INEQUALITIES ASSOCIATED TO RIESZ POTENTIALS AND NON-DOUBLING MEASURES WITH APPLICATIONS

by

MUKTA BAHADUR BHANDARI

M.A., Tribhuvan University, Nepal, 1992
M.S., Kansas State University, USA, 2005

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2010

Abstract

The main focus of this work is to study the classical Calderón-Zygmund theory and its recent developments. An attempt has been made to study some of its theory in more generality in the context of a nonhomogeneous space equipped with a measure which is not necessarily doubling.

We establish a Hedberg type inequality associated to a non-doubling measure which connects two famous theorems of Harmonic Analysis-the Hardy-Littlewood-Weiner maximal theorem and the Hardy-Sobolev integral theorem. Hedberg inequalities give pointwise estimates of the Riesz potentials in terms of an appropriate maximal function. We also establish a good lambda inequality relating the distribution function of the Riesz potential and the fractional maximal function in $(\mathbb{R}^n, d\mu)$, where μ is a positive Radon measure which is not necessarily doubling. Finally, we also derive potential inequalities as an application.

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Approved by:

Major Professor
Dr. Charles N. Moore

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We establish a Hedberg type inequality associated to a non-doubling measure which connects two famous theorems of Harmonic Analysis-the Hardy-Littlewood-Weiner maximal theorem and the Hardy-Sobolev integral theorem. Hedberg inequalities give pointwise estimates of the Riesz potentials in terms of an appropriate maximal function. We also establish a good lambda inequality relating the distribution function of the Riesz potential and the fractional maximal function in $(\mathbb{R}^n, d\mu)$, where μ is a positive Radon measure which is not necessarily doubling. Finally, we also derive potential inequalities as an application.

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Dedication

I dedicate this work to my parents

Kamal Bhandari (Father) and Dila Bhandari (Mother).

Preface

The area of this work falls in Harmonic Analysis. The space of homogeneous type has been one of the most important tools of Harmonic Analysis for more than last three decades. These spaces were formally introduced in [15] by R. Coifmann and G. Weiss. The space of homogeneous type is a metric space equipped with a measure μ satisfying the so-called "doubling condition", which means that there exists a constant $C = C(\mu) \geq 1$, such that, for every ball $B(x, r)$ of center x and radius r

$$\mu\left(B(x, 2r)\right) \leq C\mu\left(B(x, r)\right). \tag{1}$$

It was believed that space of homogeneous type was the base for the optimal level of generality to study harmonic analysis. The reason for its success is that E. M. Stein chose space of homogeneous type to develop the basic fundamental theory of harmonic analysis ([43], [42]). In recent years it has been ascertained that central results of classical Calderón-Zygmund Theory hold true in a very general situation in which the underlying measure is not necessarily doubling but only satisfies a mild condition, known as the growth condition. It came as surprise to many when F. Nazarov, S. Treil and A. Volberg announced that *"The doubling condition is superfluous for most of the classical theory of harmonic analysis"* ([50]). They meant that a rather complete theory of Calderón-Zygmund operators could be developed if, in some sense, condition (1) is replaced by the following condition: A Borel measure μ on a measure metric space $(\mathbb{X}, d)$ is said to satisfy the growth condition if

$$\mu\left(B(x, r)\right) \leq Cr^N \tag{2}$$

where the constant C is independent of x and r. This allows, in particular, non-doubling measures. Sometimes we shall refer to condition (2) by saying that the measure μ is N-dimensional. Some other prominent mathematician working on this area are X. Tolsa ([47]),

J. Verdera ([54]), C. Peréz ([39]) J. Mateu, P. Matila, A. Nicolau , J. Orobitg ([38]), Y. Sawano, H. Tanaka ([40] and José García-Cuerva, A. Eduardo Gatto ([23]).

The list of results that one can establish without resorting to the doubling condition is quite amazing and encouraging for further research in this area. For example, the T(b)-Theorem ([17] and [49]), the Calderón-Zygmund decomposition and the derivative of weak L^1 and L^p bounds, $1 < p < \infty$, from L^2 bounds ([50], [46], and [48]), Cotlar's inequality for the maximal singular integral ([50], and [45]) and many others ([38], [39], and [24]). This is the motivating factor of this work.

The first chapter deals with some basics of our work. We discuss maximal functions with their variations and introduce non-doubling measures with examples.

Chapter 2 deals with the Riesz potentials and its estimates by maximal functions. In the sequel, we generalize the well known Hedberg inequality associated to non-doubling measures. Finally, we give an application of this deriving an exponential inequality.

In Chapter 3, we review the well known good-lambda inequalities and its variations. We generalize this associated to a non-doubling measure with applications.

In Chapter 4, we conclude our work with motivation to future work.

Chapter 1

Maximal Function in Non-doubling Metric Measure Space.

1.1 Introduction.

In this chapter, we will review the history of the Hardy-Littlewood maximal operator. It was first introduced in 1930 by G. H. Hardy and J. E. Littlewood [27] in $\mathbb{R}$ in order to apply this as a tool in the theory of Complex analysis. Then N. Weiner [55] in 1939 introduced this operator in higher dimensions $\mathbb{R}^n (n > 1)$. The purpose was to apply this in Ergodic theory. Since then the operator has been widely studied and used. One of its applications is Lebesgue's differentiation theorem which can be deduced from the boundedness of the maximal operator. The generalization of the differentiation theorem to averages over a variety of families of sets leads to the definition of several variants of the Hardy-Littlewood maximal operator ([20]). Important cases are averages over balls or cubes (the usual Hardy-Littlewood maximal function), averages over rectangles with sides parallel to the coordinate axes, and averages over arbitrary rectangles. In 1971, R. Coifman and G. Weiss [15] introduced the maximal operator on a quasi-metric measure space satisfying the doubling condition which we call homogeneous space. It was in 1998, F. Nazarov, S. Treil and A. Volberg [50] introduced modified Hardy-Littlewood maximal operators on quasi-metric spaces possessing a Radon measure that do not necessarily satisfy a doubling condition which we call nonhomogeneous spaces.

In recent years it has been ascertained that the central results of classical Calderón-Zygmund Theory hold true in very general situations in which the standard doubling condition on the underlying measure is not satisfied. This came as great surprise to the authors who felt that homogeneous spaces were not only a convenient setting for developing Calderón-Zygmund Theory, but that they were essentially the right context. The T(b)-theorem ([17]), the Calderón-Zygmund decomposition and the derivation of L^1 and L^p bounds, $1 < p < \infty$, from L^2 bounds ([50], [46], [48]), Cotlar's inequality for the maximal singular integral ([50], [45]) and many others ([38], [39], [24]) are among those theorems which hold without resorting to the doubling condition.

1.2 Some Useful Definitions

Definition 1. *Let $(\mathbb{X}, \mu)$ and $(\mathbb{Y}, \nu)$ be measure spaces, and let T be an operator from $L^p(\mathbb{X}, \mu)$ into the space of measurable functions from $\mathbb{Y}$ to $\mathbb{C}$. We say that T is weak (p,q), $q < \infty$, if there exists a constant C such that for every $\lambda > 0$,*

$$\nu\left(\{y \in \mathbb{Y} : |Tf(y)| > \lambda\}\right) \leq \left(\frac{C\|f\|_p}{\lambda}\right)^q,$$

and we say that it is weak (p, ∞) if it is bounded operator from $L^p(\mathbb{X}, \mu)$ to $L^\infty(\mathbb{Y}, \nu)$. We say that T is strong (p,q) if it is bounded from $L^p(\mathbb{X}, \mu)$ to $L^q(\mathbb{Y}, \nu)$.

Remark 2. *If T is strong (p,q) then it is weak (p,q).*

Proof. Let $E_\lambda = \{y \in \mathbb{Y} : |Tf(y)| > \lambda\}$. Then

$$\nu(E_\lambda) = \int_{E_\lambda} d\nu \leq \int_{E_\lambda} \left|\frac{Tf(x)}{\lambda}\right|^q d\nu \leq \frac{\|Tf\|_q^q}{\lambda^q} \leq \left(\frac{C\|f\|_p}{\lambda}\right)^q.$$

□

If $(\mathbb{X}, \mu) = (\mathbb{Y}, \nu)$ and T is the identity operator then the weak (p, p) inequality is the classical Chebyshev inequality.

Definition 3. *A measure μ in a metric space is called doubling if balls have finite and positive measure and there is a constant $C = C(\mu) \geq 1$ such that*

$$\mu(2B) \leq C\mu(B) \tag{1.1}$$

for all balls B. The constant C is independent of the center and radius of the balls in $\mathbb{X}$. We also call a metric space $(\mathbb{X}, \mu)$ doubling or homogeneous if μ is a doubling measure. Note that if μ is a doubling measure then

$$\mu(\lambda B) \leq C(\mu, \lambda)\mu(B)$$

for all $\lambda \geq 1$.

Example 4. *Lebesgue measure in $\mathbb{R}^n$ is doubling. In general $d\mu(x) = |x|^a dx, a > -n$ is a doubling measure in $\mathbb{R}^n$.*

Proof. The condition $a > -n$ ensures that the measure under consideration is finite in $\mathbb{R}^n$. Let ν_n denote the volume of the unit ball in $\mathbb{R}^n$. We divide the balls $B(x_0, R)$ in $\mathbb{R}^n$ into two categories as follows:

$$T_1 = \{B(x_0, R) : |x_0| \geq 3R\}, \text{and}$$
$$T_2 = \{B(x_0, R) : |x_0| < 3R\}.$$

For balls in T_1 we have $|x_0| \geq 3R$. It follows that

$$|x_0| + 2R \leq 4(|x_0| - R), \quad \text{and} \quad |x_0| - 2R \geq \frac{1}{4}(|x_0| + R). \tag{1.2}$$

For balls in T_1 and $a \geq 0$,

$$\mu\left(B(x_0, 2R)\right) = \int_{B(x_0,2R)} |x|^a dx \leq \left(|x_0| + 2R\right)^a |B(x_0, 2R)| = \nu_n (2R)^n \left(|x_0| + 2R\right)^a, \tag{1.3}$$

and

$$\mu\left(B(x_0,R)\right)=\int_{B(x_0,R)}|x|^a dx \geq \left(|x_0|-R\right)^a \nu_n R^n. \tag{1.4}$$

Combining the inequalities (1.2),(1.3), and (1.4) we obtain

$$\mu\left(B(x_0,2R)\right)\leq C\mu\left(B(x_0,R)\right) \tag{1.5}$$

where $C=2^{3n}4^{|a|}$. Similarly, for the balls in T_1 and $a<0$, we obtain the following inequalities

$$\mu\left(B(x_0,2R)\right)\leq \left(|x_0|-2R\right)^a \nu_n(2R)^n,$$

and

$$\mu\left(B(x_0,R)\right)\geq \left(|x_0|+R\right)^a \nu_n R^n.$$

Combining the above two inequalities with (1.2) we obtain the same (1.5) inequality.

Note that $B(x_0,2R)\subseteq B(0,5R)$ for $|x_0|<3R$. So, for the balls in T_2 we

$$\mu\left(B(x_0,2R)\right)=\int_{B(x_0,2R)}|x|^a dx\leq \int_{|x|\leq 5R}|x|^a dx \leq (5R)^a\nu_n(5R)^n=C_nR^{n+a}.$$

Also note that the function $|x|^a$ is radially increasing for $a\geq 0$ and radially decreasing for $a<0$. So, we have

$$\int_{B(x_0,R)}|x|^a dx \geq \begin{cases} \int_{B(0,R)}|x|^a dx, & \text{when } a\geq 0,\\ \int_{B(3R\frac{x_0}{|x_0|},R)}|x|^a dx & \text{when } a<0. \end{cases} \tag{1.6}$$

For $x\in B\left(3R\frac{x_0}{|x_0|},R\right)$ we must have $|x|\geq 2R$. Thus both integrals in the inequality (1.6) are at least a multiple of R^{n+a}. This establishes the doubling condition (1.5) for the balls in T_2. This completes the proof.

$\square$

Definition 5. *Let $(\mathbb{X},\mu)$ be a metric measure space. The Hardy-Littlewood Maximal function of a locally integrable function f on $\mathbb{X}$ is defined by*

$$M(f)(x)=\sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)|d\mu(y).$$

Theorem 6. *let μ be a Lebesgue measure in $\mathbb{R}^n$. If $f \in L^1(\mathbb{R}^n, \mu)$ and is not identically 0, then $Mf \notin L^1(\mathbb{R}^n, \mu)$.*

This is true in any Euclidean space $\mathbb{R}^n$. But it is not true in arbitrary metric spaces. We can see this by the following example:

Example 7. *Let $\mathbb{X} = [0,1]$ and let μ be a Lebesgue measure on $\mathbb{X}$. Then $Mf \in L^1_{loc}(\mathbb{X}, \mu)$ for any $f \in L^1_{loc}(\mathbb{X}, \mu)$.*

The integrability of Mf in the above remark fails at infinity, and does not exclude local integrability. However, the following example shows that even local integrability can fail.

Example 8. *Let*

$$f(x) = \begin{cases} \frac{1}{x(\log x)^2}, & \text{if } 0 < x \leq 1/2; \\ 0, & \text{otherwise} \end{cases}$$

The local integrability of Mf fails for this function.

The following theorem provides a partial converse of the Theorem 6 that characterizes when Mf is locally integrable.

Theorem 9. *([19]) If f is an integrable function supported on a compact set B, then $Mf \in L^1(B)$ if and only if $f \log^+ f \in L^1(B)$.*

In a metric measure space $(\mathbb{X}, d, \mu)$ with $\mu(\mathbb{X}) < \infty$ we can find an integrable function such that its maximal function is also integrable. For example, any constant function defined on $\mathbb{X}$ has this property. It is remarkable to find such a function when $\mu(\mathbb{X}) = \infty$. Consider the following example:

Example 10. *Let $d\mu(x) = e^{|x|}dx, f(x) = \chi_{(-1,1)}(x), \quad x \in \mathbb{R}$.*

Clearly, $\mu(\mathbb{R}) = \infty$ and $f \in L^1(\mu)$. Let $x \notin (-1,1)$. We may take $x > 1$ and let $B = B(x, R)$ be a ball with center at x and radius R. For $x - R \geq 0$,

$$\mu(B) = \int_{x-R}^{x+R} e^{|y|}dy = \int_{x-R}^{x+R} e^y dy = e^{x+R} - e^{x-R}.$$

For $x - R < 0$,

$$\mu(B) = \int_{x-R}^{x+R} e^{|y|} dy = \int_{x-R}^{0} e^{-y} dy + \int_{o}^{x+R} e^{y} dye = e^{R-x} + e^{R+x} - 2.$$

Next we compute $\int_B \chi_{(-1,1)}(y) e^{|y|} dy$. For $x - R \geq 0$,

$$\int_B \chi_{(-1,1)}(y) e^{|y|} dy = \int_{x-R}^{1} e^{y} dy = e - e^{x-R}.$$

For $x - R < 0$,

$$\int_B \chi_{(-1,1)}(y) e^{|y|} dy = \int_{x-R}^{0} e^{-y} dy + \int_{0}^{1} e^{y} dy = e^{R-x} + e - 2.$$

Let

$$g(R) = \frac{1}{\mu(B)} \int_B \chi_{(-1,1)}(y) e^{|y|} dy.$$

Note that $Mf(x) = \sup_{R>0} g(R)$. We also note that $x - 1 \leq R \leq x + 1$. So, $Mf(x) = \max_{x-1 \leq R \leq x+1} g(R)$. From the above computation, we see that

$$g(R) = \begin{cases} \dfrac{e - e^{x-R}}{e^{x+R} - e^{x-R}} & \text{if } x - R \geq 0, \\ \dfrac{e^{R-x} + e - 2}{e^{R-x} + e^{R+x} - 2} & \text{if } x - R < 0. \end{cases}$$

Note that $e - e^{x-R}$, $e^{x+R} - e^{x-R}$, $e^{R-x} + e - 2$, and $e^{R-x} + e^{R+x} - 2$ are all increasing functions of R. Because of the inequality $x - 1 \leq R \leq x + 1$,

$$e - e^{x-R} \leq e - e^{x-(x+1)} = e - e^{-1},$$

$$e^{x+R} - e^{x-R} \geq e^{2x-1} - e,$$

$$e^{R-x} + e - 2 \leq 2e - 2, \text{and}$$

$$e^{R-x} + e^{R+x} - 2 \geq e^{2x-1} - 2 + e^{-1}.$$

Thus

$$g(R) = \frac{e - e^{x-R}}{e^{x+R} - e^{x-R}} \leq \frac{1 - e^{-2}}{e^{2x} - 1} \quad \text{for} \quad x - R \geq 0,$$

and

$$g(R) = \frac{e^{R-x} + e - 2}{e^{R-x} + e^{R+x} - 2} \leq \frac{2e(e-1)}{e^{2x} - 2e + 1}$$

for $x - R < 0$. We also note that

$$1 - e^{-2} \leq 2e(e-1)$$

and

$$e^{2x} - 2e + 1 \leq e^{2x} - 1.$$

Therefore, for every $R \in [x-1, x+1]$

$$g(R) \leq \frac{2e-1}{e^{2x} - 2e + 1} \leq \frac{2e(e-1)}{e^{2x} - 5}.$$

This means that

$$Mf(x) \leq \frac{2e(e-1)}{e^{2x} - 5}.$$

By symmetry

$$\int_{x \notin (-1,1)} Mf(x) d\mu(x) \leq 2 \int_1^\infty \frac{2e-1}{e^{2x} - 5} e^x dx < \infty.$$

For $x \in (-1, 1)$, let B be a ball with center at $x \in (-1, 1)$. Then,

$$\frac{1}{\mu(B)} \int_B \chi_{(-1,1)} d\mu(x) = \frac{\mu\left(B \bigcap (-1, 1)\right)}{\mu(B)} \leq 1.$$

Therefore, we conclude that $Mf \in L^1(\mu)$.

1.3 Covering and Interpolation Theorems

Dyadic Cubes in $\mathbb{R}^n$:

Definition 11. *(Dyadic Cubes in $\mathbb{R}^n$) The unit cube in $\mathbb{R}^n$, open on the right, is defined to be $[0, 1)^n$. Let $\mathcal{Q}_0$ be the collection of cubes in $\mathbb{R}^n$ which are congruent to $[0, 1)^n$ and whose* *vertices lie in* $\mathbb{Z}^n$*. Let* $\mathcal{Q}_k = \{2^{-k}Q : Q \in \mathcal{Q}_0\}, k \in \mathbb{Z}$*. That is,* $\mathcal{Q}_k$ *is the family of cubes, open on the right, whose vertices are adjacent points of the lattice* $\left(2^{-k}\mathbb{Z}\right)^n$*. The cubes in* $\mathcal{Q} := \cup_k \mathcal{Q}_k$ *are called the dyadic cubes.*

We get the following properties of the dyadic cubes from this construction:

(a) For every $x \in \mathbb{R}^n$ there exists a unique cube in $\mathcal{Q}_k$ which contains it;

(b) Any two dyadic cubes are either disjoint or one is wholly contained in the other;

(c) A dyadic cube in $\mathcal{Q}_k$ is contained in a unique cube of each family $\mathcal{Q}_j$, $j < k$, and contains 2^n dyadic cubes of $\mathcal{Q}_{k+1}$.

The covering lemmas provide the standard approach to prove that the Hardy-Littlewood maximal function in $\mathbb{R}^n$ is weak type (1,1). We state here very useful covering lemmas due to Whitney, N. Weiner, A. Besicovitch and A. P. Morse. Note that for any ball $B = B(x, r), t > 0$ we mean by tB the ball concentric to B with radius tr. That is, $tB = B(x, tr)$.

An arbitrary open set in $\mathbb{R}^n$ can be decomposed as a union of disjoint cubes whose lengths are proportional to their distance from the boundary of the open set. For a given Q in $\mathbb{R}^n$, we will denote by $\ell(Q)$ its length and by $diam(Q)$ its diameter.

Whitney Decomposition:

Theorem 12. *(Whitney Decomposition) Let* Ω *be an open nonempty proper open subset of* $\mathbb{R}^n$*. Then there exists a family of dyadic cubes* $\{Q_j\}_j$ *such that*

(a) $\Omega = \bigcup_j Q_j$ *where* Q_j*'s have disjoint interiors.*

(b) $diam(Q_j) \leq dist(Q_j, \Omega^c) \leq 4diam(Q_j)$*, for every* j*. That is,*

$$\sqrt{n}\ell(Q) \leq dist(Q_j, \Omega^c) \leq 4\sqrt{n}\ell(Q).$$

(c) If the boundaries of two cubes Q_j *and* Q_k *touch, then*

$$\frac{1}{4} \leq \frac{\ell(Q_j)}{\ell(Q_k)} \leq 4.$$

(d) For a given Q_j, there exists at most 12^n Q_k's that touch it.

The family of dyadic cubes $\{Q_j\}_j$ as in the above theorem is known as a Whitney Decomposition of Ω. ([25])

Remark 13. *Let $\mathcal{F} = \{Q_j\}_j$ be a Whitney decomposition of a proper open subset Ω of $\mathbb{R}^n$. Fix $0 < \epsilon < \frac{1}{4}$ and denote by Q_k^* the cube with the same center as Q_k but with side length $(1+\epsilon)$ times that of Q_k. Then Q_k and Q_j touch if and only if Q_k^* and Q_j intersect. Consequently, every point of Ω is contained in at most 12^n cubes Q_k^*. That is ,*

$$\sum_k \chi_{(1+\epsilon)Q}(x) \leq 12^n \chi_\Omega(x)$$

([25])

Theorem 14. *(Vitali Covering Lemma:)([20]) Let $\{B_j\}_{j\in\tau}$ be a collection of balls in $\mathbb{R}^n$. Then there exists an at most countable sub-collection of disjoint balls $\{B_k\}$ such that*

$$\bigcup_{j\in\tau} B_j \subseteq \bigcup_k 5B_k.$$

The next covering theorem is due independently to A. Besicovitch and A.P. Morse.

