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C*-algebras of labeled graphs and *-commuting endomorphisms

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C^*-ALGEBRAS OF LABELED GRAPHS AND *-COMMUTING ENDOMORPHISMS

by

Paulette Nicole Willis

An Abstract

Of a thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics in the Graduate College of The University of Iowa

May 2010

Thesis Supervisors: Professor Paul S. Muhly

ABSTRACT

My research lies in the general area of functional analysis. I am particularly interested in C^*-algebras and related dynamical systems. From the very beginning of the theory of operator algebras, in the works of Murray and von Neumann dating from the mid $1930's$, dynamical systems and operator algebras have led a symbiotic existance. Murray and von Neumann's work grew from a few esoteric, but clearly original and prescient papers, to a major river of contemporary mathematics. My work lies at the confluence of two important tributaries to this river.

On the one hand, the operator algebras that I study are C^*-algebras that are built from graphs. On the other, the dynamical systems on which I focus are symbolic dynamical systems of various types. My goal is to use dynamical systems theory to construct new and interesting C^*-algebras and to use the algebraic invariants of these algebras to reveal properties of the dynamics. My work has two fairly distinct strands: One deals with C^*-algebras built from *irreversible* dynamical systems. The other deals with group actions on graph C^*-algebras and their generalizations.

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Thesis Supervisor

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Title and Department

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Date

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Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Paulette Nicole Willis

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2010 graduation.

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Richard Baker

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Vincent Rodgers

To Jay for providing silent inspiration

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If I have forgotten anyone, please charge it to my head and not my heart–so this is for you. I would like to thank (fill in your name here) for without you this journey would have been more difficult. I am indebted to you. Thank you for believing in me.

ABSTRACT

My research lies in the general area of functional analysis. I am particularly interested in C^*-algebras and related dynamical systems. From the very beginning of the theory of operator algebras, in the works of Murray and von Neumann dating from the mid $1930's$, dynamical systems and operator algebras have led a symbiotic existance. Murray and von Neumann's work grew from a few esoteric, but clearly original and prescient papers, to a major river of contemporary mathematics. My work lies at the confluence of two important tributaries to this river.

On the one hand, the operator algebras that I study are C^*-algebras that are built from graphs. On the other, the dynamical systems on which I focus are symbolic dynamical systems of various types. My goal is to use dynamical systems theory to construct new and interesting C^*-algebras and to use the algebraic invariants of these algebras to reveal properties of the dynamics. My work has two fairly distinct strands: One deals with C^*-algebras built from *irreversible* dynamical systems. The other deals with group actions on graph C^*-algebras and their generalizations.

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CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Introduction

This dissertation describes two very different research projects. One is "complete" in the sense that it is ready for publication and the whole project builds to a single result, an isomorphism theorem, Theorem 2.6.9. Along the way, tools are developed, which are of independent interest, but their purpose for this dissertation is to yield Theorem 2.6.9. The second project contains many publishable results, but in a sense they are more scattered. Their unity of purpose is less easy to detect then that of the results in the first project. Therefore I want to begin with a bird's eye view of what I have done and I want to provide a road map to the various pieces.

Chapter 1 provides general information about the type of operators and C^*-algebras I study. It will also give the reader an idea of where my work fits in with what has been done previously. Chapter 2 discusses group actions on labeled graphs. Although Chapter 2 is self-contained, one may wish to read section 1.4 of chapter 1 to get an understanding of the general area and then return to it as necessary. Chapter 3 discusses *-commuting local homeomorphisms in many different settings. Although Chapter 3 is self-contained, one may wish to read section 1.3 of chapter 1 to get an understanding of the general area and for information that may help put the results of Chapter 3 in perspective.

1.2 General Background

An *operator algebra* is an algebra of continuous linear operators acting on a topological vector space with the multiplication given by composition of mappings. The principle examples in the literature are algebras of operators acting on Hilbert space. The algebras are assumed to be closed in the operator norm. An operator algebra on Hilbert space that is also closed under adjoints is called a *concrete*

C^*-algebra. Such an algebra may be characterized abstractly, through the work of Gelfand, Naimark, Segal and others, as an involutive Banach algebra with the property that $\|a^*a\| = \|a\|^2$ for all a. Nowadays, the distinction between concrete and abstract C^*-algebras is usually ignored, although at times it is important to make a distinction, as we shall see.

The theory of C^*-algebras is sometimes referred to as *noncommutative topology.* The reason for this is that the commutative C^*-algebras are precisely the algebras of the form $C_0(X)$, the space of continuous complex-valued functions vanishing at ∞ on a locally compact Hausdorff space X. Further, these algebras transform contravariantly under continuous (proper) maps. Familiar noncommutative examples are the $n \times n$ complex matrices and, more generally, the algebra of compact operators on Hilbert space. The full algebra of operators on Hilbert space is of course a C^*-algebra, too. In a sense, the possibilities are endless, so I won't give examples other than the ones I consider in my research and certain relatives.

Many C^*-algebras are built from generators and relations. For these, it is best to think in terms of abstract C^*-algebras. My first examples are, historically, also the first examples that connect operator algebra to dynamical systems: so-called *crossed product* or *transformation group* C^*-algebras associated to groups acting on compact spaces. Perhaps the simplest example among these occurs when one takes a single homeomorphism, say α, acting on a compact space X. The C^*-algebra, denoted by $C^*(X, \alpha)$ and $C(X) \rtimes_\alpha \mathbb{Z}$, is a natural completion of $C_c(X \times \mathbb{Z})$, the space of continuous compactly supported functions on $X \times \mathbb{Z}$, under pointwise addition and scalar multiplication, product defined by the formula

$$f * g(x, n) := \sum f(x, k) g(\alpha^{-n}(x), n - k),$$

and involution defined by the formula $f^*(x, n) = \overline{f(\alpha^{-n}(x), -n)}$. This example is generalized by replacing $C(X)$ by more general C^*-algebras and by replaceing the integers by other groups - possibly topological groups. For a second example that is very important for my work, suppose $E := (E^0, E^1, r, s)$ is a countable directed graph, meaning that E^0 and E^1 are countable sets, called the *vertices* and *edges* of E, and r and s are maps from E^1 to E^0, called the *range* and *source* maps. Then in a very natural way, one can construct a C^*-algebra, $C^*(E)$ with generators $\{P_v\}_{v\in E^0}$ and $\{S_e\}_{e\in E^1}$ subject to the following relations:

(CK1) The P_v are projections (i.e., Hermitian idempotents) in $C^*(E)$.

(CK2) The S_e are partial isometries in $C^*(E)$.

(CK3) $S_e^*S_e = P_{s(e)}$ for all $e \in E^1$; and

(CK4) $P_v = \sum_{r(e)=v} S_eS_e^*$ provided the sum is finite and $r^{-1}(v) \neq \emptyset$.

Notice that (CK2) is redundant since (CK3) already implies that S_e is a partial isometry. It is traditional, however, to include (CK2) among the axioms.

The algebra $C^*(E)$ is sometimes called a Cuntz-Krieger algebra out of homage to Cuntz and Krieger who introduced the relations (CK1) - (CK4) [5]. The C^*-algebra $C^*(E)$ encodes the properties of E, but the relation between E and $C^*(E)$ is not bijective: non-isomorphic graphs can give rise to isomorphic $C^*(E)$. Graph C^*-algebras are of interest, both for their theory and for their applications. What makes them especially attractive is the simplicity with which they are constructed. As a result, they make a marvelous environment in which to test questions about arbitrary C^*-algebras.

Of course, the connection between C^*-algebras and dynamics is explicit in the first example, but there is also a connection between examples of the graph sort and shift dynamical systems: Indeed, if $E = (E^0, E^1, r, s)$ is a *finite* graph then the infinite path space $E^\infty := \{(e_1, e_2, \ldots) \mid s(e_i) = r(e_{i+1})\}$ is a compact subset of $(E^1)^{\mathbb{N}}$ with the product topology that is invariant under the map σ defined by the equation $\sigma(e_1, e_2, \ldots) := (e_2, e_3, \ldots)$. The map σ (restricted to E^∞) is called a *shift* *of finite type*. Much of the theory of shifts of finite type has direct analogues in the structure theory of the associated graph C^*-algebras.

1.3 Dynamical systems

The theory of dynamical systems is a broad area of contemporary mathematics that includes subdisciplines such as chaos theory and ergodic theory, and has applications to all the natural sciences and engineering.

My interests are closely aligned to the theory of discrete dynamical systems, also called symbolic dynamical systems, which have been used, for example, to improve deep space communication with satellites and also in the creation of compact discs and DVD's. To understand the connection, note that the bits on the surface of a compact audio disk are written in a long sequence obeying the constraint that between successive 1's there are at least two 0's but no more than ten 0's. How can one efficiently transform arbitrary data (such as a computer program or a Beethoven symphony) into sequences that satisfy such constraints? What are the theoretical limits of a transformation? We are confronted with a space of sequences having a finite description, and we ask questions about such spaces and ways to encode and decode data from one space (the space of arbitrary sequences) to another (the space of constrained sequences). The study of symbolic dynamical systems tells us when such codes are possible, and gives us algorithms for finding them.

Shift spaces are to symbolic dynamical systems what shapes, like polygons and curves, are to geometry. Information is often represented as a sequence of discrete symbols drawn from a fixed finite set. This document, for example, is really a long sequence of letters, punctuation, and other symbols. A real number is described by the infinite sequence of symbols in its decimal expansion. Computers store data as sequences of 0's and 1's. Compact audio disks use blocks of 0's and 1's, representing signal samples, to record Beethoven symphonies.

In each of these examples, there is a finite set $\mathcal{A}$ of *symbols* which we will call the *alphabet*. Decimal expansions, for example, use the alphabet $\mathcal{A} = \{0, 1, \cdots, 9\}$.

Definition 1.3.1. If $\mathcal{A}$ is a finite alphabet, then the *full $\mathcal{A}$-shift* is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. The full $\mathcal{A}$-shift is denoted by $\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}$.

A *block* (or word) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$, which we will think of as being the *forbidden blocks*. For any such $\mathcal{F}$, define $\chi_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any block in $\mathcal{F}$.

Definition 1.3.2. A *shift space* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = \chi_{\mathcal{F}}$ for some collection of $\mathcal{F}$ of forbidden blocks over $\mathcal{A}$.

Quite generally, a *dynamical system* is a triple (X, G, α) where X is a space, G is a group, or semigroup, and α is an action of G on X. Shift spaces with the associated actions of $\mathbb{Z}$ or $\mathbb{Z}_+$ are the most prominent examples in my work, but others arise as well.

I am interested in those C^*-algebras that arise from universal constructions, in particular, those which arise from dynamical systems (X, G, α). The action of G on X is promoted to an action of G on $C_0(X)$ by transposition and the resulting system is called a C^*-dynamical system. From these, one builds in a functorial way a C^*-algebra denoted $C_0(X) \rtimes_\alpha G$ and called the *crossed product* determined by the dynamical system. Crossed products are ubiquitous in C^*-algebra and, arguably, constitute the most important examples.

The key general question is: How are the properties of the dynamical system (X, G, α) reflected in the C^*-algebra $C_0(X) \rtimes_\alpha G$?

A shift of finite type is an example of a *local homeomorphism*. Quite generally, one can always build C^*-algebras from local homeomorphisms in a fashion that mimics the crossed product construction just described. However, now there are competing possibilities. The one I have focused on in my research is based on an idea of R. Exel and the resulting C^*-algebras are called *Exel crossed products*. Suppose that $\sigma : X \to X$ is a local homeomorphism of a compact space, and let $\alpha : f \mapsto f \circ \sigma$ is the associated endomorphism of $C(X)$. The problem, of course, is that α is not invertible unless σ is a homeomorphism. How to deal with this has been the subject of some interest in the literature. Exel's key innovation [7] was to use the *transfer operator* $L : C(X) \to C(X)$, defined by the equation

$$L(f)(x) = \frac{1}{|\sigma^{-1}(x)|} \sum_{\sigma(y)=x} f(y),$$

to define a $C(X)$-valued inner product on $C(X)$. Together with naturally defined left and right actions of $C(X)$, the space $C(X)$ becomes what is known as a *Hilbert bimodule* or *C^*-correspondence* denoted M_L over $C(X)$. Exel used this bimodule to construct his crossed product, denoted $C(X) \times_{\alpha,L} \mathbb{N}$. His construction was inspired by the work of Pimsner [22], who constructed what have come to be known as Cuntz-Pimsner algebras. However, Exel's $C(X) \times_{\alpha,L} \mathbb{N}$ is not exactly a Cuntz-Pimsner algebra. (For the precise connection between Exel's work and Pimsner's see [4].) Cuntz-Pimsner algebras should be viewed as a common decendent of graph C^*-algebras and the crossed products described above.

My work concerning general local homeomorphisms focuses on the problem of building C^*-algebras from two (or more) commuting local homeomorphisms through the construction of what is known as a product system. The precise definition is somewhat technical and involved to present. Here is an imprecise but suggestive definition that may help one to understand the precise definition:

Definition 1.3.3. Let A be a C^*-algebra and let P be a semigroup. A disjoint family $\mathcal{X} := \{\mathcal{X}_t\}_{t\in P}$ of Hilbert bimodules over A is a *product system* over P if $\mathcal{X}$ is a "semigroup under tensoring" and the map $p : \mathcal{X} \to P$, defined by $p(x) = s$ if $x \in \mathcal{X}_s$, is a semigroup homomorphism.

In a bit more detail, for every $s, t \in P$, there is a map $\mathcal{X}_s \times \mathcal{X}_t \mapsto \mathcal{X}_{st}$ that extends to an isomorphism from $\mathcal{X}_s \otimes_A \mathcal{X}_t$ onto $\mathcal{X}_{st}$ in such a way that the resulting product on isomorphism classes of the $\mathcal{X}_s$'s is associative.

If $\mathcal{X}$ is a product system of Hilbert bimodules over a C^*-algebra A, then there is a way to construct a C^*-algebra $C^*(\mathcal{X})$ from $\mathcal{X}$ that mimics the construction of Cuntz-Pimsner algebras. This construction is due to Fowler [10]. Actually, there are several possibilities, but Fowler's is the one of greatest current interest. Thus to construct C^*-algebras from two commuting local homeomorphisms it suffices to build a product system over $\mathbb{N}^2$. The construction is possible, if the local homeomorphisms commute.

If we assume that the local homeomorphisms $*$-*commute* one can obtain an especially nice groupoid presentation of $C^*(\mathcal{X})$, placing the theory of $*$-commuting local homeomorphisms into another framework that is well-developed for the study of dynamical systems. This dissertation, however, will not discuss the groupoid presentation. By definition, two local homeomorphisms, μ and ν, $*$-commute if they commute and given $(y, z) \in X \times X$ such that $\mu(y) = \nu(z)$, there is a unique $x \in X$ such that $\nu(x) = y$ and $\mu(x) = z$. This somewhat strange-looking condition was introduced by V. Arzumanian and J. Renault in [1]. However, it is very interesting in special cases. For example, following up on some of Exel's and Renault's work [8], I have identitfied all the continuous functions on E^∞ that $*$-commute with the shift, σ. These, in turn, are related to certain cellular automata, but the consequences of this relation have still to be determined.

The reason $*$-commutivity is important for my work is this: When μ and ν commute, their transfer operators also commute and the bimodules over $C(X)$ have the property that

$$M_K \otimes_{C(X)} M_L \simeq M_L \otimes_{C(X)} M_K.$$

That is, the bimodules "commute up to isomorphism." This, in turn, implies that if we set $\mathcal{X}_{(n,m)} = (M_K)^{\otimes n} \otimes_{C(X)} (M_L)^{\otimes m} \ldots$, then $\{\mathcal{X}_{(n,m)}\}_{(n,m)\in\mathbb{N}^2}$ is a product system over the semigroup $\mathbb{N}^2$. If μ and ν *-commute, then the product system

$\{\mathcal{X}_{(n,m)}\}_{(n,m)\in\mathbb{N}^2}$ is compactly aligned– a condition that is very important for the general theory of product systems and their C^*-algebras. My current work on $*$-commuting local homeomorphisms, which is far from complete, is focused on trying to extract properties of $C^*(\mathcal{X})$ from the local homeomorphisms used to build $\mathcal{X}$.

1.4 Labeled graph C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a countable directed graph and let $\mathcal{A}$ be a countable set, which we call an *alphabet*. A *labeling of* E by $\mathcal{A}$ is simply a function $\mathcal{L} : E^1 \to \mathcal{A}$. The pair $(E, \mathcal{L})$ is called a *labeled graph*. In [3], Bates and Pask attach a C^*-algebra to a labled graph $C^*(E, \mathcal{L})$. It is constructed from generators consisting of projections and partial isometries that satisfy relations similar to the Cuntz-Krieger relations described above. However, the projections are no longer indexed by vertices in E, but rather by *subsets of vertices* that form a Boolean algebra. The projections constitute a sort of "spectral measure" defined on the Boolean algebra. There are other complications as well that I will not dwell upon. But I do want to point out that just as a graph C^*-algebra is connected to a shift of finite type, $C^*(E, \mathcal{L})$ is connected to a shift space - but no longer of finite type, in general. In the work I am doing with Bates and Pask, we consider discrete groups acting on $(E, \mathcal{L})$. That is, each element of G is associated to a triple of maps, one acting on the vertex space E^0, one acting the edge space E^1 and one acting on the alphabet $\mathcal{A}$ in such a way that all the structure of $(E, \mathcal{L})$ is preserved. The precise formula need not concern us here. The action of G on $(E, \mathcal{L})$ is *free*, if the only element of G that fixes any vertex or letter in the alphabet is the identity of G. The Gross-Tucker theorem for groups G acting freely on an ordinary directed graph E says that E is isomorphic to a kind of skew product that is obtained from the quotient graph E/G and a map c from E^1/G to G satisfying certain properties that allow one to identify E^1 with $(E^1/G) \times G$ and E^0 with $(E^0/G) \times G$. The function c is involved with defining range and source maps, r_c and s_c, making the pair $(E^0/G) \times G$, $(E^1/G) \times G$

into a graph: $r_c([e], g) := ([e], gc([e]))$, and $s_c([e], g) := (s([e]), g)$, where $[e]$ denotes the image of e in E^1/G. We are able to prove an analogue of the Gross-Tucker theorem for labeled graphs. Using this analouge we show that when a discrete group G acts freely on a labeled graph $(E, \mathcal{L})$ then the action induces a natural co-action δ of G on $C^*(E, \mathcal{L})$ in such a way that the co-crossed product $C^*(E, \mathcal{L}) \times_\delta G$ is naturally isomorphic to the C^*-algebra of a certain skew product labeled graph $(E \times_c G, \mathcal{L}_d)$ determined by functions c and d from E^1 to G. This analysis allows us to obtain structure theorems for C^*-algebras of labeled graphs with group actions that parallel important results of Kaliszewski, Quigg and Raeburn [13]. In particular, we are able to prove that under very general hypotheses, there is an action of G on $C^*(E \times_c G, \mathcal{L}_d)$ such that the crossed product $C^*(E \times_c G, \mathcal{L}_d) \times G$ is naturally isomorphic to $C^*(E, \mathcal{L}) \otimes \mathcal{K}(\ell^2(G))$, where $\mathcal{K}(\ell^2(G))$ denotes the compact operators on $\ell^2(G)$.

CHAPTER 2
GROUP ACTIONS ON LABELED GRAPHS AND THEIR C^*-ALGEBRAS

The research described in this chapter was conducted in collaboration with Teresa Bates and David Pask.

2.1 Introduction

Labeled graph algebras are C^*- algebras associated to a labeled graph in such a way that if the labeling is trivial then the resulting C^*-algebra is a graph algebra. We should, therefore, expect many results which hold for graph algebras to hold for labeled graph algebras in a suitably modified form. One such result is the result of Kumjian and Pask on free actions of groups on directed graphs: If α is a free action of a group G on a directed graph E, then

$$C^*(E) \times_\alpha G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G)). \tag{2.1}$$

Alternatively: $C^*(E) \times_\alpha G$ is strongly Morita equivalent to $C^*(E/G)$.

A key ingredient in this proof is the Gross-Tucker Theorem, which characterizes those directed graphs which admit a free action of G. In this chapter we set out to prove an analogous result for labeled graphs.

It has been shown that graph algebras can be considered as C^*-algebras of shifts of finite type. In a similar way, labeled graph algebras can be considered as the C^*-algebras associated to the shifts which the labeled graph presents. These are known in the literature as *sofic shifts*. The consequences of our results will therefore be new results about labeled graphs, shift spaces, and their associated C^*-algebras.

We begin by describing labeled graphs and their C^*-algebras and give a new formulation of the canonical C^*-algebra associated to a labeled graph in Theorem 2.2.16. In section 2.3 we then describe the automorphisms of a labeled graph and group-actions on labeled graphs. In Theorem 2.4.18 we show that an action of a group G on a labeled graph induces an action of G on the associated labeled graph algebra. We then describe those group-actions on labeled graphs which are free and give a version of the Gross-Tucker Theorem for labeled graphs (see Theorem 2.5.15). In the course of proving Theorem 2.5.15 we show that every labeled graph which admits a free action of G is a skew-product labeled graph of a base labeled graph which carries a pair of G-valued functions. In section 2.6 we show that the C^*-algebra of a skew-product labeled graph is isomorphic to the crossed product of the base labeled graph by a coaction of G.

The version of (2.1) for reduced crossed products will now follow as in Corollary 2.5 of [13]. The version of (2.1) for full crossed products will either follow from an argument similar to that of Theorem 3.1 in [13] or by "proper action techniques" as in [23].

We shall also explore the consequences of the Gross-Tucker Theorem in terms of shift spaces which admit group-actions. We shall also consider other consequences of Theorem 2.6.8 for the functions which take the value $1 \in \mathbf{Z}$: The resulting crossed product will be Morita equivalent to the AF-core of the C^*-algebra associated to the base labeled graph. This will also give us an alternative proof that labeled graph algebras are nuclear.

2.2 Labeled Graphs and their C^*-algebras

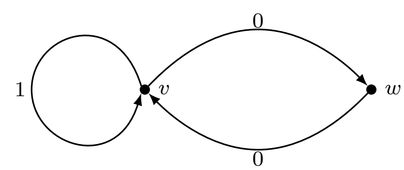
Throughout this chapter $E = (E^0, E^1, r, s)$ will denote a countable *directed graph* where $r, s : E^1 \to E^0$ are the range and source maps, respectively. A *sink* is a vertex $v \in E^0$ which does not emit any edges, that is $s^{-1}(v) = \emptyset$. A *source* is a vertex v which does not receive any edges, that is $r^{-1}(v) = \emptyset$. We shall assume throughout this chapter that all directed graphs E have no sources or sinks. We let E^n denote the set of *paths of length* n, so $E^n = \{(e_1e_2\cdots e_n)|s(e_i) = r(e_{i+1}), i = 1, 2, \cdots, n-1\}$. Let $E^* := \bigcup_{n\geq 1} E^n$. Also, we fix an alphabet $\mathcal{A}$ and a labeling $\mathcal{L} : E^1 \to \mathcal{A}$. We will assume that $\mathcal{L}$ is always surjective, i.e. $\mathcal{L}(E^1) = \mathcal{A}$. Let $\mathcal{A}^*$ be the collection of all *words* in the symbols of $\mathcal{A}$. The map $\mathcal{L}$ extends naturally to a map $\mathcal{L}^* : E^* \to \mathcal{A}^*$: for $n \geq 1$ and $\lambda = \lambda_1 \cdots \lambda_n \in E^n$ we set $\mathcal{L}^*(\lambda) = \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n)$, and we say that λ is a *representative* of $\mathcal{L}^*(\lambda)$. Note that $\mathcal{L}^*$ is not injective in general and so $\mathcal{L}^*(\lambda)$ can have many representations. If $\mathcal{L}^n(E)$ is the collection of all labeled paths in $(E, \mathcal{L})$ of length n, then $\mathcal{L}^*(E) = \cup_{n \geq 1} \mathcal{L}^n(E)$ denotes the collection of all words in the alphabet $\mathcal{A}$ which may be represented by paths in the labeled graph $(E, \mathcal{L})$.

Examples 2.2.1.

1. A directed graph is an example of a labeled graph where we let $\mathcal{A} = E^1$ and take the labeling to be the identity map $\mathcal{L} : E^1 \to E^1$. Furthermore, if $\mathcal{L} : E^1 \to \mathcal{A}$ is injective, then $(E, \mathcal{L})$ becomes a directed graph when we identify E^1 with $\mathcal{A}$.

2. The following self-explanatory diagram illustrates another example of a labeled graph $(E, \mathcal{L})$ with alphabet $\{0, 1\}$.

$(E,\mathcal{L})$:=



Definition 2.2.2. The labeled graph $(E, \mathcal{L})$ is called *left-resolving* if for all $v \in E^0$ the map $\mathcal{L}$ restricted to $r^{-1}(v)$ is injective.

Evidently Example 2.2.1 (1) and (2) are examples of a left-resolving labeled graph. The left-resolving condition ensures that for all $v \in E^0$ the labels $\{\mathcal{L}(e) : r(e) = v\}$ of all incoming edges to v are all different. In particular if $\lambda, \mu \in E^n$ satisfy $\mathcal{L}^*(\lambda) = \mathcal{L}^*(\mu)$ and $r(\lambda) = r(\mu)$ then $\lambda = \mu$. Many of the results in this paper will hold under the weaker hypothesis of *weakly left-resolving* as defined in [3, Definition 3.7]. However since we know of no examples which are weakly left-resolving but not left-resolving we shall assume all labeled graphs $(E, \mathcal{L})$ are left-resolving.

Definition 2.2.3. For $\alpha \in \mathcal{L}^*(E)$ we put $s_{\mathcal{L}}(\alpha) = \{s(\lambda) \in E^0 : \mathcal{L}^*(\lambda) = \alpha\}$ and $r_{\mathcal{L}}(\alpha) = \{r(\lambda) \in E^0 : \mathcal{L}^*(\lambda) = \alpha\}$.

We drop the subscript $\mathcal{L}$ on $r_{\mathcal{L}}$ and $s_{\mathcal{L}}$ if the context in which it is being used is clear.

Definition 2.2.4. Let $(E, \mathcal{L})$ be a labeled graph. For $A \subseteq E^0$ and $\beta \in \mathcal{L}^*(E)$ the *relative range of* β *with respect to* A is defined to be

$$r(A, \beta) = \{r(\lambda) : \lambda \in E^*, \mathcal{L}^*(\lambda) = \beta, s(\lambda) \in A\}.$$

So the relative range of β with respect to A is the set of ranges of λ where λ is a path whose label is β and whose source is in A.

Definition 2.2.5. Let $\mathcal{E}(r, \mathcal{L})$ be the smallest collection of subsets of E^0 which contains $r(\beta)$ for all $\beta \in \mathcal{L}^*(E)$ and is closed under finite unions and intersections.

Remark 2.2.6. For $\beta \in \mathcal{L}^*(E)$ and $a \in \mathcal{A}$ by [3, Remark 3.9] we have

$$r(r(\beta), a) = r(\beta a). \tag{2.2}$$

So $\mathcal{E}(r, \mathcal{L})$ is automatically closed under relative ranges for $(E, \mathcal{L})$. Hence the triple $(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$ is a labeled space as defined in [3, Definition 3.6] . By Definition 2.2.5 we know that every $A \in \mathcal{E}(r, \mathcal{L})$ is of the form $A = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m_n} r(\beta_{ij})$ for some $\beta_{ij} \in \mathcal{L}^*(E)$.

Definition 2.2.7. For $A \subseteq E^0$ and $n \geq 1$, let $L_A^n := \{\beta \in \mathcal{L}^n(E) : A \cap s(\beta) \neq \emptyset\}$ denote those labeled paths of length n whose source intersects A nontrivially.

Definition 2.2.8. Let $(E, \mathcal{L})$ be a left-resolving labeled graph. A *representation of* $(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$ consists of projections $\{p_A : A \in \mathcal{E}(r, \mathcal{L})\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ with the properties that:

(Rep1) If $A, B \in \mathcal{E}(r, \mathcal{L})$ then $p_A p_B = p_{A \cap B}$ and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ where $p_\emptyset = 0$.

(Rep2) If $a \in \mathcal{A}$ and $A \in \mathcal{E}(r, \mathcal{L})$ then $p_A s_a = s_a p_{r(A,a)}$.

(Rep3) If $a, b \in \mathcal{A}$ then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$

(Rep4) For $A \in \mathcal{E}(r, \mathcal{L})$, if L_A^1 is finite, non-empty, and A contains no sinks we have

$$p_A = \sum_{a \in L_A^1} s_a p_{r(A,a)} s_a^*. \tag{2.3}$$

Notice that (Rep1) says that $A \mapsto p_A$ is a (weakened form of a) spectral measure on $\mathcal{E}(r,\mathcal{L})$. (Rep2) is a kind of covariance equation. (Rep3) shows that each s_a is a partial isometry. (Rep4) is clearly an analogue of (CK4).

The following result can be found as [3, Theorem 4.5].

Theorem 2.2.9. *Let $(E,\mathcal{L})$ be a left-resolving labeled graph. There exists a C^*-algebra B generated by a universal represenation $\{t_a, q_A\}$ of $(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$. Furthermore t_a and q_A are nonzero for all $a \in \mathcal{A}$ and $A \in \mathcal{E}(r,\mathcal{L})$.*

Definition 2.2.10. Let $(E,\mathcal{L})$ be a left-resolving labeled graph, then $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ is the universal C^*-algebra generated by a represenation of $(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$.

Definition 2.2.11. Let $(E,\mathcal{L})$ be a left-resolving labeled graph. A *Cuntz-Krieger $(E,\mathcal{L})$-family* consists of commuting projections $\{p_{r(\beta)} : \beta \in \mathcal{L}^*(E)\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ with the properties that:

(CK1a) For all $\beta,\omega \in \mathcal{L}^*(E)$, $p_{r(\beta)}p_{r(\omega)} = 0$ if and only if $r(\beta) \cap r(\omega) = \emptyset$.

(CK1b) For all $\beta,\omega,\kappa \in \mathcal{L}^*(E)$, if $r(\beta) \cap r(\omega) = r(\kappa)$ then $p_{r(\beta)}p_{r(\omega)} = p_{r(\kappa)}$ and if $r(\beta) \cup r(\omega) = r(\kappa)$ then $p_{r(\beta)} + p_{r(\omega)} - p_{r(\beta)}p_{r(\omega)} = p_{r(\kappa)}$.

(CK2) If $a \in \mathcal{A}$ and $\beta \in \mathcal{L}^*(E)$ then $p_{r(\beta)}s_a = s_a p_{r(\beta a)}$.

(CK3) If $a, b \in \mathcal{A}$ then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$

(CK4) For $\beta \in \mathcal{L}^*(E)$, if $L^1_{r(\beta)}$ is finite, non-empty, and $r(\beta)$ contains no sinks we have

$$p_{r(\beta)} = \sum_{a \in L^1_{r(\beta)}} s_a p_{r(\beta a)} s_a^*. \tag{2.4}$$

Remark 2.2.12. Using relations (CK1a) and (CK1b) we may now unambiguously define $p_{r(\beta)\cap r(\omega)} = p_{r(\beta)}p_{r(\omega)}$ and $p_{r(\beta)\cup r(\omega)} = p_{r(\beta)} + p_{r(\omega)} - p_{r(\beta)}p_{r(\omega)}$.

If $r(\beta)\cap r(\omega) \neq \emptyset$, then we write $p_{r(\beta)}p_{r(\omega)} = p_{r(\beta)\cap r(\omega)}$, so (CK1) implies that $p_\emptyset = 0$.

Theorem 2.2.13. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph. There exists a C^*-algebra B generated by a universal Cuntz-Krieger $(E, \mathcal{L})$-family $\{s_a, p_{r(\beta)}\}$. Furthermore s_a and $p_{r(\beta)}$ are nonzero for all $a \in \mathcal{A}$ and $\beta \in \mathcal{L}^*(E)$.*

Proof. This follows from the proof of Theorem 4.5 in [3] *mutatis mutandis.* □

Definition 2.2.14. Let $(E, \mathcal{L})$ be a left-resolving labeled graph, then $C^*(E, \mathcal{L})$ is the universal C^*-algebra generated by a Cuntz-Krieger $(E, \mathcal{L})$-family.

Remark 2.2.15. By Remarks 2.2.6 and 2.2.12 a Cuntz-Krieger $(E, \mathcal{L})$-family gives rise to a representation of the labeled space $(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$ in the sense of [3, Definition 4.1], and conversely. Hence we may identify $C^*(E, \mathcal{L})$ with $C^*(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$. From [3, Theorem 4.5] we can assume that if $\{s_a, p_{r(\beta)}\}$ is a universal Cuntz-Krieger $(E, \mathcal{L})$ family and $\beta, \omega \in \mathcal{L}^*(E)$ are such that $r(\beta) \subsetneq r(\omega)$ then $p_{r(\beta)} \neq p_{r(\omega)}$.

If $\{S_a, P_{r(\beta)}\}$ is a Cuntz-Krieger $(E, \mathcal{L})$-family in a C^*-algebra B, we denote by $\pi_{S,P}$ the homomorphism $\pi_{S,P} : C^*(E, \mathcal{L}) \to B$ which satisfies $\pi_{S,P}(s_a) = S_a$, and $\pi_{S,P}(p_{r(\beta)}) = P_{r(\beta)}$. Let $\{s_a, p_{r(\beta)}\}$ be the universal Cuntz-Krieger family of $(E, \mathcal{L})$ which generates $C^*(E, \mathcal{L})$. For $z \in \mathbf{T}$, $a \in \mathcal{A}$, and $\beta \in \mathcal{L}^*(E)$ let

$$\gamma_z s_a = z s_a \qquad \text{and} \qquad \gamma_z p_{r(\beta)} = p_{r(\beta)}; \tag{2.5}$$

then the family $\{z s_a, p_{r(\beta)}\} \in C^*(E, \mathcal{L})$ is also a Cuntz-Krieger $(E, \mathcal{L})$-family. A routine $\epsilon/3$ argument shows that γ extends to a strongly continuous action

$$\gamma : \mathbf{T} \to \operatorname{Aut} C^*(E, \mathcal{L})$$

which we call the *gauge action.*

Theorem 2.2.16. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph, let $\{s_a, p_{r(\beta)}\}$ be a universal Cuntz-Krieger $(E, \mathcal{L})$-family, and let $\{t_a, q_A\}$ be a universal representation of $(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$. Then the map $\Psi : C^*(E, \mathcal{L}) \to C^*(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$ determined by*

$$\Psi(s_a) = t_a \quad \text{and} \quad \Psi(p_{r(\beta)}) = q_{r(\beta)}$$

is an isomorphism.

