# Mass Estimates, Conformal Techniques, and Singularities in General Relativity

by

Jeffrey L. Jauregui

Department of Mathematics
Duke University

Date: ________________________

Approved:

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Hubert L. Bray, Advisor

________________________
William K. Allard

________________________
Leslie D. Saper

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Mark A. Stern

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2010

# Abstract

(Mathematics)

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Abstract

In the theory of general relativity, the Riemannian Penrose inequality (RPI) provides a lower bound for the ADM mass of an asymptotically flat manifold of nonnegative scalar curvature in terms of the area of the outermost minimal surface, if one exists. In physical terms, an equivalent statement is that the total mass of an asymptotically flat spacetime admitting a time-symmetric spacelike slice is at least the mass of any black holes that are present, assuming nonnegative energy density. The main goal of this thesis is to deduce geometric lower bounds for the ADM mass of manifolds to which neither the RPI nor the famous positive mass theorem (PMT) apply. This may be the case, for instance, for manifolds that contain metric singularities or have boundary components that are not minimal surfaces.

The fundamental technique is the use of conformal deformations of a given Riemannian metric to arrive at a new Riemannian manifold to which either the PMT or RPI applies. Along the way we are led to consider the geometry of certain types non-smooth metrics. We prove a result regarding the local structure of area-minimizing hypersurfaces with respect to such metrics using geometric measure theory.

One application is to the theory of "zero area singularities," a class of singularities that generalizes the degenerate behavior of the Schwarzschild metric of negative mass. Another application deals with constructing and understanding some new invariants of the harmonic conformal class of an asymptotically flat metric.

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# List of Abbreviations and Symbols

| Abbreviations | | |
|---|---|---|
| GR | general relativity | pg. 1 |
| DEC | dominant energy condition | pg. 7 |
| AF | asymptotically flat | pg. 8 |
| ADM | Arnowitt-Deser-Misner | pg. 9 |
| PMT | positive mass theorem | pg. 12 |
| RPI | Riemannian Penrose inequality | pg. 13 |
| ZAS | zero area singularity | pg. 105 |
| Riemannian geometry | | |
| $\mathrm{dist}_g$ | distance with respect to metric $g$ | |
| $\nabla$ | gradient, with metric understood | |
| $\mathrm{div}_g$ | divergence, with respect to $g$ | |
| $\Delta_g$ | (negative spectrum) Laplacian with respect to $g$ | |
| $\nu$ | outward unit normal vector to a hypersurface | pg. 17 |
| $H$ | mean curvature of a hypersurface in direction $\nu$ | |
| $R_g$ | scalar curvature of a metric $g$ | |
| $\delta$ | flat metric | |
| Set theory and topology | | |
| $X \setminus Y$ | set difference: $X$ intersect the complement of $Y$ | |
| $B(p, r)$ | open ball of radius $r$ about $p$, with metric understood | |
| $B_g(p, r)$ | open ball of radius $r$ about $p$, with respect to metric $g$ | |
| $\overline{B}$ | closed ball, unspecified center and radius | |
| $\overline{B}^+$ | closed upper half-ball | pg. 49 |
| Measure theory | | |
| $\mathcal{H}_g^k$ | Hausdorff $k$-measure, with respect to smooth metric $g$ | pg. 160 |
| $\lvert \cdot \rvert_g$ | area with respect to $g$ — alias for $\mathcal{H}_g^2(\cdot)$ | |
| $\lvert \cdot \rvert_{g'}$ | area with respect to $g' \in \overline{\mathcal{H}(g)}$ | pg. 31 |
| $dA_g$ | area measure with respect to $g$ — alias for $d\mathcal{H}_g^2$ | |
| $dV_g$ | volume measure with respect to $g$ — alias for $d\mathcal{H}_g^3$ | |

| Asymptotically flat manifolds | | |
|---|---|---|
| $(M, g)$ | smooth asymptotically flat 3-manifold, with or without boundary | pg. 8 |
| $\Sigma$ | $\partial M$, the boundary of $M$ | pg. 9 |
| $m_{\mathrm{ADM}}(M, g)$ | ADM mass of $(M, g)$ | pg. 9 |
| $\{\lvert x\rvert = r\}$ | coordinate sphere of radius $r$ in $M$ | |
| $S_\infty$ | coordinate sphere at infinity; notation for $\lim_{r\to\infty}$ in an expression involving $\{\lvert x\rvert = r\}$ | pg. 159 |
| $B_r$ | compact region in $M$ bounded by $\{\lvert x\rvert = r\}$ | |
| $K$ | large compact set in $M$ | pg. 46 |
| $\min(\Sigma, g)$ | minimal enclosing area of boundary $\Sigma$, with respect to metric $g$ | pg. 46 |
| $\tilde{\Sigma}_g$ | outermost minimal area enclosure of $\Sigma$, with respect to metric $g$ | pg. 26, 46 |
| $K(x, y)$ | Poisson kernel for $(M, g)$ with respect to boundary $\Sigma$, with $x \in M$, $y \in \Sigma$, $x \neq y$ | pg. 30 |
| $\varphi$ | harmonic function with respect to $g$ that vanishes on boundary and is 1 at infinity | pg. 30 |
| $\alpha(A)$ | area profile function for harmonic conformal class | pg. 62, 128 |
| $\mathcal{S}_K$, $\mathcal{S}_{\min}$ | certain collections of surfaces enclosing $\Sigma$ | pg. 62, 68 |
| $\mathcal{F}_g$ | functional on asymptotically flat conformal class | pg. 143 |
| Conformal geometry | | |
| $\mathcal{H}(g)$ | harmonic conformal class of $g$ | pg. 14 |
| $\mathcal{H}_A(g)$ | subset of $\mathcal{H}(g)$ with boundary area $A$ | pg. 15 |
| $\overline{\mathcal{H}(g)}$ | generalized harmonic conformal class of $g$ | pg. 31 |
| $\overline{\mathcal{H}_A(g)}$ | generalized harmonic conformal class of $g$, with boundary area $A$ | pg. 31 |
| $[g]_{AF}$ | asymptotically flat conformal class | pg. 15 |

# Acknowledgements

First and foremost, I would like to express deep gratitude to my advisor, Dr. Hugh Bray. Aside from introducing me to mathematical relativity and to the main problems dealt with in this thesis, he has been an outstanding mentor during my time at Duke. I will always appreciate the opportunities he has given me and all of the insightful discussions we have had.

Next, I would like to thank my other committee members, Drs. William Allard, Leslie Saper, and Mark Stern, for their discussions and valuable contributions toward my completion of this project. I am also very grateful to many others with whom I have had fruitful mathematical discussions, including Michael Eichmair, Jeffrey Streets, Graham Cox, and George Lam.

I acknowledge and sincerely thank the Harvey Mudd College Department of Mathematics for creating and maintaining the fantastic environment in which I truly began my mathematical development.

I am indebted to the National Science Foundation for funding three years of my graduate study with a Graduate Research Fellowship and to the Duke University Graduate School for providing me with the James B. Duke Fellowship. I was also very fortunate to have my travel to several academic conferences funded, primarily by my advisor via NSF grants. Additionally, the American Mathematical Society provided travel grants to a Mathematical Research Communities session in Snowbird, Utah and a Joint Mathematics Meeting in San Francisco, California. I also thank the Graduate School of Duke University for partially funding my travel to an AMS sectional meeting in Boca Raton, Florida.

Most of all, I wish to thank my fiancée, Ellen Gasparovic, and my family for their endless love and support.

# Introduction

General relativity (GR) is the study of space and time and how they interact with energy and matter. First formulated nearly a century ago, GR remains the most successful physical model for describing the structure of the universe on the large scale.

GR had its origins in special relativity, which provides a framework for dealing with space and time together as a single structure known as *spacetime*. Special relativity fixes some problems associated with classical Newtonian physics but is limited to situations in which no gravitation or acceleration is present. GR avoids this limitation through Einstein's remarkable hypothesis that a spacetime has a metric — an infinitesimal measure of distances and times — that is allowed to vary from point to point. This beautiful insight naturally leads to the fundamental equation of GR, Einstein's equation, which relates the curvature of a spacetime to the distribution of energy and matter.

Over the last several decades, much research in both the mathematics and physics communities has focused on defining the *mass* of a spacetime and how various conditions on the spacetime can lead to inequalities for the mass. The keystone result in this direction was the positive mass theorem (PMT) of Schoen and Yau, first proved in 1979 [35]. Aside from its importance in GR, this theorem has had profound implications in seemingly unrelated problems in geometric analysis, such as the Yamabe problem [33], [24]. Additionally, the PMT was used to prove a sharper statement known as the Riemannian Penrose inequality (RPI) [7].

The unifying theme of the present work is the use of conformal deformations (pointwise rescalings) of a Riemannian metric to change a given manifold to one to which either the PMT or RPI applies. Such transformations consequently provide estimates for the ADM mass of the original manifold. A secondary theme is the appearance of *conformal invariants*: objects that are canonically associated to a conformal class or a harmonic conformal class of metrics.

## Overview and brief summary of results

Chapter 1 introduces our perspective on general relativity, including a discussion of energy conditions and the ubiquitous hypothesis of nonnegative scalar curvature. We recall the definitions of asymptotically flat manifolds and the ADM mass. We briefly discuss the Schwarzschild manifolds of both positive and negative mass and move on to the precise statements of the PMT and the RPI. Both the PMT and RPI are the true workhorses of our main results. We close this background chapter by recalling the harmonic conformal class of a metric, whose significance is intimately related with nonnegative scalar curvature.

The primary purpose of Chapter 2 is motivation. We ask the question of whether a given asymptotically flat manifold $(M, g)$ with compact boundary $\Sigma$ can be deformed within its harmonic conformal class to a metric for which the boundary is an outermost minimal surface. In light of the RPI, this turns out to be a perfectly natural question. If the answer is "yes," we show that a lower bound for the ADM mass immediately follows from the RPI. However, we explain why the answer in general is "no," by constructing an explicit geometric obstruction that involves both local and global data. Finally, we introduce Conjecture 7, due to Bray, that motivates much of this thesis. The conjecture essentially acknowledges that the answer to the above question may be "no" while still predicting that the ADM mass can be bounded using the RPI. The last section of Chapter 2 gives a heuristic, non-rigorous proof of this conjecture and sets the stage for Chapters 3 and 4.

Chapter 3 is primarily setup for our attack of Conjecture 7. We introduce the *generalized* harmonic conformal class, which allows for conformal factors that lack good boundary regularity. The advantage of this enlarged space is that it is compact in a natural weak topology. We explain how to measure the area of surfaces with these non-smooth metrics and prove a result on the lower semi-continuity area; this leads to a construction of area-minimizers. Finally, in a result of possible independent interest, we prove a theorem regarding the local structure of possible singularities of area-minimizers with respect to such non-smooth metrics.

In Chapter 4 we present a suitable modification of Conjecture 7 (namely Conjecture 28) and go a long way toward proving it; the proof is complete up to two technical assumptions. Roughly, the goal is to find a metric $g'$ in the generalized harmonic conformal class of a given metric $g$ such that the outermost minimal area enclosure $\tilde{\Sigma}_{g'}$ is disjoint from the boundary $\Sigma$. The key to our approach lies in a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, defined by maximizing within the harmonic conformal class of metrics the minimum area in the homology class of the boundary, subject to a boundary area constraint. More precisely, given $A > 0$, we define

$$\alpha(A) = \sup_{g'} \inf_{S} |S|_{g'},$$

where the supremum is taken over all metrics $g'$ in the harmonic conformal class for which the boundary area is at most $A$, and the infimum is taken over all surfaces $S$ enclosing the boundary. Here, $|S|_{g'}$ is the area of $S$ with respect to $g'$. Using a compactness argument, we show that the above supremum is attained. This maximizer $g' = u^4 g$ is studied in much greater detail and is our candidate for the solution of the conjecture. A variational argument shows that the conformal factor $u^4$ is bounded below by a positive constant. This leads to a proof of the existence of an outermost surface $\tilde{\Sigma}_{g'}$ attaining the minimum $g'$-area among surfaces enclosing the boundary (a nontrivial result precisely because $u$ is not necessarily continuous on the boundary). Another variational argument shows that $\tilde{\Sigma}_{g'}$ may only touch the boundary $\Sigma$ on a set of measure zero. On this intersection $\tilde{\Sigma}_{g'}$ *a priori* has no regularity. However, we invoke the theorem from the previous chapter regarding the local behavior of such singularities. At last, one final variational argument proves Conjecture 28 under the additional assumptions of boundedness of $u$ and of the existence of only finitely many area-minimizers in the homology class of the boundary.

In the next few chapters, we study the implications of Conjecture 28. Chapter 5 applies this conjecture to produce an estimate for the ADM mass of an asymptotically flat manifold with boundary in terms of the capacity of the boundary and a harmonic conformal invariant. By restricting to flat space, a new result regarding positive harmonic functions on $\mathbb{R}^3$ minus a domain follows, assuming Conjecture 28.

Chapter 6 applies this conjecture to deduce a mass estimate for asymptotically flat manifolds of nonnegative scalar curvature that contain *zero area singularities*, or ZAS. Such manifolds are typically incomplete, and so the PMT need not apply. We follow an argument of Bray to produce a possibly negative lower bound for the ADM mass of such a manifold in terms of the local geometry of its singularities. We proceed to answer some miscellaneous questions that arise in the theory of ZAS. Finally, we close by observing that the mass estimate of this chapter for ZAS in entirely equivalent to the estimate of the previous chapter for manifolds with boundary.

In Chapter 7 we adopt the perspective that since *conformal invariants* are interesting objects of study on closed manifolds (e.g. [18], [1]), invariants of the *harmonic conformal class* of an asymptotically flat manifold with boundary also deserve attention. We produce two such nontrivial objects: one is the function $\alpha$ referred to earlier; the other is a function $\mu$ defined analogously by maximizing the ADM mass. Both $\alpha$ and $\mu$ are shown to be strictly increasing, continuous functions $\mathbb{R}^+ \to \mathbb{R}$; $\mu$

is calculated explicitly for $\mathbb{R}^3$ minus a round ball. As another application of the RPI and Conjecture 28, we show that $\mu$ can be compared with $\alpha$ and find restrictions on the crossings of their graphs.

Chapter 8 is a departure from the other chapters, in that we treat asymptotically flat manifolds $M$ without boundary. Such manifolds $M$, equipped with metrics of nonnegative scalar curvature, have been extensively studied in the literature. Our purpose is to prove a type of mass estimate without a hypothesis on the scalar curvature. We explain why it is completely natural for this lower bound to become $-\infty$ in cases where the scalar curvature is "too negative." The proof of the main result is technically only for manifolds that are harmonically flat at infinity, although we conjecture that the mass estimate holds in general. Harmonic flatness at infinity allows one to construct test functions on a conformally compactified manifold. The proof is broken down into three cases, according to the sign of the Yamabe invariant of the compactification.

We close the introduction with the remark that the results presented in this thesis extend naturally to manifolds of dimension up to seven. This restriction, also characteristic of the PMT and RPI (at present), arises because of the possible non-smoothness of area-minimizing hypersurfaces in manifolds of dimension eight and higher.

# 1

# Technical Background

## 1.1 Spacetimes, Einstein's equation, and energy conditions

For our purposes, a *spacetime* is a smooth, connected 4-dimensional manifold $N$ equipped with a smooth symmetric $(0,2)$-tensor $h$ of signature $(-,+,+,+)$. Such $h$ is called a *Lorentzian metric*. A tangent vector $v$ to $N$ is called

1. *time-like* if $h(v,v) < 0$,
2. *null* (or *light-like*) if $h(v,v) = 0$, or
3. *space-like* if $h(v,v) > 0$.

We also assume that $N$ admits a smooth vector field $X$ that is everywhere time-like. Such $X$ is called a *time-orientation*, as it distinguishes time-like or null vectors $v$ as "future-pointing" (if $h(X,v) < 0$) or "past-pointing" (if $h(X,v) > 0$).

Next, we assume that $(N,h)$ satisfies Einstein's equation:

$$\mathrm{Ric}_h - \frac{1}{2}R_h h = 8\pi T, \tag{1.1}$$

where $\mathrm{Ric}_h$ and $R_h$ are respectively the Ricci curvature and scalar curvature of $h$, and $T$ is the *stress-energy-momentum tensor* of the spacetime. The particular form of the symmetric $(0,2)$-tensor $T$ is determined by the type of energy or matter that is present and its corresponding physics. If $T$ vanishes identically, then the spacetime is said to be *vacuum*.

To assure that the spacetime $(N,h)$ is physically reasonable, additional restrictions must be placed on $T$, called energy conditions. The *dominant energy condition* (DEC) stipulates that for all future time-like vectors $v$ and $w$,

$$T(v,w) \geq 0. \tag{1.2}$$

Physically, the DEC is equivalent to the statement that energy density is nonnegative and momentum does not flow faster than the speed of light. The DEC is contrasted with the *weak energy condition*, which requires only that $T(v,v) \geq 0$ for all future time-like vectors $v$. Wald's book is a good reference on the above material [40].

Finally, we assume that $(N,h)$ possesses a smooth 3-dimensional submanifold $M$ with the following properties:

1. The tangent bundle $TM$ consists of space-like vectors (viewing $TM \subset TN$), so that the restriction of $h$ to $TM$ is a Riemannian metric $g$ on $M$.
2. $M$ is totally geodesic (i.e., has zero second fundamental form) in $(N,h)$.
3. The Riemannian 3-manifold $(M,g)$ is asymptotically flat (see the next section).

Suppose $(M,g)$ has future-pointing unit normal vector $\hat{n}$ relative to $(N,h)$. The quantity $T(\hat{n},\hat{n})$ is physically interpreted as the local energy density measured by an observer living in the submanifold $M$. Moreover, by the totally geodesic assumption, the Gauss equation implies that the scalar curvature $R_g$ of $g$ equals $16\pi T(\hat{n},\hat{n})$ [7]. Up to the factor $16\pi$, scalar curvature can thus be identified with local energy density. Note that both the dominant and weak energy conditions imply that $R_g$ is nonnegative (still assuming that $M$ is totally geodesic), which is the reason why scalar curvature plays a significant role in the present work.

After this chapter we make no further mention of Lorentzian manifolds; we take the data $(M, g)$ as given. Many of the results require nonnegative scalar curvature.

## 1.2 Asymptotic flatness, ADM mass, and Schwarzschild manifolds

To remain consistent with the previous section, we consider only manifolds $M$ of dimension three. For simplicity, we restrict to oriented manifolds.

**Definition 1.** *A smooth, connected, oriented, three-dimensional Riemannian manifold $(M, g)$ is* ***asymptotically flat*** *if*

1. *there exists a compact subset $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^3 \setminus \overline{B}$ (where $\overline{B}$ is a closed ball about the origin),*
2. *in the coordinates $(x^1, x^2, x^3)$ on $M \setminus K$ induced by $\Phi$, the metric obeys the* ***decay conditions***

$$|g_{ij} - \delta_{ij}| \leq \frac{c}{|x|^p}, \qquad |\partial_k g_{ij}| \leq \frac{c}{|x|^{p+1}},$$

$$|\partial_k \partial_l g_{ij}| \leq \frac{c}{|x|^{p+2}}, \qquad |R_g| \leq \frac{c}{|x|^q},$$

   *for $|x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ sufficiently large and all $i, j, k, l = 1, 2, 3$, where $c > 0$, $p > \frac{1}{2}$, and $q > 3$ are constants, $\delta_{ij}$ is the Kronecker delta, $\partial_k = \frac{\partial}{\partial x^k}$, and $R_g$ is the scalar curvature of $g$.*

*Such $(x^i)$ are called* ***asymptotically flat coordinates****.*

We point out the first condition of asymptotic flatness is topological, while the second is geometric. The above definition is technically for an asymptotically flat manifold *with one end*, which is the only case we consider. We do allow for the possibility that $M$ has a smooth compact boundary.

To remark on the decay conditions, the Kronecker delta $\delta_{ij}$ is the representation in standard coordinates of the flat metric on $\mathbb{R}^3$. The threshold values of $\frac{1}{2}$ and 3 for $p$ and $q$, respectively, are explored in detail in Bartnik's paper [5]. We often abbreviate decay conditions with notation such as $g_{ij} - \delta_{ij} = O(r^{-p})$, where $r = |x|$. We warn the reader that several distinct definitions of "asymptotically flat" appear in the literature.

Physically, one expects asymptotically flat manifolds to arise as spacelike submanifolds of spacetimes modeling isolated systems: far from the concentration of matter and energy, the spacetime should be approximately flat.

For asymptotically flat manifolds, an associated quantity called the ADM mass, named after the physicists Arnowitt, Deser, and Misner [2], measures the "$1/r$" rate at which the metric becomes flat at infinity.

**Definition 2.** *The* ***ADM mass*** *of an asymptotically flat* 3*-manifold* $(M, g)$ *is the number*

$$m_{\text{ADM}}(M, g) = \lim_{r \to \infty} \frac{1}{16\pi} \sum_{i,j=1}^{3} \int_{\{|x|=r\}} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \frac{x^j}{r} dA,$$

*where* $(x^i)$ *are asymptotically flat coordinates and* $dA$ *is the area form on the coordinate sphere* $\{|x| = r\}$ *(induced from either the flat metric or* $g$*).*

Due to the work of Bartnik, the above limit exists, is finite, and is independent of the choice of asymptotically flat coordinates [5]. Note that none of these statements is immediately obvious from decay conditions on $g$. Sometimes the ADM mass is called the *total mass* or the *mass* when there is no chance of confusion.

The physical interpretation of the ADM mass is as follows. Suppose $(M, g)$ has ADM mass $m$ and arises as a submanifold of a spacetime as in section 1.1. Then to a good approximation, a test particle near infinity in the spacetime accelerates as if in a classical Newtonian potential well of mass $m$. Thus, the ADM mass is an analog for the Newtonian concept of mass for an isolated gravitational system.

*Examples:* The simplest example of a spacetime is Minkowski spacetime [40]: $\mathbb{R}^4$ with global coordinates $(t, x, y, z)$ equipped with the Lorentzian metric

$$h = -dt^2 + dx^2 + dy^2 + dz^2.$$

By specifying $T = 0$, this spacetime obeys Einstein's equation and the DEC. The submanifold $(M, g)$ corresponding to $\{t = 0\}$ is space-like, totally geodesic, and asymptotically flat. Indeed, $(M, g)$ is isometric to Euclidean 3-space and has zero ADM mass.

Possibly the next simplest example is the Schwarzschild spacetime [40], with metric

$$h = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $m \in \mathbb{R}$ is a parameter, $t \in \mathbb{R}$ is the time coordinate, $\theta$ and $\phi$ are standard spherical coordinates, and $r$ is a radial coordinate with range

$$r \in \begin{cases} (2m, \infty), & \text{if } m \geq 0, \\ (0, \infty), & \text{if } m < 0. \end{cases}$$

This spacetime has zero Ricci curvature and therefore obeys Einstein's equation and the DEC by specifying $T = 0$. A simple computation shows that the $t = 0$ submanifold is totally geodesic and is isometric to $M = \mathbb{R}^3 \setminus \overline{B(0, |m|/2)}$, equipped with the Riemannian metric

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$$

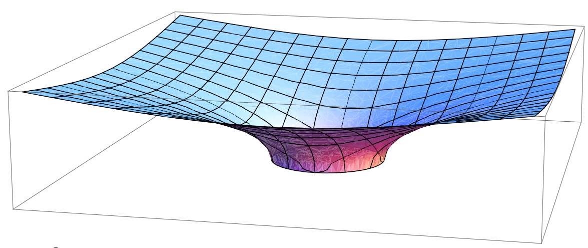
where $\delta_{ij}$ is the flat metric and $|x|$ is the Euclidean distance to the origin. We call $(M, g)$ the *Schwarzschild manifold* of mass $m$.

For $m \in \mathbb{R}$, the Schwarzschild manifold $(M, g)$ of mass $m$ is asymptotically flat with ADM mass equal to $m$. Moreover, the metric $g$ is spherically-symmetric, has zero scalar curvature, and is conformal to the flat metric via a harmonic function to the fourth power (formulas given in Appendix A assist with verifying some of these facts). Other aspects of the geometry of $(M, g)$ depend on the sign of $m$.

Case: $m > 0$. In this case, $g$ extends smoothly to $\partial M = \partial B(0, m/2)$; this boundary is a round sphere of area $16\pi m^2$ that has zero mean curvature in $(M, g)$. Physically, $(M, g)$ corresponds to a space-like slice of a static spacetime with a single black hole of mass $m$ [40]. The boundary $\partial M$ represents the *apparent horizon* of the black hole. See figure 1.1 for an illustration.

We also remark that for $m > 0$, $g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$ is smooth on all of $\mathbb{R}^3 \setminus \{0\}$; this manifold has two asymptotically flat ends, with a reflection symmetry about the minimal sphere at $|x| = m/2$.

FIGURE 1.1: The Schwarzschild manifold of positive mass



Pictured here is a diagram of $\mathbb{R}^3$ minus a ball, endowed with the Schwarzschild metric of mass $m > 0$. We have suppressed one dimension, so that, for instance, the boundary circle represents a 2-sphere. Near the outer edges, the surface flattens, indicating that the manifold is asymptotically flat.

Case: $m = 0$. For $m = 0$, $(M, g)$ can be smoothly extended across the origin and is isometric to Euclidean space.

Case: $m < 0$. For $m$ negative, the metric $g$ degenerates on $\partial M$, as the conformal factor $\left(1 + \frac{m}{2|x|}\right)^4$ is continuous and vanishes at $|x| = |m|/2$. The resulting metric singularity is called a *zero area singularity* and will be studied in more detail in Chapter 6, where a picture is also given.

## 1.3 Theorems regarding positivity of mass

In a spacetime obeying the dominant energy condition, a reasonable conjecture is that a test particle at infinity would be attracted inward, rather than repelled outward toward infinity. In light of the interpretation of ADM mass as determining the effective Newtonian potential, this statement is that the ADM mass should be nonnegative. The fundamental result in this direction, due to Schoen and Yau [35], is:

**Theorem 3** (Positive mass theorem)**.** *Let $(M, g)$ be a complete, asymptotically flat Riemannian 3-manifold (without boundary) of nonnegative scalar curvature with ADM mass $m$. Then $m > 0$, unless $(M, g)$ is isometric to $\mathbb{R}^3$ with the flat metric, in which case $m = 0$.*

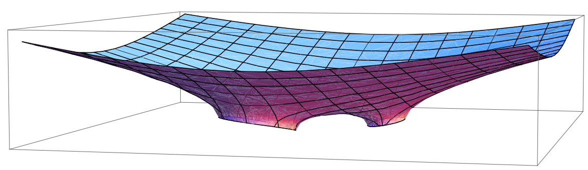
Two years later, Witten gave an alternative proof of Theorem 3 using spinors [41], whereas the original proof used minimal surfaces and a stability argument. The positive mass theorem holds for manifolds of dimension up to and including seven and for spin manifolds.

One phenomenon that contributes to the positivity of ADM mass is positive scalar curvature. Another is the presence of black holes. For our purposes, a *black hole* is a connected component of the outermost compact minimal (zero mean curvature) surface $S$ in $(M, g)$, if such a surface exists [7]. A compact minimal surface in $(M, g)$ is *outermost* if there exists no other compact minimal surface enclosing it. For physical reasons [28], [7], the total mass of all black holes in $(M, g)$ is defined to be $\sqrt{\frac{A}{16\pi}}$, where $A$ is the total area of all connected components of $S$ with respect to $g$. This definition is one way to motivate the Riemannian Penrose inequality, which predicts that the ADM mass of $(M, g)$ is at least the mass contributed by the black holes. The precise statement is as follows.

**Theorem 4** (Riemannian Penrose inequality, Theorem 19 of [7]). *Let* $(M, g)$ *be an asymptotically flat Riemannian 3-manifold with nonempty, compact smooth boundary* $\partial M$ *and nonnegative scalar curvature, with ADM mass* $m$*. Assume that* $\partial M$ *is an outermost minimal surface. Then* $m \geq \sqrt{\frac{A}{16\pi}}$*, where* $A$ *is the area of* $\partial M$*. Equality holds if and only if* $(M, g)$ *is isometric to the Schwarzschild manifold of mass* $m$*.*

Using a different approach, Huisken and Ilmanen [22] first proved a version of the Riemannian Penrose inequality (RPI) in which $A$ is replaced with the area of the largest connected component of $\partial M$. Bray's proof relied on the positive mass theorem, whereas Huisken–Ilmanen's ideas give an independent proof of the PMT. Later, Bray and Lee generalized Bray's original argument for manifolds up to dimension 7 (with the $\frac{1}{2}$ exponent and the $16\pi$ normalizing factor appropriately changed) [11]. The setup of the RPI is given schematically in figure 1.2.

FIGURE 1.2: The Riemannian Penrose inequality



Depicted above is an asymptotically flat manifold with a compact boundary consisting of a two-component outermost minimal surface. Under the assumption of nonnegative scalar curvature, Theorem 4 bounds the ADM mass from below in terms of the area of the boundary.

An alternate formulation of the RPI is found by considering an asymptotically flat manifold $(M, g)$ *without* boundary that possesses a nonempty outermost minimal surface $S$. Then the RPI as stated above can be made to apply by deleting the compact region bounded by $S$. One subtlety is that the case of equality $m = \sqrt{\frac{A}{16\pi}}$ implies only that the region exterior to $S$ in $(M, g)$ is isometric to the Schwarzschild manifold of mass $m$.

The Riemannian Penrose inequality not only motivates many of the results of this thesis but plays a key role in their proofs.

## 1.4 The harmonic conformal class

Assume $(M, g)$ is a smooth, complete, asymptotically flat, 3-manifold with nonempty compact smooth boundary $\partial M = \Sigma$. To be explicit, we require $g$ to extend smoothly to $\Sigma$. For $g'$ some other Riemannian metric on $M$, consider the relation

$$g' \sim g \qquad \Leftrightarrow \qquad g' = u^4 g,\ \Delta_g u = 0,\ u > 0,\ u \to 1 \text{ at infinity},\ u \text{ is smooth.}$$

Certainly the assumption that $\Delta_g u = 0$ (i.e., $u$ is $g$-harmonic) implies that $u$ is smooth on the interior of $M$; above, we are requiring that $u$ extends smoothly to the boundary. Using formula (A.6) of Appendix A, one can check that $\sim$ is an equivalence relation. The equivalence classes are called the *harmonic conformal classes* of $M$ [7]; the harmonic conformal class to which $g$ belongs will be denoted $\mathcal{H}(g)$. Note that every element of $\mathcal{H}(g)$ is an asymptotically flat metric on $M$ as in Definition 1; this fact follows from the existence of an expansion at infinity [5] of $u$ as

$$u(x) = 1 + \frac{a}{|x|} + O(|x|^{-2}),$$

with successively higher decay for higher derivatives, and formula (A.7), which shows that $R_g$ and $R_{g'}$ share the same asymptotic behavior. The normalization $u \to 1$ at infinity is not essential, but we find it convenient.

We point out that the harmonic conformal class is defined for asymptotically flat manifolds without boundary, but is uninteresting. In this case, the requirement of $u$ being harmonic and tending to 1 at infinity forces $u \equiv 1$ by the maximum principle.

The fundamental geometric significance of the harmonic conformal class is that the sign of the scalar curvature at each point in $M$ agrees for all metrics in $\mathcal{H}(g)$. This follows from formula (A.7) in Appendix A. In particular, if $g$ has nonnegative scalar curvature globally, then so does every metric in $\mathcal{H}(g)$. For this reason, the harmonic conformal class was crucial in Bray's proof of the Riemannian Penrose inequality [7].

We partition $\mathcal{H}(g)$ according to the area assigned to the boundary $\Sigma$. For $A > 0$, let

$$\mathcal{H}_A(g) = \left\{ u^4 g \in \mathcal{H}(g) \ : \ |\Sigma|_{u^4 g} = A \right\},$$

where $|\Sigma|_{u^4 g}$ is the area of $\Sigma$ with respect to $u^4 g$; from formula (A.4) in Appendix A this area is computed by the integral $\int_\Sigma u^4 dA_g$, where $dA_g$ is the area measure on $\Sigma$ induced by $g$. We then have the obvious decomposition:

$$\mathcal{H}(g) = \coprod_{A>0} \mathcal{H}_A(g).$$

We shall return to this discussion in Chapter 3, where we define the *generalized* harmonic conformal class.

## 1.5 The asymptotically flat conformal class

Let $(M, g)$ be a smooth, complete asymptotically flat 3-manifold with or without boundary. The relation on metrics given by

$$g' \approx g \qquad \Leftrightarrow \qquad \begin{cases} g' = \phi^4 g, \ g \text{ and } g' \text{ are asymptotically flat,} \\ \phi \to 1 \text{ at infinity, } \phi > 0, \ \phi \text{ is smooth} \end{cases}$$

is easily seen to be an equivalence relation; call the $\approx$-equivalence class of $g$ the *asymptotically flat conformal class* of $g$ and denote this set by $[g]_{AF}$. By construction

$$\mathcal{H}(g) \subset [g]_{AF}.$$

If $\phi > 0$ is a smooth function satisfying the decay conditions

$$\begin{aligned} \phi &= 1 + O(r^{-p}), & \partial_i \phi &= O(r^{-p-1}), \\ \partial_i \partial_j \phi &= O(r^{-p-2}), & \Delta_g \phi &= O(r^{-q}), \end{aligned} \tag{1.3}$$

for constants $p > \frac{1}{2}$ and $q > 3$, then $\phi^4 g$ belongs to the asymptotically flat conformally class of $g$. We return to the asymptotically flat conformal class in Chapter 8.

# 2

# A Minimal Boundary Problem in Conformal Geometry

In this chapter, let $(M, g)$ be an asymptotically flat 3-manifold with nonempty, smooth, compact boundary $\Sigma = \partial M$. Supposing $g$ has nonnegative scalar curvature, we ask the question: does there exist a metric $g'$ belonging to the harmonic conformal class of $g$ such that the Riemannian Penrose inequality applies to $g'$? In Chapter 1 we saw that $g'$ automatically has nonnegative scalar curvature. Thus the question is primarily that of boundary conditions: can one attain a zero mean curvature boundary within the harmonic conformal class *and* guarantee that the boundary is an outermost minimal surface?

In section 2.1 we explain how an affirmative answer to the above question translates to a mass estimate for the original manifold $(M, g)$, but in 2.2 we establish a geometric criterion that shows the answer can be no. We show in 2.3 how the question can be suitably modified so as to still essentially capture the mass estimate, modulo an unproven conjecture. Finally, in 2.4 we detail a strategy for proving this conjecture.

## 2.1 Motivation from the Penrose inequality

We begin by deriving an estimate for the ADM mass of $(M, g)$, assuming the existence of a particular harmonic conformal factor. Certainly we may not expect a positive lower bound for the ADM mass in general, as demonstrated by the Schwarzschild manifold of $m < 0$ with a neighborhood of its singularity deleted.

**Proposition 5.** *Let $(M, g)$ be a smooth, asymptotically flat 3-manifold with nonnegative scalar curvature and smooth compact boundary $\Sigma$. Assume there exists $g' = u^4 g \in \mathcal{H}(g)$ such that $\Sigma$ is an outermost minimal surface with respect to $g'$. Then*

$$m_{\mathrm{ADM}}(M, g) \geq \sqrt{\frac{\int_\Sigma u^4 dA_g}{16\pi}} + \frac{1}{2\pi} \int_\Sigma \nu(u) dA_g,$$

*where $\nu$ is the unit normal to $\Sigma$ (with respect to $g$) pointing toward infinity.*

In Chapters 5 and 6, we apply this type of mass estimate to deduce more concrete statements.

*Remark:* In all chapters we adopt the convention that unit normal vectors to hypersurfaces (such as $\Sigma$ or a coordinate sphere) point toward infinity.

*Proof of Proposition 5.* The asymptotically flat metric $g'$ has nonnegative scalar curvature. By the other hypotheses, the Riemannian Penrose inequality applies to $(M, g')$ to yield

$$m_{\mathrm{ADM}}(M, g') \geq \sqrt{\frac{\int_\Sigma u^4 dA_g}{16\pi}}.$$

By formula (A.9) in Appendix A,

$$m_{\mathrm{ADM}}(M, g') - m_{\mathrm{ADM}}(M, g) = -\frac{1}{2\pi} \int_{S_\infty} \nu(u) dA_g,$$

where $S_\infty$ is a coordinate sphere "at infinity," explained in Appendix A. By the harmonicity of $u$ and the divergence theorem, the integral of $\nu(u)$ is unchanged if taken over $\Sigma$ in lieu of $S_\infty$. □

We remark that if $u$ were allowed to be superharmonic with respect to $g$, then $g' = u^4 g$ would still have nonnegative scalar curvature. However, the above ADM mass estimate would be weakened when we exchange the flux integral on $S_\infty$ for a flux integral on $\Sigma$.

## 2.2 Obstructions to deforming to minimal boundary

In this section we show that there need not exist $g' \in \mathcal{H}(g)$ such that $\Sigma$ is a minimal surface with respect to $g'$, even without requiring $\Sigma$ to be outermost. Note that it is very difficult to recognize $(M, g)$ as being harmonically conformal to a manifold with minimal boundary: let $(M, g)$ be the Schwarzschild manifold of positive mass (which has minimal boundary), and apply a conformal factor $u^4$ with $\Delta_g u = 0$ and $u|_\Sigma$ a highly oscillatory, positive function. Then it would indeed be difficult to see *a priori* that $(M, u^4 g)$ is harmonically conformal to a metric with boundary of zero mean curvature.

**Proposition 6.** *Let $(M, g)$ be smooth, asymptotically flat with compact, smooth boundary $\Sigma$. Let $\varphi$ be the $g$-harmonic function that vanishes on $\Sigma$ and tends to 1 at infinity. Suppose*

$$H > 4\nu(\varphi)$$

*on $\Sigma$, where $\nu$ is the $g$-unit normal to $\Sigma$ pointing into $M$, and $H$ is the mean curvature of $\Sigma$ with respect to $g$ in the direction of $\nu$. Then every metric in $\mathcal{H}(g)$ assigns positive mean curvature to at least one point of $\Sigma$.*

By the maximum principle, $\nu(\varphi)$ is positive, so the hypothesis is a positivity condition on the mean curvature, relative to a quantity $\nu(\varphi)$ involving the global geometry of $(M, g)$.

*Proof.* First, consider conformal deformations given by $u_c^4$, where $u_c$ is $g$-harmonic, equal to a constant $c > 0$ on $\Sigma$ and 1 at infinity. We write $u_c$ explicitly as

$$u_c = c(1 - \varphi) + \varphi.$$

Using formula (A.8) from Appendix A, the mean curvature $H_c$ of $\Sigma$ with respect to $g_c := u_c^4 g$ is

$$\begin{aligned} H_c &= u_c^{-3}(H u_c + 4\nu(u_c)) \\ &= c^{-3}(Hc + 4\nu(c(1-\varphi) + \varphi)) \\ &= c^{-3}(c(H - 4\nu(\varphi)) + 4\nu(\varphi)) \end{aligned}$$

Note that $\nu(\varphi) > 0$ by the maximum principle, and $c > 0$ and $H - 4\nu(\varphi)$ are positive by hypothesis. Thus, $H_c$ is positive for all $c > 0$.

Now, let $g' = u^4 g$ be any metric in $\mathcal{H}(g)$. We must show that $g'$ assigns positive mean curvature to at least one point of $\Sigma$. Let $c > 0$ be the minimum value of $u|_\Sigma$, and let $u_c$ and $g_c$ be as given above. Note that

$$g' = u^4 g = \left(\frac{u}{u_c}\right)^4 g_c,$$

so that formula (A.8) gives the mean curvature $H'$ of $\Sigma$ with respect to $g'$:

$$H' = \left(\frac{u}{u_c}\right)^{-3} \left(H_c \left(\frac{u}{u_c}\right) + 4\nu_{g_c}\left(\frac{u}{u_c}\right)\right),$$

where $\nu_{g_c}$ is the unit normal to $\Sigma$ with respect to $g_c$. By formula (A.6), $\dfrac{u}{u_c}$ is harmonic with respect to $g_c$ and attains its global minimum value of 1 at a point $p \in \Sigma$ for which $u(p) = c$. By the maximum principle, $\nu_{g_c}\left(\frac{u}{u_c}\right)$ is nonnegative at $p$; by the previous part, $H_c$ is positive. It follows that $H'$ is positive at $p$.

□

*Remarks:*

1. Proposition 6 remains valid if $u$ is allowed to tend to $k > 0$ at infinity.
2. One may define a *binary* invariant of $\mathcal{H}(g)$ according to whether it contains an element with zero mean curvature boundary. Note that $\mathcal{H}(g)$ contains at most one such metric (by formula (A.8) and the maximum principle).
3. Proposition 6 gives a geometric obstruction to finding a metric in $\mathcal{H}(g)$ with minimal boundary. Bray gives an additional topological obstruction [8] using the positive mass theorem.

We reiterate a key point: even if there exists $g' \in \mathcal{H}(g)$ with minimal boundary, we can not automatically apply Proposition 5 to estimate the mass of $(M, g)$ in terms of the area of $\Sigma$, since the RPI only bounds the ADM mass in terms of the area of the *outermost* minimal surface. The RPI gives some positive lower bound for the ADM mass of $(M, g')$, but we have no control over the size of the bound.

## 2.3 A modified problem: a conjecture in conformal geometry

While satisfying the hypotheses of Proposition 5 is evidently too much to hope for in general, Bray conjectured the following [8]. Fix $(M, g)$ as above, with nonempty, smooth, compact boundary $\Sigma$.

**Conjecture 7.** *Let $\delta > 0$. There exists $g' \in \mathcal{H}(g)$ such that:*

1. *The areas of $\Sigma$ and $\tilde{\Sigma}_{g'}$, taken with respect to $g'$, differ by less than $\delta$.*
2. *The surfaces $\tilde{\Sigma}_{g'}$ and $\Sigma$ are disjoint.*

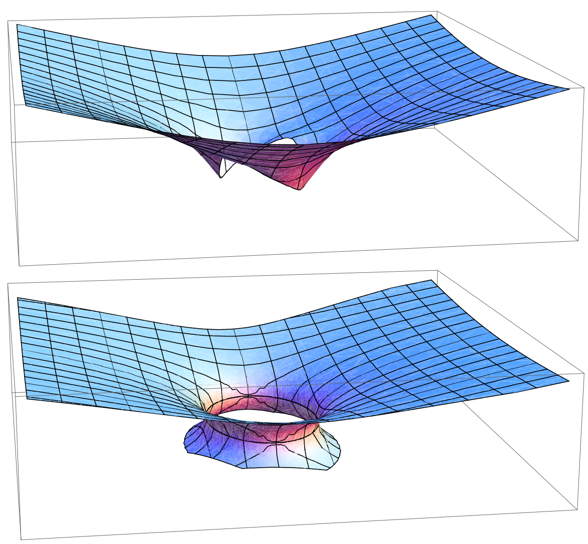
*Here, $\tilde{\Sigma}_{g'}$ is the outermost minimal area enclosure of $\Sigma$ with respect to $g'$ (see below).*

From standard theory, there exists at least one surface enclosing $\Sigma$ that has the least possible area with respect to $g'$. The surface $\tilde{\Sigma}_{g'}$ is well-defined as the *outermost* such surface. These results and the terminology are clarified in section 3.1. An illustration of Conjecture 7 is given in figure 2.1. Note that Conjecture 7 makes no hypotheses on the scalar curvature of $g$ but is typically applied in situations for which $R_g \geq 0$. By the second condition, $\tilde{\Sigma}_{g'}$ is a $C^\infty$ surface with zero mean curvature with respect to $g'$ (see Theorem 11). In particular, if we assume $R_g \geq 0$, the RPI applies to the closure of the region outside $\tilde{\Sigma}_{g'}$. (Note that $\tilde{\Sigma}_{g'}$ is not necessarily an outermost minimal surface, but any minimal surface enclosing $\tilde{\Sigma}_{g'}$ has greater area, by definition.) Given Conjecture 7, one obtains a similar mass estimate as that given in Proposition 5.

$$m_{\mathrm{ADM}}(M,g) \geq \sqrt{\frac{|\tilde{\Sigma}_{g'}|_{g'}}{16\pi}} + \frac{1}{2\pi}\int_\Sigma \nu(u)dA_g \geq \sqrt{\frac{\int_\Sigma u^4 dA_g - \delta}{16\pi}} + \frac{1}{2\pi}\int_\Sigma \nu(u)dA_g.$$

This idea is pursued further in Chapter 5.

