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Classical groups, integrals and Virasoro constraints

Da Xu
University of Iowa

CLASSICAL GROUPS, INTEGRALS AND VIRASORO CONSTRAINTS

by

Da Xu

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2010

Thesis Supervisors: Professor Lihe Wang
Professor Palle Jorgensen

ABSTRACT

First, we consider the group integrals where integrands are the monomials of matrix elements of irreducible representations of classical groups. These group integrals are invariants under the group action. Based on analysis on Young tableaux, we investigate some related duality theorems and compute the asymptotics of the group integrals for fixed signatures, as the rank of the classical groups go to infinity. We also obtain the Viraosoro constraints for some partition functions, which are power series of the group integrals. Second, we show that the proof of Witten's conjecture can be simplified by using the fermion-boson correspondence, i.e., the KdV hierarchy and Virasoro constraints of the partition function in Witten's conjecture can be achieved naturally. Third, we consider the partition function involving the invariants that are intersection numbers of the moduli spaces of holomorphic maps in nonlinear sigma model. We compute the commutator of the representation of Virasoro algebra and give a fat graph(ribbon graph) interpretation for each term in the differential operators.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2010 graduation.

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To my parents

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ABSTRACT

First, we consider the group integrals where integrands are the monomials of matrix elements of irreducible representations of classical groups. These group integrals are invariants under the group action. Based on analysis on Young tableaux, we investigate some related duality theorems and compute the asymptotics of the group integrals for fixed signatures, as the rank of the classical groups go to infinity. We also obtain the Viraosoro constraints for some partition functions, which are power series of the group integrals. Second, we show that the proof of Witten's conjecture can be simplified by using the fermion-boson correspondence, i.e., the KdV hierarchy and Virasoro constraints of the partition function in Witten's conjecture can be achieved naturally. Third, we consider the partition function involving the invariants that are intersection numbers of the moduli spaces of holomorphic maps in nonlinear sigma model. We compute the commutator of the representation of Virasoro algebra and give a fat graph(ribbon graph) interpretation for each term in the differential operators.

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CHAPTER 1
INTRODUCTION

One of the motivation of the present thesis is the following group integral

$$\int_G \rho^{\lambda^{(1)}}_{i_1 j_1} \rho^{\lambda^{(2)}}_{i_2 j_2} \cdots \rho^{\lambda^{(k)}}_{i_k j_k} \bar{\rho}^{\lambda'^{(1)}}_{i'_1 j'_1} \bar{\rho}^{\lambda'^{(2)}}_{i'_2 j'_2} \cdots \bar{\rho}^{\lambda'^{(k')}}_{i'_{k'} j'_{k'}} du, \tag{1.1}$$

where G is a compact classical Lie group and du denotes its Haar measure, and $\rho^{\lambda^{(i)}}$, $\bar{\rho}^{\lambda'^{(i)}}$ are single-valued irreducible representations and dual representations of G with signatures $\lambda^{(i)}$ and $\lambda'^{(i)}$ respectively. We shall focus on some related duality theorems first and then investigate the asymptotics of this integral as the rank of the group G goes to infinity and the signatures of the representations are fixed. These group integrals as random matrix integrals have important applications in many fields of physics.

1.1 In quantum information theory

Let us review some of the physics origins. In this section, let us review some standard definitions and facts that are used in this subsection and the whole paper. We adopt the following standard conventions of quantum mechanics: The states of an n-level quantum system (and we include the case infinity) are represented by an n-dimensional complex Hilbert space $\mathcal{H}$. In this familiar representation, the states of a composite of two systems A and B say, the first one A n-level, and the second B m-level, is then represented by the tensor product of the respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. As a result, the composite system AB is an nm-level system. This also makes sense in the infinite case where we then use standard geometry for Hilbert space, and suitable choices of orthonormal bases (ONBs). For a fixed system with Hilbert space $\mathcal{H}$, the corresponding pure quantum states are vectors in $\mathcal{H}$ of norm one, or rather equivalence classes of such vectors: Equivalent vectors v and v' in $\mathcal{H}$ yield the same rank-one projection operator P, i.e., the projection of $\mathcal{H}$ onto the one-dimensional subspace in $\mathcal{H}$ spanned by v. In Dirac's terminology, we write $P = |v\rangle\langle v|$. As per Dirac, a bra-ket is an inner product in an ambient Hilbert space $\mathcal{H}$, while a ket-bra is a rank-one operator in $\mathcal{H}$. So a bra-ket is a complex scalar, while a ket-bra is an operator. We will work with the group U(n) of d by d complex unitary matrices; the matrix entries of a matrix U will be denoted by double subscripts $U_{ij} = \langle i|U|j\rangle$, and this will refer to the standard ONB in $\mathbb{C}^n$. Normalized Haar measure will be denoted "du". If $n = 2$, i.e. $\mathcal{H} \equiv \mathbb{C}^2$, the familiar Pauli matrices offer a conventional realization of pure and mixed states. Set

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

Let ρ be a state, and consider the point

$$(\rho(\sigma_1), \rho(\sigma_2), \rho(\sigma_3)) \in \mathbb{R}^3. \tag{1.3}$$

A computation shows that

$$\rho(\sigma_1)^2 + \rho(\sigma_2)^2 + \rho(\sigma_3)^2 \leq 1, \tag{1.4}$$

and the equality holds in (3.9) if and only if the state ρ is pure. Here the pure state are represented by the points on the two-sphere $S^2 \in \mathbb{R}^3$. Identify a state ρ with

$$\rho(A) = \mathrm{Tr}(\rho A), \tag{1.5}$$

for all complex 2×2 matrices A. The state is pure if and only if $\exists v \in \mathcal{H}$, $\|v\| = 1$, such that $\rho = |v\rangle\langle v|$. Further, we recall that quantum observables are selfadjoint operators in H; and states selfadjoint positive semi-definite trace-class operators ρ on H whose trace is one, called density matrices. We can use the terminology "density matrix" even if $\mathcal{H}$ is infinite-dimensional. By the Spectral Theorem, a density matrix ρ then corresponds to a pure state if and only if it is a rank-one projection. In general, a state ρ may be mixed, in which case it is a convex combination of pure states, allowing for infinite convex combinations. When this standard formalism is applied to the tensor product (AB) of two quantum systems A and B, then states ρ in the tensor product Hilbert space have associated marginal states. They are obtained by an application of a partial trace computation: When we trace over an ONB for the second system B, i.e., a partial trace-summation applied to ρ, we obtain an associated marginal state ρ_A , where ρ_A is now a density matrix in the Hilbert space $\mathcal{H}_A$. And analogously, a partial trace summation using an ONB in $\mathcal{H}_A$ yields ρ_B , the second marginal state. Motivated by recent applications, in this paper we are concerned with the computation of von Neumann entropy of marginal states derived from pure states in composite systems (AB), formed from tensor factors A and B, A n-level, and B m-level. First, Dirac terminology is defined by

Definition 1.1. $R = |v\rangle\langle w|$ is a operator that satisfies $Rx = |v\rangle\langle w|x = \langle w|x\rangle v$, with inner product $\langle w|x\rangle \in \mathbb{C}$.

Then from the definition, $R^* = |v\rangle\langle w|^* = |w\rangle\langle v|$, and $R^2 = \langle w|v\rangle|v\rangle\langle w| = \langle w|v\rangle R$.

Definition 1.2. Let $\mathcal{H}$ be a complex Hilbert space and mapping the complex Hilbert space $\mathcal{H}_A$ on $\mathcal{H}_B$ such that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho : \mathcal{H} \to \mathcal{H}$ be a trace class operator. Pick $\{e_i\}$ ONB in $\mathcal{H}_A$, and $\{f_j\}$ ONB in $\mathcal{H}_B$. If $x_1, y_1 \in \mathcal{H}_A$, set

$$\langle x_1|\rho_A y_1\rangle_{\mathcal{H}_A} = \sum_j \langle x_1 \otimes f_j|\rho(y_1 \otimes f_j)\rangle_{\mathcal{H}}. \tag{1.6}$$

If $x_2, y_2 \in \mathcal{H}_B$, set

$$\langle x_2|\rho_B y_2\rangle_{\mathcal{H}_B} = \sum_i \langle e_i \otimes x_2|\rho(e_i \otimes y_2)\rangle_{\mathcal{H}}. \tag{1.7}$$

Definition 1.3. Let $\mathcal{H}$ be a complex Hilbert space, an set $\mathcal{T}_1(\mathcal{H})$ = all density matrices ,i.e., all $\rho : \mathcal{H} \to \mathcal{H}$ linear, ρ is a trace class, $\langle x|\rho x\rangle_{\mathcal{H}} \geq 0, \forall x \in \mathcal{H}$,and $\mathrm{Tr}(\rho) = 1$, then by the spectral theorem, the operator $\rho\ln(\rho)$ is well defined. The value $S(\rho) = -\mathrm{Tr}(\rho\ln\rho)$ is called the von Neumann entropy.

We will compute $S_A(\rho_A)$, and $S_B(\rho_B)$ with S_ρ. Further note that if $\rho \in \mathcal{T}_1(\mathcal{H})$, there are projection $P_k = |v_k\rangle\langle v_k|$, $\lambda_k \in \mathcal{R}, \lambda_k \geq 0, \sum_k \lambda_k = 1$, such that

$$\rho = \sum_k \lambda_k P_k, \tag{1.8}$$

and

$$S(\rho) = -\sum_k \lambda_k \ln \lambda_k. \tag{1.9}$$

It follows that if ρ is a pure state , then $S(\rho) = 0$.

Lemma 1.1. *If $\rho \in \mathcal{T}_1(\mathcal{H})$, then the linear functional $\phi = \phi_\rho$ defined by*

$$\phi(A) = \mathrm{Tr}(\rho A), \tag{1.10}$$

for $A \in B(\mathcal{H})$ = all bounded operators from $\mathcal{H}$ to $\mathcal{H}$, satisfies

$$\phi(A^*A) \geq 0, \tag{1.11}$$

and

$$\phi(I) = 1, \tag{1.12}$$

with the identity operator in $\mathcal{H}$. If $\rho = P_v = |v\rangle\langle v|$, for $v \in \mathcal{H}$, then

$$\mathrm{Tr}(\rho A) = \langle v|Av\rangle_{\mathcal{H}}, \tag{1.13}$$

for all $A \in B(\mathcal{H})$.

Definition 1.4. Let $\rho \in \mathcal{T}_1(\mathcal{H})$. By the spectral theorem, there is an ONB $\{e_i\}$ in $\mathcal{H}$ and an eigenvalue list $\lambda_1 \geq \lambda_2 \geq \cdots$, $\sum \lambda_i = 1$, such that

$$\rho = \sum_{i=1}^{\infty} \lambda_i |e_i\rangle\langle e_i|, \tag{1.14}$$

where $P_i = |e_i\rangle\langle e_i|$ is the projection onto $\mathbb{C}e_i$, and $\rho e_i = \lambda_i e_i$.

If *dim* $(\mathcal{H}) = \infty$, then $\lambda_i \to 0$.

Proof. An application of the spectral theorem. □

The following lemma is due to Schmit:

Lemma 1.2. *Let $\forall v \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, then there exist ONBs $\{e_i\}$ of $\mathcal{H}_A$ and $\{f_i\}$ of $\mathfrak{B}_A$, and $\xi_k \geq 0$, $\xi_k \geq 0$ such that $v = \sum_k \xi_k e_k \otimes f_k$.*

Proof. We refer the proof to page 150 in [26]. □

Remark. The conclusion from the Lemma applies to higher rank tensor products as well. It follows for example induction: Given an n-fold Hilbert tensor product H formed from Hilbert spaces $H_i, i = 1, 2, ...n$ as tensor factors; consider an arbitrary vector v in $\mathcal{H}$ (the tensor product Hilbert space). Then there are n ONBs, one in each Hilbert space $\mathcal{H}_i$, the bases depending on the given vector v, and there are numbers c with corresponding indices such that v has a representation like the case $n = 2$, but now with each tensor factor in the sum being a n-fold tensor of vectors from the respective ONBs. And the convergence with coefficients c holds in the same sense.

Lemma 1.3. *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a tensor product of Hilbert spaces as described above. Let $\rho_A \in \mathcal{T}_1(\mathcal{H}_A)$ and $\rho_B \in \mathcal{T}_1(\mathcal{H}_B)$. The the following two conditions are equivalent:*

(i)The two states ρ_A and ρ_B have the same eigenvalue list.

(ii)$\exists\rho \in \mathcal{T}_1(\mathcal{H})$ pure ,such that $\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(\rho)$ and $\rho_B = \mathrm{Tr}_{\mathcal{H}_A}(\rho)$.

Proof. From (ii) to (i). Suppose a pure state exists $\rho|v\rangle\langle v| \in \mathcal{T}_1(\mathcal{H})$ satisfying the two inequivalent conditions with respect to ρ_A and ρ_B. Then $v \in \mathcal{H}$ must satisfy $\|v\| = 1$. Since $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, by schmidt's theorem there are ONBs $\{e_i\}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, and the there are

$$\xi_k \in \mathbb{R},\ \xi_k \geq 0, \text{ such that } v = \sum_k \xi_k e_k \otimes f_k. \tag{1.15}$$

We will now compute the marginal density matrices ρ_A and ρ_B with the use of the two ONBs in (3.9). From the Schmidt decomposition. Instead if $x, y \in \mathcal{H}_A$, then

$$\begin{aligned}
\langle x|\rho_A y\rangle &= \sum_j \langle x \otimes f_j|\rho(y \otimes f_j)\rangle_{\mathcal{H}} = \sum_i \langle x \otimes f_j|v\rangle\langle v|(y \otimes f_j)\rangle_{\mathcal{H}} \\
&= \sum_{i,k} \xi_i\xi_k \langle x \otimes f_j|e_i \otimes f_i\rangle\langle e_k \otimes f_k|y \otimes f_j\rangle_{\mathcal{H}} \\
&= \sum_{j,k,k} \xi_i\xi_k \langle x \otimes e_i\rangle_{\mathcal{H}_A}\langle f_j|f_i|f_j\rangle_{\mathcal{H}_B}\langle e_k|y\rangle_{\mathcal{H}_A}\langle f_k|f_j\rangle_{\mathcal{H}_B} \\
&= \sum_{i,k} \xi_i\xi_k \sum_j \delta_{ji}\delta_{kj}\langle x|e_i\rangle_{\mathcal{H}_A}\langle e_k|y\rangle_{\mathcal{H}_A} = \sum_k \xi_k^2\langle x|e_k\rangle\langle e_k|y\rangle \\
&= \langle x|\sum_k \xi_k^2 P_k|y\rangle_{\mathcal{H}_A},
\end{aligned} \tag{1.16}$$

with $P_k = |e_k\rangle\langle e_k|$. It follows that $\{\xi_k^2\}$ is the eigenvalue list for the state

$$\rho_A = \sum_k \xi_k^2 P_k. \tag{1.17}$$

In deed we can arrange the order $\xi_1^2 \geq \xi_2^2 \geq \cdots$. Since $\|v\|_{\mathcal{H}}^2 = 1$, it follows from (3.9) that $\sum_k \xi_k^2 = 1$. The same argument shows that

$$\rho_B = \sum_k \xi_k^2|f_k\rangle\langle f_k|. \tag{1.18}$$

So the same sequence $\{\xi_k^2\}$ form the eigenvalue list of the second marginal state ρ_B.

Now let's prove $(i) \Rightarrow (ii)$.

