Spring 2010

Relative primeness

Jeremiah N. Reinkoester
University of Iowa

RELATIVE PRIMENESS

by

Jeremiah N. Reinkoester

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2010

Thesis Supervisor: Professor Dan Anderson

ABSTRACT

Recently, Dan Anderson and Andrea Frazier introduced a generalized theory of factorization. Given a relation τ on the nonzero, nonunit elements of an integral domain D, they defined a *τ-factorization* of a to be any proper factorization $a = \lambda a_1 \cdots a_n$ where $\lambda \in U(D)$ and a_i is τ-related to a_j, denoted $a_i \ \tau \ a_j$, for $i \neq j$. From here they developed an abstract theory of factorization that generalized factorization in the usual sense. They were able to develop a number of results analogous to results already known for usual factorization.

Our work focuses on the notion of τ-factorization when the relation τ has characteristics similar to those of coprimeness. We seek to characterize such τ-factorizations. For example, let D be an integral domain with nonzero, nonunit elements $a,\ b \in D$. We say that a and b are *comaximal* (resp. *v-coprime*, *coprime*) if $(a, b) = D$ (resp., $(a, b)_v = D$, $[a, b] = 1$). More generally, if $*$ is a star-operation on D, a and b are *$*$-coprime* if $(a, b)^* = D$. We then write $a \ \tau_{max} \ b$ (resp. $a \ \tau_v \ b$, $a \ \tau_{[\ ]} \ b$, or $a \ \tau_* \ b$) if a and b are comaximal (resp. v-coprime, coprime, or $*$-coprime).

Abstract Approved: ______________________________
Thesis Supervisor

____
Title and Department

____
Date

RELATIVE PRIMENESS

by

Jeremiah N. Reinkoester

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2010

Thesis Supervisor: Professor Dan Anderson

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Jeremiah N. Reinkoester

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2010 graduation.

Thesis Committee: ______________________
Dan Anderson, Thesis Supervisor

____
Frauke Bleher

____
Victor Camillo

____
Richard Dykstra

____
Paul Muhly

ACKNOWLEDGEMENTS

First and foremost, I would like to thank God (Psalm 121). Second, I would like to thank my family for their constant encouragement throughout my schooling. It hardly needs to be said that they are instrumental in everything I do in life. My wife, Christine, deserves a special acknowledgement for her patience: there were countless nights I was doing math to her neglect. She shouldered it well. I love you.

Finally, I would like to thank all of the professors and teachers that were, and continue to be, vital to my academic growth. Among those, a special acknowledgement must be made to my advisor, Professor Dan Anderson, and to my high school math teacher, Mrs. Gretchen Deutschmann. Professor Anderson had a heavy hand in the completion of my degree. His door was always open, and he was always willing to stop what he was doing for a discussion in math, religion, politics, Iowa history, and Iowa football to name a few. Mrs. Deutschmann taught me just about every math course I took in high school. Without her gifts of exuding joy in the subject and making the material seem easy I don't know that I ever would have come this far in mathematics.

ABSTRACT

Recently, Dan Anderson and Andrea Frazier introduced a generalized theory of factorization. Given a relation τ on the nonzero, nonunit elements of an integral domain D, they defined a τ-*factorization* of a to be any proper factorization $a = \lambda a_1 \cdots a_n$ where $\lambda \in U(D)$ and a_i is τ-related to a_j, denoted $a_i \ \tau \ a_j$, for $i \neq j$. From here they developed an abstract theory of factorization that generalized factorization in the usual sense. They were able to develop a number of results analogous to results already known for usual factorization.

Our work focuses on the notion of τ-factorization when the relation τ has characteristics similar to those of coprimeness. We seek to characterize such τ-factorizations. For example, let D be an integral domain with nonzero, nonunit elements $a, \ b \in D$. We say that a and b are *comaximal* (resp. v-*coprime*, *coprime*) if $(a, b) = D$ (resp., $(a, b)_v = D, \ [a, b] = 1$). More generally, if $*$ is a star-operation on D, a and b are $*$-*coprime* if $(a, b)^* = D$. We then write $a \ \tau_{max} \ b$ (resp. $a \ \tau_v \ b, \ a \ \tau_{[\ ]} \ b$, or $a \ \tau_* \ b$) if a and b are comaximal (resp. v-coprime, coprime, or $*$-coprime).

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

An effective way to understand an object is to break it down to its smallest components. Understanding the way in which these foundational components interact strengthens our understanding of the object itself. In algebra, for example, the prime numbers are the building blocks of the integers. Every nonzero, nonunit integer can be uniquely factored into a product of prime elements. We can investigate the factorization of elements in an integral domain into irreducible and prime elements, respectively. The distinction made between prime and irreducible elements already illustrates the greater difficulty when studying factorization in general integral domains. We generalize the notion of factorization even further by incorporating relations on the elements. In this thesis, we narrow our focus to a specific category of these relations, which we call general coprimeness relations. We connect these ideas to previously known results, develop some concrete examples, and seek to axiomatize the concept of a comprime relation.

In 2004, Stephen McAdam and Richard Swan [17] first introduced the notion of comaximal factorization. For a a nonzero, nonunit element of an integral domain D, then $a = a_1 \cdots a_n$, each a_i a nonunit of D, is a *comaximal factorization* of a if $(a_i, a_j) = D$ for all $i \neq j$. With respect to comaximal factorization they developed definitions analogous to irreducible, atomic, and unique factorization domains which they called *pseudo-irreducible*, *comaximal factorization domain* (CFD), and *unique comaximal factorization domain* (UCFD), respectively. They were able to develop a characterization of UCFD's in terms of CFD's. They also gave necessary and sufficient conditions in terms of D for $D[X]$ to be a UCFD.

A few years later Dan Anderson and Andrea Frazier [2] introduced a generalized theory of factorization. Given a relation τ on the nonzero, nonunit elements of an integral domain D, they defined a *τ-factorization* to be any factorization of a nonzero, nonunit element of D such that each proper factor is τ-related. From this they developed a theory of factorization that generalized factorization in the usual sense as well as the comaximal factorization of McAdam and Swan. They were able to develop a number of results analogous to results already known for usual factorization.

1.1 Definitions and Backgrounds

Throughout this thesis D will denote an integral domain, K the quotient field of D, D^* the nonzero elements of D, $U(D)$ the units of D, and $D^\#$ the nonzero, nonunit elements of D. The following section states the definitions and results about τ-factorization that we will need. For an introduction to τ-factorization, see [2].

1.1.1 τ-factorization

In this thesis, we will only discuss relations that are symmetric. Let τ be a relation on $D^\#$. For $a \in D^\#$, we define $a = \lambda a_1 \cdots a_n$, $\lambda \in U(D)$ and $a_i \in D^\#$, to be a *τ-factorization of* a if a_i is τ-related to a_j (denoted $a_i \ \tau \ a_j$) for each $i \neq j$. We say that a is a *τ-product* of the a_i's and that each a_i is a *τ-factor* of a. For $a, b \in D^\#$, we say that a *τ-divides* b, written $a \mid_\tau b$, if a is a τ-factor of b.

We call τ *multiplicative* if for $a, b, c \in D^\#$, $a \ \tau \ b$ and $a \ \tau \ c$ implies $a \ \tau \ bc$. We call τ *divisive* if for $a, a', b, b' \in D^{\#}$, $a \; \tau \; b$, $a' \mid a$, and $b' \mid b$ imply $a' \; \tau \; b'$. We say that τ is *associate-preserving* if for $a, b, b' \in D^{\#}$, $b \sim b'$ and $a \; \tau \; b$ implies $a \; \tau \; b'$.

At this point we make a few observations which help us see the motivation behind some of these definitions. If τ is associate-preserving and $a = \lambda a_1 \cdots a_n$ is a τ-factorization, then so is $a = (\lambda a_1)a_2 \cdots a_n$. Thus, when τ is associate-preserving, we can dispense with the unit λ. If τ is divisive, then τ is associate-preserving. Suppose that τ is divisive and that $a = a_1 \cdots a_n$ is a τ-factorization of a (since τ is divisive, we are omitting the unit λ). Given any τ-factorization of an a_i, say $a_i = b_1 \cdots b_m$, then $a = a_1 \cdots a_{i-1} b_1 \cdots b_m a_{i+1} \cdots a_n$ is also a τ-factorization. This second τ-factorization is called a τ-*refinement* of a.

If τ is multiplicative, then we can group the τ-factors in a τ-factorization in any way, and still have a τ-factorization. Formally, if $a = \lambda a_1 \cdots a_n$ is a τ-factorization, $\{1, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_s$ is a disjoint union with each A_i nonempty, and $b_i = \prod\{a_j \mid j \in A_i\}$, then $a = \lambda b_1 \cdots b_s$ is a τ-factorization of a.

Given $a \in D^{\#}$, we say that a is τ-*irreducible* or a τ-*atom* if it has no proper τ-factors. If every element of $D^{\#}$ has a τ-factorization into τ-atoms, then we say that D is τ-*atomic*. We say that $a \in D^{\#}$ is τ-*prime* ($|_\tau$-*prime*) if whenever a divides (τ-divides) a τ-factorization, $\lambda a_1 \cdots a_n$, then a divides (τ-divides) some τ-factor a_i of the τ-factorization. If τ is multiplicative, we can take $n = 2$ in the definition of τ-prime and $|_\tau$-prime [2]. D is said to be a τ-*unique factorization domain* (τ-UFD) if D is τ-atomic and each τ-atomic factorization of a nonzero, nonunit of D is unique up to order and associates.

1.1.2 $*$-operations

We briefly go over a few facts and definitions regarding $*$-operations as they pertain to this paper. We include well known results about the v-operation. For a more detailed account, see [12], [18], and [13] in that order. A *fractional ideal* is a D-module, I, contained in K such that $aI \subseteq D$ for some $a \in D$. So a fractional ideal is of the form $\frac{1}{a}J$ for some $a \in D$ and J an ideal of D. The set of nonzero fractional ideals of D is denoted by $F(D)$, and the set of nonzero finitely generated fractional ideals is denoted by $f(D)$.

A $*$-*operation* is a mapping of $F(D)$ into $F(D)$, denoted by $A \longrightarrow A^*$, such that for $a \in K^* = K - \{0\}$ and $A,\ B \in F(D)$ we have the following properties:

(1) $(a) = (a)^*$ and $(aA)^* = aA^*$,

(2) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$, and

(3) $(A^*)^* = A^*$.

A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$. Any non-zero intersection of $*$-ideals is a $*$-ideal. Also, given any $A,\ B \in F(D)$, we have $(AB)^* = (AB^*)^* = (A^*B^*)^*$ [12, Proposition 32.2]. The intersection property allows us to show that $A^{-1} = \{x \in K \mid xA \subseteq D\}$ is a $*$-ideal since $A^{-1} = \bigcap_{a \in A-\{0\}} (\frac{1}{a})$ [18]. A is called $*$-*invertible* if there exists a $B \in F(D)$ such that $(AB)^* = D$. The $*$-invertible, $*$-ideals of D form a group under the operation $A * B = (AB)^*$. If such an ideal B exists, then $B^* = A^{-1}$ is the unique $*$-ideal satisfying $(AB)^* = D$. First, $(AA^{-1})^* \subseteq D = (AB)^*$. Second, $(AB)^* = D \implies AB \subseteq D \implies B \subseteq A^{-1}$. So

$(AB)^* = (AA^{-1})^* = D \implies B^* = A^{-1}$.

A $*$-operation is said to be of *finite character* if for each $A \in F(D)$, $A^* = \cup\{F^* \mid F \in f(D) \text{ and } F \subseteq A\}$. Every $*$-operation induces a finite character $*$-operation, denoted by $*_s$, defined by $A^{*_s} = \cup\{F^* \mid F \in f(D) \text{ and } F \subseteq A\}$. For a finite character $*$-operation, each proper integral $*$-ideal is contained in a maximal proper integral $*$-ideal and such a maximal $*$-ideal is prime. To see that maximal $*$-ideals exist we use Zorn's Lemma. Let $\{P_\alpha\}$ be a totally ordered (by containment) set of proper integral $*$-ideals. It is clear that $\cup P_\alpha$ is a proper integral ideal. We need to show $\cup P_\alpha$ is a $*$-ideal. We have $(\cup P_\alpha)^* = \{F^* \mid F \in f(D) \text{ and } F \subseteq \cup P_\alpha\}$. Since each F is finitely generated, $F \subseteq P_{\alpha_0}$ for some $P_{\alpha_0} \in \{P_\alpha\}$. Hence, $F^* \subseteq P_{\alpha_0}$, and so $\cup P_\alpha$ is a $*$-ideal.

We now show that maximal $*$-ideals are prime. Let M be a maximal $*$-ideal. By way of contradiction assume that M is not prime. Then there exists $ab \in M$ with neither a nor b in M. So $(M, a)^* = (M, b)^* = D$. But then we have $D = ((M, a)^*(M, b)^*)^* = ((M, a)(M, b))^* \subseteq M^* = M$, a contradiction.

The simplest example of a $*$-operation is the d-*operation*, $A_d = A$. Another example of interest to us is the v-*operation*, $A_v = (A^{-1})^{-1}$. Another characterization of the v-operation is that for $A \in F(D)$, A_v is the intersection of the set of principal fractional ideals of D containing A. Also, for all $A \in F(D)$ and any $*$-operation on D, $A^* \subseteq A_v$ [12, Theorem 34.1]. The finite character $*$-operation induced by the v-operation is called the t-operation, denoted A_t.

The following lemma is from [18, Observation A]. It is frequently used so we state and prove it here.

Lemma 1.1. *Let D be an integral domain. Given that A is a nonzero integral ideal of D, then $A_v \neq D$ if and only if there exists $a,\ b \in D - \{0\}$ such that $A \subseteq \frac{a}{b}D$, and $a \nmid b$.*

Proof. ($\Longleftarrow$) If $A_v = D$, then every principal fractional ideal containing A also contains D. So $A \subseteq \frac{a}{b}D$ implies $D \subseteq \frac{a}{b}D$ and hence $\frac{a}{b} \cdot d = 1$ for some $d \in D$. Hence, $a \mid b$.

($\Longrightarrow$) If, for every $A \subseteq \frac{a}{b}D$, we have $a \mid b$, then $\frac{a}{b}D = \frac{1}{d}D$ for some $d \in D$. But then $D \subseteq \frac{a}{b}D$. Hence, $A_v = D$. ∎

1.2 Overview

Chapter 2 focuses on relations with "relatively prime" properties. First, we look at relations defined in terms of a collection of ideals. Specifically, two elements are related if they are not contained in any ideal in the set. Many relations we study are defined in terms of a set $\mathfrak{S}$ of ideals. We look at some examples of interest where $\mathfrak{S}$ is the set of maximal ideals, principal primes ideals, or minimal prime ideals, to name a few.

Our second approach is axiomatic. Our motivation arises from studying the actual relatively prime relation, comaximal factorization, and the v-operation. From here we develop what we feel are the common properties of these relations. We can then develop general results about relations that satisfy these properties. We are of course interested in the factorization properties with respect to these relations. One common theme is that such relations can be defined in terms of a collection of ideals. This of course is our tie to the first part of the chapter.

Chapter 3 is where we really begin to connect the relatively prime relation to previously known results. We are particularly interested in τ-UFD's where τ is the relatively prime relation (we denote this by $\tau_{[\ ]}$). So nonzero, nonunit elements $a,\ b \in D$ are $\tau_{[\ ]}$-related if a and b are relatively prime. When D is a weakly factorial domain, that is, every nonzero, nonunit of D is a product of primary elements, we found that D is a $\tau_{[\ ]}$-UFD if and only if D_P is a $\tau_{[\ ]}$-UFD at each of the height-one prime ideals P of D (Theorem 3.8).

As one might expect, GCD domains also provide a nice background from which to study $\tau_{[\ ]}$-UFD's. If D is a GCD domain, then D is a $\tau_{[\ ]}$-UFD if and only if $D[X]$ is a $\tau_{[\ ]}$-UFD (Theorem 3.18). We also look to connect the notion of a $\tau_{[\ ]}$-UFD to the unique comaximal factorization domains (UCFD's) of McAdam and Swan [17]. We make this connection by localizing the polynomial ring at a specific multiplicatively closed set.

In Chapter 4, we study several subrings of the ring of formal power series over a field. Specifically, we focus on $k[[X^2, X^3]]$ and $k + X^n K[[X]]$ where k is a subfield of a field K. We determine the $\tau_{[\ ]}$-atomic structure of $k[[X^2, X^3]]$, and give necessary and sufficient conditions for $k + X^n K[[X]]$ to be a $\tau_{[\ ]}$-UFD.

CHAPTER 2
GENERAL COPRIMENESS

2.1 Introduction

This chapter investigates various forms of coprimeness, and we introduce a very general form of coprimeness, called $\mathfrak{S}$-coprimeness.

For $a, b \in D$, $[a, b]$ (resp. $]a, b[$) denotes the GCD (resp. LCM) of a and b. We write $[a, b] \neq 1$ if it is not the case that $[a, b] = 1$. In other words, some nonzero, nonunit divides both a and b. So $[a, b] \neq 1$ does not require that $[a, b]$ exists. For example let R be the ring of polynomials over $\mathbb{Z}$ with even coefficients of X. Then $[4X, 2X^3] \neq 1$, but there is no GCD of $4X$ and $2X^3$ [10, pg. 253].

