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Conjugate diameters: Apollonius of Perga and Eutocius of Ascalon

Colin Bryan Powell McKinney
University of Iowa

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CONJUGATE DIAMETERS: APOLLONIUS OF PERGA AND EUTOCIUS OF
ASCALON

by

Colin Bryan Powell McKinney

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

July 2010

Thesis Supervisor: Professor Daniel D. Anderson

ABSTRACT

The *Conics* of Apollonius remains a central work of Greek mathematics to this day. Despite this, much recent scholarship has neglected the *Conics* in favor of works of Archimedes. While these are no less important in their own right, a full understanding of the Greek mathematical corpus cannot be bereft of systematic studies of the *Conics*. However, recent scholarship on Archimedes has revealed that the role of secondary commentaries is also important. In this thesis, I provide a translation of Eutocius' commentary on the *Conics*, demonstrating the interplay between the two works and their authors as what I call *conjugate*. I also give a treatment on the duplication problem and on compound ratios, topics which are tightly linked to the *Conics* and the rest of the Greek mathematical corpus. My discussion of the duplication problem also includes two computer programs useful for visualizing Archytas' and Eratosthenes' solutions.

Abstract Approved: _____
Thesis Supervisor

Title and Department

Date

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A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree in Mathematics
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July 2010

Thesis Supervisor: Professor Daniel D. Anderson

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Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Colin Bryan Powell McKinney

has been approved by the Examining Committee
for the thesis requirement for the Doctor of
Philosophy degree in Mathematics at the July 2010
graduation.

Thesis Committee: _____
Daniel D. Anderson, Thesis Supervisor

Craig A. Gibson

Paul S. Muhly

Philip C. Kutzko

Palle E. T. Jorgensen

Οὐκοῦν τοῦτο ἔτι διομολογητέον; Τὸ ποῖον; Ὡς τοῦ
ἀεὶ ὄντος γνώσεως, ἀλλ' οὐ τοῦ ποτέ τι γιγνομένου
καὶ ἀπολλυμένου. Εὐομολόγητον, ἔφη· τοῦ γὰρ ἀεὶ
ὄντος ἡ γεωμετρικὴ γνώσις ἐστίν. Ὅκλον ἄρα, ὦ
γενναῖε, ψυχῆς πρὸς ἀλήθειαν εἶη ἂν καὶ ἀπεργαστικὸν
φιλοσόφου διανοίας πρὸς τὸ ἄνω σχεῖν ἂ νυν κάτω
οὐ δέον ἔχομεν. Ὡς οἶόν τε μάλιστα, ἔφη.

ΠΛΑΤΩΝ, ΠΟΛΙΤΕΙΑ Ζ'

ABSTRACT

The *Conics* of Apollonius remains a central work of Greek mathematics to this day. Despite this, much recent scholarship has neglected the *Conics* in favor of works of Archimedes. While these are no less important in their own right, a full understanding of the Greek mathematical corpus cannot be bereft of systematic studies of the *Conics*. However, recent scholarship on Archimedes has revealed that the role of secondary commentaries is also important. In this thesis, I provide a translation of Eutocius' commentary on the *Conics*, demonstrating the interplay between the two works and their authors as what I call *conjugate*. I also give a treatment on the duplication problem and on compound ratios, topics which are tightly linked to the *Conics* and the rest of the Greek mathematical corpus. My discussion of the duplication problem also includes two computer programs useful for visualizing Archytas' and Eratosthenes' solutions.

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CHAPTER 1 INTRODUCTION AND HISTORICAL OVERVIEW

1.1 A Brief Historical Overview

This thesis approaches several periods within the history of Greek mathematics. They are all mathematically related, of course; but I focus on them as, say, mathematical vignettes.

The Greek mathematical period ranges a considerable quantity of time, from Thales of Miletus (7th century BCE), through Pythagoras of Samos (5th century BCE) and moving to Eudoxus of Cnidus, Archytas of Tarentum, and others who were associates of Plato. After the end of the Athenian hegemony and the conquest of Alexander, our story moves to the Hellenistic period, where the center of mathematical activity was the newly founded city of Alexandria on the coast of the now-Ptolemaic realm of Egypt. Euclid and Apollonius are two key figures here; we know that both lived and worked primarily in Alexandria. And while Archimedes mostly kept to himself in Syracuse, in Sicily, he was in contact with mathematicians in Alexandria such as Eratosthenes of Cyrene, near the western extent of Ptolemaic Egypt.

With the Roman conquests of mainland Greece, the cession of the Attalid empire in Anatolia to Rome, and the eventual defeat of Marcus Antonius and Cleopatra, the so-called Hellenistic era came to a close. Several notable Greek mathematicians emerge, though, after the founding of the Roman Empire: chief amongst them, of course, is Claudius Ptolemaeus, whose *Syntaxis* formed the mathematical and physical explanation for the geocentric Greco-Roman cosmos.

Another important class of authors appears during the Roman (and, after the division of the Empire, the Byzantine) period. These are the great commentators, who aimed to organize and expound upon the mathematical wisdoms of their ancestors. Notable here are Pappus of Alexandria, Theon of Alexandria and his daughter Hypatia, Proclus Lycaeus

(also called Proclus Diadochus, *the Successor*), and lastly, Eutocius of Ascalon. These mathematicians wrote commentaries on works such as the *Elements*, the Archimedean corpus, and the *Conics*; in addition, they also edited and compiled new editions of these texts.

Eutocius specifically edited the *Conics* and several of Archimedes' works; his edition and commentaries on the *Conics* form the basis for our story.

1.2 Eutocius, Apollonius, and Archimedes

In the current literature, Archimedes is by far a more popular author to study. This in itself is not surprising, for several reasons. First, Archimedean works are smaller and self-contained; each approaching a specific mathematical concept with distinct goals in mind. Second, of course, is the re-discovery of the Archimedes Palimpsest. After its purchase at auction by an anonymous buyer, the Palimpsest came to reside at the Walters Art Museum in Baltimore, where, under the careful study of the Museum's staff, the text has been thoroughly analyzed. The rediscovery of the Palimpsest has allowed Reviel Netz to publish a new English edition of *On the Sphere and the Cylinder*, together with Eutocius' commentaries thereon ([16, 24]); future works are forthcoming.

With the exception of Michael N. Fried and Sabetai Unguru's recent book ([3]), and a few papers by Ken Saito, Apollonius' *Conics* has largely been neglected in recent times: the last major work on the subject was that of H.G. Zeuthen (in [28, 29]). It was not even until Fried's dissertation that Book IV of *Conics* was available in English (now published in [22]; Books I-III had been translated from the Greek by R. Catesby Taliaferro at the request of St. John's College in Annapolis, Maryland. Books V-VII, which are extant only from the Arabic tradition, were translated by Gerald Toomer ([19, 20]). This left the very awkward situation of Book IV being the lone untranslated extant book from the Greek.

The lack of attention to Apollonius, though, is not altogether surprising. As Saito quite eloquently puts it (in [27], his review of [3]):

If we define a classic as “something that everybody wants to have read and nobody wants to read” (Mark Twain), then there is no doubt that Apollonius’

Conica is the classic of the classics in Greek mathematics. There have been few people who have gone through this long and difficult text during the 20th century; one-third of those who have gone through Archimedes may be too optimistic an estimate. The Archimedean Corpus, the only extant document comparable to Apollonius' *Conica* in the level of its mathematical content, is more readable by far, consisting of several single works of medium length (up to about 50 propositions), each of which accomplishes some definite aim. The *Conica*, in contrast, is a massive heap. Though it is divided into seven books (eight, if the lost Book VIII is taken into account), its one and only aim is to investigate the properties of conic sections—practically an endless task. There are certainly admirable propositions here and there, but their significance is often very unclear, and one can get far less satisfaction from reading Apollonius than from reading Archimedes, who at least rewards the reader's patience by arriving at some significant result at the end of each work. Thus a reader of the *Conica* is always left with a sense of being lost in a labyrinth, the intention of whose creator is obscure. No wonder this work has stimulated the intellectual appetite of modern historians far less than the Archimedean Corpus.

Netz' edition of *On the Sphere and the Cylinder* adds a new layer to this, by considering not just the Archimedean text, but also the commentary written by Eutocius. However, there is currently no complete translation of Eutocius' commentaries on the *Conics*, a situation which this thesis partially aims to rectify¹. My translation, together with some analysis of my own, is presented in Chapter 3.

1.3 Conjugate Diameters

Throughout this thesis, I make considerable use and reference to the secondary commentaries and scholia. In a large sense, I believe these commentaries are permanently linked to the texts on which they comment. For example, Eutocius himself tells us, in his commentary on Apollonius, that his aim was twofold: to produce *a new edition of the Conics itself*, and to add commentary and alternate proofs as scholia. From his commentary:

But since there are so many editions, as he himself [Apollonius] says in his letter, I considered it better to bring them together, from those available, placing the more manifest things in the text side by side for the benefit of beginning students, and to note outside in the adjacent scholia the different courses of the proofs, as seemed reasonable.

¹A French translation, together with a new critical edition of the Greek text, is underway by Roshdi Rashed, Micheline Decorps-Foulquier, and Michel Federspiel. The text is to be published in the near future by de Gruyter.

The letter to which Eutocius is referring is Apollonius' introductory letter to *Conics*

I:

...and how on arranging them in eight books we immediately communicated them in great haste because of his [Naukrates'] near departure, not revisiting them but putting down whatever came to us with the intention of a final going over. And so finding now the occasion of correcting them, one book after another, we publish them. And since it happened that some others among those frequenting us got acquainted with the first and second books before the revision, don't be surprised if you come upon them in a different form.

It is in this sense that I call Eutocius' commentary and the *Conics* itself *conjugate*, and why I place such importance on understanding these secondary commentaries. Further, given that Eutocius comments on Books I-IV of the *Conics*, and that these are precisely the books extant in Greek, one cannot help but agree with Heath (in [4, 5] in his assertion that the commentary aided the survival and transmission of the text.

1.4 Editions, Texts, Copies, Commentaries, and Scholia

It is important to make a distinction between the following terms: edition, text, copy, commentary, and scholia, and to understand what role they had to Eutocius. By edition, Eutocius refers to the different editions of the original *Conics*. We know that different editions existed even in Apollonius' time, as confirmed by his introductory letter to *Conics* I. The idea of different editions of a work holds even today, and the existence of the concept in antiquity is not surprising.

A major difference, however, is that of *copies*. In modern publishing, two copies of a given edition are largely identical. Even when not identical (such as when an edition is reprinted) the differences are usually extremely minor and in some cases these reprints are facsimiles of the source edition. These differences are more extreme with ancient texts, which were hand copied over time, often from other previously made copies. The result is that different copies disagree from each other and common ancestor source copies. The job of sorting these differences out is left to the textual critic; recent scholarship of Netz et al. has added a focus on diagrams to mathematical texts.

Eutocius essentially was a textual critic. He had at his disposal a number of source

copies, perhaps representing many different lineages from Apollonius' two editions. The differences in these copies forms the basis for many parts of edition of the *Conics*: he wishes to use the best versions of proofs in order to make the overall text more clear and more understandable to students and other non-specialists. Alternate versions of proofs found from the various copies are given in his commentary when appropriate. He also adds in the scholia found in his source copies.

A word on Eutocius' audience: in some sense there are three. First and foremost, as Eutocius himself tells us, he is concerned with students and how they will be using the text. He wants the proofs to be as clear and concise as possible for the students. Second, he is writing directly for Anthemius, who he addresses in the introduction. And third, he is writing for future mathematicians, who will use his commentary to see alternative proofs and glimpses of his own source copies.

1.5 Notation and The Structure of Greek Mathematical Arguments

In general I follow the notational style of Heath's edition of the *Elements*.² Other works use a similar format, specifically the Green Lion editions of the *Conics* ([21, 22]). A summary of notational elements is given in Table 1.1.

There are six traditional parts of a Greek mathematical proof. Extended discussions of them are given in [4, 5, 10]. A short description is given below:

1. The *protasis* (πρότασις): The enunciation of the proposition, which states the hypotheses and thing to be proved without reference to any particular diagram.
2. The *ekthesis* (ἔκθεσις): A restatement of part of the protasis, in which the hypotheses are given in terms of a constructed diagram. In Greek, ekthesis literally means "setting out." A common Latinate rendering is "exposition."
3. The *diorismos* (διορισμός): The claim about what is to be proved. In Greek this generally starts with the words "λέγω ὅτι," or, "I say that..."
4. The *kataskeuē* (κατασκευή): Completion of the diagram, constructing any additional auxiliary parts not already constructed in the *ekthesis*.

²The text of this is the same as in the Greek Lion Edition ([12]), however, that edition dispenses with the mathematical notation in favor of keeping things in word form (as in the Greek). The technical nature of the *Conics*, however, I feel justifies using this notation.

Table 1.1: Summary of Notation

tri.($AB\Gamma$)	The triangle with vertices A, B, Γ .
sq.(AB)	The square with side AB .
rect.($AB, \Gamma\Delta$)	The rectangle with sides $AB, \Gamma\Delta$.
rect.($A\Gamma$)	The rectangle about the diagonal $A\Gamma$.
quad.($A\Gamma$)	the quadrilateral about the diagonal $A\Gamma$.
pllg.($AB, \Gamma\Delta$)	The parallelogram with sides $AB, \Gamma\Delta$.
pllg.($A\Gamma$)	The parallelogram about the diagonal $A\Gamma$.
pllpd.($AB, \Gamma\Delta, EZ$)	The parallelepipedal solid with sides $AB, \Gamma\Delta, EZ$.
pllpd.($A\Delta$)	The parallelepipedal solid about the diagonal $A\Delta$.
$A : B = \Gamma : \Delta$	The ratio of A to B is equal to that between Γ and Δ .
dup.($A : B$)	The duplicate ratio of $A : B$.
trip.($A : B$)	The triplicate ratio of $A : B$.
$(A : B)$ comp. $(\Gamma : \Delta)$	The ratio compounded from the ratios $A : B$ and $\Gamma : \Delta$.
arc($AB\Gamma$)	The arc through points A, B, Γ .
section($AB\Gamma$)	The conic section through points A, B, Γ .

5. The *apodeixis* (ἀπόδειξις): This is the actual proof. The word here is from the verb ἀποδείκνυμι.
6. The *symperasma* (συμπέρασμα): The conclusion, in which the claim of the diorismos is restated as now having been proved. Apollonius occasionally adds a definition in this part, such as "...and let such a section be called a parabola."

1.6 On Translation

The style and structure of mathematics in Greek differs considerably from that of modern mathematics. The following is a short discussion of some of the differences and how they affect translators and their translations.

To start, the text is written continuously; the practice of putting "equations" on their own lines is much more modern. Greek also has the ability to leave out many more words than English due to its inflected nature. If a word is to be the direct object of a verb, for example, a pronoun will suffice when this word is mentioned again. This presents a difficulty in English, where typically a pronoun is taken to refer to the last explicitly mentioned noun. In translating Greek to English, a translator often must resupply the appropriate noun, lest the passage turn into an impenetrable pile of confusing pronouns. Since there are only archaic vestiges of inflection left in English, the only other way to keep pronouns organized would be to color them, and make the red pronoun "it" and the blue pronoun "it" different³.

For a translator, this presents a few difficulties. First, of course, is that it makes total fidelity to the Greek impossible. But it is important to recognize that Greek and English are totally different languages, with different syntax, word order restrictions, *et cetera*, so it should not be totally surprising that issues such as these come up. The second issue is more one of utility and readability. Reading a translation with resupplied nouns, for example, may falsely lead the reader into thinking that this is how it appeared in Greek. Making note of this issue, then, is absolutely necessary. The question is how and when these notes are best made, and where on the continuum between total fidelity to the Greek

³Interestingly, this can work in a more pictorial setting: Oliver Byrne's edition of the *Elements* illustrates this beautifully.

and ease of use for a modern English reader the work should be.

Further, words in mathematical Greek are sometimes completely omitted. For example, in Eutocius, the phrase "the square on AB" never actually mentions the word square. In Greek, it appears as "τὸ ἀπὸ AB", literally, "the from AB." Another example is the differences between "angles" and "rectangles": in Greek, they differ only by gender. The first appears as the feminine "ἡ ὑπὸ ABΓ", whereas the second appears as the neuter "τὸ ὑπὸ ABΓ". These phrases are essentially abbreviations; they come up often enough in mathematical works for the omission of "square" or "angle" or "rectangle" to be practical. But how do we make note of this as translators? One way would be to always reinsert the appropriate noun, such as "square," but to somehow note that it does not actually appear in the Greek. Netz takes this approach; for "τὸ ἀπὸ AB", he writes "the <square> on AB." By doing so, Netz makes very clear what actually appears in the Greek and what are his insertions. However, it does make for some laborious reading, and the presence of so many pointed brackets is, frankly, an eyesore. I prefer to take a more selective approach; even though "square" may not appear in the Greek, I prefer to put it in the English without any brackets or other distracting notation except in special situations. While this may be less transparent than Netz' approach, I feel the benefits to the reader justify my preferred practice.

The matter of vocabulary has also been a significant challenge. In most cases, I try to use the same translations of technical vocabulary as exist in the current English literature, especially the Heath, Taliaferro, and Netz translations. Heath made many contributions to mathematical uses listed in the Liddell, Scott, and Jones dictionary ([8, 9]), which has been my primary source for matters of translation. Also quite helpful has been Mugler's *Dictionnaire Historique de la Terminologie Géométrique des Grecs* ([11]).

1.7 Mathematical Vignettes

The structure of my thesis is divided as follows: in Chapter 2, I give a mathematical overview of the state of conic sections both before and after Apollonius. Chapter

3 presents my translation of Book I of Eutocius' commentary, statements of the theorems from Apollonius', and my own discussion and analysis where appropriate. For this I have translated Heiberg's edition of the Greek, from [14] and the digital version of it on Thesaurus Linguae Graecae. Chapter 4 explores the problem of duplicating a cube, a problem whose many varied solutions come to us primarily through Eutocius. Chapter 5 deals exclusively with the topic of compound ratios, as it appears in major mathematical authors and in Eutocius' work. Appendix A contains Heiberg's Greek text of Eutocius' commentary on the *Conics*, which is rather hard to find in print due to the scarcity of the century-old Teubner editions (and the unpublished status of the forthcoming Rashed, Decorps-Foulquier, and Federspiel edition). Though this text is also available on Thesaurus Linguae Graecae, I have included it here for the convenience of interested readers or those unable to access the TLG. Appendix B is included for the same reason, but presents the Greek text for various discussions of compound ratios subordinate to Chapter 5. The Greek texts for this appendix comes from [15, 14, 13], with English translations by Netz (from [16]), myself (from Chapter 3), and Knorr (from [7]).

CHAPTER 2 INTRODUCTION TO CONIC SECTIONS

2.1 Pre-Apollonian Notions

Eutocius himself gives an account of cones and conics before Apollonius. Quite before him, however, Euclid defines a (pre-Apollonian) cone in *Elements XI*, though it is in his lost *Elements of Conics* that the conics themselves are defined and examined. We know, for example, that Archimedes was familiar with this work, himself referring to it in his *Quadrature of the Parabola* [23].

Before Apollonius, a cone was constructed by means of a right triangle, say, $AB\Gamma$. Keeping one leg about the right angle fixed, the triangle was revolved, until it returned to its original position. The surface so swept out was defined as a cone; the vertex was the endpoint of the hypotenuse which remained fixed, and the base was the circle generated by the rotating leg. The axis of the cone is the fixed leg, which thus is perpendicular to the base circle and connects the vertex and the center of the base circle. A cone was said to be acute, right, or obtuse, according to whether the angle subtending the rotating leg was less than, equal to, or greater than one-half a right angle, respectively. Such is the definition in the *Elements*, and is confirmed on the authority of Geminus as seen in Eutocius' commentary on Apollonius.

Each conic, then, was defined by a particular intersection of a plane with one of the cones. In all instances, the plane intersects the cone at right angles, in the sense that it is perpendicular to the generating hypotenuse. The common section of the cutting plane and the plane of the generating triangle was called the diameter of the section. If the original cone is right, then the diameter is parallel to a generator, and the resulting section was called *the section of a right-angled cone*. Although we would call such a section a

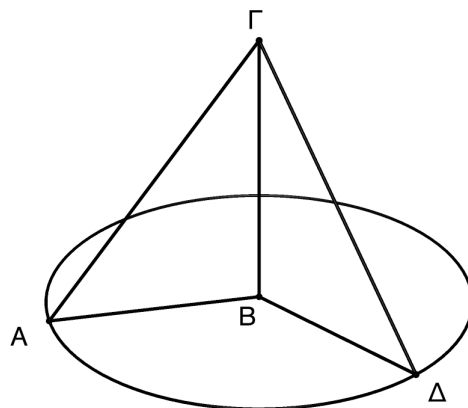


Figure 2.1: A pre-Apollonian Cone

parabola, it was Apollonius himself who first coined this term. Similarly, then, a hyperbola is the section of an obtuse angled-cone; the ellipse the section of an acute angled-cone. Even Archimedes, a mere twenty-five year elder to Apollonius, uses these "old" names for the sections.

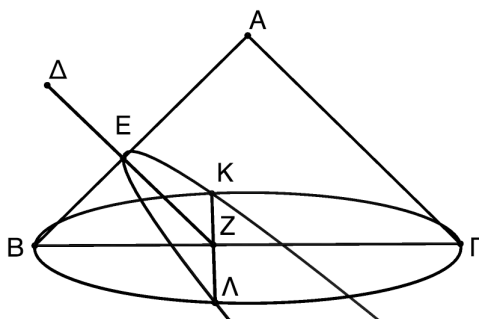


Figure 2.2: A pre-Apollonian Section of a Cone

Heath reports that the conics were first investigated by Menaechmus (a student of Eudoxus and Archytas); indeed, Eutocius' quotation of Eratosthenes' solution to the duplication of the cube calls him "three-conic cutting Menaechmus" ([16]) Why it is that Menaechmus first conceived of the conics remains a mystery: whether it was as a specific tool to solve the duplication problem or as a curious mathematical object, we cannot

know. But it should be stated that his solutions to the duplication problem are of such simplicity, assuming the principle properties of conics, that it would not be entirely unlikely that he first investigated cones expressly for this purpose. One of his teachers, Archytas (an associate of Plato), gives a wonderfully grand construction that involves intersections of rotating figures and a cylinder, yielding non-planar curved lines. Details of these constructions are given in the chapter on the duplication problem. If his teacher was capable of imagining such a bold construction, perhaps it is not so unlikely that Menaechmus thus imagined the conics.

2.2 Apollonian Notions

Apollonius, however, defines his cones and conics differently. Instead of beginning with a triangle, as did his predecessors, Apollonius considers a circle and a point not on the plane of that circle. Taking a point on the circle, and joining the two points, this line is extended both directions indefinitely. As the point on the circle is rotated about the circle, until it returns to its starting position, the infinite line sweeps a surface, really *two* surfaces, which Apollonius calls collectively the *conic surface*. The vertex is the original point not on the plane of the circle, the base is the given circle, and the portion of the surface between he defines as a cone. The conic surface, then, may be imagined as a double-napped cone, much as in the modern conception, though he often concentrates on one cone instead of the conic surface. In fact, Fried and Unguru (in [3]) believe that he tries to avoid the conic surface entirely, due to its awkward singular/dual existence. One surface or two?

Apollonius differentiates two types of cones: those which are right, and those which are not. The criteria for each depends on whether the axis of the cone meet the base at right angles. In pre-Apollonian cones, of course, this is always the case; therefore all cones before Apollonius, even the so-called acute angled or obtuse angled cones, are right as far as Apollonius is concerned. The second type of cone is the oblique¹. In essence, then,

¹We tend to use oblique rather than scalene, despite the Greek word in question being *σκαληνός*.

Apollonius' work will capture everything about the works of his predecessors, but doing so in sufficient generality so as to demonstrate new and extended results.

Apollonius defines next the *axial triangle*, which serves to give him a scaffolding with which to construct the other sections. There is a unique diameter of the (base) circle which, when a plane containing it is set up perpendicular to the base, the plane also contains the vertex. The common section of this plane and the cone is called the axial triangle²; manifestly, then, it contains the axis. The cutting plane that will yield the sections is established so that it intersects one side of the axial triangle and cuts the aforementioned diameter at right angles. Depending on whether or not it cuts the other side of the axial triangle, and where, the section is called either a parabola, an hyperbola, or an ellipse.

It is also interesting to note here the intrinsic use of the "true" form of Euclid's fifth postulate, often misquoted, as it appears in the *Elements*. For completeness, I state it here, using Heath's translation in [12]. I also give a diagram as an example.

Elements I, Postulate 5: That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

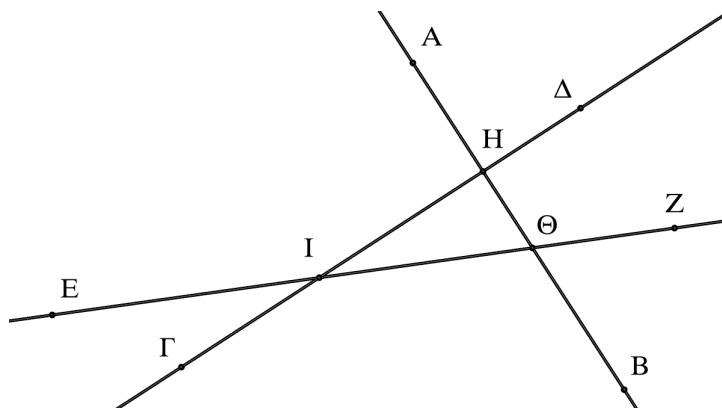


Figure 2.3: Euclid's Fifth Postulate

²The cone, properly speaking, consists of the base circle too. So the axial triangle is really a triangle, with one side the diameter of the circle, and the other two on the surface of the cone and terminating at the vertex.

In the figure, take the straight line as AB , and the two straight lines as $\Gamma\Delta$, EZ . The interior angles in question here are $I\Theta$ and $I\Theta H$, which are together less than two right angles. Accordingly, the lines $H\Delta$, ΘZ , when produced, meet on the same side of AB as these two angles. In the figure, this happens at I .

But back to the conic sections. Eutocius himself plays a bit of a word game with the names: for in a parabola, the cutting plane (and thus the diameter of the section, being the common section of the cutting plane and the axial triangle) is parallel to the other side of the axial triangle; in the ellipse it meets on the same side of the vertex as the cone, and in the hyperbola it meets beyond the vertex. Considering certain angles in the figures, he describes that the hyperbola occurs when these angles exceed ($\acute{\upsilon}\pi\epsilon\rho\beta\acute{\alpha}\lambda\lambda\omicron\nu$) two right angles, the ellipse when they are deficient ($\acute{\epsilon}\lambda\lambda\epsilon\acute{\iota}\pi\omicron\nu$), or when equal, the diameter is thus parallel. In the case of the parabola in Figure 2, for example, the angles in question are ZEA and EAG .

I say word games because this is not precisely the reason that Apollonius chooses these names, for he himself is more concerned with the excess/equality/deficiency upon the application of areas to certain lines. It is certainly possible, though, that Eutocius intended a double word play: contrasting the parallelism (arising from the perpendicular application) of the cutting planes in pre-Apollonian sections with the non-parallel cutting planes in Apollonius.

This method of cutting the cone thus differs from his predecessors in a complementary or dual way: his predecessors kept the sectioning procedure fixed for all cones, creating the varying sections by varying the type of cone which is sectioned. Apollonius, however, shows that for a fixed cone of any type whatsoever, it is possible, by means of a different application of the cutting plane, to cut any of the sections. The result of this is that we might include the circle amongst Apollonius' conic sections. Though it certainly would be clear to his predecessors that a plane cutting the cone parallel to the base will generate a circle, the sectioning procedure is not that of the other conics. Apollonius, on

the other hand, is free to section as he likes, and indeed, the fact that one may cut such a circle is the fourth proposition in the *Conics*. What is perhaps more unusual, however, is the existence of a second way to section a circle: this by means of a plane cutting subcontrariwise³. On account of this similarity, it should not be surprising that a plane that cuts the cone with this third side as a diameter should be a circle; indeed Apollonius proves this fact. It should be noted that this occurs only in the oblique cone: for in a right cone, the subcontrary plane is itself parallel to the base on account of the axial triangle being isosceles.

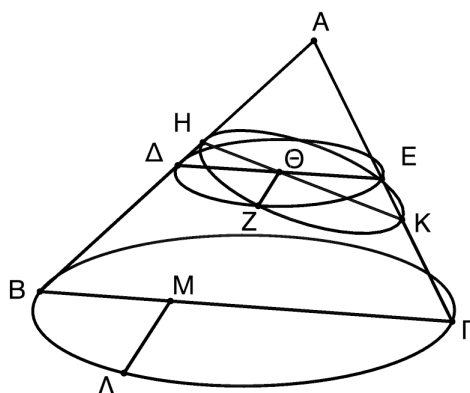


Figure 2.4: A Subcontrary Circle

2.3 The *Symptomata*

The following *symptomata* are adapted and condensed from Heiberg's edition of the Greek text and Taliaferro's subsequent translation. For brevity, I have omitted the full *ektheseis* and *kataskewai*⁴, though my diagrams retain all parts of those in Heiberg's edition. The diagrams are equivalent to those found in Taliaferro's translation, but for consistency, I retain the Greek letters.

³That is, if a triangle is situated inside the axial triangle subcontrariwise (ὕπερκειμένοι), similar but flipped, so that the third side is not parallel to the base.

⁴These are two of the six parts of most Greek proofs. See the appendix for definitions.

2.3.1 The Parabola

The square on the ordinate is equal to the rectangle contained by the abscissa and parameter (or latus rectum). In Figure 2.5, this is $\text{sq.}(K\Lambda) = \text{rect.}(\Theta Z, ZK)$, with ZK the abscissa, and $Z\Theta$ the parameter. Note that "parameter" is an abbreviation. Apollonius states in his *symperasma*: "let ΘZ be called the straight line to which the the straight lines drawn ordinatewise to the diameter ZH are applied in square ($\text{παρ' ἦν δύννται αἱ καταγόμεναι τεταγμένως ἐπὶ τὴν } ZH \text{ διάμετρον}$), and let it also be called the upright side ($\acute{\alpha}\rho\theta\acute{\iota}\alpha$)."

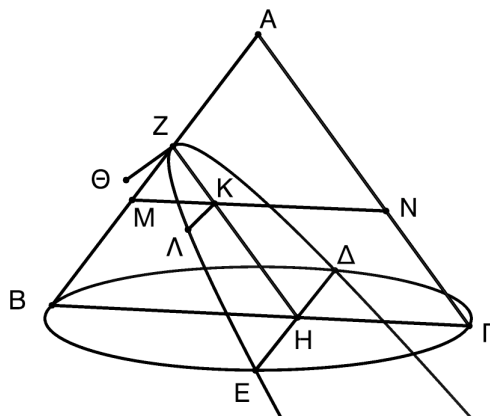


Figure 2.5: *Symptoma* of the Parabola

2.3.2 The Hyperbola

The square on the ordinate is equal to a rectangle applied to the upright side which has the abscissa as breadth, and this rectangle projects beyond the upright side by a rectangle similar to the rectangle contained by the upright and transverse sides. Again, the upright and transverse sides are abbreviations. The upright side is the abbreviation for the same Greek for which we use "parameter" as an abbreviation; the transverse side is essentially a "second" parameter. In terms of the diagram, Apollonius' *diorismos* is as follows: "I say that MN is equal in square to the parallelogram $Z\Xi$ which is applied to

$Z\Lambda$, having ZN as breadth, and projecting beyond by a figure $\Lambda\Xi$ similar to the rectangle contained by ΘZ and $Z\Lambda$."

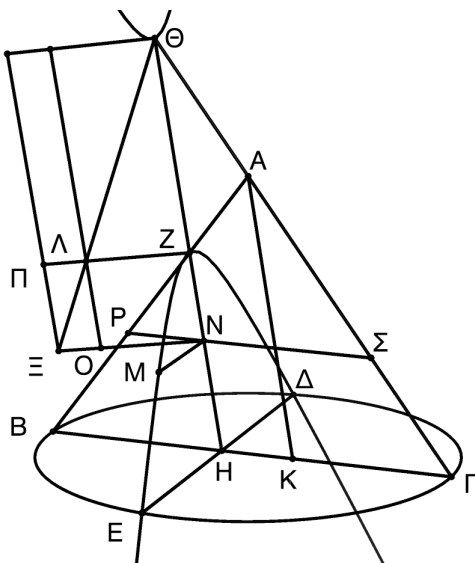


Figure 2.6: An Hyperbola

2.3.3 The Ellipse

The square on the ordinate is equal to a rectangle applied to the upright side which has the abscissa as breadth, and this rectangle is deficient by a rectangle similar to the rectangle contained by the upright and transverse sides. In terms of the figure, the ordinate is ΛM , the abscissa EM , the upright side $E\Theta$, and the transverse side $E\Delta$. The abbreviations "upright side" and "transverse side" are expressed the same as in the case of the hyperbola.

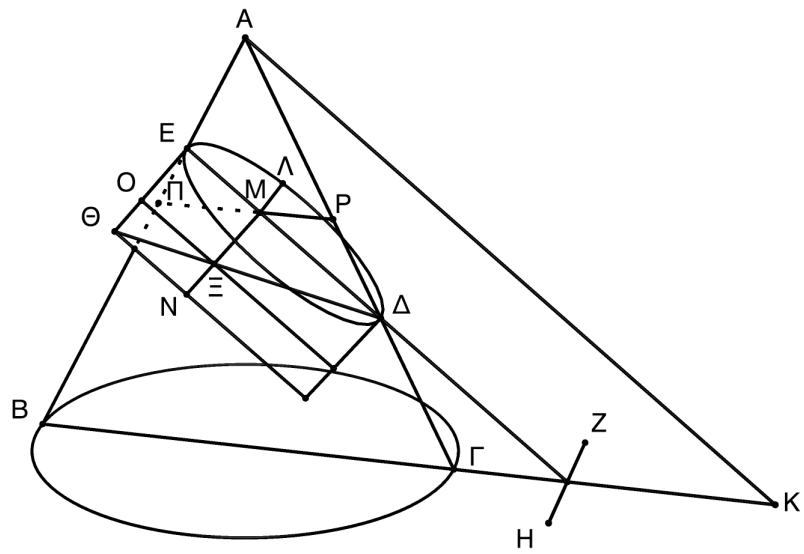


Figure 2.7: An Ellipse

CHAPTER 3

EUTOCIUS' COMMENTARY: TRANSLATION AND ANALYSIS

3.1 Eutocius' Introduction and To the First Definitions

(168.5) Apollonius the geometer, my dear friend Anthemius, was born in Perga in Pamphylia during the reign of Ptolemy Euergetes, as chronicled by Heraklius, who wrote *The Life of Archimedes*. He also says that Archimedes first thought of the conic theorems; but that Apollonius, having found them unpublished by Archimedes, made them his own. But he is not correct, in my opinion, at least. For both Archimedes seems to recall in many passages the *Elements of Conics* as more ancient, and Apollonius does not write his own thoughts: for he would not have said that he had worked these things out in full and more generally than the writings of others. But the very thing which Geminus says is true: that the ancients, defining a cone as the revolution of the right triangle, with one side about the right angle remaining fixed, naturally assumed that all cones are right, and that one section occurs in each: in the right-angled¹ cone, what is now called the parabola, in the obtuse-angled, the hyperbola, and in the acute-angled, the ellipse: and it is possible among them² to find the so-called sections. So just as the ancients theorized on the fact that in any triangle there are two right angles--first in the equilateral, in turn the isosceles, and last the scalene--their descendants proved a general theorem as follows: in every triangle, the three internal angles are equal to two right angles; likewise in the case of the sections of the cone. For the thing called a section of a right-angled cone they viewed only by means of a plane cutting orthogonal to one side of the cone, and they showed that the section of the obtuse cone occurs in an obtuse cone, and that of the acute cone in an acute one: likewise in all cones bringing planes orthogonal to one side of the cone: and he

¹It is important that we distinguish - as does Apollonius - that a cone is *right* when it is formed from the revolution of a right triangle. The angle at the vertex of the cone, which subtends the rotating leg, determines if the cone is right-angled, obtuse-angled, or acute-angled.

²i.e. the different cones

[Geminus] shows also the ancient names of the curves themselves. But later, Apollonius of Perga theorized somewhat generally, that in every cone, both the right and the oblique, all the sections are according to a different application of the cutting plane to the cone; and his contemporaries, having marveled at the wonder of the conic theorems shown by him, called him a great geometer. So Geminus says these things in the sixth book of *The Theory of Mathematics*. But what he says, we will make clear in the diagrams below.

(170.28) The Parabola. Let there be the triangle $AB\Gamma$ through the axis³, and let ΔE be drawn at right angles to AB from the random point E , and let the plane produced through ΔE cut the cone orthogonal to AB : therefore each of the angles $AE\Delta$, AEZ is right. Since the cone is right and the angle $BA\Gamma$ is clearly right, as in the first diagram, the angles $BA\Gamma$, AEZ will be equal to two right angles: so that the line ΔEZ will be parallel to $A\Gamma$. And a section is produced on the surface of the cone which is called a *parabola*; thus named on account of the line ΔEZ , which is the section common to the cutting plane and the axial triangle, being parallel to the side $A\Gamma$ of the triangle.

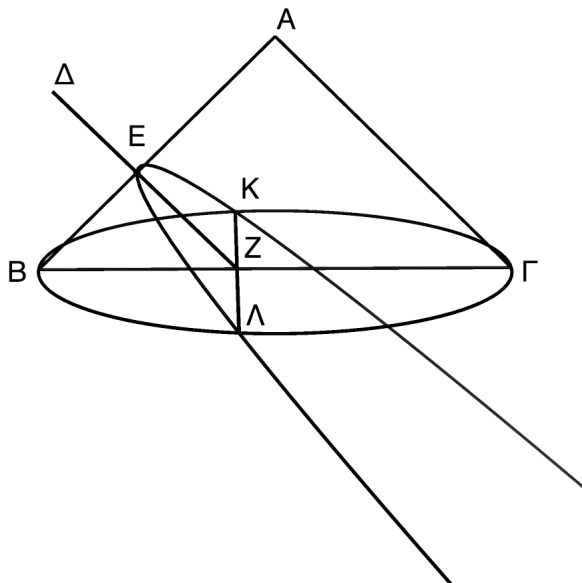


Figure 3.1: A Parabola

³i.e. the axial triangle

(172.18) The Hyperbola. But if the cone be obtuse-angled, as in the second diagram, with the angle $B\Lambda\Gamma$ manifestly being obtuse, but the angle AEZ right, the angles $B\Lambda\Gamma$, AEZ will be greater than two right angles: so that ΔEZ will not intersect the side $A\Gamma$ at the parts near Z, Γ ; but rather, at the parts near A, E , with ΓA manifestly being projected to Δ . So the cutting plane will make on the surface of the cone a section, the so-called *hyperbola*, being called thus from the fact that the aforementioned angles, i.e. AEZ , $B\Lambda\Gamma$, exceed two right angles; or on account of ΔEZ projecting beyond the vertex of the cone and meeting ΓA outside.