Proof. For $a \in \mathcal{A}$, set $S_a = t_a$, and for $\beta \in \mathcal{L}^*(E)$ let $P_{r(\beta)} = q_{r(\beta)}$. It is straight forward to check that $\{S_a, P_{r(\beta)}\}$ form a Cuntz-Krieger $(E,\mathcal{L})$-family in $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$. By the universal property of $C^*(E,\mathcal{L})$ there exists a homomorphism $\Psi = \pi_{S,P}$ from $C^*(E,\mathcal{L})$ to $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ such that

$$\Psi(s_a) = S_a \quad \text{and} \quad \Psi(p_{r(\beta)}) = P_{r(\beta)}.$$

For $a \in \mathcal{A}$, set $T_a = s_a$. For $A = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m_n} r(\beta_{ij}) \in \mathcal{E}(r,\mathcal{L})$, let $Q_A = \sum_{i=1}^{n} \prod_{j=1}^{m_n} p_{r(\beta_{ij})}$. It is a routine calculation to show that $\{T_a, Q_A\}$ is a representation of $(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ in $C^*(E,\mathcal{L})$. By the universal property of $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ there exists a homomorphism $\Phi : C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L})) \to C^*(E,\mathcal{L})$ such that

$$\Phi(t_a) = T_a \quad \text{and} \quad \Phi(q_A) = Q_A.$$

Since $\Phi(q_{r(\beta)}) = Q_{r(\beta)} = p_{r(\beta)}$ and $\Phi(t_a) = T_a = s_a$ we see that $\Psi \circ \Phi$ is the identity map on $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ and similarly $\Phi \circ \Psi$ is the identity map on $C^*(E,\mathcal{L})$ so the result follows. $\square$

Remark 2.2.17. Since we shall be working almost exclusively with labeled spaces of the form $(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ we shall only consider Cuntz-Krieger $(E,\mathcal{L})$-families and $C^*(E,\mathcal{L})$. The following is a restatement of the Gauge Invariant Uniqueness Theorem for labeled graphs.

Theorem 2.2.18 (cf.[2]). *Let $(E,\mathcal{L})$ be a left-resolving graph and $\{S_a, P_{r(\beta)}\}$ be a Cuntz-Krieger $(E,\mathcal{L})$ family on Hilbert space. Take $\pi_{S,P}$ to be the representation of $C^*(E,\mathcal{L})$ satisfying $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_{r(\beta)}) = P_{r(\beta)}$. Suppose that whenever $r(\beta) \subsetneq r(\omega)$ we have $P_{r(\beta)} \neq P_{r(\omega)}$ and that there is a strongly continuous action β of $\mathbf{T}$ on $C^*(\{S_a, P_{r(\beta)}\})$ such that for all $z \in \mathbf{T}$, $\beta_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$. Then $\pi_{S,P}$ is faithful.*

Remark 2.2.19. Note that the hypotheses of Theorem 2.2.18 differ from those of [3, Theorem 5.3]. This is due to the revision of the Gauge-Invariant Uniqueness Theorem given in [2].

2.3 Automorphisms of Labeled graphs and their C^*-algebras

We begin by defining what a graph morphism is, and use the definition to define a graph automorphism. We then define a labeled graph morphism and use the definition to define a labeled graph automorphism. Then in Theorem 2.3.7 we show that a labeled graph automorphism of $(E, \mathcal{L})$ induces an automorphism of $C^*(E, \mathcal{L})$.

Definition 2.3.1. Let E, F be a directed graphs. A *graph morphism* is a pair of maps $\phi = (\phi^0, \phi^1) : E \to F$ such that for all $e \in E^1$ we have

$$\phi^0(r(e)) = r(\phi^1(e))$$

$$\phi^0(s(e)) = s(\phi^1(e)).$$

If $\phi = (\phi^0, \phi^1)$ is bijective, then ϕ is called a *graph isomorphism*. If, in addition $F = E$, then ϕ is called a *graph automorphism*. For $A \subseteq E^0$ we define $\phi^0(A) = \{\phi^0(v) : v \in A\}$.

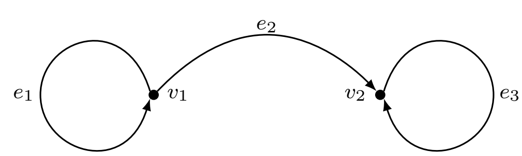
The set

$$\text{Aut}(E) := \{\phi : E \to E \ : \ \phi \text{ is a graph automorphism}\}$$

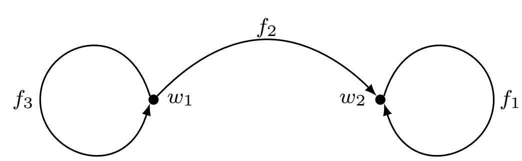
forms a group under composition.

Example 2.3.2. Let E and F be the following directed graphs.

E:=



F:=



Define $\phi^i : E^i \to F^i$ by $\phi^0(v_j) = w_j$ and

$$\phi^1(e_1) = f_3 \ , \ \phi^1(e_2) = f_2 \ , \ \phi^1(e_3) = f_1.$$

Then ϕ is bijective. Notice that

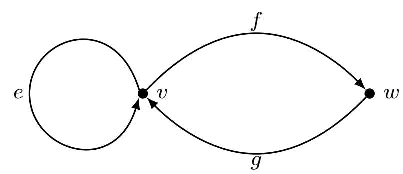
$$\phi^0(r(e_1)) = w_1 = r(\phi^1(e_1))$$

$$\phi^0(r(e_2)) = w_2 = r(\phi^1(e_2))$$

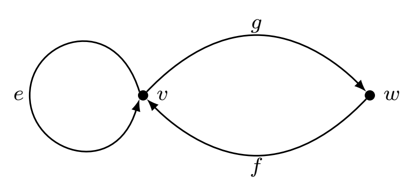
$$\phi^0(r(e_3)) = w_2 = r(\phi^1(e_3))$$

thus ϕ is a graph isomorphism.

Example 2.3.3. Let E be the directed graph below.



Define $\phi^0 : E^0 \to E^0$ be the identity map and $\phi^1 : E^1 \to E^1$ be defined by $\phi^1(e) = e$, $\phi^1(f) = g$, and $\phi^1(g) = f$.



We see that ϕ is bijective. Notice that

$$\phi^0(r(e)) = v = r(\phi^1(e))$$

$$\phi^0(r(f)) = w = r(\phi^1(f))$$

$$\phi^0(r(g)) = v = r(\phi^1(g))$$

thus ϕ is a graph automorphism.

Definition 2.3.4. Let $(E, \mathcal{L})$ and $(F, \mathcal{M})$ be labeled graphs over alphabets $\mathcal{A}_E$ and $\mathcal{A}_F$ respectively. A *labeled graph morphism* is a triple $\phi := (\phi^0, \phi^1, \phi^{\mathcal{A}_E}) : (E, \mathcal{L}) \to (F, \mathcal{M})$ such that

1. (ϕ^0, ϕ^1) is a graph morphism from E to F and

2. $\phi^{\mathcal{A}_E} : \mathcal{A}_E \to \mathcal{A}_F$ is a map such that $\mathcal{M} \circ \phi^1 = \phi^{\mathcal{A}_E} \circ \mathcal{L}$.

If (ϕ^0, ϕ^1) is a graph isomorphism and $\phi^{\mathcal{A}_E}$ is a bijection, then the triple $\phi := (\phi^0, \phi^1, \phi^{\mathcal{A}_E})$ is called a *labeled graph isomorphism*. If, in addition $F = E$ and $\mathcal{A}_E = \mathcal{A}_F$, then ϕ is called a *labeled graph automorphism*.

Let $(E, \mathcal{L})$ be a labeled graph over the alphabet $\mathcal{A}$. Then the set

$$\mathrm{Aut}(E, \mathcal{L}) := \{\phi : (E, \mathcal{L}) \to (E, \mathcal{L}) \ : \ \phi \text{ is a labeled graph automorphism}\}$$

forms a group under composition. In particular $\phi^{\mathcal{A}}$ belongs to $\mathrm{Bij}(\mathcal{A})$, the group of bijections from $\mathcal{A}$ to $\mathcal{A}$.

Example 2.3.5. Let $(E, \mathcal{L})$ be a labeled graph. If $\phi^0 : E^0 \to E^0$ and $\phi^1 : E^1 \to E^1$ are the identity maps and $\phi^{E^1} = \mathcal{L}$ then the triple $\phi = (\phi^0, \phi^1, \phi^{E^1})$ is a surjective labeled graph automorphism.

Before stating the main result of this section, Theorem 2.3.7, we collect some useful facts about a labeled graph automorphism.

Lemma 2.3.6. *Let α be a labeled graph automorphism of $(E, \mathcal{L})$. Then for all $\beta, \omega \in \mathcal{L}^*(E)$, and $a \in \mathcal{A}$ we have*

1. $\alpha^0(r(\beta) \cup r(\omega)) = \alpha^0 r(\beta) \cup \alpha^0 r(\omega)$ *and* $\alpha^0(r(\beta) \cap r(\omega)) = \alpha^0 r(\beta) \cap \alpha^0 r(\omega)$

2. $r(\alpha^{\mathcal{A}}(a)) = \alpha^0 r(a)$

3. $r(\alpha^0(r(\beta)), \alpha^{\mathcal{A}}(a)) = \alpha^0 r(\beta a)$

4. $L^1_{\alpha^0 r(\beta)} = \alpha^{\mathcal{A}} L^1_{r(\beta)}$.

Proof. Property (1) holds since is α^0 is a bijection. For (2) we see that for $e \in E^1$ with $\mathcal{L}(e) = a$, we have $\mathcal{L}(\alpha^1(e)) = \alpha^{\mathcal{A}}\mathcal{L}(e) = \alpha^{\mathcal{A}}(a)$. Thus we have

$$r(\alpha^{\mathcal{A}}(a)) = \{r(\alpha^1(e)) : \mathcal{L}(e) = a\} = \{\alpha^0 r(e) : \mathcal{L}(e) = a\} = \alpha^0 r(a).$$

For (3) notice that

$$\begin{aligned} r(\alpha^0(r(\beta)), \alpha^{\mathcal{A}}(a)) &= r(r(\alpha^{\mathcal{A}}(\beta)), \alpha^{\mathcal{A}}(a)) \\ &= r(\alpha^{\mathcal{A}}(\beta)\alpha^{\mathcal{A}}(a)) \text{ by [3, Remark 3.9]} \\ &= r(\alpha^{\mathcal{A}}(\beta a)) \\ &= \alpha^0 r(\beta a). \end{aligned}$$

Finally, we want to show that

$$L^1_{\alpha^0 r(\beta)} = \alpha^{\mathcal{A}} L^1_{r(\beta)}.$$

Suppose $a \in L^1_{\alpha^0 r(\beta)}$ and we want to show that $a \in \alpha^{\mathcal{A}} L^1_{r(\beta)}$. Then it suffices to show that $(\alpha^{\mathcal{A}})^{-1}(a) \in L^1_{r(\beta)}$. Notice that

$$r(\beta) \cap s((\alpha^{\mathcal{A}})^{-1}(a)) = r(\beta) \cap (\alpha^0)^{-1} s(a) = (\alpha^0)^{-1}(s(a) \cap \alpha^0 r(\beta)).$$

Since $\alpha^0 r(\beta) \cap s(a) \neq \emptyset$ and $(\alpha^0)^{-1}$ is an automorphism, then $r(\beta) \cap s((\alpha^{\mathcal{A}})^{-1}(a)) \neq \emptyset$. Therefore $(\alpha^{\mathcal{A}})^{-1}(a) \in L^1_{r(\beta)}$. Conversely, suppose $\alpha^{\mathcal{A}} b \in \alpha^{\mathcal{A}} L^1_{r(\beta)}$; we want to show that $\alpha^{\mathcal{A}} b \in L^1_{\alpha^0 r(\beta)}$. Notice that

$$\alpha^0 r(\beta) \cap s(\alpha^{\mathcal{A}}(b)) = \alpha^0 r(\beta) \cap \alpha^0 s(b) = \alpha^0 (s(b) \cap r(\beta)).$$

Since $s(b) \cap r(\beta) \neq \emptyset$ and α^0 is an automorphism, then $\alpha^0 r(\beta) \cap s(\alpha^{\mathcal{A}}(b)) \neq \emptyset$. Therefore $\alpha^{\mathcal{A}} b \in L^1_{\alpha^0 r(\beta)}$. □

Theorem 2.3.7. *Let α be an automorphism of a left-resolving labeled graph $(E, \mathcal{L})$ and let $\{s_a, p_{r(\beta)}\}$ be a universal Cuntz-Krieger $(E, \mathcal{L})$-family. For $a \in \mathcal{A}$ define $S_a := s_{\alpha^{\mathcal{A}} a}$, and for $\beta \in \mathcal{L}^*(E)$ define $P_{r(\beta)} := p_{\alpha^0 r(\beta)}$. Then the map $\alpha : C^*(E, \mathcal{L}) \to C^*(E, \mathcal{L})$ determined by*

$$\alpha(s_a) = S_a \qquad \text{and} \qquad \alpha(p_{r(\beta)}) = P_{r(\beta)} \tag{2.6}$$

is an automorphism.

Proof. First we will check that $\{S_a : a \in \mathcal{A}\}$, $\{P_{r(\beta)} : \beta \in \mathcal{L}^*(E)\}$ generates a Cuntz-Krieger $(E, \mathcal{L})$-family. (CK1a) and (CK1b) follow from Lemma 2.3.6 (1) and the fact that α is a homomorphism. To see (CK2), notice that

$$P_{r(\beta)} S_a = p_{\alpha^0(r(\beta))} s_{\alpha^{\mathcal{A}}(a)} = s_{\alpha^{\mathcal{A}}(a)} p_{r(\alpha^0(r(\beta)), \alpha^{\mathcal{A}}(a))} = s_{\alpha^{\mathcal{A}}(a)} p_{\alpha^0 r(\beta a)} = S_a P_{r(\beta a)}$$

by Lemma 2.3.6 (3).

In order to observe that (CK3) holds, notice that $S_a^* S_a = s^*_{\alpha^{\mathcal{A}}(a)} s_{\alpha^{\mathcal{A}}(a)} = p_{r(\alpha^0(a))} = p_{\alpha^0 r(a)} = P_{r(a)}$ by Lemma 2.3.6 (2). Also note that $S_a^* S_a = s^*_{\alpha^{\mathcal{A}}(a)} s_{\alpha^{\mathcal{A}}(b)} = 0$ unless $\alpha^{\mathcal{A}}(a) = \alpha^{\mathcal{A}}(b)$.

(CK4) holds since

$$
\begin{aligned}
P_{r(\beta)} = p_{\alpha^0 r(\beta)} &= \sum_{a \in L^1_{\alpha^0 r(\beta)}} s_a p_{r(\alpha^0 r(\beta), a)} s_a^* \\
&= \sum_{\alpha^{\mathcal{A}} b \in \alpha^{\mathcal{A}} L^1_{r(\beta)}} s_{\alpha^{\mathcal{A}} b} p_{r(\alpha^0 r(\beta), \alpha^{\mathcal{A}} b)} s^*_{\alpha^{\mathcal{A}} b} \text{ by Lemma 2.3.6 (4)} \\
&= \sum_{\alpha^{\mathcal{A}} b \in \alpha^{\mathcal{A}} L^1_{r(\beta)}} s_{\alpha^{\mathcal{A}} b} p_{\alpha^0 r(\beta b)} s^*_{\alpha^{\mathcal{A}} b} \text{ by Lemma 2.3.6 (3)} \\
&= \sum_{a \in L^1_{r(\beta)}} S_a P_{r(\beta a)} S_a^*.
\end{aligned}
$$

By the universal property of $C^*(E, \mathcal{L})$, there is a map

$$\alpha : C^*(E, \mathcal{L}) \to C^*(E, \mathcal{L})$$

given by $s_a \mapsto S_a$ and $p_{r(\beta)} \mapsto P_{r(\beta)}$. Since $\alpha^{\mathcal{A}}$, α^0, and α^1 are surjective, it follows that the map α defined in (2.6) is surjective. Since $\alpha \circ \gamma_z = \gamma_z \circ \alpha$ for all $z \in \mathbf{T}$ and $P_{r(\beta)} \neq 0$ for all $\beta \in \mathcal{L}^*(E)$, it follows that α is injective by Theorem 2.2.18. □

2.4 Skew product labeled graphs and group-actions

In this section we shall define a skew product labeled graph, and define what it means for a group to act on a labeled graph.

Skew product labeled graphs

We begin by reviewing the definition of skew product graph for directed graphs.

Definition 2.4.1. Let E be a directed graph, G be a group, and $c : E^1 \to G$ be a function. The *skew product graph* $E \times_c G$ is defined as follows:

$$
\begin{aligned}
(E \times_c G)^0 &:= E^0 \times G \\
(E \times_c G)^1 &:= E^1 \times G \\
r(e, g) &:= (r(e), gc(e)) \\
s(e, g) &:= (s(e), g).
\end{aligned}
$$

Remark 2.4.2. There are several definitions of skew product graphs in the literature (see [11, Section 2.1.1], [16, Definition 2.1], [13, Section 2], and [27, Chapter 6]). Remark 2.2 in[13] establishes an isomorphism between their version of the skew product and the version of the skew product in [16, 11]. Our version of $E \times_c G$ tallies with [16, 11] and is isomorphic to the skew product graph $E \times_{\widetilde{c}} G$ defined in [27] where $\widetilde{c}(e) = c(e)^{-1}$ for all $e \in E^1$.

Remark 2.4.3. For $\mu \in E^n$ let $c(\mu) = c(\mu_1) \cdots c(\mu_n)$, hence $c : E^1 \to G$ extends in a natural way to a functor $c : E^* \to G$ from the path category to the group.

It is important to understand how elements in the set $(E \times_c G)^*$ appear. Let us examine a path of length two $(\mu_1, g)(\mu_2, h) \in (E \times_c G)^2$. To be a path of length two we know that

$$r(\mu_1, g) = (r(\mu_1), gc(\mu_1)) = (s(\mu_2), h) = s(\mu_2, h).$$

Therefore $r(\mu_1) = s(\mu_2)$ and $gc(\mu_1) = h$. Continuing in this way we see that a path of length n in $(E \times_c G)^*$ is of a form that is most clearly expressed with the aid of additional notation: For $\mu \in E^n$ we denote $\mu' = \mu_1 \cdots \mu_{n-1}$ so that $\mu = \mu'\mu_n$. With this notation we may write a path of length n as

$$(\mu_1, g)(\mu_2, gc(\mu_1)) \cdots (\mu_n, gc(\mu')).$$

For convenience, we will define

$$(\mu, g) := (\mu_1, g)(\mu_2, gc(e_1)) \cdots (\mu_n, gc(\mu')).$$

Notice that

$$s(\mu, g) = (s(\mu), g) \qquad \text{and} \qquad r(\mu, g) = (r(\mu), gc(\mu)). \tag{2.7}$$

Definition 2.4.4. Let $(E, \mathcal{L})$ be a labeled graph and let $c, d : E^1 \to G$ be functions. The *skew product labeled graph* $(E \times_c G, \mathcal{L}_d)$ over alphabet $\mathcal{A} \times G$ is the skew product graph $E \times_c G$ together with the labeling $\mathcal{L}_d : E^1 \times G \to \mathcal{A} \times G$ given by

$$\mathcal{L}_d(e, g) := (\mathcal{L}(e), gd(e)).$$

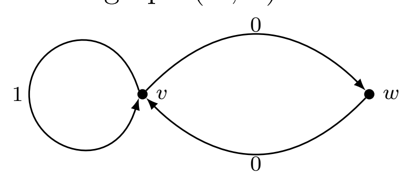
Remark 2.4.5. The definition of a skew product labeled graph has been made with a view to proving a version of the Gross-Tucker theorem for labeled graphs (see Theorem 2.5.15). In Remark 2.5.16 we shall discuss why it is necessary to introduce a separate function $d : E^1 \to G$ in order to form the skew product labeled graph.

Note that the labels received by $(v, g) \in (E \times_c G)^0$ are in one to one correspondence with the labels received by $v \in E^0$. Hence, if $(E, \mathcal{L})$ is left-resolving then so is $(E \times_c G, \mathcal{L}_d)$.

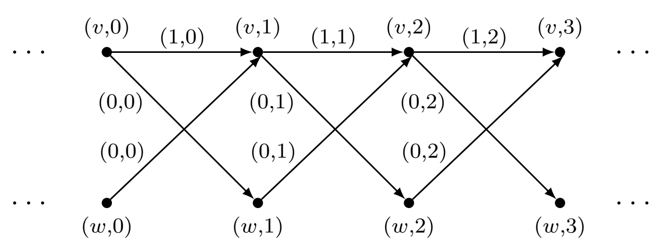
Examples 2.4.6.

1. Let $(E, \mathcal{L})$ be a directed graph with the trivial labeling, $c : E^1 \to G$ be any function, and $d : E^1 \to G$ be defined by $d(e) = 1_G$ for all $e \in E^1$. Then one can show that the skew product labeled graph $(E \times_c G, \mathcal{L}_d)$ is isomorphic to the usual skew product directed graph $E \times_c G$ with the trivial labeling.

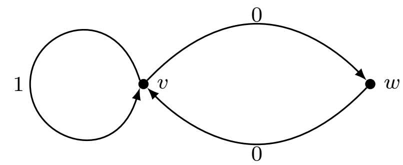
2. Consider the following labeled graph $(E, \mathcal{L})$.



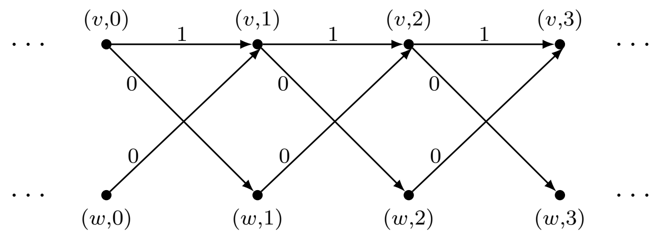
Let $G = \mathbf{Z}$ and let $c : E^1 \to \mathbf{Z}$ be the constant function $c(e) = 1$. Let $d : E^1 \to \mathbf{Z}$ be the constant function $d(e) = 0$. Then the skew-product labeled graph $(E \times_c \mathbf{Z}, \mathcal{L}_d)$ is as in the following diagram.



Remark 2.4.7. If instead of the labeling $\mathcal{L}_d$ in Definition 2.4.4 we use the *induced labeling* $\mathcal{L}_s : (E \times_c G)^1 \to \mathcal{A}$ defined by $\mathcal{L}_s(e, g) := \mathcal{L}(e)$ on $E \times_c G$ then we obtain a labeled graph in which every pair (e, g) carries the same label $\mathcal{L}(e)$. For example, the labeled graph



has induced labeling on the skew product graph defined by the function $c : E^1 \to \mathbf{Z}$ where $c(e) = 1$ for all e as shown



We denote such a graph $(E \times_c G, \mathcal{L}_s)$. Note that although the underlying directed graph is the same as in Example 2.4.6 (2) the labeling is quite different.

Observe that $\mathcal{L}_s^*((\mu, g)) = \mathcal{L}^*(\mu)$ for all $g \in G, \mu \in E^*$, and so $\mathcal{L}_s^*((E \times_c G)^*) = \mathcal{L}^*(E)$. Hence the shift spaces $\mathsf{X}_{(E \times_c G, \mathcal{L}_s)}$ and $\mathsf{X}_{(E,\mathcal{L})}$ are equal as they have the same language (see [20, §3], [3, §3]). If $\mu \in E^*$ is such that $\mathcal{L}^*(\mu) = \beta$, then $\mathcal{L}_s^*((\mu, g)) = \beta$ for all $g \in G$. Since $r(\mu, g) = (r(\mu), gc(\mu))$, it follows that

$$r_{\mathcal{L}_s}(\beta) = r_{\mathcal{L}}(\beta) \times G \tag{2.8}$$

for all $\beta \in \mathcal{L}^*(E)$. Hence every element $A = \bigcup_{i=1}^n \bigcap_{j=1}^{m_n} r_{\mathcal{L}_s}(\beta_{ij}) \in \mathcal{E}(r, \mathcal{L}_s)$ is of the form $A' \times G$ where $A' = \bigcup_{i=1}^n \bigcap_{j=1}^{m_n} r_{\mathcal{L}}(\beta_{ij}) \in \mathcal{E}(r, \mathcal{L})$.

Lemma 2.4.8. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph, let G be a group, and let $c : E^1 \to G$ be a function. Then $C^*(E \times_c G, \mathcal{L}_s) \cong C^*(E, \mathcal{L})$.*

Proof. Let $\{s_a, p_{r_{\mathcal{L}}(\beta)}\}$ be a Cuntz-Krieger $(E, \mathcal{L})$-family and $\{t_a, q_{r_{\mathcal{L}_s}(\beta)}\}$ be a Cuntz-Krieger $(E \times_c G, \mathcal{L}_s)$-family. For $a \in \mathcal{A}$, set $S_a = t_a$, and for $\beta \in \mathcal{L}^*(E)$ let $P_{r_{\mathcal{L}}(\beta)} = q_{r_{\mathcal{L}_s}(\beta)}$. Using (2.8) it is straight forward to check that $\{S_a, P_{r_{\mathcal{L}}(\beta)}\}$ form a Cuntz-Krieger $(E, \mathcal{L})$-family in $C^*(E \times_c G, \mathcal{L}_s)$. By the universal property of $C^*(E, \mathcal{L})$ there exists a homomorphism $\pi_{S,P} : C^*(E, \mathcal{L}) \to C^*(E \times_c G, \mathcal{L}_s)$ such that

$$\pi_{S,P}(s_a) = S_a \quad \text{and} \quad \pi_{S,P}(p_{r_{\mathcal{L}}(\beta)}) = P_{r_{\mathcal{L}}(\beta)}.$$

For $a \in \mathcal{A}$, set $T_a = s_a$ and for $\beta \in \mathcal{L}^*(E)$, let $Q_{r_{\mathcal{L}_s}(\beta)} = p_{r_{\mathcal{L}}(\beta)}$. It is a routine calculation, using (2.8), to show that $\{T_a, Q_{r_{\mathcal{L}_s}(\beta)}\}$ is a Cuntz-Krieger $(E \times_c G, \mathcal{L}_s)$ in $C^*(E, \mathcal{L})$. By the universal property of $C^*(E \times_c G, \mathcal{L}_s)$, there exists a homomorphism $\pi_{T,Q} : C^*(E \times_c G, \mathcal{L}_s) \to C^*(E, \mathcal{L})$ such that

$$\pi_{T,Q}(t_a) = T_a \quad \text{and} \quad \pi_{T,Q}(q_{r_{\mathcal{L}_s}(\beta)}) = Q_{r_{\mathcal{L}_s}(\beta)}.$$

Since $\pi_{T,Q}(q_{r_{\mathcal{L}_s}(\beta)}) = Q_{r_{\mathcal{L}_s}(\beta)} = p_{r_{\mathcal{L}}(\beta)}$ and $\pi_{T,Q}(t_a) = T_a = s_a$ we see that $\pi_{S,P} \circ \pi_{T,Q}$ is the identity map on $C^*(E \times_c G, \mathcal{L}_s)$ and similarly $\pi_{T,Q} \circ \pi_{S,P}$ is the identity map on $C^*(E, \mathcal{L})$ so the result follows. $\square$

We now return our attention to skew product labeled graphs as in Definition 2.4.4.

Let $(E, \mathcal{L})$ be a labeled graph over the alphabet $\mathcal{A}$, let $c, d : E^1 \to G$ be functions, and let $(E \times_c G, \mathcal{L}_d)$ be the associated skew product labeled graph. Let $\phi_c^i : (E \times_c G)^i \to E^i$ be given by $\phi_c^i(x, g) = x$ for $x \in E^i$, $i = 0, 1$, and let $\phi_d^{\mathcal{A} \times G} : \mathcal{A} \times G \to \mathcal{A}$ be given by $\phi_d^{\mathcal{A} \times G}(a, g) = a$.

Lemma 2.4.9. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph, let G be a group, and let $c, d : E^1 \to G$ be functions. Define $\phi_{c,d} : (E \times_c G, \mathcal{L}_d) \to (E, \mathcal{L})$ by*

$$\phi_c^i(x, g) = x \textit{ for } x \in E^i, i = 0, 1, \textit{ and } \phi_d^{\mathcal{A} \times G}(a, g) = a.$$

Then $\phi_{c,d}$ is a surjective labeled graph morphism.

Proof. It is clear that $\phi_{c,d} = (\phi_c^0, \phi_c^1, \phi_d^{\mathcal{A} \times G})$ is a surjective, so we begin by showing $\phi_c = (\phi_c^0, \phi_c^1)$ is a graph morphism. So for all $(e, g) \in E \times_c G$ we have

$$\phi_c^0(r(e, g)) = \phi_c^0(r(e), gc(e)) = r(e) = r(\phi_c^1(e, g))$$

and

$$\phi_c^0(s(e, g)) = \phi_c^0(s(e), g) = s(e) = s(\phi_c^1(e, g)).$$

Thus $\phi_c = (\phi_c^0, \phi_c^1)$ is a graph morphism. Now we examine the condition for $\phi_d^{\mathcal{A} \times G}$. For all $(e, g) \in (E \times_c G)^1$ we have

$$\phi_d^{\mathcal{A} \times G} \circ \mathcal{L}_d(e, g) = \phi_d^{\mathcal{A} \times G}(\mathcal{L}(e), gd(e)) = \mathcal{L}(e) = \mathcal{L} \circ \phi_c^1(e, g)$$

and the claim has been proven. $\square$

Proposition 2.4.10. *Let $(E,\mathcal{L})$ be a labeled graph over the alphabet $\mathcal{A}$, let $c,d : E^1 \to G$ be functions, and let $(E\times_c G, \mathcal{L}_d)$ be the associated skew product labeled graph. If $\mathcal{L}_d(\mu,g)=\mathcal{L}_d(\nu,h)$, then $\mathcal{L}(\mu)=\mathcal{L}(\nu)$.*

Proof. Let $(\mu,g),(\nu,h)\in(E\times_c G)^*$ have the same label, so they must have the same length. The result is manifest for paths of length 1; so we shall assume the length of μ,ν is $n\geq 2$. Hence we have

$$(\mu,g)=(\mu_{1_G},g)(\mu_2,gc(\mu_{1_G}))\cdots(\mu_n,gc(\mu'))$$

and

$$(\nu,h)=(\nu_{1_G},h)(\nu_2,hc(\nu_{1_G}))\cdots(\nu_n,hc(\nu'))$$

for some $g,h\in G$. Since the labels for $(\mu,g),(\nu,h)$ are the the same we have

$$\begin{gathered}\mathcal{L}_d(\mu_{1_G},g)=\mathcal{L}_d(\nu_{1_G},h)\\ \mathcal{L}_d(\mu_2,gc(\mu_{1_G}))=\mathcal{L}_d(\nu_2,hc(\nu_{1_G}))\\ \vdots\\ \mathcal{L}_d(\mu_n,gc(\mu'))=\mathcal{L}_d(\nu_n,hc(\nu')).\end{gathered}$$

Applying the definition of $\mathcal{L}_d$ to each equation gives us

$$\begin{gathered}(\mathcal{L}(\mu_{1_G}),gd(\mu_{1_G}))=(\mathcal{L}(\nu_{1_G}),hd(\nu_{1_G}))\\ (\mathcal{L}(\mu_2),gc(\mu_{1_G})d(\mu_2))=(\mathcal{L}(\nu_2),hc(\nu_{1_G})d(\nu_2))\\ \vdots\\ (\mathcal{L}(\mu_n),gc(\mu')d(\mu_n))=(\mathcal{L}(\nu_n),hc(\nu')d(\nu_n));\end{gathered}$$

and so $\mathcal{L}(\mu_i)=\mathcal{L}(\nu_i)$ for $i=1,\cdots,n$, as required. □

Group-Actions

Let E be a directed graph and G a group. An *graph action of G on E* is a triple (E,G,α) where $\alpha: G\to \mathrm{Aut}(E)$ is a group homomorphism. We say that the graph action (E,G,α) is *free* if $\alpha_g^0(v)=v$ for all $v\in E^0$ then we must have $g=1_G$. Two graph actions (E,G,α) and (F,G,β) are *isomorphic* or *conjugate* if there is a graph isomorphism $\phi : E \to F$ which is *equivariant* in the sense that $\phi \circ \alpha_g = \beta_g \circ \phi$ for all $g \in G$.

The following lemma (which follows from a routine argument) shows that skew product graphs provide a rich source of examples of free group-actions.

Lemma 2.4.11. *Let E be a directed graph, let $c : E^1 \to G$ be a function, and let $E \times_c G$ be the associated skew product graph. For $g \in G, i = 0, 1$ let*

$$\tau_g^i(x, h) = (x, gh) \quad \text{for } (x, h) \in (E \times_c G)^i.$$

Then $\tau_g = (\tau_g^0, \tau_g^1)$ is an automorphism of $E \times_c G$. The map $\tau = (\tau^0, \tau^1) : G \to \operatorname{Aut}(E \times_c G)$ defined by $g \to \tau_g$ is a homomorphism. Furthermore, $\tau_g^i(x, h) = (x, h)$ if and only if $g = 1_G$, hence $(E \times_c G, G, \tau)$ is a free graph action.

Definition 2.4.12. Let E be a directed graph, let $c : E^1 \to G$ be a function, and let $E \times_c G$ be the associated skew product graph. We call the map $\tau = (\tau^0, \tau^1) : G \to \operatorname{Aut}(E \times_c G)$ as described in Lemma 2.4.11 as the *left graph translation map*, and the action $(E \times_c G, G, \tau)$ the *left graph translation action*.

Definition 2.4.13. Let $c_1, c_2 : E^1 \to G$ be functions, then we say that c_1 and c_2 are *cohomologous* (and write $c_1 \sim_b c_2$) if there is a function $b : E^0 \to G$ such that $c_1(e)b(r(e)) = b(s(e))c_2(e)$ holds for all $e \in E^1$.

Remark 2.4.14. It is straightforward to check that the relation $\sim$ defined above is an equivalence relation.

The following result was stated but not proved in [25, Section 7].

Proposition 2.4.15. *Suppose that E is a directed graph and c_1, c_2 are functions such that $c_1 \sim c_2$, then the left graph translation action $(E \times_{c_1} G, G, \tau)$ is isomorphic to the left graph translation action $(E \times_{c_2} G, G, \tau)$.*

Proof. Suppose that $c_1 \sim c_2$, then there exists a function $b : E^0 \to G$ such that $c_1(e)b(r(e)) = b(s(e))c_2(e)$ for all $e \in E^1$. We define maps $\phi^i : (E \times_{c_1} G)^i \to$

$(E \times_{c_2} G)^i$ for $i = 0, 1$ by

$$\phi^0(v,g) = (v, gb(v)), \qquad \text{for } (v,g) \in (E \times_{c_1} G)^0,$$

$$\phi^1(e,g) = (e, gb(s(e))) \qquad \text{for } (e,g) \in (E \times_{c_1} G)^1.$$

We claim that $\phi = (\phi^0, \phi^1)$ is a graph isomorphism.

We begin by showing that the pair (ϕ^0, ϕ^1) is a graph morphism. Let $(e,h) \in (E \times_{c_1} G)^1$, then

$$\begin{aligned}\phi^0(r(e,h)) = \phi^0(r(e), hc_1(e)) &= (r(e), hc_1(e)b(r(e))) \\ &= (r(e), hb(s(e))c_2(e)) = r(e, hb(s(e))) = r(\phi^1(e,h))\end{aligned}$$

and

$$\phi^0(s(e,h)) = \phi^0(s(e), h) = (s(e), hb(s(e))) = s(e, hb(s(e))) = s(\phi^1(e,h)).$$

Thus we have shown that (ϕ^0, ϕ^1) is a graph morphism.

Now we need to show that (ϕ^0, ϕ^1) are bijective. To see that ϕ^0 is injective, suppose that $(v_1, h), (v_2, k) \in (E \times_{c_1} G)^0$ satisfy $\phi^0(v_1, h) = \phi^0(v_2, k)$. Then $(v_1, hb(v_1)) = (v_2, kb(v_2))$ and so $v_1 = v_2 = v$, say. Now we know $hb(v) = kb(v)$ which means $h = k$. Therefore $(v_1, h) = (v_2, k)$ and ϕ^0 is injective.

