Summer 2010

Analysis of symmetric function ideals: towards a combinatorial description of the cohomology ring of Hessenberg varieties

Abukuse Mbirika III
University of Iowa

ANALYSIS OF SYMMETRIC FUNCTION IDEALS: TOWARDS A COMBINATORIAL DESCRIPTION OF THE COHOMOLOGY RING OF HESSENBERG VARIETIES

by

Abukuse Mbirika III

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

July 2010

Thesis Supervisors: Assistant Professor Julianna Tymoczko
Professor Frederick Goodman

ABSTRACT

Symmetric functions arise in many areas of mathematics including combinatorics, topology and algebraic geometry. Using ideals of symmetric functions, we tie these three branches together. This thesis generalizes work of Garsia and Procesi in 1992 that gave a quotient ring presentation for the cohomology ring of Springer varieties.

Let R be the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. We present two different ideals in R. Both are parametrized by a Hessenberg function h, namely a nondecreasing function that satisfies $h(i) \geq i$ for all i. The first ideal, which we call I_h, is generated by modified elementary symmetric functions. The ideal I_h generalizes the work of Tanisaki who gave a combinatorial description of the ideal used in Garsia and Procesi's quotient ring. Like the Tanisaki ideal, the generating set for I_h is redundant. We give a minimal generating set for this ideal. The second ideal, which we call J_h, is generated by modified complete symmetric functions. The generators of this ideal form a Gröbner basis, which is a useful property. Using the Gröbner basis for J_h, we identify a basis for the quotient R/J_h.

We introduce a partial ordering on the Hessenberg functions, and in turn we discover nice nesting properties in both families of ideals. When $h > h'$, we have $I_h \subset I_{h'}$ and $J_h \subset J_{h'}$. We prove that I_h equals J_h when h is maximal. Since I_h is the ideal generated by the elementary symmetric functions when h is maximal, the generating set for J_h forms a Gröbner basis for the elementary symmetric functions. Moreover, the quotient R/J_h gives another description of the cohomology ring of the full flag variety.

The generators of the ring R/J_h are in bijective correspondence with the Betti numbers of certain Hessenberg varieties. These varieties are a two-parameter generalization of Springer varieties, parametrized by a nilpotent operator X and a Hessenberg function h. These varieties were introduced in 1992 by De Mari, Procesi and Shayman. We provide evidence that as h varies, the quotient R/J_h may be a presentation for the cohomology ring of a subclass of Hessenberg varieties called regular nilpotent varieties.

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The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Abukuse Mbirika III

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2010 graduation.

Thesis Committee: ______________________________
Julianna Tymoczko, Thesis Supervisor

____
Frederick Goodman, Thesis Supervisor

____
Charles Frohman

____
Jonathan Simon

____
Sriram Pemmaraju

I dedicate this thesis to some of the physically smaller things in my life that have had a BIG impact. To colleague Paulette Willis' poodle "Jay" who shared many fun adventures with me these past six years. I never knew I loved dogs until I met Jay. And to colleague Sam Schmidt and his fiance Jessica Boyd's wee pooch "Buddy Larry" who never tired of spending long nights in the math office with me, and then venturing off to enjoy the nightlife of Iowa City. And lastly, I dedicate this thesis to colleague Jeff and his wife Nora Boerner's wee daughter Greta Joy who reminds me often that rainbows and joy are among the most precious things in life.

ACKNOWLEDGMENTS

I thank my main advisor in this project Julianna Tymoczko for endless feedback at our many meetings. And I thank her kindly for the tough love and straightforward criticism that thickened my skin and made me a better mathematician with a love for research. I am also grateful to advisor Fred Goodman for very helpful comments which significantly improved this thesis. I also thank him for bringing me to MSRI for the Spring 2008 combinatorial representation theory program where I learned the fun of combinatorics. Thanks also to Megumi Harada for fruitful conversations. Colleagues Jonas Meyer, Erik Insko, and Nick Teff also gave useful input. On the non-mathematical side, I thank my heart's best friend and fellow loving rainbow seeker, Violet (formerly Sarah Klahs).

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Let R be the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. We present two different ideals in R. Both are parametrized by a Hessenberg function h, namely a nondecreasing function that satisfies $h(i) \geq i$ for all i. The first ideal, which we call I_h, is generated by modified elementary symmetric functions. The ideal I_h generalizes the work of Tanisaki who gave a combinatorial description of the ideal used in Garsia and Procesi's quotient ring. Like the Tanisaki ideal, the generating set for I_h is redundant. We give a minimal generating set for this ideal. The second ideal, which we call J_h, is generated by modified complete symmetric functions. The generators of this ideal form a Gröbner basis, which is a useful property. Using the Gröbner basis for J_h, we identify a basis for the quotient R/J_h.

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generalization of Springer varieties, parametrized by a nilpotent operator X and a Hessenberg function h. These varieties were introduced in 1992 by De Mari, Procesi and Shayman. We provide evidence that as h varies, the quotient R/J_h may be a presentation for the cohomology ring of a subclass of Hessenberg varieties called regular nilpotent varieties.

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INTRODUCTION

1.1 Motivation

Let $R = \mathbb{Q}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables. Any function in R that remains invariant upon permuting the subscripts of its variables is called a *symmetric function*. That is, the function $f(x_1, \ldots, x_n)$ in R is symmetric if $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all permutations σ in the symmetric group $\mathcal{S}_n$. Symmetric functions have a rich history and arise in a variety of areas in mathematics such as combinatorics, Galois theory, representation theory, and linear algebra. The first published work was in 1629 by Albert Girard [8], expressing symmetric functions of the form $x_1^m + \cdots + x_n^m$ in terms of the *elementary symmetric functions* in R – that is, the n functions of the form

$$e_k := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k} \tag{1.1}$$

for $k \in \{1, \ldots, n\}$. Later historical figures who explored symmetric functions include Gabriel Cramer, Isaac Newton, Edward Waring, Alexander Vandermonde, Leonard Euler, and Joseph Lagrange to name a few. More recently, the past century has offered a wealth of contributions to the theory of symmetric functions from a list of mathematicians too numerous to mention. Richard Stanley [15] writes, "the theory of symmetric functions and its connections with combinatorics is in my opinion one of the most beautiful topics in all of mathematics."

The ring of symmetric functions has a number of different bases of which two, in particular, will be a focus of this thesis: the set of elementary symmetric functions e_k (as given in (1.1)) and the set of complete symmetric functions h_k given by

$$h_k := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

for $k \in \{1, \ldots, n\}$. The main difference between these two bases is that the elementary symmetric functions allow only square-free summands. We will analyze elementary and complete symmetric functions having only a subset of the variables $x_1, \ldots, x_n$. These *modified* (or partial) *symmetric functions* arise in applications such as the Tanisaki ideal [16] (see Section 1.3 for a brief history and related details). In fact, our work can be viewed as a generalization of work by Garsia and Procesi [7] on the Springer variety.

Springer varieties $\mathfrak{S}_X$ are defined to be the set of flags stabilized by a nilpotent operator X. In 1978, Springer [14] observed that the cohomology ring of $\mathfrak{S}_X$ carries a symmetric group action, and gave a deep geometric construction of this action. Over the next two decades, many mathematicians [11, 4, 7, 16] worked to make this action more accessible. Using Tanisaki's ideal of modified elementary symmetric functions, Garsia and Procesi gave a quotient ring presentation for the cohomology ring of $\mathfrak{S}_X$ and an explicit basis of monomials $\mathcal{B}(\mu)$ for this quotient.

This thesis seeks to generalize the work of Garsia and Procesi. To this end, we connect two different perspectives of cohomology. On the one hand, there is a geometric view given by Tymockzo [17] in which the Betti numbers of a Springer variety are counted by objects which we call (h, μ)-fillings. These fillings combinatorially describe the dimensions of the cells of a paving of the Springer variety. On the other hand, we have an algebraic view given by the Garsia-Procesi basis $\mathcal{B}(\mu)$ of monomials for the quotient of a polynomial ring by an ideal generated by symmetric functions. In Chapter 2, we establish a direct bijective correspondence between the set of (h, μ)-fillings and the monomials $\mathcal{B}(\mu)$. To generalize this picture, we explore Hessenberg varieties, which are a two-parameter generalization of Springer varieties. They appear in various areas including numerical analysis, geometric representation theory, quantum cohomology, and number theory.

A natural question to ask is whether the geometric and algebraic correspondence we established in the Springer variety setting extends to the more general Hessenberg variety setting. For the class of regular nilpotent Hessenberg varieties, the answer is yes (see Chapter 6). To achieve this goal, we define and analyze two different families of ideals. One is generated by modified elementary symmetric functions generalizing the Tanisaki ideal; the other is generated by modified complete symmetric functions. In Chapters 3, 4, and 5, we explore a host of useful properties exhibited by these two families of symmetric function ideals. In Chapter 7, we provide a sufficient condition for these two families to coincide. Finally in Chapter 8, we conjecture that a quotient of a polynomial ring by one of our ideals of modified complete symmetric functions gives a presentation for the integral cohomology ring of a regular nilpotent Hessenberg variety.

1.2 Overview and main results

In the current chapter, we define a Hessenberg variety $\mathfrak{H}(X,h)$ where X is a nilpotent operator in $\mathrm{Mat}_n(\mathbb{C})$ and $h = (h_1,\ldots,h_n)$ is a Hessenberg function. Tymoczko [17] gave a combinatorial description of the dimension of each of the graded parts of the cohomology ring of $\mathfrak{H}(X,h)$ by using certain fillings of the Young diagram μ associated to X, which we call (h,μ)-fillings. In Section 1.6, we describe a map Φ from (h,μ)-fillings onto a set of monomials $\mathcal{A}_h(\mu)$ in $\mathbb{Z}[x_1,\ldots,x_n]$. This map extends to a graded vector space morphism from the formal linear span of (h,μ)-fillings, which we denote $M^{h,\mu}$, to the span of monomials $\mathcal{A}_h(\mu)$.

In Chapter 2, we focus on the Springer setting and describe the three legs of the following triangle:

$$\begin{tikzcd}
 & H^*(\mathfrak{S}_X) \arrow[dl, leftrightsquigarrow] \arrow[dr, leftrightarrow, dashed, "\cong"] & \\
\begin{array}{c}(h,\mu)\text{-fillings}\\ \text{spanning } M^{\mu}\end{array} \arrow[rr, leftrightarrow, "\Phi", "\cong"'] & & \begin{array}{c} R/I_\mu \text{ with}\\ \mathcal{A}_h(\mu)=\mathcal{B}(\mu)\\ \text{basis}\end{array}
\end{tikzcd}$$

In this case, the (h, μ)-fillings are simply the row-strict tableaux. They tableaux are the generating set for the vector space M^{μ} (see Section 2.1). The ideal I_{μ} is the famed Tanisaki ideal [16]. It turns out that the image of Φ, namely, our set of monomials $\mathcal{A}_h(\mu)$, coincides with the Garsia-Procesi basis $\mathcal{B}(\mu)$ of monomials for a polynomial quotient ring presentation R/I_{μ} for the rational cohomology of the Springer varieties. We take R to be the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Garsia and Procesi used a tree on Young diagrams to find $\mathcal{B}(\mu)$. We refine their construction and build a *modified GP-tree for* μ (see Definition 2.3.4). This refinement helps us obtain more information from their tree; the actual paths on the modified GP-tree describe how to reconstruct an (h, μ)-filling corresponding to a basis element $\mathbf{x}^{\alpha}$ in R/I_{μ}. We conclude in Section 2.3 that the map Φ is a graded vector space isomorphism.

In Chapter 3, motivated by work of Biagioli, Faridi, and Rosas [17], we generalize the Tanisaki ideal to non-Springer settings. We fix X to be a regular nilpotent operator and let the Hessenberg function h vary. For each $h = (h_1, \ldots, h_n)$, we generate a corresponding ideal I_h from a set of at most $\frac{n(n+1)}{2}$ modified elementary symmetric functions. Denote this collection of generators by $\mathfrak{C}_h$. We introduce a partial ordering on Hessenberg functions and derive nice nesting properties such as $h > h'$ implies $I_h \subset I_{h'}$. Under certain circumstances a stronger result holds, namely, one of containment of generators. That is, for certain $h > h'$ we have $\mathfrak{C}_h$ contained in $\mathfrak{C}_{h'}$. As in Tanisaki's original work, our generating set is not minimal. In Section 3.5, we give a minimal generating set for I_h with exactly n generators.

In Chapter 4, we construct another family of ideals J_h. For each Hessenberg function $h = (h_1, \ldots, h_n)$, we generate the corresponding ideal J_h from a set of exactly n modified complete symmetric functions. Analogous to the setting of the ideal I_h, we deduce a nesting property that if $h > h'$ then $J_h \subset J_{h'}$.

In Chapter 5, we discuss two key properties of the ideals J_h. In Theorem 5.3.2, we prove that the generators form a Gröbner basis. Using properties of Gröbner bases, we prove in Theorem 5.4.3 that we can easily identify a basis for R/J_h. Furthermore in Corollary 5.4.5, we show that R/J_h has finite rank. In Sections 5.1 and 5.2, we provide the necessary definitions and background from commutative algebra. (Results from this chapter hold over $\mathbb{Z}$ coefficients, not just $\mathbb{Q}$, so we may take R to be $\mathbb{Z}[x_1, \ldots, x_n]$.)

In Chapter 6, we generalize the Springer-setting results from Chapter 2 to the setting of regular nilpotent Hessenberg varieties $\mathfrak{H}(X, h)$, where X is a regular nilpotent operator and h is an arbitrary Hessenberg function. The quotient R/J_h plays the role of Garsia and Procesi's quotient R/I_μ. We prove in Section 6.3 that the monomials $\mathcal{A}_h(\mu)$ coincide with the basis $\mathcal{B}_h(\mu)$ of R/J_h. Moreover, the map Φ is a graded vector space isomorphism between $M^{h,\mu}$ (the formal linear span of (h, μ)-fillings) and the span of monomials $\mathcal{A}_h(\mu)$. In Theorem 6.3.3, we show that the generators of R/J_h describe the Betti numbers of the regular nilpotent Hessenberg varieties.

In Chapter 7, we prove that when h is maximal, namely when $h = (n, \ldots, n)$, the ideal I_h equals the ideal J_h. Since I_h in this setting is generated by precisely the n elementary symmetric functions, we obtain a Gröbner basis for the set of elementary symmetric functions.

In Chapter 8, we discuss future directions and open questions. For instance, for $n = 4$ we observed that each of the 14 ideals I_h coincided with their corresponding ideals J_h. Does $I_h = J_h$ for all Hessenberg functions h? We also conjecture that R/J_h is a presentation for the integral cohomology ring of regular nilpotent Hessenberg varieties. Recent results of Harada and Tymoczko [9] support our claim that R/J_h may indeed be a presentation for the cohomology ring of a subclass of regular nilpotent Hessenberg varieties called Peterson varieties.

1.3 Brief history of the Springer setting

Let $\mathfrak{N}(\mu)$ be the set of nilpotent elements in $\mathrm{Mat}_n(\mathbb{C})$ with Jordan blocks of sizes $\mu_1 \geq \mu_2 \ldots \geq \mu_s > 0$ so that $\sum_{i=1}^{s} \mu_i = n$. The quest began 50 years ago to find the equations of the closure $\overline{\mathfrak{N}(\mu)}$ in $\mathrm{Mat}_n(\mathbb{C})$ – that is, the generators of the ideal of polynomial functions on $\mathrm{Mat}_n(\mathbb{C})$ which vanish on $\mathfrak{N}(\mu)$. When $\mu = (n)$, Kostant [10] showed in his fundamental 1963 paper that the ideal is given by the invariants of the conjugation action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{Mat}_n(\mathbb{C})$. In 1981, De Concini and Procesi [4] proposed a set of generators for the ideals of the schematic intersections $\overline{\mathfrak{N}(\mu)} \cap T$ where T is the set of diagonal matrices and μ is an arbitrary partition of n. In 1982, Tanisaki [16] simplified their ideal; his simplification has since become known as the Tanisaki ideal I_μ. In 1992, Garsia and Procesi [7] showed that the ring $R_\mu = \mathbb{Q}[x_1, \ldots, x_n]/I_\mu$ is isomorphic to the cohomology ring of a variety called the *Springer variety associated to a nilpotent element* $X \in \mathfrak{N}(\mu)$. Much work has been done to simplify the description of the Tanisaki ideal even further, including work by Biagioli, Faridi, and Rosas [1] in 2008. Inspired by their work, we generalize the Tanisaki ideal in Chapter 3 for a subclass of a family of varieties that naturally extends Springer varieties, called Hessenberg varieties.

1.4 Definition of a Hessenberg variety

Hessenberg varieties were introduced by De Mari, Procesi, and Shayman [5] in 1992. Let $h : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ be a map subject to the constraints $i \leq h(i)$ for all i, and $h(i) \leq h(i+1)$ for all $i < n$. We call the map h a *Hessenberg function*. In this paper we will denote this function by an n-tuple $h = (h_1, \ldots, h_n)$ where $h_i = h(i)$. A flag is a nested sequence of $\mathbb{C}$-vector spaces

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n,$$

where each V_i has dimension i. The collection of all such flags is called the full flag variety $\mathfrak{F}$. Fix a nilpotent operator $X \in \mathrm{Mat}_n(\mathbb{C})$. We define a *Hessenberg variety* to be the following subvariety of the full flag variety:

$$\mathfrak{H}(X,h) = \{\text{Flags} \in \mathfrak{F} \mid X \cdot V_i \subseteq V_{h(i)} \text{ for all } i\}.$$

Since conjugating the nilpotent X will produce a variety homeomorphic to $\mathfrak{H}(X,h)$ [17, Proposition 2.7], we can assume that the nilpotent X is in Jordan canonical form, with a weakly decreasing sequence of Jordan block sizes $\mu_1 \geqslant \cdots \geqslant \mu_s > 0$ so that $\sum_{i=1}^{s} \mu_i = n$. We may view μ as a partition of n or as a Young diagram with row lengths μ_i. Thus there is a one-to-one correspondence between Young diagrams and conjugacy classes of nilpotent operators.

For a fixed nilpotent operator X, there are two extreme cases. If the Hessenberg function is $h = (1, 2, \ldots, n)$, then $\mathfrak{H}(X,h)$ is the Springer variety, which we denote $\mathfrak{S}_X$. At the other extreme if the Hessenberg function is $h = (n, \ldots, n)$, then all flags satisfy the condition $X \cdot V_i \subseteq V_{h(i)}$ for all i and hence $\mathfrak{H}(X,h) = \mathfrak{F}$. We will always take h to be in this range, so $(1, 2, \ldots, n) \leq h \leq (n, \ldots, n)$.

1.5 Using $(\boldsymbol{h}, \mu)$-fillings to compute the Betti numbers of Hessenberg varieties

In 2005, Tymoczko [17] gave a combinatorial procedure for finding the dimensions of the graded parts of $H^*(\mathfrak{H}(X,h))$. Let the Young diagram μ correspond to the Jordan canonical form of X as given in Section 1.4. Any injective placing of the numbers $1, \ldots, n$ in a diagram μ with n boxes is called a *filling of* μ. It is called an *(h-μ)-filling* if it adheres to the following rule: a horizontal adjacency $\boxed{k|j}$ is allowed only if $k \leq h(j)$. If h and μ are clear from context, then we often call this a *permissible filling.* When $h = (3,3,3)$ all permissible fillings of $\mu = (2,1)$ coincide with all possible fillings as shown below.

$$\begin{array}{|c|c|}\hline 1 & 2 \\\hline 3 \\\cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|}\hline 1 & 3 \\\hline 2 \\\cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|}\hline 2 & 3 \\\hline 1 \\\cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|}\hline 2 & 1 \\\hline 3 \\\cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|}\hline 3 & 1 \\\hline 2 \\\cline{1-1}\end{array}\;,\ \text{and}\ \begin{array}{|c|c|}\hline 3 & 2 \\\hline 1 \\\cline{1-1}\end{array}$$

Figure 1.1: The six (h, μ)-fillings for $h = (3,3,3)$ and $\mu = (2,1)$.

If $h = (1,3,3)$ then the fourth and fifth tableaux in Figure 1.1 are not (h,μ)-fillings since $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 3 & 1 \\ \hline\end{array}$ are not allowable adjacencies for this given h.

Definition 1.5.1 (Dimension pair)**.** Let h be a Hessenberg function and μ be a partition of n. The pair (a,b) is a *dimension pair* of an (h,μ)-filling T if

1. $b > a$,

2. b is below a and in the same column, or b is in any column strictly to the left of a, and

3. if some box with filling c happens to be adjacent and to the right of a, then $b \leq h(c)$.

Theorem (Tymoczko)**.** *[17, Theorem 1.1] The dimension of $H^{2k}(\mathfrak{H}(X,h))$ equals the number of (h,μ)-fillings T such that T has k dimension pairs.*

Remark 1.5.2. Tymoczko proves this theorem by providing an explicit geometric construction which partitions $\mathfrak{H}(X,h)$ into pieces homeomorphic to complex affine space. Consequently, this paving by affines determines the Betti numbers of $\mathfrak{H}(X,h)$.

Example 1.5.3. Fix $h = (1,3,3)$ and let μ have shape $(2,1)$. The figure below gives all possible (h,μ)-fillings and their corresponding dimension pairs.

$$\begin{array}{llll}
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \longleftrightarrow (1,3),(2,3) & & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \longleftrightarrow (1,2) \\
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \longleftrightarrow \text{no dimension pairs} & & \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \longleftrightarrow (2,3)
\end{array}$$

Figure 1.2: The four (h,μ)-fillings for $h = (1,3,3)$ and $\mu = (2,1)$.

We conclude H^0 has dimension 1 since exactly one filling has 0 dimension pairs. H^2 has dimension 2 since exactly two fillings have 1 dimension pair each. Lastly, H^4 has dimension 1 since the remaining filling has 2 dimension pairs.

1.6 The map Φ from $(\boldsymbol{h},\mu)$-fillings to monomials $\mathcal{A}_h(\mu)$

Let R be the polynomial ring $\mathbb{Q}[x_1,\dots,x_n]$. We introduce a map from the set of (h,μ)-fillings onto a set of monomials in R. First, we provide some notation for the set of dimension pairs.

Definition 1.6.1 (DP^T – set of dimension pairs of T)**.** Fix a partition μ of n. Let T be an (h,μ)-filling. Define DP^T to be the set of dimension pairs of T according to Section 1.5. For a fixed $y \in \{2,\dots,n\}$, define

$$\mathrm{DP}^T_y := \left\{(x,y) \mid (x,y) \in \mathrm{DP}^T\right\}.$$

The number of dimension pairs of an (h,μ)-filling T is called the *dimension of T*.

Fix a Hessenberg function h and a partition μ of n. The map Φ is the following:

$$\Phi : \{(h,\mu)\text{-fillings}\} \longrightarrow R \qquad \text{defined by} \qquad T \longmapsto \prod_{\substack{(i,j)\in \mathrm{DP}^T_j \\ 2\le j\le n}} x_j .$$

Denote the image of Φ by $\mathcal{A}_h(\mu)$. By abuse of notation we also denote the linear span of these monomials by $\mathcal{A}_h(\mu)$. Denote the formal linear span of the (h,μ)-fillings by $M^{h,\mu}$. Extending Φ linearly, we get a map on vector spaces $\Phi : M^{h,\mu} \to \mathcal{A}_h(\mu)$.

Remark 1.6.2. Any monomial $\mathbf{x}^\alpha \in \mathcal{A}_h(\mu)$ will be of the form $x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. That is, the variable x_1 can never appear in $\mathbf{x}^\alpha$ since 1 will never be the larger number in a dimension pair.

Theorem 1.6.3. *If μ is a partition of n, then Φ is a well-defined degree-preserving map from the set of (h,μ)-fillings onto the monomials $\mathcal{A}_h(\mu)$. That is, r-dimensional (h,μ)-fillings map to degree-r monomials in $\mathcal{A}_h(\mu)$.*

Proof. Let T be an (h,μ)-filling of dimension r. Then T has r dimension pairs by definition. By construction $\Phi(T)$ will have degree r. Hence the map is degree-preserving. $\square$

CHAPTER 2
SPRINGER VARIETY SETTING

In this chapter we will fill in the details of Figure 2.1. Recall that if we fix the Hessenberg function to $h = (1, 2, \ldots, n)$ and let the nilpotent operator X (equivalently, the shape μ) vary, the Hessenberg variety $\mathfrak{H}(X, h)$ obtained is the Springer variety $\mathfrak{S}_X$. Since this chapter focuses on this setting, we omit the h in our notation. For instance, the image of Φ is $\mathcal{A}(\mu)$. Similarly, the Garsia-Procesi basis will be denoted $\mathcal{B}(\mu)$ (as it is denoted in the literature [7]).

$$\begin{tikzcd}
& H^*(\mathfrak{S}_X) \arrow[dl, leftrightsquigarrow] \arrow[dr, dashed, leftrightarrow, "\cong"] & \\
\begin{array}{c}(h,\mu)\text{-fillings}\\ \text{spanning } M^{\mu}\end{array} \arrow[rr, leftrightarrow, "\Phi", "\cong"'] & & \begin{array}{c} R/I_{\mu} \text{ with} \\ \mathcal{A}(\mu) = \mathcal{B}(\mu) \\ \text{basis}\end{array}
\end{tikzcd}$$

Figure 2.1: Triangle – Springer setting.

In Section 2.1, we recast the statement of the graded vector space morphism Φ to the setting of Springer varieties. In Section 2.2, we define an inverse map Ψ from the span of monomials $\mathcal{A}(\mu)$ to the formal linear span of (h, μ)-fillings, thereby giving not only a bijection of sets but also a graded vector space isomorphism as is shown in Corollary 2.3.10. This completes the bottom leg of the triangle in Figure 2.1. In Section 2.3, we modify the work of Garsia and Procesi [7] and develop a technique to build the (h, μ)-filling corresponding to a monomial in their quotient basis $\mathcal{B}(\mu)$. We conclude $\mathcal{A}(\mu) = \mathcal{B}(\mu)$. Lastly in Section 2.4, we expose a barrier to applying the inverse map Ψ in arbitrary Hessenberg variety settings.