Besicovitch Covering Lemma:

Theorem 15. *(Besicovitch Covering Lemma:)([20]) Let A be a bounded set in $\mathbb{R}^n$, and suppose that $\{B_x\}_{x\in A}$ is a collection of balls such that $B_x = B(x, r_x), r_x > 0$. Then there exists an at most countable sub-collection of balls $\{B_j\}$ and a constant C_n, depending only on the dimension, such that*

$$A \subset \bigcup_j B_j \quad \textit{and} \quad \sum_j \chi_{B_j}(x) \leq C_n.$$

The following theorem is a version of classical Besicovitch Covering Therem.

Theorem 16. *([30]) Let A be a bounded set in a metric space $\mathbb{X}$ and $\mathcal{F}$ be a collection of balls centered at points of A. Then, there is a sub-collection $\mathcal{G} \subset \mathcal{F}$ such that*

$$A \subset \bigcup_{B\in\mathcal{G}} B$$

and that

$$\frac{1}{5}\mathcal{G} = \left\{\frac{1}{5}B : B \in \mathcal{G}\right\}$$

is a disjointed family of balls. Moreover, if $\mathbb{X}$ *carries a doubling measure, then* $\mathcal{G}$ *is countable, and if* $\mathbb{X} = \mathbb{R}^n$*, then one can choose* $\mathcal{G}$ *such that*

$$\sum_j \chi_B(x) \leq C(n) < \infty$$

for some dimensional constant $C(n)$*, where* χ_E *denotes the characteristic function of a set* E*.*

Next we state interpolation theorems. They are useful tools in proving the L^p boundedness of maximal functions for $1 < p \leq \infty$.

Interpolation:

Theorem 17. *(Riesz-Thorin Interpolation, [19]). Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$*, and for* $0 < \theta < 1$ *define* p *and* q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T *is a linear operator from* $L^{p_0} + L^{p_1}$ *to* $L^{q_0} + L^{q_1}$ *such that*

$$\|Tf\|_{q_0} \leq M_0 \|f\|_{p_0} \quad \textit{for} \quad f \in L^{p_0}$$

and

$$\|Tf\|_{q_1} \leq M_1 \|f\|_{p_1} \quad \textit{for} \quad f \in L^{p_1},$$

then

$$\|Tf\|_q \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p \quad \textit{for} \quad f \in L^p.$$

Definition 18. *Let* $(\mathbb{X}, \mu)$ *be a measure space and let* $f : \mathbb{X} \longrightarrow \mathbb{C}$ *be a measurable function. We call the function* $a_f : (0, \infty) \longrightarrow [0, \infty]$*, given by*

$$a_f(\lambda) = \mu\left(\{x \in \mathbb{X} : f(x) > \lambda\}\right),$$

the distribution function of f *associated with* μ*.*

Note that the weak inequalities measure the size of the distribution function. The following interpolation theorem which is due to Marcinkiewicz helps to deduce L^p boundedness from weak inequalities. It applies to larger class of operators than linear ones. Note that maximal functions are not linear but sublinear.

Definition 19. *An operator T from a vector space of measurable functions to measurable functions is sublinear if for every pair of measurable functions f and g and for every $\lambda \in \mathbb{C}$*

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|,$$
$$|T(\lambda f)| = |\lambda||Tf|.$$

Theorem 20. *(Marcinkiewicz Interpolation, [19]) Let $(\mathbb{X}, \mu)$ and $(\mathbb{Y}, \nu)$ be measure spaces, $1 \leq p_0 < p_1 \leq \infty$ and let T be a sublinear operator from $L^{p_0}(\mathbb{X}, \mu) + L^{p_1}(\mathbb{X}, \mu)$ to the measurable functions on $\mathbb{Y}$ that is weak (p_0, p_0) and weak (p_1, p_1). Then T is strong (p, p) for $p_0 < p < p_1$.*

Definition 21. *(Central and noncentral maximal function:) Let μ be a non-negative Borel measure in $\mathbb{R}^n, n \geq 1$. If $f \in L^1_{loc}(\mu)$, we define the maximal function*

$$M_\mu f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f|(y) d\mu(y)$$

where supremum is taken over all balls B containing x. Here B may be assumed open, closed, or containing any μ-measurable part of its boundary. In this definition, x is not necessarily at the center of the ball B. This is called noncentral maximal function of f. The maximal function defined above is called a central maximal function if the supremum is taken over only to the balls with center at x. It is denoted by M^c_μ.

A natural question to ask is: For which μ is M^c_μ a bounded operator on $L^p(\mathbb{R}^n, d\mu)$? It follows from the Besicovitch Covering lemma and Marcinkiewicz interpolation theorem that M^c_μ is L^p bounded where $1 < p \leq \infty$.

Theorem 22. *([21], [22], [18]) Let μ be a nonnegative Borel measure in $\mathbb{R}^n, n \geq 1$. Then* *(a) M^c_μ is weak type $(1,1)$. That is*

$$\mu\left(\{x \in \mathbb{R}^n : M^c_\mu f(x) > \lambda\}\right) \le \frac{C}{\lambda}\|f\|_{L^1(\mu)} \quad \forall\lambda > 0.$$

(b) M^c_μ is strong type (p,p) for $1 < p \le \infty$.

Proof. (a) Let $f \in L_{loc}(\mathbb{R}^n)$ and $\lambda > 0$ be arbitrarily chosen, and let

$$E_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}.$$

Let K be a bounded measurable set of $\mathbb{R}^n$. For every $x \in E_\lambda \bigcap K$ there exists an open ball $B_x = B(x, r_x)$ centered at x and radius r_x such that

$$\frac{1}{\mu(B_x)}\int_{B_x} |f(y)|d\mu(y) > \lambda.$$

Then the collection $\{B_x\}_{x\in E_\lambda \bigcap K}$ is an open covering of the set $E_\lambda \bigcap K$. Using the Besicovitch Covering Lemma, there exists a sub-collection $\{B_j\}_{j=1}^\infty$ of $\{B_x\}_{x\in E_\lambda \bigcap K}$, and a constant C_n, depending only on the dimension n, such that

$$E_\lambda \bigcap K \subseteq \bigcup_j B_j \quad \text{and} \quad \sum_j \chi_{B_j}(x) \le C_n.$$

Hence,

$$\begin{aligned}
\mu\left(E_\lambda \bigcap K\right) \le & \mu\left(\bigcup_j B_j\right) \le \sum_j \mu(B_j) \le \\
\le & \frac{1}{\lambda}\sum_j \int_{B_j} |f(y)|d\mu(y) \le \\
\le & \frac{1}{\lambda}\sum_j \int_{\mathbb{R}^n} \chi_{B_j}(y)|f(y)|d\mu(y) \sum_j \chi_{B_j} \\
= & \frac{1}{\lambda}\int_{\mathbb{R}^n} \sum_j \chi_{B_j}(y)\||f(y)|d\mu(y) \le \\
\le & \frac{C_n}{\lambda}\|f\|_{L^1(\mu)}.
\end{aligned}$$

Since this estimate is independent of K, we obtain

$$\mu(E_\lambda) \le \frac{C}{\lambda}|f\|_{L^1(\mu)}$$

for every $\lambda > 0$.

(b) It is clear that

$$|M_\mu^c f| \leq \|f\|_{L^\infty(\mu)}$$

for every $f \in L^\infty(\mu)$. This implies that

$$\|M_\mu^c f\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\mu)}.$$

Thus the centered Hardy-Littlewood maximal operator M_μ^c is both weak type $(1,1)$ and strong type (∞,∞). Therefore, by the Marcinkiewicz interpolation theorem, it follows that the centered Hardy-Littlewood maximal operator M_μ^c is strong type (p,p) for $1 < p \leq \infty$.

□

Remark 23. *The above theorem is not necessarily true for noncentral maximal function M_μ. The following theorem reveals this fact:*

Theorem 24. *([41])Let μ be a nonnegative Borel measure in $\mathbb{R}^n, n \geq 1$.*

(a) For $n = 1, M_\mu$ is weak type $(1,1)$ and strong type (p,p) for $1 < p \leq \infty$.

(b) For $n = 2$, there is a measure μ for which M_μ doesn't map $L^1(\mu)$ into weak $L^1(\mu)$. The situation is the same if the balls in the definition of M_μ are replaced by squares parallel to the axes.

Proof. (a) Without loss of generality, we may assume that $0 \leq f \in L^1(\mu)$ and let $\lambda > 0$. Let $E_\lambda = \{x : M_\mu f(x) > \lambda\}$. Then for every $x \in E_\lambda$ there exists an interval $I_x \ni x$ such that

$$0 < \mu(I_x) < \frac{1}{\lambda} \int_{I_x} f(y) d\mu(y).$$

Let $\mathcal{F}$ be the set of these intervals. Clearly $\mu(I) < \frac{1}{\lambda} \int_I f(y) d\mu(y)$ for every $I \in \mathcal{F}$. That is $\mu(I)$ is bounded for every $I \in \mathcal{F}$. Next we extract a family of disjoint intervals from $\mathcal{F}$ in the following way: Having chosen $I_1, \ldots, I_{j-1}, j \geq 1$ let I_j be an interval in $\mathcal{F}$ disjoint from $I_1, \ldots, I_{j-1}$, if any, such that

$$2\mu(I_j) > \sup\{\mu(I) : I \in \mathcal{F}, I \quad \text{disjoint from} \quad I_1, \ldots, I_{j-1}\}.$$

The disjoint condition is void for $j = 1$. This selection either stops at some j or gives an infinite sequence. Note that the collection $\{I_1, I_2, I_3, \dots, I_j, \dots\}$ doesn't necessarily cover E_λ. Then

$$\sum_j \mu(I_j) \le \frac{1}{\lambda} \sum_j \int_{I_j} f(y) d\mu(y) \le \frac{1}{\lambda} \int_{\mathbb{R}^n} f(y) d\mu(y) < \infty. \tag{1.7}$$

So $\mu(I_j) \longrightarrow 0$ as $n \to \infty$ in the case when there are infinitely many disjoint intervals I_j's. Now we enlarge each I_j. Let a_j and b_j denote the left and right end points of I_j respectively. Define,

$$a_j^* = \inf\{x : x \le a_j \quad and \quad \mu((x, a_j]) < 2\mu(I_j)\}$$

and

$$b_j^* = \sup\{y : y \ge b_j \quad and \quad \mu([b_j, y)) < 2\mu(I_j)\}$$

and let $I_j^* = (a_j^*, b_j^*)$. Then $I_j^* \supset I_j$ and $\mu(I_j^*) \le 5\mu(I_j)$. Let $I \in \mathcal{F}$ and $I \ne I_j$ for every j. Then I must intersect some I_j. If I_j is the first one which intersects I then $\mu(I) < 2\mu(I_j)$ because of the selection process of I_j's. But then $I \subset I_j^*$, and therefore $E_\lambda \subset \bigcup_j I_j^*$. this implies that

$$\mu(E_\lambda) \le 5 \sum_j \mu(I_j) \le 5 \frac{1}{\lambda} \int_{\mathbb{R}^n} f(y) d\mu(y).$$

Clearly $\|M_\mu f\|_{L^\infty(\mu)} \le \|f\|_{L^\infty(\mu)}$. Therefore, part (a) follows from Marcinkiewicz interpolation theorem.

(b) (Two-dimensional counter example:) Consider the standard Gaussian measure μ where $d\mu(x, y) = e^{-(x^2+y^2)/2} dxdy$. Let the maximal function M_μ be defined with disks. Then M_μ is not weak $(1, 1)$.(See [41], [5])

Proof. Here we take as μ the standard Gaussian measure $d\mu(x, y) = e^{-(x^2+y^2)/2} dxdy$. Consider that the non-centered maximal functions M_μ are defined with disks. By a simple limiting argument, a weak type $(1, 1)$ estimate for M_μ would imply a weak L^1 estimate for the maximal function $M_\mu\lambda$ of a finite measure λ, which it is enough to disprove. Let us take λ as a unit mass at $(0, a+1), a > 0$ large. Consider a unit disk B_s centered at $(s, a+1), |s| < 1$. Observe that

$$(x-s)^2 + [y-(a+1)]^2 = 1$$

implies that

$$y > a + \frac{(x-s)^2}{2}.$$

So, the ball $B_s \subseteq \{(x,y) : y > a + \frac{(x-s)^2}{2}\}$. Using also the fact that

$$\int_x^\infty e^{-t^2/2} dt \sim \frac{1}{x} e^{-x^2/2} \tag{1.8}$$

for large x, we obtain

$$\begin{aligned}\mu(B_s) &\leq \int_{-1}^{1} dx \int_{a+x^2/2}^{\infty} e^{-y^2/2} dy \leq \frac{C}{a} \int_{-1}^{1} e^{-(a+x^2/2)^2/2} dx \\ &\leq \frac{C}{a} e^{-a^2/2} \int_{-1}^{1} e^{-ax^2/2} dx \leq \frac{C}{a\sqrt{a}} e^{-a^2/2}.\end{aligned}$$

Here the constant C denote various positive constants. Hence, $M_\mu\lambda \geq Ca\sqrt{a}e^{a^2/2}$ in the set $\{(x,y) : |x| < 1, a < y < a+2\}$. This set has μ-measure at least $Ca^{-1}e^{-a^2/2}$ because of the relation (1.8). As $a \to \infty$, this disproves the weak $(1,1)$ estimate. □

□

Remark 25. *If μ satisfies the doubling condition, then M_μ is always of weak type $(1,1)$. The following theorem reveals this fact:*

Theorem 26. *Let μ be nonnegative Borel measure in $\mathbb{R}^n$ satisfying the doubling condition 1.1. Then the noncentral maximal function M_μ satisfies the weak $(1,1)$ inequality.*

Proof. Let $E_\lambda = \{x \in \mathbb{R}^n : M_\mu f(x) > \lambda\}, \lambda > 0$. Then for every $x \in E_\lambda$ there exists a ball B_x containing x such that

$$\mu(B_x) < \frac{1}{\lambda} \int_{B_x} |f|(y) d\mu(y).$$

Let $\mathcal{F}$ be the collection of such balls B_x's. Then by the Vitali-type covering lemma due to N. Wiener (Theorem 14), there exists an at most countable sub-collection of disjoint balls $\{B_k\}$ such that

$$E_\lambda \subseteq \bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_k 5B_k.$$

Then

$$\mu(E_\lambda) \leq \sum_k \mu(5B_k) < \frac{C}{\lambda}\sum_k \int_{B_k} |f|(y)d\mu(y) = \frac{C}{\lambda}\int_{\bigcup_k B_k} |f|(y)d\mu(y) \leq \frac{C}{\lambda}\|f\|_{L^1(\mu)}.$$

$\square$

We saw that, for $n = 1$, M_μ always maps $L^1(d\mu)$ into $L^{1,\infty}(d\mu)$, no matter what μ is. In $\mathbb{R}^n$, and if μ is a doubling measure, then M_μ is of weak type $(1,1)$. For $n = 2$, we saw that the maximal operator associated with the measure $d\mu(x) = e^{-|x|^2/2}dx$ does not have the same boundedness property. So, there are two questions that arise from these observations:

(1) Is there any non-doubling measure μ in $\mathbb{R}^n$, $n > 1$, such that M_μ is weak $(1,1)$?

(2) How can we know whether or not a measure μ provides a weak type $(1,1)$ operator ?

For certain kinds of measures, a weaker hypothesis than doubling implies that M_μ is weak $(1,1)$. The following theorem characterizes the measures μ for which M_μ is bounded from $L^1(d\mu)$ to $L^{1,\infty}(d\mu)$. This answers the second question.

Theorem 27. *([53]) Let μ be a rotation invariant and strictly increasing measure on $\mathbb{R}^n$ which is finite on compact sets. The following assertions are equivalent:*

(a) $\mathcal{M}_\mu : L^1(d\mu) \to L^{1,\infty}(d\mu)$ is bounded.

(b) There exists a constant C such that for all $r \leq 10a$,

$$\mu(\{a < |x| < a + 3r/2\}) \leq C\mu(\{a + r/2 < |x| < a + 2r\}).$$

(c) μ *is a doubling measure away from the origin, that is,*

$$\mu(B(x_0, 2s)) \le C\mu(B(x_0, s))$$

for all $s \le |x_0|/4$, *with* C *independent of* s *and* x_0.

Example 28. $d\mu(x) = (1 + |x|^\alpha)^{-1} dx$ *is doubling away from the origin. So,* M_μ *is of weak type* $(1,1)$. *But if* $\alpha \ge n$, $d\mu$ *is not a doubling measure. This example provides an affirmative answer to the first question.*

For the case of a measure $d\mu(x) = g(|x|)dx$ with g monotonic, (b) has even simpler statement:

Corollary 29. *(A. M. Vargas [53]) Let* $d\mu(x) = g(|x|)dx$ *be a measure in* $\mathbb{R}^n$ *with* g *monotonic and strictly positive on* $(0, \infty)$. *Then* M_μ *is of weak type* $(1,1)$ *if and only if there are some constants* $c_k > 0, k \in \mathbb{Z}$ *and* $C > 0$ *such that*

$$c_k \le g(r) \le Cc_k$$

for $2^{k-1} \le r \le 2^{k+1}$.

1.4 Non-doubling Measures

We are interested in measures weaker than doubling measures. In this section, we introduce non-doubling measures with some examples.

Definition 30. *We say that a Borel measure* μ *on a metric space* $(\mathbb{X}, d)$ *satisfies a growth condition if there exists a constant* $C > 0$ *and* $N > 0$ *such that*

$$\mu(B(x,r)) \le Cr^N, \quad x \in \mathbb{X}, \quad r > 0. \tag{1.9}$$

This inequality is known as a growth condition for μ.

Example 31. *The following are some examples of non-doubling measures:*

(i) The Lebesgue measure in $\mathbb{R}^n$.

(ii) The gaussian measure $d\mu(x) = e^{-|x|^2}dx$ *in* $\mathbb{R}^n$.

(iii) In general, $d\mu(x) = w(x)dx$ *in* $\mathbb{R}^n$ *with a bounded density* w.

(iv) Let $Q = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ *and* $I = Q \cap \mathbb{R}$. *Then* $d\mu = \chi_Q(x, y)dxdy + \chi_I(x)dx$ *is a non-doubling measure in* $\mathbb{R}^2$. *In fact, if* B *is the disk centered at* $(x, y) \in Q, y > 0$, *of radius* y, *then* $\mu(B) = \pi y^2$ *while* $\mu(2B) \approx y$.

Note that some doubling measures are non-doubling as well. For example, the Lebesgue measure in $\mathbb{R}^n$ is doubling and non-doubling as well. But every doubling measure is not necessarily non-doubling. The following examples reveals this fact:

Example 32. *Consider the doubling measure* $d\mu(x) = 2xdx$ *in* $(\mathbb{R}, |.|)$. *Let* B *be a ball with center* $y > 0$ *and radius* r *in* $\mathbb{R}$. *Then*

$$\mu(B) = \int_{y-r}^{y+r} 2xdx = 4yr.$$

Clearly $\mu(B) \leq Cr^N$ *is false for any pair of constants* C *and* N *because* $\mu(B)$ *can be made as large as possible by taking* $y > 0$ *sufficiently large.*

In this sense, the non-doubling measures are weaker than the doubling measure.

Chapter 2

Estimates for Riesz Potentials by Maximal Function

2.1 Introduction

In this chapter we generalize the Hedberg type inequalities to a metric apace $(\mathbb{X}, d, \mu)$ endowed with a Radon measure μ satisfying the following growth condition: For every ball $B(x, r)$ there exists a constant C independent of x and r such that

$$\mu(B(x, r)) \leq Cr^N, \tag{2.1}$$

This allows, in particular, non-doubling measures. Sometimes we shall refer to condition (2.1) by saying that the measure μ is N-dimensional.

Hedberg inequalities are point-wise estimates of potentials in terms of maximal functions. In the sequel, we will define modified maximal functions which are bounded operators on $L^p(\mathbb{X}, \mu)$ for $1 < p \leq \infty$ and apply these maximal functions to obtain exponential inequalities involving Riesz potentials. It was V. I. Yudovich [56] in 1961 who first announced these estimate. N. S. Trudinger [51] in 1967, J. A. Hempel, G. R. Morris and N. S. Trudinger [34] in 1970, and R.S. Strichartz [44] in 1972 provide generalization and extension of such inequality. The latest known development with correct limiting exponent is due to L. I. Hedberg [28] in 1972.

2.2 Generalization of the Hedberg Inequalities

Definition 33. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space, and let $0 < \alpha < N$. The fractional integral I_α associated to the measure μ satisfying the growth condition (2.1) is defined, for appropriate function f on $\mathbb{X}$ as*

$$I_\alpha f(x) = \int_{\mathbb{X}} \frac{f(y)}{d(x,y)^{N-\alpha}} d\mu(y).$$

This definition makes sense when f is bounded and has bounded support because the function $f \mapsto \dfrac{1}{d(x,y)^{N-\alpha}}$ is locally integrable. This follows from the lemmas that follow the next definition.

