STABILITY OF CONTINUATION AND OBSTACLE PROBLEMS IN ACOUSTIC AND ELECTROMAGNETIC SCATTERING

A Dissertation by

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DEDICATION

To my parents, my brother,
and my advisor Dr. Victor Isakov.

We could use up two Eternities in learning all that is to be learned about our own world and the thousands of nations that have arisen and flourished and vanished from it. Mathematics alone would occupy me eight million years. –Mark Twain

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ABSTRACT

Study of the Cauchy problem for Helmholtz equation is motivated by the inverse scattering theory and more generally by remote sensing. In this dissertation the increased stability of the Cauchy problem for Helmholtz equation and the Maxwell's system is investigated with varying frequency. Here it has been shown that the the stability of continuation is improving with the increasing frequency. The continuation is inside the convex hull of the surface where the Cauchy data is given. This has been demonstrated by numerical experiments with simple geometry. When we continue outside of the convex hull, the subspace of stable solutions is growing with frequency. This is also demonstrated by numerical experiments where we reconstruct the density function of the single layer potential. Another problem that is presented here is the electromagnetic obstacle scattering problem, with variable frequency. Here the existence and uniqueness of the solution to the forward problem is presented and the analytic dependence of the solution on the frequency is proved.

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CHAPTER 1

INTRODUCTION

The problem of the continuation or the Cauchy problem for partial differential equations is of fundamental theoretical interest and it is very important for practical applications, for example in control theory and inverse problems. This problem started with Holmgren-John theorem about uniqueness for equations with analytic coefficients. Finding the solution u of a partial differential equation from the given data g_j, f is the well known *Cauchy Problem.*

$$Lu = f \text{ on } \Omega,$$

$$\partial_\nu^j u = g_j,\ j \leq m-1 \text{ on } \Gamma, \tag{1.1}$$

where $Lu = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha u$, Ω is a domain in $\mathbb{R}^n$, $\Gamma \in C^{m-1}$, is a part of $\partial\Omega$, the boundary of the domain Ω, and ν is the outward normal to the boundary.

A Cauchy problem is said to be *well-posed* in the sense of Hadamard if the following conditions hold:

1. $u \in U$ exists for any $g_j \in G$, $f \in F$;

2. $u \in U$ is determined uniquely by $g_j \in G$, $f \in F$;

3. $u \in U$ depends continuously on $g_j \in G$, $f \in F$,

where U is the space of all solutions u and $G\times F$ is the space of all data g_j, f prescribed on the boundary Γ and on the domain Ω. In other words a Cauchy problem is well posed, if the operator $A : U \to G\times F$ defined as $Au = \{g,\ f\}$ has a continuous inverse from $G \times F$ onto U. U, G, and F are open subsets of classical spaces $C^k(\overline{\Omega})$, $C^k(\Gamma)$, $H^p_k(\Omega)$, $H^p_k(\Gamma)$ or their closed subspaces of finite codimension.

If one of the above three conditions is not satisfied, then such problems are called *ill posed* problems in the sense of Hadamard. For ill posed problems u may not exist. If u exists continuous dependence of u on g_j, f may not be guaranteed.

A feature of this problem for elliptic equations is its exponential instability pointed out by Hadamard in 1920's. For example, consider the classical example of Hadamard: of the Cauchy problem for the Laplace equation

$$\partial_x^2 u + \partial_y^2 u = 0 \text{ in } \mathbb{R}^2_+ = \{(x,y)|y > 0\}, \tag{1.2}$$

$$u = 0,\ \partial_y u = g_1,\text{when } y = 0. \tag{1.3}$$

If $g_1(x) = n^{-k} cos(nx)$, then u exists ($u = n^{-3} cos(nx) sinh(ny)$, k fixed). But one can note that there is no continuous dependence of u on g_1. In applications (inverse problems) this continuous dependence (stability estimates for u) is of most importance. Only this condition guarantees the convergence of the solutions u while using computational algorithms.

A stability estimate is defined [17] as a function ω such that

$$\|u - u^*\|_U \leq \omega(\|Au - Au^*\|_{G\times F}) \tag{1.4}$$

An important condition for stability estimate is that $\lim_{\tau\to 0} \omega(\tau) = 0$ and also ω is increasing monotonically. Depending on this function ω we have three kinds of stability estimates:

1. if $\omega(\varepsilon) = C\varepsilon$, then the solution u depends Lipschitz continuously on g_j, f;

2. if $\omega(\varepsilon) = C\varepsilon^{\kappa}, 0 < \kappa < 1$, then the solution u depends Hölder continuously on g_j, f;

3. if $\omega(\varepsilon) = \dfrac{C}{|log\varepsilon|}$, then the solution u depends logarithmic continuously on g_j, f, which is much weaker kind of continuity.

For applications mere continuity is not good, to develop efficient numerics we expect that u depends at least Hölder continuously on data (the best case would be Lipschitz continuity). Cauchy problems where u depends on data Hölder continuously are said to be *well behaved*. This can be achieved by assuming that the solution u and first few derivatives of u are bounded. Hence we consider a restricted solution space U_M where M is the *apriori* bound.

In 1960 John [19] showed that one has at least logarithmic stability for a wide class of partial differential operators. In the same paper he considered an important example:

$$\partial_x^2 u + \partial_y^2 u = \partial_t^2 u \text{ in } \Omega \times (-T, T), \tag{1.5}$$

$$u = g_0 \text{ and } \partial_\nu u = g_1 \text{ on } \partial\Omega \times (-T, T). \tag{1.6}$$

Let the solution u and first few derivatives be bounded and the domain Ω be given by

$$\Omega = \{(x, y) : x^2 + y^2 < 1\}. \tag{1.7}$$

It is shown that at any point inside the cylinder $\Omega \times (-T, T)$ the solution u depends Holder continuously on the Cauchy data g_0, g_1(or even Lipschitz continuously

[17] if appropriate norms are selected), but for the Cauchy problem

$$\partial_x^2 u + \partial_y^2 u = \partial_t^2 u \text{ in } \Omega_e \times (-T, T), \tag{1.8}$$

$$u = g_0 \text{ and } \partial_\nu u = g_1 \text{ on } \partial\Omega_e \times (-T, T). \tag{1.9}$$

with $\Omega_e = \{(x, y) : x^2 + y^2 > 1\}$, the dependence of the solution u on g_0, g_1 is at best logarithmically continuous. This can be inferred by considering the solution of the wave equation $\partial_x^2 u + \partial_y^2 u = \partial_t^2 u$ in polar coordinates which is given by

$$u_n = J_n(nr)e^{in(t+\theta)},$$

where $J_n(nr)$ is the Bessel function of order $n \in \mathbb{Z}^+$.

For $r < 1$, it is shown that $|u_n| = |J_n(nr)| < q^n$ and for $r > 1$, u_n decreases only like some negative powers of n, reaching maximum at $r = 1$. Also, in [19] it is shown that for any r and any $n \in \mathbb{Z}^+$,

$$|u_n| = |J_n(nr)| < An^{-1/3}.$$

These estimates for u_n show that the the best possible stability estimate is of logarithmic type.

Since,

$$v_n = e^{-int}u_n = J_n(nr)e^{in\theta}$$

solves the Helmholtz equation

$$(\Delta + n^2)v_n = 0 \text{ in } \mathbb{R}^2$$

this shows that the Cauchy problem for the Helmholtz equation is also not well behaved.

Logarithmic stability is quite damaging for numerical solution of many inverse problems. In the recent papers [3], [13], [15], [16] it was demonstrated that when one continues solutions of the Helmholtz equation from a surface Γ onto its convex hull the stability is increasing and unstable (Hölder) component of stability estimates goes to zero as the wave number is increasing. These results are summarized in chapters 2 and 3.

Stability of the continuation is crucial for stability (and hence for an efficient numerical solution) of (non linear) inverse problems. Better numerical resolution for higher wave numbers in the inverse medium and obstacle problems was observed in [6], [7] (inverse medium problems in optics), [8] (inverse electromagnetic obstacle problem), [12] (inverse source problem), and [26] (an inverse medium problem in ultrasound tomography). The Helmholtz equation is a good model for acoustics where the physically interesting wave numbers are not very high (typically less than 30). They can be really high for electromagnetic fields ($k\sqrt{\varepsilon\mu}$ up to hundreds or thousands), and then increasing stability is expected to be more dramatic.

CHAPTER 2

INCREASED STABILITY IN THE CONTINUATION FOR HELMHOLTZ EQUATION

Motivated by the inverse problems in the (acoustical, electromagnetic) wave propagation and in particular by scattering theory, we focus on the Cauchy problem for the Helmholtz equation

$$\left(\Delta + k^2\right) u = f \text{ in } \Omega,\ u \in H_{(1)}(\Omega), \tag{2.1}$$

with Cauchy data

$$u = g_0 \text{ and } \partial_\nu u = g_1 \text{ on } \Gamma, \tag{2.2}$$

where Ω is a Lipschitz bounded domain in $\mathbf{R}^n, n = 2, 3$, and Γ is a part of its boundary.

In Theorem 1.1 Ω is a subset of a cylinder $\{x : 0 < x_n < h, |x'| < r\}$, $x' = (x_1, x_2, ..., x_{n-1})$, and Γ is the open part of the boundary $\partial\Omega$ contained in the layer $\{0 < x_n < h\}$. We assume that $\Omega = \{x : 0 < x_n < \omega(x'),\ x' \in \Omega'\}$, $\omega > 0$, $\omega \in C^1(\overline{\Omega'})$. Let $\Omega(d) = \Omega \bigcap \{x : x_n > d\}$, $F = \|f\|(\Omega) + \|u\|(\Gamma) + \| \bigtriangledown u\|(\Gamma)$ and $F(k, d) = \|f\|(\Omega) + d^{-0.5}(k + d^{-1})\|u\|(\Gamma) + \| \bigtriangledown u\|(\Gamma)$, here $\|u\|_{(l)}(\Omega)$ is the norm in the Sobolev Space $H_{(l)}(\Omega)$ and $\|u\| = \|u\|_{(0)}$.

$$\|u\|_{\infty,l}(\Omega) = \sum_{|\alpha|\leq l} \|\partial^\alpha u\|_\infty(\Omega)$$

and $\|\|_p$ is the standard norm in L^p. C denote generic constants depending on Ω and Γ, any additional dependence is indicated.

In [13] they proved the following stability estimate for the solution u of (4.1) and (2.2):

$$\|u\|(\Omega(d)) \leq C \left(F + \frac{M_1^{1-\lambda} F(k, d)^\lambda}{d^{2-2\lambda} k} \right) \tag{2.3}$$

where

$$\lambda = \frac{2r^2d + \frac{3}{8}d^3}{4r^2h + h^2d + \frac{5}{4}d^2h + \frac{3}{8}d^3 + 3r^2d}. \tag{2.4}$$

One important corollary of (2.3) is that the stability and hence the resolution in the Cauchy problem in the subdomain $\Omega(d)$ increases as the frequency k grows.

Theorem 2.0.1. *Let $\|u\|_{(1)}(\Omega) \leq M_1$. Then there exists a constant C such that for any solution u to (4.1) and (2.2)*

$$\|u\|^2(\Omega(0)) \leq CM_1^2(\varepsilon^2 + \frac{1}{(-ln\varepsilon + k)^{\frac{1}{8}}}) \tag{2.5}$$

where $\varepsilon = \frac{F}{M_1}$.

Observe that the stability estimate (2.5) consists of two terms. If only the first term ε^2 is present we have the best possible Lipschitz stability, guaranteeing in particular high resolution of suitable numerical algorithms. However, at fixed k the Cauchy problem (4.1),(2.2) for the elliptic Helmholtz equation is notoriously (exponentially) ill-posed, so a Lipschitz stability is not possible. Theorem 1.1 shows that the second "logarithmically unstable" term in(2.5) is going to zero (as a power of k), and hence stability and resolution in the Cauchy problem are improving when k grows.

Theorem 1.1 combined with known theory of Sobolev spaces implies global improved stability in the exterior of an obstacle. Let Ω_0 be a bounded convex domain in $\mathbf{R^n}$ and D be an open subset of Ω_0. Let P be a half-space of $\mathbf{R}^n$ and $\Omega_P = \Omega \cap P$. Let $\mathcal{P}$ be the set of all P such that $\partial\Omega_0 \cap P \subset \Gamma$ and $\Omega(D, \Gamma)$ be the union of all sets $\Omega(P)$ over $P \in \mathcal{P}$.

In next Theorem and its Corollaries $\Omega = \Omega_0 \setminus \bar{D}$.

Theorem 2.0.2. *Let $\|u\|_{(l)}(\Omega) \leq M_l$. Then there exists a constant $C = C(l)$ such* *that for any solution u to (4.1) and (2.2)*

$$\|u\|_\infty(\Omega(D,\Gamma)) \leq CM_l(\varepsilon^2 + \frac{1}{(-ln\varepsilon + k)^{\frac{1}{8}}})^{\theta_l}, \text{ with } \theta_l = \frac{1}{2} - \frac{0.8}{l} \tag{2.6}$$

where $\varepsilon = \frac{F}{M_1}$.

If $l = 2$ we have $\theta_l = 0.1$. For large l this exponent is increasing to 0.5 showing better stability.

Corollary 2.0.1. *Let $\|u\|_{(l)}(\Omega) \leq M_l$. If Ω_0, D are convex domains and $\Gamma = \partial\Omega_0$, then the bound (2.6) holds in $\Omega(D,\Gamma) = \Omega_0 \setminus D$*

Corollary 2.0.2. *Let $\|u\|_{\infty,2}(\Omega) \leq M_{\infty,2}$. Then there exists constant C such that for any solution u to (4.1) and (2.2)*

$$\|u\|_{\infty,1}(\Omega(D,\Gamma)) \leq CM_{\infty,2}(\varepsilon^2 + \frac{1}{(-ln\varepsilon + k)^{\frac{1}{8}}})^{\theta} \tag{2.7}$$

where $\varepsilon = \frac{F}{M_1}, \theta = 0.55 \cdot 0.625 \cdot 0.25 = 0.0859....$

Observe that this corollary indicates better stability in reconstruction of the boundary coefficient b in the impedance boundary condition $\partial_\nu u + bu = 0$ on ∂D of a given convex obstacle (on "illuminated" part of nonconvex ∂D) from the Cauchy data (2.2) for a solution to (4.1). Indeed, only remaining question to resolve is to evaluate "size" of zero set of ∇u on ∂D. Unfortunately, it is not a very simple issue.

2.1 Auxiliary Trace, Embedding , and Interpolation Results.

In this section we collect some mostly known results on traces and interpolation which are needed in the proofs of section 1.2.

Lemma 2.1.1. *Let $S(d) = \Omega \cap \{x : x_n = d\}$. There exists a constant C such that*

$$\|u\|_{(0)}(S(d)) \leq C\|u\|_{(1)}(\Omega).$$

This is a know result about bound of traces, see for example [21], p. 44.

Now we mention the well know interpolation inequalities for intermediate derivatives. The main idea is that if u is bounded in $H_{(s_1)}$ and $H_{(s_2)}$ then u is bounded in all the intermediate Sobolev spaces $H_{(s)}$ where $s_1 < s < s_2$. We remind the standard interpolation inequality [23] is given by:

$$\|u\|_{(s)}(\Omega) \leq C\|u\|^{1-\theta}_{(s_1)}(\Omega)\|u\|^{\theta}_{(s_2)}(\Omega) \tag{2.8}$$

where $s = (1-\theta)s_1 + \theta s_2$, $0 < \theta < 1$ and $C = C(\Omega,\, s_1,\, s_2,\, \theta)$.

By known Sobolev embedding theorems [21], [28], p. 328

$$\|u\|_q(\Omega) \leq C\|u\|_{(s)},\ \text{when}\ n(\frac{1}{2}-\frac{1}{q}) \leq s,\ \|u\|_\infty(\Omega) \leq C(s)\|u\|_{(s)}(\Omega),\ \text{when}\ \frac{n}{2} < s. \tag{2.9}$$

Now we remind definition and properties of less standard space of functions $H_{p,s}(\Omega)$. When $\Omega = \mathbf{R}^n$ the norm of a function u in this space is

$$\|u\|_{p,s}(\mathbf{R}^n) = \|F^{-1}(1+|\xi|^2)^{\frac{s}{2}}Fu\|_p(\mathbf{R}^n)$$

where F is the Fourier transform [28], p. 177. For a bounded Ω the definition is obtained by taking minimal norm of extension onto $\mathbf{R}^n$ [28]. p.310. Referring to [28], pp. 59, 317, 328, we have the interpolations inequalities

$$\|u\|_{p,s}(\Omega) \leq C(p, s_1, s_2, \theta, \Omega)\|u\|^{1-\theta}_{p,s_1}(\Omega)\|u\|^{\theta}_{p,s_2}(\Omega) \tag{2.10}$$

where $1 < p < \infty, s_1 < s_2, 0 < \theta < 1, s = (1-\theta)s_1 + \theta s_2$. In addition, there are embedding theorems

$$\|u\|_{\infty,l}(\Omega \leq C\|u\|_{p,s}(\Omega),\ 1 < p < \infty,\ l + \frac{n}{p} < s \tag{2.11}$$

Lemma 2.1.2. *There exists a constant C such that*

$$\|u\|_\infty(\Omega) \leq C\|u\|^{1-\theta(l)}_{(0)}\|u\|^{\theta(l)}_{(2)}(\Omega),\ \theta(l) = \frac{1.6}{l}. \tag{2.12}$$

$$\|u\|_\infty(\Omega) \leq C\|u\|_{(0)}^{1/40}(\Omega)\|u\|_{\infty,1}^{39/40}(\Omega). \tag{2.13}$$

Proof:

For $n = 2, 3$ we let $s = 1.6$ in(2.9) to yield

$$\|u\|_\infty(\Omega) \leq C\|u\|_{(1.6)}(\Omega). \tag{2.14}$$

From (2.8) with $s_1 = 0, s_2 = l, \theta(l) = \frac{1.6}{l}$ we have

$$\|u\|_{(1.6)}(\Omega) \leq C\|u\|_{(0)}^{1-\theta(l)}(\Omega)\|u\|_{(2)}^{\theta(l)}(\Omega). \tag{2.15}$$

Combining (2.14) and 2.15) we obtain the first statement (2.12).