2.1 Remarks on the map Φ when $h = (1, 2, \ldots, n)$

Fix a partition μ of n. Upon consideration of the combinatorial rules governing a permissible filling of a Young diagram, we see that if $h = (1, 2, \ldots, n)$, then the (h, μ)-fillings are just the row-strict tableaux of shape μ. Suppressing the h, we denote the formal linear span of these tableaux by M^μ. This is the standard symbol for this space, commonly known as the permutation module corresponding to μ (see expository work of Fulton [6]). In this specialized setting, the map Φ is simply

$$\Phi : M^\mu \twoheadrightarrow \mathcal{A}(\mu) \qquad \text{defined by} \qquad T \longmapsto \prod_{\substack{(i,j) \in \mathrm{DP}_j^T \\ 2 \le j \le n}} x_j,$$

and hence Theorem 1.6.3 specializes to the following.

Theorem 2.1.1. *If μ is a partition of n, then Φ is a well-defined degree-preserving map from the set of row-strict tableaux in M^μ onto the monomials $\mathcal{A}(\mu)$. That is, r-dimensional tableaux in M^μ map to degree-r monomials in $\mathcal{A}(\mu)$.*

Example 2.1.2. Let $\mu = (2, 2, 2)$ have the filling $T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 6 \\ \hline 4 & 5 \\ \hline \end{array}$. Suppressing the commas for ease of viewing, the contributing dimension pairs are (23), (24), (25), (26) and (34). Observe $(23) \in \mathrm{DP}_3^T$, $(24), (34) \in \mathrm{DP}_4^T$, $(25) \in \mathrm{DP}_5^T$, and $(26) \in \mathrm{DP}_6^T$. Hence Φ takes this tableau to the monomial $x_3 x_4^2 x_5 x_6 \in \mathcal{A}(\mu)$.

In the next section we will give an explicit algorithm to recover the original row-strict tableau from any monomial in $\mathcal{A}(\mu)$. Example 2.2.10 in particular applies the inverse algorithm to the example above.

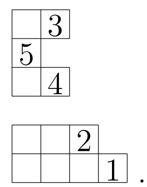
2.2 The inverse map Ψ from monomials in $\mathcal{A}(\mu)$ to $(\boldsymbol{h}, \mu)$-fillings

The map back from a monomial $\mathbf{x}^\alpha \in \mathcal{A}(\mu)$ to an (h, μ)-filling is not as transparent. We will construct the tableau by filling it in reverse order starting with the number n. The next definitions give us the language to speak about where we can place n and the subsequent numbers.

Definition 2.2.1 (Composition of n)**.** Let ρ be a partition of n corresponding to a diagram of shape $(\rho_1, \rho_2, \ldots, \rho_s)$ that need not be a proper Young diagram. That is, the sequence neither has to weakly increase nor decrease and some ρ_i may even be zero. An ordered partition of this kind is often called a *composition* of n and is denoted $\rho \models n$.

Definition 2.2.2 (Dimension-ordering of a composition)**.** We define a *dimension-ordering* on a composition ρ in the following manner. Order the boxes on the far-right of each row starting from the right-most column to the first column going from top to bottom each time.

Example 2.2.3. If $\rho = (2, 1, 0, 3, 4) \models 12$, then the ordering is



Notice that imposing a dimension-ordering on a diagram places exactly one number in the far-right box of each non-empty row.

Definition 2.2.4 (Subfillings and subdiagrams of a composition)**.** Let T be a filling of a composition ρ of n. If the values $i+1, i+2, \ldots, n$ and their corresponding boxes are removed from T, then what remains is called a *subfilling of* T and is denoted $T^{(i)}$. Ignoring the numbers in these remaining i boxes, the shape is called a *subdiagram of* ρ and is denoted $\rho^{(i)}$.

Observe that $\rho^{(i)}$ need no longer be a composition. For example, let $\rho = \boxed{}\boxed{}\boxed{}$ have filling $T = \boxed{1}\boxed{3}\boxed{2}$. Then $T^{(2)}$ is $\boxed{1}\quad\boxed{2}$ and so $\rho^{(2)}$ gives the subdiagram $\boxed{}\quad\boxed{}$ which is not a composition. The next property gives a sufficient condition on T to ensure $\rho^{(i)}$ is a composition.

Subfilling Property. A filling T of a composition ρ of n satisfies the *subfilling property* if the number i is in the right-most box of some row of the subfilling $T^{(i)}$ for each $i \in \{1, \ldots, n\}$.

Lemma 2.2.5. *Let T be a filling of a composition ρ of n. Then the following are equivalent:*

(a) T satisfies the Subfilling Property.

(b) T is a row-strict filling of ρ.

In particular, every filling of a Young diagram ρ satisfying the Subfilling Property lies in M^ρ.

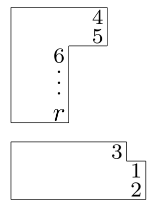
Proof. Let T be a filling of composition ρ of n. Suppose T is not row-strict. Then there exists some row in ρ with an adjacent filling of two numbers $\boxed{k}\boxed{j}$ such that $k > j$. However the subfilling $T^{(k)}$ does not have k in the right-most box of this row, so T does not satisfy the Subfilling Property. Hence (a) implies (b). For the converse, suppose T does not satisfy the Subfilling Property. Then there exists a number i such that i is not in the far-right box of some nonzero row in $T^{(i)}$. Thus there is some k in this row that is smaller and to the right of i so T is not row-strict. Hence (b) implies (a). $\square$

Lemma 2.2.6. *Let $\rho = (\rho_1, \rho_2, \ldots, \rho_s)$ be a composition of n. Suppose that r of the s entries ρ_i are nonzero. We claim:*

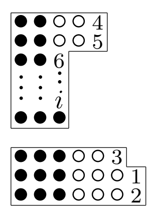
(a) There exist exactly r positions where n can be placed in a row-strict composition.

(b) Let T be a row-strict filling of ρ. If n is placed in the box of T with dimension-ordering $i \in \{1, \ldots, r\}$, then n is in a dimension pair with exactly $i-1$ other numbers; that is, $|\,\mathrm{DP}_n^T\,| = i-1$.

Proof. Suppose $\rho = (\rho_1, \rho_2, \ldots, \rho_s)$ is a composition of n where r of the s entries are nonzero. Claim (a) follows by the definition of row-strict and the fact that n is the largest number in any filling of $\rho \vDash n$. To illustrate the proof of (b), consider the following schematic for ρ:

$\rho :=$  .

Enumerate the far-right boxes of each nonempty row (as in the figure above) using dimension-ordering with the numbers $1, \ldots, r$. Let T be a row-strict filling of ρ. Suppose n lies in the box with dimension-ordering $i \in \{1, \ldots, r\}$. It suffices to count the number of dimension pairs with n, or simply $|\,\mathrm{DP}_n^T\,|$ since n is the largest value in the filling. Thus we want to count the distinct values β such that $(\beta, n) \in \mathrm{DP}_n^T$. We need not concern ourselves with boxes in the same column below or anywhere left of the i^{th} dimension-ordered box for then $(\beta, n) \in \mathrm{DP}_n^T$ would imply $\beta > n$ which is impossible (see $\bullet$-shaded boxes in figure below). We also need not concern ourselves with any boxes that are in the same column above or anywhere to the right of the i^{th} dimension-ordered box if it has a neighbor j immediately right of it (see $\circ$-shaded boxes in figure below).

$\rho :=$  .

If so, then $(\beta, n) \in \mathrm{DP}_n^T$ would imply $n \leq h(j)$ which is impossible since $h(j) = j$ and $j < n$. That leaves exactly the $i - 1$ boxes which are dimension-ordered boxes that are in the same column above n or anywhere to the right of n. Hence $|\,\mathrm{DP}_n^T\,| = i - 1$ and (b) is shown. $\square$

Lemma 2.2.7. *Suppose T is a row-strict filling of a composition ρ of n. If i lies in $\{1, \ldots, n\}$, then $|\,\mathrm{DP}_i^{T^{(i)}}\,| = |\,\mathrm{DP}_i^T\,|$.*

Proof. Consider the subfilling $T^{(i)}$. All the existing pairs $(\beta, i) \in \mathrm{DP}_i^{T^{(i)}}$ will still be valid dimension pairs in T if we restore the numbers $i + 1, \ldots, n$ and their corresponding boxes. Hence $|\,\mathrm{DP}_i^{T^{(i)}}\,| \leq |\,\mathrm{DP}_i^T\,|$. However no further pairs (β, i) can be created by restoring numbers larger than i, thus we get equality. $\square$

Lemma 2.2.8. *Fix a partition μ of n. Let T be a tableau in M^μ. Suppose $\Phi(T)$ is the monomial $\boldsymbol{x}^\alpha$. For each $i \in \{2, \ldots, n\}$, consider the subdiagram $\mu^{(i)}$ of μ corresponding to the subfilling $T^{(i)}$ of T. Then each $\mu^{(i)}$ has at least $\alpha_i + 1$ nonzero rows where α_i is the exponent of x_i in the monomial $\boldsymbol{x}^\alpha$.*

Proof. Fix a partition μ of n. Let $T \in M^\mu$ and $\mathbf{x}^\alpha \in \mathcal{A}(\mu)$ be $\Phi(T)$. By Remark 1.6.2, $\mathbf{x}^\alpha$ is of the form $x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Suppose that the claim does not hold. Then there is some $i \in \{2, \ldots, n\}$ for which $\mu^{(i)}$ has r nonzero rows and $r < \alpha_i + 1$. Lemma 2.2.6 implies the number of dimension pairs in $\mathrm{DP}_i^{T^{(i)}}$ is at most $r-1$. Thus $|\,\mathrm{DP}_i^{T^{(i)}}\,| \leq r - 1 < \alpha_i$. Since $|\,\mathrm{DP}_i^{T^{(i)}}\,| = |\,\mathrm{DP}_i^T\,|$ by Lemma 2.2.7, it follows that $|\,\mathrm{DP}_i^T\,| < \alpha_i$, contradicting the fact that variable x_i has exponent α_i. $\square$

Theorem 2.2.9 (A map from $\mathcal{A}(\mu)$ to (h,μ)-fillings)**.** *Given a partition μ of n, there exists a well-defined dimension-preserving map Ψ from the monomials $\mathcal{A}(\mu)$ to the set of row-strict tableaux in M^μ. That is, Ψ maps degree-r monomials in $\mathcal{A}(\mu)$ to r-dimensional (h,μ)-fillings in M^μ. Moreover the composition*

$$\mathcal{A}(\mu) \xrightarrow{\Psi} M^\mu \xrightarrow{\Phi} \mathcal{A}(\mu)$$

is the identity.

Proof. Fix a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of n. Let $\mathbf{x}^\alpha$ be a degree-r monomial in $\mathcal{A}(\mu)$. Remark 1.6.2 reminds us that $\mathbf{x}^\alpha$ is of the form $x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. The goal is to construct a map $\Psi : \mathcal{A}(\mu) \to M^\mu$ such that $\Psi(\mathbf{x}^\alpha)$ is an r-dimensional tableau in M^μ and $\Phi \circ \Psi(\mathbf{x}^\alpha) = \mathbf{x}^\alpha$. Recall $\mathcal{A}(\mu)$ is the image of M^μ under Φ so we know there exists some tableau $T' \in M^\mu$ with $|\,\mathrm{DP}_i^{T'}\,| = \alpha_i$ for each $i \in \{2, \ldots, n\}$.

We now construct a filling T (not a priori the same as T') by giving μ a precise row-strict filling to be described next. Lemma 2.2.5 ensures T will lie in M^μ. To construct T we iterate the algorithm below with a triple-datum of the form

$(\mu^{(i)},\ i,\ x_i^{\alpha_i})$ of a composition $\mu^{(i)}$ of i, an integer i, and the $x_i^{\alpha_i}$-part of $\mathbf{x}^\alpha$. Start with $i = n$ in which case $\mu^{(n)}$ is μ itself; then decrease i by one each time and repeat the steps below with the new triple-datum. The algorithm is as follows:

1. Input the triple-datum.
2. Impose the dimension-ordering on $\mu^{(i)}$.
3. Place i in the box with dimension-order $\alpha_i + 1$.
4. If $i \geq 2$, then remove the box with the entry i to get a new subdiagram $\mu^{(i-1)}$. Pass the new triple-datum $(\mu^{(i-1)},\ i-1,\ x_{i-1}^{\alpha_{i-1}})$ to Step 1.
5. If $i = 1$, then the final number 1 is forced in the last remaining box. Replace all $n-1$ removed numbers and call this tableau T.

We confirm that this algorithm is well-defined and produces a tableau in M^μ. Step 3 can be performed because Lemma 2.2.8 ensures the box exists. The Subfilling Property ensures that the subdiagram at Step 4 is indeed a composition. By Lemma 2.2.5, T is row-strict and hence lies in M^μ.

We are left to show Φ maps T to the original $\mathbf{x}^\alpha \in \mathcal{A}(\mu)$ from which we started. It suffices to check that if the exponent of x_i in $\mathbf{x}^\alpha$ is α_i, then $|\,\mathrm{DP}_i^T\,| = \alpha_i$ for each $i \in \{2, \ldots, n\}$. By Lemma 2.2.6, when $i = n$ we know $|\,\mathrm{DP}_n^T\,| = \alpha_n$. At each iteration after this initial step, we remove one more box from μ. At step $i = m$ for $m < n$, we placed m into $\mu^{(m)}$ in the box with dimension-order $\alpha_m + 1$ and hence $|\,\mathrm{DP}_m^{T(m)}\,| = \alpha_m$ by Lemma 2.2.6. But $|\,\mathrm{DP}_m^{T(m)}\,| = |\,\mathrm{DP}_m^T\,|$ by Lemma 2.2.7. Thus $|\,\mathrm{DP}_m^T\,| = \alpha_m$ as desired. Hence given the monomial $\mathbf{x}^\alpha \in \mathcal{A}(\mu)$, we see by construction of $T = \Psi(\mathbf{x}^\alpha)$ that T has the desired dimension pairs to map back to $\mathbf{x}^\alpha$ via the map Φ. That is, the composition $\Phi \circ \Psi$ is the identity on $\mathcal{A}(\mu)$. $\square$

Example 2.2.10. Let $\mu = (2, 2, 2)$ and consider the monomial $x_3 x_4^2 x_5 x_6$ from Example 2.1.2. We show that this monomial will map to the filling

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 6 \\ \hline 4 & 5 \\ \hline \end{array}$$

which we showed in Example 2.1.2 maps to the monomial $x_3x_4^2x_5x_6$ under Φ. We have the following flowchart:

$$\begin{array}{|c|c|}\hline & \text{a} \\ \hline & \text{b} \\ \hline & \text{c} \\ \hline \end{array} \xRightarrow{6\mapsto 2} \begin{array}{|c|c|}\hline & \text{a} \\ \hline \text{c} & 6 \\ \hline & \text{b} \\ \hline \end{array} \xRightarrow{5\mapsto 2} \begin{array}{|c|c|}\hline & \text{a} \\ \hline \text{b} & 6 \\ \hline \text{c} & 5 \\ \hline \end{array} \xRightarrow{4\mapsto 3} \begin{array}{|c|c|}\hline & \text{a} \\ \hline \text{b} & 6 \\ \hline 4 & 5 \\ \hline \end{array} \xRightarrow{3\mapsto 2} \begin{array}{|c|c|}\hline & \text{a} \\ \hline 3 & 6 \\ \hline 4 & 5 \\ \hline \end{array} \xRightarrow{2\mapsto 1} \begin{array}{|c|c|}\hline & 2 \\ \hline 3 & 6 \\ \hline 4 & 5 \\ \hline \end{array} \xRightarrow{1 \text{ forced}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 6 \\ \hline 4 & 5 \\ \hline \end{array}.$$

For clarity we label the dimension-ordered boxes at each stage in small font with letters a, b, c to mean 1st, 2nd, 3rd dimension-ordered boxes respectively. Place 6 in the second dimension-ordered box b since the exponent of x_6 is 1. Place 5 in the second dimension-ordered box b since the exponent of x_5 is 1. Place 4 in the third dimension-ordered box c since the exponent of x_4 is 2. And so on.

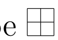
Remark 2.2.11. Since $M^\mu \xrightarrow{\Phi} \mathcal{A}(\mu) \xrightarrow{\Psi} M^\mu$ is the identity, it follows that $\mathcal{A}(\mu)$ and M^μ are isomorphic as graded vector spaces. This proof is a simple consequence of the fact that the monomials $\mathcal{A}(\mu)$ coincide with the Garsia-Procesi basis $\mathcal{B}(\mu)$. We show this in the next section in Corollary 2.3.10.

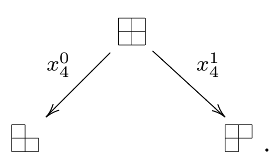
2.3 $\mathcal{A}(\mu)$ coincides with the Garsia-Procesi basis $\mathcal{B}(\mu)$

Garsia and Procesi construct a tree [7, pg.87] that we call a *GP-tree* to define their monomial basis $\mathcal{B}(\mu)$. In this section we modify this tree's construction to deliver more information. For a given monomial $\mathbf{x}^\alpha \in \mathcal{B}(\mu)$, each path on the modified tree tells us how to to construct a rwo-strict tableau T such that $\Phi(T)$ equals $\mathbf{x}^\alpha$. In other words the paths on the tree give Ψ. First we recall what Garsia and Procesi did. Then we give an example that makes this algorithm more transparent.

Definition 2.3.1 (GP-tree)**.** Although Garsia and Procesi mention nothing of a dimension-ordering (recall Definition 2.2.2), we find it clearer to explain the combinatorics of building a GP-tree using this concept. They also use French-style Ferrers diagrams, but we will use the convention of having our tableaux flush top and left.

If μ is a partition of n, then the *GP-tree of* μ is a tree with n levels constructed as follows. Let μ sit alone at the top Level n. From a subdiagram $\mu^{(i)}$ at Level i, we branch down to exactly r new subdiagrams at Level $i-1$ where r equals the number of nonzero rows of $\mu^{(i)}$. Note that this branching is *injective* – that is, no two Level i diagrams branch down to the same Level $i-1$ diagram. Label these r edges left to right with the labels $x_i^0, x_i^1, \ldots, x_i^{r-1}$. Impose the dimension-ordering on $\mu^{(i)}$. The subdiagram at the end of the edge labelled x_i^j for some $j \in \{0, 1, \ldots, r-1\}$ will be exactly $\mu^{(i)}$ with the box with dimension-ordering $j+1$ removed. If a gap in a column is created by removing this box, then correct the gap by pushing up on this column to make a proper Young diagram instead of a composition. At Level 1 there is a set of single box diagrams. Instead of placing single boxes at this level, put the product of the edge labels from Level n down to this vertex. These monomials are the basis for $\mathcal{B}(\mu)$ [7, Theorem 3.1, pg.100].

Example 2.3.2 (GP-tree for $\mu = (2,2)$)**.** Let $\mu = (2,2)$, which has shape . We start at the top Level 4 with the shape (2,2). The first branching of the (2,2)-tree is



But we make the bottom-left non-standard diagram into a proper Young diagram by pushing the bottom-right box up the column. In the Figure 2.2, we show the completed GP-tree. Observe that the six monomials at Level 1 are the Garsia-Procesi basis $\mathcal{B}(\mu)$.

Remark 2.3.3. Each time a subdiagram is altered to make it look like a proper Young diagram, we lose information that can be used to reconstruct a row-strict tabeau in M^μ from a given monomial in $\mathcal{B}(\mu)$. Our construction below will take this into account, and give the precise prescription for constructing a filling from a monomial in $\mathcal{B}(\mu)$.

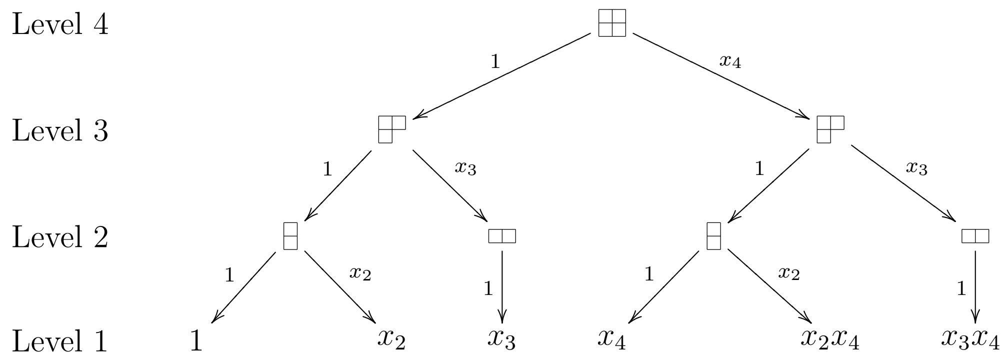


Figure 2.2: The GP-tree for $\mu = (2, 2)$.

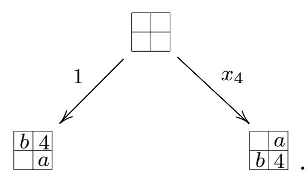
Definition 2.3.4 (Modified GP-tree)**.** Let μ be a partition of n. The *modified GP-tree for* μ is a tree with $n+2$ levels. The top is Level n with diagram μ at its vertex. The branching and edge labelling rules are the same as in the GP-tree. The crucial modification from the GP-tree is the diagram at the end of a branching edge.

- When branching down from Level i down to Level $i-1$ for $i \geq 1$, the new diagram at Level $i-1$ will be a composition $\mu^{(i-1)}$ of $i-1$ with a partial filling of the values $i, \ldots, n$ in the remaining $n-(i-1)$ boxes of μ. In the diagram at the end of the edge labelled x_i^j, instead of removing the box with dimension-ordering $j+1$ place the value i in this box.

Rename Level 0 as Level A. Label the edge from Level A down to Level B with the label 1. At each vertex at Level B, put the product of the edge labels from Level n down to this vertex.

Remark 2.3.5. Observe that we never move a box as was done in the GP-tree to create a Young diagram from a composition. There are now two sublevels below Level 1: Level A has a fillings of μ constructed through this tree, and Level B has the monomials in $\mathcal{B}(\mu)$ coming from the product of the edge labels on the paths. Theorem 2.3.7 highlights a profound relationship between these two levels.

Example 2.3.6. Again consider the shape $\mu = (2,2)$. Dimension order the Level 4 diagram to get $\begin{array}{|c|c|}\hline & a \\ \hline & b \\ \hline\end{array}$. Branch downward left placing 4 in the dimension-ordered box we labelled a. Branch downward right placing 4 in the dimension-ordered box we labelled b. Ignoring the filled box, impose dimension-orderings on the left composition of shape $(1,2)$ and the right composition of shape $(2,1)$. This gives:



In each of these subdiagrams branch down to the next Level by placing 3 in the appropriate dimension-ordered boxes. The completed tree is given in Figure 2.3.

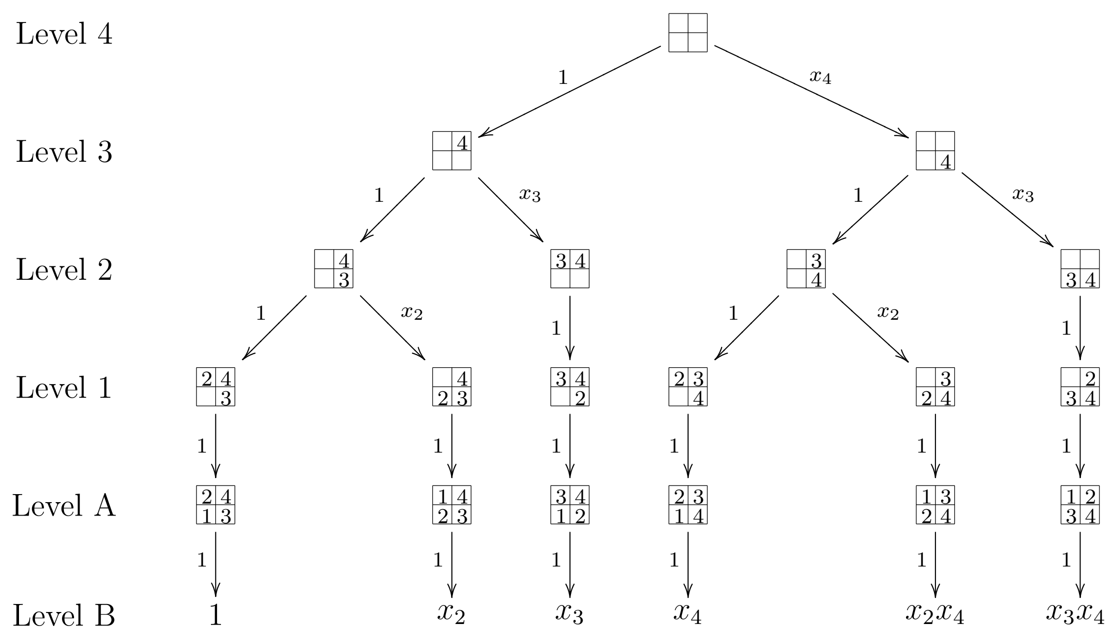


Figure 2.3: The modified GP-tree for $\mu = (2,2)$.

Theorem 2.3.7. *Let μ be a Young diagram and consider its corresponding modified GP-tree. Each of the fillings at Level A are (h,μ)-fillings. Moreover given a filling*

T, the image $\Phi(T)$ will be the monomial $\boldsymbol{x}^\alpha \in \mathcal{B}(\mu)$ at the neighbor of T in Level B.

Proof. Fix a partition μ of n. Consider a path in the modified GP-tree for μ. From Level n to Level A, the numbers n through 1 are placed in reverse-order in the dimension-ordered boxes. Finally at Level A, a filling T satisfying the Subfilling Property is completed. By Lemma 2.2.5, T is row-strict and hence is an (h, μ)-filling.

Let T be a tableau at Level A, and let $\mathbf{x}^\alpha = x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ at Level B be the monomial below T. We claim that $\Phi(T) = \mathbf{x}^\alpha$. By Lemma 2.2.6, for each i the cardinality of $\mathrm{DP}_k^{T^{(i)}}$ equals α_i where $T^{(i)}$ is a i^{th}-subfilling of T (recall Definition 2.2.4). By Lemma 2.2.7, the value $|\,\mathrm{DP}_k^{T^{(i)}}\,|$ will equal $|\,\mathrm{DP}_i^T\,|$. Hence DP_i^T has exactly α_i dimension pairs so $\Phi(T) = \mathbf{x}^\alpha$ as desired. □

A surprising application of the modified GP-tree is that it counts the elements of M^μ. The corollary gives $\mathcal{A}(\mu) = \mathcal{B}(\mu)$.