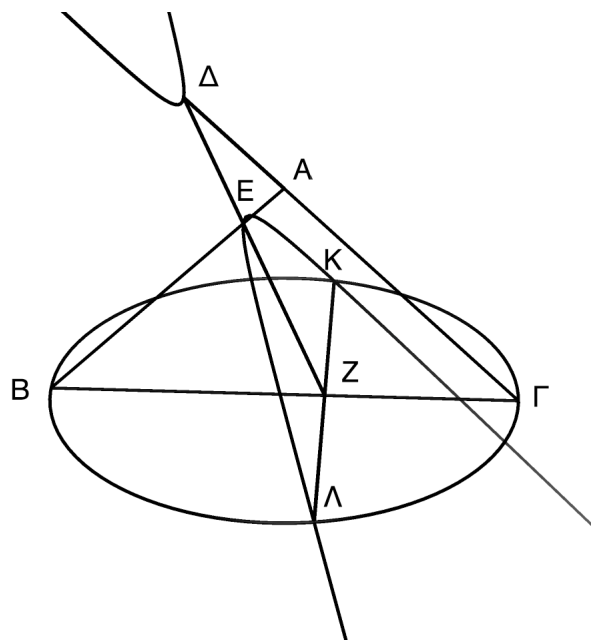


Figure 3.2: An Hyperbola, with its Opposite Section

(174.3) The Ellipse. But if the cone be acute, with the angle at $B\Lambda\Gamma$ manifestly being acute, the angles $B\Lambda\Gamma$, AEZ will be less than two right angles: so that the segments EZ , $A\Gamma$, being produced, will meet somewhere: for I am able to extend the cone. So there will be a section on the surface, which is called an *ellipse*, thus being called either on account of the aforementioned angles falling short of two right angles, or on account of the ellipse falling short of a circle.

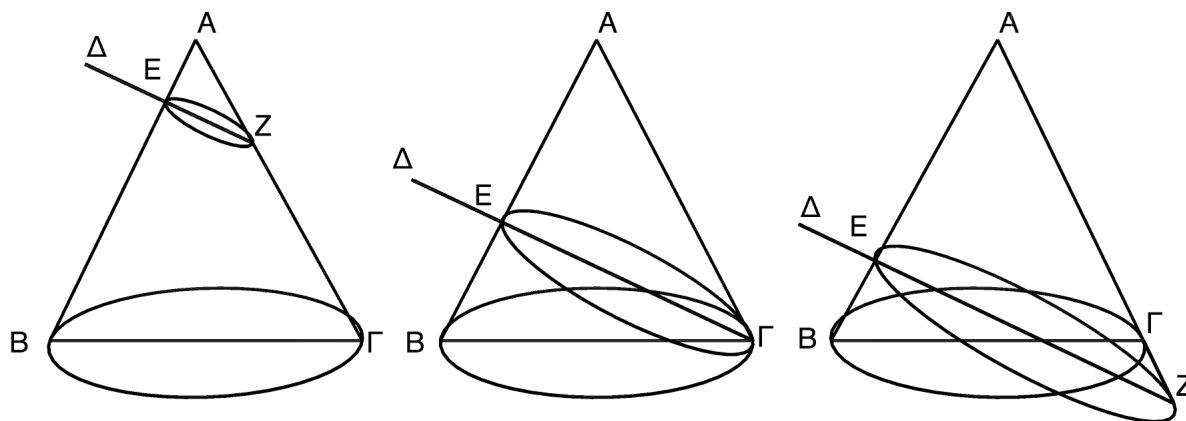


Figure 3.3: Three Possible Ellipses

(174.11) And so the ancients, having laid down the cutting plane, the one through ΔEZ , orthogonal to the side AB of the triangle through the axis of the cone⁴, even so saw the cones as different, and in each its own section. But Apollonius, having laid down the right cone and the oblique one, made the different sections by the different inclination of the cutting plane.

(174.18) Let there again be, as in the same diagrams, the cutting plane through $KE\Lambda$, and let the section common to it and the base of the cone be $KZ\Lambda$; and again, let the common section of the cutting plane $KE\Lambda$ itself and the axial triangle $AB\Gamma$ be EZ , which is called the *diameter* of the section. So of all the sections, he lays down $K\Lambda$ orthogonal to the base $B\Gamma$ of the triangle $AB\Gamma$; but it remains to produce, if EZ is parallel to $A\Gamma$, the parabolic section $KE\Lambda$ on the surface of the cone; but if EZ meets $A\Gamma$ beyond the vertex of the cone (as at Δ), to produce the hyperbolic section $KE\Lambda$; and if EZ meets $A\Gamma$ inside, to produce the elliptical section, which they also call a shield. So in general, the diameter of a parabola is parallel to one side of the triangle; but the diameter of the hyperbola intersects the side of the triangle at the parts above the vertex of the cone; and

⁴i.e. the axial triangle.

the diameter of the ellipse meets the side of the triangle on the side of the base⁵. But it is necessary to see also, that the parabola and the hyperbola are things which increase towards infinity, but the ellipse not at all; for it converges back to itself completely, like a circle.

(176.16) But since there are so many editions, as he himself says in his letter, I considered it better to bring them together, from those available, placing the more manifest things⁶ in the text side by side for the benefit of beginning students, and to note outside in the adjacent scholia the different courses of the proofs, as seemed reasonable.

(176.23) So then he says in his letter that the first four books encompass a course in the elements⁷, of which the first encompasses the origins of the three sections of the cone, and of the sections called *opposite*, and the principle properties in them. And these are all the ones that follow from their first origin⁸. But the second book encompasses the properties deduced from the diameters and the axes of the sections, and also the *asymptotes*, and other things providing a fundamental and necessary function to their *diorismoi*.

But that the *diorismos* is twofold is clear to everyone; the one setting up after the *ekthesis* what is the thing being sought; the other, not agreeing that the *protasis* is general, but saying when and how and in how many ways it is possible to set up the proposition, such as the one in the twenty-second proposition of the first book of Euclid's *Elements* does: from three straight lines, which are equal to three given straight lines, to construct a triangle: indeed it is necessary that two sides chosen in any manner be greater than the remaining side; since it has been shown that in every triangle, the two sides are greater than the remaining side, being taken in any manner.

But the third book of the *Conics*, he says, encompasses many useful and amazing

⁵i.e. below the vertex of the cone.

⁶τὰ σαφέστερα; I take this to mean the clearer versions of proofs, nicer definitions, etc.

⁷i.e. the basics, which would have been in Euclid's *Elements of Conics*.

⁸i.e. the ἀρχικά συμπτώματα follow directly from the origins of the sections.

theorems to the constructions of solid loci. It was customary for the ancient geometers to speak of planar loci whenever, in the problems, the problem arises not from one point alone, but rather from more; for example, when having been given some straight line, if someone enjoins you to find some point, from which the perpendicular being drawn to the given line makes a mean proportional of the pieces; they call things such as this a locus: for the thing making the problem is not only the one point, but rather a whole locus, which the perimeter of the circle, having the given straight line as its diameter, has. For if a semicircle is drawn around the given straight line, whatever point you take on the perimeter, and from it draw a perpendicular to the diameter, will solve the problem.

And likewise, given some straight line, if someone enjoins you to find a point outside it, from which the lines joining the endpoints of the [given] straight line will be equal to one another, in this case too, the thing making the problem is not only one point, but a locus, which the straight line being drawn at right angles at the midpoint of the given line provides. For if, having cut the given straight line in two, you also draw a straight line at right angles on the midpoint, and whatever point you take on it, will solve the assigned problem.

(180.11) And Apollonius himself writes similarly in the *Treasury of Analysis*⁹ in the case of the following.

(180.13) When there are two given points¹⁰ in a plane and an established ratio of unequal straight lines, it is possible to draw in the plane a circle so that the straight lines inflected from the given points to the perimeter of the circle have the same ratio as the given ratio.

(180.18) Let the given points be A and B, and the given ratio that of Γ to Δ with Γ being bigger: indeed it is necessary to solve the assigned problem. For let AB be joined, and let it be projected to the parts near B, and let it be contrived that, as Δ is to Γ , so too is

⁹See Heath's notes on Pappus.

¹⁰Heiberg has δύο δοθέντων [εὐθειῶν] ἐν ἐπιπέδῳ [καὶ] σημείων... There are not two given straight lines, just two given points. Note how Eutocius starts the construction in the next paragraph.

Γ to some other magnitude clearly being greater than Δ , and let, for example, it be $E\Delta$ ¹¹:

$$\Delta : \Gamma = \Gamma : E\Delta;$$

and let again it be contrived, that

$$E : AB = \Delta : BZ = \Gamma : H.$$

It is manifest that both Γ is proportionally between $E\Delta$ and Δ ¹² and H is proportionally between AZ , ZB ¹³:

$$AZ : H = H : ZB;$$

With center Z and radius H , let the circle $K\Theta$ be drawn. It is manifest indeed, that the circumference $K\Theta$ cuts orthogonal to AB : for the straight line H is proportionally between AZ , ZB . Indeed, let a random point Θ be taken on the circumference, and let ΘA , ΘB , and ΘZ be joined. So then ΘZ is equal to H , and on account of this,

$$AZ : Z\Theta = Z\Theta : ZB.$$

¹¹That is, $E = \Gamma + \Delta$.

¹²This is true by hypothesis.

¹³I hardly found this point clear. As hypotheses, we have $\Delta : \Gamma = \Gamma : E\Delta$, $E : AB = \Delta : BZ$, and $E : AB = \Gamma : H$. Eutocius claims that H is a mean proportional between AZ and BZ . From the hypothesis that $E : AB = \Delta : BZ$, I say that $AZ : E\Delta = BZ : \Delta$.

$$E : \Delta = AB : BZ \text{ (Alternando)}$$

$$E\Delta : \Delta = AZ : BZ \text{ (Componendo)}$$

$$E\Delta : AZ = \Delta : BZ \text{ (Alternando)}$$

$$AZ : E\Delta = BZ : \Delta \text{ (Invertendo)}$$

as desired. But since it was hypothesized that $H : \Gamma = BZ : \Delta$, and the result just shown is that $AZ : E\Delta = BZ : \Delta$, we have that

$$H : \Gamma = AZ : E\Delta. \text{ Therefore}$$

$$AZ : H = E\Delta : \Gamma \text{ (Alternando)}.$$

But it was hypothesis that

$$E\Delta : \Gamma = \Gamma : \Delta, \text{ so,}$$

$$AZ : H = \Gamma : \Delta, \text{ and therefore}$$

$$AZ : \Gamma = H : \Delta \text{ (Alternando)}$$

We have as hypothesis that

$$\Gamma : H = \Delta : BZ, \text{ so } ex\ aequali,$$

$$AZ : H = H : BZ.$$

Therefore H is the mean proportional between AZ and BZ .

Also, [the sides] about the same angle ΘZB are proportional: then the triangle $AZ\Theta$ is similar to the triangle ΘBZ , and the angle under $Z\Theta B$ is equal to the one under ΘAB .

Let $B\Lambda$, parallel to $A\Theta$, be drawn through B . So since

$$AZ : Z\Theta = Z\Theta : ZB,$$

as the first AZ is to the third ZB , so too is the square on AZ to the square on $Z\Theta$:

$$AZ : ZB = \text{sq.}(AZ) : \text{sq.}(Z\Theta).$$

But

$$AZ : ZB = A\Theta : B\Lambda,$$

and so

$$\text{sq.}(AZ) : \text{sq.}(Z\Theta) = A\Theta : B\Lambda.$$

Again, when the the angle $B\Theta Z$ is equal to the angle ΘAB , the angle $A\Theta B$ is also equal to the angle $\Theta B\Lambda$: for they are situated alternately¹⁴: and so the remaining angle is equal to the remaining angle, and the triangle $A\Theta B$ is similar to the triangle $B\Theta\Lambda$, and the sides about equal angles are proportional:

$$A\Theta : \Theta B = \Theta B : B\Lambda;$$

and

$$\text{sq.}(A\Theta) : \text{sq.}(\Theta B) = A\Theta : B\Lambda.$$

But it was previously shown also, that

$$A\Theta : B\Lambda = \text{sq.}(AZ) : \text{sq.}(Z\Theta) :$$

so

$$\text{sq.}(AZ) : \text{sq.}(Z\Theta) = \text{sq.}(A\Theta) : \text{sq.}(\Theta B);$$

and on account of this,

$$AZ : Z\Theta = A\Theta : \Theta B.$$

But

$$AZ : Z\Theta = E\Delta : \Gamma = \Gamma : \Delta :$$

¹⁴That is, they are alternate interior angles in a transversal formed by ΘB cutting the parallels $A\Theta$, $B\Lambda$.

and so

$$\Gamma : \Delta = A\Theta : \Theta B.$$

Similarly, all the straight lines from the points A, B, terminating at the perimeter of the circle, will be shown as having to one another the same ratio as Γ to Δ .

(184.3) Indeed, I say; that at another point not being on the perimeter, the ratio of the straight lines from the points A and B to it, does not become the same as that of Γ to Δ .

(184.7) For if possible, let it happen at the point M outside the perimeter; for indeed, if it be taken inside, the same contradiction will happen on account of the other hypotheses, and let MA, MB, MZ be joined; and let it be supposed that

$$\Gamma : \Delta = AM : MB.$$

So

$$E\Delta : \Delta = \text{sq.}(E\Delta) : \text{sq.}(\Gamma) = \text{sq.}(AM) : \text{sq.}(MB).$$

But

$$E\Delta : \Delta = AZ : ZB.$$

and so

$$AZ : ZB = \text{sq.}(AM) : \text{sq.}(MB).$$

And because of the things having been shown before, if we draw from B a parallel to AM, it will be shown that

$$AZ : ZB = \text{sq.}(AZ) : \text{sq.}(ZM).$$

But it was also shown, that

$$AZ : ZB = \text{sq.}(AZ) : \text{sq.}(Z\Theta).$$

Therefore

$$Z\Theta = ZM:$$

the very thing which is impossible.

(184.21) So much, then, for the planar loci: but the so-called solid loci received that designation from the fact that the figures (through which the problems relating to them

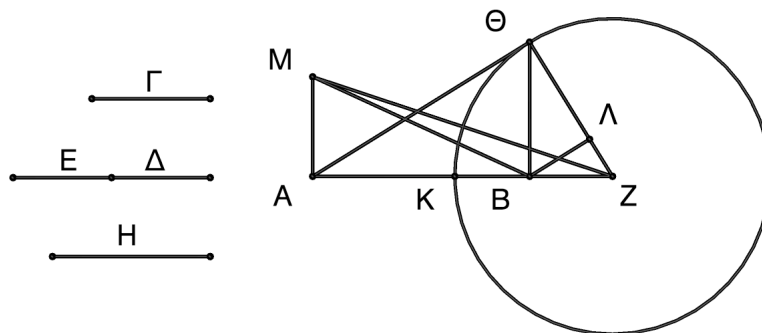


Figure 3.4: Eutocius' Locus Diagram

are drawn) have their origin from the section of solids, such as the sections of the cone and many others. But there are also other loci, being so-named according to a surface, which have their designation on account of the property concerning them.

(186.1) But he [Geminus] does not go on to blame Euclid, as Pappus and certain others believe, on account of his [Euclid] not having found two means proportional: for Euclid easily found one mean proportional, but not, as he himself says, by chance; and he absolutely did not try to investigate concerning the two means proportional in the *Elements*; and Apollonius himself seems to investigate nothing concerning two means proportional in his third book. But rather, as seems likely, he [Geminus] finds fault with another book written by Euclid concerning loci, which has not been carried down to us.

(186.11) But the subsequent things said about the fourth book are manifest. He says that the fifth book comprises things concerning the minima and maxima. For just as in the *Elements* we learned that in the case of the circle, there is a point outside, from which, when lines fall on the interior¹⁵, the one through the center is largest, and when they fall to the perimeter¹⁶, the one between the point and the diameter is least; so also he investigates the case of the sections of the cone in his fifth book. The theme of the sixth, seventh, and

¹⁵τὴν κοίλην περιφέρειαν, lit. the "concave perimeter".

¹⁶lit. the "convex perimeter".

eighth books has been stated clearly by him [Apollonius]. So much for his letter.

(186.22) Beginning with definitions, he sketches out the origin of the conic surface, but he did not give the definition of what it is: but it is permissible to take the definition from the origin itself for those wishing to do so. But what is being said by him, we will make manifest through a diagram.

"If, from some point to the circumference of a circle," etc.¹⁷ For let the circle be AB , the center of which be Γ , and let there be some point Δ not in the plane of the circle, and let ΔB , having been joined, be projected to infinity in both directions as towards E, Z . Indeed, if, with Δ being fixed, ΔB be carried around, until B , having been carried around upon the perimeter of the circle AB , is restored to its original position from whence it began to be revolved, it will generate a certain surface, which is comprised of two surfaces being joined to one another at Δ , which he calls a *conic surface*. But he says, that also it extends to infinity, on account of the fact that the straight line drawing it--for instance, ΔB --is projected infinitely. And he calls Δ the *vertex* of the surface, and $\Delta\Gamma$ the *axis*.

(188.13) And he calls a *cone* the shape which is bounded by both the circle AB and the surface, which the straight line ΔB alone draws, and calls Δ the *vertex* of the cone, and $\Delta\Gamma$ the *axis*, and the circle AB the *base*.

(188.17) And if, on the one hand, $\Delta\Gamma$ be orthogonal to the circle AB ¹⁸, he calls the cone right; but if, on the other hand, it is not orthogonal, he calls the cone oblique: but an oblique cone will be generated, whenever upon taking the circle, we set up a straight line from the center itself not orthogonal to the plane of the circle, but from the raised point of the set-up line we join a straight line to the circle, and rotate the joined straight line around the circle, with the point at the point¹⁹ on the set-up straight line remaining fixed:

¹⁷Eutocius is quoting Apollonius' definition here.

¹⁸i.e. the plane of the circle. The circle includes its interior.

¹⁹The Greek isn't terribly clear: here we have ἀπὸ δὲ τοῦ μετεώου σημείου τῆς ἀναταθείσης εὐθείας ἐπὶ τὸν κύκλον ἐπιζεύξωμεν εὐθείαν περὶ τὸν κύκλον τοῦ πρὸς τῷ μετεώρῳ σημείῳ τῆς ἀναταθείσης μένοντος. In the preceding materials, Eutocius specifically names the τοῦ μένοντος as the point Δ , even though here we have no specific names for any of the points or lines. For the genitive absolute

for the produced shape will be an oblique cone.

(190.1) But it is manifest that the rotating straight line becomes greater and smaller during this rotation, and in certain locations also will be equal at one point and another point of the circle. He proves this thus: if, straight lines be drawn from the vertex to the base of an oblique cone, of all the straight lines being drawn from the vertex to the base, one is least, and one is greatest; but two alone are equal, one on each side of the least and the greatest; but always the one nearer to the least is lesser than the one farther. Let there be an oblique cone, whose base is the circle $AB\Gamma$, and whose vertex is the point Δ . And since the line drawn perpendicular from the vertex of the oblique cone to the underlying plane²⁰ will fall either at the perimeter of the circle $AB\Gamma ZH$, or outside it, or inside it. Let ΔE first fall at the perimeter, as in the first diagram, and let the center of the circle be taken, and let it be K , and let EK from E to K be joined, and projected to B , and let $B\Delta$ be joined, and let there be taken two equal arcs EZ , EH , one on each side of E , and on each side of B as AB , $B\Gamma$, and let EZ , EH , ΔZ , ΔH , EA , $E\Gamma$, AB , $B\Gamma$, ΔA , and $\Delta\Gamma$ be joined. So since

$$EZ = EH$$

(for they are subtended by equal arcs), but also ΔE is common and perpendicular, therefore

$$\text{base } \Delta Z = \text{base } \Delta H.$$

Again, since

$$\text{arc}(AB) = \text{arc}(B\Gamma),$$

and the diameter of the circle is BE ,

$$\text{remainder arc}(EZ\Gamma) = \text{remainder arc}(EHA):$$

τοῦ μένοντος, I read (a) that it refers to the same point which we selected not on the plane of the circle, and (b) it refers to what in the previous diagram is the point Δ . It seems reasonable to me that Eutocius is doubly-referring to this point (the point at the point) in order to remind us of the previously addressed case of the right cone.

²⁰Again, the plane of the base circle.

so that also

$$AE = E\Gamma.$$

But also $E\Delta$ is common and orthogonal: therefore the base ΔA is equal to the base $\Delta\Gamma$. And likewise, all those equally distant from either ΔE or ΔB will be shown to be equal. Again, since of the triangle ΔEZ , the angle under ΔEZ is a right angle, ΔZ is greater than ΔE . And again, since the straight line EA is greater than the straight line EZ , and since the arc EZA is greater than the arc EZ , but also ΔE is common and orthogonal, ΔZ is therefore less than ΔA . Also on account of the same things, ΔA is lesser than ΔB . So since ΔE was shown to be less than ΔZ , ΔZ less than ΔA , and ΔA less than ΔB ; ΔE is the smallest, ΔB is largest largest, but always the one nearer to ΔE is less than the one further.

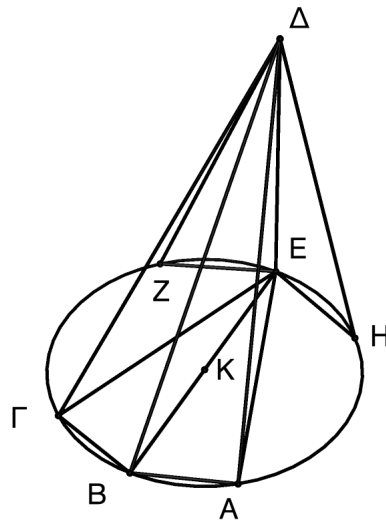


Figure 3.5: Maximum and Minimum Lines on a Cone with E on the Base Circle

(192.15) But indeed let the perpendicular ΔE fall outside of the circle $AB\Gamma HZ$, as in the second diagram, and let again the center of the circle K be taken, and let EK be joined and projected to B , and let ΔB , $\Delta\Theta$ be joined, and let two equal arcs ΘZ , ΘH be taken, one on each side of Θ , and let two equal arcs AB , $B\Gamma$ be taken, one on each side of B , and let EZ , EH , ZK , HK , ΔZ , ΔH , AB , $B\Gamma$, KA , $K\Gamma$, ΔK , ΔA , $\Delta\Gamma$ be joined.

So since the arc ΘZ is equal to the arc ΘH , the angle under ΘKZ is therefore also equal to the angle under ΘKH . So since the straight line ZK is equal to the straight line KH --for they are both radii--but KE is common, and the angle under ZKE is common to the angle under HKE , and the base ZE is equal to the base HE . So since the straight line ZE is equal to the straight line HE , and $E\Delta$ is common and orthogonal, the base ΔZ is therefore equal to the base ΔH . Again, since the arc BA is equal to the arc $B\Gamma$, the angle under AKB is also therefore equal to the angle under ΓKB , so that also the remaining angle under AKE is equal to the remaining angle under ΓKE . So since the straight line AK is equal to the straight line ΓK --for they are both radii--but KE is common, the two are equal to the two, and the angle under AKE is equal to the angle under ΓKE : and the base AE is therefore equal to the base ΓE . So since the straight line AE is equal to the straight line ΓE , and the straight line $E\Delta$ is both common and orthogonal, the base ΔA is therefore equal to the base $\Delta \Gamma$. Likewise also all the straight lines that are equidistant from ΔB or $\Delta \Theta$ will be shown to be equal. Also, since $E\Theta$ is less than EZ , but also $E\Delta$ is common and orthogonal, the base $\Delta \Theta$ is therefore less than the base ΔZ . Again, since the segment from E touching the circle is bigger than all those falling towards the arc, and it was shown in the third book of the *Elements*²¹, that the rectangle contained by AE , $E\Delta$ is equal to the square on EZ , whenever EZ is tangent, it is manifest that, as AE is to EZ , so too is EZ to $E\Delta$. But EZ is greater than $E\Delta$: for always the one nearer to the least is less than the one further: also AE is therefore greater than EZ . So since EZ is smaller than EA , and $E\Delta$ common and orthogonal, the base ΔZ is therefore smaller than the base ΔA . Again, since AK is equal to KB , but KE is common, the two straight lines AK , KE are therefore equal to EK , KB , i.e. to the whole EKB . But AK , KE are bigger than AE : and BE is therefore greater than AE . Again, since AE is smaller than EB , but $E\Delta$ is common and orthogonal, the base ΔA is therefore smaller than the base $B\Delta$. So since $\Delta \Theta$ is smaller than ΔZ , and ΔZ smaller than ΔA , and ΔA smaller than ΔB , $\Delta \Theta$ is

²¹ *Elements* III.36.

smallest and ΔB the biggest, and the nearer is always smaller than the further.

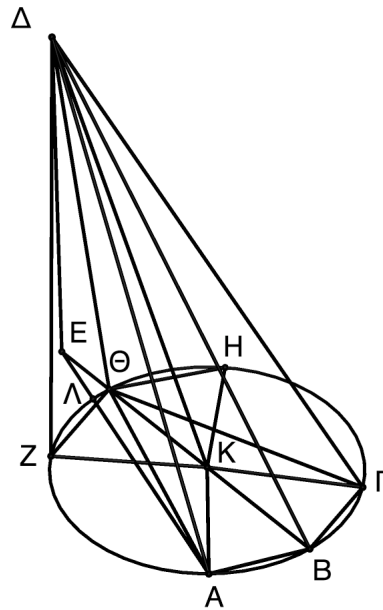


Figure 3.6: Maximum and Minimum Lines on a Cone with E outside the Base Circle

(196.7) But indeed let the perpendicular ΔE fall inside the circle $AB\Gamma HZ$ as in the third diagram, and K be taken as the center of the circle, and let EK be joined and produced in each direction to B , Θ , and let $\Delta\Theta$, ΔB be joined, and let two equal arcs ΘZ , ΘH be taken on each side of Θ , and let EZ , EH , ZK , HK , ΔZ , ΔH , KA , $K\Gamma$, EA , $E\Gamma$, ΔA , $\Delta\Gamma$, AB , $B\Gamma$ be joined. So since the arc ΘZ is equal to the arc ΘH , the angle under ΘKZ is therefore also equal to the angle under ΘKH . And since KZ is equal to HK , and KE is common, also the angle under ZKE is equal to the angle under HKE , the base ZE is therefore equal to the base HE . So since ZE is equal to HE , and ΔE is common, and the angle under $ZE\Delta$ is equal to the angle under $HE\Delta$, the base ΔZ is therefore equal to the base ΔH . Again, since the arc AB is equal to the arc $B\Gamma$, the angle under AKB is also therefore equal to the angle under ΓKB : so that also the remainder to two right angles AKE is equal to the remainder to two right angles ΓKE . So since AK is equal to $K\Gamma$, and EK is common, and the angle under AKE is equal to the angle under

ΓKE , the base AE is therefore equal to the base ΓE . So since AE is equal to ΓE , and $E\Delta$ is common, and the angle under $AE\Delta$ is therefore equal to the angle under $\Gamma E\Delta$, the base ΔA is therefore equal to the base $\Delta \Gamma$. Likewise also all the straight lines that are equidistant from either ΔB or $\Delta \Theta$ will be shown to be equal. And since, in the circle $AB\Gamma$, the point E which is not the center of the circle has been taken on the diameter, EB is greatest, $E\Theta$ is least; but always the one nearer to $E\Theta$ is less than the one farther: so that $E\Theta$ is less than EZ . And since ΘE is less than $Z E$, but also $E\Delta$ is common and orthogonal, the base $\Delta \Theta$ is therefore smaller than the base ΔZ . Again, since EZ is nearer than $E\Theta$ and AE is even further, EZ is less than AE . So since EZ is lesser than EA , and $E\Delta$ is common and orthogonal, the base ΔZ is therefore smaller than the base ΔA . Again, since AK is equal to KB , and KE is common, the two straight lines AK, KE are equal to the two straight lines BK, KE , i.e. to the whole BKE . But AK, KE are greater than AE : and EB is therefore bigger than EA . Again, since EA is less than EB , and $E\Delta$ is common and orthogonal to them, the base ΔA is therefore smaller than the base ΔB . So since $\Delta \Theta$ is smaller than ΔZ , and ΔZ smaller than ΔA , and ΔA smaller than ΔB , $\Delta \Theta$ is smallest, etc.

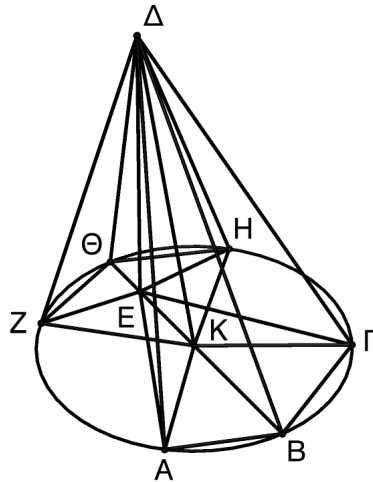


Figure 3.7: Maximum and Minimum Lines on a Cone with E inside the Base Circle

(198.26) "Of each curved figure, which is in one plane, I call the diameter," etc.²² He said "in one plane" because of the helix of the cylinder and the sphere: for they are not in one plane. But what he means is as follows: let there be a curved figure $AB\Gamma$ and in it some parallel straight lines $A\Gamma, \Delta E, ZH, \Theta K$, and let from B the straight line BA be carried cutting them in two. So he says, that the *diameter* of the curve $AB\Gamma$ I call BA , the *vertex* B, and each of the lines $A\Gamma, \Delta E, ZH, \Theta K$ have been drawn ordinatewise to BA ²³. But if BA bisects and cuts at right angles the parallels²⁴, it is called the *axis*.

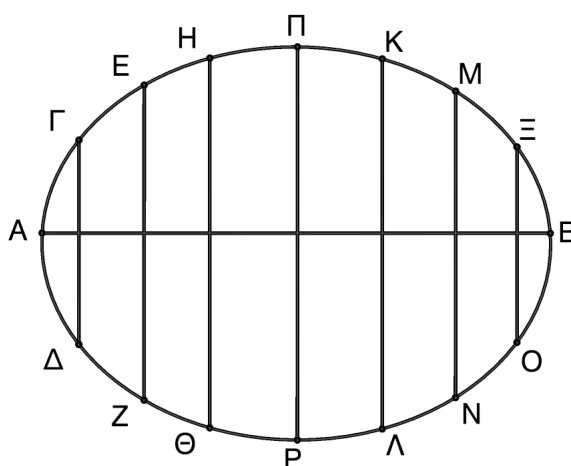


Figure 3.8: The Axis and Ordinates of an Ellipse

(200.10) "But similarly of two curved lines," etc. For if we consider the curves A, B and in them the parallels $\Gamma\Delta, EZ, H\Theta, ΚΛ, MN, \Xi O$ and the straight line AB , having been produced in each direction and cutting the parallels in two, I call AB , he says, the *transverse diameter*; the points A and B the *vertices* of the curves; and the curves having been drawn ordinatewise to AB the lines $\Gamma\Delta, EZ, H\Theta, ΚΛ, MN, \Xi O$ ²⁵. And if it (AB)

²²Eutocius is again quoting Apollonius.

²³τεταγμένως δὲ ἐπὶ τὴν BA κατῆχθαι. Taliaferro renders this as *having been drawn ordinatewise to the [diameter] BA*.

²⁴i.e. these ordinates.

²⁵In short, just *the ordinates*.

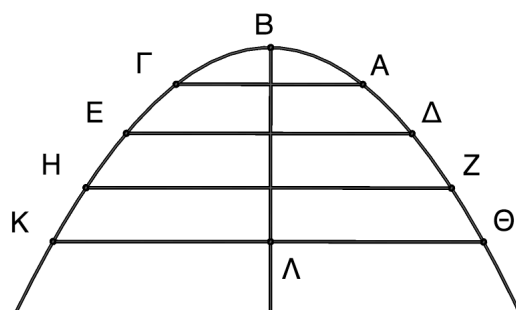


Figure 3.9: The Axis and Ordinates of a Parabola

also bisects them at right angles, it is called the *axis*. But if a certain straight line, such as ΠP , having been carried through $\Gamma \Xi$, EM , HK bisects the parallels to AB , then ΠP is called the *upright diameter*, and each of the lines $\Gamma \Xi$, EM , HK , [he says] have been drawn ordinatewise to the upright diameter ΠP . But if it bisects it at right angles, it is called *the upright axis*; and if AB , ΠP bisect the parallels of each other, they are called *the conjugate diameters*, but if they bisect at right angles, they are called the *conjugate axes*.

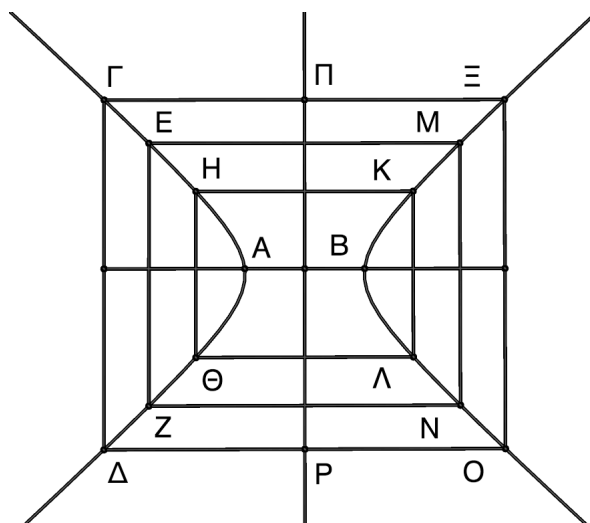


Figure 3.10: The Conjugate Axes of Two Opposite Sections

3.2 Theorem 1

(202.6) Concerning the different diagrams, one must see the many cases of the theorems, that a case is, whenever the things having been specified in the *protasis*, are given for a particular arrangement: for the taking of a different one of them, with the same *sumperasma*, makes the case. But similarly also a case occurs from the changing of the *kataskeue*. But when the theorem has many cases, the same proof fits all of them, and in the same elements save small differences; we will see this successively. For directly, the first theorem has three cases, on account of the fact that the point having been taken on the surface, i.e. B, is sometimes on the lower surface and this in two ways, either higher than the circle or lower; and sometimes on the part which, according to the vertex, lies opposite it.

He proposed this theorem to investigate that it is not true that any two points being taken on the surface, a straight line being joined between them is on the surface; rather only the inclination beyond the vertex, because of the fact that the conic surface is generated by a straight line having its boundary remaining fixed. But the second theorem shows that this is true.

3.3 Theorem 2

(202.26) The second theorem has three cases, because the chosen points Δ , E are either on the surface above the vertex, or below it in two ways: either inside or outside the circle. But it is necessary to understand that this theorem in some copies is found proved in full through the *reductio ad absurdum*.

3.4 Theorem 3

(204.7) The third theorem does not have cases. But it is necessary to understand in it that the line AB is straight on account of being the common section of the cutting plane and the surface of the cone, that which was drawn by a straight line having its boundary remaining fixed at the vertex of the surface. For the whole surface, cutting the section by a means of a plane, does not make a straight line; nor does the cone itself, unless the

cutting plane passes through the vertex.

3.5 Theorem 4

(204.16) There are three cases of this theorem, just as of both the first and the second.

3.6 Theorem 5

(204.19) The fifth theorem does not have cases. But beginning the *ekthesis* he says: "let the cone be cut by a plane through the axis at right angles to the base."²⁶

But since in an oblique cone, the triangle through the axis is perpendicular to the base according to one arrangement only, we will show this thus: taking the center of the base, we will construct from it, at right angles to the plane of the base, and projecting a plane through it and the axis, we will have the thing being sought: for it is shown in the eleventh book of Euclid's *Elements*, that, if a straight line is orthogonal to some plane, then also all the planes through it [the line] will be orthogonal to the same plane. But he set forth the oblique cone, since in the isosceles cone²⁷, the plane parallel to the base is the same as the one drawn subcontrariwise.

(206.7) Still, he says: "But let it also be cut by another plane perpendicular to the triangle through the axis, and which cuts off a triangle near the vertex which is similar to the triangle $AB\Gamma$, but which is lying subcontrariwise." But this occurs thus: for let there be the triangle through the axis $AB\Gamma$, and let a random point H be taken on AB , and let it the angle AHK be supposed equal to the angle $A\Gamma B$, i.e. the one near the straight line AH and the point H on it; therefore the triangle AHK is similar to $AB\Gamma$, but lying subcontrariwise. Indeed, on HK , let a point Z be taken at random, and from Z let $Z\Theta$ be set up at right angles to the plane of the triangle $AB\Gamma$, and let the plane through HK , ΘZ be projected. Indeed, this is orthogonal to the triangle $AB\Gamma$ on account of $Z\Theta$ and

²⁶This is not an exact quote of the Greek as it appears in Heiberg, but more of a synopsis. The important parts match.

²⁷It is interesting that he calls the non-oblique cone *isosceles* here, as opposed to *right*.

making the hypothesis.

(206.22) He says in the *sumperasma*, that on account of the similarity of the triangles ΔZH , EZK , the angle ΔZE is equal to the angle HKZ . But it is possible to show this without reference to the similarity of the triangles, saying that since each of the angles AKH , $A\Delta E$ is equal to that at B , the points Δ , H , E , K are in the same section of the circumscribed circle. And since in the circle, the two straight lines ΔE , HK cut one another at Z , the rectangle ΔZE is equal to the rectangle HZK ²⁸.