We next define three well known forms of coprimeness. For $a,\ b \in D^*$, we say that a and b are *comaximal* (resp. *v-coprime*, *coprime*) if $(a, b) = D$ (resp., $(a, b)_v = D$, $[a, b] = 1$). More generally, given a $*$-operation on D, a and b are *$*$-coprime* if $(a, b)^* = D$. The following proposition lists some well known properties and connections between these forms of coprimeness.

Proposition 2.1. *Let D be an integral domain and let $a, b \in D - \{0\}$.*

(1) a and b are comaximal $\Longleftrightarrow$ $[a, b] = 1$ and $[a, b]$ is a linear combination of a and b $\Longleftrightarrow$ a and b are d-coprime.

(2) For $a, b \in D$ the following are equivalent:

(i) $(a, b)_v = D$,

(ii) $]a,b[=ab$,

(iii) $(a)\cap(b)=(ab)$,

(iv) $(a):(b)=(a)$, *and*

(v) $(b):(a)=(b)$.

(3) $[a,b]=1$ *if and only if* $(a,b)\subseteq(t)\subseteq D \implies (t)=D$.

(4) For any $*$*-operation on* D *we have* a *and* b *comaximal* $\Longrightarrow$ a *and* b *are* $*$*-coprime* $\Longrightarrow$ a *and* b *are* v*-coprime* $\Longrightarrow$ a *and* b *are relatively prime.*

Proof. (1) and (3) are straightforward.

(2) $(i)\Longrightarrow(ii)$ Suppose $(m)\subseteq(a)\cap(b)$. Since $\frac{m}{a}$ and $\frac{m}{b}$ are in D, $(a,b)\subseteq\frac{ab}{m}D$. So by Lemma 1.1 $ab\mid m$ as desired.

$(ii)\Longrightarrow(iii)$ Let $ra=sb\in(a)\cap(b)$. Then a and b divide ra which implies $]a,b[=ab$ divides ra. So $r\in(b)$ as desired.

$(iii)\Longrightarrow(iv)$ $(a)=(ab):(b)=((a)\cap(b)):(b)=((a):(b))\cap((b):(b))=(a):(b)$.

$(v)\Longrightarrow(iii)$ $(b)\cap(a)=(((b)\cap(a)):(a))(a)=((b):(a))(a)=(b)(a)=(ba)$.

$(iii)\Longrightarrow(v)$ and $(iv)\Longrightarrow(iii)$ Interchange a and b in the previous two arguments.

$(iv)\Longrightarrow(i)$ Suppose that $(a,b)\subseteq\frac{c}{d}D$. Then $ad=cd_1$ and $bd=cd_2$ for some $d_i\in D$. So $add_2=cd_1d_2=bdd_1$ which implies $ad_2=bd_1$. Now $d_1\in(a):(b)=(a)$, say $d_1=ra$. Then $ad=cd_1=cra$ implies $d=cr$. By Lemma 1.1 $(a,b)_v=D$.

(4) It is well known that for any $*$-operation, $I^* \subseteq I_v$. Also, I_v is the intersection of all principal fractional ideals containing I. The result follows from these facts. □

We next develop a general notion of coprimeness. We are motivated by the study of comaximal, $*$-coprime, v-coprime, and coprime. We would like to study relations that lie between comaximal and coprime, that is, any relation τ such that $(a,b) = D \implies a \ \tau \ b \implies [a,b] = 1$. From Proposition 2.1, (4) we see that $*$-coprime for any $*$-operation is such a relation. We are particularly interested in what properties all such relations possess.

Definition 2.2. Let $\mathfrak{S}$ be a set of ideals of D. We say that a and b in $D^{\#}$ are $\mathfrak{S}$-*coprime* if $(a,b) \nsubseteq I$ for each $I \in \mathfrak{S}$. We write this as $[a,b]_{\mathfrak{S}} = 1$ and then have a relation $\tau_{\mathfrak{S}}$ on $D^{\#}$ given by $a \ \tau_{\mathfrak{S}} \ b \iff [a,b]_{\mathfrak{S}} = 1$.

2.2 Examples

Example 2.1. We give a few examples of $\mathfrak{S}$-coprimeness with some obvious choices for $\mathfrak{S}$.

(1) $\mathfrak{S} = \{D\}$. In this case, no two elements in $D^{\#}$ are $\mathfrak{S}$-coprime. If (D,M) is quasilocal, we can take $\mathfrak{S} = \{M\}$. Every nonzero, nonunit is a $\tau_{\mathfrak{S}}$-atom, and D is a $\tau_{\mathfrak{S}}$-UFD.

(2) $\mathfrak{S} = \{0\}$. Here any two $a, b \in D^{\#}$ are $\mathfrak{S}$-coprime, so $\tau_{\mathfrak{S}} = D^{\#} \times D^{\#}$ and we get the usual factorization, i.e., every factorization is a $\tau_{\mathfrak{S}}$-factorization.

(3) $\mathfrak{S} = max(D) :=$ {the maximal ideals of D}. Here $[a,b]_{\mathfrak{S}} = 1 \iff (a,b) = D$. So $[a,b]_{\mathfrak{S}} = 1$ if and only if a and b are comaximal. We can replace $max(D)$ by any subset $\mathfrak{S}$ not containing D with $max(D) \subseteq \mathfrak{S}$. This gives us the comaximal factorization from [17]. Since this is a common relation, we denote it by τ_{max}.

(4) Let $\mathfrak{S} = t\text{-}max(D) = \{$ the maximal t-ideals of $D\}$. So $[a,b]_{\mathfrak{S}} = 1 \iff (a,b) \nsubseteq M$ where M is a maximal t-ideal $\iff (a,b)_t = D \iff a$ and b are v-coprime.

(5) We can generalize Example (4) to any finite character $*$-operation. Let $\mathfrak{S}_* = *\text{-}max(D) = \{$ the maximal $*$-ideals of $D\}$. So $[a,b]_{\mathfrak{S}_*} = 1 \iff (a,b) \nsubseteq M$ for any maximal $*$-ideal $M \iff (a,b)^* = D$. We will denote this relation by τ_*. For $* = d$ we get Example (3) and for $* = t$ we get Example (4). We can replace $\mathfrak{S} = *\text{-}max(D)$ by $\mathfrak{S} = \{P \in Spec(D) \mid P^* = P\}$. We will mostly be concerned with τ_t from Example (4).

(6) Let D be a domain and $\mathfrak{S} = \{(p_\alpha)\}$ a set of nonzero principal primes. Then $[a,b]_{\mathfrak{S}} = 1 \iff (a,b) \nsubseteq (p_\alpha)$ for each $p_\alpha \iff$ no p_α divides both a and b.

Suppose D is a UFD. Then we can determine the $\tau_{\mathfrak{S}}$-atoms. Suppose that p is a $\tau_{\mathfrak{S}}$-atom. If p is not prime, then $p = ab$ for some $a, b \in D^{\#}$. Since p is a $\tau_{\mathfrak{S}}$-atom, $[a,b]_{\mathfrak{S}} \neq 1$, that is, a and b are both divisible by some $p_\alpha \in \mathfrak{S}$. So $p = up_\alpha^n$ for some n and some unit u of D. Hence, a $\tau_{\mathfrak{S}}$-atom is either prime or is associate to p_α^n for some $(p_\alpha) \in \mathfrak{S}$. If we take $D = \mathbb{Z}$ and $\mathfrak{S} = \{(p) \mid p$ a prime greater than $2\} = \{(3),(5),(7),(11),\ldots\}$, then $x \in D^{\#}$ is $\tau_{\mathfrak{S}}$-atom $\iff x = \pm 2$ or $x = \pm p^n$ for $p > 2,\ n \geq 1$.

(7) Let $\mathfrak{S} = \{P \mid ht(P) = 1\}$. Then $[a,b]_{\mathfrak{S}} = 1 \iff (a,b)$ is not contained in any height-one prime ideal. If D is Noetherian, then by the Principal Ideal Theorem $[a,a]_{\mathfrak{S}} \neq 1$ for all $a \in D^{\#}$. In other words, each $a \in D^{\#}$ is contained in a height-one prime ideal.

(8) Let $\mathfrak{S} = \{(t) \mid t \in D^{\#}\}$. So $[a,b]_{\mathfrak{S}} = 1 \iff (a,b) \nsubseteq (t)$ for any $(t) \subsetneq D \iff [a,b] = 1$. We will show in Corollary 3.21 if D is a GCD domain, then $\tau_{\mathfrak{S}}$ is the same relation as in Example (4).

Now $\tau_{\mathfrak{S}}$ is symmetric, but is not reflexive if $\mathfrak{S}$ contains a nonzero ideal. Also, $\tau_{\mathfrak{S}}$ is always divisive. For if $a' \mid a$ and $b' \mid b$, where $a, a', b, b' \in D^{\#}$, then $(a,b) \subseteq (a',b')$. So $(a,b) \nsubseteq I$ implies $(a',b') \nsubseteq I$.

2.2.1 Height-One Prime Ideals

In [3], Anderson and Mahaney studied domains in which every nonzero, nonunit element can be written as a product of primary elements. They called such domains *weakly factorial domains.* They showed that in a commutative ring if Q_1 and Q_2 are P-primary with Q_1 invertible, then Q_1Q_2 is P-primary. A product of primary ideals $Q_1 \cdots Q_n$, where each Q_i is P_i-primary, is a *reduced primary product representation* if $P_i \neq P_j$ for $i \neq j$. So in a weakly factorial domain each nonzero, nonunit element has a reduced primary product representation into primary elements. They further showed that two primary elements with distinct radicals have incomparable radicals. It follows that each primary element is contained in a unique height-one prime ideal. From this it is straightforward to show that each reduced primary product of primary elements is unique up to units and order. For further results about weakly factorial domains see [3].

For $\mathfrak{S} = \{P \mid ht(P) = 1\}$ (Example 2.1, (7)) we can develop a few basic results regarding weakly factorial domains. In this section, $\mathfrak{S}$ is the set of height-one prime ideals.

Lemma 2.3. *Let D be a weakly factorial domain. Then a nonunit, nonzero element a of D is a $\tau_{\mathfrak{S}}$-atom if and only if a is a primary element.*

Proof. If a is primary, then it is contained in a unique height-one prime ideal. Hence, a is a $\tau_{\mathfrak{S}}$-atom. Conversely, assume that a is a $\tau_{\mathfrak{S}}$-atom. Since D is weakly factorial, a can be written as a reduced product of primary elements. This product is also a $\tau_{\mathfrak{S}}$-factorization. So a must be primary. □

Corollary 2.4. *A weakly factorial domain is a $\tau_{\mathfrak{S}}$-UFD.*

Proof. From Lemma 2.3 we see that reduced products of primary elements and $\tau_{\mathfrak{S}}$-atomic factorizations are the same thing. □

This gives us that a CK domain is a $\tau_{\mathfrak{S}}$-UFD (see Section 4.4). When D is a one-dimensional domain, $\tau_{\mathfrak{S}}$ is the same as the comaximal factorization found in [17]. Hence, we get the following expansion of [17, Corollary 1.10].

Theorem 2.5. *Let D be a one-dimensional Noetherian domain. Then the following are equivalent:*

(1) $Pic(D) = 0$,

(2) D is a UCFD,

(3) D is a $\tau_{\mathfrak{S}}$-UFD,

(4) D is a weakly factorial domain.

Proof. (1), (2), and (3) are equivalent from the remarks preceding the theorem and [17, Corollary 1.10].

If D is a weakly factorial domain, then it follows from Corollary 2.4 that D is a UCFD. It also follows from the fact that every invertible ideal in a weakly factorial domain is principal [3, Corollary 11].

Suppose D is a UCFD. Since D is a one-dimensional Noetherian domain, each nonzero, nonunit of D can be written as a reduced product of primary ideals [3]. By [17, Corollary 1.10] each such primary ideal is principal. Hence, D is a weakly factorial domain. □

2.2.2 Grade and v-coprimeness

We quickly introduce the notion of grade. Our goal is to connect $\mathfrak{S} = t$-$max(D)$ to grade for Noetherian domains. For a complete introduction see [16, Chapter 3]. Given R a commutative ring, and A any R-module, then the ordered sequence of elements $x_1, \ldots, x_n$ of R is said to be an *R-sequence* on A if

(a) $(x_1, \ldots, x_n)A \neq A$,

(b) For $i = 1, \ldots, n,\ x_i \notin Z(A/(x_1, \ldots, x_{i-1})A)$.

For our purposes we are interested in the case when $A = R$. We define a *maximal R-sequence* in an ideal I to be an R-sequence $x_1, \ldots, x_n$ in I in which there does not exist an $x_{n+1} \in I$ such that $x_1, \ldots, x_n, x_{n+1}$ is an R-sequence. When R is Noetherian it is well know that maximal R-sequences exist and any two maximal R-sequences contained in an ideal I have the same length. This common length is denoted by $G(I)$. If we do not specify that R is Noetherian, then $G(I) > 1$ will mean I contains no maximal R-sequence of length 1.

Lemma 2.6. *Let D be an integral domain and let $a, b \in D^*$ such that $(a, b) \neq D$. Then $(a, b)_t = D$ if and only if a, b is an R-sequence.*

Proof. By way of contradiction suppose that $(a, b)_t \neq D$ and a, b is an R-sequence. Since $(a, b)_t \neq D$, by Lemma 1.1 there exists c and d in D with $(a, b) \subseteq \frac{c}{d}D$ and $c \nmid d$. So $da = cd_1$ and $db = cd_2$ for some $d_i \in D$. Then $cd_1d_2 = d_2da = d_1db$. So $d_2a = d_1b$. Since $b \notin Z(R/(a))$, $d_1 \in (a)$, say $d_1 = ax$. So $da = cd_1 \Longrightarrow da = cax \Longrightarrow d = cx$, contradicting that $c \nmid d$. So a, b is not an R-sequence.

Suppose a, b is not an R-sequence. By hypothesis $(a, b) \neq D$ so b must be in $Z(D/(a))$, say $rb = sa$ for some $r \notin (a)$ and $s \in D$. Then $(a, b) \subseteq \frac{a}{r}D$ and $a \nmid r$. So by Lemma 1.1 $(a, b)_t \neq D$. □

The hypothesis $(a, b) \neq D$ insures that part (a) of the definition of R-sequences is satisfied. We will be concerned with R-sequences contained in proper ideals, so requiring this hypothesis will not pose a problem.

Proposition 2.7. *Let D be a Noetherian domain. For a nonzero prime ideal P $P_t \neq D$ if and only if $G(P) = 1$.*

Proof. Suppose that P is a nonzero prime ideal with $P_t \neq D$. Let $0 \neq a \in P$. For any $0 \neq b \in P$, $(a,b)_t \neq D$. By Lemma 2.6 a, b is not an R-sequence. So a is a maximal R-sequence in P and hence $G(P) = 1$.

Suppose that $G(P) = 1$. Let $(x_1, \ldots, x_n) = P$ where $x_i \neq 0$. Then $(x_1, \ldots, x_n) = P \subseteq Z(D/(x_1))$ since x_1 is a maximal R-sequence in P. By [16, Theorem 80] there exists $t \notin (x_1)$ such that $t(x_1, \ldots, x_n) \subseteq (x_1) \Longrightarrow (x_1, \ldots, x_n) \subseteq (\frac{x_1}{t}) \Longrightarrow (\frac{t}{x_1}) \subseteq (x_1, \ldots, x_n)^{-1}$. Hence, $(x_1, \ldots, x_n)^{-1} \neq D$. So $P_t = (P^{-1})^{-1} \neq D$. □

Proposition 2.7 leads us to another proposition.

Proposition 2.8. *Let D be a Noetherian domain. For nonzero, nonunits a, $b \in D$ $(a,b)_t = D$ if and only if $G(P) > 1$ for every prime P containing (a,b).*

Proof. If $(a,b)_t = D$, then it follows from Proposition 2.7 that $G(P) > 1$ for any prime P containing (a,b). Conversely, let P be a prime ideal containing (a,b). Again from Proposition 2.7 $G(P) > 1$ implies $P_t = D$. Hence, $(a,b)_t = D$ or $(a,b)_t$ is a maximal t-ideal. But if $(a,b)_t$ is a maximal t-ideal, then $G((a,b)_t) > 1$ which contradicts that $(a,b)_t \neq D$. So $(a,b)_t = D$. □

Using Lemma 2.6 and Proposition 2.7 we now characterize $\tau_{\mathfrak{S}}$, $\mathfrak{S} = t\text{-}max(D)$ in terms of grade.

Example 2.2. In the case for v-coprimeness, $[a,b]_t = 1 \Longleftrightarrow (a,b)_t = D$ for a and b nonzero, nonunits of D. From Lemma 2.6 this is equivalent to $G((a,b)) > 1$. For if a_1, a_2 is an R-sequence in (a,b), then $D = (a_1, a_2)_t \subseteq (a,b)_t$. There is a corresponding height version. Let $\mathfrak{S} = X^{(1)}(D)$, the set of height-one primes of D. Then $[a,b]_{\mathfrak{S}} = 1 \Longleftrightarrow (a,b) \nsubseteq P$ for any height-one prime $P \Longleftrightarrow ht(a,b) > 1$.