Suppose two states $\rho_A \in \mathcal{T}_1(\mathcal{H}_A)$, $\rho_B \in \mathcal{T}_1(\mathcal{H}_B)$ have the same eigenvalue list λ_k, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Then there are ONBs $\{e_i\}$ in $\mathcal{H}_A$ with $\rho_A e_i = \lambda_i e_i$; and $\{f_i\}$ in $\mathcal{H}_B$ with $\rho_B f_i = \lambda_i f_i$. Set $v = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$. Then $v \in \mathcal{H}$ satisfies $\|v\|_{\mathcal{H}}^2 = 1$; and

$$\mathrm{Tr}_{\mathcal{H}_B} |v\rangle\langle v| = \rho_A,$$

$$\mathrm{Tr}_{\mathcal{H}_A} |v\rangle\langle v| = \rho_B.$$

Moreover these states satisfy (2.18) and (1.18) with $|\xi_i|^2 = \lambda_i$.

$\square$

Don N. Page in [40] considered a system AB with Hilbert dimension mn. The entropy of a pure state of the whole system is zero. The author reobtained an approximated formula by of The entropy of the subsystem A which was derived by Lubkin [33]:

$$S_{mn} \simeq \ln m - \frac{m}{2n}, \tag{1.19}$$

and also conjectured

$$S_{mn} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}. \tag{1.20}$$

This conjecture was proved later by S.K.Foong and S.Kanno [13]. Page's work is also used in the analysis of the information loss in black hole radiations. It turned out that the information comes out extremely slowly. Later there have been many work on random pure states of entanglement [17][43][1][4][33][6][20][45][38]. Therefore it is interesting to compute the average entropy of a random pure state with respect to various symmetries. The calculation of the entropy will rely on the calculation of group integrals of unitary group.

1.2 In quantum mechanics and gauge theory

Weingarten mentioned the asymptotics of the group integrals are connected to the g^{-2} expansion of the Green's functions of Wilson's formulation of gauge theory on a lattice [46]. His results showed that how fast the m-string vertices fall in Feynman diagram. Since gauge fields of various representations also involve in the interactions, so the integral (1.1) is physically important.

Also, in quantum mechanics [11], when we consider the orbital momentum and spin angular momentum of electrons, we consider the average of the product matrix elements of irreducible representations of SU(2), which is a more general integral integral:

$$\int_{\mathrm{SU}(2)} D^{J_1}_{i_1 j_1} D^{J_2}_{i_2 j_2} \cdots D^{J_k}_{i_k j_k} D^{J_1^{'}*}_{i_1^{'} j_1^{'}} D^{J_2^{'}*}_{i_2^{'} j_2^{'}} \cdots D^{J^{'}_{k^{'}}*}_{i^{'}_{k^{'}} j^{'}_{k^{'}}}\, du, \tag{1.21}$$

where du is the Haar measure of SU(2). Recall Wigner formula [11]

$$D^J_{m^{'} m}(\alpha, \beta, \gamma) = e^{-\sqrt{-1}(\alpha m^{'} + \gamma m)} d^J_{m^{'} m}(\beta), \tag{1.22}$$

where

$$\begin{aligned} & d^J_{m^{'} m}(\beta) \\ =& (-1)^{m^{'}-m} \sum_{\mu} (-1)^{\mu} \frac{\sqrt{(j+m)!(j-m)!(j+m^{'})!(j-m^{'})!}}{\mu!(j+m-\mu)!(j-\mu-m^{'})!(m^{'}+\mu-m)!} \\ & (\cos(\beta/2))^{2j+m-m^{'}-2\mu} (\sin(\beta/2))^{2\mu+m^{'}-m}, \end{aligned} \tag{1.23}$$

and μ goes over all the possible values that the denominators are defined. Note that $D^J(\alpha,\beta,\gamma)^*_{m,m^{'}} = (-1)^{m-m^{'}} D^J(\alpha,\beta,\gamma)_{-m,-m^{'}}$. Apply Wigner formula to (1.21), we get that (1.21) is equal to

$$\begin{aligned} & \frac{1}{32\pi^2} \int_0^{4\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{4\pi} \\ & e^{-\sqrt{-1}(\alpha(\sum_{p=1}^{k} i_p - \sum_{p=1}^{k^{'}} i_p^{'}) + \gamma(\sum_{p=1}^{k} j_p - \sum_{p=1}^{k^{'}} j_p^{'})} (-1)^{\sum_{p=1}^{k}(i_p - j_p) + \sum_{p=1}^{k}(i_p^{'} - j_p^{'})} \\ & (\sum_{\mu,\mu^{'}} (-1)^{\sum_{p=1}^{k}(\mu_k + \mu_k^{'})} \prod_{p=1}^{k} \frac{\sqrt{(J_k+j_k)!(J_k-j_k)!(J_k+i_k)!(J_k-i_k)!}}{\mu!(J_k+j_k-\mu)!(J_k-\mu-i_k)!(i_k+\mu-j_k)!} \end{aligned}$$

$$\cdot \frac{\sqrt{(J_k^{'}+j_k^{'})!(J_k^{'}-j_k^{'})!(J_k^{'}+i_k^{'})!(J_k^{'}-i_k^{'})!}}{\mu!(J_k^{'}+j_k^{'}-\mu)!(J_k^{'}-\mu-i_k^{'})!(i_k^{'}+\mu-j_k^{'})!}$$

$$\cdot (\cos(\beta/2))^{\sum_{p=1}^{k}(2J_k+j_k-i_k-2\mu_k)+\sum_{p=1}^{k^{'}}(2J^{'}_{k^{'}}-j^{'}_{k^{'}}+i^{'}_{k^{'}}-2\mu^{'}_{k^{'}})}$$

$$\cdot (\sin(\beta/2))^{\sum_{p=1}^{k}(-j_k+i_k+2\mu_k+\sum_{p=1}^{k^{'}}(j^{'}_{k^{'}}-i^{'}_{k^{'}}+2\mu^{'}_{k^{'}}))}$$

$$=\delta_{0,\sum_{p=1}^{k} i_p-\sum_{p=1}^{k'} i_p'}\delta_{0,\sum_{p=1}^{k} j_p-\sum_{p=1}^{k'} j_p'}$$

$$(\sum_{\mu,\mu^{'}}(-1)^{\sum_{p=1}^{k}(\mu_k+\mu_k^{'})}\prod_{p=1}^{k}\frac{\sqrt{(J_k+j_k)!(J_k-j_k)!(J_k+i_k)!(J_k-i_k)!}}{\mu!(J_k+j_k-\mu)!(J_k-\mu-i_k)!(i_k+\mu-j_k)!}$$

$$\cdot \frac{\sqrt{(J_k^{'}+j_k^{'})!(J_k^{'}-j_k^{'})!(J_k^{'}+i_k^{'})!(J_k^{'}-i_k^{'})!}}{\mu!(J_k^{'}+j_k^{'}-\mu)!(J_k^{'}-\mu-i_k^{'})!(i_k^{'}+\mu-j_k^{'})!}$$

$$\int_0^{\pi}\cos(\beta/2))^{1+\sum_{p=1}^{k}(2J_k-2\mu_k)+\sum_{p=1}^{k^{'}}(2J^{'}_{k^{'}}-2\mu^{'}_{k^{'}})}(\sin(\beta/2))^{1+\sum_{p=1}^{k}2\mu_k+\sum_{p=1}^{k^{'}}2\mu^{'}_{k^{'}}}d\beta)$$

$$=\delta_{0,\sum_{p=1}^{k} i_p-\sum_{p=1}^{k'} i_p'}\delta_{0,\sum_{p=1}^{k} j_p-\sum_{p=1}^{k'} j_p'}$$

$$(\sum_{\mu,\mu^{'}}(-1)^{\sum_{p=1}^{k}(\mu_k+\mu_k^{'})}\prod_{p=1}^{k}\frac{\sqrt{(J_k+j_k)!(J_k-j_k)!(J_k+i_k)!(J_k-i_k)!}}{\mu!(J_k+j_k-\mu)!(J_k-\mu-i_k)!(i_k+\mu-j_k)!}$$

$$\cdot \frac{\sqrt{(J_k^{'}+j_k^{'})!(J_k^{'}-j_k^{'})!(J_k^{'}+i_k^{'})!(J_k^{'}-i_k^{'})!}}{\mu!(J_k^{'}+j_k^{'}-\mu)!(J_k^{'}-\mu-i_k^{'})!(i_k^{'}+\mu-j_k^{'})!}$$

$$(\sum_{i=0}^{\sum_{p=1}^{k}J_k-\mu_k+\sum_{p=1}^{k^{'}}J^{'}_{k^{'}}-2\mu^{'}_{k^{'}}}(-1)^i\frac{2C^i_{\sum_{p=1}^{k}J_k-\mu_k+\sum_{p=1}^{k^{'}}J^{'}_{k^{'}}-2\mu^{'}_{k^{'}}}}{i+2+\sum_{p=1}^{k}2\mu_k+\sum_{p=1}^{k^{'}}2\mu^{'}_{k^{'}}})). \tag{1.24}$$

Then if we let $J_1=J_2=\cdots=J_k=J_1^{'}=\cdots=J^{'}_{k^{'}}=J$, and let $J\to\infty$, then to understand the asymptotic behavior of (1.21) seems to be a very interesting and challenging problem.

CHAPTER 2
ASYMPTOTIC BEHAVIORS OF CLASSICAL GROUP INTEGRALS

2.1 Related duality theorems

In their paper [5], Collins and Sniady offered a beautiful formula to compute the group integral

$$\int_G G_{i_1j_1}G_{i_2j_2}\cdots G_{i_pj_p}\bar{G}_{i'_1j'_1}\bar{G}_{i'_2j'_2}\cdots\bar{G}_{i'_pj'_p}dg, \tag{2.1}$$

where dg is the Haar measure of G, and $\bar{G}=(G^{-1})^T$. G can be unitary, symplectic or orthogonal group. They reobtained Weingarten's asymptotic formula for this group integral using their formulas. The main tool for their calculation are the following duality theorems.

Theorem 2.1. *([49][19][50]) V is an N dimensional complex vector space. The commutant of the representation $\mathbb{C}[\rho^k(G)]\subset \mathrm{End}(V^{\otimes k})$ of the group algebra $\mathbb{C}[G]$ on the the tensor space $V^{\otimes k}$ is*

$$\begin{cases} \sigma_k(\mathbb{C}[S_k]) & \text{if } G=\mathrm{GL(V)},\\ \psi(\mathbb{C}[P_{2k}]\theta_k) & \text{if } G=\mathrm{O(V)},\\ \psi(\mathbb{C}[P_{2k}]\theta_k) & \text{if } G=\mathrm{Sp(V)},\ \text{only for even } N,\end{cases} \tag{2.2}$$

where σ_k is the natural representation of S_k on $V^{\otimes k}$; $P_{2k}=S_{2k}/\mathfrak{B}_k$ is the Brauer algebra with $\mathfrak{B}_k=\tilde{S}_k\mathcal{R}_k$, $\tilde{S}_k\subset S_{2k}$ is the subgroup of all permutations of the set $\{(1,2),\cdots,(2k-1,2k)\}$, and $\mathcal{R}_k$ is the subgroup generated by the the transformation $2j-1\leftrightarrow 2j$ for $j=1,\cdots,k$; $\theta_k=\psi^{-1}(I_{V^{\otimes k}})$, where $\psi: V^{\otimes 2k}\to \mathrm{End}(V^{\otimes k})$ is an *isomorphism defined by*

$$\psi(v_1 \otimes v_2 \otimes \cdots \otimes v_{2k})u = \omega(u, v_2 \otimes v_4 \otimes \cdots \otimes v_{2k})v_1 \otimes v_2 \otimes \cdots \otimes v_{2k-1}, \tag{2.3}$$

where $v_i \in V$, $u \in V^{\otimes k}$, $\omega(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \prod_{i=1}^{k} \omega(u_i, v_i)$. *Here* $\omega(u_i, v_i)$ *is the nondegenerate invariant symmetric bilinear for* O(V)*, or the nondegenerate skew symmetric bilinear form for* Sp(V)*.*

When we compute the group integral (1.1), we need to consider the commutant of the group action on the corresponding vector space. Let W be a finite dimensional complex vector space. $\mathcal{A} \subset \mathrm{End}(W)$ is a semisimple algebra, $\mathcal{A}_1$ is a semisimple subalgebra of $\mathcal{A}$. By the Double Commutant theorem [19],

$$W \cong \bigoplus_i W_i \otimes U_i, \tag{2.4}$$

where W_i and U_i are irreducible modules of $\mathcal{A}$ and $\mathcal{B}$, i.e., $\mathcal{A} = \bigoplus_i \mathrm{End}(W)_i \otimes I_{U_i}$ and $\mathcal{B} = \bigoplus_i I_{W_i} \otimes \mathrm{End}(U_i)$. Restricting the $\mathcal{A}$ representation on W_i to $\mathcal{A}_1$, and applying the Double Commutant theorem again, we get

$$W \cong \bigoplus_{i,j} W_{ij} \otimes U_{ij} \otimes U_i, \tag{2.5}$$

where W_{ij}s are irreducible modules of the representation of $\mathcal{A}_1$ and U_{ij}s are the irreducible modules of the commutant of $\mathcal{A}_1$ action on V_i. Then

$$\mathcal{A}_1 = \bigoplus_{i,j} \mathrm{End}(W_{ij}) \otimes I_{U_{ij}} \otimes I_{U_i}. \tag{2.6}$$

We denote

$$\mathcal{B}_{10} = \bigoplus_{i,j} I_{V_{ij}} \otimes \mathrm{End}(U_{ij}) \otimes I_{U_i}. \tag{2.7}$$

Then (2.5) can be written as

$$W \cong \bigoplus_{\lambda} W^{\lambda} \otimes (\bigoplus_{i} U_{i\lambda} \otimes U_i), \tag{2.8}$$

where λ runs over all the irreducible representations of $\mathcal{A}_1$ up to equivalent, and $U_{i\lambda} = U_{ij}$ if $W_{ij} \cong W^{\lambda}$ ($U_{i\lambda}$ is trivial if no such W_{ij}). We choose $0 \neq u_{i\lambda} \otimes u_i \in U_{i\lambda} \otimes U_i$ for each nontrivial $U_{i\lambda} \otimes U_i$ and require that $u_{i\lambda} \otimes u_i$ is a basis element of a fixed basis of $U_{i\lambda} \otimes U_i$. Then define an algebra $\mathcal{B}_{11} = \bigoplus_{\lambda} I_{W^{\lambda}} \otimes \mathrm{End}(\mathrm{Span}\{u_{i\lambda} \otimes u_i; i\})$, and mapping any other basis elements in any $U_{i\lambda} \otimes U_i$ to zero. Then the $\mathcal{B}_{11}$ commutes with $\mathcal{A}_1$. Moreover, any nonzero element of W can be mapped to any other elements by the algebra generated by $\{\mathcal{B}, \mathcal{B}_{10}, \mathcal{B}_{11}\}$. Therefore we get the following

Lemma 2.2. *The commutant of $\mathcal{A}_1$ is generated by $\{\mathcal{B}, \mathcal{B}_{10}, \mathcal{B}_{11}\}$.*

Specifically, the commutant of $\mathbb{C}[\rho^k(\mathrm{SU(V)})]$ on the vector space $V^{\otimes k}$, where V is a N dimensional complex vector space, is $\sigma_k(\mathbb{C}[S_k])$, which is the same as the commutant of $\mathbb{C}[\rho^k(\mathrm{U(V)})]$. This is simply because any matrix $u \in \mathrm{U(V)}$ can be written as a complex number multiplying a matrix $g \in \mathrm{SU(V)}$.