To see that ϕ^0 is surjective, let $(v,h) \in (E \times_{c_2} G)^0$. Then $(v, hb^{-1}(v)) \in (E \times_c G)^0$ such that $\phi^0(v, hb^{-1}(v)) = (v, hb^{-1}(v)b(v)) = (v,h)$. So ϕ^0 is surjective.

The injectivity and surjectivity arguments are similar for ϕ^1. Therefore $\phi = (\phi^0, \phi^1)$ is a graph isomorphism.

It remains to show that ϕ is equivariant, that is $\phi \circ \tau_g = \tau_g \circ \phi$ for all $g \in G$. Notice that for all $g \in G$ and $(v,h) \in (E \times_{c_1} G)^0$ we have

$$\phi^0 \circ \tau_g(v,h) = \phi^0(v, gh) = (v, ghb(v)) = \tau_g(v, hb(v)) = \tau_g \circ \phi^0(v,h).$$

The argument is similar for ϕ^1. $\square$

Let $(E, \mathcal{L})$ be a labeled graph over the alphabet $\mathcal{A}$ and G a group. A *labeled graph action of G on $(E, \mathcal{L})$* is a triple $((E, \mathcal{L}), G, \alpha)$ where $\alpha : G \to \operatorname{Aut}(E, \mathcal{L})$ is a group homomorphism. In particular, by Definition 2.3.4 (2), for all $e \in E^1$ and

$g \in G$ we have

$$\mathcal{L}(\alpha_g^1(e)) = \alpha_g^{\mathcal{A}}(\mathcal{L}(e)). \tag{2.9}$$

We say that the label graph action $((E, \mathcal{L}), G, \alpha)$ is *free* if the graph action (E, G, α) is free and if $\alpha_g^{\mathcal{A}}(a) = a$ for any $a \in \mathcal{A}$ then we must have $g = 1_G$. Two labeled graph actions $((E, \mathcal{L}), G, \alpha)$ and $((F, \mathcal{M}), G, \beta)$ are *isomorphic* if there is a labeled graph isomorphism $\phi : (E, \mathcal{L}) \to (F, \mathcal{M})$ which is *equivariant* in the sense that $\phi : E \to F$ is equivariant and $\phi^{\mathcal{A}_E} \circ \alpha_g^{\mathcal{A}_E} = \beta_g^{\mathcal{A}_F} \circ \phi^{\mathcal{A}_E}$ for all $g \in G$.

Lemma 2.4.16. *Let $(E, \mathcal{L})$ be a labeled graph, let $c, d : E^1 \to G$ be functions, and let $(E \times_c G, \mathcal{L}_d)$ be the associated skew product labeled graph.*

1. *For $(a, h) \in \mathcal{A} \times G$, and $g \in G$ let $\tau_g^{\mathcal{A}}(a, h) = (a, gh)$, then $g \mapsto \tau_g^{\mathcal{A}}$ defines a map $\tau^{\mathcal{A}} : G \to \mathrm{Bij}(A \times G)$.*

2. *Let $(\tau^0, \tau^1) : G \to \mathrm{Aut}(E \times_c G)$ be the left graph translation map. Then for each $g \in G$, $\tau_g = (\tau_g^0, \tau_g^1, \tau_g^{\mathcal{A}})$ is a labeled graph automorphism.*

3. *The map $\tau = (\tau^0, \tau^1, \tau^{\mathcal{A}}) : G \to \mathrm{Aut}(E \times_c G, \mathcal{L}_d)$ defined by $g \to \tau_g$ is a homomorphism.*

4. *Since $\tau_g^{\mathcal{A}}(a, h) = (a, h)$ if and only if $g = 1_G$, it follows that $((E \times_c G, \mathcal{L}_d), G, \tau)$ is a free labeled graph action.*

Proof. Since $(\tau_g^{\mathcal{A}})^{-1} = \tau_{g^{-1}}^{\mathcal{A}}$ for all $g \in G$ statement (1) follows. For (2) first observe that $\tau^{\mathcal{A}}$ satisfies the compatibility condition of Equation (2.9): for $g \in G$ and $(e, h) \in (E \times_c G)^1$ we have

$$\mathcal{L}_d(\tau_g^1(e, h)) = \mathcal{L}_d(e, gh) = (\mathcal{L}(e), ghd(e)) = \tau_g^{\mathcal{A}}(\mathcal{L}_d(e, h)).$$

Second, recall that (τ_g^0, τ_g^1) is a graph automorphism from Lemma 2.4.11 and $\tau_g^{\mathcal{A}}$ is a bijection from (1). For (3) one checks that $\tau_{g_1 g_2}^{\mathcal{A}} = \tau_{g_1}^{\mathcal{A}} \tau_{g_2}^{\mathcal{A}}$ for all $g_1, g_2 \in G$. For (4) it suffices to show that $\tau_g^{\mathcal{A}}(a, h) = (a, h)$ implies $g = 1_G$, but that follows directly from the definition of $\tau^{\mathcal{A}}$. $\square$

Definition 2.4.17. Let $(E, \mathcal{L})$ be a labeled graph, let $c, d : E^1 \to G$ be functions, and let $(E \times_c G, \mathcal{L}_d)$ be the associated skew product labeled graph. We call the map $\tau = (\tau^0, \tau^1, \tau^{\mathcal{A}}) : G \to \mathrm{Aut}(E \times_c G, \mathcal{L}_d)$ as described in Lemma 2.4.16 as the *left labeled graph translation map*, and the action $((E \times_c G, \mathcal{L}_d), G, \tau)$ the *left labeled graph translation action.*

Since α is a labeled graph homomorphism and all our Cuntz-Krieger $(E, \mathcal{L})$-families are defined in terms of universal families, Theorem 2.4.18 (below) is obtained by a straightforward application of Theorem 2.3.7.

Theorem 2.4.18. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph, let $((E, \mathcal{L}), G, \alpha)$ be a labeled graph action, and let $\{s_a, p_{r(\beta)}\}$ be a universal Cuntz-Krieger $(E, \mathcal{L})$-family. Then for $h \in G$ the maps*

$$\alpha_h s_a = s_{\alpha_h^{\mathcal{A}} a} \qquad \textit{and} \qquad \alpha_h p_{r(\beta)} = p_{\alpha_h^0 r(\beta)}.$$

determine an automorphism α_h of $C^*(E, \mathcal{L})$. The map $\alpha : G \to \mathrm{Aut}(C^*(E, \mathcal{L}))$ defined by $h \mapsto \alpha_h$ is a homomorphism and so it gives an action of G on $C^*(E, \mathcal{L})$.

Proof. By Theorem 2.3.7 we know that for all fixed $h \in G$ α_h is an automorphism of $C^*(E, \mathcal{L})$ determined by

$$\alpha_h s_a = s_{\alpha_h^{\mathcal{A}} a} \qquad \text{and} \qquad \alpha_h p_{r(\beta)} = p_{\alpha_h^0 r(\beta)}$$

for all $a \in \mathcal{A}$ and $\beta \in \mathcal{L}^*(E)$. The map $h \mapsto \alpha_h$ from $G \to \mathrm{Aut}(C^*(E, \mathcal{L}))$ is a homomorphism since α is a labeled graph action on $(E, \mathcal{L})$. Therefore the proof is complete. $\square$

2.5 Gross-Tucker Theorem

In this section we prove a version of the Gross-Tucker theorem (see [11, Theorem 2.2.2]) for labeled graphs. For directed graphs, the Gross-Tucker theorem says, roughly speaking, that up to equivariant isomorphism, every free action α of a group G on a directed graph E is a left translation automorphism τ on a skew product graph $E/G \times_c G$ built from the quotient graph E/G. Our aim is to prove a similar result for labeled graphs, the new ingredient is the labeling map d found in the definition of a skew product labeled graph. Before giving our main result, Theorem 2.5.15, we introduce some notation and review the Gross-Tucker theorem for directed graphs.

Definition 2.5.1. Let (E, G, α) be a graph action. For $i = 0, 1$ and $x \in E^i$ let

$$Gx := \{\alpha_g^i(x) : g \in G\}$$

be the *orbit* of x under the action of G.

Lemma 2.5.2. *Let (E, G, α) be a graph action. For $i = 0, 1$ let $(E/G)^i = \{Gx : x \in E^i\}$ and for $Ge \in (E/G)^1$ let*

$$r(Ge) = Gr(e) \text{ and } s(Ge) = Gs(e). \tag{2.10}$$

Then $((E/G)^0, (E/G)^1, r, s)$ is a directed graph and for $i = 0, 1$ the map $q^i : E^i \to (E/G)^i$ given by $q^i(x) = Gx$ for $x \in E^i$ defines a surjective graph morphism $q = (q^0, q^1) : E \to E/G$.

Proof. First we check that r, s from Equation (2.10) are well-defined. Let $e, f \in E^1$ and suppose $Ge = Gf$, then there exists $g \in G$ such that $\alpha_g^1(e) = f$. Then $r(\alpha_g^1(e)) = \alpha_g^0(r(e)) = r(f)$. So we have

$$r(Ge) = Gr(e) = Gr(f) = r(Gf).$$

The argument for s being well-defined is similar, thus $((E/G)^0, (E/G)^1, r, s)$ is a directed graph.

To see that q is a graph morphism notice that for all $e \in E^1$ we have

$$q^0(r(e)) = Gr(e) = r(Ge) = r(q^1(e))$$

and

$$q^0(s(e)) = Gs(e) = s(Ge) = s(q^1(e)).$$

Since surjectivity is evident, we have shown that $q : E \to E/G$ is a surjective graph morphism. $\square$

Definition 2.5.3. Let (E, G, α) be a graph action. The *quotient graph* E/G is the directed graph $((E/G)^0, (E/G)^1, r, s)$ described in Lemma 2.5.2. The map $q = (q^0, q^1) : E \to E/G$ is called the *quotient map*.

Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action. For $a \in \mathcal{A}$ let

$$Ga = \{\alpha_g^{\mathcal{A}}(a) : g \in G\} \text{ and } \mathcal{A}/G = \{Ga : a \in \mathcal{A}\}.$$

Lemma 2.5.4. *Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action and $q^{\mathcal{A}}(a) = Ga$ for $a \in \mathcal{A}$. Let $\mathcal{L}/G : (E/G)^1 \to \mathcal{A}/G$ be given by*

$$(\mathcal{L}/G)(Ge) = G\mathcal{L}(e),$$

then $(E/G,\mathcal{L}/G)$ is a labeled graph over $\mathcal{A}/G$, and $q = (q^0,q^1,q^{\mathcal{A}}) : (E,\mathcal{L}) \to (E/G,\mathcal{L}/G)$ is a surjective labeled graph homomorphism.

Proof. To see that $\mathcal{L}/G$ is well-defined, let $e,f \in E^1$ and suppose $Ge = Gf$. Then there exists $g \in G$ such that $\alpha_g^1(e) = f$, so that $\mathcal{L}(f) = \mathcal{L}(\alpha_g^1(e)) = \alpha_g^{\mathcal{A}}(\mathcal{L}(e))$. Then

$$(\mathcal{L}/G)(Ge) = G\mathcal{L}(e) = G\mathcal{L}(f) = (\mathcal{L}/G)(Gf).$$

Thus $\mathcal{L}/G$ is well-defined and hence $(E/G,\mathcal{L}/G)$ is a labeled graph over $\mathcal{A}/G$.

Let $e \in E^1$. Since

$$(\mathcal{L}/G)(q^1(e)) = (\mathcal{L}/G)(Ge) = G\mathcal{L}(e) = q^{\mathcal{A}}\mathcal{L}(e),$$

it follows that $q = (q^0,q^1,q^{\mathcal{A}}) : (E,\mathcal{L}) \to (E/G,\mathcal{L}/G)$ is a labeled graph morphism. From Lemma 2.5.2 we know that $(q^0,q^1) : E \to E/G$ is a surjective graph morphism, and by definition $q^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}/G$ is a surjective map. Hence $q = (q^0,q^1,q^{\mathcal{A}})$ is surjective. $\square$

Definition 2.5.5. Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action. The *quotient labeled graph* $(E/G,\mathcal{L}/G)$ is the labeled graph described in Lemma 2.5.4 and the map $q = (q^0,q^1,q^{\mathcal{A}}) : (E,\mathcal{L}) \to (E/G,\mathcal{L}/G)$ is the *quotient labeled map.*

Proposition 2.5.6. *Let $(E,\mathcal{L})$ be a labeled graph over the alphabet $\mathcal{A}$, $c,d : E^1 \to G$ be functions and $(E \times_c G,\mathcal{L}_d)$ be the associated skew product labeled graph. Let $((E\times_c G,\mathcal{L}_d),G,\tau)$ be the left labeled graph translation action. Then there is a labeled graph isomorphism $\psi_{c,d} : ((E \times_c G)/G,\mathcal{L}_d/G) \to (E,\mathcal{L})$ such that $\psi_{c,d} \circ q = \phi_{c,d}$ where $\phi_{c,d} : (E\times_c G,\mathcal{L}_d) \to (E,\mathcal{L})$ is the labeled graph morphism from Lemma 2.4.9.*

Proof. We begin by defining $\psi_{c,d}$. For $i = 0,1$, let $\psi_c^i : ((E\times_c G)/G)^i \to E^i$ be given by $\psi_c^i(G(x,g)) = x$, where $x \in E^i$, and let $\psi_d^{(\mathcal{A}\times G)/G} : (\mathcal{A} \times G)/G \to \mathcal{A}$ be given by $\psi_d^{(\mathcal{A}\times G)/G}(G(a,g)) = a$ for $a \in \mathcal{A}$. We check ψ_c^0 is well-defined, the other cases being similar. If $G(v,g_1) = G(w,g_2)$, then there is an $h \in G$ such that $\tau_h^0(v,g_1) = (w,g_2)$; so that $(v,hg_1) = (w,g_2)$. Hence $v = w$ and $g_2 = hg_1$, which implies

$$\psi_c^0(G(v,g_1)) = v = \psi_c^0(G(v,hg_1)).$$

Now we show that (ψ_c^0, ψ_c^1) is a graph isomorphism. Let $G(e,g) \in ((E \times_c G)/G)^1$ and notice that

$$\psi_c^0(r(G(e,g))) = \psi_c^0(Gr(e,g)) = \psi_c^0(G(r(e),gc(e))) = r(e) = r(\psi_c^1(G(e,g))).$$

Similarly

$$\psi_c^0(s(G(e,g))) = \psi_c^0(Gs(e,g)) = \psi_c^0(G(s(e),g)) = s(e) = s(\psi_c^1(G(e,g))).$$

To see that ψ_c^0 is injective, let $G(v_1,g_1), G(v_2,g_2) \in ((E\times_c G)/G)^0$ and suppose $\psi_c^0(G(v_1,g_1)) = \psi_c^0(G(v_2,g_2))$. Then we have $v_1 = v_2$ and since G is a group there exists $g_3 \in G$ such that $g_1 = g_3g_2$. Thus $(v_1,g_1) = (v_1,g_3g_2) = \tau_{g_3}^0(v_1,g_2)$, which means $G(v_1,g_1) = G(v_2,g_2)$. So ψ_c^0 is injective.

Let $v \in E^0$. Since $\psi_c^0(G(v,g)) = v$ it follows that ψ_c^0 is surjective. Similar arguments show that ψ_c^1 and $\psi_d^{(\mathcal{A}\times G)/G}$ are bijective. Hence (ψ_c^0, ψ_c^1) is a graph isomorphism and $\psi_d^{(\mathcal{A}\times G)/G}$ is bijective.

It remains to show the equation from Definition 2.3.4(2): let $G(e,g) \in ((E \times_c G)/G)^1$ and notice that

$$\begin{aligned}
\psi_d^{(\mathcal{A}\times G)/G} \circ \mathcal{L}_d/G(G(e,g)) &= \psi_d^{(\mathcal{A}\times G)/G}(G(\mathcal{L}_d(e,g))) \\
&= \psi_d^{(\mathcal{A}\times G)/G}(G(\mathcal{L}(e), gd(e))) \\
&= \mathcal{L}(e) \\
&= \mathcal{L} \circ \psi_c^1(G(e,g)).
\end{aligned}$$

So we have shown that $\psi_{c,d} = (\psi_c^0, \psi_c^1, \psi_d^{(\mathcal{A}\times G)/G})$ is a labeled graph isomorphism.

Lastly, we show that $\psi_{c,d} \circ q = \phi_{c,d}$. Let $(x,g) \in (E \times_c G)^i$. Then

$$\psi_c^i \circ q^i(x,g) = \psi_c^i(G(x,g)) = x = \phi_c^i(x,g).$$

Similarly for $(a,g) \in \mathcal{A} \times G$ we have

$$\psi_d^{(\mathcal{A}\times G)/G} \circ q^{\mathcal{A}\times G}(a,g) = \psi_d^{(\mathcal{A}\times G)/G}(G(a,g)) = a = \phi_d^{\mathcal{A}\times G}(a,g).$$

Thus the result has been proven. □

Example 2.5.7. Recall the labeled graphs $(E, \mathcal{L})$ and $(E \times_c \mathbf{Z}, \mathcal{L}_d)$ from Example 2.4.6 (2). For the left labeled graph translation action $((E \times_c \mathbf{Z}, \mathcal{L}_d), \mathbf{Z}, \tau)$ we have $((E \times_c \mathbf{Z})/\mathbf{Z}, \mathcal{L}_d/\mathbf{Z}) \cong (E, \mathcal{L})$ by Proposition 2.5.6.

Definition 2.5.8. Let F, E be directed graphs. A surjective graph morphism $p : F \to E$ has *unique path lifting property* if given $u \in F^0$ and $e \in E^1$ with $s(e) = p^0(u)$ there is a unique edge $f \in F^1$ with $s(f) = u$ and $p^1(f) = e$.

Lemma 2.5.9. *Let (E, G, α) be a free graph action. Then the quotient map $q = (q^0, q^1) : E \to E/G$ has the unique path lifting property.*

Proof. Let $u \in E^0$ and $Ge \in (E/G)^1$ be such that $s(Ge) = q^0(u)$. Then

$$Gs(e) = s(Ge) = q^0(u) = Gu,$$

and since the action is free, there exists a unique $g \in G$ such that $\alpha_g^0 s(e) = u$. So $f = \alpha_g^1(e)$ has the required property, that is $s(\alpha_g^1(e)) = u$ and $q^1(\alpha_g^1(e)) = Ge$. So $q = (q^0, q^1)$ is a graph morphism with the unique path lifting property. □

Definition 2.5.10. Let (E, G, α) be a graph action and $q = (q^0, q^1) : E \to E/G$ be the quotient map. A *section* for q^i is a map $\eta^i : (E/G)^i \to E^i$ for $i = 0, 1$ such that $q^i \circ \eta^i = \mathrm{id}_{(E/G)^i}$.

Lemma 2.5.11. *Let (E, G, α) be a graph action and $q = (q^0, q^1) : E \to E/G$ be the quotient map. A section η^0 for q^0 uniquely determines a section η^1 for q^1 such that*

$$s(\eta^1(Ge)) = \eta^0(s(Ge)) \text{ for all } e \in E^1. \tag{2.11}$$

Proof. By Lemma 2.5.9, the quotient map $q = (q^0, q^1) : E \to E/G$ has the unique path lifting property. We use this property to define a section $\eta^1 : (E/G)^1 \to E^1$ for q^1 in the following manner. Fix $Gv \in (E/G)^0$, then for each $Ge \in (E/G)^1$ with $s(Ge) = Gv$ there is a unique $f \in E^1$ satisfying $q^1(f) = Ge = Gf$ and

$s(f) = \eta^0(Gv)$. Let $\eta^1(Ge) = f$, which is well-defined since the source map on E/G is well-defined. Since $q^1(\eta^1(Ge)) = q^1(f) = Ge$ it follows that η^1 is a section. Uniqueness follows from the unique path lifting property of q. □

The following is a restatement of the Gross-Tucker Theorem found in [11, Theorem 2.2.2].

Theorem 2.5.12. *Let (E, G, α) be a free graph action. For every section $\eta^0 : (E/G)^0 \to E^0$ for q^0 there is a function $c_\eta : (E/G)^1 \to G$ such that (E, G, α) is isomorphic to $(E/G \times_{c_\eta} G, G, \tau)$.*

Proof. We begin by fixing a section $\eta^0 : (E/G)^0 \to E^0$ for q^0. By Lemma 2.5.11 there is a section η^1 for q^1 such that

$$s(\eta^1(Ge)) = \eta^0(Gv) = \eta^0(s(Ge)). \tag{2.12}$$

We now define the map $c_\eta : (E/G)^1 \to G$. Suppose that $Ge \in (E/G)^1$ and $f = \eta^1(Ge)$, then

$$q^0(r(\eta^1(Ge))) = q^0 r(f) = Gr(f) = r(Gf) = r(Ge) = q^0(\eta^0(r(Ge))).$$

As (E, G, α) is free, there is a unique $h \in G$ such that $\alpha^0_h \eta^0(r(Ge)) = r(\eta^1(Ge))$ and we may set $c_\eta(Ge) = h$ which is well-defined since the range map on E/G is well-defined. In other words, for each $Ge \in (E/G)^1$, $c_\eta(Ge)$ is the unique element of G satisfying

$$\alpha^0_{c_\eta(Ge)} \eta^0(r(Ge)) = r(\eta^1(Ge)). \tag{2.13}$$

Next we show that the directed graphs $E/G \times_{c_\eta} G$ and E are isomorphic. We begin by constructing maps $\phi^i_{c_\eta} : (E/G \times_{c_\eta} G)^i \to E^i$, for $i = 0, 1$. For $(Gv, g) \in (E/G \times_{c_\eta} G)^0$ we define $\phi^0_{c_\eta}(Gv, g) = \alpha^0_g \eta^0(Gv)$ and for $(Ge, g) \in (E/G \times_{c_\eta} G)^1$ we define $\phi^1_{c_\eta}(Ge, g) = \alpha^1_g \eta^1(Ge)$. We claim that the pair $\phi_{c_\eta} = (\phi^0_{c_\eta}, \phi^1_{c_\eta})$ is a graph morphism. Let $(Ge, h) \in (E/G \times_{c_\eta} G)^1$, then by Equation (2.13)

$$\begin{aligned}\phi^0_{c_\eta}(r(Ge, h)) = \phi^0_{c_\eta}(r(Ge), hc_\eta(Ge)) &= \alpha^0_h \alpha^0_{c_\eta(Ge)} \eta^0(r(Ge)) \\ &= \alpha^0_h r(\eta^1(Ge)) = r(\alpha^1_h \eta^1(Ge)) = r(\phi^1_{c_\eta}(Ge, h))\end{aligned}$$

and by Equation (2.12)

$$\begin{aligned}\phi^0_{c_\eta}(s(Ge,h)) = \phi^0_{c_\eta}(s(Ge),h)) &= \alpha^0_h\eta^0(s(Ge)) \\ &= \alpha^0_h s(\eta^1(Ge)) = s(\alpha^1_h\eta^1(Ge)) = s(\phi^1_{c_\eta}(Ge,h)).\end{aligned}$$

Our claim is established.

Now we must show that $\phi^0_{c_\eta}$ and $\phi^1_{c_\eta}$ are bijections. To see $\phi^0_{c_\eta}$ is injective, we suppose that $(Gv_1,h),(Gv_2,k) \in (E/G \times_{c_\eta} G)^0$ satisfy $\phi^0_{c_\eta}(Gv_1,h) = \phi^0_{c_\eta}(Gv_2,k)$. Then $\alpha^0_h\eta^0(Gv_1) = \alpha^0_k\eta^0(Gv_2)$, and so $\alpha^0_{k^{-1}}\alpha^0_h\eta^0(Gv_1) = \eta^0(Gv_2)$. It follows that

$$Gv_2 = q^0(\eta^0(Gv_2)) = q^0(\alpha^0_{k^{-1}}\alpha^0_h\eta^0(Gv_1)) = q^0(\eta^0(Gv_1)) = Gv_1.$$

Now we have $\alpha^0_{k^{-1}}\alpha^0_h\eta^0(Gv_1) = \eta^0(Gv_1)$. Since (E,g,α) is free we must have $k^{-1}h = 1_G$, and so $h = k$. It follows that $(Gv_1,h) = (Gv_2,k)$ and $\phi^0_{c_\eta}$ is injective.

We check that $\phi^0_{c_\eta}$ is surjective. Let $v \in E^0$, then $q^0(v) = Gv$. It follows that $v = \alpha^0_h\eta^0(Gv)$ for some $h \in G$ and so $\phi^0_{c_\eta}(Gv,h) = \alpha^0_h\eta^0(Gv) = v$. Hence $\phi^0_{c_\eta}$ is surjective. The proof that $\phi^1_{c_\eta}$ is bijective is similar. Thus $\phi_{c_\eta} : (E/G \times_{c_\eta} G) \to E$ is an isomorphism.

To complete the proof of this theorem it remains to show that $\phi_{c_\eta} \circ \tau_g = \alpha_g \circ \phi_{c_\eta}$ for all $g \in G$. Notice that for all $(Gv,h) \in (E/G \times_{c_\eta} G)^0$ and $g \in G$ we have

$$\begin{aligned}\phi^0_{c_\eta}(\tau^0_g(Gv,h)) = \phi^0_{c_\eta}(Gv,gh) &= \alpha^0_{gh}\eta^0(Gv) \\ &= \alpha^0_g\alpha^0_h\eta^0(Gv) = \alpha^0_g\phi^0_{c_\eta}(Gv,h).\end{aligned}$$

The argument for $\phi^1_{c_\eta}$ is similar and we have established our result.

□

Corollary 2.5.13. *Let (E,G,α) be a free graph action. Suppose that $\eta^0,\kappa^0 : (E/G)^0 \to E^0$ are sections for q^0 and let $c_\eta, c_\kappa : (E/G)^1 \to G$ be as in Theorem 2.5.12. Then $c_\eta \sim c_\kappa$.*

Proof. Suppose $\eta^0,\kappa^0 : (E/G)^0 \to E^0$ are sections for $q^0 : E \to E/G$, that is $q^0(\eta^0(Gv)) = Gv = q^0(\kappa^0(Gv))$ for all $Gv \in (E/G)^0$. Then, as in the proof of Theorem 2.5.12, there are sections $\eta^1,\kappa^1 : (E/G)^1 \to E^1$ for q^1 and functions

$c_\eta, c_\kappa : (E/G)^1 \to G$ characterized by

$$\alpha^0_{c_\eta(Ge)} \circ \eta^0(r(Ge)) = r(\eta^1(Ge)) \tag{2.14}$$

$$\alpha^0_{c_\kappa(Ge)} \circ \kappa^0(r(Ge)) = r(\kappa^1(Ge)) \tag{2.15}$$

for all $Ge \in (E/G)^1$.

We now define the map $b : (E/G)^0 \to G$. Since $q^0(\kappa^0(Gv)) = q^0(\eta^0(Gv))$ and (E, G, α) is a free graph action, there is a unique $h \in G$ such that $\eta^0(Gv) = \alpha^0_h \kappa^0(Gv)$ and we may define $b(Gv) = h$. In other words, for each $Gv \in (E/G)^0$, $b(Gv)$ is the unique element of G satisfying

$$\eta^0(Gv) = \alpha^0_{b(Gv)} \kappa^0(Gv). \tag{2.16}$$

Applying Equation (2.16) with $Gv = r(Ge)$ we obtain

$$\eta^0(r(Ge)) = \alpha^0_{b(r(Ge))} \kappa^0(r(Ge))$$

so by substituting into Equation (2.14), we obtain

$$r(\eta^1(Ge)) = \alpha^0_{c_\eta(Ge)} \alpha^0_{b(r(Ge))} \kappa^0(r(Ge)). \tag{2.17}$$

Now applying Equation (2.16) with $Gv = s(Ge)$ we obtain $\eta^0(s(Ge)) = \alpha^0_{b(s(Ge))} \kappa^0(s(Ge))$. Since $s(\eta^1(Ge)) = \eta^0(s(Ge))$ and $s((\kappa^1(Ge)) = \kappa^0(s(Ge))$ we have

$$\eta^1(Ge) = \alpha^1_{b(s(Ge))}(\kappa^1(Ge)), \tag{2.18}$$

and since $\alpha_{b(s(Ge))}$ is a graph morphism we obtain

$$r(\eta^1(Ge)) = r(\alpha^1_{b(s(Ge))}(\kappa^1(Ge))) = \alpha^0_{b(s(Ge))} r(\kappa^1(Ge)). \tag{2.19}$$

So, by substituting Equation (2.15) into Equation (2.19), we obtain

$$r(\eta^1(Ge)) = \alpha^0_{b(s(Ge))} \alpha^0_{c_\kappa(Ge)} \kappa^0(r(Ge)). \tag{2.20}$$

Comparing Equations (2.17) and (2.20), we have

$$\alpha^0_{c_\eta(Ge)} \alpha^0_{b(r(Ge))} \kappa^0(r(Ge)) = \alpha^0_{b(s(Ge))} \alpha^0_{c_\kappa(Ge)} \kappa^0(r(e)).$$

Since the action of (E, G, α) is free it follows that $c_\eta(Ge) b(r(Ge)) = b(s(Ge)) c_\kappa(Ge)$ as required. □

Definition 2.5.14. Let $((E, \mathcal{L}), G, \alpha)$ be a labeled graph action and let $q : (E, \mathcal{L}) \to (E/G, \mathcal{L}/G)$ be the quotient map. A *section* for $q^{\mathcal{A}}$ is a map $\eta^{\mathcal{A}} : \mathcal{A}/G \to \mathcal{A}$ that satisfies $q^{\mathcal{A}} \circ \eta^{\mathcal{A}} = \mathrm{id}_{\mathcal{A}/G}$.

Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action and let $q:(E,\mathcal{L}) \to (E/G,\mathcal{L}/G)$ be the quotient map. By Lemma 2.5.11 a section η^0 for q^0 uniquely determines a section η^1 for q^1, which in turn can be used to form a section $\eta^{\mathcal{A}}$ for $q^{\mathcal{A}}$: For $Ga \in \mathcal{A}/G$ let $Ge \in (E/G)^1$ be such that $\mathcal{L}/G(Ge) = Ga$, then define $\eta^{\mathcal{A}}(Ga) = \mathcal{L}(\eta^1(Ge))$. One checks that $\eta^{\mathcal{A}}$ defined this way is a section. One must ask if $\eta^{\mathcal{A}}$ is uniquely determined by η^1. Notice that if the labeling is trivial, then $\eta^{\mathcal{A}}$ is uniquely determined by η^1. However, in general, $\eta^{\mathcal{A}}$ not uniquely determined by η^1: Suppose Ge and Gf have the same label Ga and $\mathcal{L}(\eta^1(Ge)) \neq \mathcal{L}(\eta^1(Gf))$. If we use Ge to define $\eta^{\mathcal{A}}$, then we get a different function from the one we get if we use Gf.

The following is a labeled graph version of the Gross-Tucker Theorem.

Theorem 2.5.15. *Let $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action. Let $\eta^0, \eta^{\mathcal{A}}$ be sections for $q^0, q^{\mathcal{A}}$ respectively. There are functions $c_\eta, d_\eta : (E/G)^1 \to G$ such that $((E,\mathcal{L}),G,\alpha)$ is isomorphic to $((E/G \times_{c_\eta} G, (\mathcal{L}/G)_{d_\eta}), G, \tau)$.*

Proof. By Theorem 2.5.12 there is a function $c_\eta : (E/G)^1 \to G$ and a graph isomorphism $\phi = (\phi^0_{c_\eta}, \phi^1_{c_\eta}) : E/G \times_{c_\eta} G \to E$ which is equivariant. Recall from the proof of Lemma 2.5.11 there is a section η^1 for q^1 such that for $Ge \in (E/G)^1$ we have $\eta^1(Ge) = f$ where $f \in E^1$ satisfies $q^1(f) = Ge$. We begin by fixing a section $\eta^{\mathcal{A}} : \mathcal{A}/G \to \mathcal{A}$ for $q^{\mathcal{A}}$. We now define the map $d_\eta : (E/G)^1 \to G$. Since

$$\begin{aligned} q^{\mathcal{A}}\eta^{\mathcal{A}}(\mathcal{L}/G(Ge)) = q^{\mathcal{A}}\eta^{\mathcal{A}}(\mathcal{L}/G(Gf)) &= q^{\mathcal{A}}\eta^{\mathcal{A}}(G\mathcal{L}(f)) \\ &= G\mathcal{L}(f) = q^{\mathcal{A}}\mathcal{L}(f) = q^{\mathcal{A}}\mathcal{L}\eta^1(Ge) \end{aligned}$$

and $((E,\mathcal{L}),G,\alpha)$ is free, there is a unique $k \in G$ such that $\alpha^{\mathcal{A}}_k \eta^{\mathcal{A}}((\mathcal{L}/G)(Ge)) = \mathcal{L}(\eta^1(Ge))$ and we may define $d_\eta(Ge) = k$. In other words, for each $Ge \in (E/G)^1$, $d_\eta(Ge)$ is the unique element of G satisfying

$$\alpha^{\mathcal{A}}_{d_\eta(Ge)} \eta^{\mathcal{A}}((\mathcal{L}/G)(Ge)) = \mathcal{L}(\eta^1(Ge)). \tag{2.21}$$

Now we shall construct a map $\phi^{\mathcal{A}/G\times G}_{d_\eta} : \mathcal{A}/G \times G \to \mathcal{A}$. For each $(Ga, g) \in \mathcal{A}/G \times G$ we define $\phi^{\mathcal{A}/G\times G}_{d_\eta}(Ga, g) = \alpha^{\mathcal{A}}_g \eta^{\mathcal{A}}(Ga)$. So it remains to show that the map $\phi_{d_\eta}^{\mathcal{A}/G\times G} : \mathcal{A}/G \times G \to \mathcal{A}$ satisfies Definition 2.3.4 (2), is a bijection and is equivariant. To check $\phi_{d_\eta}^{\mathcal{A}/G\times G} \circ (\mathcal{L}/G)_{d_\eta} = \mathcal{L} \circ \phi_{c_\eta}^1$ we note that by Equation (2.21)

$$\begin{aligned}
\phi_{d_\eta}^{\mathcal{A}/G\times G} \circ (\mathcal{L}/G)_d(Ge,h) &= \phi_{d_\eta}^{\mathcal{A}/G\times G}((\mathcal{L}/G)(Ge), hd_\eta(Ge)) \\
&= \alpha_h^{\mathcal{A}} \alpha_{d_\eta(Ge)}^{\mathcal{A}} \eta_{\mathcal{A}}(\mathcal{L}/G(Ge)) \\
&= \alpha_h^{\mathcal{A}} \mathcal{L}(\eta^1(Ge)) = \mathcal{L}(\alpha_h^1 \eta^1(Ge)) = \mathcal{L} \circ \phi_{c_\eta}^1(Ge,h)
\end{aligned}$$

for all $(Ge,h) \in (E/G \times_{c_\eta} G)^1$.

To see that $\phi_{d_\eta}^{\mathcal{A}/G\times G}$ is injective, suppose that $(Ga_1,h),(Ga_2,k) \in \mathcal{A}/G \times G$ satisfy $\phi_{d_\eta}^{\mathcal{A}/G\times G}(Ga_1,h) = \phi_{d_\eta}^{\mathcal{A}/G\times G}(Ga_2,k)$. Then $\alpha_h^{\mathcal{A}}\eta^{\mathcal{A}}(Ga_1) = \alpha_k^{\mathcal{A}}\eta^{\mathcal{A}}(Ga_2)$, so we have

$$Ga_2 = q^{\mathcal{A}}(\eta^{\mathcal{A}}(Ga_2)) = q^{\mathcal{A}}(\alpha_{k^{-1}}^{\mathcal{A}}\alpha_h^{\mathcal{A}}\eta^{\mathcal{A}}(Ga_1)) = q^{\mathcal{A}}(\eta^{\mathcal{A}}(Ga_1)) = Ga_1.$$

Now $\alpha_{k^{-1}}^{\mathcal{A}}\alpha_h^{\mathcal{A}}\eta^{\mathcal{A}}(Ga_1) = \eta^{\mathcal{A}}(Ga_1)$, but since $((E,\mathcal{L}),G,\alpha)$ is free we have $k^{-1}h = 1_G$, so $h = k$. Therefore $(Ga_1,h) = (Ga_2,k)$ which shows $\phi_{d_\eta}^{\mathcal{A}/G\times G}$ is injective.