Using Propositions 2.7 and 2.8 we can develop a precise grade version when D is Noetherian. Let $\mathfrak{S} = \{P \mid P \text{ is prime and } G(P) = 1\}$. Then by Lemma 2.6 and Proposition 2.8 $(a, b)_t = D \iff G((a, b)) > 1 \iff G(P) > 1$ for every prime P containing $(a, b) \iff [a, b]_{\mathfrak{S}} = 1$. Hence, $a \; \tau_t \; b$ is equivalent to $a \; \tau_{\mathfrak{S}} \; b$.

It is well known that for an integral domain D we may have $[a, b] = 1, \; [a, c] = 1$, but $[a, bc] \neq 1$. We use an example from the beginning of the chapter. Let R be the ring of polynomials in X with integer coefficients and even coefficient of X. Then $[2, 2X] = 1$ but $[2, 4X^2] = 2$ [10, pg. 253]. In fact, an atomic integral domain with the property that $[a, b] = [a, c] = 1$ implies $[a, bc] = 1$ is a UFD [5]. In terms of symmetric relations, the property $a \; \tau \; b, \; a \; \tau \; c \Longrightarrow a \; \tau \; bc$ has been called *multiplicative.* Hence, for $\mathfrak{S} = \{(a) \mid a \in D^{\#}\}$ and D atomic we have that $\tau_{\mathfrak{S}}$ is multiplicative if and only if D is a UFD. In Example 2.1, (1)$-$(7), $\tau_{\mathfrak{S}}$ is multiplicative. For Example 2.1, (6) and (7), $\tau_{\mathfrak{S}}$ is multiplicative by the following proposition, which gives a general condition under which $\tau_{\mathfrak{S}}$ is multiplicative.

Proposition 2.9. *Let D be an integral domain and $\mathfrak{S}$ a collection of ideals. If each ideal in $\mathfrak{S}$ is prime, then $\tau_{\mathfrak{S}}$ is multiplicative.*

Proof. We wish to show that if $[a, b]_{\mathfrak{S}} = 1$ and $[a, c]_{\mathfrak{S}} = 1$, then $[a, bc]_{\mathfrak{S}} = 1$. But if $[a, bc]_{\mathfrak{S}} \neq 1$, then $(a, bc) \subseteq I$ for some $I \in \mathfrak{S}$. But then a is in I, and b or c is in I, a contradiction. So $[a, bc]_{\mathfrak{S}} = 1$ as desired. $\square$

The converse is most certainly not true. Let D be a GCD-domain. Then $\tau_{\mathfrak{S}}$ with $\mathfrak{S} = \{(t) \mid t \in D^{\#}\}$ is multiplicative. A better question might be to ask if $\tau_{\mathfrak{S}}$ is multiplicative, does there exist a collection of prime ideals $\mathfrak{S}'$ such that $\tau_{\mathfrak{S}} = \tau_{\mathfrak{S}'}$?

And if this is not true for a general integral domain, under what conditions would the statement hold? We currently do not have a suitable answer.

2.3 Axioms of Coprimeness

It is interesting to contemplate on what properties a general coprimeness relation τ on $D^{\#}$ should satisfy. In looking at the previously discussed examples, six properties come to mind:

CP1. $a \not\tau\, a$,

CP2. $a \;\tau\; b \Longrightarrow b \;\tau\; a$,

CP3. $a \;\tau\; b$, $a' \mid a$, and $b' \mid b \Longrightarrow a' \;\tau\; b'$,

CP4. $Da + Db = D \Longrightarrow a \;\tau\; b$,

CP5. $a \;\tau\; b \Longrightarrow [a, b] = 1$,

CP6. $a \;\tau\; b$ and $(a, b) \subseteq (c, d) \;\Longrightarrow\; c \;\tau\; d$.

The following theorem shows that property CP6 is equivalent to τ having the form $\tau_{\mathfrak{S}}$ for some set of ideals $\mathfrak{S}$.

Theorem 2.10. *Let D be an integral domain. Let τ be a relation on $D^{\#}$. Then there exists a set $\mathfrak{S}$ of ideals of D with $a \;\tau\; b \Longleftrightarrow a \;\tau_{\mathfrak{S}}\; b$ if and only if τ satisfies property CP6.*

Proof. Suppose that τ satisfies property CP6. Define $\mathfrak{S} = \{(c, d) \mid c,\ d \in D^{\#}$ and $c \not\tau\, d\}$. If $a \;\tau_{\mathfrak{S}}\; b$, then $a \;\tau\; b$. Otherwise, $a \not\tau\, b$ implies that $(a, b) \in \mathfrak{S}$, a contradiction. Now suppose that $a\ \tau\ b$ and $a \not\tau_{\mathfrak{S}} b$. Then $(a, b) \subseteq (c, d)$ where $c \not\tau d$. By property CP6 $a \not\tau b$, a contradiction.

Conversely, suppose $a\ \tau\ b \Longleftrightarrow a\ \tau_{\mathfrak{S}}\ b$. Suppose that $a\ \tau\ b$ and $(a, b) \subseteq (c, d)$ for some $c,\ d \in D^{\#}$. Then $a\ \tau_{\mathfrak{S}}\ b$, so $(c, d) \notin \mathfrak{S}$. Hence, $c\ \tau\ d$. So property CP6 is satisfied. □

If we look at the six CP properties, we see that property CP6 implies CP2, CP3, and CP4. The first implication is immediate since $(b, a) \subseteq (a, b)$ for all $a, b \in D$. To prove CP3, we notice that given $a', a, b', b \in D^{\#}$ with $a' \mid a$ and $b' \mid b$, then $(a, b) \subseteq (a', b')$. Hence, if $a\ \tau\ b$, then $a'\ \tau\ b'$. Finally, CP4 follows from Theorem 2.10. Given $a, b \in D^{\#}$ with $(a, b) = D$, if $a \not\tau b$, then by CP6 no elements would be τ-related. Assuming τ is not the trivial relationship then $a\ \tau\ b$. So from Theorem 2.10 $\tau_{\mathfrak{S}}$ satisfies properties CP2, CP3, CP4, and CP6.

Property CP6 does not imply CP1 or CP5. For example, look at the ring $\mathbb{Z}$ with $\mathfrak{S} = \{(x) \mid x \in D^{\#},\ x \neq 2\}$. Then $2\ \tau_{\mathfrak{S}}\ 2$. So neither CP1 nor CP5 are satisfied. It is easy to see that CP5 implies CP1. Also, CP1 and CP6 hold if and only if CP5 and CP6 hold.

Motivated by Theorem 2.10 one might think that for a Noetherian domain a multiplicative relation τ satisfying CP1 and CP6 could be defined in terms of a $*$-operation. In other words, given a relation τ satisfying CP1 and CP6 such that $a\ \tau\ b$ and $a\ \tau\ c$ implies $a\ \tau\ bc$ for all $a, b, c \in D^{\#}$, then $a\ \tau\ b \Longleftrightarrow (a, b)^* = D$ for some $*$-operation. This does hold when $\mathfrak{S}$ is the set of proper principal ideals and when $\mathfrak{S}$ is the set of maximal ideals. When $\mathfrak{S}$ is the set of proper principal ideals this follows from Corollary 3.21 and [5, Corollary 3.6]. In this case, $a \ \tau_{\mathfrak{S}} \ b \iff (a,b)_t = D$. Clearly, when $\mathfrak{S}$ is the set of maximal ideals the d-operation works. It would seem that it might hold for all such relations "between" them. However, the following example shows that it does not hold in general. This example is from [15, Example 81].

Example 2.3. Let $R = K[X^2, Y^2, XY, X^3, Y^3, XY^2, X^2Y]$ the subring of $K[X,Y]$. Then R is a 2-dimensional, Noetherian domain. Let $N = (X,Y)$ in $K[X,Y]$. Then $M = (X,Y) \cap R$ is a maximal ideal in R. We show that $G(M_M) = 1$. Now ${X^4}_M \notin (X^3)_M$, but $M_M \cdot {X^4}_M \subseteq (X^3)_M$. So X^3 is a maximal R-sequence in M_M on R_M which implies $G(M_M) = 1$ by Proposition 2.7.

For simplicity of notation we assume that R is a local, 2-dimensional domain with maximal ideal M such that $G(M) = 1$. By Proposition 2.7 $M_t = M$. Thus, $M^* = M$ for all $*$-operations. Let $\mathfrak{S}$ be the set of height-one prime ideals of R. There exists $a, \ b \in R^{\#}$ such that (a,b) is not contained in any height-one prime ideal. Otherwise, M would be in the union of the height-one prime ideals of R and by [16, Theorem 88] there would only be finitely many height-one prime ideals. Thus, by [16, Theorem 81] M would be contained in a height-one prime ideal, a contradiction. Now $a \ \tau_{\mathfrak{S}} \ b$ but $(a,b)^* \neq R$ for any $*$-operation.

We next state a theorem relating $\tau_{\mathfrak{S}}$ for a set of ideals $\mathfrak{S}$ with $\tau_{\sqrt{\mathfrak{S}}}$ where $\sqrt{\mathfrak{S}}$ is defined as $\sqrt{\mathfrak{S}} = \{\sqrt{I} \mid I \in \mathfrak{S}\}$. As usual, for an ideal I, $\sqrt{I} = \{a \in D \mid a^n \in I$ for some $n \in \mathbb{N}\}$.

Theorem 2.11. *Let $\mathfrak{S}$ be a collection of ideals in D. Then the following are equiva-*

lent:

(1) For all nonzero, nonunits $a,\ b \in D$, $a\ \tau_{\mathfrak{S}}\ b \Longrightarrow a\ \tau_{\mathfrak{S}}\ b^2$,

(2) For all nonzero, nonunits $a,\ b \in D$, $a\ \tau_{\mathfrak{S}}\ b \Longrightarrow a^n\ \tau_{\mathfrak{S}}\ b^m$ for any $n, m \geq 1$,

(3) $\tau_{\mathfrak{S}} \equiv \tau_{\sqrt{\mathfrak{S}}}$.

Proof. Since $\tau_{\mathfrak{S}}$ is a symmetric relation, the equivalence of (1) and (2) is true by induction. It suffices to show the equivalence of (2) and (3). Suppose that (2) holds. For any nonzero, nonunit elements $a,\ b \in D$, if $a\ \tau_{\sqrt{\mathfrak{S}}}\ b$, then (a, b) is not contained in any ideal of $\sqrt{\mathfrak{S}}$. But then necessarily (a, b) is not contained in any ideal of $\mathfrak{S}$. So $a\ \tau_{\mathfrak{S}}\ b$. Conversely, suppose $a \not\tau_{\sqrt{\mathfrak{S}}}\ b$. Hence, there exists some $I \in \mathfrak{S}$ such that $(a, b) \subseteq \sqrt{I}$. So $(a^n, b^m) \subseteq I$ for some $n, m \geq 1$. So $a^n \quad \not\tau_{\mathfrak{S}}\ b^m$ as desired.

Suppose (3) holds. For any nonzero, nonunits $a,\ b \in D$, if $a\ \tau_{\mathfrak{S}}\ b$, then $a^n\ \tau_{\mathfrak{S}}\ b^m$ for any $n, m \geq 1$. Otherwise, by hypothesis there exists $n, m \geq 1$ and $I \in \mathfrak{S}$ such that $(a^n, b^m) \subseteq \sqrt{I}$. But then we have $(a, b) \subseteq I$, a contradiction. □

Definition 2.12. Given a set of ideals $\mathfrak{S}$, we say that D is $\mathfrak{S}$-*minimal* if every nonzero, nonunit element of D is contained in a prime ideal of $\mathfrak{S}$.

The following proposition and proof is just a generalization of the proposition and proof for [17, Lemma 1.1].

Proposition 2.13. *Let D be an integral domain and $\mathfrak{S}$ a collection of ideals of D. If D is $\mathfrak{S}$-minimal and every nonzero, nonunit element has only finitely many prime ideals in $\mathfrak{S}$ minimal over it, then D is a $\tau_{\mathfrak{S}}$-atomic domain.*

Proof. Given $a \in D^{\#}$, define $min(a)$ to be the finite collection of primes in $\mathfrak{S}$ minimal over a. By way of contradiction assume the hypothesis holds, but D is not a $\tau_{\mathfrak{S}}$-atomic domain. Then there is an $a \in D$ that does not have a $\tau_{\mathfrak{S}}$-atomic factorization. Within this set, let a be such that $\mid min(a) \mid$ is minimal. Since a has no $\tau_{\mathfrak{S}}$-atomic factorization, it cannot be $\tau_{\mathfrak{S}}$-atomic. Let $a = a_1 \cdots a_n$ be a $\tau_{\mathfrak{S}}$-factorization of a.

We claim that $min(a) = \cup min(a_i)$ where the union is disjoint. If $P \in min(a)$, then P contains some a_i and $P \in min(a_i)$. Now assume that $P \in min(a_i)$ for some a_i. Then P contains a. If there exists $P_0 \in \mathfrak{S}$ with $a \subseteq P_0 \subsetneq P$, then P_0 cannot contain a_i. So P_0 must contain a_j for some $j \neq i$. But then a_i and a_j are both in P, a contradiction. So $P \in min(a)$ as desired. The union must be disjoint by definition of a $\tau_{\mathfrak{S}}$-factorization. This gives us $|min(a_i)| < |min(a)|$. Hence, each a_i has a $\tau_{\mathfrak{S}}$-atomic factorization. But then this yields a $\tau_{\mathfrak{S}}$-atomic factorization of a, a contradiction. □

Since $\tau_{\mathfrak{S}}$ is divisive, the following theorem is true by [2, Theorem 2.11]. We give our own proof here. The proof is similar.

Theorem 2.14. *Let D be a UFD and $\mathfrak{S}$ a collection of ideals of D. Then D is a $\tau_{\mathfrak{S}}$-UFD.*

Proof. Let a be a nonzero, nonunit of D. Since D is a UFD, $a = p_1 \cdots p_n$ has a unique factorization into prime p_i's. If a is a $\tau_{\mathfrak{S}}$-atom, then we are done. Otherwise, we can reorder and group the primes

$$a = (p_1 \cdots p_{s_1}) \cdot (p_{(s_1+1)} \cdots p_{s_2}) \cdots (p_{(s_k+1)} \cdots p_n)$$

into a $\tau_{\mathfrak{S}}$-factorization. If this is not a $\tau_{\mathfrak{S}}$-atomic factorization, then each group of primes, $q_i = (p_{(s_i+1)} \cdots p_{(s_{i+1})})$, that is not a $\tau_{\mathfrak{S}}$-atom has a proper $\tau_{\mathfrak{S}}$-factorization. Since D is a UFD, each τ-factor of a proper $\tau_{\mathfrak{S}}$-factorization of q_i would simply be a product of some subset of $\{p_{(s_i+1)}, \ldots, p_{(s_{i+1})}\}$. Since the prime factorization of a has length n, this process of $\tau_{\mathfrak{S}}$-refining $\tau_{\mathfrak{S}}$-factorizations of a can only be repeated finitely many times. Hence, a has a $\tau_{\mathfrak{S}}$-atomic factorization.

We need to show uniqueness of $\tau_{\mathfrak{S}}$-atomic factorizations. Suppose $a_1 \cdots a_n = b_1 \cdots b_m$ are two $\tau_{\mathfrak{S}}$-atomic factorizations. We proceed by induction on n. The case when $n = 1$ is clear. Suppose $n > 1$, and by induction, if any element $c \in D^{\#}$ has a $\tau_{\mathfrak{S}}$-atomic factorization of length less than n, then that is the unique $\tau_{\mathfrak{S}}$-atomic factorization of c, up to order and units. If a_1 is prime, then $a_1 \mid b_i$ for some i, say $i = 1$. If a_1 is not prime, we will show that a_1 still divides some b_i. If a_1 is not prime, then $a_1 = p_1 \cdots p_l$ where each p_i is prime and $l > 1$. By way of contradiction, suppose that p_1 and p_2 divide b_1 and b_2, respectively. Since $\tau_{\mathfrak{S}}$ is divisive, $p_1 \ \tau_{\mathfrak{S}} \ p_2$. We can now group the remaining p_i's appropriately to form a proper $\tau_{\mathfrak{S}}$-factorization of a_1. For example, if $p_3 \mid b_1$, then $p_1 p_3 \ \tau_{\mathfrak{S}} \ p_2$. If $p_3 \mid b_i$ for $i > 2$, then p_1, p_2, and p_3 are all $\tau_{\mathfrak{S}}$-related. In this way, we can construct a proper $\tau_{\mathfrak{S}}$-factorization of a_1 (we exclude the complete construction since the notation is quite tedious). So each p_i must divide the same b_j, and each p_i divides exactly one b_j. Hence, a_1 divides some b_j, say b_1.

In either case, a_1 divides b_1, and we get that $a_2 \cdots a_n = cb_2 \cdots b_m$ where $a_1 c = b_1$ for some $c \in D$. As already shown, c has a $\tau_{\mathfrak{S}}$-atomic factorization, $c = c_1 \cdots c_k$. Since $\tau_{\mathfrak{S}}$ is divisive, $c_1 \cdots c_k b_2 \cdots b_m$ is a $\tau_{\mathfrak{S}}$-atomic factorization. By the induction hypothesis, and after reordering the a_i's, we have $n = k + m$, $a_{i+1} \sim c_i$ for $1 \leq i \leq k$, and $a_{k+1+i} \sim b_{i+1}$ for $1 \leq i < m$. Repeating the induction for m gives us $n = m$, and after reordering, $a_i \sim b_i$ for each i.