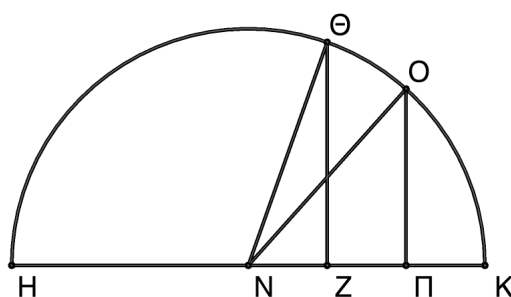


Figure 3.11: Eutocius' First Diagram for Theorem 5

(208.7) Similarly it will be shown, that also all the lines being drawn perpendicularly from the curve $H\Theta$ to HK are equal in square to the rectangle contained by the sections²⁹. The section is therefore a circle, and the diameter of it is HK . And it is possible to conclude this through *reductio ad absurdum*. For if the circle, being drawn around KH , does not pass through the point Θ , the rectangle KZ , ZH will be equal either to the square on a segment bigger than $Z\Theta$ or one smaller: the very thing which is not supposed. But we will show this also in the case of a straight line.

(208.17) Let there be some curve $H\Theta$, and let HK subtend it, and let the points Θ , O be taken at random on the curve, and from them $[\Theta, O]$ let ΘZ , $O\Pi$ be drawn to HK at

²⁸*Elements* III.36.

²⁹That is, the perpendicular meets the diameter at a point, and the rectangle is formed by the two segments of the diameter, each of which has one endpoint as an endpoint of the diameter, and the other the point where the perpendicular meets the diameter.

right angles, and let the square on $Z\Theta$ be equal to the rectangle HZK , and let the square on $O\Pi$ be equal to the rectangle $H\Pi K$. I say that the curve $H\Theta O K$ is a circle. For let HK be cut in two at N , and let $N\Theta$, NO be joined. So since the straight line HK is cut into equal segments at N , and into unequal segments at Z , the rectangle HZK together with the square on NZ is equal to the square on NK ³⁰. But the rectangle HZK was supposed equal to the square on ΘZ ; therefore the square on ΘZ together with the square on NZ is equal to the square on NK . But the squares on ΘZ , ZN are [together] equal to the square on $N\Theta$: for the segment to Z is orthogonal: therefore the square on $N\Theta$ is equal to the square on NK . Similarly we will also show that the square on NO is equal to the square on NK . Therefore the curve $H\Theta K$ is a circle, and HK is its diameter.

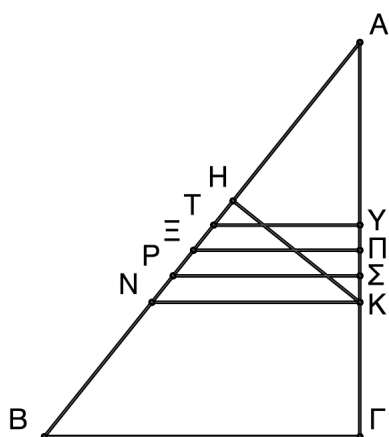


Figure 3.12: Eutocius' Second Diagram for Theorem 5

(210.8) But it is possible for the diameters ΔE , HK sometimes to be equal, other times unequal, but it is never possible for them to bisect one another. For let HK be drawn through K parallel to $B\Gamma$. So since BA is larger than $A\Gamma$, also NA is bigger than AK . Similarly also, KA [is bigger than] AH on account of the subcontrary section, so that the [line] being taken equal from AN to AK falls between the points H , N . Let it fall as $A\Xi$: therefore the line being drawn parallel to $B\Gamma$ through Ξ cuts HK . Let it cut as $\Xi O\Pi$. And

³⁰*Elements* II.5

since ΞA is equal to AK , as ΞA is to $A\Pi$, so too is KA to AH on account of the similarity of the triangles HKA , $\Xi A\Pi$, and AH is equal to $A\Pi$, also the remainder $H\Xi$ is equal to the remainder ΠK . And since the angles at Ξ , K are equal: for each of them is equal to that at B : but also the [angles] at O are equal: for they are vertical angles: therefore the triangle ΞHO is similar to the triangle ΠOK . And $H\Xi$ is equal to ΠK : so that also ΞO is equal to OK and HO to $O\Pi$ and the whole HK to $\Xi\Pi$. And it is manifest, that, if a point be taken between N , Ξ , such as P , and through P , the line $P\Sigma$ be drawn parallel to NK , it will be larger than $\Xi\Pi$; and on account of this, [it will also be bigger than] HK ; but if a point be taken between H , Ξ , such as T , and through it a parallel $T\Pi$ be drawn, it will be smaller than $\Xi\Pi$ and KH . And since the angle $\Xi\Pi K$ is bigger than the angle $A\Xi\Pi$, but the angle $O\Pi K$ is equal to the angle OHE , therefore the angle OHE is bigger than the angle $H\Xi O$. Therefore ΞO is bigger than OH , and on account of this, KO is bigger than $O\Pi$. But if one of them should ever be cut in two, the other will be cut in unequal [parts]

3.7 Theorem 6

(212.14) It is necessary to pay heed, that he set forth in the *protasis* not without purpose the necessity that the straight line drawn from the point on the surface be drawn parallel to some particular one of the straight lines in the base, being without a doubt orthogonal to the base of the triangle through the axis: for when this is not the case, it is not possible for it [the drawn straight line] to be bisected by the triangle through the axis: the very thing which is manifest from the diagram in the text. For if MN , regardless to whatever line ΔZH is parallel, is not orthogonal to $B\Gamma$, it is manifest that neither it [MN] nor $K\Lambda$ is bisected. And this follows from the same ratios, that is,

$$K\Theta : \Theta\Lambda = \Delta Z : ZH,$$

and ΔH will therefore be been cut into unequal sections at Z .

(212.27) But it is possible to show the same things of the lower circle and in the case of the surface opposite the vertex³¹.

³¹That is, the other surface of the conic surface. These two additional cases match Apollonius' diagrams very well.

3.8 Theorem 7

(214.2) The seventh theorem has four cases: for either ZH does not intersect $A\Gamma$, or it intersects in three ways: either outside the circle, or inside, or at the point Γ .

3.9 After Theorem 10

(214.6) It is necessary to realize, that these ten theorems depend on one another. But the first holds that the straight lines in the [conic] surface inclining towards the vertex stay in it. The second holds the converse. The third holds the section through the vertex of the cone; the fourth the [section] parallel to the base; the fifth, the subcontrary; the sixth, as if it were anticipating the seventh, shows that the common section of the circle and the cutting plane is in all cases necessarily at right angles to the diameter [of that circle], and that this being so, the parallels to it [the common section] are bisected by the triangle. But the seventh shows the other three sections, both that the diameter and the ordinates to it are parallel to the straight line in the base. In the eighth he shows, just as we said in the preceding materials, that the parabola and the hyperbola are of those things projecting to infinity; but in the ninth, that the ellipse, returning to itself like a circle does (on account of the cutting plane intersecting both sides of the triangle), is not a circle: for both the subcontrary section and the parallel [section] made circles. But it is necessary to understand, that the diameter of the section, in the case of the parabola, cuts one side of the [axial] triangle and the base [of it]; in the case of the hyperbola, both the side and the remaining side, being projected in a straight line from the vertex. But in the case of the ellipse, it cuts both the sides and the base. And the tenth, someone giving attention to it in a rather simple way might think that it is the same as the second, but this is not the case: for there, he says to take the two points on the whole surface, but here on the generated figure. In the following three theorems, he distinguishes more precisely each of the sections, along with specifying the principle peculiarities of them.

3.10 Theorem 11

(216.13) Let it have been made, that

$$\text{sq.}(\text{B}\Gamma) : \text{rect.}(\text{B}\text{A}\Gamma) = \Theta\text{Z} : \text{Z}\text{A}:$$

the thing being said is manifest, unless someone wants to comment on it. For let the rectangle $\text{O}\Pi\text{P}$ be equal to the rectangle $\text{B}\text{A}\Gamma$, and let it, being projected along ΠP , make $\Pi\Sigma$ as width and be equal³² to the square on $\text{B}\Gamma$; and let it have happened that

$$\text{O}\Pi : \Pi\Sigma = \text{A}\text{Z} : \text{Z}\Theta.$$

The thing being sought has therefore happened. For since it is the case, that

$$\text{O}\Pi : \Pi\Sigma = \text{A}\text{Z} : \text{Z}\Theta;$$

invertendo,

$$\Sigma\Pi : \Pi\text{O} = \Theta\text{Z} : \text{Z}\text{A}.$$

But

$$\Sigma\Pi : \Pi\text{O} = \Sigma\text{P} : \text{P}\text{O} = \text{sq.}(\text{B}\Gamma) : \text{rect.}(\text{B}\text{A}\Gamma).$$

This is useful also in the following two theorems.

(218.1) But the square on $\text{B}\Gamma$ has to the rectangle $\text{B}\text{A}\Gamma$ a ratio compounded from that which $\text{B}\Gamma$ has to ΓA and $\text{B}\Gamma$ to BA :

$$\text{sq.}(\text{B}\Gamma) : \text{rect.}(\text{B}\text{A}\Gamma) = (\text{B}\Gamma : \Gamma\text{A}) \text{ comp. } (\text{B}\Gamma : \text{B}\text{A}) :$$

it has been shown in the twenty-third theorem of the sixth book of the *Elements*, that equiangular parallelograms have to one another a ratio compounded out of that of the sides: but since it is discussed too inductively and not in the necessary manner by the commentators, we researched it; and it is written in our published work on the fourth theorem of the second book of Archimedes' *On the Sphere and the Cylinder*, and also in the scholia of the first book of Ptolemy's *Syntaxis*: but it is a good idea that this be written down here also, because readers do not always read it even in those works, and also because nearly the entire treatise of the *Conics* makes use of it.

(218.16) A ratio is said to be compounded from ratios, whenever the sizes of the

³²i.e. the applied area is equal

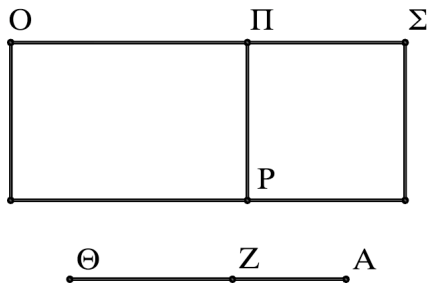


Figure 3.13: Eutocius' Diagram for Theorem 11

ratios, being multiplied into themselves, make something, with "size" of course meaning the number after which the ratio is named. So it is possible in the case of multiples that the size be a whole number, but in the case of the remaining relations it is necessary that the size must be a number plus part or parts, unless perhaps one wishes the relation to be irrational, such as are those according to the incommensurable magnitudes. But in the case of all the relations, it is manifest that the product of the size itself with the consequent of the ratio makes the antecedent.

(218.27) Accordingly, let there be a ratio of A to B, and let some mean of them be taken, as it chanced, as Γ , and let

$$\Delta = \text{size}(A : \Gamma),$$

and

$$E = \text{size}(\Gamma : B),$$

and let Δ , multiplying E, make Z. I say, that the size of the ratios A, B is Z,

$$Z = \text{size}(A : B),$$

that is that Z, multiplying B, makes A. Indeed, let Z, multiplying B, make H. So since Δ , multiplying E has made Z, and multiplying Γ has made A, therefore it is, that

$$E : Z = \Gamma : H$$

Alternando,

$$E : \Gamma = Z : H$$

But

$$E : \Gamma = Z : A$$

Therefore

$$H = A,$$

so that Z , multiplying B , has made A .

(220.17) But do not let this confuse those reading that this has been proved through arithmetic, for both the ancients made use of such proofs, being mathematical rather than arithmetical on account of the proportions, and that the thing being sought is arithmetical. For both ratios and sizes of ratios and multiplications, first begin by numbers, and through them by magnitudes, according to the speaker. As someone once said, "for these mathematical studies appear to be related."³³

3.11 Theorem 13

(222.2) It is necessary to point out, that this theorem has three diagrams, as has been said often in the case of the ellipse: for ΔE either falls on $A\Gamma$ above Γ , or at Γ itself, or meets $A\Gamma$, having been projected, outside.

3.12 Theorem 14

222.8) And it was necessary to likewise show, that,

$$\text{sq.}(A\Sigma) : \text{rect.}(B\Sigma\Gamma) = \text{sq.}(AT) : \text{rect.}(\Xi TO).$$

(222.11) For since $B\Gamma$ is parallel to ΞO ,

$$\Gamma\Sigma : \Sigma A = \Xi T : TA,$$

and on account of this,

$$A\Sigma : \Sigma B = AT : TO:$$

therefore, through equality,

$$\Gamma\Sigma : \Sigma B = \Xi T : TO,$$

³³Much ought be said of this passage; some will regrettably be left for future work. Here Eutocius retains the Doric dialect of the quotation: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν ἀδελφά. The quote is due to Archytas of Tarentum, preserved only in a fragment of his *Harmonics*. The quoted part reads: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι ἡμεν ἀδελφεία. For the full fragment, see fr. B.1 in Diels-Kranz ([6]) or Freeman's English translation ([2]). See also *Republic* VII 530d and Knorr ([7]), pp. 171 n. 22-23.

Therefore also

$$\text{sq.}(\Gamma\Sigma) : \text{rect.}(\Gamma\Sigma\text{B}) = \text{sq.}(\Xi\text{T}) : \text{rect.}(\Xi\text{T}\text{O}).$$

But on account of similar triangles,

$$\text{sq.}(\text{A}\Sigma) : \text{sq.}(\Sigma\Gamma) = \text{sq.}(\text{A}\text{T}) : \text{sq.}(\Xi\text{T}):$$

therefore, through equality,

$$\text{sq.}(\text{A}\Sigma) : \text{rect.}(\text{B}\Sigma\Gamma) = \text{sq.}(\text{A}\text{T}) : \text{rect.}(\Xi\text{T}\text{O}).$$

(222.20) And so,

$$\text{sq.}(\text{A}\Sigma) : \text{rect.}(\text{B}\Sigma\Gamma) = \Theta\text{E} : \text{E}\Pi,$$

but also,

$$\text{sq.}(\text{A}\text{T}) : \text{rect.}(\Xi\text{T}\text{O}) = \Theta\text{E} : \Theta\Pi:$$

and therefore,

$$\Theta\text{E} : \text{E}\Pi = \text{E}\Theta : \Theta\Pi.$$

Therefore

$$\text{E}\Pi = \Theta\Pi.$$

(222.24) But it does not have cases, and the investigation is manifestly continuous with the three before it: for similarly, by means of them, he investigates that the diameter of the opposite sections is the principle one, and also investigates the parameters³⁴.

3.13 Theorem 16

(224.2) The rectangle BKA is therefore equal to the rectangle AΛB--KA is therefore equal to BΛ--for since the rectangle BKA is equal to the rectangle AΛB, it will be proportionally, that

$$\text{KB} : \text{A}\Lambda = \Lambda\text{B} : \text{AK}.$$

Alternando,

$$\text{KB} : \text{B}\Lambda = \Lambda\text{A} : \text{AK}.$$

Componendo,

$$\text{K}\Lambda : \Lambda\text{B} = \Lambda\text{K} : \text{K}\Lambda.$$

³⁴τὰς παρ' ἄς δύνανται: I take this as a referring to the upright and transverse sides collectively. See Eutocius' commentary to the next theorem.

Therefore

$$KA = BA.$$

(224.8) It is necessary to understand, that in the fifteenth and sixteenth theorems he had an investigation to find the so-called second and conjugate diameters of both the ellipse and the hyperbola or the opposite sections: for the parabola does not have this type of diameter. But one must note that the diameters of the ellipse fall inside, but of those of the parabola and the opposite sections outside. But it is necessary, when drawing the figures, to arrange the parameters or the upright sides at right angles, and also manifestly those [lines] parallel to them, but those being drawn ordinatewise and the second diameters not always: for most of all in an acute angle, it is necessary to drop them, so that they be clear to those encountering them as being different from the parallels to the upright side.

3.14 Second Definitions

(224.22) After the sixteenth theorem, he sets out the definitions concerning the so-called second diameter of the hyperbola and the ellipse, which we will make clear by means of a diagram.

(224.26) For let there be a hyperbola AB , and let ΓBA be a diameter of it, and let BE be the parameter³⁵. So it is manifest that $B\Gamma$ increases to infinity on account of the section, as is shown in the eight theorem, but $B\Delta$, that is the one subtending the angle outside of the axial triangle, is finite. Indeed, bisecting it at Z and setting out from A the ordinate AH , but through Z , parallel to AH , setting out ΘZK and having made ΘZ equal to ZK , yet still also, the square on ΘK equal to to the rectangle ΔBE , we will have ΘK as the second diameter. For this is possible on account of the fact that ΘK , being outside of the section, is projected to infinity, and that it is possible to intercept one equal to a straight line extended from infinity. He calls Z the center, and names ZB and the lines similar to it which are drawn from the center Z to the section.

³⁵Fully: "that according to which the ordinates being led to $B\Gamma$ are equal in square." Henceforth I will consider this a standard abbreviation.

(226.16) So much for the cases of the hyperbola and the opposite sections: And it is clear, that each of the diameters is finite, the first obviously from the genesis of the section, the second, for the reason that it is a mean proportional between two finite straight lines, namely, the first diameter and the parameter.

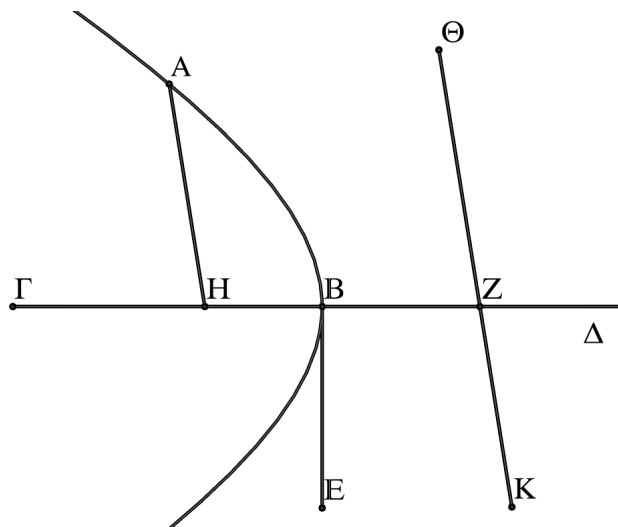


Figure 3.14: Two Diameters of an Hyperbola

(226.23) But in the case of the ellipse the thing said is not yet clear. For since it returns to itself, just as does the circle, and cuts off all the diameters inside and makes them bounded: so that not always in the case of the ellipse, the mean proportional of the sides of the figure, both being drawn through the center of the section and being bisected by the diameter, is bounded by the section.

(228.1) But it is possible to recapitulate it through the very things already said in the fifteenth theorem. For since, as it is shown there, the lines being drawn to ΔE parallel to AB are equal in square to the area applied along the third proportional to them³⁶, that is,

³⁶Conics I.15: Here the diameter is AB , the "produced straight line" is ΔE , and the third proportional ($\tau\eta\nu\ \tau\rho\iota\tau\eta\nu$) to them is some straight line AN satisfying

$$\Delta E : AB : AB : AN.$$

The lines so drawn, being parallel to AB , are equal in square to an area having this third proportional AN as its width. It is critical to note that this third is a third proportional, not a third mean proportional.

AN. Therefore on account of this, the second diameter ΔE becomes a mean proportional between the sides of the figure BA and AN.

(228.13) But it is necessary to see also this on account of the utility of the diagrams: for since the diameters AB, ΔE are unequal (for in the circle only are they equal), it is clear that the line being dropped at right angles to the lesser of them, as at ΔZ , seeing as how it is the third term of the proportion between ΔE , AB, is the larger of the two; and that the line being drawn at right angles to the greater, as at AN, on account of being the third term of the proportion between AB, ΔE , is lesser of the two, so that the four terms are in continued proportion: for as AN is to ΔE , so too is ΔE to AB and AB to ΔZ .

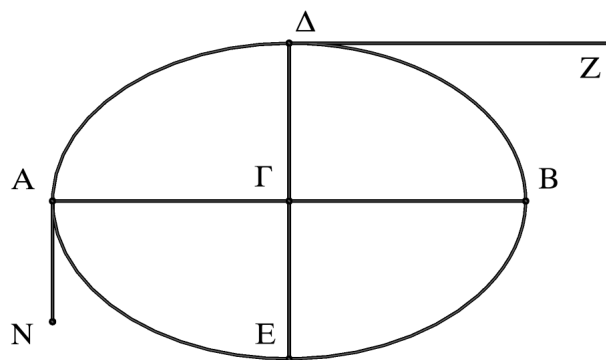


Figure 3.15: Two Diameters of an Ellipse

3.15 Theorem 17

(228.28) Euclid showed in the fifteenth theorem of the third book of the *Elements*, that the line being led at right angles from an endpoint of the diameter both falls outside and is tangent to the circle. But Apollonius in this work shows something more general, that it is possible to apply this to the the three sections of the cone and to the circle.

(230.5) The circle differs to this extent from the sections of the cone, that in this case the ordinates are drawn at right angles to the diameter: for no other straight lines parallel to themselves are bisected by the diameter of the circle, but in the case of the three sections, not always are they drawn at right angles, except to the axes alone.

3.16 Theorem 18

(230.13) In some copies, this theorem concerns a parabola and ellipse only, but it is better to have the protasis more general, unless it is because the case of the ellipse has been omitted by them as unambiguous: for $\Gamma\Delta$, being inside the section (which is finite), cuts the section in both directions.

(230.19) But it is necessary to understand, that even if AZB cuts the section, the same proof is suitable.

3.17 Theorem 20

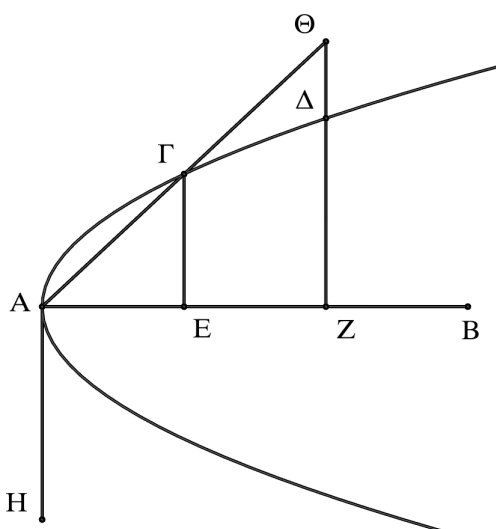


Figure 3.16: *Conics* I.20, with $\Delta\Theta$ added

(230.22) Beginning from this theorem, successively in all of them he shows the *symptomata* of the parabola that belonging to it and not to any other [section], but for the most part, he shows the same *symptomata* belonging to the hyperbola and ellipse.

(230.27) But since it does not appear useless to those making mechanical drawings, on account of the difficulty concerning instruments³⁷, also often through successive points to draw the sections of the cone in the plane, through this theorem it is possible to provide successive points, through which the parabola will be drawn by means of an application

³⁷As in mechanical instruments. "Gadgets" seems a fitting, though unacademic, word.

of a ruler. For if I set out a straight line, such as AB, and on it I take successive points, such as E, Z, and from them I make straight lines at right angles to AB, such as EΓ, ZΔ, taking on EΓ a random point Γ (if I should wish to make the parabola wider, taking Γ further from E; if narrower, closer), and I make the proportion

$$AE : AZ = \text{sq.}(E\Gamma) : \text{sq.}(Z\Delta),$$

then the points Γ and Δ will be on the section. But similarly we will also choose other

points, through which the parabola will be drawn³⁸.

3.18 Theorem 21

(232.16) The theorem is set forth manifestly, and does not have cases: nevertheless, it is necessary to understand that the parameter, that is the upright side, is equal to the

³⁸What application of what ruler? Eutocius unfortunately does not elaborate on this point. Based on his use of the word "mechanical," I can think of two possibilities for just how this "application" would be performed.

1. He means to use a marked ruler, much like a modern one. If this is the case, a degree of flexibility with arithmetic would be required, in order to solve the proportion

$$AE : AZ = \text{sq.}(\Gamma E) : \text{sq.}(Z\Delta).$$

This is certainly plausible, as similar arithmetical problems appear in the *Almagest* of Ptolemy. Certainly engineers or architects would be capable of this sort of arithmetic, and able to instruct their subordinates as to the desired measurements. It is worth recalling that Anthemius, the person to whom Eutocius addresses this work, was selected by the Byzantine emperor Justinian I to be one of the architects of the Hagia Sophia (the other being Isidore of Miletus).

2. That Eutocius has a more mechanical and mathematical procedure in mind, similar in spirit to some of the cube duplication procedures that he gives in his commentary on Archimedes.

The construction is straightforward up until Eutocius requires the proportion

$$AE : AZ = \text{sq.}(\Gamma E) : \text{sq.}(Z\Delta).$$

All of the terms in this proportion are known, with the exception of the position of Δ on $Z\Delta$. Let $Z\Delta$ be produced to Θ , and let $A\Gamma$ be joined and produced to Θ . Let $Z\Delta$ be the mean proportional between ΓE and $Z\Theta$, so that

$$\Gamma E : Z\Delta = Z\Delta : Z\Theta.$$

From the standpoint of constructing $Z\Delta$, this is easily accomplished by applying *Elements* VI.13. Therefore

$$\text{sq.}(Z\Delta) = \text{rect.}(\Gamma E, Z\Theta), \text{ and so}$$

$$\text{sq.}(\Gamma E) : \text{sq.}(Z\Delta) = \text{sq.}(\Gamma E) : \text{rect.}(\Gamma E, Z\Theta). \text{ But}$$

$$\text{sq.}(\Gamma E) : \text{rect.}(\Gamma E, Z\Theta) = \Gamma E : Z\Theta,$$

and by virtue of the similarity of the triangles $AZ\Theta$, $A\Gamma E$, we have that

$$\Gamma E : Z\Theta = AE : AZ, \text{ so}$$

$$AE : AZ = \text{sq.}(\Gamma E) : \text{rect.}(\Gamma E, Z\Theta).$$

Since $\text{sq.}(Z\Delta) = \text{rect.}(\Gamma E, Z\Theta)$, we get the required proportion

$$AE : AZ = \text{sq.}(\Gamma E) : \text{sq.}(Z\Delta).$$

It should be noted, though, that this does not give a *construction* of a parabola, but merely of the point Δ on a parabola given the position of Γ on it and the intrinsic fact that AZ is the diameter. To fully construct a parabola in this way would require an infinite number of steps, one for each point Δ , and as such is not a true Euclidean construction. For practical uses, though, constructing several points might be sufficient for a given desired degree of accuracy.

diameter in the case of a circle. For if it is, that

$$\text{sq.}(\Delta E) : \text{rect.}(AEB) = \Gamma A : AB;$$

but

$$\text{sq.}(\Delta E) = \text{rect.}(AEB)$$

in the case of the circle alone, and so

$$\Gamma A = AB.$$

(232.23) But it is necessary also to see this, that the ordinates in the perimeter of the circle are always orthogonal to the diameter, and meet at right angles the parallels to $A\Gamma$.

(232.27) But through this theorem, continuing in the same manner as those mentioned in the case of the parabola, we draw a hyperbola and an ellipse by an application of a ruler. For let a straight line AB be set out and projected towards infinity through H , and from A let $A\Gamma$ be led at right angles to this line, and let $B\Gamma$ be joined and projected, and let some points be taken on AH , such as E, H , and from E and H , let $E\Theta, HK$ be drawn parallel to $A\Gamma$, and let it be contrived that

$$ZH : AHK = \Delta E : AE\Theta:$$

for the hyperbola will have come to be through A, Δ, Z . Similarly we will construct the things in the case of the ellipse.

3.19 Theorem 23

(234.12) But it is necessary to understand, that in the protasis, by "two diameters", he means not simply random ones, but rather the so-called *conjugate* diameters, of which each is drawn ordinatewise and is a mean proportional between the sides of the figure and the conjugate diameter³⁹, and on account of this they bisect the parallels to each other, as is shown in the fifteenth theorem. For if it is not taken in this way, it will happen that the intermediate straight line of the two diameters will be parallel to the other of them: which is not supposed.

³⁹That is, the other diameter in the pair called conjugate.

(234.21) And when H is nearer than the midpoint of AB, which is Θ , and when

$$\text{rect.}(BHA) + \text{sq.}(\Theta M) = \text{sq.}(AM),$$

and

$$\text{rect.}(A\Theta B) + \text{sq.}(\Theta M) = \text{sq.}(AM).$$

But

$$\text{sq.}(\Theta M) > \text{sq.}(HM),$$

so

$$\text{rect.}(BHA) > \text{rect.}(B\Theta A).$$

3.20 Theorem 25

(236.4) In some copies there is also this proof: let some point Θ be taken on the section, and let $Z\Theta$ be joined: therefore $Z\Theta$, being projected, intersects $\Delta\Gamma$: so that also ZE , being projected, intersects $\Delta\Gamma$. Again, let it be taken, and let KZ be joined and projected: therefore it will intersect BA , being projected: so that also ZH , being projected, will intersect BA .

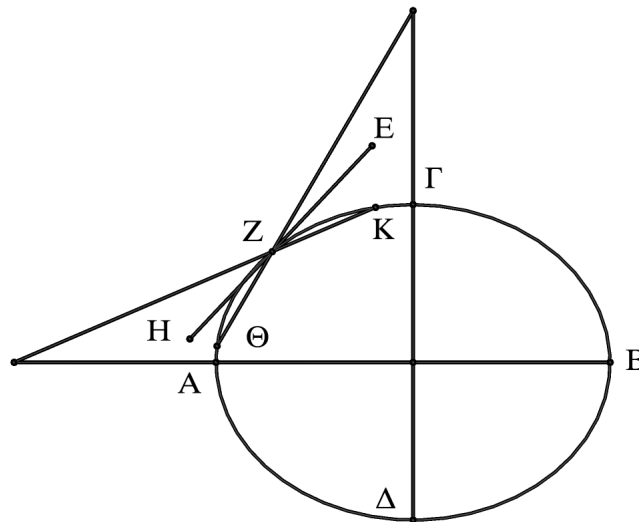


Figure 3.17: Eutocius' Figure for Theorem 25

3.21 Theorem 26

(236.11) This theorem has many cases, the first, that EZ is taken on the convex part of the section, as it is here, or on the concave part; next, that the straight line drawn ordinatewise from E, indiscriminately intersects the diameter (which is infinite) inside at one point, but when it is outside, especially in the case of the hyperbola, having this particular arrangement, it intersects either beyond B or at B or between A and B.

3.22 Theorem 27

(236.20) In some copies of the twenty-seventh theorem, the following proof is transmitted:

(236.22) Let there be a parabola, whose diameter is AB, and let it be cut by some straight line HΔ inside the section. I say, that HΔ, being projected in both directions, will intersect the section.

(238.3) For let some line AE be drawn ordinatewise through A; therefore AE will fall outside the section.

Then either HΔ is parallel to AE or not.

So if it is parallel, it has been drawn ordinatewise: so that being projected both ways, since it is bisected by the diameter, it will intersect the section. Therefore let it not be parallel to AE, but being projected, let it intersect AE at E as HΔE.

(238.11) So it is manifest that it intersects the section in the direction of E: for if projected to AE, it cuts the section long before.

(238.14) I say also that being projected in the other direction, it cuts the section.

(238.16) For let the parameter be MA, and let AZ be projected at right angles to it: therefore MA is orthogonal to AB. Let it be contrived, that

$$\text{sq.}(AE) : \text{tri.}(AE\Delta) = MA : AZ,$$

and through M and Z, let ZK and MN be drawn parallel to AB. So since ΛAΔH is a quadrilateral, and ΛA is in this particular arrangement, let ΓKB be drawn parallel to ΛA, cutting the triangle ΓKH equal to the quadrilateral ΛAΔH, and through B let ΞBN be

drawn parallel to ZAM.

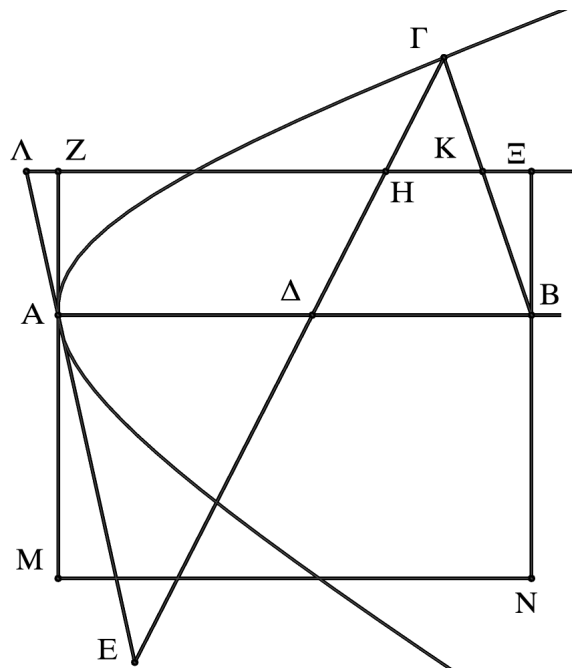


Figure 3.18: Eutocius' Figure for Theorem 27

And since

$$\text{sq.}(AE) : \text{tri.}(AE\Delta) = MA : AZ,$$

but

$$\text{sq.}(AE) : \text{tri.}(AE\Delta) = \text{sq.}(\Gamma B) : \text{tri.}(\Delta\Gamma B):$$

for AE is parallel to ΓB , and ΓE , AB join them⁴⁰. But

$$MA : AZ = \text{pll.}(AMNB) : \text{pll.}(AZ\Xi B);$$

therefore

$$\text{sq.}(\Gamma B) : \text{tri.}(\Gamma\Delta B) = \text{pll.}(AMNB) : \text{pll.}(AZ\Xi B).$$

So *alternando*,

$$\text{sq.}(\Gamma B) : \text{pll.}(AMNB) = \text{tri.}(\Gamma\Delta B) : \text{pll.}(AZ\Xi B).$$

But

$$\text{pll.}(ZABE) = \text{tri.}(\Gamma BA) :$$

⁴⁰i.e. AE, ΓB , forming a quadrilateral.

for since

$$\text{tri.}(\Gamma\text{HK}) = \text{quad.}(\Lambda\Lambda\text{H}\Delta),$$

and the quadrilateral $\text{H}\Delta\text{BK}$ is common,

$$\text{pllg.}(\Lambda\text{ABK}) = \text{tri.}(\Gamma\Delta\text{B}).$$

But

$$\text{pllg.}(\Lambda\text{ABK}) = \text{pllg.}(\text{ZAB}\Xi) :$$

for they are on the same base AB and in the same parallels AB , ZK . Therefore

$$\text{tri.}(\Gamma\Delta\text{B}) = \text{pllg.}(\text{ZAB}\Xi) :$$

so that

$$\text{sq.}(\Gamma\text{B}) = \text{pllg.}(\text{AMNB}).$$

But

$$\text{pllg.}(\text{MABN}) = \text{rect.}(\text{MAB}),$$

for MA is orthogonal to AB : therefore

$$\text{rect.}(\text{MAB}) = \text{sq.}(\Gamma\text{B}),$$

and MA is the upright side of the figure, and AB the diameter, and ΓB an ordinate (for it is parallel to AE): therefore Γ is on the section. Therefore $\Delta\text{H}\Gamma$ intersects the section at Γ : the very thing which it was necessary to show.

(240.23) Scholia to the preceding theorem.

(240.24) [Let it be contrived, that as the square on AE is to the triangle $\text{AE}\Delta$, so too is MA to AZ .] This is shown in the scholia to the eleventh theorem. For having drawn the square on AE , and being applied to its side equal to the triangle $\text{AE}\Delta$, I will have the desired property.

(242.1) To the same.

(242.2) [Since the quadrilateral is $\Lambda\Lambda\Delta\text{H}$, let ΓKB be drawn parallel to $\Lambda\Lambda$, cutting off the triangle ΓHK equal to the quadrilateral $\Lambda\Lambda\Delta\text{H}$.] We will show this as follows: for if, as we learned in the *Elements*, we construct the same ΣTY to be equal to the given rectilinear figure, the quadrilateral $\Lambda\Lambda\Delta\text{H}$, and similar to another given figure, the triangle

$AE\Delta$, so that ΣY is in the same proportion to $A\Delta$:

$$\Sigma TY : \text{tri.}(AE\Delta) = \Sigma Y : A\Delta,$$

and if we take

$$HK = \Sigma Y$$

and

$$H\Gamma = TY$$

and if we join ΓK , it will be the thing being sought. For since the angle at Y is equal to that at Δ --that is, to the angle at H --for this reason ΓHK is both equal to and similar to ΣTY . And the angle Γ is equal to E , and they are vertical angles: therefore ΓK is parallel to AE .

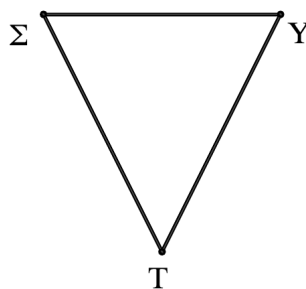


Figure 3.19: Eutocius' Figure for the Scholia of Theorem 27

(242.16) But it is manifest that, whenever AB is an axis, MA lies tangent on the section, but whenever it is not an axis, it cuts; that is, if it is led at right angles to the diameter.

3.23 Theorem 28

(242.20) That, even if $\Gamma\Delta$ cuts the hyperbola, the same things will follow, just as in the eighteenth theorem.

3.24 Theorem 30

(242.23) [And therefore, as *componendo* in the case of the ellipse, *invertendo* and *convertendo* in the case of the opposite sections.]