Now let us consider the commutant of $\mathbb{C}[\rho^k(\mathrm{SO(V)})]$ on $V^{\otimes k}$. First of all, recall that if N is even, an irreducible representation (σ, W) of O(V) is determined by a signature $\lambda_1 \geq \lambda_2 \cdots \lambda_{N/2} \geq 0$:

$$\sigma = \begin{cases} \rho_N^{\lambda} & \text{if } \lambda_{N/2} = 0, \\ \pi_N^{\lambda,\pm} & \text{if } \lambda_{N/2} \neq 0 \end{cases}, \tag{2.9}$$

where $\rho_N^{\lambda} = \mathrm{Ind}_{\mathrm{SO(V)}}^{\mathrm{O(V)}}$ and $\pi_N^{\lambda,\pm}$ are representations of SO(V) extended to O(V) satisfying $\pi_N^{\lambda,-} = \det \otimes \pi_N^{\lambda,+}$ [19]. When $\lambda_{N/2} \neq 0$, the O(V) irreducible representation ρ_N^{λ}

decomposes into two SO(V) irreducible representations with signatures $(\lambda_1, \cdots, \lambda_{N/2})$ and $(\lambda_1, \cdots, \lambda_{N/2-1}, -\lambda_{N/2})$. The map $\rho(g_0)$ exchanges the two corresponding highest weight and highest weight vectors, where g_0 exchanges the two basis elements $e_{N/2}$ and $e_{N/2+1}$ of V. By the Double Commutant theorem, the O(V) action on $V^{\otimes k}$ decomposes as

$$V^{\otimes k} \cong \bigoplus_{\lambda} \mathcal{R}^\lambda \otimes \Omega^\lambda, \tag{2.10}$$

where the $\mathcal{R}^\lambda$ is the O(V) irreducible module, and Ω^λ is the $\mathrm{P_{2k}}$ irreducible module. We define a $\tilde{\rho} \in \mathrm{End}$ by

$$\tilde{\rho}(x) = \begin{cases} \rho^k(g_0)(x) & \text{if } x \in \mathcal{R}^\lambda \otimes \Omega^\lambda \text{ with } \lambda_{N/2} \neq 0, \\ x & \text{if } x \in \mathcal{R}^\lambda \otimes \Omega^\lambda \text{ with } \lambda_{N/2} = 0. \end{cases} \tag{2.11}$$

To explicitly write the function $\tilde{\rho}(x)$, let us construct the projection operator from $V^{\otimes k}$ to Φ_λ, where $\Phi_\lambda = \mathcal{R}^\lambda \otimes \Omega^\lambda$. Let $c^{2\lambda} = \sum_{q \in C, p \in R} \mathrm{sgn}(q) qp$ be the Young symmetrizer for the Young diagram with shape $2\lambda_1 \geq 2\lambda_2 \geq 2\lambda_{[N/2]} \geq 0$, where C is the set of permutations on columns and R is the set of permutations on rows. For a tableau T of this shape, e_T denotes the vector in $V^{\otimes k}$ that corresponds T. We shall also use T itself to denote e_T. If $e_T = v_1 \otimes v_2 \otimes \cdots \otimes v_{2k}$, we require that in the tableau T, all the odd indexes are in the odd columns and v_{2i} is always right after v_{2i-1}, $1 \leq i \leq k$.

Proposition 1. $\epsilon^\lambda = \sum_{s \in S_k} \frac{1}{\mu^2} s c^\lambda s^{-1}$, *where* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{[N/2]} \geq 0$ *are nonnegative integers satisfying* $\sum_{i=1}^{[N/2]} = k$, *is the projection from* $V^{\otimes k}$ *to* $\mathcal{R}^\lambda \otimes \Omega^\lambda$.

Proof. By the property of Young symmetrizer, ϵ^λs are projections. The number of these subspaces is equal to the number of the $\mathcal{R}^\lambda \otimes \Omega^\lambda$. Since the group action of O(N) commutes with all ϵ^λs, therefore $\epsilon^\lambda(V^{\otimes k}$ is an invariant subspace under the group action of O(N). It suffices to prove that all the highest weight vectors of irreducible O(N) subspaces with signature λ are in $\epsilon^\lambda(V^{\otimes k})$. Note that the generating function for SO(V) irreducible representation with signature λ is $\alpha(g) = \prod_{i=1}^{[N/2]} \Delta^{r_i}$, where Δ_is are principal minors of $g \in$ SO(V), $r_i = \lambda_i - \lambda_{i+1}$, for $1 \leq i \leq [N/2] - 1$, and $r_{[N/2]} = \lambda_{[N/2]}$. Therefore the highest weight vector $v_h = \frac{1}{\mu} c^\lambda(T_0^\lambda)$, where tableau T_0^λ of shape $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{[N/2]} \geq 0$ with all 1s in the first row, all 2s in the second row, $\cdots$, $[N/2]$ in the $[N/2]$th row, is a highest weight vector of O(V)(other indexes are $\bar{1}$, $\bar{2}$, $\cdots$ satisfying $\omega(i, \bar{j}) = \delta_{ij}$ and $\bar{\bar{i}} = i$. For O(V), if N is odd, the index of e_N is 0 with $\bar{0} = 0$). By the double commutant theorem, the vector space spanned by all the highest weight vectors of irreducible O(N) subspaces with signature λ is

$$\mathrm{Span}\{\psi(\sigma\theta_k)\frac{1}{\mu}c(T_0^\lambda); \sigma \in P_{2k}\}. \tag{2.12}$$

Recall that for any $\sigma \in S_{2k}$, $\psi(\sigma\theta_k)$ is the product of two operators The first which is the product of some operators of the form $D_{ij}C_{ij}$, for some different pairs (i, j), $1 \leq i < j \leq k$ (the trivial case is identity operator) and the second is an operator which is the natural representation of some element of S_k on $V^{\otimes k}$ [19]. Here C_{ij} : $V^{\otimes k} \to V^{\otimes k-2}$ is called ij-contraction operator defined by

$$C_{ij}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \omega(v_i, v_j) v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \hat{v}_j \otimes \cdots \otimes v_k, \tag{2.13}$$

where v_i and v_j are omitted, and $D_{ij} : V^{\otimes k-2} \to V^{\otimes k}$ is defined by

$$D_{ij}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sum_{p=1}^{N} v_1 \otimes v_2 \otimes \cdots f_p \otimes \cdots \otimes f^p \otimes \cdots \otimes v_{k-2}, \tag{2.14}$$

where f_p and f^p are at the ith and jth positions respectively and $\{f_p\}$ is a basis and $\{f^p\}$ is its dual basis via ω. Since $c^\lambda(T_0^\lambda)$ is is a linear combination of tableaux without conjugate indexes, therefore $C_{ij}c^\lambda(T_0^\lambda) = 0$, for any i, j. Then

$$\mathrm{Span}\{\psi(\sigma\theta_k)\frac{1}{\mu}c(T_0^\lambda); \sigma \in P_{2k}\} = \mathrm{Span}\{\sigma\frac{1}{\mu}c(T_0^\lambda); \sigma \in S_k\} \tag{2.15}$$

Again by the property of central Young symmetrizer, $\epsilon(s\frac{1}{\mu}c^\lambda(T_0^\lambda)) = s\frac{1}{\mu}c^\lambda(T_0^\lambda)$, for any $s \in S_k$. Therefore we complete the proof. □

We would like to remark that this proposition and the proof apply to symplectic groups too.

Then we have proved the following

Theorem 2.3. *If N is odd, the commutants of $\mathbb{C}[\rho^k(\mathrm{SO(V)})]$ and $\mathbb{C}[\rho^k(\mathrm{O(V)})]$ are the same; if N is even, the commutant is* $\mathrm{Span}\{\psi(\mathbb{C}[P_{2k}]\theta_k), \psi(\mathbb{C}[P_{2k}]\theta_k)\tilde{\rho}\}$.

In general, denote the action of the group algebra $\mathbb{C}[G]$ on a vector space W simply by $\mathbb{C}[\rho^k(G)]$. If $\bar{W}$ is the dual vector space of W, the commutant $\mathrm{End}_G\, W$ of $\mathbb{C}[\rho^k(G)]$ can be identified as the subspace of invariant tensors in the tensor space $W \otimes \bar{W}$ [19], which is the vector space which is the direct sum of one dimensional irreducible subspaces in the irreducible decomposition of $W \otimes \bar{W}$ under the group G. The method of Z invariants will give all the irreducible subspaces [50].

2.2 Unitary group integrals

In his paper [46], Weingarten obtained the following asymptotics:

For U(N),

$$\int_{\mathrm{U(N)}} U_{i_1 j_1} U_{i_2 j_2} \cdots U_{i_q j_q} U^*_{i'_1 j'_1} U^*_{i'_2 j'_2} \cdots U^*_{i'_q j'_q} du$$
$$= \frac{1}{N^q} \sum_{\sigma \in S_q} \delta_{i_1 i'_{\sigma(1)}} \delta_{j_1 j'_{\sigma(1)}} \cdots \delta_{i_q i'_{\sigma(q)}} \delta_{j_q j'_{\sigma(q)}} + O(\frac{1}{N^{q+1}}). \tag{2.16}$$

For SO(N),

$$\int_{\mathrm{SO(N)}} U_{i_1 j_1} U_{i_2 j_2} \cdots U_{i_{2q} j_{2q}} du$$
$$= \frac{1}{N^q} \sum \delta_{i_{k_1} i_{l_1}} \cdots \delta_{i_{k_q} i_{l_q}} + O(\frac{1}{N^{q+1}}), \tag{2.17}$$

where the sum carries over all the partitions of $\{1, 2, \cdots, 2q\}$ into pairs $(k_1, l_1), \cdots, (k_q, l_q)$.

For Sp(2N),

$$\int_{\mathrm{Sp(2N)}} U^{k_1}_{i_1 j_1} U^{k_2}_{i_2 j_2} \cdots U^{k_{2q}}_{i_{2q} j_{2q}} du$$
$$= \frac{1}{(2N)^q} \sum M^{k_{l_1} k_{m_1}}_{i_{l_1} i_{m_1}} M^{k_{l_1} k_{m_1}}_{j_{l_1} j_{m_1}} \cdots M^{k_{l_q} k_{m_q}}_{i_{l_q} i_{m_q}} M^{k_{l_q} k_{m_q}}_{j_{l_q} j_{m_q}} + O(\frac{1}{(2N)^{q+1}},) \tag{2.18}$$

where $U^1_{ij} = U_{ij}$,$U^2_{ij} = U^*_{ij}$, $M^{kl}_{ij} = J_{ij}$ if $k = l$, $M^{kl}_{ij} = \delta_{ij}$ if $k \neq l$, $J_{ij} = -\delta_{i,(j-1)}$ if i is even, and $J_{ij} = \delta i, (j+1)$ if i is odd.

As we mentioned, in their paper[5], Benoit Collins and Piotr Sniady offered a method to compute the integral (2.1), i.e., for the vector representation of unitary group U(N), SO(N), and SpU(N). By the virtue of our duality theorems, this method also theoretically applies to the integrals of irreducible representations of group G. Here we offer an more transparent and elementary method to compute the group integral (2.1) for unitary, orthogonal and symplectic groups. Denote the following integral

$$\int_G \rho^\lambda_{i_1 j_1} \cdots \rho^\lambda_{i_q j_q} \bar{\rho}^\lambda_{i_1' j_1'} \cdots \bar{\rho}^\lambda_{i_q' j_q'} du, \tag{2.19}$$

by Dirac notation

$$P_G(I, J, I', J') = \int_G \langle I|\rho^\lambda(u)|J\rangle\langle J'|\rho^\lambda(u^{-1})|I'\rangle du, \tag{2.20}$$

where ρ^λ denotes the irreducible representation of G with signature λ, $|J\rangle = |e_{j_1} \otimes \cdots e_{j_q}\rangle$ with $J = (j_1, \cdots, j_q)$. Then $|J\rangle\langle I'| \in \mathrm{End}(\mathbb{C}^{N_\lambda})^{\otimes p}$, where N_λ is the dimension of the irreducible representation ρ^λ of G. we can define

Definition 2.1.

$$\Phi^\lambda_G(J, J') = \sum_{\sigma \in B} \mathrm{Tr}(|J\rangle\langle J'|\sigma^{-1})\sigma, \tag{2.21}$$

where B is the finite group that generates the group algebra $\mathcal{B}$, which is the commutant of the group algebra $\rho^{\otimes q}(\mathbb{C}[G])$, and σ also denotes the representation.

The existence of B is a straightforward consequence of the Double Commutant theorem. The conditional expectation is

$$\mathbb{E}^\lambda_G(J, J') = \int_G \rho^\lambda(u)|J\rangle\langle J'|\rho^\lambda(u^{-1}) du, \tag{2.22}$$

where du is the Haar measure of G, $\rho^\lambda_G(u) \in \mathrm{End}(\mathbb{C}^{N_\lambda})^{\otimes q}$ is the action of $u \in G$ on the tensor space $(\mathbb{C}^{N_\lambda})^{\otimes q}$, $|J\rangle = |e_{j_1} \otimes e_{j_2} \otimes \cdots e_{j_q}|$ and $\langle J'| = \langle e_{j_1'} \otimes e_{j_2'} \otimes \cdots e_{j_q'}|$. Then we can compute $P_G(I, I', J, J')$ by

Proposition 2.

$$\Phi_G^\lambda(J, J^{'}) = \mathbb{E}_G^\lambda(J, J^{'})\Phi_G^\lambda(\mathrm{Id}), \tag{2.23}$$

$\Phi_G^\lambda(\mathrm{Id})$ *has an inverse.*

Proof. Since $\mathrm{Tr} \circ \mathbb{E}_G^\lambda = \mathrm{Tr}$, it can be easily checked that Φ_G^λ is a $\mathcal{B}$ bimodule and

$$\Phi_G^\lambda(J, J^{'}) = \mathbb{E}_G^\lambda(J, J^{'})\Phi_G^\lambda(\mathrm{Id}). \tag{2.24}$$

Taking $J = J^{'} = e_1 \otimes e_2 \otimes \cdots \otimes e_q$, we get

$$\mathrm{Id} = \mathbb{E}_G^\lambda(|e_1 \otimes e_2 \otimes \cdots \otimes e_q\rangle\langle e_1 \otimes e_2 \otimes \cdots \otimes e_q|)\Phi_G^\lambda(\mathrm{Id}). \tag{2.25}$$

□

However, in order to get the asymptotics of (1.1) we need the following decomposition lemma. Let us fix the signatures and let $n \to \infty$, because $\mathrm{Par}(N, k) = \mathrm{Par}(N, k)$, when $p \geq N$. We consider $\lambda^{(i)} = (m_1^{(i)}, m_2^{(i)}, \cdots, m_{M^{(i)}}^{(i)}, 0, 0, \cdots)$ of an irreducible representation, where $m_1^{(i)} \geq m_2^{(i)}, \cdots \geq m_{M^{(i)}}^{(i)} \geq 0$ are integers, for $G =$ U(N), SO(N), or SpU(2N) when N is large. Let us consider the asymptotic behavior of the integral (1.1) with respect to N. Let us assume $G =$ U(N) first. Recall that an irreducible module of U(N)(also GL(N, $\mathbb{C}$) with signature $\lambda = (m_1, m_2, \cdots, m_M)$) can be constructed by GL standard tableaux of shape λ with elements in $\{1, 2, \cdots, N\}$ [12]. GL(N) standard tableaux are the tableaux which rows are nondecreasing and which columns are increasing. The Young symmetrizer c is defined by $c = \sum \pm qp$ (see [50]), where q ranges over the column permutation C, and p ranges over the row permutations R of the Young diagram. The sign is $+$ or $-$ according to whether q is even or odd. It is well known that the $\{c(T); T \text{ is GL(N) standard Tableaux of shape } \lambda\}$ gives a basis of the irreducible representation of U(N) with signature λ. Then we can construct an ONB of the irreducible U(N) module. For other classical groups, there are also corresponding standard tableaux. Then we get the following decomposition(not irreducible).