□

CHAPTER 3
$\tau_{[\ ]}$-UFD

In this chapter, we explore further the example of $\mathfrak{S}$-coprime where $\mathfrak{S}$ is the set of proper principal ideals. We noted in Example 2.1, (8) of Chapter 2 that $a \cdot b$ is a proper $\tau_{\mathfrak{S}}$-factorization if and only if $[a, b] = 1$. We denote this $\tau_{\mathfrak{S}}$ by $\tau_{[\ ]}$.

We start off with a basic result regarding quasilocal domains.

Theorem 3.1. *Let D be an integral domain.*

(1) Every nonzero nonunit is a τ_{max}-atom if and only if D is quasilocal.

(2) Every nonzero nonunit is a τ_v-atom if and only if D is quasilocal and for $x, y \in M - \{0\}$, $(x, y)_v \subseteq M$ where M is the maximal ideal of D.

(3) Every nonzero nonunit is a $\tau_{[\ ]}$-atom if and only if D is quasilocal and for $x, y \in M$ there exists $m \in M$ with $(x, y) \subseteq (m)$ (or equivalently, for $I \subseteq M$ finitely generated, there exists $m \in M$ such that $I \subseteq (m)$) where M is the maximal ideal of D.

Proof.

(1) ($\Longleftarrow$) If D is quasilocal, then $(a, b) \in M$ for all a and b nonunits. So there are no proper τ_{max}-factorizations.

($\Longrightarrow$) Assume there are two maximal ideals M_1 and M_2. Then there is $m_i \in M_i$ and $r_i \in D$ such that $r_1 m_1 + r_2 m_2 = 1$. But then $d = m_1 \cdot m_2$ is not a τ_{max}-atom of D.

(2) ($\Longleftarrow$) Similarly to (1) $(a,b)_v \in M$ for a and b nonunits. So there are no proper τ_v-factorizations.

($\Longrightarrow$) If every nonzero, nonunit is a τ_v-atom, then every nonzero, nonunit is a τ_{max}-atom. So from (1) we have D is quasilocal. Since $(x,y)_v \neq D$ for all nonzero, nonunit elements, $(x,y)_v$ must be contained in M.

(3) ($\Longleftarrow$) Again this direction is clear. There are no proper $\tau_{[\ ]}$-factorizations. Given $x, y \in D^{\#}$, there is an $m \in M$ such that $m|x$ and $m|y$.

($\Longrightarrow$) D is quasilocal for the same reason as in (2). The second part is clear.

Note that for the parenthetical statement in Theorem 3.1, if $I \subseteq M$ with $I = (x_1, \ldots, x_n)$, then by induction $(x_1, \ldots, x_{n-1}) \subseteq (m)$ for some $m \in M$. But then by hypothesis $(m, x_n) \subseteq (m_1)$ for some $m_1 \in M$.

□

3.1 Weakly Factorial Domains

We discussed the notion of a weakly factorial domain in Subsection 2.2.1. We showed the connection between weakly factorial domains and $\tau_{\mathfrak{S}}$-UFD's where $\mathfrak{S}$ is the set of height-one prime ideals. In this section, we study $\tau_{[\ ]}$-factorization in weakly factorial domains.

Proposition 3.2. *Let D be a weakly factorial domain with q_1 and q_2 nonzero, primary elements. Then* $\sqrt{q_1} \neq \sqrt{q_2} \Longrightarrow (q_1, q_2)_v = D$.

Proof. Let q_i be P_i-primary. We use Lemma 1.1. Suppose $(q_1, q_2) \subseteq \frac{a}{b}D$. We want to show that $a \mid b$. We have $q_i b = a d_i$ for some $d_i \in D$ $(i = 1, 2)$. This gives us $q_1q_2b = q_1ad_2 = q_2ad_1$ which implies $q_1d_2 = q_2d_1$. So $q_2d_1 \in (q_1)$ and $q_2 \notin P_1$ [3, Theorem 4] implies $d_1 \in (q_1)$, say $d_1 = dq_1$. Then $q_1b = ad_1 = adq_1$ implies $b = ad$. So $a \mid b$ as desired. □

Lemma 3.3. *Let q be P-primary. If $(q) \subseteq (b)$ for some proper principal ideal, then b is P-primary.*

Proof. Now $(q) \subseteq (b)$ gives us $rb = q$ for some $r \in D$. If $(b) \nsubseteq P$, then $r \in (q)$, say $r = sq$. Then we have $q = rb = sqb \Longrightarrow 1 = sb$, a contradiction. Hence, $b \in P$ and we get $\sqrt{(b)} = P$.

Now let $xy \in (b)$ and $y \notin P$. Let $xy = db$ for some $d \in D$. Since $rb = q$, $rxy = rdb = dq \in (q)$ which implies $rx \in (q)$, say $rx = aq$. Then $qx = rxb = aqb$ which implies $x \in (b)$ as desired. Hence, b is P-primary. □

Proposition 3.4. *Let D be a weakly factorial domain. If x is a $\tau_{[\ ]}$-atom, then x is primary.*

Proof. Since D is weakly factorial, x can be written as a reduced product of primaries, say $x = q_1 \cdots q_n$ with q_i P_i-primary. Since x is a $\tau_{[\ ]}$-atom, either $n = 1$ or there exist distinct q_i and q_j that have a common nonunit divisor. If the latter case were so, then q_i and q_j would both be contained in some proper principal ideal. So by Lemma 3.3 $P_i = P_j$, a contradiction. Hence, x is primary. □

Theorem 3.5. *Let D be a weakly factorial domain. Then q a P-primary element is a $\tau_{[\ ]}$-atom in D if and only if q is a $\tau_{[\ ]}$-atom in D_P.*

Proof. Assume that q is P-primary and is a $\tau_{[\ ]}$-atom in D. Assume by way of contradiction that $q = \frac{r_1}{s_1} \cdots \frac{r_n}{s_n}$ is a proper $\tau_{[\ ]}$-factorization in D_P. Then $(q)_P = (\frac{r_1}{s_1} \cdots \frac{r_n}{s_n})_P = (r_1 \cdots r_n)_P$. Since D is weakly factorial, each r_i has a reduced primary decomposition $r_i = x_{i,1} \cdots x_{i,k_i}$ where $x_{i,j}$ is P-primary for some j. Since $x_{i,l}$ is not P-primary for $l \neq j$, $x_{i,l} \notin P$. Hence, $(r_i)_P = (x_{i,j})_P$. For simplicity of notation let us denote $x_{i,j}$ as x_i for each r_i. Then $(q)_P = (r_1)_P \cdots (r_n)_P = (x_1)_P \cdots (x_n)_P$. Since each (x_i) is P-primary, $(q) = (x_1) \cdots (x_n)$ in D. Since q is a $\tau_{[\ ]}$-atom, there must exist x_i and x_j with a common nonunit divisor in D. But by Lemma 3.3 such a divisor must be P-primary. Hence, it is also a nonunit divisor of x_i and x_j in D_P, a contradiction.

Assume that q is P-primary and is a $\tau_{[\ ]}$-atom in D_P. Let $q = x_1 \cdots x_n$ be a proper $\tau_{[\ ]}$-factorization in D. By Lemma 3.3 each x_i is P-primary. So $(q)_P = (x_1)_P \cdots (x_n)_P$ is a product of P_P-primary ideals. Since q is a $\tau_{[\ ]}$-atom in D_P and none of the x_i's are units in D_P, there must exist x_i and x_j that have a common nonunit divisor, say r (we can assume the divisor is an element of D). Since D is weakly factorial, r can be written as a product of primary elements with one such element being P-primary. Let r_0 be this element. Then r_0 divides both x_i and x_j in D_P which is equivalent to $(x_i)_P$ and $(x_j)_P$ being contained in $(r_0)_P$. Since (x_i), (x_j), and (r_0) are all P-primary, this implies that (x_i) and (x_j) are contained in (r_0). So $[x_i, x_j] \neq 1$ in D, a contradiction. So q is a $\tau_{[\ ]}$-atom in D. □

The following corollary was proven in the proof of Theorem 3.5.

Corollary 3.6. *Let D be a weakly factorial domain. Given q_1 and q_2 P-primary for* *some prime, we have* $[q_1, q_2] = 1$ *in* D *if and only if* $[q_1, q_2] = 1$ *in* D_P.

Theorem 3.7. *Let* D *be a weakly factorial domain.* D *is* $\tau_{[\ ]}$*-atomic if and only if* D_P *is* $\tau_{[\ ]}$*-atomic at each height-one prime ideal.*

Proof. Suppose D is $\tau_{[\ ]}$-atomic. We look at D_P for a height-one prime ideal P. Let $\frac{p}{s} \in P_P$. We have $(\frac{p}{s})_P = (p)_P = (q)_P$ for some q that is P-primary. If q is a $\tau_{[\ ]}$-atom in D_P, then we are done. Otherwise, by hypothesis and Theorem 3.5 q has a $\tau_{[\ ]}$-atomic factorization in D, say $q = x_1 \cdots x_n$. As mentioned above each x_i is P-primary. From Theorem 3.5 each x_i is a $\tau_{[\ ]}$-atom in D_P. From Corollary 3.6 we also have $[x_i, x_j] = 1$ for each $i \neq j$ in D_P. Hence $\frac{p}{s} = uq = ux_1 \cdots x_n$ is a $\tau_{[\ ]}$-atomic factorization of $\frac{p}{s}$ in D_P.

Suppose that D_P is $\tau_{[\ ]}$-atomic at each height-one prime ideal P. Let x be a nonzero, nonunit element of D. Since D is weakly factorial, x can be written as a product of primary elements with radicals having height one, say $x = x_1 \cdots x_m$ with x_i being P_i-primary. By Lemma 3.3 this is a $\tau_{[\ ]}$-factorization. Then x_i has a $\tau_{[\ ]}$-atomic factorization in D_{P_i}, say $x_i = \frac{r}{s} p_{i,1} \cdots p_{i,n_i}$ with $\frac{r}{s} \in U(D_{P_i})$. We can assume that each $p_{i,j}$ is P_i-primary. From Theorem 3.5 each $p_{i,j}$ is a $\tau_{[\ ]}$-atom in D. From Corollary 3.6 we have $[p_{i,s}, p_{i,t}] = 1$ in D for $s \neq t$. Finally, $(x_i)_{P_i} = (p_{i,1})_{P_i} \cdots (p_{i,n_i})_{P_i}$ with (x_i) and each $(p_{i,j})$ being P_i-primary implies that $(x_i) = (p_{i,1}) \cdots (p_{i,n_i})$ in D. Hence $x = up_{1,1} \cdots p_{1,n_1} p_{2,1} \cdots p_{2,n_2} \cdots p_{m,1} \cdots p_{m,n_m}$ is a $\tau_{[\ ]}$-atomic factorization of x in D. □

Theorem 3.8. *Let* D *be a weakly factorial domain.* D *is a* $\tau_{[\ ]}$*-UFD if and only if* D_P *is a* $\tau_{[\ ]}$*-UFD for each height-one prime ideal* P *of* D.

Proof. From Theorem 3.7 we only need to consider the uniqueness of $\tau_{[\ ]}$-atomic factorizations.

Suppose D is a $\tau_{[\ ]}$-UFD. Let $\frac{r_1}{s_1}\cdots\frac{r_n}{s_n} = \frac{x_1}{t_1}\cdots\frac{x_m}{t_m}$ be two $\tau_{[\ ]}$-atomic factorizations in D_P. Since D is weakly factorial, we can assume that each r_i and x_j is P-primary. So we have $(r_1)_P\cdots(r_n)_P = (x_1)_P\cdots(x_m)_P$ with each r_i and x_j a $\tau_{[\ ]}$-atom in D_P and P-primary in D. Hence, $(r_1\cdots r_n) = (x_1\cdots x_m)$ in D, and by Theorem 3.5 and Corollary 3.6, $ur_1\cdots r_n = x_1\cdots x_m$, where u is a unit in D, are two $\tau_{[\ ]}$-atomic factorizations in D. By hypothesis, after reordering we have $r_i \sim x_i$ and $m = n$. Hence, D_P is a $\tau_{[\ ]}$-UFD.

Suppose D_P is a $\tau_{[\ ]}$-UFD at each height-one prime ideal P. Let $a_1\cdots a_n = b_1\cdots b_m$ be two $\tau_{[\ ]}$-atomic factorizations in D. By Proposition 3.4 each a_i and b_j is primary. We pass to D_P for some height-one prime ideal P containing the factorization. After reordering we have $(a_1)_P\cdots(a_k)_P = (b_1)_P\cdots(b_l)_P$ for some $k \leq n$ and $l \leq m$. From Theorem 3.5 and Corollary 3.6 we get $\frac{r}{s}a_1\cdots a_k = b_1\cdots b_l$ with $\frac{r}{s}\in U(D_P)$ are two $\tau_{[\ ]}$-atomic factorizations in D_P. So after reordering we get $a_i \sim b_i$ in D_P and $l = k$. But this implies that $(a_i)_P = (b_i)_P$. Since each a_i and b_i are P-primary, we have $(a_i) = (b_i)$ in D. Repeating this process at each height-one prime ideal containing the factorization gives us, after reordering, $a_i \sim b_i$ in D and $n = m$ as desired. □

Corollary 3.9. *A weakly factorial GCD domain D is a $\tau_{[\ ]}$-UFD.*

Proof. From [3, Theorem 18] D_P is a valuation domain at each height-one prime ideal P of D. Hence, every nonzero, nonunit element of D_P is a $\tau_{[\ ]}$-atom. So

D_P is a $\tau_{[\ ]}$-UFD at each height-one prime ideal P. □

3.2 GCD Domains

Lemma 3.10. *In a GCD domain, $\tau_{[\ ]}$ is a multiplicative relation.*

Proof. Since $a\ \tau_{[\ ]}\ b$ is equivalent to $[a, b] = 1$, this is just a restatement of [16, Theorem 49]. □

The following lemma is Exercise 7 from Section 1-6 of [16].

Lemma 3.11. *Let D be a GCD domain. If $[u, a] = 1$ and u divides ab, then u divides b.*

Proof. Since D is a GCD domain, $[ub, ab] = b$. Since u divides both ab and ub, u divides b. □

Proposition 3.12. *Let D be a GCD domain, and p a nonzero, nonunit element of D. Then the following our equivalent:*

(1) p is a $\tau_{[\ ]}$-atom,

(2) p is $\tau_{[\ ]}$-prime,

(3) If $p \mid ab$, where $[a, b] = 1$, then $[p, a] = 1$ or $[p, b] = 1$.

Proof. (2) $\Rightarrow$ (1) is always true.

(1) $\Rightarrow$ (2) Suppose that $p \mid ab$, say $pq = ab$, with $[a, b] = 1$. If $[p, a] = 1$ or $[p, b] = 1$, then by Lemma 3.11 $p \mid b$ or $p \mid a$, respectively. Let $[p, a] = x$ and $[p, b] = y$. Then $p = xp_1$ and $a = xa_1$ for some $p_1,\ a_1 \in D$. So $p_1 q = a_1 b$. Since $[p_1, a_1] = 1$,

$p_1 \mid b$. So $p_1 \mid b$ and $p_1 \mid p$ implies $p_1 \mid y$. Also, $y \mid p = xp_1$ and $[x, y] = 1$ implies $y \mid p_1$. Hence, $y \sim p_1$. So, $uxy = p$ for some unit u. But $[x, y] = 1$ and p is $\tau_{[\,]}$-atomic. Thus, either x or y is a unit as desired.

(2) $\Rightarrow$ (3) Suppose p is $\tau_{[\,]}$-prime and $p \mid ab$ where $[a, b] = 1$. Then p divides a or b. Suppose $p \mid a$. Then $[p, b] \mid [a, b] = 1$. Hence, $[p, b] = 1$. The same argument holds if $p \mid b$.

(3) $\Rightarrow$ (2) Suppose (3) holds, and we have $p \mid ab$ where $[a, b] = 1$. Then the result follows readily from Lemma 3.11. $\square$

Corollary 3.13. *In a GCD domain, $\tau_{[\,]}$-atomic implies $\tau_{[\,]}$-UFD.*

Proof. We need to show uniqueness of $\tau_{[\,]}$-atomic factorizations. Suppose $p_1 \cdots p_n = q_1 \cdots q_m$ are two $\tau_{[\,]}$-atomic factorizations. By Proposition 3.12 each p_i and q_j are $\tau_{[\,]}$-prime. So p_1 divides some q_j, say q_1. But then q_1 divides some p_i. Since $[p_1, p_i] = 1$ for $i \neq 1$, q_1 must divide p_1. So $(p_1) = (q_1)$ and the result follows by induction. $\square$

The following lemma is taken from [2, Lemma 2.10], and we state it here since it proves useful for us.

