zeta_-) = \left\{(r,\zeta)\,\middle|\, r_1 \le r \le r_2,\ \zeta_- \le t \le \zeta\right\}$. *The whole meridian plane is recovered as* $\lim\limits_{\zeta_+\to+1}\widetilde{\Omega}_3 = \Omega_3$ *and* $\lim\limits_{\zeta_-\to-1}\widetilde{\Omega}_4 = \Omega_4$.

$$
\begin{aligned}
&\int_{r_1}^{r_2}\left[(1-\zeta^2)\left(\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\frac{\partial\Psi(r,\zeta)}{\partial\zeta} - \frac{\mathrm{d}\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)}{\mathrm{d}\zeta}\,\Psi(r,\zeta)\right) + 2\,\zeta\,\mathcal{G}_\nu^{\frac{3}{2}}(\zeta)\,\Psi(r,\zeta)\right]\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2}\\
&= -\int_{-1}^{\zeta}\left\{\left(\overline{R}(r_2;\nu)\,\frac{\partial\Psi(r_2,t)}{\partial r} - \frac{\mathrm{d}\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\Psi(r_2,t)\right)\right.\\
&\qquad\left. - \left(\overline{R}(r_1;\nu)\,\frac{\partial\Psi(r_1,t)}{\partial r} - \frac{\mathrm{d}\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\Psi(r_1,t)\right)\right\}\mathcal{G}_\nu^{\frac{3}{2}}(t)\,\mathrm{d}t\\
&\quad + \begin{cases} -\frac{4}{\pi}\,\sin\nu\pi\,\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2} & ,\text{if}\quad \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = C_\nu^{\frac{3}{2}}(\zeta)\\ -2\cos\nu\pi\,\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2} & ,\text{if}\quad \mathcal{G}_\nu^{\frac{3}{2}}(\zeta) = D_\nu^{\frac{3}{2}}(\zeta), \end{cases}\ ,\ \nu\in\mathbb{C},\quad \nu\neq -3,-4,\ldots\,.
\end{aligned}
\tag{5.99}
$$

Subtracting equations (5.98a) multiplied by $D_\nu^{\frac{3}{2}}(\zeta)$ and (5.99b) multiplied by $C_\nu^{\frac{3}{2}}(\zeta)$ yields

$$\begin{aligned}
&(1-\zeta^2)\left(C_\nu^{\frac{3}{2}}(\zeta)\,\frac{\mathrm{d}D_\nu^{\frac{3}{2}}(\zeta)}{\mathrm{d}\zeta} - D_\nu^{\frac{3}{2}}(\zeta)\,\frac{\mathrm{d}C_\nu^{\frac{3}{2}}(\zeta)}{\mathrm{d}\zeta}\right)\int_{r_1}^{r_2}\Psi(r,\zeta)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2}\\
=&\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\left\{\left(\overline{R}(r_2;\nu)\,\frac{\partial\Psi(r_2,t)}{\partial r} - \frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\Psi(r_2,t)\right)\right.\\
&\qquad\left.-\left(\overline{R}(r_1;\nu)\,\frac{\partial\Psi(r_1,t)}{\partial r} - \frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\Psi(r_1,t)\right)\right\}\\
&+2\,\cos\nu\pi\,C_\nu^{\frac{3}{2}}(\zeta)\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2},\quad \nu\neq-3,-4,\dots,
\end{aligned}\tag{5.100}$$

where

$$\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta) := C_\nu^{\frac{3}{2}}(\zeta)\int_{-1}^{\zeta}\mathrm{d}t\,D_\nu^{\frac{3}{2}}(t) + D_\nu^{\frac{3}{2}}(\zeta)\int_{\zeta}^{+1}\mathrm{d}t\,C_\nu^{\frac{3}{2}}(t),\quad \nu\neq-3,-4,\dots,\tag{5.101}$$

is an integral operator, which we will refer to as the *Gegenbauer Integral Operator* of order $\frac{3}{2}$. Note that the integral $\int_{-1}^{\zeta}\mathrm{d}t\,D_\nu^{\frac{3}{2}}(t)$ doesn't exist. However, the integral $\int_{-1}^{\zeta}\mathrm{d}t\,(1-\zeta^2)\,D_\nu^{\frac{3}{2}}(t)$ converges.
Utilizing the Wronskian relation (3.102)

$$C_\nu^{\frac{3}{2}}(\zeta)\,\frac{\mathrm{d}D_\nu^{\frac{3}{2}}(\zeta)}{\mathrm{d}\zeta} - D_\nu^{\frac{3}{2}}(\zeta)\,\frac{\mathrm{d}C_\nu^{\frac{3}{2}}(\zeta)}{\mathrm{d}\zeta} = \frac{\Gamma(\nu+3)}{\Gamma(\nu+1)}\,\frac{1}{(1-\zeta^2)^2},\quad \nu\neq-3,-4,\dots,$$

we write equation (5.100) as follows

$$\begin{aligned}
&\int_{r_1}^{r_2}\Psi(r,\zeta)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2} = (1-\zeta^2)\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\left\{\left(\overline{R}(r_2;\nu)\,\frac{\partial\Psi(r_2,t)}{\partial r} - \frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\Psi(r_2,t)\right)\right.\\
&\left.-\left(\overline{R}(r_1;\nu)\,\frac{\partial\Psi(r_1,t)}{\partial r} - \frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\,\Psi(r_1,t)\right)\right\} + 2(1-\zeta^2)\,\cos\nu\pi\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,C_\nu^{\frac{3}{2}}(\zeta)\\
&\times\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2},\quad \nu\neq-3,-4,\dots.
\end{aligned}\tag{5.102}$$

Remark 5.7.1 *Equation* (5.102)

$$\begin{aligned}
&\left[\overline{R}(r_2;\nu)\left(\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\frac{\partial\Psi(r_2,t)}{\partial r}\right) - \frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\left(\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\Psi(r_2,t)\right)\right] - \left[\overline{R}(r_1;\nu)\left(\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\frac{\partial\Psi(r_1,t)}{\partial r}\right)\right.\\
&\left.-\frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\left(\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\Psi(r_1,t)\right)\right] = \frac{\Gamma(\nu+3)}{\Gamma(\nu+1)}\,\frac{1}{1-\zeta^2}\int_{r_1}^{r_2}\Psi(r,\zeta)\,\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2}\\
&-2\,\cos\nu\pi\,C_\nu^{\frac{3}{2}}(\zeta)\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2},\quad \nu\neq-3,-4,\dots,
\end{aligned}\tag{5.103}$$

constitutes the generalized Dirichlet-to-Neumann correspondence for the interior of a spherical shell with inner radius r_1 and outer radius r_2. Indeed, evaluating the above relation at $\zeta = 1$ we find

$$
\begin{aligned}
&C_\nu^{\frac{3}{2}}(1)\left[\overline{R}(r_2;\nu)\,\mathfrak{N}^{(2)}(\nu|D_\nu^{\frac{3}{2}}) - \frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\mathfrak{D}^{(2)}(\nu|D_\nu^{\frac{3}{2}})\right]\\
&- C_\nu^{\frac{3}{2}}(1)\left[\overline{R}(r_1;\nu)\,\mathfrak{N}^{(1)}(\nu|D_\nu^{\frac{3}{2}}) - \frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\mathfrak{D}^{(1)}(\nu|D_\nu^{\frac{3}{2}})\right]\\
&= \frac{\Gamma(\nu+3)}{\Gamma(\nu+1)}\int_{r_1}^{r_2}\left(\lim_{\zeta\to 1}\frac{\Psi(r,\zeta)}{1-\zeta^2}\right)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2} - 2\,\cos\nu\pi\,C_\nu^{\frac{3}{2}}(1)\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2}\,,
\end{aligned}
\tag{5.104}
$$

where $\mathfrak{D}^{(j)}(\nu|D_\nu^{\frac{3}{2}}), \mathfrak{N}^{(j)}(\nu|D_\nu^{\frac{3}{2}})$, $j = 1, 2$ are given by (5.68).
Since $\Psi(r,\pm 1) = 0$, the first term of the right-hand side of (5.104) *yields an indetermined form. To overcome this obstacle, set $\Psi(r,\zeta) = (1-\zeta^2)\,\widetilde{\Psi}(r,\zeta)$ and the above relation reads*

$$
\begin{aligned}
&\left[\overline{R}(r_2;\nu)\,\mathfrak{N}^{(2)}(\nu|D_\nu^{\frac{3}{2}}) - \frac{d\overline{R}(r_2;\nu)}{\mathrm{d}\,r}\,\mathfrak{D}^{(2)}(\nu|D_\nu^{\frac{3}{2}})\right]\\
&- \left[\overline{R}(r_1;\nu)\,\mathfrak{N}^{(1)}(\nu|D_\nu^{\frac{3}{2}}) - \frac{d\overline{R}(r_1;\nu)}{\mathrm{d}\,r}\mathfrak{D}^{(1)}(\nu|D_\nu^{\frac{3}{2}})\right]\\
&= 2\int_{r_1}^{r_2}\left(\widetilde{\Psi}(r,1) - \cos\nu\pi\,\widetilde{\Psi}(r,-1)\right)\overline{R}(r;\nu)\,\frac{\mathrm{d}\,r}{r^2},\quad \nu \neq -3,-4,\dots\,,
\end{aligned}
\tag{5.105}
$$

where the fact that $\frac{\Gamma(\nu+3)}{\Gamma(\nu+1)} = 2\,C_\nu^{\frac{3}{2}}(1)$ is used, and (5.67b) is recovered. In order to obtain the correspondence (5.67a), one must replace $\overline{q}(r,\zeta;\nu)$ by $\overline{R}(r;\nu)\,D_\nu^{\frac{3}{2}}(\zeta)$ in the subdomain $\widetilde{\Omega}_3$ and by $\overline{R}(r;\nu)\,C_\nu^{\frac{3}{2}}(\zeta)$ in the subdomain $\widetilde{\Omega}_4$.