Lemma 2.4. *G is a classical group. There is a decomposition under the group action of G:*

$$\begin{aligned}
&\bigotimes_{i=1}^{k} \mathbb{C}^{N_{\lambda^{(i)}}} \otimes \bigotimes_{i=1}^{k'} \mathbb{C}^{N_{\lambda'^{(i)}}} \\
&\simeq \bigotimes_{i=1}^{k} \mathrm{Span}\{c(T); T \textit{ is } G \textit{ standard Tableaux of shape } \lambda^{(i)}\} \\
&\otimes \bigotimes_{i=1}^{k'} \mathrm{Span}\{c(\bar{T}); T \textit{ is } G \textit{ standard Tableaux of shape } \lambda'^{(i)}\},
\end{aligned} \tag{2.26}$$

where N_λ is the dimension of the irreducible representation ρ_G^λ of G; $\bar{T}$ is the tableau replacing every index in T by its dual index.

Let us come back to $G = \mathrm{U(N)}$. For a signature λ, denote $\tilde{e}_T = \frac{1}{n_T} c(T)$, for $\forall T \in \mathbb{T}^\lambda$, where n_T is the normalization constant such that $\|\tilde{e}_T\| = 1$($\|\cdot\|$ is defined by the inner product $\langle \cdot, \cdot \rangle$ of the Hilbert space $(\mathbb{C}^{N_\lambda})^{\otimes k}$). Then $\{\tilde{e}_T\}$ with T runs over all GL(N) standard tableaux is an ONB of the irreducible U(N) module. Then $n_T^2 = \langle c(T), c(T) \rangle$. Since $d = \frac{1}{\mu} c$ is a projection (see chapter 8,[50]). Then $n_T^2 = \mu \langle c(T) | T \rangle$. Note that the μ and multiplicity t of the representation $\rho_{\mathrm{U(N)}}^\lambda$ in $\Phi_m \simeq (\mathbb{C}^n)^{\otimes m}$ satisfy the relation $\mu \cdot t = m!$, where $m = \sum_{i=1}^n m_i$ (Theorem 1 in Chapter 8 [50]). On the other hand, by the RSK correspondence, t is the number of the tableaux of shape λ with entries from 1 to N each occurring once [12]. A consequence of hook formula shows that $t = \frac{m!\prod_{i<j}(l_i-l_j)}{l_1!\cdot l_2!\cdots\cdot l_n!}$, where $m = \sum_{i=1}^{n} m_N$, $l_i = m_i + N - i$, $i = 1, 2, \cdots, N$. For each term of $c(T)$, unless it is T itself, its inner product with T will be zero. Moreover, the only transformations of the form pq, where $p \in R$, and $q \in C$ that fixe T are those $q = 1$ and p makes each row unchanged. Let $m_{\mu\nu}$, $1 \leq \mu \leq \nu \leq N$, denotes the Gelfand diagram that corresponds the tableau T(if (μ,ν) is not in this range, then $m_{\mu\nu}$ is set to be zero). Then the number of these transformations is

$$f = \prod_{\mu,\nu}(m_{\mu\nu} - m_{\mu,\nu-1})! \tag{2.27}$$

We use $T(i)$ to denote the map from an ONB element of an irreducible G module indexed by i to the corresponding G standard tableau.

Let W be any vector space. We generalize the Kronecker delta function to be a bilinear form on $W^{\otimes k} \times W^{\otimes k}$.

Definition 2.2. The bilinear form $\delta_{\cdot,\cdot}$ on $W^{\otimes k} \times W^{\otimes k}$ is defined by

$$\delta_{e_{i_1}\otimes e_{i_2}\cdots e_{i_k}, e_{i_1'}\otimes e_{i_2'}\cdots e_{i_k'}} = \delta_{i_1 i_1'}\delta_{i_2 i_2'}\cdots\delta_{i_k i_k'} \tag{2.28}$$

Therefore we have the following

Theorem 2.5. *If* $G = \mathrm{U(N)}$*, the group integral (1.1) with signatures* $\lambda^{(i)}$ *is equal to*

$$\prod_{p=1}^{k}\frac{1}{n_{T(i_p)}n_{T(j_p)}}\prod_{p=1}^{k'}\frac{1}{n_{T(i_p')}n_{T(j_p')}}\Big(\sum_{p_{1,i},p_{2,i},p_{1,i}',p_{2,i}'\in R,q_{1,i},q_{2,i},q_{1,i}',q_{2,i}'\in Q}\prod_{i=1}^{k}\mathrm{sgn}(q_{1,i})\,\mathrm{sgn}(q_{2,i})\cdot$$

$$\int_{\mathrm{U(N)}} du\langle q_{1,1}p_{1,1}T(i_1)|\rho_{\mathrm{U(N)}}^{\otimes m^{(1)}} q_{2,1}p_{2,1}T(j_1)\rangle\cdots\langle q_{1,k}p_{1,k}T(i_k)|\rho_{\mathrm{U(N)}}^{\otimes m^{(k)}} q_{2,k}p_{2,k}T(j_k)\rangle$$

$$\langle q_{1,1}p_{1,1}T(i_1')|\rho_{\mathrm{U(N)}}^{*\otimes m^{(1)}} q_{2,1}p_{2,1}T(j_1')\rangle\cdots\langle q_{1,k'}'p_{1,k'}'T(i_{k'}')|\rho_{\mathrm{U(N)}}^{*\otimes m^{(k')}} q_{2,k'}'p_{2,k'}'T(j_{k'}')\rangle\Big). \tag{2.29}$$

Its asymptotic behavior with respect to N is

$$\prod_{p=1}^{k}\frac{1}{n_{T(i_p)}n_{T(j_p)}}\prod_{p=1}^{k'}\frac{1}{n_{T(i'_p)}n_{T(j'_p)}}(\frac{1}{N^m}\sum_{\sigma\in S_m,p_{1,i},p_{2,i},p'_{1,i},p'_{2,i}\in R,q_{1,i},q_{2,i},q'_{1,i},q'_{2,i}\in Q}\prod_{i=1}^{k}\operatorname{sgn}(q_i)\operatorname{sgn}q'_{i'}$$

$$\delta_{q_{1,1}p_{1,1}T(i_1)\otimes\cdots\otimes q_{1,k}p_{1,k}T(i_k),\sigma q_{1,1}p_{1,1}T(i'_1)\otimes\cdots\otimes q_{1,k}p_{1,k}T(i_1)}$$

$$\delta_{q_{2,1}p_{2,1}T(j_1)\otimes\cdots\otimes q_{2,k}p_{2,k}T(j_k),\sigma q_{2,1}p_{2,1}T(j'_1)\otimes\cdots\otimes q_{2,k}p_{2,k}T(j'_{k'})}+O(\frac{1}{N^{m+1}})), \tag{2.30}$$

where $m^{(r)}=\sum_l m_l^{(r)}$,$m=\sum_{r=1}^{k}m^{(r)}$. m' is similarly defined. $m=m'$, otherwise the integral vanishes.

2.3 Orthogonal group integrals

If $G=\mathrm{O(N)}$, we need to construct a ONB for the irreducible O(N) modules. Proctor [42], King and Welsh [32] constructed irreducible O(N) modules. They defined O(N) standard tableaux satisfying the following conditions. Let $w_{\bar{i}}=e_{2i-1}$, $w_i=2i$, for $i=1,2,\cdots,[N/2]$. If N is odd, $w_0=e_N$. The orthogonal standard tableaux defined in [42] is

Definition 2.3. ([42]) Let λ be a partition of N such that $\tilde{\lambda}_1+\tilde{\lambda}_2\leq N$ and T^λ_{ab} denotes the entry of the ath row and bth column of the tableau. For $i=1,2,\cdots,r$ with $r=[N/2]$, let α_i and β_i be the numbers of the entries less than or equal to i in the first and second columns, respectively of the tableau T^λ. T^λ is O(N) standard tableau if and only if it is GL(N) standard tableau and for each $i=1,2,\cdots,r$,

(i) $\alpha_i+\beta_i\leq 2i$;

(ii)If $\alpha_i+\beta_i=2i$ with $\alpha_i>\beta_i$ and $T^\lambda_{\alpha_i,1}=\bar{i}$ and $T^\lambda_{\beta_i b}=i$ for some b then $T^\lambda_{\beta_i-1,b}=\bar{i}$;

(iii)If $\alpha_i+\beta_i=2i$ with $\alpha_i=\beta_i=i$ and $T^\lambda_{i,1}=\bar{i}$ and $T^\lambda_{\beta_i b}=i$ for some b then $T^\lambda_{i-1,b}=\bar{i}$.

If T is a O(N) standard tableau, T_0 denotes the quotient of the Young symmetrized tableau $\{T\}$ with the subpace of the form

$$\sum_{i\in\mathcal{I}} x\otimes w_i\otimes y\otimes w_{\bar{i}}\otimes z, \tag{2.31}$$

where x, y, z are arbitrary tensor powers of V. Similar to Weyl's theorem [49] for the other definition of $\mathrm{O(N)}=\{\mathrm{M};\mathrm{MM^T}=\mathrm{I},\mathrm{M}\in\mathrm{GL(N)}\}$, the tensor space $V^{\otimes m}$ can be decomposed into the direct sum of traceless subspace and its complement subspace which is spanned by all the tensors of the above form and these two subspaces are orthogonal to each other, i.e., $V^{\otimes m}=V_0^{\otimes m}+V_1^{\otimes m}$, where $V_0^{\otimes m}$ is the subspace such that for any two indexes v_i, v_j of $v=\cdots\otimes x\otimes\cdots\otimes y\otimes\cdots\in V_0^{\otimes m}$,

$$\sum_{i\in\mathcal{I}}\langle w_{\bar{i}}|v_i\rangle\langle v_j|w_i\rangle x\otimes\hat{v}_i\otimes y\otimes\hat{v}_j\otimes z=0, \tag{2.32}$$

and

$$\begin{aligned} &V_1^{\otimes m}\\ =&\mathrm{Span}\{\sum_{i\in\mathcal{I}} x\otimes w_i\otimes y\otimes w_{\bar{i}}\otimes z; x,y,z \text{ are tensor powers of } V\} \end{aligned} \tag{2.33}$$

is its complement. Therefore $T_0\in V_0^{\otimes m}$. Use T_1 to denote $T-T_0$. T_0 can be computed by taking traces for all the pairs of indexes of the equation (2.41). If T is a O(N) standard tableau, then it is straightforward to check $(cT)_0=cT_0$ and $(cT)_1=cT_1$, where c is the Young symmetrizer. Let T and $T^{'}$ be two O(N) standard tableaux. We shall compute $\langle cT_0|cT_0^{'}\rangle$.

If $T=T^{'}$, $\langle cT_0|cT_0^{'}\rangle=\mu\langle T|cT_0\rangle=\mu\langle cT-(cT)_1|T\rangle$. Therefore it suffices to compute $\langle (cT)_1|T\rangle$. By computing the traces of T, and solving a linear system, we get that

$$T_1 = \begin{cases} \frac{1}{N-1+\sum_{\mu=1}^{N/2}(s_\mu - s_{\mu-1})(s_{\bar{\mu}} - s_{\bar{\mu}-1})} \sum_{1\le i<j\le m} D_{ij}C_{ij}T & \text{if } N \text{ is even,} \\ \frac{1}{N-1+C^2_{s_n - s_{N-1}}+\sum_{\mu=1}^{(N-1)/2}(s_\mu - s_{\mu-1})(s_{\bar{\mu}} - s_{\bar{\mu}-1})} \sum_{1\le i<j\le m} D_{ij}C_{ij}T & \text{if } N \text{ is odd,} \end{cases} \tag{2.34}$$

where $s_\nu = \sum_{\mu=1}^{\nu} m_{\mu\nu}$, C_{ij} and D_{ij} are ij-contraction and ij-expansion operators. One can check that for any $p \in R$, $q \in C$, only those such that $qpT = T$ or qpT is switching two entries of T can make $\langle (qpT)_1 | T \rangle$ nonvanishing. The only qp such that $qpT = T$ are those with $q = I$, so the number of these transformations is $\prod_{\mu,\nu}(m_{\mu\nu} - m_{\mu,\nu-1})!$. The number of the transformations of switching two entries consists of two types of transformations: the first type is that switching two entries ν and $\bar{\nu}$ in the same column, which is equal to

$$\sum_{\mu} f \prod_{\nu<\bar{\nu}} (m_{(\mu+1)\bar{\nu}} - m_{\mu(\nu-1)})_+, \tag{2.35}$$

where $x_+ = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ The second type is that switching two entries ν and $\bar{\nu}$ in the same row. The second type also consists of two kinds of transformations: one is those that have identity q, and the other one is those that have nonidentity q. The number of the first kind is

$$\sum_{\mu} f \prod_{\nu_1<\bar{\nu}_1<\nu_2<\bar{\nu}_2} (m_{\mu\nu_1} - m_{\mu,(\nu_1-1)})(m_{\mu\nu_2} - m_{\mu,(\nu_2-1)}). \tag{2.36}$$

The number of the second kind is

$$\sum_{\mu} f \prod_{\nu<\bar{\nu}} (m_{(\mu+1)\bar{\nu}} - m_{\mu(\nu-1)})_{+} (m_{\bar{\mu}\nu} - m_{\mu\nu})$$

$$+ \sum_{\mu} f \prod_{\nu<\bar{\nu}} (m_{\mu\bar{\nu}} - m_{(\mu-1),(\nu-1)})_{+} (m_{\mu\nu} - m_{\mu,(\nu-1)}), \tag{2.37}$$

if the sign$(q) = -1$; and

$$\sum_{\mu} f \prod_{\nu<\bar{\nu}} (m_{(\mu+1)\bar{\nu}} - m_{\mu(\nu-1)})_{+} (m_{\mu\bar{\nu}} - m_{(\mu-1),(\nu-1)})_{+}, \tag{2.38}$$

if the sign$(q) = 1$. Then for the sum of $\langle \mathrm{sign}(q)(qpT)_1, T\rangle$ for all qpT switching two indexes ν and $\bar{\nu}$. If $qpT = T$, $\langle (qpT)_1 | T\rangle$ can be explicitly computed using (2.34). Then $\langle cT_0 | cT_0\rangle$ can be computed explicitly by plugging in all the above five formulas.

If $T \neq T^{'}$, and the sets of the entries of T and T' are the same, then there exists some $s \in S_k$, such that $T = sT'$, and $s \neq qp$ for any $q \in C$, $p \in R$. By the property of Young symmetrizer, $\langle cT_0 | cT_0^{'}\rangle = \langle cT_0 | c_s T_0\rangle = \mu \langle T_0 | cc_s T_0\rangle = 0$. If the sets of the entries of T and T' are the same except one entries, then it is easy to see that $\mu\langle T | cT_0^{'}\rangle = 0$. If the sets of the entries of T and T' are the same except two entries, if $\langle T | T_1^{'}\rangle \neq 0$, then the two entries in T and T' must be i,$\bar{i}$ and j, $\bar{j}$ respectively and $i \neq j, i \neq \bar{j}$ and we are able to compute $\langle T | cT_1^{'}\rangle = \langle T | \sum \mathrm{sgn}(q) qpT_1^{'}\rangle = \frac{1}{N} |\{p; pT^{'} = T^{'}\}|$, where $|\{p; pT^{'} = T^{'}\}|$ is the cardinality of the set $\{p; pT^{'} = T^{'}\}$(this is easy to compute depending on the positions of the two entries). Then we are able to compute $\langle cT_0 | cT_0^{'}\rangle$. If more than two elements in the sets of entries of T and $T^{'}$ are different, $\langle cT_0 | cT_0^{'}\rangle$ is always zero. After we compute all the inner products among cT_0, where T runs over O(N) standard tableaux, we are able to compute the ONB of the irreducible O(N) module. The basis elements of the ONB are denoted $\tilde{T}(i)$,

$1 \leq i \leq N_\lambda$, where N_λ is the dimension of the irreducible O(N) module. Now we have the following

Theorem 2.6. *If $G = \mathrm{O(N)}$, the group integral*

$$\int_{\mathrm{O(N)}} \rho^{\lambda^{(1)}}_{i_1 j_1} \rho^{\lambda^{(2)}}_{i_2 j_2} \cdots \rho^{\lambda^{(k)}}_{i_k j_k} du$$

with signatures $\lambda^{(i)} = (m_1^{(i)}, m_2^{(i)}, \cdots, m_n^{(i)})$ with $m_1^{(i)} \geq m_2^{(i)} \geq \cdots \geq m_n^{(i)} \geq 0$, $i = 1, \cdots, k$, is equal to

$$\int_{\mathrm{O(N)}} \langle \tilde{T}(i_1) | \rho^{m^{(1)}}_{\mathrm{O(N)}} \tilde{T}(j_1) \rangle \cdots \langle \tilde{T}(i_k) | \rho^{m^{(k)}}_{\mathrm{O(N)}} \tilde{T}(j_k) \rangle du. \tag{2.39}$$

Its asymptotic behavior with respect to N is

$$\frac{1}{N^{m/2}} \sum_{\sigma \in P_m} \delta_{\tilde{T}(i_1)\otimes\cdots\otimes\tilde{T}(i_k), \sigma\tilde{T}(i_1)\otimes\cdots\otimes\tilde{T}(i_1)} \delta_{\tilde{T}(j_1)\otimes\cdots\otimes\tilde{T}(j_k), \sigma\tilde{T}(j_1)\otimes\cdots\otimes q_{2,k}\tilde{T}(j_k)} + O(\frac{1}{N^{m/2+1}}), \tag{2.40}$$

where $m^{(r)} = \sum_l m_l^{(r)}$, and $m = \sum_{r=1}^k m^{(r)}$ is even. If m is odd, the integral vanishes.