Theorem 2.3.8. *Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of n. The number of paths in the modified GP-tree for μ is exactly $|M^\mu| = \dfrac{n!}{\mu_1! \cdots \mu_k!}$. In particular, Level A is composed of exactly all row-strict tableaux of shape μ.*

Proof. Firstly, the number of paths in the modified GP-tree is the same as in the standard GP-tree. Garsia and Procesi prove [7, Prop. 3.2] that the dimension of their quotient ring presentation equals $\frac{n!}{\mu_1! \cdots \mu_k!}$ and hence $|\mathcal{B}(\mu)| = |M^\mu|$. Each of the paths in the modified GP-tree gives a unique (h, μ)-filling at Level A by construction, and there are $|\mathcal{B}(\mu)|$ such paths. Recall that the (h, μ)-fillings in this case are the row-strict fillings. Thus the fillings at Level A are not just a subset of row-strict tableaux. Level A is exactly M^μ. □

Corollary 2.3.9. *The sets of monomials $\mathcal{A}(\mu)$ and $\mathcal{B}(\mu)$ coincide.*

Proof. This follows since the image of all (h, μ)-fillings under Φ is $\mathcal{A}(\mu)$. The image of the Level A fillings in the modified GP-tree is $\mathcal{B}(\mu)$. Theorem 2.3.8 implies that both the set of (h, μ)-fillings and the Level A fillings coincide, and hence it follows that $\mathcal{A}(\mu) = \mathcal{B}(\mu)$. $\square$

Corollary 2.3.10. *$\mathcal{A}(\mu)$ and M^μ are isomorphic as graded vector spaces.*

Proof. By Theorem 2.2.9, the composition $\Phi \circ \Psi$ is the identity on $\mathcal{A}(\mu)$. Since $\mathcal{A}(\mu) = \mathcal{B}(\mu)$, Theorem 2.3.8 implies the cardinality of $\mathcal{A}(\mu)$ equals the cardinality of the generating set of row-strict tableaux in M^μ. Also, Φ is a degree-preserving map while Ψ is a dimension-preserving map so the composition $\Phi \circ \Psi$ is a graded map. Thus $\mathcal{A}(\mu)$ and M^μ are isomorphic as graded vector spaces. $\square$

2.4 Barriers to applying this technique in non-Springer setting

What if we let h vary? Are these new monomials yielded in the image of Φ still meaningful? For other Hessenberg functions, the map Ψ no longer maps reliably back to the original filling. For example if $h = (1, 3, 3)$ then $\Phi\left(\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}\right) = x_3$, but $\Psi(x_3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$. Attempts so far to define an inverse map that work for all Hessenberg functions and all shapes μ have been unsuccessful.

However, when we fix the shape $\mu = (n) = \begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline\end{array}$ (equivalently, fix the nilpotent X to have exactly one Jordan block) and let the functions h vary, we get an important family of varieties called the regular nilpotent Hessenberg varieties. In this setting the image of Φ is indeed a meaningful set of monomials $\mathcal{A}_h(\mu)$. We will explore this in detail in Chapter 6. In the interim chapters, we need to build some machinery first. We start with generalizing the Tanisaki ideal in the following chapter.

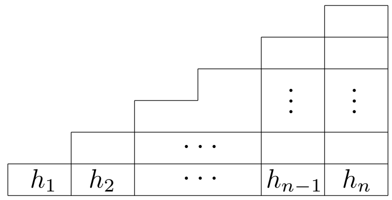
CHAPTER 3
GENERALIZING THE TANISAKI IDEAL

In 1981, De Concini and Procesi [4] claimed the cohomology ring of a Springer variety may be presented as the graded quotient of a polynomial ring. In 1982, Tanisaki [16] vastly simplified the description of the kernel of the quotient map. This kernel has come to be called the Tanisaki ideal. In 2008, Biagioli, Faridi, and Rosas [1] further simplified the description of Tanisaki's ideal. Guided by their technique, we construct a generalized Tanisaki ideal for the fixed shape $\mu = (n)$ and an arbitrary Hessenberg function h. We call this ideal I_h.

In Section 3.1, we construct the ideal I_h for each h. Analogous to Biagioli, Faridi, and Rosas's construction of the Tanisaki ideal, we build a set $\mathfrak{C}_h$ of modified elementary symmetric functions that generate I_h. In Section 3.2, we discuss a natural partial ordering on Hessenberg functions. This ordering gives rise to sequences of Hessenberg functions. In Section 3.3, we observe that for certain sequences of functions a strong nesting property of ideals exists. That is, for any $h > h'$ in a *nice* sequence we have $\mathfrak{C}_h \subset \mathfrak{C}_{h'}$. Generalizing this to arbitrary sequences, we prove in Section 3.4 that if $h > h'$ then $I_h \subset I_{h'}$, but $\mathfrak{C}_h$ is not necessarily inside $\mathfrak{C}_{h'}$. These nesting properties in turn tell us a corresponding surjection on quotients, namely, if $h > h'$ then $R/I_{h'}$ surjects onto R/I_h. Finally in Section 3.5, we exhibit a minimal generating set for I_h.

3.1 Construction of the ideal I_h

Let $\mu = (n)$ and $h = (h_1, \ldots, h_n)$ be a Hessenberg function. Consider the composition $(1, 2, \ldots, n)$ and draw its Ferrers diagram right-justified. We number our columns 1 to n from left to right. Fill the bottom row with the numbers $h_1, h_2, \ldots, h_n$ from left to right as follows:



Finally, list the numbers above each bottom box in descending order as you go up each column. We call this the *h-Ferrers diagram* for h. For example, for Hessenberg function $h = (2, 3, 3, 5, 5, 6)$ the corresponding h-Ferrers diagram is the following:

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & 1 & 2 \\ & & & 2 & 2 & 3 \\ & & 1 & 3 & 3 & 4 \\ & 2 & 2 & 4 & 4 & 5 \\ 2 & 3 & 3 & 5 & 5 & 6 \end{array}.$$

Define the *modified elementary symmetric function* $e_i(j) := e_i(\{x_1, \ldots, x_j\})$ to be the sum of all square-free monomials of degree i in the variable set $\{x_1, \ldots, x_j\}$. For example,

$$e_2(3) = e_2(\{x_1, x_2, x_3\}) = x_1x_2 + x_1x_3 + x_2x_3$$

regardless of $n \geq 3$. If the polynomial ring is fixed to have n variables, then a modified elementary symmetric function of the form $e_i(n)$ is just the standard degree-i elementary symmetric function in all n variables, so we denote such functions with the usual notation e_i.

The Reading Process

For a given Hessenberg function $h = (h_1, \ldots, h_n)$, we describe how to convert an h-Ferrers diagram into a collection $\mathfrak{C}_h$ of generators for an ideal I_h. Let $\mathfrak{C}_h$ be the collection $\{e_{h_i-j}(h_i) \mid 0 \leq j \leq i-1\}_{i=1}^{n}$. Then we say $I_h = \langle \mathfrak{C}_h \rangle$. Since $h_n = n$, the far-right column of the h-Ferrers diagram always contains the numbers 1 to n. Thus every collection $\mathfrak{C}_h$ contains the elementary symmetric functions $e_1, \ldots, e_n$.

Example 3.1.1. Let $h = (3, 3, 3, 4)$. Then the h-Ferrers diagram will look like the following:

$$\begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \cline{3-4} \multicolumn{2}{c|}{} & 1 & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 2 & 2 & 3 \\ \hline 3 & 3 & 3 & 4 \\ \hline \end{array},$$

and hence

$$\mathfrak{C}_h = \{e_1(4), e_2(4), e_3(4), e_4(4), e_1(3), e_2(3), e_3(3)\}\,.$$

The ideal I_h is generated by this collection $\mathfrak{C}_h$ and we write:

$$I_h = \left(\begin{array}{c} e_1, e_2, e_3, e_4, \\ \\ x_1x_2x_3, \\ \\ x_1x_2 + x_1x_3 + x_2x_3, \\ \\ x_1 + x_2 + x_3 \end{array} \right).$$

Observation 3.1.2. *When $h = (1, 2, \ldots, n)$, the ideal I_h coincides with the Tanisaki ideal. At the other extreme when $h = (n, \ldots, n)$, the ideal I_h is generated by the elementary symmetric functions.*

Proof. Recall that the Tanisaki ideal corresponds to the Springer setting when the Hessenberg function is $h = (1, 2, \ldots, n)$. The diagonal of the associated h-Ferrers diagram is all ones, and hence the corresponding functions $e_1(1), e_1(2), \ldots, e_1(n)$ lie in $\mathfrak{C}_h$. Thus $x_1, x_2, \ldots, x_n$ are in the ideal, and we conclude that I_h equals the polynomial ring. When $h = (n, \ldots, n)$, the corresponding h-Ferrers diagram gives exactly n distinct generators of the form $e_i(n)$ for $1 \leq i \leq n$. That is, I_h is generated by the elementary symmetric functions. □

3.2 Partial ordering on Hessenberg functions

The family of ideals I_h exhibit some nice nesting properties. To describe them, we must introduce a partial ordering on Hessenberg functions.

Definition 3.2.1 (Poset on Hessenberg functions)**.** Fix n. A *Hasse diagram on Hessenberg functions* is a directed graph whose vertices are Hessenberg functions.

The top vertex is the function $(n, \ldots, n)$, and the bottom vertex is $(1, 2, \ldots, n)$. There is an edge from $h = (h_1, \ldots, h_n)$ down to $h' = (h'_1, \ldots, h'_n)$ if exactly one entry in h' is one less than its corresponding entry in h. That is, $h'_{i_0} = h_{i_0} - 1$ for some i_0 but $h'_i = h_i$ for all $i \neq i_0$. We define a partial ordering on Hessenberg functions using this Hasse diagram. We say $h > h'$ if there is a path on the Hasse diagram connecting h down to h'. Equivalently, $h > h'$ if $h_i \geq h'_i$ for all i, and $h_{i_0} > h'_{i_0}$ for some i_0.

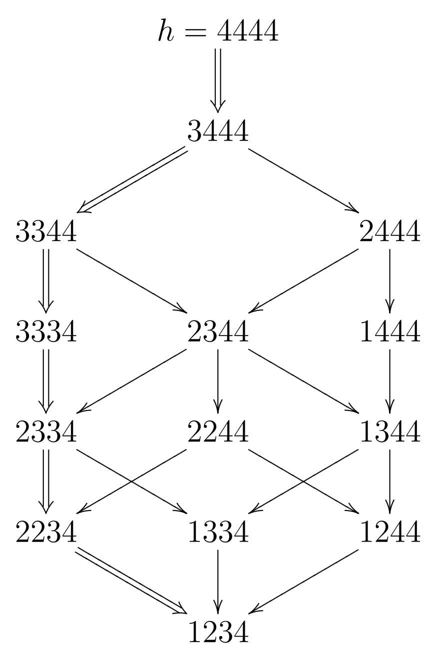


Figure 3.1: Hasse diagram on Hessenberg functions for $n = 4$.

In Figure 3.1, we denote each Hessenberg function $(h_1, \ldots, h_n)$ as the n-tuple $h_1 \cdots h_n$ for brevity.

3.3 Containment of ideals I_h (for nice sequences)

In the first case we inspect, the containment of the ideals I_h is very transparent: there exists a sequence of ideals whose actual generating sets are contained precisely inside the next ideal in the sequence. For this reason, we call these sequences *nice*.

Definition 3.3.1 (Nice sequence of Hessenberg functions)**.** Construct a sequence beginning with the maximal Hessenberg function $(n, \ldots, n)$ in the following manner. For a given tuple $\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_n)$ in this sequence, the tuple $h = (h_1, \ldots, h_n)$ is the neighbor to the right of $\tilde{h}$ if the following two conditions hold:

- $\tilde{h}_j - 1 = h_j$ and $\tilde{h}_i = h_i$ for all $i \neq j$,
- There exists no $k > j$ such that $\tilde{h}_k > \tilde{h}_j$ and $(\tilde{h}_1, \ldots, \tilde{h}_{k-1}, \tilde{h}_k - 1, \tilde{h}_{k+1}, \ldots, \tilde{h}_n)$ is a valid Hessenberg function.

We define a *nice sequence of Hessenberg functions* to be any subsequence of this constructed sequence.

Example 3.3.2. For $n = 4$ we have the following nice sequence. Suppressing the commas, we write:

$$(4444) > (3444) > (3344) > (3334) > (2334) > (2234) > (1234).$$

Theorem 3.3.3. *If $h > h'$ are adjacent neighbors in a nice sequence, then $\mathfrak{C}_h \subset \mathfrak{C}_{h'}$ and hence $I_h \subset I_{h'}$. We conclude for any $h > h'$ in a nice sequence that $I_h \subset I_{h'}$.*

Proof. Let $h > h'$ be neighbors in a nice sequence of Hessenberg functions. Suppose they differ in the j^{th} entry. Then $h = (h_1, \ldots, h_{j-1}, k, h_{j+1}, \ldots, h_n)$ and $h' = (h'_1, \ldots, h'_{j-1}, k-1, h'_{j+1}, \ldots, h'_n)$ where $h_i = h'_i$ for all $i \neq j$. The corresponding $\tilde{h}$-Ferrers diagram and h-Ferrers diagram will differ in only the j^{th} column from the left. It suffices to show that all of the elements $e_k(k), e_{k-1}(k), \ldots, e_{k-(j-1)}(k)$ corresponding to column j in the h-Ferrers diagram are also in $\mathfrak{C}_{h'}$.

The main observation is that there will always be a k to the right of the j^{th} slot in the function h'. We know h' is a Hessenberg function so $h'_j \geq j$. Since $h'_j = k - 1$, we conclude $k - 1 \geq j$. In particular, we must have $h'_k = k$ since h'_k is to the right of h'_j, and all the values $k + 1$ that could be lowered in h have already been lowered.

Observe that column k in the h'-Ferrers diagram is taller than column j in the h-Ferrers diagram. In both diagrams the bottom number is k. Hence the list of elements $e_k(k), e_{k-1}(k), \ldots, e_{k-(j-1)}(k)$ in $\mathfrak{C}_h$ are clearly also in $\mathfrak{C}_{h'}$. All the other elements of $\mathfrak{C}_h$ are also in $\mathfrak{C}_{h'}$ since the diagrams coincide in every other column, thus $I_h \subset I_{h'}$. □

3.4 Containment of ideals I_h (for arbitrary sequences)

Consider the not-as-nice Hessenberg sequence $\tilde{h} = (1, 3, 4, 4) > h = (1, 2, 4, 4)$. The collection of generators $\mathfrak{C}_{\tilde{h}}$ has $e_2(3)$ in it, which is not in the collection $\mathfrak{C}_h$. However, we can write $e_2(3) = x_1x_2 + x_1x_3 + x_2x_3$ as $x_1x_2 + x_3(x_1 + x_2)$ which equals $e_2(2) + x_3 \cdot e_1(2)$. And indeed, $e_2(2)$ and $e_1(2)$ are in $\mathfrak{C}_h$ and hence $e_2(3)$ lies in $\langle \mathfrak{C}_h \rangle$ though it is not one of the generators. This leads us to formulate the following lemma for neighboring terms in an arbitrary sequence. We will use this lemma in the subsequent theorem.

Lemma 3.4.1. *Let $e_i(l)$ be the modified elementary symmetric functions. Then*

$$e_{k-r}(k) = \begin{cases} x_k \cdot e_{k-1}(k-1) & \text{if } r = 0, \\ e_{k-r}(k-1) \quad + \quad x_k \cdot e_{k-r-1}(k-1) & \text{if } 0 < r \leq k-1. \end{cases}$$

In the case $r = k-1$, we define $e_{k-r-1} = e_{k-(k-1)-1} = e_0$ to be 1.

Proof. If $r = 0$, then $e_k(k) = x_1 \cdots x_k = x_k \cdot (x_1 \cdots x_{k-1}) = x_k \cdot e_{k-1}(k-1)$ as desired. On the other hand, suppose $0 < r \leq k-1$. Define $\mathbf{X}_j$ to be $\{x_1, \ldots, x_j\}$ for $j \leq n$. Then

$$\begin{aligned} e_{k-r}(k) &= \sum_{\{i_1 < \cdots < i_{k-r}\} \subseteq \mathbf{X}_k} x_{i_1} x_{i_2} \cdots x_{i_{k-r}} \\ &= \sum_{\{i_1 < \cdots < i_{k-r}\} \subseteq \mathbf{X}_{k-1}} x_{i_1} x_{i_2} \cdots x_{i_{k-r}} \\ &\qquad\qquad + x_k \cdot \left(\sum_{\{i_1 < \cdots < i_{k-r-1}\} \subseteq \mathbf{X}_{k-1}} x_{i_1} x_{i_2} \cdots x_{i_{k-r-1}} \right) \\ &= e_{k-r}(k-1) \quad + \quad x_k \cdot e_{k-r-1}(k-1). \end{aligned}$$

□

Remark 3.4.2. Unlike the nice sequence of Hessenberg functions, there may be several sequences between two endpoints. For instance, between the Hessenberg functions $h = (2, 3, 4, 4)$ and $h' = (1, 2, 3, 4)$ one such path is given by

$$(2, 3, 4, 4) > (1, 3, 4, 4) > (1, 3, 3, 4) > (1, 2, 3, 4).$$

In Figure 3.1, we observe five other distinct paths between these two endpoints.

Theorem 3.4.3. *(Containment of ideals I_h) If $h > h'$ then $I_h \subset I_{h'}$.*

Proof. Suppose $h = (h_1, \ldots, h_n) > h' = (h'_1, \ldots, h'_n)$. Let $h > \cdots > h'$ be any sequence of functions on the Hasse diagram on Hessenberg functions from h down to h'. It suffices to show that for any neighboring Hessenberg functions $\tilde{h} > \tilde{\tilde{h}}$ in this sequence where $h \geq \tilde{h} > \tilde{\tilde{h}} \geq h'$ that $I_{\tilde{h}} \subset I_{\tilde{\tilde{h}}}$ and hence $I_h \subset I_{h'}$. Without loss of generality, we assume h and h' are neighboring Hessenberg functions.

Let h and h' be the neighboring terms $h = (h_1, \ldots, h_{j-1}, k, h_{j+1}, \ldots, h_n)$ and $h' = (h_1, \ldots, h_{j-1}, k-1, h_{j+1}, \ldots, h_n)$. To show $I_h \subset I_{h'}$, it suffices to show that all the generators $\mathfrak{C}_h$ of I_h are in $I_{h'}$. On the level of diagrams, we notice that the h-Ferrers diagram and h'-Ferrers diagram are identical except in column j. Thus it suffices to show that the j distinct generators $e_{k-r}(k) \in \mathfrak{C}_h$ for $0 \leq r \leq j-1$ coming from column j of the h-Ferrers diagram also lie in $I_{h'}$. Lemma 3.4.1 shows that each $e_{k-r}(k)$ is a $\mathbb{Z}[x_1, \ldots, x_n]$-linear combinations of the generators in the set $\{e_{k-r}(k-1) \mid r = 0, \ldots, j-1\}$ coming from column j in the h'-Ferrers diagram. Hence all $e_{k-r}(k)$ are in $I_{h'}$ as desired, and the proof is done. $\square$

Corollary 3.4.4. *If $h > h'$ and $R = \mathbb{Z}[x_1, \ldots, x_n]$, then the quotient $R/I_{h'}$ surjects onto the quotient R/I_h.*

Proof. Since I_h is contained in $I_{h'}$, the claim follows. $\square$

3.5 Minimal generating set for I_h

Fix a Hessenberg function h. The ideal I_h is generated by at most $\frac{n(n+1)}{2}$ distinct generators arising from the h-Ferrers diagram. The generating set $\mathfrak{C}_h$ for I_h, like the original Tanisaki ideal, may be highly nonminimal and may have redundant generators. In this section we give a minimal generating set of only n generators!

This minimal generating set arises from the boxes on the anti-diagonal of the h-Ferrers diagram. As such, we call the ideal they generate the anti-diagonal ideal and denote it by I_h^{AD}. We will prove that I_h^{AD} and I_h coincide.

Definition 3.5.1 (Anti-diagonal ideal)**.** Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. The *anti-diagonal ideal* in I_h is denoted I_h^{AD} and is generated by the functions arising from the anti-diagonal fillings of the h-Ferrers diagram. That is,

$$I_h^{AD} = \left\langle e_{h_1}(h_1), e_{h_2-1}(h_2), \ldots, e_{h_i-(i-1)}(h_i), \ldots, e_{h_n-(n-1)}(h_n) \right\rangle \subseteq I_h.$$

Lemma 3.5.2. *Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. Each generator $e_{h_i}(h_i)$ in $\mathfrak{C}_h$ is a multiple of $e_{h_1}(h_1)$. We conclude that all generators $e_{h_i}(h_i)$ for $1 \leq i \leq n$ lie in I_h^{AD}.*

Proof. Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. Consider the generator $e_{h_i}(h_i)$ in $\mathfrak{C}_h$. If $i = 1$, then there is nothing to show. If $i > 1$, then

$$\begin{aligned} e_{h_i}(h_i) &= x_1 \cdots x_{h_i} \\ &= (x_1 x_2 \cdots x_{h_1}) \cdot (x_{h_1+1} \cdots x_{h_i}) \\ &= (x_{h_1+1} \cdots x_{h_i}) \cdot e_{h_1}(h_1). \end{aligned}$$

Hence $e_{h_i}(h_i)$ is a multiple of $e_{h_1}(h_1)$, and we conclude $e_{h_i}(h_i) \in I_h^{AD}$. □

The proof of Theorem 3.5.6 relies a great deal on two useful lemmas. We encountered the first one, Lemma 3.4.1, in the previous section. That lemma allowed us to take a modified elementary symmetric function in k variables and reduce it to a linear combination of two modified elementary symmetric functions in $k-1$ variables. The second useful lemma we present below.

Lemma 3.5.3. *Suppose $0 < d < r$. The modified elementary symmetric function $e_d(r)$ is a linear combination of modified elementary symmetric functions in the variables $x_1, \ldots, x_{r-j}$:*

$$e_d(r) = \sum_{t=0}^{j} e_t(x_{r-j+1}, x_{r-j+2}, \ldots, x_r) \cdot e_{d-t}(x_1, x_2, \ldots, x_{r-j}). \tag{3.1}$$

Proof. By definition, the function $e_d(r)$ is the sum of $\binom{r}{d}$ distinct summands of the form $x_{i_1} \cdots x_{i_d}$ where $1 \leq i_1 < \cdots < i_d \leq r$. The product

$$e_t(x_{r-j+1}, x_{r-j+2}, \ldots, x_r) \cdot e_{d-t}(x_1, x_2, \ldots, x_{r-j}) \tag{3.2}$$

has distinct summands each of the form $x_{i_1} \cdots x_{i_t} x_{i_{t+1}} \cdots x_{i_d}$ where the subscripts adhere to the following conditions:

$$r - j + 1 \leq i_1 < \cdots < i_t \leq r \quad \text{and} \quad 1 \leq i_{t+1} < \cdots < i_d \leq r - j.$$

Thus for each $t \in \{0, \ldots, j\}$, there are $\binom{j}{t} \cdot \binom{r-j}{d-t}$ possible summands in the expansion of the product in Equation (3.2). Hence as t goes from 0 to j, the expansion of the right hand side of Equation (3.1) yields exactly $\sum_{t=0}^{j} \binom{j}{t} \cdot \binom{r-j}{d-t}$ pairwise distinct summands. Moreover, the right hand side of Equation (3.1) recovers all possible $\binom{r}{d}$ distinct summands in the function $e_d(r)$. □

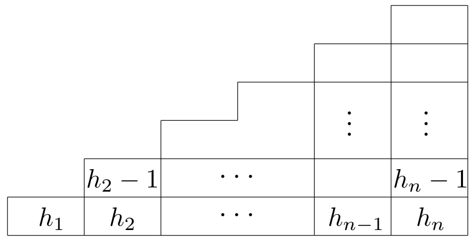
Remark 3.5.4. As t varies, not every product as given in Equation (3.2) is nonzero. The first multiplicand always appears since its degree is $t \leq j$, and there are exactly j variables. The second multiplicand, however, is zero unless $d - t \leq r - j$. Hence the sum in Equation (3.1) has all nonzero terms whenever $t \geq max\{0, d - r + j\}$. Also, we may take $0 \leq j \leq min\{r, d\}$. For $j > min\{r, d\}$, all summands are zero.

Remark 3.5.5. Lemma 3.5.3 in effect gives another proof of the well-known combinatorial identity:

$$\sum_{t=0}^{j} \binom{j}{t} \cdot \binom{r-j}{d-t} = \binom{r}{d}.$$

Theorem 3.5.6 (Minimal generating set for I_h)**.** *Let h be a Hessenberg function. Then $I_h \subseteq I_h^{AD}$. Hence I_h has a minimal generating set given by the generators of I_h^{AD}.*

Proof. Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. Consider the corresponding h-Ferrers diagram:



It suffices to show that each $e_{h_i - j}(h_i)$ for $i \in \{2, \ldots, n\}$ while $j \in \{0, \ldots, i-2\}$ lies in I_h^{AD}. These are the generators of I_h corresponding to boxes off the anti-diagonal.