Introducing $\overline{R}(r;\nu) = r^{\nu+2}$ in (5.102) we obtain

$$
\begin{aligned}
&\int_{r_1}^{r_2}\Psi(r,\zeta)\,r^\nu\,\mathrm{d}\,r = (1-\zeta^2)\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\Big\{r_2^{\nu+1}\Big(r_2\,g_{N}^{(2)}(t) - (\nu+2)\,g_{D}^{(2)}(t)\Big)\\
&- r_1^{\nu+1}\Big(r_1\,g_{N}^{(1)}(t) - (\nu+2)\,g_{D}^{(1)}(t)\Big)\Big\} + 2(1-\zeta^2)\,\cos\nu\pi\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,C_\nu^{\frac{3}{2}}(\zeta)\\
&\times\int_{r_1}^{r_2}\widetilde{\Psi}(r,-1)\,r^\nu\,\mathrm{d}\,r,\quad \nu \neq -3,-4,\dots\,,
\end{aligned}
\tag{5.106}
$$

where else, replacing $\overline{R}(r;\nu)$ in (5.102) by $r^{-\nu-1}$ yields

$$\int_{r_1}^{r_2} \Psi(r,\zeta)\, r^{-\nu-3}\,\mathrm{d}\,r = (1-\zeta^2)\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\Big\{ r_2^{-\nu-2}\Big(r_2\, g_N^{(2)}(t) + (\nu+1)\, g_D^{(2)}(t)\Big) - r_1^{-\nu-2}\Big(r_1\, g_N^{(1)}(t) + (\nu+1)\, g_D^{(1)}(t)\Big)\Big\} + 2(1-\zeta^2)\,\cos\nu\pi\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,C_\nu^{\frac{3}{2}}(\zeta) \times \int_{r_1}^{r_2} \widetilde{\Psi}(r,-1)\, r^{-\nu-3}\,\mathrm{d}\,r, \quad \nu \neq -3,-4,\ldots . \tag{5.107}$$

Equation (5.106) is recognized as the Mellin transform for the function $r\,\Psi(r,\zeta)$. The inversion formula then implies

$$\begin{aligned}\Psi(r,\zeta) = {} & \frac{1-\zeta^2}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty} \Big(\frac{r_2}{r}\Big)^{\nu+1}\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\Big(r_2\, g_N^{(2)}(t) - (\nu+2)\, g_D^{(2)}(t)\Big)d\nu \\ & - \frac{1-\zeta^2}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty} \Big(\frac{r_1}{r}\Big)^{\nu+1}\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,\mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\Big(r_1\, g_N^{(1)}(t) - (\nu+2)\, g_D^{(1)}(t)\Big)d\nu \\ & + \frac{1-\zeta^2}{\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty} r^{-\nu-1}\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+3)}\,\cos\nu\pi\, C_\nu^{\frac{3}{2}}(\zeta)\Big[\int_{r_1}^{r_2}\widetilde{\Psi}(\tau,-1)\,\tau^\nu\,d\tau\Big]d\nu\,,\end{aligned} \tag{5.108}$$

where the integral is taken over any open contour Γ, connecting the points $\mathrm{Re}\,\nu - iR$ and $\mathrm{Re}\,\nu + iR$ in the complex $\nu-$plane as R tends to infinity. The unknow boundary values, depending on the problem at hand, are evaluated with the aid of equations (5.106) and (5.107). Substituting the resulting relations into (5.108) we find

$$\begin{aligned}\Psi_D(r,\zeta) = {} & \frac{1-\zeta^2}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty} \Big(\frac{r_2}{r}\Big)^{\nu+1}\left[1-\Big(\frac{r_2}{r_1}\Big)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)} \\ & \times \mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\left[\Big(\frac{r_2}{r_1}\Big)^{\nu+2} g_D^{(1)}(t) - g_D^{(2)}(t)\right]d\nu \\ & - \frac{1-\zeta^2}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty} \Big(\frac{r_1}{r}\Big)^{\nu+1}\left[\Big(\frac{r_1}{r_2}\Big)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)} \\ & \times \mathfrak{G}_\nu^{\frac{3}{2}}(\zeta)\left[g_D^{(1)}(t) - \Big(\frac{r_1}{r_2}\Big)^{\nu+2} g_D^{(2)}(t)\right]d\nu \\ & + \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\int_{r_1}^{r_2} r^{-\nu-1}\,\Psi(\tau,\zeta)\tau^\nu d\tau\, d\nu\,,\end{aligned} \tag{5.109}$$

if Dirichlet boundary values are prescribed, or

$$\Psi_{N}(r,\zeta) = -\,r_2\,\frac{1-\zeta^2}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{r_2}{r}\right)^{\nu+1}\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)^2(\nu+2)}$$
$$\times\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[\left(\frac{r_2}{r_1}\right)^{\nu+1} g_{N}^{(1)}(t) - g_{N}^{(2)}(t)\right]d\nu$$
$$+\,r_1\,\frac{1-\zeta^2}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\left(\frac{r_1}{r}\right)^{\nu+1}\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)^2(\nu+2)}$$
$$\times\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[g_{N}^{(1)}(t) - \left(\frac{r_1}{r_2}\right)^{\nu+1} g_{N}^{(2)}(t)\right]d\nu$$
$$+\,\frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu-i\infty}^{\mathrm{Re}\,\nu+i\infty}\int_{r_1}^{r_2} r^{-\nu-1}\,\Psi(\tau,\zeta)\tau^{\nu}d\tau\,d\nu\,, \qquad (5.110)$$

if Neumann data are available.

Remark 5.7.2 *It is straightforward to show that the following equalities hold*

$$\int_{Re\,\nu-i\infty}^{Re\,\nu+i\infty}\left(\frac{r_2}{r}\right)^{\nu+1}\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)}\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[\left(\frac{r_2}{r_1}\right)^{\nu+2} g_{D}^{(1)}(t)\right.$$
$$\left.-\,g_{D}^{(2)}(t)\right]d\nu = \int_{Re\,\nu-i\infty}^{Re\,\nu+i\infty}\left(\frac{r_1}{r}\right)^{\nu+1}\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)}\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[g_{D}^{(1)}(t)\right.$$
$$\left.-\left(\frac{r_1}{r_2}\right)^{\nu+2} g_{D}^{(2)}(t)\right]d\nu\,,$$

and

$$r_2\int_{Re\,\nu-i\infty}^{Re\,\nu+i\infty}\left(\frac{r_2}{r}\right)^{\nu+1}\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)^2(\nu+2)}\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[\left(\frac{r_2}{r_1}\right)^{\nu+1} g_{N}^{(1)}(t)\right.$$
$$\left.-\,g_{N}^{(2)}(t)\right]d\nu = r_1\int_{Re\,\nu-i\infty}^{Re\,\nu+i\infty}\left(\frac{r_1}{r}\right)^{\nu+1}\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)^2(\nu+2)}\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[g_{N}^{(1)}(t)\right.$$
$$\left.-\left(\frac{r_1}{r_2}\right)^{\nu+1} g_{N}^{(2)}(t)\right]d\nu\,,$$

which imply that

$$\Psi(r,\zeta) = \frac{1}{2\pi i}\int_{Re\,\nu-i\infty}^{Re\,\nu+i\infty}\int_{r_1}^{r_2} r^{-\nu-1}\,\Psi(\tau,\zeta)\tau^{\nu}d\tau\,d\nu$$

and follows from the definition of the Mellin transform.

Remark 5.7.3 *Since $\left(\frac{\tau}{r}\right)^{\nu}$ is an entire function, the line integral*

$$\int_{Re\,\nu-i\infty}^{Re\,\nu+i\infty}\left(\frac{\tau}{r}\right)^{\nu}\,d\nu\,,$$

is independent of the path. Hence, the following equality is valid

$$\int_{Re\,\nu-i\infty}^{Re\,\nu+i\infty}\left(\frac{\tau}{r}\right)^{\nu}\,d\nu=\frac{2i\,e^{Re\,\nu\,\ln\frac{\tau}{r}}}{\ln\tau-\ln r}\lim_{R\to\infty}\sin\left(\ln\frac{\tau}{r}\,R\right),\quad \tau\neq r\,.$$

If $\tau>r$, then $\left(\frac{\tau}{r}\right)^{\nu}$ tends to zero as $Re\,\nu\to-\infty$. On the other hand, if $\tau<r$, then $\left(\frac{\tau}{r}\right)^{\nu}$ vanishes as $Re\,\nu\to+\infty$. Therefore, depending on if $Re\,\nu$ tends to $\pm\infty$, the proper choice of the contour γ, obtained by a deformation of contour process, secures that

$$\int_{\gamma}\left(\frac{\tau}{r}\right)^{\nu}\,d\nu=0\,,$$

where the interchange of order of integration is justified by properly choosing $Re\,\nu$.

The Gegenbauer functions, as ν tends to infinity, behave as (see section 3.9)

$$\left.\begin{array}{rcl} C_{\nu}^{\frac{3}{2}}(\cos\theta) &=& \sqrt{\nu}\,\left(\overline{\Theta}_1(\theta)\,e^{i\nu\theta}+\Theta_1(\theta)\,e^{-i\nu\theta}\right)+\mathcal{O}\left(\frac{1}{\nu}\right)\\ D_{\nu}^{\frac{3}{2}}(\cos\theta) &=& \sqrt{\nu}\,\left(\overline{\Theta}_2(\theta)\,e^{i\nu\theta}+\Theta_2(\theta)\,e^{-i\nu\theta}\right)+\mathcal{O}\left(\frac{1}{\nu}\right)\end{array}\right\},\quad \mathrm{Re}\,\nu>0\,, \tag{5.111}$$

$$\left.\begin{array}{rcl} C_{-\nu-3}^{\frac{3}{2}}(\cos\theta) &=& \sqrt{\nu}\,\left(\overline{\Theta}_1(\theta)\,e^{i\nu\theta}+\Theta_1(\theta)\,e^{-i\nu\theta}\right)+\mathcal{O}\left(\frac{1}{\nu}\right)\\ D_{-\nu-3}^{\frac{3}{2}}(\cos\theta) &=& \sqrt{\nu}\,\cot\nu\pi\,\left(\Theta_3(\theta)\,e^{i\nu\theta}+\overline{\Theta}_3(\theta)\,e^{-i\nu\theta}\right)\end{array}\right\},\quad \mathrm{Re}\,\nu<0\,, \tag{5.112}$$

where Θ_j are complex functions of the variable θ alone, and $\overline{\Theta}_j$ denotes complex conjugation. The fact that $\cot\nu\pi$ is bounded enables us to simplify the above asymptotic expressions as follows

$$\left.\begin{array}{rcl} C_{\nu}^{\frac{3}{2}}(\theta) &\sim& e^{i\nu\theta}+e^{-i\nu\theta}\\ D_{\nu}^{\frac{3}{2}}(\theta) &\sim& e^{i\nu\theta}+e^{-i\nu\theta}\end{array}\right\},\quad \nu\to\infty\,.$$

