MODULES AND COMODULES OVER NONARCHIMEDEAN HOPF ALGEBRAS

by

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B.A., NTUU “KPI“, Ukraine, 2000
M.A., NTUU “KPI”, Ukraine, 2002

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2010

Abstract

The purpose of this work is to study Hopf algebra analogs of constructions in the theory of p-adic representations of p-adic groups.

We study Hopf algebras and comodules, whose underlying vector spaces are either Banach or compact inductive limits of such. This framework is unifying for the study of continuous and locally analytic representations of compact p-adic groups, affinoid and $\sigma-$affinoid groups and their quantized analogs. We define the analog of Frechet-Stein structure for Hopf algebra (which play role of the function algebra), which we call CT-Stein structure. We prove that a compact type structure on a CT-Hopf algebra is CT-Stein if its dual is a nuclear Frechet-Stein structure on the dual NF-Hopf algebra. We show that for every compact p-adic group the algebra of locally analytic functions on that group is CT-Stein. We describe admissible representations in terms of comodules, which we call admissible comodules, and thus we prove that admissible locally analytic representations of compact p-adic groups are compact inductive limits of artinian locally analytic Banach space representations.

We introduce quantized analogs of algebras $U_r(\mathfrak{sl}_2, K)$ from [7] thus giving an example of infinite-dimensional noncommutative and noncocommutative nonarchimedean Banach Hopf algebra. We prove that these algebras are Noetherian. We also introduce a quantum analog of $U(\mathfrak{sl}_2, K)$ and we prove that it is a (infinite-dimensional non-commutative and non-cocommutative) Frechet-Stein Hopf algebra.

We study the cohomology theory of non-archimedean comodules. In the case of modules and algebras this was done by Kohlhasse, following the framework of J.L. Taylor. We use an analog of the topological derived functor of Helemskii to develop a cohomology theory of non-archimedean comodules (this approach can be applied to modules too). The derived functor approach allows us to discuss a Grothendieck spectral sequence (GSS) in our context.

We apply GSS theorem to prove generalized tensor identity and give an example, when this identity is nontrivial.

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Acknowledgments

It is my pleasure to thank my advisor, Zongzhu Lin, for his guidance and patience while trying to make me a good mathematician, and for his advices and help not only in mathematical questions.

I would also like to thank Yan Soibelman for initially introducing me to the subject of this thesis.

At various stages of my research I benefited from discussions with J. Teitelbaum, A. Rosenberg and G. Nagy. I would like to thank them for their expert advices.

I am grateful to the Head of our Department L. Pigno for support in my trips to schools on p-adic representations. I would also like to thank our Department for supporting me as a graduate student and our faculty and students for creating a friendly atmosphere.

The last, but by no means the least, I would like to thank David Yetter for correcting my numerous mistakes in English and Gerald Hoehn for his help with some old papers in German.

Introduction

The purpose of this work is to study Hopf algebra analogs of constructions in the theory of p-adic representations of p-adic groups.

Non-archimedean Banach Hopf algebras and comodules were first studied in the series of papers by B. Diarra. Diarra used Hopf algebras and comodules to study continuous representations of compact p-adic groups.

The breakthrough in the p-adic representation theory of p-adic groups was made in papers by P. Schneider and J. Teitelbaum. Motivated by examples of p-adic representations, which they found in their study of cohomologies of Drinfeld upper-half planes, they studied locally analytic p-adic groups and their p-adic representations in general. They studied categories of continuous representations of p-adic groups in Banach spaces over K (the orbit maps are continuous), locally-analytic representations (orbit maps are locally-analytic) of p-adic groups in topological Frechet K-vector spaces of compact type (i.e. inductive limits of p-adic Banach spaces, endowed with inductive limit topology) and relations between these two types of representations. They singled out "good" categories of representations of both types, which they called admissible representations.

Schneider and Teitelbaum's approach is to pass from a representation to its dual space, which has a structure of a module over the algebra of distributions on the group G. To the author's knowledge, in the case of continuous representations and continuous distributions this was first noticed by Diarra (see references in [5]). Schneider and Teitelbaum found a special structure on the algebra $D(G, K)$ of K-valued locally analytic distributions on a compact p-adic group, which they called a Frechet-Stein structure. This structure is the main tool in the work of M. Strauch and S. Orlik on the irreducibility of locally analytic principal series representations, in the works of Schneider and Teitelbaum and in most other works in this area.

For a compact group G, the Frechet-Stein structure on $D(G, K)$ consists of a projective system of Banach algebras $D_r(G,K)$, s.t. $D(G,K)$ is a projective limit of $D_r(G,K)$ and

i) all $D_r(G,K)$ are noetherian Banach algebras; and

ii) the transition maps in that system are flat.

In particular, the algebras $D_r(G,K)$ contain a Banach subalgebra $U_r(\mathfrak{g},K)$, which is a completion of the universal enveloping algebra $U(\mathfrak{g}_K)$ ($\mathfrak{g}_K = Lie(G) \otimes K$), and, as a $U_r(\mathfrak{g},K)$-module, $D_r(G,K)$ is finitely generated. The algebras $U_r(\mathfrak{g},K)$ also form projective system with projective limit $U(\mathfrak{g},K)$, which is called the hyper-enveloping algebra, and $U(\mathfrak{g},K)$ is a subalgebra of $D(G,K)$.

Admissible locally analytic representations on a space V are defined as those for which the topological dual V' is a separately continuous $D(G,K)$-module M ($= V'$ as a vector space), which is a projective limit of finitely generated $D_r(G,K)$-modules M_r. Schneider and Teitelbaum prove that if for all $r > s$ (s is fixed) M_r are simple modules, then V is an irreducible representation.

Attempts to introduce a quantized version of this theory started with the empirical idea (of Yan Soibelman) that, although the subject is technically difficult, introducing one more variable (q) may actually help with some difficulties and provide an insight into the non-quantized case. In the quantum case, Hopf algebras and modules and comodules give the language one must use. Thus one must understand first whether the results and constructions of locally analytic representation theory have analogs in the framework of Banach and topological Hopf algebras and comodules. To answer this question was the purpose of this work.

In Chapter 1 we study Hopf algebras and comodules whose underlying vector spaces are either Banach or compact inductive limits of such. This framework is unifying for the study of continuous and locally analytic representations of compact p-adic groups, affinoid and $\sigma-$affinoid groups and their quantized analogs. In the algebraic setting it was outlined by Z. Lin in [19]. We prove various results about Banach comodules over K-Banach Hopf algebras, including results on simplicity and finite cogeneratedness for artinian comodules over artinian Banach Hopf algebras. We define the analog of Frechet-Stein structures on Hopf algebra (which play role of the function algebra), which we call CT-Stein structures. We prove that the compact type structure on a CT-Hopf algebra is CT-Stein if its dual is a nuclear Frechet-Stein structure on dual NF-Hopf algebra. This allows us to describe admissible representations in terms of comodules, which we call admissible comodules (in a special case this description was noticed by Emmerton).

In Chapter 2 we show that the algebra of locally analytic functions on a compact p-adic group is CT-Stein. We introduce quantized analogs of $U_r(\mathfrak{sl}_2, K)$ and prove that they are Noetherian. We also introduce a quantum analog of $U(\mathfrak{sl}_2, K)$ and we prove that it is a (infinite-dimensional non-commutative and non-cocommutative) Frechet-Stein Hopf algebra.

The motivation for the study of $U_r(\mathfrak{g}, K)$ and $U(\mathfrak{g}, K)$ comes from the following: Consider a compact locally analytic group G. So far, all known examples of irreducible admissible representations of G arise as duals of simple $D(G, K)$-modules that are projective limits of simple $D_r(G, K)$-modules. While in general there might be simple $D(G, K)$-modules not of this type, this is a natural class to consider first. This motivates the study of simple $D_r(G, K)$-modules. Any compact group G has a system of neighborhoods, consisting of open normal subgroups. Fix such a subgroup H. Then we have an isomorphism of vector spaces

$$C_{r_H}(G, K) \cong \sum_{g \in G/H} C_{r_H}(gH, K)$$

and by duality

$$D_{\leq r_H}(G, K) \cong \sum_{g \in G/H} D_{\leq r_H}(gH, K),$$

where r_H is the r which corresponds to locally $H-$analytic functions. Since H is normal, by results of Kohlhasse [15] on supports of p-adic distributions (which can be extended to the case of $D_{\leq r_H}(G, K)$), $D_{\leq r_H}(G, K)$, is a G/H-graded algebra and Clifford theory gives a relation between simple $D_{\leq r_H}(G, K)-$modules and simple $D_{r_H}(H, K)-$modules. But $C_{r_H}(H, K)$ is a space of power series on H and its dual is the algebra $U_{r_H}(\mathfrak{g}_K, G)$. Thus in order to study simple $D_r(G, K)$-modules one must start with simple $U_r(\mathfrak{g}, K)$-modules. The later task is known to be extremely complicated even in the algebraic case of $U(\mathfrak{g})$-modules. The study of quantizations of these algebras may appear to be useful in view of the above and also due to other relations in the algebraic case between quantum groups and representation of their classical counterparts.

In order to prove the Frechet-Stein property of our quantized algebras we found it convenient to introduce the skew-commutative analogs of Tate algebras. We call those algebras skew-Tate algebras and we found that the Weierstrass division and preparation theorems also hold in the skew-commutative case.

In Chapter 3 we study a cohomology theory for non-archimedean comodules. In the case of modules and algebras this was done by Kohlhasse [16], following the framework of J.L. Taylor [30]. In the algebraic case the cohomology of comodules was studied by various authors. It was believed for a long time that the derived functor approach was not suitable for developing such a theory even in the case of modules. It was first done in the thesis of T. Buhler [4], who developed the derived functor approach in the case of bounded cohomology. We use an analog of the topological derived functor of Helemskii to develop a cohomology theory of non-archimedean comodules (this approach can be applied to modules too). The derived functor approach allows us to discuss the Grothendieck spectral sequence (GSS) in our context (one needs to be careful since our category is not abelian, but quasi-abelian). While the GSS theorem itself is very similar to the classical one, in the application of the situation is more complicated than in the algebraic case. We apply the GSS theorem to prove a generalized tensor identity, which is the main result of this chapter. While in the case of continuous representations of compact locally analytic groups on Banach spaces the locally analytic induction functor is exact and the above identity is just zero, in the case of comodules over Hopf algebras of rigid analytic functions the situation is different. The celebrated theorem of Noskov from the theory of bounded cohomology is also a (straightforward) consequence of our GSS theorem.

Chapter 1

Preliminaries

In this section we recall some preliminary notions and some results, mostly without proof. One may find all missing details in the references cited.

1.1 Notations

In this paper the following will mean:

K or L - a finite extension of $\mathbb{Q}_p$

$|\cdot|_K$ - the norm on K, extending the norm on $\mathbb{Q}_p$

$|K|$ - value group of K (as a set it is a set of values of $|\cdot|_K$)

o_K - its ring of integers

$\mathbb{Z}_p \subset \mathbb{Q}_p$ - the ring of p-adic integers

Δ_A- the coaction of coalgebra or comodule A

ϵ_A- the counit of coalgebra A

1.2 Some nonarchimedean functional analysis

Here we recall some definitions and facts about nonarchimedean topological vector spaces and Banach spaces. The main references are [24], but see also [6, 23, 31].

Recall that a (nonarchimedean) seminorm q on V is a function $q : V \to \mathbb{R}$, s.t.

1. $q(av) = |a|_K q(v)$ for any $a \in K$ and $v \in V$,

2. $q(v+w) \leq \max(q(v), q(w))$ for any $v, w \in V$.

From these axioms it is easy to check that $q(0) = 0$ and $q(v) \geq 0$ for any $v \in V$.

A seminorm is called a *norm* if $q(v) = 0$ implies $v = 0$.

Since any $K-$vector space is also an o_K-module, we can consider its o_K-submodules.

Definition 1.2.1. A *lattice* L in V is an o_K-submodule of V, s.t. for any $v \in V$ there exists $0 \neq a \in K : av \in L$.

The intersection of two lattices is again a lattice.

For any seminorm q we can define two o_K-submodules of V :

$$L(q) = \left\{ v \in V \middle| q(v) \leq 1 \right\} \text{ and } L^-(q) = \left\{ v \in V \middle| q(v) < 1 \right\}.$$

$L(q)$ and $L^-(q)$ are lattices.

Conversely, for any lattice L in V we define its *gauge* p_L by

$$\begin{array}{rccc} p_L : & V & \to & \mathbb{R} \\ & v & \mapsto & \inf\limits_{v \in aL} |a|_K \end{array}.$$

p_L is a seminorm on V.

Definition 1.2.2. Let $(L_j)_{j \in J}$ be a family of lattices in the $K-$vector space V such that

(lc1) for any $j \in J$ and $a \in K^\times$ there is $k \in J$ such that $L_k \subseteq aL_j$;

(lc2) for any two $i, j \in J$ there exists a $k \in J$ such that $L_k \subset L_i \cap L_j$.

Such a family form a base of a topology, which is called *locally convex topology on V defined by the family $(L_j)_{j \in J}$*.

Definition 1.2.3. Let $(q_j)_{j \in J}$ be a family of seminorms on the $K-$vector space V. *The topology defined by the family $(q_j)_{j \in J}$* is the coarsest topology on V such that

i) all $q_j : V \to \mathbb{R}$ are continuous;

ii) for any $v \in V$ the translation $v + . : V \to V$ is continuous.

For any finitely many seminorms $q_{j_1},\ldots,q_{j_r}$ in the given family $(q_j)_{j\in J}$ and any real $\epsilon > o$ define

$$V(q_{j_1},\ldots,q_{j_r};\epsilon) = \{v \in V : q_{j_1} \le \epsilon, \ldots, q_{j_r}(v) \le \epsilon\}.$$

Lemma 1.2.4. *$V(q_{j_1},\ldots,q_{j_r};\epsilon)$ is a lattice in V.*

The topology on V given by a family of seminorms $(q_j)_{j\in J}$ coincides with the topology, given by the family of lattices $V(q_{j_1},\ldots,q_{j_r};\epsilon)$.

The topology given by a family of lattices $(L_j)_{j\in J}$ can also be defined by the family of gauges $(p_{L_j})_{j\in J}$

Proof. [24, par. 4]. □

Definition 1.2.5. Let V be a topological $K-$vector space. V is said to be *locally convex* (LCVS) if the topology of V is given by a family of seminorms $(q_i)_{i\in I}$ or, equivalently, by a family of lattices.

Definition 1.2.6. Let $\mathbb{H} = (H, \tau_H)$ be a topological $K-$vector space (H is a $K-$vector space, τ_H is a topology on it). $\mathbb{H}$ is called a *$K-$Banach space* if there is a norm $\|\cdot\|$ on H, which induces the topology τ_H and H is complete w.r.t. $\|\cdot\|$.

A Banach space is an LCVS whose topology is defined by a single seminorm, which is a norm.

If we have a $K-$Banach space $(H, \|\cdot\|_1)$ then exists another norm $\|\cdot\|_2$, which is equivalent to $\|\cdot\|_1$ such that values of $\|\cdot\|_2$ lies in $|K|$.

Two important examples are [31, 3.B, 3.A]

$$c_0(X) = \left\{(c_x)_{x\in X} \in K^X | \forall\epsilon > 0 : \text{ the number of } x \text{ such that } |c_x|_K > \epsilon \text{ is finite}\right\}$$

and

$$l^\infty(X) = \left\{(c_x)_{x\in X} \in K^X | \exists C > 0 : \sup |c_x|_K < C\right\}$$

For K discretely valued, any K-Banach space is topologically isomorphic to a space $c_0(X)$ for some X.

Now let us review the concept of orthogonality in Banach spaces.

Definition 1.2.7. Let H be a $K-$Banach space with a norm $\|\cdot\|$ and M be a closed subspace. A vector $v \in H$ is called $\|\cdot\|$-orthogonal to M, if $\inf_{x\in M} \|x+v\| = \|v\|$.

In nonarchimedean case this condition is equivalent to the following:

$$\forall x \in M, \lambda, \mu \in K : \|\lambda x + \mu v\| = \max(|\lambda|_K \|x\|, |\mu|_K \|v\|)$$

A *base* in an LCVS is a linearly independent subset of vectors, such that the linear hull of this subset is dense.

Lemma 1.2.8. *Suppose that for $K-$Banach space H the values of $\|\cdot\|$ lies in $|K|$. Since K is discretely valued,*

1. *the space H has an orthogonal base;*

2. *every closed subspace in H has an orthogonal complement.*

Proof. This is a combination of results from [31]. For K discretely valued, every Banach space is of the form $c_0(X)$. This is a condition ι) of [31, Thm. 5.16] for $s \equiv 1$ (see [31, 3.H]), which is equivalent to the conditions η) of [31, Thm. 5.16] and β) of [31, Thm. 5.13]. □

Remark 1.2.9. 2 also follows from [31, 4.7] or [24, sec.10].

$V' = L(V, K)$ will denote the set of continuous linear functionals on V. We will equip V' with a topology.

Definition 1.2.10. Let V be an LCVS with the topology given by seminorms q_i. A subset $B \subset V$ is called *bounded* if $q_i(B)$ is bounded for every i.

Definition 1.2.11. Let V be a LCVS. The *strong topology* τ_b on V' is the topology, defined by family of seminorms $p_B(f) = \sup_{v\in B} |f(v)|_K$ with B running over all bounded subsets of V. We denote V' with strong topology by $V'_b = (V', \tau_b)$.

If V is a Banach space, then the strong topology on V' is the topology defined by the dual norm. By [31, p.52 and 3.Q] we have $(c_0(X))'_b \cong l^\infty(X)$.

The map $f : V \to W$ of two LCVS is called *strict* if the subspace topology on the image is equivalent to the quotient topology.

We always have $V \subseteq (V'_b)'_b$ by [24, sec. 9].

Definition 1.2.12. An LCVS V is called *reflexive* if $V = (V'_b)'_b$.

A Banach space is reflexive iff it is finite dimensional (if K is discretely valued).

Definition 1.2.13. An o_K-submodule $A \subseteq V$ is called *c-compact* if, for any decreasingly filtered family $(L_i)_{i \in I}$ of open lattices $L_i \subseteq V$ the canonical map

$$A \twoheadrightarrow \varprojlim A/(L_i \cap A)$$

is surjective.

Definition 1.2.14. A continuous map $f : V \to W$ between two LCVS is called *compact* if there is an open lattice $L \subset V$ such that the closure of the image $f(L)$ is bounded and c-compact.

If $f : V \to W$ is compact then $f' : W'_b \to V'_b$ is also compact.

Definition 1.2.15. A continuous map $f : V \to W$ between two LCVS is called *completely continuous* if it belongs to the closuse of the subspace of maps with finite-dimensional image in $L(V, W)$.

In case of Banach spaces the classes of compact and completely continuous maps are the same.

Definition 1.2.16. Consider a sequence of LCVS (V_n) with maps $\phi_{nm} : V_n \to V_m$. On inductive limit $V = \lim_{\to} V_n$ consider the strongest locally convex topology such that all inclusions $V_n \subset V$ are continuous. Equipped with this topology, V is called *locally convex* *inductive limit* of V_n. If the transition maps ϕ_{nm} are compact, the V is called a *compact limit* of the sequence V_n.

Definition 1.2.17. Let V be a LCVS. V is said to be of *compact type* (or a CT- or LS-*space*) if it is an compact limit of an inductive system (V_n, ϕ_{nm}) of Banach spaces V_n.

Remark 1.2.18. A Banach space H can be presented as a limit of the stationary sequence V_n with $V_n = H$. H is of compact type iff it is finite dimensional. This follows from the fact that only in this case the identity map is compact.

Every Banach space is an algebraic inductive limit of its finite-dimensional subspaces, but the inductive limit topology in this case is not even metrizable.

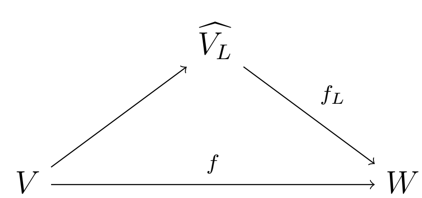
Definition 1.2.19. We call an LCVS V a *Frechet space* if it is complete and metrizable. Equivalently, the topology on V is defined by a countable family of seminorms.

Each Frechet space is countable projective limit of Banach spaces.