Again by embedding (2.9) with $n = 2, 3, q = 5$ we yield

$$\|u\|_5(\Omega) \leq C\|u\|_{(0.9)}(\Omega). \tag{2.16}$$

By interpolation inequalities (2.8) with $s_1 = 0, s_2 = 1, \theta = 0.9$

$$\|u\|_{(0.9)}(\Omega) \leq C\|u\|_{(0)}^{0.1}(\Omega)\|u\|_{(1)}^{0.9}(\Omega). \tag{2.17}$$

By more precise interpolation result (2.10)

$$\|u\|_{5,3/4}(\Omega) \leq C\|u\|_{5,0}^{0.25}(\Omega)\|u\|_{5,1}^{0.75} \leq C\|u\|_{5}^{0.25}(\Omega)\|u\|_{\infty,1}^{0.75}(\Omega) \tag{2.18}$$

Again by embedding theorems (2.11)

$$\|u\|_\infty(\Omega) \leq C\|u\|_{5,0.75}(\Omega) \tag{2.19}$$

Hence from (2.19), (2.18)

$$\|u\|_\infty(\Omega) \leq C\|u\|_{5}^{0.25}(\Omega)\|u\|_{\infty,1}^{0.75}(\Omega) \leq C\|u\|_{(0.9)}^{0.25}(\Omega)\|u\|_{\infty,1}^{0.75}(\Omega)$$

due to (2.16) and (2.17). Finally, from the last inequality and from (2.17) we yield (2.13).

The proof is complete.

Lemma 2.1.3. *Let* $\|u\|_{\infty,2}(\Omega) \leq M_{\infty,2}$. *There exists constant* C *such that*

$$\|u\|_{\infty,1}(\Omega) \leq CM_{\infty,2}^{1-\theta_1}\|u\|_{(0)}^{\theta_1}(\Omega). \tag{2.20}$$

where $\theta_1 = 0.55 \cdot 0.625 \cdot 0.5$.

Proof: Again from interpolation inequality (2.8) we have

$$\|u\|_{(0.9)}(\Omega) \leq C\|u\|_{(0)}^{0.55}(\Omega)\|u\|_{(2)}^{0.45}(\Omega) \tag{2.21}$$

By (2.10)

$$\|u\|_{5,0.75}(\Omega) \leq C\|u\|_{5}^{0.625}(\Omega)\|u\|_{5,2}^{0.375} \leq$$

$$C\|u\|_{(0.9)}^{0.625}(\Omega)\|u\|_{5,2}^{0.375} \leq C\|u\|_{(0)}^{\theta_2}(\Omega)\|u\|_{\infty,2}^{1-\theta_2}$$

where we used (2.21) and let $\theta_2 = 0.550.625$. Using in addition the standard interpolation inequality

$$\|u\|_{\infty,1}(\Omega) \leq C\|u\|_{\infty}^{0.5}(\Omega)\|u\|_{\infty,2}(\Omega)^{0.5}$$

we complete the proof of (2.20).

2.2 Proofs of stability estimates

Proof of Theorem 1.1:

First, we derive from (2.3) the following simpler upper bound

$$\|u\|(\Omega(d)) \leq C\left(F + \frac{M^{1-\lambda_1}F^{\lambda_1}}{d^2\sqrt{k}}\right),\ \lambda_1 = \frac{d}{4h} \tag{2.22}$$

assuming that $d < 1$, $F < 1$, $1 \leq M_1$, $\lambda < \frac{1}{2}$ and

$$(h^2 + 3r^2)d + \frac{5}{4}d^2h < 4r^2h. \tag{2.23}$$

Indeed from (2.4) we can conclude that

$$\lambda > \frac{2r^2d}{4r^2h + (h^2 + 3r^2)d + \frac{5}{4}d^2h} > \frac{2r^2d}{4r^2h + 4r^2h} = \frac{d}{4h}$$

due to (2.23). So (2.3) implies (2.22).

Let $F(d) = \|u\|^2(\Omega(d))$. As known (see e.g. [27], p. 77), there is

$$F'(d) = -\int_{(x',d)\in\Omega} |u(x',d)|^2\ dx' = -\|u\|^2(S(d)). \tag{2.24}$$

From the mean value theorem we have $F(d) = F(0) + F'(d^*)d$, where $0 < d^* < d$. Hence,

$$|F(0)| \leq |F(d)| + |F'(d^*)|d$$

and from (2.24), (2.22), and Lemma 2.1 we conclude that

$$\|u\|^2(\Omega(0)) \leq C\left(F^2 + M_1^2\varepsilon^{2\lambda_1}\frac{1}{kd^4} + M_1^2 d\right). \tag{2.25}$$

Let us consider the function

$$f(d) = \varepsilon^{2\lambda_1}\frac{1}{kd^4} + d. \tag{2.26}$$

and try to minimize this function with respect to $d > 0$. We choose

$$d = (E+k)^{-\frac{1}{8}},\ E = -ln\varepsilon. \tag{2.27}$$

From (2.26), (2.27),

$$f(d) = e^{\alpha} + d,\ \text{where}\ \alpha = -\frac{d}{C}E - 4lnd - lnk.$$

Due to (2.27),

$$\alpha = -\frac{E}{C(E+k)^{-\frac{1}{8}}} + \frac{1}{2}ln(E+k) - lnk \tag{2.28}$$

If $E \leq k$, then

$$\alpha \leq \frac{1}{2}ln(E+k) - lnk \leq \frac{1}{2}ln(2k) - lnk =$$

$$-\frac{1}{2}lnk + \frac{ln2}{2} \leq -\frac{1}{4}lnE - \frac{1}{4}lnk + \frac{ln2}{2}$$

If $k \leq E$, then again from (2.28)

$$\alpha \leq -\frac{E}{CE^{\frac{1}{8}}} + \frac{1}{2}ln2E - lnk \leq$$
$$-\frac{1}{C}E^{\frac{7}{8}} - lnk \leq -\ln E - lnk + C$$

where we used twice that $ClnA \leq A^{\frac{1}{8}} + C$. Finally,

$$\alpha \leq -\frac{1}{4}(lnE + lnk) + C$$

and from (2.28), (2.26) we have

$$f(d) \leq C(\frac{1}{E^{\frac{1}{4}}K^{\frac{1}{4}}} + \frac{1}{(E+k)^{\frac{1}{8}}}) \leq C\frac{1}{(E+k)^{\frac{1}{8}}}.$$

Combining with (2.25), (2.26) we complete the proof.

□

Proof of Theorem 1.2: We will apply Theorem 1.1 to domains Ω_P. First we observe that constants in lemmas 2.2, 2.3 for domains $\Omega = \Omega_P$ do not depend on P. Indeed, according to the definitions the domains Ω_P after an orthogonal co-ordinate change have the form $\Omega(0)$ where functions ω are uniformly (with respect to P) bounded in C^1, moreover they all have uniform cone property. Hence these domains can be mapped onto a standard domain Ω_0 (e.g the upper unit hemisphere in $\mathbf{R}^n$ by C^1) by diffeomorphisms which are uniformly bounded in C^1 together with their inverses. Applying Lemmas 2.2, 2.3 to Ω_0 and using inverse diffeomorphism we conclude that constants can be chosen P-independent.

Using (2.12) for $\Omega = \Omega_P, P \in \mathcal{P}$ we yield

$$\|u\|_\infty(\Omega(P)) \leq CM_l^{\theta(l)}\|u\|_{(0)}^{1-\theta(l)}(\Omega(P)) \leq M_l^{\theta(l)}M_1^{1-\theta(l)}(\varepsilon^2 + \frac{1}{(-ln\varepsilon + k)^{\frac{1}{8}}})^{0.5(1-\theta(l))}. \tag{2.29}$$

due to (2.5). Since $\Omega(D,\Gamma)$ is the union of Ω_P over $P \in \mathcal{P}$ and $M_1 \leq M_l$ we obtain (2.6).

2.3 Numerical experiments

In this section we conduct a numerical experiment to demonstrate the increasing stability of the Cauchy problem. This numerical experiment reconstructs the acoustical pressure from the knowledge of its farfield acoustical pressure in three dimensions. The reconstruction procedure is similar to the ones considered in [13] and [18].

The experimental setup for the exterior problem consists of two concentric semi spheres and a semi circle given by,

$$\begin{aligned}
\Gamma_0 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\| = r_0, \phi_1 \le \phi \le \phi_2, \theta_1 \le \theta \le \theta_2\} \\
\Gamma_1 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\| = r_1, \phi_3 \le \phi \le \phi_4, \theta_3 \le \theta \le \theta_4\} \\
\Gamma_2 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\| = r_2, \phi = \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2}\}
\end{aligned}$$

where $\phi = tan^{-1}\left(\frac{x_2}{x_1}\right)$ and $\theta = tan^{-1}\left(\frac{\sqrt{x_1^2+x_2^2}}{x_3}\right)$, $r_0 = 2$, $r_1 = 1$ and $r_2 = \frac{1}{2}$. Five acoustical sources are placed on the semicircle Γ_2, the amplitudes and positions are given by Table 1.

Amplitude	Position		
$A_1 = 1$	(0,	0,	$\frac{1}{2}$)
$A_2 = 4$	(0,	$\frac{-1}{2\sqrt{2}}$,	$\frac{1}{2\sqrt{2}}$)
$A_3 = 5$	(0,	$\frac{-1}{2}$,	0)
$A_4 = 2$	(0,	$\frac{-1}{2\sqrt{2}}$,	$\frac{-1}{2\sqrt{2}}$)
$A_5 = 3$	(0,	0,	$\frac{-1}{2}$)

Table 2.1: Amplitudes and positions of acoustical sources

Next we discretize the surfaces Γ_0 and Γ_1 by considering n angles between ϕ_1 and ϕ_2 and n angles between θ_1 and θ_2. Hence we obtain n^2 points on Γ_0 which is given by

$$\Gamma_{d0} \quad = \quad \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\| = r_0, \phi = \phi_i, \theta = \theta_i, i = 1, ..., n\}$$

where $\phi_i = \phi_1 + i\delta\phi$ and $\delta\phi = (\phi_2 - \phi_1)/n$, $\theta_i = \theta_1 + i\delta\theta$ and $\delta\theta = (\theta_2 - \theta_1)/n$. Similarly we obtain n^2 points on Γ_1 given by

$$\Gamma_{d1} \quad = \quad \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\| = r_1, \phi = \phi_j, \theta = \theta_j, j = 1, ..., n\}$$

where $\phi_j = \phi_3 + j\delta\phi$ and $\delta\phi = (\phi_4 - \phi_3)/n$, $\theta_j = \theta_3 + i\delta\theta$ and $\delta\theta = (\theta_4 - \theta_3)/n$. For this experiment $n = 10$, $\phi_1 = \phi_3 = \theta_1 = \theta_3 = 0$, $\phi_2 = \phi_4 = -\pi$ and $\theta_2 = \theta_4 = \pi$.

The acoustic pressure and its normal (radial) derivative on Γ_{d0} are calculated using

$$u(x) \quad = \quad \sum_{j=1}^{5} A_j \Phi(x, y_j) \tag{2.30}$$

$$\partial_r u(x) \quad = \quad \sum_{j=1}^{5} A_j \partial_r \Phi(x, y_j) \tag{2.31}$$

where

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

is the fundamental solution of the Helmholtz equation, while A_j and y_j are the amplitudes and positions of the acoustic sources respectively as given in Table 1.

Using equations (2.30) and (2.31) we can generate the Cauchy data on Γ_{d0} by adding some noise,

$$u_\delta \quad = \quad u + \delta\|u\|_2 \frac{\xi}{\|\xi\|_2} \tag{2.32}$$

$$\partial_r u_\delta \quad = \quad \partial_r u + \delta\|\partial_r u\|_2 \frac{\xi}{\|\xi\|_2} \tag{2.33}$$

Frequency (k)	Error of reconstruction(in %)
2.0	782.00%
4.0	544.44%
8.0	67.13%
16.0	2.79%

Table 2.2: Errors of reconstruction at various frequencies for the exterior problem

Here the noise is $\delta = 1\%$ and ξ is a vector which is uniformly distributed on $(-1, 1)$. Since u is a radiating solution of the Helmholtz equation we have from [10]

$$u(x) \approx \sum_{n=0}^{N} \sum_{m=-n}^{n} a_{n,m} h_n^{(1)}(|x|) Y_n^m \left(\frac{x}{|x|} \right) \tag{2.34}$$

The choice of N is quite important since it plays the role of a regularizer. For this experiment $N = 9, 10$ which is the best possible choice. We find the coefficients $a_{n,m}$ by matching the series expansion of the solution with the Cauchy data calculated from (4.13) and (4.14) on Γ_{d0}. This is achieved by forming a system $Ax = b$ where x is a vector of coefficients to be determined, vector b is the Cauchy data and the entries of matrix A are formed by the product of spherical Hankel functions and spherical harmonics. The solution to this system is obtained by forming the normal equations $A^*Ax = A^*b$ and by applying using conjugate gradient technique on these normal equations. The number of iterations is chosen such that the residual error is lesser that 100ϵ where ϵ is the machine epsilon. Using these coefficients $a_{n,m}$ and equation (2.34) we can reconstruct the acoustical field u_{recon} on Γ_{d1}. The error of reconstruction is given by

$$err_{recon} = \frac{\|u_{exact} - u_{recon}\|_2}{\|u_{exact}\|_2}$$

where u_{exact} is the acoustical field on Γ_1 calculated using the equation 4.1. The error of reconstruction (in percentages) for various frequencies is given in the table 2, which demonstrates the increased stability in the reconstruction of the acoustical pressure.

Also, we would like to consider the interior problem. The experimental setup for the interior problem consists of two concentric semi spheres and a semi circle given by Γ_0, Γ_1 and Γ_2 which is same as the setup for the exterior problem but with $r_0 = \frac{1}{2}$, $r_1 = 1$ and $r_2 = 2$. As before the Cauchy data is prescribed on the discretised surface Γ_0. This Cauchy data is matched with the approximate series expansion of the solution to the interior problem which is given by

$$u(x) \approx \sum_{n=0}^{N} \sum_{m=-n}^{n} a_{n,m} j_n(|x|) Y_n^m \left(\frac{x}{|x|} \right)$$

to calculate the coefficients $a_{n,m}$ and hence reconstruct the acoustic pressure on the semi sphere. For interior problem $N = 7$ is the best choice. The algorithm used in this case is precisely the same which was used for the exterior problem. The error of reconstruction for various frequencies are shown in table 3. The error of reconstruction for the interior problem increases with the increasing frequency.

Frequency (k)	Error of reconstruction(in %)
2.0	29.42%
4.0	61.84%
8.0	65.4%
16.0	117.15%

Table 2.3: Errors of reconstruction at various frequencies for the interior problem

CHAPTER 3

INCREASED STABILITY IN THE CONTINUATION FOR MAXWELL'S SYSTEM

In this chapter we consider the Cauchy problem for the stationary *Maxwell System*

$$\begin{cases} \text{curl } \boldsymbol{E} - ik\mu\boldsymbol{H} = 0 & \text{in } \Omega, \\ \text{curl } \boldsymbol{H} + (ik\varepsilon - \sigma)\boldsymbol{E} = 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

with the Cauchy data

$$\boldsymbol{E} = \boldsymbol{E}_0, \ \boldsymbol{H} = \boldsymbol{H}_0 \ \text{ on } \ \Gamma, \tag{3.2}$$

where $\boldsymbol{E}, \boldsymbol{H}$ are electrical and magnetic vectors $\boldsymbol{E}, \boldsymbol{H}$, Ω is a domain in $\mathbb{R}^3$ and Γ is a part of its boundary $\partial\Omega$. We will assume that the coefficients $\varepsilon, \mu, \sigma \in C^2(\bar{\Omega}), 0 < \varepsilon, 0 < \mu, 0 \leq \sigma$ on $\bar{\Omega}$, and the Cauchy data $\boldsymbol{E}_0, \boldsymbol{H}_0$ are given functions.

Let $\Omega \subset \{0 < x_3 < h, |x'| < r\}$ with Lipschitz $\partial\Omega$, $\bar{\Omega} \subset \{x_3 < h\}$ and $\Gamma = \partial\Omega \cap \{0 < x_3 < h\}$, $x' = (x_1, x_2)$. Let $\Omega(d) = \Omega \cap \{d < x_3\}$. $\|u\|_{(l)}(\Omega)$ is the norm in the Sobolev space $H^l(\Omega)$ and $\|u\|(\Omega) = \|u\|_{(0)}(\Omega)$. We let $F(E) = \|\boldsymbol{E}_0\|_{(1)}(\Gamma) + \|\boldsymbol{H}_0\|_{(1)}(\Gamma) + k(\|\boldsymbol{E}_0\|(\Gamma) + \|\boldsymbol{H}_0\|(\Gamma))$, and $F(E; k, d) = (k + d^{-1})(\|\boldsymbol{E}_0\|(\Gamma) + \|\boldsymbol{H}_0\|(\Gamma)) + \|\boldsymbol{E}_0\|_{(1)}(\Gamma) + \|\boldsymbol{H}_0\|_{(1)}(\Gamma)$. For later use we let $F = \|\mathbf{f}\|(\Omega) + \|\mathbf{u}_0\|_{(1)}(\Gamma) + \|\mathbf{u}_1\|_{(0)}(\Gamma)$ and $F(k, d) = \|\mathbf{f}\|(\Omega) + (k + d^{-1})\|\mathbf{u}_0\|_{(0)}(\Gamma) + \|\mathbf{u}_0\|_{(1)}(\Gamma) + \|\mathbf{u}_1\|_{(0)}(\Gamma)$. By C we denote generic constants depending only on $\varepsilon, \mu, \sigma, \Omega, \Gamma, a_0, \mathbf{B}_l, \mathbf{C}_1, \mathbf{C}_0$. Any additional dependence will be indicated.

In Theorem 1.1 we assume that $1 \leq k, \|\boldsymbol{E}\|_{(1)}(\Omega) + \|\boldsymbol{H}\|_{(1)}(\Omega) < M_1, d < 2r$.

Theorem 3.0.1. *Let*

$$0 < 2\varepsilon\mu + \nabla(\varepsilon\mu) \cdot x + \beta_3\partial_3(\varepsilon\mu), \ 0 \leq \partial_3(\varepsilon\mu) \ \text{on } \overline{\Omega} \tag{3.3}$$

for some $\beta_3 > 0$.

Then there are $C, \lambda(d) \in (0, \frac{1}{3})$ *such that*

$$\|\boldsymbol{E}\|(\Omega(d)) + \|\boldsymbol{H}\|(\Omega(d)) \leq$$

$$C(F(E) + k^{-\frac{1}{3}}(F^{\lambda_0}(E) + d^{2\lambda_0}F^{\lambda_0}(E;k,d))M^{1-\lambda_0} + k^{-\frac{1}{3}}d^{-2\lambda_0}M^{1-\lambda(d)}F^{\lambda(d)}(E;d,k)) \tag{3.4}$$

for all $\boldsymbol{E}, \boldsymbol{H}$ *solving* (5.1), (3.2)*. Here* $\lambda_0 = \frac{1}{3}$.