This implies, that the Gegenbauer integral operator (5.101) behaves as

$$\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\sim\int_0^{\pi}\mathrm{d}\phi\left(e^{i(\theta+\phi)\nu}+e^{i(\theta-\phi)\nu}+e^{-i(\theta+\phi)\nu}+e^{-i(\theta-\phi)\nu}\right),\ \phi=\arccos t\,,$$

for large values of ν.
Furthermore, the denominators present in (5.109) and (5.110) behave as

$$\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\sim\begin{cases}1 & ,\mathrm{Re}\,\nu<0\\ -\left(\frac{r_1}{r_2}\right)^{2\nu+3} & ,\mathrm{Re}\,\nu>0\end{cases},\quad \nu\to\infty\,,$$

$$\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\sim\begin{cases}\left(\frac{r_2}{r_1}\right)^{2\nu+3} & ,\mathrm{Re}\,\nu<0\\ -1 & ,\mathrm{Re}\,\nu>0\end{cases},\quad \nu\to\infty\,.$$

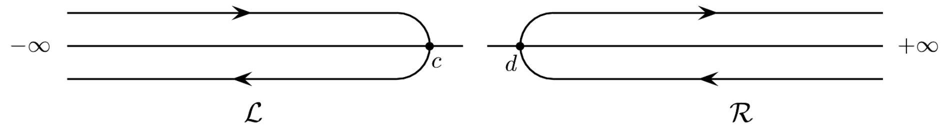

FIGURE 5.5: *The contours $\mathcal{L}$ and $\mathcal{R}$ respectively. The deformed contour $\mathcal{L}$ begins and ends in the left complex $\nu-$plane, such that Re $\nu \to -\infty$ at each end. Similar conclusions are valid for the contour $\mathcal{R}$. The constants c and d are arbitrarily chosen such that $c, d > -1$. The singularities at $\nu \neq -1, -2, \ldots$ are introduced via the Gegenbauer functions and the denominators present. Therefore, the initial contours Γ and Γ' are taken such that Re $\nu > -1$, which ensures that all singularities are to the left of the line Re $\nu = -1$.*

Analytic investigations of the inversion integral frequently depend on deforming the inversion contour to a more convenient one, therefore, replacing the contour Γ by either the contour $\mathcal{R}$ or $\mathcal{L}$ shown in Figure 5.5, depending on the boundness of the kernels, equations (5.109) and (5.110) read

$$\begin{aligned}
&\Psi_D(r,\zeta)\\
&= -\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}}\left(\frac{r_2}{r}\right)^{\nu+1}\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)}\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[\left(\frac{r_2}{r_1}\right)^{\nu+2} g_D^{(1)}(t)-g_D^{(2)}(t)\right]d\nu\\
&-\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}}\left(\frac{r_1}{r}\right)^{\nu+1}\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)}\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[g_D^{(1)}(t)-\left(\frac{r_1}{r_2}\right)^{\nu+2} g_D^{(2)}(t)\right]d\nu\,,
\end{aligned} \tag{5.113}$$

$$\begin{aligned}
&\Psi_N(r,\zeta)\\
&= r_2\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}}\left(\frac{r_2}{r}\right)^{\nu+1}\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)^2(\nu+2)}\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[\left(\frac{r_2}{r_1}\right)^{\nu+1} g_N^{(1)}(t)-g_N^{(2)}(t)\right]d\nu\\
&+r_1\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}}\left(\frac{r_1}{r}\right)^{\nu+1}\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)^2(\nu+2)}\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[g_N^{(1)}(t)-\left(\frac{r_1}{r_2}\right)^{\nu+1} g_N^{(2)}(t)\right]d\nu\,.
\end{aligned} \tag{5.114}$$

The deformed contours begin and end in the corresponding complex $\nu-$plane, such that Re ν tends to $\pm\infty$ at each end (a technique suggested by Talbot [Tal79] as part of a numerical scheme). A comprehensive list of methods of attack regarding techniques producing numerical answers, can be found in [Coh07].

Integral representations concerning the exterior domain Ω^e of the spherical shell $\mathcal{S}$, defined as

$$\Omega^e = \left\{(r,\zeta)\,\middle|\, r\in(0,r_1]\cup[r_2,\infty)\,;\,-1<\zeta<+1\right\}$$

are obtained in a similar fashion.

As r_1 tends to zero, the problem degenerates to the simplified case obtaining the stream function in the interior of a sphere of radius r_2. Thus, equations (5.109) and (5.110) rewrite as

$$\Psi^i_{\scriptscriptstyle D}(r,\zeta) = \frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}} \left(\frac{r_2}{r}\right)^{\nu+1} \frac{2\nu+3}{(\nu+1)(\nu+2)} \left(\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\, g_{\scriptscriptstyle D}(t)\right) d\nu\,, \tag{5.115}$$

for Dirichlet problems, or

$$\Psi^i_{\scriptscriptstyle N}(r,\zeta) = -r_2\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}} \left(\frac{r_2}{r}\right)^{\nu+1} \frac{2\nu+3}{(\nu+1)^2(\nu+2)} \left(\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\, g_{\scriptscriptstyle N}(t)\right) d\nu\,, \tag{5.116}$$

for Neumann data.

On the other hand, as r_2 tends to infinity, the exterior of a sphere with radius r_1 is obtained and thus, equations (5.109) and (5.110) yield

$$\Psi^e_{\scriptscriptstyle D}(r,\zeta) = \frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}} \left(\frac{r_1}{r}\right)^{\nu+1} \frac{2\nu+3}{(\nu+1)(\nu+2)} \left(\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\, g_{\scriptscriptstyle D}(t)\right) d\nu\,, \tag{5.117}$$

$$\Psi^e_{\scriptscriptstyle N}(r,\zeta) = -r_1\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}} \left(\frac{r_1}{r}\right)^{\nu+1} \frac{2\nu+3}{(\nu+1)^2(\nu+2)} \left(\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\, g_{\scriptscriptstyle N}(t)\right) d\nu\,. \tag{5.118}$$

Inverting (5.107), solutions valid in the counter part of the complex $\nu-$plane regarding (5.108) are obtained, viz

$$\begin{aligned}
&\Psi_{\scriptscriptstyle D}(r,\zeta)\\
&= \frac{1-\zeta^2}{2\pi i}\int_{\Gamma'} \left(\frac{r}{r_2}\right)^{\nu+2} \left[\left(\frac{r_1}{r_2}\right)^{2\nu+3} - 1\right]^{-1} \frac{2\nu+3}{(\nu+1)(\nu+2)}\,\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\left[\left(\frac{r_1}{r_2}\right)^{\nu+1} g^{(1)}_{\scriptscriptstyle D}(t) - g^{(2)}_{\scriptscriptstyle D}(t)\right] d\nu\\
&- \frac{1-\zeta^2}{2\pi i}\int_{\Gamma'} \left(\frac{r}{r_1}\right)^{\nu+2} \left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1} \frac{2\nu+3}{(\nu+1)(\nu+2)}\,\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\left[g^{(1)}_{\scriptscriptstyle D}(t) - \left(\frac{r_2}{r_1}\right)^{\nu+1} g^{(2)}_{\scriptscriptstyle D}(t)\right] d\nu\\
&+ \frac{1}{2\pi i}\int_{\Gamma'}\int_{r_1}^{r_2} r^{\nu+2}\,\Psi(\tau,\zeta)\tau^{-\nu-3} d\tau\, d\nu\,,
\end{aligned} \tag{5.119}$$

if Dirichlet boundary values are prescribed, or

$$\begin{aligned}
&\Psi_{\scriptscriptstyle N}(r,\zeta)\\
&= r_2\,\frac{1-\zeta^2}{2\pi i}\int_{\Gamma'} \left(\frac{r}{r_2}\right)^{\nu+2} \left[\left(\frac{r_1}{r_2}\right)^{2\nu+3} - 1\right]^{-1} \frac{2\nu+3}{(\nu+1)(\nu+2)^2}\,\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\left[\left(\frac{r_1}{r_2}\right)^{\nu+2} g^{(1)}_{\scriptscriptstyle N}(t) -, g^{(2)}_{\scriptscriptstyle N}(t)\right] d\nu\\
&- r_1\,\frac{1-\zeta^2}{2\pi i}\int_{\Gamma'} \left(\frac{r}{r_1}\right)^{\nu+2} \left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1} \frac{2\nu+3}{(\nu+1)(\nu+2)^2}\,\mathfrak{G}^{\frac{3}{2}}_{\nu}(\zeta)\left[g^{(1)}_{\scriptscriptstyle N}(t) - \left(\frac{r_2}{r_1}\right)^{\nu+2} g^{(2)}_{\scriptscriptstyle N}(t)\right] d\nu\\
&+ \frac{1}{2\pi i}\int_{\Gamma'}\int_{r_1}^{r_2} r^{\nu+2}\,\Psi(\tau,\zeta)\tau^{-\nu-3} d\tau\, d\nu\,,
\end{aligned} \tag{5.120}$$

if Neumann data are available, where Γ' is any open contour, connecting the points $-\mathrm{Re}\,\nu - iR$ and $-\mathrm{Re}\,\nu + iR$ in the complex $\nu-$plane as R tends to infinity.
Following the procedure introduced in this section, relation (5.119) becomes

$$\begin{aligned}
&\Psi_{D}(r,\zeta)\\
&= \frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}}\left(\frac{r}{r_2}\right)^{\nu+2}\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)}\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[\left(\frac{r_1}{r_2}\right)^{\nu+1}g_{D}^{(1)}(t)-g_{D}^{(2)}(t)\right]d\nu\\
&+\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}}\left(\frac{r}{r_1}\right)^{\nu+2}\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)}\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[g_{D}^{(1)}(t)-\left(\frac{r_2}{r_1}\right)^{\nu+1}g_{D}^{(2)}(t)\right]d\nu
\end{aligned} \tag{5.121}$$

if Dirichlet data are given.
In a similar fashion for Neumann data, (5.120) reads

$$\begin{aligned}
&\Psi_{N}(r,\zeta)\\
&= r_2\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}}\left(\frac{r}{r_2}\right)^{\nu+2}\left[\left(\frac{r_1}{r_2}\right)^{2\nu+3}-1\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)^2}\,\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[\left(\frac{r_1}{r_2}\right)^{\nu+2}g_{N}^{(1)}(t)-g_{N}^{(2)}(t)\right]d\nu\\
&+r_1\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}}\left(\frac{r}{r_1}\right)^{\nu+2}\left[1-\left(\frac{r_2}{r_1}\right)^{2\nu+3}\right]^{-1}\frac{2\nu+3}{(\nu+1)(\nu+2)^2}\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\left[g_{N}^{(1)}(t)-\left(\frac{r_2}{r_1}\right)^{\nu+2}g_{N}^{(2)}(t)\right]d\nu\,.
\end{aligned} \tag{5.122}$$