FIGURE 2.1: Picture of Conjecture 7



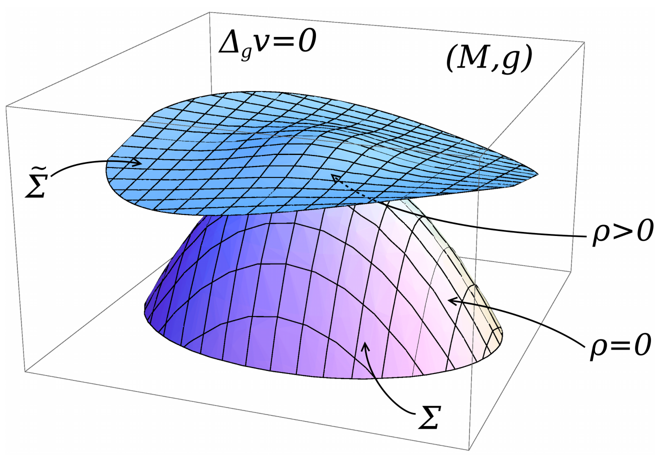
The first picture above is some given asymptotically flat manifold $(M, g)$ with boundary. The outermost minimal area enclosure of the boundary, not pictured, evidently touches the boundary. The bottom picture is the same manifold $M$ equipped with a metric $g' = u^4 g$ as given by Conjecture 7. The latter metric has the property that the outermost minimal area enclosure does not touch the boundary and has area within $\delta$ of that of the boundary. These pictures are simplifications in that a dimension is suppressed and $\Sigma$ can have multiple components that are not necessarily spheres.

## 2.4 Heuristic proof of the conjecture

Now we detail a heuristic approach to proving Conjecture 7. The key is the following calculation.

*Fundamental calculation:* Suppose that $(M, g)$ is an asymptotically flat 3-manifold with compact boundary $\Sigma$ of $g$-area equal to $A$. Assume that $\tilde{\Sigma}$, the outermost minimal area enclosure of $\Sigma$ with respect to $g$, has $g$-area strictly less than $A$ and that $\tilde{\Sigma}$ touches $\Sigma$ on a set of nonempty interior in $\Sigma$. Then there exists a smooth, nonnegative function $\rho : \Sigma \to \mathbb{R}$, supported in $\tilde{\Sigma} \cap \Sigma$, that is not identically zero. Let $v$ be the $g$-harmonic function that is equal to $\rho$ on $\Sigma$ and is zero at infinity. For a schematic of this setup, see figure 2.2.

FIGURE 2.2: Setup of the fundamental calculation




In the above picture, we have depicted an open subset of $M$ that contains the intersection of $\tilde{\Sigma}$ with $\Sigma$. The bump function $\rho$ is supported in $\tilde{\Sigma} \cap \Sigma$ and is the boundary data for a harmonic function $v$.

For $t \geq 0$, define

$$u_t = 1 + tv$$

and

$$g_t = u_t^4 g.$$

Then $g_t$ is a smooth path in $\mathcal{H}(g)$ with $g_0 = g$. We compute the instantaneous rate of change of the area $\Sigma$ along this path of metrics:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}|\Sigma|_{g_t} &= \frac{d}{dt}\Big|_{t=0}\int_\Sigma u_t^4 dA_g \\
&= 4\int_\Sigma v\, dA_g \\
&= 4\int_{\tilde{\Sigma}\cap\Sigma} \rho\, dA_g, \qquad (2.1)
\end{aligned}$$

since $v|_\Sigma = \rho$ is supported in $\tilde{\Sigma}\cap\Sigma$. Now, let $S$ be any minimal area enclosure of $\Sigma$. By definition of *outermost* minimal area enclosure, we see that

$$\tilde{\Sigma}\cap\Sigma \subset S\cap\Sigma. \qquad (2.2)$$

Let $dA_g$ also denote the area measure on $S$ induced by $g$; the rate of change of the area of $S$ along the path $g_t$ is:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}|S|_{g_t} &= \frac{d}{dt}\Big|_{t=0}\int_S u_t^4 dA_g \\
&= 4\int_S v\, dA_g \\
&= 4\int_{S\cap\Sigma} \rho\, dA_g + 4\int_{S\setminus\Sigma} v\, dA_g \\
&> 4\int_{\tilde{\Sigma}\cap\Sigma} \rho\, dA_g \qquad \text{(by (2.2))} \\
&= \frac{d}{dt}\Big|_{t=0}|\Sigma|_{g_t}, \qquad \text{(by (2.1))}
\end{aligned}$$

where the strict inequality comes from the facts that $v$ is positive on $S\setminus\Sigma$ by the maximum principle and that $S\setminus\Sigma$ is nonempty. (If $S=\Sigma$, then the minimal area needed to enclose the boundary would equal $A$, which we assume not to be the case.)

In particular, we see that under this flow of metrics, the area of all minimal area enclosures increases more rapidly than the area of $\Sigma$. We will see in Chapter 4 that it is possible to "normalize" this flow in a natural way so that $g_t$ becomes a path in $\mathcal{H}_A(g)$ (i.e., fixes the area of the boundary) and the rate of change of the areas of the minimal area enclosures is positive. Thus, on a heuristic level:

> If a metric $g' \in \mathcal{H}_A(g)$ satisfies the properties:
>
> 1. $\tilde{\Sigma}_{g'}$ has area less than $A$ with respect to $g'$, and
> 2. the minimum area needed to enclose the boundary (for $g'$) is as large as possible among all metrics in $\mathcal{H}_A(g)$,
>
> then $\tilde{\Sigma}_{g'}$ may only touch $\Sigma$ on a "small set" (one of empty interior).

Another viewpoint is that we have defined a flow $\{g_t\}$ in $\mathcal{H}(g)$ (at least infinitesimally) for which the quantity $\frac{|\tilde{\Sigma}|_{g_t}}{|\Sigma|_{g_t}}$ is monotone increasing. This motivates the following definition:

**Definition 8.** *For $A > 0$, define*

$$\alpha(A) = \sup_{g' \in \mathcal{H}(g)} \left\{ |\tilde{\Sigma}_{g'}|_{g'} \ : \ |\Sigma|_{g'} \leq A \right\},$$

*where $\mathcal{H}(g)$ is the harmonic conformal class of $g$ (see section 1.4) and $|\tilde{\Sigma}_{g'}|_{g'}$ is the $g'$-area of the outermost minimal area enclosure of $\Sigma$ with respect to $g'$.*

The essence of the function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is to maximize the area of the minimal area enclosure(s) of $\Sigma$, subject to a constraint on the area of $\Sigma$. **Warning:** Beginning in Chapter 4, we adopt a slightly different definition of $\alpha$: see Definition 30.

*Strategy:* With the above calculation in mind, we can now outline a strategy for proving Conjecture 7:

1. Given $\delta > 0$, find some $A > 0$ such that $0 < A - \alpha(A) < \delta$.
2. Find a metric $g'$ in $\mathcal{H}_A(g)$ attaining the supremum for $\alpha(A)$ as in Definition 8, and argue $|\Sigma|_{g'} = A$. Our heuristic suggests that the outermost minimal area enclosure $\tilde{\Sigma}_{g'}$ may only touch $\Sigma$ on a set of empty interior.

3. By a refinement of the above calculation, argue that $\tilde{\Sigma}_{g'}$ is actually disjoint from $\Sigma$.

This would prove the conjecture, since $0 < A - \alpha(A) = |\Sigma|_{g'} - |\tilde{\Sigma}_{g'}|_{g'} < \delta$. The first step is the easiest and is carried out in Proposition 64 in Chapter 7. Step 2 is more involved: in order to find a maximizer we must enlarge our class $\mathcal{H}_A(g)$ to allow for conformal factors with less-than-smooth boundary regularity. Such metrics are introduced in Chapter 3, where their minimal area enclosures and resulting singularities are studied. Step 3 is rather delicate and is carried out in Chapter 4 under two technical assumptions.

# 3

# Conformal Geometry with Weak Boundary Regularity

The purpose of this chapter is to lay out some of the definitions and tools to be used in the next chapter, where we return to the study of Conjecture 7. Throughout this chapter we assume $(M, g)$ is an asymptotically flat 3-manifold with nonempty, smooth, compact boundary $\Sigma$. After carefully defining surfaces and minimal area enclosures, we introduce the generalized harmonic conformal class, which allows for a type of metric that is non-smooth at the boundary. We investigate how areas of surfaces behave with respect to such metrics, and prove that minimal area enclosures of the boundary exist. In the last section, we prove a theorem regarding the local structure of singularities for area-minimizers with respect to metrics that are non-smooth at the boundary, a result of possible independent interest.

## 3.1 Surfaces and minimal area enclosures

To carry out the technical details of this and the next chapter, we need an adequate definition of a surface – we choose to work with integral currents. An $n$-current in

$M$ is a linear functional on the space $C^\infty_\circ(\Lambda^n M)$ of compactly-supported, smooth differential $n$-forms. For the relevant details on integral currents, support, boundary, rectifiable sets, Hausdorff measure, etc., see Appendix B and the references cited therein.

Let Hausdorff $k$-measure on $M$ with respect to $g$ be denoted $\mathcal{H}^k_g$. A compact, $\mathcal{H}^3_g$-measurable set $\Omega \subset M$ naturally defines a rectifiable 3-current as follows: a 3-form $\phi$ on $M$ can be written uniquely as $\phi = \rho\omega$, where $\omega$ is the oriented volume form for $(M, g)$, and $\rho$ is a smooth function. The map defined by

$$\Omega(\phi) := \int_\Omega \rho \, d\mathcal{H}^3_g$$

allows us to view $\Omega$ as a rectifiable 3-current. Moreover, $\Omega$ has multiplicity one (in general, one may attach an integer-valued multiplicity function to a rectifiable set to define a rectifiable current). We do not distinguish notationally between $\Omega$ viewed as a set or as a 3-current. In the case that $\partial\Omega$ (as a set) is 2-rectifiable and of finite $\mathcal{H}^2_g$-measure, $\Omega$ is an integral current.

We let $\Sigma$ equal $-\partial M$ as currents (but $\Sigma$ still equals $\partial M$ as sets). This is tantamount to saying that $\Sigma$ is oriented with normal vector pointing into the manifold, and $\partial M$ is oppositely oriented.

**Definition 9.** *Let $\Omega \subset M$ be an $\mathcal{H}^3_g$-measurable, compact set that defines an integral 3-current of multiplicity one (see above). A **surface enclosing** $\Sigma$ is a 2-current $S$ of the form*

$$S = \Sigma + \partial\Omega.$$

*We say $S$ encloses $\Sigma$ **properly** if the support of $S$ does not intersect $\Sigma$. We also say such $S$ is $C^{k,\alpha}$ if the support of $S$ is a $C^{k,\alpha}$ embedded submanifold of $M$. If $S_i = \Sigma + \partial\Omega_i$, $i = 1, 2$, are surfaces enclosing $\Sigma$, then we say $S_1$ **encloses** $S_2$ if the support of $\Omega_1$ contains the support of $\Omega_2$.*

We allow $\Omega$ to be empty, so that $\Sigma$ encloses itself. Note that the above definition of surface is independent of the choice of smooth metric $g$.

A surface $S = \Sigma + \partial\Omega$ enclosing $\Sigma$ is a multiplicity-one integral 2-current with zero boundary. Without loss of generality, we can view $S$ as both a 2-current or as a 2-rectifiable set with an orientation; we do not distinguish notationally between the two. By the $g$-area of $S$, we mean the mass norm of the current $S$ with respect to $g$ (c.f. Appendix B):

$$|S|_g = \sup_{\phi} \{S(\phi) \mid \phi \in C^\infty_\circ(\Lambda^2 M), \|\phi\|_g \leq 1\}, \tag{3.1}$$

where $\|\phi\|_g$ is the maximum possible value that $\phi$ takes on any 2-vector $e_1 \wedge e_2$ for orthonormal $e_1, e_2 \in TM$. The $g$-area can be more concretely realized as the Hausdorff 2-measure of $S$ with respect to $g$ (viewing $S$ as a set):

$$|S|_g = \mathcal{H}^2_g(S) = \int_S d\mathcal{H}^2_g.$$

Notationally we will distinguish between the area measures $dA_g$ and $d\mathcal{H}^2_g$ by using the former only for $C^1$ surfaces.

The mass norm with respect to $g$ for any $n$-current will also be denoted by $|\cdot|_g$ (c.f. Appendix B).

Some further remarks on notation: by the statement "$p \in S$," we mean that $p$ belongs to the support of $S$; $S \cap \Sigma$ the restriction of $S$ to the set $\Sigma$, and $S \setminus \Sigma$ is the restriction of $S$ to the set $M \setminus \Sigma$.

**Definition 10.** *A surface $S$ enclosing $\Sigma$ is a* ***minimal area enclosure*** *of $\Sigma$ (with respect to $g$) if for all surfaces $T$ enclosing $\Sigma$ we have*

$$|S|_g \leq |T|_g.$$

Possibly, many minimal area enclosures of $\Sigma$ exist, and they all obviously have the same area. The following results are well-known in the literature (e.g., [22], [7]).

**Theorem 11.** *Suppose $(M, g)$ is an asymptotically flat 3-manifold with smooth, nonempty, compact boundary. There exists a unique surface $\tilde{\Sigma}$ enclosing $\Sigma$ with the following properties:*

1. *$\tilde{\Sigma}$ is a minimal area enclosure of $\Sigma$ with respect to $g$.*
2. *If $S$ is any other minimal area enclosure of $\Sigma$, then $\tilde{\Sigma}$ encloses $S$.*

*Moreover, $\tilde{\Sigma}$ is an embedded $C^{1,1}$ surface with nonnegative (weak) mean curvature, and $\tilde{\Sigma} \setminus \Sigma$, if nonempty, is $C^\infty$ with zero mean curvature.*

The surface $\tilde{\Sigma}$ is called the *outermost minimal area enclosure* of $\Sigma$ with respect to $g$. When there is a possibility of confusion, we will denote this surface by $\tilde{\Sigma}_g$. We emphasize that in the above theorem, $g$ extends smoothly to the boundary.

*Proof.* We merely sketch the proof. Since $\Sigma$ is compact and $M$ has an asymptotically flat end, standard theory shows that there exists a minimal area enclosure of $\Sigma$ (c.f. the proof of Proposition 24). Suppose $S_i = \partial\Omega_i + \Sigma$ are minimal area enclosures, $i = 1, 2$, and let $S^* = \partial(\Omega_1 \cup \Omega_2) + \Sigma$ and $S_* = \partial(\Omega_1 \cap \Omega_2) + \Sigma$. By the identity

$$\partial(\Omega_1 \cup \Omega_2) = \partial\Omega_1 + \partial\Omega_2 - \partial(\Omega_1 \cap \Omega_2),$$

we have that

$$S^* + S_* = S_1 + S_2.$$

In particular,

$$|S^*|_g + |S_*|_g = |S_1|_g + |S_2|_g.$$

Since $S_1$ and $S_2$ are minimal area enclosures, we can read off that $S^*$ and $S_*$ are minimal area enclosures as well. In particular, $S^*$ encloses both $S_1$ and $S_2$. This observation allows one to uniquely construct the *outermost* minimal area enclosure by taking the union of the regions bounded by all minimal area enclosures.

Certainly $\tilde{\Sigma}$ has nonnegative weak mean curvature, or else an outward variation would produce a surface of less area. Finally, that $\tilde{\Sigma} \setminus \Sigma$ is $C^\infty$ with zero mean curvature is a consequence of regularity theory for minimal surfaces (Theorem 83 of Appendix B). For the $C^{1,1}$ regularity of $\tilde{\Sigma}$, see Theorem 1.3, part (iii), of Huisken and Ilmanen's paper [22] and the references therein. □

## 3.2 The generalized harmonic conformal class

Recall the definition of the harmonic conformal class $\mathcal{H}(g)$ from Chapter 1. For the purposes of maximizing $|\tilde{\Sigma}_{g'}|_{g'}$ as in Definition 8, we seek a topology on $\mathcal{H}(g)$ with good compactness properties. To achieve this, we enlarge the set $\mathcal{H}(g)$ as follows.

Let $f \geq 0$ belong to $L^4(\Sigma)$ (with respect to the measure $dA_g$ induced by $g$). (The reason for considering $L^4$ over other $L^p$ spaces is that for smooth conformal metrics $g' = u^4 g$, the area measures on hypersurfaces are related by $dA_{g'} = u^4 dA_g$.) For $x$ in the interior of $M$, define

$$u(x) = \varphi(x) + \int_\Sigma K(x,y) f(y) dA_g(y), \tag{3.2}$$

where $K(x,y)$ is the Poisson kernel for $(M,g)$ (where $x \in M$, $y \in \Sigma$ and $x \neq y$), and $\varphi(x)$ is the unique $g$-harmonic function that vanishes on $\Sigma$ and approaches one at infinity. In particular, $u$ is $g$-harmonic in the interior of $M$ and tends to one at infinity. We call $u$ the *harmonic function associated to* $f$. Since $f$ is determined uniquely by $u$ (up to almost-everywhere equivalence), we say $f$ is the function in $L^4(\Sigma)$ *determined by* $u$.

*Remarks:*

1. The function $u$ is smooth and positive on $M \setminus \Sigma$, but need not extend continuously to $\Sigma$.
2. While it is not clear that the trace of $u$ onto $\Sigma$ is defined, appropriate notions exist for the idea that $u$ "limits" to $f$ on $\Sigma$ (see Theorem 86 of Appendix C).

On $M \setminus \Sigma$, $u^4 g$ is a smooth Riemannian metric; we make the following definition.

**Definition 12.** *The* ***generalized harmonic conformal class*** *of $g$ is the set $\overline{\mathcal{H}(g)}$ of all Riemannian metrics $u^4 g$ on $M \setminus \Sigma$, where $u$ is the harmonic function associated to some nonnegative $f \in L^4(\Sigma)$ as in (3.2).*

We remark that both of the sets $\overline{\mathcal{H}(g)}$ and $L^4(\Sigma)$ are unchanged if $g$ is replaced by some other metric in $\mathcal{H}(g)$. Moreover, $\overline{\mathcal{H}(g)}$ is (non-canonically) bijective to the set of nonnegative functions in $L^4(\Sigma)$, and $\mathcal{H}(g)$ embeds (canonically) in $\overline{\mathcal{H}(g)}$. Sometimes for emphasis we will refer to $\mathcal{H}(g)$ as the *smooth* harmonic conformal class of $g$.

We also point out that every metric in $\overline{\mathcal{H}(g)}$ is asymptotically flat as in Definition 1, modulo the condition of smoothness up to the boundary.

Define the area of any surface $S$ enclosing $\Sigma$ with respect to $g' = u^4 g \in \overline{\mathcal{H}(g)}$ by:

$$|S|_{g'} = |S|_{u^4 g} = \int_{S \cap \Sigma} f^4 d\mathcal{H}^2_g + \int_{S \setminus \Sigma} u^4 d\mathcal{H}^2_g, \qquad (3.3)$$

where $\mathcal{H}^2_g$ is Hausdorff 2-measure on $M$ with respect to $g$ (see Appendix B). The above reproduces the usual notion of area in the case that $g'$ is smooth. However, in general, $|S|_{g'}$ cannot be written as a mass norm as in (3.1) for the reason that an arbitrary function $f \in L^4(\Sigma)$ is not approximated in $L^4$ *from below* by smooth functions. (See also the remark following the statement of Proposition 19.)

We partition $\overline{\mathcal{H}(g)}$ as follows. For $A \geq 0$, let

$$\overline{\mathcal{H}_A(g)} = \left\{ u^4 g \in \overline{\mathcal{H}(g)} \ : \ |\Sigma|_{u^4 g} = A \right\},$$

the metrics whose boundary area equals $A$. Trivially:

$$\overline{\mathcal{H}(g)} = \coprod_{A \geq 0} \mathcal{H}_A(g).$$

The set $\overline{\mathcal{H}_0(g)}$ contains a single element, $\varphi^4 g$.

## 3.3 Weak convergence of conformal metrics

The identification of $\overline{\mathcal{H}(g)}$ with $L^4(\Sigma)$ affords a weak topology on the former (c.f. Appendix C). While this identification depends on the choice of smooth background metric $g$, the following definition does not.

**Definition 13.** *Let $\{u_n^4 g\}$ be a sequence in $\overline{\mathcal{H}(g)}$ associated to a sequence $\{f_n\}$ of functions in $L^4(\Sigma)$, and let $u^4 g \in \overline{\mathcal{H}(g)}$ be associated to $f \in L^4(\Sigma)$. Then the sequence $\{u_n^4 g\}$* ***converges weakly*** *to $u^4 g$ if $\{f_n\}$ converges weakly to $f$ in $L^4$.*

Strong convergence could be defined in terms of $L^4$-norm convergence, but we do not need this. We will use the following fact:

**Proposition 14.** *Any sequence in $\overline{\mathcal{H}_A(g)}$ has a weakly convergent subsequence in $\overline{\mathcal{H}(g)}$. Moreover, the limit belongs to $\overline{\mathcal{H}_B(g)}$ with $B \leq A$.*

The proof is immediate from Theorems 84 and 85 in Appendix C.

Our notion of weak convergence in $\overline{\mathcal{H}(g)}$ is determined solely by boundary data. However, we must understand how the associated harmonic functions behave under weak convergence.

**Lemma 15.** *Suppose a sequence $\{u_n^4 g\}$ in $\overline{\mathcal{H}(g)}$ converges weakly to $u^4 g \in \overline{\mathcal{H}(g)}$. Then $u_n$ converges pointwise to $u$ on the interior of $M$, with uniform convergence on compact sets contained in the interior of $M$.*

*Proof.* Let $f_n$ and $f$ be the functions in $L^4(\Sigma)$ determined by $u_n^4 g$ and $u^4 g$, respectively. Let $x \in M \setminus \Sigma$, so that

$$u_n(x) - u(x) = \int_\Sigma K(x,y)(f_n(y) - f(y))dA_g(y). \tag{3.4}$$

The function on $\Sigma$ given by $y \mapsto K(x,y)$ is continuous (and in particular is in $(L^4)^* = L^{4/3}$), so (3.4) converges to zero as $n \to \infty$ by the definition of weak convergence. So $u_n \to u$ pointwise on the interior. Uniform convergence on compact subsets in the interior is automatic, since $u_n$ and $u$ are harmonic [20]. □

An immediate consequence of the above lemma is the following:

**Lemma 16.** *Let $S$ be a surface properly enclosing $\Sigma$, and suppose $\{u_n^4 g\}$ in $\overline{\mathcal{H}(g)}$ converges weakly to $u^4 g \in \overline{\mathcal{H}(g)}$. Then*

$$\lim_{n\to\infty} |S|_{u_n^4 g} = |S|_{u^4 g}.$$

*Proof.* Since $S$ encloses $\Sigma$ properly, Lemma 15 shows $u_n \to u$ uniformly on $S$, so:

$$\lim_{n\to\infty} |S|_{u_n^4 g} = \lim_{n\to\infty} \int_S u_n^4 d\mathcal{H}_g^2 = \int_S u^4 d\mathcal{H}_g^2 = |S|_{u^4 g}.$$

□

## 3.4 Convergence of surfaces and behavior of areas

We begin this section by making precise the notion that the $g'$-area of a fixed surface varies continuously under a smooth family of diffeomorphisms, where $g'$ is some metric in $\overline{\mathcal{H}(g)}$, possibly discontinuous at the boundary.

Let $X$ be a smooth, compactly supported vector field on $M$ such that $X|_\Sigma$ equals $\nu$, the unit normal vector field to $\Sigma$ pointing into the manifold. For $t \geq 0$, let $\Phi_t : M \to M$ be the flow generated by $X$; note that $\Phi_t$ is a diffeomorphism onto its image, is the identity map outside a compact set, and maps $\Sigma = \partial M$ into the interior of $M$ for $t > 0$.

**Lemma 17.** *Let $S$ be a surface enclosing $\Sigma$, and let $g' = u^4 g \in \overline{\mathcal{H}(g)}$. Suppose that $|S|_{g'} < \infty$. Then*

$$|S|_{g'} = \lim_{t\to 0^+} |\Phi_t(S)|_{g'}.$$

The claim is trivial unless $S$ intersects the boundary.

*Proof.* In this proof we view $S$ as a 2-rectifiable set, and decompose it as follows. First, consider $S \cap \Sigma$. Reparametrizing the integral,

$$|\Phi_t(S \cap \Sigma)|_{g'} = \int_{\Phi_t(S \cap \Sigma)} u^4 d\mathcal{H}^2_g = \int_{S \cap \Sigma} (u \circ \Phi_t)^4 d(\Phi_t^* \mathcal{H}^2_g), \tag{3.5}$$

where we have formed the pullback measure $\Phi_t^* \mathcal{H}^2_g$ on $S \cap \Sigma$ of the measure $\mathcal{H}^2_g$ on $\Phi_t(S \cap \Sigma)$:

$$\Phi_t^* \mathcal{H}^2_g(E) := \mathcal{H}^2_g(\Phi_t(E)),$$

for all $E \subset \Sigma$. Using the area formula (c.f. §3.2 of Federer [15], §12 of Simon [37]), we can evaluate the last integral as:

$$|\Phi_t(S \cap \Sigma)|_{g'} = \int_{S \cap \Sigma} (u \circ \Phi_t)^4 J(\Phi_t) d\mathcal{H}^2_g, \tag{3.6}$$

where $J(\Phi_t)$ is the Jacobian of the map $\Phi_t : \Sigma \to \Phi_t(\Sigma) \subset M$.

Since the $\Phi_t$ are diffeomorphisms converging smoothly to the identity as $t \to 0^+$, we have that $J(\Phi_t) : \Sigma \to \mathbb{R}^+$ converges uniformly to 1 as $t \to 0^+$. By the Fatou Theorem (Theorem 86 of Appendix C) on non-tangential convergence for harmonic functions,

$$u \circ \Phi_t|_{S \cap \Sigma} \to f|_{S \cap \Sigma}$$

pointwise almost everywhere with respect to $\mathcal{H}^2_g$. Moreover, there exists a function $M(f)$, an analog of the Hardy–Littlewood maximal function (see Theorem 87 of Appendix C), belonging to $L^4(\Sigma)$ such that

$$u \circ \Phi_t|_{\Sigma} \leq M(f)$$

for all $t$ sufficiently small.

Therefore, by Lebesgue's dominated convergence theorem and (3.6),

$$\lim_{t \to 0^+} |\Phi_t(S \cap \Sigma)|_{g'} = \int_{S \cap \Sigma} f^4 d\mathcal{H}^2_g.$$

To finish the proof, we need only show that the $g'$ area of $\Phi_t(S \setminus \Sigma)$ varies continuously in $t$. This is immediate, because $S \setminus \Sigma$ and $\Phi_t(S \setminus \Sigma)$ are subsets of the smooth Riemannian manifold $(M \setminus \Sigma, g')$, and $\Phi_t$ smoothly maps $M \setminus \Sigma$ into $M \setminus \Sigma$. However, we warn that the $g'$-area of $\Phi_t(S \setminus \Sigma)$ is not in general $C^1$ with respect to $t$ at $t = 0$, even if $S$ is smooth.

□

Next, we must understand what it means for surfaces to converge. Before proceeding, the reader is advised to consult Appendix B.

**Definition 18.** *Let $S_n = \Sigma + \partial\Omega_n$ define a sequence of surfaces enclosing $\Sigma$. We say $\{S_n\}$* ***converges*** *to $S = \Sigma + \partial\Omega$ if the sequence of 3-currents $\{\Omega_n - \Omega\}$ converges to zero in mass norm:*

$$|\Omega_n - \Omega|_g \to 0.$$

*(Equivalently, in terms of sets, convergence means $\mathcal{H}^3_g(\Omega_n \triangle \Omega) \to 0$, where $\triangle$ is the symmetric difference.) If $\{S_n\}$ and $S$ are $C^k$ surfaces, we say that $S_n$ converges to $S$ in $C^k$ if the $S_n$ are graphs over $S$ that converge in $C^k$ (see section 2.1 of [9] for more details).*

Note that the above modes of convergence depend only on the smooth topology of $M$ and not on the choice of smooth background metric $g$. This fact is significant, since we often deal with metrics with non-smooth boundary regularity. We also remark that $S_n \to S$ as above implies that the sequence of currents $\{S_n\}$ converges weakly to the current $S$ (c.f. Appendix B and §31 of Simon [37]).

We now state a result on lower semi-continuity of the area function $|\cdot|_{g'}$, which will play a fundamental role in the next chapter. Recall the standard fact that $|\cdot|_g$ is lower semi-continuous with respect to convergence of surfaces as in Definition 18 (see Appendix B); the issue addressed by the following proposition is the potential lack of boundary regularity of $g' = u^4 g$.

**Proposition 19.** *Suppose $g' = u^4 g \in \overline{\mathcal{H}(g)}$, and let $\{S_n\}$ be a sequence of surfaces enclosing $\Sigma$ that converges to some surface $S$ enclosing $\Sigma$. Then $|\cdot|_{g'}$ has the lower semi-continuity property*

$$|S|_{g'} \leq \liminf_{n\to\infty} |S_n|_{g'}.$$

We remark that the usual method of demonstrating lower semi-continuity, identifying area with the mass norm, seems not to work for metrics $g'$. The reason is that (3.1) fails in general for $g'$, since arbitrary boundary data $f \in L^4(\Sigma)$ is not approximated in $L^4$ *from below* by smooth functions. One possible fix is to extend currents to act on forms that are merely $L^1_{\text{loc}}$, say. While this fixes (3.1), the weak convergence of currents on such forms is no longer obvious.

*Proof of Proposition 19.* By Lemma 17, we may assume without loss of generality that each $S_n$ properly encloses $\Sigma$. We pass to a subsequence $\{S_{n_k}\}$ such that

$$\lim_{k\to\infty} |S_{n_k}|_{g'} = \liminf_{n\to\infty} |S_n|_{g'},$$

then replace $\{S_n\}$ with this subsequence. Now we are free to pass to subsequences in completing the argument. Below, we prove the proposition by treating progressively more general cases.

Before proceeding, we modify the background metric $g$ within the harmonic conformal class $\mathcal{H}(g)$ (and modify $u$ accordingly to preserve $g' = u^4 g$) so that $\Sigma$ has positive mean curvature with respect to $g$. This is certainly possible by formula (A.8), choosing a harmonic conformal factor that is uniformly small on $\Sigma$. In particular, a small neighborhood $U$ of $\Sigma$ is foliated by positive mean curvature surfaces. Note that we lose no generality under this modification, since Proposition 19 is a statement about $g'$. Shrinking $U$ if necessary, we also assume that the exponential map from $\Sigma$, mapping in directions normal to $\Sigma$, is injective when mapping onto $U$.

Case 1: We first consider the case in which $\{S_n\}$ converges to $\Sigma$, and each $S_n$ is a graph over $\Sigma$ in the following sense:

$$S_n = \{\exp_p\big(h_n(p)\nu(p)\big) \mid p \in \Sigma\},$$

where $h_n : \Sigma \to \mathbb{R}^+$, and $\exp_p$ is the exponential map for $(M, g)$. We also assume that each $S_n$ is supported in the neighborhood $U$ described above.

The fact that $\{S_n\}$ converges to $\Sigma$ is equivalent to the fact that $\{h_n\}$ converges to zero (strongly) in $L^1(\Sigma)$. Pass to a subsequence of the same name (for both $h_n$ and $S_n$) for which $\{h_n\}$ converges to zero pointwise almost-everywhere.

Identify $h_n : \Sigma \to \mathbb{R}^+$ with the injective map $h_n : \Sigma \to S_n \subset M$ defined in the natural way:

$$h_n(p) = \exp_p\big(h_n(p)\nu(p)\big).$$

Note that $h_n^{-1} : S_n \to \Sigma$ is Lipschitz (and therefore measurable), since it can be identified with the function on $S_n$ that measures the $g$-distance to $\Sigma$. Thus, we may define the measure $h_n^*\mathcal{H}^2_g$ as Hausdorff 2-measure on $S_n$ pulled back to $\Sigma$ via $h_n$: for $E \subset \Sigma$ an $\mathcal{H}^2_g$-measurable set,

$$h_n^*\mathcal{H}^2_g(E) = \mathcal{H}^2_g(h_n(E)).$$

Then by reparametrizing:

$$\begin{aligned} |S_n|_{g'} &= \int_{S_n} u^4 d\mathcal{H}^2_g \\ &= \int_{\Sigma} (u \circ h_n)^4 d(h_n^*\mathcal{H}^2_g). \end{aligned}$$

While tempting to evaluate the measure $d(h_n^*\mathcal{H}^2_g)$ in terms of the Jacobian of $h_n$, this is not justified since $h_n$ is not necessarily Lipschitz. However, note that $h_n^{-1} : S_n \to \Sigma$ is, by construction, the projection map from $S_n$ to $\Sigma$ along geodesics normal to $\Sigma$. Since the foliating surfaces normal to these rays have positive mean curvature by assumption, we see that $h_n^{-1}$ is area-non-increasing:

$$\mathcal{H}^2_g(h_n(E)) \geq \mathcal{H}^2_g(h_n^{-1}(h_n(E))) = \mathcal{H}^2_g(E),$$

for all measurable $E \subset \Sigma$. This shows $h_n^* \mathcal{H}^2_g \geq \mathcal{H}^2_g$ setwise as measures on $\Sigma$. Therefore:

$$|S_n|_{g'} \geq \int_\Sigma (u \circ h_n)^4 d\mathcal{H}^2_g.$$

Since $\{h_n\}$ converges to zero pointwise almost-everywhere, we have from the Fatou theorem of harmonic analysis (Theorem 86 of Appendix C) that $\{u \circ h_n\}$ converges to $f$ pointwise almost-everywhere. Then taking $\liminf$ of both sides of the above and invoking Fatou's lemma, we have

$$\liminf_{n\to\infty} |S_n|_{g'} \geq \liminf_{n\to\infty} \int_\Sigma (u \circ h_n)^4 d\mathcal{H}^2_g \geq \int_\Sigma f^4 d\mathcal{H}^2_g = |\Sigma|_{g'},$$

completing the proof of case 1.

Case 2: Next, consider the case in which $\{S_n\}$ converges to $\Sigma$, and that each $S_n$ is supported in the open neighborhood $U$. We drop the assumption that $S_n$ is a graph, but use the fact that failure to be a graph over $\Sigma$ tends to increase the areas of surfaces converging to $\Sigma$.

Since $S_n$ is supported in $U$, to each point $p \in \Sigma$ we assign the unique number $h_n(p) > 0$ such that the geodesic from $p$ with initial velocity $\nu(p)$ first intersects the support of $S_n$ at distance $h_n(p)$.

Looking at the "graph" of $h_n$, we define:

$$S_n^* = \{\exp_p\big(h_n(p)\nu(p)\big) \mid p \in \Sigma\}.$$

By construction, $h_n^{-1}$ is defined as a bijective, Lipschitz map $S_n^* \to \Sigma$. Since $S_n^* \subset S_n$ as sets, it suffices to show

$$\liminf_{n\to\infty} \int_{S_n^*} u^4 d\mathcal{H}_g^2 \geq \int_\Sigma f^4 d\mathcal{H}_g^2.$$

Certainly $\{h_n\}$ converges to zero in $L^1(\Sigma)$, so the same techniques as in case 1 complete the argument.

Case 3: Now, suppose $\{S_n\}$ converges to $\Sigma$. We show that, without loss of generality, we may assume each $S_n$ lies within the neighborhood $U$. Intuitively, this is the statement that we may excise any "tentacles" of $\Omega_n$, where $S_n = \partial\Omega_n + \Sigma$, that fall outside of $U$. The essential difficulty is that a $g'$-area-minimizer with boundary in $U$ need not lie within $U$, so we need to justify the excision.

For $r > 0$, sufficiently small, let

$$U_r = \{x \in M \mid \operatorname{dist}_g(x, \Sigma) < r\}, \text{ and}$$
$$\Sigma_r = \{x \in M \mid \operatorname{dist}_g(x, \Sigma) = r\}.$$

Take $r_0$ such that $U_{r_0} \subset U$, and let $d(\cdot) = \operatorname{dist}_g(\cdot, \Sigma)$. For almost all $r \in (0, r_0)$, the slice of $\Omega_n$ through $\Sigma_r$, namely

$$\langle \Omega_n, d, r- \rangle = \partial(\Omega_n \llcorner U_r) - \partial\Omega_n \llcorner U_r,$$

is an integral 2-current (see Appendix B for the notation). Moreover, by the slicing lemma (Lemma 79),

$$|\Omega_n|_g \geq |\Omega_n \llcorner U_{r_0}|_g \geq \int_0^{r_0} |\langle \Omega_n, d, r- \rangle|_g \, dr.$$

The left-hand side converges to zero, since $S_n \to \Sigma$; therefore $\{|\langle \Omega_n, d, r- \rangle|\}_n$ converges to zero in $L^1(0, r_0)$. We pass to a subsequence (of the same name) for which $\{|\langle \Omega_n, d, r- \rangle|\}_n$ converges to zero pointwise almost-everywhere on

$(0, r_0)$, and fix a value of $r$ for which $\{|\langle \Omega_n, d, r-\rangle|\}_n$ converges to zero. Since $g$ and $g'$ are uniformly equivalent on $\Sigma_r$, the mass with respect to $g'$ of the slice of $\Omega_n$ through $\Sigma_r$ converges to zero as well:

$$\lim_{n\to\infty} |\langle \Omega_n, d, r-\rangle|_{g'} = 0.$$

Define

$$S_n^* := \partial(\Omega_n \llcorner U_r) + \Sigma,$$

a surface enclosing $\Sigma$ that is supported in $U$. Certainly $S_n^* \to \Sigma$. Below, we see that $S_n^*$ is the result of excising the portion of $S_n$ outside of $U_r$ and replacing it with the slice of $\Omega_n$ through $\Sigma_r$. The idea is that although the $g'$-area of $S_n^*$ possibly exceeds that of $S_n$, this difference is negligible when taking $\liminf_{n\to\infty}$. To see this, we use the definition of slices and of $S_n$ and $S$:

$$\begin{aligned} S_n^* &= \partial(\Omega_n \llcorner U_r) + \Sigma \\ &= \langle \Omega_n, d, r-\rangle + \partial\Omega_n \llcorner U_r + \Sigma \\ &= \langle \Omega_n, d, r-\rangle + (S_n - \Sigma)\llcorner U_r + \Sigma \\ &= \langle \Omega_n, d, r-\rangle + S_n \llcorner U_r. \end{aligned}$$

Applying case 2 to the sequence $\{S_n^*\}$, we have

$$\begin{aligned} |S|_{g'} &\le \liminf_{n\to\infty} |S_n^*|_{g'} \\ &\le \liminf_{n\to\infty} \left(|S_n \llcorner U_r|_{g'} + |\langle \Omega_n, d, r-\rangle|_{g'}\right) \\ &\le \liminf_{n\to\infty} |S_n|_{g'} \end{aligned}$$

where on the third line we used the fact that $\{|\langle \Omega_n, d, r-\rangle|_{g'}\}$ converges to zero as $n \to \infty$. This completes the proof of the proposition for the case $S_n \to \Sigma$.

Case 4: Finally, we consider the general, most difficult, case in which $\{S_n\}$ converges to an arbitrary surface $S$ enclosing $\Sigma$. The rough idea is to decompose each

$S_n$ into portions that converge to $S \cap \Sigma$ and $S \setminus \Sigma$, but it is not immediately obvious how to do so in a useful way: in general, the portion of $S_n$ that is a graph over $S \cap \Sigma$ need not converge to $S \cap \Sigma$ in any reasonable sense.

By the same argument as in case 3, we may assume without loss of generality that each $S_n$ is supported in the open set consisting of points whose $g$-distance is less than $r_0$ from $S$, where $r_0$ is defined in case 3.

Consider the set $S \cap \Sigma$. In terms of currents, by $S \cap \Sigma$ we mean the current $S$ restricted to the set $\Sigma$. By analogy with case 2, let $h_n : S \cap \Sigma \to \mathbb{R}^+$ be defined by the property: the geodesic from $p \in S \cap \Sigma$ with initial velocity $\nu(p)$ first intersects the support of $S_n$ at distance $h_n(p)$. That $h_n(p)$ is well-defined follows from the fact that $S_n$ is supported in an $r_0$-neighborhood of $S$. Observe that $\{h_n\}$ converges to zero in $L^1(S \cap \Sigma)$. Identifying $h_n : \Sigma \to \mathbb{R}^+$ with the map $h_n : \Sigma \to M$ given by exponentiating in the normal direction, define $R_n$ by the relation:

$$S_n = h_n(S \cap \Sigma) + R_n.$$

Heuristically, one might hope to show that $\{h_n(S \cap \Sigma)\}$ converges to $S \cap \Sigma$ and $\{R_n\}$ converges to $S \setminus \Sigma$. However, this need not be true, essentially for the reason that $\{h_n\}$ need not converge uniformly to zero.

We pass to a subsequence of $\{h_n\}$ that converges to zero pointwise almost-everywhere, and to a further subsequence such that

$$\mathcal{H}^2_g(B_n) \leq \frac{1}{n^3}, \tag{3.7}$$

where

$$B_n = \left\{ x \in S \cap \Sigma : h_n(x) > \frac{1}{n} \right\}.$$

To see that this is possible, see Proposition 3.24 of Royden [32]. We pass $\{S_n\}$ and $\{R_n\}$ to the same subsequence as well. We also define

$$A_n = \left\{ x \in S \cap \Sigma : h_n(x) \leq \frac{1}{n} \right\},$$

so that $A_n \dot{\cup} B_n = S \cap \Sigma$. In other words, $\{h_n\}$ converges uniformly to zero except on the set $B_n$, which has measure rapidly converging to zero.

Observe that we can decompose $S_n$ as:

$$S_n = h_n(A_n) + h_n(B_n) + R_n. \tag{3.8}$$

Define $\overline{h}_n(x) : S \cap \Sigma \to \mathbb{R}^+$ by

$$\overline{h}_n(x) = \min\left( h_n(x), \frac{1}{n} \right) = \begin{cases} h_n(x), & x \in A_n \\ \frac{1}{n}, & x \in B_n \end{cases}.$$

We identify $\overline{h}_n(x)$ with its composition with $\exp_p(\nu(p)\cdot)$. We define a rectifiable 2-current $S_n^*$ as follows:

$$S_n^* = S_n + \overline{h}_n(B_n).$$

In other words, $S_n^*$ is obtained by adding on to $S_n$ the set sitting at height $\frac{1}{n}$ above the set $B_n$. (Note that $S_n^*$ need not be a "surface enclosing $\Sigma$.") In particular,

$$|S_n^*|_{g'} \leq |S_n|_{g'} + |\overline{h}_n(B_n)|_{g'}.$$

Next, we claim that

$$\lim_{n\to\infty} |\overline{h}_n(B_n)|_{g'} = 0. \tag{3.9}$$

Observe that every point of $\overline{h}_n(B_n)$ has distance exactly $\frac{1}{n}$ to $\Sigma$. This allows us to control how large $u$ is on this set: by Lemma 20 below, $u$ is of order $O(n^{1/2})$ on $\overline{h}_n(B_n)$, so

$$|\overline{h}_n(B_n)|_{g'} = \int_{\overline{h}_n(B_n)} u^4 d\mathcal{H}^2_g \leq O(n^2)|\overline{h}_n(B_n)|_g.$$

Therefore, it is enough to show that $|\overline{h}_n(B_n)|_g$ is of order $O(n^{-3})$. However, it is easy to see that $|\overline{h}_n(B_n)|_g$ has the same limiting behavior as $|B_n|_g$, since $\overline{h}_n(B_n)$ is the graph over $B_n$ given by the constant function $\frac{1}{n}$. Using (3.7), we have proved (3.9). It follows that:

$$\liminf_{n\to\infty} |S_n|_{g'} \geq \liminf_{n\to\infty} |S_n^*|_{g'}. \tag{3.10}$$

We now decompose $S_n^*$ into portions that converge to $S \cap \Sigma$ and $S \setminus \Sigma$, respectively, as originally sought. Using the definition of $S_n^*$, the decomposition (3.8), and the fact $A_n \dot\cup B_n = S \cap \Sigma$, we see:

$$\begin{aligned} S_n^* &= S_n + \overline{h}_n(B_n) \\ &= h_n(A_n) + h_n(B_n) + R_n + \overline{h}_n(B_n) \\ &= \underbrace{\overline{h}_n(S \cap \Sigma)}_{T_n^*} + \underbrace{h_n(B_n) + R_n}_{R_n^*}, \end{aligned}$$

where we have defined $T_n^*$ and $R_n^*$ by the underbraced terms. Next, we claim that $T_n^*$ and $R_n^*$ are disjoint (viewed as sets). Given $p \in S \cap \Sigma$, consider the geodesic $\gamma$ with initial velocity $\nu(p)$. Let $q$ be the point at which $\gamma$ first intersects the support of $S$. By definition, $q$ does not belong to $R_n$. Moreover, $q$ cannot belong to both $T_n^*$ and $h_n(B_n)$, since points in the former have distance at most $\frac{1}{n}$ to $\Sigma$, and points in the latter have distance strictly greater than $\frac{1}{n}$ from $\Sigma$, by definition of $h_n$ and $B_n$. So $T_n^*$ and $R_n^*$ are disjoint.