So in the case of the ellipse, we will say the following: since

$$\text{rect.}(AZB) : \text{sq.}(\Delta Z) = \text{rect.}(AHB) : \text{sq.}(HE);$$

but

$$\text{sq.}(\Delta Z) : \text{sq.}(Z\Gamma) = \text{sq.}(EH) : \text{sq.}(H\Gamma),$$

and so *ex aequali*,

$$\text{rect.}(AZB) : \text{sq.}(Z\Gamma) = \text{rect.}(AHB) : \text{sq.}(H\Gamma);$$

and so *componendo*,

$$\text{rect.}(AZB) + \text{sq.}(Z\Gamma) : \text{sq.}(Z\Gamma) = \text{sq.}(A\Gamma) : \text{sq.}(\Gamma Z) = \text{sq.}(\Gamma B) : \text{sq.}(\Gamma H),$$

for AB has been cut into equal sections at Γ and unequal sections at Z. And *alternando*,

$$\text{sq.}(A\Gamma) : \text{sq.}(\Gamma B) = \text{sq.}(Z\Gamma) : \text{sq.}(\Gamma H);$$

But in the case of the opposite sections: since

$$\text{rect.}(BZA) : \text{sq.}(Z\Gamma) = \text{rect.}(AHB) : \text{sq.}(\Gamma H),$$

through equality, and *invertendo*

$$\text{sq.}(Z\Gamma) : \text{rect.}(BZA) = \text{sq.}(\Gamma H) : \text{rect.}(AHB);$$

and *convertendo*,

$$\text{sq.}(Z\Gamma) : \text{sq.}(\Gamma A) = \text{sq.}(H\Gamma) : \text{sq.}(\Gamma B) :$$

for since some straight line AB has been bisected at Γ , and ZA is attached, and

$$\text{rect.}(BZA) + \text{sq.}(A\Gamma) = \text{sq.}(\Gamma Z),$$

so that the square on ΓZ exceeds the rectangle BZA by the square $A\Gamma$; and rightly it has been called *componendo*.

3.25 Theorem 31

[*Separando*, the square on ΓB has a ratio to the rectangle AHB bigger than that of the square on ΓB to the rectangle $A\Theta B$.] For since the straight line AB has been bisected at Γ , and BH is attached to it,

$$\text{rect.}(AHB) + \text{sq.}(\Gamma B) = \text{sq.}(\Gamma H) :$$

so that the square on ΓH exceeds the rectangle AHB by the square on ΓB . But for the same reason also the square on $\Gamma\Theta$ exceeds the rectangle $A\Theta B$ by the square on ΓB : so

that rightly it has been called *separando*.

3.26 Theorem 32

(246.4) In the seventeenth theorem he showed more clearly that the line having been drawn ordinatewise touches (the section), but the proof there in the *Elements* in the case of the circle alone; he shows this more generally for each section of a cone.

(246.9) But it is necessary to understand the very thing which was also shown there [*Conics* I.17], that it is perhaps in no way strange that a curved line falls between the straight line and the section, but a straight line cannot: for it itself cuts the section and is not tangent, for it is not possible for two lines to be tangent through the same point.

(246.15) But since this theorem has been proved with a lot of twists and turns in the different editions, we made the proof simpler and more manifest.

3.27 Theorem 34

(246.18) It is necessary to understand, that the ordinate to the diameter $\Gamma\Delta$, in the case of the hyperbola, defining ΔB , ΔA , leaves BA needing to be cut into the ratio of $B\Delta$ to ΔA ; but in the case of the ellipse and the circle the inverse: cutting BA into the defined ratio of $B\Delta$ to ΔA , makes us find the ratio of BE to EA : for it is in no way difficult, having been given one ratio, to provide another equal to it.

(248.1) But it is necessary also to know, that according to each section, there are two diagrams, the point Γ either being taken inside or outside with respect to the point Z : so that there are six total cases.

(248.5) But he also needs two lemmas, which we will write in turn.

(248.6) [Therefore the rectangle $AN\Xi$ is bigger than the rectangle $AO\Xi$: therefore NO has to ΞO a ratio greater than that of OA to AN .] For since

$$\text{rect.}(AN, N\Xi) > \text{rect.}(AO, O\Xi),$$

let it happen that the rectangle formed by AO and some other line $\Xi\Pi$ is equal to the rectangle $AN, N\Xi$:

$$\text{rect.}(AO, \Xi\Pi) = \text{rect.}(AN, N\Xi) :$$

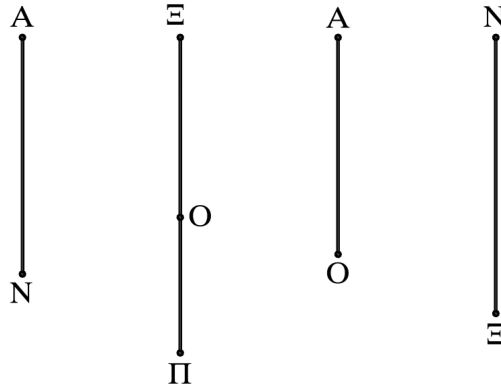


Figure 3.20: Eutocius' Figure for Theorem 34

this line $\Xi\Pi$ will be greater than ΞO : Therefore

$$OA : AN = NE : \Xi\Pi.$$

But

$$NE : EO > NE : \Xi\Pi,$$

and therefore

$$OA : AN < NE : EO.$$

(248.15) The *alternando* is also manifest, that also if

$$NE : EO > OA : AN,$$

then

$$\text{rect}(\Xi N, NA) > \text{rect}(\text{AO}, O\Xi).$$

(248.18) For let it happen, that as OA is to AN , so too is NE to some line which is manifestly greater than ΞO , say, $\Xi\Pi$:

$$OA : AN = NE : \Xi\Pi :$$

therefore

$$\text{rect}(\Xi N, NA) = \text{rect}(\text{AO}, \Xi\Pi),$$

so that

$$\text{rect}(\Xi N, NA) = \text{rect}(\text{AO}, O\Xi).$$

(248.22) To the same.

(248.23) [But as the rectangle BK, AN is to the square on ΓE , so too is the rectangle B Δ A to the square on E Δ .] So since, on account of the lines AN, E Γ , and KB being parallel,

$$AN : E\Gamma = A\Delta : \Delta E;$$

but

$$E\Gamma : KB = E\Delta : \Delta B.$$

Therefore *ex aequali*,

$$AN : KB = A\Delta : \Delta B :$$

and therefore

$$\text{sq.}(AN) : \text{rect.}(AN, KB) = \text{sq.}(A\Delta) : \text{rect.}(A\Delta B).$$

But

$$\text{sq.}(E\Gamma) : \text{sq.}(AN) = \text{sq.}(E\Delta) : \text{sq.}(\Delta A) :$$

therefore *ex aequali*,

$$\text{sq.}(E\Gamma) : \text{rect.}(AN, KB) = \text{sq.}(E\Delta) : \text{rect.}(A\Delta B);$$

and *alternando*,

$$\text{rect.}(KB, AN) : \text{sq.}(E\Gamma) = \text{rect.}(B\Delta A) : \text{sq.}(E\Delta).$$

3.28 Theorem 37

It is clear by these theorems in what way it is possible to draw a tangent through a given point on the diameter and through the vertex of the section.

3.29 Theorem 38

(250.16) In some copies this theorem is found proved only in the case of the hyperbola, but here it is shown more generally: for the same things occur in the cases of the other sections. And it seems that to Apollonius, not only the hyperbola has a second diameter, but also the ellipse, as we often heard from him in the proceeding materials.

(250.23) And with respect to the ellipse, there are not cases, but with respect to the hyperbola there are three: for the point Z, at which the tangent intersects the second diameter, is either before Δ , at Δ , or after Δ ; and for this reason the point Θ similarly will

have three possible locations, and it is necessary that one attend to the fact that either Z will fall before Δ , and Θ will be before Γ , or that Z will be on Δ , and Θ on Γ , or that Z will be after Δ , and Θ will be beyond Γ .

3.30 Theorem 41

(252.8) This theorem does not have cases for the hyperbola, but with the ellipse, if the ordinate passes through the center, and the remaining things become the same, the figure on the ordinate will be equal to the figure on the line from the center.

(252.13) For let there be an ellipse, whose diameter is AB , center Δ , and let $\Gamma\Delta$ be drawn ordinatewise, and let equiangular figures AZ , ΔH be set up on the bases $\Gamma\Delta$ and $A\Delta$, and let

$$\Delta\Gamma : \Gamma H = (A\Delta : \Delta Z) \text{ comp. (the upright : the transverse).}$$

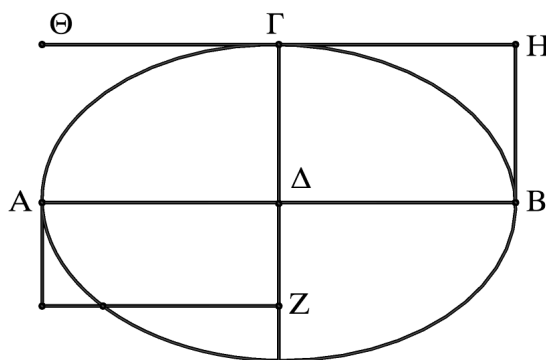


Figure 3.21: Eutocius' Figure for Theorem 41

(252.19) I say that the figure AZ is equal to the figure ΔH .

(252.20) For since it is shown in this text, that

$$\text{sq.}(A\Delta) : \text{fig.}(AZ) = \text{rect.}(A\Delta B) : \text{fig.}(\Delta H),$$

I say that also *alternando*,

$$\text{sq.}(A\Delta) : \text{rect.}(A\Delta B) = \text{fig.}(AZ) : \text{fig.}(\Delta H).$$

But

$$\text{sq.}(A\Delta) = \text{rect.}(A\Delta B),$$

therefore

$$\text{fig}(\text{AZ}) = \text{fig}(\Delta\text{H}).$$

3.31 Theorem 42

(252.26) This theorem has 11 cases, one, if Δ be taken inside of Γ : for it is clear, that also the parallels inside will fall inside of $\text{A}\Gamma\Theta$. But another five are these: if Δ be taken outside of Γ , manifestly the parallel ΔZ will fall outside of $\Theta\Gamma$; but ΔE will fall either between A and B , or at B , or between $\text{B}\Theta$, or at Θ , or outside Θ : for it is not possible for it to fall outside A ; since Δ is outside of Γ , and manifestly also the line being drawn parallel through it to $\text{A}\Gamma$. But if Δ be taken on the other [parts] of the section, either the two parallels will be terminated between Θ , B , or ΔZ inside Θ , but E at Θ , or when ΔZ likewise remaining outside of Θ , will pass through E : but again, when E falls outside, Z will either fall at Θ , so as to be the single straight line $\Gamma\Theta\Delta$, unless, strictly speaking, the peculiarity of the parallel is preserved then; or Z will fall outside of Θ . But it is necessary in the case of the proof of the final five cases to project ΔZ right up to the section and the parallel $\text{H}\Gamma$ in this way to make the proof.

(254.20) But it is possible also think of one other diagram from these things, whenever a different point is taken, the original straight lines make the thing being said; but this is a theorem rather than a case.

3.32 Theorem 43

(254.25) In some copies, the following proof of this theorem appears:

(256.1) For since

$$\text{rect}(\text{Z}\Gamma\Delta) = \text{sq}(\Gamma\text{B}),$$

therefore,

$$\text{Z}\Gamma : \Gamma\text{B} = \Gamma\text{B} : \Gamma\Delta:$$

and therefore

$$\text{fig}(\Gamma\text{Z}) : \text{fig}(\Gamma\text{B}) = \text{Z}\Gamma : \Gamma\Delta.$$

But

$$\text{sq.}(\Gamma Z) : \text{sq.}(\Gamma B) = \text{tri.}(\text{EZ}\Gamma) : \text{tri.}(\Lambda\Gamma B);$$

therefore

$$\text{tri.}(\text{E}\Gamma Z) : \text{tri.}(\text{B}\Lambda\Gamma) = \text{tri.}(\text{E}\Gamma Z) : \text{tri.}(\text{E}\Gamma\Delta).$$

Therefore

$$\text{tri.}(\text{E}\Gamma\Delta) = \text{tri.}(\text{B}\Gamma A).$$

And therefore, as in the case of the hyperbola, *convertendo*, but in the case of the the ellipse *alternando* and *separando*,

$$\text{tri.}(\text{EZ}\Gamma) : \text{quad.}(\text{E}\Lambda\text{BZ}) = \text{tri.}(\text{E}\Gamma Z) : \text{tri.}(\text{E}\Delta Z);$$

therefore

$$\text{tri.}(\text{E}\Delta Z) = \text{quad.}(\text{E}\Lambda\text{BZ}).$$

And since

$$\text{sq.}(\Gamma Z) : \text{sq.}(\Gamma B) = \text{tri.}(\text{EZ}\Gamma) : \text{tri.}(\Lambda\Gamma B),$$

in the case of the hyperbola, *convertendo*, but in the case of the ellipse *alternando*, *separando*, and *alternando* again,

$$\text{rect.}(\text{AZB}) : \text{sq.}(\text{B}\Gamma) = \text{quad.}(\text{E}\Lambda\text{BZ}) : \text{tri.}(\text{B}\Lambda\Gamma).$$

Similarly also,

$$\text{sq.}(\Gamma B) : \text{rect.}(\text{AKB}) = \text{tri.}(\Lambda\Gamma B) : \text{quad.}(\text{M}\Lambda\text{BK}) :$$

therefore *ex aequali*,

$$\text{rect.}(\text{AZB}) : \text{rect.}(\text{AKB}) = \text{quad.}(\text{E}\Lambda\text{BZ}) : \text{quad.}(\text{E}\Lambda\text{BZ}).$$

But

$$\text{rect.}(\text{AZB}) : \text{rect.}(\text{AKB}) = \text{sq.}(\text{EZ}) : \text{sq.}(\text{HK});$$

and

$$\text{sq.}(\text{EZ}) : \text{sq.}(\text{HK}) = \text{tri.}(\text{E}\Delta Z) : \text{tri.}(\text{H}\Theta\text{K}) :$$

and therefore,

$$\text{tri.}(\text{E}\Delta Z) : \text{sq.}(\text{H}\Theta\text{K}) = \text{quad.}(\text{E}\Lambda\text{BZ}) : \text{quad.}(\text{M}\Lambda\text{BK}).$$

Alternando,

$$\text{tri.}(E\Delta Z) : \text{quad.}(E\Lambda BZ) = \text{tri.}(H\Theta K) : \text{quad.}(M\Lambda BK).$$

But

$$\text{tri.}(E\Delta Z) = \text{quad.}(E\Lambda BZ)^{41} :$$

therefore also

$$\text{tri.}(H\Theta K) = \text{quad.}(M\Lambda BK).$$

Therefore the triangle $M\Gamma K$ differs from the triangle $H\Theta K$ by the triangle $\Lambda B\Gamma$.

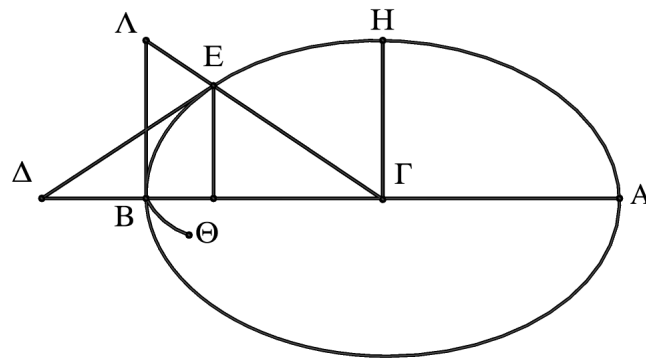


Figure 3.22: Eutocius' Figure for Theorem 43

(258.4) But it is necessary to understand by this proof: for it has a little confusion in the proportions of the ellipse: in order that we separate the things said together for the conciseness of the text, for instance--for he says: since

$$\text{sq.}(Z\Gamma) : \text{sq.}(\Gamma B) = \text{tri.}(E\Gamma Z) : \text{tri.}(\Lambda B\Gamma),$$

and *invertendo*, *convertendo* and *invertendo* again,

$$\text{sq.}(B\Gamma) : \text{sq.}(\Gamma Z) = \text{tri.}(\Lambda B\Gamma) : \text{tri.}(E\Gamma Z).$$

Convertendo, as the square on $B\Gamma$ is to the rectangle AZB , that is, the excess of the square on ΓB to the square on ΓZ on account of Γ being the midpoint of AB , so too is the triangle $\Lambda B\Gamma$ to the quadrilateral ΛBZE :

$$\text{sq.}(B\Gamma) : \text{sq.}(AZB) = \text{tri.}(\Lambda B\Gamma) : \text{quad.}(\Lambda BZE).$$

⁴¹This was shown previously.

So *invertendo*,

$$\text{rect.}(AZB) : \text{sq.}(B\Gamma) = \text{quad.}(E\Lambda BZ) : \text{tri.}(AB\Gamma).$$

(258.17) In the case of the hyperbola, it has eleven cases, as many as the one before it had for the parabola, and another one, whenever the chosen point H be the same as E: for then, it follows that the triangle EΔZ together with the triangle ΛBΓ is equal to the triangle ΓEZ: for the triangle EΔZ has been shown equal to the quadrilateral ΛBZE, but the quadrilateral ΛBZE exceeds the triangle ΓZE by the triangle ΛBΓ. But in the case of the ellipse, either H is the same as E or is taken outside of E: for it is clear, that both the parallels will fall between Δ, Z, as it is in the text. But if H be taken outside of E, and the line from it parallel to EZ falls between Z, Γ, the point Θ makes five cases: for either it falls between Δ, B or at B or between BZ or at Z or between Z, Γ. But if the parallel to the ordinate through H intersects at the center Γ, the point Θ will again make another five cases in the same manner: and it is necessary, pertaining to this, to point out that the triangle being formed from the parallels to EΔ, EZ is equal to the triangle ABΓ: for since it is, that

$$\text{sq.}(EZ) : \text{sq.}(H\Gamma) = \text{tri.}(E\Delta Z) : \text{tri.}(H\Theta\Gamma),$$

for they are similar, but

$$\text{sq.}(EZ) : \text{sq.}(H\Gamma) = \text{tri.}(BZA) : \text{rect.}(B\Gamma A),$$

that is, the square on BΓ; therefore

$$\text{tri.}(E\Delta Z) : \text{tri.}(H\Theta\Gamma) = \text{rect.}(BZA) : \text{sq.}(B\Gamma):$$

But as the rectangle BZA is to the square on BΓ, so too it has been shown that the quadrilateral ΛBZE is to the triangle ΛBΓ: and therefore

$$\text{tri.}(E\Delta Z) : \text{tri.}(H\Theta\Gamma) = \text{quad.}(\Lambda BZE) : \text{tri.}(AB\Gamma):$$

And *invertendo*. But it is also possible for those saying these things to show them in another way, that in the case of the double, parallelograms themselves, these things have been shown in the scholia to the forty-first theorem.

(260.23) But if the line being drawn through H parallel to EZ falls between Γ, A, it

will be projected until ΓE intersects it; but the point Θ will make 7 cases: for it is either between B, Δ , or falls at B , or between B, Z , or at Z , or between Z, Γ , or at Γ , or between Γ, A : and in these cases it follows that the difference between the triangles $\Lambda B\Gamma, H\Theta K$ forms a smaller triangle on the line segment AB by the line $\Lambda\Gamma$ being projected⁴².

(262.7) But if H be taken on the other parts of the section, and the line from H parallel to EZ falls between B, Z , it will be projected on account of the proof, until it cuts $\Lambda\Gamma$; but the point Θ will make seven cases: either it is between B, Z , or falling at Z , or between Z, Γ , or at Γ , or between Γ, A , or at A , or beyond A . But if the line through H parallel to EZ falls at Z , so that EZH is one straight line, the point Θ will make five cases: for either it will fall between Z, Γ , or at Γ , or between Γ, A , or at A , or beyond A . But if HK falls beyond Z, Γ , the point Θ will make five cases: for it will fall either between Z, Γ , or at Γ , or between Γ, A , or at A , or beyond A . But if HK intersects the center Γ , the point Θ will make three cases: it falling either between Γ, A or at A or beyond A : and in these cases it will again follow that the triangle $H\Theta K$ becomes equal to the triangle $\Lambda B\Gamma$. But if HK falls between Γ, A , the point Θ will fall either between Γ, A , or at A , or beyond A .

(262.28) So it follows in the case of an ellipse that there are forty-two cases in all, and in the case of the perimeter of a circle the same number; so that there are, all told, 96 cases of this theorem.

3.33 Theorem 44

(264.4) [So since the opposite sections are ZA, BE , whose diameter is AB , but the line through the center [is] $Z\Gamma E$ and the lines $ZH, \Delta E$ are tangent to the sections, ZH is parallel to $E\Delta$.] For since AZ is a hyperbola, and ZH is tangent, and ZO is an ordinate, the rectangle $O\Gamma H$ is equal to the square on ΓA by the thirty-seventh theorem: and indeed similarly, the rectangle $\Xi\Gamma\Delta$ is equal to the rectangle ΓB . Therefore it is, as the rectangle $O\Gamma H$ is to the square on $A\Gamma$, so too is the rectangle $\Xi\Gamma\Delta$ to the square on

⁴²i.e. we've made a triangle on the segment AB , whose third vertex is on the line segment $\Lambda\Gamma$.

$B\Gamma$, and *invertendo*, as the rectangle $O\Gamma H$ is to the rectangle $\Xi\Gamma\Delta$, so too is the square $A\Gamma$ to the square ΓB . But the square on $A\Gamma$ is equal to the square on ΓB : therefore also the rectangle $O\Gamma H$ is equal to the rectangle $\Xi\Gamma\Delta$. And $O\Gamma$ is equal to $\Gamma\Xi$: therefore also $H\Gamma$ is equal to $\Gamma\Delta$: but also, $Z\Gamma$ is equal to ΓE by the thirtieth theorem: therefore the [sections] $Z\Gamma H$ are equal to the [sections] $E\Gamma\Delta$. And equal angles are at Γ : for it is at a vertex, so that also ZH is equal to $E\Delta$ and the angle ΓZH is equal to the angle $\Gamma E\Delta$. And they are alternate angles: therefore ZH is parallel to $E\Delta$.

(264.22) There are twelve cases of it, just as there are in the case of the hyperbola in the thirty-third theorem, and the proof is the same.

3.34 Theorem 45

(264.25) It is necessary to understand that this theorem has very many cases. For in the case of the hyperbola there are twenty: for the point chosen instead of B is either the same as A or the same as Γ : for then it follows that the triangle on $A\Theta$, similar to the triangle $\Gamma\Delta\Lambda$, is the same as the triangle being cut by the parallels to $\Delta\Lambda\Gamma$. But if B be chosen between A , Γ , and Δ , Λ be beyond the endpoints of the second diameter, three cases occur: for Z , E are carried beyond these endpoints or at them or before them. But if the points Δ , Λ be at the endpoints of the second diameter, the points Z , E will be carried inside. But similarly also if B be chosen beyond Γ ⁴³, and $\Theta\Gamma$ is projected to Γ , it follows that another three cases thus arise: for when the point Δ is carried either beyond the endpoint of the second diameter or at it or before it, the point Z similarly being carried will make these three cases. But if the point B be taken on the other parts of the section, $\Gamma\Theta$ will be projected to Θ according to the proof, but BZ , BE make three cases, since the point Λ is either carried to the endpoint of the second diameter, or beyond it, or before it.

(266.21) But in the case of the ellipse and the perimeter of the circle, we will say nothing complicated, but only so much as was said in the preceding theorem: so that there are 104 cases of this theorem.

⁴³Heiberg notes that there is a serious textual issue here, and so this translation should be treated as tentative pending review of the manuscripts.

(268.1) But the proofs of the protasis are possible also in the case of the opposite sections.

3.35 Theorem 46

(268.4) This theorem has many cases, which we will show, paying attention to those of the forty-second [theorem].

(268.6) For the sake of an illustration, if Z falls at B, we will obviously say: since $B\Delta\Lambda$ is equal to $\Theta B\Delta M$, let the common part $NM\Delta B$ be subtracted: therefore the remainder ΛNM is equal to the remainder $N\Theta B$.

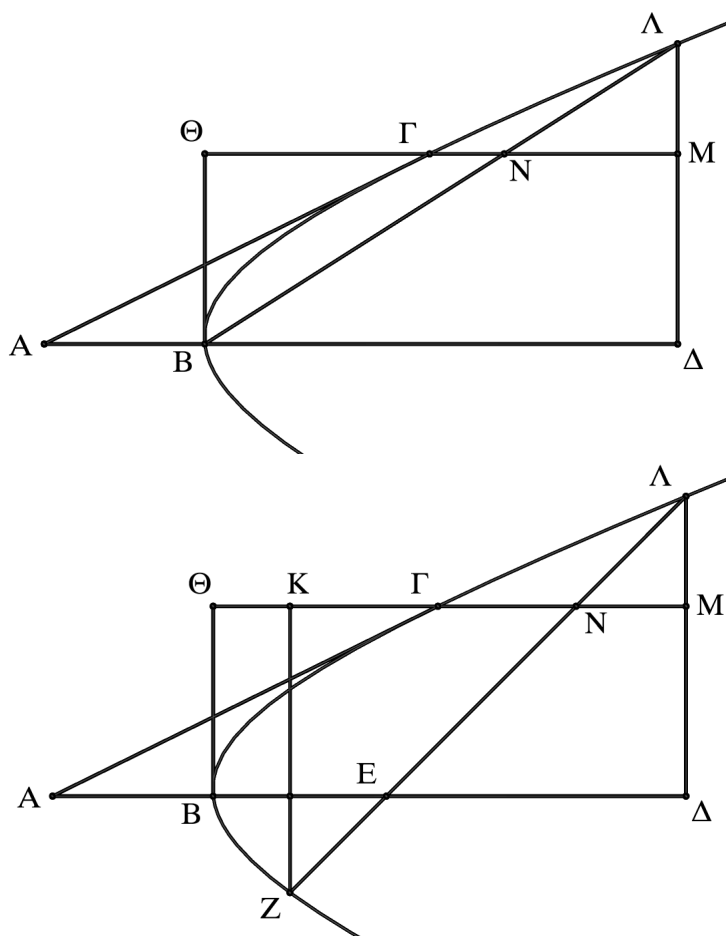


Figure 3.23: Eutocius' Figures for Theorem 46

(268.10) But in the case of the remaining, we will say: since $\Lambda E\Delta$ is equal to

$\Theta B \Delta M$, that is to $KH \Delta M$ and HZE , that is to ZKN and $NE \Delta M$, let the common part $NE \Delta M$ be subtracted: therefore also the remainder ΔNM is equal to the remainder KZN .

3.36 Theorem 47

(268.15) This theorem has cases with respect to the hyperbola, as many as the preceding theorem had for the case of the parabola, but we will show the proofs of them, paying attention to the cases of the forty-third, but also in the case of the ellipse, we will show the proofs from the cases of the forty-third theorem, for instance in the case of the diagram below, with the point H having been taken outside. Since the triangle $\Delta A \Gamma$ is equal to the triangles $\Theta H \Omega$, $\Omega \Delta M$, that is to the triangles $O \Theta \Gamma$, $O H M$, but both the triangle $\Xi \Pi \Gamma$ and the quadrilateral $\Delta \Pi \Xi$ are equal to the triangle $\Delta A \Gamma$, that is the triangle $N \Theta \Pi$ by what was shown in the forty-third theorem; and therefore the triangles $\Xi \Pi \Gamma$ and $N \Theta \Pi$ are equal to the triangles $O \Theta \Gamma$ and $O M H$. Let the common part $\Theta O \Gamma$ be subtracted: therefore the remainder $\Xi O N$ is equal to the remainder $H O M$. And $N \Xi$ is parallel to $M H$: therefore $N O$ is equal to $O H$.

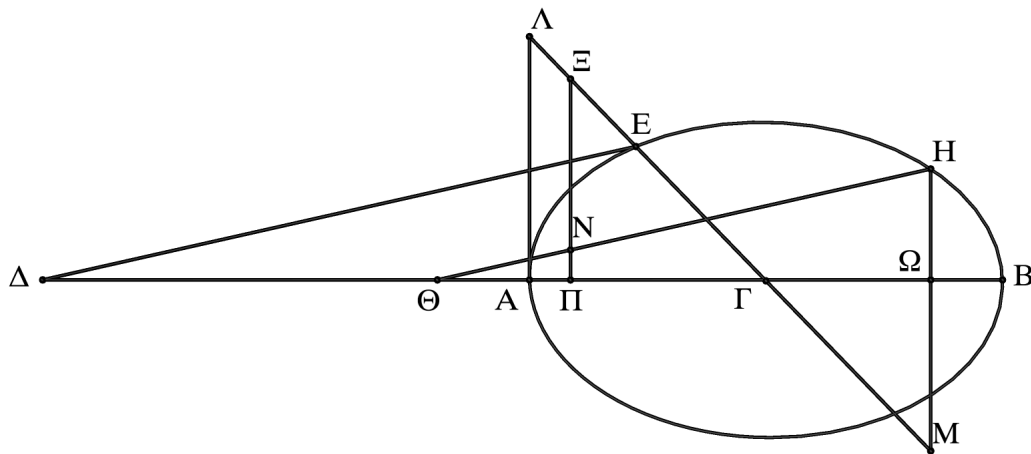


Figure 3.24: Eutocius' Figure for Theorem 47

3.37 Theorem 48

(270.15) And the cases of this theorem are similar to those of the preceding in the case of the 47th, according to the diagram of the hyperbola.

3.38 Theorem 49

(270.19) [Therefore the remainder, triangle $K\Lambda N$, is equal to the parallelogram $\Delta\Lambda\Pi\Gamma$. And the angle $\Delta\Lambda\Pi$ is equal to the angle $K\Lambda N$: therefore the rectangle $K\Lambda N$ is double the rectangle $\Lambda\Delta\Gamma$.] For let the triangle $K\Lambda N$ and the parallelogram $\Delta\Lambda\Pi\Gamma$ be set out independently. And since the triangle $K\Lambda N$ is equal to the parallelogram $\Delta\Pi$, let NP be drawn through N parallel to ΛK , and let KP be drawn through K parallel to ΛN : therefore ΛP is a parallelogram double the triangle $K\Lambda N$: so that it is also double the parallelogram $\Delta\Pi$. Indeed, let the lines $\Delta\Gamma$, $\Lambda\Pi$ be projected to Σ , T , and let $\Gamma\Sigma$ be equal to $\Delta\Gamma$, and ΠT to $\Lambda\Pi$, and let ΣT be joined: therefore ΔT is a parallelogram double $\Delta\Pi$: so that ΛP is equal to $\Lambda\Sigma$. But it is also equiangular to it, by virtue of the angles at Λ being vertical: but of the equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional: therefore it is, that as $K\Lambda$ is to ΛT , that is $\Delta\Sigma$, so too is $\Delta\Lambda$ to ΛN , and the rectangle $K\Lambda N$ is equal to the rectangle $\Lambda\Delta\Sigma$. And since $\Delta\Sigma$ is double $\Delta\Gamma$, the rectangle $K\Lambda N$ is double the rectangle $\Lambda\Delta\Gamma$.

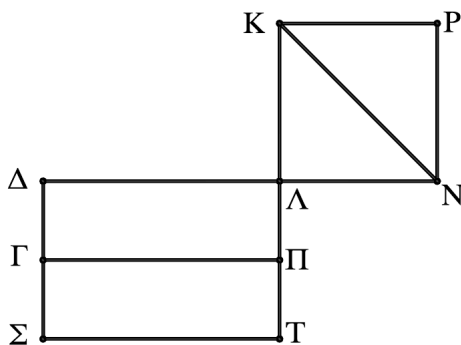


Figure 3.25: Eutocius' Figure for Theorem 49

(272.17) But if $\Delta\Gamma$ is parallel to $\Lambda\Pi$, and $\Gamma\Pi$ is parallel to $\Lambda\Delta$, it is manifest that

$\Delta\Pi\Lambda$ is a trapezoid; but I also say, that the rectangle $K\Lambda N$ is equal to the rectangle $\Delta\Lambda$ and the sum of $\Gamma\Delta$, $\Lambda\Pi$. For if ΛP be restored, as has been said before, and the lines $\Delta\Gamma$, $\Lambda\Pi$ also be projected, and $\Gamma\Sigma$ be cut equal to $\Lambda\Pi$, and ΠT be cut equal to $\Delta\Gamma$, and ΣT be joined, the parallelogram ΔT will be double the parallelogram $\Delta\Pi$, and the same proof will suffice. And this will be useful to the following [theorem].

3.39 Theorem 50

(272.28) The cases of this theorem number the same as those of the forty-third, but also similarly in the case of the fifty-first.

(274.1) To the conclusion.

(274.2) By "the original diameter," he means the section that came into being in the cone, common to the cutting plane and the triangle through the axis: but he also calls this the principal diameter. But he also says that all the demonstrated *symptomata* of the sections, where we have supposed the principle diameters as in the aforementioned theorems, are able to occur when all the other diameters are supposed, as well.

3.40 Theorem 54

(274.11) [And let a plane be erected on AB at right angles to the given plane, and in it let the circle $AEBZ$ be drawn around AB , so that the section of the diameter of the circle within the sector AEB has to the section of the diameter within the sector AZB a ratio not greater than that which AB has to $B\Gamma$.] For let there be two straight lines, AB and $B\Gamma$, and let it be required to draw a circle around AB , so that the diameter of it is cut by AB in such a way that the part of it towards Γ has a ratio to the remaining part not bigger than that of AB to $B\Gamma$.

(274.22) Now let it be supposed⁴⁴, and let AB be bisected at Δ , and through it, at right angles to AB , let $E\Delta Z$ be drawn, and let it be contrived, that as AB is to $B\Gamma$, so too is $E\Delta$ to ΔZ , and let EZ be bisected: it is manifest that, if AB is equal to $B\Gamma$ and $E\Delta$ to ΔZ , then Δ will be the midpoint of EZ ; but if AB is bigger than $B\Gamma$ and $E\Delta$ is bigger

⁴⁴i.e. that they have the same ratio.

than ΔZ , the midpoint is beyond Δ , but if AB is lesser than $B\Gamma$, the midpoint is closer.

(276.7) For now let it be beyond, at say H , and with center H and diameter HZ let a circle be drawn: it is necessary that it will pass through the points A , B , or inside, or outside. And if it goes through the points A , B , the enjoined would have occurred: but let it fall beyond A , B , and let AB , being projected both ways, intersect the perimeter at the points Θ , K , and let $Z\Theta$, OE , $E\Gamma$, KZ be joined, and through let MB be drawn through B parallel to ZK , and let BA be drawn through B parallel to KE , and let MA , AA be joined: and they themselves will be parallel to $Z\Theta$, ΘE on account of $A\Delta$ being equal to ΔB , and $\Delta\Theta$ being equal to ΔK , and $Z\Delta E$ being at right angles to ΘK . And since the angle at K is right, and MB , BA are parallel to ZK , KE , therefore the angle at B is right: and on account of the same things, the one at A is also [right], so that the circle being drawn about MA will pass through A , B . Let it be drawn as $MA\Lambda B$. And since MB is parallel to ZK , it is, that as $Z\Delta$ is to ΔM , so too is $K\Delta$ to ΔB . Similarly also, as $K\Delta$ is to ΔB , so too is $E\Delta$ to $\Delta\Lambda$. And *alternando*, as $E\Delta$ is to ΔZ , that is AB to $B\Gamma$, so too is $\Lambda\Delta$ to ΔM .

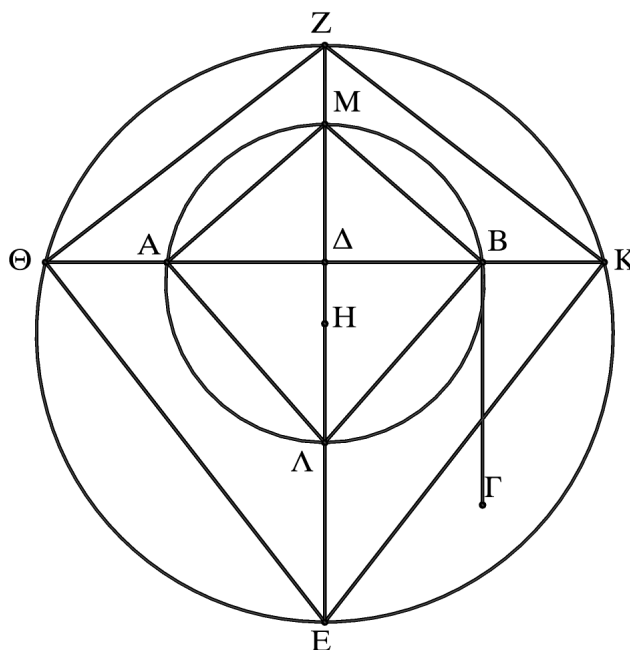


Figure 3.26: Eutocius' Figure for Theorem 54

(278.3) But similarly, even if the circle being drawn about ZE cuts AB, the same thing will be shown.

3.41 Theorem 55

(278.6) [And let the semicircle $A\Delta Z$ be drawn on $A\Delta$, and let some line ZH be drawn to the semicircle parallel to $A\Theta$, making the ratio of the square on ZH to the rectangle ΔHA the same as that of ΓA to the double of $A\Delta$ ⁴⁵. Let there be a semicircle $AB\Gamma$ on the diameter $A\Gamma$, and let the given ratio be that of EZ to ZH , and let it be required to do the enjoined.

(278.13) Let $Z\Theta$ be supposed equal to EZ , and let ΘH be bisected at K , and let some chance straight line ΓB be drawn in the circle in the angle $A\Gamma B$, and let $\Lambda\Sigma$ be drawn through from the center Λ and at right angles to it (ΓB), and being projected, let it ($\Lambda\Sigma$) intersect the circumference at N , and through N let NM be drawn parallel to ΓB : therefore it will be tangent to the circle. And let it be contrived, that as $Z\Theta$ is to ΘK , so too is $M\Xi$ to ΞN , and let it happen that NO is equal to ΞN , and let $\Lambda\Xi$, ΛO be joined, cutting the semicircle at Π , P ; and let $\Pi P A$ be joined.

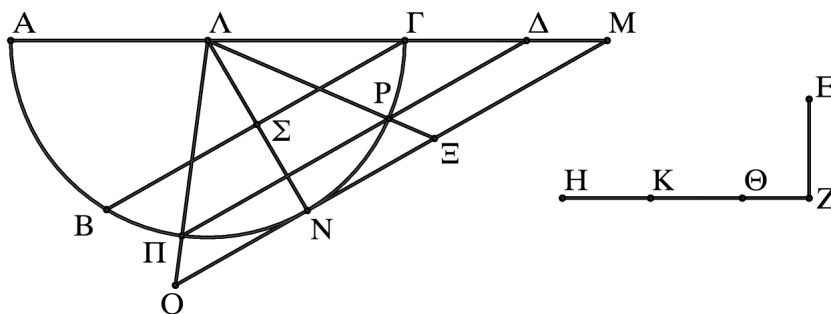


Figure 3.27: Eutocius' Figure for Theorem 55

(278.23) Then since ΞN is equal to NO , and $N\Lambda$ is common and orthogonal, therefore also ΛO is equal to $\Lambda\Xi$. But also $\Lambda\Pi$ is equal to ΛP : and therefore the remainder ΠO is equal to the remainder $\Pi\Xi$. Therefore $\Pi P A$ is parallel to MO . And it is, that as

⁴⁵i.e. to AB , since in Apollonius' proposition, Δ is the midpoint of AB .