Moreover, by a limiting procedure, solutions valid for interior or exterior problems are retrieved, i.e.

$$\Psi_{D}^{i}(r,\zeta)=\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}}\left(\frac{r}{r_2}\right)^{\nu+2}\frac{2\nu+3}{(\nu+1)(\nu+2)}\left(\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\,g_{D}(t)\right)d\nu\,, \tag{5.123}$$

$$\Psi_{D}^{e}(r,\zeta)=\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}}\left(\frac{r}{r_1}\right)^{\nu+2}\frac{2\nu+3}{(\nu+1)(\nu+2)}\left(\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\,g_{D}(t)\right)d\nu\,, \tag{5.124}$$

in the case where Dirichlet data are given, or

$$\Psi_{N}^{i}(r,\zeta)=r_2\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{R}}\left(\frac{r}{r_2}\right)^{\nu+2}\frac{2\nu+3}{(\nu+1)(\nu+2)^2}\left(\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\,g_{N}(t)\right)d\nu\,, \tag{5.125}$$

$$\Psi_{N}^{e}(r,\zeta)=r_1\,\frac{1-\zeta^2}{2\pi i}\int_{\mathcal{L}}\left(\frac{r}{r_1}\right)^{\nu+2}\frac{2\nu+3}{(\nu+1)(\nu+2)^2}\left(\mathfrak{G}_{\nu}^{\frac{3}{2}}(\zeta)\,g_{N}(t)\right)d\nu\,, \tag{5.126}$$

if Neumann boundary values are available.

Appendices

APPENDIX A

How to obtain the solution $q(x,y)$ A step-by-step guide

Utilizing the definition of the spectral functions $\rho_j(k)$, equation (2.8) is rewritten as

$$\frac{\partial q}{\partial z} = \frac{1}{4\pi} \sum_{j=1}^{4} \int_{\ell_j} e^{ikz} \left(G^{(j)}(-ik) + i\Psi^{(j)}(-ik) \right) \mathrm{d}k \,, \tag{A.1}$$

where the Dirichlet $G^{(j)}(-ik)$ and Neumann $\Psi^{(j)}(-ik)$ transforms are given as

$$G^{(j)}(-ik) = \int_{-L}^{L} e^{-ikz^{(j)}(s)} \, \partial_{T} \, q^{(j)}(s) \, \mathrm{d}s, \quad \Psi^{(j)}(-ik) = \int_{-L}^{L} e^{-ikz^{(j)}(s)} \, \partial_{N} \, q^{(j)}(s) \, \mathrm{d}s \,. \tag{A.2}$$

A suitable parametrization for the complex variable z on each side (j) of the square is

$$z^{(j)}(s) = (-i)^{j-1} \, L - (-i)^{j} \, s \,, \quad s \in [-L, L] \,.$$

Replacing the given parametrization into equations (A.2) and bearing in mind that (see sections 2.3 and 2.4)

$$\partial_{T} \, q^{(j)}(s) = \frac{\mathrm{d}}{\mathrm{d}s} f_{D}^{(j)}(s) = \frac{\mathrm{d}}{\mathrm{d}s} \sum_{n} \left[A_n^{(j)} \sin\left(\tfrac{n\pi}{L} s\right) + B_n^{(j)} \cos\left(\left(n + \tfrac{1}{2}\right) \tfrac{\pi}{L} s \right) \right]$$
$$\partial_{N} \, q^{(j)}(s) = \sum_{n} \left[\mathfrak{A}_n^{(j)} \sin\left(\tfrac{n\pi}{L} s\right) + \mathfrak{B}_n^{(j)} \cos\left(\left(n + \tfrac{1}{2}\right) \tfrac{\pi}{L} s \right) \right] ,$$

we find

$$G^{(j)}(-ik) = \frac{k}{2}\,(-i)^{j+1}\left(e^{(1-i)(-i)^j kL} - e^{(1+i)(-i)^j kL}\right)\sum_n (-1)^n\, A_n^{(j)}\left(\frac{1}{(-i)^j k + \frac{n\pi}{L}} - \frac{1}{(-i)^j k - \frac{n\pi}{L}}\right) - \frac{k}{2}\,(-i)^{j+1}\left(e^{(1-i)(-i)^j kL} + e^{(1+i)(-i)^j kL}\right)\sum_n (-1)^n\, B_n^{(j)} \times \left(\frac{1}{(-i)^j k + \frac{(n+\frac{1}{2})\pi}{L}} - \frac{1}{(-i)^j k - \frac{(n+\frac{1}{2})\pi}{L}}\right), \tag{A.3}$$

where the Fourier coefficients $A_n^{(j)}$ and $B_n^{(j)}$ are given by equations (2.12)-(2.19), as

$$\begin{array}{llll} A_n^{(1)} = a_n, & A_n^{(2)} = e_n, & A_n^{(3)} = c_n, & A_n^{(4)} = g_n \\ B_n^{(1)} = b_n, & B_n^{(2)} = f_n, & B_n^{(3)} = d_n, & B_n^{(4)} = h_n \end{array}, \tag{A.4}$$

and

$$\Psi^{(j)}(-ik) = \frac{1}{2}\left(e^{(1-i)(-i)^j kL} - e^{(1+i)(-i)^j kL}\right)\sum_n (-1)^n\, \mathfrak{A}_n^{(j)}\left(\frac{1}{(-i)^j k + \frac{n\pi}{L}} - \frac{1}{(-i)^j k - \frac{n\pi}{L}}\right) + \frac{1}{2}\left(e^{(1-i)(-i)^j kL} + e^{(1+i)(-i)^j kL}\right)\sum_n (-1)^n\, \mathfrak{B}_n^{(j)}\left(\frac{1}{(-i)^j k + \frac{(n+\frac{1}{2})\pi}{L}} - \frac{1}{(-i)^j k - \frac{(n+\frac{1}{2})\pi}{L}}\right) \tag{A.5}$$

where the Fourier coefficients $\mathfrak{A}_n^{(j)}$ and $\mathfrak{B}_n^{(j)}$ are given by equations (2.28)-(2.35), as

$$\begin{array}{llll} \mathfrak{A}_n^{(1)} \to (2.28), & \mathfrak{A}_n^{(2)} \to (2.30), & \mathfrak{A}_n^{(3)} \to (2.32), & \mathfrak{A}_n^{(4)} \to (2.34), \\ \mathfrak{B}_n^{(1)} \to (2.29), & \mathfrak{B}_n^{(2)} \to (2.31), & \mathfrak{B}_n^{(3)} \to (2.33), & \mathfrak{B}_n^{(4)} \to (2.35) \end{array}. \tag{A.6}$$

Putting everything into (A.1) we obtain

$$\frac{\partial q}{\partial z} = \frac{1}{8\pi}\sum_n (-1)^n \sum_{j=1}^{4}\int_{\ell_j}\left\{\left(k\,(-i)^{j+1}\,A_n^{(j)} + i\,\mathfrak{A}_n^{(j)}\right)\left(e^{\left(iz+(1-i)(-i)^j L\right)k} - e^{\left(iz+(1+i)(-i)^j L\right)k}\right)\left(\frac{1}{(-i)^j k + \frac{n\pi}{L}} - \frac{1}{(-i)^j k - \frac{n\pi}{L}}\right) + \left(i\,\mathfrak{B}_n^{(j)} - k\,(-i)^{j+1}\,B_n^{(j)}\right) \times \left(e^{\left(iz+(1-i)(-i)^j L\right)k} + e^{\left(iz+(1+i)(-i)^j L\right)k}\right)\left(\frac{1}{(-i)^j k + \frac{(n+\frac{1}{2})\pi}{L}} - \frac{1}{(-i)^j k - \frac{(n+\frac{1}{2})\pi}{L}}\right)\right\}\mathrm{d}k\,. \tag{A.7}$$

From this point on, due to the extensiveness of the calculations and the length of the resulting formulas, the sequence will be presented as an example.

Proposition A.0.1 *Let $q(x,y)$ satisfy the Laplace equation in the interior Ω of a square defined by*

$$\Omega = \Big\{ -L \le x \le L,\; -L \le y \le L \Big\},$$

with Dirichlet boundary values $f_D^{(1)} = \sin\frac{\pi}{L}s$ *on side* 1 *and zero on the remaining sides. The solution by means of separation of variables is then given as*

$$q(x,y) = \frac{1}{\sinh 2\pi}\,\sinh\Big(\tfrac{\pi}{L}(x+L)\Big)\,\sin\tfrac{\pi}{L}\,y\,. \tag{A.8}$$

Employing eq. (2.36) one finds

$$\begin{aligned}
\partial_N\, q^{(1)}(s) &= \frac{\pi}{L}\,\frac{\cosh 2\pi}{\sinh 2\pi}\,\sin\tfrac{\pi}{L}\,s \\
\partial_N\, q^{(2)}(s) &= -\frac{1}{L}\sum_{n=0}^{\infty}(-1)^n\frac{n}{1+n^2}\,\sin\left(\tfrac{n\pi}{L}\,s\right) + \frac{1}{L}\sum_{n=0}^{\infty}(-1)^n\frac{\left(n+\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)^2}\,\cos\Big(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}\,s\Big) \\
\partial_N\, q^{(3)}(s) &= \frac{\pi}{L}\,\frac{1}{\sinh 2\pi}\,\sin\tfrac{\pi}{L}\,s \\
\partial_N\, q^{(4)}(s) &= -\partial_N\, q^{(2)}(-s)\,.
\end{aligned}$$