We show that $\phi_{d_\eta}^{\mathcal{A}/G\times G}$ is surjective. Let $a \in \mathcal{A}$. Then there exists $e \in E^1$ such that $\mathcal{L}(e) = a$, so $Ge \in (E/G)^1$ satisfies $\phi_{c_\eta}^1(Ge,h) = e$. Since $(\mathcal{L}/G)(Ge) = Ga$ and by Equation (2.21) we have

$$\begin{aligned}
\phi_{d_\eta}^{\mathcal{A}/G\times G}(Ga, hd_\eta(Ge)) = \phi_{d_\eta}^{\mathcal{A}/G\times G}((\mathcal{L}/G)(Ge), hd_\eta(Ge)) &= \alpha_h^{\mathcal{A}}\alpha_{d_\eta(Ge)}^{\mathcal{A}}\eta^{\mathcal{A}}((\mathcal{L}/G)(Ge)) \\
&= \mathcal{L}(\alpha_h^1\eta^1(Ge)) = \mathcal{L}(e) = a.
\end{aligned}$$

Therefore $\phi_{d_\eta}^{\mathcal{A}/G\times G}$ is surjective.

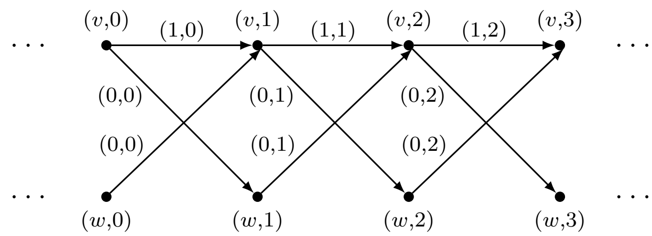
To see that $\phi_{d_\eta}^{\mathcal{A}/G\times G}$ is equivariant notice that for all $(Ge,h) \in (E/G \times G)^1$ and $g \in G$ we have

$$\phi_{d_\eta}^{\mathcal{A}/G\times G}(\tau_g^{\mathcal{A}/G\times G}(Ge,h)) = \phi_{d_\eta}^{\mathcal{A}/G\times G}(Ge,gh) = \alpha_g^{\mathcal{A}}\alpha_h^{\mathcal{A}}\eta^{\mathcal{A}}(Ge) = \alpha_g^{\mathcal{A}}\phi_{d_\eta}^{\mathcal{A}/G\times G}(Ge,h).$$

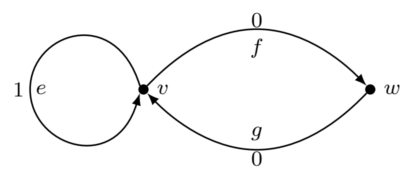
Thus $\phi_{c_\eta,d_\eta} = (\phi_{c_\eta}^0, \phi_{c_\eta}^1, \phi_{d_\eta}^{\mathcal{A}/G\times G})$ is an isomorphism of labeled graph actions as required. □

Remark 2.5.16. It is not possible to prove the Gross-Tucker theorem for labeled graphs (Theorem 2.5.15) without using the function $d_\eta : (E/G)^1 \to G$ in the definition of the skew product labeled graph $(E/G \times_{c_\eta} G, \mathcal{L}_{d_\eta})$. The function d_η is there to accommodate the case where there are two edges in E/G with the same label. If Ge and Gf have the same label Ga, then d_η describes the possibly different translations of $\mathcal{L}(\eta^1(Ge))$ and $\mathcal{L}(\eta^1(Gf))$ by $\alpha^{\mathcal{A}}$ to $\eta^{\mathcal{A}}(Ga)$ (see Equation (2.21)).

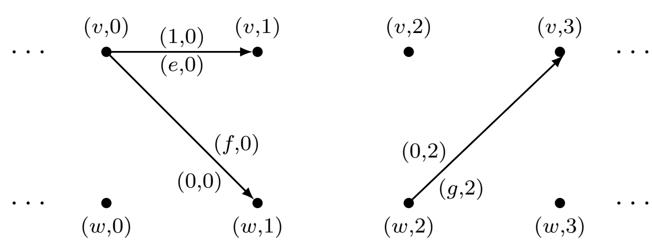
We illustrate the problems mentioned in the previous paragraph with the following concrete example. Consider the following labeled graph $(E, \mathcal{L})$.



By Lemma 2.4.16 and Proposition 2.5.6 there is a free labeled graph action $((E, \mathcal{L}), \mathbf{Z}, \alpha)$ with quotient $(E/\mathbf{Z}, \mathcal{L}/\mathbf{Z})$ as shown below.



Let $\eta^0(v) = (v, 0)$ and $\eta^0(w) = (w, 2)$, then the section η^1 as defined in Lemma 2.5.11 is given by $\eta^1(e) = (e, 0)$, $\eta^1(f) = (f, 0)$, and $\eta^1(g) = (g, 2)$ as shown below.



Note that $c_\eta(e) = 1$, $c_\eta(f) = -1$, and $c_\eta(g) = 3$.

Observe that $f, g \in (E/\mathbf{Z})^1$ have the same label, however $\mathcal{L}(\eta^1(f)) = (0, 0) \neq (0, 2) = \mathcal{L}(\eta^1(g))$. If we define $\eta^{\mathcal{A}}$ using f, then we have $\eta^{\mathcal{A}}(0) = (0, 0)$. If a skew product labeled graph is defined without using the function d, then the labeling map on $E/\mathbf{Z} \times_{c_\eta} \mathbf{Z}$ is given by $\mathcal{L}'(x,n) = (\mathcal{L}/\mathbf{Z}(x), n)$ for all $(x,n) \in (E/\mathbf{Z} \times \mathbf{Z})^1$. The map $\phi_{c_\eta} = (\phi^0_{c_\eta}, \phi^1_{c_\eta}, \phi^{\mathcal{A}/\mathbf{Z}\times\mathbf{Z}})$ fails to be a labeled graph isomorphism because $\phi^{\mathcal{A}/\mathbf{Z}\times\mathbf{Z}} \circ \mathcal{L}' = \mathcal{L} \circ \phi^1_{c_\eta}$ does not hold. For instance,

$$\phi^{\mathcal{A}/\mathbf{Z}\times\mathbf{Z}} \circ \mathcal{L}'(g,0) = \phi^{\mathcal{A}/\mathbf{Z}\times\mathbf{Z}}(\mathcal{L}/\mathbf{Z}(g),0) = \phi^{\mathcal{A}/\mathbf{Z}\times\mathbf{Z}}(0,0) = \eta^{\mathcal{A}}(0) = (0,0)$$

whereas

$$\mathcal{L} \circ \phi^1_{c_\eta}(g,0) = \mathcal{L}(\eta^1(g)) = \mathcal{L}(g,2) = (0,2).$$

The problem we have just described above came about because $\mathcal{L}(\eta^1(f)) = (0,0)$ and $\mathcal{L}(\eta^1(g)) = (0,2)$. The function d_η accounts for this difference: According to Equation (2.21) $d_\eta(g) = 2$, since $\alpha^{\mathcal{A}}_2(0,0) = (0,2)$, whereas $d_\eta(f) = 0$. Hence $d_\eta(g) \neq d_\eta(f)$ even though $\mathcal{L}(g) = \mathcal{L}(f)$.

We briefly pause to consider the effect of making a different choice for the section $q^{\mathcal{A}}$.

Definition 2.5.17. Let $c_1, c_2 : E^1 \to G$ satisfy $c_1 \sim_b c_2$. Functions $d_1, d_2 : E^1 \to G$ are *related* (and we write $d_1 \clubsuit d_2$) if there is a function $t : \mathcal{A} \to G$ such that $b(s(e))d_2(e) = d_1(e)t(\mathcal{L}(e))$ holds for all $e \in E^1$.

Corollary 2.5.18. *Consider the function d_η in the proof of Theorem 2.5.15. If we choose a different section $\kappa^{\mathcal{A}}$ for $q^{\mathcal{A}}$ then the corresponding function $d_\kappa : (E/G)^1 \to G$ is such that $d_\eta \clubsuit d_\kappa$.*

Proof. Suppose $\eta^{\mathcal{A}}, \kappa^{\mathcal{A}} : \mathcal{A}/G \to \mathcal{A}$ are sections for $q^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}/G$, that is $q^{\mathcal{A}}(\eta^{\mathcal{A}}(a)) = a = q^{\mathcal{A}}(\kappa^{\mathcal{A}}(a))$ for all $a \in \mathcal{A}$. Then, as in the proof of Theorem 2.5.12, there are sections $\eta^1, \kappa^1 : (E/G)^1 \to E^1$ and functions $d_\eta, d_\kappa : (E/G)^1 \to G$ characterized by

$$\alpha^{\mathcal{A}}_{d_\eta(Ge)} \circ \eta^{\mathcal{A}}(\mathcal{L}/G(Ge)) = \mathcal{L}(\eta^1(Ge)) \tag{2.22}$$

$$\alpha^{\mathcal{A}}_{d_\kappa(Ge)} \circ \kappa^{\mathcal{A}}(\mathcal{L}/G(Ge)) = \mathcal{L}(\kappa^1(Ge)) \tag{2.23}$$

for all $Ge \in (E/G)^1$.

We now define the map $t : \mathcal{A}/G \to G$ which implements $d_\eta \clubsuit d_\kappa$. Since $q^{\mathcal{A}}(\kappa^{\mathcal{A}}(Ga)) = q^{\mathcal{A}}(\eta^{\mathcal{A}}(Ga))$ and $\alpha^{\mathcal{A}}$ is a free action of G on $\mathcal{A}$ there is a unique

$h \in G$ such that $\eta^{\mathcal{A}}(Ga) = \alpha_h^{\mathcal{A}} \kappa^{\mathcal{A}}(Ga)$ and we set $t(a) = h$. In other words, for each $Ga \in (\mathcal{A}/G)$, $t(a)$ is the unique element of G satisfying

$$\eta^{\mathcal{A}}(Ga) = \alpha_{t(Ga)}^{\mathcal{A}} \kappa^{\mathcal{A}}(Ga). \tag{2.24}$$

Fix $Ge \in E^1/G$ and set $a = \mathcal{L}/G(Ge)$ in Equation (2.24) to obtain $\eta^{\mathcal{A}}(\mathcal{L}/G(Ge)) = \alpha_{t(\mathcal{L}/G(Ge))}^{\mathcal{A}} \kappa^{\mathcal{A}}(\mathcal{L}/G(Ge))$. Now substitute this into Equation (2.22) to obtain

$$\mathcal{L}(\eta^1(Ge)) = \alpha_{d_\eta(Ge)}^{\mathcal{A}} \alpha_{t(\mathcal{L}/G(Ge))}^{\mathcal{A}} \kappa^{\mathcal{A}}(\mathcal{L}/G(Ge)). \tag{2.25}$$

Recall Equation (2.18) which is $\eta^1(Ge) = \alpha_{b(s(Ge))}^1 \kappa^1(Ge)$ and notice that

$$\mathcal{L}(\eta^1(Ge)) = \mathcal{L}(\alpha_{b(s(Ge))}^1 \kappa^1(Ge)) = \alpha_{b(s(Ge))}^{\mathcal{A}} \mathcal{L}(\kappa^1(Ge)).$$

Now apply Equation (2.23) to obtain

$$\mathcal{L}(\eta^1(Ge)) = \alpha_{b(s(Ge))}^{\mathcal{A}} \alpha_{d_\kappa(Ge)}^{\mathcal{A}} \kappa^{\mathcal{A}}(\mathcal{L}/G(Ge)). \tag{2.26}$$

Comparing Equations (2.25) and (2.26) we have

$$\alpha_{d_\eta(Ge)}^{\mathcal{A}} \alpha_{t(\mathcal{L}/G(Ge))}^{\mathcal{A}} \kappa^{\mathcal{A}}(\mathcal{L}/G(Ge)) = \alpha_{b(s(Ge))}^{\mathcal{A}} \alpha_{d_\kappa(Ge)}^{\mathcal{A}} \kappa^{\mathcal{A}}(\mathcal{L}/G(Ge)).$$

Since the action of $\alpha^{\mathcal{A}}$ on $\mathcal{A}$ is free we have $d_\eta(Ge)t(\mathcal{L}/G(Ge)) = b(s(Ge))d_\kappa(Ge)$ as required. $\square$

2.6 Coactions on Labeled Graph Algebras

For graph algebras, a function $c : E^1 \to G$ induces a coaction δ on $C^*(E)$. One should expect, therefore, that the functions $c, d : E^1 \to G$ would induce a coaction on $C^*(E \times_c G, \mathcal{L}_d)$. However, it is not as straightforward as one might think. For further information about coactions of discrete groups see [26].

Remark 2.6.1. If $(E, \mathcal{L})$ is left-resolving, then $(E \times_c G, \mathcal{L}_d)$ is also left-resolving since for $(v, g) \in E^0 \times G$

$$r^{-1}(v, g) = \{(e, gc(e)^{-1}) : r(e) = v\}.$$

Hence $\mathcal{L}_d$ restricted to $r^{-1}(v, g)$ is injective as

$$\mathcal{L}_d(r^{-1}(v, g)) = \{(\mathcal{L}(e), gc(e)^{-1}d(e)) : r(e) = v\}.$$

Definition 2.6.2. Let $(E, \mathcal{L})$ be a labeled graph. A function $c : E^1 \to G$ is *label consistent* if whenever $e, f \in E^1$ satisfy $\mathcal{L}(e) = \mathcal{L}(f)$ we have $c(e) = c(f)$.

Notice that if a function $c : E^1 \to G$ is label consistent, it induces a well-defined function $C : \mathcal{A} \to G$ such that $C(a) = c(e)$ where $\mathcal{L}(e) = a$. The function $C : E^1 \to \mathcal{A}$ is called the map *induced* by c. For $\beta \in \mathcal{L}^n(E)$ let $C(\beta) = C(\beta_1) \cdots C(\beta_n)$ then $C : \mathcal{A} \to G$ extends to a function $C : \mathcal{A}^* \to G$. For $\beta \in \mathcal{L}^n(E)$ we denote $\beta' = \beta_1 \cdots \beta_{n-1}$ so that $\beta = \beta' \beta_n$.

Recall from Remark 2.4.3 that we write paths in $(E \times_c G)^*$ as pairs (μ, g) where $\mu \in E^*$, $g \in G$, $s(\mu, g) = (s(\mu), g)$, and $r(\mu, g) = (r(\mu), gc(\mu))$.

The following Proposition is a partial converse of Proposition 2.4.10.

Proposition 2.6.3. *Let $(E, \mathcal{L})$ be a labeled graph and $\phi_c : E \times_c G \to E$ be the surjective graph homomorphism from Lemma 2.4.9. In addition, let $\mu, \nu \in E^*$ and $c, d : E^1 \to G$ be label consistent functions. If $\mathcal{L}^*(\mu) = \mathcal{L}^*(\nu)$, then $\mathcal{L}_d^*(\mu, g) = \mathcal{L}_d^*(\nu, g)$ for all $g \in G$.*

Proof. Notice that

$$\mathcal{L}_d^*(\mu, g) = \mathcal{L}_d(\mu_1, g)\mathcal{L}_d(\mu_2, gc(\mu_1)) \cdots \mathcal{L}_d(\mu_n, gc(\mu'))$$

which equals

$$(\mathcal{L}(\mu_1), gd(\mu_1))(\mathcal{L}(\mu_2), gc(\mu_1)d(\mu_2)) \cdots (\mathcal{L}(\mu_n), gc(\mu')d(\mu_n)).$$

Similarly

$$\mathcal{L}_d^*(\nu, g) = (\mathcal{L}(\nu_1), gd(\nu_1))(\mathcal{L}(\nu_2), gc(\nu_1)d(\nu_2)) \cdots (\mathcal{L}(\nu_n), gc(\nu')d(\nu_n)).$$

Since $\mathcal{L}^*(\mu) = \mathcal{L}^*(\nu)$ we have $\mathcal{L}(\mu_i) = \mathcal{L}(\nu_i)$ for all $i = 1, \cdots, n$. As c is label consistent, this implies $c(\mu_i) = c(\nu_i)$, and so $d(\mu_i) = d(\nu_i)$ for all $i = 1, \cdots, n$. Hence $\mathcal{L}_d^*(\mu, g) = \mathcal{L}_d^*(\nu, g)$ as required. $\square$

Recall from Remark 2.4.3 that may identify $(E \times_c G)^*$ with $E^* \times G$. We shall now show that if $c, d : E^1 \to G$ are label consistent then we may identify $\mathcal{L}_d^*(E \times_c G)$ with $\mathcal{L}^*(E) \times G$.

Definition 2.6.4. Let $c, d : E^1 \to G$ be label consistent functions with corresponding induced maps $C, D : E^1 \to \mathcal{A}$. For $\beta \in \mathcal{L}^n(E)$ and $g \in G$ we define

$$(\beta, g) = (\beta_1, gD(\beta_1))(\beta_2, gC(\beta_1)D(\beta_2)) \cdots (\beta_n, gC(\beta')D(\beta_n)). \tag{2.27}$$

Remark 2.6.5. Let $\mu \in E^*$ be such that $\mathcal{L}^*(\mu) = \beta$, then since d is label consistent we have

$$\mathcal{L}_d^*(\mu, g) = (\mathcal{L}(\mu_1), gd(\mu_1))(\mathcal{L}(\mu_2), gc(\mu_1)d(\mu_2)) \cdots (\mathcal{L}(\mu_n), gc(\mu')d(\mu_n))$$

$$\mathcal{L}_d^*(\mu, g) = (\beta_1, gD(\beta_1))(\beta_2, gC(\beta_1)D(\beta_2)) \cdots (\beta_n, gC(\beta')D(\beta_n))$$

$$\mathcal{L}_d^*(\mu, g) = (\mathcal{L}^*(\mu), g) = (\beta, g). \tag{2.28}$$

Hence for $\beta \in \mathcal{L}^n(E)$ and $g \in G$ we have $(\beta, g) \in \mathcal{L}_d^*(E \times_c G)$. Equation (2.27) gives an identification between $\mathcal{L}^*(E) \times G$ and $\mathcal{L}_d^*(E \times_c G)$.

The following Lemma indicates the behavior of the range and relative range maps under the identification in Equation (2.27).

Lemma 2.6.6. *Let $c, d : E^1 \to G$ be label consistent functions. Let $a \in \mathcal{A}$, $\beta \in \mathcal{L}^*(E)$, and $g \in G$. Then under the identification of $\mathcal{L}^*(E) \times G$ with $\mathcal{L}_d^*(E \times_c G)$ we have*

1. *For $(\beta, g) \in \mathcal{L}^*(E) \times G$ we have $r(\beta, g) = (r(\beta), gC(\beta)) \in \mathcal{E}(r, \mathcal{L}) \times G$.*

2. *For $(r(\beta), g) \in \mathcal{E}(r, \mathcal{L}) \times G$ and $(a, h) \in \mathcal{A} \times G$ we have $r((r(\beta), g), (a, h)) = \emptyset$ if $g \neq h$ and $r((r(\beta), g), (a, g)) = (r(\beta a), gC(a))$ otherwise.*

3. *$\mathcal{E}(r, \mathcal{L}_d) = \mathcal{E}(r, \mathcal{L}) \times \mathcal{F}(G)$, where $\mathcal{F}(G)$ denotes the collection of finite subsets of G.*

Proof. To see (1) observe that for $(\beta, g) \in \mathcal{L}^*(E) \times G$ we have

$$r(\beta, g) = r((\beta_1, gD(\beta_1))(\beta_2, gC(\beta_1)D(\beta_2)) \cdots (\beta_n, gC(\beta')D(\beta_n))) \quad \text{by (2.27)}$$

$$r(\beta, g) = \{r((\mu_1, g)(\mu_2, gc(\mu_1)) \cdots (\mu_n, gc(\mu'))) : \mathcal{L}^*(\mu) = \beta\} \quad \text{by (2.28)}$$

$$r(\beta, g) = \{(r(\mu), gc(\mu)) : \mathcal{L}^*(\mu) = \beta\} \quad \text{by (2.7)}$$

$$r(\beta, g) = \{(r(\mu), gC(\beta)) : \mathcal{L}^*(\mu) = \beta\} \quad \text{since } c \text{ is label consistent.} \tag{2.29}$$

Hence we may identify $r(\beta, g)$ with $(r(\beta), gC(\beta))$.

For part (2) in order to compute $r((r(\beta), g)(a, g))$ we must identity the set of vertices in $(r(\beta), g)$ and source of the edges with label (a, h). Using (2.29) we have $(r(\beta), gC(\beta)^{-1}) = \{(r(\mu), g) : \mathcal{L}^*(\mu) = \beta\}$. Observe that (e, h) with $\mathcal{L}(e) = a$ is such that $\mathcal{L}_d(e, h) = (a, h)$ by (2.28). Now $s(e, h) = (s(e), h)$ by (2.7) and so $s(e, h) \in (r(\beta), g)$ if and only if $s(e) \in r(\beta)$ and $g = h$. Hence

$$\begin{aligned} r((r(\beta), g)(a, g)) &= \{r(e, g) : s(e) \in r(\beta), \mathcal{L}(e) = a\} \\ &= \{(r(e), gC(a)) : s(e) \in r(\beta), \mathcal{L}(e) = a\} \end{aligned}$$

by (2.7) and since c is label consistent

$$\begin{aligned} &= \{(r(\mu e), gC(a) : \mathcal{L}(e) = a, \mathcal{L}^*(\mu) = \beta\} \\ &= (r(\beta a), gC(a)) \end{aligned}$$

as required.

To see (3) recall that $\mathcal{E}(r, \mathcal{L}_d)$ is the is the smallest accommodating subset of $2^{E^0 \times G}$ containing $r(\beta, g)$. Notice that (1) and (2) tell us that the sets generating $\mathcal{E}(r, \mathcal{L}_d)$ are of the form $(r(\beta), g))$ for some $\beta \in \mathcal{L}^*(E), g \in G$. Since $\mathcal{E}(r, \mathcal{L}) \times \mathcal{F}(G)$ is the smallest accommodating subset of $2^{E^0 \times G}$ containing $(r(\beta), g))$ the result follows. $\square$

The following lemma is an analog of [13, Lemma 3.2].

Lemma 2.6.7. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph, let G be a discrete group, and let $c : E^1 \to G$ be a label consistent function. Then there is a coaction $\delta : C^*(E, \mathcal{L}) \to C^*(E, \mathcal{L}) \otimes C^*(G)$ such that*

$$\delta(s_a) = s_a \otimes u_{C(a)} \qquad \text{and} \qquad \delta(p_{r(\beta)}) = p_{r(\beta)} \otimes u_{1_G} \tag{2.30}$$

where $\{s_a, p_{r(\beta}\}$ is a universal Cuntz-Krieger $(E, \mathcal{L})$-family and $\{u_g : g \in G\}$ are the canonical generators of $C^*(G)$.

Proof. Since s_a and $p_{r(\beta)}$ are nondegenerate, $s_a \otimes u_{C(a)}$ and $p_{r(\beta)} \otimes u_{1_G}$ are nondegenerate. We claim that $\{s_a \otimes u_{C(a)}, p_{r(\beta)} \otimes u_{1_G}\}$ form a Cuntz-Krieger $(E, \mathcal{L})$-family. To verify (CK1a) of Definition 2.2.11 observe that for $\beta, \omega \in \mathcal{L}^*(E)$

$$(p_{r(\beta)} \otimes u_{1_G})(p_{r(\omega)} \otimes u_{1_G}) = p_{r(\beta)} p_{r(\omega)} \otimes u_{1_G}$$

it follows that $(p_{r(\beta)} \otimes u_{1_G})(p_{r(\omega)} \otimes u_{1_G}) = 0$ if and only if $r(\beta) \cap r(\omega) = \emptyset$. To verify

(CK2) of Definition 2.2.11 observe that for $\beta \in \mathcal{L}^*(E)$ and $a \in \mathcal{A}$ we have $(p_{r(\beta)} \otimes u_{1_G})(s_a \otimes u_{C(a)}) = p_{r(\beta)} s_a \otimes u_{C(a)} = s_a p_{r(\beta a)} \otimes u_{C(a)} = (s_a \otimes u_{C(a)})(p_{r(\beta a)} \otimes u_{1_G})$. The proof of (CK1b), (CK3) and (CK4) from Definition 2.2.11 follow from similar calculations. Thus the claim has been shown. So the universal property gives a nondegenerate homomorphism $\pi_{S,P} : C^*(E,\mathcal{L}) \to C^*(E,\mathcal{L}) \otimes C^*(G)$ which satisfies (2.30). Let $\delta = \pi_{S,P}$, let γ be the canonical gauge action on $C^*(E,\mathcal{L})$ defined in Equation (2.5), and define a strongly continuous action of $\mathbf{T}$ on $C^*(E,\mathcal{L}) \otimes C^*(G)$ by $\gamma \otimes 1$.[1] We claim that $\delta \circ \gamma_z = (\gamma_z \otimes \mathrm{id}) \circ \delta$ for all $z \in \mathbf{T}$. Notice that for $s_a \in C^*(E,\mathcal{L})$ we have

$$\begin{aligned}
\delta(\gamma_z(s_a)) &= \delta(z^{|a|} s_a) \\
&= z^{|a|} s_a \otimes u_{C(a)} \\
&= (\gamma_z \otimes \mathrm{id})(s_a \otimes u_{C(a)}) \\
&= (\gamma_z \otimes \mathrm{id})(\delta(s_a)).
\end{aligned}$$

The proof for $p_{r(\beta)}$ is similar, thus the claim has been shown. If $r(\beta) \subsetneq r(\omega)$ then as the family $\{s_a, p_{r(\beta)}\}$ is a universal Cuntz-Krieger $(E,\mathcal{L})$-family we have $p_{r(\beta)} \neq p_{r(\omega)}$ and so $P_{r(\beta)} \neq P_{r(\omega)}$ and so by Theorem 2.2.18 the map δ is injective. So by Theorem 2.2.18 δ is injective. Lastly, we must show that the coaction identity $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$ holds on generators s_a and $p_{r(\beta)}$. Recall that $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ is the canonical coaction given by $u_g \mapsto u_g \otimes u_g$. Then for s_a we have

$$\begin{aligned}
(\delta \otimes \mathrm{id}) \circ \delta(s_a) &= (\delta \otimes \mathrm{id})(s_a \otimes u_{C(a)}) = (s_a \otimes u_{C(a)}) \otimes u_{C(a)} \\
&= s_a \otimes (u_{C(a)} \otimes u_{C(a)}) = (\mathrm{id} \otimes \delta_G)(s_a \otimes u_{C(a)}) \\
&= (\mathrm{id} \otimes \delta_G) \circ \delta(s_a).
\end{aligned}$$

[1] For all $s_a \otimes u_g \in C^*(E,\mathcal{L}) \otimes C^*(G)$ the map $z \mapsto (z s_a, u_g)$ is continuous.

Similarly, for $p_{r(\beta)}$ we have

$$\begin{aligned}(\delta\otimes\mathrm{id})\circ\delta(p_{r(\beta)}) &= (\delta\otimes\mathrm{id})(p_{r(\beta)}\otimes u_{1_G}) = (p_{r(\beta)}\otimes u_{1_G})\otimes u_{1_G}\\ &= p_{r(\beta)}\otimes(u_{1_G}\otimes u_{1_G}) = (\mathrm{id}\otimes\delta_G)(p_{r(\beta)}\otimes u_{1_G})\\ &= (\mathrm{id}\otimes\delta_G)\circ\delta(p_{r(\beta)}).\end{aligned}$$

Since the coaction identity holds on generators, it will extend by algebra and continuity to all of $C^*(E,\mathcal{L})$ and completes the proof. □

By Theorem 2.4.18 the left labeled graph translaction action $((E\times_c G,\mathcal{L}_d),G,\tau)$ defined in Definition 2.4.17 induces an action $\tau: G\to \mathrm{Aut}\, C^*(E\times_c G,\mathcal{L}_d)$. When we identify $\mathcal{L}_d^*(E\times_c G)$ with $\mathcal{L}^*(E)\times G$ this action may be described on the generators of $C^*(E\times_c G,\mathcal{L}_d)$ as follows: For $h,g\in G$, $a\in\mathcal{A}$, and $\beta\in\mathcal{L}^*(E)$ we have

$$\tau_h(s_{a,g}) = s_{a,hg} \text{ and } \tau_h(p_{r(\beta),g}) = p_{r(\beta),hg}. \tag{2.31}$$

Theorem 2.6.8. *Let $(E,\mathcal{L})$ be a left-resolving labeled graph, let G be a discrete group, let $c,d: E^1\to G$ be label consistent functions, and let δ be the coaction from Lemma 2.6.7. Then*

$$C^*(E,\mathcal{L})\times_\delta G\cong C^*(E\times_c G,\mathcal{L}_d)$$

equivariantly for the dual action $\widehat{\delta}$ of G on $C^*(E,\mathcal{L})\times_\delta G$ and the action τ of G on $C^*(E\times_c G,\mathcal{L}_d)$ described in (2.31).

Proof. For each $g\in G$, let $C^*(E,\mathcal{L})_g=\{b\in C^*(E,\mathcal{L}) : \delta(b)=b\otimes u_g\}$ denote the corresponding spectral subspace; we write b_g to denote a generic element of $C^*(E,\mathcal{L})_g$. [2] Then $C^*(E,\mathcal{L})\times_\delta G$ is densely spanned by the set $\{(b_g,h) : b_g\in C^*(E,\mathcal{L})_g \text{ and } g,h\in G\}$, and the algebraic operations are given on this set by

$$(b_g,x)(b_h,y)=(b_gb_h,y) \quad \text{if } y=h^{-1}x \text{ (and 0 if not), and} \quad (b_g,x)^*=(b_g^*,gx).$$

If $(j_{C^*(E,\mathcal{L})},j_G)$ is the canonical covariant homomorphism of $C^*(E,\mathcal{L})$ into $M(C^*(E,\mathcal{L})\times_\delta G)$ then (b_g,x) is by definition $(j_{C^*(E,\mathcal{L})}(b_g)j_G(\chi_{\{x\}}))$

[2]This subscript convention conflicts with the standard notation for Cuntz-Krieger families: each partial isometry s_a is in $C^*(E,\mathcal{L})_{C(a)}$, and each projection $p_{r(\beta)}$ is in $C^*(E,\mathcal{L})_{1_G}$. We hope this does not cause confusion.

We aim to define a Cuntz-Krieger $(E \times_c G, \mathcal{L})$-family in $C^*(E, \mathcal{L}) \times_\delta G$ by identifying $\mathcal{L}_d^*(E \times_c G)$ with $\mathcal{L}^*(E) \times G$ and letting

$$t_{a,g} = (s_a, C(a)^{-1}g^{-1}) \qquad \text{and} \qquad q_{r(\beta),g} = (p_{r(\beta)}, g^{-1}).$$

for $a, g \in \mathcal{A} \times G$, $\beta \in \mathcal{L}^*(E)$ and $g \in G$.

Conditions (CK1a) and (CK1b) of Definition 2.2.11 follow directly from the definitions. To verify (CK2) of Definition 2.2.11 notice that by Lemma 2.6.6(2) we have

$$\begin{aligned} q_{r(\beta),g}t_{a,g} &= (p_{r(\beta)}, g^{-1})(s_a, C(a)^{-1}g^{-1}) \\ &= (p_{r(\beta)}s_a, C(a)^{-1}g^{-1}) \\ &= (s_a p_{r(\beta a)}, C(a)^{-1}g^{-1}) \\ &= (s_a, C(a)^{-1}g^{-1})(p_{r(\beta a)}, C(a)^{-1}g^{-1}) \\ &= t_{a,g}q_{r(\beta a),gC(a)} \\ &= t_{a,g}q_{r((r(\beta),g)(a,g))}. \end{aligned}$$

Note that by Lemma 2.6.6(1) we have

$$\begin{aligned} t_{a,g}^*t_{a,g} &= (s_a, C(a)^{-1}g^{-1})^*(s_a, C(a)^{-1}g^{-1}) \\ &= (s_a^*, C(a)C(a)^{-1}g^{-1})(s_a, C(a)^{-1}g^{-1}) \\ &= (s_a^*, g^{-1})(s_a, C(a)^{-1}g^{-1}) \\ &= (s_a s_a^*, C(a)^{-1}g^{-1}) \\ &= (p_{r(a)}, (gC(a))^{-1}) \\ &= q_{r(a),gC(a)} \\ &= q_{r(a,g)}. \end{aligned}$$

Therefore (CK3) of Definition 2.2.11 is satisfied.

If $r(\beta)$ contains no sinks then $r(\beta), g = r(\beta, gC(\beta)^{-1})$ contains no sinks and we have

$$
\begin{aligned}
q_{r(\beta),g} = (p_{r(\beta)}, g^{-1}) = (\sum_{a\in\mathcal{A}} s_a p_{r(\beta a)} s_a^*, g^{-1}) \\
&= \sum_{a\in\mathcal{A}} (s_a p_{r(\beta a)} s_a^*, g^{-1}) \\
&= \sum_{a\in\mathcal{A}} (s_a p_{r(\beta a)} s_a^*, C(a)C(a)^{-1}g^{-1}) \\
&= \sum_{a\in\mathcal{A}} (s_a, C(a)^{-1}g^{-1})(p_{r(\beta a)}, C(a)^{-1}g^{-1})(s_a^*, C(a)C(a)^{-1}g^{-1}) \\
&= \sum_{a\in\mathcal{A}} (s_a, C(a)^{-1}g^{-1})(p_{r(\beta a)}, C(a)^{-1}g^{-1})(s_a, C(a)^{-1}g^{-1})^* \\
&= \sum_{a\in\mathcal{A}} t_{a,g} q_{r(\beta a),gC(a)} t_{a,g}^* \\
&= \sum_{a,h\in\mathcal{A}\times G} t_{a,h} q_{r((\beta,g),(a,h))} t_{a,h}^* \text{ by Lemma 2.6.6 (2).}
\end{aligned}
$$

This shows that $\{t_{a,g}, q_{r(\beta),g}\}$ is a Cuntz-Krieger $(E \times_c G, \mathcal{L})$-family.

We use the universal property of the graph algebra to get a homomorphism $\pi_{t,q} : C^*(E \times_c G, \mathcal{L}_d) \to C^*(E, \mathcal{L}) \times_\delta G$ such that $\pi(s_{a,g}) = t_{a,g}$ and $\pi(p_{r(\beta),g}) = q_{r(\beta),g}$ and we shall prove that it is injective using Theorem 2.2.18.

From the proof of Lemma 2.6.7 the gauge automorphisms γ_z defined in Equation (2.5) commute with the coaction δ in the sense that $\delta(\gamma_z(y)) = \gamma_z \otimes \mathrm{id}(\delta(y))$ for $y \in C^*(E, \mathcal{L})$, and hence by the universal property of the crossed product induce automorphisms $\gamma_z \times_\delta G$ of $C^*(E, \mathcal{L}) \times_\delta G$. Thus there is a strongly[3] continuous action $\gamma \times_\delta G$ of $\mathbf{T}$ on $C^*(E, \mathcal{L}) \times_\delta G$ such that

$$(\gamma_z \times_\delta G)(b_g, x) = (\gamma_z(b_g), x).$$

[3] For all $(s_a, x) \in C^*(E, \mathcal{L}) \times_\delta G$ the map $z \mapsto (zs_a, x)$ is continuous.