Next we discuss nuclearity. For any o_K-submodule $A \subseteq V$ in an LCVS V we can form a $K-$vector space $V_A = K \otimes_{o_K} A$ and equip it with the topology defined by the gauge p_A. Denote by $\widehat{V_A}$ the Hausdorff completion of V_A. It is a Banach space w.r.t. the continuous extension of p_A.

If $A = L$ is an open lattice in V, then the identity map on V gives a continuous map $V \to V_L$ with dense image.

If $f : V \to W$ is a continuous linear map into a Banach space W, then there exists unique $f_L : \widehat{V_L} \to W$, where L is a preimage of the unit ball in W, such that we have a commutative diagram



When $M \subseteq L$ is a second open lattice in V then the identity map can be viewed as a continuous map $V_M \to V_L$. By taking completions and using the above universal property we get a canonical continuous map $\widehat{V_M} \to \widehat{V_L}$.

Definition 1.2.20. An LCVS V is called nuclear if for any open lattice $L \subseteq V$ there exists another open lattice $M \subseteq L$ such that the canonical map $\widehat{V_M} \to \widehat{V_L}$ is compact.

Any nuclear Frechet space (NF-space) is reflexive and its dual is a CT-space.

Definition 1.2.21. An LCVS V is called *of countable type* if there is a countable dense subset in V.

Remark 1.2.22. In [12] it is proved that any CT-space is of countable type. In [21] it is proved that any NF-space is of countable type. Thus in most places we can safely assume that our spaces are of countable type. Mostly we will not use this assumption.

Definition 1.2.23. Let V be a Banach space.

1. Let U be a closed subspace of V. We define $U^{\perp} \subset V'_b$ as

$$U^{\perp} = \left\{ \phi \in V'_b \middle| \phi(u) = 0 \ \forall u \in U \right\}.$$

2. Let W be a closed subspace of V'_b. We define

$$\operatorname{Ker}(W) = W^{\perp} \cap V = \left\{ v \in V \middle| \phi(v) = 0 \ \forall \phi \in W \right\}.$$

It follows from Hahn-Banach theorem, that if U is a proper closed subspace of V, then $U^{\perp} \neq 0$. It is also clear that

$$\operatorname{Ker}(W) = \bigcap_{\phi \in W} \operatorname{Ker}(\phi)$$

and $U^{\perp}$ and $\operatorname{Ker}(W)$ are closed subspaces.

The following lemma is obvious.

Lemma 1.2.24. *Let V and W be two CT-spaces. If $\phi : V \twoheadrightarrow W$ is a continuous surjection, then $\phi' : W'_b \to V'_b$ is a continuous injection.*

Lemma 1.2.25. *Let U be a subspace of a Banach space V. Then the largest Hausdorff quotient of V/U is isomorphic to $V/\bar{U}$.*

Proof. The topology on V/U is given by quotient seminorm under the projection $\pi : V \to V/U$. The points of $\bar{U} = \pi^{-1}(\overline{\{0\}})$, which are not in U, in this topology are infinitely close to zero, and thus if $\pi(u) \in \pi\left(\bar{U}\right)$, then $\forall v \in V/U$ the points v and $v + \pi(u)$ are not separated. In any Hausdorff quotient $\phi : V/U \twoheadrightarrow W$ if $\phi(v)$ and $\phi(v + \pi((u)))$ are separated in W with non-intersecting neighborhoods $W_{\phi(v)}$ and $W_{\phi(v+\pi((u)))}$, then $\phi^{-1}\left(W_{\phi(v)}\right)$ and $\phi^{-1}\left(W_{\phi(v+\pi((u)))}\right)$ are non-intersecting neighborhoods of v and $v + \pi(u)$.

Thus the quotient of V/U is Hausdorff iff $\pi\left(\bar{U}\right)$ goes to zero. Clearly $V/\bar{U} = (V/U)/\left(\bar{U}/U\right)$ is the largest quotient with this property. □

Corollary 1.2.26. *Let U be a subspace of a space V of compact type. Then the largest Hausdorff quotient of V/U is isomorphic to $V/\bar{U}$.*

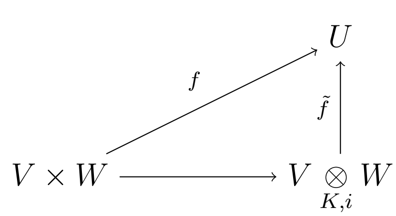
1.3 Complete tensor products

Let V and W be two LCVSs (see, for example [24, sec. 17]).

The *inductive tensor product topology* is the finest locally convex topology on $V \underset{K}{\otimes} W$, such that the canonical bilinear map

$$V \times W \to V \underset{K}{\otimes} W$$

is separately continuous. We write $V \underset{K,i}{\otimes} W$ for $V \underset{K}{\otimes} W$ equipped with this topology. The space $V \underset{K,i}{\otimes} W$ is characterized by the universal property that for any bilinear separately continuous map $f : V \times W \to U$ the induced map $\tilde{f}$

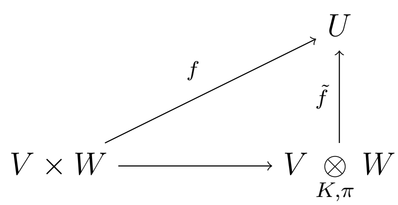


is continuous.

The *projective tensor product topology* is the finest locally convex topology on $V \underset{K}{\otimes} W$, such that the canonical bilinear map

$$V \times W \to V \underset{K}{\otimes} W$$

is (jointly) continuous. We write $V \underset{K,\pi}{\otimes} W$ for $V \underset{K}{\otimes} W$ equipped with this topology. The space $V \underset{K,\pi}{\otimes} W$ is characterized by the universal property that for any bilinear continuous map $f : V \times W \to U$ the induced map $\tilde{f}$



is continuous.

Proposition 1.3.1. *Let V and W be two LCVS.*

1. *If V and W are Frechet (in particular, Banach) spaces, then the projective and injective tensor product topologies coincide.*

2. *If V and W are CT spaces, then the projective and injective tensor product topologies coincide.*

Proof. 1) [24, 17.6]; 2)[6, 1.1.31]. □

Thus we will usually write $V \underset{K}{\otimes} W$ meaning the above topology and $V \widehat{\otimes} W$ for Hausdorff completion of $V \underset{K}{\otimes} W$.

Lemma 1.3.2. *If V and W are a) Banach; b) Frechet; c) nuclear Frechet; d) CT spaces, then $V \widehat{\otimes} W$ is also of the same type.*

Proof. [6, 1.1.28,29,32.] □

The categories of a) Banach; b) Frechet; c) nuclear Frechet; d) CT spaces are tensor categories with the tensor structure given by $\widehat{\otimes}$ (the morphisms are continuous maps).

Lemma 1.3.3. *Let W and V be both either Frechet or CT spaces and let U be a linear subspace of V.*

Then $W \widehat{\otimes} (V/U) = W \widehat{\otimes} \left(V/\bar{U}\right)$.

The following is the analog of the "integration" theorem of [26, Thm. 22]

Proposition 1.3.4. *Let A be a CT LCVS and A' is its dual NF LCVS. Then we have continuous K-linear isomorphisms*

$$\begin{aligned} &1) \quad L_b(A, V) \cong A' \underset{K,\pi}{\widehat{\otimes}} V \\ &2) \quad L_b(A', V) \cong A \widehat{\otimes} V \quad (\textit{the "integration" map}) \end{aligned}$$

for any V of compact type.

Proof. The first isomorphism is [24, 20.9]. For the second observe that, if $V = \varinjlim V_n$, then $L_b(A', V) = \varinjlim L_b(A', V_n)$ and $A \widehat{\otimes} V = \varinjlim A \widehat{\otimes} V_n$, and apply [26, Prop. 1.5]. □

The above proposition does not hold for K-Banach spaces (see [24, 18.11]).

Lemma 1.3.5. *For any K-Banach space A and any Banach space V*

$$A \widehat{\otimes} V \cong C(A', V) = CC(A', V)$$

where $C(\cdot, \cdot)$ denotes compact maps and $CC(\cdot, \cdot)$ completely continuous maps, respectively.

Lemma 1.3.6. *Let $f : V \to W$ be a strict map of Banach spaces. Then $1 \otimes f : U \widehat{\otimes} V \to U \widehat{\otimes} W$ is also strict for any Banach space U.*

Proof. $\operatorname{Im}(1 \otimes f) = U \widehat{\otimes} \operatorname{Im}(f)$. □

1.4 Seminormed spaces

We call a topological vector space E seminormed if its topology is defined by a seminorm p_E.

If E is a topological vector subspace of F and p_F is a seminorm on F, then it induces a seminorm p' on E and p'' on F/E, defined by

$$p'(x) = p(x) \text{ and } p''(x+E) = \inf_{e \in E} p(x+e).$$

Thus in the category of seminormed spaces one can take a quotient by an arbitrary linear subspace.

The seminormed space X is called complete if any net in X has an accumulation point.

Every seminormed space X can be embedded into a complete seminormed space $\tilde{X}$, which is the space of Cauchy nets in X.

1.5 Nonarchimedean topological algebras.

Let A be a locally convex algebra, that is A is an LCVS with an associative unital K-algebra structure such that the algebra multiplication $m : A \otimes A \to A$ is continuous and thus can be continued to the map $m : A \widehat{\otimes} A \to A$.

Definition 1.5.1. A seminorm q on A is called an *algebra seminorm* if there exists $c > 0$ such that

$$q(ab) \leq cq(a)\,q(b) \text{ for any } a, b \in A.$$

The seminorms, defining topology on A, are necessarily algebra seminorms.

Definition 1.5.2. An algebra seminorm is called *submultiplicative* if $c = 1$, i.e.

$$q(ab) \leq q(a)\,q(b) \text{ for any } a, b \in A$$

and unital if $q(1) = 1$.

Definition 1.5.3. A locally convex algebra A is called a *K-Banach algebra* if its topology is given by a single seminorm $\|\cdot\|_A$, which is a norm.

Proposition 1.5.4. *[27, Prop. 2.1]Let A be a (left) noetherian K-Banach algebra.*

- *Each finitely generated A-module carries a unique K-Banach space topology (called the canonical topology) such that the A-module structure map $A \times M \to M$ is continuous;*
- *every A-submodule of a finitely generated module is closed in the canonical topology; in particular, every (left) ideal in A is closed;*
- *any homomorphism of finitely generated A-modules is continuous and strict for the canonical topologies.*

Finitely generated modules over a Noetherian $K-$Banach algebra form an abelian category.

Since K is discretely valued, when values of $\|\cdot\|_A$ are a subset of the value group of K, we can equip such $K-$Banach algebra with an integral, complete, separated, decreasing filtration

$$F_n A = \left\{ a \in A \middle| \, \|a\|_A \leq p^{-n} \right\}.$$

Thus we can view A as a filtered ring.

A locally convex algebra A is called a (nuclear) Frechet algebra if it is a (nuclear) Frechet LCVS.

Definition 1.5.5. A Frechet algebra A is called Frechet-Stein algebra, if it is a projective limit of a projective system of $K-$Banach algebras (A_n, ϕ_{nm}), where $\phi_{nm} : A_n \to A_m$, such that

1. the A_n are noetherian $K-$Banach algebras;
2. the maps ϕ_{nm} are flat algebra homomorphisms, i.e. A_m is a flat A_n-module under $\phi_{nm} : A_n \to A_m$.

The system (A_n, ϕ_{nm}) is called a Frechet-Stein structure on A.

Definition 1.5.6. A module M over a Frechet-Stein algebra A is called coadmissible with respect to the Frechet-Stein structure (A_n, ϕ_{nm}), if M is isomorphic to a projective limit of a projective system (M_n, ψ_{nm}) of a finitely generated A_n-modules, such that

$$A_n \underset{A_{n+1}}{\otimes} M_{n+1} \simeq M_n.$$

Coadmissible modules over a Frechet-Stein algebra form an abelian category (see [27]).

Lemma 1.5.7. *Let I be a closed two-sided ideal in a Frechet-Stein algebra A. Then A/I is also a Frechet-Stein algebra.*

Proof. [27, Prop. 3.7]. □

Chapter 2

Nonarchimedean Hopf algebras and comodules

2.1 Nonarchimedean Hopf Algebras

Throught this chapter we assume K is a nonarchimedean discretely valued complete field. Unless specified otherwise, for any LCVS V, V' will mean the strong dual V'_b.

2.1.1 Banach and topological Hopf algebras.

Recall that the categories of Banach, NF and CT spaces are tensor categories with the tensor structure given by $\widehat{\otimes}$.

Definition 2.1.1. We say that A is a $K-$Banach Hopf algebra, if it is a Hopf algebra in the category of $K-$Banach spaces, i.e. there are continuous maps $(m_A, e_A, \Delta_A, \epsilon_A, S_A)$ satisfying all usual axioms of Hopf algebras.

If A is a K-Banach Hopf algebra with structure maps $(m_A, e_A, \Delta_A, \epsilon_A, S_A)$, then A' is also a K-Banach Hopf algebra with structure maps $(\Delta_A^*, \epsilon_A^*, m_A^*, e_A^*, S_A^*)$, where $*$ denotes the dual map.

Let $\{A_n, \phi_n\}$ be an inductive system of K-Banach Hopf algebras A_n with injective transition maps $\phi_n : A_n \to A_{n+1}$, s.t. ϕ_n are morphisms of $K-$Banach Hopf algebras. Then $A = \lim\limits_{\rightarrow} A_n$ is a Hopf algebra in the category of locally convex K-vector spaces, with Hopf algebra maps $(m, e, \Delta, \epsilon, S)$ defined by the corresponding maps $(m_n, e_n, \Delta_n, \epsilon_n, S_n)$ and universal property of inductive limit.

Definition 2.1.2. Suppose ϕ_n are compact maps. Then A is a locally convex vector space (LCVS) of compact type (CT-space or LS-space). In this case we call A a *K-Hopf algebra of compact type* (or *CT-Hopf algebra*).

The dual A' is a nuclear Frechet vector space, which is a projective limit $A' = \lim\limits_{\leftarrow} A'_n$ of K-Banach spaces A'_n, with compact transition maps $\phi_n^* : A'_{n+1} \to A'_n$. The maps ϕ_n^* are also morphisms of K-Banach Hopf algebras. Thus A' is a topological Hopf algebra with structure maps $(\Delta^*, \epsilon^*, m^*, e^*, S^*)$ and it is a compact projective limit of $K-$Banach Hopf algebras A'_n.

Definition 2.1.3. We call A a *nuclear Frechet K-Hopf algebra* (or *NF-Hopf algebra*) if it is topologically isomorphic to compact projective limit of K-Banach Hopf algebras.

So, if A is a CT-Hopf algebra, then A' is an NF-Hopf algebras. Since spaces of compact type are reflexive, by duality we have an anti-equivalence of categories

$$\{\text{CT-Hopf algebras}\} \quad \longleftrightarrow \quad \{\text{NF-Hopf algebras}\}\ .$$

Definition 2.1.4. If $A' = \lim\limits_{\leftarrow} A'_n$, we say that A'_n *defines a NF structure* on A. We say that NF structures $\{A'_n\}$, $\{B'_n\}$ are equivalent if they are equivalent (in the sense of [6, 1.2.7]) in the category of projective systems of $K-$Banach Hopf algebras.

It is known that any two NF structures are equivalent [6, 1.2.7].

Definition 2.1.5. If $A = \lim\limits_{\rightarrow} A_n$ with injective and compact transition maps we say that $\{A_n\}$ defines a compact type structure on A.

If $A = \lim\limits_{\rightarrow} A_n$ and $B = \lim\limits_{\rightarrow} B_n$ are compact type K-Hopf algebras, then $A \widehat{\otimes} B \cong \lim\limits_{\rightarrow} \left(A_n \widehat{\otimes} B_n\right)$ is a compact type K-Hopf algebra with CT-Hopf structure being inductive limit of Hopf structure on $A_n \widehat{\otimes} B_n$.

It is known that if V is an LCVS of compact type and U is a closed vector subspace, then U and V/U are also of compact type [26, Prop.1.2].

Proposition 2.1.6. *If V is a CT-Hopf algebra and U is a closed Hopf subalgebra, then U is also of compact type. If I is a closed Hopf ideal of V, then V/I is a CT-Hopf algebra.*

Proof. By the Banach-Dieudonne Theorem, $U \subset V$ is closed iff $U_n = U \cap V$ is closed $\forall n$ [13]. Thus U_n are Banach subspaces of V_n and, since U is a Hopf subalgebra, are K-Banach Hopf subalgebras of V_n. Thus $U = \lim\limits_{\rightarrow} U_n$ is a CT-Hopf algebra.

The same argument works for $V/I = \lim\limits_{\rightarrow} V_n/I_n$ with $I_n = I \cap V_n$. □

2.1.2 Normal Hopf algebras

In the algebraic theory of Hopf algebras there are different (but related) notions corresponding to normal subgroups. They arise by generalizing notions natural in two principal classical cases of Hopf algebras: coordinate function algebras and group algebras.

Since in non-archimedean analysis these two objects have different topological types, the corresponding notions, corresponding to normal subgroups, and relations between them in our context are more clear.

We assume that the reader is familiar with Sweedler notations.

Definition 2.1.7. Let $(A, m, e, \Delta, \epsilon, S)$ be a CT- or Banach Hopf algebra.

1. The *left adjoint coaction* of A on itself is a map

$$\rho_l : A \longrightarrow A \widehat{\otimes} A$$

$$\rho_l(h) = \sum h_1 S(h_3) \otimes h_2$$

2. The *right adjoint coaction* of A on itself is a map

$$\rho_r : A \longrightarrow A \widehat{\otimes} A$$

$$\rho_r(h) = \sum h_2 \otimes S(h_1) h_3$$

3. A Hopf ideal I of A is called *normal* if $\rho_l(I) \subseteq A \widehat{\otimes} I$ and $\rho_r(I) \subseteq I \widehat{\otimes} A$.

4. We say that a Hopf algebra morphism $\phi : A \to B$ is *normal* if Ker ϕ is a normal Hopf ideal of A.

One can write ρ_l as

$$\rho_l = (m \otimes id) \circ (id \otimes \sigma_{23}) \circ (id \otimes id \otimes S) \circ (id \otimes \Delta_A) \circ \Delta_A$$

and

$$\rho_r = (id \otimes m) \circ (\sigma_{12} \otimes id) \circ (S \otimes id \otimes id) \circ (\Delta_A \otimes id) \circ \Delta_A$$

(σ_{ij} denotes cyclic permutation of terms from i to j).

Remark 2.1.8. One can check (straightforward) that ρ_l is an $A-$comodule structure on A and a morphism of Hopf algebras $\phi : A \to B$ is normal if Ker ϕ is a subcomodule w.r.t. this structure.

Definition 2.1.9. Let $(A, m, e, \Delta, \epsilon, S)$ be a NF- or Banach Hopf algebra.

1. The *left adjoint action* of A on itself is a map

$$(ad_l h)(k) = \sum h_1 k (S(h_2))$$

for all $h, k \in A$.

2. The *right adjoint action* of A on itself is a map

$$(ad_r h)(k) = \sum S(h_1) k h_2$$

for all $h, k \in A$.

3. A Hopf subalgebra B of A is called *normal* if $(ad_l A)(B) \subseteq B$ and $(ad_r A)(B) \subseteq B$.

One can see easily that $ad_l,\ ad_r : A \widehat{\otimes} A \to A$ and

$$ad_l = m \circ (id \otimes m) \circ (id \otimes id \otimes S) \circ (id \otimes \sigma_{23}) \circ (\Delta_A \otimes id)$$

and

$$ad_r = m \circ (m \otimes id) \circ (S \otimes id \otimes id) \circ (\sigma_{12} \otimes id) \circ (id \otimes \Delta_A)\,.$$

Taking dual map gives

$$(\rho_{A,l})' = ad_{A',l} \text{ and } (\rho_{A,r})' = ad_{A',r}$$

and vice verse. Thus one has

Proposition 2.1.10. *Let A be a CT-Hopf algebra.*

A Hopf ideal I in A is normal (and projection $A \twoheadrightarrow A/I$ is normal) iff $(A/I)'_b$ is a normal Hopf subalgebra of A'_b.

2.2 Modules and comodules

All our modules and comodules are assumed to be continuous, i.e. multiplication and comultiplication, respectively, are continuous maps.

2.2.1 Modules and comodules.

Definition 2.2.1. Let A be a CT- (Banach) Hopf algebra and V an LCVS of compact type.