$Z\Theta$ is to ΘK , so too is $M\Xi$ to $N\Xi$: but as ΘK is to ΘH , so too is $N\Xi$ to ΞO : therefore by equality, as ΘZ is to ΘH , so too is $M\Xi$ to $\Xi\Theta$: *invertendo*, as $H\Theta$ is to ΘZ , so too is $O\Xi$ to ΞM ; *componendo*, as HZ is to $Z\Theta$, so too is $O\Xi$ to ΞM , that is, $\Pi\Delta$ to ΔP . But as $\Pi\Delta$ is to ΔP , so too is the rectangle $\Pi\Delta P$ to the square on ΔP ; but the rectangle $\Pi\Delta P$ is equal to the rectangle $A\Delta\Gamma$: therefore as HZ is to ZE , so too is the rectangle $A\Delta\Gamma$ to the square on ΔP . Therefore *invertendo*, as EZ is to ZH , so too is the square on ΔP to the rectangle $A\Delta\Gamma$.

3.42 Theorem 58

(280.13) And on AE let the semicircle AEZ have been drawn, and in it, let ZH be drawn parallel to $A\Delta$, making

$$\text{sq.}(ZH) : \text{rect.}(AHE) = \Gamma A : \text{double}(AE).$$

For let there be a semicircle $AB\Gamma$, and in it, let there be some straight line AB , and let two unequal straight lines ΔE and EZ be given, and let EZ be produced to H , and let it happen that

$$ZH = \Delta E,$$

and let the whole EH be bisected at Θ , and let the center of the circle be taken as K , and from it, at right angles to AB , let a straight line be drawn, and let it intersect the circumference at Λ , and through Λ parallel to AB let ΛM be drawn, and let KA being produced, intersect ΛM at M , and let it have been contrived, that

$$\Theta Z : ZH = \Lambda M : MN.$$

And let

$$\Lambda\Xi = AN;$$

and let NK , $K\Xi$ be joined and produced; and let the circle, being completed, cut them at Π and O ; and let $O\Pi$ be joined.

(282.7) Then since

$$Z\Theta : ZH = \Lambda M : MN,$$

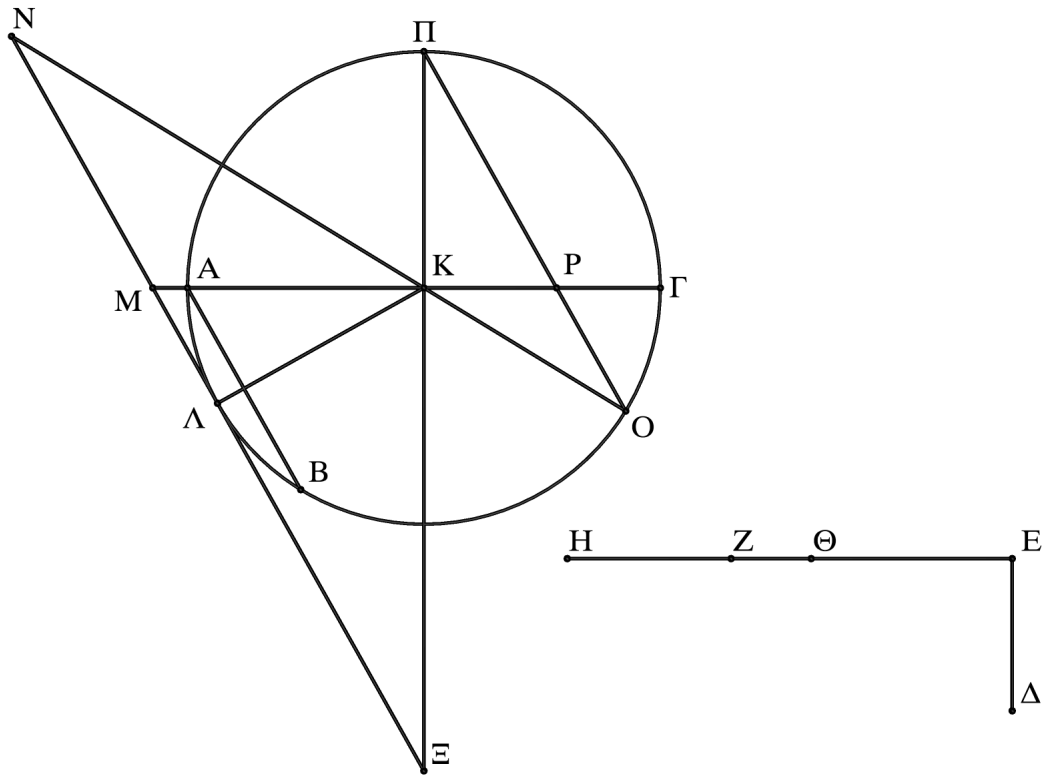


Figure 3.28: Eutocius' Figure for Theorem 58

componendo,

$$H\Theta : HZ = \Lambda N : MN;$$

invertendo,

$$ZH : H\Theta = NM : N\Lambda.$$

But

$$ZH : HE = MN : N\Xi,$$

so *separando,*

$$ZH : ZE = NM : M\Xi.$$

And since

$$N\Lambda = N\Xi,$$

and ΛK is orthogonal and common, therefore

$$KN = K\Xi.$$

But

$$KO = K\Pi \text{ also,}$$

therefore $N\Xi$ and $O\Pi$ are parallel. Therefore the triangle KMN is similar to the triangle OKP , and the triangle $KM\Xi$ is similar to the triangle ΠPK . Therefore,

$$KM : KP = MN : PO.$$

But also,

$$KM : KP = M\Xi : \Pi P,$$

therefore

$$NM : M\Xi = HZ : ZE = \Delta E : EZ;$$

but

$$OP : \Pi P = \text{sq.}(OP) : \text{rect.}(O\Pi P);$$

and therefore

$$\Delta E : EZ = \text{sq.}(OP) : \text{rect.}(O\Pi P).$$

But

$$\text{rect.}(O\Pi P) = \text{rect.}(AP\Gamma).$$

Therefore

$$\Delta E : EZ = \text{sq.}(OP) : \text{rect.}(AP\Gamma).$$

3.43 Eutocius' Epilogue to the First Book

(284.1) In the scholia after the tenth theorem, the aim of the first thirteen theorems is stated, and in the [scholia] to the sixteenth the [aim of the] following three; but it is necessary to understand, that in the sixteenth, he says that the line drawn through the vertex to an ordinate intersects [the ordinate] outside [the section]. But in the eighteenth, he says that the parallel to any tangent line, being drawn inside the section, will intersect the section.

In the nineteenth, he says that the line being drawn ordinatewise from some point on the diameter intersects the section.

In the twentieth and twenty-first, he seeks the ordinates of the sections, how they are

to one another and how the sections of the diameter being defined by them are in relation to one another.

In the twenty-second and twenty-third he speaks concerning the straight line intersecting the section at two points.

In the twenty-fourth and twenty-fifth, he speaks concerning the straight line intersecting the section at one point, that is a tangent.

In the twenty-sixth he speaks concerning the line being drawn parallel to the diameter of the parabola and the hyperbola.

In the twenty-seventh he speaks concerning the line cutting the diameter of the parabola, that it intersects section in both directions.

In the twenty-eighth he speaks concerning the line being drawn parallel to a tangent of one of the opposite sections.

In the twenty-ninth, he speaks concerning the line being projected through the center of the opposite sections.

In the thirtieth, he says that the line being projected through the center of the ellipse and the opposite sections is bisected⁴⁶.

In the thirty-first he says that in the case of the hyperbola, the tangent line cuts the diameter between the vertex and the center.

In the thirty-second, thirty-third, thirty-fourth, thirty-fifth, and thirty-sixth, he speaks concerning the tangents, he gives an account of the ratio.

In the thirty-seventh, he speaks concerning the tangents and the ordinates from the point of contact of the ellipse and the hyperbola.

In the thirty-eighth, he speaks he speaks concerning the tangents of the hyperbola and the ellipse, how they are in relation to the second diameter.

In the thirty-ninth and fortieth, concerning the same, he gives an account of the

⁴⁶i.e. the the line connecting two points on the sections, through the center, is bisected at the center.

ratio, seeking the ratios compounded from these⁴⁷.

In the forty-first, he speaks concerning the parallelograms being drawn from the ordinate and the line from the center of the hyperbola and ellipse.

In the forty-second, concerning the parabola, he says that the triangle being formed from the tangent and the ordinate is equal to the parallelogram having the same height as it, but having half the base.

In the forty-third, in the cases of the hyperbola and the ellipse, he seeks how the triangles formed by tangents and ordinates are to one another; in the forty-fourth, the same for the opposite sections; in the forty-fifth, the same in the case of the second diameter of the hyperbola and the ellipse.

In the forty-sixth, he speaks concerning the other [diameters] after the principle diameter of the parabola; in the forty-seventh concerning the other diameters of the hyperbola and the ellipse; in the forty-eighth concerning the other diameters of the opposite sections;

In the forty-ninth he speaks concerning the straight lines to which the straight lines being drawn ordinatewise to the different diameters of the parabola are equal in square; in the fiftieth concerning the same thing of the hyperbola and the ellipse; in the fifty-first concerning the same thing of the opposite sections.

Having said these things, and having appended an epilogue to the things being said, he shows a problem in the fifty-second and fifty-third, how it is possible to draw the parabola in a plane.

In the fifty-fourth and fifty-fifth he says how it is necessary to draw the hyperbola; in the fifty-sixth, fifty-seventh, and fifty-eighth, how it is necessary to draw the ellipse; in the fifty-ninth he says how it is possible to draw opposite sections.

In the sixtieth he speaks concerning the conjugate sections.

⁴⁷i.e. the diameters of the ellipse and the hyperbola.

CHAPTER 4 THE DUPLICATION PROBLEM

As in the last section, let us recall that the circle may be thought of as the locus which finds one mean proportional between two given finite straight lines. This again was well known even before Euclid, appearing in *Elements* VI.13. However, a much more difficult question is the matter of finding *two* means proportional. The desire to find this second proportional was not merely an attempt at mathematical abstraction; it came from the desire to solve this problem came from another more practical one: the duplication of the cube.

4.1 An Historical Overview of the Problem

The problem comes to us via two quotes, both of Eratosthenes. One of the quotes is given by Theon of Smyrna, the other by Eutocius in his commentary on Proposition II.1 of Archimedes' *On the Sphere and the Cylinder* ([15, 16]). Theon quotes:

Eratosthenes in his work entitled *Platonicus* relates that, when the god proclaimed to the Delians by the oracle that, if they would get rid of a plague, they should construct an altar double of the existing one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made double of a (similar) solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the god wanted an altar double the size, but that he wished, in setting them to the task, to shame the Greeks for their neglect of mathematics and their contempt for Geometry.

Eutocius, however, quotes him thus in a letter to king Ptolemy III Euergetes. Heath believed this letter to be a forgery; Netz, on the other hand, believes it is genuine.

Eratosthenes to king Ptolemy, greetings.

They say that one of the old tragic authors introduced Minos, building a tomb to Glaucos, and, hearing that it is to be a hundred cubits long in each direction, saying:

"You have mentioned a small precinct of a tomb royal;
Let it be double, and, not losing its beauty,
Quickly double each side of the tomb."

He seems, however, to have been mistaken; for, the sides doubled, the plane becomes four times¹, while the solid becomes eight times. And this was investigated by the geometers, too: in which way one could double the given solid, the solid keeping the same shape; and they called this problem "duplication of a cube:" for, assuming a cube, they investigated how to double it. And, after they were all puzzled by this for a long time, Hippocrates of Chios was the first to realize that, if it is found how to take two mean proportionals, in continuous proportion, between two straight lines (of whom the greater is double the smaller), then the cube shall be doubled, so that he converted the puzzle into another, no smaller puzzle. After a while, they say, some Delians, undertaking to fulfill an oracle demanding that they double one of their altars, encountered the same difficulty, and that they sent messengers to the geometers who were with Plato in the Academy, asking of them to find that which was asked. Of those who dedicated themselves to this diligently, and investigated how to take two mean proportionals between two given lines, it is said that Archytas of Tarentum solved this with the aid of semicylinders, while Eudoxus² did so with so-called curved lines; as it happens, all of them wrote demonstratively, and it was impossible practically to do this by hand (except Menaechmus, by the *shortness*³, and this with difficulty). But we have conceived of a certain easy mechanical way of taking two proportionals through which, given two lines, means - not only two, but as many as one may set forth - shall be found.

Interestingly enough, there is a passage in Plato (*Republic* VII) that matches these stories somewhat. While Plato does not specifically mention the cube duplication problem, he does refer to solid geometry as an area of mathematics not fully explored:

(528.A7) "After plane surfaces," said I, "we went on to solids in revolution before studying them in themselves. The right way is next in order after the second dimension to take the third. This, I suppose, is the dimension of cubes and everything that has depth." "Why, yes, it is," he said; "but this subject, Socrates, does not appear to have been investigated yet." "There are two causes of that," said I: "first, inasmuch as no city holds them in honour, these inquiries are languidly pursued owing to their difficulty. And secondly, the investigators need a director, who is indispensable for success and who, to begin with, is not easy to find, and then, if he could be found, as things are now, seekers in this field would be too arrogant to submit to his guidance. But if the state as a whole should join in superintending these studies and honour them, these specialists would accept advice, and continuous and strenuous investigation would bring out the truth. Since even now, lightly esteemed as

¹Cf. Plato, *Meno* [17].

²Lamentably, Eudoxus' solution did not pass the scrutiny of Eutocius, who derides it, and thus it does not appear in his collection of solutions. I suspect, however, that the solution Eutocius had was a corruption or forgery; though I admit that currently extant texts give no evidence either way.

³A question for future study: what is the "shortness"?

they are by the multitude and hampered by the ignorance of their students as to the true reasons for pursuing them, they nevertheless in the face of all these obstacles force their way by their inherent charm and it would not surprise us if the truth about them were made apparent." "It is true," he said, "that they do possess an extraordinary attractiveness and charm. But explain more clearly what you were just speaking of. The investigation of plane surfaces, I presume, you took to be geometry?" "Yes," said I. "And then," he said, "at first you took astronomy next and then you drew back." "Yes," I said, "for in my haste to be done I was making less speed. For while the next thing in order is the study of the third dimension or solids, I passed it over because our absurd neglect to investigate it, and mentioned next after geometry astronomy, which deals with the movements of solids."

We do know that Archytas' solution was the first, and that it involves a complicated construction involving the intersection of a torus (really just the top half of one), cylinder, and cone; though oddly, the letter here refers to multiple semicylinders, whereas the construction itself uses only one. Later, Eudoxus' and Archytas' student, Menaechmus, solved the problem in two ways, by considering the intersections of conics; other authors solve the problem by means of mechanical arguments that often depend on rulers or the like. We will consider three solutions: that of Archytas, Eratosthenes, and the first of Menaechmus. Before this, however, we should examine the relationship of duplicating a cube and finding two means proportional.

4.2 Reduction of the Problem due to Hippocrates of Chios

As Eratosthenes states in the quote above, Hippocrates of Chios reduced the problem of finding a duplicate cube into one finding two means proportional. We can see shadows of it in the propositions of *Elements* XI, echoing, in a way, similar propositions in *Elements* VI. For example, Euclid demonstrates in VI.1 that "triangles and parallelograms which are under the same height are to one another as their bases," and in VI.19-20, he shows that "similar triangles are to one another in the duplicate ratio of the corresponding sides." Of course our problem is about cubes, but this is dealt with firmly by XI.32 and 33, which are, in a sense, parallels of VI.1 and VI.19-20, respectively. XI.32 states that "parallelepipedal solids which are of the same height are to one another as their bases," while XI.33 states that "Similar parallelepipedal solids are to one another in the triplicate

ratio of their corresponding sides."

In our case, the two parallelepipedal solids are cubes, which are clearly similar to one another. It would be worth noting Euclid's porism to XI.33, which states that "from this it is manifest that, if four straight lines be [continuously] proportional, as the first is to the fourth, so will a parallelepipedal solid on the first be to the similar and similarly described parallelepipedal solid on the second, inasmuch as the first has to the fourth the triplicate ratio of that which it has to the second." The last part is really just V. def. 10. In symbols, if we have that $A : B = B : \Gamma = \Gamma : \Delta$, then the cube on A is to the cube on B as A is to Δ . In the case of our duplication, we should take Δ to be double A, so that the cube on B is double the cube on A. Thus duplicating the cube requires finding the magnitudes B and Γ . (It should be noted that finding one immediately gives the other, since the problem then is to find a single mean proportional). For the similar case within Euclid's books on number, cf. *Elements* VIII.19.

4.3 Finding Two Means Proportional

Elements XI.33 shows us that the matter of duplicating a cube is equivalent to that of finding two means proportional. (Pseudo?) Eratosthenes puts it quite right: Hippocrates exchanged one puzzle for another no less difficult. Eutocius, fortunately, collected together many ancient solutions to the problem, beginning with Archytas and ending with Pappus, in his commentary to II.1 of Archimedes' *On the Sphere and the Cylinder*. The variety of different solutions itself is striking: we have everything from a "standard" construction involving conics to excerpts from a work by Hero on the construction of missile-throwing machines to solutions so beautiful to their discoverer (Eratosthenes) that they write in epigrams and attach a model of the mechanical apparatus to a pillar dedicated to King Ptolemy. I do find it quite interesting that two solutions (Hero's and Eratosthenes') make explicit reference to the military sciences. And if we are to believe Eratosthenes, it should not be surprising that an air of mysticism and divinity should pervade the subject. For Archytas and the lot, having by their various means solved the problem, have fulfilled

the demands of the Delians by their oracle. Indeed, even, they have done the work of God himself! For if we read the *Timaeus* ([18]),

...God placed water and air in the mean between fire and earth, and made them to have the same proportion so far as was possible (as fire is to air so is air to water, and as air is to water so is water to earth); and thus he bound and put together a visible and tangible heaven. And for these reasons, and out of such elements which are in number four, the body of the world was created, and it was harmonised by proportion, and therefore has the spirit of friendship; and having been reconciled to itself, it was indissoluble by the hand of any other than the framer.

4.4 Menaechmus' Solutions

The solutions are in an analysis-synthesis form. First we suppose that the solution has been found, and then investigate necessary conditions. Then we reverse the process to construct the solution in a rigorous way. Modern mathematicians do the same thing, if not explicitly: consider, for example, how we "find" the necessary δ in a proof involving limits.

Menaechmus' First Solution

Analysis. Let the two given magnitudes be A , E . Suppose that the two means proportional have been found, say, B , Γ . Thus

$$A : B = B : \Gamma = \Gamma : E$$

In particular, $A : B = B : \Gamma$, so that

$$\text{rect.}(A, \Gamma) = \text{sq.}(B).$$

Let the line ΔH be set out, and at Δ , let ΔZ be set equal to Γ , and let $Z\Theta$ be constructed at right angles to ΔZ and equal to B . Hence it follows that

$$\text{rect.}(A, \Gamma) = \text{sq.}(Z\Theta).$$

Therefore the point Θ is on a parabola through Δ (i.e. with vertex Δ) with a parameter (or *latus rectum*) A [*Conics* I.11].

Let now ΔK , ΘK be drawn as parallels (to ΘZ , ΔZ , respectively). Again, by

hypothesis,

$$A : B = B : \Gamma = \Gamma : E,$$

so that in particular,

$$A : B = \Gamma : E,$$

whence

$$\text{rect.}(A, E) = \text{rect.}(B, \Gamma).$$

But by hypothesis, $B = Z\Theta$, $\Gamma = \Delta Z$, and $Z\Theta$ is perpendicular to ΔZ , it follows that

$$\text{rect.}(A, E) = \text{rect.}(B, \Gamma) = \text{rect.}(\Delta Z, Z\Theta).$$

Therefore Θ is on a hyperbola with asymptotes $K\Delta$, ΔZ [*Conics* II.12]. Since Θ is now given, being the intersection of a parabola and an hyperbola, Z is now also given. Hence both B , Γ are now given.

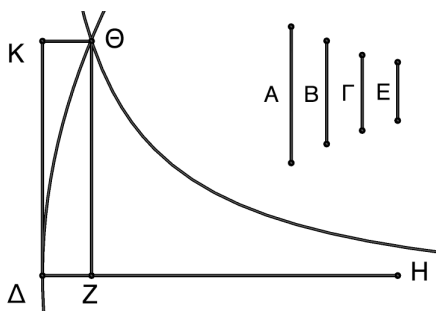


Figure 4.1: Menaechmus' First Solution

Synthesis. Let again the two given lines be A , E . Let ΔH be drawn, limited at Δ (so that H can be as far as needed from Δ). With Δ as vertex, and ΔH as axis, let a parabola be constructed whose parameter (latus rectum) is A (the construction of a parabola given the parameter is provided in *Conics* I.52). Let this parabola be $\Delta\Theta$, and let ΔK be drawn at right angles to ΔZ . Now let an hyperbola be drawn with $K\Delta$, ΔZ as asymptotes, and on this hyperbola, the lines drawn parallel to $K\Delta$, ΔZ make the rectangular area contained by them equal to the rectangle contained by the given lines A , E (*Conics* II.12). Therefore also the hyperbola will cut the parabola; let this happen at Θ . Now let $K\Theta$, ΘZ be drawn

as perpendiculars. Since Θ is on the parabola, we know that

$$\text{sq.}(Z\Theta) = \text{rect.}(A, \Delta Z),$$

and thus

$$A : Z\Theta = \Theta Z : Z\Delta.$$

Also, since

$$\text{rect.}(A, E) = \text{rect.}(\Theta Z, Z\Delta),$$

we have

$$A : Z\Theta = Z\Delta : E.$$

But then

$$A : Z\Theta = \Theta Z : Z\Delta = Z\Delta : E.$$

Let now $B = \Theta Z$, and $\Gamma = \Delta Z$, so that

$$A : B = B : \Gamma = \Gamma : E,$$

which it was required to find.

Menaechmus' Second Solution

Analysis. Let the two given lines be $AB, B\Gamma$, at right angles to one another, and let the means be $\Delta B, BE$, so that

$$\Gamma B : B\Delta = B\Delta : BE = BE : BA.$$

Let also $\Delta Z, EZ$ be drawn at right angles [to each other, and also to the given lines $AB, B\Gamma$]. Since $\Gamma B : B\Delta = B\Delta : BE$,

$$\begin{aligned} \text{rect.}(\Gamma B, BE) &= \text{sq.}(B\Delta) \\ &= \text{sq.}(EZ). \end{aligned}$$

Therefore Z lies on a parabola with axis BE , parameter $B\Gamma$, and vertex B .

Similarly, since $B\Delta : BE = BE : BA$,

$$\begin{aligned} \text{rect.}(AB, B\Delta) &= \text{sq.}(BE) \\ &= \text{sq.}(\Delta Z). \end{aligned}$$

Therefore Z also lies on a parabola with axis $B\Delta$, parameter AB , and vertex B . But since

it also lies on the other parabola, with the same vertex, axis BE , and parameter $B\Gamma$, the point Z is given. Therefore the means, EB and $B\Delta$, being perpendiculars, are also given.

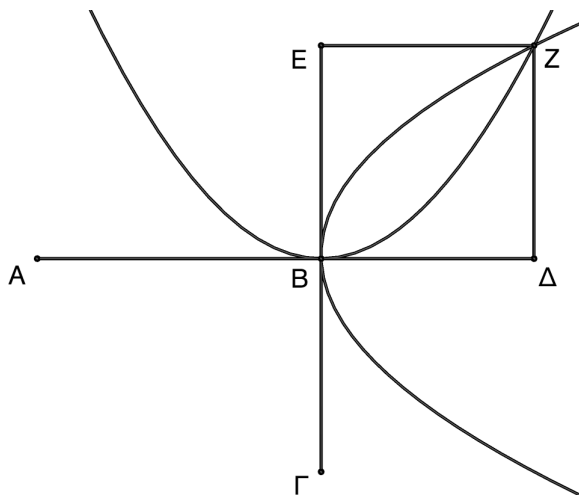


Figure 4.2: Menaechmus' Second Solution

Synthesis. Let again the given lines be AB , $B\Gamma$, set out at right angles to each other. With axis $B\Gamma$ produced, vertex B , and parameter AB , set out a parabola. Similarly, with axis AB produced, vertex B , and parameter $B\Gamma$, set out a second parabola. By construction, they will meet, for their axes are perpendicular and they share a common vertex. Let them meet at Z . Let EZ be drawn parallel to AB produced, and also $Z\Delta$ parallel to $B\Gamma$ produced. Now since Z is on each parabola, we have

$$\text{rect.}(\Gamma B, BE) = \text{sq.}(EZ)$$

and

$$\text{rect.}(AB, B\Delta) = \text{sq.}(\Delta Z).$$

Therefore

$$AB : \Delta Z = \Delta Z : B\Delta$$

and

$$\Gamma B : EZ = EZ : BE.$$

But EZ is equal to BΔ, so *invertendo*,

$$BE : B\Delta = B\Delta : \Gamma B,$$

whence

$$AB : EB = EB : B\Delta = B\Delta : \Gamma B,$$

which it was required to find. [Note the uses of *Conics* I.11 and I.52.]

4.5 Eratosthenes' Solution

Eratosthenes' solution involves the motion of three rectangles in the plane. But it is best to imagine the solution having a third dimensional component, albeit small. Netz suggests we imagine each rectangle as a door⁴. The object is to find the two means between AE and ΘΔ, and the solution will result by moving the doors into a certain position.

We begin with three rectangular doors, AZ, ΛH, and IΘ, with the diagonals also connected and parallel. The middle door will remain fixed, but the other two may move over one another, with door AZ on top, door ΛH in the middle, and door IΘ at the bottom. They still should be regarded as on the same plane, however, as if the doors themselves are made of exceedingly thin paper. Define the point B to be where the right hand edge of door one meets the diagonal of door two; so that B starts as Λ. Similarly, the point Γ is the intersection of the right hand edge of door two with the diagonal of door three; again, Γ starts as I. By moving the right hand door left, and the left hand door right, the segments AB, BΓ, and ΓΔ will eventually all be colinear, forming the straight line ABΓΔK, with point K the intersection of that line with the line EΘ.

Once the doors and lines are in this configuration, we are left with a series of similar triangles. The desired means then are ZB and HΓ. So by similar triangles,

$$AK : KB = EK : KZ = ZK : KH.$$

Similarly,

$$BK : K\Gamma = ZK : KH = HK : K\Theta.$$

⁴I have produced an interactive diagram for the case of three doors. The Processing source code is included with this thesis.

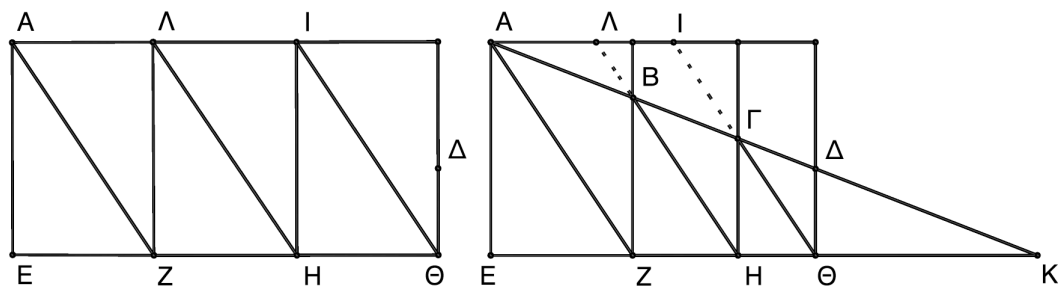


Figure 4.3: Eratosthenes' "Moving Doors" Solution

But

$$ZK : KH = EK : KZ,$$

therefore also

$$EK : KZ = ZK : KH = HK : K\Theta.$$

But

$$EK : KZ = AE : BZ,$$

and

$$ZK : KH = BZ : \Gamma H.$$

And

$$HK : K\Theta = \Gamma H : \Delta\Theta,$$

therefore

$$AE : ZB = ZB : \Gamma H = \Gamma H : \Delta\Theta,$$

which was enjoined.

The usefulness of Eratosthenes solution goes further, though. For if it is enjoined to find any number of means proportional, say n , then it suffices to carry out a similar procedure with $n + 1$ doors instead. The doors all are of course the same, with parallel diagonals, so that the argument reduces to a multitude of similar triangles, all sharing the vertex K .

The letter goes on to give a briefer description of the proof, which appeared on the pillar he dedicated to King Ptolemy III. He also mentions that this task will find multiple

means, again, by using one more "door" than means required. And lastly, he gives an epigram, which was also on the pillar. Netz' analysis of it indicates a fusion of literary style; mathematical and epic, in a sense. I reproduce the epigram below for the benefit of the reader:

Εἰ κύβον ἐξ ὀλίγου διπλήσιον, ὦγαθέ, τεύχειν
 φράζεαι ἢ στερεὴν πᾶσαν ἐς ἄλλο φύσιν
 εὖ μεταμορφῶσαι, τόδε τοι πάρα, κὰν σύ γε μάνδρην
 ἢ σιρὸν ἢ κοίλου φρεΐατος εὐρὸν κύτος
 τῆδ' ἀναμετρήσαιο, μέσας ὅτε τέρμασιν ἄκροις
 συνδρομάδας δισσῶν ἐντὸς ἔλης κανόνων.
 μηδὲ σύ γ' Ἀρχύτεω δυσμήχανα ἔργα κυλίνδρων
 μηδὲ Μεναιχμείους κωνοτομεῖν τριάδας
 διζήση, μηδ' εἴ τι θεουδέος Εὐδόξοιο
 καμπύλον ἐγ γραμμαῖς εἶδος ἀναγράφεται.
 τοῖσδε γὰρ ἐν πινάκεσσι μεσόγραφα μυρία τεύχοις
 ῥεῖά κεν ἐκ παύρου πυθμένος ἀρχόμενος.
 εὐαίων, Πτολεμαίε, πατήρ ὅτι παιδί συνηβῶν
 πάνθ', ὅσα καὶ Μούσαις καὶ βασιλεῦσι φίλα,
 αὐτὸς ἐδωρήσω· τὸ δ' ἐς ὕστερον, οὐράνιε Ζεῦ,
 καὶ σκήπτρων ἐκ σῆς ἀντιάσειε χερσός.
 καὶ τὰ μὲν ὡς τελέοιτο, λέγοι δέ τις ἄνθεμα λεύσσων
 τοῦ Κυρηναίου τοῦτ' Ἐρατοσθένεος.

If you plan, of a small cube, its double to fashion,
 Or - dear friend - any solid to change to another
 In Nature: it's yours. You can measure, as well:
 Be it byre, or corn-pit, or the space of a deep,
 Hollow well. As they run to converge, in between
 The two rulers - seize the means by their boundary-ends.
 Do not seek the impractical works of Archytas'
 Cylinders; nor the three conic-cutting Menaechmics;
 And not even that shape which is curved in the lines
 That Divine Eudoxus constructed.
 By these tablets, indeed, you may easily fashion -
 With a small base to start with - even thousands of means.
 O Ptolemy, happy! Father, as useful as son:
 You have given him all that is dear to the muses
 And to kings. In the future - O Zeus! - may *you* give him,
 From your hand, this, as well: a sceptre.
 May it all come to pass. And may him, who looks, say:
 "Eratosthenes, of Cyrene, set up this dedication."

The epigram does raise in my mind several questions. Why does Eratosthenes again mention the *cylinders* of Archytas, when in fact only one cylinder is explicitly mentioned

in his solution? We might imagine a second (semi)cylinder, which has the same radius and axis as the cone of his construction, but Archytas makes no mention of it whatsoever (as we shall see). He also refers to three conic-cutting Menaechmus; perhaps he is not explicitly referring to Menaechmus' two constructions, neither of which make use of the ellipse. And what does he mean by the lines that Divine Eudoxus constructed? Indeed, there is some dissonance here with what Eutocius himself says, for Eutocius explicitly excludes Eudoxus' solution on the grounds that it does not actually use curved lines, and falsely that a discrete proportion is a continuous one⁵. Netz, however, refers to the Eudoxus to whom Eutocius refers as pseudo-Eudoxus, apparently, then, believing that the proof Eutocius examined was a forgery. Unfortunately, of course, we do not have this proof of (pseudo?)-Eudoxus, and Netz says nothing further on the matter.

4.6 Archytas' Solution

Archytas' solution is quite a bit more bold, perhaps not surprisingly, given that it is the first known solution. It involves the following parts: a triangle, which rotates and thus sweeps out a cone; a semicircle, which rotates around one of its vertices, sweeping out a torus with no middle hole; and a cylinder. The rotating figures intersect the cylinder, forming two curves, which themselves intersect at a special point of consideration.⁶

Let the given straight lines be $A\Delta$ and Γ , and let a circle be described with diameter $A\Delta$ in the plane of reference. Let a point B be taken on the circle, and let AB be joined, so that AB is equal to Γ . Let AB be produced to Π , and the line $\Pi\Delta O$ be made tangent to the circle at Δ . Through B , let the straight line BEZ be drawn perpendicular to the diameter AB , with Z on the circumference of the circle. With diameter $A\Delta$, let a semicircle be constructed in the plane through $A\Delta$ which is perpendicular to BEZ . Lastly, let a right cylinder be constructed on the circle $A\Delta$.

⁵Netz suggests that Eutocius means that Eudoxus gave a proportion of the form $a : b = c : d$ instead of one like $a : b = b : c = c : d$.

⁶I have produced an animated 3D diagram of his construction. The Processing source code is attached to this thesis.

Now the motion begins. Let the triangle $\Pi\Delta A$ be rotated about the side $A\Delta$. Its intersection with the cylinder as it moves traces out a curved line ("Eugene's Line," under Netz's explanation), which starts at B and moves upwards, eventually to a point above Δ , and then falls back in the same manner towards Z . The semicircle rotates about the point A , sweeping out a torus; its intersection with the cylinder traces another curved line ("Tatiana's Line") which starts at Δ and moves upwards, before falling again to rest at A . These two curves will intersect at a point, say K . Let this occur when the triangle is in position $A\Delta\Lambda$, and when the semicircle in the position $A\Delta K$. Let the semicircle intersect the base circle at I . Now the motion of the point B sweeps out a semicircle on the cone. Let this semicircle intersect the other at M . Lastly, let KI and MI be joined, and let $M\Theta$ be produced through M and fall perpendicularly to Θ , the intersection of AI and BZ .

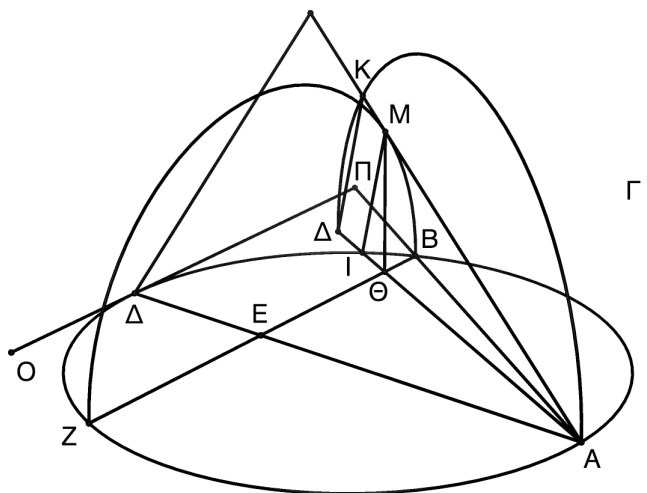


Figure 4.4: Archytas' Solution

The line KI , being on the cylinder, is perpendicular to the base plane. It is therefore parallel to the other perpendicular, $M\Theta$. Now

$$\begin{aligned} \text{rect.}(B\Theta, \Theta Z) &= \text{rect.}(A\Theta, \Theta I) \quad (\textit{Elements III.35}) \\ &= \text{sq.}(M\Theta) \quad (\textit{Elements III.31, VI.8}). \end{aligned}$$

Imagine a semicircle with diameter AI . Since $M\Theta$ was set up perpendicular to the plane,

and $\text{rect.}(A\Theta, \Theta I) = \text{sq.}(M\Theta)$, it is the case that the point M is actually on such a semi-circle. Therefore the angle $\angle AMI$ is right. For the same reason, the angle $\angle \Delta KA$ is right, so that the lines $K\Delta$ and MI are parallel. Therefore each of the triangles $\triangle AMI$, $\triangle MI\Theta$, and $\triangle MA\Theta$ are similar. On account of this,

$$\begin{aligned} \Delta A : AK &= KA : AI \\ &= IA : AM \end{aligned}$$

Therefore

$$\Delta A : AK = AK : AI = AI : AM.$$

But AM is equal to AB , since the cone is right, whence

$$\Delta A : AK = AK : AI = AI : AB.$$

But AB was constructed equal to Γ , so

$$\Delta A : AK = KA : AI = AI : \Gamma;$$

and that which was enjoined has been done.

CHAPTER 5 ON COMPOUND RATIOS

5.1 Introduction

In this chapter, I wish to revisit some of the issues surrounding compound ratios, specifically those discussed by Ken Saito in *Compound ratio in Euclid and Apollonius* ([26], itself building upon [25]). His paper, as the title indicates, examines the role of compound ratios both in the *Elements* and *Data* of Euclid, along with the *Conics* of Apollonius and selections from the *Collection* of Pappus. Throughout it, Saito shows how we should not think that compounding ratios, at least in his source texts, should be thought of in some sort of algebraic way. In my own studies of compound ratios, I began to look towards other authors, specifically Eutocius, who provides commentaries to both Archimedes' *On the Sphere and the Cylinder* and the aforementioned *Conics*. While it is certainly true that Eutocius is quite a late author within the timeline of Greek mathematics, his commentaries of the *Conics* certainly played a role in the transmission of the Greek text, and so it should be studied together the "older" text of the *Conics* itself and other "old" mathematical works. Saito's paper even invites this revisiting; he concludes with a citation of Archimedes' use of compound ratios in the analysis of *SC* II.4, saying:

I believe more overlooked peculiarities remain in ancient theories concerning geometric magnitudes. I have established only one example of the inadequacy of the algebraic interpretation. Further reexamination of the ancient technique concerning geometric magnitudes is needed to understand Greek mathematics in the proper sense of the word.