Definition 34. *Let $(\mathbb{X}, d)$ be a metric space, μ be a Borel measure on $\mathbb{X}$ and f be a locally integrable function on $\mathbb{X}$. The maximal function of $f, M(f)$, is defined by*

$$M(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y).$$

The fractional maximal function of f is defined for $0 < \alpha < N$ by

$$M_\alpha(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))^{\frac{N-\alpha}{N}}} \int_{B(x,r)} |f(y)| d\mu(y).$$

Theorem 35. *In an Euclidean space $\mathbb{R}^n$ there exists a constant C such that*

$$M_\alpha(f) \leq C I_\alpha(f)$$

for all $f \geq 0$ where

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(x-y)dy}{|y|^{n-\alpha}}.$$

Then, M_α maps L^p to L^q whenever I_α does.

Proof. Without loss of generality, we may assume that $f \geq 0$. For $|y| \leq t$ and $0 \leq \alpha < n$ it follows that

$$1 \leq \left(\frac{t}{|y|}\right)^{n-\alpha} = t^{n-\alpha}|y|^{\alpha-n}.$$

Therefore,

$$\begin{aligned}
\frac{1}{(\nu_n t^n)^{\frac{n-\alpha}{n}}}\int_{|y|\le t}|f(x-y)|dy &\le \frac{1}{(\nu_n t^n)^{\frac{n-\alpha}{n}}}\int_{|y|\le t}|f(x-y)||y|^{-n+\alpha}t^{n-\alpha}dy \\
&= \frac{1}{\nu_n^{\frac{n-\alpha}{n}}}I_\alpha(f)(x).
\end{aligned}$$

Now taking supremum over all $t>0$ yields

$$M_\alpha(f)(x) \le CI_\alpha(f)(x)$$

for every x and for every $f \ge 0$. Therefore,

$$M_\alpha(f) \le CI_\alpha(f)$$

for every $f \ge 0$. Next, suppose that I_α maps L^p to L^q. This means that

$$\|I_\alpha(f)\|_{L^q} \le C\|f\|_{L^p} \qquad f \in L^p.$$

Then for any $f \in L^p$,

$$\begin{aligned}
\|M_\alpha(f)\|_{L^p} &= \left(\int_{\mathbb{R}^n}|M_\alpha(f)(x)|^p dx\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}^n}|CI_\alpha(f)(x)|^p dx\right)^{\frac{1}{p}} \\
&= C\|I_\alpha(f)\|_{L^p} \\
&\le \tilde{C}\|f\|_{L^q}.
\end{aligned}$$

Therefore, M_α maps L^p to L^q whenever I_α does. $\square$

Now we state the following two classical theorems from Harmonic Analysis:

Theorem 36 (Hardy-Littlewood-Weiner Maximal Theorem)**.** *For every* $1 < p \le \infty$

$$M : L^p(\mathbb{R}^n, \mathbb{R}) \longrightarrow L^p(\mathbb{R}^n, \mathbb{R})$$

is continuous. That is

$$\|Mf\|_{L^p(\mathbb{R}^n,\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R}^n,\mathbb{R})}$$

For $p = 1$ the Hardy-Littlewood maximal operator M is weak type (1,1).

Theorem 37 (Hardy-Littlewood-Sobolev Fractional Integral Theorem)**.** *If* $1 < p < q < \infty, \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$ *and* $0 < \alpha < n$ *then*

$$I_\alpha : L^p(\mathbb{R}^n, \mathbb{R}) \longrightarrow L^q(\mathbb{R}^n, \mathbb{R})$$

is continuous. That is, there exists a constant $C > 0$ *such that for every* $f \in L^p(\mathbb{R}^n, \mathbb{R})$

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n,\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{R})}.$$

These two classical theorems stated above are linked by an inequality due to L. R. Hedberg [28] which states that

$$|I_\alpha f(x)| \leq A_{p,q}(\alpha)[Mf(x)]^{p/q}\|f\|_{L^p(\mathbb{R}^n,\mathbb{R})}{}^{1-p/q}, \quad x \in \mathbb{R}^n \tag{2.2}$$

for every $f \in L^p(\mathbb{R}^n, \mathbb{R})$ where $1 \leq p < q < \infty$, $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$ and the constant $A_{p,q}(\alpha)$ is independent of f. This was first introduced by L. R. Hedberg in 1972. D. R. Adams in 1975 [2] extends the Hedberg inequality for a fractional maximal operator. This is summarized in the following theorem:

Theorem 38. *[2] Let* $\alpha > 0, 1 < p < \dfrac{n}{\alpha}, \alpha \leq q \leq \infty$ *be such that* $\dfrac{1}{r} = \dfrac{1}{p} - \dfrac{\alpha}{n} + \dfrac{\alpha p}{nq}$. *Then there exists a constant* $C > 0$ *(depending on the previous parameters) such that for all positive functions* f *we have*

$$I_\alpha f(x) \leq CM_{n/p}(f)(x)^{\frac{\alpha p}{n}} M_0(f)(x)^{1-\frac{\alpha p}{n}}.$$

This yields

$$\|I_\alpha(f)\|_{L^r} \leq C\|M_{n/p}(f)\|_{L^q}^{\frac{\alpha p}{n}}\|f\|_{L^p}^{\frac{\alpha p}{n}} \tag{2.3}$$

Proof. For $f \neq 0$, set

$$\begin{aligned} I_\alpha(f)(x) &= \int_{\mathbb{R}^n} f(x-y)|y|^{-n+\alpha}dy \\ &= \int_{\mathbb{R}^n} f(y)|x-y|^{-n+\alpha}dy \\ &=: I + II, \end{aligned}$$

where,

$$\begin{aligned} I &= \int_{|x-y|\leq\delta} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \\ II &= \int_{|x-y|>\delta} \frac{f(y)}{|x-y|^{n-\alpha}} dy \end{aligned}$$

For every $k \in \mathbb{Z}$ define

$$a_k(x) = \{y : 2^k\delta \leq |x-y| < 2^{k+1}\delta\}.$$

Then,

$$\begin{aligned} |I| &= \left| \sum_{k=1}^{\infty} \int_{a_{-k}(x)} f(y)|x-y|^{-n+\alpha} dy \right| \\ &\leq \sum_{k=1}^{\infty} \int_{2^{-k}\delta\leq|x-y|<2^{-k+1}\delta} f(y)|x-y|^{-n+\alpha} dy \\ &\leq \sum_{k=1}^{\infty} (2^{-k}\delta)^{-n+\alpha} \int_{|x-y|<2^{-k+1}\delta} f(y) dy. \\ &\leq \sum_{k=1}^{\infty} (2^{-k}\delta)^{-n+\alpha} (2^{-k+1}\delta)^n M^0 f(x). \\ &= C\delta^\alpha M_0(f)(x), \end{aligned}$$

since $0 < \alpha < n$. Similarly,

$$
\begin{aligned}
|II| &\leq \left|\sum_{k=0}^{\infty}\int_{a_k(x)} f(y)|x-y|^{-n+\alpha}dy\right| \\
&= \sum_{k=0}^{\infty}\int_{2^k\delta\leq|x-y|<2^{k+1}\delta} f(y)|x-y|^{-n+\alpha}dy \\
&\leq \sum_{k=0}^{\infty}(2^k\delta)^{\alpha-n}\int_{|x-y|<2^{k+1}\delta}|f(y)|dy \\
&= \sum_{k=0}^{\infty}(2^k\delta)^{\alpha-n}(2^{k+1}\delta)^{n-\frac{n}{p}}(2^{k+1}\delta)^{-(n-\frac{n}{p})}\int_{|x-y|<2^{k+1}\delta}|f(y)|dy \\
&= 2^{n-\frac{n}{p}}\delta^{\alpha-\frac{n}{p}}\sum_{k=0}^{\infty}2^{k(\alpha-\frac{n}{p})}M_{\frac{n}{p}}(f)(x) \\
&\leq C\delta^{\alpha-\frac{n}{p}}M_{\frac{n}{p}}(f)(x).
\end{aligned}
$$

Thus,

$$I_\alpha(f)(x) \leq C\left(\delta^\alpha M_0(f)(x) + \delta^{\alpha-\frac{n}{p}}M_{\frac{n}{p}}(f)(x)\right).$$

To minimize this expression, we choose

$$\delta = \left[\frac{M_{\frac{n}{p}}(f)}{M_0(f)}\right]^{\frac{p}{n}}.$$

Then we obtain,

$$I_\alpha(f)(x) \leq CM_{\frac{n}{p}}(f)^{\frac{\alpha p}{n}}M_0(f)^{1-\frac{\alpha p}{n}}.$$

We have

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n} + \frac{\alpha p}{nq},$$

which yields

$$\frac{r\alpha p}{nq} + \frac{r(n-\alpha p)}{pn} = 1.$$

This means that $\left(\frac{nq}{r\alpha p}, \frac{pn}{r(n-\alpha p)}\right)$ is a conjugate pair. Using Hölder's inequality for this conjugate pair, we get

$$
\begin{aligned}
\|I_\alpha(f)\|_{L^r}^r &\leq C\int_{\mathbb{R}^n} M^{\frac{n}{p}}(f)^{\frac{r\alpha p}{n}}(x) M_0(f)^{r(1-\frac{\alpha p}{n})}(x)dx \\
&\leq C\|M_{\frac{n}{p}}(f)^{\frac{r\alpha p}{n}}\|_{L^{\frac{nq}{r\alpha p}}} \|M_0(f)^{r(1-\frac{\alpha p}{n})}\|_{L^{\frac{pn}{r(n-\alpha p)}}} \\
&= C\|M_{\frac{n}{p}}(f)\|_{L^q}^{\frac{r\alpha p}{n}} \|M_0(f)\|_{L^p}^{r(1-\frac{\alpha p}{n})}.
\end{aligned}
$$

Therefore,

$$
\|I_\alpha(f)\|_{L^r} \leq C\|M_{\frac{n}{p}}(f)\|_{L^q}^{\frac{\alpha p}{n}} \|M_0(f)\|_{L^p}^{(1-\frac{\alpha p}{n})}.
$$

We note that M_0 is the classical maximal function M which maps L^p to itself. Hence,

$$
\|I_\alpha(f)\|_{L^r} \leq \tilde{C}\|M_{\frac{n}{p}}(f)\|_{L^q}^{\frac{\alpha p}{n}} \|f\|_{L^p}^{(1-\frac{\alpha p}{n})}.
$$

□

Note that since $M_{n/p}f \leq C(M_n|f|^p)^{1/p}$ the inequality (2.3) becomes the familiar Sobolev inequality when $q = \infty$. M. Martin and P. Szeptycky in 1997 [32] give a generalization of the Hedberg inequality. We summarize this in the following theorem. Let $k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $k(tx) = t^{-\kappa n}k(x), x \in \mathbb{R}^n \setminus \{0\}, 0 < \kappa < 1$ and $t \in (0, \infty)$. The convolution operator T associated with k is defined as

$$
Tf(x) = k * f(x) = \int_{\mathbb{R}^n} k(y)f(x-y)dy,\ x \in \mathbb{R}^n.
$$

The maximal function associated to k is defined as

$$
\mathcal{M}f(x) = \sup_{t>0} \frac{1}{vol(tX)} \int_{tX} |f(x-y)|dy, x \in \mathbb{R}^n
$$

where

$$
\begin{aligned}
X =& \{x \in \mathbb{R}^n \setminus \{0\} : |k(x)| \geq 1\} \bigcup \{0\} \\
tX =& \{tx : x \in X\},\ t \in (0, \infty).
\end{aligned}
$$

It is assumed that $vol.(tX) \neq 0$. We observe that T and $\mathcal{M}$ become the classical Riesz potential I_α and the classical maximal function M respectively by taking $k(x) = |x|^{\alpha-n}$ where $0 < \alpha < n$. The following theorem is the generalization of the Hedberg inequality:

Theorem 39. *[32] Suppose that* $1 \leq p < (1-\kappa)^{-1}$. *Then*

$$|Tf(x)| \leq A\,(\mathcal{M}f(x))^{1-(1-\kappa)p}\,\|f\|_p^{(1-\kappa)p}, \tag{2.4}$$

for every $f \in L^p(\mathbb{R}^n)$ *and almost all* $x \in \mathbb{R}^n$, *with*

$$A = A(k,p) = \frac{1}{1-\kappa}\left[\frac{\kappa p}{1-(1-\kappa)p}vol(X)\right]^{\kappa}. \tag{2.5}$$

Moreover the inequality (2.4) *is sharp.*

Note that the Hedberg inequality (2.2) follows from (2.4) by taking $q^{-1} = p^{-1} - (1-\kappa)$. The best value of $A_{p.q}$ in (2.2) follows from (2.5) which is

$$A_{p,q} = \frac{q}{q-p}\left[\frac{pq-q+p}{p}vol.(\mathbb{B}^n)\right]^{1-p^{-1}+q^{-1}}. \tag{2.6}$$

Theorem 39 thus provides an improvement and generalization of the Hedberg inequality (2.2). The improvement amounts to determining the best value of the constant $A_{p,q}(\alpha)$, and the generalization is achieved by replacing the kernel function k_α with a broader class of homogeneous kernels, and providing the sharp form of the inequalities derived for these kernels. M. Martin and P. Szeptycki [33] in 2004 completely characterize the kernel function on $\mathbb{R}^n$ with the property that the associated convolution operators are controlled by certain maximal operators, in a way similar to Hedberg's inequality (2.2). Let $\mathbb{X}$ and $\mathbb{Y}$ be two metric measure spaces equipped with positive Borel measure. Let $K : \mathbb{X} \times \mathbb{Y} \longrightarrow [-\infty, \infty]$ be a measurable function on $\mathbb{X} \times \mathbb{Y}$. Let T_K denote the integral operator associated to K given by

$$T_K f(x) = \int_{\mathbb{Y}} K(x,y)f(y)dy, \ x \in \mathbb{X}$$

and defined for measurable function $f : \mathbb{Y} \longrightarrow \mathbb{R}$ such that the integral above exists for almost all $x \in \mathbb{X}$. Let

$$\Sigma = \{(x,y) \in \mathbb{X} \times \mathbb{Y} : |K(x,y)| = \infty\}$$

and assume that Σ has measure 0 with respect to the product measure on the product measure space $\mathbb{X} \times \mathbb{Y}$. Now for $x \in \mathbb{X}$ and $0 < t < \infty$ define the set $\Omega[x,t] = \{y \in \mathbb{Y} :$

$|K(x,y)| \geq t\}$ and let $\omega : (0,\infty) :\longrightarrow [0,\infty]$ be the function defined by $\omega(x,t) =$ measure $\Omega[x,t]$. That is $\omega(x,.) : (0,\infty) \longrightarrow [0,\infty]$ is the distribution function corresponding to the measurable function $K(x,.) : \mathbb{Y} \longrightarrow \mathbb{R}$. We always assume that $\omega(x,t) < \infty$. For every measurable function $f : \mathbb{Y} \longrightarrow \mathbb{R}$ we associate its maximal function m_K on $\mathbb{X}$ by setting

$$m_K(x) = \sup_{t>0} \frac{1}{\omega(x,t)} \int_{\Omega[x,t]} |f(y)dy, \ x \in \mathbb{X}.$$

We set $\dfrac{1}{\omega(x,t)} \int_{\Omega[x,t]} |f(y)|dy = 0$ if $\Omega[x,t] = 0$. Now we state the theorem due to M. Martin and P. Szeptycki [33] which provides Hedberg's inequality in its most general form.

Theorem 40. *[33] Suppose $1 < \kappa < \infty$. The following two statements are equivalent:*

(i) If $1 \leq p < \infty$ and $q^{-1} = p^{-1} - 1 + \kappa^{-1}$, then for every $x \in \mathbb{X}$ there exists a positive constant $A_{p,q}(K,x)$ such that

$$|T_K f(x)| \leq A_{p,q}(K,x)[m_K f(x)]^{p/q} \|f\|_p^{1-\frac{1}{q}}, \tag{2.7}$$

for every $f \in L^p(\mathbb{Y},\mathbb{R})$.

(ii) There exists a measurable function $\lambda :\longrightarrow [0,\infty]$ such that

$$\omega_K(x,t) \leq \lambda(x)t^{-\kappa}, \ x \in \mathbb{X}, \ 0 < t < \infty. \tag{2.8}$$

Whenever (i) or (ii) is true, and $\lambda(x)$ in (2.8) is defined as

$$\lambda(x) = \sup_{t>0} t^{\kappa}\omega(x,t), \ x \in \mathbb{X}, \tag{2.9}$$

a possible value of $A_{p,q}(K,x)$ in (2.7) is given by

$$A_{p,q}(K,x) = \frac{q}{q-p} \left[\frac{pq-q+p}{p} .\lambda(x) \right]^{1-p^{-1}+q^{-1}}. \tag{2.10}$$

Moreover, at points $x \in \mathbb{X}$ where (2.8) is an equality for all $t \in (0,\infty)$, the value of $A_{p,q}(K,x)$ in (2.10) is the best constant in inequality (2.7).

The above discussion is a short description of the improvement and generalization of Hedberg's inequality. Next follows the generalization of the Hedberg type inequalities presented in Chapter 3 of [4] to a metric measure space endowed with a Radon measure satisfying the growth condition (2.1). For this we start with the following two useful lemmas.

Lemma 41. *For every* $\gamma > 0$

$$\int_{B(x,r)} \frac{1}{d(x,y)^{N-\gamma}} d\mu(y) \leq Cr^{\gamma},$$

where C *is a constant.*

Proof. Suppose $N \leq \gamma$. Then $d(x,y) \leq r$ implies that $\frac{1}{d(x,y)^{N-\gamma}} \leq r^{\gamma-N}$. Using the condition (2.1) we obtain

$$\int_{B(x,r)} \frac{1}{d(x,y)^{N-\gamma}} d\mu(y) \leq r^{\gamma-N} Cr^N = Cr^{\gamma}.$$

Next suppose that $\gamma < N$ and let

$$B_k(x) = \{y : 2^{-k-1}r \leq d(x,y) < 2^{-k}r\}, k = 1, 2, 3, \dots$$

Then $B(x,r) = \bigcup_{k=0}^{\infty} B_k(x)$ a disjoint union. Also note that $B_k \subset B(x, 2^{-k}r)$ for every $k = 0, 1, 2 \dots$ Now using these remarks and the condition (2.1), we obtain

$$\begin{aligned}
\int_{B(x,r)} \frac{1}{d(x,y)^{N-\gamma}} d\mu(y) =& \sum_{k=0}^{\infty} \int_{B_k} \frac{1}{d(x,y)^{N-\gamma}} d\mu(y) \\
\leq& \sum_{k=0}^{\infty} \frac{1}{(2^{-k-1}r)^{N-\gamma}} \mu(B(x, 2^{-k}r)) \\
\leq& \sum_{k=0}^{\infty} \frac{2^{(k+1)(N-\gamma)}}{r^{N-\gamma}} C(2^{-k}r)^N \\
=& C\left(\frac{2^N}{2^{\gamma}-1}\right) r^{\gamma} = Cr^{\gamma}
\end{aligned}$$

□

We also have

Lemma 42. *For every $\gamma > 0$*

$$\int_{\mathbb{X}\setminus B(x,r)} \frac{1}{d(x,y)^{N+\gamma}} d\mu(y) \leq Cr^{-\gamma},$$

where C is a constant.