It follows that

$$|S_n^*|_{g'} = |T_n^*|_{g'} + |R_n^*|_{g'}.$$

By the superadditivity of $\liminf$, we have

$$\liminf_{n\to\infty} |S_n^*|_{g'} \geq \liminf_{n\to\infty} |T_n^*|_{g'} + \liminf_{n\to\infty} |R_n^*|_{g'}, \tag{3.11}$$

so it is enough to show lower semi-continuity separately for each term on the right-hand side. By the same argument as in case 1, since $\{\overline{h}_n\}$ converges to zero in $L^1(S \cap \Sigma)$,

$$\liminf_{n\to\infty} |T_n^*|_{g'} \geq |S \cap \Sigma|_{g'}. \tag{3.12}$$

As for the other term, we claim that $\{R_n^*\}$ converges weakly (in the sense of currents) to $S \setminus \Sigma$, viewed as currents in the open set $M \setminus \Sigma$. By definition,

$$\begin{aligned} R_n^* &= S_n^* - T_n^* \\ &= S_n + \overline{h}_n(B_n) - T_n^* \\ &= S_n - \overline{h}_n(A_n), \end{aligned}$$

where we have used the fact that $\overline{h}_n(A_n)+\overline{h}_n(B_n) = \overline{h}_n(S\cap\Sigma) = T_n^*$. Certainly $\{S_n\}$ converges weakly to $S \setminus \Sigma$ as currents in $M \setminus \Sigma$: for, let $\phi$ be a smooth 2-form compactly supported in $M \setminus \Sigma$. Then

$$\int_{S_n} \phi \to \int_S \phi = \int_{S\setminus\Sigma} \phi,$$

since $S_n \to S$ weakly in $M$. Thus, we need only show that the sequence of currents $\{\overline{h}_n(A_n)\}$ converges weakly to zero in $M \setminus \Sigma$; this is obvious, since the support of $\overline{h}_n(A_n)$ is contained in a closed $\frac{1}{n}$-neighborhood of $\Sigma$ and thus eventually leaves any compact set contained in $M \setminus \Sigma$. We conclude that $R_n^* \to S \setminus \Sigma$ weakly. We apply lower semi-continuity of $|\cdot|_{g'}$ under weak convergence in the *smooth* Riemannian manifold $(M \setminus \Sigma, g')$ to conclude

$$\liminf_{n\to\infty} |R_n^*|_{g'} \geq |S \setminus \Sigma|_{g'}. \tag{3.13}$$

The proof follows by combining (3.10), (3.11), (3.12), and (3.13).

□

**Lemma 20.** *Let $f \in L^4(\Sigma)$ be nonnegative, and suppose $u$ is the $g$-harmonic function associated to $f$. Let $r(x)$ be the $g$-distance to the boundary $\Sigma$. Then there exists a constant $c > 0$, independent of $r$, such that*

$$u(x) \leq cr(x)^{-1/2},$$

*for $x$ belonging to the interior of $M$ with $r(x)$ sufficiently small.*

*Proof.* Let $x$ belong to the interior of $M$. Then

$$\begin{aligned} u(x) - \varphi(x) &= \int_\Sigma K(x,y)f(y)dA_g(y) \\ &\leq \left(\int_\Sigma K(x,y)^{4/3}dA_g(y)\right)^{3/4}\left(\int_\Sigma f(y)^4 dA_g(y)\right)^{1/4} \\ &\leq \max_{y\in\Sigma} K(x,y)^{1/4}\left(\int_\Sigma K(x,y)dA_g(y)\right)^{3/4}\|f\|_{L^4(\Sigma)}. \end{aligned}$$

By inspection, $1 - \varphi(x) = \int_\Sigma K(x,y)dA_g(y)$, and $\varphi(x)$ extends continuously to zero on $\Sigma$. It is well-known that the Poisson kernel blows up at worst as $O(r^{-2})$ (see Theorem 4.17 of Aubin [4]), which gives the desired blowup rate of $u$. □

The following statement extends the lower-semi-continuity result slightly.

**Proposition 21.** *Let $u$ and $w$ be harmonic functions on $M \setminus \Sigma$ given by*

$$u(x) = \varphi(x) + \int_\Sigma K(x,y)f(y)dA_g(y)$$

$$w(x) = \int_\Sigma K(x,y)h(y)dA_g(y)$$

*for nonnegative $f, h \in L^4(\Sigma)$. For surfaces $S$ enclosing $\Sigma$, the function*

$$S \mapsto \int_{S\cap\Sigma} f^3 h\, d\mathcal{H}^2_g + \int_{S\setminus\Sigma} u^3 w\, d\mathcal{H}^2_g$$

*is lower semi-continuous with respect to convergence of surfaces as in Definition 18.*

The above reduces to Proposition 19 in the case that $f = h$.

*Proof.* The proof is a straightforward modification to that of Proposition 19, since $u^3 w$ is smooth in the interior of $M$ and converges along normal rays to the boundary to $f^3 h$, pointwise almost-everywhere. □

## 3.5 Minimal area enclosures for non-smooth metrics

As an application of Proposition 19, we construct an outermost minimal area enclosure of $\Sigma$, with respect to metrics $g' = u^4 g \in \overline{\mathcal{H}(g)}$ possibly lacking boundary regularity. While it is not immediately obvious that a minimal area enclosure exists in this case, we can nevertheless define the minimum area needed to enclose $\Sigma$.

**Definition 22.** *Given $g' \in \overline{\mathcal{H}(g)}$, define the **minimal enclosing area** of $\Sigma$ with respect to $g'$ to be the number*

$$\min(\Sigma, g') = \inf_S \{|S|_{g'} : S \textit{ is a surface enclosing } \Sigma\}.$$

Clearly $\min(\Sigma, g') \leq |\Sigma|_{g'}$. In the case that $g'$ is a *smooth* metric on $M$, it is clear that $\min(\Sigma, g')$ is simply the $g'$-area of any minimal area enclosure. Next, we give a simplification to the definition of minimal enclosing area.

**Lemma 23.** *There exists a compact set $K \subset M$ such that*

$$\min(\Sigma, g') = \inf_S \{|S|_{g'} : S \textit{ is a surface } \text{properly } \textit{enclosing } \Sigma \textit{ and } \operatorname{spt} S \subset K\},$$

*where $\operatorname{spt} S$ is the support of $S$. Moreover, given $A > 0$, $K$ may be chosen so that the above holds for all $g'$ for which $|\Sigma|_{g'} \leq A$.*

*Proof.* That we may consider only surfaces properly enclosing $\Sigma$ is immediate from Lemma 17. Now, observe that outside a compact set, $(M, g)$ is foliated by coordinate spheres $\{|x| = r\}$ of positive mean curvature, for $r$ greater than or equal to some value

$r_0$; in particular, any surface competing for the minimum $g$-area needed to enclose $\Sigma$ can be taken to be contained in $B_{r_0}$, the closure of the region in $M$ bounded by $\{|x| = r_0\}$. Moreover, the mean curvature $H(r)$ of $\{|x| = r\}$ is asymptotic to $\frac{2}{r}$, the value for round spheres in flat $\mathbb{R}^3$:

$$H(r) = \frac{2}{r} + O(r^{-1-p}),$$

where $p > \frac{1}{2}$ is the constant controlling the rate at which $g$ approaches the flat metric at infinity [22], [14]. Now, given any nonnegative $f \in L^4(\Sigma)$ with $\int_\Sigma f^4 dA_g \leq A$ and associated harmonic function $u$, we have the decay

$$|u - 1| = O(r^{-1})$$
$$\nu(u) = O(r^{-2}),$$

where $O(r^{-1})$ and $O(r^{-2})$ depend on $A$ but not on $f$. In particular, by formula (A.8), there exists $r_1 \geq r_0$ sufficiently large (depending on $A$ but not on $f$) so that the coordinate spheres $\{|x| = r\}$ have positive mean curvature with respect to $g'$ for all $r \geq r_1$. Choose $K = B_{r_1}$. □

A surface $S$ enclosing $\Sigma$ that attains the infimum for $\min(\Sigma, g')$ is called a *minimal area enclosure* of $\Sigma$ with respect to $g'$.

**Proposition 24.** *Suppose $g' = u^4 g \in \overline{\mathcal{H}(g)}$ and that there exists a constant $\epsilon > 0$ such that $u \geq \epsilon$. Then there exists a unique surface $\tilde{\Sigma}_{g'}$ enclosing $\Sigma$ satisfying:*

1. *$\tilde{\Sigma}_{g'}$ is a minimal area enclosure of $\Sigma$ with respect to $g'$, and*
2. *any other minimal area enclosure is enclosed by $\tilde{\Sigma}_{g'}$.*

*Moreover, $\tilde{\Sigma}_{g'} \setminus \Sigma$, if nonempty, is a $C^\infty$ embedded surface of zero mean curvature with respect to $g'$.*

We call $\tilde{\Sigma}_{g'}$ the *outermost minimal area enclosure* of $\Sigma$ with respect to $g'$.

*Proof.* The nontrivial part of the proof is constructing some minimal area enclosure; the remainder follows from standard techniques referred to in Theorem 11.

Let $S_n = \Sigma + \partial\Omega_n$ be a sequence of surfaces enclosing $\Sigma$ such that

$$|S_n|_{g'} \searrow \min(\Sigma, g').$$

By Lemma 23, we may assume without loss of generality that all $S_n$ properly enclose $\Sigma$ and are supported in some fixed compact set $K$. Since $u \geq \epsilon > 0$,

$$\begin{aligned} |S_n|_g &= \int_{S_n} u^{-4} u^4 d\mathcal{H}^2_g \\ &\leq \epsilon^{-4} |S_n|_{g'} \\ &\leq \epsilon^{-4} |S_1|_{g'} < \infty. \end{aligned}$$

This shows the sequence $\{S_n\}$ has a uniform $g$-area bound. In particular, the sequences of currents $\{\Omega_n\}$ and $\{\partial\Omega_n\}$ have uniform mass bounds. By the compactness theorem (Theorem 80 of Appendix B), we may pass to a subsequence of $\{\Omega_n\}$ (of the same name, say, also passing to the corresponding subsequence of $\{S_n\}$) such that $\{\Omega_n\}$ converges in mass norm to some integral 3-current $\Omega$. It is not difficult to see that $\Omega$ is multiplicity-one, so that

$$S := \Sigma + \partial\Omega$$

is a surface enclosing $\Sigma$. Moreover, $S_n \to S$ in the sense of Definition 18.

By Proposition 19,

$$|S|_{g'} \leq \liminf_{n\to\infty} |S_n|_{g'} = \min(\Sigma, g').$$

By the definition of $\min(\Sigma, g')$, equality must hold, so that $S$ is a minimal area enclosure of $\Sigma$ with respect to $g'$. The remainder of the proof follows arguments entirely analogous to those of Theorem 11. □

## 3.6 Singularities of minimal area enclosures

In this section we assume that $g' = u^4 g \in \overline{\mathcal{H}(g)}$ and that $\epsilon \leq u \leq C$ for positive constants $\epsilon, C$. We showed in the previous section that there exists an outermost minimal area enclosure $\tilde{\Sigma}_{g'}$ of $\Sigma$ with respect to $g'$. From standard theory, $\tilde{\Sigma}_{g'} \setminus \Sigma$ is a smooth manifold. In this section we aim to study the local behavior of $\tilde{\Sigma}_{g'}$ (and other possible minimal area enclosures) near the possible singular set $\tilde{\Sigma}_{g'} \cap \Sigma$.

We fix a minimal area enclosure $S$ of $\Sigma$ with respect to $g'$ and a point $p \in S \cap \Sigma$ (recall that this means that $p$ belongs to both $\Sigma$ and the support of $S$). Let $\overline{U}$ equal $\overline{B}(p, \rho)$, the closed $g$-metric ball about $p$ of some fixed radius $\rho > 0$ sufficiently small. Let $\overline{B}^+$ be the closed upper half of the unit ball in $\mathbb{R}^3$ centered at 0, and choose a diffeomorphism $\Phi : \overline{U} \to \overline{B}^+$ such that

1. $\Phi(p) = (0, 0, 0)$,
2. $\Phi(\overline{U} \cap \Sigma)$ equals the intersection of $\overline{B}^+$ with the $\{z = 0\}$ plane, and
3. $d\Phi_p : T_p M \to \mathbb{R}^3$ is an isometry.

We let $(x, y, z)$ be the coordinates on $\overline{U}$ induced from $\Phi$ and do not distinguish between coordinates on $\overline{U}$ and on $\overline{B}^+$. Note that the third statement above merely says that $\{\partial_x, \partial_y, \partial_z\}$ is an orthonormal frame at $p$.

One quantity we shall need is the *lower density* of $S$ at $p$:

$$\Theta(S, p) := \liminf_{r \to 0^+} \frac{|S \llcorner B(p, r)|_g}{\pi r^2},$$

where $B(p, r)$ is the $g$-metric ball centered about $p$ with radius $r$. Recall that $S \llcorner B(p, r)$ is the restriction of the current $S$ to $B(p, r)$ (c.f. Appendix B). The value of $\Theta(S, p)$ is unchanged if $B(p, r)$ is replaced with the coordinate ball of radius $r$.

For $\alpha > 0$, define

$$C_\alpha = \{(x, y, z) \in \overline{U} : (x^2 + y^2)^{1/2} \leq \alpha z\},$$

which can be viewed as the closed region in $\overline{U}$ that lies above a cone with vertex at $p$. The parameter $\alpha$ controls the opening angle: as $\alpha \to \infty$, the cone approaches a plane; as $\alpha \to 0^+$, the cone closes up. Now we can define the *lower partial density* of $S$ at $p$:

$$\Theta_\alpha(S,p) := \liminf_{r\to 0^+} \frac{|S \llcorner (B(p,r) \cap C_\alpha)|_g}{\pi r^2},$$

measuring the lower density of $S$ contributed by points lying within the cone $C_\alpha$.

We also define the *upper density* and *upper partial density* analogously, without introducing notation, by replacing the lim inf with a lim sup where appropriate. Clearly the lower (partial) density is less than or equal to upper (partial) density.

Without some assumption on the boundary regularity of $g'$, we have no obvious monotonicity formula for the $g'$-area-minimizer $S$ (see Appendix B), and thus cannot automatically deduce the existence of a tangent cone (see below). Nevertheless, we can still make a statement regarding the local structure of $S$ at $p$.

**Theorem 25.** *Let $g' = u^4 g \in \overline{\mathcal{H}(g)}$ satisfy $\epsilon \leq u \leq C$ for constants $\epsilon, C > 0$. Suppose $S$ is a minimal area enclosure of $\Sigma$ with respect to $g'$ and that $p \in S \cap \Sigma$. Then one of the following statements holds:*

1. *For some $\alpha > 0$, the lower partial density $\Theta_\alpha(S,p)$ is positive.*
2. *$S$ has a tangent plane at $p$ (see below).*

While weaker than the statement that $S$ has a tangent cone, the above theorem will be sufficient for applications in Chapter 4. We now explain what we mean by tangent plane and tangent cone in this context. Fix the following notation:

1. $D \subset \overline{B}^+$ is the closed unit disk in the $\{z = 0\}$ plane in $\mathbb{R}^3$, with unit normal pointing into to $\overline{B}^+$.
2. $H^+$ is the closed upper-hemisphere of the unit sphere in $\mathbb{R}^3$ centered at 0, with unit normal vector pointing out from $\overline{B}^+$.

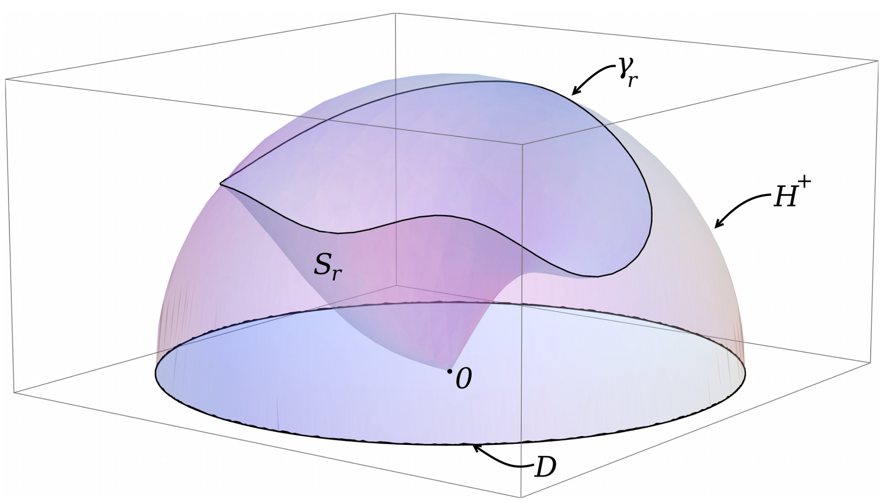
By our choice of orientations, $\partial\overline{B}^+ = H^+ - D$.

Given $r \in (0, \rho]$, we define a surface $S_r$ in $\overline{B}^+$ by "zooming in" on $S$ as follows: $S_r$ is the image of $S \cap \overline{B}(p, \rho)$ under $\Phi$, intersected with the Euclidean ball of radius $r/\rho$ in $\overline{B}^+$ about 0, followed by scaling by $\rho/r$:

$$S_r := \frac{\rho}{r} \cdot \left(\Phi(S \cap \overline{B}(p, \rho)) \cap \overline{B}_\delta(0, r/\rho)\right), \tag{3.14}$$

where $\overline{B}_\delta$ is a closed ball in $\mathbb{R}^3$ measured with respect to the flat metric $\delta$. For instance, $S_1$ is simply $\Phi(S \cap \overline{B}(p, \rho))$, and $S_r$ is a zoomed-in version of the portion of $S$ within distance $r$ of $p$. We use $S_r$ to probe the local structure of $S$ at $p$; we emphasize that $S_r$ is a subset of $\overline{B}^+ \subset \mathbb{R}^3$, not of $M$. We also take the viewpoint that $S_r$ is a rectifiable 2-current in $\overline{B}^+$. See figure 3.1 for a diagram of this setup.

FIGURE 3.1: Setup for studying the local behavior of area minimizers




Pictured above is the closed upper-half ball $\overline{B}^+$, with boundary consisting of the disk $D$ and upper hemisphere $H^+$. The subset $S_r$ corresponds to the portion of a surface $S \subset M$ contained in a ball of radius $r$ about $p \in S$. The curve $\gamma_r$ is the boundary of $\partial S_r$.

We say that $S$ has a *tangent cone* at $p$ if there exists some sequence $\{r_i\} \searrow 0$ such that $\{S_{r_i}\}$ converges to a cone in $\overline{B}^+$ with vertex at the origin. (This notion of convergence, analogous to that given in Definition 18, is clarified in the proof of

Theorem 25.) If $\{S_{r_i}\}$ converges to the disk $D$, then we say $S$ has a *tangent plane* at $p$ (corresponding to the plane $T_p\Sigma \subset T_pM$).

The first step toward proving Theorem 25 is showing that $S$ has positive lower density at $p$. The proof will be an isoperimetric argument. Before proceeding, we make the observation that under the identification of $\overline{U}$ and $\overline{B}^+$ via the diffeomorphism $\Phi$, the metric $g$ on $B(p,r)$ and the metric $\delta$ on $\overline{B}^+$ can be made arbitrarily uniformly close (modulo scaling by $r^{-2}$) as $r \to 0$. In particular,

$$\liminf_{r\to 0^+} |S_r|_\delta = \liminf_{r\to 0^+} \frac{|S \llcorner B(p,r)|_g}{r^2} \tag{3.15}$$

where $|S_r|_\delta$ is the area (mass norm) of $S_r$ with respect to the Euclidean metric $\delta$. The same statement holds with lim inf replaced with lim sup. The factor $r^2$ arises from the scaling in the definition of $S_r$, equation (3.14).

**Lemma 26.** *If $p \in S \cap \Sigma$ and $S$ is a minimal area enclosure of $\Sigma$ with respect to $g'$, then*

$$\Theta(S,p) > 0.$$

*In particular,*

$$\liminf_{r\to 0^+} |S_r|_\delta > 0.$$

*Proof.* For $0 < r \leq \rho$, define

$$m(r) = |S \llcorner B(p,r)|_g,$$

a monotone increasing function of $r$. In particular, $m'(r)$ exists for almost all $r \in [0,\rho]$. Moreover, since we interpret "$p \in S \cap \Sigma$" to mean $p$ belongs to $\Sigma$ and the support of $S$, we have $m(r) > 0$ for $r > 0$. By the slicing lemma (Lemma 79 of Appendix B), for almost all $r$,

$$\gamma_r := \partial(S \llcorner B(p,r))$$

is a rectifiable 1-current for which

$$m'(r) \geq |\gamma_r|_g, \tag{3.16}$$

where $|\gamma_r|_g$ is the mass norm of $\gamma_r$ with respect to $g$. (Note that $\gamma_r$ possibly has a nontrivial multiplicity, so it is necessary to consider its mass rather than its Hausdorff 1-measure. Also, $\partial S$ does not appear in the definition of $\gamma_r$, since $\partial S = 0$.) By the isoperimetric inequality for $(M, g)$ (Theorem 77 of Appendix B), there exists some integral 2-current $T_r \subset M$ with $\partial T_r = \gamma_r$ such that

$$|T_r|_g \leq c|\gamma_r|_g^2, \tag{3.17}$$

for some constant $c$ depending only on $(M, g)$. Now,

$$\begin{aligned} |T_r|_g &\geq C^{-4}|T_r|_{g'} && (u \leq C) \\ &\geq C^{-4}|S\llcorner B(p,r)|_{g'} && (S \text{ minimizes } g'\text{-area}) \\ &\geq C^{-4}\epsilon^4\,|S\llcorner B(p,r)|_g && (u \geq \epsilon) \\ &= C^{-4}\epsilon^4\, m(r). \end{aligned} \tag{3.18}$$

Combining inequalities (3.16), (3.17), and (3.18), we have for almost all $r$,

$$cm'(r)^2 \geq C^{-4}\epsilon^4 m(r).$$

Then for almost all $r \in [0, \rho]$,

$$\frac{d}{dr}\sqrt{m(r)} = \frac{m'(r)}{2\sqrt{m(r)}} \geq \frac{\epsilon^2}{2C^2\sqrt{c}}.$$

Note that $\sqrt{m(r)}$ is an increasing function of $r$ and $m(0) = 0$, so that for *all* $r \in [0, \rho]$

$$\begin{aligned} \sqrt{m(r)} &\geq \int_0^r \frac{d}{ds}\sqrt{m(s)}ds \\ &\geq \frac{\epsilon^2}{2C^2\sqrt{c}} r \end{aligned}$$

In particular, the mass ratio $\frac{m(r)}{\pi r^2}$ is bounded below by the positive constant $\frac{\epsilon^4}{4\pi c C^4}$ as $r \to 0^+$. By definition, it follows that the lower density of $S$ at $p$ is positive. The other claim now follows from (3.15). $\square$

The above also trivially shows that the upper density is positive. Next, we show that the upper density is finite, which implies the lower density is finite.

**Lemma 27.** *The upper density of $S$ at $p$ is finite. In particular,*

$$\limsup_{r \to 0^+} |S_r|_\delta < \infty.$$

*Proof.* For this proof, we again view $S = \Sigma + \partial\Omega$ as a 2-current. Let $d$ be the distance function from $p$, measured with respect to $g$. For $0 < r \leq \rho$, consider the slice of $S$ through the level set $\{d = r\}$:

$$\gamma_r = \langle S, d, r- \rangle = \partial(S \llcorner B(p, r)).$$

See Appendix B for the notation. Note that $\gamma_r$ is supported in $\{d = r\}$. Let $\Omega^c$ be the complement of $\Omega$ in $M$, and let $T_r = \langle \Omega^c, d, r^- \rangle$, the slice of $\Omega^c$ through $\{d = r\}$. By construction, $T_r$ is supported in $\{d = r\}$; we now show that $\partial T_r = \gamma_r$:

$$\begin{aligned} \partial T_r &= \partial \langle \Omega^c, d, r- \rangle \\ &= -\langle \partial\Omega^c, d, r- \rangle. \end{aligned}$$

Since $\Omega + \Omega^c = M$ and $\partial M = -\Sigma$ as currents, we have that

$$\partial\Omega^c = -\Sigma - \partial\Omega = -S.$$

We have therefore shown that

$$\partial T_r = \langle \partial S, d, r- \rangle = \gamma_r.$$

Now, $T_r$ and $S \llcorner B(p, r)$ have the same boundary. Since $S$ minimizes area with respect to $g'$, we have

$$|S \llcorner B(p, r)|_{g'} \leq |T_r|_{g'}.$$

Since the conformal factor $u^4$ relating $g$ and $g'$ is bounded above by a constant $C$,

$$|T_r|_{g'} \leq C^4 |T_r|_g.$$

Finally, $T_r$ is a 2-current of multiplicity one that is supported in $\{d = r\}$. The $g$-area of $\{d = r\}$ is asymptotic to $2\pi r^2$, so for $r$ sufficiently small, we have

$$|T_r|_g \leq 3\pi r^2.$$

Combining our inequalities and using $u \geq \epsilon$, we have

$$\epsilon^4 |S \llcorner B(p, r)|_g \leq 3\pi C^4 r^2.$$

Dividing both sides by $r^2$ and taking $\limsup_{r \to 0^+}$, we have shown that the upper density of $S$ at $p$ is finite. The other claim follows from (3.15), with lim inf replaced with lim sup. □

We now prove Theorem 25:

*Proof.* Let $\{r_i\}$ be a decreasing sequence of positive numbers converging to 0. Assume the first condition fails: for all $\alpha > 0$, $\Theta_\alpha(S, p) = 0$. We may pass to a subsequence of $\{r_i\}$ (of the same name) and find an increasing sequence $\{\alpha_i\}$ of positive numbers diverging to $+\infty$ such that

$$\lim_{i \to \infty} \frac{|S \llcorner (B(p, r_i) \cap C_{\alpha_i})|_g}{\pi r_i^2} = 0. \tag{3.19}$$

In this proof, whenever we pass to a subsequence of $\{r_i\}$, we also pass to the corresponding subsequence of $\{\alpha_i\}$. Note that (3.19) is preserved under taking subsequences.

Write $S = \Sigma + \partial\Omega$. We consider the sequence of surfaces $S_i := S_{r_i}$ in $\overline{B}^+$. Let $\Omega_i$ be the region in $\overline{B}^+$ that lies between $S_i$ and $D$; more precisely,

$$\Omega_i = \frac{\rho}{r} \cdot \left(\Phi(\Omega \cap \overline{B}(p, \rho)) \cap \overline{B}_\delta(0, r/\rho)\right).$$

Observe that $S_i$ and $\Omega_i$ are related as follows:

$$S_i = \partial\Omega_i + D - H^+ \cap \Omega_i,$$

where we interpret $H^+ \cap \Omega_i$ as the slice of $\Omega_i$ through the hemisphere $H^+$.

By Lemma 27, $\{S_i\}$ has uniformly bounded Euclidean area. Certainly all $S_i$ are contained in the compact set $\overline{B}^+$. Using the compactness theorem (Theorem 80, Appendix B), we pass to a subsequence such that $\{S_i\}$ converges to a 2-current $S_0$ defined by:

$$S_0 := \partial\Omega_0 + D - H^+ \cap \Omega_0, \tag{3.20}$$

in the sense that $|\Omega_i - \Omega_0|_\delta \to 0$ for some $\Omega_0 \subset \overline{B}^+$. (Note that this implies $S_i \to S_0$ weakly as currents acting on test forms with support in $\overline{B}^+$ disjoint from $H^+$.)

By the lower semi-continuity of $|\cdot|_\delta$, $S_0$ has finite mass. We view $S_0$ as a "tangent object" to $S$ at $p$.

*Claim:* $S_0$ is not zero, viewed as a 2-current. If this were the case, then from (3.20), the boundary of $\Omega_0$ is supported in $\overline{B}^+$. Applying the constancy theorem (Theorem 81) to the interior of $\overline{B}^+$, we conclude $\Omega_0 = \overline{B}^+$. In particular, $|\overline{B}^+ \setminus \Omega_i|_\delta \to 0$ as $i \to \infty$. In other words, the region in $\overline{B}^+$ lying above $S_i$ collapses to zero volume. We now show that this does not happen by constructing a positive lower bound for the region in $\overline{B}^+$ lying above $S_i$. The idea is as follows: if $|\overline{B}^+ \setminus \Omega_i|_\delta$ becomes arbitrarily small, then almost all slices of $\Omega_i$ through spheres about the origin have arbitrarily small area. Using such slices as competitors for minimizing area, we can show that lower density of $S$ is zero at $p$, contradicting Lemma 26.

In the remainder of this proof, we drop the $\delta$ subscript for the mass norm $|\cdot|$ with respect to $\delta$. For $0 < r \leq 1$, let $A(r)$ denote the mass of the slice of $\overline{B}^+ \setminus \Omega_1$ through the sphere of radius $r$ about 0 in $\overline{B}^+$: for $d$ given by the distance function to the origin,

$$A(r) = |\langle \overline{B}^+ \setminus \Omega_1, d, r+\rangle|.$$

Let $V(r)$ denote the mass of $\overline{B}^+ \setminus \Omega_1$ restricted to the ball of radius $r$ about 0:

$$V(r) = |(\overline{B}^+ \setminus \Omega_1) \llcorner \{d \leq r\}|.$$

Since $V(r)$ is monotone increasing, $V'(r)$ exists for almost all $r$. By the slicing lemma (Lemma 79), $A(r) \leq V'(r)$ for almost all $r$, so

$$\int_0^1 A(r)dr \leq V(1) = |\overline{B}^+ \setminus \Omega_1|.$$

For the purposes of these computations, it suffices to assume that $\rho = 1$. In particular, $\overline{B}^+ \setminus \Omega_i$ is obtained by restricting $\overline{B}^+ \setminus \Omega_1$ to $\overline{B}_\delta(0, r_i)$, followed by scaling by $r_i^{-1}$. It follows that

$$\int_0^{r_i} A(r)dr \leq V(r_i) = r_i^3\,|\overline{B}^+ \setminus \Omega_i|,$$

and therefore,

$$r_i^{-3} \int_{r_i/2}^{r_i} A(r)dr \leq |\overline{B}^+ \setminus \Omega_i|.$$

Then

$$\frac{1}{r_i^3} \cdot \frac{r_i}{2} \inf_{r \in [r_i/2, r_i]} A(r) \leq |\overline{B}^+ \setminus \Omega_i|. \tag{3.21}$$

Now $A(r)$ is not necessarily continuous, so this infimum need not be attained. However, there exists $s_i \in [r_i/2, r_i]$ such that

$$A(s_i) \leq r_i^3 + \inf_{r \in [r_i/2, r_i]} A(r). \tag{3.22}$$

Since $s_i \geq \frac{r_i}{2}$, inequalities (3.21) and (3.22) give:

$$\begin{aligned} \frac{s_i^{-2}}{8} A(s_i) &\leq \frac{r_i^{-2}}{2} A(s_i) \\ &\leq \frac{r_i}{2} + \frac{r_i^{-2}}{2} \inf_{r \in [r_i/2, r_i]} A(r) \\ &\leq \frac{r_i}{2} + |\overline{B}^+ \setminus \Omega_i| \end{aligned} \tag{3.23}$$

Thus, to provide a positive lower bound for $|\overline{B}^+ \setminus \Omega_i|$ as $i \to \infty$, it is enough to provide a positive lower bound for $s_i^{-2} A(s_i)$.

Consider the sequence of surfaces $\{S_{s_i}\}$ in $\overline{B}^+$ with boundary $\gamma_{s_i}$ supported in $H^+$ (i.e. zoom in on $S$ at scale $s_i$). The number $s_i^{-2} A(s_i)$ is the area of the surface in the hemisphere $H^+$ whose boundary is $\gamma_{s_i}$. Since $\delta, g$, and $g'$ are mutually uniformly equivalent and $S$ minimizes $g'$ area, we see there exists a number $\lambda > 0$, independent of $i$, such that

$$\frac{|S \llcorner B(p, s_i)|_g}{\pi s_i^2} \leq \lambda s_i^{-2} A(s_i).$$

(In other words, we are relating the areas of a $g'$-area minimizer $S_{s_i}$ with a competing surface of the same boundary.) Combining this statement with inequality (3.23),

$$\frac{|S \llcorner B(p, s_i)|_g}{\pi s_i^2} \leq 4\lambda r_i + 8\lambda \, |\overline{B}^+ \setminus \Omega_i|.$$

Taking lim inf of both sides and invoking Lemma 26, we see

$$0 < \Theta(S, p) \leq \liminf_{i \to \infty} 8\lambda \, |\overline{B}^+ \setminus \Omega_i|.$$

This demonstrates that $|\overline{B}^+ \setminus \Omega_i|$ is bounded below by a positive constant for $i$ sufficiently large, proving the claim. That is, the "tangent object" $S_0$ is nontrivial: $S_0$ is a nonzero current in $\overline{B}^+$. We proceed to examine two exhaustive, mutually-exclusive cases.

Case 1: Suppose $S_0$ is supported in the disk $D$, so $\partial S_0$ is supported in $D$ as well. But from (3.20), we see $\partial S_0$ is supported in the closed hemisphere $H^+$. This shows that $\partial S_0$ has support in $\partial D$, since the intersection of $D$ and $H^+$ is $\partial D$. Applying the constancy theorem (Theorem 81 of Appendix B) to the interior of the disk $D$, we see $S_0 = D$. Then by definition, $S$ has a tangent plane at $p$.

If case 1 fails, then:

Case 2: The support of $S_0$ intersects the interior of $\overline{B}^+$. Identify the cone $C_\alpha$ with its image under $\Phi$ in the half-ball $\overline{B}^+$. Since the support of $S_0$ intersects the interior of $\overline{B}^+$, there exists some $\alpha > 0$ such that $|S_0 \cap C_{\alpha/2}| > 0$. We claim that $\{S_i \llcorner C_\alpha\}$ converges weakly to $S_0 \llcorner C_\alpha$ as currents in the interior of $C_\alpha$. Let $\phi$ be a smooth 2-form supported in the interior of $C_\alpha$. Then

$$(S_i \llcorner C_\alpha)(\phi) = S_i(\phi) \to S_0(\phi) = (S_0 \llcorner C_\alpha)(\phi),$$

since $S_i \to S_0$ weakly.

By lower semi-continuity of $|\cdot|$ under weak convergence,

$$0 < |S_0 \llcorner C_{\alpha/2}| \leq |S_0 \llcorner C_\alpha| \leq \liminf_{i\to\infty} |S_i \llcorner C_\alpha|,$$

where we have also used the fact that $C_{\alpha/2} \subset C_\alpha$. Since $C_{\alpha_i}$ contains $C_\alpha$ for $i$ sufficiently large,

$$0 < \liminf_{i\to\infty} |S_i \cap C_\alpha| \leq \liminf_{i\to\infty} |S_i \cap C_{\alpha_i}|. \tag{3.24}$$

Finally, from (3.15) and (3.24), we see that

$$0 < \liminf_{i\to\infty} \frac{|S \llcorner (B(p, r_i) \cap C_{\alpha_i})|_g}{\pi r_i^2},$$

contradicting (3.19).

To summarize, we have shown a slightly stronger statement than that which was claimed: if $\Theta_\alpha(S, p) = 0$ for all $\alpha > 0$, then *every* sequence $\{r_i\} \searrow 0$ has a subsequence such that $S_{r_i}$ converges to the tangent plane corresponding to $T_p\Sigma$. $\square$

In Theorem 25 we have not ruled out the possibility that both alternatives 1 and 2 occur, though such a case is not expected.

# 4

# The Conformal Conjecture

In Chapter 2 we introduced Conjecture 7, a statement regarding the harmonic conformal class of an asymptotically flat manifold with boundary. Based on the heuristic proof given in section 2.4, we were motivated to consider the generalized harmonic conformal class, $\overline{\mathcal{H}(g)}$, given in Definition 12. Below we present a modified version of Conjecture 7, adapted to the generalized harmonic conformal class. Throughout this chapter, we let $(M, g)$ be an asymptotically flat 3-manifold with smooth, nonempty, compact boundary $\Sigma$.

**Conjecture 28** ("Conformal conjecture")**.** *Given $\delta > 0$, there exists $g' \in \overline{\mathcal{H}(g)}$ with outermost minimal area enclosure $\tilde{\Sigma}_{g'}$ such that:*

1. *$\tilde{\Sigma}_{g'}$ is disjoint from $\Sigma$, and*
2. *the areas of $\Sigma$ and $\tilde{\Sigma}_{g'}$, taken with respect to $g'$, differ by less than $\delta$.*

In later chapters we will see some applications of this conjecture for manifolds of nonnegative scalar curvature. Although the above is slightly weaker than Conjecture 7, it is evidently sufficient for most applications (c.f., Chapters 5, 6, 7).

Conjecture 28 remains an interesting open problem. The purpose of this chapter is to prove the following main result of this thesis: namely that the conjecture holds if we impose two further assumptions.

**Theorem 29.** *Conjecture 28 is true under the following additional assumptions:*

1. *the maximizer* $g' = u^4 g$ *of Theorem 32 below satisfies* $u \leq C$ *for a constant* $C > 0$*, and*
2. *there exist only finitely many minimal area enclosures of* $\Sigma$ *with respect to* $g'$*.*

The proof of Theorem 29 appears in section 4.3.1 after a number of preliminary results.

## 4.1 Maximizing the minimal enclosing area

Recall the function $\alpha(A)$ from Definition 8 of Chapter 2. Here we make a similar definition for the *generalized* harmonic conformal class. We ask the question: among metrics $g' = u^4 g \in \overline{\mathcal{H}(g)}$ with some fixed upper bound on the boundary area $|\Sigma|_{g'}$, how large can one make the minimal enclosing area of $\Sigma$ with respect to $g'$? The answer is captured by the following definition.

**Definition 30.** *For* $A > 0$*, define*

$$\alpha(A) = \sup_{g' \in \overline{\mathcal{H}(g)}} \left\{ \min(\Sigma, g') \;:\; |\Sigma|_{g'} \leq A \right\},$$

*where* $\overline{\mathcal{H}(g)}$ *is the generalized harmonic conformal class of* $g$ *(Definition 12), and* $\min(\Sigma, g')$ *is the minimal enclosing area (Definition 22).*

By construction, $\alpha$ defines a function $\mathbb{R}^+ \to \mathbb{R}^+$ that satisfies $\alpha(A) \leq A$. From now on, we adopt the above definition for $\alpha(A)$ in lieu of Definition 8.

Let us fix some notation: $\mathcal{S}_K$ is the collection of all surfaces *properly* enclosing $\Sigma$, supported in the compact set $K$ of Lemma 23. Note that $K$ depends on the choice of upper bound $A$ for the boundary area, which we view as fixed.

The remainder of this section is devoted to proving Theorem 32: that the supremum in Definition 30 is attained. This justifies our choice to consider the generalized harmonic conformal class: it is not clear that the supremum in Definition 8 is attained within the smooth harmonic conformal class. First, we need a lemma.

**Lemma 31.** *In Definition 30, the value of $\alpha(A)$ is unchanged if the supremum is taken over the subset $\overline{\mathcal{H}_A(g)}$ of metrics with boundary area equal to A:*

$$\alpha(A) = \sup_{g' \in \overline{\mathcal{H}(g)}} \{\min(\Sigma, g') : |\Sigma|_{g'} = A\} = \sup_{g' \in \overline{\mathcal{H}_A(g)}} \{\min(\Sigma, g')\}.$$

*Moreover, if the supremum for $\alpha(A)$ is attained by some $g' \in \overline{\mathcal{H}(g)}$, then $|\Sigma|_{g'} = A$, so that $g' \in \overline{\mathcal{H}_A(g)}$.*

*Proof.* Let $g' = u^4 g$ be any element of $\overline{\mathcal{H}(g)}$ for which $|\Sigma|_{g'} \leq A$. If $f \in L^4(\Sigma)$ is the function determined by $u$, then

$$\int_\Sigma f^4 d\mathcal{H}^2_g \leq A.$$

There exists a unique number $c \geq 0$ such that

$$\int_\Sigma (f + c)^4 d\mathcal{H}^2_g = A.$$

Let $v$ be the harmonic function associated to $f + c$, and set $g'' = v^4 g$, which belongs to $\overline{\mathcal{H}_A(g)}$. In particular, $v \geq u$ on the interior of $M$ (by the maximum principle) and $f + c \geq f$ on $\Sigma$, so for all surfaces $S$ enclosing $\Sigma$,

$$|S|_{g''} \geq |S|_{g'}.$$

We may take infimum over all such $S$ to conclude

$$\min(\Sigma, g'') \geq \min(\Sigma, g'),$$

which implies the first claim.

To prove the second claim, suppose $g'$ attains the supremum for $\alpha(A)$, but $|\Sigma|_{g'}$ is strictly less than $A$. Then the number $c$ defined above is positive, and by the maximum principle, $v-u$ is bounded below by a positive constant $\kappa$ on the compact set $K$ of Lemma 23. In particular, for $S$ belonging to the set $\mathcal{S}_K$,

$$\begin{aligned} |S|_{g''} - |S|_{g'} &= \int_S (v^4 - u^4) d\mathcal{H}^2_g \\ &\geq \int_S (v-u)^4 d\mathcal{H}^2_g \\ &\geq \kappa^4 \min(\Sigma, g). \end{aligned}$$

Therefore $|S|_{g''} - |S|_{g'}$ is bounded below by a uniform positive constant for $S \in \mathcal{S}_K$, so

$$\min(\Sigma, g'') > \min(\Sigma, g') = \alpha(A),$$

which is impossible, since $g'$ attains the supremum for $\alpha(A)$. □

We now have the techniques to prove the following theorem.

**Theorem 32.** *There exists $g' = u^4 g \in \overline{\mathcal{H}_A(g)}$ that attains the supremum in Definition 30:*

$$\min(\Sigma, g') = \alpha(A).$$

In subsequent sections, such $g' = u^4 g$ will be called a *maximizer for* $\alpha(A)$ without further comment.

*Proof.* Let $\{u_n^4 g\}$ be a maximizing sequence for $\alpha(A)$ in $\overline{\mathcal{H}(g)}$. That is, assume

$$|\Sigma|_{u_n^4 g} \leq A, \quad \text{and} \quad \min(\Sigma, u_n^4 g) \nearrow \alpha(A). \tag{4.1}$$

By Lemma 31, we may assume that each $u_n^4 g$ belongs to $\overline{\mathcal{H}_A(g)}$. By Proposition 14, we may pass to a subsequence (of the same name, say) that converges weakly to some $g' = u^4 g \in \overline{\mathcal{H}_B(g)}$ with $B \leq A$.

Let $S$ properly enclose $\Sigma$. From the definition of $\min(\Sigma, u_n^4 g)$, we have that for all $n$,

$$\min(\Sigma, u_n^4 g) \leq |S|_{u_n^4 g}.$$

The left hand side converges to $\alpha(A)$ by construction. By Lemma 16 we have that

$$\lim_{n\to\infty} |S|_{u_n^4 g} = |S|_{g'}.$$

Thus,

$$\alpha(A) \leq |S|_{g'}.$$

Since this holds for all $S$ properly enclosing $\Sigma$, we may take the infimum over all such $S$ and use Lemma 23 to conclude:

$$\alpha(A) \leq \min(\Sigma, g').$$

From the definition of $\alpha(A)$, the reverse inequality holds as well.