By examining compound ratios as seen in Eutocius, I am to add to the discussion begun by Saito.

In my study of Eutocius' commentary to *Conics* I, I came upon his discussion of compound ratios. This in itself is nothing noteworthy, but the processes of translating this commentary led me to consult the Greek (and Netz' translation) of Eutocius' commentary on *The Sphere and the Cylinder*. At first glance, the very passage on compound ratios

in Eutocius' commentary of *Conics* I.11 is, *mutatis mutandis*, that which appears in his commentary of *On the Sphere and the Cylinder* II.4. This I endeavor to analyze and compare the two passages to discover exactly what these differences are.

It should be noted however that a comparative analysis of the two passages already exists: given by Knorr in chapter 7 of *Textual Studies in Ancient and Medieval Geometry* ([7]). However, the purpose of his analysis is not strictly speaking to compare Eutocius' two works together or to concentrate on compound ratios in them. Instead, he seeks to refute the thesis of J. Mogenet, expressed in his *l'Introduction à l'Almageste*, that the unattributed text *Introduction to the Syntaxis* is a work of Eutocius. Part of this unattributed text deals with compound ratios; Knorr observes that the style and mathematical understanding displayed by the author just does not fit with Eutocius' two passages. In doing so, he places Eutocius' passages side-by-side and compares the flow of the passage in the *Introduction* with them. Part of my analysis will concentrate on just the parts of Knorr's analysis concerning Eutocius himself; in doing so, exploring a new direction with Knorr's analysis as a foundation.

I this begin with a look at how compound ratios are used in several Hellenistic authors, specifically Euclid, Archimedes, and Apollonius.

5.2 Compound Ratios as Used in Hellenistic Authors

5.2.1 Euclid

The notion of compounded ratios only appears twice in the *Elements*. In VI.23, Euclid proves that equiangular parallelograms are to one another in the ratio compounded out of the ratios of their corresponding sides; VIII.5 is an analogous proposition for the case of numbers. Disturbingly, nowhere is the idea of compounded ratios explicitly defined¹, though it could be regarded as a generalization of the duplicate and triplicate ratios. Euclid defines these concepts as follows:

V def. 9: When three magnitudes are proportional, the first is said to have to

¹At least in the non-Theonine editions of the *Elements*. We shall address this soon.

the third the *duplicate ratio* of that which it has to the second.

V def. 10: When four magnitudes are [continuously] proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually, whatever be the proportion.

In symbols, if we have that $A : B = B : \Gamma = \Gamma : \Delta$, then the ratio $A : \Gamma$ is said to be the duplicate ratio of $A : B$; similarly, the ratio $A : \Delta$ is the triplicate ratio of $A : B$. The first definition is used in *Elements* VI.19, for example, to prove that similar triangles are to one another as the duplicate ratio of their corresponding sides. Euclid extends this in VI.20 to any pair of similar rectilinear figures, by virtue of dividing the rectilinear figures into two set of similar triangles, equal in multitude, and applying VI.19.

Moving on to VI.23, we see the debut use of the idea of compounded ratios. In this proposition, as we have said, Euclid proves that equiangular parallelograms have to one another the ratio compounded of the ratios of their sides. That is, if we have two equiangular parallelograms, $AB\Gamma\Delta$ and ΓEZH , then

$$\text{pllg.}(AB\Gamma\Delta) : \text{pllg.}(\Gamma EZH) = (AB : \Gamma E)\text{comp.} (\Gamma\Gamma : \Gamma H).$$

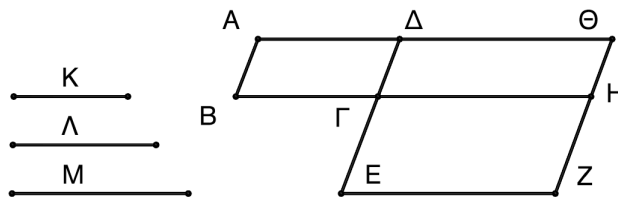


Figure 5.1: *Elements* VI.23

Now if it so happened to be the case that the parallelograms $AB\Gamma\Delta$ and ΓEZH were also similar, so that $AB : B\Gamma = \Gamma E : EZ$, then VI.20 applies, so that

$$\text{pllg.}(AB\Gamma\Delta) : \text{pllg.}(\Gamma EZH) = \text{dup.}(AB : \Gamma E).$$

But similar parallelograms are in particular equiangular, so the conclusion of VI.23 still applies. Hence VI.11 yields that

$$\text{dup.}(AB : \Gamma E) = (AB : \Gamma E)\text{comp.} (B\Gamma : \Gamma H).$$

But because the parallelograms are supposed similar, *alternando*, $AB : \Gamma E = B\Gamma : EZ$. And $EZ = \Gamma H$, so that $AB : \Gamma E = B\Gamma : \Gamma H$. Hence

$$\text{dup.}(AB : \Gamma E) = (AB : \Gamma E)\text{comp.} (AB : \Gamma E).$$

For this reason we should see the ratio compounded of two ratios as a generalization of a duplicate ratio; a similar argument using XI.33 shows that the ratio compounded of three ratios is a generalization of a triplicate ratio. This may be a reason why Euclid never formally defines compound ratio, taking it to be such a generalization of duplicate and triplicate ratios.

Saito, however, gives a reconstructed definition of compounded ratios, saying²

- (1) Let A, B, C be magnitudes of the same kind. The ratio $A : C$ is said to be compounded out of the ratios $A : B$ and $B : C$.
- (2) Further, if $A : B = D : E$ and $B : C = F : G$ then $A : C$ is said to be compounded of $D : E$ and $F : G$.

The first definition is close in spirit to the case of duplicate ratios, though notice that it is not assumed that $A : B = C : D$, only that they are two arbitrary ratios. If they are proportional, however, then the compounded ratio that Saito gives corresponds precisely to the duplicate ratio. In the second definition, the special case when $B : C = E : F$ yields that $A : C = D : F$ *ex aequali*. Therefore we note that $D : F$, and hence $A : C$, are each the ratio compounded out of the ratios $D : E$ and $E : F$.

Saito observes that Euclid really never uses compounded ratio in the *Elements*, even though VI.23 and XI.33 are about compounded ratio. And in the *Data*, he points out that in situations that could involve compounded ratio, Euclid prefers instead to reduce the matter to linear ratios by means of the application of areas. This method is first apparent

²I have preserved Saito's Latin letters in this quote.

in VI.19, where Euclid introduces a mean proportional between in order to move the given problem (about ratios of areas) to a question about linear ratios. This, of course, is necessary under his definition of duplicate ratios, since he needs a proportion of three terms in order to invoke the definition. While VI.19 by itself would not by itself point to an aversion on the part of Euclid for using compounded ratios, Saito argues that it is the case throughout the *Data* that Euclid uses a method of reduction to linear ratios instead of compounded ratio.

5.2.2 Apollonius and Archimedes

The next question, of course, is what about Apollonius? To what extent did he use compounded ratio or this Euclidean method of reduction to linear areas? For it certainly is the case that compounded ratio appears quite often in *Conics* I, as early as, for example, I.12 (the hyperbola). But Saito argues that compounded ratios here are not truly part of Apollonius' analysis; rather, he contends, Apollonius uses compounded ratio as a way to simplify arguments and expressions, where other more cumbersome expressions (such as the use of ratios *ex aequali*) would be needed. But in terms of the analysis, at least for I.41-43, Saito argues that Apollonius followed Euclid in using a reduction to linear ratios.

He contrasts this Euclidean and Apollonian aversion to compounded ratio with Pappus, who writing much later, makes far more extensive use of them. In fact, Saito says that Pappus uses compounding as an operation of sorts on ratios, and thus his arguments do have a quasi-algebraic feel that is not present in those of Euclid or Apollonius. Interestingly enough, Saito mentions in a footnote that compounded ratios are indispensable in Archimedes' analysis in *On the Sphere and the Cylinder* II.4, but nothing more on the matter is said. Fortunately we now have Netz. He also relegates the point to a footnote, though a quite interesting one:

The operation of "composition of ratios" was never fully clarified by the Greeks: see Eutocius for an honest attempt.

And later, in his translation of Eutocius' commentary, he adds (in another footnote):

What Eutocius says is that as far as the mathematical consensus is concerned, Archimedes' argument [SC II.4] is clear and even obvious. However, since the mathematical consensus itself seems to be at fault here, a commentary is required. First we had a spirit of philological enterprise, in the catalogue of two means proportionals [Eutocius' commentary to SC II.1], and now a mathematical independence. Eutocius has grown considerably since the commentary to the first book. The composition of ratios is indeed a sore point in Greek mathematics: let's see how much sense he will make out of it (Eutocius himself clearly was happy with his own discussion, and he has recycled it in his later commentary to Apollonius' *Conics*, II. pp. 218 ff.).

And so the discussion comes full circle. If anyone would have a clear idea as to the matter of compounded ratios, it should be Eutocius, who writes by far the latest. The fact that he understands Archimedes well enough to write a valuable commentary must mean that he in particular understood Archimedes' use of compounded ratios. But Netz says that he wrote his commentary to *On the Sphere and the Cylinder* before his commentary on *Conics*. And as Netz also says, the character of Eutocius' commentary grows from the commentary on SC I to that on SC II. But as Eutocius' commentary on the *Conics* is, beyond a few fragments untranslated by others, and my own translation does not yet extend beyond Eutocius' introduction, we are left at the moment with a profound and open question. *Just what does Eutocius say about compounded ratios? Just what is the definition of compound ratio?*

5.3 The Definition in Theon's Edition of the *Elements*

As I mentioned above, compound ratio is not defined in the non-Theonine editions of the *Elements*. By this, I mean editions which do not descend from the edition of Theon. The manuscript tradition, though, is quite a mess. See Heath and Heiberg for details of the manuscript tradition of the *Elements*, and Cameron ([1]) for details on the practice of editing mathematical texts.

5.3.1 Theon's Definition

In Theon's edition, we do have the following (as VI def. 5: bracketed in Heiberg, absent from the Green Lion edition):

A ratio is said to be compounded of ratios, whenever the sizes of the ratios, having been multiplied into themselves, produce something.
 Λόγος ἐκ λόγων συγκείσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα.

In many respects, this definition is disappointing, since it seems to raise more question than it answers. What precisely are the sizes (αἱ πηλικότητες) of the ratios, how are they multiplied into themselves (πολλαπλασιασθεῖσαι), and just what something (τίνα) do they produce? The definition is inherently non-geometric, and feels completely out of place given the context of Book VI. It would make much more sense if it appeared in Book V, given the nature of that book; however, since it is first *used* in Book VI, perhaps Theon felt it to be the appropriate place for the definition.

5.3.2 Theon's Impact on Eutocius

It is to be noted that Theon wrote earlier than Eutocius, and by a sufficient amount that it is most likely that Eutocius thought Theon's edition was the *Elements*. Eutocius deals with compound ratios twice in his works; the first being in his commentary on *On the Sphere and the Cylinder*, the second being that on the *Conics*. In the next section, I will show what Eutocius says on the matter in each case, and the combined weight of his words in both commentaries will show the influence that Theon had on Eutocius.

5.4 Further Definitions: Exposition and Analysis

Each subsection here will present original source material (in translation). I postpone discussion until the next major section.

5.4.1 Eutocius' Exposition in his *Commentary on On the Sphere and the Cylinder*

"Now since the ratio of PA to ΛX is combined of both: the ratio \langle ratio \rangle which PA has to $\Lambda\Delta$, and \langle of \rangle $\Lambda\Delta$ to ΛX ." It is obvious that, once $\Lambda\Delta$ is taken as a mean, the synthesis of ratios is taken (as this is taken in the *Elements*, too). Since, however,

the discussion of the subject has been somewhat confused, and not such as to make the concept satisfactory, (as can be found reading Pappus and Theon and Arcadius, who, in many treatises, present the operation not by arguments, but by examples), there will be no incongruity if we linger briefly on this subject so as to present the operation more clearly.

So: I say that if some middle term is taken between two numbers (or magnitudes), the ratio of the initially taken numbers is composed of the ratio, which the first has to the mean and of the ratio which the mean has to the third.

So first it ought to be recalled how a ratio is said to be compounded of ratios. For as in the *Elements*: "when the quantities of the ratios, multiplied, produce a certain <quantity>," where "quantity" clearly stands for "the number" whose cognate is the given ratio (as say several authors as well as Nichomachus in the first book of *On Music* and Heron as in the commentary to the *Arithmetical Introduction*) which is the same as saying: "the number which, multiplied on the consequent term of the ratio, produces the antecedent as well." And the quantity would be taken in a more legitimate way in the case of multiples, while in the case of the superparticulars, superpartients, it is no longer possible for the quantity to be taken without the unit remaining undivided - which, even if this does not belong to what is proper in arithmetic, yet it does belong to what is proper in calculation. And the unit is divided by the part of by the parts by which the ratio is called, so that (to say this in a clearer way), the quantity of the half-as-large again is, added to the unit, half the unit; and <the quantity of the> four-thirds is, added to the unit, one third the unit, so that, as has been said above as well, the quantity of the ratio, multiplied on the consequent term, produces the antecedent. For the quantity of nine to six, being the unit and the half, multiplied on 6, produces 9, and it is possible to observe the same in other cases as well.

Having clarified these first, let us return to the enunciated proposition. For let the two given numbers be A, B, and let a certain mean be taken between them, Γ . So it is to be proved that the ratio of A to B is combined of the ratio which A has to Γ , and Γ to B.

For let the quantity of the ratio A, Γ be taken, namely Δ , and let the quantity of

the ratio Γ , B be taken, namely E; therefore Γ , multiplying Δ , produces A, while B, multiplying E, produces Γ . So let Δ , multiplying E, produce Z. I say that Z is a quantity of the ratio of A to B, that is, that Z, multiplying B, produces A. For let B, multiplying Z, produce H. Now since B, multiplying Z, has produced H, and multiplying E, has produced Γ , it is therefore: as Z to E, H to Γ . Again, since Δ , multiplying E, has produced Z, while, multiplying Γ , it has produced A, it is therefore: as E to Γ , Z to A. Alternately: as E to Z, Γ to A, inversely also: as Z to E, so A to Γ . But as Z to E, H was proved to be Γ , therefore also: as H to Γ , A to Γ ; therefore A is equal to H. But B, multiplying Z, has produced H; therefore B, multiplying Z, produces A as well; therefore Z is the quantity of the ratio of A to B. And Z is: Δ , multiplied on E, that is: the quantity of the ratio A, Γ , multiplied on the quantity of the ratio Γ , B; therefore the ratio of A to B is composed of both: the ratio which A has to Γ , and Γ to B; which it was required to prove.

[Eutocius concludes this passage with numerical examples, which I omit.]

5.4.2 Eutocius' Exposition in his *Commentary on the Conics*

But the square on B Γ has to the rectangle BA Γ a ratio compounded from that which B Γ has to Γ A and B Γ to BA:

$$\text{sq.}(B\Gamma) : \text{rect.}(BA\Gamma) = (B\Gamma : \Gamma A) \text{ comp. } (B\Gamma : BA) :$$

it has been shown in the twenty-third theorem of the sixth book of the *Elements*, that equiangular parallelograms have to one another a ratio compounded out of that of the sides: but since it is discussed too inductively and not in the necessary manner by the commentators, we researched it; and it is written in our published work on the fourth theorem of the second book of Archimedes' *On the Sphere and the Cylinder*, and also in the scholia of the first book of Ptolemy's *Syntaxis*: but it is a good idea that this be written down here also, because readers do not always read it even in those works, and also because nearly the entire treatise of the *Conics* makes use of it.

(218.16) A ratio is said to be compounded from ratios, whenever the sizes of the

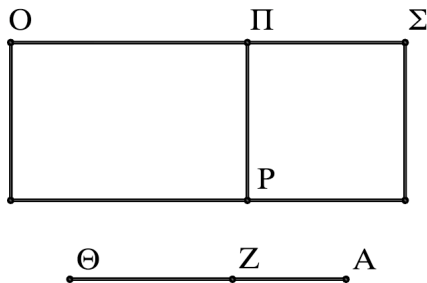


Figure 5.2: Eutocius' Diagram for Theorem 11

ratios, being multiplied into themselves, make something, with "size" of course meaning the number after which the ratio is named. So it is possible in the case of multiples that the size be a whole number, but in the case of the remaining relations it is necessary that the size must be a number plus part or parts, unless perhaps one wishes the relation to be irrational, such as are those according to the incommensurable magnitudes. But in the case of all the relations, it is manifest that the product of the size itself with the consequent of the ratio makes the antecedent.

(218.27) Accordingly, let there be a ratio of A to B, and let some mean of them be taken, as it chanced, as Γ , and let

$$\Delta = \text{size}(A : \Gamma),$$

and

$$E = \text{size}(\Gamma : B),$$

and let Δ , multiplying E, make Z. I say, that the size of the ratios A, B is Z,

$$Z = \text{size}(A : B),$$

that is that Z, multiplying B, makes A. Indeed, let Z, multiplying B, make H. So since Δ , multiplying E has made Z, and multiplying Γ has made A, therefore it is, that

$$E : Z = \Gamma : H$$

Alternando,

$$E : \Gamma = Z : H$$

But

$$E : \Gamma = Z : A$$

Therefore

$$H = A,$$

so that Z, multiplying B, has made A.

(220.17) But do not let this confuse those reading that this has been proved through arithmetic, for both the ancients made use of such proofs, being mathematical rather than arithmetical on account of the proportions, and that the thing being sought is arithmetical. For both ratios and sizes of ratios and multiplications, first begin by numbers, and through them by magnitudes, according to the speaker. As someone once said, "for these mathematical studies appear to be related."³

5.4.3 Theon's Definition and Exposition in his *Commentary on the Syntax*

Lemma⁴. One ratio is said to be compounded out of two or more ratios, whenever the sizes of the ratios, being multiplied into themselves, produce some size of a ratio.

For let the ratio of AB to $\Gamma\Delta$ be given, and that of $\Gamma\Delta$ to EZ: I say that the ratio of AB to EZ is compounded out of the ratios AB to $\Gamma\Delta$ and $\Gamma\Delta$ to EZ, that is, if the size of the ratio AB to $\Gamma\Delta$ is multiplied into that of $\Gamma\Delta$ to EZ, it makes the size of the ratio AB to EZ.

[Theon continues with numerical examples, which I omit.]

5.5 Conclusion

To conclude, we must observe a few things.

³Much ought be said of this passage; some will regrettably be left for future work. Here Eutocius retains the Doric dialect of the quotation: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν ἀδελφά. The quote is due to Archytas of Tarentum, preserved only in a fragment of his *Harmonics*. The quoted part reads: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι ἡμεν ἀδελφεά. For the full fragment, see fr. B.1 in Diels-Kranz ([6]) or Freeman's English translation ([2]). See also *Republic* VII 530d and Knorr ([7]), pp. 171 n. 22-23.

⁴This is my translation of A. Rome's edition of the Greek, in [13].

The first: to Eutocius, Theon's definition of compound ratios *was* part of the *Elements*.

The second, and much more important: note how Eutocius develops mathematically between his two passages. In his first passage, Eutocius has seemingly no conception that the "size" of a ratio might be what we call an irrational number; in essence, his understanding at that point in time is only of "rational" ratios. Or, perhaps, he simply felt that mentioning irrational magnitudes was not necessary within the context of his exposition. That irrational magnitudes existed is not new; Eutocius, as an educated mathematician, would be well versed in approaching them. His two proofs are essentially the same: nowhere does he actually need these sizes to be rational.

It is clear, too, that Eutocius takes the indefinite $\tau\acute{\iota}\nu\alpha$ to mean one of the $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\tau\alpha$, based purely on how his proofs are structured. Given that Theon's definition from the *Syntaxis* commentary adds "size of a ratio" after $\tau\acute{\iota}\nu\alpha$, and that it is entirely likely that Eutocius was aware of this commentary, I am not surprised that Eutocius reads this for $\tau\acute{\iota}\nu\alpha$. It also does not surprise me that Eutocius quoted the "simpler" definition, that is, the one from the *Elements*, though it would have certainly have been helpful for his readers (myself included!) to know how to take $\tau\acute{\iota}\nu\alpha$. For the purpose of reading Apollonius, at least in Book I, single-compounding is adequate: the multiple-compounding needed for reading the *Syntaxis* is just not relevant here.

In future work (post-graduate), I intend to further explore this area by expanding the discussion to as many Hellenistic authors as possible, with the aid of the various extant manuscripts.

APPENDIX A
THE GREEK TEXT OF EUTOCIUS' COMMENTARY ON CONICS I

- 168 **Εἰς τὸ πρῶτον.**
- Ἀπολλώνιος ὁ γεωμέτρης, ὃ φίλε ἑταίρε Ἀνθέμιε, 5
 γέγονε μὲν ἐκ Πέργης τῆς ἐν Παμφυλίας ἐν χρόνοις
 τοῦ Εὐεργέτου Πτολεμαίου, ὡς ἱστορεῖ Ἡράκλειος ὁ
 τὸν βίον Ἀρχιμήδους γράφων, ὃς καὶ φησι τὰ κωνικὰ
 θεωρήματα ἐπινοῆσαι μὲν πρῶτον τὸν Ἀρχιμήδη, τὸν
 δὲ Ἀπολλώνιον αὐτὰ εὐρόντα ὑπὸ Ἀρχιμήδους μὴ ἐκ- 10
 δοθέντα ἰδιοποιήσασθαι, οὐκ ἀληθεύων κατὰ γε τὴν
 ἐμήν. ὃ τε γὰρ Ἀρχιμήδης ἐν πολλοῖς φαίνεται ὡς
 παλαιότερας τῆς στοιχειώσεως τῶν κωνικῶν μεμνη-
 μένος, καὶ ὁ Ἀπολλώνιος οὐχ ὡς ἰδίας ἐπινοίας γράφει·
 οὐ γὰρ ἂν ἔφη ἐπὶ πλεον καὶ καθόλου μᾶλλον 15
 ἐξεργάσθαι ταῦτα παρὰ τὰ ὑπὸ τῶν ἄλλων
 γεγραμμένα. ἀλλ' ὅπερ φησὶν ὁ Γεμίνος ἀληθές
 ἐστίν, ὅτι οἱ παλαιοὶ κῶνον ὀριζόμενοι τὴν τοῦ ὀρθο-
 γωνίου τριγώνου περιφορὰν μενούσης μίας τῶν περὶ
 τὴν ὀρθὴν εἰκότως καὶ τοὺς κῶνους πάντας ὀρθοὺς 20
 ὑπελάμβανον γίνεσθαι καὶ μίαν τομὴν ἐν ἐκάστω, ἐν
 170 μὲν τῷ ὀρθογωνίῳ τὴν νῦν καλουμένην παραβολήν,
 ἐν δὲ τῷ ἀμβλυγωνίῳ τὴν ὑπερβολήν, ἐν δὲ τῷ ὀξυ-
 γωνίῳ τὴν ἔλλειψιν· καὶ ἔστι παρ' αὐτοῖς εὐρεῖν οὕτως
 ὀνομαζομένας τὰς τομάς. ὥσπερ οὖν τῶν ἀρχαίων
 ἐπὶ ἑνὸς ἐκάστου εἴδους τριγώνου θεωρησάντων τὰς 5
 δύο ὀρθὰς πρότερον ἐν τῷ ἰσοπλεύρῳ καὶ πάλιν ἐν

τῷ ἰσοσκελεῖ καὶ ὕστερον ἐν τῷ σκαληνῷ οἱ μετα-
 γενέστεροι καθολικὸν θεώρημα ἀπέδειξαν τοιοῦτο· παν-
 τὸς τριγώνου αἱ ἐντὸς τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι
 εἰσίν· οὕτως καὶ ἐπὶ τῶν τοῦ κώνου τομῶν· τὴν μὲν
 γὰρ λεγομένην ὀρθογωνίου κώνου τομὴν ἐν ὀρθο-
 γωνίῳ μόνον κώνῳ ἐθεώρουν τεμνομένῳ ἐπιπέδῳ ὀρθῷ
 πρὸς μίαν πλευρὰν τοῦ κώνου, τὴν δὲ τοῦ ἀμβλυ-
 γωνίου κώνου τομὴν ἐν ἀμβλυγωνίῳ γινομένην κώνῳ
 ἀπεδείκνυσαν, τὴν δὲ τοῦ ὀξυγωνίου ἐν ὀξυγωνίῳ,
 ὁμοίως ἐπὶ πάντων τῶν κώνων ἄγοντες τὰ ἐπίπεδα
 ὀρθὰ πρὸς μίαν πλευρὰν τοῦ κώνου· δηλοῖ δὲ καὶ
 αὐτὰ τὰ ἀρχαῖα ὀνόματα τῶν γραμμῶν. ὕστερον δὲ
 Ἀπολλώνιος ὁ Περγαῖος καθόλου τι ἐθεώρησεν, ὅτι
 ἐν παντὶ κώνῳ καὶ ὀρθῷ καὶ σκαληνῷ πᾶσαι αἱ τομαί
 εἰσι κατὰ διάφορον τοῦ ἐπιπέδου πρὸς τὸν κώνον
 προσβολήν· ὃν καὶ θαυμάσαντες οἱ κατ' αὐτὸν γενό-
 μενοι διὰ τὸ θαυμάσιον τῶν ὑπ' αὐτοῦ δεδειγμένων
 κωνικῶν θεωρημάτων μέγαν γεωμέτρην ἐκάλουν. ταῦτα
 μὲν οὖν ὁ Γεμίνος ἐν τῷ ἔκτῳ φησὶ τῆς τῶν μαθη-
 μάτων θεωρίας. ὃ δὲ λέγει, σαφὲς ποιήσομεν ἐπὶ τῶν
 ὑποκειμένων καταγραφῶν.

ἔστω τὸ διὰ τοῦ ἄξονος τοῦ κώνου τρίγωνον τὸ

172 ΑΒΓ, καὶ ἤχθω τῇ ΑΒ ἀπὸ τυχόντος σημείου τοῦ Ε
 πρὸς ὀρθὰς ἢ ΔΕ, καὶ τὸ διὰ τῆς ΔΕ ἐπίπεδον ἐκ-
 βληθὲν ὀρθὸν πρὸς τὴν ΑΒ τεμνέτω τὸν κώνον· ὀρθὴ
 ἄρα ἐστὶν ἑκατέρω τῶν
 ὑπὸ ΑΕΔ, ΑΕΖ γω-

νιῶν. ὀρθογωνίου μὲν
 ὄντος τοῦ κώνου καὶ
 ὀρθῆς δηλονότι τῆς
 ὑπὸ ΒΑΓ γωνίας ὡς
 ἐπὶ τῆς πρώτης κατα- 10
 γραφῆς δύο ὀρθαῖς ἴσαι

ἔσσονται αἱ ὑπὸ ΒΑΓ, ΑΕΖ γωνίαι· ὥστε παράλληλος
 ἔσται ἡ ΔΕΖ τῇ ΑΓ. καὶ γίνεται ἐν τῇ ἐπιφανείᾳ
 τοῦ κώνου τομὴ ἢ καλουμένη παραβολὴ οὔτω κλη-
 θείσα ἀπὸ τοῦ παράλληλον εἶναι τὴν ΔΕΖ, ἣτις ἐστὶ 15
 κοινὴ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ
 ἄξονος τριγώνου, τῇ ΑΓ πλευρᾷ τοῦ τριγώνου.

ἐὰν δὲ ἀμβλυγώνιος ᾖ ὁ κώνος ὡς ἐπὶ τῆς δευ-
 τέρας καταγραφῆς ἀμβλείας δηλονότι οὔσης τῆς ὑπὸ
 ΒΑΓ, ὀρθῆς δὲ τῆς ὑπὸ ΑΕΖ, δύο ὀρθῶν μείζους 20
 ἔσσονται αἱ ὑπὸ ΒΑΓ, ΑΕΖ γωνίαι· ὥστε οὐ συμ-
 πεσεῖται ἡ ΔΕΖ τῇ ΑΓ πλευρᾷ ἐπὶ τὰ πρὸς τοῖς Ζ, Γ
 μέρη, ἀλλὰ ἐπὶ τὰ πρὸς τοῖς Α, Ε προσεκβαλλομένης
 δηλονότι τῆς ΓΑ ἐπὶ τὸ Δ. ποιήσει οὖν τὸ τέμνον
 ἐπίπεδον ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τομὴν τὴν καλου- 25
 μένην ὑπερβολὴν οὔτω κληθείσαν ἀπὸ τοῦ ὑπερβάλ-
 λειν τὰς εἰρημένας γωνίας, τουτέστι τὰς ὑπὸ ΑΕΖ,

174 ΒΑΓ, δύο ὀρθᾶς ἢ διὰ τὸ ὑπερβάλλειν τὴν ΔΕΖ
 τὴν κορυφὴν τοῦ κώνου καὶ συμπύπτειν τῇ ΓΑ ἐκτός.

ἐὰν δὲ ὀξυγώνιος ᾖ ὁ κώνος ὀξείας δηλονότι οὔσης
 τῆς ὑπὸ ΒΑΓ, αἱ ΒΑΓ, ΑΕΖ ἔσσονται δύο ὀρθῶν
 ἐλάσσονες· ὥστε αἱ ΕΖ, ΑΓ ἐκβαλλόμεναι συμπεσοῦν- 5

ται ὅπουδήποτε· προσαυξήσαι γὰρ δύναμαι τὸν κώνον.
 ἔσται οὖν ἐν τῇ ἐπιφανείᾳ τομῇ, ἣτις καλεῖται ἔλλει-
 ψις, οὕτω κληθεῖσα ἦτοι διὰ τὸ ἐλλείπειν δύο ὀρθαῖς
 τὰς προειρημένας γωνίας ἢ διὰ τὸ τὴν ἔλλειψιν κύκλον
 εἶναι ἔλλιπῆ.

10

οὕτως μὲν οὖν οἱ παλαιοὶ ὑποθέμενοι τὸ τέμνον
 ἐπίπεδον τὸ διὰ τῆς ΔΕΖ πρὸς ὀρθὰς τῇ ΑΒ πλευρᾷ
 τοῦ διὰ τοῦ ἄξονος τοῦ κώνου τριγώνου καὶ ἔτι δια-
 φόρους τοὺς κώνους ἐθεώρησαν καὶ ἐπὶ ἐκάστου ἰδίαν
 τομῆν· ὁ δὲ Ἀπολλώνιος ὑποθέμενος τὸν κώνον καὶ
 ὀρθὸν καὶ σκαληνὸν τῇ διαφόρῳ τοῦ ἐπιπέδου κλίσει
 διαφόρους ἐποίησε τὰς τομάς.

15

ἔστω γὰρ πάλιν ὡς ἐπὶ τῶν αὐτῶν καταγραφῶν τὸ
 τέμνον ἐπίπεδον τὸ ΚΕΛ, κοινὴ δὲ αὐτοῦ τομῇ
 καὶ τῆς βάσεως τοῦ κώνου ἢ ΚΖΛ, κοινὴ δὲ πάλιν
 αὐτοῦ τοῦ ΚΕΛ ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου
 ἢ ΕΖ, ἣτις καὶ διάμετρος καλεῖται τῆς τομῆς. ἐπὶ
 πασῶν οὖν τῶν τομῶν ὑποτίθεται τὴν ΚΛ πρὸς ὀρθὰς
 τῇ ΒΓ βάσει τοῦ ΑΒΓ τριγώνου, λοιπὸν δέ, εἰ μὲν

20

176 ἢ ΕΖ παράλληλος εἶη τῇ ΑΓ, παραβολὴν γίνεσθαι
 τὴν ΚΕΛ ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τομῆν, εἰ δὲ
 συμπίπτει τῇ ΑΓ πλευρᾷ ἢ ΕΖ ἐκτὸς τῆς κορυφῆς
 τοῦ κώνου ὡς κατὰ τὸ Δ, γίνεσθαι τὴν ΚΕΛ τομῆν
 ὑπερβολὴν, εἰ δὲ ἐντὸς συμπίπτει τῇ ΑΓ ἢ ΕΖ, γί-
 νεσθαι τὴν τομῆν ἔλλειψιν, ἣν καὶ θυρεὸν καλοῦσιν.
 καθόλου οὖν τῆς μὲν παραβολῆς ἢ διαμέτρος παρ-
 ἀλληλός ἐστι τῇ μιᾷ πλευρᾷ τοῦ τριγώνου, τῆς δὲ

5

ὑπερβολῆς ἢ διάμετρος συμπίπτει τῇ πλευρᾷ τοῦ τρι-
 γώνου ὡς ἐπὶ τὰ πρὸς τῇ κορυφῇ τοῦ κώνου μέρη, 10
 τῆς δὲ ἑλλείψεως ἢ διάμετρος συμπίπτει τῇ πλευρᾷ
 τοῦ τριγώνου ὡς ἐπὶ τὰ πρὸς τῇ βάσει μέρη. κάκεινο
 δὲ χρῆ εἰδέναι, ὅτι ἢ μὲν παραβολὴ καὶ ἢ ὑπερβολὴ
 τῶν εἰς ἄπειρόν εἰσιν ἀύξανομένων, ἢ δὲ ἕλλειψις
 οὐκέτι· πᾶσα γὰρ εἰς αὐτὴν συννεύει ὁμοίως τῷ 15
 κύκλῳ.

πλειόνων δὲ οὐσῶν ἐκδόσεων, ὡς καὶ αὐτὸς φησιν
 ἐν τῇ ἐπιστολῇ, ἄμεινον ἠγησάμην συναγαγεῖν αὐτὰς
 ἐκ τῶν ἐμπιπτόντων τὰ σαφέστερα παρατιθέμενος ἐν
 τῷ ῥητῷ διὰ τὴν τῶν εἰσαγομένων εὐμάρειαν, ἕξωθεν 20
 δὲ ἐν τοῖς συντεταγμένοις σχολίοις ἐπισημαίνεσθαι
 τοὺς διαφόρους ὡς εἰκὸς τρόπους τῶν ἀποδείξεων.

φησὶ τοίνυν ἐν τῇ ἐπιστολῇ τὰ πρῶτα τέσσαρα
 βιβλία περιέχειν ἀγωγὴν στοιχειώδη· ὧν τὸ μὲν πρῶ-
 τον περιέχειν τὰς γενέσεις τῶν τριῶν τοῦ κώνου τομῶν 25
 καὶ τῶν καλουμένων ἀντικειμένων καὶ τὰ ἐν αὐταῖς
 ἀρχικὰ συμπτώματα. ταῦτα δὲ ἐστίν, ὅσα συμ-
 βαίνει παρὰ τὴν πρώτην αὐτῶν γένεσιν· ἔχουσι γὰρ
 καὶ ἕτερά τινα παρακολουθήματα. τὸ δὲ δεύτερον

178 τὰ παρὰ τὰς διαμέτρους καὶ τοὺς ἄξονας τῶν
 τομῶν συμβαίνοντα καὶ τὰς ἀσυμπώτους καὶ
 ἄλλα γενικὴν καὶ ἀναγκαίαν χρεῖαν παρεχό-
 μενα πρὸς τοὺς διορισμούς. ὁ δὲ διορισμὸς ὅτι
 διπλοῦς ἐστὶ, παντί που δῆλον, ὁ μὲν μετὰ τὴν ἔκ- 5
 θεσιν ἐφιστάνων, τί ἐστὶ τὸ ζητούμενον, ὁ δὲ τὴν

πρότασιν οὐ συγχωρῶν καθολικὴν εἶναι, λέγων δέ,
 πότε καὶ πῶς καὶ ποσαχῶς δυνατὸν συστήναι τὸ προ-
 τιθέμενον, οἷός ἐστιν ὁ ἐν τῷ εἰκοστῷ δευτέρῳ θεωρή-
 ματι τοῦ πρώτου βιβλίου τῆς Εὐκλείδου στοιχειώσεως 10
 ἐκ τριῶν εὐθειῶν, αἱ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις,
 τρίγωνον συστήσασθαι· δεῖ δὴ τὰς δύο τῆς λοιπῆς
 μείζονας εἶναι πάντη μεταλαμβανομένας, ἐπειδὴ δέ-
 δεικται, ὅτι παντὸς τριγώνου αἱ δύο πλευραὶ τῆς
 λοιπῆς μείζονές εἰσι πάντη μεταλαμβανόμεναι. τὸ δὲ 15
 τρίτον τῶν κωνικῶν περιέχειν φησὶ πολλὰ καὶ παρά-
 δοξα θεωρήματα χρήσιμα πρὸς τὰς συνθέσεις
 τῶν στερεῶν τόπων. ἐπιπέδους τόπους ἔθος τοῖς
 παλαιοῖς γεωμέτραις λέγειν, ὅταν ἐπὶ τῶν προβλημά-
 των οὐκ ἀφ' ἑνὸς σημείου μόνον, ἀλλ' ἀπὸ πλειόνων 20
 γίνεται τὸ πρόβλημα, οἷον εἰ ἐπιτάξει τις εὐθείας δο-
 θείσης πεπερασμένης εὐρεῖν τι σημεῖον, ἀφ' οὗ ἢ
 ἀχθεῖσα κάθετος ἐπὶ τὴν δοθείσαν μέση ἀνάλογον
 γίνεται τῶν τμημάτων, τόπον καλοῦσι τὸ τοιοῦτον·
 οὐ μόνον γὰρ ἐν σημείον ἐστὶ τὸ ποιοῦν τὸ πρόβλημα, 25
 ἀλλὰ τόπος ὅλος, ὃν ἔχει ἢ περιφέρεια τοῦ περι διά-
 μετρον τὴν δοθείσαν εὐθείαν κύκλου. ἐὰν γὰρ ἐπὶ
 τῆς δοθείσης εὐθείας ἡμικύκλιον γραφῆ, ὅπερ ἂν ἐπὶ
 τῆς περιφερείας λάβῃς σημεῖον καὶ ἀπ' αὐτοῦ κάθετον
 180 ἀγάγῃς ἐπὶ τὴν διάμετρον, ποιήσῃ τὸ προβληθέν.
 ὁμοίως δὲ δοθείσης εὐθείας ἐὰν τις ἐπιτάξῃ εὐρεῖν
 ἐκτὸς αὐτῆς σημεῖον, ἀφ' οὗ αἱ ἐπιζευγνύμεναι ἐπὶ τὰ
 πέρατα τῆς εὐθείας ἴσαι ἔσσονται ἀλλήλαις, καὶ ἐπὶ

τούτου οὐ μόνον ἐν σημείον ἐστι τὸ ποιούν τὸ πρό- 5
 βλημα, ἀλλὰ τόπος, ὃν ἐπέχει ἢ ἀπὸ τῆς διχοτομίας
 πρὸς ὀρθὰς ἀγομένη· ἐὰν γὰρ τὴν δοθείσαν εὐθείαν
 δίχα τεμῶν καὶ ἀπὸ τῆς διχοτομίας πρὸς ὀρθὰς ἀγά-
 γης, ὃ ἂν ἐπ' αὐτῆς λάβῃς σημείον, ποιήσει τὸ ἐπι-
 ταχθέν. 10

ὅμοιον γράφει καὶ αὐτὸς Ἀπολλώνιος ἐν τῷ Ἀνα-
 λυομένῳ τόπῳ ἐπὶ τοῦ ὑποκειμένου.