Proof. Let $B_k(x) = \{y : 2^k r \leq d(x,y) < 2^{k+1} r\}, k = 0, 1, 2, \ldots$. Then $B(x,r) = \bigcup_{k=0}^{\infty} B_k(x)$. Then

$$\begin{aligned}\int_{\mathbb{X}\setminus B(x,r)} \frac{1}{d(x,y)^{N+\gamma}} d\mu(y) =& \sum_{k=0}^{\infty} \int_{B_k(x)} \frac{1}{d(x,y)^{N+\gamma}} d\mu(y) \\ \leq& \sum_{k=0}^{\infty} \frac{\mu\left(B(x, 2^{k+1} r)\right)}{(2^k r)^{N+\gamma}} \\ \leq& C \sum_{k=0}^{\infty} \frac{(2^{k+1} r)^N}{(2^k r)^{N+\gamma}} \\ =& C \sum_{k=0}^{\infty} 2^{-\gamma k} r^{-\gamma} = Cr^{-\gamma}\end{aligned}$$

□

Lemma 43. *Let μ be a measure on a metric space $(\mathbb{X}, d)$ which satisfies the growth condition (2.1) and let f be a function on $\mathbb{X}$ which is either a nonnegative measurable function or $f \in L^1_{loc}(\mathbb{X})$. Let $0 < \alpha < N$. Then,*

(a) $\int_{d(x,y)<\delta} \frac{f(y)}{d(x,y)^{N-\alpha}} d\mu(y) = (N-\alpha) \int_0^\delta \left(\int_{B(x,r)} f(y) d\mu(y) \right) \frac{1}{r^{N-\alpha+1}} dr + \frac{1}{\delta^{N-\alpha}} \int_{B(x,\delta)} f(y) d\mu(y).$

(b) $\int_{d(x,y)<\delta} \log \frac{1}{d(x,y)} f(y) d\mu(y) = \int_0^\delta \left(\int_{B(x,r)} f(y) d\mu(y) \right) \frac{dr}{r} + \left(\int_{B(x,\delta)} f(y) d\mu(y) \right) \log(\frac{1}{\delta})$

(c) $\int_{d(x,y)\geq\delta} \frac{f(y)}{d(x,y)^{N-\alpha}} d\mu(y) = (N-\alpha) \int_\delta^\infty \left(\int_{B(x,r)} f(y) d\mu(y) \right) \frac{1}{r^{N-\alpha+1}} dr - \frac{1}{\delta^{N-\alpha}} \int_{B(x,\delta)} f(y) d\mu(y).$

Proof. (a) Using Fubini, we get

$$\begin{aligned}
&(N-\alpha)\int_0^\delta \left(\int_{B(x,r)} f(y)d\mu(y)\right)\frac{1}{r^{N-\alpha+1}}dr\\
&= (N-\alpha)\int_{B(x,\delta)} f(y)\left(\int_{d(x,y)}^\delta \frac{dr}{r^{N-\alpha+1}}\right)d\mu(y)\\
&= \int_{B(x,\delta)} f(y)\left(\frac{1}{d(x,y)^{N-\alpha}}-\frac{1}{\delta^{N-\alpha}}\right)d\mu(y)\\
&= \int_{B(x,\delta)} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}}-\frac{1}{\delta^{N-\alpha}}\int_{B(x,\delta)} f(y)d\mu(y)
\end{aligned}$$

(b) Using Fubini, we get

$$\begin{aligned}
&\int_0^\delta \left(\int_{B(x,r)} f(y)d\mu(y)\right)\frac{dr}{r}\\
&=\int_{B(x,\delta)}\left(\int_{d(x,y)}^\delta \frac{dr}{r}\right)f(y)d\mu(y)\\
&=\int_{B(x,\delta)}\left(\log\frac{1}{d(x,y)}-\log\frac{1}{\delta}\right)f(y)d\mu(y)\\
&=\int_{B(x,\delta)}\log\frac{1}{d(x,y)}f(y)d\mu(y)-\log\frac{1}{\delta}\int_{B(x,\delta)} f(y)d\mu(y).
\end{aligned}$$

(c) Using Fubini, we get

$$\begin{aligned}
&(N-\alpha)\int_\delta^\infty \left(\int_{B(x,r)} f(y)d\mu(y)\right)\frac{1}{r^{N-\alpha+1}}dr\\
=&(N-\alpha)\int_{B(x,\delta)}\left(\int_0^\delta \frac{dr}{r^{N-\alpha+1}}\right)f(y)d\mu(y)+(N-\alpha)\int_{B(x,\delta)^c}\left(\int_{d(x,y)}^\infty \frac{dr}{r^{N-\alpha+1}}\right)f(y)d\mu(y)\\
=&(N-\alpha)\int_{B(x,\delta)}\left[-\frac{1}{N-\alpha}.\frac{1}{r^{N-\alpha}}\right]_\delta^\infty f(y)d\mu(y)+(N-\alpha)\int_{B(x,\delta)^c}\left[-\frac{1}{N-\alpha}.\frac{1}{r^{N-\alpha}}\right]_{d(x,y)}^\infty f(y)d\mu(y)\\
=&\int_{B(x,\delta)}\frac{1}{\delta^{N-\alpha}}f(y)d\mu(y)+\int_{B(x,\delta)^c}\frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}}.
\end{aligned}$$

□

Proposition 44. *Let μ be a measure on a metric space $(\mathbb{X},d)$ which satisfies the growth condition (2.1). For $0<\alpha<N, 1\le p<\infty$, there exists a constant $A=A(\alpha,p,N)$ such that for any measurable function $f\ge 0$ and $x\in\mathbb{X}$*

(a) $I_\alpha f(x)\le A\|f\|_p^{\alpha p/n} Mf(x)^{1-\frac{\alpha p}{N}} \qquad 1\le p<\dfrac{N}{\alpha}.$

(b) $I_{\alpha\theta}f(x) \le A\left(I_\alpha f(x)\right)^\theta Mf(x)^{1-\theta} \quad 0<\theta<1.$

(c) $I_{\alpha\theta}f(x) \le AM_\alpha f(x)^\theta Mf(x)^{1-\theta} \qquad 0<\theta<1.$

Proof. (a) We claim that

(1) $\int_{d(x,y)<\delta} \dfrac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \le A\delta^\alpha Mf(x).$

(2) $\int_{d(x,y)\ge\delta} \dfrac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \le A\delta^{\alpha-N/p}\|f\|_{L^p}.$

Proof of (1). Using the part (a) of Lemma 43, we obtain

$$\begin{aligned}
\int_{d(x,y)<\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} =&(N-\alpha)\int_0^\delta \left(\int_{B(x,r)} f(y)d\mu(y)\right)\frac{1}{r^{N-\alpha+1}}dr + \frac{1}{\delta^{N-\alpha}}\int_{B(x,\delta)} f(y)d\mu(y)\\
\le&(N-\alpha)Mf(x)\int_0^\delta \frac{\mu(B(x,r))}{r^{N-\alpha+1}}dr + \frac{1}{\delta^{N-\alpha}}Mf(x)\mu(B(x,\delta))\\
\le&(N-\alpha)Mf(x)\int_0^\delta \frac{cr^N}{r^{N-\alpha+1}} + \frac{1}{\delta^{N-\alpha}}Mf(x)C\delta^N\\
=&(N-\alpha)Mf(x)\int_0^\delta r^{\alpha-1}dr + C\delta^\alpha Mf(x)\\
=&C(\frac{N-\alpha}{\alpha})Mf(x)\delta^\alpha + C\delta^\alpha Mf(x)\\
=&CMf(x)\delta^\alpha\left(\frac{N-\alpha}{\alpha}+1\right)\\
=&A\delta^\alpha Mf(x).
\end{aligned}$$

Proof of (2). We observe that $(N-\alpha)\dfrac{p}{p-1} = N + \left(\dfrac{N}{p}-\alpha\right)\dfrac{p}{p-1}$. Now, using Hölder's inequality and Lemma 42, we obtain

$$\begin{aligned}
&\int_{d(x,y)\ge\delta}\frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}}\\
&\qquad\le\|f\|_p\left(\int_{d(x,y)\ge\delta}\frac{d\mu(y)}{d(x,y)^{(N-\alpha)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}}\\
&\qquad\le\|f\|_p\left(\int_{d(x,y)\ge\delta}\frac{d\mu(y)}{d(x,y)^{N+(\frac{N}{p}-\alpha)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}}\\
&\qquad=\|f\|_pC\delta^{-(\frac{N}{p}-\alpha)}\\
&\qquad=A\delta^{\alpha-N/p}\|f\|_p.
\end{aligned}$$

Now, using the claims (1) and (2) we obtain

$$
\begin{aligned}
I_\alpha f(x) &= \int_{\mathbb{X}} \frac{f(y)\, d\mu(y)}{d(x,y)^{N-\alpha}} \\
&= \int_{d(x,y)<\delta} \frac{f(y)\, d\mu(y)}{d(x,y)^{N-\alpha}} + \int_{d(x,y)\geq\delta} \frac{f(y)\, d\mu(y)}{d(x,y)^{N-\alpha}} \\
&\leq A\delta^\alpha M(f)(x) + A\delta^{\alpha-N/p}\|f\|_p.
\end{aligned}
$$

Then by taking $\delta = \delta(x) = \left(\dfrac{\|f\|_p}{Mf(x)}\right)^{p/N}$ we obtain,

$$
\begin{aligned}
I_\alpha f(x) \leq A\left(\frac{\|f\|_p}{M(f)(x)}\right)^{\alpha p/N} &+ A\left(\frac{\|f\|_p}{M(f)(x)}\right)^{\frac{p}{N}(\alpha-\frac{N}{p})} \|f\|_p \\
=&A\|f\|_p^{\alpha p/N} M(f)(x)^{1-\alpha p/N} + A\|f\|_p^{\alpha p/N} M(f)(x)^{1-\alpha p/N} \\
=&C\|f\|_p^{\alpha p/N} M(f)(x)^{1-\alpha p/N}
\end{aligned}
$$

Proof of (b):

Note that $0 < \alpha\theta < N$ for $0 < \alpha < N$ and $0 < \theta < 1$. So we can replace α by $\alpha\theta$ in claim (1) of part (a) in Proposition 44 and get

$$
\int_{d(x,y)<\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} \leq A\delta^{\alpha\theta} M(f)(x). \tag{2.11}
$$

Note that

$$
N - \alpha\theta = (N-\alpha) + (\alpha - \alpha\theta).
$$

So, $d(x,y) \geq \delta$ implies that

$$
\left(\frac{1}{d(x,y)}\right)^{\alpha-\alpha\theta} \leq \left(\frac{1}{\delta}\right)^{\alpha-\alpha\theta} = \delta^{\alpha\theta-\alpha}. \tag{2.12}
$$

Now, using the inequalities (2.11) and (2.12), we obtain

$$
\begin{aligned}
I_{\alpha\theta} f(x) &= \int_X \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} \\
&= \int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} + \int_{d(x,y)<\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} \\
&\leq \delta^{\alpha\theta-\alpha} \int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} + \int_{d(x,y)<\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} \\
&\leq A\left(\delta^{\alpha\theta-\alpha} I_\alpha f(x) + \delta^{\alpha\theta} Mf(x)\right)
\end{aligned}
$$

Now we choose $\delta = \left(\frac{I_\alpha f(x)}{Mf(x)}\right)^{1/\alpha}$ and obtain,

$$
\begin{aligned}
I_{\alpha\theta} f(x) &= \int_{\mathbb{X}} \frac{f(y)\, d\mu(y)}{d(x,y)^{n-\alpha\theta}} \\
&= A\left(\frac{I_\alpha f(x)}{M(f)(x)}\right)^{\theta-1} I_\alpha f(x) + A\left(\frac{I_\alpha f(x)}{M(f)(x)}\right)^{\theta} M_\alpha(f)(x) \\
&\leq A\,(I_\alpha f(x))^\theta\, M(f)(x)^{1-\theta}
\end{aligned}
$$

Proof of (c): We claim that

$$
\int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} \leq A\delta^{\alpha\theta-\alpha} M_\alpha f(x).
$$

Proof of the Claim:

$$
\begin{aligned}
&\int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} \\
&= \int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{(N-\alpha)+(\alpha-\alpha\theta)}} \\
&\leq \delta^{\alpha\theta-\alpha}\delta^{\alpha-N} \int_{d(x,y)\geq\delta} f(y)d\mu(y) \\
&= \delta^{\alpha\theta-\alpha}\delta^{\alpha-N} \mu(B(x,\delta))^{\frac{N-\alpha}{N}} \frac{1}{\mu(B(x,\delta))^{\frac{N-\alpha}{N}}} \int_{d(x,y)\geq\delta} f(y)d\mu(y) \\
&\leq A\delta^{\alpha\theta-\alpha} M_\alpha(f)(x).
\end{aligned}
$$

We also have,

$$
\int_{d(x,y)<\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha\theta}} \leq A\delta^{\alpha\theta} M(f)(x).
$$

Now taking

$$
\delta = \left(\frac{M_\alpha(f)(x)}{M(f)(x)}\right)^{1/\alpha}
$$

we obtain the desired inequality. □

2.3 Modified Centered Maximal Function

Definition 45. *Set $(\mathbb{X}, d)$ be a metric space endowed with a Radon measure μ such that $\mu(B) > 0$ for all ball B with positive radius. Such balls are called non-degenerate. We define the k times modified centered Hardy-Littlewood maximal operator as follows:*

$$M_k f(x) = \sup_{r>0} \frac{1}{\mu\left(B(x, kr)\right)} \int_{B(x,r)} |f(y)| d\mu(y).$$

Lemma 46. *Let μ be a Radon measure which satisfies the growth condition (2.1). Then for any measurable function f on $\mathbb{X}$*

$$\int_{d(x,y)<\delta} \frac{f(y) d\mu(y)}{d(x,y)^{N-\alpha}} \leq A\delta^{\alpha} M_k f(x).$$

Proof. Without loss of generality, we may assume that $f \geq 0$. By part (a) of lemma 43,

$$\begin{aligned}
&\int_{d(x,y)<\delta} \frac{f(y) d\mu(y)}{d(x,y)^{N-\alpha}} \\
&= (N-\alpha) \int_0^{\delta} \left(\int_{B(x,r)} f(y) d\mu(y) \right) \frac{1}{r^{N-\alpha+1}} dr + \frac{1}{\delta^{N-\alpha}} \int_{B(x,\delta)} f(y) d\mu(y). \\
&= (N-\alpha) \int_0^{\delta} \mu\left(B(x, kr)\right) M_k f(x). \frac{1}{r^{N-\alpha+1}} dr + \frac{1}{\delta^{N-\alpha}} \mu\left(B(x, k\delta)\right) M_k f(x). \\
&\leq (N-\alpha) C k^N M_k f(x) \int_o^{\delta} r^N \frac{1}{N-\alpha+1} dr + \frac{1}{\delta^{N-\alpha}} C k^N \delta^N M_k f(x). \\
&= C k^N \delta^{\alpha} M_k f(x) \left(\frac{N-\alpha}{\alpha} + 1 \right). \\
&= A\delta^{\alpha} M_k f(x).
\end{aligned}$$

□

Lemma 47. *Let μ be a Radon measure which satisfies the growth condition (2.1). Then*

$$\int_{d(x,y)\geq\delta} \frac{f(y) d\mu(y)}{d(x,y)^{N-\alpha}} \leq A\delta^{\alpha - N/p} \|f\|_{L^p(\mathbb{X},\mu)}.$$

Proof. Note that $(N-\alpha)\dfrac{p}{p-1} = N + \left(\dfrac{N}{p}-\alpha\right).\dfrac{p}{p-1}$. Using the Hölder's inequality and Lemma 41, we get

$$\begin{aligned}
\int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} &\leq \|f\|_{L^p(\mathbb{X},\mu)} \left(\int_{d(x,y)\geq\delta} \frac{d\mu(y)}{d(x,y)^{(N-\alpha)(\frac{p}{p-1})}}\right)^{\frac{p-1}{p}} \\
&\leq \|f\|_{L^p(\mathbb{X},\mu)} \left(\int_{d(x,y)\geq\delta} \frac{d\mu(y)}{d(x,y)^{N+(N/p-\alpha)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}}. \\
&\leq \|f\|_{L^p(\mathbb{X},\mu)} C\delta^{-(\frac{N}{p}-\alpha)} \\
&= A\delta^{\alpha-N/p}\|f\|_{L^p(\mathbb{X},\mu)}.
\end{aligned}$$

□

Proposition 48. *For $0<\alpha<N, 1\leq p<\infty$, there exists a constant $A=A(\alpha,p,N)$ such that for any measurable function $f\geq 0$ and $x\in\mathbb{X}$*

$$I_\alpha f(x) \leq A\|f\|_p^{\alpha p/N} M_k f(x)^{1-\alpha p/N}$$

where $1\leq p<\dfrac{N}{\alpha}$.

Proof. The proof follows from the above two lemmas and by taking $\delta=\delta(x)=\left(\dfrac{\|f\|_p}{M_k f(x)}\right)^{p/N}$. □

Lemma 49. *Let $(\mathbb{X},d,\mu)$ be a metric measure space and let $W\subseteq\bigcup_{i=1}^N B(x_i,r_i)$. Then there exists a set $S\subseteq\{1,2,3,\ldots,N\}$ such that*

(a) the balls $\{B(x_i,r_i): i\in S\}$ are disjoint, and

(b) $W\subseteq\bigcup_{i\in S} B(x_i,3r_i)$.

Proof. Without loss of generality, we can reorder the balls $B_i=B(x_i,r_i)$ in such a way that $r_1\geq r_2\geq r_3\geq\ldots\geq r_N$. Let $i_1=1$ and consider the ball $B(x_{i_1},r_{i_1})$. We discard all B_j that intersect B_{i_1}. Let i_2 be the smallest index among the remaining balls. That is $B(x_{i_2},r_{i_2})$ is the largest ball disjoint from $B(x_{i_1},r_{i_1})$. Now we discard all B_j with $j>i_2$ that intersect

B_{i_2}. Let B_{i_3} be the first of the remaining balls. We proceed as long as possible. The process stops after a finite number of steps, say t, and gives $S = \{i_1, i_2, \dots, i_t\}$. Part (a) follows from the way the balls have been constructed.

Next we show that part (b) holds. For this let $x \in W$. Then $x \in B(x_j, r_j)$ for some $j \in \{1, 2, 3, \dots, N\}$. If $x \in B(x_j, r_j)$ is one of $B(x_{i_k}, r_{i_k})$ for some $i_k \in S$ then we are done. Suppose $B(x_j, r_j) \neq B(x_{i_k}, r_{i_k})$ for every $i_k \in S$. Then $B(x_j, r_j)$ is one of the balls B_j discarded at some stage. That is

$$B(x_j, r_j) \cap B(x_{i_k}, r_{i_k}) \neq \phi \quad \text{for some} \quad i_k \in S$$

with $r_{i_k} \geq r_j$. Then $B(x_j, r_j) \subseteq B(x_{i_k}, 3r_{i_k})$. That is $x \in B(x_{i_k}, 3r_{i_k})$ for some $i_k \in S$. This proves part (b). □

Theorem 50. *Let $(\mathbb{X}, d)$ be a metric space endowed with a Radon measure μ. Then the modified 3-times Hardy Littlewood maximal operator M_3 defined by*

$$M_3 f(x) = \sup_{r>0} \frac{1}{\mu(B(x, 3r))} \int_{B(x,r)} |f(y)| d\mu(y)$$

is weak-(1,1). Namely,

$$\mu\left(\{x : M_3 f(x) > \lambda\}\right) \leq \frac{1}{\lambda} \int_{B(x,r)} |f(y)| d\mu(y)$$

for every $f \in L^1(\mathbb{X}, \mu)$.

Proof. Let $E_\lambda = \{x : M_3 f(x) > \lambda\}$. We first prove that the set E_λ is open. Choose $x_0 \in E_\lambda$. Then there exists $r > 0$ such that

$$\frac{1}{\mu(B(x_0, 3r))} \int_{B(x_0,r)} |f(y)| d\mu(y) > \lambda.$$

By the absolute continuity of the integral, there exists a compact set $K \subset B(x_0, r)$ such that

$$\frac{1}{\mu(B(x_0, 3r))} \int_{K} |f(y)| d\mu(y) > \lambda.$$

If we take $\delta > 0$ sufficiently small, then for any y satisfying $|y - x_0| < \delta$, it holds that $K \subseteq B(y, r)$ and that

$$\lambda < \frac{1}{\mu(B(y, 3r))} \int_K |f(y)| d\mu(y) \leq \frac{1}{\mu(B(y, 3r))} \int_{B(y,r)} |f(y)| d\mu(y).$$

Therefore the set E_λ is an open set.

Let K be a compact set and $K \subseteq B(x, r)$ such that $\mu\left(B(x, r) \setminus K\right) < \delta$. On the other hand, given $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable set E

$$\mu(E) < \delta \quad \text{implies that} \quad \int_E f(y) d\mu(y) < \epsilon.$$

Therefore,

$$\int_{B(x,r) \setminus K} f(y) d\mu(y) < \epsilon. \tag{2.13}$$

Since $K \subseteq B(x, r)$ there exists $\eta > 0$ such that $d\left(K, B(x, r)^c\right) > \eta$. Pick a $z \in \mathbb{X}$ such that $d(z, x) < \dfrac{\eta}{2}$. Now we claim that

$$K \subseteq B(z, r - \frac{\eta}{2}) \subseteq B(x, r).$$

Note that $K \subseteq \bigcup_{n=1}^{\infty} B(x, r - \dfrac{1}{n})$ and K is compact. So there exists a natural number N such that $K \subseteq B(x, r - \dfrac{1}{N})$. Without loss of generality, we may choose η sufficiently small so that $N < \dfrac{1}{\eta}$. Thus, $K \subseteq B(x, r - \dfrac{1}{N}) \subseteq B(z, r - \dfrac{\eta}{2})$ and therefore our claim follows. Now, using these inclusions and the inequality (2.13), it follows that

$$\begin{aligned}
M_3 f(z) \geq & \frac{1}{\mu\left(B(z, 3(r - \frac{\eta}{2}))\right)} \int_{B(z, r - \frac{\eta}{2})} |f(y)| d\mu(y) \\
\geq & \frac{1}{\mu\left(B(x, 3r)\right)} \left[\int_K |f(y) d\mu(y) - \int_{B(x,r)} |f(y) d\mu(y) + \int_{B(x,r)} |f(y)| d\mu(y) \right] \\
= & - \frac{1}{\mu\left(B(x, 3r)\right)} \int_{B(x,r) \setminus K} |f(y)| d\mu(y) + \frac{1}{\mu\left(B(x, 3r)\right)} \int_{B(x,r)} |f(y)| d\mu(y) \\
> & - \frac{\epsilon}{\mu\left(B(x, 3r)\right)} + \lambda.
\end{aligned}$$

This is true for every $\epsilon > 0$. Therefore $M_3 f(z) > \lambda$. This proves that M_3 is lower semi-continuous and therefore the set E_λ is open. Finally, we prove that μ is weak-(1,1). Because

μ is Radon measure and E_λ is open

$$\mu(E_\lambda) = \sup\{\mu(K) : K \subseteq E, K \text{is compact}\}.$$

Let $K \subseteq E_\lambda$ and K be compact. Then for every $x \in K$ there exists $r_x > 0$ such that

$$\frac{1}{\mu\left(B(x, 3r_x)\right)} \int_{B(x,r_x)} |f(y)| d\mu(y) > \lambda.$$

Then the collection $\{B(x, r_x) : x \in K\}$ is an open covering of K. So there exists a finite subcover

$$B(x_j, r_j) \quad j = 1, 2, 3 \ldots N$$

which also cover K. That is $K \subseteq \bigcup_{j=1}^{j} B(x_j, r_j)$. By above lemma there exists $S \subseteq \{1, 2, 3, \ldots, N\}$ such that the collection of the ball $\{B(x_t, r_t) : t \in S\}$ is disjoint and $K \subseteq \bigcup_{t \in S} B(x_t, 3r_t)$. Then

$$\begin{aligned}
\mu(K) \leq & \sum_{t \in S} \mu\left(B(x_t, 3r_t)\right) \\
\leq & \frac{1}{\lambda} \sum_{t \in S} \int_{B(x_t, r_t)} |f(y)| d\mu(y) \\
\leq & \frac{1}{\lambda} \|f\|_{L^1(\mathbb{X}, \mu)}.
\end{aligned}$$

□

With stronger hypothesis on the metric measure space $(\mathbb{X}, \mu)$ a little more can be said. The following theorem on nonhomogeneous space is due to Y. Terasawa.