Finally, since $g'$ attains the supremum in the definition of $\alpha(A)$, Lemma 31 shows that $g'$ belongs to $\overline{\mathcal{H}_A(g)}$. □

## 4.2 Properties of the maximizer

In this section we deduce additional useful properties of the maximizer $g'$ constructed in Theorem 32 for the case $\alpha(A) < A$. For the purposes of proving Theorem 29, we need only consider the cases $\alpha(A) < A$. It is known that $\alpha(A) < A$ for all $A$ sufficiently large (Proposition 64 and Corollary 65 of Chapter 7).

We remark that the maximizer $g' = u^4 g$ does not obviously satisfy a variational equation, so there seems to be no clear way of showing the boundary data of $u$ has better than $L^4$ regularity.

*4.2.1 A lower bound*

The following fact will later allow us to use standard geometric measure theory on the smooth space $(M, g)$ to better understand the possibly singular space $(M, g')$.

**Proposition 33.** *Suppose $\alpha(A) < A$, $g' = u^4 g$ is a maximizer for $\alpha(A)$, and $f$ is the function on $\Sigma$ determined by $u$. Then there exists a number $\epsilon > 0$ such that $f \geq \epsilon$ (after possibly modifying $f$ on a set of zero $\mathcal{H}^2_g$-measure).*

The rough idea of the proof is to use the set where $f$ is small to construct a path in $\overline{\mathcal{H}_A(g)}$ through $g'$ that produces a "better" maximizer for $\alpha(A)$, i.e., a path along which the minimal enclosing area increases.

*Proof.*
*Step 1.* In this step we show that $f > 0$ almost everywhere. Assume the set $\{f = 0\}$ has positive $\mathcal{H}^2_g$-measure (or else move on to step 2). Define $\chi$ to be the characteristic function on $\Sigma$ of the set $\{f = 0\}$, which is measurable since $f$ is measurable.

For $t \geq 0$, consider the following family of nonnegative functions in $L^4(\Sigma)$:

$$f_t = f + t\chi.$$

We adopt the notation that a dot indicates a $t$-derivative, so $\dot{f}_0 = \chi$. Let $u_t$ be the harmonic function associated to $f_t$ (see equation (3.2)):

$$u_t(x) = \varphi(x) + \int_\Sigma K(x, y) f_t(y) d\mathcal{H}^2_g(y).$$

Note that $u_0 = u$ and

$$v(x) := \dot{u}_0(x) = \int_\Sigma K(x, y)\chi(y) d\mathcal{H}^2_g(y).$$

Finally, let $g_t = u_t^4 g$, a path in the space $\overline{\mathcal{H}(g)}$.

If $S$ is a surface properly enclosing $\Sigma$, then

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}|S|_{g_t} &= \frac{d}{dt}\Big|_{t=0}\int_S u_t^4 d\mathcal{H}_g^2\\ &= \int_S 4u^3 v d\mathcal{H}_g^2,\end{aligned}$$

which is positive, by the maximum principle. However, the rate of change of the area of $\Sigma$ is

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}|\Sigma|_{g_t} &= \frac{d}{dt}\Big|_{t=0}\int_\Sigma f_t^4 d\mathcal{H}_g^2\\ &= \int_\Sigma 4f^3 \chi d\mathcal{H}_g^2,\end{aligned}$$

which is zero by the definition of $\chi$. This computation provides the heuristic for the proof: for small $t > 0$, $g_t$ ought to have a larger minimal enclosing area than $g_0 = g'$. The remainder of step 1 rigorizes this idea by a) modifying the path to remain within the allowable class of metrics $\overline{\mathcal{H}_A(g)}$ and b) showing the minimal enclosing area (and not just the area of any fixed surface) actually increases along the path.

First, normalize $g_t$ to give a path in $\overline{\mathcal{H}_A(g)}$ (i.e., fix the area of $\Sigma$). To do so, define

$$A(t) = |\Sigma|_{g_t} = \int_\Sigma f_t^4 d\mathcal{H}_g^2.$$

Note that since $f\chi = 0$ almost everywhere,

$$\begin{aligned}A(t) &= \int_\Sigma (f + t\chi)^4 d\mathcal{H}_g^2\\ &= \int_\Sigma \left(f^4 + t^4\chi^4\right) d\mathcal{H}_g^2\end{aligned}$$

is a fourth-order polynomial in $t$ of the form

$$A(t) = A(1 + ct^4), \tag{4.2}$$

where $0 < c < 1$ is a constant. Next, let

$$\overline{f}_t = \frac{A^{1/4}}{A(t)^{1/4}} f_t,$$

chosen so that

$$\int_\Sigma \overline{f}_t^4 d\mathcal{H}_g^2 \equiv A.$$

Let $\overline{u}_t$ be the harmonic function associated to $\overline{f}_t$, namely

$$\begin{aligned}\overline{u}_t(x) &= \varphi(x) + \int_\Sigma K(x,y)\overline{f}_t(y) d\mathcal{H}_g^2(y) \\ &= \varphi(x) + \frac{A^{1/4}}{A(t)^{1/4}} \int_\Sigma K(x,y) f_t(y) d\mathcal{H}_g^2(y) \\ &= \varphi(x) + \frac{A^{1/4}}{A(t)^{1/4}} (u(x) - \varphi(x) + tv(x)) \qquad (4.3)\end{aligned}$$

and let $\overline{g}_t = \overline{u}_t^4 g$, the desired path in $\overline{\mathcal{H}_A(g)}$ that passes through $g'$ at $t = 0$. Since $A'(0) = 0$, the rate of change of $\overline{u}_t$ at $t = 0$ is

$$\dot{\overline{u}}_0(x) = v(x).$$

In particular, the paths $g_t$ and $\overline{g}_t$ agree to first order at $t = 0$.

Recall that $\mathcal{S}_K$ is the collection of surfaces $S$ properly enclosing $\Sigma$ that are contained in the compact set $K$ of Lemma 23. Not knowing *a priori* whether there exist minimal area enclosures with respect to $g'$, we require a notion of "near-minimal area enclosures." Let

$$\mathcal{S}_{\min} = \left\{ S \in \mathcal{S}_K : \ |S|_{g'} \leq \frac{\alpha(A) + A}{2} \right\}. \qquad (4.4)$$

The significance of this area bound is that

$$\frac{\alpha(A) + A}{2} = \alpha(A) + \frac{A - \alpha(A)}{2} = \min(\Sigma, g') + \frac{A - \alpha(A)}{2}.$$

In particular, $\mathcal{S}_{\min}$ is the nonempty set of surfaces in $\mathcal{S}_K$ whose area is close to the minimum possible value, with "closeness" being determined by the positive number $\frac{1}{2}(A - \alpha(A))$.

The goal is to argue that $\min(\Sigma, \overline{g}_t)$ is strictly increasing for $t$ sufficiently small, contradicting the fact that $g'$ is a maximizer. To carry this out, we apply an idea from calculus (Lemma 34 below) that gives sufficient criteria for the minima of a family of functions to be strictly increasing.

Consider the function on $[0,1) \times \mathcal{S}_K$ given by $(t, S) \mapsto |S|_{\overline{g}_t}$. Note that we have a Taylor expansion in $t$

$$\begin{aligned} |S|_{\overline{g}_t} &= |S|_{\overline{g}_0} + t\frac{d}{dt}\Big|_{t=0}|S|_{\overline{g}_t} + \frac{t^2}{2}\frac{d^2}{dt^2}\Big|_{t=0}|S|_{\overline{g}_t} + \dots \\ &= |S|_{g'} + 4t\int_S u^3 v\, d\mathcal{H}^2_g + R_t(S), \end{aligned} \tag{4.5}$$

where the remainder term $R_t(S)$ depends on $S$ and is $O(t^2)$ in $t$.

We will apply Lemma 34, below, to the function $|S|_{\overline{g}_t}$ and the space $X = \mathcal{S}_K$. Certainly condition 1 of the lemma holds (with $\beta = 0$), since $|S|_{g'}$ is positive.

To establish condition 2, we compute

$$\begin{aligned} |S|_{\overline{g}_t} &= \int_S \overline{u}_t^4 d\mathcal{H}^2_g \\ &= \int_S \left(\varphi + \left(\frac{A}{A(t)}\right)^{1/4}(u - \varphi + tv)\right)^4 d\mathcal{H}^2_g && \text{(by (4.3))} \\ &\geq \frac{A}{A(t)}\int_S (\varphi + (u - \varphi + tv))^4\, d\mathcal{H}^2_g && \text{(since } A(t) \geq A) \\ &\geq \frac{1}{1+ct^4}\int_S \left(u^4 + 4tu^3 v\right) d\mathcal{H}^2_g && \text{(by (4.2))} \\ &= \int_S \left(u^4 + 4tu^3 v\right) d\mathcal{H}^2_g - \frac{ct^4}{1+ct^4}\int_S \left(u^4 + 4tu^3 v\right) d\mathcal{H}^2_g \end{aligned} \tag{4.6}$$

where we have used the identity

$$\frac{1}{1+r} = 1 - \frac{r}{1+r}, \qquad \text{for } r \neq -1.$$

We can now estimate the remainder term of equation (4.5):

$$R_t(S) = |S|_{\bar{g}_t} - \int_S \left(u^4 + 4tu^3 v\right) d\mathcal{H}^2_g \qquad \text{(equation (4.5))}$$

$$\geq -\frac{ct^4}{1+ct^4}\int_S u^4 d\mathcal{H}^2_g - \frac{ct^5}{1+ct^4}\int_S 4u^3 v d\mathcal{H}^2_g. \qquad \text{(equation (4.6))}$$

Finally, since $c, t \in (0,1)$, we have

$$\frac{ct^4}{1+ct^4} < ct^4 < ct^2,$$

so we conclude

$$R_t(S) \geq -ct^2|S|_{g'} - ct^2 \int_S 4u^3 v d\mathcal{H}^2_g.$$

So condition 2 of Lemma 34 holds, taking $O(t^2) = ct^2$ and $\kappa = 1$.

Moving on to condition 3, we claim that the function $S \mapsto \int_S u^3 v d\mathcal{H}^2_g$ is bounded below by a positive constant for $S \in \mathcal{S}_{\min}$. The idea is that although the function $\int_S u^3 v d\mathcal{H}^2_g$ takes on values arbitrarily close to zero for surfaces close to $\Sigma$, such surfaces must eventually leave the set $\mathcal{S}_{\min}$ because $\alpha(A) < A$. To rigorize this, suppose there exists a sequence $\{S_i\}$ in $\mathcal{S}_{\min}$ such that

$$\int_{S_i} u^3 v d\mathcal{H}^2_g \searrow 0.$$

Since $|S_i|_g$ is bounded below by a positive constant (namely $\min(\Sigma, g)$) and the function $u^3 v$ is bounded below by a positive number on $K$ minus any neighborhood of $\Sigma$, it must be that $\{S_i\}$ converges to $\Sigma$ in the sense of Definition 18. By Proposition 19 on lower semi-continuity,

$$A = |\Sigma|_{g'} \leq \liminf_{i\to\infty} |S_i|_{g'},$$

which contradicts the assumptions that $|S_i|_{g'} \leq \frac{\alpha(A)+A}{2}$ and $\alpha(A) < A$. We conclude that there exists $\gamma > 0$ such that

$$\int_S u^3 v \geq \gamma > 0, \qquad \text{for } S \in \mathcal{S}_{\min}.$$

Therefore, condition 3 of Lemma 34 holds, taking $\delta = \frac{A-\alpha(A)}{2} > 0$ and $\gamma > 0$ as above. From Lemma 34, there exists $t > 0$ such that

$$\min(\Sigma, \overline{g}_t) > \min(\Sigma, \overline{g}_0) = \min(\Sigma, g') = \alpha(A),$$

which contradicts the assumption that $g'$ was a maximizer for $\alpha(A)$. It follows that $f > 0$, after possibly redefining on a set of measure zero.

*Step 2.* If $f$ is not bounded below almost everywhere by any positive constant, then for all $\epsilon > 0$, the set

$$E_\epsilon := \{x \in \Sigma \mid f(x) \leq \epsilon\}$$

has positive $d\mathcal{H}^2_g$-measure. By step 1, $\mathcal{H}^2_g(E_\epsilon) \to 0$ as $\epsilon \to 0$. Let $\chi = \chi_{E_\epsilon}$, the characteristic function on $\Sigma$ for $E_\epsilon$, which is measurable since $f$ is measurable. Although the construction below depends on $\epsilon$ (as of yet unchosen), we suppress $\epsilon$ in the notation.

For $t \geq 0$ define

$$f_t = f(1 + t(\chi - k))$$

where $k = \frac{1}{A}\int_\Sigma f^4 \chi d\mathcal{H}^2_g$. By step 1, $k > 0$. For all $\epsilon > 0$ sufficiently small, $k < 1$. In this case, $f_t$ is nonnegative for $t > 0$ sufficiently small, independent of $\epsilon$. Let $u_t$ be the harmonic function associated to $f_t$:

$$\begin{aligned} u_t(x) &= \varphi(x) + \int_\Sigma K(x,y) f(y)(1 + t(\chi(y) - k)) d\mathcal{H}^2_g(y) \\ &= u(x) + t w(x) - kt(u(x) - \varphi(x)) \end{aligned}$$

where $w(x)$ is $g$-harmonic, given by $f\chi$ on $\Sigma$ and zero at infinity. Next, let $g_t = u_t^4 g$, a path in $\overline{\mathcal{H}(g)}$. Note that $k$ was chosen so that the boundary area is fixed to first order:

$$\frac{d}{dt}\Big|_{t=0} |\Sigma|_{g_t} = \frac{d}{dt}\Big|_{t=0} \int_\Sigma f_t^4 d\mathcal{H}_g^2$$
$$= 4\int_\Sigma f^3(f(\chi - k))d\mathcal{H}_g^2 = 0.$$

Our strategy is the same as in step 1: show the minimum enclosing area is strictly increasing in (an appropriate normalization of) the family $g_t$. To check that this is plausible, consider $S \in \mathcal{S}_{\min}$ and compute the first variation of the area of $S$ under the metrics $g_t$:

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\Big|_{t=0} |S|_{g_t} &= \int_S u^3(w - k(u-\varphi))d\mathcal{H}_g^2 && \text{(def. of } g_t,\ u_t) \\
&= \int_S u^3 w d\mathcal{H}_g^2 - k|S|_{g'} + k\int_S u^3\varphi d\mathcal{H}_g^2 && \text{(def. of } |S|_{g'}) \\
&> \int_S u^3 w d\mathcal{H}_g^2 - \frac{\alpha(A)+A}{2}\cdot k && \text{(def. of } \mathcal{S}_{\min}) \\
&= \int_{S\setminus U_\rho} u^3 w d\mathcal{H}_g^2 - \frac{\alpha(A)+A}{2A}\cdot \int_\Sigma f^4\chi d\mathcal{H}_g^2, && \text{(def. of } k) \qquad (4.7)
\end{aligned}$$

where $U_\rho$ is the open set of points in $M$ whose $g$-distance is less than $\rho > 0$ from $\Sigma$. The motivation for excising the set $U_\rho$ is that we desire a positive lower bound for the integrand $u^3 w$. Using Lemma 35 below, we choose $\rho > 0$ such that $\mathcal{H}_g^2(S \setminus U_\rho) \geq \rho$ for all surfaces $S \in \mathcal{S}_{\min}$. Note that such $\rho$ may be chosen independently of $\epsilon$.

For $x \in K \setminus U_\rho$ (where $K \subset M$ is the compact set of Lemma 23),

$$\begin{aligned}
w(x) &= \int_\Sigma K(x,y)f(y)\chi(y)d\mathcal{H}_g^2(y) \\
&\geq \min_{x\in K\setminus U_\rho, y\in\Sigma} K(x,y)\int_\Sigma f\chi d\mathcal{H}_g^2 \\
&= w_0\int_\Sigma f\chi d\mathcal{H}_g^2,
\end{aligned}$$

where $w_0$ is a positive constant independent of $\epsilon$. Similarly, $u(x)$ is bounded below on $K \setminus U_\rho$ by a constant $u_0$ independent of $\epsilon$.

Returning to (4.7), we have for $S \in \mathcal{S}_{\min}$,

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\Big|_{t=0} |S|_{g_t} &> \int_{S\setminus U_\rho} u^3 w \, d\mathcal{H}^2_g - \frac{\alpha(A)+A}{2A}\int_\Sigma f^4 \chi \, d\mathcal{H}^2_g \\
&\geq \mathcal{H}^2_g(S\setminus U_\rho)\cdot u_0^3 w_0 \cdot \int_\Sigma f\chi \, d\mathcal{H}^2_g - \frac{\alpha(A)+A}{2A}\cdot \epsilon^3 \int_\Sigma f\chi \, d\mathcal{H}^2_g \\
&\geq \int_\Sigma f\chi \, d\mathcal{H}^2_g \left( u_0^3 w_0 \rho - \frac{\alpha(A)+A}{2A}\epsilon^3 \right),
\end{aligned}$$

where on the second line we used the fact that $f \leq \epsilon$ on the support of $\chi$ and on the third line, $\mathcal{H}^2_g(S\setminus U_\rho) \geq \rho$. Note that $\int_\Sigma f\chi$ is never zero, since $f > 0$ (by step 1) and $\chi > 0$ on a set of positive measure. Moreover, since the numbers $u_0, w_0, \rho, \alpha(A)$, and $A$ are independent of $\epsilon$, we may shrink $\epsilon > 0$ so that the above is a positive number $C(\epsilon)$:

$$\frac{1}{4}\frac{d}{dt}\Big|_{t=0} |S|_{g_t} \geq C(\epsilon).$$

From now on, we fix such a value of $\epsilon > 0$.

Now, we complete the argument by constructing a variation that shows $g'$ is not a maximizer. First, we need to normalize the path $g_t$ to be a path in $\overline{\mathcal{H}_A(g)}$. Let

$$A(t) := |\Sigma|_{g_t} = \int_\Sigma f_t^4 \, d\mathcal{H}^2_g,$$

a fourth-order polynomial in $t$ of the form

$$A(t) = A(1 + c_2 t^2 + c_3 t^3 + c_4 t^4),$$

for numbers $c_i$ depending on $\epsilon$, with $c_2 > 0$. Define

$$\overline{f}_t = \frac{A^{1/4}}{A(t)^{1/4}} f_t,$$

and let $\overline{u}_t$ be the harmonic function associated to $\overline{f}_t$:

$$\overline{u}_t = \varphi + \frac{A^{1/4}}{A(t)^{1/4}} \left(u - \varphi + tw - tk(u - \varphi)\right).$$

Let $\overline{g}_t = \overline{u}_t^4 g$, a path in $\overline{\mathcal{H}_A(g)}$. Since $A'(0) = 0$, the computations above for the first variation of the area of $S$ are identical whether carried out for $g_t$ or for $\overline{g}_t$. In particular, for $S \in \mathcal{S}_{\min}$,

$$\frac{d}{dt}\bigg|_{t=0} |S|_{\overline{g}_t} \geq C(\epsilon) \tag{4.8}$$

for a positive constant $C(\epsilon)$ independent of $S$.

As in step 1, we will apply Lemma 34 to the space $X = \mathcal{S}_K$ and the function $(t, S) \mapsto |S|_{\overline{g}_t}$. We have a Taylor expansion in $t$:

$$\begin{aligned} |S|_{\overline{g}_t} &= |S|_{\overline{g}_0} + t\frac{d}{dt}\bigg|_{t=0} |S|_{\overline{g}_t} + \frac{t^2}{2}\frac{d^2}{dt^2}\bigg|_{t=0} |S|_{\overline{g}_t} + \ldots \\ &= |S|_{g'} + t\int_S 4u^3(w - k(u-\varphi))d\mathcal{H}^2_g + R_t(S), \end{aligned} \tag{4.9}$$

where $R_t(S)$ depends on $S$ and is $O(t^2)$ in $t$.

To verify condition 1 of the lemma, note that the zeroth-order term $|S|_{g'}$ is positive, and the first-order term satisfies

$$\begin{aligned} 4\int_S u^3(w - k(u-\varphi)) &\geq -4k\int_S u^4 d\mathcal{H}^2_g \\ &= -4k|S|_{g'}. \end{aligned}$$

With $\beta = 4k$, condition 1 holds.

As for condition 2, we must find an appropriate lower bound for the remainder term in (4.9). First, we make the following calculation, using the fact that $A(t) \geq A$

for $t$ sufficiently small:

$$
\begin{aligned}
|S|_{\overline{g}_t} &= \int_S \overline{u}_t^4 d\mathcal{H}_g^2 \\
&= \int_S \left(\varphi + \frac{A^{1/4}}{A(t)^{1/4}}(u - \varphi + t(w - k(u-\varphi)))\right)^4 d\mathcal{H}_g^2 \\
&\geq \frac{A}{A(t)} \int_S (u + t(w - k(u-\varphi))^4 \, d\mathcal{H}_g^2 \\
&= \frac{1}{1 + c_2 t^2 + c_3 t^3 + c_4 t^4} \int_S (u + t(w - k(u-\varphi)))^4 \, d\mathcal{H}_g^2 \\
&\geq \int_S (u + t(w - k(u-\varphi)))^4 \, d\mathcal{H}_g^2 \\
&\quad - (c_2 t^2 + c_3 t^3 + c_4 t^4) \int_S (u + t(w - k(u-\varphi)))^4 \, d\mathcal{H}_g^2.
\end{aligned}
$$

Again, we have used the identity

$$
\frac{1}{1+r} = 1 - \frac{r}{1+r}, \qquad \text{for } r \neq -1.
$$

In the above expression, we can read off the terms that are $O(1)$ and $O(t)$ in $t$. The coefficients of terms that are order $O(t^2)$ and higher can easily be bounded, since

$$
|w - k(u-\varphi)| \leq w + ku + k\varphi \leq 3u.
$$

In particular,

$$
|S|_{\overline{g}_t} \geq \int_S u^4 d\mathcal{H}_g^2 + 4t \int_S u^3 (w - k(u-\varphi)) d\mathcal{H}_g^2 - O(t^2) \int_S u^4 d\mathcal{H}_g^2,
$$

where $O(t^2)$ is independent of $S$. By equation (4.9),

$$
R_t(S) \geq -O(t^2)|S|_{g'},
$$

so condition 2 of the lemma holds, with $\kappa = 0$.

Finally, we note condition 3 holds by (4.8), choosing $\gamma = C(\epsilon)$ and $\delta = \frac{A-\alpha(A)}{2}$. From Lemma 34, there exists $t > 0$ such that

$$\min(\Sigma, \overline{g}_t) > \min(\Sigma, \overline{g}_0) = \min(\Sigma, g') = \alpha(A),$$

which is impossible. This contradiction shows that $f \geq \epsilon > 0$, after possibly redefining $f$ on a set of measure zero.

□

The following lemma is our tool for detecting when the minimum $x$-value of a family of functions $h_t(x) > 0$ is strictly increasing in $t$. The criteria we use are:

1. $\frac{\partial}{\partial t} h_t(x)\big|_{t=0}$ must not be too negative, relative to the size of $h_0(x)$.
2. For $k \geq 2$, $\frac{\partial^k}{\partial t^k} h_t(x)\big|_{t=0}$ must not be too negative, relative to the size of $h_0(x)$ and $\frac{\partial}{\partial t} h_t(x)\big|_{t=0}$.
3. On the set $X_\delta$ consisting of points $x$ that are within $\delta$ of the minimum of $h_0(\cdot)$, the first derivative $\frac{\partial}{\partial t} h_t(x)\big|_{t=0}$ must be bounded below by a positive constant independent of $x$.

We do not assume the minimum of $h_t(x)$ is achieved. The formal statement of this result is:

**Lemma 34** (Calculus lemma)**.** *Let $X$ be a nonempty set and let $t_0 > 0$. Suppose $h : [0, t_0) \times X \to \mathbb{R}$, $(t, x) \mapsto h_t(x)$ is a function that is $C^\infty$ in $t$ for each $x$ with Taylor expansion*

$$h_t(x) = a(x) + tb(x) + R_t(x),$$

*where $R_t(x)$ is a term that is order $t^2$ for each $x$. If*

$$m(t) = \inf_{x \in X} h_t(x)$$

*is the minimum possible value of $h_t(\cdot)$, then there exists $t > 0$ so that $m(t) > m(0)$, provided all of the following criteria are met:*

1. *$a(x) > 0$ and $b(x) \geq -\beta a(x)$ (for some constant $\beta \geq 0$),*

2. *$R_t(x) \geq -(a(x) + \kappa b(x))O(t^2)$, where $O(t^2)$ is a smooth function of order $t^2$, independent of $x$, and $\kappa \geq 0$ is some constant, and*

3. *there exist $\gamma > 0, \delta > 0$ such that $b(x) \geq \gamma$ for $x$ belonging to the (nonempty) set $X_\delta := \{x \in X : a(x) \leq m(0) + \delta\}$.*

Note that $X$ need not be a topological space. To emphasize: the constants $\beta, \kappa, \gamma$, and $\delta$ are independent of $t$ and $x$.

*Proof.* For $x \in X$ and $t \in [0, t_0)$,

$$\begin{aligned} h_t(x) &= a(x) + tb(x) + R_t(x) && \text{(definition of } h_t(x)) \\ &\geq a(x) + tb(x) - (a(x) + \kappa b(x))O(t^2) && \text{(hypothesis 2)} \\ &= a(x) + b(x)(t - \kappa O(t^2)) - a(x)O(t^2) && \text{(rearranging)} \qquad (4.10) \end{aligned}$$

We look at two cases, depending on whether $x$ is or is not in the set $X_\delta$. For $x \in X_\delta$, we apply (4.10) and the definitions of $m(0)$ and $X_\delta$ to see

$$h_t(x) \geq m(0) + b(x)(t - \kappa O(t^2)) - (m(0) + \delta)O(t^2)$$

Shrinking $t_0$ if necessary, we can arrange that $t - \kappa O(t^2)$ is nonnegative for $t \in [0, t_0)$. Then by hypothesis 3:

$$h_t(x) \geq m(0) + \gamma(t - \kappa O(t^2)) - (m(0) + \delta)O(t^2).$$

By shrinking again $t_0$ if necessary, we can arrange that

$$h_t(x) \geq m(0) + \frac{\gamma}{2}t \qquad \text{for all } \; x \in X_\delta, t \in [0, t_0) \qquad (4.11)$$

Next, if $x$ belongs to the complement of $X_\delta$,

$$\begin{aligned} h_t(x) &\geq a(x) + b(x)(t - \kappa O(t^2)) - a(x)O(t^2) && \text{(equation (4.10))} \\ &\geq a(x) - \beta a(x)(t - \kappa O(t^2)) - a(x)O(t^2) && \text{(hypothesis 1)} \\ &= a(x)(1 - \beta(t - \kappa O(t^2)) - O(t^2)) && \text{(rearranging)} \qquad (4.12) \end{aligned}$$

We may shrink $t_0$ if necessary to obtain

$$1 - \beta(t - \kappa O(t^2)) - O(t^2) \geq \frac{m(0) + \delta/2}{m(0) + \delta}, \tag{4.13}$$

for $t \in [0, t_0)$. Since $x \notin X_\delta$, we have that $a(x) > m(0) + \delta$. Combining (4.12) and (4.13), we conclude

$$h_t(x) > (m(0) + \delta)\frac{m(0) + \delta/2}{m(0) + \delta} = m(0) + \delta/2 \qquad \text{for all } \; x \notin X_\delta, t \in [0, t_0) \tag{4.14}$$

Finally, combining (4.11) and (4.14), we have that for $x \in X$ and $t \in [0, t_0)$,

$$h_t(x) \geq m(0) + \min\left(t\frac{\gamma}{2}, \frac{\delta}{2}\right).$$

In particular, there exists $t \in (0, t_0)$ with $m(t) > m(0)$. □

We now prove a lemma required in the proof of step 2 of Proposition 33.

**Lemma 35.** *For $\rho > 0$, let $U_\rho$ be the open subset of $M$ consisting of points whose $g$-distance from $\Sigma$ is less than $\rho$. Suppose $g' = u^4 g \in \overline{\mathcal{H}_A(g)}$ has minimal enclosing area $B := \min(\Sigma, g')$ strictly less than $A$. There exists $\rho > 0$ such that*

$$\mathcal{H}^2_g(S \setminus U_\rho) \geq \rho$$

*for all surfaces $S$ enclosing $\Sigma$ with*

$$|S|_{g'} \leq \frac{A + B}{2}.$$

The intuition is that if $S = \partial\Omega + \Sigma$ has very little area outside $U_\rho$, then $\Omega$ has very little volume outside $U_\rho$ (by a version of the isoperimetric inequality) If $\rho$ is very small, this shows that $S$ must be close $\Sigma$ and thus cannot have area much smaller than $A$.

*Proof.* If the claim fails, there exist a sequence of numbers $\rho_i \to 0$ and a sequence of surfaces $S_i = \partial\Omega_i + \Sigma$ with area at most $\frac{1}{2}(A+B)$ such that $\mathcal{H}^2(S_i \setminus U_{\rho_i}) < \rho_i$. We proceed to show that $\mathcal{H}^3_g(\Omega_i)$ converges to 0 as $i \to \infty$.

- Certainly $\mathcal{H}^3_g(\Omega_i \cap U_{\rho_i}) \leq \mathcal{H}^3_g(U_{\rho_i})$ converges to 0 as $i \to \infty$.

- By a relative isoperimetric inequality for manifolds with boundary (see Theorem 78 in Appendix B), there exists a constant $c > 0$ such that

$$\mathcal{H}^3_g(\Omega_i \setminus U_{\rho_i})^{2/3} \leq c\mathcal{H}^2_g(S_i \setminus U_{\rho_i}),$$

  Now, $\mathcal{H}^2_g(S_i \setminus U_{\rho_i}) < \rho_i$ by assumption, so that $\mathcal{H}^3_g(\Omega_i \setminus U_{\rho_i}) \to 0$ as $i \to \infty$.

We have shown that $\mathcal{H}^3_g(\Omega_i) \to 0$ as $i \to \infty$, so that $\{S_i\}$ converges to $\Sigma$ by definition. By Proposition 19,

$$A = |\Sigma|_{g'} \leq \liminf_{i\to\infty} |S_i|_{g'},$$

which violates the assumptions that $|S_i|_{g'} \leq \frac{A+B}{2}$ and $B < A$. $\square$

*4.2.2 The minimal area enclosure for the maximizing metric*

Let $g' = u^4 g$ be a maximizer for $\alpha(A) < A$, let $f$ be the function on $\Sigma$ determined by $u$. By Proposition 33, $f$ is bounded below by a positive constant. By the maximum principle, $u$ is bounded below by a positive constant. The hypotheses of Proposition 24 of Chapter 3 are fulfilled, so there exists a unique outermost minimal area enclosure $\tilde{\Sigma}_{g'}$ of $\Sigma$ with respect to $g'$.

We generalize and rigorize the heuristic argument of section 2.4 to show that $\tilde{\Sigma}_{g'}$ may only intersect the boundary on a small subset.

**Proposition 36.** *If $\alpha(A) < A$ and $g' = u^4 g$ is a maximizer for $\alpha(A)$, then $\tilde{\Sigma}_{g'} \cap \Sigma$ has zero $\mathcal{H}^2_g$-measure.*

We do not rule out the possibility that other minimal area enclosures touch $\Sigma$ on a set of positive measure. This will not be a problem, however, because Conjecture 28 is a statement regarding the *outermost* minimal area enclosure. We remark that the condition $\alpha(A) < A$ prohibits $\Sigma$ from being its own minimal area enclosure.

*Proof.* Suppose $E = \tilde{\Sigma}_{g'} \cap \Sigma$ has positive $\mathcal{H}^2_g$-measure. Let $\chi$ be the characteristic function for $E$ on $\Sigma$. Since $E$ is closed in $\Sigma$, $\chi$ is a measurable function.

Let $f$ be the function on $\Sigma$ determined by $u$. For $t \geq 0$, define the family of functions in $L^4(\Sigma)$

$$f_t = f(1 + t(\chi - k)),$$

where $k = \frac{1}{A}\int_\Sigma f^4 \chi d\mathcal{H}^2_g$. Note that $0 < k < 1$, since $f$ is positive, $\chi$ is nontrivial, and $\Sigma$ is not its own outermost minimal area enclosure for $g'$. For $t \in [0, 1)$, $f_t \geq 0$.

Let $u_t$ be the harmonic function associated to $f_t$:

$$u_t(x) = \varphi(x) + \int_\Sigma K(x, y) f_t(y) d\mathcal{H}^2_g(y),$$

and set $g_t = u_t^4 g$. By our choice of $k$, we have that

$$\frac{d}{dt}\bigg|_{t=0} |\Sigma|_{g_t} = \int_\Sigma 4f^3(f(\chi - k)) d\mathcal{H}^2_g = 0.$$

Next, define

$$\begin{aligned} v(x) := \dot{u}_0(x) &= \int_\Sigma K(x, y) f(y)(\chi(y) - k) d\mathcal{H}^2_g(y) \\ &= w(x) - ku(x) + k\varphi(x), \end{aligned}$$

where

$$w(x) := \int_\Sigma K(x, y) f(y) \chi(y) d\mathcal{H}^2_g(y).$$

Let $S$ be any minimal area enclosure of $\Sigma$ with respect to $g'$. Since $\tilde{\Sigma}_{g'}$ encloses $S$, we have that

$$\tilde{\Sigma}_{g'} \cap \Sigma \subset S \cap \Sigma. \tag{4.15}$$

Now,

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S|_{g_t} &= \frac{1}{4}\frac{d}{dt}\Big|_{t=0}\int_{S\cap\Sigma} f_t^4 d\mathcal{H}_g^2 + \frac{1}{4}\frac{d}{dt}\Big|_{t=0}\int_{S\setminus\Sigma} u_t^4 d\mathcal{H}_g^2 \\
&= \int_{S\cap\Sigma} f^4(\chi - k) d\mathcal{H}_g^2 + \int_{S\setminus\Sigma} u^3 v d\mathcal{H}_g^2 \\
&= \int_{S\cap\Sigma} f^4\chi d\mathcal{H}_g^2 - k\int_{S\cap\Sigma} f^4 d\mathcal{H}_g^2 + \int_{S\setminus\Sigma} u^3(w - ku + k\varphi) d\mathcal{H}_g^2. \quad (4.16)
\end{aligned}$$

We throw out the positive term $\int_{S\setminus\Sigma} u^3(w + k\varphi)d\mathcal{H}_g^2$ and observe that the integrals of $f^4\chi$ over $S\cap\Sigma$ and $\Sigma$ agree, since $f^4\chi$ is supported in $S\cap\Sigma$ by (4.15). Then

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S|_{g_t} &> \int_{\Sigma} f^4\chi d\mathcal{H}_g^2 - k\int_{S\cap\Sigma} f^4 d\mathcal{H}_g^2 - k\int_{S\setminus\Sigma} u^4 d\mathcal{H}_g^2 \\
&= kA - k|S|_{g'} \\
&= k(A - \alpha(A)) > 0, \quad (4.17)
\end{aligned}$$

where we have used the definition of $k$ and the fact that $|S|_{g'} = \alpha(A)$. We will once again make use of Lemma 34 to construct a "better" maximizer for $\alpha(A)$, achieving a contradiction.

Let $A(t) = \int_\Sigma f_t^4 d\mathcal{H}_g^2$, and define a new variation

$$\overline{f}_t = \left(\frac{A}{A(t)}\right)^{1/4} f_t,$$

with associated harmonic functions $\overline{u}_t$:

$$\overline{u}_t(x) = \varphi(x) + \frac{A^{1/4}}{A(t)^{1/4}}\left(u(x) - \varphi(x) + tw(x) - tku(x) + tk\varphi(x)\right) \quad (4.18)$$

Note that $A'(0) = 0$ and $A(t)$ is a fourth-order polynomial in $t$ of the form

$$A(t) = A(1 + c_2t^2 + c_3t^3 + c_4t^4),$$

for some constants $c_i$ with $c_2 > 0$. In particular, $A(t)$ is increasing for $t$ sufficiently small. By construction, $\overline{g}_t := \overline{u}_t^4 g$ defines a path in $\overline{\mathcal{H}_A(g)}$ with $\overline{g}_0 = g'$. Since $A'(0) = 0$, the paths $\overline{g}_t$ and $g_t$ agree to first order at $t = 0$.

The function $(t, S) \mapsto |S|_{\overline{g}_t}$ for $t \in [0, 1)$ and $S \in \mathcal{S}_K$ has a Taylor expansion in $t$ given by

$$\begin{aligned} |S|_{\overline{g}_t} &= |S|_{\overline{g}_0} + t\frac{d}{dt}\Big|_{t=0}|S|_{\overline{g}_t} + \frac{t^2}{2}\frac{d}{dt}\Big|_{t=0}|S|_{\overline{g}_t} + \dots \\ &= |S|_{g'} + 4t\int_S u^3(w - ku + k\varphi)d\mathcal{H}^2_g + R_t(S), \end{aligned} \tag{4.19}$$

for some $R_t(S)$ that is $O(t^2)$ in $t$ for each $S$.

Let us verify condition 1 of Lemma 34. Certainly the zeroth-order term $|S|_{g'}$ is positive, and the first-order term satisfies

$$4\int_S u^3(w - ku + k\varphi)d\mathcal{H}^2_g \geq -4k|S|_{g'}.$$

So condition 1 holds with $\beta = 4k$.

As for condition 2, the remainder term $R_t(S)$ can be bounded below in exactly the same way as in step 2 of Proposition 33.

Last, we verify condition 3: Since $A'(0) = 0$, first derivative calculations for $g_t$ and $\overline{g}_t$ agree at $t = 0$. In particular, for any minimal area enclosure $S$ of $\Sigma$,

$$\frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S|_{\overline{g}_t} > k(A - \alpha(A)) > 0, \tag{4.20}$$

by (4.17).

Using the notation of Lemma 34, for $\delta > 0$, let $X_\delta = \{S \in \mathcal{S}_K : |S|_{g'} \leq \alpha(A) + \delta\}$, a nonempty set. Suppose criterion 3 fails: then for all positive integers $i$, there exists a surface $S_i$ in $X_{1/i}$ such that

$$\limsup_{i\to\infty} \frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S_i|_{\overline{g}_t} \leq 0 \tag{4.21}$$

The surfaces $\{S_i\}$ are contained in the compact set $K$ and have uniformly bounded $g'$-areas. Since $u$ is bounded below by a positive constant (by Proposition 33 and the maximum principle), the $g$-areas of $\{S_i\}$ are uniformly bounded above as well. By the compactness theorem (Theorem 80 of Appendix B), $\{S_i\}$ has a convergent subsequence (of the same name, say) that converges to some surface $S$ enclosing $\Sigma$. (Note that (4.21) is preserved under taking subsequences.) By Proposition 19 and the fact that $|S_i|_{g'} \leq \alpha(A) + \frac{1}{i}$, we see that $|S|_{g'} = \alpha(A) = \min(\Sigma, g')$, so $S$ is a minimal area enclosure of $\Sigma$.

Computing the first variation of area of $S_i$ along the path $\overline{g}_t$ as in (4.16),

$$\frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S_i|_{\overline{g}_t} = \underbrace{\int_{S_i\cap\Sigma} f^4\chi + \int_{S_i\setminus\Sigma} u^3(w+k\varphi)} - \underbrace{k|S_i|_{g'}}.$$

The second underbraced term converges to $k\alpha(A)$ as $i \to \infty$ by construction, and by Proposition 21 we have

$$\liminf_{i\to\infty}\left(\int_{S_i\cap\Sigma} f^4\chi + \int_{S_i\setminus\Sigma} u^3(w+k\varphi)\right) \geq \int_{S\cap\Sigma} f^4\chi + \int_{S\setminus\Sigma} u^3(w+k\varphi).$$

From this, it follows that

$$\liminf_{i\to\infty}\frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S_i|_{\overline{g}_t} \geq \frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S|_{\overline{g}_t} > k(A-\alpha(A)) > 0,$$

contradicting (4.21). So condition 3 holds.

We apply Lemma 34: there exists some $t > 0$ with $\min(\Sigma, \overline{g}_t) > \min(\Sigma, g')$, contradicting the assumption that $g'$ is a maximizer for $\alpha(A)$. □

We reiterate the point that $\tilde{\Sigma}_{g'} \setminus \Sigma$ is a smooth, embedded submanifold of $M \setminus \Sigma$.

## 4.3 Regularity of the outermost minimal area enclosure

Suppose $g' = u^4 g \in \overline{\mathcal{H}_A(g)}$ is a maximizer for $\alpha(A) < A$ as in Theorem 32, and let $f$ be the function on $\Sigma$ determined by $u$. By Proposition 33, $f \geq \epsilon > 0$. In this section we will prove Theorem 29. That is, we assume there exists a constant $C > 0$ such that $f \le C$, and that there exist only finitely many minimal area enclosures of $\Sigma$ with respect to $g'$ and will prove that $\tilde{\Sigma}_{g'}$ is disjoint from $\Sigma$.

Our approach to is refine the proof of Proposition 36. Since we have shown that $\tilde{\Sigma}_{g'} \cap \Sigma$ has zero measure, we cannot simply construct a variation in the direction of the characteristic function for $\tilde{\Sigma}_{g'} \cap \Sigma$. We emphasize that we know nothing of the structure of $\tilde{\Sigma}_{g'} \cap \Sigma$ beyond it having zero measure. A priori, this set could be empty, consist of isolated points, have positive $\mathcal{H}_g^{3/2}$-measure, etc.

Let $p \in \tilde{\Sigma}_{g'} \cap \Sigma$ (that is, $p$ belongs to $\Sigma$ and the support of $\tilde{\Sigma}_{g'}$). Recall from section 3.6 our use of coordinates $(x, y, z)$ on a neighborhood $\overline{B}(p, \rho)$ centered about $p$ and our identification of $\overline{B}(p, \rho)$ with the closed unit half-ball $\overline{B}^+ \subset \mathbb{R}^3$. For a parameter $0 < \sigma \le \rho$, define the function on $\Sigma$

$$h_\sigma = \frac{1}{\pi \sigma^2} \chi_{B(p,\sigma)}, \tag{4.22}$$

where in this context, $B(p, \sigma)$ is the metric ball in $M$ intersected with $\Sigma$. Note that

$$\lim_{\sigma \to 0^+} \int_\Sigma h_\sigma d\mathcal{H}_g^2 = 1,$$

so that for all $\sigma > 0$ sufficiently small,

$$\int_\Sigma h_\sigma d\mathcal{H}_g^2 \le \sqrt{\frac{A}{\alpha(A)}}. \tag{4.23}$$

Note that as $\sigma \to 0^+$, $\{h_\sigma d\mathcal{H}_g^2\}$ converges weakly as a sequence of measures to $\delta_p$, the unit point mass at $p$. Our approach is roughly to vary the boundary data in the direction of $h_\sigma$, described as follows. For $t \ge 0$, let

$$f_t = f + t\left(\frac{h_\sigma}{f^3} - k_\sigma f\right),$$

where

$$k_\sigma = \frac{\int_\Sigma h_\sigma d\mathcal{H}^2_g}{A}.$$

Note that $1/f^3$ is a bounded, measurable function since $f \geq \epsilon$. Also, by (4.23),

$$k_\sigma \leq \frac{1}{A}\sqrt{\frac{A}{\alpha(A)}} \tag{4.24}$$

for all $\sigma > 0$ sufficiently small. The reason for dividing by $f^3$ is clarified in Lemma 38. Let $u_t$ be the harmonic function associated to $f_t$:

$$u_t = u + t(w_\sigma - k_\sigma(u - \varphi)),$$

where $w_\sigma$ is $g$-harmonic, zero at infinity, and given by $h_\sigma/f^3$ on $\Sigma$. Let $g_t = u_t^4 g$, a path in $\overline{\mathcal{H}(g)}$ satisfying $g_0 = g'$; by our choice of $k_\sigma$, the area of $\Sigma$ is unchanged to first order at $t = 0$. Although $u_t$ and $g_t$ depend on the unspecified parameter $\sigma$, we do not indicate this in the notation.