We induct on the columns of the h-Ferrers diagram moving left to right. The base case holds since $e_{h_2}(h_2)$ lies in I_h^{AD} by Lemma 3.5.2, and generator $e_{h_2-1}(h_2)$ is by definition in I_h^{AD}. Assume for some column i that each $e_{h_i-j}(h_i)$ lies in I_h^{AD} for all $j \in \{0, \ldots, i-2\}$. It suffices to show for column $i+1$ that each $e_{h_{i+1}-j}(h_{i+1})$ lies in I_h^{AD} for all $j \in \{0, \ldots, i-1\}$. If $h_i = h_{i+1}$, then the result holds trivially. Assume $h_i < h_{i+1}$ so that the columns i and $i+1$ look like the following schematic:

	$h_{i+1} - i$
$h_i - (i-1)$	$h_{i+1} - (i-1)$
$\vdots$	$\vdots$
$h_i - s$	$h_{i+1} - s$
$h_i - (s-1)$	$\vdots$
$\vdots$	
$h_i - 1$	$h_{i+1} - 1$
h_i	h_{i+1}

Consider $h_{i+1} - s$. If $s = 0$, then $e_{h_{i+1}-s}(h_{i+1})$ lies in I_h^{AD} by Lemma 3.5.2. Assume $s \in \{1, \ldots, i-1\}$. By Lemma 3.5.3, the function $e_{h_{i+1}-s}(h_{i+1})$ equals

$$\sum_{t=0}^{j} e_t(x_{h_{i+1}-j+1}, x_{h_{i+1}-j+2}, \ldots, x_{h_{i+1}}) \cdot e_{h_{i+1}-s-t}(x_1, x_2, \ldots, x_{h_{i+1}-j}).$$

Choose j so that $h_{i+1} - j = h_i$. Then

$$e_{h_{i+1}-s}(h_{i+1}) = \sum_{t=0}^{h_{i+1}-h_i} e_t(x_{h_i+1}, x_{h_i+2}, \ldots, x_{h_{i+1}}) \cdot e_{h_{i+1}-s-t}(x_1, x_2, \ldots, x_{h_i}).$$

We are left to show that the degrees $h_{i+1} - s - t$ of the second multiplicand in each summand of the equation above make sense. That is, we need to verify that these are degrees coming from the values in the boxes of column i if the summand is nonzero. Since $t \le h_{i+1} - h_i$, we have

$$h_{i+1} - s - (h_{i+1} - h_i) \le h_{i+1} - s - t \quad \text{implies} \quad h_i - s \le h_{i+1} - s - t,$$

establishing the lower bound. By Remark 3.5.4, the second multiplicand is zero unless $h_{i+1} - s - t \le h_i$. Hence we may assume $h_{i+1} - s - h_i \le t$, and

$$h_{i+1} - s - t \le h_{i+1} - s - (h_{i+1} - s - h_i) \quad \text{implies} \quad h_{i+1} - s - t \le h_i,$$

establishing the upper bound. Hence the degrees $h_{i+1} - s - t$ are valid degrees coming from column i. By the induction hypothesis, each $e_{h_{i+1}-s-t}(h_i)$ lies in I_h^{AD} and hence $e_{h_{i+1}-s-t}(h_{i+1})$ also lies in I_h^{AD}. Thus $I_h \subseteq I_h^{AD}$. $\square$

Remark 3.5.7 (Minimal generating set for I_h is not unique)**.** Returning to Example 3.1.1, we find the anti-diagonal generators e_1, $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$, and $x_1x_2x_3$ are a minimal generating set for I_h when $h = (3, 3, 3, 4)$. However the last two generators in that list, namely $e_2(3)$ and $e_3(3)$, could have been replaced with the generators e_2 and e_3. Thus the following three ideals all coincide:

$$I = \left(\begin{array}{c} e_1, e_2, e_3, e_4, \\ x_1x_2x_3, \\ x_1x_2 + x_1x_3 + x_2x_3, \\ x_1 + x_2 + x_3 \end{array} \right) = \left(\begin{array}{c} e_1, \\ x_1x_2x_3, \\ x_1x_2 + x_1x_3 + x_2x_3, \\ x_1 + x_2 + x_3 \end{array} \right) = \left(\begin{array}{c} e_1, e_2, e_3, \\ x_1 + x_2 + x_3 \end{array} \right).$$

In the figures below, we circle the entries which give generators corresponding to a minimal generating set for I_h. Of course, the set of generators in the left figure are particularly nice. The generators from the right figure, however, arose as we explored an alternate ideal J_h. We introduce this alternate ideal in the next chapter.

(a) I_h^{AD}. (b) Alternate.

Figure 3.2: Two reduced generating sets in the same h-Ferrers diagram for $h = (3334)$.

CHAPTER 4
AN ALTERNATE IDEAL AND ITS USEFULNESS

In the last chapter we generalized the Tanisaki ideal to a family of ideals I_h. In this chapter, we construct another family of ideals J_h. Fix a Hessenberg function h. Whereas the ideal I_h was built out of modified elementary symmetric functions, we construct the ideal J_h using modified *complete* symmetric functions. In Section 4.1, we give a one-to-one correspondence between Hessenberg functions and related objects, which we call degree tuples. In Section 4.2, we use the degree tuple corresponding to h to construct the ideal J_h. Using a natural partial ordering on degree tuples, in Sections 4.3 and 4.4 we prove the nesting property that $J_h \subset J_{h'}$ if $h > h'$. Analogous to the setting of the previous chapter, a stronger result holds for nice sequences. Lastly, the ideal J_h has several useful properties which we explore in this and the next chapter:

1. For each Hessenberg function $h = (h_1, \ldots, h_n)$, the corresponding ideal J_h has exactly n generators. (Definition 4.2.2)

2. The generators of J_h form a Gröbner basis. (Theorem 5.3.2)

3. Let $R = \mathbb{Z}[x_1, \ldots, x_n]$. Then R/J_h has finite rank, and its basis can be read off easily from inspecting the degree tuple corresponding to h. (Theorem 5.4.3 and Corollary 5.4.5)

4.1 Hessenberg functions $\longleftrightarrow$ Degree tuples

From a Hessenberg function h, we construct an ideal J_h in the next section. This ideal is built using both the Hessenberg function and an object called a degree tuple. In this section we reveal the one-to-one correspondence between these two objects. We end with a few corollaries revealing very interesting combinatorial relationships between these two objects involving Catalan numbers, Dyck paths, and maximal chains on the Hasse diagram on Hessenberg functions. First we recall the structure of Hessenberg functions defined in Section 1.4. We include it to contrast it with the structure of degree tuples which we introduce subsequently.

Structure of Hessenberg functions

An n-tuple $h = (h_1, \ldots, h_n)$ is a Hessenberg function if it satisfies the conditions:

$$(a) \quad i \le h_i \le n, \quad i \in \{1, 2, \ldots, n\}$$

$$(b) \quad h_i \le h_{i+1}, \quad i \in \{1, 2, \ldots, n-1\}.$$

Structure of degree tuples

An n-tuple $\beta = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$ is a *degree tuple* if it satisfies the conditions:

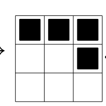
$$(a') \quad 1 \le \beta_i \le i, \quad i \in \{1, 2, \ldots, n\}$$

$$(b') \quad \beta_i - \beta_{i-1} \le 1, \quad i \in \{2, \ldots, n\}.$$

Remark 4.1.1. We call it a degree tuple because its entries are the degrees of the generating symmetric functions of our ideal J_h which we construct in the next section. The convention of listing β_i in descending subscript order in the degree tuple highlights that the i^{th} entry of the tuple corresponds to a symmetric function in exactly i variables.

One-to-one correspondences

Fix a positive integer n. For the purposes of this section, by a *partition* $(\lambda_1, \ldots, \lambda_n)$ we mean one whose Ferrers diagram fits in an n-by-n square and the λ_i satisfy $\lambda_1 \ge \ldots \ge \lambda_n \ge 0$. We draw our Ferrers diagram flush right and top. For example if $n = 3$, then

$$\lambda = (3, 1, 0) \longleftrightarrow$$ .

Definition 4.1.2 (Reverse tuple)**.** Let $t = (t_1, \ldots, t_n)$ be an n-tuple. The *reverse of* t is the n-tuple $(t_n, t_{n-1}, \ldots, t_1)$ and is denoted t^{rev}.

Definition 4.1.3 (Staircase and ample partitions)**.** Let $\rho = (n, n-1, \ldots, 1)$ denote the *staircase partition.* We call $\lambda = (\lambda_1, \ldots, \lambda_n)$ an *ample partition* when $\rho \subseteq \lambda$. In other words, the partition λ is ample whenever $\lambda_i \geq n - i + 1$ for each i.

Definition 4.1.4 (Dyck path)**.** Consider an n-by-n square with the origin at the lower left. A *Dyck path* is any path in the square from $(0, n)$ to $(n, 0)$ that lies strictly below the antidiagonal $y = -x + n$ and has only increments of $(0, -1)$ and $(1, 0)$. A traditional Dyck path runs from $(0, 0)$ to (n, n), but we rotate this standard picture by $90°$ clockwise.

Definition 4.1.5 (Conjugate of a partition)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. The *conjugate* λ' of λ is given by $\lambda'_i = \#\{k : \lambda_k \geq i\}$. Pictorially, the conjugate λ' is a reflection of λ across the diagonal line since we are drawing our Ferrers diagram flush top and right.

Lemma 4.1.6. *The set of ample partitions is closed under conjugation.*

Proof. This holds trivially since $\rho \subseteq \lambda$ implies $\rho \subseteq \lambda'$. □

Lemma 4.1.7. *The following sets are in one-to-one correspondence:*

(1) The set of ample partitions.

(2) The set of Hessenberg functions.

(3) The set of degree tuples.

(4) The set of Dyck paths.

Proof. We prove that each set (2), (3), and (4) is in bijection with ample partitions. Let $h = (h_1, \ldots, h_n)$ be an n-tuple. Its corresponding reverse tuple is $h^{rev} = (h_n, h_{n-1}, \ldots, h_1)$. The tuple h satisfies Hessenberg function structure rule (b) when h^{rev} is a partition. Furthermore, structure rule (a) is satisfied precisely when this partition h^{rev} is ample.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an n-tuple. Define the n-tuple $\beta = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$ by $\beta_i = \lambda_i - \rho_i + 1 = \lambda_i + i - n$. Observe that β satisfies degree tuple structure rule (b') only when λ is a partition. Furthermore, structure rule (a') is satisfied precisely when this partition λ is ample.

Finally, the *boundary path* of a partition is the path between the partition and its complement in the n-by-n square. A partition is determined by its boundary path. It is clear that a boundary path is a Dyck path precisely when its corresponding partition is ample. $\square$

Using the Lemmas 4.1.6 and 4.1.7 we can define a bijective map from Hessenberg functions to degree tuples. Define a composition of bijective maps from a Hessenberg function h to its corresponding ample partition $\lambda = h^{rev}$, then to conjugate λ' of this ample partition, and finally to degree tuple $\lambda' - \rho + 1$ as follows:

$$F : h \longmapsto \lambda = h^{rev} \longmapsto \lambda' \longmapsto \lambda' - \rho + 1 \longmapsto (\lambda' - \rho + 1)^{rev}$$

where ρ is the staircase partition $(n, n-1, \ldots, 1)$, and we define 1 to be the partition $(1, \ldots, 1)$. The last composition $\lambda' - \rho + 1 \mapsto (\lambda' - \rho + 1)^{rev}$ ensures that the degree tuple follows our convention of descending subscripts of the β_i as noted in Remark 4.1.1.

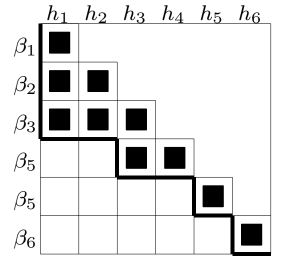
Theorem 4.1.8. *The map F is a bijection between Hessenberg functions and degree tuples.*

Proof. This holds since all the maps involved in F are bijective. $\square$

Pictorially, the maps defined in F yield a Ferrers diagram used to quickly read off entries of either a Hessenberg function or its corresponding degree tuple. This diagram will come into play again in Chapter 6.

Definition 4.1.9 (Hessenberg diagram)**.** Let h be a Hessenberg function. The composition $h \mapsto (h^{rev})'$ gives a partition in which the i^{th} column from the left has length h_i for all i. Shade these boxes in the Ferrers diagram. Furthermore, the map $(h^{rev})' \mapsto (h^{rev})' - \rho + 1$ effectively removes the staircase above the antidiagonal – namely, the partition $(n-1, n-2, \ldots, 1, 0)$. The diagram that remains is called a *Hessenberg diagram.*

Example 4.1.10. Let $h = (3, 3, 4, 4, 5, 6)$. Then we have the following Hessenberg diagram:



In this diagram, we highlighted the corresponding Dyck path in bold. Visually, we see the value of h_i is $i-1$ plus the number of shaded boxes in column i. Furthermore, β_i is the number of shaded boxes in row i. In fact, we see β_i equals i minus the number of columns left of column i whose shaded boxes do not reach the i^{th} row – namely, the value i minus the cardinality of the set $\{h_k | h_k < i\}$. This leads to the following remark.

Remark 4.1.11 (Simple formula to compute β from h)**.** The map F yields a very simple formula which we use often to calculate the degree tuple for the ideal J_h. Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. Define $\beta = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$ where $\beta_i = i - \#\{h_k | h_k < i\}$. Then β is the corresponding degree tuple for h.

There are some other interesting combinatorial corollaries that arise from Lemma 4.1.7. We mention these to show how our work fits into the existing literature.

Corollary 4.1.12. *Since the number of possible Hessenberg functions equals the number of possible Dyck paths from $(0, n)$ to $(n, 0)$ in an n-by-n box, it follows that*

$$\#\{\textit{Hessenberg functions}\} = \#\{\textit{Degree tuples}\} = \textit{Catalan}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Corollary 4.1.13. *The number of maximal chains on the Hasse diagram on Hessenberg functions (respectively, degree tuples) equals the number of maximal Dyck path chains which is the following number:*

$$\frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-1}}.$$

Richard Stanley [15] has collected a wealth of combinatorial interpretations of Catalan numbers – one of which counts Dyck paths. Another could be added to his list – namely, one that counts the number of Hessenberg functions or degree tuples. Regarding Corollary 4.1.13, we can view the maximal Dyck chains as all possible paths from the top down to the bottom vertices of the Hasse diagram (as in Figure 3.1).

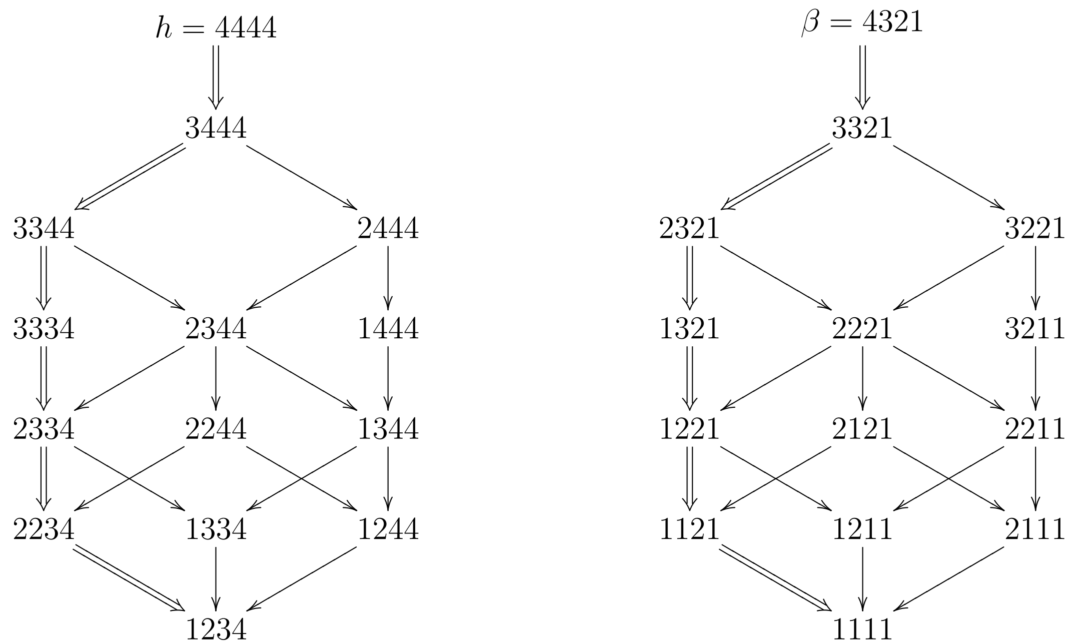
Lastly, to show the containment properties in the family of ideals J_h, we rely heavily on degree tuples. It will be helpful to define the Hasse diagram on degree tuples. Although this definition is very similar to its Hessenberg function analog, for completeness we state it below.

Definition 4.1.14 (Poset on degree tuples)**.** Fix n. A *Hasse diagram on degree tuples* is a directed graph whose vertices are degree tuples. The top vertex is the degree tuple $(n, n-1, \ldots, n)$, and the bottom tuple is $(1, \ldots, 1)$. There is an edge from $\beta = (\beta_n, \ldots, \beta_1)$ down to $\beta' = (\beta'_n, \ldots, \beta'_1)$ if exactly one entry in β' is one less than its corresponding entry in β. That is, $\beta'_{i_0} = \beta_{i_0} - 1$ for some i_0 but $\beta'_i = \beta_i$ for all $i \neq i_0$. We define a partial ordering on degree tuples using this Hasse diagram. We say $\beta > \beta'$ if there is a path on the Hasse diagram connecting β down to β'. Equivalently, $\beta > \beta'$ if $\beta_i \geq \beta'_i$ for all i, and $\beta_{i_0} > \beta'_{i_0}$ for some i_0.

Remark 4.1.15. The right diagram in Figure 4.1 gives an example of a Hasse diagram on degree tuples. The diagram shows there are exactly 14 nodes (which is the 4^{th} Catalan number as expected by Corollary 4.1.12). We compute the number of maximal chains of degree tuples, namely chains from the top vertex down to the bottom vertex, using Corollary 4.1.13:

$$\frac{\binom{4}{2}!}{\prod_{i=1}^{4-1}(2i-1)^{4-1}} \;=\; 16.$$

Similarly, there are 16 maximal chains of Hessenberg functions.



(a) Hessenberg functions. (b) Degree tuples.

Figure 4.1: Hasse diagrams on Hessenberg functions and degree tuples for $n = 4$.

4.2 Construction of the ideal J_h

Recall that the generators of the ideals I_h are modified *elementary* symmetric functions. The generators of the ideals J_h will also be symmetric functions, but they are modified *complete* symmetric functions in this new setting.

Define the *modified complete symmetric function* $\tilde{e}_i(x_j, x_{j+1}, \ldots, x_n)$ to be the sum of all monomials of degree i in the variable set $\{x_j, x_{j+1}, \ldots, x_n\}$ for some $j \leq n$. We sometimes adopt the conventions of writing $\tilde{e}_i(x_j, x_{j+1}, \ldots, x_n)$ as $\tilde{e}_i(x_{j,j+1,\ldots,n})$, or even $\tilde{e}_i(j)$ for the most brevity. For example if $n = 4$, then

$$\tilde{e}_2(3) = \tilde{e}_2(x_{34}) = \tilde{e}_2(x_3, x_4) = x_3^2 + x_3x_4 + x_4^2.$$

Remark 4.2.1 (Careful!)**.** The shorthand notation $\tilde{e}_i(j)$ will be used sparingly since it bears a close resemblance to the notation of the modified elementary symmetric function $e_i(j)$ whose variable set is $\{x_1, \ldots, x_j\}$. Furthermore, unless the number of variables in the ring R is clearly stated, the symbol $\tilde{e}_i(j)$ can be ambiguous.

Definition 4.2.2 (The ideal J_h)**.** Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. Let $\beta = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$ be its corresponding degree tuple, so it follows that $\beta_i = i - \#\{h_k | h_k < i\}$. We define the ideal J_h as follows:

$$J_h := \langle \tilde{e}_{\beta_n}(x_n), \tilde{e}_{\beta_{n-1}}(x_{n-1}, x_n), \ldots, \tilde{e}_{\beta_1}(x_1, \ldots, x_n) \rangle.$$

Example 4.2.3. Consider the Hessenberg function $h = (3, 3, 3, 4)$. We calculate the degree tuple to be $\beta = (1, 3, 2, 1)$, yielding the ideal

$$J_h = \langle \tilde{e}_1(x_4), \tilde{e}_3(x_3, x_4), \tilde{e}_2(x_2, x_3, x_4), \tilde{e}_1(x_1, x_2, x_3, x_4) \rangle.$$

Alternately, we write

$$J_h = \left(\begin{array}{c} x_4, \\ x_3^3 + x_3^2 x_4 + x_3 x_4^2 + x_4^3, \\ x_2^2 + x_2 x_3 + x_2 x_4 + x_3^2 + x_3 x_4 + x_4^2, \\ x_1 + x_2 + x_3 + x_4 \end{array} \right).$$

Remark 4.2.4 (A look ahead to Gröbner bases)**.** By observation, we can quickly simplify the set of generators above. For example since x_4 is in J_h, we can write the generators as x_4, x_3^3, $x_2^2 + x_2 x_3 + x_3^2$, and $x_1 + x_2 + x_3$. In fact, this is the unique reduced Gröbner basis for J_h! Gröbner bases are useful for ideal equality questions. For instance, we know that the ideal I_h from Example 3.1.1 equals J_h because they both have the same unique reduced Gröbner bases.

The lemma below is very straight forward and will play a similar role as Lemma 3.4.1 played in the setting of I_h. We use this lemma extensively in the next two sections to prove a nesting properties of the ideals J_h.

Lemma 4.2.5. *Fix $\{a_i\}_{i=1}^m \subseteq \{1, \ldots, n\}$ such that $a_1 < a_2 < \ldots < a_m$. Then*

$$\tilde{e}_r(x_{a_1,\ldots,a_m}) = x_{a_1} \cdot \tilde{e}_{r-1}(x_{a_1,\ldots,a_m}) + \tilde{e}_r(x_{a_2,\ldots,a_m}).$$

Proof. This proof follows simply by breaking up the left hand side into the set of monomials that contain the variable x_{a_1} and the complement of that set. We observe:

$$\sum_{b_1+\cdots+b_m=r} x_{a_1}^{b_1} x_{a_2}^{b_2} \cdots x_{a_m}^{b_m} = x_{a_1} \cdot \sum_{\substack{b_1+\cdots+b_m=r \\ b_1\neq 0}} x_{a_1}^{b_1-1} x_{a_2}^{b_2} \cdots x_{a_m}^{b_m} + \sum_{b_2+\cdots+b_m=r} x_{a_2}^{b_2} x_{a_3}^{b_3} \cdots x_{a_m}^{b_m}.$$

The far left term is $\tilde{e}_r(x_{a_1,\ldots,a_m})$ by definition. Furthermore, the two summands on the right give $x_{a_1} \cdot \tilde{e}_{r-1}(x_{a_1,\ldots,a_m}) + \tilde{e}_r(x_{a_2,\ldots,a_m})$ by definition, and the lemma is proven. $\square$

4.3 Containment of ideals J_h (for nice sequences)

In this section we prove a nesting property in the family of ideals J_h. Specifically we show that $J_h \subset J_{h'}$ whenever $h > h'$. In the setting of the ideals I_h, it was useful to look at sequences of Hessenberg functions. We proved in Section 4.1 that there is a one-to-one correspondence between Hessenberg functions and degree tuples, so we often refer to J_h as J_β, and we use sequences of degree tuples corresponding to sequence of Hessenberg functions.

We begin by giving a motivating example of the essence of the proof of the main theorem of this section. The following example employs an iterative use of Lemma 4.2.5.

Example 4.3.1. Fix n=5. Consider the neighbors $\beta > \beta'$ in a nice sequence of degree tuples where $\beta = (1, 1, 3, 2, 1)$ and $\beta' = (1, 1, 2, 2, 1)$. These yield the ideals

$$J_\beta = \langle \tilde{e}_1(x_5), \tilde{e}_1(x_{45}), \tilde{e}_3(x_{345}), \tilde{e}_2(x_{2345}), \tilde{e}_1(x_{12345}) \rangle \text{ and}$$

$$J_{\beta'} = \langle \tilde{e}_1(x_5), \tilde{e}_1(x_{45}), \tilde{e}_2(x_{345}), \tilde{e}_2(x_{2345}), \tilde{e}_1(x_{12345}) \rangle\,,$$

respectively. To show $J_\beta \subset J_{\beta'}$, it suffices to show the generator $\tilde{e}_3(x_{345})$ lies in $J_{\beta'}$. Using Lemma 4.2.5 repeatedly, wherein we denote its use with brackets [...], we easily compute:

$$\begin{aligned}
\tilde{e}_3(x_{345}) &= [x_3\tilde{e}_2(x_{345}) + \tilde{e}_3(x_{45})] \\
&= x_3\tilde{e}_2(x_{345}) + [x_4\tilde{e}_2(x_{45}) + \tilde{e}_3(x_5)] \\
&= x_3\tilde{e}_2(x_{345}) + x_4 \cdot [x_4\tilde{e}_1(x_{45}) + \tilde{e}_2(x_5)] + \tilde{e}_3(x_5) \\
&= x_3\tilde{e}_2(x_{345}) + x_4^2\tilde{e}_1(x_{45}) + x_4\tilde{e}_2(x_5) + \tilde{e}_3(x_5) \\
&= x_3\tilde{e}_2(x_{345}) + x_4^2\tilde{e}_1(x_{45}) + x_4x_5\tilde{e}_1(x_5) + x_5^2\tilde{e}_1(x_5) \\
&= x_3\tilde{e}_2(x_{345}) + x_4^2\tilde{e}_1(x_{45}) + (x_4x_5 + x_5^2)\tilde{e}_1(x_5).
\end{aligned}$$

Since $\tilde{e}_2(x_{345})$, $\tilde{e}_1(x_{45})$, $\tilde{e}_1(x_5)$ are generators in $J_{\beta'}$, we see $\tilde{e}_3(x_{345}) \in J_\beta$ also sits in $J_{\beta'}$. The essense of the proof of the main theorem is that if $\beta > \beta'$ are neighbors in a nice sequence, then we can write the symmetric function corresponding to the entry in β that was decreased as a linear combination of the generating functions of $J_{\beta'}$.

The definition of a nice sequence of Hessenberg functions can be found in Definition 3.3.1. We define a *nice sequence of degree tuples* to be any sequence of degree tuples that correspond to a nice sequence of Hessenberg functions. For example for $n = 4$, the maximal nice sequence of degree tuples is

$$(4321) > (3321) > (2321) > (1321) > (1221) > (1121) > (1111).$$

Theorem 4.3.2. *If $\beta > \beta'$ are adjacent neighbors in a nice sequence of degree tuples, then $J_\beta \subset J_{\beta'}$. We conclude if $\beta > \beta'$ is linked by a nice sequence, then $J_\beta \subset J_{\beta'}$.*

Proof. An arbitrary tuple in a nice sequence of degree tuples has the form

$$\beta := (1, \ldots, 1, r, k, k-1, \ldots, 2, 1)$$

where $2 \leq r \leq k + 1$. The neighboring lower tuple has the form

$$\beta' := (1, \ldots, 1, r-1, k, k-1, \ldots, 2, 1).$$

Since all entries are the same *except* in the lowered slot, it suffices to show that the corresponding generator $\tilde{e}_r(x_{k+1,k+2,\ldots,n}) \in J_\beta$ can be written as a linear combination of the generators of the ideal $J_{\beta'}$. By Lemma 4.2.5, we have

$$\tilde{e}_r(x_{k+1,k+2,\ldots,n}) = x_{k+1} \cdot \tilde{e}_{r-1}(x_{k+1,k+2,\ldots,n}) + \tilde{e}_r(x_{k+2,k+3,\ldots,n}).$$

The first summand lies in $J_{\beta'}$ since $\tilde{e}_{r-1}(x_{k+1,k+2,\ldots,n}) \in J_{\beta'}$. So it suffices to show the second summand is also in $J_{\beta'}$. We assert that repeated use of Lemma 4.2.5 on the second summand and its decomposed pieces can only yield modified complete symmetric functions with degree strictly less than r and/or a smaller variable set $\{x_j, x_{j+1}, \ldots, x_n\}$ where $j \geq k+2$. Applying the lemma once to this second summand, we get

$$\tilde{e}_r(x_{k+2,k+3,\ldots,n}) = x_{k+2} \cdot \tilde{e}_{r-1}(x_{k+2,k+3,\ldots,n}) + \tilde{e}_r(x_{k+3,k+4,\ldots,n}).$$

And applying the lemma to both of these decomposed pieces, we get

$$\begin{aligned}
\tilde{e}_r(x_{k+2,k+3,\ldots,n}) = x_{k+2} &\cdot [x_{k+2} \cdot \tilde{e}_{r-2}(x_{k+2,k+3,\ldots,n}) + \tilde{e}_{r-1}(x_{k+3,k+4,\ldots,n})] \\
&+ [x_{k+3} \cdot \tilde{e}_{r-1}(x_{k+3,k+4,\ldots,n}) + \tilde{e}_r(x_{k+4,k+5,\ldots,n})] \\
= x_{k+2}^2 &\cdot \tilde{e}_{r-2}(x_{k+2,k+3,\ldots,n}) + (x_{k+2} + x_{k+3}) \cdot \tilde{e}_{r-1}(x_{k+3,k+4,\ldots,n}) \\
&+ \tilde{e}_r(x_{k+4,k+5,\ldots,n}).
\end{aligned}$$

We then repeat this process on all three summands until one of the following conditions hold:

- The degree of each modified complete symmetric function is 1.
- The modified symmetric function has only an x_n variable in it.