Theorem 51. *Let $\mathbb{X}$ be a metric space possessing a non-degenerate Radon measure μ such that $\mu(B(x, r))$ is continuous in the variable $r > 0$ when $x \in \mathbb{X}$ is fixed. Then $M_k f(x) = \sup_{r>0} \frac{1}{\mu\left(B(x, kr)\right)} \int_{B(x,r)} |f(y)| d\mu(y)$ is weak-$(1, 1)$ bounded with constant 1 when $k \geq 2$. Namely,*

$$\mu\left(\{x : M_k f(x) > \lambda\}\right) \leq \frac{1}{\lambda} \int_{\mathbb{X}} |f(y)| d\mu(y)$$

for any $f \in L^1(\mathbb{X}, \mu)$ when $k \geq 2$.

Remark 52. *Obviously,* $M_k : L^\infty(\mathbb{X},\mu) \to L^\infty(\mathbb{X},\mu)$*.We know from above theorem that* $M_k (k \geq 2)$ *is weak-*$(1,1)$*. Therefore by the Marcinkiewicz interpolation theorem it follows that* $M_k : L^p(\mathbb{X},\mu) \to L^p(\mathbb{X},\mu)$ *is continuous for* $k \geq 2$ *and* $1 < p \leq \infty$*, that is, there exists a constant* $C = C(p,k)$ *such that*

$$\|M_k(f)\|_{L^p(\mathbb{X},\mu)} \leq C\|f\|_{L^p(\mathbb{X},\mu)}.$$

Theorem 53. *Let* $\mathbb{X}$ *be a metric space ,* $0 < \alpha < N, 1 < p < q < \infty$*, and* $1/q = 1/p - \alpha/N$ *where* N *is the dimension of the growth condition* (2.1)*. Then there exists a constant* $A = A(p,k)$ *such that*

$$\|I_\alpha(f)\|_{L^q(\mathbb{X},\mu)} \leq A\|f\|_{L^p(\mathbb{X},\mu)}.$$

Proof. We observe that $q\left(1 - \dfrac{\alpha p}{N}\right) = p$. Then by using Proposition 48 and the Remark 52 we obtain,

$$\begin{aligned}
\|I_\alpha(f)\|^q_{L^q} &= \int_{\mathbb{X}} |I_\alpha(f)(x)|^q d\mu(x)\\
&\leq \int_{\mathbb{X}} |A\|f\|_p^{\alpha p/N} M_k f(x)^{1-\alpha p/N}|^q d\mu(x)\\
&=A^q \int_{\mathbb{X}} \|f\|_p^{q\alpha p/N} M_k f(x)^{q(1-\alpha p/N)} d\mu(x)\\
&=A^q \|f\|_{L^p}^{\frac{\alpha pq}{N}} \int_{\mathbb{X}} M_k(f)(x)^p d\mu(x)\\
&=A^q \|f\|_{L^p}^{\frac{\alpha pq}{N}} \|M_k f\|^p_{L^p}\\
&\leq CA^q \|f\|_{L^p}^{\frac{\alpha p}{N}q} \|f\|^p_{L^p}\\
&=CA^q \|f\|_{L^p}^{q-p} \|f\|^p_{L^p}\\
&=CA\|f\|^q_{L^p}.
\end{aligned}$$

Therefore the theorem follows. ∎

Proposition 54. *Let* $(\mathbb{X}, d)$ *be a metric space and* μ *be a Radon measure on* $\mathbb{X}$ *with*

$$\mu\left(B(x,r)\right) \leq Cr^N, \quad \textit{for every} \quad r > 0, x \in X, \quad 1 \leq p < \infty, \quad \alpha p = N.$$

Then there exists a constant $A = A(\alpha, p)$ such that for every $\epsilon > 0$ and any nonnegative function $f \in L^p(\mathbb{X}, \mu)$ with support in the ball $B(x_0, R)$ and $\|f\|_{L^p(\mu)} = 1$,

$$(I_\alpha f(x) - \epsilon)_+^{p'} \leq C\left(1 + \log_+ \frac{AR^N M_k(f)(x)^p}{\epsilon^p}\right), \quad x \in B(x_0, R).$$

Proof. Let $x \in B(0, R)$ and $\delta < 2R$. For $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $\alpha p =N$ it follows that $(N - \alpha)p'$ =N. Now using this and Hölder's inequality we get

$$\begin{aligned}
\int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} &= \int_{\delta\leq d(x,y)\leq 2R} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \\
&\leq \left(\int_{\delta\leq d(x,y)\leq 2R} \frac{d\mu(y)}{d(x,y)^{(N-\alpha)p'}}\right)^{1/p'} \left(\int_{\delta\leq d(x,y)\leq 2R} |f|^p d\mu(y)\right)^{1/p} \\
&\leq \left(\int_{\delta\leq d(x,y)\leq 2R} \frac{d\mu(y)}{d(x,y)^N}\right)^{1/p'} \|f\|_p \\
&= \left(\int_{\delta\leq d(x,y)\leq 2R} \frac{d\mu(y)}{d(x,y)^N}\right)^{1/p'}
\end{aligned}$$

Now,

$$\begin{aligned}
\int_\delta^{2R} \frac{\mu(B(x,r))}{r^N}\frac{dr}{r} &= \int_\delta^{2R} \frac{1}{r^N}\int_{B(x,r)} d\mu(y)\frac{dr}{r} \\
&= \int_\delta^{2R}\int_{B(x,r)} \frac{1}{r^{N+1}} d\mu(y)dr \\
&= \int_{B(x,\delta)}\left(\int_\delta^{2R} \frac{1}{r^{N+1}}dr\right)d\mu(y) + \int_{B(x,2R)\setminus B(x,\delta)}\left(\int_{d(x,y)}^{2R} \frac{1}{r^{N+1}}dr\right)d\mu(y) \\
&= \int_{B(x,\delta)}\left(\frac{1}{N\delta^N} - \frac{1}{N(2R)^N}\right)d\mu(y) + \int_{B(x,2R)\setminus B(x,\delta)} \frac{1}{N}\left(\frac{1}{d(x,y)^N} - \frac{1}{(2R)^N}\right)d\mu(y) \\
&= \frac{\mu(B(x,\delta))}{N\delta^N} - \left(\frac{\mu(B(x,\delta)) + \mu(B(x,2R)\setminus B(x,\delta))}{N(2R)^N}\right) + \frac{1}{N}\int_{B(x,2R)\setminus B(x,\delta)} \frac{d\mu(y)}{d(x,y)^N} \\
&= \frac{\mu(B(x,\delta))}{N\delta^N} - \frac{\mu(B(X,2R))}{N(2R)^N} + \frac{1}{N}\int_{B(x,2R)\setminus B(x,\delta)} \frac{d\mu(y)}{d(x,y)^N}
\end{aligned}$$

Therefore,

$$\int_{B(x,2R)\setminus B(x,\delta)} \frac{d\mu(y)}{d(x,y)^N} = N\int_\delta^{2R} \frac{\mu(B(x,r))}{r^N}\frac{dr}{r} - \frac{\mu(B(x,\delta))}{\delta^N} + \frac{\mu(B(X,2R))}{(2R)^N}.$$

Because $\mu(B(x,r)) \leq CR^N$, it follows that

$$\int_{B(x,2R)\setminus B(x,\delta)} \frac{d\mu(y)}{d(x,y)^N} \leq C + CN\log(\frac{2R}{\delta}).$$

From Lemma 46 we know that if $k \geq 2$ then

$$\int_{d(x,y)<\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \leq A_1\delta^\alpha M_k(f)(x).$$

Let

$$\delta^\alpha = \min\{\frac{\epsilon}{A_1 M_k(f)(x)}, (2R)^\alpha\}.$$

Suppose $\delta^\alpha = \dfrac{\epsilon}{A_1 M_k(f)(x)}$. Then,

$$\begin{aligned}\int_{B(x_0,R)} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} &= \int_{d(x,y)<\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} + \int_{d(x,y)\geq\delta} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \\ &\leq A_1\delta^\alpha M_k(f)(x) + \left(C + CN\log(\frac{2R}{\delta})\right)^{1/p'} \\ &= \epsilon + C^{1/p'}\left(1 + \log_+ \frac{AR^N M_k(f)(x)^p}{\epsilon^p}\right)^{1/p'}\end{aligned}$$

Therefore,

$$(I_\alpha f(x) - \epsilon)_+^{p'} \leq C\left(1 + \log_+ \frac{AR^N M_k(f)(x)^p}{\epsilon^p}\right).$$

Next suppose $\delta^\alpha = (2R)^\alpha \leq \dfrac{\epsilon}{A_1 M(f)(x)}$. Then $\delta = 2R$. This yields,

$$\int_{B(x_0,R)} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \leq \epsilon + C^{1/p'}$$

This implies that

$$(I_\alpha f(x) - \epsilon)^{p'} \leq C.$$

Therefore the desired inequality follows because $\log_+\left(\dfrac{AR^N M_k(f)(x)^p}{\epsilon^p}\right) \geq 0$.

$\square$

Theorem 55. *(a) Let $f \in L^1(\mathbb{X})$, and $0 < \alpha < N$. Then*

$$\mu\left(\{x : |I_\alpha f(x)| > \lambda\}\right) \leq A\left(\lambda^{-1}\|f\|_{L^1}\right)^{N/N-\alpha}$$

(b) Let $f \in L^p(\mathbb{X}), 0 < \alpha < N, 1 < p < N/\alpha,.$ Set $p^* = Np/(N - \alpha p)$. Then

$$\|I_\alpha f\|_{L^{p^*}} \leq A\|f\|_p.$$

Theorem 56. *Let $f \in L^p(\mathbb{R}^\mathbb{N}, \mu), 0 < \alpha < N, p = N/\alpha$ where the measure μ is non-doubling satisfying the growth condition (2.1). Assume that $supp(f) \subseteq B(0, R)$, and $\|f\|_{L^p(\mu)} = 1$. Then*

$$\int_{B(0,R)} \exp\left(\beta|I_\alpha f(x)|^p\right) d\mu(x) \leq AR^N,$$

whenever $\beta < \beta_0 = \dfrac{\gamma_\alpha^{-p'} N}{\omega_{N-1}}$ where ω_{N-1} is the area of $(N-1)$-dimensional unit sphere.

Proof. We know that for any $c < 1$ and $\epsilon > 0$ there is a constant A such that

$$cx^{p'} \leq (x - \epsilon)^{p'} + A \qquad \text{for all } x \geq \epsilon. \tag{2.14}$$

First, assume that $I_\alpha f(x) \geq \epsilon$. Then from the inequality (2.14) for given $c = \dfrac{\beta}{\beta_0} < 1$ and $\epsilon > 0$ there exists a constant A such that

$$\begin{aligned}
c|I_\alpha f(x)|^{p'} &\leq (I_\alpha f(x) - \epsilon)_+^{p'} + A \\
\beta|I_\alpha f(x)|^{p'} &\leq \beta_0 \left(I_\alpha f(x) - \epsilon\right)_+^{p'} + A\beta_0 \\
&\leq \beta_0 \left(1 + \log_+ \frac{AR^N M_k(f)(x)^p}{\epsilon^p}\right) + A\beta_0 \\
&= \log_+ \left(\frac{AR^N M_k(f)(x)^p}{\epsilon^p}\right)^{\beta_0} + (A+1)\beta_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{B(0,R)} \exp\left(\beta|I_\alpha f(x)|^{p'}\right) d\mu(x) &\leq \int_{B(0,R)} \exp\left(1 + AC^{-1}\right) \left[\frac{AR^N M_k(f)(x)^p}{\epsilon^p}\right] d\mu(x) \\
&\leq AR^N \|f\|_{L^p(\mu)}^p = AR^N.
\end{aligned}$$

Finally, assume that $I_\alpha f(x) < \epsilon$. Then

$$\int_{B(0,R)} \exp\left(\beta|I_\alpha f(x)|^{p'}\right) d\mu(x) \leq \int_{B(0,R)} e^{\beta\epsilon^{p'}} d\mu(x) \leq AR^N.$$

$\square$

Chapter 3

Good Lambda Inequality

3.1 Introduction

A good λ inequality is a principle which allows us to derive a local or pointwise estimate of one operator in terms of another provided they satisfy an a *priori* relation of measure theoretic nature. In this chapter, we establish a good lambda inequality relating the distribution functions of Riesz potential and the fractional maximal function on $\mathbb{R}^n, d\mu$, where μ is a positive Radon measure which doesn't necessarily satisfy a doubling condition. This is extended to weights w in $A_\infty(\mu)$ associated to the measure μ. We also derive potential inequalities as an application.

Definition 57. *Let μ be a positive, regular Borel measure on $\mathbb{R}^n$ and let T_1 and T_2 be positive sublinear operators on $\mathbb{R}^n$. We say that T_1 and T_2 satisfy the good λ inequality if the following two conditions hold:*

(i) $\{T_1 f > t\}$ is an open set of finite Lebesgue measure for each f in $C_0^\infty(\mathbb{R}^n)$ and $t > 0$.

(ii) If a ball B contains a set $\{x : T_1 f(x) \leq \lambda\}$, then to each $0 < \gamma < 1$ there exists $\epsilon = \gamma(T_1, T_2, \epsilon)$ independent of λ, B and f so that $\mu(\{y \in B : T_1 f > \lambda, T_2 f(y) \leq \epsilon\lambda\}) \leq \gamma\mu(B)$.

The so called "good-λ inequality" was first used by D. L. Burkholder and R. F. Gundy in 1970 ([12]). Let X_t be a continuous martingale starting at 0 and set $X_t^* = \sup_{0<s<t} |X_s|$, $\quad X^* =$

$\sup_{t>0}|X_t|$, $S_t(X) = \sqrt{\langle X\rangle_t}$ and $S(X) = \sqrt{\langle X\rangle_\infty}$, where $\langle X\rangle_t$ is the quadratic variation process of X_t at time t. In their ground-breaking paper ([12]), D. L. Burkholder and R. F. Gundy showed that the random variables $X^\star$ and $S(X)$ satisfy certain inequalities relating their distributions. These are now commonly called *good-λ inequalities*. In the harmonic function setting the first good-λ inequalities were also proved by Burkholder and Gundy [13]. They were subsequently improved and refined by, among others, Burkholder [11], Dahlberg [16], Fefferman, Gundy, Silverstein and Stein [26]. The variations of good-λ inequalities and their applications can be found in the work of B. Muckenhoupt and R. L. Wheeden ([37]), S. D. Jaka ([31]), D. L. Burkholder ([10]), R. Bañuelos ([7]), and R. F. Bass ([6]). A fair amount of deal about such inequalities is found in the book by Rodrigo Bañuelos and Charles N. Moore ([8]). Let us recall the classical good lambda inequalities of Burkholder and Gundy [12] for continuous time martingales.

Theorem 58. *Let X_t be a continuous time martingale with maximal function $X^\star$ and square function $S(X)$. Then for all $0<\epsilon<1$, $\delta>0$ and $\lambda>0$,*

$$P\{X^\star>\delta\lambda, S(X)\le\epsilon\lambda\}\le\frac{\epsilon^2}{(\delta-1)^2}P\{X^\star>\lambda\} \tag{3.1}$$

and

$$P\{S(X)>\delta\lambda, X^\star\le\epsilon\lambda\}\le\frac{\epsilon^2}{(\delta-1)^2}P\{S(X)>\lambda\} \tag{3.2}$$

The most commonly used form of the good lambda inequality is stated below:

Definition 59. *A pair of nonnegative measurable functions f and g defined on $\mathbb{R}$ are said to satisfy a good-λ inequality if there exists constants $\delta>1$, $0<\epsilon_0\le 1$ such that for every $\lambda>0, 0<\epsilon<\epsilon_0$, we have*

$$|\{x\in\mathbb{R}: f(x)>\delta\lambda, g(x)<\epsilon\lambda\}|\le b(\epsilon)|\{x\in\mathbb{R}: f(x)>\lambda\}|, \tag{3.3}$$

where $b(\epsilon)\to 0$ as $\epsilon\to 0$.

As they are expressed here, these are actually a refinement, due to Burkholder ([9]), of the inequalities of [12]. The usefulness of such inequalities is already amply demonstrated by the following lemma, which is but one of the many applications of these type of inequalities. For this lemma we consider a non-decreasing function Φ defined on $[0, \infty]$ with $\Phi(0) = 0$, Φ is not identically 0, and which satisfies the condition $\Phi(2\lambda) \leq c\Phi(\lambda)$ for every $\lambda > 0$, where c is a fixed constant. This lemma is from [9].

Lemma 60. *Suppose that f and g are nonnegative measurable functions on a measurable space $(\mathbb{Y}, \mathcal{A}, \mu)$, and $\delta > 1$, $0 < \epsilon < 1$, and $0 < \gamma < 1$ are real numbers such that*

$$\mu\{g > \delta\lambda, f \leq \epsilon\lambda\} \leq \gamma\mu\{g > \lambda\} \tag{3.4}$$

for every $\lambda > 0$. Let ρ and ν be real numbers which satisfy

$$\Phi(\delta\lambda) \leq \rho\Phi(\lambda), \qquad \Phi(\epsilon^{-1}\lambda) \leq \nu\Phi(\lambda) \tag{3.5}$$

for every $\lambda > 0$. Finally, suppose $\rho\gamma < 1$ and $\int \Phi(\min\{1, g\})d\mu < \infty$. Then

$$\int_{\mathbb{Y}} \Phi(g)d\mu \leq \frac{\rho\nu}{1 - \rho\gamma} \int_{\mathbb{Y}} \Phi(f)d\mu. \tag{3.6}$$

Proof. We first claim that we may assume that $\int_{\mathbb{Y}} \Phi(g)d\mu < \infty$. In order to establish this claim, consider

$$h_n(x) = \min\{n, g(x)\} \quad \text{for every} \quad n \in \mathbb{N}.$$

If $\lambda \geq n$ then

$$\mu\{x \in \mathbb{R} : h_n(x) > \lambda\} = 0,$$

and

$$\mu\{x \in \mathbb{R} : h_n(x) > \delta\lambda, f(x) \leq \epsilon\lambda\} = 0$$

for $\delta > 1$. Next, let $0 < \lambda < n$. Then $g(x) > \lambda$ if and only if $h_n(x) > \lambda$. So,

$$\mu\{x \in \mathbb{R} : h_n(x) > \delta\lambda, f(x) \leq \epsilon\lambda\} = \mu\{x \in \mathbb{R} : g(x) > \delta\lambda, f(x) \leq \epsilon\lambda\}$$

and

$$\mu\{x \in \mathbb{R} : g(x) > \lambda\} = \mu\{x \in \mathbb{R} : h_n(x) > \lambda\}.$$

Also note that $\int_{\mathbb{Y}} \Phi\left(\min\{n,g\}\right) d\mu < \infty$ for every $n \in \mathbb{N}$ since $\int_{\mathbb{Y}} \Phi\left(\min\{1,g\}\right) d\mu < \infty$. Using the fact that $\lim_{n\to\infty} h_n(x) = g(x)$ and an application of Fatou's lemma it follows that the inequality (3.6) is true for f and g whenever it holds for h_n and f.