For any minimal area enclosure $S$ of $\Sigma$, the first variation of its area along the path $g_t$ is, for $\sigma > 0$ sufficiently small,

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\Big|_{t=0}|S|_{g_t} &= \int_{S\cap\Sigma} f^3\left(\frac{h_\sigma}{f^3} - k_\sigma f\right) d\mathcal{H}^2_g + \int_{S\setminus\Sigma} u^3\left(w_\sigma - k_\sigma u + k_\sigma\varphi\right) d\mathcal{H}^2_g \\
&= \int_{S\cap\Sigma} h_\sigma d\mathcal{H}^2_g + \int_{S\setminus\Sigma} u^3 w_\sigma d\mathcal{H}^2_g \\
&\quad - k_\sigma\left(\int_{S\cap\Sigma} f^4 d\mathcal{H}^2_g + \int_{S\setminus\Sigma} u^4 d\mathcal{H}^2_g\right) + k_\sigma \int_{S\setminus\Sigma} u^3\varphi d\mathcal{H}^2_g \\
&> \int_{S\cap\Sigma} h_\sigma d\mathcal{H}^2_g + \int_{S\setminus\Sigma} u^3 w_\sigma d\mathcal{H}^2_g - \sqrt{\frac{\alpha(A)}{A}},
\end{aligned} \tag{4.25}$$

where we have used equation (4.24), $|S|_{g'} = \alpha(A)$, and the fact that $k_\sigma u^3 \varphi > 0$.

Our goal is to show (4.25) can be made positive by choosing $\sigma > 0$ sufficiently small. More to the point, we must choose a single value of $\sigma > 0$ that makes (4.25) positive simultaneously for all minimal area enclosures $S$.

Since $p$ belongs to the intersection of $\Sigma$ with the outermost minimal area enclosure, $p$ belongs to $S$ as well. Recall the definitions of lower density, lower partial density, etc., from section 3.6, as well as Theorem 25 on the local structure of $S$ at $p$: either $S$ has positive lower partial density $\Theta_\alpha(S,p)$ for some $\alpha > 0$, or else $S$ has a tangent plane at $p$. We study how (4.25) behaves in these two cases as $\sigma \to 0^+$.

**Lemma 37.** *Suppose that for some $\alpha > 0$, the lower partial density $\Theta_\alpha(S,p)$ is positive. Then*

$$\liminf_{\sigma \to 0^+} \int_{S \setminus \Sigma} u^3 w_\sigma d\mathcal{H}^2_g = +\infty.$$

*In particular, (4.25) is arbitrarily large for $\sigma > 0$ sufficiently small.*

*Proof.* Let $v_\sigma$ be the $g$-harmonic function $M$ given by $h_\sigma$ on $\Sigma$ and 0 at infinity. Observe that $u^3 w_\sigma \geq C^{-3}\epsilon^3 v_\sigma$, so it suffices to show

$$\liminf_{\sigma \to 0^+} \int_{S \setminus \Sigma} v_\sigma d\mathcal{H}^2_g = +\infty.$$

Since $\{h_\sigma d\mathcal{H}^2_g\}$ converges weakly as measures to $\delta_p$ as $\sigma \to 0^+$, we see that $v_\sigma$ converges pointwise on the interior of $M$ to $K(x,p)$, the Poisson kernel based at $p$. (To see this, observe that $K(x,p)$ is the unique $g$-harmonic function given as the unit point mass at $p$ on $\Sigma$ and 0 at infinity). By Fatou's lemma,

$$\liminf_{\sigma \to 0^+} \int_{S \setminus \Sigma} v_\sigma d\mathcal{H}^2_g \geq \int_{S \setminus \Sigma} K(\cdot, p) d\mathcal{H}^2_g(\cdot).$$

For now, assume that the Poisson kernel $K(\cdot,p)$ is given locally in our choice of coordinates by $\frac{z}{(x^2+y^2+z^2)^{3/2}}$, the Euclidean Poisson kernel for half-space, based at $(0,0,0)$. We show that $K(\cdot,p)$ integrates to $+\infty$ on $S \setminus \Sigma$. (The motivation for why this is plausible is that $K(\cdot,p)$ integrates to $+\infty$ on the cone

$$\{(x,y,z) : \alpha z = \sqrt{x^2+y^2}, \sqrt{x^2+y^2+z^2} \leq r\},$$

for any $\alpha, r > 0$.) For all $0 < r \leq \rho$,

$$\begin{aligned}
\int_{(S\setminus\Sigma)\cap B(p,r)} K(\cdot,p)d\mathcal{H}^2_g &\geq \int_{S\cap B(p,r)\cap C_\alpha} \frac{z}{(x^2+y^2+z^2)^{3/2}}d\mathcal{H}^2_g \\
&\geq \int_{S\cap B(p,r)\cap C_\alpha} \frac{z}{(\alpha^2 z^2+z^2)^{3/2}}d\mathcal{H}^2_g \\
&= \int_{S\cap B(p,r)\cap C_\alpha} \frac{1}{z^2(\alpha^2+1)^{3/2}}d\mathcal{H}^2_g \\
&\geq \int_{S\cap B(p,r)\cap C_\alpha} \frac{1}{r^2(\alpha^2+1)^{3/2}}d\mathcal{H}^2_g \\
&= \frac{\pi}{(\alpha^2+1)^{3/2}} \cdot \frac{\mathcal{H}^2_g(S\cap B(p,r)\cap C_\alpha)}{\pi r^2},
\end{aligned}$$

where we have used the fact that in the set $C_\alpha \cap B(p,r)$, $\alpha z \geq \sqrt{x^2+y^2}$ and $z \leq r$. By assuming $r$ is sufficiently small and using the definition of lower partial density $\Theta_\alpha(S,p)$,

$$\frac{\mathcal{H}^2(S\cap B(p,r)\cap C_\alpha)}{\pi r^2} \geq \frac{1}{2}\Theta_\alpha(S,p),$$

which is positive by hypothesis. This shows that $\int_{(S\setminus\Sigma)\cap B(p,r)} K(\cdot,p)d\mathcal{H}^2_g$ is bounded below by a positive constant independent of $r$, for all $r$ sufficiently small. It follows that $\int_{S\setminus\Sigma} K(\cdot,p)d\mathcal{H}^2_g(\cdot)$ is $+\infty$.

To complete the proof, we justify our use of the Euclidean Poisson kernel in place of the Poisson kernel for $g$. By our choice of coordinates, $g_{ij}(p) = \delta_{ij}(p)$, so on a ball of radius $r$ about $p$, the two metrics $g$ and $\delta$ become arbitrarily uniformly close as $r \to 0$. Under the identification of a ball of radius $r$ about $p$ in $M$ with the unit half-ball $\overline{B}^+$ in $\mathbb{R}^3$, the Poisson kernel $K(\cdot,p)$ for $g$ converges pointwise to the Euclidean Poisson kernel as $r \to 0^+$. By Fatou's lemma, replacing an integral of $K(\cdot,p)$ on $B(p,r)$ with the Euclidean Poisson kernel (as we have done) only decreases its value when taking $\liminf_{r\to 0^+}$. □

In the variational argument (4.25), we have shown how to handle the minimal area enclosures that have positive lower partial density at $p$.

We now consider the case in which $\Theta_\alpha(S,p) = 0$ for all $\alpha > 0$. By Theorem 25, $S$ has a tangent plane at $p$. In this case the above argument fails for the reason that the Poisson kernel does not blow up sufficiently fast along *tangential* directions to conclude that its integral over $S \setminus \Sigma$ is infinite.

**Lemma 38.** *Suppose $S$ has zero lower partial density for all $\alpha > 0$. Then*

$$L := \liminf_{\sigma \to 0^+} \left( \int_{S \cap \Sigma} h_\sigma d\mathcal{H}^2_g + \int_{S \setminus \Sigma} u^3 w_\sigma d\mathcal{H}^2_g \right) \geq 1.$$

*In particular, (4.25) can be made positive for an appropriate choice of $\sigma$.*

*Proof.* Choose a sequence $\{\sigma_i\} \searrow 0$ such that:

$$\lim_{i \to \infty} \left( \int_{S \cap \Sigma} h_{\sigma_i} d\mathcal{H}^2_g + \int_{S \setminus \Sigma} u^3 w_{\sigma_i} d\mathcal{H}^2_g \right) = L \tag{4.26}$$

By Theorem 25 and its proof, we may pass to a subsequence of $\{\sigma_i\}$ (of the same name), preserving (4.26), and find a sequence $\{\alpha_i\} \nearrow \infty$ such that

$$\lim_{i \to \infty} \frac{\mathcal{H}^2_g(S \cap B(p, \sigma_i) \cap C_{\alpha_i})}{\pi \sigma_i^2} = 0,$$

and $\{S_{\sigma_i}\}$ converges to the disk $D$ in $\overline{B}^+$ as $i \to \infty$. (Here, $S_{\sigma_i} \subset \overline{B}^+$ is defined as in (3.14) – a zoomed-in version of $S$ on the scale $\sigma_i$.) In this proof we are viewing $S$ and $S_{\sigma_i}$ as rectifiable sets rather than currents. The above can equivalently be written as

$$\lim_{i \to \infty} \mathcal{H}^2_\delta(S_{\sigma_i} \cap C_{\alpha_i}) = 0, \tag{4.27}$$

since $\sigma_i^{-2} g$ becomes arbitrarily uniformly close to $\delta$ (c.f. equation (3.15)). As usual, we implicitly identify the cone $C_{\alpha_i}$ with its image in $\overline{B}^+$.

The first term on the left-hand side of (4.26) is:

$$\int_{S\cap\Sigma} h_{\sigma_i} d\mathcal{H}^2_g = \int_{S\cap\Sigma\cap B(p,\sigma_i)} \frac{1}{\pi\sigma_i^2} d\mathcal{H}^2_g. \tag{4.28}$$

Next, we analyze the second term on the left-hand side of (4.26). Recall that $w_{\sigma_i}$ is the $g$-harmonic function on $M$ that vanishes at infinity and equals $\dfrac{\chi_{B(p,\sigma_i)}}{\pi\sigma_i^2 f^3}$ on $\Sigma$. Let $\overline{w}_{\sigma_i}$ be $g$-harmonic, vanishing at infinity and given by $\dfrac{1-\chi_{B(p,\sigma_i)}}{\pi\sigma_i^2 f^3}$ on $\Sigma$. Then $w_{\sigma_i}$ and $\overline{w}_{\sigma_i}$ are complementary in the sense that they add up to the $g$-harmonic function that is 0 at infinity and $\dfrac{1}{\pi\sigma_i^2 f^3}$ on $\Sigma$. By Lemma 39 below,

$$u^3\left(\pi\sigma_i^2(w_{\sigma_i}+\overline{w}_{\sigma_i})+\varphi\right)\geq 1.$$

In particular,

$$u^3\left(w_{\sigma_i}+\overline{w}_{\sigma_i}\right)\geq \frac{1}{\pi\sigma_i^2}\left(1-u^3\varphi\right). \tag{4.29}$$

Now,

$$\begin{aligned}
\int_{S\setminus\Sigma} u^3 w_{\sigma_i} d\mathcal{H}^2_g &\geq \int_{(S\setminus\Sigma)\cap B(p,\sigma_i)} u^3 w_{\sigma_i} d\mathcal{H}^2_g \\
&= \int_{(S\setminus\Sigma)\cap B(p,\sigma_i)} u^3(w_{\sigma_i}+\overline{w}_{\sigma_i}) d\mathcal{H}^2_g - \int_{(S\setminus\Sigma)\cap B(p,\sigma_i)} u^3\overline{w}_{\sigma_i} d\mathcal{H}^2_g \\
&\geq \int_{(S\setminus\Sigma)\cap B(p,\sigma_i)} \frac{1}{\pi\sigma_i^2}\left(1-C^3\varphi\right) d\mathcal{H}^2_g - C^3 \underbrace{\int_{(S\setminus\Sigma)\cap B(p,\sigma_i)} \overline{w}_{\sigma_i} d\mathcal{H}^2_g}_{E_i},
\end{aligned} \tag{4.30}$$

where we have used (4.29) and the bound $u \leq C$. $E_i$ is some error term we will dispense with later. Since $\varphi$ is continuous and zero on $\Sigma$ and $S$ has finite upper density at $p$ (by Lemma 27), we may ignore the $C^3\varphi$ term (since we will take the limit $i\to\infty$). Adding together (4.28) and (4.30), we get

$$\int_{S\cap\Sigma} h_{\sigma_i} d\mathcal{H}^2_g + \int_{S\setminus\Sigma} u^3 w_{\sigma_i} d\mathcal{H}^2_g \geq \int_{S\cap B(p,\sigma_i)} \frac{1}{\pi\sigma_i^2} d\mathcal{H}^2_g - C^3 E_i$$

Taking lim inf of both sides, we get:

$$
\begin{aligned}
L &\geq \liminf_{i\to\infty}\left(\int_{S\cap B(p,\sigma_i)} \frac{1}{\pi\sigma_i^2} d\mathcal{H}_g^2 - C^3 E_i\right) \\
&\geq \frac{1}{\pi}\liminf_{i\to\infty}\left(\int_{S\cap B(p,\sigma_i)} \frac{1}{\sigma_i^2} d\mathcal{H}_g^2\right) - C^3 \limsup_{i\to\infty} E_i \\
&\geq \frac{1}{\pi}\liminf_{i\to\infty} |S_{\sigma_i}|_\delta - C^3 \limsup_{i\to\infty} E_i \\
&\geq \frac{1}{\pi}|D|_\delta - C^3 \limsup_{i\to\infty} E_i, \qquad (4.31)
\end{aligned}
$$

where on the last line we have used the fact that $\{S_{\sigma_i}\}$ converges to the disk $D$ and the lower semi-continuity of $|\cdot|_\delta$. Above, we also have used the fact that as $i \to \infty$, we may identify $\frac{1}{\sigma_i^2}\mathcal{H}_g^2(S\cap B(p,\sigma_i))$ with $|S_{\sigma_i}|_\delta$. The $\delta$-area of the disk $D$ equals $\pi$, so we are done once we show the error term $E_i$ converges to zero.

Since $f \geq \epsilon > 0$, it suffices to replace $\overline{w}_{\sigma_i}$ with the $g$-harmonic function given by $\frac{1-\chi_{B(p,\sigma_i)}}{\sigma_i^2}$ on $\Sigma$. Also, since the metrics $\delta$ and $\frac{1}{\sigma_i^2}g$ become arbitrarily uniformly close on $\overline{B}^+$ as $i \to \infty$, it suffices to prove that

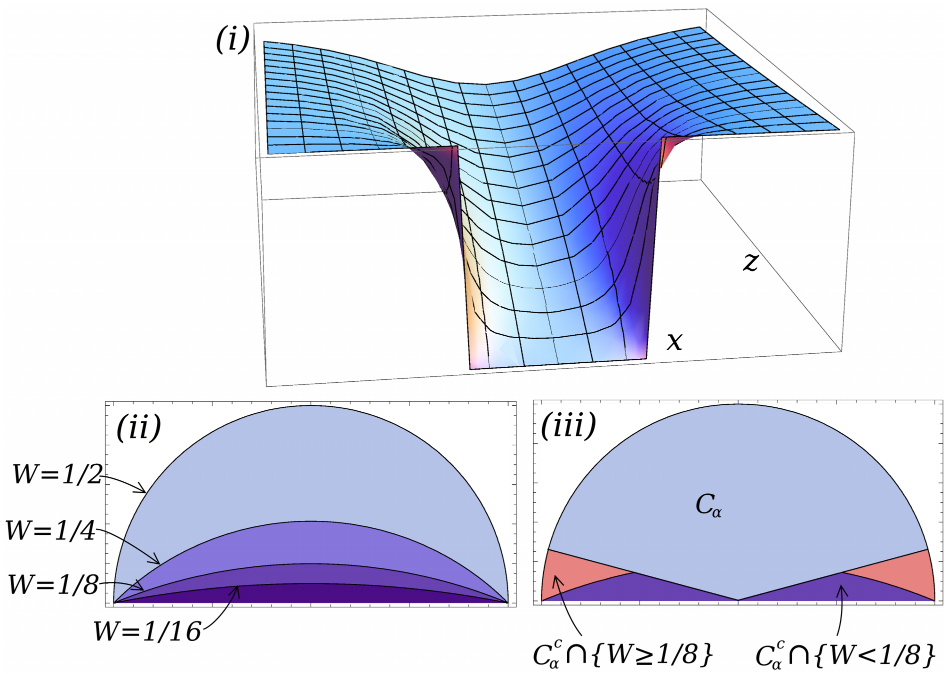
$$
\lim_{i\to\infty}\int_{S_{\sigma_i}} W dA_\delta = 0, \qquad (4.32)
$$

where $W$ is the $\delta$-harmonic function on closed upper-half-space $\mathbb{R}^3_+$ that is zero on the unit disk $D$, one on $\mathbb{R}^2 \setminus D$ and zero at infinity. See panels (i)–(ii) of figure 4.1. We decompose this integral as follows:

$$
\int_{S_{\sigma_i}} W dA_\delta = \int_{S_{\sigma_i}\cap C_{\alpha_i}} W dA_\delta + \int_{S_{\sigma_i}\cap C^c_{\alpha_i}\{W<i^{-1}\}} W dA_\delta + \int_{S_{\sigma_i}\cap C^c_{\alpha_i}\{W\geq i^{-1}\}} W dA_\delta, \qquad (4.33)
$$

where $C^c_{\alpha_i}$ is the complement of $C_{\alpha_i}$ in $\overline{B}^+$ and $\{W < i^{-1}\}$, $\{W \geq i^{-1}\}$ are sub- and super-level sets, respectively, of $W$ at height $i^{-1}$. See panel (iii) of figure 4.1. (The

FIGURE 4.1: Dealing with the error term in Lemma 38




Panel (i) is a plot of the $y = 0$ slice of the harmonic function $W(x, y, z)$ on upper-half space $\mathbb{R}^3_+$ that is zero at infinity and equal to the characteristic function of $\mathbb{R}^2$ minus the unit disk on the $z = 0$ plane. Panel (ii) shows several level sets of $W$ in the $x$–$z$ plane. Used in equation (4.33), panel (iii) demonstrates the decomposition of the half-ball into three regions: the solid cone $C_\alpha$, a super-level set of $W$ intersected with the complement of $C_\alpha$, and a sub-level set of $W$ intersected with the complement of $C_\alpha$. Observe that for fixed $i$, the boundary of $C_\alpha^c \cap \{W \geq 1/i\}$ can be made to have arbitrarily small $\mathcal{H}^2_\delta$ measure by taking $\alpha$ sufficiently large.

reason for considering the sub- and super-level sets is to handle the case in which a positive fraction of the area of $S_{\sigma_i}$ concentrates in regions for which $W$ does not approach zero.)

The first integral on the right-hand side of (4.33) converges to zero as $i \to \infty$, since $W \leq 1$ and $\mathcal{H}^2_\delta(S_{\sigma_i} \cap C_{\alpha_i})$ converges to zero by (4.27). The second integral can be bounded above as follows:

$$\int_{S_{\sigma_i} \cap C^c_{\alpha_i} \{W < i^{-1}\}} W dA_\delta \leq \frac{1}{i} \mathcal{H}^2_\delta(S_{\sigma_i}).$$

The right-hand side converges to zero since $\limsup_{i \to \infty} \mathcal{H}^2_\delta(S_{\sigma_i})$ is finite by Lemma 27.

Finally, we deal with the third integral in the right-hand side of (4.33), which can be bounded above by the $\mathcal{H}^2_\delta$-measure of the portion of $S_{\sigma_i}$ that lies beneath the cone $C_{\alpha_i}$ and within the $i^{-1}$ super-level set of $W$. Observe that given $i \geq 1$, we may find $\alpha > 0$ such that

$$\mathcal{H}^2_\delta\big(\partial(C^c_\alpha \cap \{W \geq i^{-1}\})\big) < \frac{1}{i},$$

as explained in the third panel of figure 4.1. Thus, we may pass to a common subsequence of $\{\sigma_i\}$ and $\{\alpha_i\}$ such that

$$\mathcal{H}^2_\delta(\partial(C^c_{\alpha_i} \cap \{W \geq i^{-1}\})) \to 0.$$

(Passing to such a subsequence affects none of the earlier arguments in this proof.) Now, since $S$ is $g'$ area-minimizing and $g'$ is uniformly equivalent to $g$, the isoperimetric inequality shows that

$$\mathcal{H}^2_\delta(S_{\sigma_i} \cap C^c_{\alpha_i} \cap \{W \geq i^{-1}\}) \to 0.$$

We have finally shown (4.32), completing the proof.

□

**Lemma 39.** *Let $f \in L^4(\Sigma)$, and suppose $f$ is bounded below by a positive constant. Let $u$ and $w$ be the harmonic functions associated to $f$ and $f^{-3}$, respectively. Then $u^3 w \geq 1$ in the interior of $M$.*

*Proof.* Note that $f^{-1}$ belongs to $L^\infty(\Sigma) \subset L^4(\Sigma)$. Let $v$ be the harmonic function associated to $f^{-1}$. By the maximum principle, $v$ is positive, and it is easy to check that $1/v$ is subharmonic. Since $u$ and $\frac{1}{v}$ have the same boundary data, the maximum principle proves $\frac{1}{v} \leq u$ on the interior of $M$. (One subtlety is that $f$ need not be continuous, but the argument can be made to work by considering a sequence of continuous functions converging to $f$ in $L^4$.) Similarly, one can check that $v^3$ is subharmonic and has the same boundary data as the harmonic function $w$. Then $v^3 \leq w$ in the interior of $M$. Combining this with the above establishes the claim. □

We now combine Lemmas 37 and 38 to prove a unified statement for all minimal area enclosures.

**Proposition 40.** *Suppose $\alpha(A) < A$, and let $g' = u^4 g$ be a maximizer for $\alpha(A)$. If $S$ is a minimal area enclosure of $\Sigma$ with respect to $g'$, then*

$$\liminf_{\sigma \to 0^+} \left( \int_{S \cap \Sigma} h_\sigma d\mathcal{H}^2_g + \int_{S \setminus \Sigma} u^3 w_\sigma d\mathcal{H}^2_g \right) \geq 1,$$

*as in (4.25). Now suppose $\Sigma$ has only finitely many minimal area enclosures with respect to $g'$. Then there exists $\sigma > 0$ such that the right-hand side of (4.25) satisfies*

$$\int_{S \cap \Sigma} h_\sigma d\mathcal{H}^2_g + \int_{S \setminus \Sigma} u^3 w_\sigma d\mathcal{H}^2_g - \sqrt{\frac{\alpha(A)}{A}} \geq \frac{1}{2} \left( 1 - \sqrt{\frac{\alpha(A)}{A}} \right) > 0$$

*for all minimal area enclosures $S$.*

*Proof.* If $S$ has positive lower partial density $\Theta_\alpha(S, p)$ for some $\alpha > 0$, then Lemma 37 proves the first statement. If not, then $S$ has zero lower partial density for all $\alpha > 0$, and Lemma 38 proves the first statement. By the definition of lim inf, there exists $\sigma_S > 0$ (depending on $S$) such that

$$\int_{S \cap \Sigma} h_\sigma d\mathcal{H}^2_g + \int_{S \setminus \Sigma} u^3 w_\sigma d\mathcal{H}^2_g - \sqrt{\frac{\alpha(A)}{A}} \geq \frac{1}{2} \left( 1 - \sqrt{\frac{\alpha(A)}{A}} \right)$$

for all $\sigma \leq \sigma_S$. Since there are only finitely many such $S$, there exists a single value of $\sigma > 0$ that satisfies the claim.

□

### *4.3.1 Proof of the main theorem*

At last we prove Theorem 29:

*Proof.* We recall the hypotheses that maximizers for $\alpha(A)$ have bounded boundary data and only finitely many minimal area enclosures.

Let $\delta > 0$. Proposition 64, which is logically independent of this chapter, shows that $\alpha(A)$ is continuous in $A$, satisfies $\alpha(A) \leq A$ and $\alpha(A) \not\equiv A$. Consequently we may fix $A > 0$ such that $0 < A - \alpha(A) < \delta$. For this choice of $A$, let $g' = u^4 g \in \overline{\mathcal{H}_A(g)}$ be the maximizer given by Theorem 32, and let $f$ be the function on $\Sigma$ determined by $u$. By Proposition 24 and the fact that $\alpha(A) < A$, there exists an outermost minimal area enclosure $\tilde{\Sigma}_{g'}$ of $\Sigma$ with respect to $g'$, with $|\tilde{\Sigma}_{g'}|_{g'} = \alpha(A)$. In particular, $|\Sigma|_{g'} = A$ and $|\tilde{\Sigma}_{g'}|_{g'} = \alpha(A)$ differ by less than $\delta$. To complete the proof, we must show $\tilde{\Sigma}_{g'}$ is disjoint from $\Sigma$.

If not, there exists $p \in \tilde{\Sigma}_{g'} \cap \Sigma$. Since we assume $\Sigma$ has only finitely many minimal area enclosures with respect to $g'$ and that $f$ is bounded, we may choose $\sigma > 0$ in accordance with Proposition 40. In particular, for the path $g_t$ in $\overline{\mathcal{H}(g)}$ defined at the beginning of section 4.3, the area of all minimal area enclosures increases to first order at a uniformly positive rate (see (4.25)), and the area $\Sigma$ is unchanged to first order. Following the same logic as Proposition 36, we may normalize to define a path $\overline{g}_t$ in $\overline{\mathcal{H}_A(g)}$ for which the minimal enclosing area strictly increases for $t > 0$ small. This contradicts the assumption that $\overline{g}_0 = g'$ is a maximizer for $\alpha(A)$, completing the proof. □

We close with the conjecture that $\tilde{\Sigma}_{g'}$ is disjoint from $\Sigma$ without the additional assumptions of Theorem 29.

**Conjecture 41.** *Suppose $\alpha(A) < A$. For a maximizer $g' \in \overline{\mathcal{H}_A(g)}$ of $\alpha(A)$, the outermost minimal area enclosure $\tilde{\Sigma}_{g'}$ is disjoint from $\Sigma$.*

By the previous proof, this conjecture would imply Conjecture 28.

# 5

# Estimating the ADM Mass with Harmonic Data

The purpose of this chapter is to give some applications of Conjecture 28. We show how to estimate from below the ADM mass of any asymptotically flat manifold $(M, g)$ of nonnegative scalar curvature, having a smooth, nonempty, compact boundary $\Sigma$. Next, as a particular case, we prove an inequality regarding harmonic functions on the complement of a bounded domain in $\mathbb{R}^3$. In this chapter, we assume Conjecture 28 holds (or the hypotheses of Theorem 29 are always satisfied).

## 5.1 An ADM mass estimate

Bray pointed out that Conjecture 7 can be used in conjunction with the Riemannian Penrose inequality to estimate the ADM mass of $(M, g)$ (c.f. Proposition 5) [8]. Below, we carry this out in detail, using the slightly weaker Conjecture 28 instead.

Recall that the *capacity* of $\Sigma$ in $(M, g)$ is the positive number

$$C_g(\Sigma) := \frac{1}{4\pi} \int_\Sigma \nu(\varphi) dA_g,$$

where $\nu$ is the $g$-unit normal to $\Sigma$ pointing into $M$, and $\varphi$ is the $g$-harmonic function vanishing on $\Sigma$ and tending to one at infinity [12]. By harmonicity of $\varphi$ and the divergence theorem, the flux integral may alternatively be taken over $S_\infty$ or any coordinate sphere. The capacity may also be interpreted as minus the coefficient of the $\frac{1}{r}$ term in the expansion at infinity of $\varphi$. We emphasize that $C_g(\Sigma)$ depends on the global geometry of $(M, g)$.

We define the function

$$W(y) = W_g(y) = -\frac{1}{4\pi} \int_{S_\infty} \nu_x K(x,y) dA_g(x), \tag{5.1}$$

where $\nu_x$ denotes differentiation in the direction $\nu$ with respect to the variable $x$. One may interpret $W(y)$ as the coefficient of the $\frac{1}{r}$ term in the expansion at infinity of the harmonic function $K(\cdot, y)$, with $y$ fixed.

**Lemma 42.** *Given $(M, g)$ asymptotically flat with compact boundary $\Sigma$,*

$$W = \frac{1}{4\pi} \nu(\varphi). \tag{5.2}$$

*Moreover, if $f \in L^4(\Sigma)$ and $u$ is the associated harmonic function:*

$$u(x) = \varphi(x) + \int_\Sigma K(x,y) f(y) dA_g(y),$$

*then*

$$\frac{1}{4\pi} \int_{S_\infty} \nu(u) dA_g = C_g(\Sigma) - \int_\Sigma W f dA_g. \tag{5.3}$$

Observe that the left-hand side of (5.3) is minus the $\frac{1}{r}$ term in the expansion at infinity of $u$; the lemma shows how this term can be computed in terms of the boundary data $f$.

*Proof.* The first claim follows from the symmetry of the Green's function $G(x, y)$ for

$(M, g)$. Recall that $K(x, y) = \nu_y G(x, y)$ for $y \in \Sigma$, $x \in M$, and $x \neq y$. Then:

$$
\begin{aligned}
-4\pi W(y) &= \int_{\Sigma} \nu_x K(x, y) dA_g(x) \\
&= \int_{\Sigma} \nu_x \nu_y G(x, y) dA_g(x) \\
&= \int_{\Sigma} \nu_y \nu_x G(y, x) dA_g(x),
\end{aligned}
$$

where on the first line we used the divergence theorem and the harmonicity of $K(\cdot, y)$ to exchange the integral over $S_\infty$ for an integral over $\Sigma$. Now, the derivative $\nu_y$ may be pulled out of the integral, and we have

$$
\begin{aligned}
-4\pi W(y) &= \nu_y \int_{\Sigma} K(y, x) dA_g(x) \\
&= \nu_y (1 - \varphi(y)),
\end{aligned}
$$

which is simply $-\nu(\varphi)(y)$.

To prove the second claim, suppose $f \in L^p(\Sigma)$ for any $p \geq 1$, and let $u$ be the associated harmonic function. Then

$$
\begin{aligned}
-4\pi \int_{\Sigma} W f dA_g &= \int_{\Sigma} \left( \int_{S_\infty} \nu_x K(x, y) dA_g(x) \right) f(y) dA_g(y) \\
&= \int_{S_\infty} \nu_x \left( \int_{\Sigma} K(x, y) f(y) dA_g(y) \right) dA_g(x) \\
&= \int_{S_\infty} \nu(u - \varphi) dA_g \\
&= \int_{S_\infty} \nu(u) dA_g - 4\pi C_g(\Sigma),
\end{aligned}
$$

where we have used the definition of $u$ and of capacity. □

We now state the following main result.

**Theorem 43.** *Assume Conjecture 28 holds. If $(M, g)$ is an asymptotically flat 3-manifold with nonnegative scalar curvature and nonempty, smooth, compact boundary $\Sigma$, then*

$$m_{\mathrm{ADM}}(M,g) \geq \inf_{u^4 g \in \mathcal{H}(g)} \left\{ \sqrt{\frac{\int_\Sigma u^4 dA_g}{16\pi}} + \frac{1}{2\pi} \int_{S_\infty} \nu(u) dA_g \right\} \tag{5.4}$$

$$= 2\int_\Sigma W dA_g - 4\sqrt{\pi} \left( \int_\Sigma W^{4/3} dA_g \right)^{3/2}, \tag{5.5}$$

*where $W : \Sigma \to \mathbb{R}^+$ is given by (5.1) (or, more concretely, by (5.2)). The infimum in (5.4) is finite and is achieved by a unique metric $u_0^4 g$ belonging to the smooth harmonic conformal class $\mathcal{H}(g)$.*

Our interpretation of Theorem 43 is that the ADM mass can be bounded below in terms of "harmonic data" of $(M, g)$. This is similar in spirit to results of Bray and Miao [12] that relate ADM mass to the capacity of the boundary.

We remark that, by the above theorem, the metric $u_0^4 g$ is a canonical representative of the harmonic conformal class.

*Proof.*

*Step 1.* The first step is to write the braced term of (5.4) purely in terms of the boundary data; this is immediate by Lemma 42. Given $f \in L^4(\Sigma)$, not necessarily nonnegative, with associated harmonic function $u$, the braced term of (5.4) can be written equivalently as the following functional defined on $L^4(\Sigma)$:

$$E(f) = \sqrt{\frac{\int_\Sigma f^4 dA_g}{16\pi}} + 2C_g(\Sigma) - 2\int_\Sigma W f dA_g. \tag{5.6}$$

*Step 2.* Next, we show $E(\cdot)$ has a global minimum. We begin by proving the existence of a critical point using the Euler–Lagrange equation for $E(\cdot)$. For $h$

belonging to $L^4(\Sigma)$, and $t \in \mathbb{R}$, consider

$$\begin{aligned} \frac{d}{dt}E(f+th)\Big|_{t=0} &= \frac{1}{\sqrt{16\pi}}\frac{1}{2}\frac{\int_\Sigma 4f^3 h dA_g}{\left(\int_\Sigma f^4 dA_g\right)^{1/2}} - 2\int_\Sigma W h dA_g. \\ &= \int_\Sigma \left( \frac{f^3}{2\sqrt{\pi}\left(\int_\Sigma f^4 dA_g\right)^{1/2}} - 2W \right) h dA_g \end{aligned}$$

If $f$ is a critical point, then the above expression vanishes for all choices of $h$; in particular, the integrand vanishes identically:

$$\frac{f^3}{\left(\int_\Sigma f^4 dA_g\right)^{1/2}} = 4\sqrt{\pi}W. \tag{5.7}$$

Raising both sides of the above to the power of $\frac{4}{3}$ and integrating over $\Sigma$, we see

$$\left(\int_\Sigma f^4 dA_g\right)^{1/3} = (16\pi)^{2/3}\int_\Sigma W^{4/3} dA_g.$$

Substituting back into (5.7) and rearranging, we find

$$f = 4\sqrt{\pi}\left(\int_\Sigma W^{4/3} dA_g\right)^{1/2} W^{1/3}. \tag{5.8}$$

On the other hand, if we define $f_0$ by equation (5.8), then the above calculations show that $f_0$ is a critical point of $E(\cdot)$. Moreover, since $E(\cdot)$ is the sum of a strictly convex functional, a constant, and a linear functional, we see that $E(\cdot)$ is strictly convex. In particular, $f_0$ is the unique global minimum of $E(\cdot)$. Therefore, $E(\cdot)$ is bounded below, and the infimum is attained. Since $W$ is smooth and positive, so is $f_0$. If $u_0$ is the harmonic function determined by $f_0$, then $u_0^4 g$ lies in the smooth harmonic conformal class $\mathcal{H}(g)$, achieves the infimum (5.4), and is the unique such metric.

*Step 3.* Next, we prove inequality (5.4). Given $\delta > 0$, Conjecture 28 and the Riemannian Penrose inequality give the existence of $g' = u^4 g \in \overline{\mathcal{H}(g)}$ such that

$$m_{\mathrm{ADM}}(M, g') \geq \sqrt{\frac{|\tilde{\Sigma}_{g'}|_{g'}}{16\pi}}, \qquad \text{and} \tag{5.9}$$

$$0 < |\Sigma|_{g'} - |\tilde{\Sigma}_{g'}|_{g'} < \delta, \tag{5.10}$$

where $\tilde{\Sigma}_{g'}$ is the outermost minimal area enclosure of $\Sigma$ with respect to $g'$. Let $f$ be the function on $\Sigma$ determined by $u$. By (5.9), (5.10), and formula (A.9),

$$\begin{aligned} m_{\mathrm{ADM}}(M, g) &> \sqrt{\frac{\int_\Sigma f^4 dA_g - \delta}{16\pi}} + \frac{1}{2\pi} \int_{S_\infty} \nu(u) dA_g \\ &= \sqrt{\frac{\int_\Sigma f^4 dA_g}{16\pi}} + \frac{1}{2\pi} \int_{S_\infty} \nu(u) dA_g - E(\delta), \end{aligned}$$

where $E(\delta)$ is the error term

$$\sqrt{\frac{\int_\Sigma f^4 dA_g}{16\pi}} - \sqrt{\frac{\int_\Sigma f^4 dA_g - \delta}{16\pi}}.$$

Since $f_0$ is a minimizer of $E(\cdot)$ as in the previous step, we conclude

$$m_{\mathrm{ADM}}(M, g) > \sqrt{\frac{\int_\Sigma f_0^4 dA_g}{16\pi}} + \frac{1}{2\pi} \int_{S_\infty} \nu(u_0) dA_g - E(\delta).$$

Although $f$ depends on the choice of $\delta$, the error term $E(\delta)$ can be made arbitrarily small by choosing $\delta$ sufficiently small. This proves inequality (5.4).

*Step 4.* We prove (5.5) by explicitly computing the infimum. To do so, we merely evaluate $E(f_0)$ by summing the three terms. First,

$$\sqrt{\frac{\int_\Sigma f_0^4 dA_g}{16\pi}} = 4\sqrt{\pi} \left( \int_\Sigma W^{4/3} dA_g \right)^{3/2}.$$

Next, we know from (5.3) that $C_g(\Sigma)$ equals the integral of $W$ over $\Sigma$. Last,

$$2\int_\Sigma W f_0 dA_g = 8\sqrt{\pi}\left(\int_\Sigma W^{4/3} dA_g\right)^{3/2}.$$

It follows that

$$E(f_0) = 2\int_\Sigma W dA_g - 4\sqrt{\pi}\left(\int_\Sigma W^{4/3}\right)^{3/2}.$$

□

Based on our use of the Riemannian Penrose inequality, we conjecture that equality holds in (5.4) if and only if $(M, g)$ belongs to the harmonic conformal class of flat $\mathbb{R}^3$ minus a round ball (which is the same harmonic conformal class as a Schwarzschild manifold of positive mass).

The above discussion is closely related to two invariants of the harmonic conformal class defined in the following corollary.

**Corollary 44.** *Suppose $(M, g)$ is an asymptotically flat 3-manifold with nonempty, smooth, compact boundary $\Sigma$, and let $W_g$ be the function (5.1) determined by the metric $g$. Then each of the quantities*

$$\mathcal{M}_1(g) := m_{ADM}(M, g) - 2C_g(\Sigma), \text{ and}$$

$$\mathcal{M}_2(g) := 4\sqrt{\pi}\left(\int_\Sigma W_g^{4/3} dA_g\right)^{3/2}$$

*are invariants of the harmonic conformal class $\mathcal{H}(g)$ of $(M, g)$. Now, suppose $g$ (or, equivalently, any element of $\mathcal{H}(g)$) has nonnegative scalar curvature everywhere. Then*

$$\mathcal{M}_1(g) + \mathcal{M}_2(g) \geq 0.$$

*Proof.* Recall from (A.9) that if $\overline{g} = u^4 g \in \mathcal{H}(g)$, then the ADM masses of $\overline{g}$ and $g$ differ by the flux term $-\frac{1}{2\pi}\int_{S_\infty} \nu(u) dA_g$. To show $\mathcal{M}_1$ is an invariant, we must show that twice the capacities obey the same law. Observe that $\overline{\varphi} = \varphi/u$ is the $\overline{g}$-harmonic function that vanishes on $\Sigma$ and is one at infinity, by formula (A.6). Also note that unit normals and area measures for $g$ and $\overline{g}$ agree on $S_\infty$, since $u \to 1$ at infinity. Then:

$$\begin{aligned}
2C_{\overline{g}}(\Sigma) &= \frac{1}{2\pi}\int_{S_\infty} \nu(\overline{\varphi})dA_g \\
&= \frac{1}{2\pi}\int_{S_\infty} \nu(\varphi/u)dA_g \\
&= \frac{1}{2\pi}\int_{S_\infty} \big(\nu(\varphi) - \nu(u)\big)dA_g \\
&= 2C_g(\Sigma) - \frac{1}{2\pi}\int_{S_\infty} \nu(u)dA_g,
\end{aligned}$$

where we have used the facts that $u \to 1$ and $\varphi \to 1$ at infinity. This shows $\mathcal{M}_1(g) = \mathcal{M}_1(\overline{g})$.

Next, we show the analogous statement for $\mathcal{M}_2$. With the same notation as above, observe that on $\Sigma$,

$$\begin{aligned}
\overline{\nu}(\overline{\varphi}) &= u^{-2}\nu\left(\frac{\varphi}{u}\right) \\
&= u^{-2}\left(\frac{\nu(\varphi)}{u} - \frac{\varphi\nu(u)}{u^2}\right) \\
&= u^{-3}\nu(\varphi),
\end{aligned}$$

where $\overline{\nu}$ is the unit normal to $\Sigma$ with respect to $\overline{g}$, computed using formula (A.3). On the second line, we used the fact that $\varphi$ vanishes on $\Sigma$. By Lemma 42, this shows

$$W_{\overline{g}} = u^{-3}W_g.$$

Since the area forms are related by $dA_{\overline{g}} = u^4 dA_g$, we have shown that $W_{\overline{g}}^{4/3}dA_{\overline{g}}$ agrees with $W_g^{4/3}dA_g$ as differential 2-forms on $\Sigma$. In particular, the integral of this form is an invariant of the harmonic conformal class of $g$. Finally, the inequality $\mathcal{M}_1(g) + \mathcal{M}_2(g) \geq 0$ follows from Theorem 43. $\square$

## 5.2 Application to Euclidean space

A special case of Theorem 43 deserves attention: that in which $M$ is the unbounded component of $\mathbb{R}^3 \setminus \Omega$, where $\Omega$ is a smooth, bounded domain (of possibly several components, with nontrivial topology). Let $\delta$ be the flat metric. Since $(M, \delta)$ has zero ADM mass, Theorem 43 gives the following upper bound for the capacity of $\Sigma = \partial M$:

$$C_\delta(\Sigma) = \int_\Sigma W dA_\delta \leq 2\sqrt{\pi} \left( \int_\Sigma W^{4/3} dA_\delta \right)^{3/2}. \tag{5.11}$$

In other words, the capacity of a closed surface $\Sigma$ in $\mathbb{R}^3$ can be bounded above in terms some invariant of the harmonic conformal class of $\delta$ restricted to the region exterior to $\Sigma$. We remind the reader that this inequality depends on an unproven conjecture.

To verify a case of equality, recall that the Poisson kernel on $\mathbb{R}^3 \setminus B(0, R)$ is given by

$$K(x, y) = \frac{|x|^2 - R^2}{4\pi R |x - y|^3}$$

for $x \in \mathbb{R}^3 \setminus B(0, R)$ and $y \in \partial B(0, R)$. For large $|x|$, $K(x, y)$ has a multipole expansion; the leading term is $\frac{1}{4\pi R|x|}$, independent of $y$. This shows $W \equiv \frac{1}{4\pi R}$. In this case, equality holds in (5.11).

# 6

# Zero Area Singularities

Recall that the Schwarzschild manifold of positive mass is the unique manifold that attains equality in the Riemannian Penrose inequality (Theorem 4). Moreover, flat $\mathbb{R}^3$ is the unique manifold that attains equality in the positive mass theorem (Theorem 3), and flat space is the Schwarzschild manifold of zero mass. Bray asked the question: is there some geometric inequality regarding asymptotically flat manifolds of nonnegative scalar curvature such that the Schwarzschild manifold of *negative mass* attains equality [8]? To answer such a question, the following steps are needed.

1. Formulate an appropriate notion of "singularity" that includes the singularity of the Schwarzschild manifold of negative mass.
2. Define the "mass" of such singularities.
3. Prove the inequality: the ADM mass is at least the total masses of the singularities, with equality only for Schwarzschild manifolds of negative mass.

Step 1 leads to the idea of *zero area singularities*, or ZAS, considered by Bray [8], Bray's student Robbins [30], [31], and Bray–Jauregui [9]. Step 2 will be explained in section 6.1.2. Finally, step 3 uses the Riemannian Penrose inequality and some version of Conjecture 7. Moreover, the investigation of zero area singularities provided the original motivation for Conjecture 7.

Our presentation closely follows the joint paper [9], particularly in section 6.1.

## 6.1 Introduction to ZAS

Following [9], we recall the definition of a zero area singularity. We shall abbreviate both the singular and plural by "ZAS."

**Definition 45.** *Let $(M, g)$ be a 3-manifold with smooth, nonempty, compact boundary $\Sigma$. Assume that $g$ is smooth on $M \setminus \Sigma$. A connected component $\Sigma^0$ of $\Sigma$ is a* ***zero area singularity (ZAS)*** *of $g$ if for every sequence of smooth surfaces $\{S_n\}$ properly enclosing $\Sigma^0$ and converging in $C^1$ to $\Sigma^0$, the areas of $S_n$ measured with respect to $g$ converge to zero.*

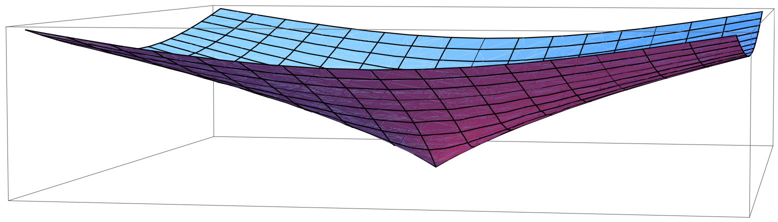
We emphasize that $g$ need not be defined on $\Sigma$, so, for instance, it is possible that $g$ does not extend continuously to the boundary as a symmetric $(0, 2)$-tensor. Recall from the remark following Definition 18 that convergence of $\{S_n\}$ to $\Sigma^0$ in $C^1$ is independent of the choice of metric. Topologically, a ZAS is a boundary surface in $M$, not a point. However, it is often convenient to think of a ZAS as a point formed by shrinking the metric to zero. For example, the boundary sphere $\{r = |m|/2\}$ of the Schwarzschild manifold of mass $m < 0$,

$$g = \left(1 + \frac{m}{2r}\right)^4 \delta, \qquad r > \frac{|m|}{2},$$

is a ZAS; see figure 6.1. Another example is that a point deleted from a smooth manifold is a zero area singularity with spherical topology. Yet another example is any conical singularity.