For each newly created modified symmetric function, if neither condition is passed, then we again apply the process to this new function. This process repeats on each $\tilde{e}_i(x_{j\cdots n})$ until either $i = 1$ or $j = n$. In the former case, we have $\tilde{e}_1(x_{j\cdots n}) \in J_{\beta'}$ since $\beta'_j = 1$ for all $j > k$. And in the latter case $\tilde{e}_i(x_n) \in J_{\beta'}$ since $\tilde{e}_i(x_n) = x_n^{i-1}\tilde{e}_1(x_n)$ and $\beta'_n = 1$. Since this process clearly terminates at these two cases, we see that $\tilde{e}_r(x_{k+2,k+3,\ldots,n})$ lies in $J_{\beta'}$ since

$$\tilde{e}_r(x_{k+2,k+3,\ldots,n}) \in \left\langle \tilde{e}_1(x_{j\cdots n}) \right\rangle_{j=k+2}^{n}.$$

Hence $\tilde{e}_r(x_{k+1,k+2,\ldots,n}) \in J_\beta$ lies in the ideal $J_{\beta'}$ since

$$\tilde{e}_r(x_{k+1,k+2,\ldots,n}) \in \left\langle \tilde{e}_{r-1}(x_{k+1,k+2,\ldots,n}), \tilde{e}_1(x_{j\cdots n}) \right\rangle_{j=k+2}^{n} \subset J_{\beta'}.$$

Thus $J_\beta \subset J_{\beta'}$. □

Corollary 4.3.3. *The elementary symmetric functions lie in all ideals J_h that arise from a nice sequence.*

Remark 4.3.4. In Observation 3.1.2, we learned the ideal I_h corresponding to the function $h = (n, \ldots, n)$ is generated by the elementary symmetric functions. In Chapter 7, we prove the powerful assertion that the generators of the ideal J_h corresponding to $h = (n, \ldots, n)$ form a Gröbner basis for I_h. A corollary to this is that $I_h = J_h$ when $h = (n, \ldots, n)$.

4.4 Containment of ideals J_h (for arbitrary sequences)

In this section, we generalize the nesting property that $h > h'$ implies $J_h \subset J_{h'}$ to the general setting of arbitrary sequences of degree tuples. This setting is far less transparent than the previous nice sequence setting. The idea of the proof, however, is similar and involves applying Lemma 4.2.5 iteratively.

Definitions of *test-index* and *testable* symmetric functions

We say a modified complete symmetric function $\tilde{e}_r(x_{j\cdots n})$ has *test-index* $r + n - j$. Applying Lemma 4.2.5 to $\tilde{e}_r(x_{j\cdots n})$, we get two new symmetric functions $\tilde{e}_{r-1}(x_{j\cdots n})$ and $\tilde{e}_r(x_{j+1\cdots n})$ of test index $(r-1)+n-j$ and $r+n-(j+1)$ respectively. That is, if we apply Lemma 4.2.5, then the test-index decreases by 1 on each new summand.

We say $\tilde{e}_r(x_{j\cdots n})$ is *testable for* β if $\beta_j \leq r$ and $1 \leq r \leq j$. The "test" is essentially a procedure for testing the membership of $\tilde{e}_r(x_{j\cdots n})$ in the ideal J_β. The outline of the test is as follows. If $\beta_j = r$, then we are done. Otherwise apply Lemma 4.2.5 and test the two new modified complete symmetric functions (ignoring the coefficients) output by the lemma. Iterate this process as necessary on the newly created symmetric functions. For ease in the main proof, we illustrate this procedure in Figure 4.2.

The Omega Algorithm

For shorthand denote $\tilde{e}_i(x_{j\cdots n})$ as simply $\tilde{e}_i(j)$ from now on. The reader is reminded of the warning in Remark 4.2.1 regarding this notation. We denote $\Omega(\tilde{e}_i(j))$ as the procedure to test whether $\tilde{e}_i(j) \in J_\beta$.

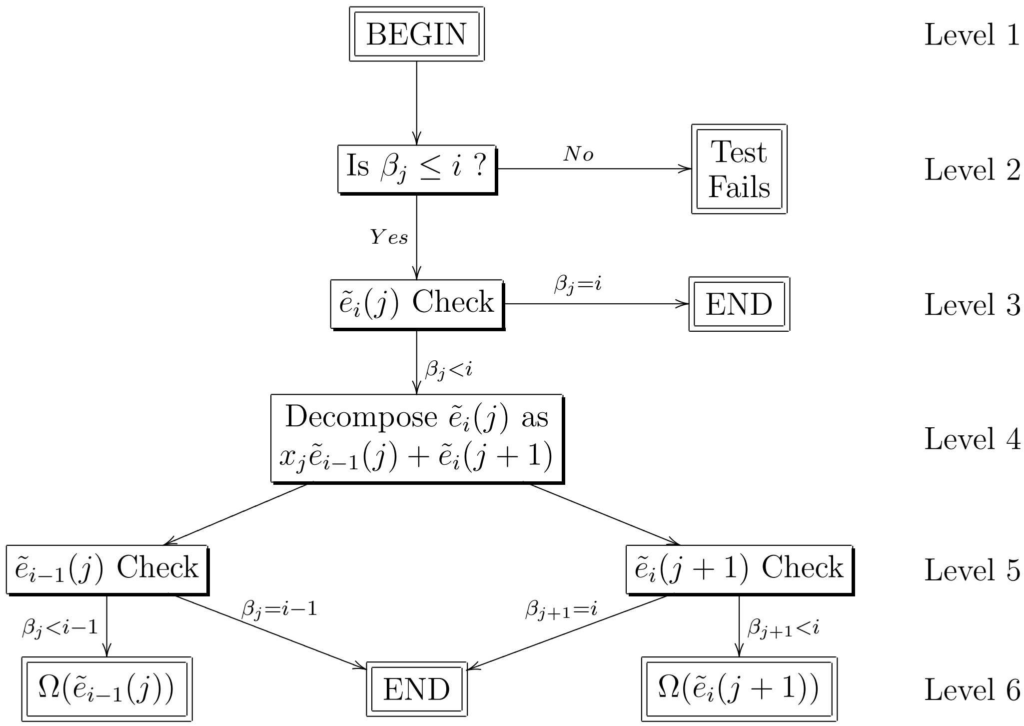


Figure 4.2: Omega algorithm.

To begin the Ω-process on any $\tilde{e}_{i_0}(j_0)$, we must first check that $\beta_{j_0} \leq i_0$. If END is reached at any point, then we say $\tilde{e}_{i_0}(j_0) \in J_\beta$. The bottom Ω boxes mean we reiterate this process for the new given symmetric function. We prove that this process should terminate since each time the test-index is lowered by one.

The following lemma will be useful in the main theorem below. It gives a relationship between the testability of a symmetric function and its test-index.

Lemma 4.4.1. *Let β be a degree tuple. If $\tilde{e}_r(j)$ is testable for β, then it follows that $1 \leq TI(\tilde{e}_r(j)) \leq n$ where $TI(\tilde{e}_r(j))$ is the test-index of $\tilde{e}_r(j)$. In particular, TI*

is always positive.

Proof. Suppose $\tilde{e}_r(j)$ is testable for β. Then by definition $\beta_j \leq r$ and $1 \leq r \leq j$. So TI is greatest when r is maximal (that is, $r = j$) whence we have

$$TI = r + n - j \leq j + n - j = n.$$

On the other hand TI is least when j is maximal (that is, $j = n$) and r is smallest (that is, $r = 1$) whence $TI = 1 + n - n = 1$. Hence $1 \leq TI(\tilde{e}_r(j)) \leq n$ follows. $\square$

Theorem 4.4.2. *If $\beta > \beta'$ then $J_\beta \subset J_{\beta'}$.*

Proof. Since $\beta > \beta'$, there is a path on the Hasse diagram on degree tuples from β to β'. Without loss of generality, assume β and β' differ only in the j^{th} component where $\beta_j = i$ and $\beta'_j = i - 1$. It suffices to show the generator $\tilde{e}_i(j)$ of the ideal J_β also lies in the ideal $J_{\beta'}$. Clearly $\tilde{e}_i(j)$ is testable for β' since $\beta'_j = i - 1 \leq i$ and by definition of a degree tuple, we have $1 \leq i \leq j$. Applying Lemma 4.2.5, we see that the first summand of $\tilde{e}_i(j) = x_j\tilde{e}_{i-1}(j) + \tilde{e}_i(j+1)$ is in $J_{\beta'}$ since $\beta'_j = i - 1$. So END is reached if $\beta_{j+1} = i$. Note $\beta'_j = i - 1$ and the degree tuple condition (b') from Section 4.1 give

$$\beta_{j+1} - \beta'_j \leq 1 \implies \beta_{j+1} - (i-1) \leq 1 \implies \beta_{j+1} \leq i.$$

Thus the second summand $\tilde{e}_i(j+1)$ is testable too and has test-index one less than $\tilde{e}_i(j)$. So we perform $\Omega(\tilde{e}_i(j+1))$ yielding $x_{j+1}\tilde{e}_{i-1}(j+1) + \tilde{e}_i(j+2)$. END is reached if $\beta_{j+1} = i - 1$ and $\beta_{j+2} = i$. Otherwise, one or both of the latter conditions failed. The former failing means $\beta_{j+1} < i - 1$ thus $\beta_{j+1} \leq i - 1$ and the Level 2 check for $\Omega(\tilde{e}_{i-1}(j+1))$ will pass. Similarly, the latter failing means $\beta_{j+2} \leq i$ and hence the Level 2 check for $\Omega(\tilde{e}_i(j+2))$ will pass.

We repeat this process on any $\tilde{e}_{i_0}(j_0)$ that fails the check at Level 5, noting the test-index will lower by one each time Ω is performed. By induction we will show that any $\Omega(\tilde{e}_{i_0}(j_0))$ that is reached at Level 6 will pass the Level 2 test when this process is reiterated. Assume that such an $\Omega(\tilde{e}_{i_0}(j_0))$ has been reached. Then one of two cases has occurred:

Case 1: $\tilde{e}_{i_0}(j_0)$ came from the left summand in the decomposition at Level 4 Then $\tilde{e}_{i_0+1}(j_0)$ was decomposed and by induction passed the Level 2 check. Hence $\beta_{j_0} \leq i_0 + 1$. But since $\Omega(\tilde{e}_{i_0}(j_0))$ is being performed, the $\tilde{e}_{i_0+1}(j_0)$ check at Level 5 failed – that is, $\beta_{j_0} < i_0$.

Case 2: $\tilde{e}_{i_0}(j_0)$ came from the right summand in the decomposition at Level 4 Then $\tilde{e}_{i_0}(j_0 - 1)$ was decomposed and by induction passed the Level 2 check. Hence $\beta_{j_0-1} \leq i_0$. But since $\Omega(\tilde{e}_{i_0}(j_0))$ is being performed, the $\tilde{e}_{i_0+1}(j_0)$ check at Level 5 failed – that is, $\beta_{j_0} < i_0$.

We claim this process will terminate since the test-index starts out as positive and lowers by one each time implying that eventually one of two cases is reached for a given $\tilde{e}_{i_0}(j_0)$:

- $\beta_{j_0} = i_0$ (compare this to the "degree=1" case in the nice sequence).
- $j_0 = n$ (compare this to the second case in the nice sequence).

Observe that TI starts out positive by Lemma 4.4.1. Each time we reach Level 4 of the omega algorithm, two new summands are produced each with TI-value lowered by one. After a sufficient number of iterations on each summand, either one of the final cases above must be reached and we are done. □

Corollary 4.4.3. *The elementary symmetric functions lie in each ideal J_h.*

Proof. See Remark 4.3.4. □

CHAPTER 5
BASIS FOR POLYNOMIAL QUOTIENT RING

This chapter proves key properties of R/J_h starting with the necessary background of commutative algebra. For much of the chapter, we treat the more general case of an ideal I in the polynomial ring $R = k[x_1, \dots, x_n]$. A main goal is to define a basis for the quotient R/I. We take k to be the field of rationals $\mathbb{Q}$. However in the main application in Section 5.4, the integers $\mathbb{Z}$ will suffice.

This chapter gives the algebraic background needed to describe the quotient R/I. The tool we use is a *Gröbner basis* for the ideal I. In Section 5.1, we define Gröbner bases and give a brief background on how these bases are used to solve some fundamental problems in commutative algebra. In Section 5.2, we list without proof all the statements of theorems needed to construct a basis for the quotient R/I. The interested reader can find the proofs of these results in the appendix. In Section 5.3, we prove that the generators of the ideal J_h form a Gröbner basis. Using this fact and results from Section 5.2, we present a basis $\mathcal{B}_h(\mu)$ for R/J_h in Section 5.4. Finally, we end with an elaborative example in Section 5.5.

5.1 Background and definitions

In this section we give the necessary definitions needed to define a Gröbner basis. We also provide a brief sketch of the story behind two fundamental problems in commutative algebra – namely, the problem with the division algorithm in multiple variables and the ideal membership problem. Solutions to both will be needed to construct a basis for R/J_h.

Definition 5.1.1 (Monomial orderings)**.** Let the monomials $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $\mathbf{x}^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ be in R, where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are the exponents of variables $x_i \in \mathbf{x}^\alpha$ and $x_j \in \mathbf{x}^\beta$ respectively. In the *lexicographic* ordering, if the vector difference $\alpha - \beta \in \mathbb{Z}^n$ has a positive leftmost nonzero entry, then we say $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta$. For example, since $\alpha - \beta = (1, -17)$, we say $x_1 >_{lex} x_2^{17}$. Similarly since $\alpha - \beta = (0, 1)$, we say $x_1 x_2^2 >_{lex} x_1 x_2$. Define the total degree of $\mathbf{x}^\alpha$ to be $|\alpha| := \sum \alpha_i$. In the *graded lexicographic* ordering

- if $|\alpha| > |\beta|$, or
- if $|\alpha| = |\beta|$ and $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta$,

then $\mathbf{x}^\alpha >_{grlex} \mathbf{x}^\beta$. For example, $x_2^{17} >_{grlex} x_1$ or $x_1^2 x_2 >_{grlex} x_1 x_2^2$. Monomials of the same total degree in graded lexicographic ordering are settled by ordinary lexicographic ordering.

Definition 5.1.2 (Multi-degree, Leading coefficient (LC), Leading monomial (LM), and Leading term (LT))**.** Let $f \in \sum_\alpha k_\alpha \mathbf{x}^\alpha$ be a nonzero polynomial in R and let $>$ be a monomial ordering. The *multi-degree* of f, denoted $multideg(f)$, is the maximum of the exponents α, where the maximum is taken with respect to the ordering $>$. The *leading coefficient* of f is $LC(f) = k_{multideg(f)} \in k$. The *leading monomial* of f is $LM(f) = \mathbf{x}^{multideg(f)}$. Note that the leading monomial always has coefficient 1 for every nonzero f. The *leading term* of f is $LT(f) = LC(f) \cdot LM(f)$. For example, if $f = 3xy^2 + 4y^4$ then $LT(f) = 3 \cdot xy^2$ under the *lex*-ordering and $LT(f) = 4 \cdot y^4$ under the *grlex*-ordering.

Remark 5.1.3. The main objects of our interest will be the generators of the ideal J_h. These generators are homogeneous symmetric functions. Hence, the leading monomial is the same whether we use *lex* or *grlex* orderings.

Division Algorithm in R

Generalizing the division algorithm to a multivariate setting is a classical result whose proof we do not reproduce here but can be found in [3, pg.64]. Fix a monomial ordering $>$ on $\mathbb{Z}^n_{\geqslant 0}$, and let $F = (f_1, \ldots, f_t)$ be an ordered t-tuple of polynomials in R. Let f be a polynomial in R. Then f can be written as

$$q_1 f_1 + q_2 f_2 + \cdots + q_t f_t + r \tag{5.1}$$

where the quotients q_i and remainder r all lie in R, and either $r = 0$ or r is a k-linear combination of monomials none of which is divisible by any of the $LT(f_1), \ldots, LT(f_t)$. Furthermore, if $q_i f_i \neq 0$, then $multideg(f) \geqslant multideg(q_i f_i)$.

A Problem with the Division Algorithm and Ideal Membership

If the remainder in Equation 5.1 is zero, then we can see that f lies in the span of the f_i above. However, the division algorithm does not yield consistent results in polynomial rings with several variables. With the current order of polynomials $f_1, \ldots, f_t$ we may find that $f = q_1 f_1 + q_2 f_2 + \cdots + q_t f_t + r$ and $r = 0$. However upon reordering as $\{f_{\sigma(i)}\}_{i=1}^{t}$ for some σ in the symmetric group $\mathfrak{S}_t$, we may get $f = p_1 f_{\sigma(1)} + p_2 f_{\sigma(2)} + \cdots + p_t f_{\sigma(t)} + r'$ and $r' \neq 0$. This is the ideal membership problem.

Introduction to Gröbner bases

Let I be an ideal in R. Let $\langle LT(I) \rangle$ denote the ideal generated by the leading terms of each element in I. If I is generated by a finite number of generators, say $f_1, \ldots, f_s$. Then it follows that $\langle LT(f_1), \ldots, LT(f_s) \rangle$ is always contained in $\langle LT(I) \rangle$. However, unless the f_i are a special collection of generators, $\langle LT(I) \rangle$ is strictly larger than $\langle LT(f_1), \ldots, LT(f_s) \rangle$. Any special collection of this kind is called a Gröbner basis.

Definition 5.1.4 (A Criterion for a Gröbner basis)**.** $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for an ideal I in R if and only if $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$.

Fact 5.1.5 (Buchberger's algorithm)**.** *Every ideal in R has a Gröbner basis.*

A Solution to the Ideal Membership Problem

Let I be an ideal in R. The Hilbert Basis Theorem guarantees that I has a finite generating set, so $I = \langle f_1, \ldots, f_s \rangle$ for some $f_1, \ldots, f_s \in I$. The division algorithm for this set of generators may not determine conclusively whether a polynomial $f \in R$ lies in the ideal I. However, Buchberger's algorithm produces a special kind of generating set for this ideal. This set is a Gröbner basis $G = \{g_1, \ldots, g_t\}$, and $I = \langle g_1, \ldots, g_t \rangle$. Gröbner bases solve some of the problems with the division algorithm above: no matter how we order the elements in G, the division algorithm now produces a unique remainder. Only some of the problems are solved because reordering the elements of G produces different quotients q_i in the expression

$$f = q_1 g_1 + q_2 g_2 + \cdots + q_t g_t + r.$$

For the purposes of an ideal membership criterion, however, this remaining problem does not matter.

Remark 5.1.6 (The notation $\overline{f}^G$)**.** Let $G = \{g_1, \ldots, g_t\}$ be a set of polynomials in R. If f is a polynomial in R, then we use the symbol $\overline{f}^G$ to denote the remainder upon division of f by G.

5.2 Commutative algebra machinery

Below, we list without proof some well-known results from commutative algebra that we use to build a basis for the quotient R/J_h. The interested reader can find their proofs in Appendix A.2.

Theorem 5.2.1 (Division by a Gröbner basis gives a unique remainder)**.** *Assume $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for an ideal I in R and suppose $f \in R$. Then there exists a unique $r \in R$ such that:*

(i) No term of r is divisible by any $LT(g_1), \ldots, LT(g_t)$.

(ii) There exists $g \in I$ such that $f = g + r$.

In particular, r is the unique remainder upon division of f by G no matter how the elements of G are listed.

Corollary 5.2.2 (Ideal Membership Criterion)**.** *A polynomial $f \in R$ is in an ideal I of R if and only if the remainder upon division by a Gröbner basis of I is zero.*

Theorem 5.2.3. *Given $f \in R$ and an ideal I in R, f is congruent modulo I to a unique polynomial r. This polynomial r is a k-linear combination of monomials in the complement of $\langle LT(I) \rangle$.*

Theorem 5.2.4. *[3, Exer.1, pg.237]. Let I be an ideal in R. Then, the set*

$\{\boldsymbol{x}^\alpha \mid \boldsymbol{x}^\alpha \notin \langle LT(I)\rangle\}$ *is linearly independent modulo* I. *That is, if* $\sum_\alpha c_\alpha \boldsymbol{x}^\alpha \equiv 0$ *modulo* I *and each* $\boldsymbol{x}^\alpha \notin \langle LT(I)\rangle$, *then* $c_\alpha = 0$ *for all* α.

Lemma 5.2.5 (Strengthened Version of Theorem 5.2.1)**.** *[3, Exer.1, pg.87]. Let* $G = \{g_1, \ldots, g_t\}$ *be a Gröbner basis for an ideal* I *in* R *and suppose* $f \in R$. *Then there exists a unique* $r \in R$ *such that:*

(*i*) *No term of* r *is divisible by any element of* $LT(I)$.

(*ii*) *There exists* $g \in I$ *such that* $f = g + r$.

So r *is the unique remainder upon division of* f *by* I. *Furthermore, this remainder coincides with the remainder given in Theorem 5.2.1.*

Lemma 5.2.6. *[3, Exer.12a, pg.88]. If* f *and* g *are polynomials in* R, *then* $\overline{f}^G = \overline{g}^G$ *if and only if* $f - g \in I$.

Lemma 5.2.7. *[3, Exer.12b, pg.88]. If* f *and* g *are polynomials in* R, *then* $\overline{f+g}^G = \overline{f}^G + \overline{g}^G$.

Lemma 5.2.8. *[3, Exer.12c, pg.88]. If* f *and* g *are polynomials in* R, *then* $\overline{fg}^G = \overline{\overline{f}^G + \overline{g}^G}^G$.

Theorem 5.2.9 (A Basis for R/I)**.** R/I *is isomorphic to the* k*-span of the set* $\{\boldsymbol{x}^\alpha | \boldsymbol{x}^\alpha \notin \langle LT(I)\rangle\}$ *as* k*-vector spaces.*

Observation 5.2.10. *The quotient* R/I *is almost – but not! – ring isomorphic to the* k*-span of* $\{\boldsymbol{x}^\alpha | \boldsymbol{x}^\alpha \notin \langle LT(I)\rangle\}$.

5.3 Generators of J_h form a Gröbner basis

We briefly recall the construction of the ideals J_h from Section 4.2. Consider the map $h = (h_1, h_2, \ldots, h_n) \longmapsto \beta = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$ from Hessenberg functions to degree tuples where $\beta_i = i - \#\{h_k | h_k < i\}$. The ideal J_h is defined as

$$J_h := \langle \tilde{e}_{\beta_n}(x_n), \tilde{e}_{\beta_{n-1}}(x_{n-1}, x_n), \ldots, \tilde{e}_{\beta_1}(x_1, \ldots, x_n)\rangle$$

where each $\tilde{e}_{\beta_i}(x_i, x_{i+1}, \ldots, x_n)$ is the modified complete symmetric function introduced in Section 4.2.

There are many criteria to show that a set $G = \{g_1, \ldots, g_t\}$ of polynomials forms a Gröbner basis. One criterion was given in Definition 5.1.4. A sufficient criterion is given in the following theorem in [3, pg 104]. Recall that two polynomials $f_1 \neq f_2$ are relatively prime if $LCM(LM(f_1), LM(f_2)) = LM(f_1) \cdot LM(f_2)$, where LCM is the least common multiple.

Theorem 5.3.1. *Let $G = \{g_1, \ldots, g_t\}$ be a set of polynomials. If the leading monomials of the polynomials in G are pairwise relatively prime, then G is a Gröbner basis.*

Theorem 5.3.2. *The generating set of an ideal J_h is a Gröbner basis with respect to the lexicographic.*

Proof. Let $\langle \tilde{e}_{\beta_n}(x_n), \tilde{e}_{\beta_{n-1}}(x_{n-1}, x_n), \ldots, \tilde{e}_{\beta_1}(x_1, \ldots, x_n)\rangle$ be the ideal J_h for a given Hessenberg function h. For brevity, let us name the generators

$$\begin{aligned} f_n &:= \tilde{e}_{\beta_n}(x_n) \\ f_{n-1} &:= \tilde{e}_{\beta_{n-1}}(x_{n-1}, x_n) \\ &\vdots \\ f_1 &:= \tilde{e}_{\beta_1}(x_1, \ldots, x_n). \end{aligned}$$

To show the set $G = \{f_1, f_2, \ldots, f_n\}$ is a Gröbner basis, it suffices to show that the leading monomials of f_i and f_j are relatively prime for all $i \neq j$, and hence the claim holds by Theorem 5.3.1. Since the leading monomial of each f_i is $x_i^{\beta_i}$, it follows that if $i \neq j$, then we have

$$LCM(LM(f_i), LM(f_j)) = LCM(x_i^{\beta_i}, x_j^{\beta_j}) = x_i^{\beta_i} x_j^{\beta_j} = LM(f_i) \cdot LM(f_j).$$

We conclude that G is a Gröbner basis for the ideal J_h. Observe that we could have used the graded lexicographic monomial ordering instead. Remark 5.1.3 pointed out that both orderings give the same leading monomials for homogenous polynomials like the f_i. $\square$

5.4 Monomial basis $\mathcal{B}_h(\mu)$ for R/J_h

By Theorem 5.2.9, if I is an ideal in $R = k[x_1, \dots, x_n]$, then R/I has basis $\{\mathbf{x}^\alpha \mid \mathbf{x}^\alpha \notin \langle LT(I)\rangle\}$. To find a basis for R/J_h we must understand the ideal generated by the leading terms of the elements in the ideal J_h, namely, the ideal of leading terms $\langle LT(J_h)\rangle$.