If ε, μ *are constants and* $\sigma = 0$*, then*

$$\|\boldsymbol{E}\|(\Omega) + \|\boldsymbol{H}\|(\Omega) \leq C(F(E) + \frac{M}{(-ln\delta_1 + k)^{\frac{1}{16}}}), \tag{3.5}$$

where $\delta_1 = \frac{F(E)}{M}$.

The condition (3.3) guarantees absence of trapped rays in the corresponding dynamical (time dependent) problem. Presence of trapped (disjoint with Γ) rays makes the improving stability estimate (3.4) impossible, as shown in [19]. It is related to monotonicity of the speed of the propagation $(\varepsilon\mu)^{-\frac{1}{2}}$ with respect to x_3 in the dynamical case. The monotonicity condition is in particular very well known in the (geophysical) inverse seismic problem. Its violation can even result in non uniqueness of the continuation for the dynamical equations (see [14], p. 70, and related references in this book). If the speed of the propagation decreases with respect to x_3 (i.e. increases in the direction of the continuation), then the condition (3.3) can be achieved by choosing large β_3.

For the dynamical Maxwell system sharp uniqueness of the continuation results and stability estimates were obtained in the paper [11]. In the dynamical case the Cauchy data are given on $\Gamma \times (0, T)$. The best possible Lipschitz type stability in [11] requires Γ to be more than "one half" of $\partial\Omega$, it needs some boundary data on $\partial\Omega \setminus \Gamma$ and T to be sufficiently large. So the conditions in [11] are global. In our result Γ

can be any part of the boundary of a larger domain extending Ω, which means much more flexible local data.

The assumptions on Ω can be relaxed, moreover Theorem 1.1 implies increasing stability of the continuation from the boundary of a convex domain Ω_1 onto $\Omega_1 \setminus D$ where D is a convex subdomain of Ω. Indeed, complement of a convex D can represented as the union of half-spaces P, as on Figure 2. Applying Theorem 1.1 to $\Omega_1 \cap P$ and combining the estimates over the union of P-s we complete the argument. For more detail we refer to [3], [13].

First we reduce (5.1), (3.2) to the Cauchy problem for a system of the equations of second order with the diagonal Helmholtz operator in its principal part. Next we obtain energy type estimates in the low frequency zone, which are vectorial versions of estimates in [16] and combine them to obtain a conditional Lipschitz stability estimate with an additional energy type term. An additional (to [16]) difficulty is that coefficients of low order terms can depend on k. Another new ingredient of our results is a removal of an additional artificial small parameter in [15], [16]. Finally, we use a scalar Carleman type estimate in [16], [4] to obtain Hölder type stability estimates of the energy type term and complete the proof of stability estimate for principally diagonal auxiliary system.

3.1 A reduction to a second order system

We start with a simple result which reduces the Maxwell's system to a vectorial Helmholtz equation. This reduction is well known, but we need details of it which are essential for transformation of the Cauchy data.

Lemma 3.1.1. *The Cauchy Problem for Maxwell's System* (5.1),(3.2) *implies the* *following Cauchy problem*

$$\Delta\boldsymbol{E}+k^2\varepsilon\mu\boldsymbol{E}+\nabla((\varepsilon+ik^{-1}\sigma)^{-1}\nabla(\varepsilon+i\sigma k^{-1})\cdot\boldsymbol{E})+\mu^{-1}\nabla\mu\times curl\boldsymbol{E}+\mu\sigma ik\boldsymbol{E}=0 \ \ in \ \ \Omega,$$

$$\Delta\boldsymbol{H}+k^2\varepsilon\mu\boldsymbol{H}+(\varepsilon+i\sigma k^{-1})^{-1}(\nabla(\varepsilon+i\sigma k^{-1})\times\boldsymbol{H})+\nabla(\mu^{-1}\nabla\mu\cdot\boldsymbol{H})+\mu\sigma ik\boldsymbol{H}=0 \ \ in \ \ \Omega, \tag{3.6}$$

with the Cauchy data

$$\boldsymbol{E}=\boldsymbol{E}_0, \partial_\nu\boldsymbol{E}=\boldsymbol{E}_1, \ \boldsymbol{H}=\boldsymbol{H}_0, \partial_\nu\boldsymbol{H}=\boldsymbol{H}_1 \ on\,\Gamma, \tag{3.7}$$

where

$$|\boldsymbol{E}_1|+|\boldsymbol{H}_1|\leq|\nabla_{tan}\boldsymbol{E}_0|+|\nabla_{tan}\boldsymbol{H}_0|+C(k(|\boldsymbol{E}_{0tan}|+|\boldsymbol{H}_{0tan}|)+|\boldsymbol{E}_0|+|\boldsymbol{H}_0|). \tag{3.8}$$

Proof:

From the first equation in (5.1) we have

$$0=\text{curl (curl }\boldsymbol{E})-ik\text{curl }\mu\boldsymbol{H}=-\Delta\boldsymbol{E}+\nabla div\boldsymbol{E}-ik(\nabla\mu\times\boldsymbol{H})-ik\mu\text{curl }\boldsymbol{H}.$$

From the second equation in (5.1) it follows that $div((ik\varepsilon-\sigma)\boldsymbol{E})=0$ and hence

$$div\boldsymbol{E}=-(ik\varepsilon-\sigma)^{-1}\nabla(ik\varepsilon-\sigma)\cdot\boldsymbol{E}. \tag{3.9}$$

Expressing, in addition, $\boldsymbol{H}$ from the first set of equations in (5.1) and $curl\mathbf{H}$ from the second set in (5.1) we yield the first set (5.6). The second set is obtained by a similar argument.

To bound the Cauchy data we consider any $x_0\in\Gamma$ and use an orthonormal coordinate system where the direction x_3-axis is the normal $\nu(x_0)$. Using the invariance of the Maxwell system with respect to these coordinates we will have

$$\partial_2E_3-\partial_3E_2=ik\mu H_1, \ \partial_3E_1-\partial_1E_3=ik\mu H_2;$$

and therefore

$$\partial_3 E_1 = \partial_2 E_3 + ik\mu H_2,$$

$$\partial_3 E_2 = \partial_2 E_3 - ik\mu H_1$$

at x_0. From (3.9) we have

$$\partial_3 E_3 = -\partial_1 E_1 - \partial_2 E_2 - (\varepsilon + i\sigma k^{-1})^{-1}\nabla(\varepsilon + ik^{-1}\sigma) \cdot \boldsymbol{E}).$$

Observing that E_1, E_2 are tangential components of $\boldsymbol{E}$ at x_0 and ∂_1, ∂_2 are tangential differentiations at x_0 from three previous equalities we yield (3.8) for $\boldsymbol{E}_1$. The bound for $\boldsymbol{H}_1$ is obtained similarly.

The proof is complete.

We observe that the data (3.2) are overdetermined. Indeed, in the same coordinate system at $x_0 \in \Gamma$ from (5.1) we have $\partial_1 E_2 - \partial_2 E_1 = ik\mu H_3$, so the normal components H_3 and similarly E_3 are linear combinations of tangential derivatives of tangential components $\boldsymbol{E}_{tan}, \boldsymbol{H}_{tan}$ and it is sufficient to prescribe on Γ only tangential components of electromagnetic vector. However, then H^1 norms of normal components will be bounded by H^2 norms of tangential components. When studying stability we prefer to use more natural Sobolev norms for all components by prescribing the overdetermined Cauchy data (3.2).

Let $m \times m$ matrix functions $\mathbf{B}_l, l = 1, 2, 3,$ $\mathbf{C} = \mathbf{C}_1 k + \mathbf{C}_0 \in C^1(\bar{\Omega})$ and a positive function $a_0 \in C^2(\bar{\Omega})$. We will consider the Cauchy problem for the (more general than (5.6)) principally diagonal system

$$(\Delta + a_0^2 k^2 + \sum_{l=1}^{3} \mathbf{B}_l \partial_l + \mathbf{C})\mathbf{u} = \mathbf{f} \text{ in } \Omega, \tag{3.10}$$

$$\mathbf{u} = \mathbf{u}_0, \partial_\nu \mathbf{u} = \mathbf{u}_1 \text{ on } \Gamma. \tag{3.11}$$

Theorem 3.1.1. *Let us assume the condition*

$$0 < a_0 + \nabla a_0 \cdot x + \beta_3 \partial_3 a_0,\ 0 \leq \partial_3 a_0 \ on\ \overline{\Omega}. \tag{3.12}$$

Then there are $C, \lambda(d) \in (0, \frac{1}{3})$ *such that*

$$\|\mathbf{u}\|(\Omega(d)) \leq C(F + k^{-\frac{1}{3}}(F^{\lambda_0} + d^{2\lambda_0} F^{\lambda_0}(k,d))M_1^{1-\lambda_0} + k^{-\frac{1}{3}} d^{-2\lambda_0} M_1^{1-\lambda(d)} F^{\lambda(d)}(k,d)) \tag{3.13}$$

for all $\mathbf{u}$ *solving* (3.10), (3.11) *provided* $\|\mathbf{u}\|_{(1)}(\Omega) \leq M_1$. *Here* $\lambda_0 = \frac{1}{3}$.

This result will be proven in section 4.

The bound (3.4) of Theorem 1.1 immediately follows from Lemma 2.1 and Theorem 2.1. The bound (3.5) follows from Lemma 2.1 and Theorem in [3] for the Helmholtz equation.

3.2 Energy type estimates in low frequency zone

We will obtain some auxiliary results imitating the standard energy estimate for hyperbolic initial value problems

In Lemmas 3.1-3.4 $a \in C^1([0,h])$ is a scalar function, $a = a(x_3)$, $\mathbf{B}_l(3), \mathbf{C}_1(3), \mathbf{C}_0(3) \in C^1([0,h]), l = 1,2,3$ be $m \times m$ matrices depending only on x_3, $\mathbf{C}(3) = \mathbf{C}_1(3)k + \mathbf{C}_0(3)$ and vector functions $\mathbf{v}_j \in C^2(\bar{\Omega}^*)$ (with values in $\mathbb{R}^m$) are zero outside $\bar{\Omega}$. In this section we let $\Omega^*(d) = \{x : d < x_3 < h\}$ and denote by $\mathbf{V}(\xi, x_3)$ the Fourier transform of the function $\mathbf{v}(x', x_3)$ with respect to x'.

Lemma 3.2.1. *Let a vector function* $\mathbf{v}_j, j = 1,2,3$ *solve the initial value problem*

$$(\Delta + a^2k^2 + \sum_{l=1}^{3} \mathbf{B}_l(3)\partial_l + \mathbf{C}(3))\mathbf{v}_j = \partial_j \mathbf{f}_j \ in\ \Omega^*(d),\ j = 1,2,$$

$$\mathbf{v}_j = 0 \ on\ \Omega^*(h_1) \tag{3.1}$$

for some $h_1 \in (d,h)$, $\mathbf{f}_j \in C^\infty(\bar\Omega^*(d))$, $\mathbf{f}_j = 0$ *on* $\Omega^*(h_1)$, *and*

$$\mathbf{V}_j(\xi, x_3) = 0 \text{ when } \frac{a^2(x_3)}{2}k^2 < |\xi|^2 \tag{3.2}$$

Then there is constant C *depending only on* $h, sup(|\mathbf{B}|_l + |\mathbf{C}_1| + |\mathbf{C}_0| + |\partial_3 a| + |a|), supa^{-1}$ *over* $(0,h)$, *such that*

$$\|\mathbf{v}_j\|(\Omega^*(d)) \leq C\|\mathbf{f}_j\|(\Omega^*(d)). \tag{3.3}$$

Proof. Due to Parseval's identity it suffices to show that the solution to the initial value problem

$$\partial_3^2\mathbf{V}_j + (a^2k^2 - |\xi|^2)\mathbf{V}_j + \mathbf{B}_3(3)\partial_3\mathbf{V}_j - \sum_{l=1}^{2} i\mathbf{b}_l(3)\xi_l\mathbf{V}_j + (\mathbf{a}_1(3)k + \mathbf{a}_0(3))\mathbf{V}_j =$$

$$-i\xi_j\mathbf{F}_j \text{ on } (d,h), j = 1,...,n-1, \tag{3.4}$$

with the zero final conditions

$$\mathbf{V}_j = 0,\ \mathbf{F}_j = 0 \text{ on } (h_1, h), \tag{3.5}$$

satisfies the bound

$$\int_d^h |\mathbf{V}_j|^2(\xi,s)ds \leq C\int_d^h |\mathbf{F}_j|^2(\xi,s)ds,\ j = 1,...,n-1. \tag{3.6}$$

Taking the inner product of the both sides of (3.4) and of $\partial_3\bar{\mathbf{V}}_j$, taking complex conjugate and adding results we yield

$$(\partial_3^2\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}_j + (\partial_3^2\bar{\mathbf{V}}_j)\cdot\partial_3\mathbf{V}_j + (a^2k^2 - |\xi|^2)(\mathbf{V}_j\cdot\partial_3\bar{\mathbf{V}}_j + \bar{\mathbf{V}}_j\cdot\partial_3\mathbf{V}_j) +$$

$$(\mathbf{B}_3(3)\partial_3\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}_j + (\mathbf{b}_3(3)\partial_3\bar{\mathbf{V}}_j)\cdot\partial_3\mathbf{V}_j - \sum_{l=1}^{2} i((\mathbf{B}_l(3)\xi_l\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}} - (\mathbf{B}_l(3)\xi_l\bar{\mathbf{V}}_j)\cdot\partial_3\mathbf{V})$$

$$((\mathbf{C}(3))\mathbf{V}_j\cdot\partial_3\bar{\mathbf{V}}_j + ((\mathbf{C}(3))\bar{\mathbf{V}}_j\cdot\partial_3\mathbf{V}_j = i\xi_j(\mathbf{F}_j\cdot\partial_3\bar{\mathbf{V}}_j - \bar{\mathbf{F}}_j\cdot\partial_3\mathbf{V}_j).$$

Observing that $\partial_3|\mathbf{V}|^2 = \mathbf{V}\cdot\partial_3\bar{\mathbf{V}} + \partial_3\mathbf{V}\bar{\mathbf{V}}$ and multiplying by -$e^{\tau x_3}$ we will have

$$-(\partial_3|\partial_3\mathbf{V}|^2)e^{\tau x_3} - (a^2k^2 - |\xi|^2)\partial_3|\mathbf{V}_j|^2 e^{\tau x_3} - 2\Re(\mathbf{b}_3(3)\partial_3\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}_j)e^{\tau x_3} +$$

$$2\Re(\sum_{l=1}^{2} i((\mathbf{B}_l(3)\xi_l \mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}})e^{\tau x_3} - 2\Re((\mathbf{C}(3))\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}_j)e^{\tau x_3} =$$

$$-2\Re(i\xi_j(\mathbf{F}_j\cdot\partial_3\bar{\mathbf{V}}_j)e^{\tau x_3}.$$

Integrating by parts over the interval (x_3, h) with use of (3.5) we obtain

$$|\partial_3\mathbf{V}_j|^2(x_3)e^{\tau x_3} + (a^2k^2 - |\xi|^2)|\mathbf{V}_j|^2(x_3)e^{\tau x_3} + \int_{x_3}^{h}(\tau - 2\Re(\mathbf{B}_3(3)\partial_3\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}_j(s))e^{\tau s}ds +$$

$$\int_{x_3}^{h}(\tau(a^2(k^2-|\xi|^2) + 2a\partial_3 ak^2)|\mathbf{V}_j|^2(s)e^{\tau s}ds +$$

$$\int_{x_3}^{h} 2\Re(\sum_{l=1}^{2} i((\mathbf{B}_l(3)\xi_l\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}) - 2\Re((\mathbf{C}(3))\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}_j)(s)e^{\tau s}ds =$$

$$-\Re(i\xi_j\int_{x_3}^{h}(\mathbf{F}_j\partial_3\cdot\bar{\mathbf{V}}_j)(s)e^{\tau s}ds). \tag{3.7}$$

By elementary inequalities

$$|\int_{x_3}^{h} 2\Re(\sum_{l=1}^{2} i((\mathbf{B}_l(3)\xi_l\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}) - 2\Re((\mathbf{C}(3))\mathbf{V}_j)\cdot\partial_3\bar{\mathbf{V}}_j)(s)e^{\tau s}ds| \leq$$

$$C\int_{x_3}^{h}|\partial_3\mathbf{V}_j|^2(s)e^{\tau s}ds + \int_{x_3}^{h}|\xi|^2|\mathbf{V}_j|^2(s)e^{\tau s}ds + \int_{x_3}^{h}k^2|\mathbf{V}_j|^2(s)e^{\tau s}ds,$$

and

$$|-\Re(i\xi_j\int_{x_3}^{h}(\mathbf{F}_j\partial_3\cdot\bar{\mathbf{V}}_j)(s)e^{\tau s}ds| \leq \int_{x_3}^{h}|\xi|^2|\mathbf{F}_j|^2(s)e^{\tau s}ds + \int_{x_3}^{h}|\partial_3\mathbf{V}_j|^2(s)e^{\tau s}ds,$$

so using that $|\xi| < Ck$, due to condition (3.2), and dropping first two terms on the left side of (3.7) we yield

$$\int_{x_3}^{h}(\tau - C)|\partial_3\mathbf{V}_j|^2(s)e^{\tau s}ds +$$

$$\int_{x_3}^{h}(\tau(a^2k^2 - |\xi|^2) - Ck^2)|\mathbf{V}_j|^2(s)e^{\tau s}ds \leq C\int_{x_3}^{h}k^2|\mathbf{F}_j|^2(s)e^{\tau s}ds. \tag{3.8}$$

Again due to the condition (3.2), $a^2\frac{k^2}{2} \leq a^2k^2 - |\xi|^2$ and choosing τ (depending on the same parameters as C) we achieve that the first integral in (3.7) is non negative, so we obtain

$$\int_{x_3}^{h} (\tau a_2 k^2 - Ck^2)|\mathbf{V}_j|^2(s)e^{\tau s}ds \le C \int_{x_3}^{h} k^2|\mathbf{F}_j|^2(s)e^{\tau s}ds$$

and finally choosing τ large but depending on the same parameters as C again we arrive at (3.3) and complete the proof.