δύο δοθέντων [εὐθειῶν] ἐν ἐπιπέδῳ [καὶ] σημείων
 καὶ λόγου δοθέντος ἀνίσων εὐθειῶν δυνατὸν ἐστὶν
 ἐν τῷ ἐπιπέδῳ γράψαι κύκλον ὥστε τὰς ἀπὸ τῶν 15
 δοθέντων σημείων ἐπὶ τὴν περιφέρειαν τοῦ κύκλου
 κλωμένας εὐθείας λόγον ἔχειν τὸν αὐτὸν τῷ δοθέντι.

ἔστω τὰ μὲν δοθέντα σημεία τὰ Α, Β, λόγος δὲ
 ὁ τῆς Γ πρὸς τὴν Δ μείζονος οὔσης τῆς Γ· δεῖ δὴ
 ποιῆσαι τὸ ἐπιταχθέν. ἐπεζεύχθω ἡ ΑΒ καὶ ἐκβε- 20
 βλήσθω ἐπὶ τὰ πρὸς τῷ Β μέρη, καὶ γεγονέτω, ὡς ἡ
 Δ πρὸς τὴν Γ, ἡ Γ πρὸς ἄλλην τινὰ μείζονα δηλον-
 ὅτι τῆς Δ, καὶ ἔστω, εἰ τύχοι, πρὸς τὴν ΕΔ, καὶ
 πάλιν γεγονέτω, ὡς ἡ Ε πρὸς τὴν ΑΒ, ἡ Δ πρὸς
 τὴν ΒΖ καὶ ἡ Γ πρὸς τὴν Η. φανερόν δὴ, ὅτι ἡ 25
 τε Γ μέση ἀνάλογόν ἐστι τῆς ΕΔ καὶ τῆς Δ καὶ ἡ

182 Η τῶν ΑΖ, ΖΒ. καὶ κέντρῳ μὲν τῷ Ζ διαστήματι
 δὲ τῇ Η κύκλος γεγράφθω ὁ ΚΘ. φανερόν δὴ, ὅτι
 τέμνει ἡ ΚΘ περιφέρεια τὴν ΑΒ εὐθείαν· ἡ γὰρ Η
 εὐθεῖα μέση ἀνάλογόν ἐστι τῶν ΑΖ, ΖΒ. εἰλήφθω
 δὴ ἐπὶ τῆς περιφερείας τυχὸν σημείον τὸ Θ, καὶ ἐπε-

ζεύχθωσαν αἱ ΘA , ΘB , ΘZ . ἴση ἄρα ἐστὶν ἡ ΘZ
 τῇ H , καὶ διὰ τοῦτό ἐστιν, ὡς ἡ AZ πρὸς τὴν $Z\Theta$,
 ἢ $Z\Theta$ πρὸς ZB . καὶ περὶ τὴν αὐτὴν γωνίαν τὴν
 ὑπὸ ΘZB ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ $AZ\Theta$
 τῷ ΘBZ τριγώνῳ, καὶ ἴση ἡ ὑπὸ $Z\Theta B$ γωνία τῇ
 ὑπὸ $\Theta A B$. ἤχθω δὲ διὰ τοῦ B τῇ $A\Theta$ παράλληλος
 ἢ BA . ἐπεὶ οὖν ἐστὶν, ὡς ἡ AZ πρὸς $Z\Theta$, ἢ ΘZ
 πρὸς ZB , καὶ ὡς ἄρα πρώτη ἢ AZ πρὸς τρίτην τὴν
 ZB , τὸ ἀπὸ AZ πρὸς τὸ ἀπὸ $Z\Theta$. ἀλλ' ὡς ἡ AZ
 πρὸς ZB , ἢ $A\Theta$ πρὸς BA . καὶ ὡς ἄρα τὸ ἀπὸ AZ
 πρὸς τὸ ἀπὸ $Z\Theta$, ἢ $A\Theta$ πρὸς BA . πάλιν ἐπεὶ ἴση
 ἐστὶν ἡ ὑπὸ $B\Theta Z$ τῇ ὑπὸ $\Theta A B$, ἔστι δὲ καὶ ἡ ὑπὸ
 $A\Theta B$ τῇ ὑπὸ $\Theta B A$ ἴση· ἐναλλάξ γάρ· καὶ ἡ λοιπὴ
 ἄρα τῇ λοιπῇ ἴση ἐστίν, καὶ ὅμοιόν ἐστι τὸ $A\Theta B$
 τῷ $B\Theta A$, καὶ ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς
 ἴσας γωνίας, ὡς ἡ $A\Theta$ πρὸς ΘB , ἢ ΘB πρὸς BA ,
 καὶ ὡς τὸ ἀπὸ $A\Theta$ πρὸς τὸ ἀπὸ ΘB , ἢ $A\Theta$ πρὸς
 BA . ἦν δὲ καὶ, ὡς ἡ $A\Theta$ πρὸς BA , τὸ ἀπὸ AZ
 πρὸς τὸ ἀπὸ $Z\Theta$. ὡς ἄρα τὸ ἀπὸ AZ πρὸς τὸ ἀπὸ
 $Z\Theta$, τὸ ἀπὸ $A\Theta$ πρὸς τὸ ἀπὸ ΘB , καὶ διὰ τοῦτο, ὡς
 ἢ AZ πρὸς $Z\Theta$, ἢ $A\Theta$ πρὸς ΘB . ἀλλ' ὡς ἡ AZ
 πρὸς $Z\Theta$, ἢ $E\Delta$ πρὸς Γ καὶ ἢ Γ πρὸς Δ . καὶ ὡς
 ἄρα ἢ Γ πρὸς Δ , ἢ $A\Theta$ πρὸς ΘB . ὁμοίως δὲ δειχ-
 θήσονται πάσαι αἱ ἀπὸ τῶν A , B σημείων ἐπὶ τὴν
 184 περιφέρειαν τοῦ κύκλου κλώμεναι τὸν αὐτὸν
 ἔχουσαιλόγον ταῖς Γ , Δ .

λέγω δὴ, ὅτι πρὸς ἄλλῳ σημείῳ μὴ ὄντι ἐπὶ τῆς

περιφερείας οὐ γίνεται λόγος τῶν ἀπὸ τῶν A, B σημείων ἐπ' αὐτὸ ἐπιζευγνυμένων εὐθειῶν ὁ αὐτὸς τῷ τῆς Γ πρὸς Δ .

5

εἰ γὰρ δυνατόν, γεγονέτω πρὸς τῷ M ἐκτὸς τῆς περιφερείας· καὶ γὰρ εἰ ἐντὸς ληφθείη, τὸ αὐτὸ ἄτοπον συμβήσεται καθ' ἑτέραν τῶν ὑποθέσεων· καὶ ἐπεξεύχθωσαν αἱ MA, MB, MZ , καὶ ὑποκείσθω, ὡς ἡ Γ πρὸς Δ , οὕτως ἡ AM πρὸς MB . ἔστιν ἄρα, ὡς ἡ $E\Delta$ πρὸς Δ , οὕτως τὸ ἀπὸ $E\Delta$ πρὸς τὸ ἀπὸ Γ καὶ τὸ ἀπὸ AM πρὸς τὸ ἀπὸ MB . ἀλλ' ὡς ἡ $E\Delta$ πρὸς Δ , οὕτως ὑπόκειται ἡ AZ πρὸς ZB · καὶ ὡς ἄρα ἡ AZ πρὸς ZB , τὸ ἀπὸ AM πρὸς τὸ ἀπὸ MB . καὶ διὰ τὰ προδειχθέντα, ἐὰν ἀπὸ τοῦ B τῇ AM παράλληλον ἀγάγωμεν, δειχθήσεται, ὡς ἡ AZ πρὸς ZB , τὸ ἀπὸ AZ πρὸς τὸ ἀπὸ ZM . ἐδείχθη δὲ καί, ὡς ἡ AZ πρὸς ZB , τὸ ἀπὸ AZ πρὸς τὸ ἀπὸ $Z\Theta$. ἴση ἄρα ἡ $Z\Theta$ τῇ ZM · ὅπερ ἀδύνατον.

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τόποι οὖν ἐπίπεδοι λέγονται τὰ τοιαῦτα· οἱ δὲ λεγόμενοι στερεοὶ τόποι τὴν προσωνυμίαν ἐσχήκασιν ἀπὸ τοῦ τὰς γραμμᾶς, δι' ὧν γράφονται τὰ κατ' αὐτοὺς προβλήματα, ἐκ τῆς τομῆς τῶν στερεῶν τὴν γένεσιν ἔχειν, οἷαί εἰσιν αἱ τοῦ κώνου τομαὶ καὶ ἕτεραι πλείους. εἰσὶ δὲ καὶ ἄλλοι τόποι πρὸς ἐπιφανείαν λεγόμενοι, οἱ τὴν ἐπωνυμίαν ἔχουσιν ἀπὸ τῆς περὶ αὐτοὺς ιδιότητος.

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μέμφεται δὲ ἐξῆς τῷ Εὐκλείδῃ οὐχ, ὡς οἶεται Πάππος καὶ ἕτεροί τινες, διὰ τὸ μὴ εὐρηκέναι δύο

μέσας ανάλογον· ὁ τε γὰρ Εὐκλείδης ὑγιῶς εὖρε
 τήνμίαν μέσσην ανάλογον, ἀλλ' οὐχ ὡς αὐτός φησιν οὐκ
 εὐτυχῶς, καὶ περὶ τῶν δύο μέσων οὐδὲ ὅλως ἐπεχεί- 5
 ρησε ζητῆσαι ἐν τῇ στοιχειώσει, αὐτὸς ὁ τε Ἀπολλώ-
 νιος οὐδὲν περὶ τῶν δύο μέσων ανάλογον φαίνεται
 ζητῆσαι ἐν τῷ τρίτῳ βιβλίῳ· ἀλλ', ὡς ἔοικεν, ἐτέρῳ
 βιβλίῳ περὶ τόπων γεγραμμένῳ τῷ Εὐκλείδῃ ἐπισκήπ-
 τει, ὅπερ εἰς ἡμᾶς οὐ φέρεται. 10

τὰ δὲ ἐφεξῆς περὶ τοῦ τετάρτου βιβλίου λεγόμενα σαφῆ
 ἐστίν. τὸ δὲ πέμπτον φησὶ περιέχειν τὰ περὶ τῶν ἐλαχίστων
 καὶ μεγίστων. ὥσπερ γὰρ ἐπὶ τοῦ κύκλου ἐμάθομεν ἐν
 τῇ στοιχειώσει, ὅτι ἔστι τι σημεῖον ἐκτός, ἀφ' οὗ τῶν
 μὲν πρὸς τὴν κοίλην περιφέρειαν προσιπτουσῶν με- 15
 γίστη ἐστὶν ἢ διὰ τοῦ κέντρου, τῶν δὲ πρὸς τὴν κυρ-
 τὴν ἐλαχίστη ἐστὶν ἢ μεταξὺ τοῦ σημείου καὶ τῆς
 διαμέτρου, οὕτως καὶ ἐπὶ τῶν τοῦ κώνου τομῶν ζητεῖ
 ἐν τῷ πέμπτῳ βιβλίῳ. τοῦ δὲ ἔκτου καὶ ἑβδόμου καὶ
 ὀγδόου σαφῶς ἢ πρόθεσις ὑπ' αὐτοῦ εἴρηται. καὶ 20
 ταῦτα μὲν περὶ τῆς ἐπιστολῆς.

Ἀρχόμενος δὲ τῶν ὄρων γένεσιν ὑπογράφει κωνι-
 κῆς ἐπιφανείας, ἀλλ' οὐ τὸν τί ἔστι διορισμὸν παρα-
 δέδωκεν· ἔξεστι δὲ τοῖς βουλομένοις ἐκ τῆς γενέσεως
 αὐτῆς τὸν ὄρον λαμβάνειν. τὸ δὲ λεγόμενον ὑπ' αὐ- 25
 τοῦ διὰ καταγραφῆς σαφὲς ποιήσομεν·
 ἐὰν ἀπὸ τινος σημείου πρὸς κύκλου περι-
 φέρειαν καὶ τὰ ἐξῆς. ἔστω κύκλος ὁ ΑΒ, οὗ κέν-

ζευχθείσα ἡ ΔΒ ἐκβεβλήσθω εἰς ἄπειρον ἐφ' ἐκάτερα
 μέρη ὡς ἐπὶ τὰ Ε, Ζ. ἐὰν δὴ μένοντος τοῦ Δ ἡ ΔΒ
 φέρεται, ἕως ἂν τὸ Β ἐνεχθὲν κατὰ τῆς τοῦ ΑΒ
 κύκλου περιφερείας ἐπὶ τὸ αὐτὸ πάλιν ἀποκατασταθῆ,
 ὅθεν ἤρξατο φέρεσθαι, γεννήσει ἐπιφάνειάν τινα,
 ἣτις σύγκειται ἐκ δύο ἐπιφανειῶν ἀπτομένων ἀλλή-
 λων κατὰ τὸ Δ, ἣν καὶ καλεῖ κωνικὴν ἐπιφάνειαν.
 φησὶ δέ, ὅτι καὶ εἰς ἄπειρον αὐξεται διὰ τὸ καὶ τὴν
 γράφουσαν αὐτὴν εὐθείαν οἷον τὴν ΔΒ εἰς ἄπειρον
 ἐκβάλλεσθαι. κορυφὴν δὲ τῆς ἐπιφανείας λέγει τὸ Δ,
 ἄξονα δὲ τὴν ΔΓ.

κώνον δὲ λέγει τὸ περιεχόμενον σχῆμα ὑπὸ τε τοῦ
 ΑΒ κύκλου καὶ τῆς ἐπιφανείας, ἣν μόνη γράφει ἡ
 ΔΒ εὐθεῖα, κορυφὴν δὲ τοῦ κώνου τὸ Δ, ἄξονα δὲ
 τὴν ΔΓ, βάσιν δὲ τὸν ΑΒ κύκλον.

καὶ ἐὰν μὲν ἡ ΔΓ πρὸς ὀρθὰς ᾖ τῷ ΑΒ κύκλῳ,
 ὀρθὸν καλεῖ τὸν κώνον, ἐὰν δὲ μὴ πρὸς ὀρθὰς, σκα-
 ληγόν· γενήσεται δὲ κώνος σκαληγός, ὅταν λαβόντες
 κύκλον ἀπὸ τοῦ κέντρου αὐτοῦ ἀναστήσωμεν εὐθείαν
 μὴ πρὸς ὀρθὰς τῷ ἐπιπέδῳ τοῦ κύκλου, ἀπὸ δὲ τοῦ
 μετεώρου σημείου τῆς ἀναταθείσης εὐθείας ἐπὶ τὸν
 κύκλον ἐπιζεύξωμεν εὐθείαν καὶ περιαγάγωμεν τὴν
 ἐπιζευχθείσαν εὐθείαν περὶ τὸν κύκλον τοῦ πρὸς τῷ
 μετεώρῳ σημείῳ τῆς ἀναταθείσης μένοντος· τὸ γὰρ
 προσληφθὲν σχῆμα κώνος ἔσται σκαληγός.

190 δῆλον δέ, ὅτι ἡ περιαγομένη εὐθεῖα ἐν τῇ περι-
 αγωγῇ μείζων καὶ ἐλάττων γίνεται, κατὰ δὲ τινὰς θέσεις

και ἴση πρὸς ἄλλο καὶ ἄλλο σημεῖον τοῦ κύκλου.
 ἀποδείκνυται δὲ τοῦτο οὕτως· ἐὰν κώνου σκαληνοῦ
 ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν ἀχθῶσιν εὐθεῖαι, 5
 πασῶν τῶν ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν ἀχθει-
 σῶν εὐθειῶν μία μὲν ἐστὶν ἐλαχίστη μία δὲ μεγίστη,
 δύο δὲ μόναι ἴσαι παρ' ἐκάτερα τῆς ἐλαχίστης καὶ
 τῆς μεγίστης, αἰεὶ δὲ ἡ ἕγγιον τῆς ἐλαχίστης τῆς ἀπώ-
 τερόν ἐστὶν ἐλάσσων. ἔστω κώνος σκαληνός, οὗ βάσις 10
 μὲν ὁ ΑΒΓ κύκλος, κορυφή δὲ τὸ Δ σημεῖον. καὶ ἐπεὶ
 ἡ ἀπὸ τῆς κορυφῆς τοῦ σκαληνοῦ κώνου ἐπὶ τὸ ὑπο-
 κείμενον ἐπίπεδον κάθετος ἀγομένη ἦτοι ἐπὶ τῆς περι-
 φερείας τοῦ ΑΒΓΖΗ κύκλου πεσεῖται ἢ ἐκτὸς ἢ ἐν-
 τὸς, ἐμπιπτέτω πρότερον ἐπὶ τῆς περιφερείας ὡς ἐπὶ 15
 τῆς πρώτης καταγραφῆς ἡ ΔΕ, καὶ εἰλήφθω τὸ κέν-
 τρον τοῦ κύκλου καὶ ἔστω τὸ Κ, καὶ ἀπὸ τοῦ Ε ἐπὶ
 τὸ Κ ἐπεζεύχθω ἡ ΕΚ καὶ ἐκβεβλήσθω ἐπὶ τὸ Β, καὶ
 ἐπεζεύχθω ἡ ΒΔ, καὶ εἰλήφθωσαν δύο ἴσαι περιφέ-
 ρειαι παρ' ἐκάτερα τοῦ Ε αἱ ΕΖ, ΕΗ, καὶ παρ' 20
 ἐκάτερα τοῦ Β αἱ ΑΒ, ΒΓ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ,
 ΕΗ, ΔΖ, ΔΗ, ΕΑ, ΕΓ, ΑΒ, ΒΓ, ΔΑ, ΔΓ. ἐπεὶ οὖν
 ἴση ἐστὶν ἡ ΕΖ εὐθεῖα τῇ ΕΗ εὐθείᾳ· ἴσας γὰρ
 περιφερείας ὑποτείνουσιν· κοινὴ δὲ καὶ πρὸς ὀρθὰς
 ἡ ΔΕ, βάσις ἄρα ἡ ΔΖ τῇ ΔΗ ἐστὶν ἴση. πάλιν 25
 ἐπεὶ ἡ ΑΒ περιφέρεια τῇ ΒΓ ἐστὶν ἴση, καὶ διάμετρος
 192 ἡ ΒΕ, λοιπὴ ἄρα ἡ ΕΖΓ τῇ ΕΗΑ ἐστὶν ἴση· ὥστε
 καὶ ἡ ΑΕ τῇ ΕΓ. κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ·
 βάσις ἄρα ἡ ΔΑ τῇ ΔΓ ἐστὶν ἴση. ὁμοίως δὲ καὶ

πάσαι δειχθήσονται αἰ ἴσον ἀπέχουσαι τῆς ΔΕ ἢ τῆς
 ΔΒ ἴσαι. πάλιν ἐπεὶ τριγώνου τοῦ ΔΕΖ ὀρθή ἐστι 5
 γωνία ἢ ὑπὸ ΔΕΖ, μείζων ἐστὶν ἢ ΔΖ τῆς ΔΕ.
 καὶ πάλιν ἐπεὶ μείζων ἐστὶν ἢ ΕΑ εὐθεία τῆς ΕΖ,
 ἐπεὶ καὶ περιφέρεια ἢ ΕΖΑ τῆς ΕΖ περιφερείας,
 κοινὴ δὲ καὶ πρὸς ὀρθᾶς ἢ ΔΕ, ἢ ΔΖ ἄρα τῆς ΔΑ
 ἐλάσσων ἐστίν. διὰ τὰ αὐτὰ καὶ ἢ ΔΑ τῆς ΔΒ 10
 ἐλάσσων ἐστίν. ἐπεὶ οὖν ἢ ΔΕ τῆς ΔΖ ἐλάσσων
 ἐδείχθη, ἢ δὲ ΔΖ τῆς ΔΑ, ἢ δὲ ΔΑ τῆς ΔΒ, ἐλα-
 χίστη μὲν ἐστὶν ἢ ΔΕ, μεγίστη δὲ ἢ ΔΒ, αἰεὶ δὲ
 ἢ ἔγγιον τῆς ΔΕ τῆς ἀπώτερον ἐλάσσων ἐστίν.
 ἀλλὰ δὴ ἢ κάθετος πιπτέτω ἐκτὸς τοῦ ΑΒΓΗΖ κύ- 15
 κλου ὡς ἐπὶ τῆς δευτέρας καταγραφῆς ἢ ΔΕ, καὶ
 εἰλήφθω πάλιν τὸ κέντρον τοῦ κύκλου τὸ Κ, καὶ ἐπε-
 ζεύχθω ἢ ΕΚ καὶ ἐκβεβλήσθω ἐπὶ τὸ Β, καὶ ἐπεξεύχ-
 θωσαν αἰ ΔΒ, ΔΘ, καὶ εἰλήφθωσαν δύο ἴσαι περι-
 φέρειαι παρ' ἑκάτερα τοῦ Θ αἰ ΘΖ, ΘΗ καὶ παρ' 20
 ἑκάτερα τοῦ Β αἰ ΑΒ, ΒΓ, καὶ ἐπεξεύχθωσαν αἰ ΕΖ,
 ΕΗ, ΖΚ, ΗΚ, ΔΖ, ΔΗ, ΑΒ, ΒΓ, ΚΑ, ΚΓ, ΔΚ,
 ΔΑ, ΔΓ. ἐπεὶ οὖν ἴση ἐστὶν ἢ ΘΖ περιφέρεια τῇ
 ΘΗ, καὶ γωνία ἄρα ἢ ὑπὸ ΘΚΖ τῇ ὑπὸ ΘΚΗ ἐστὶν
 ἴση. ἐπεὶ οὖν ἢ ΖΚ εὐθεία τῇ ΚΗ ἐστὶν ἴση· ἐκ 25
 κέντρον γάρ· κοινὴ δὲ ἢ ΚΕ, καὶ γωνία ἢ ὑπὸ ΖΚΕ
 τῇ ὑπὸ ΗΚΕ ἴση, καὶ βάσις ἢ ΖΕ τῇ ΗΕ ἴση. ἐπεὶ
 οὖν ἢ ΖΕ εὐθεία τῇ ΗΕ ἐστὶν ἴση, κοινὴ δὲ καὶ
 πρὸς ὀρθᾶς ἢ ΕΔ, βάσις ἄρα ἢ ΔΖ τῇ ΔΗ ἐστὶν
 ἴση. πάλιν ἐπεὶ ἴση ἐστὶν ἢ ΒΑ περιφέρεια τῇ ΒΓ,

καὶ γωνία ἄρα ἡ ὑπὸ ΑΚΒ τῆ ὑπὸ ΓΚΒ ἐστὶν ἴση· 5
ὥστε καὶ λοιπὴ εἰς τὰς δύο ὀρθὰς ἡ ὑπὸ ΑΚΕ λοιπὴ
εἰς τὰς δύο ὀρθὰς τῆ ὑπὸ ΓΚΕ ἐστὶν ἴση. ἐπεὶ οὖν
ἡ ΑΚ εὐθεῖα τῆ ΓΚ ἐστὶν ἴση· ἐκ κέντρου γάρ· κοινὴ
δὲ ἡ ΚΕ, δύο δυσὶν ἴσαι, καὶ γωνία ἡ ὑπὸ ΑΚΕ
τῆ ὑπὸ ΓΚΕ· καὶ βάσις ἄρα ἡ ΑΕ τῆ ΓΕ ἐστὶν ἴση. 10
ἐπεὶ οὖν ἴση ἡ ΑΕ εὐθεῖα τῆ ΓΕ, κοινὴ δὲ ἡ ΕΔ
καὶ πρὸς ὀρθὰς, βάσις ἄρα ἡ ΔΑ τῆ ΔΓ ἴση. ὁμοί
ως δὲ καὶ πᾶσαι δειχθήσονται αἱ ἴσον ἀπέχουσαι τῆς
ΔΒ ἢ τῆς ΔΘ ἴσαι. καὶ ἐπεὶ ἡ ΕΘ τῆς ΕΖ ἐστὶν
ἐλάσσων, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ, βάσις ἄρα 15
ἡ ΔΘ βάσεως τῆς ΔΖ ἐστὶν ἐλάσσων. πάλιν ἐπεὶ ἡ
ἀπὸ τοῦ Ε ἐφαπτομένη τοῦ κύκλου πασῶν τῶν
πρὸς τὴν κυρτὴν περιφέρειαν προσπιπτουσῶν μείζων ἐστίν,
ἐδείχθη δὲ ἐν τῷ γ' τῆς στοιχειώσεως τὸ ὑπὸ ΑΕ,
ΕΛ ἴσον τῷ ἀπὸ τῆς ΕΖ, ὅταν ἡ ΕΖ ἐφάπτηται, 20
ἔστιν ἄρα, ὡς ἡ ΑΕ πρὸς ΕΖ, ἡ ΕΖ πρὸς ΕΛ. μεί-
ζων δὲ ἐστὶν ἡ ΕΖ τῆς ΕΛ· αἰεὶ γὰρ ἡ ἔγγιον τῆς
ἐλαχίστης τῆς ἀπώτερόν ἐστὶν ἐλάσσων· μείζων ἄρα
καὶ ἡ ΑΕ τῆς ΕΖ. ἐπεὶ οὖν ἡ ΕΖ τῆς ΕΑ ἐστὶν
ἐλάσσων, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ, βάσις ἄρα 25
ἡ ΔΖ τῆς ΔΑ ἐστὶν ἐλάσσων. πάλιν ἐπεὶ ἴση ἐστὶν
ἡ ΑΚ τῆ ΚΒ, κοινὴ δὲ ἡ ΚΕ, δύο ἄρα αἱ ΑΚ, ΚΕ
ταῖς ΕΚ, ΚΒ, τουτέστιν ὅλη τῆ ΕΚΒ, εἰσιν ἴσαι.
ἀλλ' αἱ ΑΚ, ΚΕ τῆς ΑΕ μείζονές εἰσιν· καὶ ἡ ΒΕ
196 ἄρα τῆς ΑΕ μείζων ἐστίν. πάλιν ἐπεὶ ἡ ΑΕ τῆς
ΕΒ ἐστὶν ἐλάσσων, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ,

βάσις ἄρα ἡ ΔA τῆς $B\Delta$ ἐστὶν ἐλάσσων. ἐπεὶ οὖν
 ἡ $\Delta\Theta$ τῆς ΔZ ἐστὶν ἐλάσσων, ἡ δὲ ΔZ τῆς ΔA , ἡ
 δὲ ΔA τῆς ΔB , ἐλαχίστη μὲν ἐστὶν ἡ $\Delta\Theta$, μεγίστη
 δὲ ἡ ΔB , αἰεὶ δὲ ἡ ἔγγιον καὶ τὰ ἐξῆς.

5

ἀλλὰ δὴ ἡ κάθετος πιπτέτω ἐντὸς τοῦ $ABGHZ$
 κύκλου ὡς ἐπὶ τῆς τρίτης καταγραφῆς ἡ ΔE , καὶ εἰ-
 λήφθω τὸ κέντρον τοῦ κύκλου τὸ K , καὶ ἐπεζεύχθω
 ἡ EK καὶ ἐκβεβλήσθω ἐφ' ἐκάτερα τὰ μέρη ἐπὶ τὰ
 B , Θ , καὶ ἐπεζεύχθωσαν αἱ $\Delta\Theta$, ΔB , καὶ εἰλήφθωσαν
 δύο ἴσαι περιφέρειαι παρ' ἐκάτερα τοῦ Θ αἱ ΘZ ,
 ΘH καὶ παρ' ἐκάτερα τοῦ B αἱ AB , $B\Gamma$, καὶ ἐπεζεύχ-

10

θωσαν αἱ EZ , EH , ZK , HK , ΔZ , ΔH , KA , $K\Gamma$,
 EA , $E\Gamma$, ΔA , $\Delta\Gamma$, AB , $B\Gamma$. ἐπεὶ οὖν ἴση ἡ ΘZ
 περιφέρεια τῇ ΘH , καὶ γωνία ἄρα ἡ ὑπὸ ΘKZ γωνία
 τῇ ὑπὸ ΘKH ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ KZ

15

τῇ HK , κοινὴ δὲ ἡ KE , καὶ γωνία ἡ ὑπὸ ZKE
 γωνία τῇ ὑπὸ HKE ἐστὶν ἴση, βάσις ἄρα ἡ ZE τῇ
 HE ἐστὶν ἴση. ἐπεὶ οὖν ἡ ZE τῇ HE ἐστὶν ἴση,

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κοινὴ δὲ ἡ ΔE , καὶ γωνία ἡ ὑπὸ ZED γωνία τῇ
 ὑπὸ $HE\Delta$ ἐστὶν ἴση, βάσις ἄρα ἡ ΔZ τῇ ΔH ἐστὶν
 ἴση. πάλιν ἐπεὶ ἴση ἐστὶν ἡ AB περιφέρεια τῇ $B\Gamma$,
 καὶ γωνία ἄρα ἡ ὑπὸ AKB γωνία τῇ ὑπὸ ΓKB

25

ἐστὶν ἴση· ὥστε καὶ λοιπὴ εἰς τὰς δύο ὀρθὰς ἡ ὑπὸ
 AKE λοιπὴ εἰς τὰς δύο ὀρθὰς τῇ ὑπὸ ΓKE ἐστὶν
 ἴση. ἐπεὶ οὖν ἡ AK τῇ $K\Gamma$ ἐστὶν ἴση, κοινὴ δὲ ἡ
 EK , καὶ γωνία ἡ ὑπὸ AKE γωνία τῇ ὑπὸ ΓKE

ἐστὶν ἴση, βάσις ἄρα ἡ AE τῇ ΓE ἐστὶν ἴση. ἐπεὶ

οὖν ἡ AE τῇ GE ἐστὶν ἴση, κοινὴ δὲ ἡ ED , καὶ
 γωνία ἡ ὑπὸ $AEΔ$ τῇ ὑπὸ $GEΔ$ ἴση, βάσις ἄρα ἡ
 $ΔA$ τῇ $ΔΓ$ ἐστὶν ἴση. ὁμοίως δὲ καὶ πᾶσαι δειχ-
 θήσονται αἱ ἴσον ἀπέχουσαι ἢ τῆς $ΔB$ ἢ τῆς $ΔΘ$ 5
 ἴσαι. καὶ ἐπεὶ ἐν κύκλῳ τῷ $ABΓ$ ἐπὶ τῆς διαμέτρου
 εἴληπται σημεῖον τὸ E μὴ ὄν κέντρον τοῦ κύκλου,
 μεγίστη μὲν ἡ EB , ἐλαχίστη δὲ ἡ $EΘ$, ἀεὶ δὲ ἡ ἕγ-
 γιον τῆς $EΘ$ τῆς ἀπώτερόν ἐστὶν ἐλάσσων· ὥστε ἡ
 $EΘ$ τῆς EZ ἐστὶν ἐλάσσων. καὶ ἐπεὶ ἡ $ΘE$ τῆς ZE 10
 ἐλάσσων ἐστίν, κοινὴ δὲ καὶ πρὸς ὀρθὰς αὐταῖς ἡ
 ED , βάσις ἄρα ἡ $ΔΘ$ βάσεως τῆς $ΔZ$ ἐλάσσων ἐστίν.
 πάλιν ἐπεὶ ἡ μὲν EZ ἕγγιόν ἐστι τῆς $EΘ$, ἡ δὲ AE
 πορρωτέρω, ἐλάσσων ἐστὶν ἡ EZ τῆς AE . ἐπεὶ οὖν
 ἐλάσσων ἡ EZ τῆς EA , κοινὴ δὲ καὶ πρὸς ὀρθὰς 15
 ἐστὶν αὐταῖς ἡ ED , βάσις ἄρα ἡ $ΔZ$ βάσεως τῆς $ΔA$
 ἐστὶν ἐλάσσων. πάλιν ἐπεὶ ἴση ἡ AK τῇ KB , κοινὴ
 δὲ ἡ KE , δύο αἱ AK, KE δύο ταῖς BK, KE , τουτ-
 ἐστὶν ὅλη τῇ BKE , εἰσιν ἴσαι. ἀλλ' αἱ AK, KE
 τῆς AE μείζονές εἰσιν· καὶ ἡ EB ἄρα τῆς EA μεί- 20
 ζων ἐστίν. πάλιν ἐπεὶ ἡ EA τῆς EB ἐλάσσων ἐστίν,
 κοινὴ δὲ καὶ πρὸς ὀρθὰς αὐταῖς ἡ ED , βάσις ἄρα ἡ
 $ΔA$ βάσεως τῆς $ΔB$ ἐστὶν ἐλάσσων. ἐπεὶ οὖν ἡ $ΔΘ$
 τῆς $ΔZ$ ἐλάσσων, ἡ δὲ $ΔZ$ τῆς $ΔA$, ἡ δὲ $ΔA$ τῆς
 $ΔB$, ἐλαχίστη μὲν ἐστὶν ἡ $ΔΘ$ καὶ τὰ ἐξῆς. 25

Πάσης καμπύλης γραμμῆς, ἥτις ἐστὶν ἐν ἐνὶ
 ἐπιπέδῳ, διάμετρον καλῶ καὶ τὰ ἐξῆς. τὸ ἐν
 ἐνὶ ἐπιπέδῳ εἶπε διὰ τὴν ἕλικα τοῦ κυλίνδρου καὶ

200 τῆς σφαίρας· αὐταὶ γὰρ οὐκ εἰσὶν ἐν ἐνὶ ἐπιπέδῳ. ὁ
 δὲ λέγει, τοιοῦτόν ἐστιν· ἔστω καμπύλη γραμμὴ ἢ
 ΑΒΓ καὶ ἐν αὐτῇ εὐθείαι τινες παράλληλοι αἰ ΑΓ,
 ΔΕ, ΖΗ, ΘΚ, καὶ διήχθω ἀπὸ τοῦ Β εὐθεῖα ἢ ΒΛ
 δίχα αὐτὰς τέμνουσα. φησὶν οὖν, ὅτι τῆς ΑΒΓ γραμ- 5
 μῆς διάμετρον μὲν καλῶ τὴν ΒΛ, κορυφὴν δὲ τὸ Β,
 τεταγμένως δὲ ἐπὶ τὴν ΒΛ κατῆχθαι ἐκάστην τῶν
 ΑΓ, ΔΕ, ΖΗ, ΘΚ. εἰ δὲ ἡ ΒΛ δίχα καὶ πρὸς
 ὀρθὰς τέμνει τὰς παραλλήλους, ἄξων καλεῖται.

Ὅμοίως δὲ καὶ δύο καμπύλων γραμμῶν καὶ 10
 τὰ ἐξῆς. ἐὰν γὰρ νοήσωμεν τὰς Α, Β γραμμὰς καὶ
 ἐν αὐταῖς τὰς ΓΔ, ΕΖ, ΗΘ, ΚΛ, ΜΝ, ΞΟ παραλ-
 λήλους καὶ τὴν ΑΒ διηγμένην ἐφ' ἐκάτερα καὶ τέμ-
 νουσαν τὰς παραλλήλους δίχα, τὴν μὲν ΑΒ καλῶ,
 φησὶν, πλαγίαν διάμετρον, κορυφὰς δὲ τῶν γραμμῶν 15
 τὰ Α, Β σημεία, τε-
 ταγμένως δὲ ἐπὶ τὴν
 ΑΒ τὰς ΓΔ, ΕΖ,
 ΗΘ, ΚΛ, ΜΝ, ΞΟ.
 εἰ δὲ δίχα καὶ πρὸς 20
 ὀρθὰς αὐτὰς τέμνει,
 ἄξων καλεῖται. ἐὰν δὲ διαχθεῖσά τις εὐ-
 θεῖα ὡς ἢ ΠΡ τὰς ΓΞ, ΕΜ, ΗΚ παραλλήλους
 τῇ ΑΒ δίχα τέμνει, ὀρθία μὲν διάμετρος καλεῖται ἢ
 ΠΡ, τεταγμένως δὲ κατῆχθαι ἐπὶ τὴν ΠΡ διάμετρον 25
 ἐκάστη τῶν ΓΞ, ΕΜ, ΗΚ. εἰ δὲ δίχα καὶ πρὸς
 ὀρθὰς αὐτὴν τέμνει, ἄξων ὀρθός, ἐὰν δὲ αἰ ΑΒ, ΠΡ

202 δίχα τέμνουσι τὰς ἀλλήλων παραλλήλους, λέγονται συζυγείς διάμετροι, ἐὰν δὲ δίχα καὶ πρὸς ὀρθάς, συζυγείς ἄξονες ὀνομάζονται.

Εἰς τὸ α'.