Lemma 5.4.1. *Given the ideal J_h with Hessenberg function $h = (h_1, \dots, h_n)$, the ideal $\langle LT(J_h)\rangle$ is the monomial ideal $\langle x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_n^{\beta_n}\rangle$.*

Proof. The set $\{\tilde{e}_{\beta_n}(x_n), \tilde{e}_{\beta_{n-1}}(x_{n-1}, x_n), \dots, \tilde{e}_{\beta_1}(x_1, \dots, x_n)\}$ is the generating set of the ideal J_h. By Theorem 5.3.2, this set forms a Gröbner basis. For brevity, denote $f_i := \tilde{e}_{\beta_i}(x_i, x_{i+1}, \dots, x_n)$. Thus $\{f_1, \dots, f_n\}$ is our Gröbner basis, and by the definition of a Gröbner basis it follows that $\langle LT(J_h)\rangle = \langle LT(f_1), \dots, LT(f_n)\rangle$. Since $LT(f_i) = x_i^{\beta_i}$ for each i, the ideal $\langle LT(f_1), \dots, LT(f_n)\rangle$ is just the monomial ideal $\langle x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_n^{\beta_n}\rangle$. $\square$

Corollary 5.4.2. *If $\boldsymbol{x}^\alpha \notin \langle LT(J_h)\rangle$, then none of the $x_i^{\beta_i}$ divides $\boldsymbol{x}^\alpha$.*

Proof. Suppose some $x_i^{\beta_i}$ divides $\mathbf{x}^\alpha$. Then $\mathbf{x}^\alpha = p_i \cdot x_i^{\beta_i}$ for some $p_i \in R$. So $\mathbf{x}^\alpha \in \langle x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_n^{\beta_n}\rangle = \langle LT(J_h)\rangle$. This proves the claim. $\square$

Theorem 5.4.3 (A Basis for R/J_h)**.** *Let J_h be the ideal corresponding to the Hessenberg function h. Then R/J_h has the basis*

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \,|\, 0 \le \alpha_i \le \beta_i - 1, i = 1, \dots, n\}.$$

Proof. By Theorem 5.2.9, the quotient R/J_h has basis $\{\mathbf{x}^\alpha | \mathbf{x}^\alpha \notin \langle LT(J_h)\rangle\}$. By Corollary 5.4.2, $\mathbf{x}^\alpha \notin \langle LT(J_h)\rangle$ implies none of the $x_i^{\beta_i}$ divides $\mathbf{x}^\alpha$. Thus if x_i appears in the monomial $\mathbf{x}^\alpha$, then its exponent cannot exceed $\beta_i - 1$. So $\mathbf{x}^\alpha$ must be of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ such that each exponent $\alpha_i \in \{0, 1, \dots, \beta_i - 1\}$. The claim follows. $\square$

Definition 5.4.4 (Monomial basis $\mathcal{B}_h(\mu)$)**.** Let h be a Hessenberg function. The monomials given in Theorem 5.4.3 form a *monomial basis* $\mathcal{B}_h(\mu)$ for the ring R/J_h.

Corollary 5.4.5. *For every ideal J_h, the rank of R/J_h equals* $\prod_{i=1}^{n} \beta_i$. *In particular, the quotient R/J_h has finite rank.*

Proof. By definition of a degree tuple $(\beta_n, \ldots, \beta_1)$, each β_i is positive. We count all basis elements of R/J_h, namely as all monomials $\mathbf{x}^\alpha$ with exponents α_i in the range $\{0, 1, \ldots, \beta_i - 1\}$. This gives a total of $\prod_{i=1}^{n} \beta_i$ distinct monomials. Therefore, the rank of R/J_h equals $\prod_{i=1}^{n} \beta_i$, which is finite. $\square$

Remark 5.4.6 (Interesting fact about $\prod_{i=1}^{n} \beta_i$ above)**.** As a consequence of Theorem 6.2.3, the number of all possible (h, μ)-fillings of a Young diagram of shape μ equals the rank of R/J_h. This will be useful to us later in Chapter 6.

5.5 Elaborative example

Let $h = (3, 3, 3, 4)$ be a Hessenberg function. Then $\beta = (1, 3, 2, 1)$, yielding the ideal $J_h = \langle \tilde{e}_1(x_4), \tilde{e}_3(x_3, x_4), \tilde{e}_2(x_2, x_3, x_4), \tilde{e}_1(x_1, x_2, x_3, x_4) \rangle$. We write

$$J_h = \begin{pmatrix} x_4 \\ x_3^3 + x_3^2 x_4 + x_3 x_4^2 + x_4^2 \\ x_2^2 + x_2 x_3 + x_2 x_4 + x_3^2 + x_3 x_4 + x_4^2 \\ x_1 + x_2 + x_3 + x_4 \end{pmatrix}.$$

The leading term ideal $\langle LT(J_h) \rangle = \langle x_4, x_3^3, x_2^2, x_1 \rangle$ since the generators of J_h form a Gröbner basis. Thus the quotient R/J_h has basis

$$\left\{ x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} \;\middle|\; \begin{array}{c} \alpha_1 = 0, \;\; \alpha_2 = 0, 1 \\ \alpha_3 = 0, 1, 2, \;\; \alpha_4 = 0 \end{array} \right\}.$$

So $\mathcal{B}_h(\mu)$ equals $\{1,\ x_2,\ x_3,\ x_2 x_3,\ x_3^2,\ x_2 x_3^2\}$. As per Corollary 5.4.5, it has $\prod_{i=1}^{4} \beta_i = 1 \cdot 2 \cdot 3 \cdot 1 = 6$ elements. By Remark 5.4.6, there are exactly 6 possible (h, μ)-fillings. Below we give these fillings, their dimension pairs, and the corresponding monomials given by the function Φ.

$$
\begin{array}{cccccc}
\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array} & \begin{array}{|c|c|c|c|}\hline 2&1&3&4\\\hline\end{array} & \begin{array}{|c|c|c|c|}\hline 1&3&2&4\\\hline\end{array} & \begin{array}{|c|c|c|c|}\hline 2&3&1&4\\\hline\end{array} & \begin{array}{|c|c|c|c|}\hline 3&1&2&4\\\hline\end{array} & \begin{array}{|c|c|c|c|}\hline 3&2&1&4\\\hline\end{array} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
\text{no pairs} & (12) & (23) & (12),(13) & (13),(23) & (12),(13),(23) \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
1 & x_2 & x_3 & x_2x_3 & x_3^2 & x_2x_3^2
\end{array}
$$

To our delight, we get the same monomials from the (h,μ)-fillings as the basis of R/J_h. The goal of Chapter 6 is to prove this always happens.

CHAPTER 6
REGULAR NILPOTENT HESSENBERG VARIETY SETTING

As promised in Chapter 2, we now provide an analogy of what was done in the setting of Springer varieties. To this end, we explain the basic set-up and goal as given in Figure 6.1. We remind the reader that in the setting of regular nilpotent Hessenberg varieties, we fix the nilpotent operator X (equivalently, fix the shape $\mu = (n)$). Recall that the dimensions of the graded parts of $H^*(\mathfrak{H}(X,h))$ are combinatorially described by the (h,μ)-fillings. This gives the geometric view of the cohomology ring denoted by the left edge of the triangle. The formal linear span of the (h,μ)-fillings is denoted $M^{h,\mu}$. The map Φ is a graded vector space morphism from $M^{h,\mu}$ to the span of monomials $\mathcal{A}_h(\mu)$. In Section 6.3, we show that Φ is actually a graded isomorphism, completing the bottom leg of the triangle. In Theorem 6.3.3, we see that the generators of degree i in R/J_h correspond to (h,μ)-fillings of dimension i and hence to the i^{th} Betti number of the regular nilpotent Hessenberg varieties. This gives the algebraic view of $H^*(\mathfrak{H}(X,h))$.

$$\begin{tikzcd} & H^*(\mathfrak{H}(X,h)) \arrow[dl, leftrightsquigarrow] \arrow[dr, dashed, leftrightarrow, "\cong ?"] & \\ \begin{array}{c}(h,\mu)\text{-fillings}\\ \text{spanning } M^{h,\mu}\end{array} \arrow[rr, leftrightarrow, "\xrightarrow{\Phi}", "\xleftarrow{\Psi_h}"'] & & \begin{array}{c} R/J_h \text{ with}\\ \mathcal{A}_h(\mu)=\mathcal{B}_h(\mu)\\ \text{basis}\end{array} \end{tikzcd}$$

Figure 6.1: Triangle – Regular nilpotent Hessenberg setting.

In Section 6.1, for a given Hessenberg function h we build an h-tableau tree. This tree assumes the role that the modified GP-tree filled in Chapter 2. In Section 6.2, we construct the inverse map Ψ_h from the span of the monomials $\mathcal{A}_h(\mu)$ to the the vector space $M^{h,\mu}$. Finally in Section 6.3, we show that the monomials

$\mathcal{A}_h(\mu)$ coincide with the basis $\mathcal{B}_h(\mu)$ (recall Definition 5.4.4) of the quotient R/J_h.

6.1 Constructing an h-tableau-tree

Analogous to the Springer case, we first build a tree that we call an *h-tree* whose leaves give the basis $\mathcal{B}_h(\mu)$ for the quotient ring R/J_h. As with the modified GP-tree, we then take these leaves and describe how to construct the corresponding (h, μ)-filling. We label the vertices of the h-tree to produce a graph that we call an *h-tableau-tree*. In this chapter, we repeatedly use the notion of degree tuples from Section 4.1. For completeness, we restate its definition below.

Definition 6.1.1 (Degree tuple)**.** Let $h = (h_1, h_2, \ldots, h_n)$ be a Hessenberg function. Let $\beta_i = i - \#\{h_k \mid h_k < i\}$ for each $1 \leq i \leq n$. The *degree tuple* corresponding to h is $\beta = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$.

Definition 6.1.2 (h-tree)**.** Given a Hessenberg function $h = (h_1, h_2, \ldots, h_n)$, the corresponding *$h$-tree* has $n + 1$ levels labeled from the top Level 1 to the bottom Level $n + 1$. We start with one vertex at Level 1. For $i \in \{2, \ldots, n\}$, we go from Level $i - 1$ to Level i by traversing exactly β_i distinct vertices from each vertex at Level $i - 1$ injectively (that is, no two Level $i - 1$ vertices share an edge with the same Level i vertex). Label the β_i edges going down from each vertex on Level $i - 1$ with the labels $\{x_i^{\beta_i - 1}, x_i^{\beta_i - 2}, \ldots, x_i^2, x_i, 1\}$ going left to right. Let each vertex at Level n branch down to a unique vertex at Level $n + 1$. Label this connecting edge with the number 1. Label the leaf at Level $n + 1$ with the product of the edge labels connecting the top Level 1 vertex with this leaf.

Theorem 6.1.3. *Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. Then*

1. *The number of leaves at Level $n + 1$ equals $\prod_{i=1}^{n} \beta_i$.*

2. *The collection of the $\prod_{i=1}^{n} \beta_i$ leaf labels at Level $n + 1$ of the h-tree is exactly the basis of monomials $\mathcal{B}_h(\mu)$ of R/J_h from Section 5.4.*

Proof. Obvious. $\square$

Example 6.1.4. Let $h = (2, 3, 3)$ be a Hessenberg function. It has corresponding degree tuple $\beta = (2, 2, 1)$. Figure 6.2 shows the corresponding h-tree.

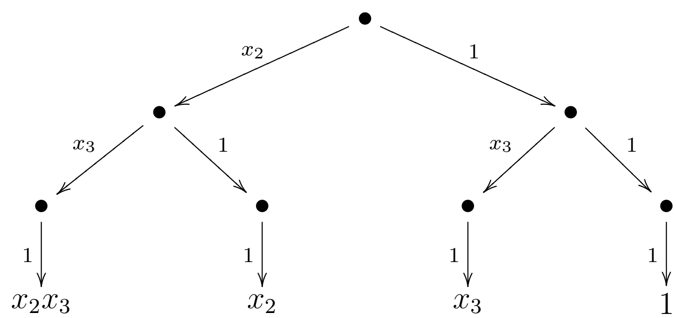


Figure 6.2: The h-tree for $h = (2, 3, 3)$.

Before we give the precise construction of an h-tableau-tree, we define a barless tableau and give a lemma that instructs us how to fill this object to construct a tableau.

Definition 6.1.5 (Barless tableau)**.** Fix n. A *barless tableau* is the following diagram $\boxed{\quad\quad\quad\quad}$ filled with some proper subset of $\{1, \ldots, n\}$ without any bars.

Remark 6.1.6 (Using a barless tableau to build an (h, μ)-filling)**.** We place the values $1, \ldots, n$ in increasing order into the barless tableau satisfying an h-permissibility condition. Suppose we have placed the numbers $1, \ldots, i-1$. We say the $i-1$ fillings are in *h-permissible* positions if each horizontal adjacency adheres to the rule: k is immediately left of j if and only if $k \leq h_j$. The lemma below allows us to predict how many h-permissible positions are available for the placement of the next value, i. When the n^{th} number is placed, we replace the bars in the barless tableau.

Lemma 6.1.7. *Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. If a barless tableau is filled with $1, 2, \ldots, i-1$, then the number of h-permissible positions for i in this tableau is exactly β_i, where $(\beta_n, \beta_{n-1}, \ldots, \beta_1)$ is the degree tuple corresponding to h.*

Proof. Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function and $\beta = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$ be its corresponding degree tuple. Suppose a barless tableau is filled with $1, 2, \ldots, i-1$. Consider β_i. By definition $\beta_i = i - \#\{h_k \mid h_k < i\}$ and so $\#\{h_k \mid h_k < i\}$ equals $i - \beta_i$. Since each h_k is at least k, only the values $h_1, \ldots, h_{i-1}$ can possibly lie in the set $\{h_k \mid h_k < i\}$. The remaining $(i-1)-(i-\beta_i) = \beta_i - 1$ of the $h_1, \ldots, h_{i-1}$ satisfy $i \leq h_k$ which is the h-permissibility condition for the descent such as $\boxed{i}\boxed{k}$. Hence i can be place to the immediate left of an of these $\beta_i - 1$ values. This gives $\beta_i - 1$ positions that are h-permissible positions. In addition, the value i can be placed to the right of the far-right entry since i is larger than any number $1, \ldots, i-1$ in the barless tableau. This yields a total of $(\beta_i - 1) + 1 = \beta_i$ possible h-permissible positions for i. $\square$

Definition 6.1.8 (h-tableau-tree)**.** Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function and $\beta = (\beta_n, \beta_{n-1}, \ldots, 1)$ be its corresponding degree tuple. The *h-tableau-tree* is the h-tree together with an assignment of barless tableaux to label each vertex on Levels 1 to n. The top is Level 1 and has a single barless tableau with the entry 1. Remark 6.1.6 explained how we build an (h, μ)-filling. Given a barless tableau T at Level $i-1$ with fillings $1, \ldots, i-1$, we obtain the β_i different Level i barless tableaux by the following algorithm:

- Place a bullet at each of the h-permissible positions in the barless tableau T. Lemma 6.1.7 asserts there will be exactly β_i bullets going right to left. The diagram at Level i joined by the edge x_i^j is found by replacing the $(j+1)^{th}$ bullet (counting right to left) with the number i and erasing all other bullets.

When we reach Level n, each barless tableau will contain the numbers $1, \ldots, n$. We may now place the bars into this tableau yielding a filling of μ.

Example 6.1.9. In Figure 6.3, we give an example of an h-tableau-tree when $h = (3, 3, 3, 4)$. The corresponding degree tuple is $\beta = (1, 3, 2, 1)$. For ease of viewing, we omit the barless tableaux's rectangular boundaries and just give the

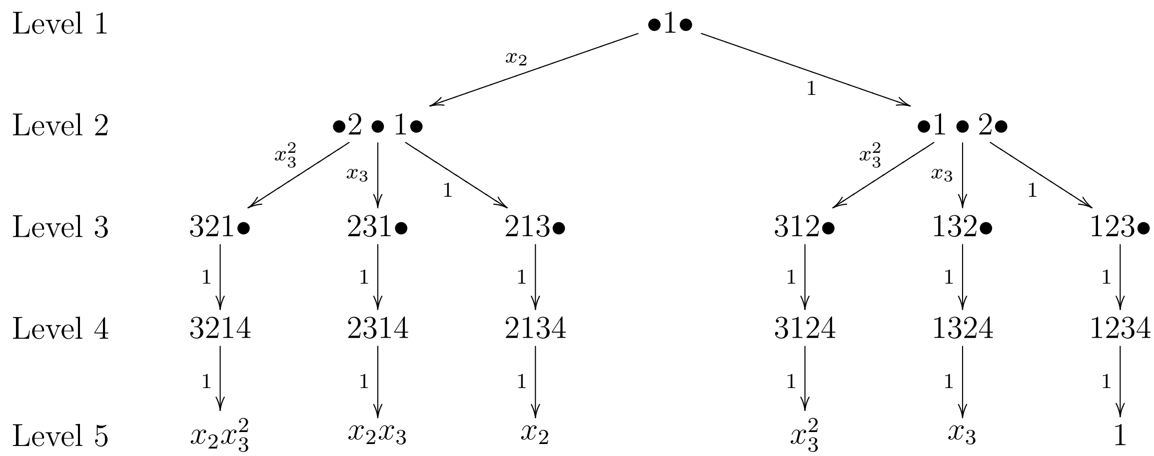


Figure 6.3: The h-tableau-tree for $h = (3, 3, 3, 4)$.

fillings. Observe that the six Level 4 tableaux are (h, μ)-fillings. There are only six possible (h, μ)-fillings for this particular Hessenberg function and hence these are *all* the (h, μ)-fillings. Further, the function Φ maps each one to the monomial in $\mathcal{B}_h(\mu)$ on Level 5. We conclude that the set of monomials $\mathcal{A}_{(3,3,3,4)}(\mu)$ coincides with the monomial basis $\mathcal{B}_{(3,3,3,4)}(\mu)$ for R/J_h when using this regular nilpotent shape $\mu = (n)$. We generalize these points in the next section, where we exhibit the inverse map to Φ in the setting of regular nilpotent Hessenberg varieties. Compare this with the elaborative example from Section 5.5.

6.2 The inverse map Ψ_h from monomials in $\mathcal{B}_h(\mu)$ to $(\boldsymbol{h}, \mu)$-fillings

Recall from Section 1.6, the function Φ from (h, μ)-fillings onto the set $\mathcal{A}_h(\mu)$ of monomials is given by the map

$$T \longmapsto \prod_{\substack{(i,j)\in \mathrm{DP}_j^T \\ 2\leq j\leq n}} x_j.$$

In the Springer setting, we first constructed the inverse map Ψ from $\mathcal{A}(\mu)$ to (h, μ)-fillings, then proved $\mathcal{A}(\mu) = \mathcal{B}(\mu)$. In the regular nilpotent Hessenberg setting we will again prove that Φ is a graded vector space isomorphism by first constructing an inverse map Ψ_h from $\mathcal{B}_h(\mu)$ and then verifying $\mathcal{A}_h(\mu) = \mathcal{B}_h(\mu)$. In this new setting this plan of attack makes more sense since we know more about the structure of $\mathcal{B}_h(\mu)$ (see Theorem 5.4.3), whereas in the Springer setting the basis $\mathcal{B}(\mu)$ was given via a recursion formula [7, Equation 1.2]. As the remarks in Example 6.1.9 disclosed, we will show the following:

1. The Level n fillings in the h-tableau-tree are distinct (h, μ)-fillings.

2. The number of (h, μ)-fillings equals the number of leaves of the h-tableau-tree.

3. The Level n fillings are *all* possible (h, μ)-fillings.

4. The function Φ maps each of these fillings to the monomial $\mathbf{x}^\alpha \in \mathcal{B}_h(\mu)$ below it at Level $n + 1$.

5. The set $\mathcal{A}_h(\mu)$ coincides with the set $\mathcal{B}_h(\mu)$.

Theorem 6.2.1. *The Level n fillings of an h-tableau-tree for $h = (h_1, \ldots, h_n)$ are distinct (h, μ)-fillings.*

Proof. When going down from Level $i-1$ down to i, the value i is placed immediately to the left of a number $k \in \{1, \ldots, i-1\}$ only if $i \leq h(k)$. That is, all fillings in the tree are h-permissible and hence the Level n fillings are (h, μ)-fillings. Branching rules ensure all are distinct. $\square$

The proof of Theorem 6.2.3 relies on combinatorial facts about the two numbers in question, namely the cardinalities of the set of possible (h, μ)-fillings and the set of leaves of an h-tableau-tree. The former number is given by the following theorem.

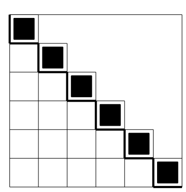
Theorem 6.2.2 (Sommers-Tymoczko [13])**.** *Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. The number of (h, μ)-fillings of a one-row diagram of shape (n) equals $\prod_{i=1}^{n} \nu_i$ where $\nu_i = h_i - i + 1$.*

Fix a Hessenberg function $h = (h_1, \ldots, h_n)$. Let A_h denote the multiset $A_h := \{\nu_i\}_{i=1}^{n}$. Theorem 6.1.3 shows that the number of leaves of the h-tree (and consequently of the h-tableau-tree) is $\prod_{i=1}^{n} \beta_i$ where $\beta_i = i - \#\{h_k < i\}$. Let B_h denote the multiset $B_h := \{\beta_i\}_{i=1}^{n}$. The sets A_h and B_h are *multisets*. In these types of sets order is ignored, but multiplicity matters. For example $\{1,2,3\} = \{2,1,3\}$ but $\{1,1,2\} \neq \{1,2\}$.

Theorem 6.2.3. *The number of (h,μ)-fillings equals the number of leaves in the h-tableau-tree.*

Proof. Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function. It suffices to show the multisets A_h and B_h are equal. Represent the function h pictorially by its corresponding Hessenberg diagram (see Definition 4.1.9). We may view the elements of A_h as a vector difference $(\nu_i)_{i=1}^{n} = (h_1, \ldots, h_n) - (0, 1, \ldots, n-1)$. So ν_i equals the number of shaded boxes on or below the diagonal in column i. Regarding set B_h, observe that $\#\{h_k \mid h_k < i\}$ equals the number of empty boxes in row i – these are the h_k that never touch row i. So β_i is the number of shaded boxes on or left of the diagonal in row i. Thus it suffices to show each column length ν_i corresponds to exactly one row length β_j. We induct on the Hessenberg function.

Consider the minimal Hessenberg function $h = (1, 2, \ldots, n)$. This gives the following Hessenberg diagram:

 .

Each shaded box contributes to both an ν_i and a β_i of length 1. It follows that $A_h = B_h = \{1, 1, \ldots, 1\}$, proving the base case holds.

Assume for some fixed Hessenberg function $h = (h_1, \ldots, h_n)$ that $A_h = B_h$. Add a shaded box to its Hessenberg diagram in a position (i_0, j_0) so that the new function $\tilde{h} = (h_1, \ldots, h_{j_0-1}, i_0, h_{j_0+1}, \ldots, h_n)$ is a Hessenberg function, namely so $i_0 \leq h_{j_0+1}$. We claim that the multisets $A_{\tilde{h}} = \{\tilde{\nu}_i\}_{i=1}^{n}$ and $B_{\tilde{h}} = \{\tilde{\beta}_i\}_{i=1}^{n}$ will coincide.

Every box above (i_0, j_0) in column j_0 must be shaded, up to the shaded diagonal box (j_0, j_0). This shaded column length is $\tilde{\nu}_{j_0}$. And since $\tilde{h}$ is a Hessenberg function, every box to the right of (i_0, j_0) is shaded up to the shaded diagonal box (i_0, i_0). This shaded row length is $\tilde{\beta}_{i_0}$. A reminder that no other box in row i_0 or column j_0 below the diagonal is shaded because h is a Hessenberg function. Clearly $\tilde{\nu}_{j_0} = \nu_{j_0} + 1 = (h_{j_0} - j_0 + 1) + 1 = h_{j_0} - j_0 + 2$. The value $\tilde{\beta}_{i_0}$ is just the number of boxes in row i_0 from the position (i_0, j_0) to the diagonal (i_0, i_0) which we count as $i_0 - j_0 + 1$. Observe $i_0 = h_{j_0} + 1$ implies that $h_{j_0} + 2 = i_0 + 1$. Hence $h_{j_0} - j_0 + 2 = i_0 - j_0 + 1$. We conclude $\tilde{\nu}_{j_0} = \tilde{\beta}_{i_0}$, and the claim holds since

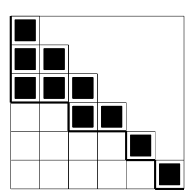
(1) $\nu_{j_0} = \beta_{i_0}$ necessarily in the original Hessenberg diagram for h,

(2) ν_{j_0} and β_{i_0} both increase by 1 in the new Hessenberg diagram for $\tilde{h}$, and

(3) no other ν_i or β_j in the original diagram for h will change in the diagram for $\tilde{h}$.

This completes the induction step, and we conclude that the multisets $A_{\tilde{h}}$ and $B_{\tilde{h}}$ are equal. $\square$

Example 6.2.4 (Clarifying example for the induction step above)**.** Let h be the Hessenberg function $(3, 3, 4, 4, 5, 6)$. The corresponding Hessenberg diagram is



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In this example $A = \{3, 2, 2, 1, 1, 1\}$ and $B = \{1, 2, 3, 2, 1, 1\}$ reading the column lengths and row lengths, respectively. At the induction step in the proof above, there are only three legal places to add a box: the positions $(4, 2)$, $(5, 4)$, or $(6, 5)$. Adding the $(4, 2)$-box changes ν_2 from 2 to 3 and changes β_4 from 2 to 3 also. Moreover, adding the $(4, 2)$-box did not affect any other ν_i or β_j values in A_h or B_h respectively.

Corollary 6.2.5. *The Level n fillings are all possible (h,μ)-fillings*

Proof. Level n fillings are distinct (h,μ)-fillings by Theorem 6.2.1. The claim follows immediately from the previous theorem together with Theorem 6.2.2 of Sommers-Tymoczko. □

We now need a lemma similar to Lemma 2.2.8. This will be useful in building the inverse map Ψ_h.