1. We say that V is a right CT-comodule (Banach comodule) over A if exists $\rho : V \to V \widehat{\otimes} A$, a K-linear continuous map such that

$$\begin{array}{c} (id_V \bar{\otimes} \epsilon_A) \circ \rho_V = id_V \\ (\rho_V \otimes id_A) \circ \rho_V = (id_V \otimes \Delta_A) \circ \rho_V \end{array}.$$

   Right CT (Banach) comodules form a category $Comod_{CT} - A$ ($Comod_{Ban} - A$) with morphisms being continuous morphisms of comodules.

2. By duality, V'_b is an NF- (Banach) space which is a continuous right A'_b−module. We will say that V'_b is a module, dual to comodule V. Denote the category of right continuous A'-modules that are NF spacesby $mod_{NF} - A'$.

3. On V there is also a left $A'-$module structure

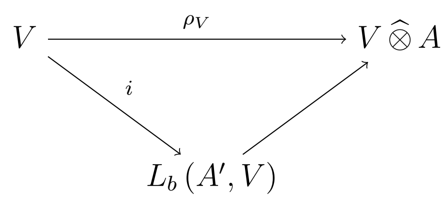
$$\begin{array}{rcl} m: \ A' \underset{K,\pi}{\widehat{\otimes}} V \to V & & \\ \lambda \otimes v \mapsto \lambda \cdot v & = & \left(id_V \bar{\otimes} \lambda\right) \circ \rho_V\left(v\right). \end{array}$$

Such modules (with A'_b-module structure coming from the comodule structure on V) are called *rational*.

4. In Banach case, left $A'-$module structure on V gives a continuous left $A-$comodule structure on V'. Equipped with this comodule structure, we will call V' a dual comodule.

5. Similarly, if W is a Banach space, one can give a structure of a continuous right $A'-$module to the space $L_b\left(V, W\right)$.

Now if V is a left $A'-$module and a compact type LCVS, then "integration" theorem implies existence of a map ρ_V



(where $i\left(v\right)\left(\lambda\right) = \lambda \cdot v$), such that the $A'-$module structure on V is exactly $\lambda \cdot v = \left(id_v \otimes \lambda\right) \circ \rho_V\left(v\right)$. The module axioms for V imply that ρ_V satisfies the right comodule axioms ([29, 2.1.1]). Thus we have

Proposition 2.2.2. *All continuous $A'-$modules on an LCVS of compact type are rational (in the sense of [29, 2.1]).*

Let A be a Banach K-Hopf algebra and M a left continuous Banach module over A'. Following [29, 2.1], we define

$$\begin{array}{cccll} \rho: & M & \to & L_b\left(A', M\right) & \\ & m & \mapsto & \rho\left(m\right): & \rho\left(m\right)\left(c^*\right) = c^* m \end{array}. \tag{2.2.1}$$

There is a natural embedding

$$\begin{array}{ccc} M\widehat{\otimes}A & \hookrightarrow & L_b(A',M) \\ m\otimes a & \longmapsto & f(m\otimes a): \quad f(m\otimes a)(c^*) = c^*(a)\cdot m \end{array}.$$

Now Prop.1.3.5 says that we have

Proposition 2.2.3. *M is rational if $\rho(M)\subset C(A',M)$,i.e. $\rho(m)$ is a compact map for every $m\in M$.*

Now let V be a compact type comodule over A. The dual V'_b is a nuclear Frechet LCVS. Since taking dual invert all arrows in diagrams, expressing comodule properties of V, V'_b is a right module over A' which is continuous. All together, we have an equivalence of categories

$$comod_{CT}-A \;\sim\; A'-mod_{CT}$$

and anti-equivalence of categories

$$comod_{CT}-A \;\sim\; mod_{NF}-A' \;.$$

Note that if the antipode of a Hopf algebra A' is involutive (i.e. $S^2_{A'}=Id$), then the categories of left and right modules are equivalent.

Proposition 2.2.4. *Let V be a right CT-comodule over A and U a closed subcomodule of A. Then U and V/U are right A-comodules of compact type and the exact sequence*

$$0\to U\to V\to V/U\to 0$$

give rise to the exact sequence of right $A'-$modules

$$0\to (V/U)'_b\to V'_b\to U'_b\to 0$$

with strict maps.

Definition 2.2.5. We call a topological comodule *simple*, if it does not have any closed subcomodules.

Proposition 2.2.4 gives

Corollary 2.2.6. *CT A-comodule V is simple if and only if V'_b does not have closed proper $A'-$submodules.*

Lemma 2.2.7. *Let V be a Banach A-comodule and V' be an $A'-$module.*

1. *If U is a closed subcomodule of V then $U^{\perp}$ is a closed submodule of V'.*

2. *If U' is a closed submodule of V' then $U = Ker(U')$ is a closed subcomodule of V.*

Proof. (1) If $\phi \in U^{\perp}$ and $\lambda \in A'$, then for any $u \in U$

$$(\lambda\phi)(u) = (\lambda\bar{\otimes}\phi) \circ \Delta_A(u) = \sum \lambda(u_0)\phi(u_1) = 0$$

since $u_1 \in U$.

(2) Let $u \in U$. Then $\Delta_A(u) = \sum u_0 \otimes u_1$ can be written as $\Delta_A(u) = \sum_{i\in I} a_i \otimes u_i$, where $\{a_i\}_{i\in I}$ is an orthogonal base of A. Let λ_i be a functional, dual to a_i. But then $(\lambda_i\phi)(u) = \phi(u_i)$ for any $\phi \in V'$. Since U' is a submodule, $\forall\phi \in U' : (\lambda_i\phi)(u) = \phi(u_i) = 0$ and thus $u_i \in U$. So U is a subcomodule. □

Lemma 2.2.8. *Let A be a Banach Hopf algebra and V be a Banach A-comodule. Then*

1. *V is simple if V' is simple;*

2. *V is simple if and only if it is simple as an $A'-$module.*

Proof. (1) If $0 \to M \hookrightarrow V \twoheadrightarrow V/M \to 0$ is an exact sequence of $A-$comodules, then, by lemma 1.2.24, $(V/M)'_b$ is a closed submodule of V'_b.

(2) is proved in [5]. □

The following is the analog of [26, Lemma 3.9]

Proposition 2.2.9. *Let V be a compact type comodule over A, which is an compact inductive limit of Banach comodules V_n over A_n and $\rho_V|_{V_n} = \rho_{V_{n+1}}|_{V_n} = \rho_{V_n}$. Suppose there exists $N > 0$ such that V_n are simple for all $n \geq N$. Then V is simple.*

Proof. Let $U \subset V$ be a proper closed subcomodule. Since V is of compact type, $U = \lim\limits_{\rightarrow} U_n$, such that for each n U_n is a closed subspace of V_n. Since U and V_n are subcomodules of V, U_n is A_n-subcomodules of V_n. Since U is a proper subspace of V, U_n must be proper subspaces of V_n for all $n > N$ for some $N > 0$ and this completes the proof. □

Remark 2.2.10. [26, Lemma 3.9] is proved for coadmissible modules over Frechet-Stein algebra, i.e. the duals of V_n are required to be finitely generated over A'_n and A'_n are required to be Noetherian. These assumptions are not required in our result. On the other hand, our result is for CT-comodules, which on the dual side mean a nuclear Frechet module, and nuclearity is not required in [26, Lemma 3.9].

For K-Banach spaces an operator is compact iff it is completely continuous. Since the later are limits of finite-dimensional operators, they motivate the following

Definition 2.2.11. Let M be a Banach module over A'.

- $m \in M$ is called *finite* if $A'm$ is finite dimensional.
- $N \in comod_{CT} - A$. $n \in N$ is called *finite* if $\rho_N(n)$ is a finite sum in $N \widehat{\otimes} A$ (i.e. $\rho_N(n) \in N \underset{K}{\otimes} A$)

Proposition 2.2.12. *$n \in N$ is finite as an element of the comodule N if and only if n is finite as a vector of the $A'-$module N iff $\rho(n)$ (see 2.2.1) is a finite-dimensional operator.*

Proof. Let $\rho_N(n) = a_1 \otimes n_1 + \cdots + a_k \otimes n_n$. Then for any $\phi \in A'$ $\phi{\cdot}n = \phi(a_1)\, n_1 + \cdots + \phi(a_k)\, n_n$. Taking as ϕ the functional, dual to a_i, gives $n_i \in A'n$. Thus $A'n = Kn_1 + \cdots + Kn_2$ and this proves the first part. The second part is clear. □

Example 2.2.13. Examples from representation theory:

- If $N = A = C(Z_p, K)$ - the continuous functions on Z_p, then all polynomials are finite elements and they are dense in A.

- If $N = Ind_P^{GL_2(Z_p)}(\chi)$ – the continuous principal series representation from [25] (viewed as comodule over $C(GL_2(Z_p), K)$) with dominant integral χ, then the only finite elements are the elements of the space of corresponding rational representation and they are not dense.

- If, as above, χ is such that the representation is topologically irreducible, then there are no finite elements in N.

2.2.2 Induction

Let A and B be Banach or compact type $K-$Hopf algebras. Let (M, ρ_M) be a right $A-$comodule and let (N, τ_N, ρ_N) be an $A - B-$comodule with left $A-$coaction τ_N and right $B-$coaction ρ_N.

Definition 2.2.14. The space $M\underset{A}{\widehat{\boxtimes}}N = Ker(\rho_M \otimes id_N - id_M \otimes \tau_N)$ is called *the cotensor product of M and N over A*. It has a right $B-$comodule structure with coaction $id_M \otimes \rho_N$.

Since $M\underset{A}{\widehat{\boxtimes}}N$ is a kernel of a continuous map, it is a closed subspace of $M \widehat{\otimes} N$. Thus if M and N are both Banach or CT spaces, then $M\underset{A}{\widehat{\boxtimes}}N$ is Banach or CT respectively.

It is straightforward to check (using Hahn-Banach Theorem) the following

Lemma 2.2.15. $\left(M\underset{A}{\widehat{\boxtimes}}N\right)_b' = M_b'\underset{A_b'}{\widehat{\otimes}}N_b'$, $\left(M_b'\underset{A_b'}{\widehat{\otimes}}N_b'\right)_b' = M''\underset{A''}{\widehat{\boxtimes}}N''$.

Let $\pi : A \to B$ be a morphism of topological Hopf algebras (either Banach or compact type). Then A is a left and right $B-$comodule via maps

$$\rho_{l\pi} = (\pi \otimes id_A) \circ \Delta_A$$

and

$$\rho_{r\pi} = (id_A \otimes \pi) \circ \Delta_A.$$

Denote those comodules by ${}_\pi A$ and A_π respectively.

Definition 2.2.16. Let $V \in comod. - B$. Then $V\underset{B}{\widehat{\boxtimes}}_{\pi}A$ is called *induced* A-comodule. We have $V\underset{B}{\widehat{\boxtimes}}_{\pi}A \in comod.-A$. We also denote the induced right A-comodule by V^{π}. The functor $(-)^{\pi}$ is called *induction*.

Definition 2.2.17. For $V \in comod. - A$ V_{π} will denote the $B-$comodule V with coaction $V \overset{\rho_V}{\to} V \widehat{\otimes} A \overset{id\otimes\pi}{\to} V \widehat{\otimes} B$. The functor $(-)_{\pi}$ is called *restriction*.

In order to justify the names of our functors we need to prove Frobenius reciprocity, i.e. that induction functor is a left adjoint to restriction.

Proposition 2.2.18. *(Frobenius reciprocity)*

Let $\pi : A \to B$ be a continuous morphism of CT K-Hopf algebras, M be a CT-comodule over B and N be a CT-comodule over A.

There is a topological isomorphism

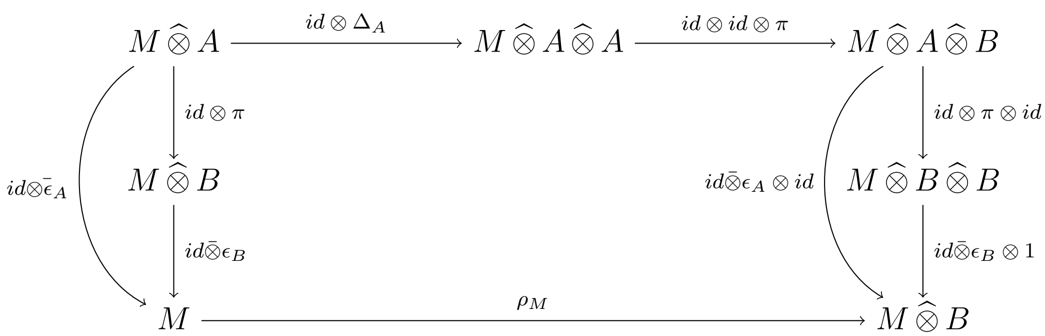
$$Comod_{CT} - A\left(N, M\underset{B}{\widehat{\boxtimes}}_{\pi}A\right) \simeq Comod_{CT} - B\left(N_{\pi}, M\right),$$

where $Comod_{CT} - A\,(V, W)$ is a space of continuous comodule morphisms between V and W, with the topology induced from $L_b\,(V, W)$.

First we prove a technical

Lemma 2.2.19. *The map $id\otimes\epsilon_A : M\underset{B}{\widehat{\boxtimes}}_{\pi}A \longrightarrow M$ sending $m\otimes a \longmapsto \epsilon_A\,(a)\,m$ is a morphism of B-comodules $\left(M\underset{B}{\widehat{\boxtimes}}_{\pi}A\right)_{\pi}$ and M.*

Proof. We need to prove the commutativity of the following diagram:



Let $m\otimes a\in M\widehat{\boxtimes}_{B}{}_{\pi}A$. This means that we have

$$\sum m_{(0)}\otimes m_{(1)}\otimes a=\sum m\otimes\pi\left(a_{(1)}\right)\otimes a_{(2)}.$$

The left and bottom parts composed are clearly $\sum m\otimes a\mapsto\sum m_{(0)}\otimes m_{(1)}\epsilon_{A}\left(a\right).$

The top and right parts composed are giving

$$\sum m\otimes a\mapsto\sum m\otimes\epsilon_{B}\left(\pi\left(a_{(1)}\right)\right)\pi\left(a_{(2)}\right)=\sum m\otimes\epsilon_{B}\left(\pi\left(a\right)_{(1)}\right)\pi\left(a\right)_{(2)}=$$

$$=\sum m\otimes\pi\left(a\right)_{(1)}\epsilon_{B}\left(\pi\left(a\right)_{(2)}\right)==\sum m\otimes\pi\left(a\right)_{(1)}\epsilon_{A}\left(a_{(2)}\right)=\sum m_{(0)}\otimes m_{(1)}\epsilon_{A}\left(a\right).$$

□

Proof of prop.2.2.18. Define morphisms

$$\begin{array}{rccll} & \phi & \longmapsto & (id\bar{\otimes}\epsilon_{A})\circ\phi & =\tilde{\phi}=(id\bar{\otimes}\left(\epsilon_{B}\circ\pi\right))\circ\phi \\ (\psi\otimes id)\circ\rho_{N}= & \bar{\psi} & \longleftarrow & (\psi:N\to M) & \end{array}.$$

The fact that $\tilde{\phi}$ is a morphism of $B-$comodules follows from equality

$$\left(\tilde{\phi}\otimes id\right)\circ\rho_{N_{\pi}}=(id\bar{\otimes}\epsilon_{B}\otimes id)\circ(id\otimes\pi\otimes id)\circ(\phi\otimes id)\circ(id\otimes\pi)\circ\rho_{N}=$$

$$=(id\bar{\otimes}\epsilon_{B}\otimes id)\circ(id\otimes\pi\otimes\pi)\circ(\phi\otimes id)\circ\rho_{N}=(id\bar{\otimes}\epsilon_{B}\otimes id)\circ(id\otimes\pi\otimes\pi)\circ(id\otimes\Delta_{A})\circ\phi=$$

$$\overset{\#\#}{=}\rho_{M}\circ(id\bar{\otimes}\epsilon_{B})\circ(id\otimes\pi)\circ\phi=\rho_{M}\circ\tilde{\phi},$$

where (##) is given by Lemma 2.2.19.

The identity

$$(\rho_{M}\otimes id)\circ\tilde{\phi}=(\rho_{M}\otimes id)\circ(\psi\otimes id)\circ\rho_{N}=(\psi\otimes id\otimes id)\circ(\rho_{N_{\pi}}\otimes id)\circ\rho_{N}=$$

$$=(\psi\otimes id\otimes id)\circ(id\otimes\pi\otimes id)\circ(\rho_{N}\otimes id)\circ\rho_{N}=$$

$$=(\psi\otimes id\otimes id)\circ(id\otimes\pi\otimes id)\circ(id\otimes\Delta_{A})\circ\rho_{N}=$$

$$=(id\otimes\pi\otimes id)\circ(id\otimes\Delta_{A})\circ(\psi\otimes id)\circ\rho_{N}=(id\otimes\pi\otimes id)\circ(id\otimes\Delta_{A})\circ\tilde{\psi}$$

means that the image of $\tilde{\psi}$ belongs to $M\widehat{\boxtimes}_{B}{}_{\pi}A$.

The identity

$$\rho_{M\widehat{\boxtimes}_{B}{}_{\pi}A}\circ\tilde{\psi}=(id\otimes\Delta_A)\circ(\psi\otimes id)\circ\rho_N=(\psi\otimes id\otimes id)\circ(id\otimes\Delta_A)\circ\rho_N=$$

$$=(\psi\otimes id\otimes id)\circ(\rho_N\otimes id)\circ\rho_N=\left(\tilde{\psi}\otimes id\right)\circ\rho_N$$

shows that $\tilde{\psi}$ is a morphism of $A-$comodules from N to $M\widehat{\boxtimes}_{B}{}_{\pi}A$.

Let us show that $\tilde{\tilde{\phi}}=\phi$ and $\tilde{\tilde{\psi}}=\psi$.

$$\tilde{\tilde{\phi}}=\left(\tilde{\phi}\otimes id\right)\circ\rho_N=(id\otimes\epsilon_A\otimes id)\circ(\psi\otimes id)\circ\rho_N=(id\otimes\epsilon_A\otimes id)\circ\rho_{M\widehat{\boxtimes}_{B}{}_{\pi}A}\circ\phi=$$

$$=(id\otimes\epsilon_A\otimes id)\circ(id\otimes\Delta_A)\circ\phi=\phi$$

$$\tilde{\tilde{\psi}}=(id\bar{\otimes}\epsilon_A)\circ(\psi\otimes id)\circ\rho_N=(id\bar{\otimes}\epsilon_A)\circ\rho_M\circ\psi=\psi.$$

Since the topologies on the spaces $Comod_{CT}-A\left(N,M\widehat{\boxtimes}_{B}{}_{\pi}A\right)$ and $Comod_{CT}-B\left(N_\pi,M\right)$ are induced from $L_b\left(N,M\widehat{\otimes}A\right)$ and $L_b\left(N,M\right)$ respectively, the continuity of our linear bijections follows from the argument same as in [24, sec.18]. Namely, composition with linear continuous map $W\to U$ is a linear continuous map $L_b\left(V,W\right)\to L_b\left(V,U\right)$. Since our $\tilde{(\cdot)}$ maps are compositions of continuous maps, they are continuous. □

Remark 2.2.20. Proposition 2.2.18 is also true for Banach comodules over Banach $K-$Hopf algebras. The proof is the same.

2.2.3 Tensor Identities

One can define the tensor product of topological comodules as in the algebraic case.

Definition 2.2.21. Let (M,ρ_M) and (N,ρ_N) be two right Banach or CT comodules over a Banach or CT Hopf algebra A respectively. Then on $M\widehat{\otimes}N$ there is an right A-comodule structure

$$\rho_{M\widehat{\otimes}N}=(id_M\otimes id_N\otimes m_A)\circ(id_M\ \otimes\sigma_{23}\otimes id_A)\circ(\rho_N\otimes\rho_N),$$

making $M\widehat{\otimes}N$ an A-comodule of appropriate type.