For $G = \mathrm{SO(N)}$, the formula can be derived using the SO(N) standard tableau in [32].

2.4 Symplectic group integrals

If $G = \mathrm{SpU(2N)}$, similar to the orthogonal group, the entries of tableaux are ordered as $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{N} < N$. The Sp(2N) standard tableau is defined such that if the entries are increasing in each column., non-decreasing in each row and if, in addition, the elements of row i are all greater than or equal to $\bar{i}$, for each i [30][2]. The ONB $\tilde{T}_i$, $1 \leq i \leq N_\lambda$, where N_λ is the dimension of the irreducible Sp(2N) module, can be computed following similar steps as the orthogonal group case, noting that the tensor space $V^{\otimes m}$ decomposes into different subspaces: $V^{\otimes m} = V_{\tilde{0}}^{\otimes m} + V_{\tilde{1}}^{\otimes m}$, where $V_{\tilde{0}}^{\otimes m}$ is the traceless subspace such that for any two indexes v_1, v_2 of $v = \cdots \otimes x \otimes \cdots \otimes y \otimes \cdots \in V_{\tilde{0}}^{\otimes m}$,

$$\sum_{i=1}^{n}(\cdots \otimes \langle w_{\bar{i}}|v_1\rangle \otimes \cdots \otimes \langle v_2|w_i\rangle \otimes \cdots - \cdots \otimes \langle w_i|v_1\rangle \otimes \cdots \otimes \langle v_2|w_{\bar{i}}\rangle \otimes \cdots = 0,$$

and

$$V_{\tilde{1}}^{\otimes m}$$

$$= \mathrm{Span}\{\sum_{i\in\mathcal{I}} x \otimes w_i \otimes y \otimes w_{\bar{i}} \otimes z - x \otimes w_{\bar{i}} \otimes y \otimes w_i \otimes z; x, y, z \text{ are tensor powers of } V\}$$

is its complement.

Let W be any vector space. We define a bilinear form on $W^{\otimes k} \times W^{\otimes k}$:

Definition 2.4. The bilinear form $M_{\cdot,\cdot}$ on $W^{\otimes k} \times W^{\otimes k}$ is defined by

$$M(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, e_{i_1'} \otimes e_{i_2'} \otimes \cdots \otimes e_{i_k'}) = \prod_{p=1}^{k} M(e_{i_p}, e_{i_p'}),$$

where

$$M_{e_{i_p}, e_{i_p'}} = \begin{cases} J_{i_p, i_p'} & \text{if } i_p \text{ and } i_p' \text{ are both covariant or contravariant,} \\ \delta_{i_p, i_p'} & \text{otherwise.} \end{cases}$$

Then we can state the following

Theorem 2.7. *If $G = \mathrm{SpU(N)}$, the group integral (1.1) with signatures $\lambda^{(i)} = (m_1^{(i)}, m_2^{(i)}, \cdots, m_n^{(i)})$ with $m_1^{(i)} \geq m_2^{(i)} \geq \cdots \geq m_n^{(i)} \geq 0$ is equal to*

$$\int_{\mathrm{SpU(N)}} du \langle \tilde{T}(i_1)|\rho_{\mathrm{SpU(2N)}}^{m^{(1)}} \tilde{T}(j_1)\rangle \cdots \langle \tilde{T}(i_k)|\rho_{\mathrm{SpU(2N)}}^{m^{(k)}} \tilde{T}(j_k)\rangle)$$

$$\langle \tilde{T}(i_1')|\rho_{\mathrm{SpU(2N)}}^{*m^{(1)}} \tilde{T}(j_1')\rangle \cdots \langle \tilde{T}(i_{k'}')|\rho_{\mathrm{SpU(2N)}}^{*m^{(k')}} \tilde{T}(j_{k'}')\rangle. \tag{2.41}$$

Its asymptotic behavior with respect to N is

$$\frac{1}{(2N)^{m/2}}\sum_{\sigma\in P_m} M_{\tilde{T}(i_1)\otimes\cdots\otimes\tilde{T}(i'_{k'}),\sigma\tilde{T}(i_1)\times\cdots\otimes\tilde{T}(i'_{k'})} M_{\tilde{T}(j_1)\otimes\cdots\otimes\tilde{T}(j'_{k'}),\sigma\tilde{T}(j_1)\times\cdots\otimes\tilde{T}(j'_{k'})}+$$
$$O(\frac{1}{(2N)^{m/2+1}}), \tag{2.42}$$

where $m^{(r)}=\sum_l m_l^{(r)}$, and $m=\sum_{r=1}^{k} m^{(r)}+\sum_{r=1}^{k'} m^{'(r)}$ is even. If m is odd, the integral vanishes when $n>m$.

Then we complete the computations of the asymptotic behaviors of (1.1) for fixed signatures.

2.5 Virasoro constraints

For each of the integral, the calculation is complicated. However, inspired by [31], we propose that the partition function involving the above group integrals is annihilated by a (half) representation of Virasoro algebra. At the end, inspired by Virasoro conjecture, we propose a conjectures regarding the group integrals we have discussed in the first three chapters. Inspired by the Virasoro constraints on the partition function of random Hermitian matrices in [31], we have the following

Theorem 2.8. *If $G=\mathrm{U(N)}$, $\mathrm{SO(N)}$ or $\mathrm{SpU(2N)}$, there exist differential operators L_n satisfying*

$$[L_n,L_m]=(n-m)L_{n+m},\ n\geq -1,$$

such that the matrix integral

$$Z_N=\int_G \exp(\sum_{-\infty}^{\infty} t_n\,\mathrm{Tr}(G^n))d\mu, \tag{2.43}$$

where $d\mu$ is the Haar measure of G, is annihilated by L_n, and t_is are parameters.

Proof. If $G = \mathrm{U(N)}$, we denote the expectation of the function $\mathcal{O}(G)$ of G with respect to the partition function by $\langle \mathcal{O} \rangle$. The Haar measure of $U(N)$ is a constant times $\prod_{i<j}(e^{2\pi\sqrt{-1}\theta_j} - e^{2\pi\sqrt{-1}\theta_j})^2 d\theta_1 \cdots d\theta_N$, where $\theta \in [0, 2\pi]^N$. Let $x_i = e^{\sqrt{-1}\theta_i}$, $i = 1, 2, \cdots, N$. Inserting the partial derivative $\partial/\partial x_i$ to the expectation of $\frac{1}{z-x_i}$, $i = 1, 2, \cdots, N$, since $G = \mathrm{U(N)}$ is compact(if the integrand in the partition function is a Schwartz function, we don't need to assume that G is compact), we have

$$\langle \sum_{i=1}^{N} (\frac{\partial}{\partial x_i} + 2\sum_{j \neq i} \frac{1}{x_i - x_j} + \sum_{n \in \mathbb{Z}} n t_n x_i^{n-1})(\frac{1}{z - x_i}) \rangle = 0. \tag{2.44}$$

This is equivalent to

$$\langle W^2(z) + \sum_{i=1}^{N} \frac{1}{z - x_i} (\sum_{-\infty}^{\infty} n t_n x_i^{n-1}) \rangle = 0, \tag{2.45}$$

where $W(z) = \sum_{i=1}^{N} \frac{1}{z-x_i}$. On the other hand, $W(z)$ can be considered as a operator $\tilde{W}(z)$:

$$\tilde{W}(z) Z_N = \sum_{n \geq 0} z^{-n-1} \partial_n, \tag{2.46}$$

where $\partial_n = \frac{\partial}{\partial t_n}$. The second summand in (2.45) is also an operator

$$\sum_{m \geq 0, n \in \mathbb{Z}} z^{-m-1} n t_n \partial t_{m+n-1}. \tag{2.47}$$

Therefore (2.45) is

$$\sum_{m \geq -1, 0 \leq n_1 \leq m, n_2 \in \mathbb{Z}} (\partial_{n_1} \partial_{m-n_1} + n_2 t_{n_2} \partial_{m+n_2}) \frac{1}{z^{m+2}} Z_N = 0, \tag{2.48}$$

or $\sum_{m \geq -1} L_m \frac{1}{z^{m+2}} Z_N = 0$. It is straightforward to check these L_ms, $m \geq -1$ satisfy (2.8). The cases of other compact Lie groups can be proved similarly. Recall that for

SpU(2N), the Haar measure is proportional to

$$\Delta(x_1, \cdots, x_N, x_1^{-1}, \cdots, x_N^{-1}) \prod_{i=1}^{N} (x_i - x_i^{-1}); \tag{2.49}$$

for SO(2N) or SO(2N + 1), the Haar measure is proportional to

$$\Delta(x_1, \cdots, x_N, x_1^{-1}, \cdots, x_N^{-1}) \prod_{i=1}^{N} (x_i^{-1} - x_i)^{-1}. \tag{2.50}$$

□

CHAPTER 3
MODULI SPACE INVARIANTS

In this chapter we consider a problem from string theory whose solution involves moduli spaces from algebraic geometry, unitary representations of infinite-dimensional Lie groups arising as central extensions. We shall adopt standard terminology from algebraic geometry, for example, "a moduli space of a Riemann surface of genus g with n points punched" is a geometric space which is the collection of the complex structure with the n points on the Riemann surface [39]. Such spaces arise generally (as in our present analysis) as solutions to classification problems: For example, if one can show that a collection of smooth algebraic curves of a fixed genus can be given the structure of a geometric space, this then leads to a new parametrization; so an object viewed as an entirely separate space. This in turn is accomplished by introducing coordinates on the resulting space. In this context, the term "modulus" is used synonymously with "parameter"; moduli spaces are understood as spaces of parameters rather than as spaces of objects.

Examples: The real projective space $\mathbb{R}P^n$ is a moduli space. It is the space of lines in $\mathbb{R}^{n+1}$ which pass through the origin. Similarly, complex projective space is the space of all complex lines in $\mathbb{C}^{n+1}$. More generally, the Grassmannian $Gk(V)$ of a vector space V over a field F is the moduli space of all k-dimensional linear subspaces of V. The Hilbert scheme $\mathrm{Hilb}(X)$ is a moduli scheme. Every closed point of $\mathrm{Hilb}(X)$ corresponds to a closed subscheme of a fixed scheme X, and every closed subscheme is represented by such a point.

Moduli spaces are defined more generally in terms of the moduli functors, and spaces representing them, as is the case for the classical approaches and problems using Teichmüller spaces in complex analytical geometry. Our presentation here will take place within this generally framework of moduli spaces.

On the physics side, due to intensive study of string theory, in particular, nonlinear sigma model, the intersection number of moduli spaces play a fundamental role, see [39], where the readers can find other references. Let us recall some definitions in this subsection.

$\bar{\mathcal{M}}_{g,n}$ is the compactification of the parameter space of the complex structures of a Riemann surface with genus g and n points punched out.

$\mathcal{L}_i$ denotes the line bundle on $\bar{\mathcal{M}}_{g,n}$,which fiber at point $\{C; x_1, x_2, \cdots, x_n\}$ is the cotangent space $T^*_{x_i}(C)$.

$c_1(\mathcal{L}_i)$ denotes the first chern class of the line bundle $\mathcal{L}_i$.

When $n = 0$, $\bar{\mathcal{M}}_{g,0}$ is the quotient space of the space of metrics $\mathcal{G}_g$ by the group action of (Diff $\times$ Weyl, where Diff and Weyl denotes diffeomorphism and Weyl transformations [41].

The compactification of $\mathcal{M}_{g,n}$ is denoted by $\bar{\mathcal{M}}_{g,n}$ [8].

The following definitions are often used in this paper:

Amplitude:

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \int_{\bar{\mathcal{M}}_{g,n}} \Pi_{i=1}^{n} (c_i(\mathcal{L}_i))^{d_i}.$$

Another notation that will be used is

$$\langle \tau_0^{r_0} \tau_1^{r_1} \tau_2^{r_2} \cdots \rangle := \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle,$$

where r_0 of the d_is are equal to 0;r_1 of them are equal to 1,etc.

Free energy of genus g:

$$F_g(t_0, t_1, \cdots) = \langle \exp(\sum_{i=0}^{\infty} t_i \tau_i) \rangle = \sum_{(k)} \langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle \prod_{i=0}^{\infty} \frac{t_i^{k_i}}{k_i!}, \tag{3.1}$$

Partition function:

$$Z(t_*) = \exp \sum_{g=0}^{\infty} F_g(t_*), \tag{3.2}$$

where the free energy $F_g(t_*)$. Witten's conjecture (1990) asserts that

the partition function $Z(t_*) = \exp \sum_{g=0}^{\infty} F_g(t_*)$ *[47] is the τ-function of the KdV hierarchy.*

A τ-function for the KdV hierarchy means from the τ-function, we can construct the solution of the KdV equation:

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}, \tag{3.3}$$

where $U = \frac{\partial^2 \ln Z(t_*)}{\partial t_0^2}$.

The Korteweg-de Vires equations(KdV) have their origins in the story of water waves in a shallow channel, and have numerous of applications different from their origin. If x is the space variable, the standard form of KdV is

$$u_t + u_{xxx} + 6uu_x = 0. \tag{3.4}$$

This equation is known to have soliton solutions, i.e., $u(x,t) = f(x-ct)$ where f satisfies

$$-cf^{'} + f^{'''} + 6ff^{'} = 0. \tag{3.5}$$

so,

$$f = \frac{c}{2\cosh^2(\frac{\sqrt{c}}{2}(x-a))},$$

where a and c are constants. Since this initial discovery, other after invariants have been found. The first such sequence of invariants came in 1960s from P.Lax's commutator method: Let $u = u(x,t)$ be a function in two variables, and consider

$$(L_u(t)f)(x) = -f^{''} + u(x,t)f(x),$$

i.e., the Schrödinger operator

$$L_u(t) = -(\frac{d}{dx})^2 + u(x,t) \tag{3.6}$$

acting on function $f(x)$.

Lax (1968) found that if

$$A = 4(\frac{\partial}{\partial x})^3 - 3(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u).$$

Then

$$\frac{\partial}{\partial t}L_u(t) = [L_u, A] = L_uA - AL_u, \tag{3.7}$$

and this accounts for one infinite family of "integrals" or invariants. Specifically with $u = u(x,t)$, consider $L = L_u$ as in (3.6), we then obtain that eigenvalues of (3.6) yield invariants, and the relevant u in (3.7) is from the solutions to the KdV equation (3.4).