Lemma 3.14. *Let D be an integral domain and let τ be a divisive relation on $D^{\#}$. Let $a_1 \cdots a_n$ be a τ-atomic factorization. Then for $i \neq j$, either $[a_i, a_j] = 1$ or $a_i \sim a_j$ are atoms.*

Lemma 3.15. *Let D be a τ-atomic GCD domain where τ is a divisive relation on D. Then the $\tau_{[\,]}$-atoms of D are τ-atoms or elements of the form up^n where u is a* *unit, $n \geq 1$, and p is prime.*

Proof. Let x be a $\tau_{[\ ]}$-atom in D. Since D is τ-atomic, x has a τ-atomic factorization $x = a_1 \cdots a_n$. If $[a_i, a_j] = 1$ for some $i \neq j$, then by Lemma 3.14 we get a proper $\tau_{[\ ]}$-factorization of x by grouping the elements that are not relatively prime together. Hence, if $n > 1$, then by Lemma 3.14 each a_i must be associative atoms. Moreover, since D is a GCD domain, each a_i is actually prime. So, $x = up^n$ as desired. □

Theorem 3.16. *Suppose D is a GCD, $\tau_{[\ ]}$-atomic domain; and τ is a divisive, multiplicative relation on D. If D is τ-atomic, then D is a τ-UFD.*

Proof. By Corollary 3.13 D is a $\tau_{[\ ]}$-UFD. We will use this fact along with Lemmas 3.14 and 3.15 throughout this proof without further comment.

We have only to show the uniqueness of τ-atomic factorizations. Suppose that $b_1 \cdots b_m = c_1 \cdots c_n$ are two τ-atomic factorizations. If $m = 1$ or $n = 1$, then we are done. If $[b_i, b_j] \neq 1$ for some $i \neq j$, then we can group all such τ-atoms, and after grouping and reordering we can write $b_1 \cdots b_m = b_1 \cdots b_{m'} {p_1}^{m_1} \cdots {p_{m''}}^{m_{m''}}$ where each p_i is prime, $p_i \ \tau \ p_i$, and any two factors on the right are relatively prime. We can group the c_i's in a similar manner to get $b_1 \cdots b_{m'} {p_1}^{m_1} \cdots {p_s}^{m_s} = c_1 \cdots c_{n'} {q_1}^{n_1} \cdots {q_t}^{n_t}$. Note that $b_i \not\sim {q_j}^{n_j}$ for any j and $1 \leq i \leq m'$ since $q_j \ \tau \ q_j$. Similarly, $c_i \not\sim {p_j}^{m_j}$ for any j and $1 \leq i \leq n'$.

Now this element has a $\tau_{[\ ]}$-atomic factorization, say

$$b_1 \cdots b_{m'} {p_1}^{m_1} \cdots {p_s}^{m_s} = a_1 \cdots a_k = c_1 \cdots c_{n'} {q_1}^{n_1} \cdots {q_t}^{n_t} \tag{3.1}$$

Also, any pair of elements in the factorization on the left are relatively prime. Likewise, for the factorization on the right. Hence, since D is a $\tau_{[\,]}$-UFD, any b_i or c_i in Equation (3.1) is a product of a subset of the a_i's, and each ${p_i}^{m_i}$ or ${q_i}^{n_i}$ is equal to some a_i. So both $b_1 \cdots b_{m'} {p_1}^{m_1} \cdots {p_s}^{m_s}$ and $c_1 \cdots c_{n'} {q_1}^{n_1} \cdots {q_t}^{n_t}$ have $\tau_{[\,]}$-factorizations of the form

$$(a_{1,1} \cdots a_{1,s_1})(a_{2,1} \cdots a_{2,s_2}) \cdots (a_{v,1} \cdots a_{v,s_v}) \tag{3.2}$$

where, for example, $b_1 = (a_{1,1} \cdots a_{1,s_1})$, $b_2 = (a_{2,1} \cdots a_{2,s_2})$,...,${p_s}^{m_s} = (a_{v,1} \cdots a_{v,s_v})$ (in this instance $s_v = 1$). Let us assume that Equation (3.2) is a factorization of $b_1 \cdots b_{m'} {p_1}^{m_1} \cdots {p_s}^{m_s}$. If $c_1 \cdots c_{n'} {q_1}^{n_1} \cdots {q_t}^{n_t}$ has the same such factorization, then $m = n$ and after reordering $b_i \sim c_i$. We claim that they both must have the same such factorization of the a_i's. If the grouping of factors differs for $c_1 \cdots c_{n'} {q_1}^{n_1} \cdots {q_t}^{n_t}$, then there exists $a_{i,j}$ that is no longer in the same grouping of factors. If $a_{i,j}$ is with a new grouping of factors, then using multiplicativity and the fact that $a_{i,j} \ \tau \ a_{s,t}$ for $i \neq s$ we would have a proper τ-factorization of one of the c_i's, a contradiction. If $a_{i,j}$ is not with a new grouping of factors, then $a_{i,j}$ is in a grouping that is a subset of the grouping $a_{i,j}$ was in for $b_1 \cdots b_{m'} {p_1}^{m_1} \cdots {p_s}^{m_s}$. But then b_i has a proper τ-factorization, a contradiction. □

We state here some facts that will be useful. Given a GCD domain D, each nonconstant $f \in D[X]$ is uniquely expressible, to within unit factors in D, as $f = a \cdot g$ where g is a primitive polynomial and $a \in D$. Also, each nonconstant primitive polynomial in $D[X]$ is a finite product of prime polynomials in $D[X]$ [12, Theorem 34.10].

Lemma 3.17. *Let D be a GCD domain. Suppose that $f \in D[X]$ is a nonconstant $\tau_{[\ ]}$-atomic element. Given $f = ag$, where g is primitive and $a \in D$, then $f \sim g$. Hence, every nonconstant $\tau_{[\ ]}$-atom is primitive.*

Proof. Since g is primitive, $[a, g] = 1$. So a must be a unit in D. □

Theorem 3.18. *If D is a GCD domain, then D is a $\tau_{[\ ]}$-UFD if and only if $D[X]$ is a $\tau_{[\ ]}$-UFD.*

Proof. If $D[X]$ is a $\tau_{[\ ]}$-UFD, then it is straightforward to show that D is a $\tau_{[\ ]}$-UFD. Suppose that D is a $\tau_{[\ ]}$-UFD. We first show that $D[X]$ is $\tau_{[\ ]}$-atomic. We do this by induction on the degree of an element. The base case is covered by the hypothesis. Let $f \in D[X]$ be a nonzero, nonunit element with $deg(f) = n$ where $n \geq 1$. We can write $f = a \cdot g$ where g is primitive, $a \in D$, and $[a, g] = 1$. If a is a nonunit, then since D is a $\tau_{[\ ]}$-UFD, there exists $a = a_1 \cdots a_n$, a $\tau_{[\ ]}$-atomic factorization of a. If g is a $\tau_{[\ ]}$-atom, then $a_1 \cdots a_n \cdot g$ is a $\tau_{[\ ]}$-atomic factorization of f. Otherwise, we have $g = g_1 \cdot g_2$ a $\tau_{[\ ]}$-factorization of g. Since g is primitive, $deg(g_i) < deg(g)$ for each i. So each g_i has a $\tau_{[\ ]}$-atomic factorization by the induction hypothesis. This gives a $\tau_{[\ ]}$-atomic factorization of f.

By Corollary 3.13 we have that $D[X]$ is a $\tau_{[\ ]}$-UFD.

□

Proposition 3.19. *A pre-Schreier $\tau_{[\ ]}$-atomic domain D is a $\tau_{[\ ]}$-UFD.*

Proof. We show that in a pre-Schreier domain $\tau_{[\ ]}$-atoms are $\tau_{[\ ]}$-primes. Let p be a $\tau_{[\ ]}$-atom, and suppose $p \mid a_1 \cdots a_n$ where $[a_i, a_j] = 1$ for each $i \neq j$. Then

$p = p_1 \cdots p_n$ where $p_i \mid a_i$ for each i. But then $[p_i, p_j] = 1$ for each $i \neq j$. Since p is a $\tau_{[\ ]}$-atom, only one p_i is a nonunit. Thus, $p \mid a_i$ and so each $\tau_{[\ ]}$-atom is $\tau_{[\ ]}$-prime. The proof is now exactly like Corollary 3.13. □

Let R be a commutative ring. Recall that for any $f \in R[X]$, $c(f)$ is defined to be the ideal in R generated by the coefficients of f. Also, given a GCD domain D and a finite character $*$-operation, we define $N_* = \{f \in D[X] \mid c(f)^* = D\}$. By [12, Lemma 32.6] N_* is multiplicatively closed if $*$ is endlich arithmetisch brauchbar. By [12, Proposition 34.8] N_t is multiplicatively closed if D is integrally closed. So by [16, Theorem 50] N_t is multiplicatively closed when D is a GCD domain.

Lemma 3.20. *Let D be a GCD domain. For $a_1, \ldots, a_n \in D$ then $(a_1, \ldots, a_n)_t = (a)$ where $[a_1, \ldots, a_n] = a$.*

Proof. Since a divides each a_i and (a) is a t-ideal, $(a_1, \ldots, a_n)_t \subseteq (a)$. We must show the reverse inclusion. We have $(a_1, \ldots, a_n)_t = \cap(\frac{c}{d})$, the intersection being taken over all principal fractional ideals containing $(a_1, \ldots, a_n)$. Suppose $(a_1, \ldots, a_n) \subseteq (\frac{c}{d})$. Since D is a GCD domain, we can assume $[c, d] = 1$. This implies $c \mid a_i$ for each i and so $c \mid a$. Hence, $(a) \subseteq (c) \subseteq (\frac{c}{d})$. So, $(a) = (a_1, \ldots, a_n)_t$ as desired. □

For our purposes we are interested in the implications of Lemma 3.20 where $(a_1, \ldots, a_n) = c(f)$ for some $f \in D[X]$. Thinking of $(c(f))_t$ as the ideal generated by the greatest common divisor of the coefficients of f gives us a nice relationship between $\tau_{[\ ]}$-factorization in D and comaximal factorization in $D[X]_{N_t}$. In fact, they turn out to be the same. We state here a special case of Lemma 3.20. Notice that this corollary says that the $\tau_{[\ ]}$ relation and the τ_t relation are the same in a GCD domain.

Corollary 3.21. *Let D be a GCD domain. Then for $a,\ b \in D^{\#}$ we have $(a,b)_t = D \iff [a,b] = 1$.*

Recall in Lemma 2.6 we showed that for two elements a and b in D that are not comaximal then $(a,b)_t = D$ if and only if $a,\ b$ is an R-sequence. We state here the obvious corollary of Lemma 2.6 and Corollary 3.21.

Corollary 3.22. *Let D be a GCD domain. For nonzero elements $a,\ b \in D$ with $(a,b) \neq D$ the following are equivalent:*

1. $[a,b] = 1$,
2. $(a,b)_t = D$,
3. $a,\ b$ *is an R-sequence.*

Lemma 3.23. *Let D be a GCD domain. Then $[a,b] = 1$ if and only if $(a,b)D[X]_{N_t} = D[X]_{N_t}$.*

Proof. Suppose $[a,b] = 1$ in D. Then by Lemma 3.20 $f := aX + b \in N_t$. Hence, $(aX+b)\frac{1}{f} = 1 \iff a \cdot \frac{X}{f} + b \cdot \frac{1}{f} = 1$ in $D[X]_{N_t}$. So $(a,b)D[X]_{N_t} = D[X]_{N_t}$ as desired.

Suppose $[a,b] \neq 1$. Then $(a,b)D \subseteq rD \subsetneq D$ for some $r \in D$. So $(a,b)D[X]_{N_t} \subseteq (r)D[X]_{N_t} \subsetneq D[X]_{N_t}$. Otherwise, $r \cdot \frac{f}{g} = 1$ for some $\frac{f}{g} \in D[X]_{N_t}$, and so $D = c(g)_t = c(rf)_t = r(c(f)_t) \subseteq rD$, a contradiction. So $(a,b)D[X]_{N_t} \neq D[X]_{N_t}$. □

Lemma 3.24. *Let D be a GCD domain. Then $\frac{f}{g} \in D[X]_{N_t}$ is a τ_{max}-atom if and only if $c(f)_t = (a)$ is a $\tau_{[\ ]}$-atom in D.*

Proof. Suppose $\frac{f}{g} \in D[X]_{N_t}$ is a τ_{max}-atom with $c(f)_t = (a)$. If $a = a_1 \cdots a_n$ is a proper $\tau_{[\ ]}$-factorization of a in D, then by Lemma 3.23 $\frac{f}{g} = a_1 \cdots a_n \frac{f'}{g}$ is a proper τ_{max}-factorization of $\frac{f}{g}$ in $D[X]_{N_t}$ with $\frac{f'}{g} \in N_t$, a contradiction.

Conversely, suppose a is a $\tau_{[\ ]}$-atom of D. If $\frac{f}{g} = \frac{f_1}{g_1} \cdots \frac{f_n}{g_n}$ is a proper τ_{max}-factorization in $D[X]_{N_t}$ with $c(f_i)_t = (a_i)$, then we have $a\frac{f'}{g} = a_1\frac{f_1'}{g_1} \cdots a_n\frac{f_n'}{g_n} = a_1 \cdots a_n \frac{f_1'}{g_1} \cdots \frac{f_n'}{g_n}$ for some f' and f_i''s in N_t. So from Lemmas 3.20 and 3.23 we get $(a) = c(af'g_1 \cdots g_n)_t = c(a_1 \cdots a_n g f_1' \cdots f_n')_t = (a_1) \cdots (a_n)$ forms a proper $\tau_{[\ ]}$-factorization of a, a contradiction.

□

Theorem 3.25. *Let D be a GCD domain. Then D is a $\tau_{[\ ]}$-UFD if and only if $D[X]_{N_t}$ is a UCFD.*

Proof. Suppose that D is a $\tau_{[\ ]}$-UFD. Let $\frac{f}{g}$ be a nonzero, nonunit element in $D[X]_{N_t}$. Let $c(f)_t = (a)$. Let $a = a_1 \cdots a_n$ be a $\tau_{[\ ]}$-atomic factorization of a in D. So by Lemma 3.20 $\frac{f}{g} = a_1 \cdots a_n \frac{f'}{g}$ where $\frac{f'}{g}$ is a unit in $D[X]_{N_t}$. From Lemma 3.23 a_i and a_j are comaximal for $i \neq j$. We must show each a_i is a τ_{max}-atom. Suppose $a_i = \frac{f_1}{g_1}\frac{f_2}{g_2}$ is a proper τ_{max}-factorization of a_i with $c(f_i)_t = (b_i)$ in $D[X]_{N_t}$. Then by Lemma 3.20 $(b_1 b_2) = c(f_1)_t c(f_2)_t = c(f_1 f_2)_t = c(a_i g_1 g_2)_t = a_i c(g_1 g_2)_t = (a_i)$. But by Lemma 3.23 $b_1 b_2$ is a proper $\tau_{[\ ]}$-factorization in D, a contradiction. So a_i is a τ_{max}-atom.

We must show uniqueness. Suppose $\frac{f_1}{g_1} \cdots \frac{f_n}{g_n} = \frac{h_1}{k_1} \cdots \frac{h_m}{k_m}$ are two τ_{max}-atomic factorizations in $D[X]_{N_t}$ with $c(f_i)_t = (a_i)$ and $c(h_j)_t = (b_j)$. From the argument in the previous paragraph and Lemma 3.24, we see this implies $ua_1 \cdots a_n = vb_1 \cdots b_m$, where $u,\ v \in U(D)$, are two $\tau_{[\ ]}$-atomic factorizations in D. By hypothesis we get, after reordering, $(a_i) = (b_i)$ and $n = m$ in D. But then $a_i D[X]_{N_t} = b_i D[X]_{N_t}$. Hence, $(\frac{f_i}{g_i}) = (\frac{h_i}{k_i})$ as desired.

Suppose that $D[X]_{N_t}$ is a UCFD. By Corollary 3.13 it suffices to show that D is $\tau_{[\ ]}$-atomic. Let a be a nonzero, nonunit element of D. Then $a = \frac{f_1}{g_1} \cdots \frac{f_n}{g_n}$ has a τ_{max}-atomic factorization in $D[X]_{N_t}$ with $c(f_i)_t = (a_i)$. Then $a = a_1 \cdots a_n \frac{f}{g}$ for some unit $\frac{f}{g}$ in $D[X]_{N_t}$. By Lemmas 3.23 and 3.24 $a_1 \cdots a_n$ is a $\tau_{[\ ]}$-atomic factorization in D. Also, $(a_1 \cdots a_n) = c(f_1)_t \cdots c(f_n)_t = c(f_1 \cdots f_n)_t = c(ag)_t = ac(g)_t = (a)$. Hence, a has a $\tau_{[\ ]}$-atomic factorization in D. □

From [17, Corollary 1.10] we get this immediate corollary.

Corollary 3.26. *Suppose D is a GCD domain such that every ideal of $D[X]_{N_t}$ is contained in only finitely many maximal ideals of $D[X]_{N_t}$. Then D is a $\tau_{[\ ]}$-UFD if and only if every invertible ideal of $D[X]_{N_t}$ is principal.*

For a GCD domain D, the hypothesis of Corollary 3.26 is satisfied when every ideal of $D[X]$ disjoint from N_t is contained in only finitely many prime ideals maximal with respect to being disjoint from N_t. An ideal I of $D[X]$ is disjoint from N_t precisely when each $f \in I$ has $c(f)_t \subsetneq D$. From Lemma 3.20 this is equivalent to the coefficients of f having a common divisor for each $f \in I$.