Since

$$\begin{aligned}(\pi_{t,q})(\gamma_z(s_{a,g})) &= \pi_{t,q}(z(s_{a,g}))\\ &= (zs_{a,g}, C(a)^{-1}g^{-1})\\ &= (\gamma_z \times_\delta G)(s_{a,g}, C(a)^{-1}g^{-1})\\ &= (\gamma_z \times_\delta G)(\pi_{t,q}(s_{a,g})),\end{aligned}$$

and

$$(\pi_{t,q})(\gamma_z(p_{r(\beta),g}) = (p_{r(\beta)}, g^{-1}) = (\gamma_z \times_\delta G)(\pi_{t,q}(p_{r(\beta),g})),$$

it is straightforward to check that $\pi_{t,q} \circ \gamma_z = (\gamma_z \times_\delta G) \circ \pi_{t,q}$ for all $z \in \mathbf{T}$.

If $(r(\beta), g) \subsetneq (r(\omega), h)$, then as the family $\{s_{a,g}, p_{r(\beta),g}\}$ is a universal Cuntz-Krieger $(E\times_c, \mathcal{L}_d)$-family we have $p_{r(\beta),g} \neq p_{r(\omega),h}$ and so $q_{r(\beta),g} \neq q_{r(\omega),h}$ and so by Theorem 2.2.18 it follows that $\pi_{t,q}$ is injective.

It remains to show that $\pi_{t,q}$ is surjective. Observe that $C^*(E, \mathcal{L}) \times_\delta G$ is generated by (s_a, g) and $(p_{r(\beta)}, h)$. Since

$$\pi_{t,q}(s_{a,g^{-1}C(a)^{-1}}) = t_{a,g^{-1}C(a)^{-1}} = (s_a, C(a)^{-1}C(a)g)$$

and $\pi_{t,q}(p_{r(\beta),h^{-1}}) = (p_{r(\beta)}, h)$, we see that $\pi_{t,q}$ is surjective. Hence $\pi_{t,q} : C^*(E \times_c G, \mathcal{L}_d) \to C^*(E, \mathcal{L}) \times_\delta G$ is the desired isomorphism.

The C^*-algebra $C^*(E \times_c G, \mathcal{L}_d)$ carries an free labeled graph action $((E \times_c G, \mathcal{L}_d), G, \tau)$. It also carries an action of $\mathbf{T}$, namely the gauge action γ_z. The C^*-algebra $C^*(E, \mathcal{L}) \times_\delta G$ carries an action $\widehat{\delta}$ of G via the formula $\widehat{\delta}_h(b_g, x) = (b_g, xh^{-1})$. It also carries an action of $\mathbf{T}$, namely $\gamma_z \times_\delta G$ defined above.

We need to check that $\pi_{t,q}$ is equivariant for the G actions, we claim that $\pi_{t,q} \circ \tau_g = \widehat{\delta}_g \circ \pi_{t,q}$ for all $g \in G$. Notice that for all $s_{a,h} \in C^*(E \times_c G, \mathcal{L}_d)$
$\pi_{t,q} \circ \tau_g(s_{a,h}) = \pi_{t,q}(s_{a,gh}) = (s_a, C(a)^{-1}h^{-1}g^{-1}) = \widehat{\delta}_g(s_a, C(a)^{-1}h^{-1}) = \widehat{\delta}_g \circ \pi_{t,q}(s_{a,h})$
and similarly $\pi_{t,q} \circ \tau_g(p_{r(\beta)}, h)) = \widehat{\delta}_g \circ \pi_{t,q}(p_{r(\beta)}, h))$ for $p_{r(\beta),h} \in C^*(E \times_c G, \mathcal{L}_d)$. Since we have checked $\pi_{t,q} \circ \tau_g = \widehat{\delta}_g \circ \pi_{t,q}$ on the generators, the claim holds.

We claim that $\pi_{t,q}$ is equivariant for the $\mathbf{T}$ actions, that is $\pi_{t,q} \circ \gamma_z = (\gamma_z \times$

$G) \circ \pi_{t,q}$ for all $z \in \mathbf{T}$. Notice that for all $s_{a,h} \in C^*(E \times_c G, \mathcal{L}_d)$

$$\pi_{t,q} \circ \gamma_z(s_{a,h}) = \pi_{t,q}(zs_{a,h}) = (zs_a, C(a)^{-1}h^{-1}) = (\gamma_z \times G)(s_a, C(a)^{-1}h^{-1}) = (\gamma_z \times_\delta G) \circ \pi_{t,q}(s_{a,h})$$

and similarly $\pi_{t,q} \circ \gamma_z(p_{r(\beta}, h)) = (\gamma_z \times G) \circ \pi_{t,q}(p_{r(\beta}, h))$ for $p_{r(\beta),h} \in C^*(E \times_c G, \mathcal{L}_d)$. Since we have checked $\pi_{t,q} \circ \gamma_z = (\gamma_z \times G) \circ \pi_{t,q}$ on the generators the claim holds. □

Theorem 2.6.9. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph, let G be a discrete group, let $c, d : E^1 \to G$ be label consistent functions, and let τ the action of G on $C^*(E \times_c G, \mathcal{L}_d)$ from* (2.31). *Then*

$$C^*(E \times_c G, \mathcal{L}_d) \times_{\tau,r} G \cong C^*(E, \mathcal{L}) \otimes \mathcal{K}(\ell^2(G)).$$

Proof. Since the isomorphism of $C^*(E \times_c G, \mathcal{L}_d)$ with $C^*(E, \mathcal{L}) \times_\delta G$ is equivariant for the G-actions $\tau, \widehat{\delta}$, respectively, it follows that

$$C^*(E \times_c G, \mathcal{L}_d) \times_{\tau,r} G \cong C^*(E, \mathcal{L}) \times_\delta G \times_{\widehat{\delta},r} G.$$

Following the argument in [13, Corollary 2.5], Katayama's duality theorem [14] give us that $C^*(E, \mathcal{L}) \times_\delta G \times_{\widehat{\delta},r} G$ is isomorphic to $C^*(E, \mathcal{L}) \otimes \mathcal{K}(\ell^2(G))$, as required. □

2.7 Additional Remarks

Free actions

If $((E, \mathcal{L}), \alpha, G)$ is a free labeled graph action then the Gross-Tucker Theorem for labeled graphs, Theorem 2.5.15 says that there are functions $c_\eta, d_\eta : (E/G)^1 \to G$ such that $((E/G \times_{c_\eta} G, (\mathcal{L}/G)_{d_\eta}), \tau, G) \cong ((E, \mathcal{L}), \alpha, G)$. We would like use Theorem 2.6.8 and Corollary 2.6.9 to conclude that $C^*(E, \mathcal{L}) \times_{\alpha,r} G$ is isomorphic to $C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G))$. However, as yet we do not know when the functions $c_\eta, d_\eta : (E/G)^1 \to G$ are label consistent. We know this is not true in general: In the example given in Remark 2.5.16 we have $(\mathcal{L}/G)(f) = 0 = (\mathcal{L}/G)(g)$, however $c_\eta(f) = -1 \neq 3 = c_\eta(g)$. We must therefore examine conditions on the free labeled graph action $((E, \mathcal{L}), G, \alpha)$ which ensure that the functions c_η, d_η from Theorem 2.5.15 are label consistent.

Definition 2.7.1. Let (E, G, α) be a free graph action. The set $T \subseteq E^0$ is a *fundamental domain for* (E, G, α) if for every $v \in E^0$ there exists $g \in G$ and a unique $w \in T$ such that $v = \alpha_g^0 w$.

A free graph action (E, α, G) always has a fundamental domain (see [6]) as the image of every cross section η^0 for q^0 describes a fundamental domain.

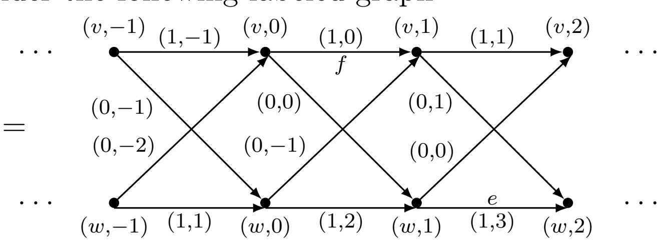
Definition 2.7.2. Let $((E, \mathcal{L}), G, \alpha)$ be a free labeled graph action. A *fundamental domain for* $((E, \mathcal{L}), G, \alpha)$ is a fundamental domain for (E, G, α) such that for every $e, f \in E^1$ we have

1. if $r(e), r(f) \in T$, and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$ and

2. if $s(e), s(f) \in T$, and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$.

The following example illustrates that there exists labeled graphs that do not have fundamental domains.

Example 2.7.3. Consider the following labeled graph

$(E, \mathcal{L}) :=$ 

The group $\mathbf{Z}$ acts freely on $(E, \mathcal{L})$ by addition in the second coordinate of the vertices, edges and labels as indicated in the picture above; call this action α. Let $T = \{(v, 0), (w, 1)\}$, then T is a fundamental domain for the restricted action $(E, \alpha, \mathbf{Z})$. However when considering the labeled graph action $((E, \mathcal{L}), \alpha, \mathbf{Z})$ the set T does not satisfy Definition 2.7.2 (2). Consider the edges e, f as shown above with $\mathcal{L}(e) = (1, 3)$ and $\mathcal{L}(f) = (1, 0)$ respectively. We have $s(e) = (w, 1) \in T$ and $s(f) = (v, 0) \in T$ and $\mathbf{Z}\mathcal{L}(e) = \mathbf{Z}\mathcal{L}(f) = \{(1, n) : n \in \mathbf{Z}\}$, however $\mathcal{L}(e) = (1, 3) \neq (1, 0) = \mathcal{L}(f)$. Indeed any fundamental domain for the restricted action $(E, \alpha, \mathbf{Z})$ will have a similar flaw.

Proposition 2.7.4. *Let* $c, d : E^1 \to G$ *be label consistent functions, let* $(E \times_c G, \mathcal{L}_d)$ *be a skew product labeled graph, and let* $((E \times_c G, \mathcal{L}_d), G, \tau)$ *be the associated left* *labeled graph translation action. Then* $T = \{(v, 1_G) : v \in E^0\}$ *is a fundamental domain for* $((E \times_c G, \mathcal{L}_d), G, \tau)$.

Proof. Let $T = \{(v, 1_G); v \in E^0\}$ and notice that for $(w, g) \in (E \times_c G)^0$ we see that $(w, 1_G)$ is the unique element in T such that $\tau_g^0(w, 1_G) = (w, g)$, so T is a fundamental domain for $(E \times_c G, G, \tau)$. Let $(e, g_1), (f, g_2) \in (E \times_c G)^1$ be such that $r(e, g_1), r(f, g_2) \in T$ and $G\mathcal{L}_d(e, g_1) = G\mathcal{L}_d(f, g_2)$. Notice that $r(e, g_1) = (r(e), c(e)^{-1})$, so we have $g_1 = c(e)^{-1}$ and $g_2 = c(f)^{-1}$. Therefore

$$G\mathcal{L}_d(e, g_1) = G\mathcal{L}_d(f, g_2)$$

$$G\mathcal{L}_d(e, c(e)^{-1}) = G\mathcal{L}_d(f, c(f)^{-1})$$

$$\{(\mathcal{L}(e), h_1 c(e)^{-1} d(e)) : h_1 \in G\} = \{(\mathcal{L}(f), h_2 c(f)^{-1} d(f)) : h_2 \in G\}$$

and so we have $\mathcal{L}(e) = \mathcal{L}(f)$. Hence since c and d are label consistent we have

$$\mathcal{L}_d(e, c(e)^{-1}) = (\mathcal{L}(e), c(e)^{-1} d(e)) = (\mathcal{L}(f), c(f)^{-1} d(f)) = \mathcal{L}_d(f, c(f)^{-1}).$$

The argument for condition (2) of Definition 2.7.2 follows in a similar and more straightforward manner. Therefore the result follows. □

Remark 2.7.5. Let T be a fundamental domain for the free group-action (E, G, α). Then for every $Gv \in (E/G)^0$ there exists a unique $w \in T$ such that $Gw = Gv$. Hence if we define $\eta_T^0(Gv) = w$, then $\eta_T^0 : (E/G)^0 \to T$ is a section for q^0 since

$$q^0 \circ \eta^0(Gv) = q^0(w) = Gw = Gv.$$

Proposition 2.7.6. *Let* $((E, \mathcal{L}), G, \alpha)$ *be a free labeled graph action which admits a fundamental domain. Then there exists label consistent functions* $c, d : (E/G)^1 \to G$ *such that* $((E, \mathcal{L}), G, \alpha) \cong ((E/G \times_c G, (\mathcal{L}/G)_d), G, \tau)$.

Proof. Let T be a fundamental domain for $((E, \mathcal{L}), G, \alpha)$, and let η_T^0 be a section for q^0 as described in Remark 2.7.5. Then define η_T^1 and c_{η_T} as in Theorem 2.5.12, and $\eta_T^{\mathcal{A}}$ and d_{η_T} as in Theorem 2.5.15. We want to show that c_{η_T} and d_{η_T} are label consistent. Suppose $Ge, Gf \in (E/G)^1$ such that $(\mathcal{L}/G)(Ge) = (\mathcal{L}/G)(Gf) = Ga \in \mathcal{A}/G$, say. Let $b = \eta_T^{\mathcal{A}}(Ga) \in \mathcal{A}$, $d_{\eta_T}(Ge) = k \in G$, and $d_{\eta_T}(Gf) = l \in G$. Then by the definition of d_{η_T} we have

$$\mathcal{L}(\eta^1_T(Ge)) = \alpha^{\mathcal{A}}_k \eta^{\mathcal{A}}_T(\mathcal{L}/G)(Ge) = \alpha^{\mathcal{A}}_k b \tag{2.32}$$

$$\mathcal{L}(\eta^1_T(Gf)) = \alpha^{\mathcal{A}}_l \eta^{\mathcal{A}}_T(\mathcal{L}/G)(Gf) = \alpha^{\mathcal{A}}_l b. \tag{2.33}$$

This implies that $G\mathcal{L}(\eta^1_T(Ge)) = Ga = G\mathcal{L}(\eta^1_T(Gf))$ and so $\mathcal{L}(\eta^1_T(Ge)) = \mathcal{L}(\eta^1_T(Gf))$ since $s(\eta^1_T(Ge)), s(\eta^1_T(Gf)) \in T$. From Equations (2.32) and (2.33) we have $\alpha^{\mathcal{A}}_k b = \alpha^{\mathcal{A}}_l b$ and so $k = l$ since the G action on $\mathcal{A}$ is free. Therefore d_{η_T} is label consistent.

Suppose $Ge, Gf \in (E/G)^1$ are such that $(\mathcal{L}/G)(Ge) = (\mathcal{L}/G)(Gf) = Ga \in \mathcal{A}/G$, say. Let $b = \eta^{\mathcal{A}}_T(Ga) \in \mathcal{A}$, $c_{\eta_T}(Ge) = k \in G$, and $c_{\eta_T}(Gf) = l \in G$. Then by the definition of c_{η_T} we have

$$r(\eta^1_T(Ge)) = \alpha^0_k \eta^0_T(r(Ge)) \tag{2.34}$$

$$r(\eta^1_T(Gf)) = \alpha^0_l \eta^0_T(r(Gf)). \tag{2.35}$$

Then if we let $e = \alpha^1_{-k}(\eta^1_T(Ge))$ and $f = \alpha^1_{-l}(\eta^1_T(Gf))$ we have $e, f \in E^1$ with $r(e) = \eta^0_T(r(Ge)), r(f) = \eta^0_T(r(Gf)) \in T$ and $G\mathcal{L}(e) = G\mathcal{L}(f)$. Since T is a fundamental domain $\mathcal{L}(e) = \mathcal{L}(f)$ and hence

$$\alpha^{\mathcal{A}}_{-k}(\mathcal{L}(\eta^1_T(Ge))) = \mathcal{L}(e) = \mathcal{L}(f) = \alpha^{\mathcal{A}}_{-l}(\mathcal{L}(\eta^1_T(Gf))).$$

Since $\mathcal{L}(\eta^1_T(Ge)) = \mathcal{L}(\eta^1_T(Gf))$ by the above argument (which shows d_{η_T} is label consistent) we can conclude that $k = l$. Therefore c_{η_T} is label consistent and our result is established. $\square$

Taken together, Proposition 2.7.4 and Proposition 2.7.6 demonstrate the relationship between fundamental domains and label consistent functions.

Corollary 2.7.7. *Let $(E, \mathcal{L})$ be a left-resolving labeled graph and let $((E, \mathcal{L}), G, \alpha)$ a free labeled graph action which admits a fundamental domain. Then*

$$C^*(E, \mathcal{L}) \times_{\alpha,r} G \cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G)).$$

Proof. By Proposition 2.7.6 there are label consistent functions $c, d : E^1/G \to G$

such that

$$((E,\mathcal{L}),G,\alpha) \cong ((E/G \times_c G,(\mathcal{L}/G)_d),G,\tau).$$

Hence

$$C^*(E,\mathcal{L}) \times_{\alpha,r} G \cong C^*(E/G \times_c G,(\mathcal{L}/G)_d) \times_{\tau,r} G.$$

But

$$C^*(E/G \times_c G,(\mathcal{L}/G)_d) \times_{\tau,r} G \cong C^*(E/G,\mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G))$$

by Corollary 2.6.9 and the result follows. □

Dual actions

Now we consider the case when we have a left resolving labeled graph $(E,\mathcal{L})$ and built a skew-product labeled graph $(E \times_c G, \mathcal{L}_d)$ using functions $c,d :\to G$, where G is abelian. As in [16, Corollary 2.5] we may use Pontryagin duality to describe an action of $\widehat{G}$ on $C^*(E,\mathcal{L})$ such that the resulting crossed product is isomorphic to $C^*(E \times_c G, \mathcal{L}_d)$.

The method for proving our next result is developed from [28, Lemma 3.1] for graph algebras, with $G = \mathbf{Z}$ and $c(e) = 1$ for all $e \in E$. Before we prove this theorem observe that we can apply the integrated form of the embedding $i_{\widehat{G}} : \widehat{G} \to M(C^*(E,\mathcal{L}) \times_\alpha \widehat{G})$ to the functions $\chi \mapsto \langle \chi, g\rangle$ in $L^1(\widehat{G})$ to get a family of mutually orthogonal projections $\{X_g : g \in G\}$ in $M(C^*(E,\mathcal{L}) \times_\alpha \widehat{G})$. Specifically

$$X_g = \int \langle \chi, g\rangle i_{\widehat{G}}(\chi) d\lambda(\chi) \tag{2.36}$$

where λ is Haar measure on $\widehat{G}$.

Theorem 2.7.8. *Let $(E,\mathcal{L})$ be a left-resolving labeled graph over the alphabet $\mathcal{A}$ and $c,d : E^1 \to G$ be label consistent functions, where G is abelian. Let $\{s_a, p_{r(\beta)}\}$ be the canonical generating family of $C^*(E,\mathcal{L})$.*

1. *There is an action α of $\widehat{G}$ on $C^*(E,\mathcal{L})$ determined by*

   $$\alpha_\chi(s_a) = \langle \chi, C(a)\rangle s_a \quad \alpha(p_{r(\beta)}) = p_{r(\beta)}$$

   where $a \in \mathcal{A}$, $\beta \in \mathcal{L}^*(E)$ and $\chi \in \widehat{G}$.

2. *There is an isomorphism from $C^*((E\times_c G,\mathcal{L}_d))$ to $C^*(E,\mathcal{L})\times_\alpha \widehat{G}$ which carries the action τ of G on $C^*((E\times_c G,\mathcal{L}_d))$ to the dual action $\widehat{\alpha}$ of G on $C^*(E,\mathcal{L})\times_\alpha \widehat{G}$.*

Proof. For each $\chi\in\widehat{G}$, $a\in\mathcal{A}$ and $\beta\in\mathcal{L}^*(E)$ define $S_a=\langle\chi,C(a)\rangle s_a$ and $P_{r(\beta)}=p_{r(\beta)}$ then one checks that the collection $\{S_a,P_{r(\beta)}\}$ forms a Cuntz-Krieger $(E,\mathcal{L})$-family in $C^*(E,\mathcal{L})$. Hence by the universal property of $C^*(E,\mathcal{L})$ there is a homomorphism $\pi_{S,P}:C^*(E,\mathcal{L})\to C^*(E,\mathcal{L})$ such that $\pi_{S,P}(s_a)=S_a$ and $\pi_{S,P}(p_{r(\beta)})=P_{r(\beta)}$. Similarly, for $\chi\in\widehat{G}$, $a\in\mathcal{A}$ and $\beta\in\mathcal{L}^*(E)$ define $T_a=\overline{\langle\chi,C(a)\rangle}s_a$ and $Q_{r(\beta)}=p_{r(\beta)}$ then one checks that the collection $\{T_a,Q_{r(\beta)}\}$ forms a Cuntz-Krieger $(E,\mathcal{L})$-family in $C^*(E,\mathcal{L})$. Hence by the universal property of $C^*(E,\mathcal{L})$ there is a homomorphism $\pi_{T,Q}:C^*(E,\mathcal{L})\to C^*(E,\mathcal{L})$ such that $\pi_{S,P}(s_a)=T_a$ and $\pi_{S,P}(p_{r(\beta)})=Q_{r(\beta)}$. Since $\pi_{S,P}\circ\pi_{T,Q}$ is the identity map, it follows that $\alpha_\chi=\pi_{S,P}$ is an automorphism of $C^*(E,\mathcal{L})$. The map $\chi\mapsto\alpha_\chi$ is a homomorphism since $\chi\mapsto\langle\chi,C(a)\rangle$ is a homomorphism for all $a\in\mathcal{A}$, this completes the proof of (1).

For $a\in\mathcal{A}$ and $\beta\in\mathcal{L}^*(E)$ let $t_a=i_{C^*(E,\mathcal{L})}(s_a)$ and $q_{r(\beta)}=i_{C^*(E,\mathcal{L})}(p_{r(\beta)})$. Using the covariance relation for $i_{C^*(E,\mathcal{L})}$ and $i_{\widehat{G}}$ in $C^*(E,\mathcal{L})\times_\alpha\widehat{G}$ and the definition of X_G given in (2.36), one checks that

$$X_g t_a=t_a X_{gC(a)} \text{ and } X_g q_{r(\beta)}=q_{r(\beta)}X_g \tag{2.37}$$

for all $g\in G$, $a\in\mathcal{A}$ and $r(\beta)\in\mathcal{L}^*(E)$.

For $(a,h)\in\mathcal{A}\times G$ let $T_{(a,h)}=t_aX_h$ and $Q_{r(\beta),g}=q_{r(\beta),g}X_g$ for $(\beta,g)\in\mathcal{E}(r,\mathcal{L})\times G$. Using (2.37) and the identification of $\mathcal{L}_d^*(E\times_c G)$ with $\mathcal{L}(E)\times G$ given in (2.27) one checks that $\{T_{(a,h)},Q_{r(\beta),g}\}$ forms a Cuntz-Krieger $(E\times_c G,\mathcal{L}_d)$-family. Hence by the universal property of $C^*((E\times_c G,\mathcal{L}_d))$ there is a map $\pi_{T,Q}:C^*((E\times_c G,\mathcal{L}_d))\to C^*(E,\mathcal{L})\times_\alpha\widehat{G}$ given by $\pi_{T,Q}(t_{(a,h)})=T_{(a,h)}$ and $\pi_{T,Q}(q_{r(\beta),g})=Q_{r(\beta),g}$. Let $\beta:\mathbf{T}\to\operatorname{Aut}(C^*(E,\mathcal{L})\times_\alpha\widehat{G})$ be the strongly continuous $\mathbf{T}$-action on $C^*(E,\mathcal{L})\times_\alpha\widehat{G}$ induced by the usual gauge action on $C^*(E,\mathcal{L})$ so that $\beta_z(T_a))=zT_a$ and $\beta_z(Q_{r(\beta)})=Q_{r(\beta)}$ and $\beta_z(i_{\widehat{G}}(\chi))=i_{\widehat{G}}(\chi)$. One checks that $\beta_z\circ\pi_{T,Q}=\pi_{T,Q}\circ\gamma_z$

where γ_z is the canonical gauge action on $C^*((E \times_c G, \mathcal{L}_d))$ and $z \in \mathbf{T}$. Suppose that $(\beta, g), (\omega, h) \in \mathcal{L}^*(E) \times G$ are such that $r(\beta, g) \subsetneq r(\omega, h)$. Since this implies that $r(\beta) \subsetneq r(\omega)$ and so $p_{r(\beta)} \neq p_{r(\omega)}$ it follows that $Q_{r(\beta),g} \neq Q_{r(\omega),h}$. Hence by the gauge invariant uniqueness theorem 2.2.18, we may deduce that $\pi_{T,Q}$ is injective.

An application of the Stone-Weierstrass Theorem shows that the functions $\chi \mapsto \langle \chi, g \rangle$ span a dense $|\cdot|_1$-subspace of $C(\widehat{G})$. It then follows that the elements

$$\{t_\alpha q_A t_\beta^* X_g : \alpha, \beta \in \mathcal{L}^*(E^*), A \in \mathcal{E}(r, \mathcal{L}) \times G, g \in G\}$$

span a dense subspace of $C^*(E, \mathcal{L}) \times_\alpha \widehat{G}$. Hence, by continuity, $\pi_{T,Q}$ is surjective.

The dual action $\widehat{\alpha}$ of G on $C^*(E, \mathcal{L}) \times_\alpha \widehat{G}$ is characterized by $\widehat{\alpha}_g(i_{\widehat{G}}(\chi)) = \langle \chi, g \rangle i_{\widehat{G}}(\chi)$. Hence $\widehat{\alpha}_g(X_h) = X_{gh}$. Since $\widehat{\alpha}$ fixes $\{t_a, q_{r(\beta)}\}$ the last assertion follows in a straightforward manner. $\square$

CHAPTER 3
IMPLICATIONS OF *-COMMUTATIVITY IN C^*-ALGEBRAS

3.1 Introduction to *-commutativity

As indicated in Chapter 1, our goal is to associate a C^*-algebra to two commuting local homeomorphisms. For this purpose, it suffices to build a product system over $\mathbb{N}^2$. This is always possible. That is, with two local homeomorphisms, one can always build a product system over $\mathbb{N}^2$. However without additional conditions the product system and its C^*-algebra may be difficult to handle. Our first objective, then, is to show that if the local homeomorphisms *-commute, then the product system is *compactly aligned*. This somewhat technical condition is very important for the theory (see [9, 10]). It seems to be precisely the condition that makes the theory run smoothly. Then we set out to explore the notion of *-commutativity. In section 3.3 we identify the continuous maps that *-commute with the shift and we study the local homeomorphisms that *-commute with the shift. In section 3.4 we identify when the two shifts determined by a 2-graph *-commute. In a sense, it is evident that there is a lot more to do, but these two sections show that *-commutativity is a rich notion that arises sufficiently often to warrant further investigation.

There are two equivalent definitions of "*-commutativity." We begin by recalling them and proving their equivalence.

Definition 3.1.1. (Arzumanian, Renault [1, Definition 5.6]) Let X be a set. Two functions $S, T : X \to X$ **-commute* if the map $f : x \mapsto (T(x), S(x))$ is a bijection from X onto the set $X_S *_T X := \{(y, z) \in X \times X : S(y) = T(z)\}$.

Definition 3.1.2. (Exel, Renault [8, Definition 10.1]) Let X be a set. Two functions $S, T : X \to X$ **-commute* if they commute under composition and given $(y, z) \in X \times X$ such that $S(y) = T(z)$ there exists a unique $x \in X$ such that $T(x) = y$ and $S(x) = z$.

$$
\begin{tikzcd}
& x \arrow[dl, dashed, "T"'] \arrow[dr, dashed, "S"] & \\
y \arrow[dr, "S"'] & & z \arrow[dl, "T"] \\
& S(y)=T(z) &
\end{tikzcd}
$$

Lemma 3.1.3. *Let $S, T : X \to X$ be functions. Then S and T *-commute in the sense of Exel and Renault if and only if S and T *-commute in the sense of Arzumanian and Renault. In particular, if S and T *-commute in the sense of Arzumanian and Renault, then S and T commute as maps of X under composition.*

Proof. Suppose S and T Exel-Renault *-commute. Let $y, z \in X$ be given such that $(y, z) \in X_S *_T X$, then $S(y) = T(z)$. We will show that there exists a unique $x \in X$ such that $f(x) = (y, z)$. Since $S(y) = T(z)$, by Exel-Renault *-commutativity there is a unique $x \in X$ such that $T(x) = y$ and $S(x) = z$. That is $f(x) = (T(x), S(x)) = (y, z)$, so f is surjective. Since this x is unique, f is injective. Therefore S and T Arzumanian-Renault *-commute.

Conversely, suppose S and T Arzumanian-Renault *-commute. Let $x \in X$. Then $f(x) = (T(x), S(x))$ and since the pair $(T(x), S(x)) \in X_S *_T X$ this means $S(T(x)) = T(S(x))$. Thus S and T commute. Now let $(y, z) \in X \times X$ be given such that $S(y) = T(z)$. Then $(y, z) \in X_S *_T X$ and since f is surjective there exists $x \in X$ such that $f(x) = (T(x), S(x)) = (y, z)$. Since f is injective this x is unique. Therefore S and T Exel Renault *-commute. $\square$

Remark 3.1.4. We shall use Exel and Renault's definition of *-commute.

3.2 Exel systems which *-commute

This section is based on joint work with Iain Raeburn.

3.2.1 Introduction

In operator theory, we say that two operators on Hilbert space α and β *-commute if they commute and $\alpha\beta^* = \beta^*\alpha$. This notion will be related to our work, as we shall see shortly. We consider local homeomorphisms $\phi, \psi : X \to X$, where X is a compact Hausdorff space, and use these functions to generate an action of $\mathbb{N}^2$ by endomorphisms of $C(X)$. This leads us to look for a crossed product $C(X) \rtimes \mathbb{N}^2$. The Exel crossed product seems the most suitable because the endomorphisms are unital. We use the version due to Larsen in [18] which is based on product systems of Hilbert bimodules introduced by Fowler in [10].

We build a product system of Hilbert bimodules (correspondences) over $\mathbb{N}^2$ using transfer operators. We then characterize the dynamical systems which arise from *-commuting functions as those Exel systems for which the transfer operators commute with the endormorphism it is not a transfer operator for. Since the transfer operators K, L for endormorphisms μ, ν, respectively, are almost left inverses (i.e. $K \circ \mu = id$), the relation $K \circ \nu = \nu \circ K$ looks a lot like $\alpha\beta^* = \beta^*\alpha$.

In section 3.2.2 we construct Exel and Exel-Larsen systems (which are a type of dynamical system). In section 3.2.3 we build Hilbert bimodules (correspondences) over $\mathbb{N}^2$ from Exel-Larsen systems. In section 3.2.4 we create a product system of Hilbert bimodules and show that the Toeplitz representations of the product system are completely determined by the Toeplitz-covariant representations of the related Exel systems. In section 3.2.5 we examine the additional properties that occur when the endormorphisms used to construct the Exel systems *-commute.

3.2.2 Exel systems and Exel-Larsen systems

In order to construct an Exel or Exel-Larsen system we must first define a transfer operator for an endomorphism of a C^*-algebra. We will only consider unital C^*-algebras.

Definition 3.2.1. Let A be a unital C^*-algebra and $\mu \in \mathrm{End}(A)$. Then a *transfer operator for* μ is a positive continuous linear function $K : A \to A$ such that $K(\mu(a)b) = aKb$ for all $a, b \in A$.

Throughout this section let X be a compact Hausdorff space and suppose $\phi, \psi : X \to X$ are surjective local homeomorphisms which commute with each other. Define $\widetilde{\phi}, \widetilde{\psi} : C(X) \to C(X)$ by $\widetilde{\phi}(f) = f \circ \phi$ and $\widetilde{\psi}(f) = f \circ \psi$ for all $f \in C(X)$. Then $\widetilde{\phi}$ and $\widetilde{\psi}$ are commuting unital endomorphisms of $C(X)$ that are injective, since ϕ and ψ are surjective.

Lemma 3.2.2. *For every surjective local homeomorphism $\phi : X \to X$, the equation $K(f)(x) := \sum_{\phi(t)=x} f(t)$, where $f \in C(X)$ and $x \in X$, defines a function $K : C(X) \to C(X)$ which is a transfer operator for $\widetilde{\phi}$.*

Proof. We see that K is a function from $C(X)$ to $C(X)$ by [15, Lemma 1.5]. By definition, $K(C(X))_+ \subseteq C(X)_+$. To see that K is linear, let $f, g \in C(X)$, $x \in X$, and $z \in \mathbb{C}$ denote the function from X to $\mathbb{C}$ such that $y \mapsto z$ for all $y \in X$. Then we have

$$\begin{aligned} K(zf+g)(x) &= \sum_{\phi(t)=x} zf + g(t) \\ &= z \sum_{\phi(t)=x} f(t) + g(t) \\ &= z \sum_{\phi(t)=x} f(t) + \sum_{\phi(t)=x} g(t) \\ &= zK(f)(x) + K(g)(x). \end{aligned}$$

K is continuous since it is a positive linear map on a unital C^*-algebra. To see that

K is a transfer operator for $\widetilde{\phi}$ notice that for $f, g \in C(X)$ and $x \in X$ we have

$$\begin{aligned} f \cdot K(g)(x) &= f(x) \sum_{\phi(t)=x} g(t) \\ &= \sum_{\phi(t)=x} f(\phi(t))g(t) \\ &= \sum_{\phi(t)=x} \widetilde{\phi}(f)(t)g(t) \\ &= K(\widetilde{\phi}(f) \cdot g)(x). \end{aligned}$$

□

Define $K(f)(x) := \sum_{\phi(t)=x} f(t)$ and $L(f)(x) := \sum_{\psi(t)=x} f(t)$, then K and L are transfer operators for $\widetilde{\phi}$ and $\widetilde{\psi}$ respectively. Note that K and L commute with each other since ϕ and ψ commute.

Remark 3.2.3. We defined a transfer operator K for the endomorphism $\widetilde{\phi}$ as $K(f)(x) := \sum_{\phi(t)=x} f(t)$ for all $f \in C(X)$ and $x \in X$, and we defined the transfer operator L for $\widetilde{\psi}$ similarly. Those who are familiar with transfer operators may wonder why we did not use the normalized transfer operator. That is, $K(f)(x) = \frac{1}{\#\{t:\phi(t)=x\}} \sum_{\phi(t)=x} f(t)$. The reason we did not use the normalized transfer operator is because with normalization KL need not equal LK. In order to construct the product system over the abelian semigroup $\mathbb{N}^2$, K and L must commute.

Remark 3.2.4. Let A be a C^*-algebra. Notice that if $T, T' : A \to A$ are transfer operators for $\mu, \nu \in \mathrm{End}(A)$, respectively, then $T \circ T'$ is a positive, continuous, linear function such that for all $a, b \in A$ we have

$$a \cdot (T \circ T')(b) = T(\mu(a) \cdot T'(b)) = (T \circ T')(\nu \circ \mu(a) \cdot b).$$

Thus $T \circ T'$ is a transfer operator for $\nu \circ \mu \in \mathrm{End}(A)$.