Kontsevich's proof mainly consists of three steps:

1.Based on a theorem of Strebel, the one to one correspondence of the space $\bar{\mathcal{M}}_i \times \mathbb{R}_+^n$ with the "fat graphs". And then the main identity is proved:

$$\sum_{d_*:\sum d_i=d} \langle \tau_{d_1} \cdots \rangle \prod_{i=1}^{n} \frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}} = \sum_{\Gamma \in G_{g,n}} \frac{2^{-v(\Gamma)}}{|Aut\Gamma|} \prod_{e\in E} \frac{2}{\tilde{\lambda}(e)}. \tag{3.8}$$

2.By the main identity and the Feynman diagram techniques, the partition function which is the exponential of the free energy $F(t_0(\Lambda), t_1(\Lambda), \cdots)$, where

$$t_i(\Lambda) = -(2i-1)!!tr\ (\Lambda^{-(2i+1)})$$

is the asymptotic expansion of the random matrix integral

$$I_N(\Lambda) = \int \exp(\frac{\sqrt{-1}}{6} tr\ (M^3)) d\mu_{(\Lambda)}(M), \tag{3.9}$$

where the measure $d\mu_{(\Lambda)}(M) = \frac{-e^{\frac{tr\ (M^2\Lambda)}{2}} dM}{\int e^{-\frac{tr\ (M^2\Lambda)}{2}} dM}$.

3. By expansion of matrix Airy function and some properties of τ-functions of KdV hierarchy,the author shows the integral (3.9) is the τ-function is the asymptotic expansion of (3.9) as the rank of Λ goes to infinity.

3.1 Matrix integral to fat graph

A *quadratic differential* ϕ on a Riemann surface C of finite type is a holomorphic section of the line bundle $(T^*)^{\otimes 2}$. A nonzero quadratic differential defines a metric in a local coordinates z:

$$|\phi(z)|^2|dz|^2, \ where \ \phi = \phi(z)dz^2. \tag{3.10}$$

A *horizontal trajectory* of a quadratic differential is a curve along which $\phi(z)dz^2$ is real and positive. JS quadratic differentials are those for which the union of nonclosed trajectories has measure zero. In 1960s, Strebel proved the following

Theorem 3.1. *(Strebel, 1960s) For any connected Riemann surface C and n distinct points $x_1, \cdots, x_n \in C, n > 0, n > \chi(C)$ and n positive real numbers $p_1, p_2, \cdots, p_n$ there exists a unique JS quadratic differential on $C/\{x_1, \cdots, x_n\}$ whose maximal ring domains are n punctured disks D_i surrounding points x_i with circumference p_i.*

Based on Strebel's theorem, Kontsevich found the one to one correspondence between the fat graphs, which are formed by the closed horizontal trajectories, which end at the zeros of the JS forms. and the product space $\bar{\mathcal{M}}_{g,n} \times \mathbb{R}_+^n$. Each graph carries the following structures

(1)for each vertex a cyclic order on the set of germs of edges meeting this vertex is fixed;

(2) to each edge is attached a positive real number, its length(which is determined by the metric);

(3)the valency of each vertex of a fat graph is three(we can derive that each valency is at least 3 by changing to polar system);

(4) the loops of the graph is numbered by $1, 2, \cdots, n$;

(5) we make these graphs double-line graphs(this is not required by the one to one

correspondence theorem).

For a fat graph, denote l_e the length of a edge(double) e and for each face f, the perimeter $p_f = \sum_{e\subset f} l_e$. Then we have

$$E - n - V = 2g - 2, \tag{3.11}$$

$$2E = 3V. \tag{3.12}$$

Kontsevich proved the first chern class $c_1(\mathcal{L}_i)$ can be written as (this step is not hard)

$$\omega_i = \sum_{a,b\in f_i} d(l_1/p_i) \wedge d(l_b/p_i), \tag{3.13}$$

and we can define a volume form on the fat graph space which is $\Omega^d/d!$, where $\Omega = \sum_i p_i^2\omega_i$.

Now since the volume form on the fat graph space M^{comb} is defined, we can compute the Laplace transform with respect to $p_1, p_2, \cdots, p_n$

$$\begin{aligned}&\prod_{i=1}^{n}\Big(\int_0^\infty e^{-\lambda_i p_i} dp_i\Big) \int \frac{(\sum_i p_i^2\omega_i)^d}{(d)!}\\ =&2^d \sum_{d_1+d_2+\cdots+d_n=d} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i - 1)!!\lambda_i^{-2d_i-1}.\end{aligned} \tag{3.14}$$

On the other hand, by a very delicate argument on complex cohomology, one has

$$1/d! \prod_i dp_i \wedge \Big(\sum_i \sum_{a,b\subset f_i} dl_1 \wedge dl_b\Big)^d = 2^{5g-5+2n} dl_1 \wedge dl_2 \cdots dl_E. \tag{3.15}$$

In the right hand side, we endow a orientation.

In the factor $2^{5g-5+2n}$ in the right hand side of (3.15) can be derived by computing the torsions of chain complexes that are constructed based on the structure of (3.15)[27](for the definition and theory of torsion, see [44]). Alternatively, we can prove it in by induction on the number of edges and the genus.

3.2 KP and KdV hierarchy

In [25], Itzykson and Zuber proved that

$$I_N(\Lambda) = \begin{vmatrix} \sum_{n\geq 0} c_n^{(0)} S_{3n}(\theta.) & \cdots & \sum_{n\geq 0} c_n^{(0)} S_{3n+N-1}(\theta.) \\ \vdots & \vdots & \vdots \\ \sum_{n\geq 0} c_n^{(N-1)} S_{3n-N+1}(\theta.) & \cdots & \sum_{n\geq 0} c_n^{(N-1)} S_{3n}(\theta.) \end{vmatrix}, \tag{3.16}$$

where $c_n^{(i)}$s are constants; $\theta_k = \frac{1}{k}\operatorname{Tr}(\Lambda^{-k})$ and $p_n(\theta.)$s are Schur functions defined by

$$\sum_0^\infty p_k(\theta.)u^k = e^{\sum_1^\infty \theta_n u^n}. \tag{3.17}$$

Then we shall show that (3.16) satisfies KdV equation.

We shall follow Palle Jorgensen's book [26] and Kac's book [27] to derive KP hierarchy. We offer a simple philosophy to get the the KP hierarchy , which is a set of infinite many PDEs. Let $\mathcal{A}$ be the Heisenberg algebra, the complex Lie algebra with a basis $\{a_n, n \in \mathbb{Z}; \hbar\}$, with the commutation relations

$$[\hbar, a_n] = 0,\ \ (n \in \mathbb{Z}),$$

$$[a_m, a_n] = m\delta_{m,-n}\hbar\ (m, n \in \mathbb{Z}). \tag{3.18}$$

, and $\{L_n\}$ denotes the Virasoro algebra with central extension c, i.e.,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\delta_{m,-n}}{12}(m^3 - m)c. \tag{3.19}$$

One thing that is nice in the context of our two infinite systems of operators a_n and L_k below in the following is the following close analogue to an important family of unitary representations of Lie groups. It is in fact a natural extension of what was first realized for finite dimensional groups as the Weil-Segal-Shale representations of the metaplectic groups. In fact, consider the following setting for a finite-dimensional Heisenberg group H. Let G be the corresponding group of automorphisms of H which fix the center. Then G is a finite dimensional Lie group, the metaplectic group. Pick a Schröedinger representation of H, and compose it with an automorphism, so an element in G. The result is a second representation of H. By the Stone-von Neumann uniqueness theorem the two representations are unitarily equivalent, and so the equivalence is implemented by a unitary operator $U(g)$. By passing to a double cover of $\tilde{G}$ one can show that $U(g)$ in fact then defines a unitary representation of $\tilde{G}$. If we now pass to the corresponding Lie algebras $L(H)$ and $L(G)$ we see that $L(H)$ is normalized by $L(G)$. Moreover $L(H)$ in the Schröedinger representation is spanned by Heisenberg's canonical operators P, Q, and the one-dimensional center; here we write P for momentum and Q for position, possibly with several degrees of freedom. By comparison, in the Weil representation, the Lie algebra $L(G)$ is then spanned by all the quadratic polynomials in the Ps and the Qs. Now the Stone-von Neumann uniqueness theorem is not valid for an infinite number of degrees of freedom, but nonetheless, the representations of the two Lie algebras we present by the infinite systems of operators a_n and L_k in sections 4 present themselves as a close analogy to the Weil representations in the case of Lie groups, i.e., the case of a finite number of degrees of freedom. Our infinite-dimensional Virasoro Lie algebra spanned by the infinite system $\{a_n\}$, and it is a central extension; hence a direct analogue of the Heisenberg Lie algebra. Similarly, our infinite-dimensional Lie algebra of quantum fields spanned by $\{L_k\}$ normalizes the Virasoro Lie algebra, and so it is a direct analogue of the Lie algebra of operators $L(G)$ in the finite-dimensional case. In both of these cases of representations, the operators in the respective Lie algebras are unbounded but densely defined in the respective infinite-dimensional Hilbert spaces. We show that our representations of the Lie algebra of quantum fields spanned by $\{L_k\}$ may be obtained with the use of highest weight vectors, and weights.

In Kac and Raina's presentation[28], V is a infinite dimensional complex vector space with a fixed basis $\{v_j; j \in \mathbb{Z}\}$, i.e., $V = \oplus_{j\in\mathbb{Z}}\mathbb{C}v_j$. The Lie group GL_∞ is defined by

$$\mathrm{GL}_\infty = \{(\mathrm{a_{ij}})_{\mathrm{i,j}\in\mathbb{Z}};\ \text{invertible and all but finite numbers of } a_{ij} - \delta_{ij} \text{ are } 0\}, \quad (3.20)$$

the Lie algebra is defined by

$$gl_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}};\ \text{all but a finite number of the } a_{ij} \text{ are } 0\}, \quad (3.21)$$

and the bigger Lie algebra $\bar{a}_\infty$ is defined by

$$\bar{a}_\infty = \{(a_{ij}); i, j \in \mathbb{Z}, a_{ij} = 0 \text{ for } |i - j| \ll 0\}. \quad (3.22)$$

Obviously, all of them have a representation on infinite wedge space $\wedge^\infty V$.

The shift operator $\Lambda_k \in \bar{a}_\infty$ is defined by

$$\Lambda_k v_j = v_{j-k}, \quad (3.23)$$

i.e.,

$$\Lambda_k = \sum_{i\in\mathbb{Z}} E_{i,i+k}. \quad (3.24)$$

The fermion space $F^{(m)}$ is the linear span of semi-infinite monomials of the form

$$\phi = v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots , \tag{3.25}$$

where

$$i_m > i_{m-1} > \cdots \tag{3.26}$$

and

$$i_k = k + m \quad \text{for } k \ll 0. \tag{3.27}$$

Then the representation r of gl_∞ on $F^{(m)}$ is defined by

$$\begin{aligned} &r(a)(v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots) \\ =&av_{i_m} \wedge v_{i_{m-1}} \wedge \cdots + v_{i_m} \wedge av_{i_{m-1}} \wedge \cdots + \cdots . \end{aligned} \tag{3.28}$$

$\hat{r}$ is defined by

$$\begin{aligned} &\hat{r}(E_{ij}) = r_m(E_{ij}) \quad \text{if } i \neq j \text{ or } i = j > 0, \\ &\hat{r}(E_{ii}) = r_m(E_{ii}) - I \text{ if } i \leq 0. \end{aligned} \tag{3.29}$$

The boson space $B^{(m)}$ is $\mathbb{C}[x_1, x_2, \cdots]$. There is an isomorphism $\sigma_m : F^{(m)} \to B^{(m)}$ determined by

$$\hat{r}_m^B(\Lambda_k) = \frac{\partial}{\partial x_k},$$

$$\hat{r}_m^B(\Lambda_{-k}) = kx_k, \ \hat{r}_m^B(\Lambda_0) = m. \tag{3.30}$$

.

By the fermion-boson correspondence, GL_∞ has a representation on the space $B = \mathbb{C}[x_1, x_2, \cdots]$, the polynomial ring of infinite many variables. Then we have a representation of GL_∞ on B. Then the KP hierarchy is the orbit Ω of the the vacuum 1 in B under the action of GL_∞, i.e., $\Omega = \mathrm{GL}_\infty \cdot 1$. Also, Dirac's positron theory can be given a representation-theoretic interpretation and used to obtain highest weight representations of these Lie algebras. The following are the relevant definitions and theorems.

It was proved that any element $\Omega \subset B$ satisfies the Kadomtzev-Petviashvili(KP) equation:

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right). \tag{3.31}$$

If u doesn't depend on y, then (3.31) becomes KdV equation. Therefore I_N satisfies KdV equation.

3.3 Virasoro constraints on Ω

The τ-function of the KdV hierarchy is annihilated by a sequence of differential operators, which form a half branch of the Virasoro algebra. ([8],[14], and [29]). For the partition function $I_N(\Lambda)$, the Virasoro constraints arise in the following manner. According to [27], the Virasoro algebra with central charge c_β has representations on $F = \oplus_{m\in\mathbb{Z}} F^{(m)}$

$$L_i = \hat{r}(d_i) \text{ if } i \neq 0,$$

$$L_0 = \hat{r}(d_0) + h_0, \tag{3.32}$$

where $c_\beta = -12\beta^2 + 12\beta - 2$, $h_m = \frac{1}{2}(\alpha - m)(\alpha + 2\beta - 1 - m)$; and

$$d_{n-k}(v_k) = -(k + \alpha + \beta + \beta(n-k))v_n. \tag{3.33}$$

If $\alpha = 0$, it is easy to check that $i \geq -1$ annihilates 1 in B. Therefore for any element in Ω, there are corresponding representation of Virasoro algebra still denoted by L_n such that it is annihilated by L_n, $n \geq -1$. The explicit expression of the differential operators of L_is are determined by the following method: In fact, when $i \neq 0$,

$$\begin{aligned} L_i =& \hat{r}(d_i) \\ =& \hat{r}(\sum_{k\in\mathbb{Z}}(k - \alpha - \beta(i+1))E_{k-i,k}) \\ =& \sum_{k\in\mathbb{Z}}(k - \alpha - \beta(i+1))\hat{r}(E_{k-i,k}) \\ =& \sum_{k\in\mathbb{Z}}(k - \alpha - \beta(i+1))r(E_{k-i,k}), \end{aligned} \tag{3.34}$$

and when $i = 0$,

$$\begin{aligned} L_0 &= \hat{r}(d_0) + h_0 \\ &= \hat{r}(\sum_{k\in\mathbb{Z}}(k - \alpha - \beta)E_{k,k}) + h_0 \\ &= \sum_{k\in\mathbb{Z}}(k - \alpha - \beta)\hat{r}(E_{k,k}) + h_0 \\ &= \sum_{k>0}(k - \alpha - \beta)(r(E_{k,k}) - I) + \sum_{k\leq 0}(k - \alpha - \beta)r(E_{k,k}) + h_0. \end{aligned} \tag{3.35}$$

Moreover, $\hat{r}(E_{ij})$ is determined by

$$\sum_{i,j\in\mathbb{Z}} u^i v^{-j} r^B(E_{ij}) \equiv \sigma_m(X(u)X^*(v))\sigma_m^{-1} = \frac{(u/v)^m}{1-(v/u)}\Gamma(u,v), \tag{3.36}$$

where $\Gamma(u,v)$ is the vertex operator

$$\Gamma(u,v) = \exp(\sum_{j\geq 1}(u^j - v^j)x_j)\exp(-\sum_{j\geq 1}\frac{u^{-j}-v^{-j}}{j}\frac{\partial}{\partial x_j}). \tag{3.37}$$

3.4 Difficulty of a conjecture of Kontsevich

Kontsevich also proposed some conjectures in [27]. Let us see the some of them that are concerned with the KdV hierarchies. First of all, one can introduce variables s :

$$Z(t_0, t_1, \cdots, ; s_0, s_1, \cdots) = \exp(\sum_{n_* m_*}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{m_0,m_1,\cdots}\prod_{i=0}^{\infty}\frac{t_i^{n_i}}{n_i!}\prod_{j=0}^{\infty}s_j^{m_j}). \tag{3.38}$$

It can be shown ([27]) that that $Z(t_*, s_*)$ is an asymptotic expansion of

$$I_N(\Lambda) = \int \exp(\sqrt{-1}\sum_{j=0}^{\infty}(-1/2)^j s_j \frac{\mathrm{Tr}\ M^{2j+1}}{2j+1})d\mu_\Lambda(M). \tag{3.39}$$

Then we can list the statements of these conjectures are

1. Z is a τ-function for KdV-hierarchy in variables $T_{2i+1} := \frac{t_i}{(2i+1)!!}$ for arbitrary s.

2. Z is a τ-function for KdV-hierarchy in variables $T_{2i+1} := \frac{s_i}{(2i+1)!!}$ for arbitrary t.

3. Let T be any formal τ-function for the KdV-hierarchy considered as a matrix function. Then $\int T(X)d\mu_\Lambda(X)$ is a matrix τ-function for the KdV-hierarchy in Λ.