From the definitions of the Dirichlet and Neumann transforms of the boundary data, we obtain

$$\begin{aligned}
G^{(1)}(k) &= \int_{-L}^{L} e^{ks}\,\partial_T\, q^{(1)}(s)\mathrm{d}\,s = \int_{-L}^{L} e^{ks}\,\mathrm{d}\left(\sin\tfrac{\pi}{L}s\right) \\
&= -\frac{\pi}{2L}\,\left(e^{kL}-e^{-kL}\right)\left(\frac{1}{k+i\frac{\pi}{L}}+\frac{1}{k-i\frac{\pi}{L}}\right)
\end{aligned}$$

$$\Psi^{(1)}(k) = -\frac{\pi}{2iL}\,\frac{\cosh 2\pi}{\sinh 2\pi}\,\left(e^{kL}-e^{-kL}\right)\left(\frac{1}{k+i\frac{\pi}{L}}-\frac{1}{k-i\frac{\pi}{L}}\right)\,,$$

$$\begin{aligned}
\Psi^{(2)}(-i\,k) = &-\frac{\pi}{2iL}\,\frac{1}{k-i\frac{\pi}{L}}\left[e^{-ikL}\left(1-\frac{\cosh 2\pi}{\sinh 2\pi}\right)+\frac{e^{ikL}}{\sinh 2\pi}\right] \\
&+\frac{\pi}{2iL}\,\frac{1}{k+i\frac{\pi}{L}}\left[\frac{e^{ikL}}{\sinh 2\pi}-e^{-ikL}\left(\frac{\cosh 2\pi}{\sinh 2\pi}+1\right)\right]\,,
\end{aligned}$$

$$\Psi^{(3)}(-k) = \frac{\pi}{2iL}\,\frac{e^{kL}-e^{-kL}}{\sinh 2\pi}\,\left(\frac{1}{k+i\frac{\pi}{L}}-\frac{1}{k-i\frac{\pi}{L}}\right)\,,$$

$$\begin{aligned}
\Psi^{(4)}(i\,k) = &-\frac{\pi}{2i\,L}\,\frac{1}{k+i\frac{\pi}{L}}\left[\frac{e^{ikL}}{\sinh 2\pi}-e^{-ikL}\left(\frac{\cosh 2\pi}{\sinh 2\pi}+1\right)\right] \\
&+\frac{\pi}{2i\,L}\,\frac{1}{k-i\frac{\pi}{L}}\left[\frac{e^{ikL}}{\sinh 2\pi}+e^{-ikL}\left(1-\frac{\cosh 2\pi}{\sinh 2\pi}\right)\right]\,,
\end{aligned}$$

which substituted into eq. (A.1) yields

$$\begin{aligned}
4\pi\,\frac{\partial q}{\partial z} = &\int_0^{-i\infty} e^{(i\,z-i\,L)k}\,\Big(G^{(1)}(k)+i\Psi^{(1)}(k)\Big)\,\mathrm{d}k + i\int_0^{\infty} e^{(i\,z-L)k}\,\Psi^{(2)}(k)\mathrm{d}k \\
&+ i\int_0^{i\infty} e^{(i\,z+i\,L)k}\,\Psi^{(3)}(k)\mathrm{d}k + i\int_0^{-\infty} e^{(i\,z+L)k}\,\Psi^{(4)}(k)\mathrm{d}k \quad\Leftrightarrow
\end{aligned}$$

$$8L\,\frac{\partial q}{\partial z} = -\left(\frac{\cosh 2\pi}{\sinh 2\pi}+1\right)\mathcal{I}_1 - \left(1-\frac{\cosh 2\pi}{\sinh 2\pi}\right)\mathcal{I}_2 - \frac{1}{\sinh 2\pi}\,\mathcal{I}_3$$

and $\mathcal{I}_k,\ k=1,2,3$ the following integrals to be evaluated

$$\begin{aligned}\mathcal{I}_1 = &\int_0^{-i\infty}\frac{e^{(iz+L-iL)k}}{k+i\frac{\pi}{L}}\mathrm{d}k - \int_0^{-\infty}\frac{e^{(iz+L-iL)k}}{k+i\frac{\pi}{L}}\mathrm{d}k\\ &-\int_0^{-i\infty}\frac{e^{(iz-L-iL)k}}{k+i\frac{\pi}{L}}\mathrm{d}k + \int_0^{\infty}\frac{e^{(iz-L-iL)k}}{k+i\frac{\pi}{L}}\mathrm{d}k\,,\end{aligned}$$

$$\begin{aligned}\mathcal{I}_2 = &\int_0^{-i\infty}\frac{e^{(iz-iL+L)k}}{k-i\frac{\pi}{L}}\mathrm{d}k - \int_0^{-\infty}\frac{e^{(iz-iL+L)k}}{k-i\frac{\pi}{L}}\mathrm{d}k\\ &-\int_0^{-i\infty}\frac{e^{(iz-iL-L)k}}{k-i\frac{\pi}{L}}\mathrm{d}k + \int_0^{\infty}\frac{e^{(iz-iL-L)k}}{k-i\frac{\pi}{L}}\mathrm{d}k\,,\end{aligned}$$

$$\begin{aligned}\mathcal{I}_3 = &\int_0^{\infty}\frac{e^{(iz-L+iL)k}}{k-i\frac{\pi}{L}}\mathrm{d}k - \int_0^{i\infty}\frac{e^{(iz-L+iL)k}}{k-i\frac{\pi}{L}}\mathrm{d}k\\ &-\int_0^{\infty}\frac{e^{(iz-L+iL)k}}{k+i\frac{\pi}{L}}\mathrm{d}k + \int_0^{i\infty}\frac{e^{(iz+iL-L)k}}{k+i\frac{\pi}{L}}\mathrm{d}k\\ &+\int_0^{i\infty}\frac{e^{(iz+iL+L)k}}{k-i\frac{\pi}{L}}\mathrm{d}k - \int_0^{-\infty}\frac{e^{(iz+iL+L)k}}{k-i\frac{\pi}{L}}\mathrm{d}k\\ &-\int_0^{i\infty}\frac{e^{(iz+iL+L)k}}{k+i\frac{\pi}{L}}\mathrm{d}k + \int_0^{-\infty}\frac{e^{(iz+iL+L)k}}{k+i\frac{\pi}{L}}\mathrm{d}k\,.\end{aligned}$$

Grouping the above integrals and applying Cauchy's theorem combined with the calculus of residues, it is straightforward to show that

$$\begin{aligned}\mathcal{I}_1 &= 2i\pi\,e^{-\pi}\,e^{\frac{\pi}{L}\,z}\\ \mathcal{I}_2 &= 0\\ \mathcal{I}_3 &= -2i\pi\,e^{-\pi}\,e^{-\frac{\pi}{L}\,z}\,,\end{aligned}$$

and therefore

$$\partial_z\,q(z) = i\frac{\pi}{4L}\,\frac{e^{-\pi}}{\sinh 2\pi}\left(e^{-\frac{\pi}{L}\,z} - e^{2\pi}\,e^{\frac{\pi}{L}\,z}\right)\,. \tag{A.9}$$

Integrating the above expression yields

$$q(z) = \frac{1}{2\sinh 2\pi}\,\sinh\left(\tfrac{\pi}{L}(x+L)\right)\,\sin\tfrac{\pi}{L}\,y - i\frac{1}{2\sinh 2\pi}\,\cosh\left(\tfrac{\pi}{L}(x+L)\right)\,\cos\tfrac{\pi}{L}\,y \tag{A.10}$$

Applying, at last, the reality condition $q(x,y) = 2\mathrm{Re}\,q(z)$, equation (A.8) is recovered.

Remark A.0.4 *Note, that the imaginary part of* (A.10) *is also a eigenfunction of the Laplacian operator.*

APPENDIX B

Evaluation of the limit $(1-x^2)\,\frac{d}{\mathrm{d}x}\,Q_\nu(x)$ as x tends to -1^+

The Legendre functions of the second kind admit the series expansion [Leb72, p.179]

$$Q_\nu(x) = \frac{\Gamma(\frac{\nu}{2}+1)\,\sqrt{\pi}\,\cos\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+1}{2})}\,x\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right) - \frac{\Gamma(\frac{\nu+1}{2})\,\sqrt{\pi}\,\sin\frac{\pi\nu}{2}}{2\Gamma(\frac{\nu+2}{2})}\,F\left(\frac{\nu+1}{2},-\frac{\nu}{2},\frac{1}{2};x^2\right)\,, \tag{B.1}$$

valid in the interval $-1 < x < +1$, for every complex $\nu \neq -1, -2, \dots$.
Differentiating the above expression with respect to the argument and using the chain rule

$$\frac{d}{\mathrm{d}x} = 2x\frac{d}{\mathrm{d}x^2}\ ,$$

we arrive at

$$\frac{d}{\mathrm{d}x}\,Q_\nu(x) = \frac{\Gamma(\frac{\nu}{2}+1)\,\sqrt{\pi}\,\cos\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+1}{2})}\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right) + \frac{\Gamma(\frac{\nu}{2}+1)\,\sqrt{\pi}\,\cos\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+1}{2})}\,\frac{(1-\nu)(\nu+2)}{3}\,x^2\,F\left(\frac{3-\nu}{2},\frac{\nu+4}{2},\frac{5}{2};x^2\right) + \frac{\Gamma(\frac{\nu+1}{2})\,\sqrt{\pi}\,\sin\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+2}{2})}\,\frac{\nu(\nu+1)}{2}\,x\,F\left(\frac{\nu+3}{2},\frac{2-\nu}{2},\frac{3}{2};x^2\right),\quad \nu\neq -1,-2,\dots\ , \tag{B.2}$$

where the property

$$\frac{d}{\mathrm{d}x}F\left(\alpha,\beta,\gamma;x\right) = \frac{\alpha\,\beta}{\gamma}\,F\left(\alpha+1,\beta+1,\gamma+1;x\right) \tag{B.3}$$

has been used. Employing the formula

$$F\left(\alpha,\beta,\gamma;z\right)=(1-z)^{\gamma-\alpha-\beta}\,F\left(\gamma-\alpha,\gamma-\beta,\gamma;z\right), \tag{B.4}$$

valid in the domain

$$|\arg\left(1-z\right)|<\pi\;, \tag{B.5}$$

on the last two terms of the right-hand side of (B.2), we obtain

$$F\left(\frac{3-\nu}{2},\frac{\nu+4}{2},\frac{5}{2};x^2\right)=\frac{1}{1-x^2}\,F\left(\frac{2+\nu}{2},\frac{1-\nu}{2},\frac{5}{2};x^2\right) \tag{B.6}$$

$$F\left(\frac{\nu+3}{2},\frac{2-\nu}{2},\frac{3}{2};x^2\right)=\frac{1}{1-x^2}\,F\left(-\frac{\nu}{2},\frac{1+\nu}{2},\frac{3}{2};x^2\right)\;. \tag{B.7}$$