Define $\gamma : \mathbb{N}^2 \to \mathrm{End}(C(X))$ by $\gamma_{m,n} = \widetilde{\phi}^m \widetilde{\psi}^n$. Note that since ϕ and ψ commute, $\widetilde{\phi}$ and $\widetilde{\psi}$ commute. Now define H from $\mathbb{N}^2$ to the positive linear maps on $\mathrm{End}(C(X))$ by $H_{m,n}(f)(y) = K^m L^n(f)(y) = \sum_{\phi^m \psi^n(t)=y} f(t)$. By Remark 3.2.4 $H_{m,n}$ is a transfer operator for $\gamma_{m,n}$.

Definition 3.2.5. An *Exel system* is a triple (A, μ, K) where A is a C^*-algebra,

$\mu \in \mathrm{End}(A)$ and $K : A \to A$ is a transfer operator for μ.

We have shown that $(C(X), \widetilde{\phi}^m, K^m)$, $(C(X), \widetilde{\psi}^n, L^n)$ and $(C(X), \gamma_{s,t}, H_{s,t})$ are Exel systems for all $n, m \in \mathbb{N}$ and $(s,t) \in \mathbb{N}^2$.

Definition 3.2.6. An *Exel-Larsen system* is a quadruple (A, S, μ, K) where A is a C^*-algebra, S is an abelian semigroup, $\mu : S \to \mathrm{End}(A)$ is an action and K is a map from S to the positive linear maps on A such that K_s is a transfer operator for μ_s for all $s \in S$.

Clearly Exel-Larsen systems are a family of Exel systems "organized" by a semigroup. Throughout we will have to pass back and forth between and among individual Exel systems, viewed as parts of an Exel-Larsen system, and the entire Exel-Larsen system.

Proposition 3.2.7. *Let A be a C^*-algebra and $R, T : A \to A$ be endomorphisms that commute, that is $RT = TR$. For $(m,n) \in \mathbb{N}^2$ define $\eta_{m,n} = R^m T^n$. Then $\eta : \mathbb{N}^2 \to \mathrm{End}(A)$ is an action, i.e. a semigroup homomorphism.*

Our discussion to this point clearly proves Proposition 3.2.7. If γ and H are as just defined, then $(C(X), \mathbb{N}^2, \gamma, H)$ is an Exel-Larsen system.

3.2.3 Building Hilbert bimodules from Exel systems

We will review the process of building Hilbert bimodules from Exel systems. To see a generalization of this process see [18, Section 2.2].

Assume that (A, S, μ, K) is an Exel-Larsen system with A unital. Then (A, μ_m, K_m) is an Exel system for each $m \in S$. We endow $A_{K_m} := A\mu_m(1)$ with a right A-module structure given by

$$g \cdot f := g\mu_m(f) \text{ for } f \in A \text{ and } g \in A_{K_m}.$$

Also define an A-valued (possibly degenerate) inner-product $\langle \cdot, \cdot \rangle_{K_m}$ by

$$\langle g, h \rangle_{K_m} := K_m(g^* h), \text{ for all } g, h \in A_{K_m}.$$

Upon modding out vectors of norm zero and completing we get a right Hilbert A-module which we denote M_{K_m}. We denote the quotient map by $q_{K_m} : A_{K_m} \to M_{K_m}$ and note that $q_{K_m}(A_{K_m})$ is dense in M_{K_m}.

For all $f \in A$ and $g \in A_{K_m}$ we have

$$\left\|\langle fg, fg\rangle_{K_m}\right\| = \|K_m(g^* f^* fg)\| \leq \|f\|^2 \|K_m(g^* g)\| = \|f\|^2 \left\|\langle g, g\rangle_{K_m}\right\|$$

(by linearity and positivity of K_m) so left multiplication by f on A_{K_m} extends to a bounded operator $\theta_{K_m}(f) : M_{K_m} \to M_{K_m}$, which is adjointable by a computation using

$$\langle h, g\rangle = K_m(h^* g)$$

for $h, g \in A$. Thus we obtain a *-homomorphism $\theta_{K_m} : A \to \mathcal{L}(M_{K_m})$. Therefore M_{K_m} is a Hilbert bimodule over A with actions

$$q_{K_m}(g) \cdot f = q_{K_m}(g\mu_m(f)) \text{ and } \theta_{K_m}(f)(q_{K_m}(g)) = q_{K_m}(fg)$$

for $f \in A$ and $g \in A_{K_m}$.

By this method we see that from Exel systems $(C(X), \widetilde{\phi}^m, K^m)$, $(C(X), \widetilde{\psi}^n, L^n)$ and $(C(X), \gamma_{s,t}, H_{s,t})$ we can build Hilbert bimodules M_{K^m}, M_{L^n}, and $M_{H_{s,t}}$, respectively, over $C(X)$.

Remark 3.2.8. Since ϕ and ψ commute, $\widetilde{\psi}\widetilde{\phi} = \widetilde{\phi}\widetilde{\psi} = \gamma_{1,1}$, $LK = KL = H_{1,1}$. So we have $A_{LK} = A_{KL} = A_{H_{1,1}}$ and $LK(g^* h) = KL(g^* h) = H_{1,1}(g^* h)$ for all $g, h \in A$, hence $q_{KL} = q_{LK}$.

Remark 3.2.9. Note that for $f \in A$ we have

$$\begin{aligned}
\|f - f\mu_m(1)\|^2 &= \langle f - f\mu_m(1), f - f\mu_m(1)\rangle \\
&= K_m((f - f\mu_m(1))^*(f - f\mu_m(1))) \\
&= K_m(f^* f - \mu_m(1) f^* f - f^* f\mu_m(1) + \mu_m(1) f^* f\mu_m(1)) \\
&= K_m(f^* f) - 1K_m(f^* f) - K_m(f^* f)1 + 1K_m(f^* f)1 \\
&= 0.
\end{aligned}$$

Therefore $f = f\mu_m(1)$ in M_{K_m}.

3.2.4 Toeplitz representations of the Product system

We begin by creating a product system from Hilbert bimodules which come from Exel systems. We then show that the Toeplitz-covariant representations of Exel systems completely determine the Toeplitz representations of the product system.

Lemma 3.2.10. *For any Exel systems (A, μ, K) and (A, ν, L) such that $KL = LK$* [1]*, there is a well-defined inner-product preserving bimodule isomorphism $\Phi : M_K \otimes_A M_L \to M_{KL}$ such that $\Phi(q_K(a) \otimes q_L(b)) = q_{KL}(a\mu(b))$, where $a, b \in A$.*

Proof. Consider $\Psi : M_K \times_A M_L \to M_{KL}$ well-defined by $\Psi(q_K(a), q_L(b)) = q_{KL}(a\mu(b))$. Then Ψ is a bilinear map and extends to a linear map $\Phi : M_K \odot M_L \to M_{KL}$ such that $\Phi(q_K(a) \odot q_L(b)) = q_{KL}(a\mu(b))$.

Now we will show that inner products are preserved. Let $f, f' \in M_K$ and $g, g' \in M_L$, we will show that $\langle f \otimes g, f' \otimes g' \rangle = \langle f\mu(g), f'\mu(g') \rangle$

$$\begin{aligned}
\langle f \otimes g, f' \otimes g' \rangle &= \langle \langle f', f \rangle g, g' \rangle \\
&= \langle K((f')^* f)g, g' \rangle \\
&= L(g^* K(f^* f')g') \\
&= L(g^* K(f^* f' \mu(g'))) \\
&= LK(\mu(g^*) f^* f' \mu(g')) \\
&= KL(\mu(g^*) f^* f' \mu(g')) \\
&= \langle f\mu(g), f'\mu(g') \rangle .
\end{aligned}$$

It is enough to check that the function preserves the inner-product of elementary tensors since addition passes through the inner-product and all elements may be written as a linear combination of elementary tensors.

Now we complete the algebraic tensor product $M_K \odot M_L$ in the inner product

[1] If we do not assume $KL = LK$, then we have $M_K \otimes_A M_L \cong M_{LK}$ where M_{LK} is not necessarily isomorphic to M_{KL}. Then we would not be able to define a product system over $\mathbb{N}^2$.

defined on elementary tensors by

$$\langle q_K(a_1) \otimes q_L(b_1), q_K(a_2) \otimes q_L(b_2) \rangle_{KL} := \langle \langle q_K(a_2), q_K(a_1) \rangle_K \cdot q_L(b_1), q_L(b_2) \rangle_L .$$

Since completing includes modding out by vectors of length zero, it balances the tensor product by making $(q_K(a_1) \cdot a) \otimes q_L(b) = q_K(a_1) \otimes (a \cdot q_L(b))$ and this extension is isometric (since the inner-product is preserved), therefore this function is injective.

Since the range of Φ is the range of q_{KL} which is dense in M_{KL}, Φ has dense range. So by [29, Remark 3.27] we have shown that Φ is a bimodule homomorphism.

$\square$

Definition 3.2.11. Let A be a C^*-algebra and S be a semigroup. A family of Hilbert bimodules $\mathcal{X} = \{\mathcal{X}_s\}_{s \in S}$ over A is a *product system over* S if there is a map $p : \mathcal{X} \to S$, with $\mathcal{X}_s := p^{-1}(s)$, and the following condition holds: for every $s, t \in S$, the map $\mathcal{X}_s \times \mathcal{X}_t \mapsto \mathcal{X}_{st}$ extends to an isomorphism $\mathcal{X}_s \otimes_A \mathcal{X}_t \cong \mathcal{X}_{st}$, such that after identifing $\mathcal{X}_{st}$ with $\mathcal{X}_s \otimes_A \mathcal{X}_t$ via this isomorphism, $p(x \otimes y) = p(x)p(y)$ for all $x \in \mathcal{X}_s$ and $t \in \mathcal{X}_t$.

Let $(C(X), \mathbb{N}^2, \gamma, H)$ be an Exel-Larsen system. Fix $(m, n) \in \mathbb{N}^2$ and consider the Exel system $(C(X), \gamma_{m,n}, H_{m,n})$. Define $\mathcal{X}_{m,n} := M_{H_{m,n}}$. With the operation $q_{m,n}(a)q_{k,l}(b) = q_{m+k,n+l}(a\gamma_{m,n}(b))$, the family $\mathcal{X} = \bigsqcup \mathcal{X}_{m,n}$ is a product system over $\mathbb{N}^2$ by [18, Proposition 2.1]. This is the product system that we associate with the Exel-Larsen system $(C(X), \mathbb{N}^2, \gamma, H)$.

Recall that a *representation* π of a C^*-algebra A in a C^*-algebra B is simply a *-homomorphism of A to B. We will assume that our representations are unital if A and B are unital.

Definition 3.2.12. Let B be a C^*-algebra, (A, μ, K) be an Exel system, let $\pi : A \to B$ be a representation, and let $V \in B$. The pair (V, π) is a *Toeplitz-covariant representation of* (A, μ, K) in B (in the sense of Brownlowe and Raeburn [4]) if , for every $a \in A$:

(TCR1) $V\pi(a) = \pi(\mu(a))V$,

(TCR2) $V^*\pi(a)V = \pi(K(a))$.

Proposition 3.2.13. *Let (V,π), (W,π) be Toeplitz-covariant representations of(A,μ,K) and (A,ν,L), respectively, such that $VW = WV$. Then the pair (Z,π), where $Z = VW$, is a Toeplitz-covariant representation of $(A,\mu\nu,LK)$.*

Proof. We already know that π is a representation, so we will check that (TCR1) and (TCR2) are satisfied. Observe that for every $a \in A$ we have

$$\begin{aligned} Z\pi(a) &= VW\pi(a) \\ &= V\pi(\nu(a))W \\ &= \pi(\mu\nu(a))VW \end{aligned}$$

and

$$\begin{aligned} Z^*\pi(a)Z &= (VW)^*\pi(a)VW \\ &= W^*V^*\pi(a)VW \\ &= W^*\pi(K(a))W \\ &= \pi(LK(a)). \end{aligned}$$

□

So by Proposition 3.2.13 we see that if (V,π) is a Toeplitz-covariant representation of the Exel system (A,μ,K) in B, then (V^m,π) is a Toeplitz-covariant representation of the Exel system (A,μ^m,K^m) in B for all $m \in \mathbb{N}$.

Now we wish to examine the relationship between Toeplitz representations of the product system $\mathcal{X}$ that we built and Toeplitz-covariant representations of $(C(X),\widetilde{\phi},K)$ and $(C(X),\widetilde{\psi},L)$.

Definition 3.2.14. Let $\mathcal{X}$ be a product system over a semigroup S. A map $\lambda : \mathcal{X} \to B$ into a C^*-algebra B is called a *Toeplitz representation of* $\mathcal{X}$ if $\lambda(u)\lambda(v) = \lambda(uv)$ for all $u, v \in \mathcal{X}$, and the pair $(\lambda_s,\lambda_0) := (\lambda|_{\mathcal{X}_s},\lambda|_{\mathcal{X}_0})$ is a Toeplitz representation of the bimodule $\mathcal{X}_s$, that is λ_s is linear, $\lambda_0 : A \to B$ is a homomorphism, and the conditions

(TR1) $\lambda_s(u \cdot a) = \lambda_s(u)\lambda_0(a)$,

(TR2) $\lambda_s(u)^*\lambda_s(v) = \lambda_0(\langle u, v\rangle_s)$,

(TR3) $\lambda_s(a \cdot u) = \lambda_0(a)\lambda_s(u)$

are satisfied for all $s \in S$, $a \in A$, and $u, v \in \mathcal{X}_s$.

Notice that the representation of an Exel system has a covariance condition (TCR2) and the representation of a product system does not.

We will state [4, Lemma 3.2] since it will be very useful momentarily.

Lemma 3.2.15 (Brownlowe-Raeburn)**.** *Given a Toeplitz-covariant representation (V, π) of the Exel system (A, μ, K) in a C^*-algebra B, there exists a linear map $\delta_V : M_K \to B$ such that $\delta_V(q_K(a)) = \pi(a)V$ and the pair (δ_V, π) is a Toeplitz representation of M_K in B. Conversely, if (δ, π) is a Toeplitz representation of M_K in B and π is unital, then the pair $(\delta(q_K(1)), \pi)$ is a Toeplitz-covariant representation of (A, μ, K) and $\delta_{\delta(q_K(1))} = \delta$.*

The Brownlowe-Raeburn Lemma tells us that given a Toeplitz-covariant representation $(Z_{m,n}, \pi)$ of the Exel system $(A, \mu^m\nu^n, L^nK^m)$ in B, there exists a linear map $\lambda^{Z_{m,n}} : M_{K^mL^n} \to B$ such that $(\lambda^{Z_{m,n}}, \pi)$ is a Toeplitz representation of $M_{K^mL^n}$ in B. So when we look at the product system, $(\lambda^{Z_{m,n}}, \pi)$ is a Toeplitz representation of $\mathcal{X}_{m,n}$ in B. We also have the converse, that is, given a Toeplitz representation (λ, π) of $\mathcal{X}_{m,n}$ in B, if π is unital then $(Z^\lambda_{m,n}, \pi)$ is a Toeplitz-covariant representation of $(A, \mu^m\nu^n, L^nK^m)$ (where $Z^\lambda_{m,n} := \lambda(q_{K^mL^n}(1)))$ and $\lambda^{Z^\lambda_{m,n}} = \lambda$.

Theorem 3.2.16. *Let (V, π) and (W, π) be Toeplitz-covariant representations of (A, μ, K) and (A, ν, L) (where $KL = LK$), respectively, in a C^*-algebra B such that $VW = WV$. Then for the product system $\mathcal{X}$ over $\mathbb{N}^2$ where $\mathcal{X}_{m,n} := M_{K^mL^n}$, $\lambda^{VW} : \mathcal{X} \to B$ where $\lambda^{V^sW^t} : \mathcal{X}_{s,t} \to B$ defined by $\lambda^{V^sW^t}(q_{K^sL^t}(a)) = \pi(a)V^sW^t$ and $\lambda^{V^0W^0} := \pi$, is a Toeplitz representation of the product system $\mathcal{X}$ in B and $\lambda^{V^0W^0}$ is unital.*

Conversely, if λ *is a Toeplitz representation of the product system* $\mathcal{X}$ *(where* $\mathcal{X}_{m,n} := M_{K^m L^n}$ *for commuting transfer operators* K *and* L*) in* B *such that* $\lambda_{0,0}$ *is unital, then* (V^λ, π^λ) *and* (W^λ, π^λ) *are Toeplitz-covariant representations of* (A, μ, K) *and* (A, ν, L)*, respectively, where* $V^\lambda := \lambda_{1,0}(q_K(1))$ *and* $W^\lambda := \lambda_{0,1}(q_L(1))$*, and* $\pi^\lambda := \lambda_{0,0}$*. Furthermore* V^λ *commutes with* W^λ*,* $\lambda^{V^\lambda W^\lambda} = \lambda$*,* $(V^{\lambda^{VW}}, \pi^{\lambda^{VW}}) = (V, \pi)$*, and* $(W^{\lambda^{VW}}, \pi^{\lambda^{VW}}) = (W, \pi)$*.*

Proof. Let (V, π) and (W, π) be Toeplitz-covariant representations of (A, μ, K) and (A, ν, L), respectively, in a C^*-algebra B such that $VW = WV$. Proposition 3.2.13 gives us that for all $s, t \in \mathbb{N}$ and $(m, n) \in \mathbb{N}^2$, (V^s, π), (W^t, π), and $(V^m W^n, \pi)$ are Toeplitz-covariant representations of (A, μ^s, K^s), (A, ν^t, L^t), and $(A, \mu^m \nu^n, K^m L^n)$ respectively. The Brownlowe-Raeburn Lemma gives us that $(\lambda^{V^m W^n}, \pi)$ is a Toeplitz representation of $\mathcal{X}_{m,n}$ in B. We must prove that $\lambda^{VW} : \mathcal{X} \to B$ is a homomorphism, that is $\lambda^{VW}(u)\lambda^{VW}(v) = \lambda^{VW}(uv)$ for $u, v \in \mathcal{X}$. Let $a, b \in A$ and $(s, t), (m, n) \in \mathbb{N}$, then we have

$$\begin{aligned}
\lambda^{V^s W^t}(q_{K^s, L^t}(a))\lambda^{V^m W^n}(q_{K^m, L^n}(b)) &= \pi(a)V^s W^t \pi(b) V^m W^n \\
&= \pi(a)\pi(\mu^s \nu^t(b)) V^{s+m} W^{t+n} \\
&= \pi(a\mu^s \nu^t(b)) V^{s+m} W^{t+n} \\
&= \lambda^{V^{s+m} W^{t+n}}(q_{K^{s+m} L^{t+n}}(a\mu^s \nu^t(b)) \\
&= \lambda^{V^{s+m} W^{t+n}}(q_{K^s L^t}(a) \otimes q_{K^m L^n}(b)).
\end{aligned}$$

Therefore λ^{VW} is a Toeplitz representation of the product system $\mathcal{X}$ in B. Since π is a representation, it is unital, therefore $\lambda^{V^0 W^0}$ is unital.

Conversely, given λ, a Toeplitz representation of the product system $\mathcal{X}$ in B, such that $\lambda_{0,0}$ is unital and define $\pi^\lambda := \lambda_{0,0}$, we know that $(\lambda_{m,n}, \pi^\lambda)$ is a Toeplitz representation of $\mathcal{X}_{m,n}$ for all $(m, n) \in \mathbb{N}^2$. So in particular, $(\lambda_{1,0}, \pi^\lambda)$, and $(\lambda_{0,1}, \pi^\lambda)$ are Toeplitz representations of $\mathcal{X}_{1,0} = M_K$ and $\mathcal{X}_{0,1} = M_L$, respectively. Now define $V^\lambda := \lambda_{1,0}(q_K(1))$ and $W^\lambda := \lambda_{0,1}(q_L(1))$. The Brownlowe-Raeburn Lemma gives us that (V^λ, π^λ) and (W^λ, π^λ) are Toeplitz-covariant representations of (A, μ, K) and (A, ν, L), respectively. Notice that

$$\begin{aligned} V^\lambda W^\lambda &= \lambda_{1,0}(q_K(1))\lambda_{0,1}(q_L(1)) \\ &= \lambda_{1,1}(q_K(1) \otimes q_L(1)) \\ &= \lambda_{1,1}(q_{KL}(1)) \\ &= \lambda_{1,1}(q_{LK}(1)) \\ &= \lambda_{0,1}(q_L(1))\lambda_{1,0}(q_K(1)) \\ &= W^\lambda V^\lambda. \end{aligned}$$

To see that $\lambda^{V^\lambda W^\lambda} = \lambda$, observe that

$$\begin{aligned} \lambda^{V^\lambda W^\lambda}_{m,n}(q_{K^m L^n}(a)) &= \pi^\lambda(a)(V^\lambda)^m (W^\lambda)^n \\ &= \lambda_{0,0}(a)\lambda_{m,0}(q_{K^m}(1))\lambda_{0,n}(q_{L^n}(1)) \\ &= \lambda_{0,0}(a)\lambda_{m,n}(q_{K^m L^n}(1)) \\ &= \lambda_{m,n}(a \cdot q_{K^m L^n}(1)) \\ &= \lambda_{m,n}(q_{K^m L^n}(a)). \end{aligned}$$

Lastly, notice that $\pi^{\lambda^{VW}} = \lambda^{VW}_{0,0} = \pi$, and since π is unital we have

$$V^{\lambda^{VW}} = \lambda^{V^1 W^0}(q_K(1)) = \pi(1)V^1 W^0 = V$$

and

$$W^{\lambda^{VW}} = \lambda^{V^0 W^1}(q_L(1)) = \pi(1)V^0 W^1 = W.$$

□

So Toeplitz-covariant representations of Exel systems $(C(X), \widetilde{\phi}, K)$ and $(C(X), \widetilde{\psi}, L)$ completely determine Toeplitz representations of the product system $\mathcal{X}$ over $\mathbb{N}^2$ defined by $\mathcal{X}_{m,n} := M_{K^m L^n}$.

3.2.5 Exel systems that *-commute

We examine the additional properties that emerge from Exel systems (A, μ, K) and (A, ν, K) when μ and ν *-commute.

Lemma 3.2.17. *Suppose $\phi, \psi : X \to X$ are functions. Then the following are equivalent:*

1. *ϕ and ψ *-commute.*

2. *For every $y \in X$, $\psi : \{x : \phi(x) = y\} \to \{t : \phi(t) = \psi(y)\}$ is a bijection.*

3. *For every $t \in X$, $\phi : \{x : \psi(x) = t\} \to \{y : \phi(t) = \psi(y)\}$ is a bijection.*

Proof. To prove that (1) implies (2) we begin by showing $\psi(\{x : \phi(x) = y\}) \subseteq \{t : \phi(t) = \psi(y)\}$. Fix $y \in X$ and let $x \in \{x : \phi(x) = y\}$. Then $\phi(\psi(x)) = \psi(\phi(x)) = \psi(y)$, therefore $\psi(x) \in \{t : \phi(t) = \psi(y)\}$. Now we want to show that $\psi|_{\{x:\phi(x)=y\}}$ is surjective, so let $t \in \{t : \phi(t) = \psi(y)\}$. Then $\phi(t) = \psi(y)$, thus there exists $x \in X$ such that $\phi(x) = y$ (which says that $x \in \{x : \phi(x) = y\}$) and $\psi(x) = t$. Hence $\psi|_{\{x:\phi(x)=y\}}$ is surjective. To see that $\psi|_{\{x:\phi(x)=y\}}$ is injective, suppose $x_1, x_2 \in \{x : \phi(x) = y\}$ such that $\psi(x_1) = \psi(x_2) = t$. Notice that $\phi(x_1) = \phi(x_2) = y$, but since ϕ and ψ *-commute there is only one x such that $\phi(x) = y$ and $\psi(x) = t$. Therefore $x_1 = x_2$.

Conversely, to see that (2) implies (1) let $x \in X$. Let $y = \phi(x)$, so $x \in \{x : \phi(x) = y\}$. Since $\psi(x) \in \{t : \phi(t) = \psi(y)\}$ we have $\phi(\psi(x)) = \psi(y) = \phi(\psi(x))$, hence ϕ and ψ commute. Now fix $(y, t) \in X \times X$ such that $\phi(t) = \psi(y)$. Then $t \in \{t : \phi(t) = \psi(y)\}$ and since $\psi|_{\{x:\phi(x)=y\}}$ is bijective there exists a unique $x \in \{x : \phi(x) = y\}$ such that $\psi(x) = t$. Hence ϕ and ψ *-commute.

A similar argument shows (1) if and only if (3). $\square$

Proposition 3.2.18. *Suppose $\phi, \psi : X \to X$ are commuting surjective local homeomorphisms and let $(C(X), \widetilde{\phi}, K)$, $(C(X), \widetilde{\psi}, L)$ be the corresponding Exel systems. Then the following are equivalent:*

1. *ϕ and ψ *-commute.*

2. *$\widetilde{\psi} \circ K = K \circ \widetilde{\psi}$.*

3. $\widetilde{\phi} \circ L = L \circ \widetilde{\phi}$.

Proof. To see that (1) implies (2) let $f \in C(X)$ and $y \in X$. Observe that

$$\widetilde{\psi} \circ K(f)(y) = K(f)(\psi(y)) = \sum_{\phi(t)=\psi(y)} f(t).$$

By Lemma 3.2.17 we have

$$\sum_{\phi(t)=\psi(y)} f(t) = \sum_{\phi(x)=y} f(\psi(x)) = \sum_{\phi(x)=y} \widetilde{\psi}(f(x)) = K \circ \widetilde{\psi}(f)(y).$$

Conversely, suppose that ϕ and ψ do not *-commute. Then by Lemma 3.2.17 $\psi|_{\{x:\phi(x)=y\}}$ is not bijective. Suppose $\psi|_{\{x:\phi(x)=y\}}$ is not injective, so for some $w \in \{t : \phi(t) = \psi(y)\}$ there exists $x_1, x_2 \in \{x : \phi(x) = y\}$ such that $x_1 \neq x_2$ and $\psi(x_1) = \psi(x_2) = w$. Fix $f \in C(X)$ such that $f(w) = 1$ and $f(z) = 0$ for all $z \in \{t : \phi(t) = \psi(y)\} \setminus \{w\}$. Then we have

$$\widetilde{\psi} \circ K(f)(y) = K(f)(\psi(y)) = \sum_{\phi(t)=\psi(y)} f(t) = f(w) = 1$$

and

$$K \circ \widetilde{\psi}(f)(y) = \sum_{\phi(x)=y} \widetilde{\psi}(f(x)) = \sum_{\phi(x)=y} f(\psi(x)) = f(\psi(x_1)) + f(\psi(x_2)) = 2.$$

Therefore $\widetilde{\psi} \circ K \neq K \circ \widetilde{\psi}$. Now suppose $\psi|_{\{x:\phi(x)=y\}}$ is not surjective, so there exists $w \in \{t : \phi(t) = \psi(y)\}$ such that $\psi(x) \neq w$ for all $x \in \{x : \phi(x) = y\}$. Fix $f \in C(X)$ such that $f(w) = 1$ and $f(z) = 0$ for all $z \in \{t : \phi(t) = \psi(y)\} \setminus \{w\}$. Then we have

$$\widetilde{\psi} \circ K(f)(y) = K(f)(\psi(y)) = \sum_{\phi(t)=\psi(y)} f(t) = f(w) = 1$$

and

$$K \circ \widetilde{\psi}(f)(y) = \sum_{\phi(x)=y} \widetilde{\psi}(f(x)) = \sum_{\phi(x)=y} f(\psi(x)) = 0.$$

Therefore $\widetilde{\psi} \circ K \neq K \circ \widetilde{\psi}$.

It is a similar argument to show (1) if and only if (3). $\square$

For any Hilbert module Y_A and any $x, y \in Y$, the function $\Theta_{x,y} : z \mapsto x \cdot \langle y, z \rangle_A$ is an adjointable operator on Y, with adjoint $\Theta_{y,x}$. The closed span of the operators $\Theta_{x,y}$ is an ideal $\mathcal{K}(Y)$ in the C^*-algebra $\mathcal{L}(Y)$. Elements of $\mathcal{K}(Y)$ are called compact operators (see [27, Example 8.4]). We will denote $\Theta^K_{1,1} := \Theta_{q_K(1),q_K(1)}$.

Definition 3.2.19. Let (A, μ, K) and (A, ν, L) be Exel systems. Then we say Exel systems **-commute* if $\mu\nu = \nu\mu$, $KL = LK$, and $\mu L = L\mu$.

Theorem 3.2.20. *Given a pair (A, μ, K), (A, ν, L) of *-commuting Exel systems, then the operators $\Theta^K_{1,1} \in \mathcal{K}(M_K)$, $\Theta^L_{1,1} \in \mathcal{K}(M_L)$ satisfy*

$$(\Theta^L_{1,1} \otimes 1)(\Theta^K_{1,1} \otimes 1) = \Theta^{LK}_{1,1}. \tag{3.38}$$

Notice that $\Theta^K_{1,1} : M_K \to \mathcal{K}(M_K)$ is defined by

$$\begin{aligned} q_K(a) \mapsto & \, q_K(1) \cdot \langle q_K(1), q_K(a) \rangle \\ & = q_K(1\mu(\langle q_K(1), q_K(a) \rangle) \\ & = q_K(\mu(K(1^*a))) \\ & = q_K(\mu K(a)). \end{aligned}$$

First we want to be clear on what we mean by $\Theta^L_{1,1} \otimes 1$ and $\Theta^K_{1,1} \otimes 1$ as elements in $\mathcal{K}(M_{LK})$. Recall the isomorphism Φ from Lemma 3.2.10. We shall denote $\Phi_{KL} : M_K \otimes_A M_L \to M_{KL}$ and $\Phi_{LK} : M_L \otimes_A M_K \to M_{LK}$ and note that $M_{KL} = M_{LK}$. So the operators on the left-hand side of Equation (3.38) are really

$$\Theta^L_{1,1} \otimes 1 = \Phi_{LK} \circ (\Theta^L_{1,1} \otimes 1) \circ \Phi^{-1}_{LK} \quad \text{and} \quad \Theta^K_{1,1} \otimes 1 = \Phi_{KL} \circ (\Theta^K_{1,1} \otimes 1) \circ \Phi^{-1}_{KL}.$$

Proof of Theorem 3.2.20. The right hand side of Equation (3.38) yields

$$\begin{aligned} \Theta^{LK}_{1,1}(q_{LK}(a)) &= q_{LK}(1) \cdot \langle q_{LK}(1), q_{LK}(a) \rangle \\ &= q_{LK}(1\mu\nu(\langle q_{LK}(1), q_{LK}(a) \rangle)) \\ &= q_{LK}(\mu\nu LK(1^*a)) \\ &= q_{LK}(\mu\nu LK(a)). \end{aligned}$$

The operator $\Theta^K_{1,1} \otimes 1$ on the left hand side applies to an element $q_{LK}(a)$ as

$$\begin{aligned}
\Phi_{KL} \circ (\Theta^K_{1,1} \otimes 1) \circ \Phi^{-1}_{KL}(q_{LK}(a)) &= \Phi_{KL} \circ (\Theta^K_{1,1} \otimes 1) \circ \Phi^{-1}_{KL}(q_{KL}(a)) \text{ by Remark 3.2.8} \\
&= \Phi_{KL} \circ (\Theta^K_{1,1} \otimes 1)(q_K(a) \otimes q_L(1)) \\
&= \Phi_{KL}(q_K(\mu K(a)) \otimes q_L(1)) \\
&= q_{KL}(\mu K(a)\mu\nu(1)) \\
&= q_{LK}(\mu K(a)\mu\nu(1)) \text{ by Remark 3.2.8} \\
&= q_{LK}(\mu K(a)) \text{ by Remark 3.2.9}
\end{aligned}$$

.

The operator $\Theta^L_{1,1} \otimes 1$ on the left hand side applies to an element $q_{LK}(\mu K(a))$ as

$$\begin{aligned}
\Phi_{LK} \circ (\Theta^L_{1,1} \otimes 1) \circ \Phi^{-1}_{LK}(q_{LK}(\mu K(a))) &= \Phi_{LK} \circ (\Theta^L_{1,1} \otimes 1)(q_L(\mu K(a)) \otimes q_K(1)) \\
&= \Phi_{LK}(q_L(\nu L\mu K(a)) \otimes q_K(1)) \\
&= q_{LK}(\nu L\mu K(a)\nu\mu(1)) \\
&= q_{LK}(\nu L\mu K(a)) \text{ by Remark 3.2.9} \\
&= q_{LK}(\mu\nu LK(a)).
\end{aligned}$$

So the left hand side is of Equation (3.38) yields

$$\begin{aligned}
(\Theta^L_{1,1} \otimes 1)(\Theta^K_{1,1} \otimes 1)(q_{LK}(a)) &= (\Theta^L_{1,1} \otimes 1)(q_{LK}(\mu K(a))) \\
&= q_{LK}(\mu\nu LK(a)).
\end{aligned}$$

□

Definition 3.2.21. A semigroup G is said to be *lattice ordered* if there exists a partial ordering $<$ of the elements of G satisfying:

1. $g < h$ implies $fg < fh$ and $gf < hf$ for all $f, g, h \in G$, and
2. every finite set has a least upper bound and a greatest lower bound.

Remark 3.2.22. $\mathbb{N}^2$ is an example of a lattice ordered semigroup.

Definition 3.2.23. [9, Definition 1.5] Suppose P is a lattice ordered semigroup and E is a product system over P. We say that E is *compactly aligned* if whenever

$s, t \in P$ have a common upper bound and S and T are compact operators on E_s and E_t, respectively, the operator $(S \otimes 1)(T \otimes 1)$ is a compact operator on $E_{s \vee t}$.

Theorem 3.2.24. *If $\mathcal{X}$ is a product system over $\mathbb{N}^2$ defined from two *-commuting Exel systems, then $\mathcal{X}$ is compactly aligned.*

Proof. Observe that $\mathbb{N}^2$ is a lattice ordered semigroup by Remark 3.2.22. By Theorem 3.2.20 $\mathcal{X}$ is compactly aligned. □

We do not know if the converse holds. That is, we do not know whether if $\mathcal{X}$ is compactly aligned, then the two commuting Exel systems must *-commute.

3.3 Surjective Local Homeomorphisms which *-commute with the Shift

3.3.1 Introduction

In this section we were interested in classifying surjective local homeomorphisms which *-commute with the unilateral shift. This project was in collaboration with Iain Raeburn and benefited from information shared by Ruy Exel.

In section 3.3.3 we describe the continuous functions that commute with the shift. Our result is a one-sided analog of an old theorem of Hedlund in [12]. In section 3.3.4 we introduce the concept of regressive and prove that it characterizes continuous functions that *-commute with the shift. In section 3.3.5 we try to describe which of the functions in section 3.3.4 are local homeomorphisms. We identify some sufficient conditions but do not yet have a complete answer. Futhermore, in section 3.3.6 we show that covering maps (i.e. local homeomorphisms) which commute with the shift are surjective and k to 1 for some $k \in \mathbb{N}$.

3.3.2 Background

Let A be a finite alphabet. We denote A^n to be all the words of length n, $A^* := \bigcup_{n \geq 1} A^n$, and A^∞ to be all the sequences of infinite length. Since A is a compact Hausdorff space, A^∞ is also a compact Hausdorff space by Tychonoff's Theorem. For $\mu \in A^*$ we define $Z(\mu) := \{x \in A^\infty : x_1 \cdots x_{|\mu|} = \mu\}$. Observe that the family $\{Z(\mu) : \mu \in A^*\}$ is a basis for A^∞. We use this fact repeatedly.

Remark 3.3.1. We will use σ to denote the unilateral shift and assume A, throughout this section ,that is a finite alphabet.