We point out that a manifold may possess a ZAS and still be complete (in the sense that Cauchy sequences converge) if, for instance, the boundary is infinitely far away.

Figure 6.1: The Schwarzschild manifold of negative mass



Pictured here is a diagram of $\mathbb{R}^3$ minus a ball, endowed with the Schwarzschild metric of mass $m < 0$. As usual, we have suppressed one dimension, so that level sets of the graph are 2-spheres. The cusp is topologically a sphere but appears as a point because its area has been collapsed to zero.

We argue that the Schwarzschild metric of negative mass and ZAS in general are natural objects to study in light of the following considerations:

1. Physics: ZAS are dual to black holes under the correspondence $m \mapsto -m$ for the Schwarzschild family of metrics and thus give a notion of "black hole of negative mass."
2. Uniqueness: The only spherically symmetric, maximally–extended, asymptotically–flat metrics on $S^2 \times I$ of zero scalar curvature are Schwarzschild manifolds of mass $m \in \mathbb{R}$. Such metrics have ZAS for $m < 0$. Here, $I$ is an open interval in $\mathbb{R}$.
3. Geometry: Given an asymptotically flat manifold $(M, g)$ of nonnegative scalar curvature and compact nonempty boundary $\Sigma$, consider the problem of gluing a compact region $\Omega$ to $M$ along $\Sigma$ such that $M \cup_\Sigma \Omega$ is smooth, complete, has no boundary, and has nonnegative scalar curvature. By the positive mass theorem, this problem has no solution in general. For instance, if $(M, g)$ has negative mass, any such "fill-in" $\Omega$ must have singularities in the sense of metric incompleteness. However, we conjecture that fill-ins exist in general, with the only singularities being a special class of zero area singularities, such as regular or harmonically regular ZAS.

#### 6.1.1 Regular and harmonically regular ZAS

We recall the special class of *regular* zero area singularities [8].

**Definition 46.** *Let $\Sigma^0$ be a ZAS of $g$. Then $\Sigma^0$ is* ***regular*** *if there exists a smooth, nonnegative function $\overline{\varphi}$ and a smooth Riemannian metric $\overline{g}$, both defined on a neighborhood $U$ containing $\Sigma^0$, such that*

1. *$\overline{\varphi}$ vanishes precisely on $\Sigma^0$,*
2. *$\overline{\nu}(\overline{\varphi}) > 0$ on $\Sigma^0$, where $\overline{\nu}$ is the unit normal to $\Sigma^0$ (taken with respect to $\overline{g}$ and pointing into the manifold), and*
3. *$g = \overline{\varphi}^4 \overline{g}$ on $U \setminus \Sigma^0$.*

*A pair $(\overline{g}, \overline{\varphi})$ is called a* ***local resolution*** *of $\Sigma^0$.*

In other words, the local geometry of a regular ZAS is conformal to a smooth metric, by a conformal factor that vanishes and is non-degenerate on the boundary. Note that not all ZAS are regular, and regular ZAS necessarily lead to metric incompleteness.

*Remark on notation:* For this chapter only, the symbols $\varphi, \overline{\varphi}$, etc. take on the indicated meaning, not necessarily that of a harmonic function that vanishes on $\Sigma$ and tends to one at infinity (as is the case in other chapters).

Much of our work utilizes an even more specialized class of singularities: those for which the resolution function can be chosen to be harmonic [9].

**Definition 47.** *A regular ZAS $\Sigma^0$ of $g$ is said to be* ***harmonically regular*** *if there exists a local resolution $(\overline{g}, \overline{\varphi})$ such that $\overline{\varphi}$ is harmonic with respect to $\overline{g}$. A pair $(\overline{g}, \overline{\varphi})$ is called a* ***local harmonic resolution****.*

For example, the Schwarzschild metric of negative mass has a harmonically regular ZAS, with local harmonic resolution $\left(\delta, 1 + \frac{m}{2r}\right)$. From an explicit example, it is known that not all regular ZAS are harmonically regular [9].

If several components of $\partial M$ are (harmonically) regular ZAS, then there is a natural notion of a local (harmonic) resolution of the union $\Sigma$ of these components: in Definitions 46 and 47, simply replace $\Sigma^0$ with $\Sigma$.

We also define resolutions that are globally defined:

**Definition 48.** *Suppose all components of $\Sigma = \partial M$ are harmonically regular ZAS. The pair $(\overline{g}, \overline{\varphi})$ is a* ***global harmonic resolution*** *of $\Sigma$ if*

1. *$\overline{g}$ is a smooth, asymptotically flat metric on $M$,*
2. *$\overline{\varphi}$ is the $\overline{g}$-harmonic function on $M$ vanishing on $\Sigma$ and tending to one at infinity, and*
3. *$g = \overline{\varphi}^4 \overline{g}$ on $M \setminus \Sigma$.*

*6.1.2 Defining the mass of ZAS*

In this section, we define the mass of a collection of ZAS [8]. For simplicity, we assume all components of $\Sigma = \partial M$ are ZAS of $g$ and define their total mass. We begin by restricting to regular ZAS.

**Definition 49.** *Suppose $(M, g)$ has boundary $\Sigma$, every component of which is a regular ZAS. Let $(\overline{g}, \overline{\varphi})$ be a local resolution of $\Sigma$. Then the* ***regular mass*** *of $\Sigma$ is*

$$m_{\text{reg}}\left(\Sigma\right) = -\frac{1}{4}\left(\frac{1}{\pi}\int_{\Sigma} \overline{\nu}(\overline{\varphi})^{4/3} d\overline{A}\right)^{3/2}, \tag{6.1}$$

*where $\overline{\nu}$ is the unit normal to $\Sigma$, pointing into the manifold, and $d\overline{A}$ is the area measure, both with respect to $\overline{g}$.*

In the case that $\Sigma$ has multiple components $\Sigma_1, \ldots, \Sigma_k$, each of which are regular ZAS, then the above formula leads to the addition rule:

$$m_{\text{reg}}(\Sigma) = \left(\sum_{i=1}^{k} m_{\text{reg}}(\Sigma_i)^{2/3}\right)^{3/2}.$$

We point out some properties of the definition of regular mass.

**Proposition 50.** *In Definition 6.1, the regular mass of* $\Sigma$

1. *is independent of the choice of local resolution,*
2. *depends only on the local geometry of* $(M, g)$ *near* $\Sigma$*, and*
3. *equals* $m$ *for the Schwarzschild ZAS metric of ADM mass* $m < 0$*.*

*Proof.* Claim 1 has been previously established [8], [30], [9], [31], as has claim 2 [9], [31]. Claim 3 is a straightforward exercise, using the local resolution $\left(\delta, 1 + \frac{m}{2r}\right)$. □

Next, we present the definition of the mass of arbitrary ZAS used in earlier works, which essentially approximates a given collection of ZAS with harmonically regular ZAS. Suppose $\{\Sigma_n\}$ is a sequence of smooth surfaces properly enclosing $\Sigma$ and converging in $C^1$ to $\Sigma$. Let $\Omega_n$ be the open region bounded by $\Sigma_n$, and let $\varphi_n$ be the unique $g$-harmonic function that vanishes on $\Sigma_n$ and tends to one at infinity. Then the conformal metric $\varphi_n^4 g$ is an asymptotically flat metric on the manifold $M_n := M \setminus \Omega_n$; moreover the boundary $\Sigma_n$ of $M_n$ consists of harmonically regular ZAS of the conformal metric $\varphi_n^4 g$ with tautological resolution $(g|_{TM_n}, \varphi_n)$. Thus it makes sense to consider the regular mass of $\Sigma_n$ in $\varphi_n^4 g$, which we denote by $m_{\text{reg}}(\Sigma_n)$.

**Definition 51.** *Suppose an asymptotically flat manifold* $(M, g)$ *has boundary* $\Sigma$ *consisting of ZAS. The* ***mass*** *(or* ***ZAS mass****) of* $\Sigma$ *is*

$$m_{\text{ZAS}}(\Sigma) := \sup_{\{\Sigma_n\}} \left( \limsup_{n \to \infty} m_{\text{reg}}(\Sigma_n) \right)$$

$$= \sup_{\{\Sigma_n\}} \left( \limsup_{n \to \infty} -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma_n} \nu(\varphi_n)^{4/3} dA \right)^{3/2} \right),$$

*where the supremum is taken over all sequences* $\{\Sigma_n\}$ *converging in* $C^1$ *to* $\Sigma$*.*

While the regular mass of a regular ZAS is a negative real number, $m_{\text{ZAS}}(\Sigma)$ takes values in $[-\infty, 0]$. Examples with ZAS mass $-\infty$ and 0 are known [9]. Some other properties enjoyed by the ZAS mass are:

**Proposition 52.** *The definition of the mass of* $\Sigma$ *depends only on the geometry of any neighborhood of* $\Sigma$*. Moreover, if the components* $\Sigma$ *are* harmonically regular*, then the two definitions of mass agree:*

$$m_{\text{ZAS}}(\Sigma) = m_{\text{reg}}(\Sigma).$$

The first claim was proved by Robbins [30] and the second by Bray–Jauregui [9].

## 6.2 The Riemannian ZAS inequality

The Riemannian ZAS inequality, given below, is an analog of the Riemannian Penrose inequality for zero area singularities.

**Theorem 53** (The Riemannian ZAS inequality, global harmonically regular case)**.** *Suppose* $g$ *is an asymptotically flat metric on* $M \setminus \partial M$ *of nonnegative scalar curvature such that all components of the boundary* $\Sigma = \partial M$ *are ZAS. Assume there exists a global harmonic resolution* $(\overline{g}, \overline{\varphi})$ *of* $\Sigma$*. If Conjecture 28 holds, then*

$$m_{\text{ADM}}(M, g) \geq m_{\text{ZAS}}(\Sigma), \tag{6.2}$$

*where* $m_{\text{ZAS}}(\Sigma)$ *is the ZAS mass of* $\Sigma$*.*

This theorem was proved by Bray assuming a stronger version of Conjecture 7 [8]. In that case, equality in (6.2) is proven to hold only for Schwarzschild manifolds of negative mass [9]. Moreover, the inequality is known for the case that $\Sigma$ is connected by Robbins' proof using inverse mean curvature flow [30].

*Proof.* While the inequality can be proven directly (c.f. [8], [9]), we invoke Theorem 43 of the previous chapter (which assumes Conjecture 28) to give a shorter argument.

Let $(\overline{g}, \overline{\varphi})$ be a global harmonic resolution of $\Sigma$. By Theorem 43,

$$m_{\mathrm{ADM}}(M, \overline{g}) \geq 2 \int_\Sigma \overline{W}\, \overline{dA} - 4\sqrt{\pi} \left( \int_\Sigma (\overline{W})^{4/3} \overline{dA} \right)^{3/2},$$

where $\overline{W} = W_{\overline{g}}$ is defined by equation (5.1). By Lemma 42, $\overline{W} = \frac{1}{4\pi}\overline{\nu}(\overline{\varphi})$, so the above can be rearranged as

$$m_{\mathrm{ADM}}(M, \overline{g}) - \frac{1}{2\pi} \int_\Sigma \overline{\nu}(\overline{\varphi}) \overline{dA} \geq -4\sqrt{\pi} \left( \int_\Sigma \left( \frac{1}{4\pi} \overline{\nu}(\overline{\varphi}) \right)^{4/3} \overline{dA} \right)^{3/2}.$$

The left-hand side is the ADM mass of $(M, g)$ by formula (A.9). (Note that the flux integral over $\Sigma$ can be exchanged with the flux integral over $S_\infty$ by harmonicity of $\overline{\varphi}$.) The right-hand side is the regular mass of the ZAS $\Sigma$, by definition. By Proposition 52, the regular mass of $\Sigma$ equals the ZAS mass of $\Sigma$. □

Bray demonstrated that Theorem 53 immediately implies an analogous inequality for arbitrary ZAS [8]:

**Theorem 54** (The Riemannian ZAS inequality, general case)**.** *Suppose $g$ is an asymptotically flat metric on $M \setminus \partial M$ of nonnegative scalar curvature such that all components of the boundary $\Sigma = \partial M$ are ZAS. Assume that Conjecture* 28 *holds. Then*

$$m_{\mathrm{ADM}}(M, g) \geq m_{\mathrm{ZAS}}(\Sigma).$$

It is conjectured that the unique case of equality in Theorem 54 is the Schwarzschild manifold of non-positive mass, modulo zero area singularities that are deleted points [9].

## 6.3 Results on the mass of ZAS

The purpose of this section is to prove some results regarding the mass of ZAS referred to in a previous paper [9]. We first show that in the definition of the ZAS mass of $\Sigma$, there exists a sequence of surfaces converging to $\Sigma$ that attains the supremum and is such that the limsup may be replaced with a limit. While this is practically obvious from the definition, we spell it out here in detail.

**Proposition 55.** *Suppose the boundary components $\Sigma$ of $(M, g)$ are ZAS. Then there exists a sequence of surfaces $\{\Sigma_n^*\}$ converging in $C^1$ to $\Sigma$ such that*

$$\lim_{n\to\infty} m_{\text{reg}}(\Sigma_n^*) = m_{\text{ZAS}}(\Sigma).$$

*Proof.* First, consider the case $m_{\text{ZAS}}(\Sigma) = -\infty$. Then for any sequence $\Sigma_n \xrightarrow{C^1} \Sigma$, we have $\limsup_{n\to\infty} m_{\text{reg}}(\Sigma_n) \to -\infty$. But then $\lim_{n\to\infty} m_{\text{reg}}(\Sigma_n) = -\infty = m_{\text{ZAS}}(\Sigma)$.

Now we may assume $m_{\text{ZAS}}(\Sigma)$ is finite, possibly zero. Then there exists a collection $\{\Sigma_n^{(i)}\}_{i,n=1}^\infty$ of surfaces in $M$ such that:

- For each fixed $i$, the $C^1$ norm of $\{\Sigma_n^{(i)}\}_n$ (viewed as a graph over $\Sigma$ with respect to some choice of smooth background metric) is strictly decreasing to zero as $n \to \infty$.
- The numbers $a_i := \limsup_{n\to\infty} m_{\text{reg}}(\Sigma_n^{(i)})$ form an increasing sequence with limit $m_{\text{ZAS}}(\Sigma)$.

Now we construct the desired sequence. Define $\Sigma_1^*$ to be $\Sigma_1^{(1)}$. Choose $\Sigma_i^*$ among the $\{\Sigma_n^{(i)}\}_n$ by the conditions:

1. the $C^1$ norm of $\Sigma_i^*$ is less than $2^{-i}$, and
2. the regular mass of $\Sigma_i^*$ is within $2^{-i}$ of $a_i$.

That such a surface $\Sigma_i^*$ exists for each $i \geq 2$ is clear from our earlier assumptions. Now $\Sigma_i^*$ converges in $C^1$ to $\Sigma$ as $i \to \infty$ by condition 1. Moreover, the sequence formed by taking the regular masses of $\{\Sigma_i^*\}_i$ converges to $m_{\text{ZAS}}(\Sigma)$ by condition 2. $\square$

Suppose $\Sigma$ is a collection of regular ZAS. Currently an unresolved question is whether the definitions of regular mass and ZAS mass of $\Sigma$ agree. However, we have an inequality:

**Proposition 56.** *Suppose the boundary components $\Sigma$ of $(M, g)$ are regular ZAS. Then $m_{\text{ZAS}}(\Sigma) \leq m_{\text{reg}}(\Sigma)$. In particular, the ZAS mass of such $\Sigma$ is strictly negative.*

*Proof.* Let $(\overline{g}, \overline{\varphi})$ be a local resolution of $g$ such that $\overline{\nu}(\overline{\varphi}) \equiv 1$ on $\Sigma$. That such a resolution exists is proved in Proposition 13 of Bray–Jauregui [9]. Extend $\overline{g}$ and $\overline{\varphi}$ smoothly to all of $M$ with the conditions that $g = \overline{\varphi}^4 \overline{g}$ on $M$, and $\overline{\varphi} \equiv 1$ outside of some compact set $K$. Suppose $\Sigma_n \to \Sigma$ in $C^1$, and let $\varphi_n$ be $g$-harmonic, 0 on $\Sigma_n$ and 1 at infinity. Each $\varphi_n$ is a function on $M_n$, the closure of the region exterior to $\Sigma_n$. Consider the regular mass of $\Sigma_n$, viewed as ZAS as in Definition 51:

$$\begin{aligned} m_{\text{reg}}(\Sigma_n) &= -\frac{1}{4}\left(\frac{1}{\pi}\int_{\Sigma_n} \nu(\varphi_n)^{4/3} dA\right)^{3/2} && \text{(definition of regular mass)} \\ &= -\frac{1}{4}\left(\frac{1}{\pi}\int_{\Sigma_n} \left(\overline{\varphi}\,\overline{\nu}(\varphi_n)\right)^{4/3} d\overline{A}\right)^{3/2} \\ &= -\frac{1}{4}\left(\frac{1}{\pi}\int_{\Sigma_n} \left(\overline{\nu}(\overline{\varphi}\varphi_n)\right)^{4/3} d\overline{A}\right)^{3/2}. \end{aligned} \tag{6.3}$$

On the second line, we used formulas (A.3) and (A.4) to compare the unit normals and area forms for the conformal metrics $g$ and $\overline{g}$. The third line used the fact that $\varphi_n$ vanishes on $\Sigma_n$ by definition. To complete the proof, it is enough to show

$$\int_{\Sigma} \overline{\nu}(\overline{\varphi})^{4/3} d\overline{A} \leq \liminf_{n\to\infty} \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}\varphi_n)^{4/3} d\overline{A}. \tag{6.4}$$

To see why this is sufficient, take $\limsup\limits_{n\to\infty}$ of both sides of (6.3), followed by the supremum over all sequences $\{\Sigma_n\}$. We would be done if we knew that $\overline{\nu}(\overline{\varphi}\varphi_n)$ converged uniformly to $\overline{\nu}(\overline{\varphi})$ (under an identification of $\Sigma_n$ with $\Sigma$), but this is not known.

Since we wish to compare flux integrals of $\overline{\varphi}\varphi_n$ and $\overline{\varphi}$, we are motivated to look at the function

$$f_n = \overline{\varphi}\varphi_n - \overline{\varphi}$$

defined on the region $M_n$ and its Laplacian

$$\overline{\Delta} f_n = \overline{\Delta}(\overline{\varphi}\varphi_n) - \overline{\Delta}\overline{\varphi} = \overline{\Delta}\overline{\varphi}(\varphi_n - 1).$$

On the last line we used formula (A.6) and the fact that $\varphi_n$ is $g$-harmonic. On one hand,

$$\lim_{n\to\infty} \int_{M_n} \overline{\Delta} f_n d\overline{A} = 0 \tag{6.5}$$

from the dominated convergence theorem, since $\overline{\Delta}\overline{\varphi}$ is bounded and of compact support and $(\varphi_n - 1)$ converges pointwise almost everywhere to zero. ($\varphi_n$ converges pointwise to 1, except on $\Sigma$ [9].)

On the other hand, by the divergence theorem,

$$\begin{aligned} \int_{M_n} \overline{\Delta} f_n d\overline{A} &= \int_{\Sigma_n} \overline{\nu}(f_n) d\overline{A} + \int_{S_\infty} \overline{\nu}(f_n) d\overline{A} \\ &= \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}\varphi_n - \overline{\varphi}) d\overline{A} + \int_{S_\infty} \nu(\varphi_n) d\overline{A} \end{aligned} \tag{6.6}$$

where we have used the fact that $\overline{\varphi} \equiv 1$ outside of a compact set. The last term (representing the capacity of $\Sigma_n$) converges to zero, since the ZAS $\Sigma$ are regular [9]. Moreover, $\int_{\Sigma_n} \overline{\nu}(\overline{\varphi}) d\overline{A}$ converges to $\int_{\Sigma} \overline{\nu}(\overline{\varphi}) d\overline{A}$, since $\Sigma_n \to \Sigma$ in $C^1$. Combining equations (6.5) and (6.6), we have now shown

$$\lim_{n\to\infty} \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}\varphi_n) d\overline{A} = \lim_{n\to\infty} \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}) d\overline{A} \tag{6.7}$$

We finally prove (6.4). Repeatedly using the fact that $\overline{\nu}(\overline{\varphi}) \equiv 1$ on $\Sigma$, we have:

$$
\begin{aligned}
\int_\Sigma \overline{\nu}(\overline{\varphi})^{4/3} d\overline{A} &= \int_\Sigma \overline{\nu}(\overline{\varphi}) d\overline{A} \\
&= \lim_{n\to\infty} \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}) d\overline{A} && (\Sigma_n \to \Sigma \text{ in } C^1) \\
&= \lim_{n\to\infty} \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}\varphi_n) d\overline{A} && (\text{by (6.7)}) \\
&\leq \liminf_{n\to\infty} \left( \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}\varphi_n)^{4/3} d\overline{A} \right)^{3/4} \left( \int_{\Sigma_n} 1 d\overline{A} \right)^{1/4} \\
&= \liminf_{n\to\infty} \left( \int_{\Sigma_n} \overline{\nu}(\overline{\varphi}\varphi_n)^{4/3} d\overline{A} \right)^{3/4} \left( \int_{\Sigma} \nu(\overline{\varphi})^{4/3} \right)^{1/4},
\end{aligned}
$$

where on the second to last line we used Hölder's inequality and on the last line, the fact that $\overline{\nu}(\overline{\varphi}) \equiv 1$. The proof follows by dividing both sides by $\left(\int_\Sigma \nu(\overline{\varphi})^{4/3}\right)^{1/4}$, then raising to the power of 4/3. □

*A remark on the Bartnik mass of ZAS:* It may appear undesirable that the ZAS mass can possibly be $-\infty$. Such is the case, for instance, for spherically-symmetric metrics for which the cross-sectional area of spheres is proportional to the distance to the singularity (Table 2, [9]). However, below we use the idea of Bartnik mass to show that this phenomenon is completely natural.

Following a slight modification of [9], define $m_{\mathrm{B}}(\Sigma)$ (the Bartnik mass of ZAS $\Sigma$) as the infimum of the ADM mass of all asymptotically flat manifolds of nonnegative scalar curvature that contain an isometric copy of some neighborhood of $\Sigma$ and no other singularities. If $f(\Sigma)$ is any definition of the "mass" of $\Sigma$ that 1) depends only on the geometry in any neighborhood of $\Sigma$ and 2) estimates the ADM mass from below, we get an inequality:

$$m_{\mathrm{B}}(\Sigma) \geq f(\Sigma).$$

The following proposition shows that $m_{\mathrm{B}}(\Sigma) = -\infty$ for a particular class of ZAS.

**Proposition 57.** *Consider the metric on* $U = (0, 1] \times S^2$ *given by:*

$$g = ds^2 + \beta s d\sigma^2$$

*for* $0 < s \leq 1$ *and a constant* $\beta > 0$*, and* $d\sigma^2$ *the round metric of area* $4\pi$ *on* $S^2$*. Note that* $g$ *has a ZAS* $\Sigma$ *by extension to* $s = 0$*. Then* $m_{\mathrm{B}}(\Sigma) = -\infty$*. In particular, any definition of the ZAS mass of such a singularity that depends only on the local geometry and bounds the ADM mass from below must equal* $-\infty$*.*

Note that the above metric $g$ has positive scalar curvature for $s > 0$.

*Proof.* We will construct an asymptotically flat "extension" of a neighborhood of $\Sigma$ that has nonnegative scalar curvature and arbitrarily negative ADM mass. Let $m < 0$, and consider the Schwarzschild metric of mass $m$:

$$g_m = \left(1 + \frac{m}{2r}\right)^4 \delta$$

for $r > |m|/2$. Let $\Sigma_s$ be the concentric round sphere in $U$ of radius $s$. The area and mean curvature of $\Sigma_s$ with respect to $g$ are $4\pi\beta s$ and $1/s$, respectively. On the other hand, the area and mean curvature of the coordinate sphere $S_r$ in $g_m$ are respectively

$$4\pi r^2 \left(1 + \frac{m}{2r}\right)^4$$

and

$$\frac{2}{r} \cdot \frac{r - m/2}{r + m/2}.$$

By setting the areas of $S_r$ and $\Sigma_t$ equal, we obtain

$$s = \frac{r^2}{\beta}\left(1 + \frac{m}{2r}\right)^4.$$

By choosing $r$ sufficiently close to $|m|/2$, $s$ belongs to $(0, 1]$. For such $s$ and $r$, $S_r$ and $\Sigma_s$ are isometric, as they are round spheres of equal areas. In particular, we may glue the region in $U$ bounded by $\Sigma_s$ to the region in the Schwarzschild manifold outside $S_r$ to obtain a manifold that is Lipschitz along the gluing hypersurface $S_r \cong \Sigma_s$. By choosing $r$ sufficiently close to $|m|/2$, we can guarantee that the mean curvature of the inner gluing surface $\Sigma_s$ in $U$ is greater than the mean curvature of the outer gluing surface $S_r$ in the Schwarzschild manifold. This assures that the glued manifold has nonnegative distributional scalar curvature across the hypersurface [6], [26]. In particular, for any $m < 0$, there exists an asymptotically flat extension of nonnegative scalar curvature and mass $m$ of a neighborhood of $\Sigma$; therefore, $m_B(\Sigma) = -\infty$. □

## 6.4 Existence of global harmonic resolutions and Green's functions

A question that naturally arises in the study of ZAS is: do harmonically regular ZAS admit global harmonic resolutions? Here we give an affirmative answer.

**Theorem 58.** *Suppose $(M, g)$ is asymptotically flat with boundary $\Sigma$ consisting of harmonically regular ZAS of $g$ (i.e., assume there exists a local harmonic resolution of $\Sigma$). Then there exists a global harmonic resolution of $\Sigma$.*

*Proof.*

*Step 1: Reducing to Green's functions.* Define a *Green's function* for $(M, g, \Sigma)$ to be a smooth function $G$ on $M \setminus \Sigma$ such that $(G^4 g, G^{-1})$ is a global harmonic resolution of $\Sigma$. Equivalently, $G$ is a function satisfying the properties:

1. $\Delta_g G = 0$ in $M \setminus \Sigma$,
2. $G \to +\infty$ on $\Sigma$,
3. $G \to 1$ at infinity,
4. $G^4 g$ extends smoothly to $\Sigma$ as a Riemannian metric $\tilde{g}$ on $M$, and
5. $G^{-1}$ extends smoothly to $\Sigma$ and satisfies $\tilde{\nu}(G^{-1}) > 0$ on $\Sigma$, where $\tilde{\nu}$ is the $\tilde{g}$-unit normal to $\Sigma$.

By the maximum principle, properties 1–3 imply that $G > 0$. Thus, to construct a global harmonic resolution of $\Sigma$, it is sufficient to construct a Green's function.

*Step 2: Constructing an approximate Green's function.* In this section we construct an approximate Green's function $G_0$, namely a function $G_0$ satisfying properties 2–5 of the previous section along with:

1a. $\Delta_g G_0 \leq 0$ on $M \setminus \Sigma$, and

1b. $\Delta_g G_0 < 0$ only on a set compactly contained in the interior of $M$.

By hypothesis, there exists a local harmonic resolution $(\overline{g}, \overline{\varphi})$ defined on a neighborhood $U$ of $\Sigma$. Let $G_3 = \frac{1}{\overline{\varphi}}$, a positive $g$-harmonic function on $U \setminus \Sigma$ (by formula (A.6)) that blows up on $\Sigma$. For some number $L$ sufficiently large, the level set $\Sigma_L := G_3^{-1}(L)$ is a smooth surface enclosing $\Sigma$ and is contained entirely in $U$. Let $h_L$ be the unique $g$-harmonic function, defined outside $\Sigma_L$, that equals $L$ on $\Sigma_L$ and tends to 0 at infinity. By the maximum principle, $\nu(h_L) < 0$ on $\Sigma_L$. Thus, there exists a number $C > 0$ sufficiently large so that

$$\nu(Ch_L) < \nu(G_3) \qquad \text{on } \Sigma_l, \tag{6.8}$$

since the right hand side is bounded. Now, consider the following function on $M \setminus \Sigma$:

$$G_2(x) = \begin{cases} G_3(x) + CL - L, & \text{inside } \Sigma_L \\ Ch_L, & \text{outside } \Sigma_L. \end{cases} \tag{6.9}$$

By construction, the function $G_2$ is smooth and $g$-harmonic on $M \setminus (\Sigma \cup \Sigma_L)$ and is Lipschitz continuous across $\Sigma_L$. By the condition (6.8) on normal derivatives, $G_2$ is weakly superharmonic with respect to $g$, and thus there exists a smooth, $g$-superharmonic function $G_1$ defined on $M \setminus \Sigma$ that agrees with $G_2$ except on a neighborhood of $\Sigma_L$ that is compactly contained in $M$. To remedy the boundary condition at infinity, define $G_0 := G_1 + 1$. We claim that $G_0$ satisfies all of the desired properties of our approximate Green's function.

Now, $G_0$ satisfies properties 1a and 1b since $\Delta_g G_1 \leq 0$ and is nonzero only on a compact neighborhood of $\Sigma_L$ (in the interior of $M$) and $G_0 - G_1$ is a constant. Properties 2 and 3 are clear from the construction. For properties 4 and 5, define $\tilde{g} = G_0^4 g$. On a small neighborhood $U'$ of $\Sigma$, note that

$$G_0 = \frac{1}{\overline{\varphi}} + CL - L + 1,$$

so on $U'$

$$\begin{aligned}\tilde{g} := G_0^4 g &= \left(\overline{\varphi}^{-1} + CL - L + 1\right)^4 \overline{\varphi}^4 \overline{g} \\ &= \left(1 + \overline{\varphi}(CL - L + 1)\right)^4 \overline{g},\end{aligned}$$

extends smoothly to $\Sigma$ as a Riemannian metric, since $\overline{\varphi}$ is smooth and vanishes on the boundary. Let $\tilde{\varphi} = \frac{1}{G_0}$. On the neighborhood $U'$,

$$\tilde{\varphi} = \frac{1}{\frac{1}{\overline{\varphi}} + CL - L + 1} = \frac{\overline{\varphi}}{1 + \overline{\varphi}(CL - L + 1)},$$

which extends smoothly to zero on $\Sigma$. Note that $\tilde{g}$ and $\overline{g}$ agree on tangent vectors to $M$ based on $\Sigma$, since the ratio $\dfrac{\tilde{\varphi}}{\overline{\varphi}}$ is 1 on $\Sigma$. Then, on $\Sigma$,

$$\begin{aligned}\tilde{\nu}(\tilde{\varphi}) &= \overline{\nu}\left(\tilde{\varphi}\right) \\ &= \overline{\nu}\left(\frac{\overline{\varphi}}{1 + \overline{\varphi}(CL - L + 1)}\right) \\ &= \overline{\nu}(\overline{\varphi}) > 0.\end{aligned}$$

We have shown that $G_0$ satisfies properties 1a, 1b, and 2–5.

*Step 3: Solving for a Green's function.* Consider the function $G_0$ from step 2. We wish to add a bounded function $\psi \in C^\infty(M)$ to $G_0$ to obtain a globally $g$-harmonic function:

$$\Delta_g \left(G_0 + \psi\right) = 0, \tag{6.10}$$

with boundary conditions merely that $\psi \to 0$ at infinity and $\psi$ is bounded on $\Sigma$. If such $\psi$ exists and we define $G = G_0 + \psi$, it is straightforward to check that $G$ is a Green's function, completing the proof of the theorem.

Note that (6.10) is simply Poisson's equation for $\psi$ in terms of the given function $f_0 := -\Delta_g G_0$; the only difficulty is that $g$ is singular on $\Sigma$. Let $(\tilde{g}, \tilde{\varphi})$ be the resolution determined by $G_0$ as in the previous step. (That is, $\tilde{g} = G_0^4 g$ and $\tilde{\varphi} = G_0^{-1}$.) Write the unknown function $\psi$ in terms of a new unknown $\tilde{\psi}$ via the equation

$$\psi = \frac{\tilde{\psi}}{\tilde{\varphi}}.$$

(The motivation is that since $\tilde{\varphi}$ vanishes on the boundary and has positive normal derivative there, we can be sure that $\psi$ is a smooth, bounded function by requiring $\tilde{\psi}$ to be smooth with zero boundary conditions on $\Sigma$.) We now cast our original problem (6.10) on the singular space in terms of an equivalent problem on the smooth space $(M, \tilde{g})$.

We have the conformal relation $g = \tilde{\varphi}^4 \tilde{g}$. Applying formula (A.6),

$$\Delta_{\tilde{g}} \tilde{\psi} = \tilde{\varphi}^5 \Delta_g (\psi) + \tilde{\psi} \frac{\Delta_{\tilde{g}} \tilde{\varphi}}{\tilde{\varphi}}. \tag{6.11}$$

Substituting (6.10) and the definition of $f_0$, we get the equivalent problem

$$\begin{cases} (\Delta_{\tilde{g}} - f_1)\, \tilde{\psi} = \tilde{\varphi}^5 f_0 & \text{in } M \\ \tilde{\psi} = 0 & \text{on } \Sigma \\ \tilde{\psi} \to 0 & \text{at infinity} \end{cases} \tag{6.12}$$

where $f_1 := \dfrac{\Delta_{\tilde{g}} \tilde{\varphi}}{\tilde{\varphi}}$. (To see that $f_1$ is a smooth function on $M$, use the fact that $\Delta_{\overline{g}} \overline{\varphi}$ vanishes on a neighborhood of $\Sigma$ by hypothesis, the fact that $\tilde{g}$ and $\overline{g}$ are harmonically conformal on the neighborhood $U'$ of $\Sigma$, and formula (A.6).) The first key is that problem (6.12) is purely a PDE with smooth data and boundary conditions on the smooth Riemannian manifold $(M, \tilde{g})$. The second key is that by our construction of $G_0$ as a $g$-superharmonic function, $\tilde{\varphi}$ is $\tilde{g}$-subharmonic: from formula (A.6),

$$\begin{aligned}
0 &= \Delta_{\tilde{g}} \left( \tilde{\varphi} \cdot \frac{1}{\tilde{\varphi}} \right) \\
&= \tilde{\varphi}^5 \Delta_g \left( \frac{1}{\tilde{\varphi}} \right) + \frac{1}{\tilde{\varphi}} \Delta_{\tilde{g}}(\tilde{\varphi}) \\
&= \tilde{\varphi}^5 \Delta_g (G_0) + G_0 \Delta_{\tilde{g}}(\tilde{\varphi}).
\end{aligned}$$

Since $G_0$ and $\varphi$ are nonnegative and $\Delta_g (G_0) \leq 0$, we see that $f_1 = \Delta_{\tilde{g}}(\tilde{\varphi}) \geq 0$. In particular, it follows from the maximum principle that the operator $\Delta_{\tilde{g}} - f_1$ (with zero boundary conditions at infinity and on $\Sigma$) has trivial kernel. Since $f_0$ and $f_1$ each have compact support, it is now standard [5] that there exists a smooth solution $\tilde{\psi}$ to (6.12). Then $\psi := \dfrac{\tilde{\psi}}{\tilde{\varphi}}$ is smooth on $M$ and solves (6.10). In particular, $G_0 + \psi$ is the desired Green's function. $\square$

# 7

# Invariants of the Harmonic Conformal Class

Given an asymptotically flat manifold $(M, g)$ with compact boundary $\Sigma$, in Chapters 4, 5, and 6, we have utilized the function

$$\alpha(A) = \sup_{g' \in \overline{\mathcal{H}(g)}} \{\min(\Sigma, g') \ : \ |\Sigma|_{g'} \leq A\},$$

given in Definition 30. Recall that $\overline{\mathcal{H}(g)}$ is the generalized harmonic conformal class of $g$, $\min(\Sigma, g')$ is the minimal enclosing area of $\Sigma$ with respect to $g'$, and $|\Sigma|_{g'}$ is the area of $\Sigma$ with respect to $g'$, all defined in Chapter 3. The value of $\alpha(A)$ is the answer to the question: how large can the minimal enclosing area be for metrics in the harmonic conformal class of $g$, given an upper bound $A$ on the area of the boundary? By construction, $\alpha$ is a *harmonic conformal invariant*, since it is defined by optimizing a geometric quantity subject to a geometric constraint over the (generalized) harmonic conformal class. In this chapter, we investigate $\alpha$ and other such functions that are canonically associated to a harmonic conformal class through a geometric optimization process.

In section 7.1 we define two functions $\mu$ and $\nu$ through a process of optimizing the ADM mass within the harmonic conformal class. These "mass profile functions" enjoy some nice properties such as continuity and monotonicity and are similar in nature to the "area profile functions" $\alpha$ and $\beta$ defined in section 7.2. We shall see that the functions $\beta$ and $\nu$ are constant and thus uninteresting, while the functions $\mu$ and $\alpha$ are nontrivial. In section 7.3 we give an example for the harmonic conformal class of flat space minus a round ball. Finally, we point out some interesting relationships between $\mu$ and $\alpha$ in section 7.4 coming from the Riemannian Penrose inequality and Conjecture 41.

## 7.1 Mass profile functions

The following two functions are constructed by optimizing the ADM mass in a harmonic conformal class, subject to an area constraint. For $A \geq 0$, define

$$\mu(A) = \sup_{g' \in \overline{\mathcal{H}(g)}} \{m_{\text{ADM}}(M, g') \ : \ |\Sigma|_{g'} \leq A\}, \quad \text{and}$$

$$\nu(A) = \inf_{g' \in \overline{\mathcal{H}(g)}} \{m_{\text{ADM}}(M, g') \ : \ |\Sigma|_{g'} \leq A\}.$$

By construction, $\mu$ and $\nu$ are canonically associated to the harmonic conformal class. We refer to these functions of $A$ as "mass profile functions." Without the area constraint, there is no upper bound for the ADM mass among metrics in the generalized harmonic conformal class $\overline{\mathcal{H}(g)}$. In section 7.5, we discuss possible alternate definitions of $\mu$ and $\nu$.

We will show that the above supremum and infimum are attained.

**Theorem 59.** *For each $A \geq 0$, there exists a unique metric $g' \in \overline{\mathcal{H}_A(g)}$ such that $\mu(A) = m_{\text{ADM}}(M, g')$.*

*Proof.* To prove existence, let $\{u_n^4 g\}$ be a maximizing sequence for $\mu(A)$: that is, assume $|\Sigma|_{u_n^4 g} \leq A$ for each $n$ and

$$m_{\text{ADM}}(M, u_n^4 g) \nearrow \mu(A).$$

Increasing $f_n$, the function on $\Sigma$ determined by $u_n$, by adding a constant only increases ADM mass (by formula (A.9) and the maximum principle), so we may assume without loss of generality that $|\Sigma|_{u_n^4 g} = A$. Using Proposition 14, we pass to a weakly convergent subsequence, of the same name, with limit $u^4 g \in \overline{\mathcal{H}_C(g)}$ with $C \leq A$. Since $u_n \to u$ as harmonic functions uniformly on compact subsets on the interior of $M$, we see from formula (A.9) that

$$m_{\text{ADM}}(M, u_n^4 g) \to m_{\text{ADM}}(M, u^4 g),$$

which shows that $g' = u^4 g$ attains the supremum for $\mu(A)$. If $C < A$, then by adding a constant to $f$, the function determined by $u$, we could construct a larger maximizer.

The uniqueness of $g' = u^4 g$ as the maximizer follows from a concavity argument. Let $v$ be a nonnegative $g$-harmonic function that vanishes at infinity. The ADM mass of the path $(u + tv)^4 g$ in $\overline{\mathcal{H}(g)}$ is an affine function of $t$:

$$m_{\text{ADM}}(M, (u + tv)^4 g) = m_{\text{ADM}}(M, u^4 g) - \frac{t}{2\pi} \int_{S_\infty} \nu(v) dA_g,$$

by formula (A.9). To normalize $(u + tv)^4 g$ to be a path with fixed boundary area $A$, we must divide the boundary data by the number $|\Sigma|^{1/4}_{(u+tv)^4 g}$, which is a convex function of $t$. By considering a path between $u^4 g$ and any other maximizer, we are led to a contradiction. $\square$

A simple observation follows: $\mu(A)$ is finite for each $A \geq 0$. Next, we demonstrate some basic regularity for the function $\mu$:

**Proposition 60.** *$\mu(A)$ is a strictly increasing, continuous function of $A$.*

*Proof.* Let $0 < A < B$, and suppose $g_A = u_A^4 \in \overline{\mathcal{H}_A(g)}$ attains the supremum for $\mu(A)$. Let $f_A \in L^4(\Sigma)$ be the function determined by $u_A$. Then for some unique

$c > 0$,

$$\int_\Sigma (f_A + c)^4 dA_g = B.$$

Let $u$ be the harmonic function associated to $f_A + c$. By formula (A.9) and the maximum principle,

$$m_{\text{ADM}}(M, g_A) < m_{\text{ADM}}(M, u^4 g).$$

The left-hand side equals $\mu(A)$ by construction, and the right-hand side is at most $\mu(B)$, since $u^4 g$ is a valid test metric in the definition of $\mu(B)$. So $\mu(A) < \mu(B)$.

Next, we show continuity by controlling how much $\mu(\cdot)$ could jump from $A$ to $B$. Let $g_B = u_B^4 \in \overline{\mathcal{H}_B(g)}$ attain the supremum for $\mu(B)$, so in particular

$$m_{\text{ADM}}(M, g_B) = \mu(B). \tag{7.1}$$

Let $f_B \in L^4(\Sigma)$ be the function determined by $u_B$, and choose $\lambda \in (0, 1)$ such that

$$\int_\Sigma (\lambda f_B)^4 dA_g = A,$$

namely $\lambda = \left(\frac{A}{B}\right)^{1/4}$. Let $u_\lambda$ be the harmonic function associated to $\lambda f_B$, which we write in a convenient form:

$$\begin{aligned} u_\lambda &= \varphi + \int_\Sigma K(x, y) \lambda f_B(y) dA_g(y) \\ &= \varphi + \lambda(u_B - \varphi) \\ &= u_B + (1 - \lambda)(\varphi - u_B). \end{aligned} \tag{7.2}$$

Recall the definition of the capacity of $\Sigma$ in $(M, g)$:

$$C_g(\Sigma) = \frac{1}{4\pi} \int_{S_\infty} \nu(\varphi) dA_g,$$

a positive number. Now we do a computation:

$$\begin{aligned}
\mu(A) &\geq m_{\mathrm{ADM}}(M, u_\lambda^4 g) \\
&= m_{\mathrm{ADM}}(M, g) - \frac{1}{2\pi}\int_{S_\infty} \nu(u_\lambda) dA_g \\
&= \underbrace{m_{\mathrm{ADM}}(M, g) - \frac{1}{2\pi}\int_{S_\infty} \nu(u_B) dA_g}_{} - \underbrace{\frac{(1-\lambda)}{2\pi}\int_{S_\infty} \nu(\varphi - u_B) dA_g}_{}.
\end{aligned}$$

The inequality comes from the definition of $\mu(A)$, since $u_\lambda^4 g$ is a valid test metric for $\mu(A)$. The next two equalities are formula (A.9) and equation (7.2). Now, the first underbraced term equals the ADM mass of $(M, u_B^4 g)$, which is $\mu(B)$ by construction. As for the second underbraced term,

$$\begin{aligned}
\frac{(1-\lambda)}{2\pi}\int_{S_\infty} \nu(\varphi - u_B) dA_g &= (1-\lambda)\left(2C_g(\Sigma) + m_{ADM}(M, u_B^4 g) - m_{ADM}(M, g)\right) \\
&= (1-\lambda)\left(2C_g(\Sigma) + \mu(B) - m_{ADM}(M, g)\right).
\end{aligned}$$

Putting it all together,

$$\begin{aligned}
\mu(A) &\geq \mu(B) - (1-\lambda)\left(2C_g(\Sigma) + \mu(B) - m_{ADM}(M, g)\right)) \\
&= \lambda\mu(B) - (1-\lambda)\left(2C_g(\Sigma) - m_{ADM}(M, g)\right)).
\end{aligned}$$

Note that $\lambda$ converges to 1 as $B \to A$ or $A \to B$, and that $C_g(\Sigma)$ and $m_{ADM}(M, g)$ are independent of $A$ and $B$. Taking appropriate limits inferior and limits superior,

$$\mu(A) \geq \limsup_{B \to A^+} \mu(B)$$

$$\liminf_{A \to B^-} \mu(A) \geq \mu(B).$$

Along with the fact that $\mu(A) < \mu(B)$, it follows that $\mu$ is continuous from both the left and right. $\square$

Moving on, we demonstrate that the function $\nu$ is trivial.