CHAPTER 4
SOME CHARACTERIZATIONS OF $\tau_{[\ ]}$-UFD'S

We further motivate our study of $\tau_{[\ ]}$-UFD's with some examples. We first look at $\mathbb{Z}_2[[X^2, X^3]]$ to give us an idea of the $\tau_{[\ ]}$-atomic structure of $k[[X^2, X^3]]$ for a general field k. Let $\overline{D}$ denote the integral closure of D.

4.1 $\tau_{[\ ]}$-atomic structure of $\mathbb{Z}_2[[X^2, X^3]]$

Let $D = \mathbb{Z}_2[[X^2, X^3]]$, so $\overline{D} = \mathbb{Z}_2[[X]]$, and $U(D) = \{1 + a_2X^2 + \cdots \mid a_n \in \mathbb{Z}_2\}$. An element of $U(\overline{D})$ is of the form $v = 1 + a_1X + a_2X^2 + \cdots$ with $a_i \in \mathbb{Z}_2$. If $a_1 = 0$, then $v \in U(D)$. If $a_1 = 1$, then we can factor $1 + X$ out to get $v \in (1 + X)U(D)$. Hence, $U(\overline{D})/U(D) = \{1U(D), (1 + X)U(D)\}$. Elements of $D^{\#}$ have the form ωX^n with $\omega \in U(\overline{D})$ and $n \geq 2$. Put $u = 1 + X$, from our observation about $U(\overline{D})/U(D)$ we see elements of $D^{\#}$ have the form λX^n or $\lambda u X^n$ where $\lambda \in U(D)$, $n \geq 2$.

The following theorem gives us some of the $\tau_{[\ ]}$-factorization characteristics of D.

Theorem 4.1. *Let $D = \mathbb{Z}_2[[X^2, X^3]]$ and $u = 1 + X$. Then the following properties regarding $\tau_{[\ ]}$-factorization hold.*

(1) The atoms of D are λX^n and $\lambda u X^n$ where $n = 2, 3$ and $\lambda \in U(D)$. So each atom is associate to one of X^2, uX^2, X^3, uX^3,

(2) Let $\alpha, \beta \in \{1, u\}$. Then $[\alpha X^n, \beta X^m] = 1$ is equivalent to $\alpha \neq \beta$ and $2 \leq n, m \leq 3$; or $\{m, n\} = \{3, 4\}$ or $\{2, 3\}$,

(3) $\tau_{[\,]}$ *is divisive, but not multiplicative,*

(4) D *is* $\tau_{[\,]}$*-atomic with the following nonassociate* $\tau_{[\,]}$*-atoms:*

$$X^2, uX^2, X^3, uX^3, X^4, X^6, X^9, uX^9, X^n, uX^n$$

for $n \geq 11$*,*

(5) D *is not a* $\tau_{[\,]}$*-UFD,*

(6) We can not define $a \in D$ *to be a* $\tau_{[\,]}$*-atom if* $a \neq bc$ *for any* $b,\ c \in D^{\#}$ *where* $[b, c] = 1$*. In other words, since* $\tau_{[\,]}$ *is not multiplicative, it is not necessarily true that the* $\tau_{[\,]}$*-atom definition can be given with just a* $\tau_{[\,]}$*-factorization of length* 2*. And, in fact, it is not true in this case,*

Proof. (1) Given nonunit $y = X^n + a_{n+1}X^{n+1} + \cdots$ in D with $n \geq 4$, it is clear that X^2 is a proper factor of y. If $n = 2$ and $a_{n+1} = 0$, then y is associate to X^2. If $n = 2$ and $a_{n+1} = 1$, then y is associate to uX^2. A similar argument holds for when $n = 3$.

(2) Suppose that $[\alpha X^n, \beta X^m] = 1$. Without loss of generality, let $m \leq n$. If $m + 1 < n$, then we get $X^m \mid X^n$, a contradiction. Also, if $n, m > 3$, then they are both divisible by X^2, a contradiction. So $m = n$ or $n = m + 1$, and either n or m is less than or equal to 3. Clearly, if $\alpha \neq \beta$, then we can have $n = m$, and if $\alpha = \beta$, then we must have $\{m, n\} = \{3, 4\}$ or $\{2, 3\}$.

From what we just discussed the converse is clear.

(3) We already know that $\tau_{[\,]}$ is divisive. Since $[X^2, X^3] = [X^2, X^3] = 1$, but $[X^2, X^6] \neq 1$, then $\tau_{[\,]}$ is not multiplicative.

(4) Let $v = (1+X)^{-2}$. So $v \equiv 1 \mod U(D)$. The following $\tau_{[\ ]}$-factorizations prove (4) for elements of order less than or equal to 10:

$$\begin{aligned} uX^4 &= X^2 \cdot uX^2 \\ X^5 = X^2 \cdot X^3 &= uX^2 \cdot uvX^3 \\ uX^5 = X^2 \cdot uX^3 &= uX^2 \cdot X^3 \\ uX^6 &= X^3 \cdot uX^3 \\ X^7 = vX^2 \cdot uX^2 \cdot uX^3 &= X^3 \cdot X^4 = uX^3 \cdot uvX^4 \\ uX^7 = X^2 \cdot uX^2 \cdot X^3 &= uX^3 \cdot X^4 = X^3 \cdot uX^4 \\ X^8 &= uX^2 \cdot vX^3 \cdot uX^3 \\ uX^8 &= X^2 \cdot X^3 \cdot uX^3 \\ X^{10} &= vX^3 \cdot uX^3 \cdot uX^4 \\ uX^{10} &= X^3 \cdot uX^3 \cdot X^4 \end{aligned}$$

That these are proper $\tau_{[\ ]}$-factorizations follows from (2). For $X^n,\ uX^n$, where $n \geq 11$, we must consider a few things. Any element of order 4 or greater is divisible by X^2. Also, given elements $f(X)$ and $g(X)$ of D with order of $f(X)$ greater than order of $g(X)$ by 2 or more, $g(X)$ divides $f(X)$. This will be proven in more generality in Lemma 4.3. Hence the above list is an exhaustive list of proper $\tau_{[\ ]}$-factorizations.

(5) We have $uX^5 = X^2 \cdot uX^3 = uX^2 \cdot X^3$ from the proof of (4). From (2) these are both $\tau_{[\ ]}$-atomic factorizations.

(6) X^8, uX^8 are not $\tau_{[\ ]}$-atoms, but do not have a $\tau_{[\ ]}$-factorization of length 2.

□

4.2 $\tau_{[\ ]}$-atomic structure of $k[[X^2, X^3]]$

4.2.1 Atoms

Let $D = k[[X^2, X^3]]$ for a field k. Any element of the form $a_0 + a_2X^2 + \cdots$ with $a_0 \neq 0$ is a unit [14, Proposition III.5.9].

So let us look at elements of the forms

$$f(X) = a_2X^2 + a_3X^3 + \cdots \tag{4.1}$$

Now any element with a proper factorization must have order at least four, and since $a_4X^4 + a_5X^5 + \cdots = X^2(a_4X^2 + a_5X^3 + \cdots)$ is a proper factorization, the atoms of D are precisely the elements in Equation (4.1) with $a_2 \neq 0$ or $a_3 \neq 0$.

4.2.2 Associates

Let us determine the associate classes of D. The following lemma allows us to find a nice finite sum to represent each associate class.

Lemma 4.2. *For $n \geq 2$*

$$X^n + a_{n+1}X^{n+1} + \cdots = (X^n + a_{n+1}X^{n+1})(b_0 + b_2X^2 + b_3X^3 + \cdots) \tag{4.2}$$

for some $(b_0 + b_2X^2 + b_3X^3 + \cdots)$ with $b_0 \neq 0$. Further, we get $X^n + aX^{n+1} \sim X^n + bX^{n+1}$ if and only if $b = a$.

Proof. Equation (4.2) is true if and only if the following system of equations hold:

$$\begin{aligned} b_0 &= 1 \\ b_0 a_{n+1} &= a_{n+1} \\ b_2 &= a_{n+2} \\ b_2 a_{n+1} + b_3 &= a_{n+3} \\ &\vdots \\ b_k a_{n+1} + b_{k+1} &= a_{n+k+1} \\ &\vdots \end{aligned}$$

A simple induction shows

$$b_{k+1} = a_{n+k+1} - a_{n+1}a_{n+k} + a_{n+1}^2 a_{n+k+1} - \cdots (-a_{n+1})^{k-1} a_{n+2}$$

Suppose $X^n + aX^{n+1} = (X^n + bX^{n+1})(c_0 + c_2X^2 + c_3X^3 + \cdots)$. Then the following system of equations must hold:

$$\begin{aligned} c_0 &= 1 \\ c_0 b &= a \\ c_2 &= 0 \\ c_2 b + c_3 &= 0 \\ &\vdots \\ c_k b + c_{k+1} &= 0 \\ &\vdots \end{aligned}$$

Hence, $b = a$. The converse is obvious. □

We can conclude the associate classes of D are precisely $\{X^n + aX^{n+1} \mid n > 1, a \in k\}$. So the atoms of D are, up to associates,

$$\{X^2 + aX^3,\ X^3 + bX^4 \mid a, b \in k\}. \tag{4.3}$$

4.2.3 $\tau_{[\ ]}$-atoms

We know atoms are $\tau_{[\ ]}$-atoms. So we only need to look at elements of D with order greater than or equal to 4. The following lemma sheds some light onto which elements are $\tau_{[\ ]}$-related.

Lemma 4.3. *For $a,\ b \in k$ and $n,\ k \geq 2$ there always exists c_i's in k that satisfy the following equation:*

$$X^{n+k} + aX^{n+k+1} = (X^n + bX^{n+1})(c_0 + c_2X^2 + \cdots) \tag{4.4}$$

Proof. We look at two case: first when $k = 2$ and second when $k > 2$. As in Lemma 4.2, Equation (4.4) holds if and only if the following system of equations has a solution:

Case when $k = 2$	Case when $k > 2$
$c_0 = 0$	$c_0 = 0$
$c_0 b = 0$	$c_0 b = 0$
$c_2 = 1$	$c_2 = 0$
$c_2 b + c_3 = a$	$\vdots$
$c_3 b + c_4 = 0$	$c_{k-1} b + c_k = 1$
$c_4 b + c_5 = 0$	$c_k b + c_{k+1} = a$
$\vdots$	$\vdots$
$c_m b + c_{m+1} = 0$	$c_{k+m} b + c_{k+m+1} = 0$
$\vdots$	$\vdots$

Both cases follow by induction. □

Lemma 4.3 shows the factors in any $\tau_{[\ ]}$-factorization must have orders within 1 of each other. Note, X^4 is the only element, up to units, of order 4 that can possibly be a $\tau_{[\ ]}$-atom; this only occurs when the characteristic of k is 2. Consider any $X^4 + aX^5$ where $a \neq 0$. Then $X^2(X^2 + aX^2)$ is a proper $\tau_{[\ ]}$-factorization by Lemma 4.2. For X^4 we look at $(X^2+aX^3)(X^2+bX^3) = X^4+(a+b)X^5+abX^6$. From the conclusion of Lemma 4.2 this is a proper $\tau_{[\ ]}$-factorization of X^4, up to units, if and only if $a = -b$ and $a \neq b$. So such a proper $\tau_{[\ ]}$-factorization occurs if and only if k does not have characteristic 2.

This all leads us to the following corollary.

Corollary 4.4. *Suppose k has characteristic 2. Up to units, proper $\tau_{[\ ]}$-atomic factorizations in D have one of the following two forms:*

$$f(X) \ := \ (X^2 + a_1X^3) \cdots (X^2 + a_sX^3)(X^3 + b_1X^4) \cdots (X^3 + b_tX^4) \tag{4.5}$$

or

$$f(X) \ := \ (X^3 + b_1X^4) \cdots (X^3 + b_sX^4) \cdot X^4 \tag{4.6}$$

where $b_i \neq b_j$ and $a_i \neq a_j$ for $i \neq j$.

If k has characteristic other than 2, then proper $\tau_{[\ ]}$-atomic factorizations have only the form in Equation (4.5).

Proof. We restate the facts that make this corollary true. By Lemma 4.3 elements in a $\tau_{[\ ]}$-factorization can differ in order by at most 1. By Lemma 4.2 $X^n + aX^{n+1} \sim X^n + bX^{n+1}$ if and only if $a = b$, and elements of this form make up the associate classes. Finally, X^4 is a $\tau_{[\,]}$-atom of order 4 if and only if k has characterstic 2. □

It is important to note the word "proper" in Corollary 4.4. Factors in Equations (4.5) and (4.6) are not necessarily the only $\tau_{[\,]}$-atoms. However, from our previous considerations any possible τ-atoms of higher degree are not $\tau_{[\,]}$-related to any nonzero, nonunit elements. This leads us to ponder when we can completely characterize the $\tau_{[\,]}$-atoms. Theorem 4.5 gives us an answer for when k is an infinite field with characteristic not equal to 2.

Let k have characteristic other than 2. Assume we have a $\tau_{[\,]}$-atomic factorization of an element, $X^n + aX^{n+1}$, of order $n \geq 4$; and allow some carelessness with units. Then we have:

$$\begin{aligned} X^n + aX^{n+1} &= (X^{m_1} + a_1X^{m_1+1}) \cdots (X^{m_k} + a_kX^{m_k+1})u \\ &= (X^{m_1+\cdots+m_k} + (a_1 + \cdots + a_k)X^{m_1+\cdots+m_k+1})u_1 \\ &= (X^{m_1+\cdots+m_k} + (a_1 + \cdots + a_k)X^{m_1+\cdots+m_k+1})(c_0 + c_2X^2 + \cdots) \end{aligned} \tag{4.7}$$

where $u_1 = (c_0 + c_2X^2 + \cdots)$ is a unit. For this to be a proper $\tau_{[\,]}$-atomic factorization we need $m_1 + \cdots + m_k = n$ with $m_i \in \{2, 3\}$ for each i, and $a_1 + \cdots + a_k = a$ with $a_i \neq a_j$ for any $m_i = m_j$.

Theorem 4.5. *Given $D = k[[X^2, X^3]]$ with k an infinite field not of characteristic 2, then the $\tau_{[\,]}$-atoms of D coincide with the atoms of D.*

Proof. From Equation (4.7) it suffices to show given $a \in k$ and $n \geq 4$, then there exists $\{m_i\}$, $\{a_i\}$, and $t > 1$ such that $m_1 + \cdots + m_t = n$ with $m_i \in \{2, 3\}$, and

$a_1 + \cdots + a_t = a$ with $a_i \neq a_j$ for any $m_i = m_j$.

We already know that a collection of $\{m_i\}$ such that $m_1 + \cdots + m_t = n$ with $m_i \in \{2, 3\}$ and $t > 1$ exists. So it suffices to show that a corresponding collection $\{a_i\}$ exists with $a_1 + \cdots + a_t = a$ and $a_i \neq a_j$ for $i \neq j$. We can break this up into two case:

(1) If t is odd we let $a_1 = a$, and choose $\frac{t-1}{2}$ distinct elements $\{b_i\}$ of k that are not equal to a or $-a$ and such that $b_i \neq -b_i$. Then we have $a + b_1 + \cdots + b_{\frac{t-1}{2}} - b_1 - \cdots - b_{\frac{t-1}{2}} = a$ as desired.

(2) If t is even and $a \neq 0$, then we let $\{a_1, a_2\} = \{0, a\}$ and then it follows similarly to case (1). If $a = 0$, then we simply have $b_1 + \cdots + b_{\frac{t}{2}} - b_1 - \cdots - b_{\frac{t}{2}} = 0$ for a collection $\{b_i\}$ similar to those in case (1).

In both cases, we have constructed the necessary sums. Therefore, given an element with order greater than or equal to 4, we can construct a proper $\tau_{[\ ]}$-atomic factorization of the element. □

With the algorithm in place from the proof of Theorem 4.5 we can characterize the $\tau_{[\ ]}$-atoms of D when k is a finite field not of characteristic 2. By Corollary 4.4 we can find a positive integer n such that elements of order greater than n are $\tau_{[\ ]}$-atoms. Thus, that leaves us finitely many cases left to mull over.

Corollary 4.6. *Let k be the finite field of order p^m for some prime greater than 2 (so $k = \mathbb{F}_{p^m}$). Let $f(X) \in D$ with order n. Then, up to units, we can say the following with regards to $\tau_{[\ ]}$-atomic factorizations of $f(X)$:*

(1) If $n = 2,\ 3,$ *or* $n > 5p^m$*, then* $f(X)$ *is a* $\tau_{[\ ]}$*-atom,*

(2) If $3 < n \leq 5p^m - 2$*, then* $f(X)$ *is not a* $\tau_{[\ ]}$*-atom,*

(3) $f(X) = X^{5p^m}$ *is the only element of order* $5p^m$ *with a proper* $\tau_{[\ ]}$*-factorization,*

(4) If $n = 5p^m - 1$*, then* $f(X)$ *is a* $\tau_{[\ ]}$*-atom.*

Proof. All proper $\tau_{[\ ]}$-atomic factorization have the form of Equation (4.5). We will simply be evaluating the various possibilities for such a factorization. We will continue with the notation in Equation (4.5). In other words, a_i represents a coefficient of a factor with order 2 and b_j represents a coefficient of a factor with order 3. Also, all results are up to associates. For the rest of the proof we will not continue to mention this fact.