3.3.3 Continuous Functions which Commute with the Shift

In [12, Theorem 3.4] Hedlund proved that continuous functions on bisequences that commute with the shift are of the form $\sigma^k \tau_d$ where $k \in \mathbb{Z}$ and τ_d is notation that will be defined in this section. Here we show that continuous functions on sequences that commute with the shift are of the form τ_d, thus the one-sided case is a simplified version of Hedlund's two-sided case.

Definition 3.3.2. For a fixed $n \in \mathbb{N}$ let $d : A^n \to A$ be a function. Then we define a function $\tau_d : A^\infty \to A^\infty$ by $\tau_d(x)_i = d(x_i \cdots x_{i+n-1})$ and say that τ_d is defined by d or d defines τ_d.

Sometimes τ_d is called a *sliding block code*.

Lemma 3.3.3. *For a fixed $n \in \mathbb{N}$ let $d : A^n \to A$ be a function which defines τ_d. Then τ_d is continuous and commutes with σ.*

Remark 3.3.4. To show that τ_d is continuous it is important to recall the following fact: Let X, Y be topological spaces, $f : X \to Y$ be a function, and $\mathbb{B}$ is a basis for Y. If $f^{-1}(U)$ is open in X for all $U \in \mathbb{B}$, then f is continuous.

Proof of Lemma 3.3.3. Fix $\mu \in A^k$ for some k. We want to show that $\tau_d^{-1}(Z(\mu))$ is open, so suppose $x \in \tau_d^{-1}(Z(\mu))$. Then $\tau_d(x) \in Z(\mu)$, so

$$\tau_d(x) = d(x_1 \cdots x_n) d(x_2 \cdots x_{n+1}) \cdots = \mu_1 \mu_2 \cdots .$$

Now consider $w \in Z(x_1 \cdots x_{k+n-1})$. We want to show that $Z(x_1 \cdots x_{k+n-1}) \subseteq$

$\tau_d^{-1}(Z(\mu))$.

$\tau_d(w) = \tau_d(x_1 \cdots x_{k+n-1} w_{k+n} \cdots) = d(x_1 \cdots x_n) \cdots d(x_k \cdots x_{k+n-1}) \cdots = \mu_1 \cdots \mu_k \cdots$.

So $\tau_d(w) \in Z(\mu)$ which means $w \in \tau_d^{-1}(Z(\mu))$. Thus we have shown that the inverse image of a base element is open. So by Remark 3.3.4 we have τ_d is continuous.

Let $x \in A^\infty$ and observe

$$\tau_d \sigma(x) = \tau_d(x_2 x_3 x_4 \cdots) = d(x_2 \cdots x_{n+1}) d(x_3 \cdots x_{n+2}) \cdots$$

and

$\sigma \tau_d(x) = \sigma(d(x_1 \cdots x_n) d(x_2 \cdots x_{n+1}) d(x_3 \cdots x_{n+2}) \cdots) = d(x_2 \cdots x_{n+1}) d(x_3 \cdots x_{n+2}) \cdots$.

Therefore τ_d and σ commute. □

Example 3.3.5. Define $d : A^2 \to A$ by $d(a_1 a_2) = a_2$. Then $\tau_d : A^\infty \to A^\infty$ is defined to be $\tau_d(a_1 a_2 a_3 \cdots) = a_2 a_3 a_4 \cdots$. So by Lemma 3.3.3, τ_d is a continuous function which commutes with σ. For this particular example, $\tau_d = \sigma$.

Remark 3.3.6. It is important to note that there is not a unique function d which defines τ_d. For example, define $d' : A^3 \to A$ by $d'(a_1 a_2 a_3) = a_2$. Then $\tau_{d'}$ equals the τ_d from Example 3.3.5.

Lemma 3.3.7. *Let $\phi : A^\infty \to A^\infty$ be a continuous function which commutes with σ. Then there exists $n \in \mathbb{N}$ such that*

$(*)$ *for every $\mu \in A^n$ there exists a unique $a \in A$ such that $Z(\mu) \subseteq \phi^{-1}(Z(a))$.*

Choose the smallest $n \in \mathbb{N}$ with property $(*)$ and define $d : A^n \to A$ by $d(\mu) = a$ where $\phi(Z(\mu)) \subseteq Z(a)$. Then $\tau_d = \phi$.

Proof. Consider $\{Z(a) | a \in A\}$. Notice that $Z(a)$ are open, closed, and disjoint sets which cover A^∞. Define $V_a = \phi^{-1}(Z(a))$. Then V_a are also open, closed, and disjoint sets which cover A^∞. Since the closed subset of a compact space is compact (see [21, Theorem 26.2]) and A^∞ is compact, V_a is compact. Therefore V_a is the union of a finite number of basis elements. That is, $V_a = \bigcup_{i=1}^k Z(x_1^i \cdots x_{n_j}^i)$. Denote $m = \max\{n_j\}$. We may write $V_a = \bigcup_{i=1}^k Z(\mu^i)$ where $\mu^i \in A^m$, so m has property

$(*)$. Denote n to be the smallest of all such m's. We define the function $d : A^n \to A$ by $d(x_1 \cdots x_n) = a$ where $Z(x_1 \cdots x_n) \subseteq V_a$. This function is well-defined since the V_a are disjoint.

We will now show that $\tau_d = \phi$. Let $k \in \mathbb{N}$ and $x \in A^\infty$. We will show that $\phi(x)_k = \tau_d(x)_k$. Notice that $\sigma^k(x) \in Z(x_{k+1} \cdots x_{k+n}) \subseteq V_a = \phi^{-1}(Z(a))$ for some $a \in A$. Then $\phi(\sigma^k(x))_1 = a$. Also, we have $\tau_d(\sigma^k(x))_1 = d(\sigma^k(x)_1 \cdots \sigma^k(x)_n) = d(x_{k+1} \cdots x_{k+n}) = a$. Thus

$$\phi(x)_k = \sigma^k(\phi(x))_1 = \phi(\sigma^k(x))_1 = a = \tau_d(\sigma^k(x))_1 = \sigma^k(\tau(x))_1 = \tau_d(x)_k.$$

Since this holds for all $k \in \mathbb{N}$, $\phi(x) = \tau_d(x)$ and since x was arbitrary, $\tau_d = \phi$. □

Corollary 3.3.8. *Let $\phi : A^\infty \to A^\infty$ be a continuous function which commutes with σ. Let n be the smallest natural number with property $(*)$ and define $d : A^n \to A$ such that $\tau_d = \phi$. Suppose $d' : A^m \to A$ is a function such that $\tau_{d'} = \phi$, then $d'(x_1 \cdots x_m) = d(x_1 \cdots x_n)$ and $m \geq n$.*

Proof. Since $\tau_d = \phi = \tau_{d'}$, for all $i \in \mathbb{N}$ and $x \in A^\infty$ we have

$$d(x_i \cdots x_{n+i-1}) = \tau_d(x)_i = \phi(x)_i = \tau_{d'}(x)_i = d'(x_i \cdots x_{m+i-1}).$$

Now suppose that $m < n$. We will arrive at a contradiction by showing that m has property $(*)$. For all $\mu \in A^m$ observe that

$$d(\mu x_{m+1} \cdots x_n) = a = d'(\mu)$$

for some $a \in A$, thus $Z(\mu) \subseteq \phi^{-1}(Z(a))$. Now suppose there also exists $b \in A$ such that $Z(\mu) \subseteq \phi^{-1}(Z(b))$, then $d'(\mu) = b$. So $b = a$ otherwise d' is not a function, thus a is unique. So m satisfies property $(*)$, but n is the smallest natural number which satisfies property $(*)$ hence we have reached the desired contradiction. □

Example 3.3.9. Fix $a \in A$ and define $aaa \cdots = x \in A^\infty$. Consider $\phi : A^\infty \to A^\infty$ defined by $\phi(y) = x$ for all $y \in A^\infty$. Then ϕ is a continuous function which commutes with σ. Notice that $\phi^{-1}(Z(b)) = \emptyset$ unless $b = a$, thus we have $\phi^{-1}(Z(a)) = V_a = A$ and $n = 1$. So by Lemma 3.3.7 $d : A \to A$ is defined by $d(b) = a$ for all $b \in A$ and

$\tau_d = \phi$.

3.3.4 Continuous functions which *-commute with the Shift

In this section, we classify continuous functions τ_d which *-commute with the shift by requiring d to be regressive.

Definition 3.3.10. Fix $n \in \mathbb{N}$ and let $d : A^n \to A$ be a function. Then d is *regressive* if for each fixed $x_1 \cdots x_{n-1} \in A^{n-1}$ the function $r_d^{x_1 \cdots x_{n-1}} : A \to A$ defined by $r_d^{x_1 \cdots x_{n-1}}(a) = d(ax_1 \cdots x_{n-1})$ is bijective .

Example 3.3.11. Let $B = \{a, b, c, d\}$ and define $d : B^2 \to B$ by

$$\begin{array}{llll} d(aa) = a & d(ab) = a & d(dc) = a & d(dd) = a \\ d(da) = b & d(db) = b & d(ac) = b & d(ad) = b \\ d(ca) = c & d(cb) = c & d(bc) = c & d(bd) = c \\ d(ba) = d & d(bb) = d & d(cc) = d & d(cd) = d. \end{array}$$

Notice that when the second coordinate is fixed d is bijective, therefore d is regressive.

Example 3.3.12. Let $B = \{0, 1, 2, 3\}$ and define $d : B^2 \to B$ by

$$\begin{array}{llll} d(00) = 0 & d(01) = 0 & d(02) = 1 & d(03) = 1 \\ d(10) = 2 & d(11) = 2 & d(12) = 3 & d(13) = 3 \\ d(20) = 0 & d(21) = 0 & d(22) = 1 & d(23) = 1 \\ d(30) = 2 & d(31) = 2 & d(32) = 3 & d(33) = 3. \end{array}$$

is not regressive.

In conversations with Ruy Exel, he told us that τ_d *-commutes with σ if and only if it is defined from a regressive function d. This result has not been published, therefore we prove it.

Theorem 3.3.13. *Fix $n \in \mathbb{N}$ and let $d : A^n \to A$ be a function. Then d is regressive if and only if τ_d *-commutes with σ.*

Proof. By Lemma 3.3.3, τ_d commutes with σ. Suppose we have $y, z \in A^\infty$ such that $\sigma(y) = \tau_d(z)$. Since d is regressive there exists a unique $x_1 \in A$ such that $r_d^{z_1 \cdots z_{n-1}}(x_1) = d(x_1 z_1 \cdots z_{n-1}) = y_1$. Notice that $y_{i+1} = \sigma(y)_i = \tau_d(z)_i = \tau_d(x_1 z)_{i+1}$. So we have

$$\tau_d(x_1 z) = d(x_1 z_1 \cdots z_{n-1})\tau_d(x_1 z)_2 \tau_d(x_1 z)_3 \cdots = y_1 y_2 y_3 \cdots = y$$

and $\sigma(x_1 z) = z$. To see that $x_1 z$ is unique suppose there exists $w \in A^\infty$ such that $\tau_d(w) = y$ and $\sigma(w) = z$. Then $w = az$ for some $a \in A$. Notice that

$$y_1 = \tau_d(w)_1 = \tau_d(az)_1 = d(az_1 \cdots z_{n-1}).$$

Since d is regressive and we know $d(x_1 z_1 \cdots z_{n-1}) = y_1$, so then $a = x_1$. Hence $x_1 z$ is unique, therefore τ_d *-commutes with σ.

Conversely, let $x_1 \cdots x_{n-1} \in A^{n-1}$. To see that $r_d^{x_1 \cdots x_{n-1}}$ is injective suppose for $a_1, a_2 \in A$ we have $r_d^{x_1 \cdots x_{n-1}}(a_1) = r_d^{x_1 \cdots x_{n-1}}(a_2)$. Then let $z \in Z(x_1 \cdots x_{n-1})$ and observe that

$$\tau_d(a_1 z)_1 = d(a_1 z_1, \cdots z_{n-1}) = r_d^{x_1 \cdots x_{n-1}}(a_1) = r_d^{x_1 \cdots x_{n-1}}(a_2) = d(a_2 z_1, \cdots z_{n-1}) = \tau_d(a_2 z)_1.$$

For $i \geq 2$ we have $\tau_d(a_1 z)_i = \tau_d(z)_{i-1} = \tau_d(a_2 z)_i$. So $\tau_d(a_1 z) = \tau_d(a_2 z)$ and $\sigma(a_1 z) = z = \sigma(a_2 z)$. Since τ_d *-commutes with σ we have $a_1 z = a_2 z$ which implies $a_1 = a_2$. Hence $r_d^{x_1 \cdots x_{n-1}}$ is injective.

Let $x_1 \cdots x_{n-1} \in A^{n-1}$. To see that $r_d^{x_1 \cdots x_{n-1}}$ is surjective, let $a \in A$. Suppose $z \in Z(x_1 \cdots x_{n-1})$ and define $w = \tau_d(z)$. Then $aw, z \in A^\infty$ satisfy $\sigma(aw) = \tau_d(z)$. Since τ_d and σ *-commute there exists a unique $v \in A^\infty$ such that $\sigma(v) = z$ and $\tau_d(v) = aw$. Notice that since $\sigma(v) = z$, there exists $b \in A$ such that $v = bz$. So we have

$$a = \tau_d(v)_1 = \tau_d(bz)_1 = d(bz_1 \cdots z_{n-1}) = d(bx_1 \cdots x_{n-1}).$$

So $b \in A$ such that $r_d^{x_1 \cdots x_{n-1}}(b) = d(bx_1 \cdots x_{n-1}) = a$, therefore $r_d^{x_1 \cdots x_{n-1}}$ is surjective. Thus d is regressive. $\square$

Corollary 3.3.14. *Suppose $\phi : A^\infty \to A^\infty$ is continuous and commutes with σ. Then ϕ *-commutes with σ if and only if there exists a regressive function $d : A^n \to$*

A such that $\phi = \tau_d$.

Proof. Since ϕ is continuous and commutes with σ, by Lemma 3.3.7, there exists $d : A^n \to A$ such that $\phi = \tau_d$. Since ϕ *-commutes with σ, by Theorem 3.3.13 d is regressive. The converse is part of Theorem 3.3.13. □

Example 3.3.15. The function $d : A^2 \to A$ defined by $d(a_1 a_2) = a_2$ which defines σ is not regressive. Fix $x \in A$, then $r_d(a) = d(ax) = x$ for any $a \in A$. Hence r_d is not injective. Therefore d is not regressive, so σ does not *-commute with itself.

3.3.5 Local Homeomorphisms which Commute with the Shift

In this section, we will examine local homeomorphisms which commute with σ. We know these functions must be of the form τ_d, but we have not yet found a necessary and sufficient condition to describe the subset of τ_d which are local homeomorphisms.

Definition 3.3.16. Let X, Y be topological spaces. A function $f : X \to Y$ is a *homeomorphism* if it is bijective, continuous, and the inverse is continuous.

Definition 3.3.17. Let X, Y be topological spaces. A continuous function $f : X \to Y$ is a *local homeomorphism* if for every point $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is open in Y and $f|_U : U \to f(U)$ is a homeomorphism.

Definition 3.3.18. Fix $n \in \mathbb{N}$ and let $d : A^n \to A$ be a function. Then d is *progressive* if for each fixed $x_1 \cdots x_{n-1} \in A^{n-1}$, the function $p^{x_1 \cdots x_{n-1}} : A \to A$ defined by $p^{x_1 \cdots x_{n-1}}(a) = d(x_1 \cdots x_{n-1} a)$ is bijective .

Example 3.3.19. The function $d : A^2 \to A$ defined by $d(a_1 a_2) = a_2$ (see Example 3.3.15) is progressive. Fix $a \in A$, let $a_1, a_2 \in A$ and suppose $p^a(a_1) = p^a(a_2)$. Then we have

$$a_1 = d(aa_1) = p^a(a_1) = p^a(a_2) = d(aa_2) = a_2,$$

so p^a is injective. Note that $b = p^a(b)$ for all $b \in A$ so p^a is surjective. Hence d is progressive.

Exel and Renault prove in [8, Theorem 14.3] that if d is progressive, then τ_d is a local homeomorphism. However, this does not characterize all local homeomorphisms which commute with the shift. The next example shows a function d such that d is not progressive, yet τ_d is a local homeomorphism. This example is a simplified version of one shown to us by Exel.

Example 3.3.20. The function $d : B^2 \to B$ defined by

$$\begin{array}{llll} d(aa) = a & d(ba) = d & d(ca) = c & d(da) = b \\ d(ab) = a & d(bb) = d & d(cb) = c & d(db) = b \\ d(ac) = b & d(bc) = c & d(cc) = d & d(dc) = a \\ d(ad) = b & d(bd) = c & d(cd) = d & d(dd) = a. \end{array}$$

from Example 3.3.11 is not progressive. With a little work one can check that

$$\tau_d(Z(a)) = Z(a) \cup Z(b)$$

$$\tau_d(Z(b)) = Z(c) \cup Z(d)$$

$$\tau_d(Z(c)) = Z(c) \cup Z(d)$$

$$\tau_d(Z(d)) = Z(a) \cup Z(b)$$

and that τ_d is a homeomorphism on $Z(a), Z(b), Z(c), Z(d)$. So τ_d is a local homeomorphism.

We generalize the idea of progressive, called weakly progressive, and show that this implies local homeomorphism. However even this generalization is not a necessary condition.

Definition 3.3.21. Fix $n, m \in \mathbb{N}$ and let $d : A^n \to A$ have the property that for every $\mu \in A^n$ and every $\nu \in A^m$ such that $d(\mu) = \nu_1$ there exists a unique $a \in A$ such that $p_m^{\mu_1 \cdots \mu_{n-1}}(a\alpha) = d(\mu_1 \cdots \mu_{n-1} a) d(\mu_2 \cdots \mu_{n-1} a \alpha_1) \cdots = \nu$ has a solution

$\alpha \in A^{m-1}$. Then we say that d is *weakly progressive* of order m. [2]

Example 3.3.22. The function d from Example 3.3.20 is weakly progressive of order 2.

Example 3.3.23. The function $d : B^2 \to B$ defined by

$$\begin{array}{llll} d(00) = 0 & d(10) = 2 & d(20) = 0 & d(30) = 2 \\ d(01) = 0 & d(11) = 2 & d(21) = 0 & d(31) = 2 \\ d(02) = 1 & d(12) = 3 & d(22) = 1 & d(32) = 3 \\ d(03) = 1 & d(13) = 3 & d(23) = 1 & d(33) = 3. \end{array}$$

from Example 3.3.12 is weakly progressive of order 2.

Proposition 3.3.24. *Let $d : A^n \to A$ be a function and fix $x_1 \cdots x_{n-1} \in A^{n-1}$. If d is weakly progressive, then $\tau_d : Z(x_1 \cdots x_{n-1}) \to \bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$ is bijective.*

Proof. Fix m such that d is weakly progressive of order m. Notice that for $x \in Z(x_1 \cdots x_{n-1})$, $\tau_d(x) \in Z(d(x_1 \cdots x_n))$ which is contained in $\bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$. So the range of $\tau_d|_{Z(x_1 \cdots x_{n-1})} \subseteq \bigcup_{a \in A} Z(d(x_1 \cdots x_{n-1}a))$. Let $y \in Z(d(x_1 \cdots x_{n-1}a))$ for some $a \in A$. We want to show that there exists a unique $x \in Z(x_1 \cdots x_{n-1})$ such that $\tau_d(x) = y$. Notice that $x_1 \cdots x_{n-1}a \in A^n$ and $y_1 \cdots y_m \in A^m$ satisfy $d(x_1 \cdots x_{n-1}a) = y_1$. So since d is weakly progressive there exists a unique $a_1 \in A$ such that

$$p_m^{x_1 \cdots x_{n-1}}(a_1 \alpha) = d(x_1 \cdots x_{n-1}a_1)d(x_2 \cdots x_{n-1}a_1\alpha_1) \cdots = y_1 \cdots y_m$$

for some $\alpha \in A^{m-1}$. Now consider $x_2 \cdots x_{n-1}a_1\alpha_1 \in A^n$ and $y_2 \cdots y_{m+1} \in A^m$ such that $d(x_2 \cdots x_{n-1}a_1\alpha_1) = y_2$. Since d is weakly progressive there exists a unique $a_2 \in A$ such that

$$p_m^{x_1 \cdots x_{n-1}}(a_2 \beta) = d(x_2 \cdots x_{n-1}a_1a_2)d(x_3 \cdots x_{n-1}a_1a_2\beta_1) = y_2 \cdots y_{m+1}$$

for some $\beta \in A^{m-1}$.. We may continue in this manner to construct $x = x_1 \cdots x_{n-1}a_1a_2 \cdots$ such that $\tau_d(x) = y$, hence the function is surjective. Since each a_i was unique,

[2] If d is progressive, then d is weakly progressive of order 1.

$\tau_d|(Z(x_1 \cdots x_{n-1}))$ is injective. [3] $\square$

Theorem 3.3.25. *Let $d : A^n \to A$ be a function. If d is weakly progressive then τ_d is a local homeomorphism.*

Proof. By Proposition 3.3.24 $\tau_d(Z(x_1 \cdots x_{n-1}))$ is open since it equals $\bigcup_{a\in A} Z(d(x_1 \cdots x_{n-1}a))$. Since τ_d is continuous, its restriction to $Z(x_1 \cdots x_{n-1})$ is also continuous. Also since A^∞ is Hausdorff and $Z(x_1 \cdots x_{n-1})$ is a subspace of A^∞, then $Z(x_1 \cdots x_{n-1})$ is Hausdorff (see [15, Theorem 31.2 (a)]). So we know that $\tau_d|Z(x_1 \cdots x_{n-1})$ is a continuous bijective function from the compact space $Z(x_1 \cdots x_{n-1})$ to the Hausdorff space $\bigcup_{a\in A} Z(d(x_1 \cdots x_{n-1}a))$. By [15, Theorem 5.8], $\tau_d|_{Z(x_1 \cdots x_{n-1})}$ is a homeomorphism. Therefore τ_d is a local homeomorphism. $\square$

We have not yet been able to completely characterize the local homeomorphisms which commute with σ; that is we do not know if τ_d local homeomorphism implies d is weakly progressive.

3.3.6 Covering Maps

Most ideas in this section come directly from collaboration and generous help from Ruy Exel.

We prove that if a local homeomorphism ϕ *-commutes with σ then ϕ is a k-fold covering map.

Definition 3.3.26. Let ϕ be a function and $k \in \mathbb{N}$. We define the sets $Z_k^\phi := \{y : |\phi^{-1}(y)| = k\}$ and $Z_{\geq k}^\phi := \{y : |\phi^{-1}(y)| \geq k\}$.

Lemma 3.3.27. *Let ϕ *-commute with σ. Then for all $k \in \mathbb{N}$ we have $\sigma(Z_k^\phi) = (Z_k^\phi)$, that is $\sigma(y) \in Z_k^\phi$ if and only if $y \in Z_k^\phi$.* [4]

---

[3] Observe that if d is progressive, then $\bigcup_{a\in A} Z(d(x_1 \cdots x_{n-1}a)) = A^\infty$. Thus τ_d is $|A|^{n-1}$ to 1.

[4] This can be generalized to the statement: If ϕ and ψ *-commute, then $\psi(Z_k^\phi) = Z_k^\phi$ and $\phi(Z_l^\psi) = Z_l^\psi$ for all $k, l \in \mathbb{N}$.

Proof.

$$
\begin{tikzcd}
& x^i \arrow[dl, "\phi"'] \arrow[dr, "\sigma"] & \\
y \arrow[dr, "\sigma"'] & & z^i \arrow[dl, "\phi"] \\
& \sigma(y) &
\end{tikzcd}
$$

Let $y \in A^\infty$ and fix $k \in \mathbb{N}$ such that $\sigma(y) \in Z_k^\phi$. Then $\sigma(y)$ has k preimages under ϕ and we define $\{z^i\}_{i=1}^k = \phi^{-1}(\sigma(y))$. Since σ and ϕ *-commute, for each z^i there exists a unique x^i such that $\phi(x^i) = y$ and $\sigma(x^i) = z^i$. So $|\phi^{-1}(y)| \geq k$, but we want to show that $|\phi^{-1}(y)| = k$. Suppose $x \in \phi^{-1}(y)$. Then $\phi(\sigma(x)) = \sigma(\phi(x)) = \sigma(y)$. So $\sigma(x) \in \phi^{-1}(\sigma(y)) = \{z^i\}_{i=1}^k$, and there exists i such that $\sigma(x) = z^i$. Hence $\phi(x) = y$, $\sigma(x) = z^i$, but x^i is the unique element with those properties thus $x = x^i$. Therefore $|\phi^{-1}(y)| = k$ which means $y \in Z_k^\phi$.

$$
\begin{tikzcd}
& x^i \arrow[dl, "\phi"'] \arrow[dr, "\sigma"] & \\
y \arrow[dr, "\sigma"'] & & \sigma(x^i) \arrow[dl, "\phi"] \\
& \sigma(y) &
\end{tikzcd}
$$

Conversely, let $y \in A^\infty$ and fix $k \in \mathbb{N}$ such that $y \in Z_k^\phi$. Then define $\{x^i\}_{i=1}^k = \phi^{-1}(y)$. Suppose $w \in \phi^{-1}(\sigma(y))$. Since σ and ϕ *-commute there exists x such that $\phi(x) = y$ and $\sigma(x) = w$. However $\{x^i\}_{i=1}^k = \phi^{-1}(y)$ so $x = x^i$ for some i. So for each $w \in \phi^{-1}(\sigma(y))$, $w = \sigma(x^i)$ for some i. So $|\phi^{-1}(\sigma(y))| \leq k$. Suppose $|\phi^{-1}(\sigma(y))| < k$, then there exists $x^i, x^j \in \phi^{-1}(y)$ with $i \neq j$ such that $\sigma(x^i) = \sigma(x^j) = z$, say. So we have $y, z \in A^\infty$ such that $\sigma(y) = \phi(z)$ and $x^i \neq x^j$ such that $\phi(x^i) = y = \phi(x^j)$

and $\sigma(x^i) = z = \sigma(x^j)$. Since ϕ and σ *-commute $x^i = x^j$, which is a contradiction. Therefore $|\phi^{-1}(\sigma(y))| = k$ which means $\sigma(y) \in Z_k^\phi$. □

Lemma 3.3.28. *If $\phi : A^\infty \to A^\infty$ is a local homeomorphism, then for all $k \in \mathbb{N}$ $Z_{\geq k}$ is open in A^∞.*

Proof. Let $y \in A^\infty$ and fix $k \in \mathbb{N}$ such that $y \in Z_{\geq k}$. Then $y \in Z_l$ for some $l \geq k$. Then $\phi^{-1}(y) = \{x_i\}_{i=1}^l$ where each x_i is distinct. Since ϕ is a local homeomorphism, for each x_i there exists a neighborhood U_i containing x_i such that x_j is not an element of U_i for $i \neq j$, $y \in \phi(U_i)$ for each i, and $\phi(U_i)$ is open in A^∞. Since A^∞ is Hausdorff, let $x_i \in V_i$ for each i and $V_i \cap V_j = \emptyset$ for each $i \neq j$. Then define $W_i = V_i \cap U_i$. Thus the set $\{W_i\}_{i=1}^l$ are pairwise disjoint. Now let $W = \bigcap_{i=1}^l \phi(W_i)$. Then W is open in A^∞ and $y \in W$. We want to show that $W \subseteq Z_{\geq k}$. Let $z \in W$. Then $z \in \phi(W_i)$ for each i, so there exists $w_i \in W_i$ such that $\phi(w_i) = z$. Since $\{W_i\}_{i=1}^l$ is pairwise disjoint, the w_i are distinct. Therefore z has l preimages. Hence $z \in Z_l \subseteq Z_{\geq k}$, thus $W \subseteq Z_{\geq k}$. Therefore $Z_{\geq k}$ is open. □

Remark 3.3.29. It is important to recall that for A^∞, the only open sets U in A^∞ such that $\sigma(U) = U$ are $U = \emptyset$ or $U = A^\infty$. Any open set $U \neq \emptyset$ contains a basic open set $Z(x_1, \cdots, x_l)$. If we assume $\sigma(U) = U$, then $\sigma^l(U) = U$ for any $l \in \mathbb{N}$. So $\sigma^l(Z(x_1, \cdots, x_l)) = A^\infty \subseteq U$. So $U = A^\infty$.

Lemma 3.3.30. *If $\phi : A^\infty \to A^\infty$ is a local homeomorphism, then there exists an $M \in \mathbb{N}$ such that $\{Z(\mu) : \mu \in A^M\}$ is finite covering of disjoint sets of A^∞ such that ϕ is a homeomorphism on each $Z(\mu)$.*

Remark 3.3.31. When considering a covering $\{W_\alpha\}$ of A^∞ such that ϕ is a homeomorphism on each set, it suffices to consider a covering of basic sets $\{Z(\mu)\}$ such that ϕ is a homeomorphism on each set. This is because each W_α which is not a basic open set is an arbitrary union of basic sets, that is $W_\alpha = \bigcup_{\lambda \in \Lambda} Z_\lambda(\mu)$. So in the covering we may replace the set in question with the collection of basic open sets. Since each $Z_\lambda(\mu) \subseteq W_\alpha$, $\phi|_{Z_\lambda(\mu)}$ is still a homeomorphism.

Proof of Theorem 3.3.30. Let $\{W_\alpha\}$ be a covering basic open sets of A^∞ such that ϕ is a homeomorphism on each set and let $\{W_i\}_{i=1}^n$ be a finite subcover. So $W_i = Z(x_1^i \cdots x_{n_i}^i)$. Let $M = \max\{n_i\}$. Notice that

$$Z(x_1^i \cdots x_{n_i}^i) = \bigcup_{y_{n_i+1}\cdots y_M} Z(x_1^i \cdots x_{n_i}^i y_{n_i+1} \cdots y_M)$$

where this is a finite disjoint union. Now we have a finite cover $\{Z(\mu) : \mu \in A^M\}$ such that ϕ is a homeomorphism on each $Z(\mu)$. □

Proposition 3.3.32. *If ϕ is a local homeomorphism and *-commutes with σ, then there exists $k \in \mathbb{N}$ such that ϕ is k to 1 and surjective.*

Proof. Let $l \in \mathbb{N}$ and consider Z_l^ϕ. Notice that $Z_l^\phi = \emptyset$ if $l > |A|^M$ where M is from Lemma 3.3.30, so there exists $k \in \mathbb{N}$ such that $k = max\{l : Z_l^\phi \neq \emptyset\}$. By Lemma 3.3.27 we have $\sigma(Z_k^\phi) = Z_k^\phi$. Since $Z_l^\phi = \emptyset$ for $l > k$, we have $Z_k^\phi = Z_{\geq k}^\phi$. By Lemma 3.3.28 Z_k^ϕ is open and by Remark 3.3.29 $Z_k^\phi = A^\infty$ [5]. Thus every element of A^∞ has exactly k preimages under ϕ.

If ϕ is not surjective, then there exists $y \in Z_0$. Then $k = 0$ which means ϕ is not defined for any element in A^∞ which does not make sense. Thus ϕ must be surjective. □

Definition 3.3.33. Let $p : E \to B$ be a continuous surjective function. The open set U of B is said to be *evenly covered* by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U.

Definition 3.3.34. Let $p : E \to B$ be a continuous surjective function. If every point b of B has a neighborhood U that is evenly covered by p, the p is called a *covering map* and E is said to be a *covering space* of B. If $p^{-1}(b)$ has k elements

[5]Remark 3.3.29 which is used to show that $Z_k^\phi = A^\infty$ is the only part of the proof that relies on σ.

for every $b \in B$, then E is called a *k-fold covering* of B.

Remark 3.3.35. A covering map p is automatically a local homeomorphism (in general the converse does not hold), so we have shown that if p is a covering map which *-commutes with σ, then p is a k-fold covering map for some $k \in \mathbb{N}$. We have yet to find an example of a local homeomorphism which *-commutes with σ yet is not a covering map.

Example 3.3.36. Let A be a finite alphabet and A^∞ be the one-sided infinite words. Let $\sigma : A^\infty \to A^\infty$ be the unilateral shift (i.e. $\sigma(x_0x_1x_2\cdots) = x_1x_2\cdots$) and denote

$$Z(a) = \{x \in A^\infty : x = ay, y \in A^\infty\}.$$

Notice that $\sigma^{-1}(A^\infty) = \sqcup_{a\in A} Z(a)$ and $\sigma(Z(a)) = A^\infty$ for all $a \in A$. Therefore σ is a covering map. In particular, σ is a $|A|$-fold covering map.

Example 3.3.37. Let $B = \{a, b, c, d\}$ and define $d : B^2 \to B$ by

$$\begin{array}{llll} d(aa) = a & d(ba) = d & d(ca) = c & d(da) = b \\ d(ab) = a & d(bb) = d & d(cb) = c & d(db) = b \\ d(ac) = b & d(bc) = c & d(cc) = d & d(dc) = a \\ d(ad) = b & d(bd) = c & d(cd) = d & d(dd) = a, \end{array}$$

which is taken from Example 3.3.11. Define $V = Z(a) \cup Z(d)$ and $W = Z(b) \cup Z(c)$. With a little work one can check that

$$\begin{array}{ll} \tau_d^{-1}(V) = Z(a) \cup Z(d) & \tau_d^{-1}(W) = Z(b) \cup Z(c) \\ \tau_d(Z(a)) = V = \tau_d(Z(d)) & \tau_d(Z(b)) = W = \tau_d(Z(c)). \end{array}$$

So τ_d is a 2-fold covering map. We know that this map *-commutes with σ.

Example 3.3.38. Let $B = \{0, 1, 2, 3\}$ and define $d : B^2 \to B$ by

$$\begin{array}{llll} d(00) = 0 & d(10) = 2 & d(20) = 0 & d(30) = 2 \\ d(01) = 0 & d(11) = 2 & d(21) = 0 & d(31) = 2 \\ d(02) = 1 & d(12) = 3 & d(22) = 1 & d(32) = 3 \\ d(03) = 1 & d(13) = 3 & d(23) = 1 & d(33) = 3, \end{array}$$

which is taken from Example 3.3.12. Define $V = Z(0) \cup Z(1)$ and $W = Z(2) \cup Z(3)$. With a little work one can check that

$$\begin{array}{ll} \tau_d^{-1}(V) = Z(0) \cup Z(2) & \tau_d^{-1}(W) = Z(1) \cup Z(3) \\ \tau_d(Z(0)) = V = \tau_d(Z(2)) & \tau_d(Z(1)) = W = \tau_d(Z(3)). \end{array}$$

So τ_d is a 2-fold covering map. We know that this map does not *-commute with σ.

Remark 3.3.39. An important result from Proposition 3.3.32 that is useful to prove the next lemma is that if $\phi : A^\infty \to A^\infty$ is a local homeomorphism and *-commutes with σ, then there exists $k \in \mathbb{N}$ such that ϕ is k to 1 and surjective.