**Proposition 61.** *For each $A \geq 0$, the unique metric $\varphi^4 g \in \overline{\mathcal{H}_0(g)}$ attains the infimum for $\nu(A)$, so $\nu(A) \equiv m_{\mathrm{ADM}}(M, \varphi^4 g)$. In particular, the function $\nu(A)$ is constant, given by the number $m_{\mathrm{ADM}}(M, g) - 2C_g(\Sigma)$, where $C_g(\Sigma)$ is the capacity of $\Sigma$ in $(M, g)$.*

Note that $m_{\mathrm{ADM}}(M, g) - 2C_g(\Sigma)$ is the numerical invariant $\mathcal{M}_1$ we encountered in Corollary 44.

*Proof.* Let $u^4 g \in \overline{\mathcal{H}_A(g)}$ with $A > 0$. By formula (A.9) and the maximum principle, the ADM mass can always be made smaller by scaling down the boundary value of $u$. From this, we see that the unique metric $\varphi^4 g \in \overline{\mathcal{H}_0(g)}$ attains the infimum for $\nu(A)$, for all $A \geq 0$. Moreover, by formula (A.9)

$$\begin{aligned} \nu(A) = \nu(0) &= m_{\mathrm{ADM}}(M, \varphi^4 g) \\ &= m_{\mathrm{ADM}}(M, g) - \frac{1}{2\pi} \int_{S_\infty} \nu(\varphi) dA_g, \end{aligned}$$

and the last term is twice the capacity of $\Sigma$. □

We make the trivial observation that $\nu(0) = \mu(0)$. Observing that the metric $\varphi^4 g$ has zero area singularities on $\Sigma$, we use the Riemannian ZAS inequality of Chapter 6 to estimate $\mu(\cdot)$ from below.

**Proposition 62.** *Assume Conjecture 28 holds. Let $\varphi$ be the $g$-harmonic function that vanishes on $\Sigma$ and limits to 1 at infinity. Then*

$$\mu(A) \geq \mu(0) \geq -\frac{1}{4} \left( \frac{1}{\pi} \int_\Sigma \nu(\varphi)^{4/3} dA_g \right)^{3/2}.$$

*Proof.* That $\mu$ is increasing gives the first inequality. The second inequality is Theorem 53, noting that $\mu(0)$ is the ADM mass of $(M, \varphi^4 g)$ and the right hand side is the definition of the regular mass of the collection of zero area singularities $\Sigma$. □

## 7.2 Area profile functions

Above we extremized the ADM mass within a harmonic conformal class. Now we focus on optimizing another geometric quantity: the minimal enclosing area. For $A \geq 0$, define

$$\alpha(A) = \sup_{g' \in \overline{\mathcal{H}(g)}} \{\min(\Sigma, g') \; : \; |\Sigma|_{g'} \leq A\}, \quad \text{and}$$

$$\beta(A) = \inf_{g' \in \overline{\mathcal{H}(g)}} \{\min(\Sigma, g') \; : \; |\Sigma|_{g'} \leq A\}.$$

We refer to $\alpha$ and $\beta$ as "area profile functions." As with the case of the mass profile function $\mu(A)$, we remark that in the absence of an area constraint, there is no upper bound for the minimal enclosing area $\min(\Sigma, g')$ among metrics in the generalized harmonic conformal class $\overline{\mathcal{H}(g)}$. We explore the possibility of alternate definitions of $\alpha$ and $\beta$ in section 7.5.

By construction, the functions $\alpha$ and $\beta$ depend only the harmonic conformal class $\mathcal{H}(g)$. Both the supremum and infimum are attained (for $\alpha$ and $\beta$, respectively):

**Theorem 63.**

1. *For each $A \geq 0$, there exists $g' \in \overline{\mathcal{H}_A(g)}$ such that $\alpha(A) = \min(\Sigma, g')$.*
2. *The unique metric $\varphi^4 g \in \overline{\mathcal{H}_0(g)}$ satisfies $\min(\Sigma, \varphi^4 g) = 0$. In particular, $\beta(A) \equiv 0$.*

*Proof.* For $A > 0$, the first claim was proved in Theorem 32. The rest follows from the observation that the minimal enclosing area of $\Sigma$ with respect to $\varphi^4 g$ is zero. □

Whether the maximizer for $\alpha(A)$ is unique is unclear; the concavity argument used in Theorem 59 fails here, since the quantity we optimize, area, is a convex function of the conformal factor $u$.

Although $\beta$ is trivial, the following result shows that $\alpha$ is nontrivial. This is essential in our proof of the main result, Theorem 29 of Chapter 4.

**Proposition 64.**

1. *$\alpha(A)$ is a strictly increasing, Lipschitz continuous function that satisfies $0 < \alpha(A) \leq A$ for $A > 0$.*
2. *There exists $A > 0$ such that $\alpha(A) < A$.*

*In particular, given $\delta > 0$, there exists $A > 0$ such that $0 < A - \alpha(A) < \delta$.*

*Proof.*

*Step 1.* The bounds $0 < \alpha(A) \leq A$ follow immediately from the definition of $\alpha$. That $\alpha(\cdot)$ is nondecreasing follows from the definition of supremum. The proof that $\alpha(\cdot)$ is strictly increasing is essentially the same as that for $\mu(\cdot)$ and is omitted.

Now, let $0 < A < B$. By Theorem 32, there exists a maximizer $g_B = u_B^4 g \in \overline{\mathcal{H}_B(g)}$ for $\alpha(B)$. Let $f_B \in L^4(\Sigma)$ be the function determined by $u_B$, and choose $0 < \lambda < 1$ so that

$$\int_\Sigma (\lambda f_B)^4 \, dA_g = A,$$

namely, $\lambda = \left(\frac{A}{B}\right)^{1/4}$. Let $u_\lambda$ be the harmonic function associated to $\lambda f_B$, and let $g_\lambda = u_\lambda^4 g$. In particular, $g_\lambda$ measures the boundary area to be $A$, so that

$$\alpha(A) \geq \min(\Sigma, g_\lambda),$$

by the definition of $\alpha(A)$. Next,

$$\begin{aligned} u_\lambda(x) &= \varphi(x) + \int_\Sigma K(x,y)\lambda f_B(y) dA_g(y) \\ &= \varphi(x) + \lambda(u_B(x) - \varphi(x)) \\ &\geq \lambda u_B(x), \end{aligned}$$

since $\lambda < 1$. In particular, the minimal enclosing area for $g_\lambda$ is at least $\lambda^4$ times that for $g_B$:

$$\min(\Sigma, g_\lambda) \geq \lambda^4 \min(\Sigma, g_B).$$

But we have chosen $g_B$ so that its minimal enclosing area is precisely $\alpha(B)$. Stringing our inequalities together, we have

$$\alpha(A) \geq \lambda^4 \alpha(B) = \frac{A}{B}\alpha(B).$$

It follows that the function $\frac{\alpha(A)}{A}$ is non-increasing as a function of $A$. Combined with the fact that $\alpha(A)$ is increasing, one can easily show that $\alpha$ is Lipschitz continuous with Lipschitz constant at most 1.

*Step 2.* To show that $\alpha(A) \not\equiv A$, we will produce a number $A > 0$ and a surface $S$ enclosing $\Sigma$ that has the property that

$$|S|_{g'} < 0.99|\Sigma|_{g'}, \quad \text{for all metrics } g' \in \overline{\mathcal{H}_A(g)}. \tag{7.3}$$

This is sufficient, since then

$$\begin{aligned} \min(\Sigma, g') &\leq |S|_{g'} \\ &< 0.99|\Sigma|_{g'} = 0.99A, \end{aligned}$$

so taking the supremum over all $g' \in \overline{\mathcal{H}_A(g)}$ gives $\alpha(A) \leq 0.99A < A$. The argument given works with 0.99 replaced by any number between 0 and 1.

Without loss of generality, assume $|\Sigma|_g = 1$. In this proof, we use the following notation:

- $\psi$ is any harmonic function on $M$ tending to zero at infinity, with boundary data on $\Sigma$ given by a nonnegative function whose $L^4$ norm equals 1.
- $\lambda > 0$ is a constant.
- $\psi_\lambda = \lambda\psi + \varphi$; this is a positive harmonic function that tends to 1 at infinity and equals $\lambda\psi$ on $\Sigma$.
- $g_\lambda = \psi_\lambda^4 g$, a metric in $\overline{\mathcal{H}_A(g)}$, where $A = \lambda^4$.

The point is that any metric in the generalized harmonic conformal class $\overline{\mathcal{H}_A(g)}$ can be uniquely written in the form $g_\lambda = \psi_\lambda^4 g$ for some $\psi$ as above and $\lambda = A^{1/4}$.

Identifying $x$ with an asymptotically flat coordinate chart, any $\psi$ as above satisfies

$$\begin{aligned}
\psi(x) &= \int_\Sigma K(x,y)\psi(y)dA_g(y)\\
&\leq \frac{c}{r}\int_\Sigma \psi(y)dA_g(y)\\
&\leq \frac{c}{r}\left(\int_\Sigma \psi^4 dA_g\right)^{1/4}|\Sigma|_g^{3/4} \qquad \text{(by Hölder's inequality)}\\
&= \frac{c}{r}
\end{aligned}$$

where $r = |x|$ and $c > 0$ is a constant depending only on $(M, g)$. We have used the fact that the Poisson kernel is $O(r^{-1})$ in $x$ for large $r = |x|$, by asymptotic flatness.

The $g$-area of the coordinate sphere $\{|x| = r\}$ in $M$ is less than $5\pi r^2$ for $r$ sufficiently large. In particular, the quantity

$$\int_{\{|x|=r\}} \psi^4 dA_g \leq 5\pi r^2 \cdot \frac{c^4}{r^4}$$

can be made less than 0.9 by choosing $r$ sufficiently large, *independently of* $\psi$. Fix such a value of $r$, and let $S = \{|x| = r\}$.

By construction, the area of $\Sigma$ with respect to any $g_\lambda$ as above is $\lambda^4$. Let us compute the area of $S$ in the metric $g_\lambda$:

$$\begin{aligned}
|S|_{g_\lambda} &= \int_S \psi_\lambda^4 dA_g\\
&= \int_S (\lambda\psi + \varphi)^4 dA_g\\
&= \lambda^4\int_S \psi^4 + 4\lambda^3\int_S \psi^3\varphi + 6\lambda^2\int_S \psi^2\varphi^2 + 4\lambda\int_S \psi\varphi^3 + \int_S \varphi^4,
\end{aligned}$$

where we have dropped the "$dA_g$" notation for convenience. The above is a fourth degree polynomial in $\lambda$. The leading coefficient is less than 0.9 by our earlier choice of $r$, and the other coefficients can be bounded independently of $\psi$ via Hölder's inequality and the fact $\varphi \leq 1$. Then for some value of $\lambda$ sufficiently large,

$$|S|_{g_\lambda} \leq 0.99\lambda^4 = 0.99|\Sigma|_{g_\lambda}$$

for all choices of $\psi$. We have shown (7.3), since every $g'$ is of the form $g_\lambda$ for some $\lambda$ and $\psi$. □

**Corollary 65.** *Suppose that $\alpha(A) < A$ for some value of $A$. Then $\alpha(B) < B$ for all $B \geq A$.*

*Proof.* We showed above that $\frac{\alpha(A)}{A}$ is non-increasing as a function of $A$. □

## 7.3 An example with spherical symmetry

In this section we determine the functions $\mu$ and $\alpha$ for the manifold $M = \mathbb{R}^3 \setminus B(0,1)$ equipped with the flat metric $g = \delta$. Let $\Sigma = \partial M$, the round unit sphere. On $M$, the unique spherically symmetric harmonic function $u_A$ satisfying $u_A \to 1$ at infinity and $\int_\Sigma u_A^4 dA_g = A$ is given by

$$u_A(r) = 1 + \frac{\left(\frac{A}{4\pi}\right)^{1/4} - 1}{r}.$$

For now, we assume the maximizers for $\mu(A)$ and $\alpha(A)$ are spherically symmetric. (This is justified below for $\mu(A)$; for $\alpha(A)$ the justification uses Conjecture 41.) So we assume that $u_A^4\delta$ is the maximizer for both $\mu(A)$ and $\alpha(A)$. The ADM mass of $u_A^4\delta$ is computed from equation (A.9) as

$$\mu(A) = m_{\text{ADM}}(M, u_A^4\delta) = 2\left(\frac{A}{4\pi}\right)^{1/4} - 2,$$

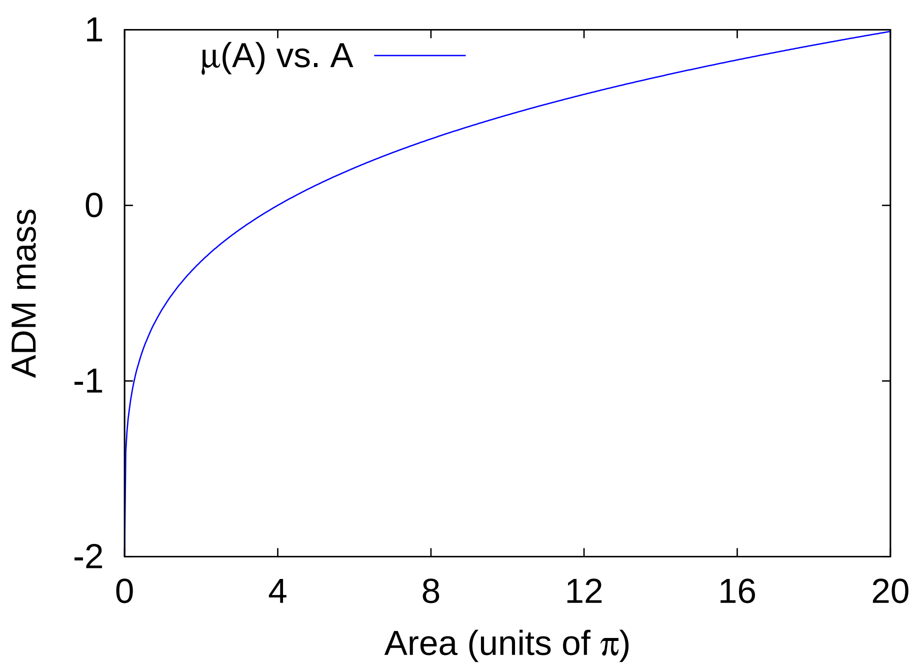
and the minimal enclosing area of $\Sigma$ for $u_A^4\delta$ is

$$\alpha(A) = \min(\Sigma, u_A^4\delta) = \begin{cases} A, & 0 < A \leq 64\pi \\ 64\pi\left(\left(\frac{A}{4\pi}\right)^{1/4} - 1\right)^2, & A \geq 64\pi. \end{cases}$$

To see the latter, note that $(M, u_A^4\delta)$ is a subset of the two-ended Schwarzschild manifold of positive mass. For $A < 64\pi$, $(M, u_A^4\delta)$ does not include the horizon (i.e., the round minimal 2-sphere). Moreover, $\Sigma$ has positive mean curvature and $M$ is foliated by positive mean curvature surfaces, so $\Sigma$ is its own outermost minimal area enclosure. For $A > 64\pi$ $(M, u_A^4\delta)$ includes the horizon (which is the outermost minimal area enclosure of the boundary); the area of this horizon is easily computed from the ADM mass of $(M, u_A^4\delta)$, since the Schwarzschild manifold gives equality in the RPI. That is, for $A \geq 64\pi$, $\mu(A) = \sqrt{\frac{\alpha(A)}{16\pi}}$. The value $A = 64\pi$ corresponds to the case in which $(M, u_A^4\delta)$ is precisely the Schwarzschild manifold whose boundary is the horizon.

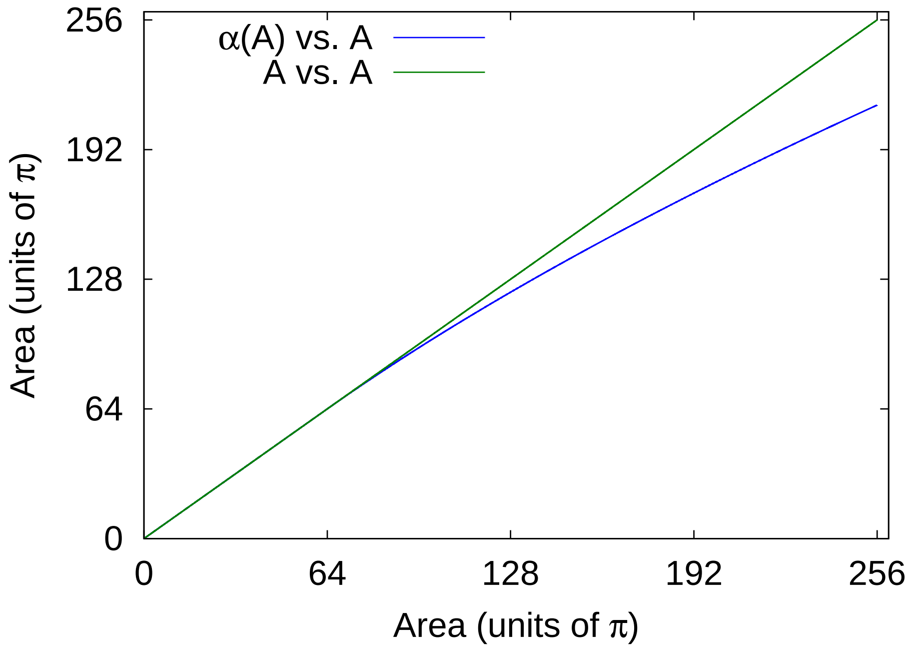
In figures 7.1 and 7.2 we have included plots of the functions $\mu$ and $\alpha$ for this example. Note the different scales on the $x$-and $y$-axes.

FIGURE 7.1: Plot of mass profile function




Above is a plot of $\mu(A)$ vs. $A$ for the harmonic conformal class of $\mathbb{R}^3$ minus a unit ball.

FIGURE 7.2: Plot of area profile function




Above is a plot of $\alpha(A)$ vs. $A$ (in blue) for the harmonic conformal class of $\mathbb{R}^3$ minus a unit ball, overlaid with a plot of $A$ vs. $A$ (in green) for comparison. The two functions agree precisely on the interval $[0, 64\pi]$. As explained in the text, this computation of $\alpha(A)$ depends on an unproven conjecture.

*Are the maximizers for $\mu(A)$ and $\alpha(A)$ are spherically symmetric?* First we deal with $\mu(A)$. By the concavity established by the proof of Theorem 59, it suffices to show that the ADM mass of $(M, u_A^4\delta)$ is stationary under perturbations: consider a path in the space $\overline{\mathcal{H}}(\delta)$ given by

$$g_t = (u_A + tv)^4\delta,$$

where $v$ is a bounded harmonic function with respect to the flat metric on $M$ that approaches zero at infinity. To fix the boundary area to first order at $t = 0$, require that $\int_\Sigma 4u_A^3 v dA_g = 0$, which, by spherical symmetry, is equivalent to $\int_\Sigma v dA_g = 0$. Expanding $v$ in spherical harmonics, we have

$$v(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{a_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi),$$

where $(\theta, \varphi)$ are spherical coordinates on $S^2 = \Sigma$, $a_{lm}$ are constants, and $\{Y_{lm}\}$ are the standard spherical harmonics on $S^2$. Since $\int_\Sigma Y_{lm}(\theta,\varphi)dA_g$ is zero for $l \geq 1$ and $\int_\Sigma v = 0$, we must have that $a_{00} = 0$ (i.e., $v$ carries no $1/r$ term). But by formula (A.9), we have

$$\frac{d}{dt} m_{\text{ADM}}(M, g_t) = -\frac{1}{2\pi} \int_{S_\infty} \nu(v) dA_g = 2a_{00} = 0.$$

This shows that $u_A^4\delta$ is a critical point; by strict concavity, $u_A^4\delta$ is the unique maximizer for $\mu(A)$.

Without a concavity property for $\alpha(A)$, the above type of argument shows only that the minimal enclosing area for $u_A^4\delta$ is stationary under perturbations. We resort to the following trick. By Proposition 64, there exists $A$ sufficiently large so that $\alpha(A) < A$. In Theorem 66 below, we show that by virtue of the Riemannian Penrose inequality and Conjecture 41,

$$\mu(A) \geq \sqrt{\frac{\alpha(A)}{16\pi}}.$$

By definition, $\alpha(A)$ is at least the minimal enclosing area of $u_A^4\delta$, which we computed above. So we have:

$$\mu(A) \geq \sqrt{\frac{\alpha(A)}{16\pi}} \geq \sqrt{\frac{\min(\Sigma, u_A^4\delta)}{16\pi}} = 2\left(\frac{A}{4\pi}\right)^{1/4} - 2.$$

Now, the right-hand side equals $\mu(A)$, so both inequalities are equalities. This shows that $\alpha(A)$ equals the minimal enclosing area of $u_A^4\delta$ whenever $\alpha(A) < A$. By the continuity of $\alpha(\cdot)$, we see $u_A^4\delta$ is a maximizer for $\alpha(A)$ for all $A > 64\pi$. For $A \leq 64\pi$, it is clear that $\alpha(A) = A$.

## 7.4 Relating the mass and area profile functions

In summary, we have defined four canonical functions associated to $\mathcal{H}(g)$, two of which are constant functions. The following result gives a relationship between the nonconstant functions $\mu$ and $\alpha$:

**Theorem 66.** *Suppose the assumptions of Theorem 29 hold (alternatively, assume that Conjecture 41 is true). Let $(M, g)$ be an asymptotically flat 3-manifold of nonnegative scalar curvature, with nonempty, smooth, compact boundary $\Sigma$. If $\alpha(A) < A$, then*

$$\mu(A) \geq \sqrt{\frac{\alpha(A)}{16\pi}}.$$

*In particular, $\mu(A)$ is positive for sufficiently large $A$. Moreover, one of the following holds:*

*(i) the graphs of $\mu(A)$ and $\sqrt{\frac{\alpha(A)}{16\pi}}$ do not intersect for any value of $A > 0$*

*(ii) the graphs of $\mu(A)$ and $\sqrt{\frac{\alpha(A)}{16\pi}}$ intersect at some $A_0$ for which $\alpha(A_0) = A_0$.*

*(iii) $\overline{\mathcal{H}(g)}$ contains a metric that makes $M$ into an appended Schwarzschild manifold (see below).*

Call $(M, g')$ an *appended Schwarzschild manifold* if the region exterior to $\tilde{\Sigma}_{g'}$ is isometric to a Schwarzschild manifold. Remark: if $\mu(0) < 0$, then the curves must intersect by the previous theorem and continuity.

*Proof.* Suppose $\alpha(A) < A$. By Theorem 29, given the two assumptions, a maximizer $g' \in \overline{\mathcal{H}_A(g)}$ for $\alpha(A)$ fulfills the hypotheses of the Riemannian Penrose inequality in the region exterior to $\tilde{\Sigma}_{g'}$. Then

$$\mu(A) \geq m_{\text{ADM}}(M, g') \geq \sqrt{\frac{|\tilde{\Sigma}_{g'}|_{g'}}{16\pi}} = \sqrt{\frac{\alpha(A)}{16\pi}}, \tag{7.4}$$

where we have used the definition of $\mu(A)$, the RPI, and the fact that $g'$ is a maximizer for $\alpha(A)$.

Next, suppose that for some (possibly different from above) value of $A > 0$,

$$\mu(A) = \sqrt{\frac{\alpha(A)}{16\pi}}.$$

(If no such $A$ exists, we are done.) If $\alpha(A) = A$, we are likewise done, so we assume $\alpha(A) < A$. Let $g' \in \overline{\mathcal{H}}_A(g)$ attain the supremum for $\alpha(A)$. Then equality holds throughout (7.4), so equality holds in the Riemannian Penrose inequality. It follows that $(M, g')$ minus the open region bounded by $\tilde{\Sigma}_{g'}$ is isometric to the Schwarzschild manifold of mass $\mu(A)$. In particular, $(M, g')$ is an appended Schwarzschild manifold.

□

## 7.5 Further remarks

*Alternative definitions:* First, we give some justification for the definitions that we have adopted in this chapter. We point out that without the parameter $A$ giving an upper bound on the boundary area, the suprema of $\min(\Sigma, g')$ and $m_{\text{ADM}}(\Sigma, g')$ in the generalized harmonic conformal class are both $+\infty$, while the infima are unchanged. Moreover, even with a lower bound for the boundary area in lieu of an upper bound, the definitions of $\beta$ and $\nu$ are still unchanged. To see this, let $A > 0$ and consider a sequence of continuous, nonnegative functions $\{f_n\}$ on $\Sigma$ such that the supports of $\{f_n\}$ shrink down to a point and $\int_\Sigma f_n^4 dA_g = A$ for all $n$. By Hölder's inequality, the sequence $\{f_n\}$ converges weakly to the zero function on $\Sigma$. Then the associated harmonic functions $\{u_n\}$ converge to $\varphi$. Since $\varphi^4 g$ is the global minimizer in $\overline{\mathcal{H}(g)}$ for both minimal enclosing area and ADM mass, it is not difficult to see that $\beta$ and $\nu$ are unchanged even if modified to have a lower bound on boundary area.

Next, suppose $(M, g)$ has nonnegative scalar curvature. We could define the functions $\alpha, \beta, \mu, \nu$ by optimizing over the class of conformal metrics that have nonnegative scalar curvature. However, it is not difficult to see that under this modification, $\beta$ and $\nu$ are unchanged, and $\alpha(A) \equiv A$, and $\mu(A) \equiv +\infty$. Thus, optimizing within the (generalized) harmonic conformal class is evidently more interesting. Moreover, the definitions presented in this chapter make sense without assumptions on the sign of scalar curvature.

Finally, we address the issue of whether $\alpha$ and $\mu$ could be defined by considering conformal metrics $u^4 g$ such that $u$ is $g$-harmonic and tends to a positive constant $k$ at infinity. While such $u^4 g$ does not belong to the harmonic conformal class of $g$ as we have defined it, we could just as easily consider the rescaled metric $k^4 g$ with harmonic function $u/k$ that tends to one at infinity. In other words, no new information is gleaned by dropping the normalization $u \to 1$ at infinity.

*Conjectures and questions:* We state a few conjectures regarding the functions $\alpha$ and $\mu$. Much of the motivation comes from the spherically symmetric examples.

1. For all $A$ sufficiently small, $\alpha(A) = A$.
2. Assuming nonnegative scalar curvature, if $\mu(0) > 0$, then the graphs of $\mu$ and $\sqrt{\alpha/16\pi}$ never intersect. In general, they intersect at most once.
3. The asymptotic behavior of $\alpha(A)$ and $\mu(A)$ is the same as in the spherically symmetric example – a power law of $A^{1/2}$ and $A^{1/4}$, respectively.

We close this chapter with some questions:

1. Is the maximizer for $\alpha(A)$ unique?
2. To what extent do the invariants $\mu$ and $\alpha$ determine the harmonic conformal class?
3. Can other interesting invariants be found through an analogous process of optimizing geometric quantities?

# 8

# Scalar Curvature Lower Bounds for the ADM Mass

The previous chapters have centered on the study of asymptotically flat manifolds with boundary. Here, we digress to asymptotically flat manifolds without boundary and will eventually restrict to manifolds that are *harmonically flat at infinity.* From our perspective the main significance of dealing with manifolds without boundary is that the harmonic conformal class is trivial, as harmonic functions approaching one at infinity are necessarily constant by the maximum principle. To remain consistent with the other chapters, we treat only the case of three dimensional manifolds, although analogous results hold in all dimensions $n$ for which the positive mass theorem is known, $3 \leq n \leq 7$.

## 8.1 Motivation from the positive mass theorem

Under the identification of scalar curvature $R_g$ with $16\pi$ times the energy density (c.f. section 1.1), it is tempting to naïvely define the total mass of $(M, g)$ by $\frac{1}{16\pi}$ times the integral of scalar curvature. However, this integral can both under- and overestimate the ADM mass, which is the well-accepted definition of total mass. The integral of scalar curvature ignores gravitational potential energy, the energy contributed by gravitational waves, and contributions from black holes (outermost minimal surfaces). Furthermore, $\int_M R_g$ depends on more than the geometry at infinity, which is undesirable for a definition of mass. Nevertheless, there are some interesting connections between the integral of scalar curvature and the ADM mass, such as the fact that they are simultaneously finite [5]. Bartnik also shows that for metrics $g$ on $\mathbb{R}^3$ of nonnegative scalar curvature that are close to the flat metric in a suitable sense, the following inequality holds [5]:

$$m_{\mathrm{ADM}}(M,g) \geq \frac{1}{16\pi} \int_{\mathbb{R}^3} \left( \frac{1}{8} |\partial g|^2 + R_g \right) dx, \tag{8.1}$$

where $dx$ denotes Lebesgue measure for the flat metric and

$$|\partial g|^2 = g^{ij} g^{kl} g^{pq} \partial_i g_{kp} \partial_j g_{lq},$$

in an appropriate choice of asymptotically flat coordinates for $g$. We also remark that Witten's proof of the positive mass theorem yields an exact expression for the ADM mass in terms of a certain integral involving scalar curvature and a particular choice of spinor on $M$ [41].

The following observation is a known consequence of the positive mass theorem [26] and gives a relationship between the ADM mass and a *weighted* integral of scalar curvature, similar in spirit to (8.1).

**Proposition 67.** *Let $(M,g)$ be a complete, asymptotically flat 3-manifold without boundary. Assume there exists a solution $u > 0$ to*

$$\Delta_g u = \frac{1}{8} R_g u, \qquad u \to 1 \text{ at infinity.} \tag{8.2}$$

*Then*

$$m_{ADM}(M,g) \geq \frac{1}{16\pi} \int_M \left( 8|\nabla u|_g^2 + R_g u^2 \right) dV_g, \tag{8.3}$$

*with equality holding if and only if $(M,g)$ is conformal to $(\mathbb{R}^3, \delta)$.*

In this chapter, we use the notation $dV_g$ for $d\mathcal{H}^3_g$.

*Proof.* The complete Riemannian metric $u^4 g$ has zero scalar curvature, by formula (A.7), and is asymptotically flat. To see asymptotic flatness, either assume $R_g$ vanishes outside a compact set (so that $u$ is harmonic outside of a compact set), or use weighted elliptic estimates to show sufficient decay of $u$ [5]. Applying the positive mass theorem to $(M, u^4 g)$, we have

$$\begin{aligned} 0 &\leq m_{\text{ADM}}(M, u^4 g) && \text{(positive mass theorem)} \\ &= m_{\text{ADM}}(M, g) - \frac{1}{2\pi} \int_{S_\infty} \nu(u) dA_g. && \text{(formula A.9)} \end{aligned}$$

Next, we evaluate this flux integral in terms of the geometry of $(M, g)$. Let $B_r$ be the region in $M$ bounded by the coordinate sphere $\{|x| = r\}$. Then

$$\begin{aligned} \lim_{r\to\infty} \int_{\{|x|=r\}} \nu(u) dA_g &= \lim_{r\to\infty} \int_{\{|x|=r\}} u\nu(u) dA_g && (u \to 1 \text{ at infinity}) \\ &= \lim_{r\to\infty} \int_{B_r} \text{div}_g(u \nabla u) dV_g && \text{(divergence theorem)} \\ &= \int_M (|\nabla u|_g^2 + u \Delta_g u) dV_g && \text{(expanding)} \\ &= \int_M \left( |\nabla u|_g^2 + \frac{1}{8} R_g u^2 \right) dV_g && \text{(substituting (8.2))} \end{aligned}$$

On the second to last line, note that the integrand is indeed integrable on $M$, since $|\nabla u|_g^2$ is $O(r^{-4})$ and $u\Delta_g u$ is $O(r^{-q})$ for $q > 3$. Rearranging, we get the desired estimate for the ADM mass of $(M, g)$. If equality holds, then by the positive mass theorem $(M, u^4 g)$ is isometric to $(\mathbb{R}^3, \delta)$, which completes the proof. □

In the above, there is no requirement that the scalar curvature is nonnegative. However, in the case of nonnegative, not identically zero, scalar curvature, Proposition 67 provides a sharper estimate than the positive mass theorem alone.

In general, positive solutions to $\Delta_g u = \frac{1}{8} R_g u$ need not exist, particularly when the scalar curvature becomes "too negative." A natural conjecture, however, is:

**Conjecture 68.** *If* $(M, g)$ *is a complete, asymptotically flat 3-manifold without boundary, then*

$$m_{ADM}(M, g) \geq \frac{1}{16\pi} \inf_{\phi} \left\{ \int_M \left( 8|\nabla \phi|_g^2 + R_g \phi^2 \right) dV_g \right\} \tag{8.4}$$

*where the infimum is taken over all smooth, positive functions* $\phi$ *that tend to 1 at infinity. Moreover, equality holds in (8.4) if and only if* $(M, g)$ *is conformal to flat Euclidean 3-space* $(\mathbb{R}^3, \delta)$.

We point out that the above conjecture would become trivial if the following statement were known: "if the infimum (8.4) is finite, then this infimum is achieved by a positive function." Indeed, in the next section we shall see that the existence of a critical point of $\mathcal{F}_g$ immediately implies the hypotheses of Proposition 67. The fact that this statement is not known is what makes the problem interesting. We remark that Conjecture 68 is known in the case of nonnegative scalar curvature [26], and more generally in the case that the $L^{3/2}$ norm of the negative part of the scalar curvature is sufficiently small; see section 8.4.

The primary purpose of this chapter is to prove Conjecture 68 for the class of harmonically flat manifolds (Theorem 71 below). We demonstrate that such manifolds admit a canonical *conformal compactification*. We then consider the infimum (8.4) on the compactified manifold; the sign of the Yamabe invariant determines whether the infimum is attained on the original manifold, or whether it is $-\infty$.

Although at first glance it may seem undesirable that the mass lower bound can trivially be $-\infty$, a second thought suggests that such a phenomenon is to be expected. Indeed, under the philosophy that negative scalar curvature metrics are generic [25], the presence of too much negative scalar curvature does not constrain the asymptotics of the metric sufficiently to expect to deduce anything at all regarding the ADM mass. In other words, without some hypothesis on scalar curvature, there simply is not enough information to bound the ADM mass from below.

## 8.2 Preliminaries and setup

*Euler–Lagrange formulation:* Let $(M, g)$ be a complete, asymptotically flat 3-manifold without boundary. Recall the definition of the *asymptotically flat conformal class* of $g$, denoted $[g]_{AF}$, from section 1.5. For smooth functions $\phi > 0$ satisfying $\phi^4 g \in [g]_{AF}$, define

$$\mathcal{F}_g(\phi) = \frac{1}{16\pi} \int_M \left(8|\nabla \phi|_g^2 + R_g \phi^2\right) dV_g. \tag{8.5}$$

The requirement that $\phi^4 g \in [g]_{AF}$ ensures $\phi \to 1$ at infinity with sufficient decay for $\mathcal{F}_g(\phi)$ to be finite. A straightforward computation shows that the Euler–Lagrange equation for a critical point $u$ of $\mathcal{F}_g$ is described precisely by the second-order, linear, elliptic PDE (8.2). As remarked in the proof of Proposition 67, if $u$ is a positive solution to (8.2), then $u^4 g \in [g]_{AF}$.

*Conformal transformation law for $\mathcal{F}_g$:* We now determine how the functional $\mathcal{F}_g$ defined in (8.5) behaves under conformal changes of the metric. First, observe that for a smooth function $\phi$, the quantities $|\nabla \phi|_g$ and $|d\phi|_g$ area equal, where $\nabla \phi$ is the gradient of $\phi$ with respect to (the Levi–Civita connection for) $g$. This observation simplifies our calculations, since $d\phi$ is independent of the metric or connection.

Consider the conformal metric $u^4 g \in [g]_{AF}$. Note that the domains of $\mathcal{F}_g$ and $\mathcal{F}_{u^4 g}$ agree, from the definition of the asymptotically flat conformal class. For $\phi$ belonging to this domain, we have

$$16\pi\mathcal{F}_{u^4g}(\phi) = \int_M \left(8|d\phi|^2_{u^4g} + R_{u^4g}\phi^2\right) dV_{u^4g}$$
$$= \int_M \left(8|d\phi|^2_g u^{-4} + u^{-5}(-8\Delta_g u + R_g u)\phi^2\right) u^6 dV_g$$
$$= \int_M \left(8|d\phi|^2_g u^2 - 8\phi^2 u\Delta_g u + R_g u^2\phi^2\right) dV_g,$$

where, on the second line, we have used formulas (A.2), (A.7), and (A.5). Continuing the string of equalities,

$$16\pi\mathcal{F}_{u^4g}(\phi) = \int_M \left(8|d\phi|^2_g u^2 + 8\phi^2|du|^2_g + 16u\phi\left\langle d\phi, du\right\rangle_g + R_g u^2\phi^2\right) dV_g$$
$$- 8\int_M \operatorname{div}_g(\phi^2 u\nabla u) dV_g$$
$$= \int_M \left(8|d(u\phi)|^2_g + R_g(u\phi)^2\right) dV_g - 8\int_{S_\infty} \phi^2 u\nu(u) dA_g$$
$$= 16\pi\mathcal{F}_g(u\phi) - 8\int_{S_\infty} \nu(u) dA_g,$$

having used the divergence theorem and the fact that $\phi$ and $u$ tend to one at infinity. To summarize,

$$\mathcal{F}_{u^4g}(\phi) = \mathcal{F}_g(u\phi) - \frac{1}{2\pi}\int_{S_\infty} \nu(u) dA_g. \qquad (8.6)$$

*Harmonically flat manifolds:* We recall from the literature a special class of asymptotically flat manifolds [35], [34], [7].

**Definition 69.** *A smooth, connected* 3*-manifold* $(M, g)$ *is* ***harmonically flat at infinity*** *(or* ***harmonically flat****) if there exists a compact subset* $K \subset M$ *and a positive function* $v : \mathbb{R}^3 \setminus \overline{B} \to \mathbb{R}$*, harmonic with respect to the flat metric* $\delta$ *and approaching a positive constant at infinity, such that* $(M \setminus K, g)$ *is isometric to* $(\mathbb{R}^3 \setminus \overline{B}, v^4\delta)$*.*

By standard theory of harmonic functions on Euclidean space, the harmonic function $v$ has an expansion at infinity of the form

$$v(x) = a + \frac{b}{|x|} + O\left(|x|^{-2}\right), \tag{8.7}$$

for constants $a > 0$ and $b$, with successively higher decay for higher derivatives of $v$. Moreover, from formula (A.7), we see that manifolds $(M, g)$ that are harmonically flat at infinity have vanishing scalar curvature outside of a compact set. These facts show that $(M, g)$ is asymptotically flat as in Definition 1. While not needed, we recall that the ADM mass of $(M, g)$ as above equals $2ab$.

In general, the utility of harmonically flat manifolds is that they can approximate asymptotically flat manifolds in a desirable way [35], [34], [7]. For our purposes, their utility lies in the fact that they can be conformally compactified.

## 8.3 Conformal compactification

For the rest of this chapter, assume $(M, g)$ is harmonically flat at infinity, without boundary. In this section we show how to "conformally compactify" $(M, g)$ in a canonical way. Note that for an arbitrary asymptotically flat manifold, there are known obstructions to conformal compactifications in general [23]; therefore the assumption of harmonic flatness at infinity is crucial.

**Proposition 70.** *Given $(M, g)$ as above, there exists a smooth, closed 3-manifold $\overline{M}$ with smooth Riemannian metric $\overline{g}$ and a point $p_\infty \in \overline{M}$ such that $(\overline{M} \setminus p_\infty, \overline{g})$ is conformal to $(M, g)$. Moreover, the manifold $\overline{M}$ is unique up to diffeomorphism, and the metric $\overline{g}$ is unique up to conformal transformations.*

*Proof.* By harmonic flatness at infinity, there exists a compact set $K \subset M$, a closed ball $\overline{B} \subset \mathbb{R}^3$, a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^3 \setminus \overline{B}$, and a harmonic function $v : \mathbb{R}^3 \setminus \overline{B} \to \mathbb{R}$ such that $g = (v \circ \Phi)^4 \Phi^* \delta$ on $M \setminus K$.

Recall that inverse stereographic projection on $\mathbb{R}^3$ induces a conformal factor carrying the flat metric to a round (constant positive sectional curvature) metric on a punctured 3-sphere. In particular, choosing

$$\sigma(r) = \frac{1}{(16 + r^2)^{1/2}}$$

makes $(\mathbb{R}^3, \sigma^4 \delta)$ isometric to the unit 3-sphere minus a point.

Let $\Gamma : M \setminus K \to \mathbb{R}^+$ be the function $\dfrac{\sigma \circ \Phi}{v \circ \Phi}$. Extend $\Gamma$ to a smooth, positive function on $M$. Then the Riemannian metric $\Gamma^4 g$ on $M$ is, outside a compact set, isometric to a spherical cap with a deleted point.

By setting $\overline{M} = M \cup \{p_\infty\}$ so as to fill in the deleted point on the sphere, we see that $\overline{M}$ naturally becomes a smooth manifold, and the metric $\Gamma^4 g$ extends smoothly to $\overline{M}$. It is clear that $\overline{M}$ is compact, without boundary, and $(\overline{M} \setminus p_\infty, g)$ is conformal to $(M, g)$. Uniqueness of the topology and conformal structure of $(\overline{M}, \overline{g})$ are clear from the construction. □

As a consequence of uniqueness, it makes sense to refer to "the" conformal compactification of $(M, g)$. We will use the notation $M$ and $\overline{M} \setminus p_\infty$ interchangeably (which is justified, since these manifolds are diffeomorphic).

Recall the Yamabe problem [42]: on a closed Riemannian manifold, find a conformal metric of constant scalar curvature. Such a metric can be found by minimizing within the conformal class the *Yamabe energy*, namely the integral of scalar curvature divided by an appropriate power of volume. By the work of Yamabe [42], Trudinger [39], Aubin [3], and Schoen [33], a smooth metric attaining the infimum of Yamabe energy exists in every conformal class. The Yamabe energy of such a metric is called the *Yamabe invariant* of the conformal class.

We now state the main result of the present chapter. Recall that $[g]_{AF}$ is the asymptotically flat conformal class of $g$, defined in section 1.5.

**Theorem 71.** *Let $(M, g)$ be a 3-manifold without boundary that is harmonically flat at infinity. Then*

$$m_{\mathrm{ADM}}(M, g) \geq \frac{1}{16\pi} \inf_{\phi} \left\{ \int_M \left(8|\nabla\phi|_g^2 + R_g\phi^2\right) dV_g \right\}, \tag{8.8}$$

*where the infimum is taken over all functions $\phi$ such that $\phi^4 g \in [g]_{AF}$. If equality holds, then $(M, g)$ is conformal to $(\mathbb{R}^3, \delta)$. Let $Y$ be the Yamabe invariant of the conformal compactification of $(M, g)$.*

1. *If $Y > 0$, the infimum in (8.8) is finite, achieved by some unique $u^4 g \in [g]_{AF}$; moreover, $u^4 g$ is the only critical point of $\mathcal{F}_g$.*
2. *If $Y \leq 0$, the infimum (8.8) is equal to $-\infty$ (and is therefore not achieved), and $\mathcal{F}_g$ has no critical points.*

Note that equality in (8.8) is possible only in the $Y > 0$ case, since the ADM mass is always finite under our definition of asymptotic flatness [5].