In both cases (B.6) and (B.7) we notice that

$$\mathrm{Re}(\gamma-\alpha-\beta)>0.$$

Furthermore, note that condition (B.5) excludes the ray $[1,+\infty)$ from the complex $\nu-$plane. Eq. (B.2), multiplied by $1-x^2$, then rewrites

$$\begin{aligned}(1-x^2)\frac{d}{\mathrm{d}x}\,Q_\nu(x)=&\frac{\Gamma(\frac{\nu}{2}+1)\sqrt{\pi}\,\cos\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+1}{2})}\,(1-x^2)\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right)\\&+\frac{\Gamma(\frac{\nu}{2}+1)\sqrt{\pi}\,\cos\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+1}{2})}\,\frac{(1-\nu)(\nu+2)}{3}\,x^2\,F\left(\frac{2+\nu}{2},\frac{1-\nu}{2},\frac{5}{2};x^2\right)\\&+\frac{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\,\sin\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+2}{2})}\,\frac{\nu(\nu+1)}{2}\,x\,F\left(-\frac{\nu}{2},\frac{1+\nu}{2},\frac{3}{2};x^2\right)\;,\\&\qquad -1<x<+1\;,\quad \nu\neq-1,-2,\ldots\;.\end{aligned} \tag{B.8}$$

As x tends to -1^+, and bearing in mind that

$$\lim_{x\to-1^+}F\left(\alpha,\beta,\gamma;x^2\right)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\;,\quad \mathrm{Re}(\gamma-\alpha-\beta)>0, \tag{B.9}$$

equation (B.8) deduces to

$$\begin{aligned}\lim_{x\to-1^+}(1-x^2)\frac{d}{\mathrm{d}x}\,Q_\nu(x)=&\frac{\Gamma(\frac{\nu}{2}+1)\sqrt{\pi}\,\cos\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+1}{2})}\,\lim_{x\to-1^+}(1-x^2)\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right)\\&+\frac{\Gamma(\frac{\nu}{2}+1)\sqrt{\pi}\,\cos\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+1}{2})}\,\frac{(1-\nu)(\nu+2)}{3}\,\frac{\Gamma(\frac{5}{2})\,\Gamma(1)}{\Gamma(\frac{3-\nu}{2})\,\Gamma(\frac{\nu+4}{2})}\\&-\frac{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\,\sin\frac{\pi\nu}{2}}{\Gamma(\frac{\nu+2}{2})}\,\frac{\nu(\nu+1)}{2}\,\frac{\Gamma(\frac{3}{2})\,\Gamma(1)}{\Gamma(\frac{3+\nu}{2})\,\Gamma(\frac{2-\nu}{2})}\;,\\&\qquad -1<x<+1\;,\quad \nu\neq-1,-2,\ldots\;.\end{aligned} \tag{B.10}$$

The above relation simplifies further, using the identities

$$\Gamma(1+\nu) = \nu\,\Gamma(\nu),$$
$$\Gamma(1+\nu)\,\Gamma(1-\nu) = \frac{\pi\nu}{\sin\pi\nu}\ ,$$

as well as the well known values

$$\Gamma(1) = 1,$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ .$$

Thus, we find that

$$\lim_{x\to -1^+}(1-x^2)\frac{d}{\mathrm{d}x}\,Q_\nu(x) = \frac{\Gamma(\frac{\nu}{2}+1)\,\sqrt{\pi}\,\cos\frac{\nu\pi}{2}}{\Gamma(\frac{\nu+1}{2})}\lim_{x\to -1^+}(1-x^2)$$
$$\times\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right) + \cos\pi\nu, \quad -1 < x < +1\ , \quad \nu \neq -1,-2,\dots\ . \tag{B.11}$$

Next we evaluate the limit

$$\lim_{x\to -1^+}(1-x^2)\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right),$$

considering first the hypergeometric function $F\left(-\frac{\nu+1}{2},\frac{\nu}{2},\frac{1}{2};x^2\right)$.
Differentiating the hypergeometric function with respect to the argument, we immediately obtain the relation

$$\frac{d}{d(x^2)}F\left(-\frac{\nu+1}{2},\frac{\nu}{2},\frac{1}{2};x^2\right) = -\frac{\nu\,(\nu+1)}{2}\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right)\ . \tag{B.12}$$

Integrating over the interval $[0,x^2]$ the above expression we obtain

$$F\left(-\frac{\nu+1}{2},\frac{\nu}{2},\frac{1}{2};x^2\right) = 1 - \frac{\nu\,(\nu+1)}{2}\int_0^{x^2} F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right)\,d(x^2)\ . \tag{B.13}$$

As x tends to -1^+ the left-hand side of (B.13) remains bounded and so must the right-hand side. This implies that $F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right)$ as x tends to -1^+, enjoys the asymptotic behavior

$$F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right) \sim \frac{1}{(1-x^2)^p}, \quad 0 < p < 1\ , \tag{B.14}$$

and

$$\lim_{x\to -1^+}(1-x^2)\,F\left(\frac{1-\nu}{2},\frac{\nu+2}{2},\frac{3}{2};x^2\right) = 0.$$

Putting everything together we obtain

$$\lim_{x\to -1^+}(1-x^2)\,\frac{d}{\mathrm{d}x}\,Q_\nu(x) = \cos\pi\nu, \quad \nu \neq -1,-2,\dots\quad .$$

Remark B.0.5 *Following a similar procedure, it can be shown that*

$$\lim_{x\to+1^-}(1-x^2)\,\frac{d}{\mathrm{d}x}\,Q_\nu(x)=1,\quad \nu\neq -1,-2,\ldots \quad ,$$

and

$$\lim_{x\to-1^+}(1-x^2)\frac{d}{\mathrm{d}x}\,P_\nu(x)=\frac{2}{\pi}\,\sin\pi\nu\,.$$

APPENDIX C

Evaluation of certain Integrals

C.1 The integral $\int_{\nu_R-i\infty}^{\nu_R+i\infty} e^{\alpha\nu}\, \mathcal{F}(\nu)\,\mathrm{d}\nu$

Consider the integral

$$\oint_{\mathcal{C}} e^{\alpha\,\nu}\, \mathcal{F}(\nu)\,\mathrm{d}\nu, \quad \alpha \in \mathbb{R}\,,$$

where $\mathcal{F}(\nu)$ is a rational function with only poles located on the negative real axis and α is a real parameter.
Since $\mathcal{F}(\nu)$ is a rational function, we have

$$\mathcal{F}(\nu) = \frac{a_n\,\nu^n + \ldots + a_1\,\nu + a_0}{b_m\,\nu^m + \ldots + b_1\,\nu + b_0} = \frac{a_n}{b_m}\,\frac{1}{\nu^{m-n}}\,\frac{1+\ldots+\frac{a_0}{a_n\,\nu^n}}{1+\ldots+\frac{b_0}{b_m\,\nu^m}}\,, \tag{C.1}$$

and therefore

$$|\mathcal{F}(\nu)| = \left|\frac{a_n}{b_m}\right|\,\frac{1}{|\nu|^{m-n}}\,\left|\frac{1+\ldots+\frac{a_0}{a_n\,\nu^n}}{1+\ldots+\frac{b_0}{b_m\,\nu^m}}\right|,$$

from which

$$|\mathcal{F}(\nu)| \leq \frac{M}{|\nu|^{m-n}}\,, \tag{C.2}$$

for all sufficiently large $|\nu|$, where M is a positive constant. If $a_0 = 0$ or $b_0 = 0$ (or both), (C.1) has to be modified.
In the sequence, the case where $m = n$ is examined (for $m - n \geq 2$ see [Mar85]). If $m = n$, then from (C.2) obviously

$$|\mathcal{F}(\nu)| \leq M\,.$$

Integrating over a rectangle as shown on Fig. C.1, taking ϵ and c sufficiently large, the integral rewrites

$$\oint_{\mathcal{C}} e^{\alpha\,\nu}\, \mathcal{F}(\nu)\,\mathrm{d}\nu = \left(\int_{\nu_R-ic}^{\nu_R+ic} + \int_{\mathcal{P}_1\cup\mathcal{P}_2} + \int_{\ell} + \int_{\mathcal{P}_3\cup\mathcal{P}_4}\right) e^{\alpha\,\nu}\, \mathcal{F}(\nu)\,\mathrm{d}\nu\,, \tag{C.3}$$

where ℓ is a contour in the left complex $\nu-$plane avoiding all singularities. If ℓ is chosen in such a way, the integral (C.3) equals zero, since no singularities are encircled by the contour $\mathcal{C}$.

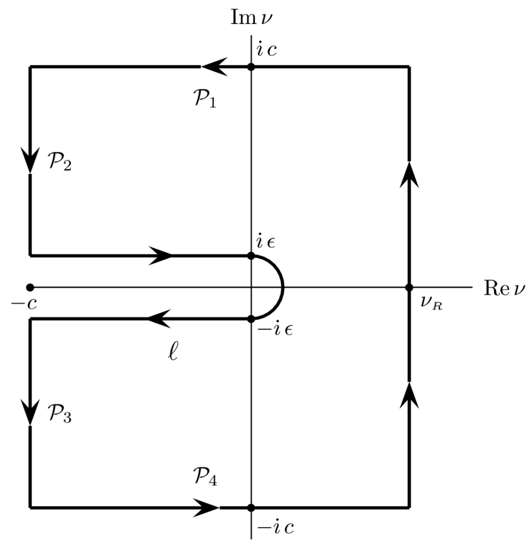

FIGURE C.1: *The closed contour $\mathcal{C}$ appropriate in the case where $\alpha > 0$.*

Thus,

$$\begin{aligned} I(c) &= \left| \left(\int_{\mathcal{P}_1} + \int_{\mathcal{P}_2} + \int_{\mathcal{P}_3} + \int_{\mathcal{P}_4} \right) e^{\alpha\,\nu}\, \mathcal{F}(\nu)\,\mathrm{d}\nu \right| \\ &\leq \int_{\mathcal{P}_1} e^{\alpha\,x}\, |\mathcal{F}(\nu)|\,\mathrm{d}x + \int_{\mathcal{P}_2} e^{-\alpha\,c}\, |\mathcal{F}(\nu)|\,\mathrm{d}y + \int_{\mathcal{P}_3} e^{-\alpha\,c}\, |\mathcal{F}(\nu)|\,\mathrm{d}y + \int_{\mathcal{P}_4} e^{\alpha\,x}\, |\mathcal{F}(\nu)|\,\mathrm{d}x \\ &\leq M\, 2e^{-\alpha\,c}\,(\epsilon - c)\,. \end{aligned}$$

Letting $c \to \infty$, implies $I \to 0$ if $\alpha > 0$, and, furthermore, $\ell \to \mathcal{L}$.
From (C.3) it follows then, that (schematically)

$$\int_{\nu_R - i\infty}^{\nu_R + i\infty} = - \int_{\mathcal{L}} \to 0\,,$$

which comes from the fact that the integrand is *analytic* and *bounded*, due to the presence of the exponential $e^{\alpha\,\nu}$, $\alpha > 0$, in the left complex $\nu-$plane.