Lemma 3.3.40. *Let A be a finite alphabet. If $\phi : A^\infty \to A^\infty$ is a local homeomorphism and *-commutes with σ, then ϕ is a (k-fold) covering map.*

Proof. Since ϕ is a local homeomorphism it is continuous and by Proposition 3.3.32 ϕ is surjective. Let $y \in A^\infty$, then $\phi^{-1}(y) = \{x_i\}_{i=1}^k$ for some $k \in \mathbb{N}$ by Proposition 3.3.32. Since ϕ is a local homeomorphism there exists an open neighborhood W_i of x_i such that $\phi : W_i \to \phi(W_i)$ is a homeomorphism (and $\phi(W_i)$ is open in A^∞) for $i = 1, \cdots, k$. Since A^∞ is Hausdorff we may define open sets W_i' such that $x_i \in W_i' \subseteq W_i$ and $\{W_i'\}_{i=1}^k$ are pairwise disjoint. Notice that $\phi|_{W_i'}$ is a homeomorphism onto its image and $\phi(W_i')$ is open (since it is open in $\phi(W_i)$ which is open in A^∞). Let $U = \cap_{i=1}^k \phi(W_i')$ which is an open set containing y. Define $V_i = \phi^{-1}(U) \cap W_i'$, which is open and non-empty since x_i is in $\phi^{-1}(U)$ and W_i'. Also notice that $\{V_i\}_{i=1}^k$ are pairwise disjoint. So $\phi^{-1}(U) = \sqcup_{i=1}^k V_i$, $\phi|_{V_i}$ is a homeomorphism (since $V_i \subseteq W_i'$

and $\phi|_{W_i'}$ is a homeomorphism), and

$$\phi(V_i) = \phi(\phi^{-1}(U) \cap W_i') = U \cap \phi(W_i') = U$$

hence $\phi(V_i)$ is onto U. Therefore A^∞ is evenly covered by ϕ. Since y was arbitrary, ϕ is a (k-fold) covering map. $\square$

Theorem 3.3.41. *If ϕ is a local homeomorphism of A^∞ that *-commutes with σ, then there exists $n \in \mathbb{N}$ and a regressive function $d : A^n \to A$ such that $\tau_d = \phi$. Further, ϕ is a k-fold covering map for some $k \in \mathbb{N}$. Conversely, if $\tau_d = \phi$ for some regressive function d and if ϕ is a k-fold covering map, then ϕ is a local homeomorphism (by definition) on A^∞ that *-commutes with σ.*

Proof. Suppose ϕ is a local homeomorphism that *-commutes with σ. Then ϕ is continuous and commutes with σ, therefore by Lemma 3.3.7 there exists $n \in \mathbb{N}$ and $d : A^n \to A$ such that $\tau_d = \phi$. Now by Lemma 3.3.40 ϕ is a k-fold covering map.

Conversely, suppose there exists $n \in \mathbb{N}$ and a regressive function $d : A^n \to A$ such that $\tau_d = \phi$ is a k-fold covering map. Since ϕ is a covering map, it is a local homeomorphism. Since d is regressive, by Theorem 3.3.13 $\tau_d = \phi$ *-commutes with σ. $\square$

3.4 Characterization of 2-graphs whose shifts *-commute

This section is based on collaboration with Ben Maloney, David Pask, and Iain Raeburn.

We introduce a concept of coalignment, and prove that two surjective local homeomorphisms *-commute on the infinite path space of a 2-graph, in the sense of Exel and Renault, if and only if the 2-graph is 1-coaligned. We go on to classify completely all 2-graphs defined from *tile basic data* on which the one-sided unilateral shift maps *-commute.

Definition 3.4.1. A *k-graph* is a pair (Λ, d) consisting of a countable category Λ and a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the *factorization* *property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. See [24].

Definition 3.4.2. For $k \geq 1$, Ω_k is a category with unit space $\Omega_k^0 = \mathbb{N}^k$, morphism space $\Omega_k^* = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$, range map $r(m, n) = m$, and source map $s(m, n) = n$. Let $d : \Omega \to \mathbb{N}^k$ be defined by $d(m, n) = m - n$, then (Ω_k, d) is a k-graph, which we denote by Ω_k.

Definition 3.4.3. Let (Λ, d) be a k-graph. Then

$$\Lambda^\infty := \{x : \Omega_k \to \Lambda : x \text{ is a } k\text{-graph morphism}\}$$

is the *infinite path space* of Λ.

We denote sections of these paths with range $m \in \mathbb{N}^k$ and source $n \in \mathbb{N}^k$ by $x(m, n)$. For $p \in \mathbb{N}^k$, we define the map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m, n) = x(m + p, n + p)$ for $(m, n) \in \Omega_k^*$.

The following proposition was proved by Kumjian and Pask, [17, Proposition 2.3]. We will rely on it more than once.

Proposition 3.4.4. *Let Λ be a k-graph. For all $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $x(0, 0) = s(\lambda)$, there is a unique $y \in \Lambda^\infty$ such that $x = \sigma^{d(\lambda)}y$ and $\lambda = y(0, d(\lambda))$. We will write $y = \lambda x$. For every $x \in \Lambda^\infty$ and $p \in \mathbb{N}^k$, $x = x(0, p)\sigma^p x$.*

Remark 3.4.5. Let X, Y be topological spaces and $f : X \to Y$ be a function. If the image of every basis element in X is open in Y, then f is an open map. Similarly, if every preimage of a base element of Y is open in X, then f is continuous.

Definition 3.4.6. Given $\lambda, \mu \in \Lambda$, the *minimal common extension* of λ and μ is defined $MCE(\lambda, \mu) = \{(\alpha, \beta) : \lambda\alpha = \mu\beta \text{ and } d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$.

Remark 3.4.7. Observe that if $\lambda \neq \mu$ with $d(\lambda) = d(\mu)$, then the minimal common extension of λ, μ is empty. Additionally, $MCE(\lambda, \mu)$ is empty if $r(\lambda) \neq r(\mu)$, as $\lambda\alpha = \mu\beta$ implies $r(\lambda\alpha) = r(\mu\beta)$ which is true only if $r(\lambda) = r(\mu)$.

The following definition of a cylinder set is standard (see Kumjian and Pask [17, Definition 2.4]).

Definition 3.4.8. Suppose Λ is a k-graph. For $\lambda \in \Lambda$, we define the *cylinder set of* λ,

$$Z(\lambda) = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}. \tag{3.39}$$

Lemma 3.4.9. *For $\lambda, \mu \in \Lambda$, $Z(\lambda) \bigcap Z(\mu) = \bigcup_{(\alpha,\beta)\in MCE(\lambda,\mu)} Z(\lambda\alpha)$.*

Proof. We will prove this using a double containment argument, but first we shall deal with the trivial case: if $d(\lambda) \geq d(\mu)$, then $Z(\lambda) \subseteq Z(\mu)$, and so $Z(\lambda) \bigcap Z(\mu) = Z(\lambda)$. Hence degree-zero morphisms are in $MCE(\lambda, \mu)$, and $\bigcup_{(\alpha,\beta)\in MCE(\lambda,\mu)} Z(\lambda\alpha) = Z(\lambda)$. By symmetry, if $d(\lambda) \leq d(\mu)$, then $Z(\lambda) \bigcap Z(\mu) = Z(\mu)$.

Suppose $x \in \bigcup_{(\alpha,\beta)\in MCE(\lambda,\mu)} Z(\lambda\alpha)$. Then $x(0, d(\lambda)) = \lambda$, so $x \in Z(\lambda)$. Also, since $\lambda\alpha = \mu\beta$, $x(0, d(\mu)) = \mu$, so $x \in Z(\mu)$. Hence $\bigcup_{(\alpha,\beta)\in MCE(\lambda,\mu)} Z(\lambda\alpha) \subseteq Z(\lambda) \bigcap Z(\mu)$.

Suppose $x \in Z(\lambda) \bigcap Z(\mu)$, so $x(0, d(\lambda)) = \lambda$ and $x(0, d(\mu)) = \mu$. To avoid the trivial case, suppose $d(\lambda) \not\leq d(\mu)$ and $d(\mu) \not\leq d(\lambda)$. Let $d(\lambda) \vee d(\mu) = m$, say. The unique factorization property of the k-graph Λ implies that $x(0, m) = \lambda x(d(\lambda), m) = \mu x(d(\mu), m)$. Then $(x(d(\lambda), m), x(d(\mu), m))$ a pair in the minimal common extension, demonstrating $Z(\lambda) \bigcap Z(\mu) \subseteq \bigcup_{(\alpha,\beta)\in MCE(\lambda,\mu)} Z(\lambda\alpha)$. Hence we have $Z(\lambda) \bigcap Z(\mu) = \bigcup_{(\alpha,\beta)\in MCE(\lambda,\mu)} Z(\lambda\alpha)$. □

Corollary 3.4.10. *If $\lambda \neq \mu$ with $d(\lambda) = d(\mu)$, then $Z(\lambda) \bigcap Z(\mu) = \emptyset$.*

Proof. Apply Lemma 3.4.9. By Remark 3.4.7, if $\lambda \neq \mu$ with $d(\lambda) = d(\mu)$, then $MCE(\lambda, \mu)$ is empty. □

Lemma 3.4.11. *Let Λ be a row-finite k-graph with no sources or sinks.*

1. *For $\lambda \in \Lambda$, the compact sets $Z(\lambda)$ form a subbasis for a locally compact Hausdorff topology on Λ^∞.*

2. *For $i = 1, \ldots, k$, the maps $\sigma^{e_i} : \Lambda^\infty \to \Lambda^\infty$ defined by*

$$(\sigma^{e_i} x)(m, n) := x(m + e_i, n + e_i) \tag{3.40}$$

are surjective local homeomorphisms such that $\sigma^{e_i}\sigma^{e_j} = \sigma^{e_j}\sigma^{e_i}$ for $0 \leq i, j \leq k$.

Proof. For assertion (1), observing that for all $x \in \Lambda^\infty$, $x \in Z(r(x(0,0)))$, the set $\{Z(\lambda) : \lambda \in \Lambda\}$ is a subbasis for Λ^∞. The set $\{Z(r(x(0,0)))\}$ is compact for all $\lambda \in \Lambda$ by [17, Lemma 2.6]. Thus Λ^∞ is a locally compact space. To show that Λ^∞ is Hausdorff, suppose $x \neq y \in \Lambda^\infty$; then there exists $(m,n) \in \Omega_k$ such that $x(m,n) \neq y(m,n)$. Hence $x \in Z(x(0,n))$, $y \in Z(y(0,n))$, but since $d(x(0,n)) = d(y(0,n))$, by Corollary 3.4.10, $Z(x(0,n)) \bigcap Z(y(0,n)) = \emptyset$. Therefore, the topology generated by taking arbitrary unions of finite intersections of elements of the form $Z(\lambda)$ is a locally compact Hausdorff topology.

For assertion (2) of the lemma, we begin by claiming σ^{e_i} is surjective. Take $y \in \Lambda^\infty$; since Λ has no sinks, we can take $e \in \Lambda^{e_i} r(y)$ so that $s(e) = r(y)$. Then we use Proposition 3.4.4 to see that there exists a unique $z \in \Lambda^\infty$ such that $z(0, e_i) = e$ and $z(m + e_i, n + e_i) = y(m,n)$ for all $m, n \in \Omega_k$. Then $\sigma^{e_i} z = y$ by definition.

We now show σ^{e_i} is continuous. We need to show that the image by $(\sigma^{e_i})^{-1}$ of an open set is open. By Remark 3.4.5, it suffices to show that $(\sigma^{e_i})^{-1} Z(\lambda) = \bigsqcup_{e \in \Lambda^{e_i} r(\lambda)} Z(e\lambda)$. Suppose $x \in Z(e\lambda)$ for some e, then $\sigma^{e_i} x \in Z(\lambda)$. Hence $(\sigma^{e_i})^{-1} Z(\lambda) \supseteq \bigsqcup_{e \in \Lambda^{e_i} r(\lambda)} Z(e\lambda)$. Now suppose $x \in (\sigma^{e_i})^{-1} Z(\lambda)$, so $x = x(0, e_i)\lambda\sigma^{d(\lambda)}(x)$ where $x(0, e_i) \in \Lambda^{e_i} r(\lambda)$. Hence $x \in \bigsqcup_{e \in \Lambda^{e_i} r(\lambda)} Z(e\lambda)$. Therefore $(\sigma^{e_i})^{-1} Z(\lambda) \subseteq \bigsqcup_{e \in \Lambda^{e_i} r(\lambda)} Z(e\lambda)$. So σ^{e_1} is continuous.

Proving that σ^{e_i} is a local homeomorphism: let $x \in \Lambda^\infty$; by Proposition 3.4.4 we know $x = x(0, e_i)\sigma^{e_i}(x)$. We denote $x(0, e_i) = e_x$ and show that $Z(e_x)$ is a neighbourhood of x such that $\sigma^{e_i}(Z(e_x)) = Z(s(e_x))$ is open and $\sigma^{e_i} : Z(e_x) \to Z(s(e_x))$ is a homeomorphism. It is immediate that $x \in Z(e_x)$ and that $\sigma^{e_i}(x) \in Z(s(e_x))$, hence $\sigma^{e_i}(Z(e_x)) \subseteq Z(s(e_x))$. Now suppose $y \in Z(s(e_x))$; since Λ has no sources, there exists $z \in \Lambda^\infty$ such that $z = z(0, e_i)y$. Then $y \in \sigma^{e_i}(Z(e_x))$. Therefore $\sigma^{e_i}(Z(e_x)) = Z(s(e_x))$ is open and onto its range, and the map is therefore surjective. To see that $\sigma^{e_i}|_{Z(e_x)}$ is injective: let $y, z \in Z(e_x)$ such that $\sigma^{e_i}(y) = \sigma^{e_i}(z)$. Then by Proposition 3.4.4, we have $y = x(0, e_i)\sigma^{e_i}(y) = x(0, e_i)\sigma^{e_i}(z) = z$.

The restriction $\sigma^{e_i}|_{Z(e_x)}$ is continuous because it is the restriction of a continuous map, and has a continuous inverse because it is a bijection from a compact space to a Hausdorff space. Hence $\sigma^{e_i}|_{Z(e_x)}$ is a homeomorphism, and by definition $\sigma^{e_i} : \Lambda^\infty \to \Lambda^\infty$ is a local homeomorphism. For $x \in \Lambda^\infty$,

$$(\sigma^{e_i}\sigma^{e_j}x)(m,n) = \sigma^{e_i}x(m+e_j, n+e_j) = x(m+e_j+e_i, n+e_j+e_i)$$

$$(\sigma^{e_j}\sigma^{e_i}x)(m,n) = \sigma^{e_j}x(m+e_i, n+e_i) = x(m+e_i+e_j, n+e_i+e_j).$$

However, addition in $\mathbb{N}^2$ is commutative, so $\sigma^{e_i}\sigma^{e_j}(x) = \sigma^{e_j}\sigma^{e_i}(x)$, which demonstrates the last part of the lemma. □

We wish to provide necessary and sufficient conditions for $\sigma^{e_1}, \sigma^{e_2} : \Lambda^\infty \to \Lambda^\infty$ to be *-commuting.

Lemma 3.4.12. *Let (Λ, d) be a 2-graph and let σ^{e_i} to be defined as equation* (3.40)*. If for each pair $(e^1, e^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ with $s(e^1) = s(e^2)$ there is a unique pair $(f^1, f^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ such that $f^1e^2 = f^2e^1$, then $\sigma^{e_1}, \sigma^{e_2} : \Lambda^\infty \to \Lambda^\infty$ *-commute.*

Proof. By Lemma 3.4.11, $\sigma^{e_1}, \sigma^{e_2}$ commute. Suppose $y, z \in \Lambda^\infty$ such that $\sigma^{e_1}(y) = \sigma^{e_2}(z) = w$, say. Take $y(0, e_1) = e^1$ and $z(0, e_2) = e^2$, which is a pair in $\Lambda^{e_1} \times \Lambda^{e_2}$, and associate to it the unique pair (f^1, f^2) such that $f^1e^2 = f^2e^1$ is a path in Λ of degree $e_1 + e_2$. Define $x := f^2e^1w$ (equivalently equals f^1e^2w), then $\sigma^{e_1}(x) = \sigma^{e_1}(f^1e^2w) = e^2w = z$ and $\sigma^{e_2}(x) = \sigma^{e_2}(f^2e^1w) = e^1w = y$. Hence $\sigma^{e_1}, \sigma^{e_2} : \Lambda^\infty \to \Lambda^\infty$ *-commute. □

The converse is the challenge.

Lemma 3.4.13. *If $\sigma^{e_1}, \sigma^{e_2} : \Lambda^\infty \to \Lambda^\infty$ are *-commuting maps, then for each pair $(e^1, e^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ with $s(e^1) = s(e^2)$, there exists a unique pair $(f^1, f^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ such that $f^1e^2 = f^2e^1$.*

Proof. Fix $(e^1, e^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ with $s(e^1) = s(e^2)$. Since the cylinder set $Z(s(e^1))$ is non-empty, there exists $w \in Z(s(e^1))$. Define $y := e^1w$ and $z := e^2w$. Since $\sigma^{e_1}(y) = w = \sigma^{e_2}(z)$, there exists a unique $x \in \Lambda^\infty$ such that $\sigma^{e_1}(x) = z$ and

$\sigma^{e_2}(x) = y$. Define $f^1 := x(0, e_1)$ and $f^2 := x(0, e_2)$. So Proposition 3.4.4 gives us

$$x = x(0, e_1)\sigma^{e_1}(x) = f^1 z = f^1 e^2 w \text{ and } x = x(0, e_2)\sigma^{e_2}(x) = f^2 y = f^2 e^1 w.$$

Hence $(f^1, f^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ satisfies $f^1 e^2 = f^2 e^1$.

To demonstrate uniqueness, suppose there exists a pair $(g^1, g^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ such that $g^1 e^2 = g^2 e^1$. Then $g^1 e^2 w = g^2 e^1 w$ and $\sigma^{e_1}(g^1 e^2 w) = e^2 w = z$ and $\sigma^{e_2}(g^2 e^1 w) = e^1 w = y$. Since $\sigma^{e_1}, \sigma^{e_2}$ *-commute, $g^1 e^2 w = x = g^2 e^1 w$. By Proposition 3.4.4, $g^1 = x(0, e_1) = f^1$ and $g^2 = x(0, e_2) = f^2$. □

Definition 3.4.14. A k-graph Λ is 1-*coaligned* if for every pair $(e^1, e^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ with $s(e^1) = s(e^2)$ there exists a unique pair $(f^1, f^2) \in \Lambda^{e_1} \times \Lambda^{e_2}$ such that $f^1 e^2 = f^2 e^1$.

We are now able to present our classification theorem.

Theorem 3.4.15. *Suppose (Λ, d) is a 2-graph, and $\sigma^{e_i} : \Lambda^\infty \to \Lambda^\infty$ is the one-sided unilateral shift in the direction e_i. The maps $\sigma^{e_1}, \sigma^{e_2}$ *-commute if and only if (Λ, d) is 1-coaligned.*

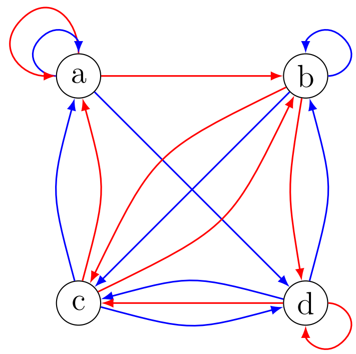
Proof. Apply Lemma 3.4.12 and Lemma 3.4.13. □

We consider an example motivated by work by Ledrappier [19], the details of which are provided by Pask, Raeburn and Weaver [24, 31].

Example 3.4.16. Tile the plane with socks. Let the sum of the entries of each sock tile equal zero modulo two. So of the eight possible permutations, only four exist that meet this criteria. That is,

$$\begin{array}{|c|cc}\hline 0 & & a \\ \hline 0 & \multicolumn{1}{c|}{0} & \\ \hline \end{array} \qquad \begin{array}{|c|cc}\hline 0 & & b \\ \hline 1 & \multicolumn{1}{c|}{1} & \\ \hline \end{array} \qquad \begin{array}{|c|cc}\hline 1 & & c \\ \hline 0 & \multicolumn{1}{c|}{1} & \\ \hline \end{array} \qquad \begin{array}{|c|cc}\hline 1 & & d \\ \hline 1 & \multicolumn{1}{c|}{0} & \\ \hline \end{array}$$

We can then draw the 1-skeleton of the associated 2-graph.



With a little work, one can check that this example is 1-coaligned, therefore the unilaterial shifts $\sigma^{e_1}, \sigma^{e_2}$ *-commute on this particular 2-graph.

We expand our field of interest to consider a general 2-graph defined from tiles that obey some basic rules (see [24]).

Definition 3.4.17. A subset T of $\mathbb{N}^2$ is *hereditary* if for $j \in T$, each i such that $0 \leq i \leq j$, $i \in T$.

Definition 3.4.18. There are four variables that make up the *basic data*:

- a finite hereditary subset of $\mathbb{N}^2$, called the *tile*, T.
- an *alphabet* $\{0, \ldots, q-1\}$, identified with $\mathbb{Z}_q$. [6]
- an element t of the alphabet called the *trace*.
- a weight function $w : T \to \{0, \ldots, q-1\}$ called the *rule*.

The basic data is denoted (T, q, t, w).

From the basic data, we aim to define a 2-graph, $\Lambda(T, q, t, w)$. We combine the four elements of the basic data to define the vertex set of $\Lambda(T, q, t, w)$:

$$\Lambda^0 := \{v : T \to \mathbb{Z}_q : \sum_{i \in T} w(i)v(i) \equiv t \pmod{q}\}. \tag{3.41}$$

There is a simple condition to be satisfied to create a 2-graph from basic data.

Definition 3.4.19. Let $(c_1, c_2) := \bigvee\{i : i \in T\}$. The rule w has *invertible corners* if $w(c_1e_1)$ and $w(c_2e_2)$ are invertible elements of $\mathbb{Z}_q$.

---

[6]We use $\mathbb{Z}_q$ to denote the ring $\mathbb{Z}/q\mathbb{Z}$. Recall that for $w \in \mathbb{Z}_q$, the principal ideal generated by w is denoted (w).

To describe paths of tiles, some more notation is required. We follow the definitions and notations of [24, §3]. For $S \subset \mathbb{Z}^2$ and $n \in \mathbb{Z}^2$, define the translate of S by n by $S + n := \{i + n : i \in S\}$. Set $T(n) := \bigcup_{0 \le m \le n} T + m$. From a function $f : S \to \mathbb{Z}_q$ defined on a subset S of $\mathbb{N}^2$ containing $T + n$, we define $f|_{T+n} : T \to \mathbb{Z}_q$ by $f_{T+n}(i) = f(i + n)$ for $i \in T$. A *path of degree* n is a function $\lambda : T(n) \to \mathbb{Z}_q$ such that $\lambda|_{T+m}$ is a vertex for $0 \le m \le n$, with source $s(\lambda) = \lambda|_{T+n}$ and range $r(\lambda) = \lambda|_T$. We defined Λ^0 previously; define $\Lambda^* := \bigcup_{n \in \mathbb{N}^2} \Lambda^n$.

A result of Pask, Raeburn and Weaver [24, Theorem 3.4] proves that the existence of invertible corners for the rule is a sufficient condition to be able to define a 2-graph from basic data.

Theorem 3.4.20 (Pask, Raeburn, Weaver)**.** *Suppose we have basic data* (T, q, t, w) *and the rule* w *has invertible corners. Say that* $\mu \in \Lambda^m$ *and* $\nu \in \Lambda^n$ *are composable if* $s(\mu) = r(\nu)$ *and the composition is the unique path* λ *satisfying* $\lambda(0, m) = \mu$ *and* $\lambda(m, m + n) = \nu$*. Define* $d : \Lambda \to \mathbb{N}^2$ *by* $d(\lambda) = n$ *for* $\lambda \in \Lambda^n$*. Then with* $\Lambda^0, \Lambda^*, r, s$ *defined previously,* $\Lambda(T, q, t, w) := ((\Lambda^0, \Lambda^*, r, s), d)$ *is a 2-graph.*

Suppose the tile T has invertible corners. Our goal is to give a condition regarding the basic data that guarantees the one-sided unilateral shifts *-commute. We would like a condition that allows us to apply Theorem 3.4.15.

Lemma 3.4.21. *Let* $w \in \mathbb{Z}_q$*. Then* $wx \equiv m \pmod{q}$ *has a unique solution* x *for each* $m \in \mathbb{Z}_q$ *if and only if* w *is invertible in* $\mathbb{Z}_q$*.*

Proof. Fix $w \in \mathbb{Z}_q$ and suppose $wx \equiv m \pmod{q}$ has a unique solution $x \in \mathbb{Z}_q$ for each $m \in \mathbb{Z}_q$. We want to show that w is invertible in $\mathbb{Z}_q$. If q is prime, then w must be invertible. Suppose q is not prime; we know $(w) = \mathbb{Z}_q$ since $m \in (w)$ for all $m \in \mathbb{Z}_q$. So in particular $1 \in (w)$, hence w is invertible.

Conversely, suppose w is invertible in $\mathbb{Z}_q$. Define $x_m := w^{-1}m$. Then $wx_m = ww^{-1}m = m$. To see that x_m is the unique solution, suppose $wy \equiv m \pmod{q}$. Then $w(y - x) = wy - wx = m - m = 0$. Since w is invertible, w is not a zero divisor, therefore $y - x \equiv 0$; hence $y = x$ in $\mathbb{Z}_q$. $\square$

Before we present our classification theorem, we set up some notation. Take $n = (n_1, n_2)$. Given e^b, e^r with common source, we aim to define a path $\lambda : T(e_1 + e_2) \to \mathbb{Z}_q$. Recall that $T(e_1+e_2) = T \cup (T+e_1) \cup (T+e_2) \cup (T+e_1+e_2)$. For $n \in T$, define $v : T\backslash\{0,0\} \to \mathbb{Z}_q$:

$$v(n) = \begin{cases} e^b(n-e_1) & \text{if } n_1 > 0 \\ e^r(n-e_2) & \text{if } n_2 > 0. \end{cases} \tag{3.42}$$

From the basic data, equation (3.41), we define:

$$v(0,0) = w(0,0)^{-1}(t - \sum_{i \in T\backslash\{0,0\}} w(i)v(i)). \tag{3.43}$$

If $v(0,0) \in \mathbb{Z}_q$, then v is a vertex in Λ^0.

For $n \in T(e_1 + e_2)$, define

$$\lambda(n) = \begin{cases} e^b(n-e_1) & : n_1 > 0 \\ e^r(n-e_2) & : n_2 > 0 \\ v(0,0) & : n = (0,0). \end{cases} \tag{3.44}$$

We wish to show that λ is a path of degree $e_1 + e_2$, and it is the only one with $\lambda(e_1, e_1+e_2) = e^r$ and $\lambda(e_2, e_1+e_2) = e^b$. To see that λ is a path, it is sufficient to show $\lambda|_T = v$ is a vertex, $\lambda|_{T+e_1} = r(e^r)$, $\lambda|_{T+e_2} = r(e^b)$ and $\lambda|_{T+e_1+e_2} = s(e^r) = s(e^b)$. That $\lambda|_{T+e_1} = r(e^r)$, $\lambda|_{T+e_2} = r(e^b)$ and $\lambda|_{T+e_1+e_2} = s(e^r) = s(e^b)$ are immediate from the definition of λ, equation (3.44). The challenge is to show that $\lambda|_T = v$ is a vertex.

Theorem 3.4.22. *Suppose $\Lambda(T, q, t, w)$ is a 2-graph with invertible corners. Then $\Lambda(T, q, t, w)$ is 1-coaligned if and only if $w(0,0)$ is invertible in $\mathbb{Z}_q$ for $0 \in T$.*

Proof. Suppose $w(0,0)$ is invertible in $\mathbb{Z}_q$. We wish to show that Λ is 1-coaligned. We need to show there exists a pair $(f^b, f^r) \in \Lambda^{e_1} \times \Lambda^{e_2}$ such that $f^b e^r = f^r e^b$, and then demonstrate uniqueness. Since $w(0,0)$ is invertible, there exists a vertex v defined by equation (3.42) and equation (3.43). For $n \in T$, we have that $\lambda(n) = v(n)$, and so $\lambda|_T = v$. If we define $f^b = \lambda(0, e_1)$ and $f^r(0, e_2)$, then $f^r e^b = f^b e^r$, because they are both degree $e_1 + e_2$ factorisations of λ. By applying Lemma 3.4.21 to Equation (3.43) we know that $v(0,0)$ is unique, this implies that f^b and f^r are unique.

Conversely, suppose that $w(0,0)$ not invertible in $\mathbb{Z}_q$. We wish to show either there exist multiple possibilities for the path λ, or no path at all. The path λ is defined for all $n \in T(e_1 + e_2)\backslash\{0,0\}$ by the fixed edges e^b, e^r, so the existence and uniqueness of λ is dependent on the existence and uniqueness of $v(0,0)$. Since $w(0,0)$ is assumed not to be invertible, Lemma 3.4.21 implies that Equation (3.43) does not have a unique solution contradicting the uniqueness requirement for Λ to be 1-coaligned. $\square$

Corollary 3.4.23. *The one-sided unilateral shifts $\sigma^{e_1}, \sigma^{e_2}$ on $\Lambda(T,q,t,w)^\infty$ *-commute if and only if $w(0,0)$ is invertible in $\mathbb{Z}_q$.*

Proof. Applying Proposition 3.4.22, Lemma 3.4.21, and Theorem 3.4.15, we have the one-sided unilateral shifts *-commute on $\Lambda(T,w,t,q)$ if and only if $\Lambda(T,w,t,q)$ is 1-coaligned, which is true if and only if $w(0,0)$ is invertible in $\mathbb{Z}_q$. $\square$

This generates a host of examples

- Given basic data (T,q,t,w) and invertible corners, whenever $w(0,0) = 1$, the shift maps will *-commute.

- Given any alphabet of cardinality prime q, for any non-zero rule, the shift maps will *-commute.

If we wish to generalize these results further we first need a formulation of aperiodicity; this one is due to Robertson and Sims [30, §2]

Definition 3.4.24. A 2-graph Λ is *aperiodic* if for every $v \in \Lambda^0$ and $m,n \in \mathbb{N}^2$ with $m \neq n$, there is a path $\lambda \in \Lambda$ satisfying $r(\lambda) = v$, $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

An interesting subset of our examples are 1-coaligned 2-graphs formed from basic data $(T,q,0,w)$. According to Pask, Raeburn and Weaver [24, Theorem 5.2],

$\Lambda(T, q, 0, w)$ is aperiodic. The associated C^*-algebra $C^*(\Lambda(T, q, 0, w))$ is unital, nuclear, simple, purely infinite, and belongs to the bootstrap class $\mathcal{N}$ by Pask, Raeburn and Weaver [24, Theorem 6.1]. This implies that $C^*(\Lambda)$ is determined up to isomorphism by its K-theory (according to Kirchberg and Phillips).

A further restriction on the rule w is required to assure aperiodicity: w has *three invertible corners* if $w(0,0), w(c_1e_1), w(c_2e_2)$ are all invertible in $\mathbb{Z}_q$. Remark 5.5 of [24] indicates that we may reduce to the $t = 0$ case provided there is a constant vertex – that is, there exists v_0 such that $v_0(m) = c$ for all $m \in T$. This is equivalent to the existence of $c \in \mathbb{Z}_q$ such that $c(\sum_{i\in T} w(i)) = t \pmod{q}$.

REFERENCES

[1] V. Arzumanian and J. Renault. Examples of pseudogroups and their C*-algebras. *Operator Algebras and Quantum Field Theory*, pages 93–104, 1996.

[2] T. Bates, D. Gow, and D. Pask. Corrigendum to C*-algebras of labeled graphs. in preparation.

[3] T. Bates and D. Pask. C*-algebras of labelled graphs. *Journal of Operator Theory*, 57:207–226, 2007.

[4] N. Brownlowe and I. Raeburn. Exel's crossed product and relative Cuntz-Pimsner algebras. *Math. Proc. Camb. Phil. Soc*, 141:497–508, 2006.

[5] J. Cuntz and W. Krieger. A class of C*-algebras and topological Markov chains. *Invent. Math.*, 56:251–268, 1980.

[6] W Dicks and M. Dunwoody. *Groups Acting on Graphs.* Cambridge University Press, group actions, directed graphs 1989.

[7] R. Exel. A new look at the crossed-product of a C*-algebra by an endomorphism. *Ergod. Th. & Dynam. Sys.*, 23:1737–1771, 2003.

[8] R. Exel and I. Raeburn. Semigroups of local homeomorphisms and interaction groups. *Ergod. Th. & Dynam. Sys.*, 27:1–18, 2007.

[9] N. Fowler. Compactly-aligned discrete product systems and generalization of $\mathcal{O}_\infty$. *Internat J. Math.*, 10(6):721–738, 1999.

[10] N. Fowler. Discrete product systems of Hilbert bimodules. *Pacific J. Math.*, 204:335–375, 2002.

[11] J. L. Gross and T. W. Tucker. *Topological graph theory.* Series in Discrete Mathematics and Optimization. Wiley International, 1987.

[12] G. A. Hedlund. Endomorphisms and automorphisms of the shift dynamical systems. *Math. Systems Theory*, 3:320–375, 1969.

[13] S. Kaliszewski, J. Quigg, and I. Raeburn. Skew products and crossed products by coactions. *J. Operator Theory*, 46:411–433, 2001.

[14] Y. Katayama. Takesaki's duality for a non-degenerate co-action. *Math. Scand.*, 55:141–151, 1985.

[15] T. Katsura. A class of C*-algebras generlizing both graph algebras and homeomorphism c^*-algebras i. *Trans. Amer. Math. Soc.*, 356(11):4287–4322, 2004.

[16] A. Kumjian and D. Pask. C*-algebras of directed graphs and group actions. *Ergod. Th. & Dynam. Sys.*, 19:1503–1519, 1999.

[17] A. Kumjian and D. Pask. Higher rank graph C*-algebras. *New York J. Math.*, 6:1–20, 2000.

[18] N. Larsen. Crossed products by abelian semigroups via transfer operators. *Cambridge University Press*, July 2009.

[19] F. Ledrappier. Un champ markovian peut être d'entropie nulle et malangént. *C.R. Acad. Sci. Paris Sér. I Math.*, 287:561–562, 1978.

[20] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding.* Cambridge University Press, 1995.

[21] J. R. Munkres. *Topology.* Prentice Hall, 2nd edition, 1975.

[22] W.L. Paschle. The crossed product of a C*-algebra by an endomorphism. *Proc. Amer. Math. Soc.*, 80:113–118, 1980.

[23] D. Pask and I. Raeburn. Symmetric imprimivity theorems for graph C*-algebras. *Internat J. Math.*, 12:609–623, 2001.

[24] D. Pask, I. Raeburn, and N. Weaver. A family of 2-graphs arising from two-dimensional subshifts. *Ergod. Th. & Dynam. Sys.*, 29(5):1613–1639, 2009.

[25] D. Pask and S-J Rho. Some intrinsic properties of simple graph C*-algebras. pages 325–340. Proceedings of the conference on Operator Algebras and Mathematical Physics, Constanta 2001, Theta, Bucharest, 2003.

[26] J. Quigg. Discrete C*-coactions and C*-algebraic bundles. *J. Austral. Math. Soc.*, 60(A):204–221, 1996.

[27] I. Raeburn. Graph algebras. volume 103 of *CBMS Regional Conference Series in Mathematics.* Amer. Math. Soc., 2005.

[28] I. Raeburn and W. Szymański. Cuntz-Krieger algebra of infinite graphs and matrices. *Trans. Amer. Math. Soc.*, 356:39–59, 2003.

[29] I. Raeburn and D. P. Williams. *Morita equivalence and continuous-trace C*-algebras*, volume 60. Amer. Math. Soc., Providence, 1998.

[30] D. Roberston and A. Sims. Simplicity of C*-algebras associated to higher-rank graphs. *Bull. Lond. Math. Soc*, 39(2):337–344, 2007.

[31] N. Weaver. An aperiodic 2-graph: Honours thesis. Master's thesis, University of Newcastle, 2005.