Proposition 2.2.22. *(Tensor identities) Let $\pi : A \to B$ be a continuous morphism of topological K-Hopf algebras (either Banach or compact type), and let W be a comodule over B and V a comodule over A (of appropriate type), then*

(i) $V \widehat{\otimes} \left(W \underset{B}{\widehat{\boxtimes}}_{\pi} A \right) \cong \left(V_{\pi} \widehat{\otimes} W \right) \underset{B}{\widehat{\boxtimes}}_{\pi} A,$

(ii) $\left(W \underset{B}{\widehat{\boxtimes}}_{\pi} A \right) \widehat{\otimes} V \cong \left(W \widehat{\otimes} V_{\pi} \right) \underset{B}{\widehat{\boxtimes}}_{\pi} A,$

both isomorphisms being as topological A-comodules.

Proof. We include a complete proof, since it is not present in any source known to us.

(i) Both $V \widehat{\otimes} \left(W \underset{B}{\widehat{\boxtimes}}_{\pi} A \right)$ and $\left(V_{\pi} \widehat{\otimes} W \right) \underset{B}{\widehat{\boxtimes}}_{\pi} A$ are embedded into $V \widehat{\otimes} W \widehat{\otimes} A$. Their elements satisfy the following identities in respective order:

$$(*) \qquad \sum v \otimes w_{(0)} \otimes w_{(1)} \otimes h = \sum v \otimes w \otimes \pi(h_{(1)}) \otimes h_{(2)} \tag{2.2.2}$$

$$(**) \qquad \sum v_{(0)} \otimes w_{(0)} \otimes \pi(v_{(1)}) w_{(1)} \otimes h = \sum v \otimes w \otimes \pi(h_{(1)}) \otimes h_{(2)} \tag{2.2.3}$$

in $V \widehat{\otimes} W \widehat{\otimes} B \widehat{\otimes} A$.

Let ϕ be the map

$$\begin{array}{cccc} \phi : & V \widehat{\otimes} W \widehat{\otimes} A & \to & V \widehat{\otimes} W \widehat{\otimes} A \\ & \sum v \otimes w \otimes h & \longmapsto & \sum v_{(0)} \otimes w \otimes v_{(1)} h \end{array}.$$

Suppose that $\sum v \otimes w \otimes h$ satisfies (*). For $\sum v_{(0)} \otimes w \otimes v_{(1)} h$ (**) takes the following form

$$\sum v_{(0)(0)} \otimes w_{(0)} \otimes \pi(v_{(0)(1)}) w_{(1)} \otimes v_{(1)} h \underset{(***)}{=} \sum v_{(0)} \otimes w \otimes \pi(v_{(1)(1)}) \pi(h_{(1)}) \otimes v_{(1)(2)} h_{(2)}.$$

A direct check shows that (***) is obtained from (*) by applying to the left and right hand side the maps enlisted below

$$\begin{array}{c}
\sum v\otimes w_{(0)}\otimes w_{(1)}\otimes h\\
\Big\downarrow {\scriptstyle \rho_v\otimes id\otimes id\otimes id}\\
\sum v_{(0)}\otimes v_{(1)}\otimes w_{(0)}\otimes w_{(1)}\otimes h\\
\Big\downarrow {\scriptstyle \rho_v\otimes id\otimes id\otimes id\otimes id}\\
\sum v_{(0)(0)}\otimes v_{(0)(1)}\otimes v_{(1)}\otimes w_{(0)}\otimes w_{(1)}\otimes h\\
\Big\downarrow {\scriptstyle (id\otimes id\otimes m_B\otimes id)\circ\sigma_{23}\circ(id\otimes\pi\otimes id\otimes id\otimes m_A)\circ\sigma_{35}}\\
\sum v_{(0)(0)}\otimes w_{(0)}\otimes\pi\left(v_{(0)(1)}\right)w_{(1)}\otimes v_{(1)}h
\end{array}
\qquad
\begin{array}{c}
\sum v\otimes w\otimes\pi\left(h_{(1)}\right)\otimes h_{(2)}\\
\Big\downarrow {\scriptstyle \rho_v\otimes id\otimes id\otimes id}\\
\sum v_{(0)}\otimes v_{(1)}\otimes w\otimes\pi\left(h_{(1)}\right)\otimes h_{(2)}\\
\Big\downarrow {\scriptstyle id\otimes\Delta_A\otimes id\otimes id\otimes id}\\
\sum v_{(0)}\otimes v_{(1)(1)}\otimes v_{(1)(2)}\otimes w\otimes\pi\left(h_{(1)}\right)\otimes h_{(2)}\\
\Big\downarrow {\scriptstyle (id\otimes id\otimes m_B\otimes id)\circ\sigma_{23}\circ(id\otimes\pi\otimes id\otimes id\otimes m_A)\circ\sigma_{35}}\\
\sum v_{(0)}\otimes w\otimes\pi\left(v_{(1)(1)}\right)\pi\left(h_{(1)}\right)\otimes v_{(1)(2)}h_{(2)}
\end{array}$$

which can be easily seen to be identical (σ_{ij} means cyclic permutation from the j^{th} to the i^{th} entry).

Thus ϕ maps $V\widehat{\otimes}\left(W\underset{B}{\widehat{\boxtimes}}_{\pi}A\right)$ to $\left(V_{\pi}\widehat{\otimes}W\right)\underset{B}{\widehat{\boxtimes}}_{\pi}A$.

Let us show that ϕ is a morphism of right $A-$comodules, i.e.

$$(\phi\otimes id_A)\circ\rho_{V\widehat{\otimes}\left(W\underset{B}{\widehat{\boxtimes}}_{\pi}A\right)}=\rho_{\left(V_{\pi}\widehat{\otimes}W\right)\underset{B}{\widehat{\boxtimes}}_{\pi}A}\circ\phi.$$

We have

$$\begin{array}{ccc}
& \sum v\otimes w\otimes h & \\
\swarrow {\scriptstyle \phi} & & \searrow {\scriptstyle \rho_{V\widehat{\otimes}\left(W\underset{B}{\widehat{\boxtimes}}_{\pi}A\right)}}\\
\sum v_{(0)}\otimes w\otimes v_{(1)}h & & \sum v_{(0)}\otimes w\otimes h_{(1)}\otimes v_{(1)}h_{(2)}\\
\Big\downarrow {\scriptstyle \rho_{\left(V_{\pi}\widehat{\otimes}W\right)\underset{B}{\widehat{\boxtimes}}_{\pi}A}} & & \Big\downarrow {\scriptstyle \phi\otimes id}\\
\sum v_{(0)}\otimes w\otimes v_{(1)(1)}h_{(1)}\otimes v_{(1)(2)}h_{(2)} & & \sum v_{(0)(0)}\otimes w\otimes v_{(0)(1)}h_{(1)}\otimes v_{(1)}h_{(2)}
\end{array}$$

The equality follows from

$$\left(\rho_V\otimes id\otimes\Delta_A\right)\left(\sum v\otimes w\otimes h\right)=\sum v_{(0)}\otimes v_{(1)}\otimes w\otimes h_{(1)}\otimes h_{(2)}=$$

$$=\left(\rho_V\otimes id\otimes id\otimes id\otimes id\right)\left(\sum v_{(0)}\otimes v_{(1)}\otimes w\otimes h_{(1)}\otimes h_{(2)}\right)=$$

$$=\sum v_{(0)(0)}\otimes v_{(0)(1)}\otimes v_{(1)}\otimes w\otimes h_{(1)}\otimes h_{(2)}=$$

$$= (id \otimes \Delta_A \otimes id \otimes id \otimes id) \left(\sum v_{(0)} \otimes v_{(1)} \otimes w \otimes h_{(1)} \otimes h_{(2)} \right) =$$

$$= \sum v_{(0)} \otimes v_{(1)(1)} \otimes v_{(1)(2)} \otimes w \otimes h_{(1)} \otimes h_{(2)}$$

Now let

$$\begin{array}{cccc} \psi : & V \widehat{\otimes} W \widehat{\otimes} A & \longrightarrow & V \widehat{\otimes} W \widehat{\otimes} A \\ & \sum v \otimes w \otimes h & \longmapsto & \sum v_{(0)} \otimes w \otimes S_A \left(v_{(1)} \right) h \end{array} .$$

Suppose $\sum v \otimes w \otimes h$ satisfies (**). For $\sum v_{(0)} \otimes w \otimes S(v_{(1)})h$ (*) takes the form:

$$\sum v_{(0)} \otimes w_{(0)} \otimes w_{(1)} \otimes S(v_{(1)})h \overset{?}{=} \sum v_{(0)} \otimes w \otimes \left\{ (\pi \otimes id) \circ \left(S_A \left(v_{(1)} \right) h \right) \right\} =$$

$$= \sum v_{(0)} \otimes w \otimes \pi \left(S_A \left(v_{(1)(2)} \right) h_{(2)} \right) .$$

The last part of the above formula equals

$$(id \otimes id \otimes id \otimes m_A) \circ \sigma_{24} \circ (id \otimes id \otimes id \otimes m_b \otimes id) \circ \sigma_{34} \circ$$

$$\circ (id \otimes id \otimes \pi \otimes id \otimes id \otimes id) \circ (id \otimes S_A \otimes S_A \otimes id \otimes id \otimes id) \circ$$

$$\circ (\underbrace{(id \otimes \Delta_A) \circ \rho_V}_{\substack{|| \\ (\rho_V \otimes id_A) \circ \rho_V}} \otimes id \otimes id \otimes id) \left(\underbrace{\sum v \otimes w \otimes \pi \left(h_{(1)} \right) \otimes h_{(2)}}_{\substack{|| \text{ by (**)} \\ \sum v_{(0)} \otimes w_{(0)} \otimes \pi \left(v_{(1)} \right) w_{(1)} \otimes h}} \right) =$$

$$= \sum v_{(0)(0)(0)} \otimes w_{(0)} \otimes \pi \left(S \left(v_{(0)(1)} \right) \right) \pi \left(v_{(1)} \right) w_{(1)} \otimes S \left(v_{(0)(0)(1)} \right) h =$$

$$= \sum v_{(0)} \otimes w_{(0)} \otimes w_{(1)} \otimes S \left(v_{(1)} \right) h,$$

where the last equality is due to

$$\sum v_{(0)(0)} \otimes \pi \left(S_A \left(v_{(0)(1)} \right) \right) \pi \left(v_{(1)} \right) = \sum v_{(0)} \otimes \pi \left(S_A \left(v_{(1)(1)} \right) v_{(1)(2)} \right) =$$

$$= \sum v_{(0)} \otimes \epsilon_A \left(v_{(1)} \right) = v \otimes 1.$$

Thus ψ maps $\left(V_\pi \widehat{\otimes} W \right) \widehat{\underset{B}{\boxtimes}}_\pi A$ to $V \widehat{\otimes} \left(W \widehat{\underset{B}{\boxtimes}}_\pi A \right)$.

We have

$$\sum v \otimes w \otimes h \overset{\psi}{\longrightarrow} \sum v_{(0)} \otimes w \otimes S \left(v_{(1)} \right) h \overset{\rho_{V \widehat{\otimes} \left(W \widehat{\underset{B}{\boxtimes}}_\pi A \right)}}{\longrightarrow}$$

$$\sum v_{(0)(0)} \otimes w \otimes S\left(v_{(1)(2)}\right) h_{(1)} \otimes v_{(0)(1)} S\left(v_{(1)(1)}\right) h_{(2)} \stackrel{\#}{=}$$

$$= \sum v_{(0)(0)} \otimes w \otimes S\left(v_{(1)}\right) h_{(1)} \otimes \underbrace{v_{(0)(1)(1)} S\left(v_{(0)(1)(2)}\right)}_{\substack{|| \\ \epsilon\left(v_{(0)(1)}\right)}} h_{(2)} =$$

$$= \sum v_{(0)(0)} \epsilon\left(v_{(0)(1)}\right) \otimes w \otimes S\left(v_{(1)}\right) h_{(1)} \otimes h_{(2)},$$

where the (#) equality is due to

$$(\rho_V \otimes \Delta_A) \circ \rho_V = (id \otimes \Delta_A \otimes id) \circ (\rho_V \otimes id) \circ \rho_V.$$

On the other hand

$$\sum v \otimes w \otimes h \stackrel{\rho_{(V_\pi \widehat{\otimes} W) \underset{B}{\widehat{\boxtimes}}_\pi A}}{\longrightarrow} \sum v \otimes w \otimes h_{(1)} \otimes h_{(2)} \stackrel{\psi \otimes id_A}{\longmapsto} \sum v_{(0)} \otimes w \otimes S\left(v_{(1)}\right) h_{(1)} \otimes h_{(2)} =$$

$$= \sum v_{(0)(0)} \epsilon\left(v_{(0)(1)}\right) \otimes w \otimes S\left(v_{(1)}\right) h_{(1)} \otimes h_{(2)}.$$

Thus ψ is morphism of right $A-$comodules.

The verification that $\phi \circ \psi = id_{(V_\pi \widehat{\otimes} W) \underset{B}{\widehat{\boxtimes}}_\pi A}$ and $\psi \circ \phi = id_{V \widehat{\otimes} \left(W \underset{B}{\widehat{\boxtimes}}_\pi A\right)}$ is trivial. Since all involved maps are continuous, ϕ and ψ are topological isomorphisms.

The proof of *(ii)* is the same up to permutation of some terms in tensor products. □

Corollary 2.2.23. *If $B = K$, $\pi = \epsilon_A$ and $W = K$ then we have a comodule isomorphism*

$$V \widehat{\otimes} A \cong V_{tr} \widehat{\otimes} A,$$

where V_{tr} means the underlying vector space of V with trivial comodule structure.

2.3 Admissible comodules

Recall [6, 1.2.8] that a module M over A' is called coadmissible (w.r.t. a fixed nuclear Frechet structure A'_n on A'), if we have the following data:

1. a sequence of finitely generated Banach modules M_n over A'_n;

2. an isomorphism of Banach A'_n-modules $A'_n \underset{A'_{n+1}}{\widehat{\otimes}} M_{n+1} \simeq M_n$;

3. an isomorphism of topological $A-$modules $M \simeq \lim\limits_{\leftarrow} M_n$ (projective limit is taken w.r.t. transition maps $M_{n+1} \longrightarrow M_n$, induced by 2).

It is known that if M is coadmissible w.r.t. one nuclear Frechet structure on A, then it is coadmissible w.r.t. any [6, 1.2.9].

Definition 2.3.1. Let $V \in comod_{CT} A$. We call V admissible, if exists a sequence $\{V_n\}_n$ of comodules over A_n, s.t.

1. V_n are Banach right comodules over A_n and there is an embedding of V_n into a finite product of copies of A_ns;
2. we have an isomorphism of A_n-comodules $V_{n+1} \underset{A_{n+1}}{\widehat{\boxtimes}} A_n \simeq V_n$ (A_n is a left and right $A_{n+1}-$comodule, so we can take completed cotensor product);
3. we have an isomorphism of topological comodules $V \simeq \lim\limits_{\rightarrow} V_n$.

Proposition 2.3.2. *Let $\phi : A \to B$ be a morphism of compact type $K-$Hopf algebras and suppose that A is admissible as both a left and right $B-$comodule. Then:*

- *If V is an admissible A-comodule then V_ϕ is an admissible B-comodule;*
- *If W is an admissible B-comodule then W^ϕ is an admissible A-comodule.*

Proof. In both cases we have to check the three conditions of admissibility.

- V_ϕ case

  1. If V_n is embedded into A_n^k and A_n is embedded into B_n^m, then we have an embedding $V_n \hookrightarrow B_n^{km}$.
  2. We have $V_{n+1} \underset{A_{n+1}}{\widehat{\boxtimes}} A_n \cong V_n$ and $(A_{n+1})_\phi \underset{B_{n+1}}{\widehat{\boxtimes}} B_n \cong (A_n)_\phi$. Then

$$(V_{n+1})_\phi \underset{B_{n+1}}{\widehat{\boxtimes}} B_n \cong \left(V_{n+1} \underset{A_{n+1}}{\widehat{\boxtimes}} A_{n+1} \right)_\phi \underset{B_{n+1}}{\widehat{\boxtimes}} B_n = V_{n+1} \underset{A_{n+1}}{\widehat{\boxtimes}} (A_{n+1})_\phi \underset{B_{n+1}}{\widehat{\boxtimes}} B_n =$$

$$= V_{n+1} \underset{A_{n+1}}{\widehat{\boxtimes}} (A_n)_\phi \cong \left(V_{n+1} \underset{A_{n+1}}{\widehat{\boxtimes}} A_n \right)_\phi \cong (V_n)_\phi$$

3. $(V)_\phi = \lim\limits_{\rightarrow} (V_n)_\phi$ follows from $V = \lim\limits_{\rightarrow} V_n$.

- W^ϕ case:

1. If $W_n \hookrightarrow B_n^m$, then $W_n \widehat{\boxtimes}_{B_n} {}_\phi A_n \hookrightarrow B_n^m \widehat{\boxtimes}_{B_n} {}_\phi A_n = \left(B_n \widehat{\boxtimes}_{B_n} {}_\phi A_n \right)^m \cong A_n^m$.

2. We have

$$\left(W_{n+1} \widehat{\boxtimes}_{B_{n+1}} {}_\phi A_{n+1} \right) \widehat{\boxtimes}_{A_{n+1}} A_n \cong W_{n+1} \widehat{\boxtimes}_{B_{n+1}} {}_\phi A_n \cong W_{n+1} \widehat{\boxtimes}_{B_{n+1}} \left(B_n \widehat{\boxtimes}_{B_n} {}_\phi A_n \right) \cong$$

$$\cong \left(W_{n+1} \widehat{\boxtimes}_{B_{n+1}} B_n \right) \widehat{\boxtimes}_{B_n} {}_\phi A_n \cong W_n \widehat{\boxtimes}_{B_n} {}_\phi A_n$$

with the last isomorphism due to the admissibility of W_n.

3. $W^\phi = W \widehat{\boxtimes}_B {}_\phi A = \lim\limits_{\rightarrow} \left(W_n \widehat{\boxtimes}_{B_n} {}_\phi A_n \right)$ (clear).

□

Definition 2.3.3. Let M be a topological left $A-$comodule.

- M is called *s-coflat*, if the functor $- \widehat{\boxtimes}_A M : \ comod. - A \rightarrow LCVS_K$ is exact.

- M is call *s-cofree* if $M = V \widehat{\otimes} A$ for some vector space V (Banach or compact type).

Clearly all s-cofree modules are s-coflat.

2.4 Compact Type-Stein Hopf algebras

In this section we define the structure dual to the structure of Frechet-Stein algebra and prove analogs of some results from [27] in terms of comodules.

2.4.1 Artinian comodules

Let A be a Banach Hopf algebra. Recall that a (co)module is called Noetherian if any increasing chain of its sub(co)modules stabilises and Artinian if any decreasing chain stabilises In our case we restrict to closed sub(co)modules.

Proposition 2.4.1. *Let V be an A-comodule. Then*

V is Artinian if V_b' is a Noetherian $A_b'-$module.

If V_b' is a Noetherian $A_b'-$module then $(V_b')_b'$ is Artinian $(A_b')_b'-$comodule.

Proof. Let V' be Noetherian and let $V \supseteq X_1 \supseteq X_2 \supseteq \dots$ be a descending chain of closed subcomodules in V. Then $X_i^{\perp}$ form an ascending chain of closed submodules of V'. Since V' is Noetherian, there exists $N : X_n^{\perp} = X_N^{\perp}$ for all $n \geq N$. But, since $(X_n/X_{n+1})' = X_{n+1}^{\perp}/X_n^{\perp} = 0$, this imply that $X_n = X_{n+1}$ for all $n \geq N$.

The proof of the second assertion is similar. □

In Noetherian modules over Noetherian rings every submodule is finitely generated. To prove similar property for Artinian comodules first we need to prove the next simple lemma.

Lemma 2.4.2. *Let A be a coalgebra, $B \subset A$ be a subcoalgebra and M be a subspace of $A^n = \bigoplus_{i=1}^{n} A$, such that $\Delta(M) \subset M \widehat{\otimes} B$.*

Then $M \subset B^n$.

Proof. It is enough to prove this in the case $n = 1$. If $\Delta_A(M) \subset M \widehat{\otimes} B$ then $(\epsilon_A \otimes 1) \circ \Delta_A(M) \subset K \widehat{\otimes} B = B$. □

Proposition 2.4.3. *Let A be an Artinian Banach Hopf algebra and let V be a Banach A-comodule. Then*

V is Artinian if and only if for any quotient $V \twoheadrightarrow M$ there is an embedding $M \hookrightarrow A^n$ for some n.