We shall explain the difficulty of the first conjecture in this subsection. We recall the Harish-Chandra formula [23].

Lemma 3.2. *If Φ is a conjugacy invariant function on the space of hermitian $N\times N$-matrices, then for any diagonal hermitian matrix Y,*

$$\int \Phi(X)e^{-\sqrt{-1}\,\mathrm{Tr}\,XY}dX = (-2\pi\sqrt{-1})^{N(N-1)/2}(V(Y))^{-1}\int \Phi(D)e^{-\sqrt{-1}\,\mathrm{Tr}\,DY}V(D)dD, \tag{3.40}$$

where the last integral is taken over the space of diagonal hermitian matrices D;V is the Vandermonde Polynomial determinant which is defined by

$$V(diag(X_1, X_2, \cdots, X_n)) = \prod_{i<j}(X_j - X_i) = \det(X_i^{j-1}). \tag{3.41}$$

Harish-Chandra generalized the above fact[21]:

Let G be a compact simple Lie group, L its Lie algebra of order N and rank n, W the Weyl group of L,R_+ the set of positive roots, and $m_i = d_i - 1$ its Coxerter indexes. Also X and Y elements of L. Let (X, Y) be a bilinear form which is invariant under G,i.e.,$(gX, gY) = (X, Y)$,for $\forall g \in G$. Then

$$\int_{g\in G} \exp(c(X, gYg^{-1})dg = \text{const} \sum_{w\in W} \epsilon_w \exp(c(X, wY)/ \prod_{\alpha\in R_+} (\alpha, X)(\alpha, Y). \tag{3.42}$$

We need the following

Lemma 3.3. *$I_N(\Lambda)$ is symmetric with respect to Λ and $I_N(\Lambda)$ is in the field $\mathbb{C}(\Lambda^{-1})$, which is the field of polynomial ring of Λ.*

Proof. We apply Harish-Chandra's result to the unitary group U(N), then

$$\begin{aligned}
&\int \Phi(X)e^{-\frac{1}{2}\operatorname{Tr}\Lambda X^2}dX \\
=&\text{const}\int \Phi(D)(\int e^{-\frac{1}{2}\operatorname{Tr}\Lambda U D^2 U^{-1}}dU)V^2(D)dD \\
=&\text{const}\int \Phi(D)
\end{aligned}$$

$$\cdot \Big(\sum_{w\in S_N} \operatorname{sign}(w) e^{-\frac{1}{2}\operatorname{Tr}(\Lambda U^{-1}D^2 U)}\Big/ \prod_{1\le i<j\le N} \operatorname{Tr}((\epsilon_j-\epsilon_i)\Lambda)\operatorname{Tr}((\epsilon_j-\epsilon_i)D^2) V^2(D)dD$$

$$= \mathrm{const}\int \Phi(D)$$

$$\cdot \Big(\sum_{w\in S_N} \operatorname{sign}(w) e^{-\frac{1}{2}\operatorname{Tr}(\Lambda U^{-1}D^2 U)}\Big/ \prod_{1\le i<j\le N} \operatorname{Tr}((\epsilon_j-\epsilon_i)\Lambda)\operatorname{Tr}((\epsilon_j-\epsilon_i)D^2) V^2(D)dD$$

$$= \mathrm{const}\int \Phi(D)V(D)\frac{\sum_{w\in S_N}\operatorname{sign}(w)e^{-\frac{1}{2}\operatorname{Tr}(\Lambda w(D^2))}}{V(\Lambda)\prod_{i<j}(D_i+D_j)}dD. \tag{3.43}$$

By direct computation,

$$\int \exp(-\frac{1}{2}\operatorname{Tr}\Lambda M^2)dM = 2^{\frac{N(N-1)}{2}}(2\pi)^{\frac{N^2}{2}}\prod_{r=1}^{N}\lambda_r^{-\frac{1}{2}}\prod_{i<j}(\lambda_i+\lambda_j)^{-1}. \tag{3.44}$$

Hence we have

$$\mathrm{const}\int V(D)\frac{\sum_{w\in S_N}\operatorname{sign}(w)e^{-\frac{1}{2}\operatorname{Tr}(\Lambda w(D^2))}}{\prod_{i<j}(D_i+D_j)}dD = V(\Lambda)\prod_{r=1}^{N}\lambda_r^{-\frac{1}{2}}\prod_{i<j}(\lambda_i+\lambda_j)^{-1}. \tag{3.45}$$

Acting on both sides by operator $\sum_{i=1}^{N}\frac{\partial}{\partial\lambda_i}$ and multiplying

$$\frac{\prod_{r=1}^{N}\lambda_r^{\frac{1}{2}}\prod_{i<j}(\lambda_i+\lambda_j)}{V(\Lambda)},$$

we get

$$\mathrm{const}\frac{\prod_{r=1}^{N}\lambda_r^{\frac{1}{2}}\prod_{i<j}(\lambda_i+\lambda_j)}{V(\Lambda)}$$

$$\cdot\int(\sum_{i=1}^{N}D_i^2)V(D)\frac{\sum_{w\in S_N}\operatorname{sign}(w)e^{-\frac{1}{2}\operatorname{Tr}(\Lambda w(D^2))}}{\prod_{i<j}(D_i+D_j)}dD$$

$$= -\sum_{r=1}^{N}\frac{1}{\lambda_r}-\sum_{r<k}\frac{2}{\lambda_r+\lambda_k}. \tag{3.46}$$

$\square$

Therefore $\langle\sum_{i=1}^{N} D_i^2\rangle$ is in $\mathbb{C}(\Lambda^{-1})$, but not in $\mathbb{C}[\Lambda^{-1}]$. By induction, we can prove that for any symmetric polynomial $P(\Lambda^{-1})$ of Λ^{-1}, $\langle P(\Lambda^{-1})\rangle \in \mathbb{C}(\Lambda^{-1})$, but not in $\mathbb{C}[\Lambda^{-1}]$. This is the main reason why the first conjecture is hard to prove, since the τ-function of KP hierarchy is in $\mathbb{C}[\Lambda^{-1}]$.

CHAPTER 4
VIRASORO CONTRAINTS AND RANDOM MATRICES

4.1 Virasoro conjecture and matrix model

Let us recall the definition of $\bar{\mathcal{M}}_{g,n}(M,\beta)$ and some properties [39].

Definition 4.1. Let M be a non-singular projective variety. A morphism f from a pointed nodal curve to X is a stable map if every genus 0 contracted component of Σ has at least three special points, and every genus 1 contracted component has at least one special point.

Definition 4.2. A stable map represents a homology class $\beta \in H_2(M,\mathbb{Z})$ if $f_*(C) = \beta$.

The moduli space of stable maps from n-pointed genus g nodal curves to M representing the class β is denoted $\bar{\mathcal{M}}_{g,n}(M,\beta)$. The moduli space $\bar{\mathcal{M}}_{g,n}(M,\beta)$ is a Deligne-Mumford stack. It has the following properties:

(1)There is an open subset $\bar{\mathcal{M}}_{g,n}(M,\beta)$ corresponding maps from non-singular curves.

(2)$\bar{\mathcal{M}}_{g,n}(M,\beta)$ is compact.

(3)There are n "evaluation maps" $\mathrm{ev}_i : \bar{\mathcal{M}}_{g,n}(M,\beta) \to M$ defined by

$$\mathrm{ev}_i(\Sigma, p_1, \cdots, p_n, f) = f(p_i),\ 1 \le i \le n. \tag{4.1}$$

(4)If $n_1 \ge n_2$, there is a "forgetful morphism"

$$\bar{M}_{g,n_1}(M,\beta) \to \bar{M}_{g,n_2}(M,\beta). \tag{4.2}$$

so long as the space on the right exists.

(5)There is a "universal map" over the moduli space:

$$(\tilde{\Sigma}, \tilde{p_1}, \cdots, \tilde{p_n}) \xrightarrow{\bar{f}} M,$$

$$(\tilde{\Sigma}, \tilde{p_1}, \cdots, \tilde{p_n}) \xrightarrow{\pi} \bar{\mathcal{M}}_{g,n}(M, \beta). \tag{4.3}$$

(6)Given a morphism $g : X \to Y$, there is an induced morphism

$$\bar{\mathcal{M}}_{g,n}(X, \beta) \to \bar{M}_{g,n_2}(Y, g_*\beta), \tag{4.4}$$

so long as the space on the right exists.

(7)Under certain nice circumstances, if M is convex, $\bar{M}_{0,n}(M, \beta)$ is non-singular of dimension

$$\int_\beta c_1(T_M) + \dim M + n - 3. \tag{4.5}$$

Definition 4.3. At each point $[\Sigma, p_1, \cdots, p_n, f]$ of $\bar{\mathcal{M}}_{g,n}(X, \beta)$, the cotangent line to σ at point p_i is a one dimensional vector space, which gives a line bundle $\mathbb{L}_i$, called the ith tautological line bundle.

Given classes $\gamma_1, \gamma_2, \cdots, \gamma_k \in H^*(M, \mathbb{Q})$, the gravitational descendant invariants are defined by

$$\begin{aligned}
&\langle \tau_{n_1}(\gamma_1)\tau_{n_2}(\gamma_2) \cdots \tau_{n_k}(\gamma_k)\rangle \\
&= \sum_{A \in H_2(M,\mathbb{Z})} q^A \int_{[\bar{M_{g,k}}(M,A)]^{\mathrm{Virt}}} c_1(\mathbb{L}_1)^{n_1} \cup \mathrm{ev}_1^*(\gamma_1) \cup c_1(\mathbb{L}_2)^{n_2} \cup \mathrm{ev}_2^*(\gamma_2) \cdots \\
&c_1(\mathbb{L}_k)^{n_k} \cup \mathrm{ev}_1^*(\gamma_k)
\end{aligned} \tag{4.6}$$

The free energy F_g^M can be written as

$$F_g^M(t) := \langle \exp(\sum_{n\geq 0, 1\leq\alpha\leq N} t_n^\alpha \tau_n(\mathcal{O}_\alpha))\rangle_g, \tag{4.7}$$

where $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_N$ form a basis of $H^*(M, \mathbb{Q})$; α ranges from 1 to N; n ranges over nonnegative integers; only finite t_n^α are nonzero.

In 1997,T. Eguchi, K. Hori and C. Xiong and S.Katz conjectured that the partition function

$$Z^M(t) = \exp(\sum_{g\geq 0} \lambda^{2-2g} F_g^M(t)). \tag{4.8}$$

is annihilated by L_n, $n \geq -1$, which forms part of Virasoro algebra with central charge $c = \chi(M)$,i.e., $\{L_n\}$ satisfy

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\delta_{m,-n}}{12}(m^3 - m)\cdot\chi(M), \tag{4.9}$$

for $m, n \in \mathbb{Z}$. This conjecture is called Virasoro conjecture in many literatures. Since it was proposed, there have been lots of efforts on it. It has been confirmed up to genus 2 [34] and there have been good results [35][15][36][37] and etc. The representation of L_n, $n \geq -1$ is

$$\begin{aligned}
L_{-1} &= \sum_{\alpha=0}^{N}\sum_{m=1}^{\infty} m t_m^\alpha \partial_{m-1,\alpha} + \frac{1}{2\lambda^2}\sum_{\alpha=0}^{N} t^\alpha t_\alpha, \\
L_0 &= \sum_{\alpha=1}^{N}\sum_{m=0}^{\infty}(m+b_\alpha)t_m^\alpha \partial_{m,\alpha} + (N+1)\sum_{\alpha=1}^{N}\sum_{m=0}^{\infty} m t_m^\alpha \partial_{m-1,\alpha+1} \\
&\quad + \frac{1}{2\lambda^2}\sum_{\alpha=1}^{N}(N-1)t^\alpha t_{\alpha+1} - \frac{1}{48}(N-1)(N+1)(N+3), \\
L_n &= \sum_{m=0}^{\infty}\sum_{\alpha,\beta}\sum_{j} C_\alpha^{(j)}(m,n)(\mathcal{C}^j)_\alpha^\beta t_m^\alpha \partial_{m+n-j,\beta} \\
&\quad + \frac{\lambda^2}{2}\sum_{\alpha,\beta}\sum_{j}\sum_{m=0} D_\alpha^{(j)}(\mathcal{C})_\alpha^\beta \partial_m^\alpha \partial_{n-m-j-1,\beta} + \frac{1}{2\lambda^2}\sum_{\alpha,\beta}(\mathcal{C}^{n+1})_\alpha^\beta t^\alpha t_\beta,
\end{aligned} \tag{4.10}$$

$$b_\alpha = q_\alpha - \frac{d-1}{2},\ \omega_\alpha = H^{2q_\alpha}(M), \tag{4.11}$$

where d is the complex dimension of the manifold M;

$$\mathcal{C}_\alpha^\beta = \int_M c_1(M)\wedge\omega_\alpha\wedge\omega^\beta,\ \eta_{\alpha\beta} = \delta_{\alpha+\beta,N}, \tag{4.12}$$

and $\mathcal{C}^j$ is the j-th power of the matrix $\mathcal{C}$;

$$\begin{aligned} C_\alpha^{(j)}(m,n) =& \frac{(b_\alpha+m)(b_\alpha+m+1)\cdots(b_\alpha+m+n)}{(m+1)(m+2)\cdots(m+n-j)} \\ & \sum_{m\le l_1<l_2<\cdots<l_j\le m+n}\prod_j(\frac{1}{b_\alpha+l_j}); \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} D_\alpha^j(m,n) =& \frac{b^\alpha(b^\alpha+1)\cdots(b^\alpha+m)b_\alpha(b_\alpha+1)\cdots(b_\alpha+n-m-1)}{m!(n-m-1)!} \\ & \sum_{-m\le l_1<l_2<\cdots<l_j\le n-m-1}\prod_j(\frac{1}{b_\alpha+l_j}) \end{aligned} \tag{4.14}$$

The operators (4.10) form a Virasoro algebra with a central charge $c = \sum_\alpha 1 = \chi(M)$, if the following condition is satisfied

$$\frac{1}{4}\sum_\alpha b^\alpha b_\alpha = \frac{1}{24}(\frac{3-\mathrm{M}}{2}\chi(M) - \int_M c_1(M)\wedge c_{d-1}(M)). \tag{4.15}$$