Lemma 3.2.2. *Let $\mathbf{v}_3$ solve the initial value problem*

$$(\Delta + a^2k^2 + \sum_{l=1}^{3} \mathbf{B}_l(3)\partial_l + \mathbf{C}(3))\mathbf{v}_3 = \partial_3 \mathbf{f}_3 \text{ in } \Omega^*(d),$$

$$\mathbf{v}_3 = 0 \text{ on } \Omega^*(h_1) \tag{3.9}$$

for some $h_1 \in (d,h)$, $\mathbf{f}_3 \in C^\infty(\bar{\Omega}^*(d))$, $\mathbf{f}_3 = 0$ on $\Omega^*(h_1)$, and

$$\mathbf{V}_3(\xi, x_3) = 0 \text{ when } \frac{a^2(x_3)}{2}k^2 < |\xi|^2. \tag{3.10}$$

Then there is constant C depending only on $h, sup(|\mathbf{B}|_l + |\mathbf{C}_1| + |\mathbf{C}_0| + |\partial_3 a| + |a|), supa^{-1}$ over $(0,h)$, such that

$$\|\mathbf{v}_3\|(\Omega^*(d)) \le C\|\mathbf{f}_3\|(\Omega^*(d)). \tag{3.11}$$

Lemma 3.2.3. *Let $\mathbf{v}_4$ solve the initial value problem*

$$(\Delta + a^2k^2 + \sum_{l=1}^{3} \mathbf{B}_l(3)\partial_l + \mathbf{C}(3))\mathbf{v}_4 = k\mathbf{f}_4 \text{ in } \Omega^*(d),$$

$$\mathbf{v}_4 = 0 \text{ on } \Omega^*(h_1) \tag{3.12}$$

for some $h_1 \in (d,h)$, $f_3 \in C^\infty(\bar{\Omega}^*(d))$, $f_4 = 0$ on $\Omega^*(h_1)$, and

$$\mathbf{V}_4(\xi, x_3) = 0 \text{ when } \frac{a^2(x_3)}{2}k^2 < |\xi|^2. \tag{3.13}$$

Then there is constant C depending only on $h, sup(|\mathbf{B}|_l + |\mathbf{a}_1| + |\mathbf{a}_0| + |\partial_3 a| + |a|), supa^{-1}$ over $(0,h)$, such that

$$\|\mathbf{v}_4\|(\Omega^*(d)) \le C\|\mathbf{f}_4\|(\Omega^*(d)). \tag{3.14}$$

Lemma 3.2.4. *Let* $\mathbf{v}_0$ *solve the initial value problem*

$$(\Delta + a^2k^2 + \sum_{l=1}^{3} \mathbf{B}_l(3)\partial_l + (\mathbf{C}(3))\mathbf{v}_0 = k^2\mathbf{f}_0 \ in\ \Omega^*(d),$$

$$\mathbf{v}_0 = 0 \ on\ \Omega^*(h_1) \tag{3.15}$$

for some $h_1 \in (d,h)$, $f_3 \in C^\infty(\bar{\Omega}^*(d))$, $f_4 = 0$ *on* $\Omega^*(h_1)$, *and*

$$\mathbf{V}_0(\xi, x_3) = 0 \ when\ \frac{a^2(x_3)}{2}k^2 < |\xi|^2. \tag{3.16}$$

Then there is constant C *depending only on* $h, sup(|\mathbf{B}|_l + |\mathbf{C}_1| + |\mathbf{C}_0| + |\partial_3 a| + |a|), supa^{-1}$ *over* $(0,h)$, *such that*

$$\|\mathbf{v}_0\|(\Omega^*(d)) \leq C(\|\mathbf{f}_0\|(\Omega^*(d) + \|\partial_3\mathbf{f}_0\|(\Omega^*(d))). \tag{3.17}$$

Now by using Lemmas 3.1-3.4, freezing coefficients with respect to x' and partitioning the unity, we will obtain energy type estimates.

Let $\delta > 0$. By $X'(j)$ we denote points in $\mathbb{R}^2$ with integer coordinates. Let $x'(j), j = 1, ..., J$ be points $\delta X'(j)$ which are contained in $\Omega' = \{x' : x \in \Omega\}$. It is clear that $J \leq C\delta$. The balls $B'(x(j);\delta)$ form an open covering of $\overline{\Omega'}$. We define $\Omega_j = B'(x(j);\delta) \times (d,h)$. Let $\chi(x';j)$ be partition of the unity subordinated to this covering. We can assume that

$$0 \leq \chi(;j) \leq 1,\ |\nabla\chi(;j)| \leq C\delta^{-1},\ |\Delta\chi(;j)| \leq C\delta^{-2}. \tag{3.18}$$

.

We will introduce a "low frequency" projector $\mathbf{v}_1 = P\mathbf{v}$ of a function $\mathbf{v}$. Let us introduce a function $\chi \in C^\infty(\mathbb{R})$ such that $\chi = 1$ on $(0, 1/2)$, $\chi = 0$ on $(3/4, \infty)$, $0 \leq \chi \leq 1$. Let $\chi_j(x_3;\xi) = \chi(k^{-1}a_0^{-1}(x(j), x_3)|\xi|)$. We define

$$\mathbf{v}(;j) = \chi(;j)\mathbf{v},\ P_j\mathbf{v}(;j) = \mathcal{F}^{-1}\chi_j\mathcal{F}\mathbf{v}(;j),\ \mathbf{v}_1 = \sum_{j=1}^{J} P_j\mathbf{v}(;j). \tag{3.19}$$

where $\mathcal{F}$ the the Fourier transform with respect to x'.

For brevity in Lemma 3.5 we let $\|\mathbf{v}\|(d) = \|\mathbf{v}\|_{(0)}(\Omega^*(d))$.

Lemma 3.2.5. *Let* $\mathbf{v} \in C^2(\bar{\Omega}^*(d))$, $\mathbf{v} = 0$ *on* $\Omega^*(d)\setminus\Omega$, *solve the initial value problem*

$$(\Delta + a_0^2 k^2 + \sum_{l=1}^{3} \mathbf{B}_l \partial_l + \mathbf{C})\mathbf{v} = \partial_1 \mathbf{f}_1 + ... + \partial_3 \mathbf{f}_3 + k\mathbf{f}_4 + k^2 \mathbf{f}_0 \ \textit{in}\ \Omega^*(d),$$

$$\mathbf{v} = 0 \ \textit{on}\ \Omega^*(h_1) \tag{3.20}$$

for some $h_1 < h$.

Then there is a constant C *such that*

$$\|\mathbf{v}\|(d) \le C((1 + \delta^{-1}k^{-1})(\|\mathbf{f}_1\|(d) + ... + \|\mathbf{f}_3\|(d)) + \|\mathbf{f}_4\|(d) + \|\mathbf{f}_0\|(d) + \|\partial_3 \mathbf{f}_0\|(d) +$$

$$\delta^{-2}k^{-1}\|\mathbf{v}\|_{(1)}(\Omega^*(d)) + \delta(\|\mathbf{v}\|(d) + \|\partial_3 \mathbf{v}\|(d))). \tag{3.21}$$

Proof.

From (3.19) and from the Leibniz formula we have

$$(\Delta + k^2 a_0^2 + \sum_{l=1}^{3} \mathbf{B}_l \partial_l + \mathbf{C})\mathbf{v}(;j) =$$

$$\chi(;j)(\partial_1 \mathbf{f}_1 + ... + \partial_3 \mathbf{f}_3 + k\mathbf{f}_4 + k^2 \mathbf{f}_0) + 2\nabla\chi(;j)\cdot\nabla\mathbf{v} + (\sum_{l=1}^{3} \mathbf{B}_l \cdot \partial_l \chi(;j) + \Delta\chi(;j))\mathbf{v},$$

so

$$(\Delta + k^2 a_0^2(x'(j), x_3) + \sum_{l=1}^{3} \mathbf{B}_l(x'(j), x_3)\partial_l + \mathbf{C})(x'(j), x_3)\mathbf{v}(;j) =$$

$$\partial_1(\chi(;j)\mathbf{f}_1) + ... + \partial_3(\chi(;j)\mathbf{f}_3) - \partial_1\chi(;j)\mathbf{f}_1 - ... - \partial_3\chi(;j)\mathbf{f}_3 + k\chi(;j)\mathbf{f}_4 + k^2\chi(;j)\mathbf{f}_0 +$$

$$2\nabla\chi(;j)\cdot\nabla\mathbf{v} + (\sum_{l=1}^{3} \mathbf{B}_l \partial_l \chi(;j) + \Delta\chi(;j))\mathbf{v} + (k^2((a_0^2(x'(j), x_3) - a_0^2) +$$

$$\sum_{l=1}^{3} (\mathbf{B}_l(x'(j), x_3) - \mathbf{B}_l)\partial_l + (\mathbf{C}(x'(j), x_3) - \mathbf{C})\mathbf{v}(;j).$$

Applying the low frequency projector P_j to the both parts we yield

$$(\Delta + k^2 a_0^2(x'(j), x_3) + \sum_{l=1}^{3} \mathbf{B}_l(x'(j), x_3)\partial_l + \mathbf{C}(x'(j), x_3))P_j\mathbf{v}(;j) =$$

$$\partial_1 P_j(\chi(;j)\mathbf{f}_1) + ... + \partial_2 P_j(\chi(;j)\mathbf{f}_2) - P_{j,3}(\chi(;j)\mathbf{f}_3) -$$

$$P_j((\partial_1\chi(;j))\mathbf{f}_1) - ... - P_2((\partial_2\chi(;j))\mathbf{f}_2) + kP_j(\chi(;j)\mathbf{f}_4) + k^2 P_j(\chi(;j)\mathbf{f}_0) +$$

$$\mathcal{F}^{-1}\partial_3^2\chi_j\mathcal{F}\mathbf{v}(;j) + 2\mathcal{F}^{-1}\partial_3\chi_j\mathcal{F}\partial_3\mathbf{v}(;j) + \mathbf{B}_3(x'(j);)\mathcal{F}^{-1}\partial_3\chi_j\mathcal{F}\mathbf{v}(;j) +$$

$$+P_j((\Delta\chi(;j) + \sum_{l=1}^{3}\mathbf{B}_l\partial_l\chi(;j))\mathbf{v}) + P_j(2\nabla'\chi(j)\cdot\nabla\mathbf{v}) +$$

$$k^2 P_j((a_0^2(x'(j),x_3) - a_0^2) + P_j\sum_{l=1}^{3}(\mathbf{B}_l(x'(j),x_3) - \mathbf{B}_l)\partial_l + (\mathbf{C}(x'(j),x_3)) - \mathbf{C})\mathbf{v}(;j),$$

where $P_{j,3}(\mathbf{f}) = \mathcal{F}^{-1}\partial_3\chi_j\mathcal{F}\mathbf{f}$. Observing that

$$|(a^2(x'(j);) - a_0^2)| + |\partial_3(a^2(x'(j);) - a_0^2)| + |\mathbf{B}(x'(j),) - \mathbf{B}| \le C\delta,\ |\mathbf{C}(x'(j),) - \mathbf{C}| \le Ck\delta$$

on support of $\mathbf{v}(;j)$, that $\|P_j\mathbf{f}\| \le \|\mathbf{f}\|$, using (3.18), and applying Lemmas 3.1-3.4 we obtain

$$\|P_j\mathbf{v}(;j)\|^2(d) \le C(\|\chi(;j)\mathbf{f}_1\|^2(d) + ... + \|\chi(;j)\mathbf{f}_3\|^2(d) + \delta^{-2}k^{-2}(\|\mathbf{f}_1\|^2(\Omega_j) + ... + \|\mathbf{f}_3\|^2(\Omega_j)) +$$

$$\|\chi(;j)\mathbf{f}_4\|^2(d) + \|\chi(;j)\mathbf{f}_0\|^2(d) + \|\chi(;j)\partial_3\mathbf{f}_0\|^2(d) +$$

$$\delta^{-2}k^{-2}\|\nabla\mathbf{v}\|^2(\Omega_j) + \delta^{-4}k^{-2}\|\mathbf{v}\|^2(\Omega_j) + \delta^2(\|\mathbf{v}\|^2(\Omega_j) + \|\partial_3\mathbf{v}\|^2(\Omega_j)). \tag{3.22}$$

Now, summing local estimates (3.22) we will obtain a bound for $\mathbf{v}_1$ given by (3.19). Support of $\mathbf{v}(;j)$ intersects at most $2^3 = 8$ supports of other $\mathbf{v}(;k)$, but this is not true for $P_j\mathbf{v}(;j)$. To make certain that constants C be δ independent, we will use that $(I - P_j)\mathbf{v}(;j)$ is a high frequency component of $\mathbf{v}(;j)$ as defined by (3.19), hence

$$\|(I - P_j)\mathbf{v}(;j)\|^2(d) \le Ck^{-2}\|\mathbf{v}(;j)\|^2_{(1)}(d)$$

and

$$\|\mathbf{v}(;j)\|^2(d) = \|P_j\mathbf{v}(;j)\|^2(d) + \|(I-P_j)\mathbf{v}(;j)\|^2(d) \le \|P_j\mathbf{v}(;j)\|^2(d) + Ck^{-2}\|\mathbf{v}(;j)\|^2_{(1)}(\Omega_j).$$

Using that multiplicity of covering Ω_j is at most 2^3 and summing (3.22) over $j = 1, ..., J$ we yield

$$\|\mathbf{v}\|^2(d) \leq C \sum_{j=1}^{J} \|\mathbf{v}(;j)\|^2(d) \leq$$

$$C(\|\mathbf{f}_0\|^2(d) + ... + \|\mathbf{f}_4\|^2(d) + \|\partial_3 \mathbf{f}_0\|^2(d) +$$

$$\delta^{-2}k^{-2}(\|\mathbf{f}_1\|^2(d) + ... + \|\mathbf{f}_3\|^2(d)) + \delta^{-4}k^{-2}\|\mathbf{v}\|^2_{(1)}(\Omega^*(d)) + \delta^2(\|\mathbf{v}\|^2(d) + \|\nabla \mathbf{v}\|^2(d))).$$

where we also used that $\chi^2(;1) + ... + \chi^2(;J) \leq 1$ and that multiplicity of the covering Ω_j is less than 2^3. From the last bound we obtain (3.21) and complete the proof of Lemma 3.5.

3.3 Carleman estimates and a proof of stability

Theorem 3.3.1. *Let the condition (3.12) be satisfied.*

Then there are $C, \lambda_1(d) \in (0,1)$ such that

$$\|\mathbf{u}\|_{(1)}(\Omega(d)) \leq C(d^2 F(k,d) + d^{-2} M_1^{1-\lambda_1(d)} F^{\lambda_1(d)}(k,d)) \tag{3.23}$$

for all $\mathbf{u}$ solving (3.10), (3.11).

In the proofs we will use the following Carleman type estimate.

Let

$$w(x;\tau) = \int_{-1}^{1} exp(2\tau e^{\sigma(|x-\beta|^2 - \theta^2 t^2)})dt, \ \beta = (0,0,\beta_3).$$

Lemma 3.3.1. *Let the condition (3.3) be satisfied.*

Then there are constants C, θ such that

$$\int_{\Omega_1} ((\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2) w(,\tau) \leq$$

$$C(\int_{\Omega_1} |(\Delta + a_0^2 k^2)u|^2 w(,\tau) + \int_{\partial\Omega_1} ((\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2) w(,\tau))$$

for all functions $u \in H^2(\Omega_1)$ and all $\tau > C$.

A proof based on the known Carleman type estimates for hyperbolic equations is given in [16]

Proof of Theorem 4.1.

We will choose $\beta_3 = -(\frac{2r^2}{d} - \frac{3}{8}d), \beta = (0, 0, \beta_3)$ and we introduce the notation $\Omega_d = \Omega \cap \{(d-\beta_3)^2 < |x-\beta|^2\}$. We will assume that $3d^2 < 16r^2$, so $\beta_3 < 0$. Using our choice of β and considering the intersection of level surface $|x-\beta|^2 = (\frac{1}{2}d - \beta_3)^2$ with the lateral wall $\{|x'| = r\}$ of the cylindrical domain one can be convinced that the boundary layer $\{x_3 < \frac{1}{4}d\} \cap \Omega$ does not intersect $\Omega_{\frac{d}{2}}$. Indeed, if (x', x_3^*) is a point of the intersection of this cylindrical domain and of the boundary of $\Omega_{\frac{d}{2}}$ then

$$r^2 + (x_3^* - \beta_3)^2 = (d-\beta_3)^2 = (\frac{d}{8} + \frac{2r^2}{d})^2,$$

$(x_3^* - \beta_3)^2 = (\frac{d}{8} - \frac{2r^2}{d})^2$, and $x_3^* - \beta_3 = \frac{2r^2}{d} - \frac{d}{8}$, which gives $x_3^* = \frac{d}{4}$. Hence there is a cut-off function χ which is 1 on $\Omega_{\frac{d}{2}}$, zero near $\partial\Omega \cap \{x_3 = 0\}$ and which satisfy the bounds $|\nabla\chi| \leq Cd^{-1}, |\Delta\chi| \leq Cd^{-2}$.