Proof. "Only if" part: If the statement holds then in the dual $A'-$module V' every submodule is finitely generated. Since A' is Noetherian, V' is Noetherian and by proposition 2.4.1 V is Artinian.

"If" part: Let V be an Artinian $A-$comodule and let $V \twoheadrightarrow M$ be a quotient of V. Then V' is a Noetherian $A'-$module and M' is an $A'-$submodule of V'. Since A' is Noetherian we have a surjection $M' \twoheadleftarrow (A')^n$. By duality, we have an embedding $\phi : (M')' \hookrightarrow \left((A')'\right)^n$.

Since ϕ is a morphism of comodules and M is a closed $A-$subcomodule of M'', the coaction of $\left((A')'\right)^n$ sends M to $\left((A')'\right)^n \widehat{\otimes} A$. By lemma 2.4.2, $\phi : M \hookrightarrow A^n$. $\square$

2.4.2 Flatness and Coflatness. CT-Stein structure.

Proposition 2.4.4. *Let A be a Banach K-Hopf algebra and V be a Banach A-comodule. Then*

1. *V is a coflat A-comodule if V'_b is a flat A'-module;*
2. *V' is a flat A-module if V''_b is a coflat A''-comodule.*

Proof. Let

$$0 \to M \to N \to L \to 0$$

is a short exact sequence in Comod-A. By Corollary 4.1.3, which will be proved independently, exactness of the above sequence is equivalent to the exactness of the sequence

$$0 \to L'_b \to N'_b \to M'_b \to 0.$$

Since V'_b is a flat A'-module, we have an exact sequence

$$0 \to L' \underset{A'}{\widehat{\otimes}} V' \to N' \underset{A'}{\widehat{\otimes}} V' \to M' \underset{A'}{\widehat{\otimes}} V' \to 0.$$

of Banach spaces. By lemma 2.2.15 this sequence is exactly

$$0 \to \left(L \underset{A}{\widehat{\boxtimes}} V\right)'_b \to \left(N \underset{A}{\widehat{\boxtimes}} V\right)'_b \to \left(M \underset{A}{\widehat{\boxtimes}} V\right)'_b \to 0$$

The same Corollary 4.1.3 give us the exactness of the sequence

$$0 \to M \underset{A}{\widehat{\boxtimes}} V \to N \underset{A}{\widehat{\boxtimes}} V \to L \underset{A}{\widehat{\boxtimes}} V \to 0$$

of Banach spaces.

The proof of (2) is similar. $\square$

Recall that

Definition. An NF-Hopf algebra $A = \varprojlim A_n$ is called *nuclear Frechet-Stein* (NFS) if

1. A_n are Noetherian;

2. A_n is a flat $A_{n+1}-$module.

Definition 2.4.5. A CT-Hopf algebra $A = \varinjlim A_n$ is called *compact type-Stein* (CTS) if

1. A_n are Artinian;

2. A_n is coflat $A_{n+1}-$comodule.

Propositions 2.4.1 and 2.4.4 together give us the following

Theorem 2.4.6. *Let A be a CT-Hopf algebra and let $\{A_n\}$ be a CT-structure on A. Then*

1. *$\{A_n\}$ is CTS-structure for A if $\{A'_n\}$ is a NFS-structure for A'_b.*

2. *$\{A'_n\}$ is a NFS-structure for A'_b if $\{A''_n\}$ is a CTS-structure for A.*

3. *Let A be such that $\{A_n\}$ be a CTS-structure for A and $\{A'_n\}$ is a NFS-structure for A'_b. Then an A-comodule V is admissible if V' is coadmissible A'-module.*

Conjecture 2.4.7. *The above theorem is true for "iff" statement.*

Chapter 3

Examples

3.1 Examples from representation theory.

3.1.1 Finite dimension.

Any finite group G can be thought as p-adic Lie group with finite dimensional algebra of $K-$valued functions equal to the group algebra $K[G]$. It is both a Banach and CT-Stein $K-$Hopf algebra.

More generally, any finite-dimensional Hopf algebra over K is a Banach and CT-Stein Hopf algebra.

3.1.2 Locally analytic compact groups.

Let G be a uniform compact locally analytic group. Then

$$G \cong \mathbb{Z}_p^d$$

as locally analytic manifolds. The space of locally analytic functions $C^{la}(G,K)$ is isomorphic to the space of Mahler series

$$C^{la}(G,K) \cong C^{la}\left(\mathbb{Z}_p^d, K\right) = \left\{ \sum_{|n|=0}^{\infty} f_n \binom{x}{n},\ f \in K \middle|\ \forall r > 1 :\ |f_n|_K\, r^{|n|} \to 0 \right\}.$$

The space of locally analytic distributions $D^{la}(G,K)$ on G thus can be described as

$$D^{la}(G,K) \cong D^{la}\left(\mathbb{Z}_p^d, K\right) = \left\{ d = \sum_{|n|=0}^{\infty} d_n b_n | \,\forall r > 1 :\ \limsup_n |d_n|_K\, r^{|n|} < \infty \right\},$$

where $\{b_n\}$ is the dual basis for Mahler polynomials $\binom{x}{n}$.

Consider the space of locally analytic functions of order h (denoted $C^{la}_h(G,K)$). These are the functions, whose restriction to any ball of radius p^h is a power series. By the Amice Approximation Theorem [1, III.10, Cor. 3] we have the following description

$$C^{la}_{R_h}(G,K) \cong C^{la}_{R_h}\left(\mathbb{Z}_p^d, K\right) = \left\{ \sum_{|n|=0}^{\infty} f_n \binom{x}{n},\ f \in K \middle|\ |f_n|_K R_h^{|n|} \to 0 \right\}, \tag{3.1.1}$$

where $R_h = \inf\left\{ R \middle| \forall n \in \mathbb{N} : \left| \left(\left[\frac{n}{p^h}\right]! \right)^{-1} \right|_K \leq R^n \right\} = \liminf_n \left(\sqrt[n]{1/\left|\left[\frac{n}{p^h}\right]!\right|_K} \right) > 1$. $C^{la}_{R_h}(G,K)$ is a Banach space w.r.t to the norm

$$\|f\|_{R_h} = \max_n |f_n|_K R_h^{|n|}.$$

We have inclusions $C^{la}_{R_h}(G,K) \hookrightarrow C^{la}_{R_s}(G,K)$ for $h < s$ and these inclusions are compact maps of Banach spaces. Thus we have an topological isomorphism

$$C^{la}(G,K) \cong \lim_{\rightarrow} C^{la}_{R_h}(G,K)$$

and the spaces $C^{la}_{R_h}(G,K)$ with give $C^{la}(G,K)$ a compact type structure. $C^{la}_{R_h}(G,K)$ and $C^{la}(G,K)$ are topological Hopf algebras with comultiplication, counit and antipode induced from the group operations. Thus each $C^{la}_{R_h}(G,K)$ is a commutative Banach Hopf algebra and $C^{la}(G,K)$ is a commutative CT-Hopf algebra.

The space $\left(C^{la}_{R_h}(G,K)\right)'$ can be described as

$$D^{la}_{\leq R_h}(G,K) = \left(C^{la}_{R_h}(G,K)\right)' \cong \left\{ d = \sum_{|n|=0}^{\infty} d_n b_n \middle|\ \limsup_n |d_n|_K R_h^{-|n|} < \infty \right\}.$$

By duality, $D^{la}_{\leq R_h}(G,K)$ are cocommutative Banach Hopf algebras and $D^{la}(G,K)$ is a cocommutative NF-Hopf algebra.

Proposition 3.1.1. *The Banach Hopf algebras* $\left\{C^{la}_{R_h}(G,K)\right\}$ *give* $C^{la}(G,K)$ *a CT-Stein structure.*

Proof. In [27, 4] it is proved that the algebras $D^{la}_{\leq R_h}(G,K)$ are Noetherian and the transition maps between them are flat. Thus $\left\{D^{la}_{\leq R_h}(G,K)\right\}$ is a NFS-structure on $D^{la}(G,K)$ and by theorem 2.4.6 $\left\{C^{la}_{R_h}(G,K)\right\}$ is a CTS-structure. □

Similarly to [27, 5] one prove the above proposition for any locally analytic compact group. Thus for all compact locally analytic groups the Hopf algebras of locally analytic functions are CT-Stein.

Algebras of germs and hyperenveloping algebras

For G as above, the algebra of germs of locally analytic functions at identity $C^{\omega}_1(G,K) = \varinjlim C^{an}_r(G,K)$ from [26] is also a compact type $K-$Hopf algebra. Its dual $C^{\omega}_1(G,K)' =: U(\mathfrak{g},K)$ is a "hyperenveloping algebra" ($\mathfrak{g} = Lie(G)$). It is a nuclear Frechet-Stein $K-$Hopf algebra and $U(\mathfrak{g},K) = \varprojlim U_r(\mathfrak{g},G) = \varprojlim U_{\leq r}(\mathfrak{g},G)$. Since $(C^{an}_r(G,K))' \cong U_{\leq r}(\mathfrak{g},G)$, $C^{\omega}_1(G,K)$ is CT-Stein. It is known that $U(\mathfrak{g},K)$ does not depend on G.

3.2 Examples from quantum enveloping algebras

Here we construct an example of noncommutative and noncocommutative NFS-Hopf algebra, by completing the quantum enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. In this section the base field is denoted by L.

3.2.1 Preliminaries on a quantum group $SL_q(2,L)$.

Here we will recall some notions from theory of quantum groups. There are numerous references on that subject, we will use [14]. For any unknown notation in this section one should look in [14].

Through this paper the words **quantum group** mean the *quantized function algebra* on a corresponding group G. Quantum enveloping algebras will be referred to as is QEA.

The quantum matrix algebra $M_q(2,L)$ is a bialgebra, defined as a quotient of free algebra $L\langle a,b,c,d\rangle$ by the following relations

$$ab = qba,\ ac = qca,\ bd = qdb,\ cd = qdc,\ bc = cb,$$

$$ad - da = \left(q - q^{-1}\right) bc\ .$$

The comultiplication is given by formulas

$$\begin{array}{ll} \Delta(a) = a \otimes a + b \otimes c & \Delta(b) = a \otimes b + b \otimes d \\ \Delta(c) = c \otimes a + d \otimes c & \Delta(d) = c \otimes b + d \otimes d \end{array}\ .$$

The counit is given by

$$\epsilon(a) = 1 \quad \epsilon(b) = 0 \quad \epsilon(c) = 0 \quad \epsilon(d) = 1 \quad .$$

The quantum determinant $det_q = ad - qbc$ is a central group-like element in this algebra.

The quantum group $SL_q(2, L)$ is a quotient $SL_q(2, L) = M_q(2, L) \diagup (det_q = 1)$.

The set $\{a^{n_a} b^{n_b} c^{n_c}, b^{n_b} c^{n_c} d^{n_d}\}$, $n_i \in \mathbb{N}$ is a vector space basis for $SL_q(2, L)$.

It is a Hopf algebra with the antipode

$$S(a) = d,\ S(b) = -q^{-1} b,\ S(c) = -qc,\ S(d) = a\ .$$

The transposition morphism $\theta_{\alpha,\beta}$ is an automorphism of $SL_q(2, L)$, given by the following formulas

$$\theta_{\alpha,\beta}(a) = \alpha a,\ \theta_{\alpha,\beta}(b) = \beta c,\ \theta_{\alpha,\beta}(c) = \beta^{-1} b,\ \theta_{\alpha,\beta}(d) = \alpha^{-1} d\ .$$

$U_q(\mathfrak{sl}_{2,L})$.

The QEA $U_q(\mathfrak{sl}_{2,L})$ is the associative algebra over the field $L(q)$ with generators E, F, K and K^{-1} and the following relations

$$\begin{array}{c} K \cdot K^{-1} = K^{-1} \cdot K = 1\ , \\ KE = q^2 EK\ , \\ KF = q^{-2} FK\ , \\ EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}\ . \end{array} \tag{3.2.1}$$

The algebra $\breve{U}_q(\mathfrak{sl}_{2,L})$ has the same generators but different relations

$$\begin{gathered} K\cdot K^{-1}=K^{-1}\cdot K=1\ , \\ KE=qEK\ , \\ KF=q^{-1}FK\ , \\ EF-FE=\frac{K^2-K^{-2}}{q-q^{-1}}\ . \end{gathered}$$

The set $\left\{E^nF^lK^m\right\}$, $n,l>0$ and $m\in\mathbb{Z}$, is a basis for both algebras.

There is an injective algebra homomorphism

$$\phi:\begin{array}{ccc} U_q(\mathfrak{sl}_{2,L}) & \longrightarrow & \breve{U}_q(\mathfrak{sl}_{2,L}) \\ E & \longmapsto & EK \\ F & \longmapsto & K^{-1}F \\ K & \longmapsto & K^2 \end{array}\ ,$$

but the two algebras are not isomorphic.

Both algebras admit an automorphism

$$\theta_\alpha:\begin{array}{ccc} E & \longmapsto & \alpha E \\ F & \longmapsto & \alpha^{-1}F \\ K & \longmapsto & K \end{array}\ .$$

The Hopf algebra structure on $U_q(\mathfrak{sl}_{2,L})$ is given by comultiplication

$$\begin{gathered} \Delta(E)=E\otimes K+1\otimes E \\ \Delta(F)=F\otimes 1+K^{-1}\otimes F\ , \\ \Delta(K)=K\otimes K \end{gathered}$$

counit

$$\epsilon(K)=1,\quad \epsilon(F)=0,\quad \epsilon(E)=0,$$

and antipode

$$S(F)=-KF,\quad S(E)=-EK^{-1},\quad S(K)=K^{-1}\ .$$

Duality between $U_q(\mathfrak{sl}_{2,L})$ and $SL_q(2,L)$.

There is a non-degenerate pairing $\langle\cdot,\cdot\rangle$ between $U_q(\mathfrak{sl}_{2,L})$ and $SL_q(2,L)$, see [14, I.4.4].

There is also a pairing $\langle\cdot,\cdot\rangle^{\smile}$ for $\breve{U}_q(\mathfrak{sl}_{2,L})$ and $SL_q(2,L)$, [14, I.4.4 Prop.22]:

$$\left\langle K^mE^nF^l,d^sc^rb^t\right\rangle^{\smile}=q^{(n-r)^2}\begin{bmatrix} s \\ n-r \end{bmatrix}_{q^2}\gamma_{mnt}^{srt}$$

if $0\le n-r=l-t\le s$, $\left\langle K^mE^nF^l,d^sc^rb^t\right\rangle^{\smile}=0$ otherwise, and

$$\left\langle K^mE^nF^l,a^sc^rb^t\right\rangle^{\smile}=\delta_{rn}\delta_{tl}\gamma_{mnt}^{-srt}\ ,$$

where

$$\gamma^{srt}_{mnt} = \frac{q^{m(s+r-t)/2}q^{-s(n+l)/2}}{q^{n(n-1)/2}q^{l(l-1)/2}} \frac{\left(q^2;q^2\right)_l\left(q^2;q^2\right)_n}{\left(1-q^2\right)^{l+n}} \ ,$$

$$(a;q)_n = (1-a)(1-aq)\ldots\left(1-aq^{n-1}\right) \ .$$

There is a relation between these pairings

$$\langle x,y\rangle = \left\langle \phi(x), \theta_{1,q^{-1/2}}(y)\right\rangle^{\smile} = \left\langle \left(\phi\circ\theta_{q^{1/2}}\right)(x), y\right\rangle^{\smile}.$$

Remark 3.2.1. A direct check shows that if $|1-q|_L < 1$, then $|\gamma^{srt}_{mnt}|_L = 1$. This mean that any $K^mE^nF^l$ is a bounded linear functional on $SL_q(2,L)$ with norm 1. In the case $|1-q|_L = 1$ any $K^mE^nF^l$ is just a bounded linear functional. The condition $|1-q|_L < 1$ corresponds to the case when $q = \exp(h)$ for some $h \in \mathbb{Z}_p$ s.t. $\exp(h)$ converges, i.e. the case when $SL_q(2,L)$ and $U_q(\mathfrak{sl}_{2,L})$ are deformations of $SL(2,L)$ and $U(\mathfrak{sl}_{2,L})$ respectively.

3.2.2 Skew-Tate algebras.

Let A be a $K-$Banach algebra, $A^0 = \{x \in A|\, \|x\|_A \le 1\}$ and $A^{00} = \{x \in A|\, \|x\|_A < 1\}$. A^0 and A^{00} are complete K^0-submodules of A and $\bar{A} = A^0/A^{00}$ is a $k-$vector space. Denote the image of $f \in A^0$ in $\bar{A}$ by $\bar{f}$.

First we recall the definition of Ore extension.

Definition 3.2.2. Let A be a ring, $\alpha : A \to A$ is an injective homomorphism and $\delta : A \to A$ is an α-derivation of A. The *Ore extension* $A[x,\alpha,\delta]$ is a ring of polynomials $A[x]$ with a new multiplication, defined by the relation $x \cdot a = \alpha(a)x + \delta(a)$, $a \in A$.

If α is an automorphism and A is left Noetherian, then $A[x,\alpha,\delta]$ is also left Noetherian.

Lemma 3.2.3. *Let $B = A[x,\alpha,\delta]$ be an Ore extension of A ($\alpha : A \to A$ automorphism of A and δ is an $\alpha-$derivation of A). Consider the "Gauss R-norm" on B: for $f \in B$, $f = \sum f_n x^n$, then $\|f\|_{Gauss,R} = \max_n \|f_n\|_A R^n$ ($R \in \mathbb{R}$). Suppose $\|\alpha\| \le 1$, $\|\delta\| \le 1$ and $|R| \ge 1$. Then $\|\cdot\|_{Gauss,R}$ is a submultiplicative non-archimedean algebra norm on B.*

Proof. It is clear that $\|\cdot\|_{Gauss,R}$ is a $K-$vector space norm (as in commutative case).

Let us prove submultiplicativity, i.e. $\|fg\|_{Gauss,R} \leq \|f\|_{Gauss,R} \|g\|_{Gauss,R}$.

One can prove (using induction) that $x^n \cdot a = \sum_{k=0}^{n} c_{nk}(a) x^k$, where $c_{nk}(a)$ is the sum of all words with $k-$letters α and $(n-k)-$letters δ, applied to a. Since $\|\alpha\| \leq 1$, $\|\delta\| \leq 1$, then $\|c_{nk}(a)\|_A \leq \|a\|_A$. Now let $f = \sum_{n=0}^{l} f_n x^n$ and $g = \sum_{k=0}^{s} g_k x^k$. Then

$$\|fg\|_{Gauss,R} = \left\| \left(\sum_{n=0}^{l} f_n x^n \right) \left(\sum_{k=0}^{s} g_k x^k \right) \right\|_{Gauss,R} = \left\| \sum_{n=0}^{l} f_n \left(\sum_{k=0}^{s} (x^n g_k) x^k \right) \right\|_{Gauss,R} =$$

$$= \left\| \sum_{n=0}^{l} \sum_{k=0}^{s} \left(f_n \sum_{i=0}^{n} c_{ni}(g_k) x^i \right) x^k \right\|_{Gauss,R} = \left\| \sum_{n=0}^{l} f_n \sum_{i=0}^{n} \left(\sum_{k=0}^{s} c_{ni}(g_k) x^{k+i} \right) \right\|_{Gauss,R} \leq$$

$$\leq \max_n \left(\left\| \sum_{i=0}^{n} \sum_{k=0}^{s} f_n c_{ni}(g_k) x^{k+i} \right\|_{Gauss,R} \right) = \max_n \max_i \max_k \left(\|f_n\|_A \|c_{ni}(g_k)\|_A R^{k+i} \right) \leq$$

$$= \max_n \max_i \max_k \left(\|f_n\|_A R^i \right) \left(\|g_k\|_A R^k \right) \leq \|g\|_{Gauss,R} \cdot \max_{n,i} \|f_n\|_A R^i =$$

$$= \|g\|_{Gauss,R} \cdot \max_n \|f_n\|_A R^n = \|f\|_{Gauss,R} \|g\|_{Gauss,R}.$$

□

Denote the completion of $B = A[x, \alpha, \delta]$ w.r.t. $\|\cdot\|_{Gauss,R}$ by $A\{x/R, \alpha, \delta\}$.

Remark 3.2.4. it is clear that if $|s|_K = R$, then $A\{x/R, \alpha, \delta\} \cong A\{z, \alpha, s^{-1}\delta\}$ with x mapped to sz.