Let $d\Phi$ denote the Lebesgue-Stieltjes measure satisfying $\int_{[a,b]} d\Phi(\lambda) = \Phi(b) - \Phi(a)$ whenever $0 \le a \le b \le \infty$. This is a positive σ- finite Borel measure on $[0,\infty)$. Furthermore, an elementary Fubini theorem argument shows that if $h \ge 0$ is measurable function on $\mathbb{Y}$ then

$$\int_{\mathbb{Y}} \Phi(h) d\mu = \int_0^\infty \mu\{h > \lambda\} d\Phi(\lambda).$$

From inequality (3.4) we get

$$\begin{aligned}\mu\{g > \delta\lambda\} &= \mu\{g > \delta\lambda, f \le \epsilon\lambda\} + \mu\{g > \delta\lambda, f > \epsilon\lambda\} \\ &\le \gamma\mu\{g > \lambda\} + \mu\{f > \epsilon\lambda\}.\end{aligned}$$

Consequently,

$$\int_{\mathbb{Y}} \Phi\left(\frac{g}{\delta}\right) d\mu \le \gamma \int_{\mathbb{Y}} \Phi(g) d\mu + \int_{\mathbb{Y}} \Phi\left(\frac{f}{\epsilon}\right) d\mu. \tag{3.7}$$

But using the inequality (3.5) we have

$$\int_{\mathbb{Y}} \Phi(g) d\mu = \int_{\mathbb{Y}} \Phi\left(\delta\left(\frac{g}{\delta}\right)\right) d\mu \le \rho \int_{\mathbb{Y}} \Phi\left(\frac{g}{\delta}\right) d\mu. \tag{3.8}$$

Combining the inequalities (3.5), (3.7) and (3.8) we obtain

$$\int_{\mathbb{Y}} \Phi(g) d\mu \le \rho \int_{\mathbb{Y}} \Phi\left(\frac{g}{\delta}\right) d\mu \le \rho\gamma \int_{\mathbb{Y}} \Phi(g) d\mu + \rho\nu \int_{\mathbb{Y}} \Phi(f) d\mu.$$

Now subtracting the finite quantity $\rho\gamma \int_{\mathbb{Y}} \Phi(g) d\mu$ from both sides and dividing by $1 - \rho\gamma$ we obtain the desired inequality (3.6). This proves the lemma. □

Next, we look at some examples:

Example 61. *Let $\mathbb{Y} = \mathbb{R}, \mu =$ Lebesgue measure in $\mathbb{R}$ and $\Phi(\lambda) = \lambda^p$ for $0 < p < \infty$.*

Then Φ satisfies all the condition of the Lemma 60. From the conditions in inequality (3.5) we may choose $C = 2^p, \rho = \delta^p$ where $\delta > 1$. We usually obtain $\gamma = \gamma(\epsilon) = c_1 e^{-\frac{c_2}{\epsilon^2}}$

for some positive constants c_1 and c_2. We may choose ϵ small enough so that $\gamma\rho < 1$. A straightforward computation with $\delta = 2c_1$ yields $\epsilon = \dfrac{C}{\sqrt{p+1}}$ where the constant C depends on c_1, c_2 and δ. For,

$$\rho\gamma = \delta^p e^{-\frac{c_2}{\epsilon^2}} \leq \frac{1}{2} < 1$$

implies that

$$e^{-\frac{c_2}{\epsilon^2}} \leq \frac{1}{2\delta^p c_1}.$$

Taking natural log on both sides and $\delta = 2c_1$, we obtain,

$$\epsilon^2 \leq \frac{c_2}{(p+1)\ln\delta}$$

Therefore,

$$\epsilon = \frac{C}{\sqrt{p+1}}.$$

Thus if f and g satisfy the good lambda inequality (3.4) then the Lemma 60 yields

$$||g||_p \leq C||f||_p.$$

Definition 62. *The cone of aperture $\alpha > 0$ and vertex at $x \in \mathbb{R}^n$ is defined by*

$$\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t\}.$$

Let u be a harmonic function defined on $\mathbb{R}^{n+1}_+$. We set

$$N_\alpha u(x) = sup\{|u(t,y)| : (t,y) \in \Gamma_\alpha(x)\}.$$

If u is of the form $u(x,t) = \varphi_t * f(x)$ for some $\varphi \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, we will often write $N_\alpha f$ instead of $N_\alpha u$. This is called the non-tangential maximal function of u (or f).

Definition 63. *Let $\alpha > 0$ and u be a harmonic function on $\mathbb{R}^{n+1}_+$. For $x \in \mathbb{R}^n$ we define*

$$A_\alpha u(x) = \left(\int_{\Gamma_\alpha(x)} |\bigtriangledown u(s,t)|^2 t^{1-n} ds dt\right)^{\frac{1}{2}}.$$

The function $A_\alpha f$ is called the Lusin area function of f, or the square function of f.

Example 64. *D. L. Burkholder and R. F. Gundy in [13] proved that there exists a good-λ inequality between $N_\alpha u(x)$ and $A_\alpha u(x)$. Consequently,*

$$||N_\alpha u||_p \leq C||A_\alpha u||_p,$$

and

$$||A_\alpha u|| \leq C||N_\alpha u||_p.$$

Definition 65. *A function $u(x,t)$ defined on $\mathbb{R}^{n+1}$ is said to be caloric function if it satisfies the heat equation $u_t = \triangle u$. Caloric functions are also sometimes called parabolic functions. For $x \in \mathbb{R}^n$ we define the parabolic cone in $\mathbb{R}^{n+1}_+$ with vertex at x and aperture $\alpha > 0$ by*

$$\mathcal{P}_\alpha(x) = \{(y,s) \in \mathbb{R}^{n+1}_+ : |x-y|^2 < \alpha^2 s\}$$

and for a caloric function u its parabolic Lusin area function and parabolic non-tangential maximal functions are respectively given by

$$\mathcal{P}A_\alpha u(x) = \left(\int_{\mathcal{P}_\alpha(x)} s^{-\frac{n}{2}} \nabla_y u(y,s)|^2 dyds\right)^{\frac{1}{2}}$$

and

$$\mathcal{P}N_\alpha u(x) = \sup\{|u(x,s)| : (y,s) \in \mathcal{P}_\alpha(x)\}.$$

Example 66. *A. P. Calderon and A. Torchinsky in [14] proved that good-λ inequalities exist between $\mathcal{P}N_\alpha u(x)$ and $\mathcal{P}A_\alpha u(x)$.*

Good λ inequality and Potentials in $\mathbb{R}^n$:

Theorem 67. *Let μ be a positive Radon measure on $\mathbb{R}^n$ and $I_\alpha\mu(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}$. Then there exists $a > 1$ and $b > 0$ such that for every $\lambda > 0$ and for every $\epsilon, 0 < \epsilon \leq 1$,*

$$|\{x : I_\alpha\mu(x) > a\lambda\}| \leq b\epsilon^{n/n-\alpha}|\{x : I_\alpha\mu(x) > \lambda\}| + |\{x : M_\alpha\mu(x) > \epsilon\lambda\}|.$$

Equivalently,

$$|\{x : I_\alpha f > a\lambda, M_\alpha f < \epsilon\lambda\}| \leq C\epsilon^{\overline{N-\alpha}}{}^{\!\!N}|\{x : I_\alpha f(x) > \lambda\}|.$$

Proof. Let $E_\lambda = \{x : I_\alpha \star \mu(x) > \lambda\}$ and observe that this set is open. So, there exists a family of dyadic cubes $\{Q_j\}_1^\infty$, called Whitney cubes such that

$$E_\lambda = \bigcup_{j=1}^{\infty} Q_j,$$

and

$$diam(Q_j) \leq dist(Q_j, E_\lambda^c) \leq 4diam(Q_j) \tag{3.9}$$

for every j. This means that for every j there exists $x \in E_\lambda^c$ such that

$$d(x, Q_j) \leq 4diam(Q_j) \quad \text{and} \quad I_\alpha \star \mu(x) \leq \lambda.$$

Fix a cube $Q \in \{Q_j\}$ and $a > 1$. Consider the set $\{x \in Q : I_\alpha\mu(x) > a\lambda\}, \lambda > 0$. Let $M = \{x : M_\alpha\mu(x) \leq \epsilon\lambda\}$ and let $x_0 \in Q \bigcap M$. Let P be the ball concentric to the cube Q with radius $6diam(Q)$. Let μ_1 be the restriction of μ on P, and $\mu_2 = \mu - \mu_1$. Note that $M_\alpha\mu(x) \leq I_\alpha\mu(x)$. Therefore by the Theorem 55, we obtain

$$\left|\left\{x : I_\alpha\mu_1(x) > \frac{a\lambda}{2}\right\}\right| \leq \left(\frac{1}{a\lambda}\int_{\mathbb{R}^n} d\mu_1\right)^{\frac{n}{n-\alpha}}. \tag{3.10}$$

Let $x_0 \in Q \bigcap M$. Let B denote the ball with center at x_0 and radius $8diam(Q)$. Then $P \subseteq B$,and

$$\int_{\mathbb{R}^n} d\mu_1 = \int_P d\mu \leq \int_B d\mu \leq AM_\alpha\mu(x_0)|B|^{\frac{n}{n-\alpha}} \leq A\epsilon\lambda|B|^{\frac{n}{n-\alpha}} \tag{3.11}$$

Combining the above two inequalities we see that there exists a constant b such that

$$\left|\left\{x : I_\alpha\mu_1(x) > \frac{a\lambda}{2}\right\}\right| \leq b\epsilon^{\frac{n-\alpha}{n}}. \tag{3.12}$$

The inequality 3.10 and the construction of the ball P allows us to choose a point x_1 in P with $d(x_1, Q) \leq 4diam(Q)$. The for every $x \in Q$ and $y \in P^c$

$$|x_1 - y| \leq |x_1 - x| + |x - y| \leq Cdiam(Q) + |x - y| \leq L|x - y|. \tag{3.13}$$

If, in addition, $I_\alpha\mu(x_1) \leq \lambda$, then

$$\begin{aligned}
I_\alpha\mu_2(x) &= \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} d\mu_2(y) \\
&\leq L^{n-\alpha} \int_{\mathbb{R}^n} \frac{1}{|x_1-y|^{n-\alpha}} d\mu(y) \\
&\leq L^{n-\alpha} I_\alpha\mu_2(x_1) \\
&\leq L^{n-\alpha}\lambda.
\end{aligned}$$

Thus, if a is chosen so that $a \geq 2L^{n-\alpha}$, then $I_\alpha\mu_2(x) \leq \frac{a\lambda}{2}$. Hence if $I_\alpha\mu(x) > a\lambda$, it follows that

$$I_\alpha\mu_1(x) = I_\alpha\mu(x) - I_\alpha\mu_2(x) > a\lambda - \frac{a\lambda}{2} = \frac{a\lambda}{2}.$$

This implies that

$$\{x \in Q : I_\alpha\mu(x) > a\lambda\} \subseteq \left\{x : I_\alpha\mu_1(x) > \frac{a\lambda}{2}\right\}. \tag{3.14}$$

Thus either

$$Q \subseteq \{x : M_\alpha\mu(x) > \epsilon\lambda\}$$

or,

$$\{x \in Q : I_\alpha\mu(x) > a\lambda\} \subseteq \left\{x : I_\alpha\mu(x) > \frac{a\lambda}{2}\right\}.$$

This implies that

$$|\{x \in Q : I_\alpha\mu(x) > a\lambda\}| \leq \left|\left\{x : I_\alpha\mu(x) > \frac{a\lambda}{2}\right\}\right| + |\{x : M_\alpha\mu(x) > \epsilon\lambda\}|.$$

That is,

$$|\{x \in Q : I_\alpha\mu(x) > a\lambda\}| \leq b\epsilon^{\frac{n}{n-\alpha}}|Q| + |\{x : M_\alpha\mu(x) > \epsilon\lambda\}|. \tag{3.15}$$

The desired inequality follows by summing the above inequality over all $Q \in \{Q_j\}_1^\infty$. $\square$

Note that in the above theorem we do not require that the Radon measure μ satisfy the doubling condition or the growth condition (2.1). It is also noteworthy that the same inequality is true if $d\mu(y)$ is replaced by $f(y)dm(y)$ where m is a Lebesgue measure. We formulate this fact in the following theorem:

Theorem 68. *Let $0 < \alpha < N$. For measurable function f on $\mathbb{R}^n$ define $I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{N-\alpha}}$ and $M_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{\frac{N-\alpha}{N}}} \int_{B(x,r)} |f(y)|dy$. Then there exists constants $a > 1$ and b such that for every $\lambda > 0$ and $0 < \epsilon \leq 1$,*

$$|\{x : I_\alpha f(x) > a\lambda, M_\alpha f(x) < \epsilon\lambda\}| \leq b\epsilon^{\frac{N}{N-\alpha}} |\{x : I_\alpha f(x) > \lambda\}| .$$

In general the inequality is true for any doubling Radon measure μ in $\mathbb{R}^n$. This is established in the following theorem.

Some Notations:

By a "cube" $Q \subset \mathbb{R}^n$ we mean a closed cube having sides parallel to the axes. Its side length will be denoted by $\ell(Q)$. By Q_x we also denote the cube centered at x and side length $\ell(Q)$ unless otherwise specified. For $\rho > 0, \rho Q$ means a cube concentric to Q with side length $\rho\ell(Q)$.

Theorem 69. *Let μ be a positive Radon measure satisfying the doubling condition (1.1) and $0 < \alpha < N, \dfrac{1}{q} = 1 - \dfrac{\alpha}{N}$. For measurable function f on $\mathbb{R}^n$ define*

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)d\mu(y)}{|x-y|^{N-\alpha}}$$

Then there exists constants $a > 1, b > 0$ (independent of f) such that whenever $0 < \epsilon \leq 1$ and $\lambda > 0$ we have

$$\mu(\{x : I_\alpha f(x) > a\lambda\}) \leq b\epsilon^{N/N-\alpha}\mu(\{x : I_\alpha f(x) > \lambda\}) + \mu(\{x : M_\alpha f(x) > \epsilon\lambda\}) .$$

Proof. We first observe that $I_\alpha f$ is lower semi-continuous whenever $f \geq 0$. Indeed, if $x_j \to x$, then $|x_j - y|^{\alpha-N} f(y) \to |x-y|^{\alpha-N} f(y)$ for every $y \in \mathbb{R}^n$. Then by Fatou's lemma

$$\begin{aligned}
\lim_{j\to\infty} I_\alpha f(x_j) &\geq \liminf_{j\to\infty} \int_{\mathbb{R}^n} \frac{f(y)d\mu(y)}{|x_j - y|^{N-\alpha}} \\
&\geq \int_{\mathbb{R}^n} \liminf_{j\to\infty} \frac{f(y)d\mu(y)}{|x_j - y|^{N-\alpha}} \\
&= \int_{\mathbb{R}^n} \frac{f(y)d\mu(y)}{|x-y|^{N-\alpha}} = I_\alpha f(x).
\end{aligned}$$

Therefore, $E_\lambda := \{x : I_\alpha f(x) > \lambda\}$ is open. Then by Whitney Decomposition (Theorem 12) there exists a countable collection of dyadic cubes $\{Q_j\}_j^\infty$ with disjoint interiors, such that $E_\lambda = \bigcup_j Q_j$ and

$$diam(Q_j) \le dist\,(Q_j, E_\lambda^c) \le 4diam(Q_j) \tag{3.16}$$

for every j.

Fix a Whitney cube $Q \in \{Q_j\}_j^\infty$ and a constant $a > 1$. Consider the set $\{x \in Q : I_\alpha f(x) > a\lambda\}$. Suppose that $Q \bigcap \{x : M_\alpha(f)(x) \le \epsilon\lambda\}$ is nonempty. Let P be a ball concentric to the cube Q with radius $6diam(Q)$.

Now we decompose f as $f = f_1 + f_2$ where $f_1 = \chi_P f$. Then $I_\alpha f_1$ is weak type $(1,1)$ and

$$\mu\left(\left\{x : I_\alpha f_1(x) > \frac{a\lambda}{2}\right\}\right) \le \left(\frac{1}{a\lambda}\int_{\mathbb{R}^n} |f_1(y)|d\mu(y)\right)^{N/N-\alpha}. \tag{3.17}$$

Next we estimate the integral on the right side of the inequality (3.17). For this, choose a point $x_0 \in Q \bigcap \{x : M_\alpha(f)(x) \le \epsilon\lambda\}$. Consider a ball B center at x_0 and radius $8diam(Q)$. Then $B \supseteq P$ and $P \supseteq Q$. We observe that there exists a constant C depending on n such that $CQ \supseteq B \supseteq Q$. It suffices to take $C \ge 17\sqrt{n} + 1$. Then by monotonicity and the doubling condition (1.1) we get $\mu\,(B) \le C\mu\,(Q)$. Now,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f_1(y)|d\mu(y) &\le \int_P |f(y)|d\mu(y) \\
&\le \int_B |f(y)|d\mu(y) \\
&\le \mu(B)^{N/N-\alpha} M_\alpha f(x_0) \\
&\le \mu\left(B\left(x_0, 8diam(Q)\right)\right)^{N-\alpha/N} \epsilon\lambda.
\end{aligned}$$

Applying the above inequality to (3.17) yields

$$\begin{aligned}
\mu\left(\left\{x : I_\alpha f_1(x) > \frac{a\lambda}{2}\right\}\right) &\le \left(\frac{1}{a\lambda}\int_{\mathbb{R}^n} |f_1(y)|d\mu(y)\right)^{N/N-\alpha} \\
&\le \left(\mu\left(B(x_0, 8diam(Q))\right)\epsilon\lambda\right)^{N/N-\alpha} \\
&\le \left(\frac{\epsilon}{a}\right)^{\frac{N}{N-\alpha}} C\mu(Q) \\
&\le C\epsilon^{N/N-\alpha}\mu(Q).
\end{aligned}$$

On the other hand, let $x_1 \in P \bigcap E_\lambda^c$ such that

$$d(x_1, Q) \leq 4diam(Q).$$

This is possible by the choice of P. Then for every $x \in Q$ and $y \in P^c$ and using the inequality (3.16) there exists a constant L depending on the dimension n such

$$|x_1 - y| \leq L|x - y|.$$

Therefore,

$$\frac{1}{|x-y|} \leq \frac{L}{|x_1 - y|}.$$

Note that $I_\alpha f(x_1) < \lambda$ because $x_1 \in E_\lambda^c$. Then

$$\begin{aligned} I_\alpha f_2(x) &= \int_{\mathbb{R}^n} \frac{f_2(y)}{|x-y|^{N-\alpha}} d\mu(y) \\ &\leq L^{N-\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x_1-y|^{N-\alpha}} d\mu(y) \\ &= L^{N-\alpha} I_\alpha f(x_1) \\ &\leq L^{N-\alpha} \lambda. \end{aligned}$$

Now, choose a so that $a \geq 2L^{N-\alpha}$. Then $I_\alpha f_2(x) \leq \frac{a\lambda}{2}$. Hence if $I_\alpha f(x) > a\lambda$, it follows that $I_\alpha f_1(x) > \frac{a\lambda}{2}$. This implies that

$$\{x \in Q : I_\alpha f(x) > a\lambda\} \subseteq \left\{x : I_\alpha f_1(x) > \frac{a\lambda}{2}\right\}.$$

whenever $Q \bigcap \{x : M_\alpha(f)(x) > \epsilon\lambda\}$ is nonempty. Thus either

$$Q \subseteq \{x : M_\alpha(f)(x) > \epsilon\lambda\},$$

or,

$$\{x \in Q : I_\alpha f(x) > a\lambda\} \subseteq \left\{x : I_\alpha f_1(x) > \frac{a\lambda}{2}\right\}.$$

This implies that

$$\begin{aligned} \mu\left(\{x \in Q : I_\alpha f(x) > a\lambda\}\right) \leq& \mu\left(\left\{x \in Q : I_\alpha f_1(x) > \frac{a\lambda}{2}\right\}\right) \\ &+ \mu\left(\{x : M_\alpha(f)(x) > \epsilon\lambda\}\right) \\ \leq& b\epsilon^{\frac{N}{N-\alpha}} \mu(Q) + \mu\left(\{x : M_\alpha(f)(x) > \epsilon\lambda\}\right). \end{aligned}$$

That is,

$$\mu\left(\{x \in Q : I_\alpha f(x) > a\lambda, M_\alpha(f)(x) > \epsilon\lambda\}\right) \leq b\epsilon^{N/N-\alpha}\mu(Q)$$

for every $Q \in \{Q_j\}_j^\infty$. Finally, the desired inequality follows from summing over all $Q \in \{Q_j\}_j^\infty$. □

Definition 70. *(X. Tolsa [48]) Let $\mathcal{Q}(\mu) = \{Q \subset \mathbb{R}^n : \mu(Q) > 0\}$. Given $\alpha > 1$ and $\beta > \alpha^N$, we say that some $Q \subset \mathbb{R}^n$ is (α, β)-doubling if $\mu(\alpha Q) \leq \beta\mu(Q)$. If α and β are not specified then by a doubling cube we mean a $(2, 2^{N+1})$-doubling cube.*

Here we state some remarks about the existence of doubling cubes. (X. Tolsa [48], [47])

Remark 71. *Due to the fact that μ satisfies the growth condition (2.1), there are lots of "big" doubling cubes. Precisely speaking, given any point $x \in$ supp (μ) and $c > 0$, there exists some (α, β)-doubling cube Q centered at x with $\ell(Q) \geq c$. This follows from (2.1) and the fact that $\beta > \alpha^N$. Indeed, if there are no doubling cubes centered at x with $\ell(Q) \geq c$, then $\mu(\alpha^m Q) > \beta^m \mu(Q)$ for every m, and letting $m \to \infty$ it follows that the growth condition (2.1) can not hold.*

Remark 72. *There are a lot of "small" doubling cubes too. Precisely speaking, if $\beta > \alpha^N$, then for μ-a.e $x \in \mathbb{R}^n$ there exists a sequence of (α, β)-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. This is a property that any Radon measure on $\mathbb{R}^n$ satisfies (the growth condition (2.1) is not necessary in this argument).*

The next theorem extends the above with non-doubling measure μ.