We have

$$(\Delta + k^2a_0^2)(\chi\mathbf{u}) = \chi(\Delta + k^2a_0^2)\mathbf{u} + 2\nabla\chi\cdot\nabla\mathbf{u} + \Delta\chi\mathbf{u} =$$

$$\chi(\mathbf{f} - \sum_{l=1}^{3}\mathbf{B}\partial_l\mathbf{u}) - \mathbf{C}(\chi\mathbf{u}) + 2\nabla\chi\cdot\nabla\mathbf{u} + \Delta\chi\mathbf{u},$$

due to (3.10). Applying Lemma 4.1 to each of m components of $\chi\mathbf{u}$ instead of $\mathbf{u}$ and adding bounds for components we yield

$$\int_\Omega((\tau^3 + \tau k^2)|\chi\mathbf{u}|^2 + \tau|\nabla(\chi\mathbf{u})|^2)w(;\tau) \leq$$

$$C(\int_\Omega |\mathbf{f}|^2 w(;\tau) + \int_\Omega(|\nabla\mathbf{u}|^2 + k^2|\chi\mathbf{u}|^2)w(;\tau) + \int_{\Omega\setminus\Omega_{\frac{d}{2}}} |2\nabla\chi\cdot\nabla\mathbf{u} + (\Delta\chi)\mathbf{u}|^2 w(;\tau) +$$

$$\int_\Gamma((\tau^3 + \tau k^2)|\mathbf{u}|^2 + \tau|\nabla\mathbf{u}|^2 + \tau|\nabla\mathbf{u}|^2)w(;\tau))).$$

By choosing $\tau > C$ we can remove the term with $k^2|\chi\mathbf{u}|^2$ on the right side absorbing it by the left side. Using that $\chi = 1$ on $\Omega_{\frac{d}{2}}$ and choosing τ large once more we can similarly replace the integration domain in the second integral of the right side by $\Omega \setminus \Omega_{\frac{d}{2}}$. Shrinking the integration domain in the left side and using the choice of χ we yield

$$\int_{\Omega_d} ((\tau^3 + \tau k^2)|\mathbf{u}|^2 + \tau|\nabla\mathbf{u}|^2)w(;\tau) \leq$$
$$C(\int_{\Omega} |\mathbf{f}|^2 w(;\tau) + \int_{\Omega\setminus\Omega_{\frac{d}{2}}} (d^{-2}|\nabla\mathbf{u}|^2 + d^{-4}|\mathbf{u}|^2)w(;\tau)+$$
$$\int_{\Gamma}((\tau^3 + \tau k^2 + \tau d^{-2})|\mathbf{u}|^2 + \tau|\nabla\mathbf{u}|^2)w(;\tau)). \tag{3.24}$$

Let

$$b = e^{\sigma X^2},\ b_1 = e^{\sigma|d-\beta_3|^2},\ b_2 = e^{\sigma|\frac{d}{2}-\beta_3|^2},$$

where $X = sup|x - \beta|$ over $x \in \Omega$,

$$W(\tau) = \int_{-1}^{1} e^{2\tau b e^{-\sigma\theta^2 t^2}} dt,\ w_1(\tau) = \int_{-1}^{1} e^{2\tau b_1 e^{-\sigma\theta^2 t^2}} dt,\ w_2(\tau) = \int_{-1}^{1} e^{2\tau b_2 e^{-\sigma\theta^2 t^2}} dt.$$

Observing that $w_1 \leq w$ on Ω_d, $w \leq W$ on Ω, and $w \leq w_2$ on $\Omega \setminus \Omega_{\frac{d}{2}}$ and replacing w by its minimal value in the left side and by maximal values on the right side of (3.24) we yield

$$(\tau^3 + \tau k^2)w_1(\tau)\|\mathbf{u}\|^2(\Omega_d)) + \tau w_1(\tau)\|\nabla\mathbf{u}\|^2(\Omega_d) \leq$$

$$C(W(\tau)(\|\mathbf{f}\|^2(\Omega) + (\tau^3 + \tau(k^2 + d^{-2}))\|\mathbf{u}\|^2(\Gamma) + \tau\|\nabla\mathbf{u}\|^2(\Gamma))+$$

$$d^{-4}w_2(\tau)(\|\nabla\mathbf{u}\|^2(\Omega) + \|\mathbf{u}\|^2(\Omega))).$$

Dividing the both parts of this inequality by w_1 we obtain

$$(\tau^3 + k^2\tau)\|\mathbf{u}\|^2(\Omega_d) + \tau\|\nabla\mathbf{u}\|^2(\Omega_d) \leq$$

$$C(W(\tau)w_1^{-1}(\tau)(\|\mathbf{f}\|^2(\Omega) + (\tau^3 + \tau(k^2 + d^{-2}))\|\mathbf{u}\|^2(\Gamma) + \tau\|\nabla\mathbf{u}|^2(\Gamma))+$$

$$d^{-4}w_2(\tau)w_1^{-1}(\tau)(\|\nabla\mathbf{u}\|^2(\Omega)+\|\mathbf{u}\|^2(\Omega))). \tag{3.25}$$

Obviously,

$$W(\tau)w_1^{-1}(\tau)\leq Ce^{C(d)\tau}.$$

An important observation is that

$$w_2(\tau)w_1^{-1}(\tau)\leq Ce^{-\frac{\tau}{C}}.$$

Indeed, from the definition of b_j and β by elementary calculations

$$b_1-b_2=e^{\sigma(2r^2-\frac{d^2}{8}+(\frac{2r^2}{d}-\frac{3d}{8})^2)}(e^{\sigma(\frac{3d^2}{8}+2r^2)}-1)\geq C^{-1},$$

and therefore

$$w_1(\tau)\geq\int_{-1}^{1}e^{2\tau b_2e^{-\theta^2t^2}}e^{2\tau(b_1-b_2)e^{-\theta^2}}dt\geq w_2(\tau)e^{2\tau/C}.$$

Hence from (3.25) we have

$$k^2\|\mathbf{u}\|^2(\Omega_d)+\|\nabla\mathbf{u}\|^2(\Omega_d)\leq$$

$$C(e^{C(d)\tau}\tau^3F^2(k,d)+e^{-\tau C(d)^{-1}}d^{-4}M_1^2),\ \text{when}\ C<\tau. \tag{3.26}$$

By increasing C we can eliminate τ^3 in the right side.

To use (3.26) we need τ to be large. If $M_1\leq Cd^2F(k,d)$ for some C, then we have the Lipschitz bound (3.23). Otherwise we can equalize two terms in (3.26) by letting

$$\tau=\frac{C(d)}{C^2(d)+1}2ln\frac{M_1}{d^2F(k,d)}.$$

Then the right side in (3.26) is getting

$$Cd^{-2}F^{2\lambda_1}(k,d)M_1^{2(1-\lambda_1)}$$

with $\lambda_1=\lambda_1(d)=\frac{1}{C^2(d)+1}$, and using that $\Omega(d)\subset\Omega_d$ we obtain (3.23).

The proof is complete.

Proof of Theorem 2.1.

Since Γ is Lipschitz, by known extension theorems there is a function $\mathbf{u}^*$ such that $\mathbf{u} = \mathbf{u}^*, \nabla \mathbf{u} = \nabla \mathbf{u}^*$ on Γ and

$$\|\mathbf{u}^*\|_{(1)}(\Omega^*(0)) \leq C(\|\mathbf{u}\|(\Gamma) + \|\nabla \mathbf{u}\|(\Gamma)) \leq CF, \tag{3.27}$$

where we used the definition of F. Let $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$ on Ω and $\mathbf{v} = 0$ on $\Omega^*(0) \setminus \Omega$. It suffices to obtain the bound (3.13) for $\mathbf{v}$ instead of $\mathbf{u}$. Observe that

$$(\Delta + a_0^2 k^2 + \sum_{l=1}^{3} \mathbf{B}_l \partial_l + \mathbf{C})\mathbf{v} = \mathbf{f} - \sum_{l=1}^{3} \partial_l(\partial_l + (\mathbf{B}_l)\mathbf{u}^*) - k^2 a_0^2 \mathbf{u}^* + \sum_{l=1}^{3} (\partial_l \mathbf{B}_l - \mathbf{C})\mathbf{u}^* \text{ in } \Omega^*(0). \tag{3.28}$$

Since $\mathbf{v}$ is zero outside some cylinder by using known results about H^1-approximation of energy solutions by H^2-solutions we can assume that $\mathbf{v} \in H^2(\mathbb{R}^2 \times (0, h))$ and hence $f^* = \partial_1 f_1 + ... + \partial_3 f_3 + f_4$ with $\|f_j\| \leq CF$. By (3.28) and Lemma 3.5

$$\|\mathbf{v}\|(\mathbb{R}^2 \times (d, h)) \leq$$

$$C(k^{-1}\|\mathbf{f}\| + (1 + \delta^{-1}k^{-1})(\|\nabla \mathbf{u}^*\| + \|\mathbf{u}^*\|) +$$

$$\delta^{-2}k^{-1}(\|\mathbf{u}\|_{(1)}(\Omega(d)) + \|\mathbf{u}^*\|_{(1)}(\Omega(d))) + \delta(\|\mathbf{u}\|_{(1)}(\Omega(d)) + \|\mathbf{u}^*\|_{(1)}(\Omega(d)))) \leq$$

$$C(F + \delta^{-2}k^{-1}F + \delta^{-2}k^{-1}\|\mathbf{u}\|_{(1)}(\Omega(d)) + \delta M_1) \leq$$

$$C(F + \delta^{-2}k^{-1}(F + \|\mathbf{u}\|_{(1)}(\Omega(d))) + \delta M_1) \tag{3.29}$$

where we used that $\|\mathbf{v}\|_{(1)} \leq \|\mathbf{u}\|_{(1)} + F$ due to (3.27). The minimum of $\delta^{-2}A + \delta M_1$ with respect to δ is

$$CA^{\frac{1}{3}}M_1^{\frac{2}{3}}.$$

From this observation and from (3.29) we conclude that

$$\|\mathbf{v}\|(\Omega(d)) \leq C(F + k^{-\frac{1}{3}}(F + \|\mathbf{u}\|_{(1)}(\Omega(d)))^{\frac{1}{3}} M_1^{\frac{2}{3}}) \leq$$

$$C(F+k^{-\frac{1}{3}}F^{\frac{1}{3}}M_1^{\frac{2}{3}}+k^{-\frac{1}{3}}(d^2F(k,d)+d^{-2}M_1^{1-\lambda_1(d)}F^{\lambda_1(d)}(k,d))^{\frac{1}{3}}M_1^{\frac{2}{3}})\leq$$

$$C(F+k^{-\frac{1}{3}}(F^{\frac{1}{3}}+d^{\frac{2}{3}}F^{\frac{1}{3}}(k,d))M_1^{\frac{2}{3}}+k^{-\frac{1}{3}}d^{-\frac{2}{3}}M_1^{1-\lambda}F^{\lambda}(k,d))$$

where we used Theorem 4.1 and the elementary inequality $(a+b)^p \leq a^p+b^p, 0<p<1$ and let $\lambda = \frac{\lambda_1(d)}{3}$.

The proof is complete.

CHAPTER 4

NUMERICAL EXPERIMENTS

4.1 Introduction

This chapter demonstrates the increased stability in the recovery of density by conducting some numerical experiments. We know that the single layer representation of any solution to Helmholtz equation

$$(\Delta + k^2)u = 0 \text{ in } D \subset \mathbb{R}^3 \tag{4.1}$$

is given by

$$u(x) = \int_{\partial D} g(y)\Phi(x,y)ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{4.2}$$

where

$$\Phi(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

is the fundamental solution of the Helmholtz equation and g is the surface density or the amplitudes of the acoustic point sources given by $\Phi(x,y)$.

The double layer representation is given by

$$\frac{\partial u}{\partial \nu}(x) = \int_{\partial D} g(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D. \tag{4.3}$$

The inverse problem is to find the surface density function g from the given Cauchy data u and $\partial_\nu u$. For simplicity we assume that the density is constant with respect to frequency k. Recovering density from the measured noisy Cauchy data is very important problem in applications and is an ill-posed problem.

For numerical experiments $D = B_R = \{x : |x| < R\}$ and the Cauchy data u and $\partial_\nu u$ is given on the Unit sphere and the density g is defined on a sphere of radius

$R > 1$. This problem is important and interesting since we like to continue outside of the convex hull of the surface where the Cauchy data is prescribed. Here we assume that $g \in L_2(\partial B_R)$ and hence $u \in H^{(1)}(S^1)$.

4.2 Spherical Harmonics

From [10] we know that the function g, u and v can be expressed in terms of spherical harmonics. Since $g \in L_2(\partial B_R)$, we can write

$$g(y) = \frac{1}{R}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m Y_n^m(\hat{y}), \quad y \in \partial B_R \tag{4.4}$$

where $Y_n^m(\theta,\phi) = A_n^m P_n^m(\cos\theta)e^{im\phi}$. These coefficients a_n^m can be computed by using

$$a_n^m = K_n^m \int_0^{\pi}\int_0^{2\pi} g(\theta,\phi)P_n^m(\cos\theta)e^{-im\phi}d\phi\sin\theta d\theta,$$

where $K_n^m = RA_n^m$. Using the single layer representation for the solution u,

$$u(x) = \int_{\partial B_R}\Phi(x,y)g(y)ds(y), \ x \in S^1.$$

The fundamental solution $\Phi(x,y)$ is given by

$$\Phi(x,y) = ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n} j_n(k|x|)Y_n^m\left(\frac{x}{|x|}\right)h_n^{(1)}(k|y|)\overline{Y_n^m\left(\frac{y}{|y|}\right)}, \quad |y| > |x|.$$

Therefore

$$u(x) = ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m j_n(k|x|)h_n^{(1)}(k|y|)Y_n^m\left(\frac{x}{|x|}\right), \tag{4.5}$$

or

$$u(x) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_n^m Y_n^m(\hat{x}),$$

where

$$u_n^m = ika_n^m j_n(kr)h_n^{(1)}(Rk).$$

Differentiating (4.5) with respect to r we have

$$\partial_r u(x) = ik^2\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m j_n'(kr)h_n^{(1)}(kR)Y_n^m(\hat{x}). \tag{4.6}$$

4.3 Spectral Analysis

In this section we analyze the spectrum of the forward operator which maps g in to the Cauchy data u and $\partial_r u$. First we start by looking at the singular values of the single layer operator. Since $g \in L_2(\partial B_R)$, it can be shown that $u \in H_{(1)}(S^1)$. We can write (4.2) as

$$(A_1 g)(x) = \int_{\partial B_R} g(y)\Phi(x,y)ds(y), \quad x \in S^1, \tag{4.7}$$

Here the operator A_1 maps $g \in L_2(\partial B_R)$ in to $u \in H_{(1)}(S^1)$. From (4.5) the operator A_1 has the singular values

$$\sigma_n(A_1) = k\sqrt{1+n(n+1)}|j_n(kr)h_n^{(1)}(kR)|.$$

Similarly we can write (4.3) as

$$(A_2 g)(x) = \int_{\partial B_R} g(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}ds(y), \quad x \in S^1. \tag{4.8}$$

Here the operator A_2 maps $g \in L_2(\partial B_R)$ in to $u \in L_2(S^1)$. From (4.6) the operator A_2 has the singular values

$$\sigma_n(A_2) = k^2|j_n'(kr)h_n^{(1)}(kR)|.$$

Therefore the forward operator is given by $A = [A_1, A_2]^T$, which maps the density function $g \in L_2(\partial B_R)$ into the Cauchy data pair $(u, \partial_r u) \in H_{(1)}(S^1) \times L_2(S^1)$. The singular values of the operator A are given by

$$\sigma_n(A) = k|h_n^{(1)}(kR)|\sqrt{(1+n(n+1))|j_n(kr)|^2 + k^2|j_n'(kr)|^2}.$$

Using these singular values we can compute the condition numbers of this operator for different values of N and k. These condition numbers are given in Fig. 4.1. By looking at these condition numbers we can observe that when $N \approx K$ the condition numbers are low.

	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
K=1	3.7332	18.632	78.473	318.381	1281.301	5141.56	20604.92	82518.8	330344	1322143	5290822	2.1E+07	8E+07	3.4E+08	1.356E+09	5.424E+09
2	2.1803	8.2964	51.882	250.526	1092.364	4599.21	19039.35	78061.3	318122	1291188	5225712	2.1E+07	9E+07	3.4E+08	1.379E+09	5.545E+09
4	2.0011	2.0011	6.15987	43.6654	327.6686	1886.3	9267.603	42513.4	187885	811190	3447457	1.4E+07	6E+07	2.5E+08	1.03E+09	4.229E+09
6	5.6761	5.6761	5.67612	5.67612	29.63339	245.322	2007.252	12899.6	69426.6	342327	1605332	7284679	3E+07	1.4E+08	605773677	2.58E+09
8	3.1209	3.2101	3.21008	3.21008	3.981667	20.2675	148.8893	1332.1	11502.5	79830.8	461699	2418326	1E+07	5.7E+07	261610222	1.181E+09
10	1.3817	2.3268	2.32683	2.43858	2.43858	3.69774	16.54569	105.566	867.337	8143.11	73038.8	536932	3E+06	1.8E+07	93819372	463814034
12	2.4526	4.2686	4.26861	4.26861	4.719397	4.7194	4.719397	14.6306	82.5253	602.698	5318.01	51632	476653	3665583	23600870	136012433
14	10.343	10.343	10.3433	10.3433	10.3433	11.7159	11.71593	11.7159	13.1479	66.874	437.115	3494.17	32404	322593	3047143	24304628
16	2.2934	2.2934	2.46066	2.46066	2.460658	2.48601	3.059667	3.05967	3.44583	11.977	55.8378	331.054	2397.5	20449.2	196760.93	1997669.4
18	1.2025	2.1046	3.61511	3.61511	3.615114	3.61511	3.758532	4.57801	4.57801	4.57801	11.0329	47.7589	260.12	1721.51	13446.81	120440.19
20	4.0733	4.4074	4.40742	4.40742	4.407425	4.40742	4.407425	4.66277	5.5712	5.5712	5.5712	10.257	41.648	210.65	1286.3275	9249.3782
22	9.1191	9.1191	9.11912	9.11912	9.119124	9.11912	9.119124	9.11912	10.0347	11.7218	11.7218	11.7218	11.722	36.8945	174.84907	993.92012
24	1.8718	1.8718	2.08726	3.28065	3.280649	3.28065	3.280649	3.28065	3.28065	3.89301	4.46651	4.46651	4.4665	9.10512	33.279939	148.87772
26	1.545	2.9785	3.77471	3.77471	3.774707	3.77471	3.774707	3.77471	3.77471	3.77471	4.55651	5.16359	5.1636	5.16359	8.7158122	30.488531
27	2.8299	2.8299	2.82985	3.18515	3.185147	3.18515	3.185147	3.18515	3.18515	3.18515	3.18515	4.47099	4.471	4.47099	5.002479	15.407799
28	4.8453	10.143	10.1429	10.1429	10.14293	10.1429	10.14293	10.1429	10.1429	10.1429	10.1429	12.6674	14.181	14.1809	14.180907	14.180907
29	1.1153	1.7921	4.53875	4.6424	4.642404	4.6424	4.642404	4.6424	4.6424	4.6424	4.6424	4.74707	6.6641	6.66413	6.664134	6.664134
30	5.1337	5.1337	5.13366	5.13366	5.133655	5.13366	5.133655	5.13366	5.13366	5.13366	5.13366	5.13366	6.7239	7.44009	7.4400867	7.4400867
31	2.7747	4.8324	5.51965	5.51965	5.519647	5.51965	5.519647	5.51965	5.51965	5.51965	5.51965	5.51965	5.7345	7.92014	7.9201366	7.9201366
32	1.3549	1.3549	2.64293	3.33679	3.33679	3.33679	3.33679	3.33679	3.33679	3.33679	3.33679	3.33679	3.3368	4.44689	4.8672143	4.8672143
34	1.8888	2.7642	6.20415	6.20415	6.204154	6.20415	6.204154	6.20415	6.20415	6.20415	6.20415	6.20415	6.2042	6.20415	8.4749427	9.182809
36	19.53	19.53	19.5304	19.5304	19.53044	19.5304	19.53044	19.5304	19.5304	19.5304	19.5304	19.5304	19.53	19.5304	19.530435	27.110225
40	1.0766	1.3642	2.18709	4.97073	4.970731	4.97073	4.970731	4.97073	4.97073	4.97073	4.97073	4.97073	4.9707	4.97073	4.9707311	5.0081254
44	14.895	14.895	14.8954	14.8954	14.89537	14.8954	14.89537	14.8954	14.8954	14.8954	14.8954	14.8954	14.895	14.8954	14.895373	14.895373
48	1.3827	1.8835	3.61511	5.61047	5.61047	5.61047	5.61047	5.61047	5.61047	5.61047	5.61047	5.61047	5.6105	5.61047	5.61047	5.61047

Figure 4.1: Condition Numbers of the operator A for different values of k and N, with $r = 1$ and $R = 2$.