Definition 3.2.5. An algebra of the form $K\{x_1, \alpha_1, \delta_1\} \dots \{x_n, \alpha_n, \delta_n\}$ with $\|\cdot\|_{Gauss} = \|\cdot\|_{Gauss,1}$ will be called a skew-Tate algebra.

For Tate algebras there are Weierstrass division and preparation theorems. We now prove corresponding results for skew-Tate algebras.

Definition 3.2.6. An element $f \in A\{x, \alpha, \delta\}$ with $\|f\|_{Gauss} = 1$ is called *regular of degree* d if $\bar{f}$ has the form $\lambda z^d + \sum_{i=0}^{d-1} c_i z^i$ with $\lambda \in k^*$ and $c_i \in \bar{A}$.

Theorem 3.2.7. *(Weierstrass division and preparation)*

1. *(Division) Let f be a regular element of $A\{z,\alpha,\delta\}$ of degree d. Then for any g in $A\{z,\alpha,\delta\}$ there exists unique q and r such that $g=qf+r$ and degree of r is less then d. Moreover, $\|g\|_{Gauss} = \max\left(\|q\|_{Gauss}, \|r\|_{Gauss}\right).$*

2. *(Preparation) Let f be a regular element of $A\{x,\alpha,\delta\}$ of degree d. Then there exists $w \in A[x,\alpha,\delta]$, s.t. $f = w \cdot e$, where e is a unit in B, and w is regular of degree d. If $f \in A[x,\alpha,\delta]$ then $e \in A[x,\alpha,\delta]$ also.*

Proof. (1) We have $f = f_0 - D$, where $f_0 = \lambda z^d + \sum_{i=0}^{d-1} c_i z^i$, $c_i = A^0$ and $\|D\|_{Gauss} < 1$. Let us prove that the statement of (1) is true for f_0. Let us first prove the statement for powers of z, i.e. $z^i = q_i f_0 + r_i$. We clearly have $z^d = \lambda^{-1}\left(\lambda z^d + \sum_{i=0}^{d-1} c_i z^i - \sum_{i=0}^{d-1} c_i z^i\right) = q_d f_0 + r_d$ with $q_d = \lambda^{-1}$ and $r_d = \lambda^{-1}\left(\sum_{i=0}^{d-1} c_i z^i\right)$ and for $i < d$ $q_i = 0$. Now for z^{n+1} we have $z^{n+1} = z \cdot z^n = z \cdot (q_n f_0 + r_n) = (z \cdot q_n) f_0 + z \cdot r_n$. If $r_n = \sum_{i=0}^{d-1} c_{ni} z^i$, then from commutation relations we get

$$z \cdot r_n = \sum_{i=0}^{d-1} (z \cdot c_{ni}) z^i = \sum_{i=0}^{d-1} (\alpha(c_{ni}) z + \delta(c_{ni})) z^i =$$

$$= \alpha\left(c_{n(d-1)}\right) z^d + \sum_{i=0}^{d-1} \left(\alpha\left(c_{n(i-1)}\right) + \delta(c_{ni})\right) z^i =$$

$$= \alpha\left(c_{n(d-1)}\right)\left(\lambda^{-1} f_0 + r_d\right) + \sum_{i=0}^{d-1} \left(\alpha\left(c_{n(i-1)}\right) + \delta(c_{ni})\right) z^i =$$

$$= \alpha\left(c_{n(d-1)}\right)\lambda^{-1} f_0 + \alpha\left(c_{n(d-1)}\right) r_d + \sum_{i=0}^{d-1} \left(\alpha\left(c_{n(i-1)}\right) + \delta(c_{ni})\right) z^i.$$

Thus $z^{n+1} = q_{n+1} f_0 + r_{n+1}$, where $q_{n+1} = z \cdot q_n + \lambda^{-1}\alpha\left(c_{n(d-1)}\right)$ and $r_{n+1} = \alpha\left(c_{n(d-1)}\right) r_d + \sum_{i=0}^{d-1} \left(\alpha\left(c_{n(i-1)}\right) + \delta(c_{ni})\right) z^i$. It is clear that in this formulas the norms of the coefficients do not increase, and thus for any $g = \sum_{n=0}^{\infty} g_n z^n$ we have

$$g = \left(\sum_{n=0}^{\infty} g_n q_n\right) f_0 + \left(\sum_{n=0}^{\infty} g_n r_n\right)$$

with both sums being convergent in $A\{z,\alpha,\delta\}$.

Now let us prove the division property for f. We have $f_0 = f + D$ and for any g we have

$$g = q_0 f_0 + r_0 = q_0 f + g_1 + r_0$$

where $g_1 = q_0 D$. Since the norm is submultiplicative, we have

$$\|g_1\|_{Gauss} \leq \|q_0\|_{Gauss} \|D\|_{Gauss} \leq \|g\|_{Gauss} \|D\|_{Gauss}.$$

Next

$$g_1 = q_1 f_0 + r_1 = q_1 f + g_2 + r_1,$$

where $g_2 = q_1 D$ and

$$\|g_2\|_{Gauss} \leq \|q_1\|_{Gauss} \|D\|_{Gauss} \leq \|g_1\|_{Gauss} \|D\|_{Gauss} \leq \|g\|_{Gauss} \|D\|^2_{Gauss}.$$

Continuing by induction, we have zero sequences g_n, q_n and r_n s.t. $g_n = q_n f + g_{n+1} + r_n$. Adding up all these recurrence relations gives

$$g = \left(\sum_{n=0}^{\infty} q_n\right) f + \left(\sum_{n=0}^{\infty} r_n\right)$$

and

$$\|g\|_{Gauss} = \max\left(\left\|\sum_{n=0}^{\infty} q_n\right\|_{Gauss}, \left\|\sum_{n=0}^{\infty} r_n\right\|_{Gauss}\right).$$

Now let us prove uniqueness. If we have $g = q_1 f + r_1 = q_2 f + r_2$, we have $0 = (q_1 - q_2) f + (r_1 - r_2)$. Since the norm of f is one, we have $\|q_1 - q_2\|_{Gauss} = \|r_1 - r_2\|_{Gauss}$ and multiplication by an appropriate number makes both norms equal 1. But then in $A\{z,\alpha,\delta\}^0 / A\{z,\alpha,\delta\}^{00}$ we have $\overline{q_1 - q_2} \cdot \bar{f} = \overline{r_1 - r_2}$, and this is impossible, since on left hand side we have a skew-polynomial of degree $\geq d$ and on the right hand side $< d$.

(2) Since f is distinguished, by (1) there exists e' and r' s.t. $x^d = e' f + r$ and $\deg(r) < d$. Define $\omega = x^d - r$. We have $\omega = e' f$. Since $\|r\|_{Gauss} \leq \left\|x^d\right\|_{Gauss} = 1$, we have $\|\omega\|_{Gauss} = 1$ and ω is distinguished of degree d. Then in $A\{z,\alpha,\delta\}^0 / A\{z,\alpha,\delta\}^{00}$ we have $\overline{\omega} = \overline{e'}\overline{f}$ with $\bar{\omega}$ and $\bar{f}$ being unitary skew-polynomials of the same degree. This means that $\bar{e'}$ is a unit in $A\{z,\alpha,\delta\}^0 / A\{z,\alpha,\delta\}^{00}$ and e' is a unit in $A\{z,\alpha,\delta\}$. If f is a polynomial then so must e be. □

3.2.3 Completion of $U_q(\mathfrak{sl}_{2,L})$

Consider the QEA $U_q(\mathfrak{sl}_{2,L})$. It is generated by elements F, E and $K^{\pm 1}$ subject to relations 3.2.1 . We want to define a completion of $U_q(\mathfrak{sl}_{2,L})$ with respect to certain norm.

In order to do so, let us recall [11, Prop. 6.1.4] that $U_q(\mathfrak{sl}_{2,L})$ is a Noetherian algebra, obtained by a sequence of Ore extensions

$$\begin{gathered} L\left[K, K^{-1}\right] = A_0 \hookrightarrow A_1 = A_0\left[F, \alpha_0, 0\right] \\ \alpha_0(K) = q^2 K \end{gathered}$$

and

$$\begin{gathered} A_1 \hookrightarrow A_2 = A_1\left[E, \alpha_1, \delta\right] \\ \alpha_1\left(F^j K^l\right) = q^{-2l} F^j K^l \\ \delta(F) = \frac{K-K^{-1}}{q-q^{-1}} \\ \delta\left(F^j K^l\right) = \sum_{i=0}^{j-1} F^{j-1}\delta(F)\left(q^{-2i}K\right)K^l \\ \delta(K) = 0 \end{gathered} . \tag{3.2.2}$$

Let $\hat{A}_0$ be the algebra of bidirectional Laurant series in K,

$$\hat{A}_0 = \left\{ \sum_{n\in Z} f_n K^n \middle| \lim_{n\to\pm\infty} |f_n| R_K^n = 0 \right\}$$

with fixed R_K. It is a Banach $K-$algebra w.r.t.

$$\|f\|_R = \max_{n\in Z} |f_n|_L R_K^n.$$

Let $|q|_L = 1$. Then

$$\alpha_0 : \hat{A}_0 \to \hat{A}_0, \ \alpha_0(K) = q^2 K$$

is an automorphism of norm 1. Then, by lemma 3.2.3, the algebra $\hat{A}_1 = \hat{A}_0\left\{\frac{F}{R_F}, \alpha_0, 0\right\}$ is a Banach $\hat{A}_0-$algebra

$$\hat{A}_1 = \left\{ \sum_{n=0}^{\infty} a_n F^n \middle| a_n \in \hat{A}_0, \text{ s.t. } \lim_{n\to\infty} |a_n|_L R_F^n = 0, \right\}$$

or a $K-$Banach algebra of skew-commutative convergent series in F, $K^{\pm 1}$ with radius of convergence (R_F, R_K).

Consider $A_2 = \hat{A}_1 [E, \alpha_1, \delta]$. Since $|q|_L = 1$, $\|\alpha\| = 1$. In order to apply lemma 3.2.3, we need $\|\delta\| \le 1$. From formulas 3.2.2 we see that $\|\delta\| \le 1$ if $\left|\frac{1}{q-q^{-1}}\right|_L R_K \le R_F$. So, under this condition, the Gauss R_E-norm is a norm on A_2 and $\hat{A}_2$ (the completion of A_2) is a $K-$Banach algebra.

We denote by $U_q(\mathfrak{sl}_{2,L})(R_K, R_F, R_E)$ or just $U_{q,R}(\mathfrak{sl}_{2,L})$ the algebra $\hat{A}_2$.

Note that, due to symmetry between F and E, instead of condition $\left|\frac{1}{q-q^{-1}}\right|_L R_K \le R_F$, we can take $\left|\frac{1}{q-q^{-1}}\right|_L R_K \le R_E$ (and first extend $\hat{A}_0$ by E instead of F).

From the formulas in the section 3.2.1 for comultiplication, counit and antipode one can see that $(\Delta,\ S,\ \epsilon)$ are bounded maps only if $R_K = 1$. So, in case $R_K = 1$, $U_q(\mathfrak{sl}_{2,L})(R_F, R_E) := U_q(\mathfrak{sl}_{2,L})(1, R_F, R_E)$ is a Banach $K-$Hopf algebra (R_F or $R_E \ge \left|(q-q^{-1})^{-1}\right|_L$).

The projective limit $U_q(\mathfrak{sl}_{2,L}, L) = \varprojlim U_{q,R}(\mathfrak{sl}_{2,L})$ is a noncommutative and noncocommutative weak Frechet-Stein algebra. The topology on $U_q(\mathfrak{sl}_{2,L}, L)$ is given by the family of norms $\nu_R :\ \nu_R\left(\sum c_{nml} F^n K^m E^l\right) = \sup\left(|c_{nml}|_L R_F^n R_K^m R_E^l\right)$.

Equivalently, one can take the family of norms $\nu'_R = \sup\left(|c_{nml}|_L \left|[n]_q!\right|_L \left|[l]_q!\right|_L R_F^n R_K^m R_E^l\right)$ (similarly to [15, 1.2.8]). This is possible due to an estimate (4.1.1.1) from [32], which implies that $\exists C \ge 1$:

$$\left|\frac{1}{[n]_q!}\right|_L \le C^n p^{\frac{n}{p-1}}$$

(note, that in [32], $[n]_q = 1 + q + ... + q^{n-1} = [[n]]_q$ in notation of [14], and one needs to use $[n]_q = q^{-n+1}[[n]]_q$, [11, 6.1.1. (1.7)]).

Remark 3.2.8. Let us describe the corresponding completion of $SL_q(2, L)$.

The pairing from 3.2.1 gives the following pairing between $U_q(\mathfrak{sl}_{2,L}, L)$ and $SL_q(2, L)$:

$$\left\langle K^m E^n F^l, d^s c^r b^t \right\rangle = q^{(n-r)^2} \begin{bmatrix} s \\ n-r \end{bmatrix}_{q^2} \gamma^{srt}_{(m+n-l)nt} \cdot q^{-\left(\frac{n(n+1)}{2} + \frac{l(l-1)}{2} + ln\right)}$$
$$\text{if } 0 \le n-r = l-t \le s,$$

$$\left\langle K^m E^n F^l, d^s c^r b^t \right\rangle = 0 \text{ otherwise, and}$$

$$\left\langle K^m E^n F^l, a^s c^r b^t \right\rangle^{\smile} = \delta_{rn}\delta_{tl}\gamma^{-srt}_{(m+n-l)nt} \cdot q^{-\left(\frac{n(n+1)}{2} + \frac{l(l-1)}{2} + ln\right)}.$$

The norm on $SL_q(2,L)$ as on algebra of continuous functionals on $U_q\left(\mathfrak{sl}_{2,L},L\right)$ is given by

$$\alpha\in SL_q(2,K):\ \left\|\alpha\right\|_R^*=\sup_x\left|\langle x,\alpha\rangle\right|_L,\ x\in U_{q,R}\left(\mathfrak{sl}_{2,L}\right):\left\|x\right\|_R\le 1.$$

Since (from [14, 2.1.1. (3)])

$$\left(q^2,q^2\right)_m=[m]_q!\cdot\left(1-q^2\right)^m\cdot q^{\frac{m(m-1)}{2}},$$

then in the case $|q|_L=1$, we have

$$\left|\gamma_{xnt}^{srt}\right|_L=\left|[n]_q!\right|_L\left|[l]_q!\right|_L.$$

Thus

$$\left|\left\langle K^mE^nF^l,a^sc^rb^t\right\rangle\right|_L=\delta_{rn}\delta_{tl}\left|[n]_q!\right|_L\left|[l]_q!\right|_L$$

and the norm of $a^sc^rb^t$ as of functional on $U_{q,R'}\left(\mathfrak{sl}_{2,L}\right)$ (the completion of $U_{q,R}\left(\mathfrak{sl}_{2,L}\right)$ w.r.t. ν_R') is equal to $R_E^{-r}R_F^{-t}$.

For elements of the form $d^sc^rb^t$ we have

$$\left|\left\langle K^m\frac{E^n}{[n]_q!}\frac{F^l}{[l]_q!},d^sc^rb^t\right\rangle\right|_L=\left|\begin{bmatrix}s\\n-r\end{bmatrix}_{q^2}\right|_L\le 1$$

due to [11, 6.1.1 (1.8)] and [32, 4.1.1.2]. Thus the norm of $d^sc^rb^t$ as of functional on $U_{q,R'}\left(\mathfrak{sl}_{2,L}\right)$ is equal to $R_E^{-r}R_F^{-t}$.

We get a completion of $SL_q(2,L)$, which we denote by $C_R^{an}\left(SL_q\left(2\right)\right)$, which consist of series $\sum\alpha_{nmk}a^nb^mc^k+\sum\beta_{mkl}b^mc^kd^l$ with

$$\lim_{n,m,k\to\infty}\left|\alpha_{nmk}\right|_L\left(\frac{1}{R_E}\right)^k\left(\frac{1}{R_F}\right)^m=0,$$

$$\lim_{m,k,l\to\infty}\left|\beta_{mkl}\right|_L\left(\frac{1}{R_E}\right)^k\left(\frac{1}{R_F}\right)^m=0.$$

It is a Banach algebra w.r.t. "sup"-R norm. The comultiplication, counit and antipode are also bounded, so it is an $L-$Banach Hopf algebra.

The injective limit $C^\omega\left(SL_q\left(2\right)\right)=\lim\limits_{\rightarrow}C_R^{an}\left(SL_q\left(2\right)\right)$ is an LCVS of compact type (similarly to [24, 16.11]). So, $C^\omega\left(SL_q\left(2\right)\right)$ is a noncommutative and noncocommutative $L-$Hopf algebra of compact type.

Remark 3.2.9. More generally, one can take R_a, R_b, R_c, $R_d < 1$ and define a completion of $SL_q(2, L)$ in a similar way. The norm will not be submultiplicative (i.e. multiplication is not continuous w.r.t. Gauss-R norm), but comultiplication will be bounded, so in this case we get a noncocommutative coalgebra. Moreover, the completion can be done if $|q|_L < 1$ and in this case the Haar functional of $SL_q(2, L)$ [14, 4.2.6] is bounded.

Remark 3.2.10. We have a relation between R_K and R_F. It does not exist for completion of the 2-parameter quantum group $U_{p,q}(\mathfrak{sl}_{2,L})$ of Takeuchi with $|p - q^{-1}|_L = 1$.

Remark 3.2.11. The subalgebras $\hat{A}_0$ and $\hat{A}_1$ of $U_{q,R}(\mathfrak{sl}_{2,L})$ are also Banach $L-$Hopf algebras. One can obtain corresponding subalgebras $\tilde{A}_0$ and $\tilde{A}_1$ of $U_q(\mathfrak{sl}_{2,L}, L)$.

3.2.4 Frechet-Stein property for $U_q(\mathfrak{sl}_{2,L}, L)$

We will show that $U_q(\mathfrak{sl}_{2,L}, L)$ is a Frechet-Stein algebra. In order to do this we note that in the same way as in the previous section, one can form a projective system of algebras $A_R = k\{K, M\}\{F/R_F, \alpha_0, 0\}\{E/R_E, \alpha_1, \delta_1\}$, with δ_1 defined similarly to 3.2.2 by $\delta(F) = \frac{K-M}{q-q^{-1}}$ and take a projective limit $A = \varprojlim A_R$. Note that $U_q(\mathfrak{sl}_{2,L})(R_F, R_E) \cong A_R/\overline{(KM-1)}$, and $U_q(\mathfrak{sl}_{2,L}, L) = A/\overline{(KM-1)}$. Thus to show that $U_q(\mathfrak{sl}_{2,L}, L)$ is Frechet-Stein, by lemma 1.5.7 it is enough to show that A is Frechet-Stein.

Proposition 3.2.12. *The algebras A_R are Noetherian if $R_F \cdot R_E > \left|(q - q^{-1})^{-1}\right|_L$.*

Proof. Let $\tilde{F}$ and $\tilde{E}$ be the images of elements F and E in the skew-Tate algebra B, isomorphic to A_R (see remark 3.2.4). Then, if $R_F \cdot R_E > \left|(q - q^{-1})^{-1}\right|_L$, then in the residue algebra $\bar{B}$ of B $\bar{\tilde{F}}$ and $\bar{\tilde{E}}$ commute. Since $|1 - q|_L < 1$, then in the residue field $\bar{q} = 1$. So in this case $\bar{\tilde{F}}$ and $\bar{\tilde{E}}$ commute with $\bar{\tilde{K}}$ and $\bar{\tilde{L}}$. All together this gives commutativity of the residue algebra $\overline{B}$ and $\bar{B} \cong l[x_1, x_2, x_3, x_4]$. Since $\bar{B}$ is a residue algebra of the Tate algebra T_4, any ideal of $\bar{B}$ has an image of a regular element of some degree d. But then a preimage of this element in B is a regular element of degree d. This proves that any ideal I in B has a regular element f_I of some degree.