Lemma 6.2.6. *Fix n and let h be an arbitrary Hessenberg function. Let $\boldsymbol{x}^\alpha \in \mathcal{B}_h(\mu)$, and consider the h-tableau-tree corresponding to h. Then*

(i) The monomial $\boldsymbol{x}^\alpha$ is of the form $x_2^{\alpha_2}\cdots x_n^{\alpha_n}$.

(ii) Any barless tableau at Level $i-1$ has at least α_i+1 bullet positions available.

Proof. Let $h=(h_1,\ldots,h_n)$ be a Hessenberg function with corresponding degree tuple $\beta=(\beta_n,\beta_{n-1},\ldots,\beta_1)$. Let $\mathbf{x}^\alpha\in\mathcal{B}_h(\mu)$. By Theorem 5.4.3, $\mathbf{x}^\alpha$ is of the form $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ where each α_i satisfies $0\le\alpha_i\le\beta_i-1$. Since $\beta_1=1$ by definition, we have $\alpha_1=0$ for all h, proving part (i). Lemma 6.1.7 ensures that a Level $i-1$ barless tableau will have β_i bullets. Since $\alpha_i+1\le\beta_i$, this proves (ii). □

Theorem 6.2.7 (A map from $\mathcal{B}_h(\mu)$ to (h,μ)-fillings)**.** *Given $h=(h_1,\ldots,h_n)$ and $\mu=(n)$, there exists a well-defined dimension-preserving map Ψ_h from the monomials $\mathcal{B}_h(\mu)$ to the set of (h,μ)-fillings. That is, degree-r monomials in $\mathcal{B}_h(\mu)$ map to r-dimensional (h,μ)-fillings. Moreover the composition*

$$\mathcal{B}_h(\mu)\xrightarrow{\Psi_h}\{(h,\mu)\text{ -fillings}\}\xrightarrow{\Phi}\mathcal{B}_h(\mu)$$

is the identity.

Proof. Let $\mathbf{x}^\alpha\in\mathcal{B}_h(\mu)$ have degree r. Consider the (h,μ)-filling T sitting at Level n directly above $\mathbf{x}^\alpha$. Define $\Psi_h(\mathbf{x}^\alpha):=T$.

Lemma 6.2.6 says $\mathbf{x}^\alpha$ has the form $x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $\alpha_2+\alpha_3+\cdots+\alpha_n=r$ since $\mathbf{x}^\alpha$ has degree r. It suffices to show that the cardinality of DP_k^T equals α_k for each $k\in\{2,\ldots,n\}$. We check this by examining the path on the h-tableau-tree from

Level 1 down to $\mathbf{x}^\alpha$ at Level $n+1$. Fix $k \in \{2, \ldots, n\}$. Let $T_1, T_2, \ldots, T_{n-1}$ be the barless tableaux on this path at Levels $1, 2, \ldots, n-1$ respectively. At the $(k-1)^{th}$ step in this path, the number k is placed in the $(\alpha_k + 1)^{th}$ bullet from the right in T_{k-1}. This bullet exists by Lemma 6.2.6. The barless tableau T_{k-1} has the form

$$\boxed{\cdots \bullet B_{\alpha_k} \bullet \cdots \bullet B_2 \bullet B_1 \bullet}$$

where each block B_i is a string of numbers. The numbers $1, \ldots, k-1$ are distributed without repetition amongst all B_i. We claim there exists exactly one b_i in each B_i-block to the right of k in T_k such that $(b_i, k) \in \mathrm{DP}_k^T$.

Since all fillings in an h-tableau-tree are h-permissible, each block B_i is an ordered string of γ_i numbers $b_{i,1} b_{i,2} \cdots b_{i,\gamma_i}$ in $\{1, \ldots, k-1\}$ such that $b_{i,r} \leq h(b_{i,r+1})$ for each $r < \gamma_i$. We claim k forms a dimension pair with only the far-right entry b_{i,γ_i} of each block B_i to its right. Recall to be a dimension pair $(b, k) \in \mathrm{DP}_k^T$ in the one-row case, we must have

(i) b is to the right of k and $k > b$ holds, and

(ii) if there exists a j immediately right of b, then $k \leq h(j)$ holds also.

Since k is larger than every entry in the Level $k-1$ barless tableau, condition (i) holds. If there exists no j to the right of b_{i,γ_i}, then (ii) holds vacuously. If some j is eventually placed immediately right of b_{i,γ_i} then $j \geq k+1$. Thus $k < k+1 \leq h(j)$ and so (ii) holds. Lastly, if no j is placed right of b_{i,γ_i} and there exists a block B_{i-1} immediately right of B_i in the final tableau T, then the element $b_{i-1,1}$ is immediately right of b_{i,γ_i}. But $k \leq h(b_{i-1,1})$ since T_{k-1} had a bullet placed left of the block B_{i-1}. Thus in every case, (ii) holds and $|\,\mathrm{DP}_k^T\,|$ equals α_k as desired.

Hence the map $\Psi_h : \mathcal{B}_h(\mu) \to \{(h, \mu)\text{-fillings}\}$ takes degree-r monomials to r-dimensional (h, μ)-fillings, and $\Phi \circ \Psi_h$ is the identity on $\mathcal{B}_h(\mu)$. $\square$

6.3 $\mathcal{A}_h(\mu)$ coincides with the basis of monomials $\mathcal{B}_h(\mu)$ for R/J_h

Corollary 6.3.1. *For a given h, the set of monomials $\mathcal{A}_h(\mu)$ and $\mathcal{B}_h(\mu)$ are equal.*

Proof. The Level n fillings are (h, μ)-fillings by Theorem 6.2.1. In fact they are all the possible (h, μ)-filings by Corollary 6.2.5. Since the image of all (h, μ)-fillings under Φ is $\mathcal{A}_h(\mu)$, we have $\mathcal{A}_h(\mu) = \mathcal{B}_h(\mu)$. □

Corollary 6.3.2. *$\mathcal{A}_h(\mu)$ and $M^{h,\mu}$ are isomorphic as graded vector spaces.*

Proof. By Theorem 6.2.7, the composition $\Phi \circ \Psi_h$ is the identity on $\mathcal{B}_h(\mu)$. Since $\mathcal{A}_h(\mu)$ coincides with $\mathcal{B}_h(\mu)$ and the number of paths in the h-tableau-tree is exactly $\prod_{i=1}^{n} \beta_i = |M^{h,\mu}|$, the cardinality of $\mathcal{A}_h(\mu)$ equals the cardinality of the generating set of (h, μ)-fillings in $M^{h,\mu}$. Thus $\mathcal{A}(\mu)$ and $M^{h,\mu}$ are isomorphic as graded vector spaces. □

We are now ready to state the theorem that ties the algebraic view of the $H^*(\mathfrak{H}(X, h))$ with the geometric view of this same cohomology ring.

Theorem 6.3.3. *Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function with corresponding ideal J_h. The generators of the quotient R/J_h are in bijective graded correspondence with the (h, μ)-fillings. In particular, the generators of R/J_h give the Betti numbers of the regular nilpotent Hessenberg varieties.*

Proof. In Corollary 6.3.2, we proved that the map Φ is a graded vector space isomorphism from $M^{h,\mu}$ to $\mathcal{A}_h(\mu)$. In particular it a bijective graded correspondence between the set of (h, μ)-fillings and the set of monomials $\mathcal{A}_h(\mu)$. By Corollary 6.3.1, the sets $\mathcal{A}_h(\mu)$ and $\mathcal{B}_h(\mu)$ coincide. Hence the set of generators of degree i in R/J_h correspond directly to the i-dimensional (h, μ)-fillings. By Tymoczko [17, Theorem 1.1], the cardinality of the set of i-dimensional (h, μ)-fillings equals the dimension of the degree-$2i$ part of $H^*(\mathfrak{H}(X, h))$. Therefore, these degree i generators of R/J_h give the $2i^{th}$ Betti number of $\mathfrak{H}(X, h)$. □

Example 6.3.4. Fix $h = (2, 4, 4, 5, 5)$ and its corresponding $\beta = (2, 3, 2, 2, 1)$. The degree tuple β tells us that the monomial $x_2 x_4^2 x_5$ lies in $\mathcal{B}_h(\mu)$. Without drawing the whole filled tableau tree, we can construct the unique path that gives the corresponding (h, μ)-filling. Omitting the barless tableau frames, we get

$$\bullet 1 \bullet \xrightarrow{x_2^1} \bullet 21 \bullet \xrightarrow{x_3^0} \bullet 21 \bullet 3 \bullet \xrightarrow{x_4^2} \bullet 4213 \bullet \xrightarrow{x_5^1} 54213.$$

Thus $\Psi_h(x_2 x_4^2 x_5) = T$ where T is the (h, μ)-filling $\boxed{5|4|2|1|3}$. Conversely to recover the corresponding monomial from this filling T, we calculate the dimension-pairs. We write all pairs (ik) where k is left of i and $i < k$. We then eliminate pairs (ik) that do not satisfy the additional dimension pair condition that if j is immediately right of i, then $k \leq h(j)$. We get the following:

$$(12) \in \mathrm{DP}_2^T, \quad (14), \cancel{(24)}, (34) \in \mathrm{DP}_4^T, \quad \text{and} \quad \cancel{(15)}, \cancel{(25)}, (35), \cancel{(45)} \in \mathrm{DP}_5^T.$$

Thus Φ takes the filling T to the monomial $x_2 x_4^2 x_5$ as desired.

If we considered the Hessenberg function $h' = (2, 3, 5, 5, 5)$, then the same filling $T = \boxed{5|4|2|1|3}$ would be a permissible filling of h', but now the dimension pair $(15) \in \mathrm{DP}_5^T$ is *not* canceled since $5 \leq h'(3)$. Thus the map Φ takes T to the monomial $x_2 x_4^2 x_5^2$. Conversely, the inverse map $\Psi_{h'}$ now takes the new degree tuple into account and from this *different* monomial we will get the same T as we had gotten before. The only thing that changes is the extra bullet before the last arrow:

$$\bullet 1 \bullet \xrightarrow{x_2^1} \bullet 21 \bullet \xrightarrow{x_3^0} \bullet 21 \bullet 3 \bullet \xrightarrow{x_4^2} \bullet 421 \bullet 3 \bullet \xrightarrow{x_5^2} 54213.$$

In particular, the algorithm for Ψ_h is depends on the choice of the Hessenberg function.

CHAPTER 7
TOWARDS AN EQUALITY OF TWO FAMILIES OF IDEALS

In this chapter we prove that the ideals I_h and J_h coincide when h is maximal, namely, for $h = (n, \ldots, n)$. When h is maximal, the ideal I_h is generated by the elementary symmetric functions. Therefore, proving equality of I_h and J_h in this case effectively gives a Gröbner basis for the set of elementary symmetric functions. Not suprisingly, this has been a well-studied problem. Our claim in Theorem 7.1.2, up to change of variables, rediscovers an identity proven in 2003 by Mora and Sala [12]. In conversations with Teo Mora, we learned that his result with Sala is itself a rediscovery of a result of Valibouze [18] from 1995 in her thesis. Valibouze then told us that in 1840 Cauchy [2] gave a result similar to hers in an example for $n = 4$. Theorem 7.1.2 concludes that $I_h \subseteq J_h$ when h is maximal. We then prove in Corollary 7.1.3 that $J_h \subseteq I_h$, and hence the two ideals coincide.

It is a future goal to show $I_h = J_h$ for all h. This essentially provides a Gröbner basis presentation for the family of ideals I_h, which is otherwise a difficult and unsolved problem.

7.1 The maximal Hessenberg function setting

Let $h = (n, \ldots, n)$. The ideal I_h is $\langle e_1, e_2, \ldots, e_n\rangle$ where the e_i are elementary symmetric functions. We compute the ideal J_h to be

$$J_h = \langle \tilde{e}_{\beta_n}(x_n), \tilde{e}_{\beta_{n-1}}(x_{n-1}, x_n), \ldots, \tilde{e}_{\beta_1}(x_1, \ldots, x_n)\rangle.$$

For this particular Hessenberg function, each $\beta_i = i - \#\{h_k \mid h_k < i\} = i$ since $h_k = n$ for all k. Hence the generators have the form $\tilde{e}_i(x_i, \ldots, x_n)$. Recall a shorthand notation given in Section 4.2. The modified complete symmetric function $\tilde{e}_i(x_i, \ldots, x_n)$ will be denoted $\tilde{e}_i(x_{i,i+1,\ldots,n})$.

Example 7.1.1 ($I_h \subseteq J_h$ for $n = 4$)**.** Fix $n = 4$. Using the well-defined Gröbner division by the four generators of J_h, the elementary functions $e_1, \ldots, e_4$ can be written:

$$e_1 = e_0(x_{234}) \cdot \tilde{e}_1(x_{1234})$$

$$e_2 = e_1(x_{234}) \cdot \tilde{e}_1(x_{1234}) - e_0(x_{34}) \cdot \tilde{e}_2(x_{234})$$

$$e_3 = e_2(x_{234}) \cdot \tilde{e}_1(x_{1234}) - e_1(x_{34}) \cdot \tilde{e}_2(x_{234}) + e_0(x_4) \cdot \tilde{e}_3(x_{34})$$

$$e_4 = e_3(x_{234}) \cdot \tilde{e}_1(x_{1234}) - e_2(x_{34}) \cdot \tilde{e}_2(x_{234}) + e_1(x_4) \cdot \tilde{e}_3(x_{34}) - e_0(\varnothing) \cdot \tilde{e}_4(x_4)$$

where e_0 is defined to be 1. It is clear the expansions of the elementary symmetric functions as linear combination of generators of J_h are following a pattern. This pattern is proved in Theorem 7.1.2.

Theorem 7.1.2. *For $1 \leq r \leq n$, the elementary summetric function e_r has the following presentation:*

$$e_r = \sum_{i=1}^{r} (-1)^{i+1} e_{r-i}(x_{(i+1)\cdots n}) \cdot \tilde{e}_i(x_{i\cdots n}).$$

We conclude that if $h = (n, \ldots, n)$, then the corresponding ideal $I_h = \langle e_1, \ldots, e_n \rangle$ is contained in the ideal $J_h = \langle \tilde{e}_i(x_{i\cdots n}) \rangle_{i=1}^{n}$.

Proof. Let $\mathbf{X}_i$ be the variable set $\{x_1, \ldots, x_i\}$. Mora and Sala define the function $g_d(\mathbf{X}_n) := h_d(\mathbf{X}_{n-d+1})$ where $h_d(\mathbf{X}_r)$ is the standard complete symmetric function

$$\sum_{d_1+\cdots+d_r=d} x_1^{d_1} \cdots x_r^{d_r} \text{ where the } d_i \geq 0.$$

In [12, Prop 2.1] under the lexicographic ordering $x_1 < \cdots < x_n$, they give the following identity:

$$e_d + \sum_{i=1}^{d-1} (-1)^i g_i(\mathbf{X}_n) e_{d-i}(\mathbf{X}_{n-i}) + (-1)^d g_d(\mathbf{X}_n) = 0.$$

Their main result [12, Prop 2.2] implies that the identity holds for any term ordering $x_{\pi(1)} < \cdots < x_{\pi(n)}$ for $\pi \in S_n$. Hence we conclude that

$$e_r = \sum_{i=1}^{r} (-1)^{i+1} e_{r-i}(x_{(i+1)\cdots n}) \cdot \tilde{e}_i(x_{i\cdots n}) \text{ for } 1 \leq r \leq n$$

if we let $x_1 > x_2 > \cdots > x_n$. Hence the ideal I_h is contained in the ideal J_h in the $h = (n, \ldots, n)$ case. $\square$

Using this theorem, we prove that $I_h = J_h$ when $h = (n, \ldots, n)$.

Corollary 7.1.3. *For the Hessenberg function $h = (n, \ldots, n)$, the generators of the ideal J_h are linear combinations of the elementary symmetric functions. In particular for $1 \leq r \leq n$, the generator $\tilde{e}_r(x_{r \cdots n})$ has the following presentation:*

$$\tilde{e}_r(x_{r \cdots n}) = \sum_{j=1}^{r} (-1)^{j+1} \tilde{e}_{r-j}(x_{r \cdots n}) \cdot e_j$$

and hence $J_h \subseteq I_h$. We conclude $I_h = J_h$ when h is maximal.

Proof. Construct the lower triangular matrix $B = (f_{ij})$ where

$$f_{ij} = \begin{cases} (-1)^{j+1} e_{i-j}(x_{(j+1) \cdots n}) & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

which gives the transformation from the set $\{e_i\}_{i=1}^n$ to the set $\{\tilde{e}_i(x_{i \cdots n})\}_{i=1}^n$. Since B by construction is always invertible, we can find its inverse $B^{-1} = (g_{ij})$ where

$$g_{ij} = \begin{cases} (-1)^{j+1} \tilde{e}_{i-j}(x_{i \cdots n}) & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

We can then write $\tilde{e}_r(x_{r \cdots n})$ as a linear combination of elementary symmetric functions by reading the r^{th} row of B^{-1}. Thus $\tilde{e}_r(x_{r \cdots n}) = \sum_{j=1}^{r} (-1)^{j+1} \tilde{e}_{r-j}(x_{r \cdots n}) \cdot e_j$ as desired. □

Example 7.1.4 (The n=4 example revisited)**.** In the $n = 4$ case from Example 7.1.1 the corresponding tranformation matrices are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ e_1(x_{234}) & -1 & 0 & 0 \\ e_2(x_{234}) & -e_1(x_{34}) & 1 & 0 \\ e_3(x_{234}) & -e_2(x_{34}) & e_1(x_4) & -1 \end{pmatrix}$$

(a) The B matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde{e}_1(x_{234}) & -1 & 0 & 0 \\ \tilde{e}_2(x_{34}) & -\tilde{e}_1(x_{34}) & 1 & 0 \\ \tilde{e}_3(x_4) & -\tilde{e}_2(x_4) & \tilde{e}_1(x_4) & -1 \end{pmatrix}$$

(b) The B^{-1} matrix.

Reading off the rows of the inverse matrix as coefficients of the e_i-terms, we get

$$\begin{aligned}
\tilde{e}_1(x_{1234}) &= \tilde{e}_0(x_{1234}) \cdot e_1 \\
\tilde{e}_2(x_{234}) &= \tilde{e}_1(x_{234}) \cdot e_1 - \tilde{e}_0(x_{234}) \cdot e_2 \\
\tilde{e}_3(x_{34}) &= \tilde{e}_2(x_{34}) \cdot e_1 - \tilde{e}_1(x_{34}) \cdot e_2 + \tilde{e}_0(x_{34}) \cdot e_3 \\
\tilde{e}_4(x_4) &= \tilde{e}_3(x_4) \cdot e_1 - \tilde{e}_2(x_4) \cdot e_2 + \tilde{e}_1(x_4) \cdot e_3 - \tilde{e}_0(x_4) \cdot e_4
\end{aligned}$$

where $\tilde{e}_0$ is defined to be 1.

CHAPTER 8
FUTURE DIRECTION AND SOME OPEN QUESTIONS

In their own right, the family of ideals I_h and J_h are interesting objects of study. As an extra bonus, the quotient R/J_h has applications to calculations of cohomology rings. For the fixed shape $\mu = (n)$ and an arbitrary Hessenberg function h, the set of (h, μ)-fillings combinatorially describe the Betti numbers of the cohomology ring of the regular nilpotent Hessenberg vareties. On the other hand, the set of monomials $\mathcal{B}_h(\mu)$ that form a basis for the quotient R/J_h algebraically determine these same Betti numbers. This was shown in Theorem 6.3.3.

In this chapter, we explore future directions and open questions about the families of ideals I_h and J_h. A main conjecture is the following.

Conjecture 8.0.1. *Fix $\mu = (n)$ and let h be a Hessenberg function. The quotient R/J_h is a presentation for the cohomology ring of the regular nilpotent Hessenberg variety $\mathfrak{H}(X, h)$. Moreover, this gives the cohomology ring with integer coefficients.*

8.1 Peterson varieties

The family of regular nilpotent Hessenberg varieties contains a subclass of varieties called Peterson varieties. These are the $\mathfrak{H}(X, h)$ for which X is a regular nilpotent operator (equivalently, μ has shape (n)) and the Hessenberg function is defined as $h(i) = i + 1$ for $i < n$ and $h(n) = n$. Harada and Tymoczko [9] recently gave the first general computation of this cohomology ring in terms of generators and relations. Their presentation is given via a Monk-type formula. Although computable, the presentation is computationally heavy. Computer software such as Macaulay 2 is needed to produce small examples and exhibit a basis (via Gröbner basis reduction). For small n, we explored the relationship between their presentation and mine. Thus far, the two are isomorphic as rings. Besides its ease of computation, a further advantage of my conjectural presentation is that it generalizes to all regular nilpotent Hessenberg varieties.

8.2 Open questions

Question 1:

We showed in Section 2.2 that we have an inverse map Ψ if we fix $h = (1, 2, \ldots, n)$ and let μ vary. We showed in Section 6.2 that we have an inverse map Ψ_h if we fix $\mu = (n)$ and let h vary. Is there an inverse map $\Psi_{h,\mu}$ which incorporates both the h-function and the shape μ?

Question 2:

So far we have proven that the ideals I_h and J_h coincide for the maximal Hessenberg functions $h = (n, \ldots, n)$. For $n = 4$, we computed the unique reduced Gröbner basis for all other thirteen ideals I_h. Each ideal coincided with its corresponding ideal J_h for each h. Do $I_h = J_h$ for arbitrary h?

Question 3:

Is there a direct topological proof that our ring R/J_h is the cohomology ring of the regular nilpotent Hessenberg varieties?

APPENDIX
SOME LOOSE ENDS

A.1 Lattices of ideals I_h and J_h

At the end of this appendix, we present a variety of lattices representing the inclusion of ideals I_h and J_h as given in Chapters 3 and 4, where we proved that $h > h'$ implies $I_h \subset I_{h'}$ and $J_h \subset J_{h'}$. As with the Hasse diagrams on Hessenberg functions (or degree tuples), each edge of these lattices represent an inclusion of the object above the edge into the object below the edge. Figure A.1 gives all possible 14 ideals I_h corresponding to the 14 distinct Hessenberg functions when $n = 4$. Figure A.2 gives the h-Ferrers diagrams used to construct the generating sets for the ideals I_h. The circled numbers denote the anti-diagonal generators as given in the lattice of anti-diagonal ideals I_h^{AD} presented in the subsequent Figure A.3. Note in the latter figure, we use the shorthand $e_d(r)$ to denote the modified elementary symmetric function of degree d in variables $x_1, \ldots, x_r$. Figure A.4 gives the lattice of anti-diagonal ideals I_h^{AD} without shorthand notation. Similarly, Figures A.5 and A.6 give abridged and unabridged versions of the lattice of the 14 ideals J_h. In Figure A.5, we use the shorthand $\tilde{e}_d(r, \ldots, n)$ to denote the modified complete symmetric function of degree d in variables $x_r, \ldots, x_n$, where $r \leq n$.

A.2 Proofs of the claims from Section 5.2

Many of the statements made in Section 5.2 are well-known results from commutative algebra. Although much of this can be found in classical texts in the subject, we provide the proofs for all statements made in Section 5.2. To avoid overlapping references, Theorem 5.2.x from Section 5.2 will be called Theorem A.2.x here.

The proofs of Theorems A.2.1, A.2.2, and A.2.3 were sketched by Cox, Little, and O'Shea [3], but we provide all missing details. Proofs for Theorem A.2.4 and

Theorem A.2.9 were very weakly sketched and left much for the reader. For those theorems, the proofs below are completely our own as are the proofs of the four lemmas (which were exercises).

Theorem A.2.1 (Division by a Gröbner basis gives unique remainder)**.** *Assume $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for an ideal I in R and suppose $f \in R$. Then there exists a unique $r \in R$ such that:*

(i) No term of r is divisible by any $LT(g_1), \ldots, LT(g_t)$.

(ii) There exists $g \in I$ such that $f = g + r$.

In particular, r is the unique remainder upon division of f by G no matter how the elements of G are listed.

Proof. Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal I in R and suppose $f \in R$. By the division algorithm $f = q_1 g_1 + q_2 g_2 + \cdots + q_t g_t + r$ such that no term of r is divisible by any $LT(g_i)$. This proves (i). Letting $g = \sum_{i=1}^{t} q_i g_i$ we see that $g \in I$ and $f = g + r$. This proves (ii).

We now prove that r is unique. Suppose $f = g + r = g' + r'$ where $g, g' \in I$ and r, r' satisfy (i) above. Then $r - r' = g - g' \in I$. Suppose by contradiction that $r - r' \neq 0$. Then $LT(r - r') \in \langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$ since G is a Gröbner basis for I. This implies that $LT(r - r')$ is divisible by at least one of the $LT(g_i)$. This contradicts statement (i) that no term of r or r' is divisible by any $LT(g_i)$. Hence $r = r'$ and we conclude this remainder is unique. $\square$

We remark that listing the g_i in a different order produces different quotients q_i in the $f = \sum q_i g_i + r$ expansion. However by uniqueness, the remainder r must remain the same.

Corollary A.2.2 (Ideal Membership Criterion)**.** *A polynomial $f \in R$ is in an ideal I of R if and only if the remainder upon division by a Gröbner basis of I is zero.*

Proof. Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal I in R and suppose

$f \in R$. Then it suffices to show that $f \in I$ if and only if $\overline{f}^G = 0$, where the symbol $\overline{f}^G$ is as defined in Remark 5.1.6.

First assume $f \in I$. Then $f = f + 0$ satisfies conditions (i) and (ii) in Theorem A.2.1. Thus 0 is the unique remainder of f upon division by G, so $\overline{f}^G = 0$. Now assume $\overline{f}^G = 0$. Then $f = q_1 g_1 + q_2 g_2 + \cdots + q_t g_t$ and so $f \in I$. □

Theorem A.2.3. *Given $f \in R$ and an ideal I in R, f is congruent modulo I to a unique polynomial r. This polynomial r is a k-linear combination of monomials in the complement of $\langle LT(I) \rangle$.*

Proof. Let $f \in R$ and suppose I is an ideal in R. Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for I. By Theorem A.2.1, we know that $f = g + r$ where $g \in I$ and r is the unique remainder after division by G. So $f - r = g$ implies $f \equiv r$ modulo I. Lastly, by (i) of Theorem A.2.1, no term of r is divisible by any $LT(g_i)$. Thus the remainder r lies in the complement of $\langle LT(g_1), \ldots, LT(g_t) \rangle$. But since G is a Gröbner basis, we know $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$, and we are done. □

Theorem A.2.4. *[3, Exer.1, pg.237]. Let I be an ideal in R. Then, the set $\{\boldsymbol{x}^\alpha \mid \boldsymbol{x}^\alpha \notin \langle LT(I) \rangle\}$ is linearly independent modulo I. That is, if $\sum_\alpha c_\alpha \boldsymbol{x}^\alpha \equiv 0$ modulo I and each $\boldsymbol{x}^\alpha \notin \langle LT(I) \rangle$, then $c_\alpha = 0$ for all α.*

Proof. Let I be an ideal in R, and suppose $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for I. Assume $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ where each $\mathbf{x}^\alpha \notin \langle LT(I) \rangle$ and $f \equiv 0$ modulo I. We claim the c_α are all zero.