(1) This is clear. If $n > 5p^m$, then there is not enough elements of k to form a proper $\tau_{[\ ]}$-factorization. There would have to exist an $i \neq j$ such that $a_i = a_j$ or $b_i = b_j$.

(2) We will first look at the case when $n < 5p^m - 6$. In this case, we can construct a factorization of order n of the form of Equation (4.5) where the number of factors of order 2 and of order 3 are each less than p^m. We will then construct $\tau_{[\ ]}$-atomic factorizations for $f(X)$ in the cases when $n = 5p^m - 2,\ 5p^m - 3,\ 5p^m - 4,\ 5p^m - 5,$ and $5p^m - 6$. The cases when $n < 5p^m - 6$ will follow from these higher order cases.

Suppose $n < 5p^m - 6$. Let y_2 be the number of factors of order 2 and y_3 be the number of factors of order 3 in a factorization of the form of Equation (4.5). It is important to note that for any order n we can choose y_2 and y_3 so they differ by no more than 2 (i.e., $\mid y_2 - y_3 \mid \leq 2$). Of course, at this point we are not concerned with what the product equals. We are just interested in evaluating the order of the element. Obviously, rearranging the values for y_2 and y_3 could change the value of the product. This does not matter to us at this point since we have not determined values for the a_i's and b_j's.

Suppose $y_3 \leq y_2$. Without loss of generality, we can assume $y_2 \leq y_3 + 2$. We then have the following:

$$5y_2 - 6 = 2y_2 + 3(y_2 - 2) \leq 2y_2 + 3y_2 = n \implies y_2 \leq \frac{n+6}{5} < p^m$$

If we reverse the roles of y_2 and y_3 a similar argument yields $y_3 < p^m - \frac{2}{5} \iff y_3 < p^m$. We have shown we can create a factorization of order n with the desired values for y_2 and y_3. More specifically, we have shown for $n < 5p^m - 6$ we can create a factorization with $y_2 \leq p^m - 1$ and $y_3 \leq p^m - 1$ whose product has order n. We now proceed with the actual construction of $\tau_{[\ ]}$-atomic factorizations for the aforementioned higher order cases.

Suppose $f(X) = X^n + aX^{n+1}$ and $n = 5p^m - 2$. We may choose $y_2 = p^m - 1$ and $y_3 = p^m$ for a factorization of the form of Equation (4.5) of order n. We now need to properly select the a_i and b_j coefficients to make this a $\tau_{[\ ]}$-atomic factorization of $f(X)$. We need $a_1 + \cdots + a_s + b_1 + \cdots + b_t = a$ where $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Choose $\{a_1, \ldots, a_s\}$ to be the set of all elements in k other than $-a$, and choose $\{b_1, \ldots, b_t\}$ to be the set of all elements of k. Since $\sum_{c \in k} c = 0$ then we have

$$\sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j = \sum_{i=1}^{s} a_i = \sum_{\substack{c \in k \\ c \neq -a}} = a$$

Thus, we have constructed a $\tau_{[\ ]}$-atomic factorization of $f(X)$. The same proof holds for $n = 5p^m - 3$ with $y_3 = p^m - 1$ and $y_2 = p^m$.

Now suppose $n = 5p^m - 5$. We may choose $y_2 = p^m - 1$ and $y_3 = p^m - 1$ for a factorization of the form of Equation (4.5) of order n. If $a = 0$ we simply choose $\{a_1, \ldots, a_s\} = \{b_1, \ldots, b_t\}$ to be the set of nonzero elements of k (note that $s = t$). If $a \neq 0$ we let $a_1 = b_1 = \frac{a}{2}$. Then we choose $\{a_2, \ldots, a_s\}$ to be a collection of $p^m - 2$ distinct elements of k not equal to $\frac{a}{2}$ or $-\frac{a}{2}$, and $\{b_2, \ldots, b_t\} = \{-a_2, \ldots, -a_s\}$. This yields a $\tau_{[\ ]}$-atomic factorization of $f(X)$ as desired.

If $n = 5p^m - 4$, then we choose $y_2 = p^m - 2$ and $y_3 = p^m$. Clearly, $\{b_1, \ldots, b_t\}$ must be the collection of all elements of k if we want Equation 4.5 to be a $\tau_{[\ ]}$-atomic factorization. For the set $\{a_1, \ldots, a_s\}$ we can choose any distinct $p^m - 2$ collection from k and still have a $\tau_{[\ ]}$-atomic factorization. Maybe a better way to think about it is we can exclude two distinct elements of k and still have a $\tau_{[\ ]}$-atomic factorization. Let b be an element of k with $b \neq a - b$ (we know such an element exists; otherwise $2b = a$ for each $b \in k$). Then $-b$ and $b - a$ are distinct elements of k. So choose $\{a_1, \ldots, a_s\}$ to be the elements of k distinct from $-b$ and $b - a$. We get

$$\sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j = \sum_{i=1}^{s} a_i = \sum_{\substack{c \in k \\ c \neq -b, (b-a)}} = a$$

Hence, we can construct a $\tau_{[\ ]}$-atomic factorization of $f(X)$. The proof for $n = 5p^m - 6$ is the same by letting $y_2 = p^m$ and $y_3 = p^m - 2$.

When $3 < n < 5p^m - 6$ we can choose y_2 and y_3 so that they either differ by 1, are equal, or differ by 2. Then we can construct a $\tau_{[\ ]}$-atomic factorization as in the three cases above. Since the order is lower we will clearly have the existence of the necessary a_i's and b_j's to construct the $\tau_{[\,]}$-atomic factorizations.

(3) Any proper $\tau_{[\,]}$-atomic factorization of $f(X)$ with $n = 5p^m$ must have $y_2 = y_3 = p^m$. For an equation of the form of Equation (4.5) of order n to be a $\tau_{[\,]}$-atomic factorization we must have $\{a_1, \ldots, a_s\} = \{b_1, \ldots, b_t\}$ be the set of all elements of k. In this case, the sum of the a_i's and b_j's is 0. Hence, $f(X) = X^{5p^m}$ is the only element of order $n = 5p^m$ that is not a $\tau_{[\,]}$-atom.

(4) To have a factorization of the form of Equation (4.5) for an element of order $p^m - 1$ either y_2 or y_3 must be greater than p^m. So such a factorization will not be a $\tau_{[\,]}$-atomic factorization. □

4.3 Bezout Domains

Recall that a ring is *indecomposable* if it can not be written as the direct sum of proper ideals.

Lemma 4.7. *Given a Bezout domain D, then for a nonzero, nonunit element $a \in D$ $D/(a)$ is indecomposable if and only if a is a $\tau_{[\,]}$-atom.*

Proof. Suppose a is a $\tau_{[\,]}$-atom and $D/(a) = B/(a) \oplus C/(a)$. There exists some $b' \in B$ and $c' \in C$ such that $b' + c' + (a) = 1 + (a)$. Since D is a Bezout domain, $(b') + (a) = (b)$ and $(c') + (a) = (c)$ for some $b \in B$ and $c \in C$. So we have $D/(a) = (b)/(a) \oplus (c)/(a)$ with $bc \in (a)$ and $[b, c] = 1$. By Proposition 3.12 a is $\tau_{[\,]}$-prime. So $a \mid b$ or $a \mid c$, a contradiction.

Conversely, if $a = bc$ with b and c nonzero, nonunits and $[b, c] = 1$, then

$D/(a) = D/(b) \oplus D/(c)$ [14, Proposition 2.1]. □

Proposition 4.8. *Let D be a Bezout domain. Then the following are equivalent:*

(1) D is a CFD,

(2) D is a UCFD,

(3) D is a $\tau_{[\ ]}$-UFD,

(4) For each nonzero, nonunit $a \in D$, $D/(a)$ is a finite direct product of indecomposable ideals.

Proof. The equivalence of (1) and (2) follows from [17, Theorem 1.7]. (2) is equivalent to (3) since in a Bezout domain $[a, b] = 1$ if and only if $(a, b) = D$.

Assume (1) and let a be a nonzero, nonunit with $a = p_1 \cdots p_n$ a $\tau_{[\ ]}$-atomic factorization of a. Then we have $D/(a) = D/(p_1) \oplus \cdots \oplus D/(p_n)$ and from Lemma 4.7 each sum is indecomposable.

Assuming (4) we have $D/(a) = R_1 \oplus \cdots \oplus R_n$ with each R_i indecomposable. Under this isomorphism, denote it by $\phi = (\phi_i)_{i=1}^n$, we look at the image of a, say $\phi(a) = (p_1, \ldots, p_n)$. So we have each R_i is of the form $D/(p_i)$, and $(a) = (p_1) \cap \cdots \cap (p_n)$. Suppose $[p_i, p_j] = x$ for some $i \neq j$. Under the isomorphism let y be an element of $D/(a)$ that is mapped to $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is in the j^{th} position. Then $\phi_j(y) - 1 = y - 1 \in (p_j) \subseteq (x)$ and $\phi_i(y) = y \in (p_i) \subseteq (x)$. Hence, x is a unit, and so $[p_i, p_j] = 1$ for $i \neq j$. From Lemma 4.7 $(a) = (p_1 \cdots p_n)$ gives a $\tau_{[\ ]}$-atomic factorization of g. From Corollary 3.13 D is a $\tau_{[\ ]}$-UFD. □

4.4 CK Domains

A *Cohen-Kaplansky domain (CK domain)* is an atomic domain with only finitely many irreducible elements. A CK domain is one-dimensional, semilocal, and each irreducible element is contained in a unique maximal ideal. For our study it will also be important to note that a local CK domain cannot contain exactly 2 nonassociate atoms. For further study of CK domains see [9] and [4].

We begin our study of CK domains with a corollary that follows readily from Theorem 3.8.

Corollary 4.9. *Let D be a CK domain. Then D is a $\tau_{[\ ]}$-UFD if and only if D_P is a $\tau_{[\ ]}$-UFD for each prime ideal P in D.*

Proof. From [7, pg. 7] a CK domain is a one-dimensional, weakly factorial domain. The result then follows from Theorem 3.8. □

Given a ring R, a set S is *universal* if every atom of R divides every element of S. The following lemma follows from considerations in [9].

Lemma 4.10. *Let (D, M) be a quasilocal domain with M^2 universal. Then for nonassociate atoms a_i, $i = \{1, 2, 3\}$, $a_1a_2 = pa_3$ for some atom p, nonassociate to a_1 and a_2.*

Proof. We have $a_1a_2 \in M^2$. So $a_1a_2 \in (a_3)$, say $a_1a_2 = pa_3$. Clearly, p is not a unit, and $(p) \nsubseteq (a_i)$ for $i = \{1, 2\}$. Also, p is not in M^2. Otherwise, a_1 divides p, a contradiction. Hence, p is an atom nonassociate to a_1 and a_2 as desired. □

By [9, Theorem 8] a local domain (D, M) with three atoms has M^2 universal.

Lemma 4.11. *Let (D, M) be a local CK domain with exactly one or three nonassociate atoms. Then D is a $\tau_{[\ ]}$-UFD.*

Proof. First off, if there is only one, nonassociate atom, then D is a DVR. So D is a $\tau_{[\ ]}$-UFD. Suppose a_1, a_2, and a_3 are the nonassociate atoms of D. From Lemma 4.10 we have $a_i a_j = u{a_k}^2$ for $i \neq j$ and $k \neq i, j$. Then $[x, y] = 1$ for some $x, y \in D^{\#}$ is equivalent to $x = ua_i$ and $y = va_j$ for $i \neq j$ and units u, v of D. So any $\tau_{[\ ]}$-atomic factorization is of the form $x_1 \cdots x_n$ where $n \leq 3$ and each x_i is associate to either a_1, a_2, or a_3. Then the only proper $\tau_{[\ ]}$-atomic factorizations, up to units, are $a_1 \cdot a_2$, $a_1 \cdot a_3$, $a_2 \cdot a_3$, and $a_1 \cdot a_2 \cdot a_3$. From here it is easy to see that $\tau_{[\ ]}$-atomic factorizations are unique. □

Notice that $[x, y] = 1$ is equivalent to $x = ua_i$ and $y = va_j$ for $i \neq j$ and units u, v of D is immediate from the universality of M^2.

Proposition 4.12. *Let (D, M) be a quasilocal domain with M^2 universal. Then D is a CK domain with exactly one or three nonassociate atoms if and only if D is a $\tau_{[\ ]}$-UFD.*

Proof. The forward direction is just Lemma 4.11.

Suppose D is a $\tau_{[\ ]}$-UFD. Let $\{p_i \mid i \in I\}$ be the set of nonassociate atoms of D. By Lemma 4.10 $p_i p_j = r_k p_k$ for distinct i, j, and k and some atom r_k nonassociate to p_i and p_j. If $r_k \not\sim p_k$, then $p_i p_j = r_k p_k$ are two distinct $\tau_{[\ ]}$-atomic factorizations, a contradiction.

So we must have $p_i p_j = u{p_k}^2$ for any distinct i, j, and k. If D contains more than three nonassociate atoms, then $p_1 p_2 = u{p_4}^2 = vp_2 p_3$ where u and v are units in D. So we arrive at the contradiction $p_1 \sim p_3$. Hence, D has exactly one or three nonassociate atoms. □

From [4, Theorem 5.1] we have that in a quasilocal atomic domain D, the following are equivalent.

(1) M^2 is universal.

(2) $M \neq M^2$ and for $a \in M - M^2$, $M^2 \subseteq (a)$.

(3) For atoms $a_1, \ldots, a_n \in D$, $a_1 \cdots a_n M = M^{n+1}$.

(4) M is strongly prime, that is, for $xy \in M$ $(x,\ y \in K)$ either $x \in M$ or $y \in M$.

Hence, we can replace the hypothesis that M^2 is universal in Proposition 4.12 by any one of these statements.

Corollary 4.13. *Given a local CK domain (D, M) with n nonassociate atoms, where n is a prime number greater than 3, then D is not a $\tau_{[\ ]}$-UFD.*

Proof. By [9, Theorem 11] we have M^2 is universal. By Proposition 4.12 D is not a $\tau_{[\ ]}$-UFD. □

We next look at a local domain whose integral closure is a DVR. We develop equivalences for $\tau_{[\ ]}$-UFD's. Recall that the *group of divisibility* is defined as $G(D) := K^*/D^*$ where K is the quotient field of D. As previously stated, $\overline{D}$ is the integral closure of D.

If $G(D)$ is finitely generated, then the exact sequence

$$0 \longrightarrow U(\overline{D})/U(D) \longrightarrow G(D) \longrightarrow G(\overline{D}) \longrightarrow 0$$

splits. So we have $G(D) \cong G(\overline{D}) \oplus U(\overline{D})/U(D)$. It was shown by B. Glastad and J. Mott that if $G(D)$ is finitely generated, then $U(\overline{D})/U(D)$ is finite, and $\overline{D}$ is a finitely generated D-module [4, Theorem 3.1].

Definition 4.14. A *square-free UFD* (SQFUFD) is an atomic domain such that given two atomic factorizations $a_1 \cdots a_n = b_1 \cdots b_m$ with $a_i \nsim a_j$ and $b_i \nsim b_j$ for all $i \neq j$, then $n = m$ and after reordering $a_i \sim b_i$.

Lemma 4.15. *Given quasilocal domains $(R, N) \subseteq (D, M)$ with $U(R) = U(D)$, then $R = D$.*

Proof. Given $m \in M$, $m - 1 \in U(D) = U(R)$. Hence, $m = (m-1) + 1 \in R$. ∎

Lemma 4.16. *Let (D, M) be a local domain with integral closure $(\overline{D}, (\pi))$ a DVR. If $G(D) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ for some prime p, then there are no rings properly between D and $\overline{D}$.*

Proof. Suppose that $D \subsetneq R \subsetneq \overline{D}$. By [16, Theorem 44] $R \subseteq \overline{D}$ satisfies LO (lying over) and so (R, N) is a quasilocal domain. We have

$$0 = U(D)/U(D) \subseteq U(R)/U(D) \subseteq U(\overline{D})/U(D) \cong \mathbb{Z}_p$$

So $U(R)/U(D) = 0$ or $U(R)/U(D) = U(\overline{D})/U(D)$ which implies $U(R) = U(D)$ or $U(R) = U(\overline{D})$, respectively. Now since D, R, and $\overline{D}$ are all quasilocal then by Lemma 4.15 $R = D$ or $R = \overline{D}$, respectively. ∎

Lemma 4.17. *Let D be an integral domain with integral closure $\overline{D}$. Then $[D : \overline{D}]$ is the largest set that is an ideal of both D and $\overline{D}$.*

Proof. Since $1 \in \overline{D}$, clearly $[D : \overline{D}] \subseteq D \subseteq \overline{D}$. We first show it is an ideal of D. Let $r \in [D : \overline{D}]$. For any $d \in D$ we have $rd\overline{D} = dr\overline{D} \subseteq dD \subseteq D$. Hence, $rd \in [D : \overline{D}]$. The other properties of an ideal follow similarly. Also, the proof that $[D : \overline{D}]$ is an ideal of $\overline{D}$ is also similar.