Theorem 73. *Let μ be a positive measure on $\mathbb{R}^n$ which satisfies the growth condition (2.1). Then there exist constants k, C such that for every $\lambda > 0$ and $0 < \epsilon < 1$,*

$$\mu\left(\{x : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}\right) \leq C\epsilon^{\frac{N}{N-\alpha}}\mu\left(\{x : I_\alpha f(x) > \lambda\}\right).$$

Proof. Note that the set $\Omega_\lambda = \{x \in \mathbb{R}^n : I_\alpha f(x) > k\lambda\}$ is open because of the lower semi-continuity of the Riesz potential $I_\alpha f$. Then it has a Whitney decomposition into a family of dyadic cubes $\{Q_j\}_{j=1}^{\infty}$ with disjoint interiors such that for every j

$$diam(Q_j) \le dist(Q_j, \Omega_\lambda^c) \le 4diam(Q_j).$$

For such a Whitney cube Q let $\ell(Q)$ denote its side length. Fix $0 < \delta < \dfrac{1}{4}$. In particular, we may choose $\delta = \frac{1}{8}$. Then $(1+\delta)Q = \frac{9}{8}Q \subseteq \Omega_\lambda$, and

$$\sum_Q \chi_{\frac{9}{8}Q}(x) \le 12^n \chi_{\Omega_\lambda}(x)$$

for every $x \in \Omega_\lambda$. This implies that

$$\sum_Q \mu\left(\frac{9}{8}Q\right) \le 12^n \mu(\Omega_\lambda). \tag{3.18}$$

Fix a dyadic cube Q. Suppose $x \in Q$ and that $M_\alpha f(x) < \epsilon\lambda$. Let Q_x be the cube with side length $\delta\ell(Q)$ centered at x. Then $Q_x \subseteq (1+\delta)Q$ and $16Q_x \supseteq 2Q$. Let $k = 2, \beta = 2^{n+\epsilon}, \epsilon > 0$. Consider $\dfrac{1}{2}Q_x, \dfrac{1}{2^2}Q_x, \dfrac{1}{2^3}Q_x, \ldots$. Take first which is doubling. Let the first doubling cube in the sequence be $\widehat{Q}_x := \dfrac{1}{2^{m+1}}Q_x$ where m depends on x. So $\widehat{Q}_x \subseteq \dfrac{1}{2}Q_x$ and

$$\mu(2\widehat{Q}_x) = \mu\left(2.\frac{1}{2^{m+1}}Q_x\right) \le \beta\mu\left(\frac{1}{2^{m+1}}Q_x\right) = \beta\mu(\widehat{Q}_x)$$

whereas

$$\mu\left(2.\frac{1}{2^{j+1}}Q_x\right) \ge \beta\mu\left(\frac{1}{2^{j+1}}Q_x\right) \text{ for every } j < m.$$

Thus,

$$\bigcup_{x\in\{M_\alpha f<\epsilon\lambda\}} \widehat{Q}_x \supseteq \{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}$$

The Besicovitch Covering Lemma implies that there exists $\widehat{Q}_{x_j}, j = 1, 2, 3, \ldots$ such that

$$\bigcup_j \widehat{Q}_{x_j} \supseteq \{x \in Q : I_\alpha f > k\lambda, M_\alpha f < \epsilon\lambda\}$$

and $\sum \chi_{\widehat{Q}_{x_j}} \le 4^n \chi_{4Q}$. Let $f = f_1 + f_2$ where $f_1 = \chi_{2Q} f$ and $f_2 = f - f_1$. Then

$$\{x \in Q : I_\alpha f_1(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\} \subseteq \bigcup_j \{x \in \widehat{Q}_{x_j} : I_\alpha f_1(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}.$$

Fix a Q_{x_j}. Write $f_1 = \chi_{2\widehat{Q}_{x_j}} f_1 + f_1 \chi_{(2\widehat{Q}_{x_j})^c} = f_{11} + f_{12}$. Then,

$$\begin{aligned}
\mu\left(\{x \in 2\widehat{Q}_{x_j} : I_\alpha f_{11} > k\lambda\}\right) &\leq \left(\frac{1}{k\lambda}\|f_{11}\|_1\right)^{N/N-\alpha} \\
&= \left(\frac{1}{k\lambda}\int_{2\widehat{Q}_{x_j}} |f_1(x)| d\mu(x)\right)^{N/N-\alpha} \\
&= \left(\frac{1}{k\lambda}\mu(2\widehat{Q}_{x_j})^{\frac{N-\alpha}{N}} \frac{1}{\mu(2\widehat{Q}_{x_j})^{\frac{N-\alpha}{N}}} \int_{2\widehat{Q}_{x_j}} |f_1(x)| d\mu(x)\right)^{N/N-\alpha} \\
&= \left(\frac{1}{k\lambda}\right)^{\frac{N}{N-\alpha}} \mu(2\widehat{Q}_{x_j}) M_\alpha f_1(x_j)^{N/N-\alpha} \\
&\leq C\epsilon^{N/N-\alpha}\mu(\widehat{Q}_{x_j}).
\end{aligned}$$

Now since for any cube $Q, \mu(Q) \leq c\ell(Q)^N$, then $\left(\frac{c}{\mu(Q)}\right)^{\frac{N-\alpha}{N}} \geq \frac{1}{\ell(Q)^{N-\alpha}}$. Also,

$$\mu(Q_{x_j}) \geq \beta\mu\left(\frac{1}{2}Q_{x_j}\right) \geq \ldots > \beta^i \mu\left(\frac{1}{2^i}Q_{x_j}\right)$$

where $i < m$. This follows by the choice of our Q_{x_j}'s. So, if $x \in \widehat{Q}_{x_j}$

$$
\begin{aligned}
I_\alpha f_{12}(x) &= \int_{(2\widehat{Q}_{x_j})^c} \frac{f_1(y)}{d(x,y)^{N-\alpha}} d\mu(y) \\
&\leq \sum_{k=0}^{m-1} \int_{\frac{1}{2^k}Q_{x_j} \setminus \frac{1}{2^{k+1}}Q_{x_j}} \frac{|f_1(y)|}{d(x,y)^{N-\alpha}} d\mu(y) + \int_{2Q \setminus Q_{x_j}} \frac{|f_1|(y) d\mu(y)}{d(x,y)^{N-\alpha}} \\
&\leq \sum_{k=0}^{m-1} \frac{C}{\ell(\frac{1}{2^k}Q_{x_j})^{N-\alpha}} \int_{\frac{1}{2^k}Q_{x_j} \setminus \frac{1}{2^{k+1}}Q_{x_j}} |f_1(y)| d\mu(y) + \frac{C}{\ell(8Q_{x_j})^{N-\alpha}} \int_{2Q \setminus Q_{x_j}} f_1(y) d\mu(y) \\
&\leq \sum_{k=0}^{m-1} \frac{2^{k(N-\alpha)}}{\ell(Q_{x_j})^{N-\alpha}} \int_{\frac{1}{2^k}Q_{x_j} \setminus \frac{1}{2^{k+1}}Q_{x_j}} |f_1(y)| d\mu(y) + \frac{C}{\mu(8Q_{x_j})^{(N-\alpha)/N}} \int_{2Q \setminus Q_{x_j}} f_1(y) d\mu(y) \\
&\leq \sum_{k=0}^{m-1} C 2^{k(N-\alpha)} \frac{1}{\mu(Q_{x_J})^{\frac{N-\alpha}{N}}} \int_{\frac{1}{2^k}Q_{x_j} \setminus \frac{1}{2^{k+1}}Q_{x_j}} |f_1(y)| d\mu(y) + C M_\alpha f_1(x_j) \\
&\leq \sum_{k=0}^{m-1} C 2^{k(N-\alpha)} \frac{1}{\beta^k \mu(\frac{1}{2^k}Q_{x_j})^{\frac{N-\alpha}{N}}} \int_{\frac{1}{2^k}Q_{x_j} \setminus \frac{1}{2^{k+1}}Q_{x_j}} |f_1(y)| d\mu(y) + C\lambda \\
&\leq \sum_{k=0}^{m-1} C \frac{2^{k(N-\alpha)}}{\beta^{\frac{k}{N}(N-\alpha)}} M_\alpha f_1(x_j) + C\lambda \\
&= C \sum_{k=0}^{m-1} \left(\frac{2^{N-\alpha}}{2^{N-\alpha} 2^{\epsilon(\frac{N-\alpha}{N})}} \right)^k M_\alpha f_1(x_j) + C\lambda \\
&= C \sum_{k=0}^{m-1} 2^{-\epsilon(\frac{N-\alpha}{N})k} M_\alpha f_1(x_j) + C\lambda \\
&= C_{N,\alpha} M_\alpha f_1(x_j) \leq C_{N,\alpha} = C\lambda.
\end{aligned}
$$

Note that the constant C in different occurrences above are not necessarily the same. We also have

$$\sum \chi_{\widehat{Q}_{x_j}}(x) \leq 4^n \chi_{\frac{9}{8}Q}$$

This implies that

$$\mu\left(\bigcup_j \widehat{Q}_{x_j}\right) = \int_Q \sum \chi_{\widehat{Q}_{x_j}}(x) d\mu \leq 4^n \int_Q \chi_{2Q} d\mu = 4^n \mu(Q).$$

Take a point $x \in Q$. Consider the ball $B = B(x, 6diam(Q))$ which contains $2Q$. Let x_0 be a point in $B \cap \Omega_\lambda^c$ such that

$$diam(Q) \leq dist(Q, x_0) \leq 4diam(Q).$$

Then for any $y \in (2Q)^c$

$$d(x_0, y) \leq d(x, x_0) + d(x, y) \leq Cdiam(Q) + d(x, y) \leq Cd(x, y) + d(x, y).$$

That is , for some constant C,

$$d(x_0, y) \leq Cd(x, y).$$

So,

$$I_\alpha f_2(x) = \int_{(2Q)^c} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \leq C \int_{(2Q)^c} \frac{f(y)d\mu(y)}{d(x_0,y)^{N-\alpha}} \leq CI_\alpha f(x_0) < C\lambda$$

Now choose $\tilde{k}$ such that $\tilde{k} - 2C > k$. Then, summing over all Whitney cubes we obtain

$$\begin{aligned}
&\mu\left(\{x : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}\right) \\
&= \sum_Q \mu\left(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}\right) \\
&= \sum_{\substack{Q:\exists x \in Q \\ M_\alpha f < \epsilon\lambda}} \mu\left(\{x \in Q : I_\alpha f_1 + I_\alpha f_2 > k\lambda, M_\alpha f < \epsilon\lambda\}\right) \\
&\leq \sum_{\substack{Q:\exists x \in Q \\ M_\alpha f < \epsilon\lambda}} \mu\left(\{x \in Q : I_\alpha f_1 > (k - C)\lambda, M_\alpha f < \epsilon\lambda\}\right) \\
&= \sum_Q \sum_j \mu\left(\{x \in \widehat{Q}_{x_j} : I_\alpha f_1 > (k - C)\lambda, M_\alpha f < \epsilon\lambda\}\right) \\
&= \sum_Q \sum_j \mu\left(\{x \in \widehat{Q}_{x_j} : I_\alpha f_{11} + I_\alpha f_{12} > (k - C)\lambda, M_\alpha f < \epsilon\lambda\}\right) \\
&< \sum_Q \sum_j \mu\left(\{x \in \widehat{Q}_{x_j} : I_\alpha f_{11} > (k - 2C)\lambda, M_\alpha f < \epsilon\lambda\}\right) \\
&\leq \sum_Q \sum_j C\epsilon^{N/N-\alpha} \mu(\widehat{Q}_{x_j}) \\
&\leq \sum_Q C\epsilon^{N/N-\alpha} 4^n \mu(\frac{9}{8}Q) \\
&= 4^N 12^n C\epsilon^{N/N-\alpha} \mu(\{I_\alpha f > \lambda\}).
\end{aligned}$$

Finally,

$$\begin{aligned}\mu\left(\{x : I_\alpha f > k\lambda\}\right) &\le \mu\left(\{I_\alpha f > k\lambda, M_\alpha f \le \epsilon\lambda\}\right) + \mu\left(\{M_\alpha f > \epsilon\lambda\}\right) \\ &\le C\epsilon^{N/N-\alpha}\mu\left(\{I_\alpha f > \lambda\}\right) + \mu\left(\{M_\alpha f > \epsilon\lambda\}\right).\end{aligned}$$

□

Remark 74. *We observe From the proof of the above theorem that for every Whitney cube Q, we have*

$$\mu\left(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha f(x) < \epsilon\lambda\}\right) \le C\epsilon^{N/(N-\alpha)}\mu\left(\frac{9}{8}Q\right).$$

Potentials and Maximal functions:

Note that for $0 < \alpha < N, f \ge 0$ and $r > 0$

$$\int_X \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \ge \int_{d(x,y)\le r} \frac{f(y)d\mu(y)}{d(x,y)^{N-\alpha}} \ge \frac{1}{r^{N-\alpha}}\int_{B(x,r)} |f(y)|d\mu(y)$$

Theorem 75. *Let $1 < p < \infty$ and $0 < \alpha < N$. Then there is a constant A such that for any Lebesgue measurable function f on X and any non-doubling measure μ*

$$A^{-1}\|M_\alpha f\|_p \le \|I_\alpha f\|_p \le A\|M_\alpha f\|_p.$$

Proof. We assume that μ has compact support. The right hand inequality is a consequence of "good lambda inequality". Multiplying the good lambda inequality by λ^{p-1}, we obtain for any positive R,

$$\begin{aligned}&\int_0^R \mu(\{x : I_\alpha f(x) > k\lambda\})\lambda^{p-1}d\lambda \\ &\qquad \le b\epsilon^{N/N-\alpha}\int_0^R \mu(\{x : I_\alpha f(x) > \lambda\})\lambda^{p-1}d\lambda + \int_0^R \mu(\{x : M_\alpha(f)(x) > \epsilon\lambda\})\lambda^{p-1}d\lambda.\end{aligned}$$

After changing variables we obtain,

$$\begin{aligned}&k^{-p}\int_0^{kR} \mu(\{I_\alpha f(x) > \lambda\})\lambda^{p-1}d\lambda \\ &\qquad \le \frac{1}{2}k^{-p}\int_0^R \mu(\{I_\alpha f(x) > \lambda\})\lambda^{p-1}d\mu + \epsilon^{-p}\int_0^{kR} \mu(\{M_\alpha(f)(x) > \lambda\})\lambda^{p-1}d\lambda.\end{aligned}$$

That is,

$$k^{-p}\int_0^{kR}\mu(\{I_\alpha f(x)>\lambda\})\lambda^{p-1}d\lambda \le 2\epsilon^{-p}\int_0^{kR}\mu(\{M_\alpha(f)(x)>\lambda\})\lambda^{p-1}d\lambda.$$

Letting $R\to\infty$ and using the definition $\int_X |f|^p d\mu = p\int_0^\infty t^{p-1}\mu(\{x: f(x)>t\})dt$ we obtain,

$$k^{-p}\int_X |I_\alpha f(x)|^p d\mu(x) \le 2\epsilon^{-p}\int_X |M_\alpha(f)(x)|^p d\mu$$

This yields the right hand inequality. If μ doesn't have compact support, we let μ_n be the restriction of μ to the ball $B(x_0,n)$ for $n=1,2,3,\ldots$ where x_0 is some point in X. Then $\|I_\alpha f\|_{L^p(X,\mu_n)} \le A\|M_\alpha f\|_{L^p(X,\mu_n)}$ for all n, where A is independent of n. The theorem then follows by the Monotone Convergence theorem. $\square$

3.2 The A_p Weights

A weight is a nonnegative locally integrable function w on $\mathbb{R}^n$ such that $w(x)\in(0,\infty)$ a.e. $x\in\mathbb{R}^n$. Every weight w gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will also be denoted by w. Thus, $w(E)=\int_E w(x)dx$ for every measurable subset of $\mathbb{R}^n$.

Definition 76. *Let w be a weight, and let Ω be an open subset of $\mathbb{R}^n$. For $0<p<\infty$, we define $L^p_w(\Omega)$ as the set of measurable functions f on Ω such that*

$$||f||_{L^p_w(\Omega)} = \left(\int_\Omega |f|^p w dx\right)^{1/p} < \infty.$$

We also define weak-$L^1_w(\Omega)$ as the set of all measurable functions f on Ω satisfying

$$||f||_{wk-L^1_w(\Omega)} = \sup_{\lambda>0}\lambda w\left(\{x\in\Omega : |f(x)|>\lambda\}\right) < \infty.$$

The class of A_p weights was introduced by B. Muckenhoupt in [35], where it is shown that the A_p weights are precisely those weights w for which the Hardy-Littlewood maximal operator is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$, when $1<p<\infty$, and from $L^1_w(\mathbb{R}^n)$ to $wk-L^1_w(\mathbb{R}^n)$, when $p=1$.

Definition 77. *Let $1 \leq p < \infty$. A weight w is said to be an A_p weight, if there exists a positive constant A such that, for every ball $B \subset \mathbb{R}^n$,*

$$\left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}\right)^{p-1} \leq A,$$

if $p > 1$, or

$$\left(\frac{1}{|B|}\int_B w dx\right) \operatorname*{ess.sup}_{x\in B} \frac{1}{w(x)} \leq A,$$

if $p = 1$. The infimum over all such constants A is called the A_p constant of w. We denote by $A_p, 1 \leq \infty$, the set of all A_p weights.

Definition 78. *([19]) We say that a weight w is an A_∞ weight if there exists two positive constants C and δ such that*

$$\frac{w(E)}{w(Q)} \leq C\left(\frac{|E|}{|Q|}\right)^{\delta}$$

for every cube Q and every measurable subset E of Q. The constants C and δ are called A_∞ constants of w and the set A_∞ weights is denoted by A_∞.

The relationship between A_p and A_∞ is given by ([35] and [36])

$$A_\infty = \bigcup_{1\leq p<\infty} A_p.$$

Thus, if w satisfies the A_p condition then W satisfies the A_∞ condition. As an application of the Theorem 68, we prove the following theorem:

Theorem 79. *Let $d\mu(x) = w(x)dx$ satisfies the A_∞ condition. Let $0 < \alpha < N$, $\frac{1}{q} = 1 - \frac{\alpha}{N}$. For measurable function f on $\mathbb{R}^n$ define $I_\alpha f(x) = \int_{\mathbb{R}^n} \dfrac{f(y)d\mu(y)}{|x-y|^{N-\alpha}}$. Then there exists constants $a > 1, b > 0$ (independent of f) such that whenever $0 < \epsilon \leq 1$ and $\lambda > 0$ we have*

$$\mu\left(\{x : I_\alpha f(x) > a\lambda\}\right) \leq b\epsilon^{N/(N-\alpha)}\mu\left(\{x : I_\alpha f(x) > a\lambda\}\right) + \mu\left(\{x : M_\alpha f(x) > \epsilon\lambda\}\right).$$

Proof. From the inequality 3.15 , it follows that

$$\frac{|\{x \in Q_j : I_\alpha f > a\lambda, M_\alpha f < \epsilon\lambda\}|}{|Q_j|} \leq b\epsilon^{N/(N-\alpha)}$$

where the Q_j's are the dyadic cubes from the proof of the theorem 73. From the definition 78 of A_∞ weight, it follows that there exists two constants C and δ such that

$$\frac{\mu\left(\{x \in Q_j : I_\alpha f > a\lambda, M_\alpha f < \epsilon\lambda\}\right)}{\mu\left(Q_j\right)} \leq C\left(b\epsilon^{N/(N-\alpha)}\right)^\delta$$

The desired inequality follows by taking the sum over all $Q_j \in \{Q_j\}_1^\infty$ in the above inequality.

$\square$

Next we discuss A_p weights adapted to more general measure μ ([39]). We denote by Q a dyadic cube sides parallel to the coordinate axes such that $\mu(\partial Q) = 0$ where μ is a non-doubling measure that satisfies a Calderón-Zygmund decomposition. (Lemma 2.1 in [39]).

Definition 80. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We say that a weight w satisfies the $A_p(\mu)$ condition if there exists a constant K such that for all cubes Q*

$$\left(\frac{1}{\mu(B)}\int_B w d\mu\right)\left(\frac{1}{\mu(B)}\int_B w^{-\frac{1}{p-1}} d\mu\right)^{p-1} \leq K.$$

We define the $A_\infty(\mu)$ class as $A_\infty(\mu) = \bigcup_{p>1} A_p(\mu)$. We also say that w satisfies $A_1(\mu)$ condition if there exists a constant K such that such that for all cubes Q,

$$\frac{1}{\mu(Q)}\int_Q w d\mu \leq K \operatorname*{ess.sup}_{x \in Q} w(x).$$

Finally, we define the $A_\infty(\mu) = \bigcup_{p>1} A_p(\mu)$.

Remark 81. *$A_1(\mu) \subset A_p(\mu)$ for every $p > 1$ and $A_p(\mu) \subset A_q(\mu)$ for every $p < q$.*

Note that we use the standard notation $w(E) = \int_E w d\mu$, for every measurable set E.

Lemma 82. *([39]) For a weight w the following conditions are equivalent:*

(a) $w \in A_\infty(\mu)$.

(b) For every cube Q

$$\frac{1}{\mu(Q)}\int_Q w d\mu \approx exp\left(\frac{1}{\mu(Q)}\int_Q \log w d\mu\right).$$

(c) There are constants $0 < \alpha, \beta < 1$ such that for every cube Q

$$\mu\left(\{x \in Q : w(x) \leq \beta w_Q\}\right) \leq \alpha\mu(Q).$$

(d) There are positive constants C and β such that for every cube Q and for every $\lambda > w_Q$

$$w\left(\{x \in Q : w(x) > \lambda\}\right) \leq C\lambda\mu\left(\{x \in Q : w(x) > \beta\lambda\}\right).$$

(e) w satisfies a reverse Holder inequality. Namely, there are positive constants c and δ such that for every cube Q and measurable set E contained in Q,

$$\left(\frac{1}{\mu(Q)}\int_Q w^{1+\delta}d\mu\right)^{\frac{1}{1+\delta}} \leq \frac{c}{\mu(Q)}\int_Q wd\mu.$$

(f) There are positive constants c and ρ such that, for any cube Q and any measurable set E contained in Q,

$$\frac{w(E)}{w(Q)} \leq c\left(\frac{\mu(E)}{\mu(Q)}\right)^{\rho}.$$

(g) w satisfies the following condition: there exists positive constants $\alpha, \beta, 1$ such that whenever E is a measurable set of a cube Q

$$\frac{\mu(E)}{\mu(Q)} < \alpha \qquad \text{implies} \qquad \frac{w(E)}{w(Q)} < \beta.$$

The following theorem is an extension of the good lambda inequality adapted to weight w to a more general measure discussed in this section.