4.4 Inverse Problem

The inverse problem is to find the surface density function g from the given Cauchy data u and v. First we start with a know density function for example:

$$g(\theta, \phi) = \begin{cases} 1 & \text{for} \quad \theta \leq \frac{\pi}{2} \\ 0 & \text{for} \quad elsewhere \end{cases}$$

Using this density function g first solve the forward problem to compute the Cauchy data. This Cauchy data plus noise gives us the required data for the inverse problem. Using this noisy Cauchy data we try to reconstruct the density function g. With increasing frequency we can notice that the error in reconstruction of g decreases.

To this end we will assume that the noisy Cauchy data is given on the unit sphere and reconstruct density g on a sphere of radius R. Here $R = 2$, although g does not depend on the radius, but is a function of θ and ϕ, which is given by

$$g(\theta, \phi) = \frac{1}{2} \sum_{n=0}^{N} \sum_{m=-n}^{n} a_n^m Y_n^m (\theta, \phi) ,$$

and the Cauchy data

$$u(x) = ik \sum_{n=0}^{N} \sum_{m=-n}^{n} a_n^m j_n(k) h_n^{(1)}(2k) Y_n^m (\theta, \phi) , \tag{4.9}$$

$$\partial_r u(x) = ik^2 \sum_{n=0}^{N} \sum_{m=-n}^{n} a_n^m j_n'(k) h_n^{(1)}(2k) Y_n^m (\theta, \phi). \tag{4.10}$$

Theorem 4.4.1. *Lipschitz Stability: Let $k^2 \geq 3N(N+1)$, then*

$$R^{-1} \|\partial_r u\|^2 (\partial B_R) + k^2 R^{-1} \|u\|^2 (\partial B_R) - R^{-3} \| \left(\sqrt{\Delta_{(\phi,\theta)}}\right) u\|^2 (\partial B_R)$$

$$\leq \|\partial_r u\|^2 (S^1) + k^2 \|u\|^2 (S^1) - \| \left(\sqrt{\Delta_{(\phi,\theta)}}\right) u\|^2 (S^1).$$

Proof. From (4.9) we can write

$$u(x) = \sum_{n=0}^{N} \sum_{m=-n}^{n} u_n^m(r) Y_n^m(\theta, \phi), \tag{4.11}$$

where $u_n^m(r) = ika_n^m j_n(kr) h_n^{(1)}(kR)$, solves the differential equation

$$\partial_r(r^2 \partial_r u_n^m) + (k^2 r^2 - n(n+1)) u_n^m = 0. \tag{4.12}$$

Multiplying this equation by $r^{-3}\partial_r u_n^m$ and integrating by parts we have

$$0 = \int_1^R (\partial_r(r^2\partial_r u_n^m) + (k^2r^2 - n(n+1))u_n^m) r^{-3}\partial_r u_n^m dr$$

$$= \int_1^R \frac{r^{-5}}{2}\partial_r(r^2\partial_r u_n^m)^2 + (k^2r^{-1} - r^{-3}n(n+1))\frac{\partial_r(u_n^m)^2}{2}dr$$

$$= \frac{r^{-1}}{2}(\partial_r u_n^m)^2|_1^R + \frac{5}{2}\int_1^R r^{-6}(r^2\partial_r u_n^m)^2 dr + \frac{1}{2}(k^2r^{-1} - r^{-3}n(n+1)(u_n^m)^2)|_1^R$$
$$+\frac{1}{2}\int_1^R (k^2r^{-2} - 3r^{-4}n(n+1))(u_n^m)^2 dr$$

$$= \frac{1}{2}R^{-1}(\partial_r u_n^m(R))^2 - \frac{1}{2}(\partial_r u_n^m(1))^2 + \frac{1}{2}(k^2R^{-1} - R^{-3}n(n+1))(u_n^m(R))^2 - \frac{1}{2}(k^2 - n(n+1))(u_n^m(1))^2$$

$$+\frac{1}{2}\int_1^R (5r^{-2}(\partial_r u_n^m)^2 + (k^2r^{-2} - 3r^{-4}n(n+1))(u_n^m)^2) dr.$$

$k^2 \geq 3N(N+1)$ implies that $k^2r^{-2} - 3r^{-4}n(n+1) \geq 0$ and therefore we have the following inequality

$$R^{-1}(\partial_r u_n^m(R))^2 + (k^2R^{-1} - R^{-3}n(n+1))(u_n^m(R))^2 \leq (\partial_r u_n^m(1))^2 + (k^2 - n(n+1))(u_n^m(1))^2).$$

Using Parseval's identity and the fact that $\Delta_{(\phi,\theta)}Y_n^m(\theta,\phi) = -n(n+1)Y_n^m(\theta,\phi)$ we can conclude the proof.

$\square$

4.5 Numerical Results

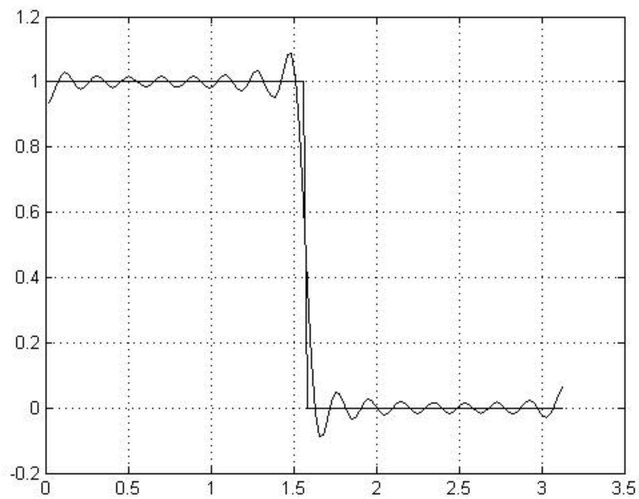

Figure 4.2: g and g_{approx} for $N = 32$

In this section we provide some pictures and errors of reconstruction of the density function. Here we let $R = 2$. First we start with a know density function g which has a jump discontinuity, given by

$$g(\theta,\phi) = \begin{cases} 1 & \text{for} \quad \theta \leq \frac{\pi}{2}, \\ 0 & \text{for} \quad elsewhere \end{cases} .$$

Using this density function g first solve the forward problem to compute the Cauchy data, given by (4.9) and (4.10). First for the given function g, we compute the Fourier type coefficients a_n^m using

$$a_n^m = K_n^m \int_0^{\pi} \int_0^{2\pi} g(\theta,\phi) P_n^m(\cos\theta) e^{-im\phi} d\phi \sin\theta d\theta.$$

The above integral is computed by using Gauss-Legendre quadrature rule as explained in [5].

Let

$$g_{approx}(\theta, \phi) = \frac{1}{2} \sum_{n=0}^{N} \sum_{m=-n}^{n} a_n^m Y_n^m (\theta, \phi) .$$

The graphs of g and g_{approx}, for $N = 32$, are given in figures 4.2, 4.3 and 4.4.

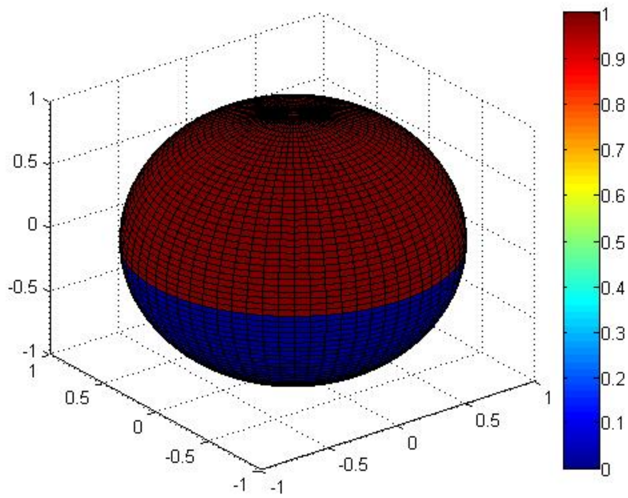

Figure 4.3: $g(\phi, \theta)$

Next using these coefficients compute the the Cauchy data using (4.9) and (4.10). Adding noise to this Cauchy data gives us the required data for the inverse problem. Noisy Cauchy data is given by

$$u_\delta = u + \delta \|u\|_2 \frac{\xi}{\|\xi\|_2}, \qquad (4.13)$$

$$\partial_r u_\delta = \partial_r u + \delta \|\partial_r u\|_2 \frac{\xi}{\|\xi\|_2}, \qquad (4.14)$$

where δ is the percentage of noise. Using this noisy Cauchy data we try to reconstruct the density function g. This can be achieved by solving the system of equations

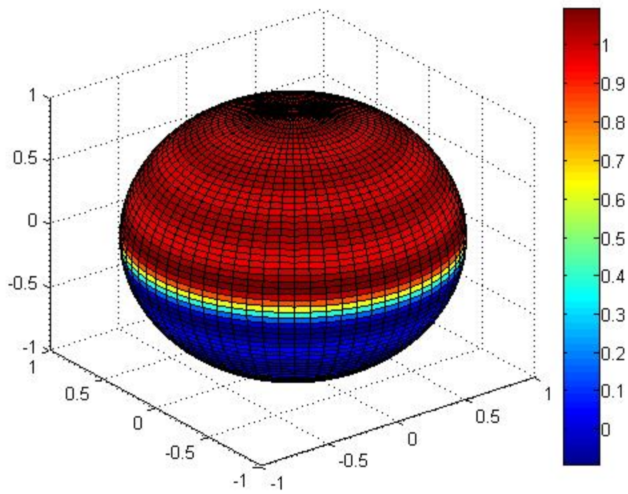

Figure 4.4: g_{approx} for $N = 32$

(4.9) and (4.10) for the coefficients a_n^m using least squares algorithm. The results are depicted in the following pictures and tables. Hence the reconstructed density function is given by

$$g_{recon}(\theta, \phi) = \frac{1}{2} \sum_{n=0}^{N} \sum_{m=-n}^{n} a_n^m Y_n^m (\theta, \phi) .$$

The errors of reconstruction given in the tables below is given by

$$error1 = \frac{\|g_{approx} - g_{recon}\|_2}{\|g_{approx}\|_2}$$

and

$$error2 = \frac{\|g - g_{recon}\|_2}{\|g\|_2}$$

Observing these pictures and errors of reconstruction of density, one can notice that with the increasing frequency the error is reducing and especially in the region where $k \approx N$ we have better stability.

K	2	4	8	16	32	64
error 1	0.696	0.258	0.0415	0.00711	0.00782	0.0162
error 2	0.689	0.256	0.114	0.113	0.1134	0.1143

Table 4.1: Errors of reconstruction for $N = 16$ with 1% noise

K	2	4	8	16	32	64
error 1	2.806E+00	1.508E+00	2.898E-01	3.543E-02	3.903E-02	8.149E-02
error 2	2.719E+00	1.257E+00	1.826E-01	1.185E-01	1.196E-01	1.388E-01

Table 4.2: Errors of reconstruction for $N = 16$ with 5% noise

K	2	4	8	16	32	64
error 1	5.188	3.044	0.436	0.0711	0.0779	0.1626
error 2	5.144	3.019	0.4327	0.1334	0.137	0.1969

Table 4.3: Errors of reconstruction for $N = 16$ with 10% noise

K	2	4	8	16	32	64
error 1	214903.1654	174707.1571	31858.44881	32.68496732	0.007921545	0.016294205
error 2	214060.6646	174022.2402	31733.55186	32.55683011	0.079494187	0.080752598

Table 4.4: Errors of reconstruction for $N = 32$ with 1% noise

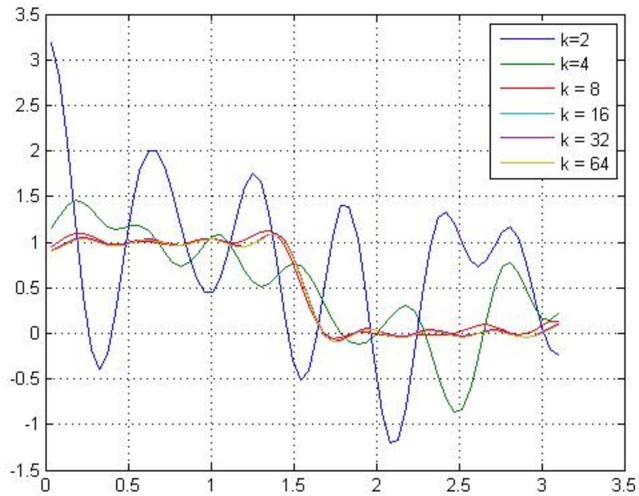

Figure 4.5: g_{recon} for $N = 16$ and various frequencies, with 1% noise

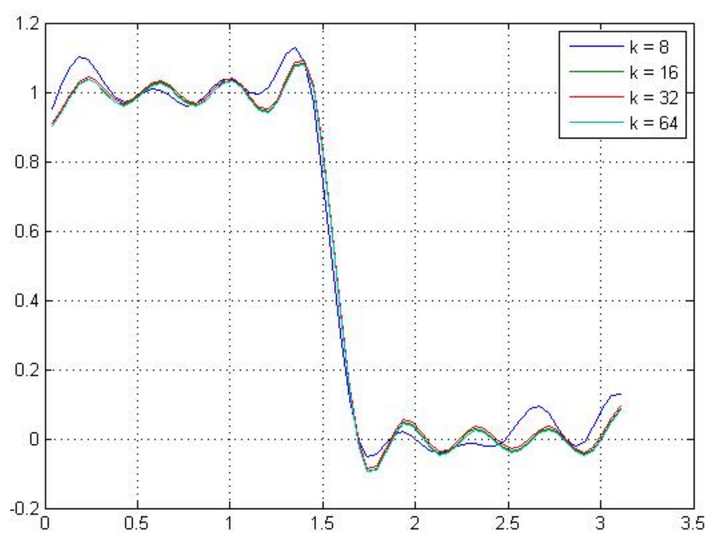

Figure 4.6: g_{recon} for $N = 16$ and various frequencies, with 1% noise

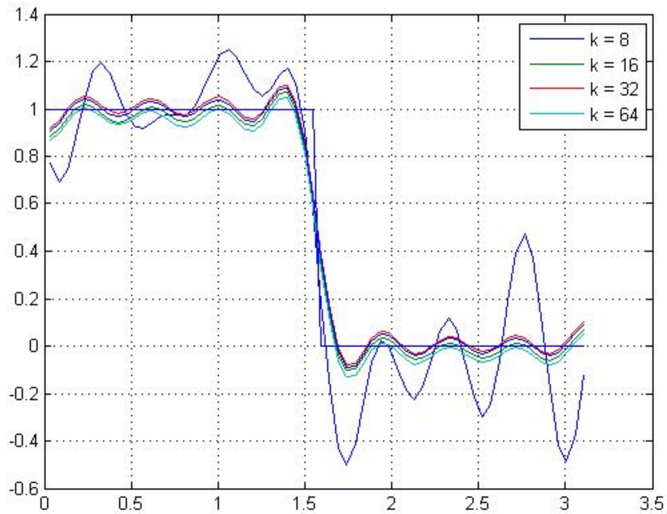

Figure 4.7: g_{recon} for $N = 16$ and various frequencies, with 5% noise

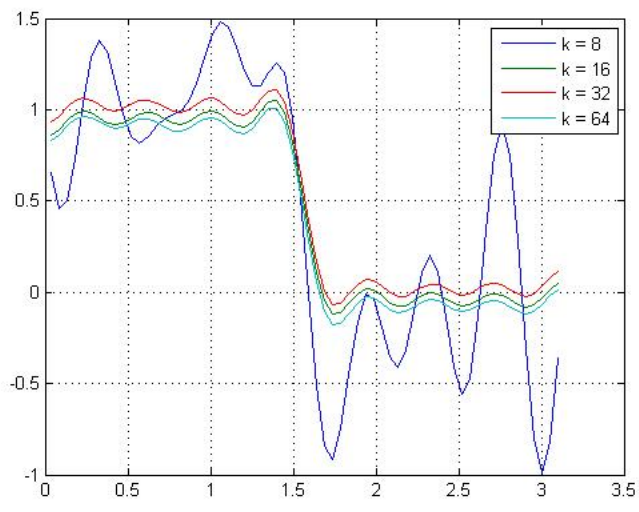

Figure 4.8: g_{approx} for $N = 16$ and various frequencies, with 10% noise

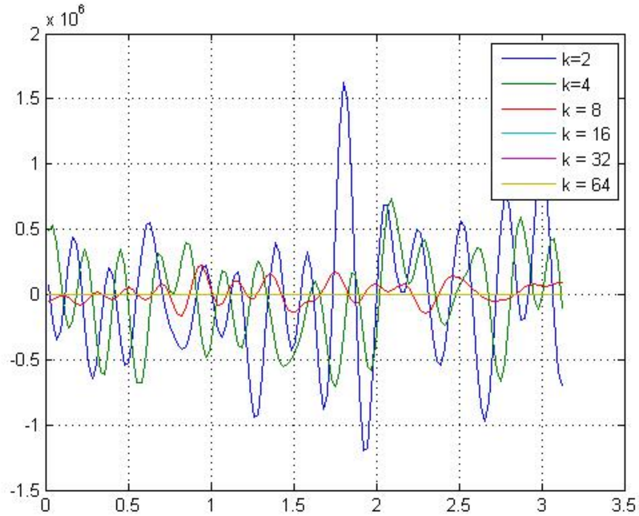

Figure 4.9: g_{approx} for $N = 32$ and various frequencies, with 1% noise

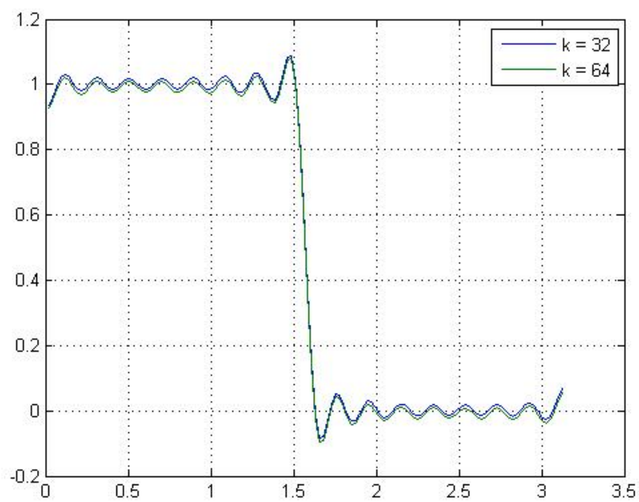

Figure 4.10: g_{recon} for $N = 32$ and various frequencies, with 1% noise

CHAPTER 5

DIRECT SCATTERING PROBLEM

5.1 Introduction

In this paper we investigate the scattering of time harmonic electromagnetic waves by an obstacle, D in $\mathbb{R}^3$, with the most general impedance boundary condition known as the Leontovich boundary condition. We assume that the boundary of the obstacle ∂D is of class C^2 and let $D_e = (\mathbb{R}^3 \setminus \overline{D})$. We consider the exterior boundary value problem for the *Maxwell's System*

$$\begin{cases} \operatorname{curl} \boldsymbol{E} - ik\boldsymbol{H} = 0 \quad \text{in } D_e, \\ \operatorname{curl} \boldsymbol{H} + ik\boldsymbol{E} = 0 \quad \text{in } D_e, \end{cases} \tag{5.1}$$

with the most general impedance boundary condition known as the Leontovich boundary condition

$$\nu \times \boldsymbol{H} - \lambda(\nu \times \boldsymbol{E}) \times \nu = 0 \text{ on } \partial D, \tag{5.2}$$