C.2 THE INTEGRAL $\int_{\nu_R - i\infty}^{\nu_R + i\infty} e^{\alpha\,\nu}\, \cos\nu\pi\, P_\nu(x)\,\mathrm{d}\nu$

Consider next the integral

$$\oint_{\mathcal{C}} e^{\alpha\,\nu}\, \cos\nu\pi\, P_\nu(x)\,\mathrm{d}\nu, \quad \alpha \in \mathbb{R}\,.$$

Remark C.2.1 *The Legendre functions of the first kind are defined as*

$$\begin{aligned} P_\nu(x) &= F\left(-\nu, \nu+1, 1; \tfrac{1-x}{2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-\nu)_n\,(\nu+1)_n}{(1)_n} \left(\frac{1-x}{2}\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\nu+n+1)}{n!\,\Gamma(\nu-n+1)} \left(\frac{1-x}{2}\right)^n , \end{aligned}$$

from which it is obvious that $P_\nu(x)$ has poles at $\nu + n = -1, -2 \ldots$.

Remark C.2.2 *The Legendre functions of the first kind for large values of ν behave as*

$$P_\nu(x) \sim \frac{1}{\sqrt{\nu}}\Big(A(\theta)\, e^{i\theta\nu} + B(\theta)\, e^{-i\theta\nu}\Big), \quad x = \cos\theta ,$$

where $A(\theta)$ and $B(\theta)$ the complex functions of the variable θ alone.

From the above asymptotic expansion it can be seen that the dominant factors are the exponentials, i.e.

$$\frac{1}{\sqrt{\nu}}\Big(A(\theta)\, e^{i\theta\nu} + B(\theta)\, e^{-i\theta\nu}\Big) \sim \begin{cases} e^{i\theta\nu} & , \mathrm{Im}\,\nu \to -\infty \\ e^{-i\theta\nu} & , \mathrm{Im}\,\nu \to +\infty \end{cases} . \tag{C.4}$$

Integrating over a rectangle as shown on Fig. C.1, taking ϵ and c sufficiently large, the integral rewrites

$$\oint_{\mathcal{C}} e^{\alpha\,\nu}\, \cos\nu\pi\, P_\nu(x)\, \mathrm{d}\nu = \left(\int_{\nu_R - ic}^{\nu_R + ic} + \int_{\mathcal{P}_1 \cup \mathcal{P}_2} + \int_{\ell} + \int_{\mathcal{P}_3 \cup \mathcal{P}_4}\right) e^{\alpha\,\nu}\, \cos\nu\pi\, P_\nu(x)\, \mathrm{d}\nu , \tag{C.5}$$

which equals zero since no singularities are encircled by the contour $\mathcal{C}$.
Evaluating the above integrals one finds

$$\begin{aligned} I =& \left| \left(\int_{\mathcal{P}_1} + \int_{\mathcal{P}_2} + \int_{\mathcal{P}_3} + \int_{\mathcal{P}_4}\right) e^{\alpha\,\nu}\, \cos\nu\pi\, P_\nu(x)\, \mathrm{d}\nu \right| \\ &\leq 2e^{-\alpha\, c} \left[-\frac{1}{\pi+\theta}\, e^{-(\pi+\theta)\epsilon} - \frac{1}{\pi-\theta}\, e^{-(\pi+\theta)\epsilon} + \frac{1}{\pi-\theta}\, e^{(\pi-\theta)\epsilon} + \frac{1}{\pi+\theta}\, e^{(\pi+\theta)\epsilon} \right] \\ &+ 2e^{-\alpha\, c} \left[\frac{1}{\pi+\theta}\, e^{-(\pi+\theta)\, c} + \frac{1}{\pi-\theta}\, e^{-(\pi+\theta)\, c} - \frac{1}{\pi-\theta}\, e^{(\pi-\theta)\, c} - \frac{1}{\pi+\theta}\, e^{(\pi+\theta)\, c} \right] . \end{aligned}$$

As c tends to infinity, we see that $I \to 0$, if $\alpha \geq 2\pi$, emanating from the fact that, since $0 < \theta < \pi$, $0 < \pi - \theta < \pi$ and $\pi < \pi + \theta < 2\pi$. Therefore, we obtain (schematically)

$$\int_{\nu_R - i\infty}^{\nu_R + i\infty} = -\int_{\mathcal{L}} \to 0 .$$

Similar conclusions hold for a contour $\mathcal{C}$ closed in the right complex $\nu-$plane, if $\alpha \leq -2\pi$.

C.3 The integral $\int_{\nu_R-i\infty}^{\nu_R+i\infty} e^{\alpha\,\nu}\,\mathcal{F}(\nu)\,\mathfrak{P}_\nu(\zeta)\,g_X(t)\,\mathrm{d}\nu$

Consider the integral

$$\oint_{\mathcal{C}} e^{\alpha\,\nu}\,\mathcal{F}(\nu)\,\mathfrak{P}_\nu(\zeta)\,g_X(t)\,\mathrm{d}\nu\,,$$

where we defined the Legendre integral operator $\mathfrak{P}_\nu(\zeta)$ (section 4.6) as

$$\mathfrak{P}_\nu(\zeta) \equiv P_\nu(\zeta)\int_{-1}^{\zeta}\mathrm{d}t\,Q_\nu(t) + Q_\nu(\zeta)\int_{\zeta}^{1}\mathrm{d}t\,P_\nu(t)\,.$$

and $\mathcal{F}(\nu)$ is replaced with $2\nu+1$ if $X = D$ or with $\frac{2\nu+1}{\nu+1}$ in the case where $X = N$. By properly choosing ν_R, the interchange of the integrals is justified, and therefore

$$\begin{aligned}\int_{\nu_R-i\infty}^{\nu_R+i\infty} e^{\alpha\,\nu}\,\mathcal{F}(\nu)\,\mathfrak{P}_\nu(\zeta)\,g_X(t)\,\mathrm{d}\nu = &\int_{-1}^{\zeta}\left[\int_{\nu_R-i\infty}^{\nu_R+i\infty} e^{\alpha\,\nu}\,\mathcal{F}(\nu)\,P_\nu(\zeta)\,Q_\nu(t)\right]g_X(t)\,\mathrm{d}t\\ &+\int_{\zeta}^{1}\left[\int_{\nu_R-i\infty}^{\nu_R+i\infty} e^{\alpha\,\nu}\,\mathcal{F}(\nu)\,Q_\nu(\zeta)\,P_\nu(t)\right]g_X(t)\,\mathrm{d}t\end{aligned}$$

Remark C.3.1 *The Legendre functions of the second kind for large values of ν behave as*

$$Q_\nu(x) \sim \frac{1}{\sqrt{\nu}}\Big(C(\theta)\,e^{i\theta\nu} + D(\theta)\,e^{-i\theta\nu}\Big)\,,\quad x = \cos\theta\,,$$

for every Re$\nu > 0$, and

$$Q_\nu(x) \sim \frac{\cot\nu\pi}{\sqrt{\nu}}\Big(C(\theta)\,e^{i\theta\nu} + D(\theta)\,e^{-i\theta\nu}\Big)\,,\quad x = \cos\theta\,,$$

for every Re$\nu < 0$, where $C(\theta)$ and $D(\theta)$ complex functions of the variable θ alone.

Utilizing eq. (C.4) together with the fact that $\cot\nu\pi$ remains bounded as ν tends to infinity, gives

$$\mathfrak{P}_\nu(\cos\theta) \sim \mathfrak{P}_\nu(\theta) \sim \int_0^\pi \mathrm{d}\phi\,\Big(e^{i(\theta+\phi)\nu} + e^{i(\theta-\phi)\nu} + e^{-i(\theta+\phi)\nu} + e^{-i(\theta-\phi)\nu}\Big)\,,$$

and the following bounds are valid (see Fig. C.2)

$$0 < \theta+\phi < 2\pi,\qquad -\pi < \theta-\phi < \pi\,. \tag{C.6}$$

Letting $\nu = x + i\,y$ it is easy shown that

$$\begin{aligned}&\left|e^{\alpha\,\nu}\,\Big(e^{i(\theta+\phi)\nu} + e^{i(\theta-\phi)\nu} + e^{-i(\theta+\phi)\nu} + e^{-i(\theta-\phi)\nu}\Big)\right|\\ &\le e^{\alpha\,x-(\theta+\phi)y} + e^{\alpha\,x-(\theta-\phi)y} + e^{\alpha\,x+(\theta-\phi)y} + e^{\alpha\,x+(\theta+\phi)y}\end{aligned}$$

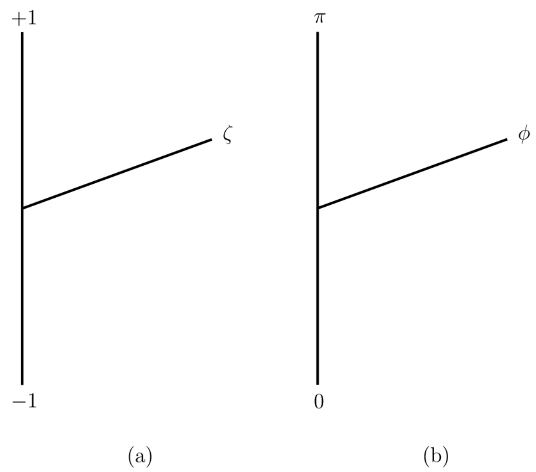

FIGURE C.2: *(a) $t-$plane and (b) $\phi-$plane*

and therefore, following the procedure introduced earlier, viz. integrating over a rectangle as shown on Fig. C.1, we obtain

$$\begin{aligned} I &= \left| \left(\int_{\mathcal{P}_1} + \int_{\mathcal{P}_2} + \int_{\mathcal{P}_3} + \int_{\mathcal{P}_4} \right) e^{\alpha\,\nu}\, \mathcal{F}\,(\nu)\, \mathfrak{P}_\nu(\zeta)\, g_X(t)\,\mathrm{d}\nu \right| \\ &\leq 2e^{-\alpha\,c} \left[-\frac{1}{\theta+\phi}\, e^{-(\theta+\phi)\epsilon} - \frac{1}{\theta-\phi}\, e^{-(\theta-\phi)\epsilon} + \frac{1}{\theta-\phi}\, e^{(\theta-\phi)\epsilon} + \frac{1}{\theta+\phi}\, e^{(\theta+\phi)\epsilon} \right] \\ &+ 2e^{-\alpha\,c} \left[\frac{1}{\theta+\phi}\, e^{-(\theta+\phi)c} + \frac{1}{\theta-\phi}\, e^{-(\theta-\phi)c} - \frac{1}{\theta-\phi}\, e^{(\theta-\phi)c} - \frac{1}{\theta+\phi}\, e^{(\theta+\phi)c} \right]. \end{aligned}$$