Let I be an ideal in $B = L\{x_1, \alpha_1, \delta_1\} \dots \{x_4, \alpha_4, \delta_4\}$. By the Weierstrass Division Theorem I is generated by f_I and by the ideal $J = I \cap L\{x_1, \alpha_1, \delta_1\} \dots \{x_3, \alpha_3, \delta_3\}[x_4, \alpha_4, \delta_4]$. Since $L\{x_1, \alpha_1, \delta_1\} \dots \{x_3, \alpha_3, \delta_3\}[x_4, \alpha_4, \delta_4]$ is an Ore extension of $L\{x_1, \alpha_1, \delta_1\} \dots \{x_3, \alpha_3, \delta_3\}$, it is Noetherian if $L\{x_1, \alpha_1, \delta_1\} \dots \{x_3, \alpha_3, \delta_3\}$ is Noetherian. If we continue in this way, we reduce the Noetherianess of $L\{x_1, \alpha_1, \delta_1\} \dots \{x_3, \alpha_3, \delta_3\}[x_4, \alpha_4, \delta_4]$ to that of L, which is a field. So the ideal J is finitely generated and I is also finitely generated.

This implies that B is Noetherian and thus A_R is also Noetherian. □

Corollary 3.2.13. *The category of finitely generated (Banach)* $U_q(\mathfrak{sl}_{2,L})(R_F, R_E)-$*modules is abelian.*

Proposition 3.2.14. *Let* $A_{R_1} \hookrightarrow A_{R_2}$ *be the inclusion map for* $R_1 > R_2$*. Then* A_{R_2} *is a flat* $A_{R_1}-$*module.*

Proof. The proof follows the idea from [27]. We view our Banach algebras as complete filtered rings with the filtration induced by the norm. By [27, Prop. 1.2] the map between two such rings is flat if the associated graded rings are Noetherian and the associated map of graded rings is flat.

As in [27] we factor our map $A_{R_1} \hookrightarrow A_{R_2}$ into $A_{R_1} \hookrightarrow A_{\leq R_1} \hookrightarrow A_{R_2}$, where

$$A_{\leq R_1} = \left\langle \sum a_n K^{n_K} M^{n_M} F^{n_F} E^{n_E} \middle| a_n \in A_{R_1} : \sup \|a_n\|_{A_{R_1}} \leq \infty \right\rangle$$

is a module with the same relations as for A_{R_1}. $A_{\leq R_1}$ is a Banach algebra w.r.t. supremum norm and A_{R_1} is a closed subalgebra. Easy to see that the associated graded ring of A_{R_1} is the ring of polynomials $l[x_1, x_2, x_3, x_4]$ and the associated graded ring of $A_{\leq R_1}$ is the ring of formal power series $l[[x_1, x_2, x_3, x_4]]$. Since both rings are Noetherian and inclusion of polynomials into power series is a flat map, the inclusion $A_{R_1} \hookrightarrow A_{\leq R_1}$ is flat.

For the second inclusion note that $A_{\leq R_1} \cong L \widehat{\otimes} A^0_{\leq R_1}$ and $A_{\leq R_1} \hookrightarrow A_{R_2}$ is flat iff $A^0_{\leq R_1} \hookrightarrow A_{R_2}$ is flat. It follows from the strong triangle inequality that $A^0_{\leq R_1}$ is a closed subset of A_{R_2} and thus it is complete w.r.t. the norm filtration of A_{R_2}. So one can apply [27, Prop.

1.2] in this case too. Similarly to [27, Thm. 4.9] one can show that the map of associated graded rings of $A^0_{\leq R_1}$ and A_{R_2} is a localization and thus is flat. This proves that the second inclusion is also flat. □

Remark 3.2.15. Similarly to the $U_q(\mathfrak{sl}_{2,L})$ case, one can consider arbitrary Drinfeld-Jimbo QEA $U_q(\mathfrak{g}_L)$ for any semisimple Lie algebra $\mathfrak{g}_L$. One can form the Banach algebras $U_q(\mathfrak{g}_L, R)$, similarly to the $\mathfrak{sl}_{2,L}$ case and $U_q(\mathfrak{g}_L, L)$. The above proof of Frechet-Stein property works for $U_q(\mathfrak{g}_L, L)$ with the only difference that we quotient the corresponding algebra A also by quantum Serre relations.

Chapter 4

Cohomology

4.1 Cohomology of nonarchimedean Hopf algebras

In order to do homological algebra for topological algebras one must work in relative context [8, 9, 30]. That is, one must consider complexes with certain restriction on maps. For topological algebras the best class of maps is given by strong maps. For nonarchimedean distribution algebras the relative homological algebra was worked out in [16].

In comodule theory one must impose the same assumption.

Recall that a $K-$linear map between two LCVS is called *strong* if it is strict, with closed image and both the kernel and the image of the map are complemented subspaces.

4.1.1 Banach Hopf algebras.

In this section we will restrict our attention to the Banach Hopf algebras case. In a Banach space strictness is equivalent to closedness of the image since every closed subspace is complemented. This also allows us to generalize results of [8, 0.5.2].

Let A be a Banach Hopf algebra.

Lemma 4.1.1. *Let $\phi : X \to Y$ be an injective morphism of (co)modules with dense image. If W is another (co)module and if $L(\phi) : L_b(Y, W) \to L_b(X, W)$ is surjective, then so is ϕ.*

Proof. Since $\phi : X \to Y$ is injective with dense image, $L(\phi)$ is injective. By the Open Mapping Theorem, $L(\phi)$ is a topological isomorphism. Any $\psi \in W'$ gives us surjective maps $L_b(X,W) \twoheadrightarrow X'$ and $L_b(Y,W) \twoheadrightarrow Y'$ such that the diagram

$$\begin{array}{ccc} L_b(Y,W) & \xrightarrow{L(\phi)} & L_b(X,W) \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\phi'} & X' \end{array}$$

is commutative. Since vertical arrows and $L(\phi)$ are surjections, ϕ' is also a surjection and, by discussion above, a topological isomorphism. But then $\phi'' : X'' \to Y''$ is also a topological isomorphism and so is ϕ, as restriction of ϕ'' to the closed subspace $X \subset X''$. $\square$

Consider a sequence of Banach $A-$comodules

$$\ldots \longrightarrow V_{n+1} \xrightarrow{d_n} V_n \xrightarrow{d_{n-1}} V_{n-1} \longrightarrow \ldots \tag{4.1.1}$$

For any Banach space W it gives rise to a sequence of Banach $A'-$modules

$$\ldots \longleftarrow L_b(V_{n+1},W) \xleftarrow{L(d_n)} L_b(V_n,W) \xleftarrow{L(d_{n-1})} L_b(V_{n-1},W) \longleftarrow \ldots \tag{4.1.2}$$

Lemma 4.1.2. *Let V and W be as above.*

1. *If 4.1.1 is exact at V_{n-1} and V_n, then 4.1.2 is exact at $L_b(V_n,W)$;*
2. *if 4.1.2 is exact at $L_b(V_{n+1},W)$, then Im d_n is closed in V_n;*
3. *if 4.1.2 is exact at $L_b(V_n,W)$, then Im d_n is a dense subset of Ker d_{n-1}.*

Proof. 1. We have $L(d_{n-1} \circ d_n) = L(d_n) \circ L(d_{n-1}) = 0$ and thus $\mathrm{Im} L(d_{n-1}) \subset \mathrm{Ker} L(d_n)$. Let $f \in L_b(V_n,W)$ and $L(d_n)(f) = 0$. Then $f = 0$ on $\mathrm{Im} d_n$ and, by exactness of 4.1.1, $f = 0$ on $\mathrm{Ker} d_{n-1}$. Define $g_0 : \mathrm{Im} d_{n-1} \to W$ by $g(d_{n-1}(x)) = f(x)$. By exactness of 4.1.1, $\mathrm{Im} d_{n-1} = \mathrm{Ker} d_{n-2}$ and thus $\mathrm{Im} d_{n-1}$ is closed. By the Open Mapping Theorem d_{n-1} is an open map on its image, and thus continuity of f implies continuity of g_0. Since $\mathrm{Im} d_{n-1}$ is complemented, we can extend g_0 to the whole V_{n-1} and denote this extension by g. For any $x \in V_{n-1}$ we have $g(d_{n-1}(x)) = f(x)$, i.e. $L(d_{n-1})(g) = f$. Thus $\mathrm{Im} L(d_{n-1}) = \mathrm{Ker} L(d_n)$.

2. Consider the quotient norm $\|\cdot\|_q$ on Im d_n. Im d_n is a Banach space with respect to $\|\cdot\|_q$, which is isomorphic to $V_n/\text{Ker } d_n$ and d_n is continuous w.r.t. $\|\cdot\|_q$. We have a natural continuous embedding $\phi : V_n/\text{Ker } d_n \to \text{Im } d_n$. By lemma 4.1.1 we need to show that $L(\phi)$ is surjective, i.e. that for all $f_0 \in L_b(V_n/\text{Ker } d_n, W)$ if f_0 is continuous w.r.t. $\|\cdot\|_q$ it is also continuous w.r.t. $\|\cdot\|_{V_n}$. If $z \in V_{n+1}$, let $h(z) = f_0(d_n z)$. Clearly $h \in L_b(V_{n+1}, W)$ and $L(d_{n+1})(h) = 0$. Exactness of 4.1.2 at $L_b(V_{n+1}, W)$ implies that $h = L(d_n) f$ for some $f \in L_b(V_n, W)$. We have $f(d_n z) = h(z) = f_0(d_n z)\ \forall z \in V_{n+1}$, i.e. $f = f_0$ on Im d_n.

3. $L(d_n) \circ L(d_{n-1}) = L(d_{n-1} \circ d_n) = 0$ implies Im $d_n \subset$ Ker d_{n-1}. Since every closed subspace is complemented, it is enough to show that every map $f \in L_b(V_n, W)$, which is zero on Im d_n, is also zero on Ker d_{n-1} (otherwise, Im d_n is not dense in Ker d_{n-1}, so we can construct a map which is zero on Im d_n, but is non-zero on Ker d_{n-1}). If f is zero on Im d_n, then $L(d_n)(f) = 0$. Since 4.1.2 is exact at $L_b(V_n, W)$, $f = L(d_{n-1})(g)$ for some $g \in L_b(V_{n-1}, W)$. But this mean that $\forall x \in V_n : f(x) = g(d_{n-1}x)$ and thus $\forall x \in \text{Ker } d_{n-1} : f(x) = g(d_{n-1}x) = 0$.

□

Corollary 4.1.3. *The sequence 4.1.1 is exact if and only if the sequence is 4.1.2 is exact.*

4.1.2 Injective resolutions

Recall that a map is called strong if its kernel and image are closed and complemented. In a Banach space strictness is equivalent to closeness of the image and every closed subspace is complemented. Thus in the Banach space case strict and strong maps are the same.

We call a map of Banach (or any topological) (co)modules strong, if it is strong as a map of the corresponding spaces.

Definition 4.1.4. We call a chain complex of Banach (CT) $A-$comodules

$$\ldots \longrightarrow V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \longrightarrow \ldots$$

is an *s-complex* if it is a chain complex in which all maps d_n are strong maps of Banach (CT) spaces (equivalently, strict maps of $A-$comodules). An equivalent requirement is the existence of a contracting homotopy in the category of Banach (CT) spaces [8, 16, 30].

One defines cochain s-complexes similarly.

Definition 4.1.5. $M \in \text{Comod} - A$ is called *s-injective* if the functor Comod-$A(-, M)$ sends short s-exact sequences of comodules to short exact sequences of $K-$vector spaces.

One can show by a standard argument that this definition is equivalent to the one in terms of diagrams, i.e. s-injective comodules are s-injective objects in Comod-A.

Lemma 4.1.6. *For any V a Banach (CT) K-vector space, $V \widehat{\otimes} A$ is a right Banach (CT) $A-$comodule with coaction $\rho_{V \widehat{\otimes} A} = id_V \otimes \Delta_A$. Moreover*

$$\begin{array}{rccc} \alpha: & Comod - A\left(W, V \widehat{\otimes} A\right) & \longrightarrow & L_b\left(W, V\right) \\ & f & \longmapsto & \left(id_V \bar{\otimes} \epsilon_A\right) \circ f \end{array}$$

is a topological isomorphism.

Proof. First statement is obvious. For the second α^{-1} is given by

$$(f : W \to V) \overset{\alpha^{-1}}{\mapsto} \left((f \otimes id_A) \circ \rho_W : W \to V \widehat{\otimes} A\right).$$

□

Corollary 4.1.7. $V \widehat{\otimes} A$ *is s-injective. Every comodule V is embedded into the s-injective comodule $V \widehat{\otimes} A$ with embedding $\rho_V : V \to V \widehat{\otimes} A$.*

Proof. Follows from the definition 4.1.5 and corollary 4.1.3. □

Thus the category of $A-$comodules has enough s-injectives.

For every comodule V one can construct an s-injective resolution, called the CoBar resolution. One takes

$$\text{Cob}^{-1}(V) = V,$$

$$\mathrm{Cob}^n(V) = \mathrm{Cob}^{n-1}(V) \widehat{\otimes} A, \; n \geq 0.$$

The differentials $d^n : \mathrm{Cob}^n(V) \to \mathrm{Cob}^{n+1}(V)$ are given by

$$d^{-1} = \rho_V,$$

$$d^n = d^{n-1} \otimes id_A + (-1)\, id_{\mathrm{Cob}^n(V)} \otimes \Delta_A.$$

The contracting homotopy for the CoBar complex is given by

$$s^n : \mathrm{Cob}^{n+1}(V) \to \mathrm{Cob}^n(V),$$

$$s^n = id_V \bar{\otimes} \epsilon_A \otimes id_{A^{\widehat{\otimes}\, n}},$$

which imply that the maps d^n are strong.

It is standard exercise to prove that any two s-injective resolutions are homotopy equivalent.

4.1.3 The Helemskii approach to derived functor

Let A and B be Banach Hopf algebras.

Consider an additive functor $F : \text{Comod-}A \to \text{Comod-}B$. For any object $V \in \text{Comod-}A$ take an s-injective resolution $V \to I^{\cdot}$ and apply the functor F to the complex $0 \to I^{\cdot}$.

Definition 4.1.8. The n-th cohomology of the complex $0 \to F(I^{\cdot})$ is called n-th right derived functor of F and is denoted by $R^n F$.

$R^n F$ takes value in the category of seminormed comodules over B, i.e. $\forall M \in \text{Comod-}A :$ $\rho_M(R^n F(M)) \subseteq R^n F(M) \tilde{\otimes} B$, where $\tilde{\otimes}$ is the completion of the tensor product of seminormed spaces.

$R^n F$ does not depend on the choice of s-injective resolution and thus one can safely take as $I^{\cdot}$ the CoBar resolution.

The following statements are proved similarly to algebraic case.

Proposition 4.1.9. *[8, III.3.2]Let* $0 \to X \to Y \to Z \to 0$ *be a short s-exact sequence in Comod-A. Then we have a long s-exact sequence*

$$\ldots \longleftarrow R^i(X) \longleftarrow R^i(Y) \longleftarrow R^i(Z) \longleftarrow R^{i+1}(X) \longleftarrow \ldots$$

Lemma 4.1.10. *[8, III.3.5]If* F *is left s-exact then* R^0F *is isomorphic to* F*.*

4.1.4 Strict derived functors

For an additive functor F : Comod-$A \to$ Comod-B consider the functors $R_s^n F$: Comod-$A \to$ Comod-B, $R_s^n F(V)$ is defined as the largest Hausdorff quotient of $R^n F(V)$. In general, these functors do not make a short s-exact sequence into a long exact sequence and are not equal to $R^n F$.

However these functors can be defined purely in terms of the categories Comod-$A \to$ Comod-B and, more generally, they are direct analogs of algebraic derived functors in quasi-abelian categories. Also, some identities are naturally expressed with their help, so we find them worthwhile to discuss.

Much of the below discussion follows from works of Kuzminov and Kopylov.

Quasi-abelian categories

Definition 4.1.11. An additive category is called *quasi-abelian* if

1. Every morphism has kernel and cokernel;

2. The pull-back of a cokernel along arbitrary morphism is a cokernel and the push-forward of a kernel along arbitrary morphism is a kernel.

Kernels are also called strict monics and cokernels are also called strict epics.

Definition 4.1.12. A morphism in a quasi-abelian category is called *strict* if it can be factors as a composition of a monic with an epic.

Example 4.1.13. Examples

1. The category of topological abelian groups is quasi-abelian. The monics are injective maps, the epics are the maps with dense image. The strict maps are the one with closed image.

2. The categories of Banach, Frechet, LS- and LF-spaces are quasi-abelian (in view of example 1).

3. The categories of Banach or Frechet modules over a Banach Hopf algebra are quasi-abelian.

4. The category of Banach comodules over a Banach Hopf algebra is quasi-abelian.

5. The category of CT-comodules over CT-Hopf algebra is quasi-abelian.

In examples 3-5 the class of strict maps is the same.

Definition 4.1.14. We call an additive functor between two quasi-abelian categories *strict* if it preserves strict morphisms.

Remark 4.1.15. In [28] such functors are called regular.

A complex in a quasi-abelian category is called *strict* or *s-complex* if all maps are strict.

Definition 4.1.16. Let $V_. = \quad \ldots \quad \longrightarrow \quad V_{n+1} \quad \overset{d_{n+1}}{\longrightarrow} \quad V_n \quad \overset{d_n}{\longrightarrow} \quad V_{n-1} \quad \longrightarrow \quad \ldots$ be a complex in a quasi-abelian category. One has a canonical morphism $\alpha_n : V_{n+1} \rightarrow \text{Ker } d_n$. The n-th cohomology of $V_.$ is $H^n(V_.) = \text{Coker } (\alpha_n)$.

One can prove in the standard way that if two complexes are homotopic then their cohomologies are isomorphic.

Lemma 4.1.17. *[17]Let A, B and C be s-complexes bounded below in a quasi-abelian category and $0 \rightarrow A \overset{\phi}{\rightarrow} B \overset{\psi}{\rightarrow} C \rightarrow 0$ be a short exact sequence such that ϕ^n and ψ^n are strict. Then we have a long s-exact sequence of cohomologies.*

Remark 4.1.18. The class of kernel-cokernel pairs in a quasi-abelian category form an exact structure. Thus any quasi-abelian category is an exact category and strict morphisms are admissible (strict) with respect to that exact structure.

One can define s-injective (s-projective) objects and resolutions. The proof that any two s-injective resolutions are homotopy equivalent is standard.

Strict derived functors

Let C be a quasi-abelian category with enough s-injectives and let F be an additive functor to a quasi-abelian category D. For any $V \in C$ an s-injective resolution $0 \to V \to I^{\cdot}$ is an s-exact sequence in C and $0 \to F(V) \to F(I^{\cdot})$ is a complex in D.

Definition 4.1.19. The functor $R_s^n F : C \to D$ with values $R_s^n F(V) = H^n(F(I^{\cdot}))$ is called the n-th strict right derived functor of F.

Since any two s-injective resolutions are homotopy equivalent, $R_s^n F$ does not depend on the choice of $I^{\cdot}$.

Lemma 4.1.20. *If F is left s-exact, then $R_s^0 F$ is isomorphic to F.*

Lemma 4.1.21. *Let $0 \to A \to B \to C \to 0$ be a short s-exact sequence. If F is strict, then we have a long s-exact sequence*

$$0 \longleftarrow R_s^0 F(A) \longleftarrow R_s^0 F(B) \longleftarrow R_s^0 F(B) \longleftarrow R_s^1 F(A) \ldots$$

Remark 4.1.22. (Derived functor of Schneiders) Since any quasi-abelian category is an exact category, one can consider derived categories and functors in exact sense. This was worked out in detail in [28].

4.1.5 The cohomology of comodules

Consider the fixed-point functor $(-)^A$ on $\mathrm{Comod} - A$,

$$M \in \mathrm{Comod} - A : \; M^A = \{m \in M : \; \rho_M(m) = m \otimes 1\} =$$

$$= \operatorname{Ker}\left(M \underset{id_M \otimes 1_A}{\overset{\rho_M}{\rightrightarrows}} M \widehat{\otimes} A\right).$$

Since M^A is a kernel, it is closed and $M^A \in \text{Comod} - A$.