Since G is a Gröbner basis for I, we know $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$ and $\mathbf{x}^\alpha \notin \langle LT(g_1), \ldots, LT(g_t) \rangle$. None of the $LT(g_i)$ divide any term of f since $\langle LT(I) \rangle$ is a monomial ideal. By Theorem A.2.1, we can write $f = 0 + f$ where this remainder f is unique since it satisfies the conditions of the theorem. But by assumption, $f \equiv 0$ modulo I and hence $f \in I$. So we can also write this expression of f as $f = f + 0$ which also satisfies the conditions of Theorem A.2.1. By uniqueness of remainder we must have $f = 0$, and we are done. $\square$

Lemma A.2.5 (Strengthened Version of Theorem A.2.1)**.** *[3, Exer.1, pg.87]. Let $G = \{g_1, \dots, g_t\}$ be a Gröbner basis for an ideal I in R and suppose $f \in R$. Then there exists a unique $r \in R$ such that:*

(i) No term of r is divisible by any element of $LT(I)$.

(ii) There exists $g \in I$ such that $f = g + r$.

So r is the unique remainder upon division of f by I. Furthermore, this remainder coincides with the remainder given in Theorem A.2.1.

Proof. By Theorem A.2.1, we can write $f = g+r$ where $g \in I$ and no term of r is divisible by any of the $LT(g_i)$. Assume instead that some term T of r is divisible by an element of $LT(I)$. That would force $T \in LT(I) \subseteq \langle LT(I)\rangle = \langle LT(g_1), \dots, LT(g_t)\rangle$. Hence, some $LT(g_i)$ would divide T contradicting the hypothesis that no term of r is divisible by any of the $LT(g_i)$. $\square$

Lemma A.2.6. *[3, Exer.12a, pg.88]. If f and g are polynomials in R, then $\overline{f}^G = \overline{g}^G$ if and only if $f - g \in I$.*

Proof. First assume that $\overline{f}^G = \overline{g}^G$. By Theorem A.2.1, we can write $f = i_1 + \overline{f}^G$ and $g = i_2 + \overline{g}^G$ where $i_1, i_2 \in I$. Then

$$f - g = (i_1 - i_2) + (\overline{f}^G - \overline{g}^G) = (i_1 - i_2) + (\overline{f}^G - \overline{f}^G) = i_1 - i_2.$$

Thus $f - g \in I$.

Now assume $f - g \in I$. By the Lemma A.2.5, we can write $f = i_1 + r_1$ and $g = i_2 + r_2$ where $i_1, i_2 \in I$ and no term of r_1 or r_2 is divisible by any element of $LT(I)$. Thus, no term of $r_1 - r_2$ is divisible by any element of $LT(I)$ either. Hence

$$f - g = (i_1 - i_2) + (r_1 - r_2)$$

where $i_1 - i_2 \in I$. By Lemma A.2.5, we know $r_1 - r_2$ is the unique remainder of $f - g$ upon division by I. We may also write $f - g = (f - g) + 0$ which also satisfies the conditions of Lemma A.2.5. By uniqueness of remainder $r_1 - r_2 = 0$, and hence

$r_1 = r_2$. Thus $\overline{f}^G = \overline{g}^G$. $\square$

Lemma A.2.7. *[3, Exer.12b, pg.88]. If f and g are polynomials in R, then $\overline{f+g}^G = \overline{f}^G + \overline{g}^G$.*

Proof. By Lemma A.2.5, we can write $f = i_1 + r_1$ and $g = i_2 + r_2$ where $i_1, i_2 \in I$ and no term of r_1 or r_2 is divisible by any element of $LT(I)$. Consider $f+g$. We have $f + g = (i_1 + i_2) + (r_1 + r_2)$ where $i_1 + i_2$ and $r_1 + r_2$ satisfy the conditions of Lemma A.2.5. Thus $r_1 + r_2$ is the unique remainder of $f+g$ upon division by G. Hence $\overline{f+g}^G = r_1 + r_2$. But $r_1 = \overline{f}^G$ and $r_2 = \overline{g}^G$ so we are done. $\square$

Lemma A.2.8. *[3, Exer.12c, pg.88]. If f and g are polynomials in R, then $\overline{fg}^G = \overline{\overline{f}^G + \overline{g}^G}^G$.*

Proof. Again use Lemma A.2.5 to write $f = i_1 + r_1$ and $g = i_2 + r_2$ where $i_1, i_2 \in I$ and no term of r_1 or r_2 is divisible by any element of $LT(I)$. Consider $f \cdot g$. We have

$$f \cdot g = (i_1 + r_1)(i_2 + r_2) = (i_1 i_2 + r_1 i_2 + i_1 r_2) + r_1 r_2.$$

Since $i_1, i_2 \in I$, the sum $i' := (i_1 i_2 + r_1 i_2 + i_1 r_2) \in I$. Also $\overline{f}^G \cdot \overline{g}^G = r_1 r_2$. Thus $f \cdot g = i' + \overline{f}^G \cdot \overline{g}^G$ holds and hence $f \cdot g - \overline{f}^G \cdot \overline{g}^G = i' \in I$. By Lemma A.2.6, we have $\overline{fg}^G = \overline{\overline{f}^G + \overline{g}^G}^G$ as desired. $\square$

Theorem A.2.9 (A Basis for R/I)**.** *R/I is isomorphic to the k-span of the set $\{\boldsymbol{x}^\alpha | \boldsymbol{x}^\alpha \notin \langle LT(I) \rangle\}$ as k-vector spaces.*

Proof. Define a map $\phi : R/I \to$ k-span-$\{\mathbf{x}^\alpha | \mathbf{x}^\alpha \notin \langle LT(I) \rangle\}$ by the rule that $\phi([f]) = \overline{f}^G$ where $\overline{f}^G$ is the unique remainder guaranteed by Theorem A.2.3. This map is well-defined and injective since $[f] = [g]$ if and only if $f - g \in I$, and $f - g \in I$ if and only if $\overline{f}^G = \overline{g}^G$ by Lemma A.2.6, and finally $\overline{f}^G = \overline{g}^G$ if and only if $\phi([f]) = \phi([g])$. The map is surjective since any element $\mathbf{x}^\alpha$ in the codomain is the image of the class $[\mathbf{x}^\alpha]$ in the domain. Thus ϕ is a bijective correspondence. It suffices then to show that ϕ preserves the vector space operations. The following string of equalities shows ϕ preserves addition:

$$\begin{aligned}
\phi([f]+[g]) &= \phi([f+g]) \\
&= \overline{f+g}^G \\
&= \overline{f}^G + \overline{g}^G \qquad \text{by Lemma A.2.7} \\
&= \phi([f]) + \phi([g]).
\end{aligned}$$

We now prove ϕ preserves scalar multiplication. We have $\phi(c[f]) = \phi([cf]) = \overline{cf}^G$ for each $c \in k$. We claim $\overline{cf}^G = c\overline{f}^G$. Indeed, by Theorem A.2.1, we can write $cf = i + \overline{cf}^G$ for some $i \in I$. Also by Theorem A.2.1, we know $f = i' + \overline{f}^G$ for some $i' \in I$. Hence $cf = ci' + c\overline{f}^G$. By uniqueness of remainder we must have $\overline{cf}^G = c\overline{f}^G$. We conclude that $\phi(c[f]) = c\overline{f}^G = c \cdot \phi([f])$ as desired. Therefore ϕ is a linear vector space isomorphism. Since the set $\{\mathbf{x}^\alpha | \mathbf{x}^\alpha \notin \langle LT(I) \rangle\}$ is linearly independent by Theorem A.2.4, we conclude this set is a basis for R/I.

$\square$

Observation A.2.10. *The quotient R/I is almost – but not! – ring isomorphic to the k-span of $\{\boldsymbol{x}^\alpha | \boldsymbol{x}^\alpha \notin \langle LT(I) \rangle\}$.*

In the proof of Theorem A.2.9, we showed that $\phi([f]+[g]) = \phi([f]) + \phi([g])$. However, the product $\phi([f]) \cdot \phi([g]) = \overline{f}^G \cdot \overline{g}^G$ is not necessarily in the k-span of $\{\mathbf{x}^\alpha | \mathbf{x}^\alpha \notin \langle LT(I) \rangle\}$. It would be if we divided this product by G once again. But then Lemma A.2.8 states $\overline{fg}^G = \overline{\overline{f}^G \cdot \overline{g}^G}^G$ giving

$$\phi([f] \cdot [g]) = \phi([fg]) = \overline{fg}^G = \overline{\overline{f}^G \cdot \overline{g}^G}^G = \overline{\phi([f]) \cdot \phi([g])}^G.$$

This is almost – but not! – a ring isomorphism between R/I and the k-span of $\{\mathbf{x}^\alpha | \mathbf{x}^\alpha \notin \langle LT(I) \rangle\}$.

$$
\begin{tikzcd}[column sep=small]
& \begin{pmatrix} e_1, e_2, e_3, e_4 \end{pmatrix}^{h=(4,4,4,4)} \arrow[d] & \\
& \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2x_3 \end{pmatrix}^{h=(3,4,4,4)} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2x_3 \\ x_1x_2 + x_1x_3 + x_2x_3 \end{pmatrix}^{h=(3,3,4,4)} \arrow[d] \arrow[dr] & & \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2 \end{pmatrix}^{h=(2,4,4,4)} \arrow[dl] \arrow[d] \\
\begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2x_3,\ x_1 + x_2 + x_3 \\ x_1x_2 + x_1x_3 + x_2x_3 \end{pmatrix}^{h=(3,3,3,4)} \arrow[d] & \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2x_3,\ x_1x_2 \\ x_1x_2 + x_1x_3 + x_2x_3 \end{pmatrix}^{h=(2,3,4,4)} \arrow[dl] \arrow[d] \arrow[dr] & \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1 \end{pmatrix}^{h=(1,4,4,4)} \arrow[d] \\
\begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2x_3,\ x_1 + x_2 + x_3 \\ x_1x_2 + x_1x_3 + x_2x_3 \\ x_1x_2 \end{pmatrix}^{h=(2,3,3,4)} \arrow[d] \arrow[dr] & \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2,\ x_1 + x_2 \end{pmatrix}^{h=(2,2,4,4)} \arrow[dl] \arrow[dr] & \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1,\ x_1x_2x_3 \\ x_1x_2 + x_1x_3 + x_2x_3 \end{pmatrix}^{h=(1,3,4,4)} \arrow[dl] \arrow[d] \\
\begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2x_3,\ x_1 + x_2 + x_3 \\ x_1x_2 + x_1x_3 + x_2x_3 \\ x_1x_2,\ x_1 + x_2 \end{pmatrix}^{h=(2,2,3,4)} \arrow[dr] & \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1,\ x_1x_2x_3 \\ x_1x_2 + x_1x_3 + x_2x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}^{h=(1,3,3,4)} \arrow[d] & \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2,\ x_1 + x_2 \\ x_1 \end{pmatrix}^{h=(1,2,4,4)} \arrow[dl] \\
& \begin{pmatrix} e_1, e_2, e_3, e_4 \\ x_1x_2x_3,\ x_1 + x_2 + x_3 \\ x_1x_2 + x_1x_3 + x_2x_3 \\ x_1x_2,\ x_1 + x_2 \\ x_1 \end{pmatrix}^{h=(1,2,3,4)} &
\end{tikzcd}
$$

Figure A.1: Lattice of ideals I_h for $n = 4$.

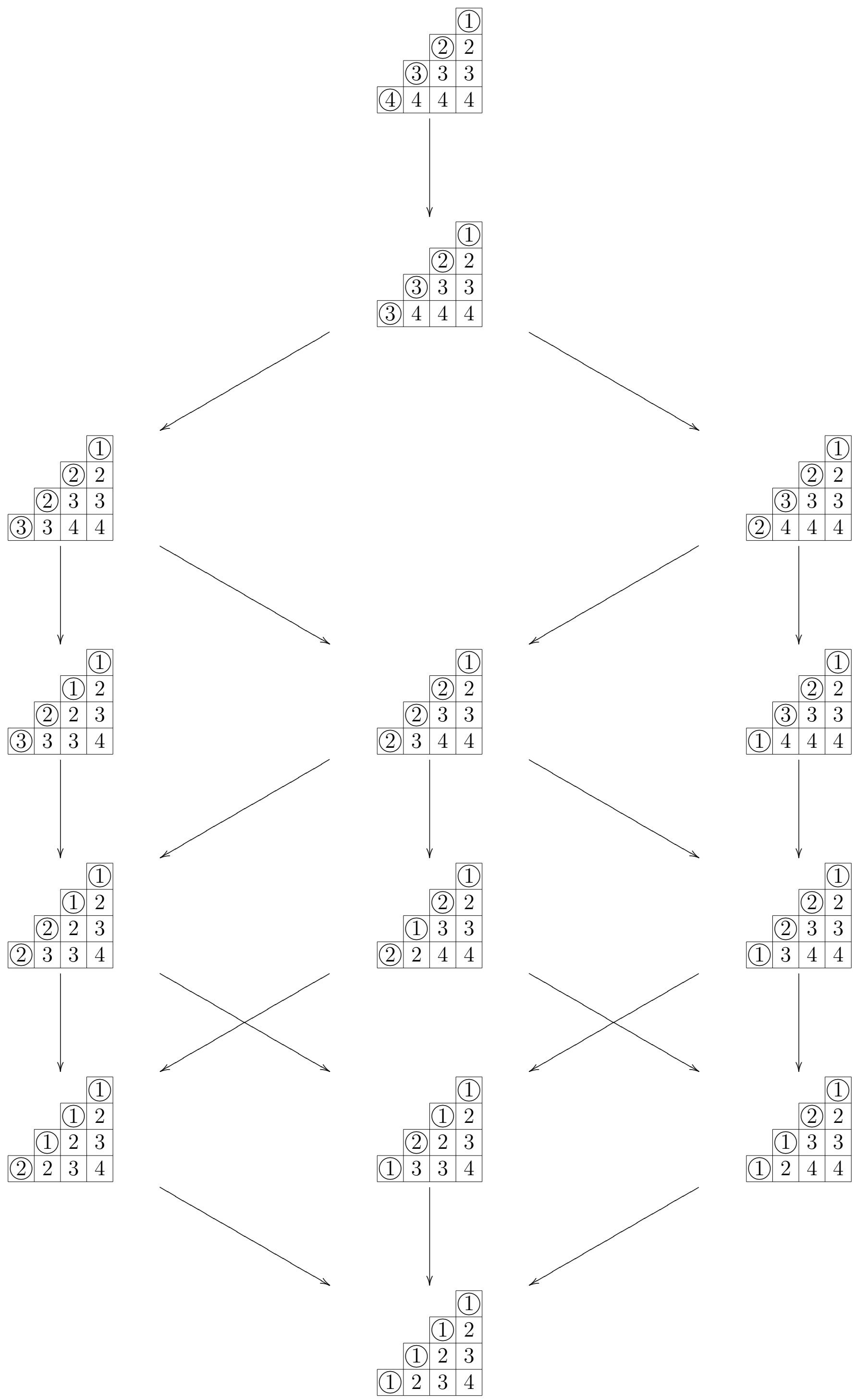

Figure A.2: Lattice of h-Ferrers diagrams for $n = 4$.

$$
\begin{tikzcd}[column sep=tiny]
& \begin{pmatrix} h=(4,4,4,4) \\ e_4(4),\ e_3(4),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[d] & \\
& \begin{pmatrix} h=(3,4,4,4) \\ e_3(3),\ e_3(4),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} h=(3,3,4,4) \\ e_3(3),\ e_2(3),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[d] \arrow[dr] & & \begin{pmatrix} h=(2,4,4,4) \\ e_2(2),\ e_3(4),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(3,3,3,4) \\ e_3(3),\ e_2(3),\ e_1(3),\ e_1(4) \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(2,3,4,4) \\ e_2(2),\ e_2(3),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[dl] \arrow[d] \arrow[dr] & \begin{pmatrix} h=(1,4,4,4) \\ e_1(1),\ e_3(4),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[d] \\
\begin{pmatrix} h=(2,3,3,4) \\ e_2(2),\ e_2(3),\ e_1(3),\ e_1(4) \end{pmatrix} \arrow[d] \arrow[dr] & \begin{pmatrix} h=(2,2,4,4) \\ e_2(2),\ e_1(2),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[dl] \arrow[dr] & \begin{pmatrix} h=(1,3,4,4) \\ e_1(1),\ e_2(3),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(2,2,3,4) \\ e_2(2),\ e_1(2),\ e_1(3),\ e_1(4) \end{pmatrix} \arrow[dr] & \begin{pmatrix} h=(1,3,3,4) \\ e_1(1),\ e_2(3),\ e_1(3),\ e_1(4) \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(1,2,4,4) \\ e_1(1),\ e_1(2),\ e_2(4),\ e_1(4) \end{pmatrix} \arrow[dl] \\
& \begin{pmatrix} h=(1,2,3,4) \\ e_1(1),\ e_1(2),\ e_1(3),\ e_1(4) \end{pmatrix} &
\end{tikzcd}
$$

Figure A.3: Lattice of ideals I_h^{AD} for $n = 4$ (abridged).

$$
\begin{tikzcd}[column sep=small]
& \begin{pmatrix} h=(4,4,4,4) \\ e_1,\ e_2,\ e_3,\ e_4 \end{pmatrix} \arrow[d] & \\
& \begin{pmatrix} h=(3,4,4,4) \\ e_1,\ e_2,\ e_3 \\ x_1x_2x_3 \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} h=(3,3,4,4) \\ e_1,\ e_2 \\ x_1x_2x_3 \\ x_1x_2+x_1x_3+x_2x_3 \end{pmatrix} \arrow[d] \arrow[dr] & & \begin{pmatrix} h=(2,4,4,4) \\ e_1,\ e_2,\ e_3 \\ x_1x_2 \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(3,3,3,4) \\ e_1 \\ x_1x_2x_3,\ x_1+x_2+x_3 \\ x_1x_2+x_1x_3+x_2x_3 \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(2,3,4,4) \\ e_1,\ e_2 \\ x_1x_2 \\ x_1x_2+x_1x_3+x_2x_3 \end{pmatrix} \arrow[dl] \arrow[d] \arrow[dr] & \begin{pmatrix} h=(1,4,4,4) \\ e_1,\ e_2,\ e_3 \\ x_1 \end{pmatrix} \arrow[d] \\
\begin{pmatrix} h=(2,3,3,4) \\ e_1 \\ x_1+x_2+x_3 \\ x_1x_2+x_1x_3+x_2x_3 \\ x_1x_2 \end{pmatrix} \arrow[d] \arrow[dr] & \begin{pmatrix} h=(2,2,4,4) \\ e_1,\ e_2 \\ x_1x_2,\ x_1+x_2 \end{pmatrix} \arrow[dl] \arrow[dr] & \begin{pmatrix} h=(1,3,4,4) \\ e_1,\ e_2 \\ x_1 \\ x_1x_2+x_1x_3+x_2x_3 \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(2,2,3,4) \\ e_1 \\ x_1+x_2+x_3 \\ x_1x_2,\ x_1+x_2 \end{pmatrix} \arrow[dr] & \begin{pmatrix} h=(1,3,3,4) \\ e_1 \\ x_1 \\ x_1x_2+x_1x_3+x_2x_3 \\ x_1+x_2+x_3 \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(1,2,4,4) \\ e_1,\ e_2 \\ x_1,\ x_1+x_2 \end{pmatrix} \arrow[dl] \\
& \begin{pmatrix} h=(1,2,3,4) \\ e_1 \\ x_1+x_2+x_3 \\ x_1+x_2 \\ x_1 \end{pmatrix} &
\end{tikzcd}
$$

Figure A.4: Lattice of ideals I_h^{AD} for $n=4$ (unabridged).

$$
\begin{tikzcd}[column sep=small]
& \begin{pmatrix} h=(4,4,4,4)\ \&\ \beta=(4,3,2,1) \\ \tilde{e}_4(4),\tilde{e}_3(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[d] & \\
& \begin{pmatrix} h=(3,4,4,4)\ \&\ \beta=(3,3,2,1) \\ \tilde{e}_3(4),\tilde{e}_3(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} h=(3,3,4,4)\ \&\ \beta=(2,3,2,1) \\ \tilde{e}_2(4),\tilde{e}_3(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[d] \arrow[dr] & & \begin{pmatrix} h=(2,4,4,4)\ \&\ \beta=(3,2,2,1) \\ \tilde{e}_3(4),\tilde{e}_2(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(3,3,3,4)\ \&\ \beta=(1,3,2,1) \\ \tilde{e}_1(4),\tilde{e}_3(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(2,3,4,4)\ \&\ \beta=(2,2,2,1) \\ \tilde{e}_2(4),\tilde{e}_2(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[dl] \arrow[d] \arrow[dr] & \begin{pmatrix} h=(1,4,4,4)\ \&\ \beta=(3,2,1,1) \\ \tilde{e}_3(4),\tilde{e}_2(3,4) \\ \tilde{e}_1(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[d] \\
\begin{pmatrix} h=(2,3,3,4)\ \&\ \beta=(1,2,2,1) \\ \tilde{e}_1(4),\tilde{e}_2(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[d] \arrow[dr] & \begin{pmatrix} h=(2,2,4,4)\ \&\ \beta=(2,1,2,1) \\ \tilde{e}_2(4),\tilde{e}_1(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[dl] \arrow[dr] & \begin{pmatrix} h=(1,3,4,4)\ \&\ \beta=(2,2,1,1) \\ \tilde{e}_2(4),\tilde{e}_2(3,4) \\ \tilde{e}_1(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(2,2,3,4)\ \&\ \beta=(1,1,2,1) \\ \tilde{e}_1(4),\tilde{e}_1(3,4) \\ \tilde{e}_2(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[dr] & \begin{pmatrix} h=(1,3,3,4)\ \&\ \beta=(1,2,1,1) \\ \tilde{e}_1(4),\tilde{e}_2(3,4) \\ \tilde{e}_1(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(1,2,4,4)\ \&\ \beta=(2,1,1,1) \\ \tilde{e}_2(4),\tilde{e}_1(3,4) \\ \tilde{e}_1(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} \arrow[dl] \\
& \begin{pmatrix} h=(1,2,3,4)\ \&\ \beta=(1,1,1,1) \\ \tilde{e}_1(4),\tilde{e}_1(3,4) \\ \tilde{e}_1(2,3,4),\tilde{e}_1(1,2,3,4) \end{pmatrix} &
\end{tikzcd}
$$

Figure A.5: Lattice of ideals J_h for $n = 4$ (abridged).

$$
\begin{tikzcd}[column sep=small]
 & \begin{pmatrix} h=(4,4,4,4)\ \text{\&}\ \beta=(4,3,2,1) \\ x_4^4,\quad x_3^3+x_3^2x_4+x_3x_4^2+x_4^3 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[d] & \\
 & \begin{pmatrix} h=(3,4,4,4)\ \text{\&}\ \beta=(3,3,2,1) \\ x_4^3,\quad x_3^3+x_3^2x_4+x_3x_4^2+x_4^3 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} h=(3,3,4,4)\ \text{\&}\ \beta=(2,3,2,1) \\ x_4^2,\quad x_3^3+x_3^2x_4+x_3x_4^2+x_4^3 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[d] \arrow[dr] & & \begin{pmatrix} h=(2,4,4,4)\ \text{\&}\ \beta=(3,2,2,1) \\ x_4^3,\quad x_3^2+x_3x_4+x_4^2 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(3,3,3,4)\ \text{\&}\ \beta=(1,3,2,1) \\ x_4^1,\quad x_3^3+x_3^2x_4+x_3x_4^2+x_4^3 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(2,3,4,4)\ \text{\&}\ \beta=(2,2,2,1) \\ x_4^2,\quad x_3^2+x_3x_4+x_4^2 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[dl] \arrow[d] \arrow[dr] & \begin{pmatrix} h=(1,4,4,4)\ \text{\&}\ \beta=(3,2,1,1) \\ x_4^3,\quad x_3^2+x_3x_4+x_4^2 \\ x_2+x_3+x_4 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[d] \\
\begin{pmatrix} h=(2,3,3,4)\ \text{\&}\ \beta=(1,2,2,1) \\ x_4,\quad x_3^2+x_3x_4+x_4^2 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[d] \arrow[dr] & \begin{pmatrix} h=(2,2,4,4)\ \text{\&}\ \beta=(2,1,2,1) \\ x_4^2,\quad x_3+x_4 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[dl] \arrow[dr] & \begin{pmatrix} h=(1,3,4,4)\ \text{\&}\ \beta=(2,2,1,1) \\ x_4^2,\quad x_3^2+x_3x_4+x_4^2 \\ x_2+x_3+x_4 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[dl] \arrow[d] \\
\begin{pmatrix} h=(2,2,3,4)\ \text{\&}\ \beta=(1,1,2,1) \\ x_4,\quad x_3+x_4 \\ x_2^2+x_2x_3+x_2x_4+x_3^2+x_3x_4+x_4^2 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[dr] & \begin{pmatrix} h=(1,3,3,4)\ \text{\&}\ \beta=(1,2,1,1) \\ x_4,\quad x_3^2+x_3x_4+x_4^2 \\ x_2+x_3+x_4 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[d] & \begin{pmatrix} h=(1,2,4,4)\ \text{\&}\ \beta=(2,1,1,1) \\ x_4^2,\quad x_3+x_4 \\ x_2+x_3+x_4 \\ x_1+x_2+x_3+x_4 \end{pmatrix} \arrow[dl] \\
 & \begin{pmatrix} h=(1,2,3,4)\ \text{\&}\ \beta=(1,1,1,1) \\ x_4,\quad x_3+x_4 \\ x_2+x_3+x_4 \\ x_1+x_2+x_3+x_4 \end{pmatrix} &
\end{tikzcd}
$$

Figure A.6: Lattice of ideals J_h for $n = 4$ (unabridged).

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