As c tends to infinity, $I \to 0$, if $\alpha \geq 2\pi$, due to (C.6).
Hence,

$$\int_{\nu_R - i\infty}^{\nu_R + i\infty} e^{\alpha\,\nu}\, \mathcal{F}\,(\nu)\, \mathfrak{P}_\nu(\zeta)\, g_X(t)\,\mathrm{d}\nu = -\int_{\mathcal{L}} e^{\alpha\,\nu}\, \mathcal{F}\,(\nu)\, \mathfrak{P}_\nu(\zeta)\, g_X(t)\,\mathrm{d}\nu\,.$$

Since the Gegenbauer functions of order $\frac{3}{2}$ exhibit similar behavior as the Legendre functions, the same conclusions hold.

APPENDIX D

Compatibility Condition

Let $\Psi(r,\zeta)$ satisfy the Stokes operator for irrotanional flow, namely $E^2\,\Psi(r,\zeta) = 0$. Integrating over a volume $\mathcal{V}$, we find

$$\int\limits_{\mathcal{V}} E^2\,\Psi(\mathbf{r})\,du(\mathbf{r}) = 0\,. \tag{D.1}$$

The operator E^2 is closely related to the Laplacian operator Δ, i.e.

$$E^2 = \Delta - \frac{2}{r}\left(\frac{\partial}{\partial r} - \frac{\zeta}{r}\frac{\partial}{\partial \zeta}\right)\,. \tag{D.2}$$

Combining (D.1) with (D.2) one obtains

$$\int\limits_{\mathcal{V}} \Delta\Psi(\mathbf{r})\,du(\mathbf{r}) = 2\int\limits_{\mathcal{V}}\left(\frac{1}{r}\,\frac{\partial\Psi(\mathbf{r})}{\partial r} - \frac{\zeta}{r^2}\,\frac{\partial\Psi(\mathbf{r})}{\partial\zeta}\right)du(\mathbf{r})\,.$$

Applying the divergence theorem, the latter relation simplifies as

$$\int\limits_{\partial\mathcal{V}} \frac{\partial\Psi(\mathbf{r})}{\partial r}\,ds(\mathbf{r}) = 2\int\limits_{\mathcal{V}} r\,\frac{\partial\Psi(\mathbf{r})}{\partial r}dr\,d\zeta\,d\phi - 2\int\limits_{\mathcal{V}}\zeta\,\frac{\partial\Psi(\mathbf{r})}{\partial\zeta}dr\,d\zeta\,d\phi\,. \tag{D.3}$$

The domain in question is a spherical shell, hence separable, and thus the above equation yields, after integrating by parts once and bearing in mind that the stream function has to vanish along the axis of revolution, i.e. $Z(\pm 1) = 0$

$$\int\limits_{\partial\mathcal{V}} \frac{\partial\Psi(\mathbf{r})}{\partial r}\,dS(\mathbf{r}) = 4\pi\left(r\,R(r)\right)\Big|_r\int_{-1}^{+1} Z(\zeta)\,d\zeta\,. \tag{D.4}$$

Distinguish three cases, namely **(i)** $r \leq r_1$, **(ii)** $r_1 \leq r \leq r_2$ and **(iii)** $r \geq r_2$.
(i) In the first case where $r \leq r_1$ (D.4) becomes

$$\int\limits_{\partial\mathcal{V}(r_1)} g_{\scriptscriptstyle N}(\mathbf{r})\,ds(\mathbf{r}) = 4\pi\, r_1\, R(r_1) \int_{-1}^{+1} Z(\zeta) d\zeta\,. \tag{D.5}$$

The only constraints on $Z(\zeta)$ are that it has to vanish as $\zeta \to \pm 1^{\mp}$ and to satisfy the ODE

$$(1-\zeta^2)Z''(\zeta) + \alpha\, Z(\zeta) = 0, \quad \alpha \in \mathbb{C}.$$

Choosing $Z(\zeta) = C_n^{-\frac{1}{2}}(\zeta)$, the constraints are satisfied if $n \geq 2$ and $\alpha = n(n-1)$. Furthermore, the following result is valid [HB86]

$$\int_{-1}^{+1} C_n^{-\frac{1}{2}}(\zeta) d\zeta = \begin{cases} 2, & n=0 \\ 0, & n=1 \\ \frac{2}{3}, & n=2 \\ 0, & n>2 \end{cases}.$$

Finally

$$\int\limits_{\partial\mathcal{V}(r_1)} g_{\scriptscriptstyle N}(\mathbf{r})\,ds(\mathbf{r}) = \begin{cases} \frac{8}{3}\pi\, r_1^3, & n=2 \\ 0, & n>2 \end{cases} \tag{D.6}$$

where we used the fact that in the interval $r \in (0, r_1]$ a solution in the $r-$direction bounded at $r=0$ is $R(r) = r^n$.
(ii) When $r_1 \leq r \leq r_2$, we find

$$\int\limits_{\partial\mathcal{V}(r_1)\cup\partial\mathcal{V}(r_2)} \sum_{j=1}^{2} g_{\scriptscriptstyle N}^{(j)}(\mathbf{r})\,ds(\mathbf{r}) = \begin{cases} \frac{8}{3}\pi\left(r_2^3 - r_1^3\right), & n=2 \\ 0, & n>2 \end{cases}, \tag{D.7}$$

wherelse, if **(iii)** $r \geq r_2$, (D.4) becomes

$$\int\limits_{\partial\mathcal{V}(r_2)} g_{\scriptscriptstyle N}(\mathbf{r})\,dS(\mathbf{r}) = \begin{cases} -\frac{8}{3}\pi, & n=2 \\ 0, & n>2 \end{cases} \tag{D.8}$$

where we used the fact that $R(r)$ must remain finite as r tends to infinity.

APPENDIX E

Spherical Coordinates (r, ζ, ϕ). Unit vectors and derivatives

The Cartesian coordinates (x, y, z) are related to the spherical coordinates (r, ζ, ϕ) by

$$
\begin{aligned}
x &= r\,\sqrt{1-\zeta^2}\,\cos\phi \\
y &= r\,\sqrt{1-\zeta^2}\,\sin\phi \\
z &= r\,\zeta\,,
\end{aligned}
$$

and thus, the scale factors are

$$
\begin{aligned}
h_r &= 1 \\
h_\zeta &= \frac{r}{\sqrt{1-\zeta^2}} \\
h_\phi &= r\,\sqrt{1-\zeta^2}\,.
\end{aligned}
$$

The area and the volume element are then given as

$$
\begin{aligned}
\mathrm{d}A &= r^2\,\mathrm{d}\zeta\,\mathrm{d}\phi \\
\mathrm{d}V &= r^2\,\mathrm{d}r\,\mathrm{d}\zeta\,\mathrm{d}\phi\,,
\end{aligned}
$$

respectively.
The unit vectors in spherical coordinates (r, ζ, ϕ) are related to the Cartesian basis $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ by

$$
\begin{aligned}
\hat{\mathbf{r}} &= \sqrt{1-\zeta^2}\,\cos\phi\,\hat{\mathbf{i}} + \sqrt{1-\zeta^2}\,\sin\phi\,\hat{\mathbf{j}} + \zeta\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\zeta}} &= -\zeta\,\cos\phi\,\hat{\mathbf{i}} - \zeta\,\sin\phi\,\hat{\mathbf{j}} + \sqrt{1-\zeta^2}\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}\,.
\end{aligned}
$$

On the other hand, the Cartesian unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ in terms of $(\hat{\mathbf{r}}, \hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\phi}})$ are given below

$$
\begin{aligned}
\hat{\mathbf{i}} &= \sqrt{1-\zeta^2}\,\cos\phi\,\hat{\mathbf{r}} - \zeta\,\cos\phi\,\hat{\boldsymbol{\zeta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\
\hat{\mathbf{j}} &= \sqrt{1-\zeta^2}\,\sin\phi\,\hat{\mathbf{r}} - \zeta\,\sin\phi\,\hat{\boldsymbol{\zeta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\
\hat{\mathbf{k}} &= \zeta\,\hat{\mathbf{r}} + \sqrt{1-\zeta^2}\,\hat{\boldsymbol{\zeta}}\,.
\end{aligned}
$$

The derivatives of the unit vectors are

$$
\begin{aligned}
\frac{\partial\hat{\mathbf{r}}}{\partial r} &= 0 \\
\frac{\partial\hat{\mathbf{r}}}{\partial\zeta} &= \frac{1}{\sqrt{1-\zeta^2}}\,\hat{\boldsymbol{\zeta}} \\
\frac{\partial\hat{\mathbf{r}}}{\partial\phi} &= \sqrt{1-\zeta^2}\,\hat{\boldsymbol{\phi}} \\
\frac{\partial\hat{\boldsymbol{\zeta}}}{\partial r} &= 0 \\
\frac{\partial\hat{\boldsymbol{\zeta}}}{\partial\zeta} &= -\frac{1}{\sqrt{1-\zeta^2}}\,\hat{\mathbf{r}} \\
\frac{\partial\hat{\boldsymbol{\zeta}}}{\partial\phi} &= -\frac{\zeta^2}{\sqrt{1-\zeta^2}}\,\hat{\mathbf{r}} - \zeta\,\hat{\boldsymbol{\zeta}} - \zeta\,\hat{\boldsymbol{\phi}} \\
\frac{\partial\hat{\boldsymbol{\phi}}}{\partial r} &= 0 \\
\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\zeta} &= 0 \\
\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} &= -\sqrt{1-\zeta^2}\,\hat{\mathbf{r}} + \zeta\,\hat{\boldsymbol{\zeta}}\,.
\end{aligned}
$$

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