Theorem 83. *Let $w \in A_\infty(\mu)$, and let $\alpha > 0$. Then there exists a positive constant $a > 1$ such that for every $0 < \epsilon \leq 1$, there exists $\eta > 0$ such that the inequality*

$$w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > a\lambda\}\right) \leq \eta w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}\right) + w\left(\{x \in \mathbb{R}^n : M_\alpha f(x) > \epsilon\lambda\}\right)$$

holds for every $\lambda > 0$.

Proof. Let $E_\lambda = \{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}, \lambda > 0$. Then E_λ is open because $I_\alpha f$ is lower semi-continuous. Then there exists a family of dyadic cubes $\{Q_j\}$, called Whitney cubes, such that $E_\lambda = \bigcup_j Q_j$ and

$$diam(Q_j) \leq dist(Q_j, E_\lambda) \leq 4diam(Q_j).$$

Because $w \in A_\infty(\mu)$, it follows that, for every $\eta > 0$, there is a δ such that if Q is a cube and E is a measurable subset of Q then there is a constant C such that

$$\frac{w(E)}{w(Q)} \leq C\left(\frac{\mu(E)}{\mu(Q)}\right)^\delta.$$

That is for every $Q \in \{Q_j\}$, using the Remark 74, we get

$$\begin{aligned}\frac{w\left(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha(f)(x) < \epsilon\lambda\}\right)}{w(Q)} &\leq \left(\frac{\mu\left(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha(f)(x) < \epsilon\lambda\}\right)}{\mu(Q)}\right)^\delta \\ &\leq \left(C\epsilon^{N/N-\alpha}\right)^\delta.\end{aligned}$$

This implies that

$$w\left(\{x \in Q : I_\alpha f(x) > k\lambda, M_\alpha(f)(x) < \epsilon\lambda\}\right) \leq C\epsilon^{(N/N-\alpha)\delta} w(Q).$$

It then follows that

$$w\left(\{x \in Q : I_\alpha f(x) > k\lambda\}\right) \leq C\epsilon^{(N/N-\alpha)\delta} w(Q) + w\left(\{x \in Q : M_\alpha(f)(x) > \epsilon\lambda\}\right).$$

The theorem now follows from summing over all $Q \in \{Q_j\}$. $\square$

Next we establish a norm inequality for fractional integrals and maximal function.

Theorem 84. *Let $w \in A_\infty(\mu), 0 < \alpha < N$, and let $0 < p < \infty$. Then there exists a positive constant C, that only depends on N, p, and $A_\infty(\mu)$ constants of w, such that*

$$\int_{\mathbb{R}^n} |I_\alpha f|^p w d\mu \leq C \int_{\mathbb{R}^n} \left(M_\alpha(f)\right)^p w d\mu$$

for every measurable function f.

Proof. Without loss of generality, we may assume that $f \geq 0$. From theorem 83 we have weighted good lambda inequality given by

$$w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > a\lambda\}\right) \leq \eta w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}\right) + w\left(\{x \in \mathbb{R}^n : M_\alpha f(x) > \epsilon\lambda\}\right)$$

We multiply this inequality by $p\lambda^{p-1}$ and integrate from 0 to $R \quad (R > 0)$ with respect to λ a to obtain

$$\begin{aligned}
&\int_0^R w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > a\lambda\}\right) p\lambda^{p-1} d\lambda \\
&\quad \leq \eta \int_0^R w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}\right) p\lambda^{p-1} d\lambda + \int_0^R w\left(\{x \in \mathbb{R}^n : M_\alpha f(x) > \epsilon\lambda\}\right) p\lambda^{p-1} d\lambda.
\end{aligned}$$

Applying change of variable and $a > 1$ yields

$$\begin{aligned}
a^{-p} \int_0^{aR} w\left(\{x \in \mathbb{R}^N : I_\alpha f(x) > \lambda\}\right) \lambda^{p-1} d\lambda \\
&\leq \eta \int_0^{aR} w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}\right) \lambda^{p-1} d\lambda \\
&+ \epsilon^{-p} \int_0^{\epsilon R} w\left(\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}\right) \lambda^{p-1} d\lambda.
\end{aligned}$$

Now we choose $\eta \leq \frac{1}{2} a^{-p}$. This yields

$$\begin{aligned}
a^{-p} \int_0^{aR} w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}\right) \lambda^{p-1} d\lambda \\
&\leq 2\epsilon^{-p} \int_0^{\epsilon R} w\left(\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}\right) \lambda^{p-1} d\lambda. \qquad (3.19)
\end{aligned}$$

Now let

$$\chi(x, \lambda) = \begin{cases} 1, & \text{if } I_\alpha f(x) > \lambda > 0; \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned}
\int_0^{aR} w\left(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}\right) \lambda^{p-1} d\lambda. &= \int_0^{aR} \left(\int_{\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}} w(x) d\mu(x) \right) \lambda^{p-1} d\lambda. \\
&= \int_0^{aR} \left(\int_{\mathbb{R}^n} \chi(x, \lambda) w(x) d\mu(x) \right) \lambda^{p-1} d\lambda. \\
&= \int_{\mathbb{R}^n} w(x) \int_0^{\min\{aR, I_\alpha f(x)\}} \lambda^{p-1} d\lambda d\mu \\
&= \int_{\mathbb{R}^n} \left(\min\{aR, I_\alpha f(x)\}\right)^p w(x) d\mu(x).
\end{aligned}$$

Similarly,

$$\int_0^{\epsilon R} w\left(\{x\in\mathbb{R}^n : M_\alpha(f)(x)>\lambda\}\right)\lambda^{p-1}d\lambda = \int_{\mathbb{R}^n}\left(\min\{\epsilon R, M_\alpha(f)(x)\}\right)^p w(x)d\mu(x).$$

Therefore,

$$\int_0^{aR} w\left(\{x\in\mathbb{R}^n : I_\alpha f(x)>\lambda\}\right)\lambda^{p-1}d\lambda \le C\int_{\mathbb{R}^n}\left(\min\{aR, I_\alpha f(x)\}\right)^p w(x)d\mu(x).$$

Using Fatou's lemma,

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(I_\alpha f(x)\right)^p w(x)d\mu(x) &= \int_{\mathbb{R}^n}\liminf_{R\to\infty}\left(\min\{aR, I_\alpha f(x)\}\right)^p w(x)d\mu(x)\\
&\le \liminf_{R\to\infty}\int_{\mathbb{R}^n}\left(\min\{aR, I_\alpha f(x)\}\right)^p w(x)d\mu(x)\\
&\le C\liminf_{R\to\infty}\int_{\mathbb{R}^n}\left(\min\{\epsilon R, M_\alpha(f)(x)\}\right)^p w(x)d\mu(x)\\
&\le \int_{\mathbb{R}^n}\left(M_\alpha(f)(x)\right)^p w(x)d\mu(x).
\end{aligned}$$

This completes the proof. □

Chapter 4

Future Motivation

In this chapter we indicate some application and future motivation of our work.

Generalization of the Sobolev Imbedding Theorem:

Definition 85. *Suppose f and g are two locally integrable functions on $\mathbb{R}^n$. Then we say that $\frac{\partial^\alpha f}{\partial x^\alpha} = g$ in the weak sense if*

$$\int_{\mathbb{R}^n} f(x)\frac{\partial^\alpha \varphi}{\partial x^\alpha}(x)dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n} g(x)\varphi(x)dx, \quad \textit{for all} \quad \varphi \in \mathcal{D}$$

where $\mathcal{D}$ denote the space of indefinitely differentiable functions with compact support, $\dfrac{\partial^\alpha}{\partial x^\alpha} = \dfrac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} + \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$, and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

Definition 86. *For any non-negative integer k, the Sobolev space $L_k^p(\mathbb{R}^n)$, is defined as the space of functions f, with $f \in L_k^p(\mathbb{R}^n)$ and where where all $\frac{\partial^\alpha f}{\partial x^\alpha}$ exists and $\frac{\partial^\alpha f}{\partial x^\alpha} \in L^p(\mathbb{R}^n)$ in the weak sense whenever $|\alpha| \leq k$.*

The usual version of the Sobolev Imbedding theorem is:

Theorem 87. *([42]) Suppose α is a positive integer, and $1/q = 1/p - \alpha/n$. Then*

(i) If $q < \infty$ (i.e. $p < n/\alpha$), then $L_k^p(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$ and the natural inclusion map is continuous.

(ii) If $q = \infty$ (i.e. $p = n/\alpha$), then the restriction of an $f \in L^p_k(\mathbb{R}^n)$ to a compact subset of $\mathbb{R}^n$ belongs to $L^r(\mathbb{R}^n)$, for every $r < \infty$.

(iii) If $p > n/\alpha$, then every $f \in L^p_k(\mathbb{R}^n)$ can be modified on a set of zero measure so that the resulting function is continuous.

B. Muckenhoupt and R. L. Wheeden [37] establish a good-λ inequality associated to Lebesgue measure and prove as an application a weighted version of a part of the Sobolev Imbedding theorem stated below:

Theorem 88. *If $1 < p < n, 1/q = 1/p - 1/n$ and $V(x)$ is a nonnegative function satisfying*

$$\left(\frac{1}{|Q|}\int_Q [V(x)]^q\right)^{1/q}\left(\frac{1}{|Q|}\int_Q [V(x)]^{p/(p-1)}dx\right)^{1-1/p} \leq K,$$

then

$$\|f(x)V(x)\|_q \leq C\left(\|f(x)V(x)\|_p + \sum_{j=1}^{n}\left\|\frac{\partial f(x)}{\partial x_j}V(x)\right\|_p\right)$$

where C is independent of f and $\partial f(x)/\partial x_j$ is taken in the sense of distributions.

Following their footprints, it is expected to obtain the Sobolev imbedding theorem in the more general context associated to a Radon measure which satisfies either the growth condition (2) or the doubling condition (1). This will open a wide range of spectrum for the future research on this line.

Wolff Type Inequality

Definition 89. *Let $0 < \alpha < n$, μ be a measure on $\mathbb{R}^n$, and $0 < \rho < \infty$. Then the inhomogeneous Riesz potential of μ, denoted by $I_\alpha\mu$, is given by*

$$I_{\alpha,\rho}\mu(x) = \int_{B(x,\rho)} \frac{d\mu(y)}{|x-y|^{N-\alpha}}$$

and the inhomogeneous maximal function, denoted by $M_{\alpha,\rho}\mu$, is defined by

$$M_{\alpha,\rho}\mu(x) = \sup_{0<r<\rho}\frac{1}{r^{N-\alpha}}\int_{B(x,r)} d\mu(y).$$

If $du = f dx$ where f is a measurable function on $\mathbb{R}^n$, then the inhomogeneous Riesz potential of μ and the inhomogeneous maximal function of μ are respectively denoted by $I_\alpha f$ and $M_\alpha f$.

There is a good-λ inequality associated to inhomogeneous Riesz potential ([52]). The proof is a modification of the proof of the Theorem 67. The Riesz potential of a function is controlled in norm by the corresponding fractional maximal function. The corresponding inequality between inhomogeneous Riesz potential and inhomogeneous maximal function is the main ingredient in the proof of the following Wolff type inequality ([52]).

$$C_1 \int \mathbb{R}^n V_\rho^\mu d\mu \leq \int_{\mathbb{R}^n} W_\rho^\mu d\mu \leq C_2 \int_{\mathbb{R}^n} V_\rho^\mu d\mu$$

where C_1 and C_2 are constants,

$$V_\rho^\mu(x) = I_{\alpha,\rho}\left((I_{\alpha,\rho}\mu)^{1/p-1}\, w^{-1/p-1}\right)$$

is the nonlinear potential of measure μ on $\mathbb{R}^n$, and

$$W_\rho^\mu(x) = \int_0^\rho \left(\frac{t^{\alpha\rho}\mu(B_t(x))}{w(B_t(x))}\right)^{1/p-1} \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$

It can be expected to extend the good-λ inequality relating the inhomogeneous Riesz potential and inhomogeneous maximal function associated to a measure doubling or non-doubling. This will lead to the potential inequalities between inhomogeneous Riesz potential and the inhomogeneous maximal function. This, in turn, can be applied to extend the Wolff type inequality for such measures. See, for example, D. R. Adams ([1], [3]), and L. I. Hedberg and T. H. Wolff ([29]).

Riesz and Bessel Capacities

Definition 90. *Let $0 < \alpha < n, 1 < p < \infty$, and $0 < \rho < \infty$, and let $E \subset \mathbb{R}^n$. We define the Riesz capacity of E, denoted by $R_{\alpha.p;\rho}(E)$, by*

$$R_{\alpha.p;\rho}(E) = \inf\left\{\|f\|^p_{L^p(\mathbb{R}^n)} : f \geq 0 \quad \textit{and} \quad I_{\alpha,\rho}(x) \geq 1 \quad \textit{for every} \quad x \in E\right\}$$

and the Bessel capacity of E, denoted by $B_{\alpha,p}(E)$, is defined by

$$B_{\alpha,p}(E) = \inf \left\{ \|f\|^p_{L^p(\mathbb{R}^n)} : f \in L^p(\mathbb{R}^n)^+ \quad \textit{and} \quad \mathcal{G}_\alpha f(x) \geq 1 \quad \textit{for every} \quad x \in E \right\}.$$

If we replace $L^p(\mathbb{R}^n)$ by $L^p_w(\mathbb{R}^n)$ (w being a weight function on $\mathbb{R}^n$) in the above definitions then these are respectively known as the weighted Riesz capacity and the weighted Bessel capacity and are denoted by $R^w_{\alpha,p;\rho}$ and $B^w_{\alpha,p}$ respectively.

It is well known that Bessel and Riesz capacities occur naturally in the study of deeper properties of Sobolev space $W^{m,p}_w(\mathbb{R}^n)$ for $1 < p < \infty$. In the case of Lebesgue measure, it has been shown that that the Bessel and Riesz capacities are equivalent (B. O. Turesson [52]). We can expect to establish Riesz potential type inequalities between Bessel potentials and inhomogeneous maximal functions. We may go further to establish that the Bessel capacity and the inhomogeneous Riesz capacity are equivalent.

Bibliography

[1] D. R. Adams, *Traces of Potentials II*, Indiana University Math. J. **22** (1973.), 907–918.

[2] ———, *A Note on Riesz Potentials*, Duke Mathematical Journal **42** (1975), 765–778.

[3] ———, *Weighted non-linear potential theory*, Trans. Amer. math. Soc. **297** (1986.), 73–94.

[4] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Springer, 1996.

[5] T. Amdeberhan, *The 'Uncentered' maximal function and L^p-boundedness*, Internet, Department of Mathematics, Temple University.

[6] R. F. Bass, *L^p inequalities for functionals of Brownian motion*, Séminaire de probabilités(Strasbourg) **21** (1987.), 206–217.

[7] R. Bañuelos, *A sharp good-λ inequality with an application to Riesz transforms*, Michigan Math. J. **35** (1988.), 117–125.

[8] R. Bañuelos and C. N. Moore, *Probabilistic Behaviour of Harmonic Functions*, Progress in Mathematics, Birkhäuser Verlag **175** (1999.).

[9] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. of Prob. **1** (1973.), 19–42.

[10] ———, *Exit times of Brownian motion, harmonic majorization, and Hardy spaces*, Advances in Mathematics **26** (1977.), 182–205.

[11] ———, *Martingale theory and harmonic analysis in Euclidean spaces*, Proc. of Symposia in Pure Mathematics **32** (1979.), 283–301.

[12] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasilinear operators on martingales*, Acta Math **124** (1970), 249–304.

[13] D. L. Burkholder and R. F. Gundy, *Distribution functions inequalities for the area integrals*, Studia Math. **44** (1972.), 527–544.

[14] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with distributions*, Adv. in Math **16** (1975.), 1–64.

[15] R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur certains Homogenes, Lecture Notes in Mathematics, 242*, Springer-Verlag, Berlin, 1971.

[16] B. E. J. Dahlberg, *Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for functions harmonic in Lipschitz domain*, Studia Math. **67** (1980.), 297–314.

[17] G. David, *Uncertifiable 1-sets have vanishing analytic capacity*, Rev. Mat. Iberoamericana **14(2)** (1998.), 369–479.

[18] M. de Guzman, *Differentiation of Integrals in* $\mathbb{R}^n$., Springer-Verlag, 1967.

[19] J. Duoandikoetxea, *Fourier Analysis, Graduate Studies in Mathematics, Volume 20*, American Mathematical Society, Province, Rhode Island, 2000.

[20] ______, *The Hardy-Littlewood maximal function and some of its variants*, Advance courses of mathematical analysis; Proceedings of the second International School, Granada, Spain (2004), 20–24.

[21] R. Fefferman, *Strong differentiation with respect to measures*, Amer. J. Math. **103** (1980), 33–40.

[22] O. Frostman and H. Wallin, *Riesz Potentials on Manifolds. Potential theory, Lecture Notes in math. 787*, Springer, 1980.

[23] J. García-Cuerva and A. Eduardo Gatto, *Boundedness properties of fractional integral operators associated to non-doubling measures*, Studia Mathematica **162(3)** (2004.).

[24] J. García-Cuerva and J. M. Martell, *Weighted inequalities and vector-valued Calderón-Zygmund operators on non-homogeneous spaces*, Publ. Mat. **44(2)** (2000.), 613–640.

[25] L. Grafakos, *Classical and Modern Fourier Analysis*, Prentice Hall, 2003.

[26] M. Silverstein, R. Fefferman, R. F. Gundy, and E. M. Stein, *Inequalities for ratios of functionals of harmonic functions*, Proc. Nat. Acad. Sci. U. S. A. **79** (1982.), 7958–7960.

[27] G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Mathematica **54**.

[28] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[29] L. I. Hedberg and T. H. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **33** (1983.), 161–187.

[30] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer, 2001.

[31] S. D. Jaka, *A note on good lambda inequality*, Lecture Notes in Mathematics, Séminaire de Probabilits XXIII (1989.), 57–65.

[32] M. Martin and P. Szeptycki, *Sharp inequalities for convolution operators with homogeneous kernels and applications*, Indiana Univ. Math. **46** (1997), 975–988.

[33] ———, *Integral transforms controlled by maximal functions*, Operator Theory: Advances and Applications **153** (2004), 169–184.

[34] J. A. Hempel, G. R. Morris, and N. S. Trudinger, *On the sharpness of the limiting case of the Sobolev imbedding theorem*, Bull. Austr. Math. Soc. **3** (1970), 369–373.

[35] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal functions*, Trans. Amer. Math. Soc. **165** (1972.), 207–226.

[36] ______, *The equivalence of two conditions for weight functions*, Studia Math. **49** (1974.), 101–106.

[37] B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974.), 261–274.

[38] J. Mateu, P. Matilla, A. Nicolau, and J. Orobitg, *BMO for non-doubling measures*, Duke Mathematical Journal **102** (2000.), 533–565.

[39] J. Orobitg and C. Perez, *Ap weights for nondoubling measures in* $\mathbb{R}^n$ *and applications*, Trans. of the American Mathematical Society **354** (2002.), 2013–2033.

[40] Y. Sawano and H. Tanaka, *Morrey spaces for non-doubling measures*, Acta Mathematics Sinica, English Series **15** (2000.), 1–11.

[41] P. Sjögren, *A remark on the maximal function for measure in* $\mathbb{R}^n$, Amer. J. Math **105** (1983), 1231–1233.

[42] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.

[43] ______, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.

[44] R. S. Strichartz, *A note on Trudinger's extension of Sobolev's inequalities*, Indiana Univ. Math. J. **21** (1972), 841–842.

[45] X. Tolsa, *Cotlar's inequality without the doubling condition and existence of principal values for the Cauchy integral of measures*, J. Reine Angew Math. **502** (1998.), 199–235.

[46] ______, *L^2 boundedness of Cauchy integral operators for continuous measures*, Duke Math. Journal **98(2)** (1999.), 269–304.

[47] ______, *BMO, H^1, and Calderón-Zygmund operators for non doubling measures.*, Math. Ann. **319** (2001.), 89–149.

[48] ______, *A proof of the weak (1,1) inequality for singular integrals with non doubling measures based on a Calderón-Zygmund decomposition*, Publ. Mat. **45** (2001.), 163–174.

[49] F. Nazarov, S. Treil and A. Volberg, *Tb theorems for non-homogeneous spaces*, Preprint.

[50] ______, *Weak type estimates and Cotlar inequalities for Cáalderon-Zygmund operators on nonhomogeneous space*, Internet Math. Res. Notices **9** (1998), 463–487.

[51] N. S. Trudinger, *On embedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[52] B. O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces, Lecture Notes in Mathematics 1736*, Springer, 2000.

[53] A. Vargas, *On the maximal function for rotation invariant measure in $\mathbb{R}^n$*, Studia Math. **110**, 9–17.

[54] J. Verdera, *The fall of the doubling condition in Calderón-Zygmund theory*, Publ. Mat., Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations, El Escorial (2002.), 275–292.

[55] N. Wiener, *The Ergodic Theorem*, Duke Math. J. **5** (1939), 1–18.

[56] V. I. Yudovich, *English translation: Some estimates connected with integral operators and with solutions of elliptic equations*, Soviet Math **2** (1961), 746–749.

Appendix A

Notations

$\mathbb{N}$: Natural Numbers

$\mathbb{Z}$: Integers

$\mathbb{R}$: RealNumbers

$\mathbb{R}^{\mathbb{N}}$: Euclidean N-space

$|E|, dm, dx$: Lebesgue measure

χ_E: Characteristic function of E

ω_{N-1}: area of $(N-1)$-dimensional unit sphere.