*Notation:* For the remainder of this chapter, $(M, g)$ is a manifold without boundary that is harmonically flat at infinity. Applying the solution of the Yamabe problem to the conformal compactification $(\overline{M}, \overline{g})$, we can and will assume that the metric $\overline{g}$ is of unit volume with constant scalar curvature $Y$ equal to the Yamabe invariant of $(\overline{M}, \overline{g})$. Let $\alpha > 0$ be the function defined on $M$ for which $\overline{g} = \alpha^4 g$ on $TM$. It is clear that $\alpha \to 0$ at infinity (or, equivalently, near $p_\infty$). From now on, we drop subscripts for the geometric quantities $R$, $\Delta$, $dV$, etc. for $g$, and use $\bar{R}, \bar{\Delta}, \overline{dV}$, etc. for the corresponding quantities for $\overline{g}$.

*Proof of Theorem 71; Yamabe positive case.* Suppose the closed manifold $(\overline{M}, \overline{g})$ has Yamabe constant $Y > 0$. Let $\bar{\mathcal{G}}(x)$ denote the Green's function of the conformal Laplacian

$$\bar{L} = -8\bar{\Delta} + \bar{R}$$

of the metric $\overline{g}$ based at the point $p_\infty$ [33], [21], and [24].

The Green's function $\bar{\mathcal{G}}(x)$ is smooth (by elliptic theory) and positive (by the maximum principle) on $\overline{M} \setminus p_\infty$ and blows up at $p_\infty$. Moreover, the conformal metric $\bar{\mathcal{G}}^4 \overline{g}$ is complete, asymptotically flat, and has zero scalar curvature. (Asymptotic flatness is automatic from this construction for manifolds of dimension 3, 4, and 5; in higher dimensions, the local conformal flatness of $\overline{g}$ near $p_\infty$ implies asymptotic flatness of $\bar{\mathcal{G}}^4 \overline{g}$ [33], [21].) In particular, there exists a metric $u^4 g$ in the asymptotically flat conformal class $[g]_{AF}$ that has zero scalar curvature.

Recall from equation (8.6) that $\mathcal{F}_g$ and $\mathcal{F}_{u^4 g}$ differ by an additive constant. Since $u^4 g$ is scalar-flat, the functional $\mathcal{F}_{u^4 g}$ reduces to the standard Dirichlet energy:

$$\mathcal{F}_{u^4 g}(\phi) = \frac{1}{2\pi} \int_M |d\phi|^2_{u^4 g} dV_{u^4 g}.$$

The infimum for $\mathcal{F}_{u^4 g}$ is achieved by $\phi \equiv 1$, and so from (8.6) we see that the infimum for $\mathcal{F}_g$ is achieved by $u$. Moreover, $\phi \equiv 1$ is the unique critical point of $\mathcal{F}_{u^4 g}$, so $u$ is the unique critical point of $\mathcal{F}_g$. In particular, $u$ solves (8.2), so the mass estimate (8.8) and case of equality follow from Proposition 67.

□

Technically we did not require a solution to the Yamabe problem for the case $Y > 0$, but the above use of the Green's function of the conformal Laplacian was crucial in Schoen's proof of the positive case of the Yamabe problem [33].

Before continuing to the Yamabe nonpositive cases, we make the following observation to write $\mathcal{F}_g$ in terms of $\overline{g}$. (It is tempting to apply the derivation of (8.6) with $\alpha$ in place of $u$ to relate $\mathcal{F}_g$ to some functional on $(\overline{M}, \overline{g})$. But this is not justified, since the flux integral term

$$8 \int_{S_\infty} \phi^2 \alpha \nu(\alpha) dA_g$$

is necessarily infinite.)

**Lemma 72.** *Suppose $\phi$ belongs to the domain of $\mathcal{F}_g$, i.e. $\phi^4 g \in [g]_{AF}$. Then if we set $\bar{\phi} = \phi/\alpha$,*

$$\int_M \left(8|d\phi|_g^2 + R\phi^2\right) dV = \int_M \left(-8\bar{\phi}\bar{\Delta}\bar{\phi} + \bar{R}\bar{\phi}^2\right) \overline{dV} + \int_{S_\infty} \nu(\phi) dA. \tag{8.9}$$

*Proof.* We prove this formula by working backwards. For $B_r$ equal to the region in $M$ bounded by the coordinate sphere of radius $r$, using the conformal relation $\overline{g} = \alpha^4 g$,

$$\begin{aligned}\int_{B_r} \left(-8\bar{\phi}\bar{\Delta}\bar{\phi} + \bar{R}\bar{\phi}^2\right) \overline{dV} &= \int_{B_r} \left(-8\bar{\phi}\bar{\Delta}\bar{\phi} + \alpha^{-5}\left(-8\Delta\alpha + R\alpha\right)\bar{\phi}^2\right) \alpha^6 dV \\ &= \int_{B_r} \left(-8\alpha^6\bar{\phi}\bar{\Delta}\bar{\phi} - 8\alpha\bar{\phi}^2\Delta\alpha + R\alpha^2\bar{\phi}^2\right) dV.\end{aligned}$$

From formula (A.6),

$$\Delta(\alpha\bar{\phi}) = \alpha^5\bar{\Delta}\bar{\phi} + \bar{\phi}\Delta\alpha,$$

which we use to continue the above string of equalities:

$$\begin{aligned}&= \int_{B_r} \left(-8\alpha\bar{\phi}\Delta(\alpha\bar{\phi}) + 8\alpha\bar{\phi}^2\Delta\alpha - 8\alpha\bar{\phi}^2\Delta\alpha + R\alpha^2\bar{\phi}^2\right) dV \\ &= \int_{B_r} \left(-8\phi\Delta\phi + R\phi^2\right) dV \\ &= \int_{B_r} \left(-8\mathrm{div}(\phi d\phi) + 8|d\phi|_g^2 + R\phi^2\right) dV \\ &= \int_{B_r} \left(8|d\phi|_g^2 + R\phi^2\right) dV - 8\int_{\{|x|=r\}} \phi\nu(\phi) dA. \end{aligned} \tag{8.10}$$

Both terms on the right-hand side of (8.10) have a limit as $r \to \infty$. In particular, the limit as $r \to \infty$ of the left-hand side exists, proving the desired identity. □

Before proving the remaining cases of Theorem 71, we introduce a technical lemma that will simplify the argument.

**Lemma 73.** *Suppose it is known that the infimum of $\mathcal{F}_g$ is $-\infty$ when taken over the class of smooth functions $\phi > 0$ satisfying the decay conditions*

$$\begin{cases} \phi(x) & = 1 + O(|x|^{-1}) \\ |\nabla\phi(x)|_g & = O(|x|^{-2}) \end{cases} \tag{8.11}$$

*for $i = 1, 2, 3$. Then the infimum of $\mathcal{F}_g$ is also $-\infty$ when taken over the class of $\phi$ such that $\phi^4 g \in [g]_{AF}$.*

As a result, we may without loss of generality prove the $Y \leq 0$ cases of Theorem 71 by considering test functions with decay (8.11) at infinity. Note that the decay on the gradient of $\phi$ is equivalent to assuming $\partial_i\phi(x) = O(|x|^{-2})$ for $i = 1, 2, 3$. Also observe that the above decay conditions are neither weaker nor stronger than those given by equations (1.3).

*Proof.* Fix a large number $\Lambda > 0$. By hypothesis, there exists a smooth, positive function $\phi$ satisfying (8.11) such that $\mathcal{F}_g(\phi) \leq -2\Lambda$. Choose $r_0$ large so that for all $r \geq r_0$,

$$\int_{B_r} \left(8|\nabla\phi|_g^2 + R\phi^2\right) dV \leq -\Lambda.$$

By (8.11), we may assume that on $M \setminus B_{r_0}$, $\phi$ satisfies the conditions

$$|\phi - 1| \leq \frac{c_1}{r}$$

$$|\nabla\phi|_g \leq \frac{c_2}{r^2}$$

for constants $c_1, c_2$ independent of $r$. In particular, there exists a smooth, positive function $\tilde{\phi}_r$ on $M$ satisfying

$$\tilde{\phi}_r = \begin{cases} \phi, & \text{on } B_r \\ 1, & \text{on } M \setminus B_{2r} \end{cases}$$

$$|\tilde{\phi}_r - 1| \leq \frac{c_1}{r} \text{ on } B_{2r} \setminus B_r$$

$$|\nabla \tilde{\phi}_r|_g \leq \frac{2c_2}{r^2} \text{ on } B_{2r} \setminus B_r.$$

We think of $\tilde{\phi}_r$ as an approximation to $\phi$ that is identically 1 outside a compact set. Now, compute $\mathcal{F}_g(\tilde{\phi}_r)$ by evaluating the integrand over the disjoint regions $B_r$, $B_{2r} \setminus B_r$ and $M \setminus B_{2r}$. First,

$$\int_{B_r} \left(8|\nabla \tilde{\phi}_r|_g^2 + R\tilde{\phi}_r^2\right) dV \leq -\Lambda,$$

since $\tilde{\phi}_r = \phi$ on $B_r$. Next,

$$\int_{B_{2r}\setminus B_r} \left(8|\nabla \tilde{\phi}_r|_g^2 + R\tilde{\phi}_r^2\right) dV \leq \frac{32c_2^2}{r^4}\mathcal{H}_g^3(B_{2r} \setminus B_r) + 2\int_{B_{2r}\setminus B_r} |R| dV.$$

The above can be made less than $\Lambda^{-1}$ by choosing $r$ sufficiently large, since $\mathcal{H}_g^3(B_{2r})$ is $O(r^3)$ and $R$ is integrable. Finally, since $\tilde{\phi}_r$ is constant outside $B_{2r}$, the quantity

$$\int_{M\setminus B_{2r}} \left(8|\nabla \tilde{\phi}_r|_g^2 + R\tilde{\phi}_r^2\right) dV \leq \int_M 2|R| dV$$

can also be made less than $\Lambda^{-1}$ since $R$ is integrable. Thus,

$$\mathcal{F}_g(\tilde{\phi}_r) \leq -\Lambda + 2\Lambda^{-1}.$$

In particular, $\mathcal{F}_g$ takes on arbitrarily negative values on smooth, positive functions that are identically 1 outside a compact set; such $\phi$ certainly satisfy $\phi^4 g \in [g]_{AF}$. $\square$

Now we prove the remainder of Theorem 71.

*Proof of Theorem 71; Yamabe negative case.* Assume $Y < 0$. We show the infimum of $\mathcal{F}_g(\phi)$ over the class of $\phi$ satisfying (8.11) is $-\infty$, which is sufficient by Lemma 73. Choose the test function

$$\phi = 1 + \Lambda\alpha$$

for a constant $\Lambda > 0$. We must show that $\phi$ has appropriate decay at infinity, i.e., $\alpha(x)$ is $O(r^{-1})$ and $|\nabla\alpha|_g$ is $O(r^{-2})$ for $r = |x|$, where $(x^i)$ is an asymptotically flat coordinate system. For $r$ sufficiently large we may write $\alpha$ as

$$\alpha(x) = \frac{\psi(x)}{v(x)(16+r^2)^{1/2}},$$

where $\psi$ is a smooth function on $\overline{M}$. This immediately follows from the construction of the conformal compactification in Proposition 70, where $\psi$ is a solution to the Yamabe problem. Since $\psi$ and $v$ are bounded at infinity, we have $\alpha(x)$ is $O(r^{-1})$. Now,

$$|\nabla\alpha|_g = \underbrace{\frac{|\nabla\psi|_g}{(16+r^2)^{1/2}v}} - \underbrace{\frac{r\psi}{(16+r^2)^{3/2}v}} - \underbrace{\frac{\psi|\nabla v|_g}{(16+r^2)^{1/2}v^2}},$$

where, without loss of generality, we have identified the $g$-gradient with the $\partial_r$ coordinate derivative for the function $(16+r^2)^{-1/2}$. By the conformal relation $\overline{g} = \alpha^4 g$, the first underbraced term can be rewritten as $\alpha^2|d\psi|_{\overline{g}}(1+r^2)^{-1/2}v(x)^{-1}$, which is $O(r^{-3})$ since $|d\psi|_{\overline{g}}$ is bounded as $r \to \infty$. The second and third terms are seen to be $O(r^{-2})$ and $O(r^{-3})$, respectively. It follows that $\phi$ satisfies (8.11).

Moving on, from equation (8.9), if we substitute $\bar{\phi} = \phi/\alpha = 1/\alpha + \Lambda$,

$$\int_M \left(8|\nabla\phi|_g^2 + R\phi^2\right) dV = \int_M \left(-8\left(\alpha^{-1}+\Lambda\right)\bar{\Delta}\alpha^{-1} + Y(\alpha^{-1}+\Lambda)^2\right)\overline{dV}$$
$$+ \Lambda\int_{S_\infty} \nu(\alpha) dA,$$

since $\overline{g}$ has constant scalar curvature equal to $Y$. Now, the right-hand-side is a polynomial in $\Lambda$ with leading term $Y\Lambda^2\int_M \overline{dV}$. Since $Y < 0$, the right-hand-side can be made arbitrarily negative by choosing $\Lambda$ sufficiently large. It follows that the infimum (8.8) is $-\infty$.

If $u^4 g \in [g]_{AF}$ is a critical point of $\mathcal{F}_g$, then $u^4 g$ has zero scalar curvature, which implies $\mathcal{F}_{u^4 g}$ is bounded below by zero. By equation (8.6), $\mathcal{F}_g$ is bounded below by some constant, a contradiction. $\square$

*Proof of Theorem 71; Yamabe zero case.* Assume $Y = 0$; note that the argument given for the $Y < 0$ case no longer applies, except the proof that $\mathcal{F}_g$ has no critical points. For $r$ large, note that the coordinate sphere $\{|x| = r\}$ in $M$ corresponds to a small sphere about $p_\infty$ in $\overline{M}$ (not necessarily a metric sphere). Fix some $r_0$ large. Then on the complement of $B_{r_0}$ in $\overline{M}$, there exists a nonnegative smooth function $\overline{G}$, harmonic with respect to $\overline{g}$, vanishing on $\{x = r_0\}$ and blowing up at $p_\infty$. ($\overline{G}$ is a Green's function for the Laplacian of $\overline{g}$ on a domain; see Theorem 4.17 of Aubin [4].) Extend $\overline{G}$ by zero over the rest of $\overline{M}$, producing a Lipschitz function. For $\Lambda > 0$, consider

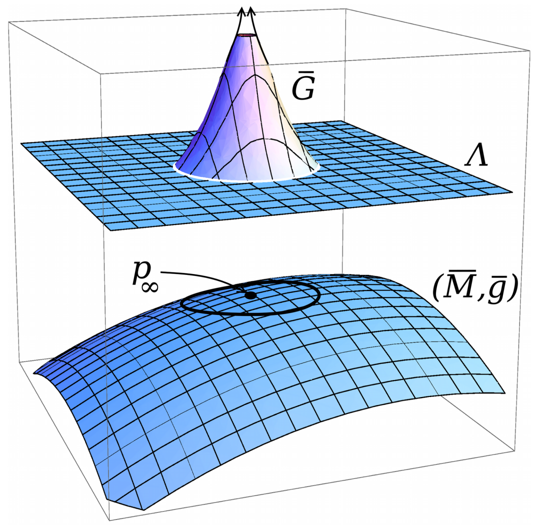
$$\bar{\phi}(x) = \max(\overline{G}(x), \Lambda)$$

a Lipschitz function that is weakly subharmonic with respect to $\overline{g}$. This construction is demonstrated in figure 8.1.

It is possible to perturb $\bar{\phi}$ on an annulus about $p_\infty$ to produce a smooth function (of the same name, say) satisfying $\bar{\Delta}\bar{\phi} \geq 0$ and $\bar{\phi} \geq \Lambda$. Let $\phi = \bar{\phi}\alpha$, and assume for now that $\phi$ is a valid test function. Then by (8.9), and the divergence theorem,

$$\begin{aligned}\int_M \left(8|\nabla\phi|_g^2 + R\phi^2\right) dV &= \int_M \left(-8\bar{\phi}\bar{\Delta}\bar{\phi}\right) \overline{dV} + \int_{S_\infty} \nu(\phi) dA \\ &\leq -8\Lambda \int_M \bar{\Delta}\bar{\phi}\, \overline{dV} + \int_{S_\infty} \nu(\phi) dA \\ &\leq -8\Lambda \lim_{r\to\infty} \int_{\{|x|=r\}} \overline{\nu}(\bar{\phi}) \overline{dA} + \int_{S_\infty} \nu(\phi) dA.\end{aligned}$$

FIGURE 8.1: Test function for the Yamabe zero case




In the lower part of the diagram is a region in the compact manifold $(\overline{M}, \overline{g})$ containing the distinguished point $p_\infty$. Pictured above is the graph of the test function $\bar{\phi}$, given pointwise as the maximum of $\overline{G}$, which blows up at $p_\infty$, and a large constant $\Lambda$. The contour curve around $p_\infty$ represents the $\Lambda$-level set of $\overline{G}$.

Observing that $\bar{\phi}(x) = \overline{G}(x)$ for $r = |x|$ sufficiently large, we conclude

$$\int_M \left(8|\nabla\phi|_g^2 + R\phi^2\right) dV \leq -8\Lambda \int_{S_\infty} \overline{\nu}(\overline{G})\overline{dA} + \int_{S_\infty} \nu(\overline{G}\alpha) dA.$$

The right-hand side is of the form $-8a\Lambda + b$, with $a, b$ independent of $\Lambda$. But $a$ is positive since $\bar{\phi}$ blows up at $p_\infty$ and is harmonic in a neighborhood of $p_\infty$, so it follows that $\mathcal{F}_g$ takes on arbitrarily negative values.

It remains to justify that $\phi = \bar{\phi}\alpha$ has decay (8.11) as in Lemma 73. Since $\bar{\phi}$ agrees with $\overline{G}$ for large $r$, we need only look at $\overline{G}\alpha$. Based on the asymptotics of $\overline{G}$ near $p_\infty$ (see [4]) and the relationship between $g$ and $\overline{g}$, we see that $\overline{G}$ is of the form

$$\overline{G}(x) = cr + O(1),$$

where $c > 0$ is a constant and $O(1)$ is some function that is smooth across $p_\infty$ (and is in particular bounded). We can read off that $\overline{G}\alpha$ tends to a constant at infinity; by rescaling $\overline{G}$ if necessary, we may assume that $\overline{G}\alpha$ tends to 1 at infinity. We can also read off that $\overline{G}\alpha - 1$ is $O(r^{-1})$. Finally, it is now straightforward to check that $|\nabla(\overline{G}\alpha)|_g$ is $O(r^{-2})$. □

## 8.4 Relation to the Schoen–Yau proof of the positive mass theorem

If $g$ has nonnegative scalar curvature, Conjecture 68 and Theorem 71 reproduce the positive mass theorem. A natural question is: what are weaker conditions on the scalar curvature that guarantee the ADM mass is positive, or at least bounded below by a computable quantity? Below we give two such conditions that are well-known, though we point out that an interesting open problem is to find other scalar curvature conditions that lead to nontrivial lower bounds for the ADM mass.

*Pointwise lower bound for scalar curvature:* In the proof of the positive mass theorem for spacetimes [36], Schoen and Yau considered the condition on scalar curvature

$$R \geq 2|X|^2 - 2\text{div}X \tag{8.12}$$

for a vector field $X$ that decays rapidly at infinity (or is of compact support). Here, we show that Conjecture 68 predicts the ADM mass is positive when condition (8.12) holds. For a function $\phi$ in the domain of $\mathcal{F}_g$,

$$\begin{aligned}\int_M \phi^2 \text{div}X dV &= \int_M \text{div}(\phi^2 X) - g(\nabla(\phi^2), X) dV \\ &= -2\int_M \phi g(\nabla\phi, X) dV,\end{aligned}$$

by the divergence theorem. We use this to compute $\mathcal{F}_g(\phi)$:

$$\begin{aligned}\int_M 8|\nabla\phi|^2 + R\phi^2 dV &\geq \int_M 8|\nabla\phi|^2 + 2|X|^2\phi^2 - 2\phi^2 \text{div}X dV \\ &= \int_M 8|\nabla\phi|^2 + 2|X|^2 u^2 + 4u\left\langle \nabla\phi, X\right\rangle dV \\ &= \int_M 6|\nabla\phi|^2 + 2|\phi X + \nabla\phi|^2 dV \\ &\geq 0,\end{aligned}$$

for all $\phi$, so the infimum of $\mathcal{F}_g$ over all $\phi$ is nonnegative. Assuming either Conjecture 68 or assuming harmonic flatness at infinity and using Theorem 71, we see that the ADM mass of $(M, g)$ is nonnegative.

*$L^p$ upper bound for negative scalar curvature:* Let $R^- = \max(0, -R)$, an integrable, Lipschitz function on $M$. Schoen and Yau prove that if the $L^{3/2}$ norm of $R^-$ is less than some constant $\epsilon_0$, then $g$ can be conformally deformed to zero scalar curvature (see Lemma 3.2, [35], and Lemma 4.1, [26]). From this it is readily shown that the hypotheses of Proposition 67 are satisfied, so we have the desired mass estimate. The significance of Theorem 71 is that it bounds the ADM mass from below in the case that the $L^{3/2}$ norm of $R^-$ exceeds $\epsilon_0$. If this norm becomes too large, however, it is expected that the lower bound is $-\infty$.

## 8.5 An invariant of the asymptotically flat conformal class

We give a straightforward observation:

**Proposition 74.** *Let $(M, g)$ be a complete, asymptotically flat manifold without boundary. The quantity*

$$\mathcal{I} := m_{\mathrm{ADM}}(M, g) - \inf_{\phi^4 g \in [g]_{AF}} \left\{ \frac{1}{16\pi} \int_M \left(8|\nabla\phi|_g^2 + R_g \phi^2\right) dV_g \right\},$$

*possibly infinite, is an invariant of the asymptotically flat conformal class $[g]_{AF}$.*

*Proof.* Let $u^4 g \in [g]_{AF}$. Using equation (8.6),

$$\begin{aligned} \inf_{\phi^4 u^4 g \in [u^4 g]_{AF}} \mathcal{F}_{u^4 g}(\phi) &= \inf_{\phi^4 g \in [g]_{AF}} \mathcal{F}_g(u\phi) - \frac{1}{2\pi} \int_{S_\infty} \nu(u) dA_g \\ &= \inf_{\phi^4 g \in [g]_{AF}} \mathcal{F}_g(\phi) - \frac{1}{2\pi} \int_{S_\infty} \nu(u) dA_g, \end{aligned}$$

so that both infima are simultaneously finite. If both are infinite, we are done. Else, by (A.9),

$$m_{\mathrm{ADM}}(M, u^4 g) - m_{\mathrm{ADM}}(M, g) = -\frac{1}{2\pi}\int_{S_\infty} \nu(u) dA_g,$$

and the claim follows by rearranging. □

Note that as a corollary to Conjecture 68, we would have that the invariant $\mathcal{I}$ is nonnegative (possibly $+\infty$) and vanishes precisely on the conformal class of flat $\mathbb{R}^3$. In any case, we have established this result for manifolds that are harmonically flat at infinity.

# Appendix A

## Formulas in Conformal Geometry

Here we collect some formulas on how certain geometric quantities behave under conformal transformations. Assume $g_1$ and $g_2$ are smooth Riemannian metrics on a 3-manifold with $g_2 = u^4 g_1$ for a smooth function $u > 0$.

*Tangent vectors and co-vectors:* If $v$ is a tangent vector and $\omega$ is a 1-form, then

$$|v|_{g_2} = u^2 |v|_{g_1} \tag{A.1}$$

$$|\omega|_{g_2} = u^{-2} |\omega|_{g_1} \tag{A.2}$$

Therefore, if $\nu_1$ is a unit normal vector to a hypersurface with respect to $g_1$, then

$$\nu_2 = u^{-2} \nu_1 \tag{A.3}$$

is a unit normal vector to the hypersurface with respect to $g_2$.

*Hausdorff measure:* If $dA_{g_i}$ (for $i = 1, 2$) are the respective area measures for hypersurfaces with respect to $g_i$, then

$$dA_{g_2} = u^4 dA_{g_1}. \tag{A.4}$$

More generally, if $d\mathcal{H}^k_{g_i}$ is Hausdorff $k$-measure for $i = 1, 2,$, then

$$d\mathcal{H}^k_{g_2} = u^{2k} d\mathcal{H}^k_{g_1}. \tag{A.5}$$

*Laplacian:* For any $C^2$ function $\phi$,

$$\Delta_1(u\phi) = u^5 \Delta_2(\phi) + \phi \Delta_1(u), \tag{A.6}$$

where $\Delta_1$ and $\Delta_2$ are the (negative spectrum) Laplace operators for $g_1$ and $g_2$, respectively [7]. In particular, if $\Delta_1 u = 0$, then $\Delta_2(1/u) = 0$.

*Scalar curvature:* If $R_1$ and $R_2$ are the respective scalar curvatures of $g_1$ and $g_2$, then

$$R_2 = u^{-5}(-8\Delta_1 u + R_1 u). \tag{A.7}$$

In particular, if $u$ is harmonic with respect to $g_1$, then $g_1$ and $g_2$ have the same pointwise sign of scalar curvature.

*Mean curvature:* If $S$ is a hypersurface of mean curvature $H_i$ with respect to $g_i$ (and unit normal $\nu_i$), $i = 1, 2$, then

$$H_2 = u^{-2} H_1 + 4u^{-3} \nu_1(u). \tag{A.8}$$

*ADM mass:* If $g_1$ and $g_2$ are asymptotically flat metrics, with respective ADM masses $m_1$ and $m_2$, such that $g_2 = u^4 g_1$ with $u \to 1$ at infinity, then

$$m_2 - m_1 = -\frac{1}{2\pi} \lim_{r\to\infty} \int_{\{|x|=r\}} \nu(u) dA, \tag{A.9}$$

where, in some asymptotically flat coordinate system $(x^i)$ (for $g_1$ or $g_2$), $\nu$ is the outward unit normal to the coordinate sphere $\{|x| = r\}$. The result is the same for $\nu$ and the area measure $dA$ taken with respect to $g_1$, $g_2$, or the coordinate chart. The limit of the flux integral is also denoted by $\displaystyle\int_{S_\infty} \nu(u) dA$.

# Appendix B

## Geometric Measure Theory

The purpose of this appendix is to recall, without proof, some definitions and results from geometric measure theory that were used in earlier chapters. We make no attempt to give a comprehensive introduction to the subject, and strive to keep the discussion as elementary as possible. Our primary reference is Simon's book [37], but the results can also be found in the works of Federer [15] and Morgan [27].

*Hausdorff measure and rectifiability:* We recall the definition of Hausdorff measure $\mathcal{H}^n$ on $\mathbb{R}^{n+k}$ §2, [37]

**Definition 75.** *The **Hausdorff** $n$**-measure** of a set $A \subset \mathbb{R}^{n+k}$ is:*

$$\mathcal{H}^n(A) = \lim_{\delta \to 0+} \inf_{\{C_j\}} \sum_{j=1}^{\infty} \omega_n \left( \frac{\operatorname{diam} C_j}{2} \right)^n ,$$

*where the infimum is taken over all countable collections of closed sets $\{C_j\}$ that cover $A$ and satisfy $\operatorname{diam} C_j \leq \delta$; $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.*

Note that $\mathcal{H}^n(A)$ is well-defined for all $A \subset \mathbb{R}^{n+k}$, possibly $+\infty$, and that Hausdorff measure is an outer measure.

Next, we define rectifiable sets, a useful generalization of submanifolds §11, [37].

**Definition 76.** *A subset $M \subset \mathbb{R}^{n+k}$ is $n$-**rectifiable** if*

1. *$M$ is $\mathcal{H}^n$-measurable, and*
2. *$M \subset \bigcup_{j=0}^{\infty} N_j$, where $\mathcal{H}^n(N_0) = 0$, and for $j \geq 1$, $N_j$ is an $n$-dimensional embedded $C^1$ submanifold of $\mathbb{R}^{n+k}$.*

We identify $n$-rectifiable sets that differ by a set of zero $\mathcal{H}^n$-measure.

*Currents:* An *$n$-current* in $\mathbb{R}^{n+k}$ is a linear, real-valued functional on the space $C^\infty_\circ(\Lambda^n\mathbb{R}^{n+k})$ of compactly-supported, smooth, differential $n$-forms on $\mathbb{R}^{n+k}$. For instance, an embedded, oriented $C^1$ submanifold $S \subset M$ of dimension $n$ defines an $n$-current in a natural way: for $\phi \in C^\infty_\circ(\Lambda^n\mathbb{R}^{n+k})$,

$$S(\phi) = \int_S \phi.$$

Currents therefore generalize the notion of oriented submanifolds. Given an open set $U \subset \mathbb{R}^{n+k}$, an $n$-current in $U$ is a linear functional on $C^\infty_\circ(\Lambda^n U)$.

Given an $n$-current $S$, $n \geq 1$, define its *boundary* $\partial S$ to be the $(n-1)$-current defined by

$$\partial S(\phi) = S(d\phi).$$

This definition is designed to be compatible with Stokes' theorem. Note that $\partial^2 S$ is the zero current.

The *support* of an $n$-current $S$, denoted $\operatorname{spt} S$, is the intersection of all closed sets $C$ for which

$$\operatorname{spt}\phi \cap C = \varnothing \quad \text{implies} \quad S(\phi) = 0,$$

for all $\phi \in C^\infty_\circ(\Lambda^n\mathbb{R}^{n+k})$, where $\operatorname{spt}\phi$ is the support of $\phi$.

Next, we define the *mass norm* of an $n$-current $S$ to be

$$|S| = \sup_{\phi}\{S(\phi) : \|\phi\| \leq 1\},$$

where $\|\phi\|$ is the largest value that $\phi$ takes on unit, simple $n$-vectors $e_{i_1} \wedge \ldots \wedge e_{i_n}$. (This is the convention used by Federer and Morgan, but not by Simon. See, however, §26 of Simon [37].) Note that the same notation $|\cdot|$ is used for all $n$.

A sequence of $n$-currents $\{S_i\}$ *converges weakly* to an $n$-current $S$, provided

$$S_i(\phi) \to S(\phi),$$

for all $\phi \in C^{\infty}_{\circ}(\Lambda^n\mathbb{R}^{n+k})$. It is straightforward to check that the mass norm is lower semi-continuous with respect to weak convergence of currents: that is,

$$|S| \leq \liminf |S_i|,$$

whenever $S_i \to S$ weakly.

Roughly, a *rectifiable $n$-current* is an $n$-current whose action is given by integrating an $n$-form $\phi$ over an $n$-rectifiable set endowed with an integer-valued multiplicity function and a measurable choice of orientation. Simon gives a precise definition §27, [37]. An *integral $n$-current* is a rectifiable current $S$ for which $\partial S$ is also a rectifiable current.

An $\mathcal{H}^{n+k}$-measurable set $\Omega \subset \mathbb{R}^{n+k}$ naturally defines a rectifiable $(n+k)$-current as follows: any $\phi \in C^{\infty}_{\circ}(\Lambda^{n+k}\mathbb{R}^{n+k})$ can be written uniquely as $\phi = \rho\omega$, where $\omega$ is the oriented volume form for $\mathbb{R}^{n+k}$, and $\rho$ is a smooth function of compact support. The map defined by

$$\Omega(\phi) := \int_{\Omega} \rho \, d\mathcal{H}^{n+k}$$

allows us to view $\Omega$ as a rectifiable $(n+k)$-current of multiplicity one.

In general, given a multiplicity-one rectifiable $n$-current $S$, the mass norm can be computed as the Hausdorff measure of the underlying rectifiable set:

$$|S| = \mathcal{H}^n(S).$$

*Isoperimetric inequalities:* The following well-known theorem generalizes the classical isoperimetric inequality for regions in the plane.

**Theorem 77** (Isoperimetric inequality, §30, [37])**.** *For $n, k \geq 1$, let $T$ be an integral $n$-current in $\mathbb{R}^{n+k}$ with compact support and zero boundary. Then there exists an integral $(n+1)$-current $R$ with compact support such that $\partial R = T$ and*

$$|R|^{\frac{n}{n+1}} \leq c|T|,$$

*for some constant $c$ depending only on $n$ and $k$. In particular, if $k = 1$, then for all bounded, $\mathcal{H}^{n+1}$-measurable sets $\Omega \subset \mathbb{R}^{n+1}$*

$$\mathcal{H}^{n+1}(\Omega)^{\frac{n}{n+1}} \leq c\mathcal{H}^n(\partial\Omega).$$

We state a type of "relative" isoperimetric inequality for the codimension-one case.

**Theorem 78** (Isoperimetric inequality on half-space)**.** *Let $\mathbb{R}^{n+1}_+$ be the closed upper-half-space $\{x^{n+1} \geq 0\}$. There exists a constant $c > 0$ (depending on $n$) such that for all bounded, $\mathcal{H}^{n+1}$-measurable sets $\Omega \subset \mathbb{R}^{n+1}_+$,*

$$\mathcal{H}^{n+1}(\Omega)^{\frac{n}{n+1}} \leq c\mathcal{H}^n(\partial\Omega \setminus \{x^{n+1} = 0\}).$$

That is, we may neglect the contribution of $\partial\Omega \cap \{x_n = 0\}$ to the right-hand side of the isoperimetric inequality. The proof is to "double" $\Omega$ by reflecting across the $\{x^{n+1} = 0\}$ plane and applying Theorem 77 [13].

*Slicing:* References for this section are §28 of Simon [37] and §4.2.1 of Federer [15]. Suppose $S$ is an integral $n$-current in $\mathbb{R}^{n+k}$. Given an $\mathcal{H}^n$-measurable set $E \subset \mathbb{R}^{n+k}$, the restriction of $S$ to $E$, denoted $S \llcorner E$, is defined in the obvious way. We can now define the positive and negative slices of $S$ through the $r$-level set of a Lipschitz function $d : \mathbb{R}^{n+k} \to \mathbb{R}$:

$$\langle S, d, r+\rangle = \partial(S \llcorner \{d \leq r\}) - \partial S \llcorner \{d \leq r\}$$

$$\langle S, d, r-\rangle = \partial(S \llcorner \{d < r\}) - \partial S \llcorner \{d < r\}$$

Note that $\langle S, d, r+\rangle$ and $\langle S, d, r-\rangle$ are $(n-1)$-currents, but not necessarily rectifiable. However, we state the non-obvious fact that if $S$ is an integral $n$-current, then almost every slice is an integral $(n-1)$-current. We also have the relations

$$\partial\langle S, d, r\pm\rangle = -\langle \partial S, d, r\pm\rangle.$$

Moreover, the slices $\langle S, d, r+\rangle$ and $\langle S, d, r-\rangle$ agree for almost all values of $r$, so generally it does not matter which slice is used. Of particular concern is the case for which $d$ is the distance function from some subset of $\mathbb{R}^{n+k}$.

**Lemma 79** (Slicing lemma, §28, [37])**.** *Suppose $d : \mathbb{R}^{n+k}$ is a Lipschitz function with Lipschitz constant at most 1. Define*

$$m(r) = |S \llcorner \{d \leq r\}|,$$

*a nondecreasing function. Then for almost every $r$,*

$$m'(r) \geq |\langle S, d, r+\rangle|.$$

*Compactness and constancy theorems:* The following compactness theorem for integral currents, due to Federer and Fleming [17], is one of the fundamental results in geometric measure theory.

**Theorem 80** (Compactness theorem, §27, [37])**.** *Let $\{S_i\}$ be a sequence of integral $n$-currents in $\mathbb{R}^{n+k}$, and suppose*

1. *there exists a compact set $K$ such that $\operatorname{spt} S_i \subset K$ for all $i$, and*
2. *there exists a number $C$ such that $|S| + |\partial S| \leq C$ for all $i$.*

*Then there exists a subsequence of $\{S_i\}$ that converges weakly to some integral $n$-current $S$. In the case that $k = 0$, we may assume that the subsequence $\{S_i\}$ converges to $S$ in mass norm: $|S_i - S|_g \to 0$.*

For the last statement, see §31 of Simon [37]. The following theorem is also very useful.

**Theorem 81** (Constancy theorem, §26, [37])**.** *Let $U \subset \mathbb{R}^n$ be a nonempty, connected open set. Suppose $S$ is an $n$-current in $U$ with zero boundary. Then $S$ is an integer multiple of the multiplicity-one current determined by $U$.*

*Regularity for minimizing currents:* We restrict to the codimension-one case.

**Definition 82.** *Let $S$ be an $n$-current in $\mathbb{R}^{n+1}$, and let $p \in \operatorname{spt} S$. Then the* ***mass ratio*** *of $S$ at $p$ is the function*

$$r \mapsto \frac{|S \llcorner B(p,r)|}{\omega_n r^n},$$

*for $r > 0$, where $\omega_n$ is the volume of the unit $n$-ball, and $B(p,r)$ is the open ball of radius $r$ about $p$. The limit of the mass ratio as $r \to 0^+$, if it exists, is called the* ***density*** *of $S$ at $p$.*

Note that the mass ratio is finite for every value of $r > 0$. It is straightforward to check that if $S$ is the multiplicity-one current associated to an oriented, $C^1$ embedded hypersurface in $\mathbb{R}^{n+1}$, then $S$ has density equal to 1 at all points in its support.

Let $S$ be an integral $n$-current in $\mathbb{R}^{n+1}$, and let $U \subset \mathbb{R}^{n+1}$ be an open set. Then $S$ is *minimizing in $U$* if for all open sets $W \subset U$ with compact closure in $U$,

$$|S \llcorner W| \leq |T \llcorner W|,$$

for all integral $n$-currents $T$ with $\partial S = \partial T$ and $\operatorname{spt}(S - T)$ a compact subset of $W$. An essential fact is that for minimizing currents, a *monotonicity formula* holds; that is, the mass ratio is non-decreasing as a function of $r$. Consequently, the density is well-defined for such surfaces. A far deeper result is the following.

**Theorem 83** (Regularity of minimizing currents)**.** *Suppose $1 \leq n \leq 6$, and let $S$ be a minimizing integral $n$-current in $U \subset \mathbb{R}^{n+1}$. Then $\operatorname{spt} S \setminus \operatorname{spt} \partial S$ is a smooth, embedded submanifold of $\mathbb{R}^{n+1}$ of zero mean curvature.*

For further details, see §37 of Simon [37], or §5.4.15 of Federer [15]. The above regularity theorem was first proved by Fleming for the case $n = 2$ [19], later extended by Federer [16].

*Extension to manifolds:* We close with the remark that the above results can be carried over to smooth Riemannian manifolds $(M, g)$ of dimension $n + k$ in place of $\mathbb{R}^{n+k}$. Hausdorff measure and rectifiable sets are defined analogously (note that rectifiability is independent of the choice of metric). Currents and the basic definitions generalize in a natural, straightforward way. Lower semi-continuity of the mass norm, the slicing lemma, the compactness theorem, constancy theorem, and the regularity theorem extend naturally to $(M, g)$. The isoperimetric inequality, including the relative version, holds on asymptotically flat manifolds with compact boundary.

# Appendix C

## Theorems from Analysis

In this appendix we recall some results from functional and harmonic analysis.

*Functional analysis:* Let $X$ be a Banach space. The dual space $X^*$ is naturally a Banach space under the operator norm: for $\lambda \in X^*$,

$$\|\lambda\|_{X^*} = \sup\{\lambda(x) : \ x \in X, \|x\|_X \leq 1\}.$$

A sequence $\{\lambda_n\}$ in $X^*$ is said to converge in weak-$*$ to $\lambda \in X^*$, if, for all $x \in X$,

$$\lambda_n(x) \to \lambda(x).$$

**Theorem 84** (Banach–Alaoglu theorem [29])**.** *The closed unit ball in $X^*$ is compact in the weak-$*$ topology. In particular, norm-bounded sequences in $X^*$ have weak-$*$ convergent subsequences.*

**Theorem 85.** *If $\{\lambda_n\}$ is a sequence in $X^*$ that converges in weak-$*$ to $\lambda$, then*

$$\|\lambda\|_{X^*} \leq \liminf_{n\to\infty} \|\lambda_n\|_{X^*}.$$

Theorem 85 follows from the definition of the operator norm on $X^*$.

We are concerned with the above theorems only in the case in which $X$ is the $L^p$ space of a Riemannian manifold, $p \in (1, \infty)$. Under the identification of $(L^p)^*$ with $L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, the previous theorems assert that a norm-bounded sequence $\{f_n\}$ in $L^p$ gives rise to a subsequence (of the same name, say) and an element $f \in L^p$ for which

$$\int f_n \phi \to \int f \phi,$$

for all $\phi \in L^q$. Moreover,

$$\|f\|_{L^p} \leq \liminf_{n \to \infty} \|f_n\|_{L^p}.$$

*Harmonic analysis:* Below we state without proof some results from harmonic analysis on $\mathbb{R}^n$, generalized in a natural way to Riemannian manifolds.

**Theorem 86** (Fatou theorem for manifolds)**.** *Let $(M, g)$ be a smooth, complete Riemannian manifold with boundary $\Sigma$, and suppose $(M, g, \Sigma)$ has Poisson kernel $K(x, y)$ (for $x \in M$, $y \in \Sigma$, $x \neq y$). If $f \in L^1(\Sigma)$, and*

$$u(x) = \int_\Sigma K(x, y) f(y) dA_g(y),$$

*a harmonic function on the interior of $M$, then for almost all $y \in \Sigma$,*

$$\lim_{t \to 0^+} u(\gamma_y(t)) = f(y),$$

*for all paths $\gamma_y(t)$ in $M$ that emanate from $y$ with $\gamma_y'(0)$ transverse to $T_y\Sigma$.*

The classical statement of this theorem is for upper half-space $\mathbb{R}^n_+$ and the notion of non-tangential convergence [38].

Recall the (Hardy–Littlewood) *maximal* function $M(f)$ associated to a function $f \in L^p(\mathbb{R}^n)$ [38]:

$$M(f)(x) = \sup_{r>0} \frac{\int_{B(x,r)} |f(y)| dy}{\omega_n r^n},$$

where $\omega_n$ is the volume of the unit $n$-ball. Evidently $M(f)(x)$ measures the largest possible average value of $|f|$ on balls centered at $x$. If $p > 1$, then $M(f)$ belongs to $L^p(\mathbb{R}^n)$, and, moreover, $f \mapsto M(f)$ is a bounded linear map $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ (see Chapter I of Stein [38]).

In relation to harmonic functions, the significance of the maximal function is as follows. Consider upper half-space $\mathbb{R}^{n+1}_+$ with coordinates $(x^1, \ldots, x^{n+1})$, where $x^{n+1} \geq 0$. Identify $\mathbb{R}^n$ with the hyperplane $\{x^{n+1} = 0\}$. If $u$ is the harmonic function on open upper half-space given by convolving the Poisson kernel with $f \in L^p(\mathbb{R}^n)$, then

$$|u(x^1, \ldots, x^{n+1})| \leq M(f)(x^1, \ldots, x^n),$$

for all $(x^1, \ldots, x^{n+1})$ (see Chapter III of Stein [38]). In other words, the size of the maximal function of the boundary data at a point controls the size of the harmonic function along normal rays from that point.

We assume a similar result for general manifolds with boundary:

**Theorem 87.** *Let $(M, g)$ be a smooth, complete Riemannian manifold with boundary $\Sigma$ with unit normal vector field $\nu$ pointing into $M$. Suppose $(M, g, \Sigma)$ has a Poisson kernel $K(x, y)$. Let $f \in L^p(\Sigma)$, with $p \geq 1$, and for $x \in M \setminus \Sigma$, let*

$$u(x) = \int_\Sigma K(x, y) f(y) dA_g(y).$$

*Then there exists a function $M(f)$ belonging to $L^p(\Sigma)$ with the following property: if $\Phi_t$ is a 1-parameter family of diffeomorphisms generated by a smooth vector field $Y$ on $M$ with $Y|_\Sigma = \nu$, then*

$$|u \circ \Phi_t(y)| \leq M(f)(y),$$

*for all $t$ sufficiently small and all $y \in \Sigma$.*

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# Biography

Jeffrey Loren Jauregui was born in Lodi, California on November 15, 1983 to parents Larry and Rebecca Jauregui. Most recently, Jauregui received Ph.D. and M.A. degrees in mathematics from Duke University in Durham, North Carolina. Prior to that, he received a B.S. degree from Harvey Mudd College in Claremont, California, with majors in mathematics and physics.

While a graduate student at Duke, Jauregui was a recipient of the James B. Duke Fellowship and the National Science Foundation Graduate Research Fellowship. In August 2009 he received the L.P. and Barbara Smith Award for excellence in teaching from the Duke University Department of Mathematics. In Fall 2010 Jauregui will be joining the faculty of the University of Pennsylvania Department of Mathematics as a lecturer.