Proposition 4.1.23. $(-)^A$ *preserves strict monomorphisms (and thus left s-exact).*

Proof. If $f : M \to N$ is a strict monomorphism in $\text{Comod}-A$, then $f^A : M^A \to N^A$ is just a restriction of f to M^A. The image of f^A is embedded in $N^A \cap \operatorname{Im}(f)$, since if $\rho_M(v) = v \otimes 1$, then

$$\rho_N(f(v)) = (f \otimes id_A) \circ \rho_V(v) = f(v) \otimes 1.$$

We want to prove that if $f(v) \in N^A$ then $v \in M^A$. If $f(v) \in N^A$ then we have an identity

$$f(v) \otimes 1 = \rho_N(f(v)) = (f \otimes id_A) \circ \rho_M(v) = (f \otimes id_A) \circ \left(\sum v_{(0)} \otimes v_{(1)}\right) = \sum f\left(v_{(0)}\right) \otimes v_{(1)}$$

in $N \widehat{\otimes} A$. Since f is injective, $f \otimes id_A$ is also injective and this implies

$$\rho_M(v) = v \otimes 1.$$

Thus the image of f^A is equal to $N^A \cap \operatorname{Im}(f)$ and closed, making $(-)^A$ preserve strict monics and thus left s-exact.

The CT case follows from the Banach one, since if $M = \lim\limits_{\to} M_n$ then $M^A = \lim\limits_{\to} M_n^A$ and $\operatorname{Im}\left(f^A\right) = \lim\limits_{\to} \operatorname{Im}\left(f^A|_{M_n^A}\right)$. □

Definition 4.1.24. The functors $H^n(A, -) := R^n(-)^A$ are called n-th cohomology functors.

The functors $H_s^n(A, -) := R_s^n(-)^A$ are called n-th strict cohomology functors.

Since the fixed point functor is left s-exact, for any $M \in \text{Comod}-A$ we have $H^0(A, M) = H_s^0(A, M) = M$.

4.2 The Grothendieck Spectral Sequence

4.2.1 Fully s-injective resolutions

In case of abelian category the key element in proving the convergence of the Grothendieck spectral sequence is the existence of fully injective resolution. Similar results holds in quasi-abelian case.

Definition 4.2.1. Consider an s-complex C in a quasi-abelian category. An s-injective resolution of C is an s-exact sequence of complexes

$$0 \quad \to C \quad \to \quad I^0 \quad \to \quad I^1 \quad \to \quad I^2 \quad \to \quad \dots \tag{4.2.1}$$

such that complexes $I^n = \quad \dots \quad \to \quad I^{n,i} \quad \overset{d^{n,i}}{\to} \quad I^{n,i+1} \quad \to \quad \dots$ consists of s-injective objects with strict differentials. Thus

$$0 \to C^n \to I^{0,n} \to I^{1,n} \to \dots$$

is an s-injective resolution of C^n.

Define complexes

$$0 \to Z^i\left(C\right) \to Z^{0,i} \to Z^{1,i} \to \dots$$

$$0 \to B^i\left(C\right) \to B^{0,i} \to B^{1,i} \to \dots$$

$$0 \to H^i\left(C\right) \to H^{0,i} \to H^{1,i} \to \dots$$

with $Z^{j,i} = \text{Ker }\left(d^{j,i}\right)$, $B^{ji} = \text{Im }\left(d^{j,i-1}\right) = \text{Ker }\left(\text{Coker }\left(d^{j,i-1}\right)\right)$ and $H^{j,i} = H^i\left(I^{\cdot,j}\right)$.

Definition 4.2.2. The resolution 4.2.1 is called fully s-injective, if the above complexes for $Z^i\left(C\right)$, $B^i\left(C\right)$ and $H^i\left(C\right)$ are s-injective resolutions.

Lemma 4.2.3. *For any s-complex in a quasi-abelian category with enough s-injectives there exists a fully s-injective resolution.*

Proof. The proof is same as in algebraic case (see [18, XX.9.5]) with the application of the Horseshoe lemma in exact categories ([4, 12.8]). □

4.2.2 Spectral sequences

Proposition 4.2.4. *Let F : Comod-$A \to$ Comod-B and G : Comod-$B \to$ Comod-C be an additive functors between categories of Banach comodules over A, B and C, such that*

- *A and B have enough s-injectives and,*
- *F is strict and maps injective objects into G-acyclic objects.*

Then for each $M \in A$ there exists a spectral sequence $E_r^{p,q}(A)$ in the quasi-abelian category of seminormed comodules over C, such that

$$E_2^{p,q}(A) = R^q G(R^p F(A)) \Rightarrow R^{p+q}(G \circ F)(A).$$

If G is also strict then $E_2^{p,q}(A)$ is in Comod-C and thus there is a low-degree terms exact sequence in Comod-C

$$0 \to \left(R^1 G\right)(F(A)) \to R^1(GF)(A) \to G\left(R^1 F(A)\right) \to R^2 G(F(A)) \to R^2(GF)(A).$$

Proof. The proof repeats the classical one. If G is strict, it will map a fully s-injective resolution of the s-complex

$$0 \to R^{\cdot} F(A) \to I^{\cdot}$$

into a double s-complex. Since the elements of the first page of spectral sequence, corresponding to the row filtration of the total complex, are just elements of $G(I^{\cdot})$ with horizontal differentials (which are strict by full s-injectivity of $I^{\cdot}$ and strictness of G), the elements of the second page belong to Comod-C. □

Corollary 4.2.5. *Under the assumptions of proposition 4.2.4,*

1. *if F is s-exact, then $R^n(G \circ F) \cong R^n G \circ F$;*
2. *if G is s-exact, then $R^n(G \circ F) \cong G \circ R^n F$.*

4.2.3 Applications

The Generalized Tensor Identity

Let $\pi : A \to B$ be a continuous morphism of Banach K-Hopf algebras.

Denote by $R^n_s\left(\widehat{\boxtimes}_{B}{}_{\pi} A\right)$ the n-th strict right derived functor of the induction functor $\left(-\widehat{\boxtimes}_{B}{}_{\pi} A\right)$.

Proposition 4.2.6. *Let W be a Banach comodule over B and V be a Banach comodule over A.*

Then

$$R^n_s\left(\widehat{\boxtimes}_{B}{}_{\pi} A\right)\left(V_\pi \widehat{\otimes} W\right) \cong V \widehat{\otimes} R^n_s\left(\widehat{\boxtimes}_{B}{}_{\pi} A\right)(W). \tag{4.2.2}$$

Remark 4.2.7. Since $V \widehat{\otimes} R^n_s\left(\widehat{\boxtimes}_{B}{}_{\pi} A\right)(W) \cong V \widehat{\otimes} R^n\left(\widehat{\boxtimes}_{B}{}_{\pi} A\right)(W)$ one also has

$$R^n_s\left(\widehat{\boxtimes}_{B}{}_{\pi} A\right)\left(V_\pi \widehat{\otimes} W\right) \cong V \widehat{\otimes} R^n\left(\widehat{\boxtimes}_{B}{}_{\pi} A\right)(W).$$

Proof. The tensor identity 2.2.22 give us an isomorphism of functors

$$\left(-\widehat{\boxtimes}_{B}{}_{\pi} A\right) \circ \left(V_\pi \widehat{\otimes} -\right) \cong \left(V \widehat{\otimes} -\right) \circ \left(-\widehat{\boxtimes}_{B}{}_{\pi} A\right)$$

Since the functor $\left(V_\pi \widehat{\otimes} -\right)$ - the tensor product over K is strict s-exact, the left hand side is obtained from 4.2.5.1 by taking Hausdorff completion.

The functor $\left(-\widehat{\boxtimes}_{B}{}_{\pi} A\right)$ is not strict, so one cannot simply apply 4.2.5.2. For any W consider CoBar resolution of $V \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right)$:

$$0 \to V \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right) \to V \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right) \widehat{\otimes} A \to V \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right) \widehat{\otimes} A \widehat{\otimes} A \to \ldots.$$

By Corollary 2.2.23 one can replace it with resolution

$$0 \to V \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right) \to V_{tr} \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right)_{tr} \widehat{\otimes} A \to V_{tr} \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right)_{tr} \widehat{\otimes} A \widehat{\otimes} A \to \ldots. \tag{4.2.3}$$

The comodule $V \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right)$ is a subcomodule of a comodule $V \widehat{\otimes} W_{tr} \widehat{\otimes} A \cong V_{tr} \widehat{\otimes} W_{tr} \widehat{\otimes} A$ and thus

$$V \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right) \cong V_{tr} \widehat{\otimes} \left(W\widehat{\boxtimes}_{B}{}_{\pi} A\right).$$

Putting these observations together, we see that the complex 4.2.3 is isomorphic to the complex

$$0 \to V_{tr} \widehat{\otimes} \left(W \widehat{\underset{B}{\boxtimes}}_{\pi} A \right) \to V_{tr} \widehat{\otimes} \left(W \widehat{\underset{B}{\boxtimes}}_{\pi} A \right) \widehat{\otimes} A \to V_{tr} \widehat{\otimes} \left(W \widehat{\underset{B}{\boxtimes}}_{\pi} A \right) \widehat{\otimes} A \widehat{\otimes} A \to \dots$$

and we have an isomorphism

$$H_s^n \left(V \widehat{\otimes} \left(W \widehat{\underset{B}{\boxtimes}}_{\pi} A \right) \right) \cong V \widehat{\otimes} H_s^n \left(W \widehat{\underset{B}{\boxtimes}}_{\pi} A \right)$$

which gives the right hand side of 4.2.2. □

Example 4.2.8. It is known that for locally analytic or continuous representations of compact p-adic groups an induction functor is exact. Thus all of its derived functors are zero and the generalized tensor identity is trivial. Here we give an example where this is not the case.

Consider a Banach Hopf algebra of affinoid functions

$$A = C^{an} \left(SL \left(2, \mathbb{Z}_p \right), K \right) = K \left\{ a, b, c, d \right\} / \overline{\left(ad - bc - 1 \right)}$$

on $SL \left(2, \mathbb{Z}_p \right)$ and a Hopf algebra of affinoid functions on its Borel subgroup $C = C^{an} \left(B, K \right) = C^{an} \left(SL \left(2, \mathbb{Z}_p \right), K \right) / \overline{(c)}$. We refer to the functor $A \widehat{\underset{C}{\boxtimes}} -$ as affinoid induction.

Any analytic character λ of the maximal torus of $SL \left(2, \mathbb{Z}_p \right)$ gives a 1-dimensional comodule over C, which we also denote by λ. It is easy to see that all these characters are algebraic (also see [3]). Thus we know that if λ is dominant then $A \widehat{\underset{C}{\boxtimes}} \lambda$ is non-zero, since it contains an algebraic induced comodule (actually, this inclusion is an equality). Also it is clear that, similarly to the case of algebraic groups (see [10, I.5]), $A \widehat{\underset{C}{\boxtimes}} \lambda$ is equal to the 0-cohomology (global sections) of the sheaf $L \left(\lambda \right)$ on $\mathbb{P}_K^1$ (rigid-analytic projective line), associated to λ. Since $\mathbb{P}_K^1$ is a proper rigid-analytic space, we can apply Serre duality [2]. Thus, similarly to the case of algebraic groups (see details in [10, II.4,5]), $A \widehat{\underset{C}{\boxtimes}} \lambda \cong R^1 \left(A \widehat{\underset{C}{\boxtimes}} \right) \left(-\lambda \right)$ and we see that in general affinoid induction is not exact.

Hochshild-Serre for bounded cohomology

A celebrated theorem of Noskov in the theory of bounded cohomology immediately follows from proposition 4.2.4.

Let G be a (discrete) group.

Theorem. *[20]Let N be a normal subgroup of G and V is a bounded G-module such that $H_b^*(N;V)$ is Hausdorff. Then $H_b^*(N;V)$ is a bounded G/N-module and there exists a spectral sequence (E_r) such that*

$$E_2^{pq} = H_b^p(G/N; H_b^q(N;V)) \Rightarrow H_b^{p+q}(G;V).$$

Proof. The condition that $H_b^*(N;V)$ is Hausdorff means that differentials in CoBar resolution of V_N are strict. □

The Hochshild-Serre spectral sequence for comodules

In the algebraic context the Hochshild-Serre spectral sequence was established in [22]. We give a simplified treatment of the theory in topological context. We need to make some assumptions on our Hopf algebras, which are satisfied, for example, in the case of finite groups.

Let A and B be a topological Hopf algebras and $\pi : A \twoheadrightarrow B$ be a normal surjective map. Consider $\rho_l : A \to A \widehat{\otimes} A$, the left adjoint coaction on A. By definition of ρ_l, the diagram

$$\begin{array}{ccccccc} & A & \xrightarrow{\rho_l} & A & \widehat{\otimes} & A & \\ \pi & \downarrow & & & \downarrow & & \pi \otimes id \\ & B & \longrightarrow & B & \widehat{\otimes} & A & \\ & & \rho_B & & & & \end{array}$$

defines an $A-$comodule structure on B.

Since $(A_\pi)^B$ is a left $B-$homogeneous, $(A_\pi)^B \subseteq A \widehat{\otimes} (A_\pi)^B$.

Lemma 4.2.9. *The normality of π implies*

1. *$(A_\pi)^B$ is a subcoalgebra of A.*

2. *For any* $(M, \rho_M) \in Comod{-}A$ $\rho_M (M_\pi)^B \subseteq (M_\pi)^B \widehat{\otimes} (A_\pi)^B$, $(M_\pi)^B$ *is a* $(A_\pi)^B$ *−comodule.*

3. *We have* $\left((M_\pi)^B\right)^{(A_\pi)^B} = M^A$.

Proof. 1 and 2 - [22, 3.4, 3.6, 3.8]. For 3, it is clear that $M^A \subseteq \left((M_\pi)^B\right)^{(A_\pi)^B}$ and also, since the action on M_π^B is ρ_M, the inverse inclusion holds. □

Since all maps involved in [22, 3.4, 3.6, 3.8] are continuous, one has

Lemma 4.2.10. *If* $H^i (B, M_\pi)$ *are Hausdorff (Banach spaces), then* $H^i (B, M_\pi)$ *are* $(A_\pi)^B$ *−comodules.*

Remark 4.2.11. If $H^i (B, M_\pi)$ are not Hausdorff, then the coaction from preceding lemma maps $H^i (B, M_\pi)$ into $H^i (B, M_\pi) \tilde{\otimes}_{K,\pi} (A_\pi)^B$ see section 1.4. The Hausdorff completion of this space is $H^i (B, M_\pi) \widehat{\otimes} (A_\pi)^B$.

Proposition 4.2.12. *Under the assumptions of this section, let M be such that all* $H^i (B, M_\pi)$ *are Hausdorff and* A_π *is an s-injective B-comodule. Then there exists a spectral sequence with*

$$E_2^{p,q} = H^p \left(A_\pi^B, H^q (B, M_\pi)\right) \Rightarrow H^{p+q} (A, M) .$$

Proof. Lemma 4.2.9.3 means that $(-)^{A_\pi^B} \circ (-)^B = (-)^A$. By lemma 4.2.10 $(-)^B$ take values in Comod$-A_\pi^B$. Thus we have a functor $((-)_\pi)^B : Comod - A \to Comod - A_\pi^B$. Since A_π is injective $B-$comodule, for any $M \in Comod - A$, $(-)_\pi$ sends the CoBar resolution of M into an s-injective resolution of M_π, thus we have $R^n ((-)_\pi)^B = R^n (-)^B \circ (-)_\pi$. Since both $(-)^B$ and $(-)_\pi$ preserve s-injectives and under $(-)^B \circ (-)_\pi$ the $A-$comodule structure is mapped to A_π^B structure, $(-)^B \circ (-)_\pi$ maps s-injective to s-injectives. Proposition 4.2.4 gives the spectral sequence. □

The following example shows that assumption “A_π is an s-injective B-comodule” is natural.

Example 4.2.13. Let G be a p-adic compact group and H be any locally analytic subgroup of G. It is known that

$$C^{la}(G,K) \cong C^{la}(G/H,K)\widehat{\otimes} C^{la}(H,K).$$

Thus, if $\pi : C^{la}(G,K) \to C^{la}(H,K)$ then $C^{la}(G,K)_{\pi}$ is injective $C^{la}(H,K)$ −comodule.

Bibliography

[1] Y. Amice. Interpolation p-adique. *Bulletin de la S. M. F.*, pages 117–180, 1964.

[2] P. Beyer. On serre duality for coherent sheaves on rigid-analytic spaces. *Manuscripta Math.*, 93:219–245, 1997.

[3] S. Bosch. k-affinoide tori. *Mathematische Annalen*, 192:1–16, 1971.

[4] T. Buhler. Exact categories. *Expo. Math.*, 28:1–69, 2010.

[5] B. Diarra. On reducibility of ultrametric almost periodic linear representations. *Glasgow Math. J.*, 37:83–98, 1995.

[6] M. Emerton. Locally analytic vectors in representations of locally analytic groups. http://arxiv.org/abs/math/0405137, 2004.

[7] H. Frommer. The locally analytic principal series representation of split reductive groups. Preprintreihe SFB 478 available at http://wwwmath1.uni-muenster.de/sfb/about/publ/. Munster 2003.

[8] A.Ya. Helemskii. *The Homology of Banach and Topological Algebras.* Springer, 1989.

[9] A.Ya. Helemskii. *Banach and Locally Convex Algebras.* Oxford Science Publications, Calderon Press, Oxford, 1993.

[10] J.C. Jantzen. *Representations of algebraic groups.* American Mathematical Society, 2 edition, 2003.

[11] C. Kassel. *Quantum groups.* Graduate Texts in Mathematics. Springer, 1994.

[12] N. De Grande-De Kimpe, J. Kakol, C. Perez-Garcia, and W.H. Schikhof. p-adic locally convex inductive limits. *in Lecture Notes in Pure and Applied Mathematics*, 192:129–222, 1997.

[13] N. De Grande-De Kimpe, J. Kakol, C. Perez-Garcia, and W.H. Schikhof. The p-adic banach-dieudonne theorem and semicompact inductive limits. *in Lecture Notes in Pure and Applied Mathematics*, 207:295–307, 1999.

[14] A. Klymuk and K. Schmudgen. *Quantum groups and their representations*. Theoretical and Mathematical Physics. Springer, 1998.

[15] J. Kohlhaase. Invariant distributions on p-adic analytic groups. *Duke Math. J.*, 137:19–62, 2007.

[16] J. Kohlhaase. The cohomology of locally analytic representations. http://wwwmath.uni-muenster.de/sfb/about/publ/heft491.pdf, 2008.

[17] Ya.A. Kopylov and V.I. Kuz'minov. Exactness of the cohomology sequence of the short exact sequence of complexes in a semiabelian category. *Siberian Advances in Mathematics*, 13(3):72–80, 2003.

[18] S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer, 3 edition, 2002.

[19] Z. Lin. Rational representations of hopf algebras. In *Proceedings of Symposia in Pure Mathematics, part II*, volume 56, pages 81–91, 1994.

[20] G. Noskov. The hochschild-serre spectral sequence for bounded cohomology. In *Proceedings of the International Conference on Algebra. Dedicated to the Memory of A.I. Mal'cev. Part 1*, volume 131 of *Contemporary Mathematics*, pages 613–629, 1992.

[21] C. Perez-Garcia and W.H. Schikhof. Tensor product and p-adic vector valued continuous functions. In *p-Adic Functional Analysis. Proceedings of the 2nd International Conference on p-adic Functional Analysis*, pages 111–120, 1994.

[22] W.R.F. Santon. Cohomology of comodules. *Pacific J. Math.*, 109:179–213, 1983.

[23] W.H. Schikhof. Banach spaces over nonarchimedean valued fields. Report N9937, Department of Mathematics, University of NIJMEGEN, 1999.

[24] P. Schneider. *Nonarchimedean functional analysis*. Springer Monographs in Mathematics. Springer, 2002.

[25] P. Schneider and J. Teitelbaum. Banach space representations and iwasawa theory. *Isr. J. Math.*, 127:359–380, 2002.

[26] P. Schneider and J. Teitelbaum. Locally analytic distributions and p-adic representation theory with applications to gl_2. *J. Am. Math. Soc.*, 15:443–468, 2002.

[27] P. Schneider and J. Teitelbaum. Algebras of p-adic distributions and admissible representations. *Inventiones Mathematicae*, 153(1):145–196, 2003.

[28] P. Schneiders. Quasi-abelian categories and sheaves. `http://www.analg.ulg.ac.be/jps/rec/qacs.pdf`.

[29] M. Sweedler. *Hopf algebras*. Addison-Wesley Pub Co, 1969.

[30] J. Taylor. Homology and cohomology for topological algebras. *Advances in Math.*, 9:137–182, 1972.

[31] A.C.M. van Rooij. *Non-Archimedean Functional Analysis*. Marcel Dekker, 1973.

[32] L. Di Vizio. Arithmetic theory of q-difference equations. *Inventiones Mathematicae*, 150:517–578, 2002.