Now suppose I is an ideal in both D and $\overline{D}$. Then $I\overline{D} \subseteq I \subseteq D$. Hence, $I \subseteq [D : \overline{D}]$ as desired. □

Lemma 4.18. *Let (D, M) be a local domain with integral closure $(\overline{D}, (\pi))$ a DVR. If $G(D) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ for some prime p, then $M = [D : \overline{D}] = \pi^n \overline{D}$ and $n \leq 2$.*

Proof. We look at the ring $D + \overline{D}M$. By Lemma 4.16 $D + \overline{D}M = \overline{D}$ or $D + \overline{D}M = D$. Since $\overline{D}$ is a finitely generated D-module, in the first case we get the contradiction that $D = \overline{D}$ by Nakayama's Lemma.

So $D + \overline{D}M = D$ which implies $\overline{D}M \subseteq D$. Hence $M \subseteq [D : \overline{D}]$. Since $[D : \overline{D}]$ is an ideal of D, we have $M = [D : \overline{D}]$. Since M is an ideal of $\overline{D}$, $M = [D : \overline{D}] = \pi^n \overline{D}$ for some n.

To show $n \leq 2$ we look at the ring $D + \overline{D}\pi^k$ for $k \in \mathbb{Z}^+$. By Lemma 4.16 we have $D = D + \overline{D}\pi^k$ or $\overline{D} = D + \overline{D}\pi^k$. If $k < n$, then $\overline{D} = D + \overline{D}\pi^k$. Otherwise, $D = D + \overline{D}\pi^k$ which implies $\overline{D}\pi^k \subseteq D$. But then $\pi^k \in [D : \overline{D}] = \pi^n \overline{D}$, a contradiction. Hence, if $n > 2$, then $D + \overline{D}\pi^2 = D + \overline{D}\pi = \overline{D}$. Let $\pi = d + b\pi^2$ for some $d \in D$ and $b \in \overline{D}$. If $d \in U(D) \subseteq U(\overline{D})$, then $\pi \in U(\overline{D})$, a contradiction. If $d \in M$, then

$d = c\pi^n$ for some $c \in \overline{D}$. But then $\pi = d + b\pi^2 = c\pi^n + b\pi^2 = \pi^2(c\pi^{n-2} + b)$. Again we come to the contradiction that $\pi \in U(\overline{D})$. So $n \leq 2$. □

Theorem 4.19. *Let (D, M) be a local domain with integral closure $(\overline{D}, (\pi))$ a DVR. Suppose that $M\overline{D} = \pi^k\overline{D}$ and $[D : \overline{D}] = \pi^n\overline{D}$. Then the following are equivalent:*

(1) D has exactly 1 or 3 atoms.

(2) D is a $\tau_{[\ ]}$-UFD with $G(D) \cong \mathbb{Z} \oplus \mathbb{Z}_p$.

(3) D is a SQFUFD with $G(D) \cong \mathbb{Z} \oplus \mathbb{Z}_p$.

Proof. (1) $\Longrightarrow$ (2) By Lemma 4.11 D is a $\tau_{[\ ]}$-UFD. That $G(D) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ follows from [9].

(2) $\Longrightarrow$ (3) is straightforward.

(3) $\Longrightarrow$ (1) First, from Lemma 4.18 we have $M = [D : \overline{D}] = \pi^n\overline{D}$. By [4, Corollary 5.6] the number of nonassociate atoms of D is $n\cdot \mid U(\overline{D})/U(D) \mid$. Since $G(D) \cong \mathbb{Z} \oplus U(\overline{D})/U(D)$, $U(\overline{D})/U(D) \cong Z_p$. Hence, the number of nonassociate atoms is $n \cdot p$.

If $n \geq 2$, then π^n, $\pi^n + \pi^{n+1}$, π^{n+1}, and $\pi^{n+1} + \pi^{n+2}$ are nonassociate atoms. We have $\pi^n \cdot (\pi^{n+1} + \pi^{n+2}) = \pi^{n+1} \cdot (\pi^n + \pi^{n+1})$ are two atomic factorizations. This contradicts the hypothesis that D is a SQFUFD. So $n = 1$.

If $\mid U(\overline{D})/U(D) \mid = p > 3$, then there exists distinct elements $\overline{u}$, $\overline{v}$, and $1 \in U(\overline{D})/U(D)$ with $\overline{uv} \neq \overline{u}$, $\overline{v}$, or 1. We get $u\pi^n \cdot v\pi^n = uv\pi^n \cdot \pi^n$ are two distinct atomic square-free factorizations, a contradiction. So $p \leq 3$. Hence, D has 1 or 3 atoms. □

The following corollary follows immediately from Theorem 4.19.

Corollary 4.20. *Let (D, M) be a local domain with integral closure $(\overline{D}, (\pi))$ a DVR. Suppose that $M\overline{D} = [D : \overline{D}] = \pi^n\overline{D}$. Then the following are equivalent:*

(1) D has exactly 1 or 3 atoms,

(2) D is a $\tau_{[\ ]}$-UFD,

(3) D is a SQFUFD.

4.5 Conditions for $k + X^nK[[X]]$ to be a $\tau_{[\ ]}$-UFD

We state here Brandis' Theorem as found in [11, pg. 234]. It will be used in the theorem to follow.

Theorem 4.21 (Brandis' Theorem)**.** *Let K be an infinite field and L an extension field. Moreover, let K^* and L^* denote their respective groups of units. If L^*/K^* is finitely generated, then $K = L$.*

The following theorem looks at when the domain $D = k + X^nK[[X]]$, where $k \subseteq K$ are fields, is a $\tau_{[\ ]}$-UFD. D is always a BFD and hence is $\tau_{[\ ]}$-atomic [1].

Theorem 4.22. *Let $k \subseteq K$ be fields, $n \geq 1$ and $D = k + X^nK[[X]]$. D is a $\tau_{[\ ]}$-UFD is equivalent to the following:*

(1) $k = K$ and $n = 1$(so $D = K[[X]]$ is a DVR) or

(2) $k = \mathbb{Z}_2$, $K = GF(2^2)$ and $n = 1$(so $D = \mathbb{Z}_2 + XGF(2^2)[[X]]$).

Proof. Assume D is a $\tau_{[\ ]}$-UFD. Suppose $n > 1$. Then $X^n, X^n + X^{n+1}, X^{n+1}$, and $X^{n+1}+X^{n+2}$ are nonassociate atoms and $X^n \cdot (X^{n+1}+X^{n+2}) = (X^n+X^{n+1}) \cdot X^{n+1}$ are distinct $\tau_{[\ ]}$-atomic factorizations of $X^{2n+1} + X^{2n+2}$. So n must be 1. Suppose $|K^*/k^*| > 3$. So there exists $u, v \in K^*$ with $\overline{uv} \neq \overline{u}, \overline{v}, \overline{1}$ in K^*/k^*. Consider X, uX, vX, uvX in D. They are nonassociate atoms in D. so $uvX^2 = uX \cdot vX = X \cdot uvX$ are two distinct $\tau_{[\ ]}$-atomic factorizations in D. Suppose $|K^*/k^*| \leq 3$ and $k \neq K$. Then by Theorem 4.21 K is finite, so $|K^*| = p^{nm} - 1$ and $|k^*| = p^m - 1$. Hence $|K^*/k^*| = \frac{p^{nm}-1}{p^m-1} = \frac{(p^m-1)((p^m)^{(n-1)}+\cdots+p^m+1)}{p^m-1} = ((p^m)^{(n-1)} + \cdots + p^m + 1)$. So $p = 2$, $m = 1$, and $n = 2$.

In (1), D is a UFD and hence a $\tau_{[\ ]}$-UFD. Consider $D = \mathbb{Z}_2 + X \cdot GF(2^2)[[X]]$. Let $GF(2^2)^* = \langle \delta \rangle = \{1, \delta, \delta^2\}$. For $b = b_0 + b_1X + b_2X^2 + \cdots \in U(GF(2^2)[[X]])$, $b = b_0(1 + {b_0}^{-1}b_1X + {b_0}^{-1}b_2X^2 + \cdots) \in b_0U(D)$. So the atoms of D up to associates are X, δX, and $\delta^2 X$ where $\lambda \in U(D)$.

Now $[aX^n, bX^m] = 1 \iff n = m = 1$ and $aU(D) \neq bU(D)$ in $U(\overline{D})/U(D) \cong GF(2^2)^*/{\mathbb{Z}_2}^* \cong GF(2^2)^* \cong (\mathbb{Z}_3, +)$. So the other $\tau_{[\ ]}$-atomic factorizations up to units of D are $\delta X^2 = X \cdot \delta X$, $X \cdot \delta^2 X = \delta^2 X^2$, $\delta X \cdot \delta^2 X = X^2$, and $X \cdot \delta X \cdot \delta^2 X = X^3$. So it is clearly checked that D is a $\tau_{[\ ]}$-UFD. □

In [4, Theorem 7.1], Anderson and Mott showed that for a finite field K with subfield k then $R = k + K[[X]]X$ is a complete local CK domain. By Theorem 4.22 we see that for $|K^*| > 3$ R is a CK domain that is not a $\tau_{[\ ]}$-UFD.

We now give an example of a local $\tau_{[\ ]}$-UFD that is not a CK domain. This example is taken from [15, Example 94]. Let $R = K(U)[[X, Y, Z]]$ where K is a field.

Let $f = X^2 + Y^3 + UZ^6$. Let $T = R/(f)$. Then T is a 2-dimensional, complete, local UFD. By Theorem 2.14 T is a $\tau_{[\ ]}$-UFD. Since T is 2-dimensional, it is not a CK-domain.

CHAPTER 5
CONCLUSION

We conclude our paper with a summary of the major results, as well as ideas for future work.

5.1 Results

In Chapter 2, we initiated our study with several examples. We looked closely at the set $\mathfrak{S}$ of height-one prime ideals, and we also developed the connection between grade and v-coprimeness. In Corollary 2.4, we showed a weakly factorial domain is a $\tau_{\mathfrak{S}}$-UFD, where $\mathfrak{S}$ is the set of height-one prime ideals; and in Theorem 2.5, we expanded [17, Corollary 1.10]. In Example 2.2, we showed for a Noetherian domain D, given $a,\ b \in D^{\#}$, $[a,b]_t = 1$ is equivalent to $G(P) > 1$ for every prime ideal P containing (a,b). Hence, $[a,b]_t = 1$ is equivalent to $[a,b]_{\mathfrak{S}} = 1$ where $\mathfrak{S} = \{P \mid P$ is prime and $G(P) = 1\}$.

Also, in Chapter 2, we studied the properties of a general comprimeness relation. In Theorem 2.10, we showed if a relation τ satisfied $a \not\tau\ a$ (CP1) and $a\ \tau\ b$ with $(a,b) \subseteq (c,d) \implies c\ \tau\ d$ (CP6) for elements $a,\ b,\ c$, and $d \in D^{\#}$, then τ was equivalent to $\tau_{\mathfrak{S}}$ where $\mathfrak{S} = \{(c,d) \mid c,\ d \in D^{\#}$ and $c \not\tau d\}$. In Theorem 2.11, we showed for a set of ideals $\mathfrak{S}$ in D, $\tau_{\mathfrak{S}} \equiv \tau_{\sqrt{\mathfrak{S}}}$ is equivalent to $a\ \tau_{\mathfrak{S}}\ b \implies a\ \tau_{\mathfrak{S}}\ b^2$ for all elements $a,\ b \in D^{\#}$.

Chapter 3 focused on $\tau_{[\ ]}$-UFD's. We developed the connection between weakly factorial domains and $\tau_{[\ ]}$-UFD's, and GCD domains and $\tau_{[\ ]}$-UFD's. In Theorem 3.8, we showed for a weakly factorial domain D, D is a $\tau_{[\ ]}$-UFD if and only if D_P is a $\tau_{[\ ]}$-UFD for each height-one prime ideal P of D.

We proved several results regarding GCD domains. In Corollary 3.13, we showed if D is a GCD domain, then $\tau_{[\ ]}$-atomic implies $\tau_{[\ ]}$-UFD. In Theorem 3.18, we showed if D is a GCD domain, then D is a $\tau_{[\ ]}$-UFD if and only if $D[X]$ is a $\tau_{[\ ]}$-UFD. Also, Therem 3.25 stated that given D a GCD domain, D is a $\tau_{[\ ]}$-UFD if and only if $D[X]_{N_t}$ is a UCFD.

In Chapter 4, we looked at several examples with respect to the $\tau_{[\ ]}$ relation. In Theorem 4.5 and Corollary 4.6, we classified the $\tau_{[\ ]}$-atoms of $k[[X^2, X^3]]$ where k is either an infinite field not of characteristic 2 or any finite field. In Theorem 4.22, we showed $k + X^n K[[X]]$, where $k \subseteq K$ are fields, is a $\tau_{[\ ]}$-UFD if and only if $k = K$ and $n = 1$ or $k = \mathbb{Z}_2$, $K = GF(2^2)$ and $n = 1$.

We also studied Bezout domains and CK domains with respect to $\tau_{[\ ]}$-UFD's. In Proposition 4.8, we showed a Bezout domain D is $\tau_{[\ ]}$-UFD if and only if $D/(a)$ is a finite direct product of indecomposable ideals for each $a \in D^{\#}$. Of course, in a Bezout domain, $\tau_{max} \equiv \tau_{[\ ]}$. The relations between the atoms in a quasilocal domain (D, M) when M^2 is universal led us to an equivalence between CK domains and $\tau_{[\ ]}$-UFD's. Specifically, in Proposition 4.12, we showed a quasilocal domain (D, M) with M^2 universal is a CK domain with exactly one or three nonassociate atoms if and only if D is a $\tau_{[\ ]}$-UFD.

5.2 Future Work

Further development of the axioms of coprimeness is of interest. We are particularly interested in the following question: Under what conditions is a general τ relation equivalent to a τ_* relation for some $*$-operation? This question arose in our study of τ_v and τ_{max}. Intuitively, it seemed if τ satisfied Properties CP1 and CP6, and τ was multiplicative, then we could define a $*$-operation such that $\tau_* \equiv \tau$. However, Example 2.3 showed otherwise.

Another area of interest is the domains that lie between atomic domains and UFD's. In [1] D.D. Anderson, D.F. Anderson, and Zafrullah studied HFD's, FFD's, idf-domains, BFD's, and ACCP. In [2] Anderson and Frazier generalized these domains using the notion of τ-factorization, and produced analogous results to those in [1]. It would be of interest to study $\tau_{[\ ]}$-HFD's, $\tau_{[\ ]}$-FFD's, $\tau_{[\ ]}$-idf domains, $\tau_{[\ ]}$-BFD's, and $\tau_{[\ ]}$-ACCP.

REFERENCES

[1] D.D. Anderson, D.F. Anderson, and M. Zafrullah. Factorization in integral domains. *J. Pure Appl. Algebra*, 69(1):1–19, 1990.

[2] D.D. Anderson and A. M. Frazier. On a general theory of factorization in integral domains. *Rocky Mountain J. Math.*, to appear.

[3] D.D. Anderson and L.A. Mahaney. On primary factorizations. *J. Pure Appl. Algebra*, 54(2–3):141–154, 1988.

[4] D.D. Anderson and J.L. Mott. Cohen-Kaplansky domains: integral domains with a finite number of irreducible elements. *J. Algebra*, 148(1):17–41, 1992.

[5] D.D. Anderson and R.O. Quinterro. Some generalizations of GCD-domains. In *Factorization in integral domains (Iowa City, IA, 1996)*, volume 189 of *Lecture Notes in Pure and Appl. Math.*, pages 189–195. Dekker, New York, 1997.

[6] D.D. Anderson and M. Zafrullah. Independent locally-finite intersections of localizations. *Houston J. Math.*, 25(3):433–452, 1999.

[7] D.D. Anderson and M. Zafrullah. Almost bezout domains. III. *Bull. Math. Soc. Sci. Math. Roumanie* 51(99), No. 1:3–9, 2008.

[8] S.T. Chapman and S. Glaz. *Non-noetherian commutative ring theory*, volume 520 of *Math. Appl.*, Kluwer Academic Publishers, Dordrecht, 2000.

[9] I.S. Cohen and I. Kaplansky. Rings with a finite number of primes. I. *Trans. Amer. Math. Soc.*, 60:468–477, 1946.

[10] P.M. Cohn. Bezout rings and their subrings. *Proc. Camb. Phil. Soc.*, 64:251–264, 1968.

[11] B. Glastad and J.L. Mott. Finitely generated groups of divisibility. *Contemp. Math.*, 8:231–247, 1982.

[12] R. Gilmer. *Multiplicative ideal theory*, volume 90 of *Queen's Papers in Pure and Appl. Math.*, Queen's University, Kingston, ON, 1992. Corrected reprint of the 1972 edition.

[13] F. Halter-Koch. *Ideal systems–an introduction to ideal theory*, volume 211 of *Monographs and Textbooks in Pure and Appl. Math.*, Marcel Dekker, Inc., New York, 1998.

[14] T. Hungerford. *Algebra*. Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1974.

[15] H.C. Hutchins. *Examples of commutative rings*. Polygonal Publishing House, Passaic, NJ, 1981.

[16] I. Kaplansky. *Commutative rings*. The University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[17] S. McAdam and R.G. Swan. Unique comaximal factorization. *J. Algebra*, 276(1):180–192, 2004.

[18] M. Zafrullah. Putting t-invertibility to use. In *Non-Noetherian commutative ring theory*, volume 520 of *Math. Appl.*, pages 429–457. Kluwer Acad. Publ., Dordrecht, 2000.