It was found that the above definition really forms a Virasoro algebra [10]. It is instructive to verify it really forms an algebra here:

$$\begin{aligned} & [L_{n_1}, L_n] \\ =& [\sum_{m_1=0}^{\infty}\sum_{\alpha_1,\beta_1}\sum_{j_1} C_{\alpha_1}^{(j_1)}(m_1,n_1)(\mathcal{C}^{j_1})_{\alpha_1}^{\beta_1} t_{m_1}^{\alpha_1}\partial_{m_1+n_1-j_1,\beta_1} \\ & + \frac{\lambda^2}{2}\sum_{\alpha_1,\beta_1}\sum_{j_1}\sum_{m_1=0} D_{\alpha_1}^{(j_1)}(m_1,n_1)(\mathcal{C}^{j_1})_{\alpha_1}^{\beta_1}\partial_{m_1}^{\alpha}\partial_{n_1-m_1-j_1-1,\beta} \end{aligned}$$

$$+\frac{1}{2\lambda^2}\sum_{\alpha_1,\beta_1}(\mathcal{C}^{n_1+1})^{\beta_1}_{\alpha_1}t^{\alpha_1}t_{\beta_1},\sum_{m=0}^{\infty}\sum_{\alpha,\beta}\sum_{j}C^{(j)}_{\alpha}(m,n)(\mathcal{C}^j)^{\beta}_{\alpha}t^{\alpha}_m\partial_{m+n-j,\beta}$$

$$+\frac{\lambda^2}{2}\sum_{\alpha,\beta}\sum_{j}\sum_{m=0}^{\infty}D^{(j)}_{\alpha}(m,n)(\mathcal{C}^j)^{\beta}_{\alpha}\partial^{\alpha}_m\partial_{n-m-j-1,\beta}+\frac{1}{2\lambda^2}\sum_{\alpha,\beta}(\mathcal{C}^{n+1})^{\beta}_{\alpha}t^{\alpha}t_{\beta}]$$

$$=\sum_{m_1=0}^{\infty}\sum_{\alpha_1,\beta_1}\sum_{j_1}\sum_{m=0}^{\infty}\sum_{\alpha,\beta}\sum_{j}C^{(j_1)}_{\alpha_1}(m_1,n_1)$$

$$(\mathcal{C}^{j_1})^{\beta_1}_{\alpha_1}C^{(j)}_{\alpha}(m,n)(\mathcal{C}^j)^{\beta}_{\alpha}(\delta_{\alpha,\beta_1}\delta_{m_1+n_1-j_1,m}t^{\alpha_1}_{m_1}\partial_{m+n-j,\beta}-\delta_{m+n-j,m_1}\delta_{\beta,\alpha_1}t^{\alpha}_m\partial_{m_1+n_1-j_1,\beta_1})$$

$$-\sum_{m_1=0}^{\infty}\sum_{\alpha_1,\beta_1}\sum_{j_1}\sum_{\alpha,\beta}\sum_{j}\sum_{m=0}^{\infty}\frac{\lambda^2}{2}C^{(j_1)}_{\alpha_1}(m_1,n_1)$$

$$(\mathcal{C}^{j_1})^{\beta_1}_{\alpha_1}D^{(j)}_{\alpha}(m,n)(\mathcal{C}^j)^{\beta}_{\alpha}(\eta^{\alpha\gamma}\delta_{m_1,m}\delta_{\gamma,\alpha_1}\partial_{n-m-j-1,\beta}\partial_{m_1+n_1-j_1,\beta_1}$$

$$+\delta_{n-m-j-1,,m_1}\delta_{\alpha_1,\beta}\partial^{\alpha}_m\partial_{m_1+n_1-j_1,\beta_1})$$

$$+\sum_{m_1=0}^{\infty}\sum_{\alpha_1,\beta_1}\sum_{j_1}\sum_{\alpha,\beta}\frac{1}{2\lambda^2}C^{(j_1)}_{\alpha_1}(m_1,n_1)(\mathcal{C}^{j_1})^{\beta_1}_{\alpha_1}(\mathcal{C}^{n+1})^{\beta}_{\alpha}(t^{\alpha_1}_{m_1}(\delta_{\beta_1,\alpha}t_{m_1+n_1-j_1,\beta}$$

$$+\eta_{\beta,\gamma}t^{\alpha}_{m_1+n_1-j_1}\delta_{\beta_1,\gamma}))$$

$$+\sum_{m_1=0}^{\infty}\sum_{\alpha_1,\beta_1}\sum_{j_1}\sum_{\alpha,\beta}\sum_{j}\sum_{m=0}^{\infty}\frac{\lambda^2}{2}C^{(j)}_{\alpha}(m,n)(\mathcal{C}^j)^{\beta}_{\alpha}D^{(j_1)}_{\alpha_1}(m_1,n_1)(\mathcal{C}^{j_1})^{\beta_1}_{\alpha_1}$$

$$(\eta^{\alpha\gamma}\delta_{m,m_1}\delta_{\gamma,\alpha_1}\partial_{n_1-m_1-j_1-1,\beta}\partial_{m+n-j,\beta_1}+\delta_{n_1-m_1-j_1-1,m}\delta_{\alpha_1,\beta}\partial^{\alpha}_{m_1}\partial_{m+n-j,\beta_1})$$

$$+\sum_{\alpha_1,\beta_1}\sum_{j_1}\sum_{m_1=0}\sum_{\alpha,\beta}\frac{1}{4}D^{(j_1)}_{\alpha_1}(m_1,n_1)(\mathcal{C}^{j_1})^{\beta_1}_{\alpha_1}(\mathcal{C}^{n+1})^{\beta}_{\alpha}(\delta_{\alpha,\beta}\delta_{m_1,n_1-m_1-j_1-1}$$

$$+\eta^{\alpha\alpha}\eta_{\beta\beta}\delta_{m_1,n_1-m_1-j_1-1}$$

$$+\delta_{\alpha,\beta}t_{n_1-m_1-j_1-1}\partial^{\alpha}_{m_1}+t^{\alpha}_{n_1-m_1-j_1-1}\eta_{\beta\beta}\partial^{\alpha}_{m_1}+\eta^{\alpha\alpha}t_{m_1,\beta}\partial_{n_1-m_1-j_1-1,\beta}$$

$$+\delta_{\alpha,\beta}t^{\beta}_{m_1}\partial_{n_1-m_1-j_1-1,\beta})$$

$$-\sum_{m=0}^{\infty}\sum_{\alpha_1,\beta_1}\sum_{j}\sum_{\alpha,\beta}\frac{1}{2\lambda^2}C^{(j)}_{\alpha}(m,n)(\mathcal{C}^j)^{\beta}_{\alpha}(\mathcal{C}^{n_1+1})^{\beta_1}_{\alpha_1}(t^{\alpha}_m(\delta_{\beta,\alpha}t_{m+n-j,\beta}+\eta_{\beta_1,\gamma}t^{\alpha_1}_{m+n-j}\delta_{\beta,\gamma}))$$

$$-\sum_{\alpha_1,\beta_1}\sum_{j}\sum_{m=0}\sum_{\alpha,\beta}\frac{1}{4}D^{(j)}_{\alpha}(m,n)(\mathcal{C}^j)^{\beta}_{\alpha}(\mathcal{C}^{n_1+1})^{\beta_1}_{\alpha_1}(\delta_{\alpha_1,\beta_1}\delta_{m,n-m-j-1}+\eta^{\alpha_1\alpha_1}\eta_{\beta_1\beta_1}\delta_{m,n-m-j-1}$$

$$+ \delta_{\alpha_1,\beta_1} t_{n-m-j-1}\partial_m^{\alpha_1} + t_{n-m-j-1}^{\alpha_1}\eta_{\beta_1\beta_1}\partial_m^{\alpha_1} + \eta^{\alpha_1\alpha_1} t_{m,\beta_1}\partial_{n-m-j-1,\beta_1} + \delta_{\alpha_1,\beta_1} t_m^{\beta_1}\partial_{n-m-j-1,\beta_1})$$

$$= (n_1 - n)(\sum_{m_1=0}^{\infty}\sum_{\alpha_1,\beta_1}\sum_{j_1} C_{\alpha_1}^{(j_1)}(m_1, n_1+n)(\mathcal{C}^{j_1})_{\alpha_1}^{\beta_1} t_{m_1}^{\alpha_1}\partial_{m_1+n_1+n-j_1,\beta_1}$$

$$+ \frac{\lambda^2}{2}\sum_{\alpha_1,\beta_1}\sum_{j_1}\sum_{m_1=0} D_{\alpha_1}^{(j_1)}(m_1, n+n_1)(\mathcal{C}^{j_1})_{\alpha_1}^{\beta_1}\partial_{m_1}^{\alpha}\partial_{n_1+n-m_1-j_1-1,\beta}$$

$$+ \frac{1}{2\lambda^2}\sum_{\alpha_1,\beta_1}(\mathcal{C}^{n_1+n+1})_{\alpha_1}^{\beta_1} t^{\alpha_1} t_{\beta_1})$$

$$= (n - n_1)L_{n+n_1}. \tag{4.16}$$

Similar to [10], we have used identities:

$$\sum_{j_1=0}^{j}(\mathcal{C}^{j-j_1})_{\alpha_1}^{\beta_1} C_{\alpha_1}^{(j-j_1)}(m_1, n_1) C_{\beta_1}^{(j_1)}(m_1 + n_1 - j_1, n)$$

$$= (\mathcal{C}^{j-j_1})_{\alpha_1}^{\beta_1}((b_{\alpha_1} + m_1 + n_1) C_{\alpha_1}^{(j)}(m_1, n_1 + n)$$

$$+ (m_1 + n + n_1 - j - j_1 + 1) C_{\alpha_1}^{(j-1)}(m_1, n_1 + n)); \tag{4.17}$$

and

$$\sum_{j_1=0}^{j}(\mathcal{C}^{j-j_1})_{\alpha}^{\alpha_1} D_{\alpha}^{(j-j_1)}(m, n) C_{\alpha_1}^{(j_1)}(n - m - j + j_1, n_1)$$

$$= (\mathcal{C}^{j-j_1})_{\alpha}^{\alpha_1}((b_\alpha + n - m - 1) D_{\alpha}^{(j)}(k, n + n_1) + (n + n_1 - m - j) D_{\alpha}^{(j-1)}(m, n + n_1)). \tag{4.18}$$

4.2 Fat graph and Virasoro constraints

From a point of view of physics, in conformal field theory, the partition function is invariant under the conformal transformation that be realized by local coordinate transformation

$$z' = z + \epsilon v(z) = z + \sum_{n=-1}^{\infty}\epsilon_n z^{n+1}.$$

The constraints $L_{-1}Z = 0$ and $L_0 Z$ have been obtained in [9] and [23]. Since for a path integral, there is a Feynman diagram for the expansion of the path integral. Inspired by this fact and the fat graph theory for the following matrix model (4.19), we would like to propose a conjecture that the Feynman diagram is a fat graph and there is a planar graphic interpretation for $L_n Z = 0$, $n \geq 1$. Two dimensional quantum gravity have been proved to be equivalent to hermitian matrix theories. It is well known, a very general one matrix model is in fact a fat graph theory [22][4][16],

$$\begin{aligned}
Z_N(t_1, t_2, \cdots) &= \langle e^{N \sum_{i\geq 1} \mathrm{Tr}(M^i/i)} \rangle \\
&= \sum_{n_1, n_2, \cdots \geq 0} \prod_{i\geq 1} \frac{(Nt_i)^{n_i}}{i^{n_i} n_i!} \langle \prod_{i\geq 1} \mathrm{Tr}(M^i)^{n_i} \rangle \\
&= \sum_{n_1, n_2, \cdots \geq 0} \prod_{i\geq 1} \frac{(Nt_i)^{n_i}}{i^{n_i} n_i!} \\
&\qquad \sum_{\text{all labeled fat graphs } \Gamma \text{ with } n_i \text{ } i\text{-valent vertices}} N^{-E(\Gamma)} N^{F(\Gamma)} \\
&= \sum_{\text{fat graphs } \Gamma} \frac{N^{V(\Gamma)-E(\Gamma)+F(\Gamma)}}{\mathrm{Aut}(\Gamma)} \prod_{i\geq 1} g_i^{n_i(\Gamma)},
\end{aligned} \tag{4.19}$$

where $\langle \cdot \rangle = \frac{\int \cdot e^{-N \,\mathrm{Tr}\, \frac{M^2}{2}} dM}{\int e^{-N \,\mathrm{Tr}\, \frac{M^2}{2}} dM}$, and dM is the Lebesgue measure on the space of N by N Hermitian matrices; $n_i(\Gamma)$ denotes the total number of i-valent vertices of Γ and $V(\Gamma) = \sum_i n_i(\Gamma)$ is the total number of vertices of Γ. We can derive the Virasoro constraints for Z_N similarly to Theorem 2.8.

At the end of the chapter, we would like to conjecture that the Virasoro constraints arise in a fat graph. To show that the partition function (4.8) is annihilated by L_n, we will show that L_n is the generator of increasing the target weight by n in the conjectural graph.

There are three terms in the right hand side of (4.10), we claim that the first term corresponds the annihilation of a vertex and creation of a vertex, such that the weight on the target increases n. There are two factors to realize this: once a edge with target weight $(m+n-j)$ is annihilated and a particle with target weight m is created, from the graph, it is in fact a $(m+n-j)$ valent vertex becomes a m valent vertex. This makes $(n-j)$ to the contribution of the n extra target weight. The rest j target weight comes from directly from the factor $(\mathcal{C}^j)^\beta_\alpha$. The factor $D^j(m,n)$ is the ratio of the process from the graph:when $2(n-j)$ edges are decoupled from a $(2m+2n-2j)$ target weight vertex, to form the new graph, we need to divide it by $\frac{1}{(2m+2)(2m+4)\cdots(2m+2n-2j)}$. Mean while, when the new vertex is created, those gluons get new freedoms to couple(it is different from decoupling. When decoupling happens, there is no restriction) to the new vertex and the extra weights, we need a factor which is $(2q_\alpha+2m-(N-1))(2q_\alpha+2m+2-(N-1))\cdots(2q_\alpha+2m+2n-(N-1))(\sum_{m\leq l_1<l_2\cdots l_j\leq m+n}\prod(\frac{1}{2q_\alpha+2l_j}-(N-1))$(there is $(N-1)$ but not $2(N-1)$ in these parenthesis because the maps in the moduli spaces are holomorphic). That is the interpretation of the first term.

For the second term, the process is that two vertices form one more genus. This annihilation increases the target weight $(m+n-j-1+1)$. The last 1 in the parenthesis is because two vertices are annihilated.The rest of the factors have the similar interpretation except the factor λ^2. In fact, this is because when the two vertices are annihilated,the Euler number increases by 2. The denominator 2 under λ^2 is because of the symmetry of the vertices. The third term's interpretation can be realized similarly to the second term, by splitting a genus into two vertices. But the target weight increases totally from the $\mathcal{C}^j$. Since Z is invariant under conformal transformation $z^{'} = z + \epsilon_n z^{n+1}$, which corresponds increasing the targest weight by n in the graph, and according to our analysis above, L_n, $n \geq 1$ is the representation of generator of this conformal transformation. Therefore Z is annihilated by L_n, i.e., $L_n Z = 0$.

CONCLUSION

The group integrals, where integrands are the monomials of matrix elements of irreducible representations of classical groups, and their asymptotic behaviors as the rank N of the group becomes large, and the related duality theorems can be obtained by the combinatorics of the corresponding Young tableaux of the irreducible representations. Moreover there are some partition functions, which are power series of these group integrals, satisfy Virasoro constraints, i.e., they are annihilated by differential operators that form a (half) representation of Virasoro algebra. The proof of Witten's conjecture can be simplified by using the fermion-boson correspondence, i.e., the KdV hierarchy and Virasoro constraints of the partition function in Witten's conjecture can be achieved naturally. Inspired by the fat graph of random matrix, by a complicated calculation of the commutator of the differential operators, which are the representation of Virasoro algebra in Virasoro conjecture and discovering the fat graph interpretation for each term in those differential operators, we have shown an important evidence that the partition function of Virasoro conjecture corresponds a fat graph.

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