$\lambda \geq 0$, $\lambda \in C^1(\partial D)$ and the scattered fields $\boldsymbol{E}^s$, $\boldsymbol{H}^s$ satisfying the *Silver-Müller* radiation conditions

$$\begin{cases} \lim\limits_{r\to\infty} (\boldsymbol{H}^s \times x - r\boldsymbol{E}^s) = 0, \\ \\ \lim\limits_{r\to\infty} (\boldsymbol{E}^s \times x + r\boldsymbol{H}^s) = 0, \end{cases} \tag{5.3}$$

and from [10] we know that the *Silver-Müller* radiation conditions are equivalent to the Sommerfeld radiation condition for the Cartesian components,

$$\begin{cases} \lim\limits_{r\to\infty} r \left(\dfrac{\partial \boldsymbol{E}^s}{\partial r} - ik\boldsymbol{E}^s \right) = 0, \\ \\ \lim\limits_{r\to\infty} r \left(\dfrac{\partial \boldsymbol{H}^s}{\partial r} - ik\boldsymbol{H}^s \right) = 0, \end{cases} \tag{5.4}$$

Also, from [10, pg 164] we have that the scattered fields have the following asymptotic behavior

$$\begin{cases} \boldsymbol{E}^s(x) = \dfrac{e^{ikr}}{r}\left\{\boldsymbol{E_\infty}(\hat{x}) + O\left(\dfrac{1}{r}\right)\right\}, & r \to \infty, \\ \\ \boldsymbol{H}^s(x) = \dfrac{e^{ikr}}{r}\left\{\boldsymbol{H_\infty}(\hat{x}) + O\left(\dfrac{1}{r}\right)\right\}, & r \to \infty, \end{cases} \tag{5.5}$$

where $\boldsymbol{E_\infty}$ and $\boldsymbol{H_\infty}$ are known as the far field pattern or the scattering amplitude and $\boldsymbol{H_\infty}(\hat{x}) = \hat{x} \times \boldsymbol{E_\infty}(\hat{x})$.

Here $\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s$, $\boldsymbol{H} = \boldsymbol{H}^i + \boldsymbol{H}^s$ and

$$\begin{cases} \boldsymbol{E}^i(x) := \frac{i}{k}\text{curl curl } \boldsymbol{p}e^{ik\boldsymbol{x}\cdot\boldsymbol{d}} = ik(\boldsymbol{d} \times \boldsymbol{p}) \times de^{ik\boldsymbol{x}\cdot\boldsymbol{d}}, \\ \\ \boldsymbol{H}^i(x) := \text{curl } \boldsymbol{p}e^{ik\boldsymbol{x}\cdot\boldsymbol{d}} = ik\boldsymbol{d} \times \boldsymbol{p}e^{ik\boldsymbol{x}\cdot\boldsymbol{d}}, \end{cases}$$

are the incident electric and magnetic fields, d is a unit vector which gives the direction of propagation and p is the polarization vector.

In the direct problem we are looking for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ in the space $\boldsymbol{H}^2(B_\rho \setminus \overline{D})$ for some $\rho > \rho_0 > 0$, $\overline{D} \subset B_\rho$, where $\boldsymbol{H}^k(D)$ is $(H^{(k)}(D))^3$ the three dimensional product of standard Sobolev space. Also, $\boldsymbol{E}$, $\boldsymbol{H} \in \boldsymbol{C}^\infty(\mathbb{R}^3 \setminus D)$ with $\boldsymbol{E}^s$, $\boldsymbol{H}^s$ satisfying the *Silver-Müller* radiation condition.

5.2 Forward Problem

Lemma 5.2.1. *The scattering problem for Maxwell's System (5.1),(5.2) and (5.3) is equivalent to the scattering problem for vector Helmholtz equation (5.6)*

$$\begin{cases} \Delta \boldsymbol{E} + k^2\boldsymbol{E} = 0 & \textit{in } D_e, \\ \textit{div } \boldsymbol{E} = 0 & \textit{on } \partial D, \\ \nu \times \textit{curl } \boldsymbol{E}^s - ik\lambda(\nu \times \boldsymbol{E}^s) \times \nu = \boldsymbol{g} & \textit{on } \partial D, \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial \boldsymbol{E}^s}{\partial r} - ik\boldsymbol{E}^s\right) = 0, \end{cases} \tag{5.6}$$

and the equation

$$\boldsymbol{H} = -\frac{i}{k} \mathit{curl}\ \boldsymbol{E}, \tag{5.7}$$

$\boldsymbol{E}^s \in \boldsymbol{H}^2(B_\rho \setminus \overline{D})$, $\boldsymbol{E}^s \in \boldsymbol{C}^\infty(\mathbb{R}^3 \setminus D)$. *The boundary data is given by* $\boldsymbol{g} := -\nu \times \mathit{curl}\ \boldsymbol{E}^i + ik\lambda(\nu \times \boldsymbol{E}^i) \times \nu$.

Proof: From the equations (5.1) we note that curl (curl $\boldsymbol{E}$)$-ik$curl $\boldsymbol{H}$ = curl (curl $\boldsymbol{E}$)$-k^2\boldsymbol{E} = -(\Delta + k^2)\boldsymbol{E} + \nabla$div $\boldsymbol{E}$. But div (curl $\boldsymbol{H}$) $+ ik$(div $\boldsymbol{E}$) $= 0$, so div $\boldsymbol{E} = 0$, and by taking traces we have that div $\boldsymbol{E} = 0$ on the boundary ∂D. Hence the scattering problem for *Maxwell's System* implies to the scattering problem for vector Helmholtz equation.

On the other hand (5.6) along with $\boldsymbol{H} = -\frac{i}{k}$curl $\boldsymbol{E}$ implies (5.1) . To this end we let $v =$ div $\boldsymbol{E}$, but div $\boldsymbol{E}^i = 0$ which implies that $v =$ div $\boldsymbol{E}^s$, due to (5.6) v is a solution of the following boundary value problem

$$\Delta v + k^2 v = 0 \text{ in } D_e,$$

$$v = 0 \text{ on } \partial D,$$

and has the asymptotic behavior

$$v(x) = \frac{e^{ikr}}{r}\left\{\text{div } \boldsymbol{E}_{\boldsymbol{\infty}}(\hat{x}) + O\left(\frac{1}{r}\right)\right\}, \qquad r \to \infty,$$

which satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial v}{\partial r} - ikv\right) = 0$$

By uniqueness in the exterior Dirichlet problem for the Helmholtz equation $v = 0$ in D_e which implies that div $\boldsymbol{E}^s = 0$ in D_e. Also, if $\boldsymbol{H} = -\frac{i}{k}$curl $\boldsymbol{E}$ then curl $\boldsymbol{H} = -\frac{i}{k}$curl curl $\boldsymbol{E} = -ik\boldsymbol{E}$, which follows from (5.6). Hence if $\boldsymbol{E}$ solves the scattering problem for Helmholtz equation, then $\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s$ along with $\boldsymbol{H} = -\frac{i}{k}\text{curl } \boldsymbol{E}$ solves the scattering problem for the Maxwell's system.

$\square$

Now we prove existence, uniqueness and analyticity of the solution with respect to k of the following non-homogeneous boundary value problem for the vector Helmholtz equation

$$\begin{aligned} \Delta \boldsymbol{v} + k^2 \boldsymbol{v} &= \boldsymbol{f} \quad \text{in } \Omega, \\ \left.\begin{aligned} \nu \times \text{curl } \boldsymbol{v} - ik\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div } \boldsymbol{v} \end{aligned}\right\} &= \boldsymbol{g} \quad \text{on } \partial D, \\ \boldsymbol{v} &= 0 \quad \text{on } \partial B_R, \end{aligned} \tag{5.8}$$

$\boldsymbol{v} \in \boldsymbol{H}^2(B_\rho \setminus \overline{D})$, $\boldsymbol{f} \in \boldsymbol{H}^0(B_\rho \setminus \overline{D})$ and $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial D)$

First we show that the above boundary value problem (5.8) is elliptic in the *Agmon-Douglis-Nirenberg* sense, or elliptic in the general sense. Consider the following boundary value problem:

$$Au = f \text{ on } M, \;\; Bu = g \text{ on } \Gamma. \tag{5.9}$$

If the Shapiro-Lopatinskij condition holds then the boundary value problem (5.8) is called elliptic in the *Agmon-Douglis-Nirenberg* sense, or elliptic in the general sense. From [1] the Shapiro-Lopatinskij condition is equivalent to the following condition: the rows of the matrix

$$b_0(\xi', \zeta)a^0(\xi', \zeta)$$

are linearly independent modulo the polynomial $a_0^+(\xi', \zeta)$. Here $a_0(x, \xi)$ is the principal symbol of A, $b_0(x, \xi)$ is the principal symbol of B, a^0 is the matrix of cofactor of the elements of the matrix a_0 and $a_0^+(\xi', \zeta) = (\zeta - \zeta_1(\xi'))...(\zeta - \zeta_q(\xi'))$, where $\zeta_1(\xi')...\zeta_q(\xi')$ roots of the polynomial det $a_0(\xi', \zeta) = 0$ lying in the upper half plane.

For the boundary value problem (5.8) $A = \Delta + k^2$,

$$B = \begin{cases} \nu \times \text{curl } \; - ik\lambda(\nu\times) \times \nu \\ \text{div} \end{cases}$$

Let $x_0 \in \partial D$ be any point on the boundary, choose a coordinate system such that $x_0 = (0,0,0)$ and that the outward unit normal is $\nu = (0,0,1)$. In these coordinate system the principal symbols of the partial differential operator and the boundary operator are given by:

$$a_0(\xi', \zeta) = \begin{bmatrix} \zeta^2 + \xi_1^2 + \xi_2^2 & 0 & 0 \\ 0 & \zeta^2 + \xi_1^2 + \xi_2^2 & 0 \\ 0 & 0 & \zeta^2 + \xi_1^2 + \xi_2^2 \end{bmatrix}$$

$$b_0(\xi', \zeta) = \begin{bmatrix} -i\zeta & -i\xi_1 & 0 \\ 0 & -i\zeta & -i\xi_2 \\ -i\xi_1 & -i\xi_2 & -i\zeta \end{bmatrix}$$

and the cofactor matrix of $a_0(\xi', \zeta)$ is

$$a^0(\xi', \zeta) = \begin{bmatrix} (\zeta^2 + \xi_1^2 + \xi_2^2)^2 & 0 & 0 \\ 0 & (\zeta^2 + \xi_1^2 + \xi_2^2)^2 & 0 \\ 0 & 0 & (\zeta^2 + \xi_1^2 + \xi_2^2)^2 \end{bmatrix}$$

One can verify through simple calculation that the rows of the matrix

$$b_0(\xi', \zeta)a^0(\xi', \zeta) = (\zeta + |\xi|^2)^2 \begin{bmatrix} -i\zeta & -i\xi_1 & 0 \\ 0 & -i\zeta & -i\xi_2 \\ -i\xi_1 & -i\xi_2 & -i\zeta \end{bmatrix}$$

are linearly independent modulo the polynomial $a_0^+(\xi', \zeta) = (\zeta - i|\xi'|)^3$. Therefore the boundary value problem (5.8) is elliptic in the *Agmon-Douglis-Nirenberg* sense, or elliptic in the general sense. Let $\mathcal{A}$ be the operator corresponding to the elliptic boundary value problem (5.8) and $\mathcal{A} : \boldsymbol{H}^2(\Omega) \to \boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial D)$. Since the boundary value problem (5.8) is elliptic from [2] we have that the operator $\mathcal{A}$ is Fredholm.

For an operator $\mathcal{T} \in C(X, Y)$, where X and Y are Banach spaces, to be Fredholm means that

- The kernel Ker $\mathcal{T} = \{x \in X : \mathcal{T}x = 0\}$ is finite dimensional,
- The range $R(\mathcal{T})$ is closed in Y,
- The cokernel Coker $\mathcal{T} = Y \setminus R(\mathcal{T})$ is finite dimensional.

The index of the Fredholm operator is given by

$$\text{ind } \mathcal{T} = \text{dim Ker } \mathcal{T} - \text{dim Coker } \mathcal{T}.$$

Theorem 5.2.1. *Let $\mathcal{T}$, $\mathcal{S} \in C(X, Y)$ and let T be Fredholm, then there exists a $\delta > 0$ such that $\|T - S\| < \delta$ implies ind $T =$ ind S.*

Theorem 5.2.2. *Let $\mathcal{T}(\kappa)$ be a family of compact operators in X holomorphic for $\kappa \in D_0$. Call κ a singular point if 1 is an eigenvalue of $T(\kappa)$. Then either all $\kappa \in D_0$ are singular points or there are only a finite number of singular points in each compact subset of D_0.*

Let S be the set of all such k which are singular points of the operator $\mathcal{A}$ corresponding to the elliptic boundary value problem (5.8).

Theorem 5.2.3. *If $k \in \mathbb{C} \setminus S$ then there exists a unique solution to the boundary value problem (5.8) in $\boldsymbol{H}^2(\Omega)$.*

Proof. Let $k_1 = ik$ for some real $k \neq 0$. We now show that the solution to the following homogeneous elliptic boundary value problem is identically zero.

$$\begin{aligned} \Delta \boldsymbol{v} + k_1^2 \boldsymbol{v} = 0 \quad & \text{in } \Omega \\ \left.\begin{aligned} \nu \times \text{curl } \boldsymbol{v} - ik_1\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div } \boldsymbol{v} \end{aligned}\right\} = 0 \quad & \text{on } \partial D \\ \boldsymbol{v} = 0 \quad & \text{on } \partial B_R \end{aligned}$$

Since $div\boldsymbol{v} = 0$, $-(\Delta + k_1^2)\boldsymbol{v} = (\text{curl curl } - k_1^2)\boldsymbol{v}$. Multiplying this with $\overline{\boldsymbol{v}}$ and integrating over the bounded domain $\Omega = B_\rho \setminus \overline{D}$

$$\int_\Omega (\text{curl curl } \boldsymbol{v} - k_1^2 \boldsymbol{v}) \cdot \overline{\boldsymbol{v}} dx = 0.$$

Now integrating by parts using Green's first vector theorem (see [10, pg 155]) and using one of the vector identity $(a \times b) \cdot c = -(a \times c) \cdot b$ we have,

$$\int_\Omega (|\text{curl } \boldsymbol{v}|^2 - k_1^2 |\boldsymbol{v}|^2) dx + \int_{\partial\Omega} (\nu \times \text{curl } \boldsymbol{v}) \cdot \overline{\boldsymbol{v}} ds = 0. \tag{5.10}$$

Here $\partial\Omega = \partial D \cup S_\rho$, hence

$$\int_{\partial\Omega} (\nu \times \text{curl } \boldsymbol{v}) \cdot \overline{\boldsymbol{v}} ds = -\int_{\partial D} (\nu \times \text{curl } \boldsymbol{v}) \cdot \overline{\boldsymbol{v}} ds + \int_{S_\rho} (\nu \times \text{curl } \boldsymbol{v}) \cdot \overline{\boldsymbol{v}} ds,$$

using the boundary condition we have

$$\begin{aligned} \int_{\partial D} (\nu \times \text{curl } \boldsymbol{v}) \cdot \overline{\boldsymbol{v}} ds &= -ik_1 \lambda \int_{\partial D} ((\nu \times \boldsymbol{v}) \times \nu) \cdot \overline{\boldsymbol{v}} ds \\ &= -ik_1 \lambda \int_{\partial D} |(\nu \times \boldsymbol{v})|^2 ds, \end{aligned}$$

Plugging the above equation in (5.10) with $(\nu \times \boldsymbol{v}) = \boldsymbol{v}_T$ the tangential component and $k_1 = ik$, we have

$$\int_{B_\rho \setminus \overline{D}} (|\text{curl } \boldsymbol{v}|^2 + k^2 |\boldsymbol{v}|^2) dx + k\lambda \int_{\partial D} |\boldsymbol{v}_T|^2 ds = 0. \tag{5.11}$$

which implies that $v = 0$.

Let $\mathcal{A}(\theta)$ be the operator corresponding to the following boundary value problem.

$$\begin{aligned} \Delta\tilde{\boldsymbol{v}} &= \boldsymbol{f} \quad &&\text{in } \Omega \\ \partial_3\tilde{v}_1 - \theta\partial_1\tilde{v}_3 &= g_1 \quad &&\text{on } \partial D \\ \partial_3\tilde{v}_2 - \theta\partial_2\tilde{v}_3 &= g_2 \quad &&\text{on } \partial D \\ \theta\partial_1\tilde{v}_1 + \theta\partial_2\tilde{v}_2 + \partial_3\tilde{v}_3 &= g_0 \quad &&\text{on } \partial D \end{aligned}$$

For the above we have written only the principal part and for some point $\tilde{x} \in \partial D$ and $\nu = (0,0,1)$, $0 \leq \theta \leq 1$. if $\theta = 1$, then $\mathcal{A}(\theta)$ corresponds to the elliptic boundary value problem (5.8). Also, the above boundary value problem is elliptic and hence the operator $\mathcal{A}(\theta)$ is Fredholm. Notice that $\mathcal{A}(\theta)$ is continuous for all $0 \leq \theta \leq 1$, hence $\mathcal{A}(\theta)$ is a connected curve in the family of operators. Also, ind is a continuous function .Therefore, ind $\mathcal{A}(\theta)$ is constant for any $0 \leq \theta \leq 1$. By setting $\theta = 0$, we have the Neumann boundary value problem,

$$\begin{aligned} \Delta\tilde{\boldsymbol{v}} &= \boldsymbol{f} \quad &&\text{in } \Omega \\ \partial_3\tilde{v}_1 &= g_1 \quad &&\text{on } \partial D \\ \partial_3\tilde{v}_2 &= g_2 \quad &&\text{on } \partial D \\ \partial_3\tilde{v}_3 &= g_0 \quad &&\text{on } \partial D \end{aligned}$$

It is well known that the index of the Neumann boundary value problem is zero i.e., ind $\mathcal{A}(0) = 0$, which implies that the index of the elliptic boundary value problem (5.8) is zero or ind $\mathcal{A}(1) =$ ind $\mathcal{A} = 0$. Therefore the elliptic boundary value problem (5.8) is uniquely solvable for $k_1 = ik$.

Next for any $k \in \mathbb{C}$ we have the following boundary value problem which is a compact perturbation of the boundary value problem (5.8),

$$
\begin{aligned}
\Delta \boldsymbol{v} + k_1^2 \boldsymbol{v} - (k_1^2 - k^2)\boldsymbol{v} &= \boldsymbol{f} \quad \text{in } \Omega \\
\left.\begin{aligned} \nu \times \text{curl } \boldsymbol{v} - ik_1\lambda(\nu \times \boldsymbol{v}) \times \nu + i(k_1^2 - k^2)\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div } \boldsymbol{v} \end{aligned}\right\} &= \boldsymbol{g} \quad \text{on } \partial D \\
\boldsymbol{v} &= 0 \quad \text{on } \partial B_R
\end{aligned}
$$

Let $\mathcal{A}_1$ be the operator corresponding to the above boundary value problem, then there exists ϵ such that $\|\mathcal{A}_1 - \mathcal{A}\| < \epsilon$, which implies that ind $\mathcal{A}_1 = 0$ or the index of the elliptic boundary value problem (5.8) is zero.

We can write the above elliptic boundary value problem in terms of operators, $\mathcal{A}_1 : \boldsymbol{H}^2(\Omega) \to \boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial D)$ and $\mathcal{B} : \boldsymbol{H}^2(\Omega) \to \boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^{\frac{3}{2}}(\partial D)$ and $(\mathcal{A}_1 + \mathcal{B})\boldsymbol{v} = \boldsymbol{f}$, $\boldsymbol{v} + \mathcal{A}_1^{-1}\mathcal{B}\boldsymbol{v} = \boldsymbol{f}$

and

$$\mathcal{A}_1^{-1}\mathcal{B} : \boldsymbol{H}^2 \to \boldsymbol{H}^3 \to \boldsymbol{H}^2$$

and the embedding $\boldsymbol{H}^3 \to \boldsymbol{H}^2$ is compact and hence $\mathcal{A}_1^{-1}\mathcal{B} : \boldsymbol{H}^2 \to \boldsymbol{H}^2$ is compact.

□

Let $\boldsymbol{E}^* \in \boldsymbol{H}^2(B_\rho \setminus \overline{D})$ solve the following elliptic boundary value problem

$$
\begin{aligned}
\Delta \boldsymbol{E}^* + k^2 \boldsymbol{E}^* &= 0 \quad \text{in } \Omega \\
\left.\begin{aligned} \nu \times \text{curl } \boldsymbol{E}^* - ik\lambda(\nu \times \boldsymbol{E}^*) \times \nu \\ \text{div } \boldsymbol{E}^* \end{aligned}\right\} &= \boldsymbol{g} \quad \text{on } \partial D \\
\boldsymbol{E}^* &= 0 \quad \text{on } \partial B_R
\end{aligned} \tag{5.12}
$$

Let ϕ be a C^∞cutoff function that is 1 near $\overline{D}$ and 0 in $B_R \setminus \Omega_0$. Then $\boldsymbol{E}_* = \boldsymbol{E} - \phi\boldsymbol{E}^*$, $\boldsymbol{E}_* \in \boldsymbol{H}^2(B_\rho \setminus \overline{D})$, solves the following scattering problem

$$
\begin{aligned}
\Delta \boldsymbol{E}_* + k^2 \boldsymbol{E}_* = \boldsymbol{f}_* &\quad \text{in } D_e \\
\text{div } \boldsymbol{E}_* = 0 &\quad \text{on } \partial D \\
\nu \times \text{curl } \boldsymbol{E}_* - ik\lambda(\nu \times \boldsymbol{E}_*) \times \nu = 0 &\quad \text{on } \partial D \\
\lim_{r\to\infty} r\left(\frac{\partial \boldsymbol{E}^s{}_*}{\partial r} - ik\boldsymbol{E}^s{}_*\right) = 0 &
\end{aligned}
\tag{5.13}
$$

where $\boldsymbol{f}_* = -(\Delta + k^2)(\phi \boldsymbol{E}_*)$, $\boldsymbol{f}_* \in \boldsymbol{H}^0(B_\rho \setminus \overline{D})$

Theorem 5.2.4. *If $k \in \mathbb{R}$ and $k \neq 0$ then there exists a unique solution to the scattering problem (5.6) in $\boldsymbol{H}^2(B_\rho \setminus \overline{D})$.*

Proof. First, as in [9], we show that the solution to the homogeneous scattering problem

$$
\begin{aligned}
\Delta \boldsymbol{E} + k^2 \boldsymbol{E} = 0 &\quad \text{in } D_e \\
\text{div } \boldsymbol{E} = 0 &\quad \text{on } \partial D \\
\nu \times \text{curl } \boldsymbol{E} - ik\lambda(\nu \times \boldsymbol{E}) \times \nu = 0 &\quad \text{on } \partial D \\
\lim_{r\to\infty} r\left(\frac{\partial \boldsymbol{E}^s}{\partial r} - ik\boldsymbol{E}^s\right) = 0 &
\end{aligned}
\tag{5.14}
$$

is identically zero. Since $div\boldsymbol{E} = 0$, $-(\Delta + k^2)\boldsymbol{E} = \text{curl curl } \boldsymbol{E} - k^2\boldsymbol{E}$. Multiplying this with $\overline{\boldsymbol{E}}$ and integrating over some bounded domain $\Omega = B_\rho \setminus \overline{D}$

$$
\int_\Omega (\text{curl curl } \boldsymbol{E} - k^2\boldsymbol{E}) \cdot \overline{\boldsymbol{E}} dx = 0.
$$

Now integrating by parts using Green's first vector theorem (see [10, pg 155]) we have,

$$
\int_\Omega (|\text{curl } \boldsymbol{E}|^2 - k^2|\boldsymbol{E}|^2) dx + \int_{\partial\Omega} (\nu \times \text{curl } \boldsymbol{E}) \cdot \overline{\boldsymbol{E}} ds = 0. \tag{5.15}
$$

Here $\partial\Omega = \partial D \cup S_\rho$, hence

$$
\int_{\partial\Omega} (\nu \times \text{curl } \boldsymbol{E}) \cdot \overline{\boldsymbol{E}} ds = -\int_{\partial D} (\nu \times \text{curl } \boldsymbol{E}) \cdot \overline{\boldsymbol{E}} ds + \int_{S_\rho} (\nu \times \text{curl } \boldsymbol{E}) \cdot \overline{\boldsymbol{E}} ds,
$$

using the boundary condition and the Maxwell's equations we have

$$\int_{\partial\Omega} (\nu \times \text{curl } \boldsymbol{E}) \cdot \overline{\boldsymbol{E}} ds = -ik\lambda \int_{\partial D} ((\nu \times \boldsymbol{E}) \times \nu) \cdot \overline{\boldsymbol{E}} ds + ik \int_{S_\rho} (\nu \times \boldsymbol{H}) \cdot \overline{\boldsymbol{E}} ds$$

$$= -ik\lambda \int_{\partial D} |(\nu \times \boldsymbol{E})|^2 ds - ik \int_{S_\rho} (\nu \times \overline{\boldsymbol{E}}) \cdot \boldsymbol{H} ds,$$

Plugging the above equation in (5.15) with $(\nu \times \boldsymbol{E}) = \boldsymbol{E}_T$ the tangential component, we have

$$\int_{B_\rho \setminus \overline{D}} (|\text{curl } \boldsymbol{E}|^2 - k^2 |\boldsymbol{E}|^2) dx - ik \int_{S_\rho} (\nu \times \overline{\boldsymbol{E}}) \cdot \boldsymbol{H} ds - ik\lambda \int_{\partial D} |\boldsymbol{E}_T|^2 ds = 0. \tag{5.16}$$

Using the scattering data for non-zero real k, by taking the imaginary part of the above equation we have

$$Re \int_{S_\rho} (\nu \times \overline{\boldsymbol{E}}) \cdot \boldsymbol{H} ds = -\lambda \int_{\partial D} |\boldsymbol{E}_T|^2 ds \leq 0.$$

Hence uniqueness follows from [10].

We use the Lax-Phillips method to show that the scattering problem (5.13) is uniquely solvable. Consider domain Ω_0 containing $\overline{D}$ such that $\overline{\Omega}_0$ lies in B_R. Let ϕ be a C^∞cutoff function that is 1 near $\overline{D}$ and 0 in $B_R \setminus \Omega_0$. Let $\Omega = D_e \cap B_R$ be the bounded exterior domain. Now we look for a solution $\boldsymbol{E}_*$ to (5.13) which is of the form

$$\boldsymbol{E}_* = \boldsymbol{w} - \phi(\boldsymbol{w} - \boldsymbol{v}) \tag{5.17}$$

where $\boldsymbol{v}(:, \boldsymbol{f}^*)$ is a solution to the following elliptic boundary value problem

$$\begin{aligned} \Delta \boldsymbol{v} + k^2 \boldsymbol{v} &= \boldsymbol{f} \quad \text{in } \Omega \\ \left.\begin{aligned} \nu \times \text{curl } \boldsymbol{v} - ik\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div } \boldsymbol{v} \end{aligned}\right\} &= \boldsymbol{g} \quad \text{on } \partial D \\ \boldsymbol{v} &= 0 \quad \text{on } \partial B_R \end{aligned} \tag{5.18}$$

where $\boldsymbol{f}^* \in \boldsymbol{H}^0(B_R)$, $\boldsymbol{f}^* = 0$ in D and $\boldsymbol{w}(:, \boldsymbol{f}^*)$ is a solution to the Helmholtz equation in free space

$$\Delta \boldsymbol{w} + k^2 \boldsymbol{w} = \boldsymbol{f}^* \tag{5.19}$$

which satisfies the radiation condition. Hence

$$\Delta \boldsymbol{E}_* + k^2 \boldsymbol{E}_* = \Delta \boldsymbol{w} + k^2 \boldsymbol{w} - \Delta\phi(\boldsymbol{w} - \boldsymbol{v})$$

$$-2\nabla\phi \cdot \nabla(\boldsymbol{w} - \boldsymbol{v}) - \phi(\Delta(\boldsymbol{w} - \boldsymbol{v}) + k^2(\boldsymbol{w} - \boldsymbol{v}))$$

$$= \boldsymbol{f}^* + K\boldsymbol{f}^*$$

where $K\boldsymbol{f}^* = -\Delta\phi(\boldsymbol{w} - \boldsymbol{v}) - 2\nabla\phi \cdot \nabla(\boldsymbol{w} - \boldsymbol{v})$ in $\Omega_0 \backslash D$. $\boldsymbol{E}_*$ solves the equation $\Delta \boldsymbol{E}_* + k^2 \boldsymbol{E}_* = \boldsymbol{f}_*$ if and only if $\boldsymbol{f}_*$ solves the equation

$$\boldsymbol{f}_* = \boldsymbol{f}^* + K\boldsymbol{f}^* = (I + K)\boldsymbol{f}^* \tag{5.20}$$

The operator K is compact from $\boldsymbol{H}^0(\Omega)$ into itself . Thus equation (5.20) is Fredholm and hence its uniqueness implies solvability. To this end we let $\boldsymbol{f}_* = 0$ then $\boldsymbol{E}_*$ is a solution to the homogeneous scattering problem and therefore $\boldsymbol{E}_* = 0$, which implies that $\boldsymbol{w} = \phi(\boldsymbol{w} - \boldsymbol{v})$. From the equations for $\boldsymbol{w}$ and $\boldsymbol{v}$ one can notice that $\boldsymbol{w} - \boldsymbol{v}$ solves the homogeneous Helmholtz equation in B_R, also $\boldsymbol{w} = 0$, $\boldsymbol{v} = 0$ on ∂B_R. We choose R such that $-k_0^2$ is not an eigenvalue for (5.8) in Ω. Hence $\boldsymbol{w} - \boldsymbol{v} = 0$ and therefore $\boldsymbol{w} = 0$, which implies that $\boldsymbol{f}^* = 0$.

Given any $\boldsymbol{f}$ and $\boldsymbol{g}$, we can find $\boldsymbol{f}_*$ and solve (5.20) for $\boldsymbol{f}^*$ and hence find $\boldsymbol{w}$ by solving $\Delta \boldsymbol{w} + k^2 \boldsymbol{w} = \boldsymbol{f}^*$ and $\boldsymbol{v}$ by solving the (5.8). Therefore we can find the unique solution of the scattering problem (5.6) using (5.17).

□

Let S be the set of all $k \in \mathbb{C}$ such that $-k^2$ is the eigenvalue of (5.8) inside the bounded domain Ω. It is well known that S is a discrete set. From now on we assume that $k \in \mathbb{C} \setminus S$. From [20] we need the following lemma.

Lemma 5.2.2. *If $T(\kappa) \in \mathcal{B}(X, Y)$ is holomorphic and $T(\kappa_0)^{-1} \in \mathcal{B}(Y, X)$ exists, then $T(\kappa)^{-1}$ exists, belongs to $\mathcal{B}(Y, X)$ and is holomorphic for sufficiently small $|\kappa - \kappa_0|$*

Let $B(k)$ be the operator which maps the solution of the boundary value problem (5.8) from $\boldsymbol{H}^2(\Omega)$ into $\boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$. Let $\boldsymbol{V}(k, :)$ be the inverse of $B(k)$ which maps $\boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ into $\boldsymbol{H}^2(\Omega)$. Let $A(k)$ be the operator which maps the solution of the scattering problem (5.6) from $\boldsymbol{H}^2(B_\rho \setminus D)$ into $\boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$.

Lemma 5.2.3. *The map $\boldsymbol{V}(k, :) : \boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \to \boldsymbol{H}^2(\Omega)$ is analytic with respect to frequency $k \in \mathbb{C} \setminus S$.*

Proof. The operator $B(k)$ is analytic with respect to k. From the above discussion we know that the operator is invertible for some $k_0 \in \mathbb{R}$. From lemma (5.2.2) we have that $\boldsymbol{V}(k, :)$ is also analytic and there exists ϵ such that $B(k_0)$ is invertible for all k where $|k - k_0| < \epsilon$. From the well-known elliptic estimates we know that $\boldsymbol{V}(k, :)$ is a linear continuous operator from $\boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ into $\boldsymbol{H}^2(\Omega)$ and $\boldsymbol{f}^*(k, :)$is analytic with respect to k. Therefore the solution $\boldsymbol{v}$ of (5.8) which is given by $\boldsymbol{V}(k, :)\boldsymbol{f}^*(k, :) = \boldsymbol{v}(k, :)$ is analytic with respect to k. □

Lemma 5.2.4. *The solution $E(;, k)$ to the scattering problem 5.6 can be analytically continued onto a complex neighborhood of the frequency $k_0 \in \mathbb{R}$.*

Proof. We repeat the proof of theorem (5.2.4) tracing analytic dependence on k. Consider domain Ω_0 containing $\overline{D}$ such that $\overline{\Omega}_0$ lies in B_R. Let ϕ be a C^∞cutoff function that is 1 near $\overline{D}$ and 0 in $B_R \setminus \Omega_0$. Let $\Omega = D_e \cap B_R$ be the bounded exterior domain. Now we look for a solution $\boldsymbol{E}_*$ to (5.13) which is of the form

$$\boldsymbol{E}_* = \boldsymbol{w} - \phi(\boldsymbol{w} - \boldsymbol{v}) \tag{5.21}$$

where $\boldsymbol{v}(:, k)$ is a solution to the following elliptic boundary value problem

$$\begin{aligned} \Delta\boldsymbol{v} + k^2\boldsymbol{v} &= \boldsymbol{f} \quad \text{in } \Omega \\ \left.\begin{aligned} \nu \times \text{curl } \boldsymbol{v} - ik\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div } \boldsymbol{v} \end{aligned}\right\} &= \boldsymbol{g} \quad \text{on } \partial D \\ \boldsymbol{v} &= 0 \quad \text{on } \partial B_R \end{aligned} \tag{5.22}$$

where $\boldsymbol{f}^* \in \boldsymbol{H}^0(B_R)$, $\boldsymbol{f}^* = 0$ in D and from 5.2.3 we have that $\boldsymbol{v}(;, k)$ is analytic in a complex neighborhood of k_0. Since $\boldsymbol{w}(:, k)$ is a solution to the Helmholtz equation in free space

$$\Delta\boldsymbol{w} + k^2\boldsymbol{w} = \boldsymbol{f}^* \tag{5.23}$$

which satisfies the radiation condition, $\boldsymbol{w}(:, k)$ is analytic in a complex neighborhood of k_0. Hence

$$\Delta\boldsymbol{E}_* + k^2\boldsymbol{E}_* = \Delta\boldsymbol{w} + k^2\boldsymbol{w} - \Delta\phi(\boldsymbol{w} - \boldsymbol{v})$$

$$-2\nabla\phi \cdot \nabla(\boldsymbol{w} - \boldsymbol{v}) - \phi(\Delta(\boldsymbol{w} - \boldsymbol{v}) + k^2(\boldsymbol{w} - \boldsymbol{v}))$$

$$= \boldsymbol{f}^* + K\boldsymbol{f}^*$$

where $K\boldsymbol{f}^* = -\Delta\phi(\boldsymbol{w} - \boldsymbol{v}) - 2\nabla\phi \cdot \nabla(\boldsymbol{w} - \boldsymbol{v})$ in $\Omega_0 \setminus D$. $\boldsymbol{E}_*$ solves the equation $\Delta\boldsymbol{E}_* + k^2\boldsymbol{E}_* = \boldsymbol{f}_*$ if and only if $\boldsymbol{f}_*$ solves the equation

$$\boldsymbol{f}_* = \boldsymbol{f}^* + K\boldsymbol{f}^* = (I + K)\boldsymbol{f}^* \tag{5.24}$$

The operator K is analytic with respect to k. The given data $\boldsymbol{f}_*$ is analytic with respect to k and hence $\boldsymbol{f}^*$ is also analytic with respect to k. Therefore $\boldsymbol{E}_*$ is analytic with respect to k.

□

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