Περὶ τῶν διαφόρων καταγραφῶν ἦτοι πώσεων 5
τῶν θεωρημάτων τοσοῦτον ἰστέον, ὅτι πῶσις μὲν
ἐστίν, ὅταν τὰ ἐν τῇ προτάσει δεδομένα τῇ θέσει ἢ
δοθέντα· ἢ γὰρ διάφορος αὐτῶν μετάληψις τοῦ αὐτοῦ
συμπεράσματος ὄντος ποιῆ τὴν πῶσιν. ὁμοίως δὲ
καὶ ἀπὸ τῆς κατασκευῆς μετατιθεμένης γίνεται πῶσις. 10
πολλὰς δὲ ἐχόντων τῶν θεωρημάτων πάσαις ἢ αὐτῇ
ἀπόδειξις ἀρμόζει καὶ ἐπὶ τῶν αὐτῶν στοιχείων πλὴν
βραχέων, ὡς ἐξῆς εἰσόμεθα· εὐθύς γὰρ τὸ πρῶτον
θεώρημα τρεῖς πῶσεις ἔχει διὰ τὸ τὸ λαμβανόμενον
σημεῖον ἐπὶ τῆς ἐπιφανείας, τουτέστι τὸ Β, ποτὲ μὲν 15
εἰς τὴν κατωτέρω ἐπιφάνειαν εἶναι καὶ τοῦτο διχῶς
ἢ ἀνωτέρω τοῦ κύκλου ἢ κατωτέρω, ποτὲ δὲ ἐπὶ τῆς
κατὰ κορυφὴν αὐτῇ ἐπικειμένης. τοῦτο δὲ τὸ θεώρημα
προέθετο ζητῆσαι, ὅτι οὐκ ἐπὶ πάντα δύο σημεῖα ἐπὶ
τῆς ἐπιφανείας λαμβανόμενα ἐπιζευγνυμένη εὐθεῖα 20
ἐπὶ τῆς ἐπιφανείας ἐστίν, ἀλλ' ἢ νεύουσα μόνον ἐπὶ
τὴν κορυφὴν, διὰ τὸ καὶ ὑπὸ εὐθείας τὸ πέρασ ἐχού-
σης μόνον γεγενῆσθαι τὴν κωνικὴν ἐπιφάνειαν. ὅτι
δὲ τοῦτο ἀληθές, τὸ δεύτερον θεώρημα δηλοῖ.

Εἰς τὸ β'. 25

Τὸ δεύτερον θεώρημα τρεῖς ἔχει πῶσεις διὰ τὸ
204 τὰ λαμβανόμενα σημεῖα τὰ Δ, Ε ἢ ἐπὶ τῆς κατὰ κο-

ρουφήν εἶναι ἐπιφανείας ἢ ἐπὶ τῆς κάτω διχῶς ἢ ἐσωτέρω τοῦ κύκλου ἢ ἐξωτερῶ. δεῖ δὲ ἐφιστάνειν, ὅτι τοῦτο τὸ θεώρημα εὐρίσκεται ἔν τισιν ἀντιγράφοις ὅλον διὰ τῆς εἰς ἀδύνατον ἀπαγωγῆς δεδειγμένον.

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Εἰς τὸ γ'.

Τὸ γ' θεώρημα πῶσιν οὐκ ἔχει. δεῖ δὲ ἐν αὐτῷ ἐπιστήσαι, ὅτι ἡ AB εὐθεία ἐστὶ διὰ τὸ κοινὴ τομὴ εἶναι τοῦ τέμνοντος ἐπιπέδου καὶ τῆς ἐπιφανείας τοῦ κώνου, ἥτις ὑπὸ εὐθείας ἐγράφη τὸ πέρασ ἐχούσης μένον πρὸς τῇ κορυφῇ τῆς ἐπιφανείας. οὐ γὰρ πᾶσα ἐπιφάνεια ὑπὸ ἐπιπέδου τεμνομένη τὴν τομὴν ποιεῖ εὐθείαν, οὐδὲ αὐτὸς ὁ κώνος, εἰ μὴ διὰ τῆς κορυφῆς ἔλθῃ τὸ τέμνον ἐπίπεδον.

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Εἰς τὸ δ'.

Αἱ πῶσεις τούτου τοῦ θεωρήματος τρεῖς εἰσιν ὥσπερ καὶ τοῦ πρώτου καὶ δευτέρου.

Εἰς τὸ ε'.

Τὸ πέμπτον θεώρημα πῶσιν οὐκ ἔχει. ἀρχόμενος δὲ τῆς ἐκθέσεώς φησιν· τετμήσθω ὁ κώνος ἐπιπέδῳ διὰ τοῦ ἄξονος ὀρθῶ πρὸς τὴν βάσιν. ἐπειδὴ δὲ ἐν τῷ σκαληνῷ κώνῳ κατὰ μίαν μόνον θέσιν τὸ διὰ τοῦ ἄξονος τρίγωνον ὀρθόν ἐστὶ πρὸς τὴν βάσιν, τοῦτο ποιήσομεν οὕτως· λαβόντες τὸ κέντρον τῆς βάσεως ἀναστήσομεν ἀπ' αὐτοῦ τῷ ἐπιπέδῳ τῆς βάσεως πρὸς ὀρθὰς καὶ δι' αὐτῆς καὶ τοῦ ἄξονος ἐκβάλλοντες ἐπίπεδον ἕξομεν τὸ ζητούμενον· δέδεικται

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- 206 γὰρ ἐν τῷ α' τῆς Εὐκλείδου στοιχειώσεως, ὅτι, ἐὰν εὐθεία ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾗ, καὶ πάντα τὰ δι' αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται. τὸν δὲ κῶνον σκαληνὸν ὑπέθετο, ἐπειδὴ ἐν τῷ ἰσοσκελεῖ τὸ παράλληλον τῇ βάσει ἐπίπεδον τῷ ὑπεναντίως ἡγμένῳ τὸ αὐτό ἐστίν. 5
- ἔτι φησὶν· τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ πρὸς ὀρθὰς μὲν τῷ διὰ τοῦ ἄξονος τριγώνῳ, ἀφαιρῶντι δὲ πρὸς τῇ κορυφῇ τρίγωνον ὅμοιον μὲν τῷ $AB\Gamma$ τριγώνῳ, ὑπεναντίως δὲ κείμενον. 10
- τοῦτο δὲ γίνεται οὕτως· ἔστω τὸ διὰ τοῦ ἄξονος τρίγωνον τὸ $AB\Gamma$, καὶ εἰλήφθω ἐπὶ τῆς AB τυχὸν σημείον τὸ H , καὶ συνεστάτω πρὸς τῇ AH εὐθεία καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ H τῇ ὑπὸ AGB γωνία ἴση ἢ ὑπὸ AHK · τὸ AHK ἄρα τρίγωνον τῷ $AB\Gamma$ ὅμοιον μὲν ἐστίν, ὑπεναντίως δὲ κείμενον. εἰλήφθω δὲ ἐπὶ τῆς HK τυχὸν σημείον τὸ Z , καὶ ἀπὸ τοῦ Z τῷ τοῦ $AB\Gamma$ τριγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω ἢ $Z\Theta$, καὶ ἐκβεβλήσθω τὸ διὰ τῶν HK , ΘZ ἐπίπεδον. τοῦτο δὲ ὀρθὸν ἐστὶ πρὸς τὸ $AB\Gamma$ τρίγωνον διὰ τὴν $Z\Theta$ καὶ ποιοῦν τὸ προκείμενον. 15
- ἐν τῷ συμπεράσματι φησιν, ὅτι διὰ τὴν ὁμοιότητα τῶν ΔZH , EZK τριγώνων ἴσον ἐστὶ τὸ ὑπὸ ΔZE τῷ ὑπὸ HZK . δυνατὸν δὲ ἐστὶ τοῦτο δεῖξαι καὶ δίχα τῆς τῶν τριγώνων ὁμοιότητος λέγοντα, ὅτι, ἐπειδὴ 20
- 208 ἑκατέρω τῶν ὑπὸ AKH , $A\Delta E$ γωνιῶν ἴση ἐστὶ τῇ πρὸς τῷ B , ἐν τῷ αὐτῷ τμήματι εἰσι τοῦ περιλαμ-

βάνοντος κύκλου τὰ Δ, Η, Ε, Κ σημεία. καὶ ἐπειδὴ
 ἐν κύκλῳ δύο εὐθείαι αἱ ΔΕ, ΗΚ τέμνουσιν ἀλλή-
 λας κατὰ τὸ Ζ, τὸ ὑπὸ ΔΖΕ ἴσον ἐστὶ τῷ ὑπὸ
 ΗΖΚ.

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ὁμοίως δὴ δειχθήσεται, ὅτι καὶ πᾶσαι αἱ ἀπὸ τῆς
 ΗΘ γραμμῆς ἐπὶ τὴν ΗΚ κάθετοι ἀγόμενοι ἴσον δύ-
 νανται τῷ ὑπὸ τῶν τμημάτων. κύκλος ἄρα ἐστὶν ἡ
 τομή, διάμετρος δὲ αὐτοῦ ἡ ΗΚ. καὶ δυνατὸν μὲν
 ἐστὶν ἐπιλογίσασθαι τοῦτο διὰ τῆς εἰς ἀδύνατον ἀπα-
 γωγῆς. εἰ γὰρ ὁ περὶ τὴν ΚΗ γραφόμενος κύκλος
 οὐχ ἦξει διὰ τοῦ Θ σημείου, ἔσται τὸ ὑπὸ τῶν ΚΖ,
 ΖΗ ἴσον ἢτοι τῷ ἀπὸ μείζονος τῆς ΖΘ ἢ τῷ ἀπὸ
 ἐλάσσονος· ὅπερ οὐχ ὑπόκειται. δείξομεν δὲ αὐτὸ καὶ
 ἐπ' εὐθείας.

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ἔστω τις γραμμὴ ἡ ΗΘ, καὶ ὑποτεινέτω αὐτὴν ἡ
 ΗΚ, εἰλήφθω δὲ καὶ ἐπὶ τῆς γραμμῆς τυχόντα σημεία
 τὰ Θ, Ο, καὶ ἀπ' αὐτῶν ἐπὶ τὴν ΗΚ κάθετοι ἦχθω-
 σαν αἱ ΘΖ, ΟΠ, καὶ ἔστω τὸ μὲν ἀπὸ ΖΘ ἴσον τῷ
 ὑπὸ ΗΖΚ, τὸ δὲ ἀπὸ ΟΠ τῷ ὑπὸ ΗΠΚ ἴσον. λέγω,
 ὅτι κύκλος ἐστὶν ἡ ΗΘΟΚ γραμμὴ. τεμήσθω γὰρ
 ἡ ΗΚ δίχα κατὰ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΝΘ,
 ΝΟ. ἐπεὶ οὖν εὐθεῖα ἡ ΗΚ τέτμηται εἰς μὲν ἴσα
 κατὰ τὸ Ν, εἰς δὲ ἄνισα κατὰ τὸ Ζ, τὸ ὑπὸ ΗΖΚ (25)
 μετὰ τοῦ ἀπὸ ΝΖ ἴσον ἐστὶ τῷ ἀπὸ ΝΚ. τὸ δὲ

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210 ὑπὸ ΗΖΚ ἴσον ὑπόκειται τῷ ἀπὸ ΘΖ· τὸ ἄρα ἀπὸ
 ΘΖ μετὰ τοῦ ἀπὸ ΝΖ ἴσον ἐστὶ τῷ ἀπὸ ΝΚ. ἴσα
 δὲ ἐστὶ τὰ ἀπὸ ΘΖ, ΖΝ τῷ ἀπὸ ΝΘ· ὀρθὴ γὰρ ἐστὶν

ἢ πρὸς τῷ Ζ· τὸ ἄρα ἀπὸ ΝΘ ἴσον ἐστὶ τῷ ἀπὸ ΝΚ.

ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ ἀπὸ ΝΟ ἴσον ἐστὶ
τῷ ἀπὸ ΝΚ. κύκλος ἄρα ἐστὶν ἡ ΗΘΚ γραμμὴ, διά-
μετρος δὲ αὐτοῦ ἡ ΗΚ.

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δυνατὸν δὲ ἐστὶ τὰς ΔΕ, ΗΚ διαμέτρους ποτὲ
μὲν ἴσας, ποτὲ δὲ ἀνίσους εἶναι, οὐδέποτε μέντοι δίχα

τέμνουσιν ἀλλήλας. ἤχθω γὰρ διὰ τοῦ Κ τῆ ΒΓ
παράλληλος ἡ ΝΚ. ἐπεὶ οὖν μείζων ἐστὶν ἡ ΒΑ τῆς

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ΑΓ, μείζων ἄρα καὶ ἡ ΝΑ τῆς ΑΚ. ὁμοίως δὲ καὶ
ἡ ΚΑ τῆς ΑΗ διὰ τὴν ὑπεναντίαν τομήν. ὥστε ἡ
τῆ ΑΚ ἀπὸ τῆς ΑΝ ἴση λαμβανομένη μεταξὺ πίπτει

τῶν Η, Ν σημείων. πίπττω ὡς ἡ ΑΞ· ἢ ἄρα διὰ
τοῦ Ξ τῆ ΒΓ παράλληλος ἀγομένη τέμνει τὴν ΗΚ.

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τεμνέτω ὡς ἡ ΞΟΠ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΞΑ τῆ
ΑΚ, ὡς δὲ ἡ ΞΑ πρὸς ΑΠ, ἡ ΚΑ πρὸς ΑΗ διὰ

τὴν ὁμοιότητα τῶν ΗΚΑ, ΞΑΠ τριγώνων, ἡ ΑΗ
τῆ ΑΠ ἐστὶν ἴση καὶ λοιπὴ ἡ ΗΞ τῆ ΠΚ. καὶ ἐπεὶ

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αἱ πρὸς τοῖς Ξ, Κ γωνίαι ἴσαι εἰσίν· ἑκατέρα γὰρ
αὐτῶν ἴση ἐστὶ τῆ Β· εἰσὶ δὲ καὶ αἱ πρὸς τῷ Ο ἴσαι·
κατὰ κορυφήν γὰρ· ὅμοιον ἄρα ἐστὶ τὸ ΞΗΟ τρίγω-

νον τῷ ΠΟΚ τριγώνω. καὶ ἴση ἐστὶν ἡ ΗΞ τῆ

ΠΚ· ὥστε καὶ ἡ ΞΟ τῆ ΟΚ καὶ ἡ ΗΟ τῆ ΟΠ καὶ

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212 ὅλη ἡ ΗΚ τῆ ΞΠ. καὶ φανερόν, ὅτι, ἐὰν μεταξὺ
τῶν Ν, Ξ ληφθῆ τι σημεῖον ὡς τὸ Ρ, καὶ διὰ τοῦ Ρ
τῆ ΝΚ παράλληλος ἀχθῆ ἡ ΡΣ, μείζων ἔσται τῆς ΞΠ
καὶ διὰ τοῦτο καὶ τῆς ΗΚ, ἐὰν δὲ μεταξὺ τῶν Η, Ξ
ληφθῆ τι σημεῖον οἶον τὸ Τ, καὶ δι' αὐτοῦ παράλλη-

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λος ἀχθῆ ἢ ΤΠ, ἐλάττων ἔσται τῆς ΞΠ καὶ τῆς ΚΗ.
καὶ ἐπεὶ ἡ ὑπὸ ΞΠΚ γωνία μείζων ἐστὶ τῆς ὑπὸ
ΑΞΠ, ἴση δὲ ἡ ὑπὸ ΟΠΚ τῆ ὑπὸ ΟΗΞ, μείζων
ἄρα καὶ ἡ ὑπὸ ΟΗΞ τῆς ὑπὸ ΗΞΟ. ἡ ΞΟ ἄρα τῆς
ΟΗ μείζων καὶ διὰ τοῦτο καὶ ἡ ΚΟ τῆς ΟΠ. ἐὰν
δέ ποτε ἡ ἑτέρα αὐτῶν δίχα διαιρεθῆ, ἡ λοιπὴ εἰς
ἄνισα τμηθήσεται.

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Εἰς τὸ ζ΄.

Προσέχειν χρῆ, ὅτι οὐ μάτην πρόσκειται ἐν τῇ
προτάσει τὸ δεῖν τὴν ἀγομένην εὐθείαν ἀπὸ τοῦ ἐν
τῇ ἐπιφανείᾳ σημείου παράλληλον μιᾷ τινι τῶν ἐν τῇ
βάσει εὐθειῶν πρὸς ὀρθὰς οὔση πάντως τῇ βάσει τοῦ
διὰ τοῦ ἄξονος τριγώνου ἄγεσθαι παράλληλον· τού-
του γὰρ μὴ ὄντος οὐ δυνατόν ἐστὶν αὐτὴν δίχα τέμ-
νεσθαι ὑπὸ τοῦ διὰ τοῦ ἄξονος τριγώνου· ὅπερ ἐστὶ
φανερόν ἐκ τῆς ἐν τῷ ῥητῷ καταγραφῆς. εἰ γὰρ ἡ
ΜΝ, ἥτινι παράλληλός ἐστὶν ἡ ΔΖΗ, μὴ πρὸς ὀρθὰς
εἴη τῇ ΒΓ, δῆλον, ὅτι οὐδὲ δίχα τέμνεται οὐδὲ ἡ
ΚΛ. καὶ διὰ τῶν αὐτῶν λόγων συνάγεται, ὅτι ἐστίν,
ὡς ἡ ΚΘ πρὸς ΘΛ, οὕτως ἡ ΔΖ πρὸς ΖΗ· καὶ ἡ
ΔΗ ἄρα εἰς ἄνισα τμηθήσεται κατὰ τὸ Ζ.

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δυνατόν δὲ κατωτέρω τοῦ κύκλου καὶ ἐπὶ τῆς
κατὰ κορυφὴν ἐπιφανείας τὰ αὐτὰ δείκνυσθαι.

214

Εἰς τὸ ζ΄.

Τὸ ζ΄ θεώρημα πτώσεις ἔχει τέσσαρας· ἡ γὰρ οὐ
συμβάλλει ἡ ΖΗ τῇ ΑΓ ἢ συμβάλλει τριχῶς ἢ ἐκτὸς
τοῦ κύκλου ἢ ἐντὸς ἢ ἐπὶ τοῦ Γ σημείου.

Μετὰ τὸ ι'.

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Χρη̄ ἐπιστῆσαι, ὅτι τὰ ι ταῦτα θεωρήματα ἀλλή-
 λων ἔχονται. ἀλλὰ τὸ πρῶτον ἔχει, ὅτι αἱ ἐν τῇ ἐπι-
 φανείᾳ εὐθείᾳ νεύουσαι ἐπὶ τὴν κορυφὴν ἐν ταύτῃ
 μένουσιν, τὸ δὲ δεύτερον τὸ ἀνάπαλιν, τὸ δὲ τρίτον
 ἔχει τὴν διὰ τῆς κορυφῆς τοῦ κώνου τομήν, τὸ δὲ
 τέταρτον τὴν παράλληλον τῇ βάσει, τὸ πέμπτον τὴν
 ὑπεναντίαν, τὸ ἕκτον ὡσανεὶ προλαμβάνεται τοῦ ἐβ-
 δόμου δεικνύον, ὅτι καὶ πρὸς ὀρθὰς ὀφείλει πάντως
 εἶναι τῇ διαμέτρῳ τοῦ κύκλου ἢ κοινῇ τομῇ αὐτοῦ
 καὶ τοῦ τέμνοντος ἐπιπέδου, καὶ ὅτι τούτου οὕτως
 ἔχοντος αἱ παράλληλοι αὐτῇ διχοτομοῦνται ὑπὸ τοῦ
 τριγώνου, τὸ δὲ ἕβδομον τὰς ἄλλας τρεῖς τομὰς ἔδειξε
 καὶ τὴν διάμετρον καὶ τὰς ἐπ' αὐτὴν καταγομένας
 παραλλήλους τῇ ἐν τῇ βάσει εὐθείᾳ. ἐν δὲ τῷ ὀγδόῳ
 δείκνυσιν, ὅπερ ἐν τοῖς προλεγομένοις εἶπομεν, ὅτι
 ἡ παραβολὴ καὶ ἡ ὑπερβολὴ τῶν εἰς ἄπειρόν εἰσιν
 αὐξομένων, ἐν δὲ τῷ ἐνάτῳ, ὅτι ἡ ἔλλειψις συννεύ-
 ούσα εἰς ἑαυτὴν ὁμοίως τῷ κύκλῳ διὰ τὸ τὸ τέμνον
 ἐπίπεδον συμπύπτειν ἀμφοτέραις ταῖς πλευραῖς τοῦ
 τριγώνου οὐκ ἔστι κύκλος· κύκλους γὰρ ἐποιοῦν ἢ τε
 ὑπεναντία τομῇ καὶ ἡ παράλληλος· καὶ δεῖ ἐπιστῆσαι,
 ὅτι ἡ διάμετρος τῆς τομῆς ἐπὶ μὲν τῆς παραβολῆς

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216 τὴν μίαν πλευρὰν τοῦ τριγώνου τέμνει καὶ τὴν βάσιν,
 ἐπὶ δὲ τῆς ὑπερβολῆς τὴν τε πλευρὰν καὶ τὴν ἐπ'
 εὐθείας τῇ λοιπῇ πλευρᾷ ἐκβαλλομένην πρὸς τὴν κο-
 ρυφῇ, ἐπὶ δὲ τῆς ἐλλείψεως καὶ ἑκατέραν τῶν πλευ-

ρῶν καὶ τὴν βᾶσιν. τὸ δὲ δέκατον ἀπλούστερον μὲν 5
 τις ἐπιβάλλον ἴσως ἂν οἰηθείη ταύτων εἶναι τῷ δευ-
 τέρῳ, τοῦτο μὲντοι οὐχ ὡς ἔχει· ἐκεῖ μὲν γὰρ ἐπὶ
 πάσης τῆς ἐπιφανείας ἔλεγε λαμβάνεσθαι τὰ δύο
 σημεῖα, ἐνταῦθα δὲ ἐπὶ τῆς γενομένης γραμμῆς. ἐν
 δὲ τοῖς ἑξῆς τρισὶν ἀκριβέστερον ἐκάστην τῶν τομῶν 10
 τούτων διακρίνει μετὰ τοῦ λέγειν καὶ τὰ ιδιώματα
 αὐτῶν τὰ ἀρχικά.

Εἰς τὸ ια΄.

Πεποιήσθω, ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ
 ΒΑΓ, οὕτως ἢ ΘΖ πρὸς ΖΑ· σαφὲς μὲν ἐστὶ τὸ 15
 λεγόμενον, πλὴν εἴ τις καὶ ὑπομνησθῆναι βούλεται.
 ἔστω τῷ ὑπὸ ΒΑΓ ἴσον τὸ ὑπὸ ΟΠΡ, τῷ δὲ ἀπὸ
 ΒΓ ἴσον παρὰ τὴν ΠΡ παραβληθὲν πλάτος ποιείτω
 τὴν ΠΣ, καὶ γεγονέτω, ὡς ἢ ΟΠ πρὸς ΠΣ, ἢ ΑΖ
 πρὸς ΖΘ· γέγονεν ἄρα τὸ ζητούμενον. ἐπεὶ γὰρ ἐστίν, 20
 ὡς ἢ ΟΠ πρὸς ΠΣ, ἢ ΑΖ πρὸς ΖΘ, ἀνάπαλιν ὡς
 ἢ ΣΠ πρὸς ΠΟ, ἢ ΘΖ πρὸς ΖΑ. ὡς δὲ ἢ ΣΠ
 πρὸς ΠΟ, τὸ ΣΡ πρὸς ΡΟ, τουτέστι τὸ ἀπὸ ΒΓ
 πρὸς τὸ ὑπὸ ΒΑΓ. τοῦτο χρησιμεύει καὶ τοῖς ἑξῆς
 δύο θεωρήμασιν.

218 Τὸ δὲ ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ λόγον
 ἔχει τὸν συγκείμενον ἐκ τοῦ ὄν ἔχει ἢ ΒΓ πρὸς
 ΓΑ καὶ ἢ ΒΓ πρὸς ΒΑ· δέδεικται μὲν ἐν τῷ ἕκτῳ
 βιβλίῳ τῆς στοιχειώσεως ἐν τῷ εἰκοστῷ τρίτῳ θεωρή-
 ματι, ὅτι τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα 5
 λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ἐπεὶ δὲ

ἐπακτικώτερον μάλλον καὶ οὐ κατὰ τὸν ἀναγκαῖον
τρόπον ὑπὸ τῶν ὑπομνηματιστῶν ἐλέγετο, ἐζητήσαμεν
αὐτὸ καὶ γέγραπται ἐν τοῖς ἐκδεδομένοις ἡμῖν εἰς τὸ
τέταρτον θεώρημα τοῦ δευτέρου βιβλίου τῶν Ἀρχιμή- 10
δους περὶ σφαίρας καὶ κυλίνδρου καὶ ἐν τοῖς σχολίοις τοῦ
πρώτου βιβλίου τῆς Πτολεμαίου συντάξεως· οὐ χεῖρον
δὲ καὶ ἐνταῦθα τοῦτο γραφῆναι διὰ τὸ μὴ πάντως τοὺς
ἀναγινώσκοντας κάκείνοις ἐντυγχάνειν, καὶ ὅτι σχεδὸν
τὸ ὅλον σύνταγμα τῶν κωνικῶν κέχρηται αὐτῷ. 15

λόγος ἐκ λόγων συγκείσθαι λέγεται, ὅταν αἱ τῶν
λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποι-
ῶσί τινα, πηλικότητος δηλονότι λεγομένης τοῦ ἀριθ-
μοῦ, οὗ παρώνυμός ἐστιν ὁ λόγος. ἐπὶ μὲν οὖν τῶν
πολλαπλασίων δυνατόν ἐστιν ἀριθμὸν ὀλόκληρον εἶναι 20
τὴν πηλικότητα, ἐπὶ δὲ τῶν λοιπῶν σχέσεων ἀνάγκη
τὴν πηλικότητα ἀριθμὸν εἶναι καὶ μόριον ἢ μόρια, εἰ μὴ
ἄρα τις ἐθέλοι καὶ ἀρρήτους εἶναι σχέσεις, οἷαί εἰσιν
αἱ κατὰ τὰ ἄλογα μεγέθη. ἐπὶ πασῶν δὲ τῶν σχέσεων
δῆλον, ὅτι αὐτὴ ἡ πηλικότης πολλαπλασιαζομένη ἐπὶ 25
τὸν ἐπόμενον ὄρον τοῦ λόγου ποιεῖ τὸν ἡγούμενον.

ἔστω τοίνυν λόγος ὁ τοῦ Α πρὸς τὸν Β, καὶ εἰ-

220 λήφθω τις αὐτῶν μέσος, ὡς ἔτυχεν, ὁ Γ, καὶ ἔστω
τοῦ Α, Γ λόγου πηλικότης ὁ Δ, τοῦ δὲ Γ, Β ὁ Ε,
καὶ ὁ Δ τὸν Ε πολλαπλασιάσας τὸν Ζ ποιεῖτω. λέγω,
ὅτι τοῦ λόγου τῶν Α, Β πηλικότης ἐστὶν ὁ Ζ, τουτ-
ἐστὶν ὅτι ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α ποιεῖ.
ὁ δὲ Ζ τὸν Β πολλαπλασιάσας τὸν Η ποιεῖτω. ἐπεὶ 5

οὖν ὁ Δ τὸν μὲν Ε πολλαπλασιάσας τὸν Ζ πεποίηκεν,
τὸν δὲ Γ πολλαπλασιάσας τὸν Α πεποίηκεν, ἔστιν
ἄρα, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α. πάλιν
ἐπεὶ ὁ Β τὸν Ε πολλαπλασιάσας τὸν Γ πεποίηκεν, 10
τὸν δὲ Ζ πολλαπλασιάσας τὸν Η πεποίηκεν, ἔστιν
ἄρα, ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Γ πρὸς τὸν Η. ἐναλλάξ,
ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Η. ἦν δέ, ὡς
ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α· ἴσος ἄρα ὁ Η
τῷ Α. ὥστε ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α 15
πεποίηκεν.

μη ταραπτέτω δὲ τοὺς ἐντυγχάνοντας τὸ διὰ τῶν
ἀριθμητικῶν δεδειχθαι τοῦτο· οἷ τε γὰρ παλαιοὶ κέ-
χρηται ταῖς τοιαύταις ἀποδείξεσι μαθηματικαῖς μάλλον
οὔσαις ἢ ἀριθμητικαῖς διὰ τὰς ἀναλογίας, καὶ ὅτι 20
τὸ ζητούμενον ἀριθμητικόν ἐστιν. λόγοι γὰρ καὶ
πηλικότητες λόγων καὶ πολλαπλασιασμοὶ τοῖς ἀριθμοῖς
πρώτως ὑπάρχουσι καὶ δι' αὐτῶν τοῖς μεγέθεσι, κατὰ
τὸν εἰπόντα· ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν
ἀδελφά.

222 **Εἰς τὸ ιγ'.** Δεῖ σημειώσασθαι, ὅτι τοῦτο τὸ θεώρημα τρεῖς
ἔχει καταγραφάς, ὡς καὶ πολλάκις εἴρηται ἐπὶ τῆς
ἐλλείψεως· ἢ γὰρ ΔΕ ἢ ἀνωτέρω τοῦ Γ συμπίπτει
τῇ ΑΓ ἢ κατ' αὐτοῦ τοῦ Γ ἢ ἐξωτέρω ἐκβαλλομένη
τῇ ΑΓ συμπίπτει. 5

Εἰς τὸ ιδ'.

Δυνατὸν ἦν καὶ οὕτως δεῖξαι, ὅτι, ὡς τὸ ἀπὸ ΑΣ
πρὸς τὸ ὑπὸ ΒΣΓ, οὕτως τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ

ΞΤΟ.

ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΒΓ τῇ ΞΟ, ἔστιν, 10
ὡς ἡ ΓΣ πρὸς ΣΑ, ἡ ΞΤ πρὸς ΤΑ, καὶ διὰ τὰ
αὐτά, ὡς ἡ ΑΣ πρὸς ΣΒ, ἡ ΑΤ πρὸς ΤΟ· δι' ἴσου
ἄρα, ὡς ἡ ΓΣ πρὸς ΣΒ, ἡ ΞΤ πρὸς ΤΟ. καὶ ὡς
ἄρα τὸ ἀπὸ ΓΣ πρὸς τὸ ὑπὸ ΓΣΒ, τὸ ἀπὸ ΞΤ πρὸς
τὸ ὑπὸ ΞΤΟ. ἔστι δὲ διὰ τὴν ὁμοιότητα τῶν τρι- 15
γώνων, ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ἀπὸ ΣΓ, τὸ ἀπὸ ΑΤ
πρὸς τὸ ἀπὸ ΞΤ· δι' ἴσου ἄρα, ὡς τὸ ἀπὸ ΑΣ πρὸς
τὸ ὑπὸ ΒΣΓ, τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΞΤΟ.
καὶ ἐστιν, ὡς μὲν τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ,
ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ 20
ΞΤΟ, ἡ ΘΕ πρὸς ΘΡ· καὶ ὡς ἄρα ἡ ΘΕ πρὸς ΕΠ,
ἡ ΕΘ πρὸς ΘΡ. ἴση ἄρα ἐστὶν ἡ ΕΠ τῇ ΘΡ.
πῶσιν μὲν οὖν οὐκ ἔχει, φανερός δὲ ἐστὶν ὁ
σκοπὸς συνεχῆς ὧν τοῖς πρὸ αὐτοῦ τρισίν· ὁμοίως γὰρ
ἐκείνοις τὴν διάμετρον τῶν ἀντικειμένων ζητεῖ τὴν 25
ἀρχικὴν καὶ τὰς παρ' ἃς δύνανται.

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Εἰς τὸ ις'.

Ἴσον ἄρα τὸ ὑπὸ ΒΚΑ τῷ ὑπὸ ΑΛΒ· ἴση
ἄρα ἐστὶν ἡ ΚΑ τῇ ΒΛ· ἐπεὶ γὰρ τὸ ὑπὸ ΒΚΑ
τῷ ὑπὸ ΑΛΒ ἐστὶν ἴσον, ἀνάλογον ἔσται, ὡς ἡ ΚΒ
πρὸς ΑΛ, ἡ ΛΒ πρὸς ΑΚ. καὶ ἐναλλάξ, ὡς ἡ ΚΒ 5
πρὸς ΒΛ, ἡ ΛΑ πρὸς ΑΚ· καὶ συνθέντι, ὡς ἡ ΚΛ
πρὸς ΛΒ, ἡ ΛΚ πρὸς ΚΑ· ἴση ἄρα ἡ ΚΑ τῇ ΒΛ.

δεῖ ἐπιστήσαι, ὅτι ἐν τῷ πεντεκαιδεκάτῳ καὶ ἐκ-
καιδεκάτῳ θεωρήματι σκοπὸν ἔσχε ζητῆσαι τὰς καλου-

μένας δευτέρας καὶ συζυγείς διαμέτρους τῆς ἐλλείψεως 10
 καὶ τῆς ὑπερβολῆς ἦτοι τῶν ἀντικειμένων· ἡ γὰρ
 παραβολὴ οὐκ ἔχει τοιαύτην διάμετρον. παρατηρητέον
 δέ, ὅτι αἱ μὲν τῆς ἐλλείψεως διάμετροι ἐντὸς ἀπολαμ-
 βάνονται, αἱ δὲ τῆς ὑπερβολῆς καὶ τῶν ἀντικειμένων
 ἐκτὸς. καταγράφοντας δὲ δεῖ τὰς μὲν παρ' ἃς δύναν- 15
 ται ἦτοι τὰς ὀρθίας πλευρᾶς πρὸς ὀρθὰς τάττειν καὶ
 δηλονότι καὶ τὰς παραλλήλους αὐταῖς, τὰς δὲ τεταγ-
 μένως καταγομένας καὶ τὰς δευτέρας διαμέτρους οὐ
 πάντως· μάλιστα γὰρ ἐν ὀξείᾳ γωνίᾳ δεῖ κατάγειν
 αὐτάς, ἵνα σαφεῖς ὣσιν τοῖς ἐντυγχάνουσιν ἕτεροι 20
 οὔσαι τῶν παραλλήλων τῇ ὀρθίᾳ πλευρᾷ.
 Μετὰ τὸ ἐκκαιδέκατον θεώρημα ὅρους ἐκτίθεται
 περὶ τῆς καλουμένης δευτέρας διαμέτρου τῆς ὑπερ-
 βολῆς καὶ τῆς ἐλλείψεως, οὓς διὰ καταγραφῆς σαφεῖς
 ποιήσομεν. 25

ἔστω ὑπερβολὴ ἡ AB , διάμετρος δὲ αὐτῆς ἔστω
 ἡ $ΓΒΔ$, παρ' ἣν δὲ δύνανται αἱ ἐπὶ τὴν $ΒΓ$ κατ-

226 ἀγόμεναι ἡ $ΒΕ$. φανερόν οὖν, ὅτι ἡ μὲν $ΒΓ$ εἰς ἄπει-
 ρον αὔξεται διὰ τὴν τομήν, ὡς δέδεικται ἐν τῷ ὀγδόῳ
 θεωρήματι, ἡ δὲ $ΒΔ$, ἣτις ἐστὶν ἡ ὑποτείνουσα τὴν
 ἐκτὸς τοῦ διὰ τοῦ ἄξονος τριγώνου γωνίαν πεπέρασ-
 ται. ταύτην δὴ διχοτομοῦντες κατὰ τὸ Z καὶ ἀγα- 5
 γόντες ἀπὸ τοῦ A τεταγμένως κατηγμένην τὴν AH ,
 διὰ δὲ τοῦ Z τῇ AH παράλληλον τὴν $ΘΖΚ$ καὶ ποι-
 ῆσαντες τὴν $ΘΖ$ τῇ ZK ἴσην, ἔτι μέντοι καὶ τὸ ἀπὸ
 $ΘΚ$ ἴσον τῷ ὑπὸ $ΔΒΕ$, ἔξομεν τὴν $ΘΚ$ δευτέραν διά-

μετρον. τούτο γὰρ δυνατὸν διὰ τὸ τὴν ΘΚ ἐκτὸς 10
οὔσαν τῆς τομῆς εἰς ἄπειρον ἐκβάλλεσθαι καὶ δυνα-
τὸν εἶναι ἀπὸ τῆς ἀπείρου προτεθείση εὐθεία ἴσην
ἀφελεῖν. τὸ δὲ Ζ κέντρον καλεῖ, τὴν δὲ ΖΒ καὶ τὰς
ὁμοίως αὐτῇ ἀπὸ τοῦ Ζ πρὸς τὴν τομὴν φερομένας ἐκ
τοῦ κέντρου. 15

ταῦτα μὲν ἐπὶ τῆς ὑπερβολῆς καὶ τῶν ἀντικειμέ-
νων· καὶ φανερόν, ὅτι πεπερασμένη ἐστὶν ἑκατέρα τῶν
διαμέτρων, ἡ μὲν πρώτη αὐτόθεν ἐκ τῆς γενέσεως
τῆς τομῆς, ἡ δὲ δευτέρα, διότι μέση ἀνάλογόν ἐστι
πεπερασμένων εὐθειῶν τῆς τε πρώτης διαμέτρου καὶ 20
τῆς παρ' ἣν δύνανται αἱ καταγόμεναι ἐπ' αὐτὴν τε-
ταγμένως.

ἐπὶ δὲ τῆς ἐλλείψεως οὕτω δῆλον τὸ λεγόμενον.
ἐπειδὴ γὰρ εἰς ἑαυτὴν συννεύει, καθάπερ ὁ κύκλος,
καὶ ἐντὸς ἀπολαμβάνει πάσας τὰς διαμέτρους καὶ 25
ὠρισμένας αὐτὰς ἀπεργάζεται· ὥστε οὐ πάντως ἐπὶ
τῆς ἐλλείψεως ἡ μέση ἀνάλογον τῶν τοῦ εἴδους πλευ-
ρῶν καὶ διὰ τοῦ κέντρου τῆς τομῆς ἀγομένη καὶ ὑπὸ
τῆς διαμέτρου διχοτομουμένη ὑπὸ τῆς τομῆς περατοῦται·

228 δυνατὸν δὲ αὐτὴν συλλογίζεσθαι δι' αὐτῶν τῶν εἰρη-
μένων ἐν τῷ πεντεκαιδεκάτῳ θεωρήματι. ἐπεὶ γὰρ, ὡς
ἐκεῖ δέδεικται, αἱ ἐπὶ τὴν ΔΕ καταγόμεναι παράλληλοι
τῇ ΑΒ δύνανται τὰ παρακείμενα παρὰ τὴν τρίτην αὐταῖς
ἀνάλογον γινομένην, τουτέστι τὴν ΖΔ, ἔστιν, ὡς ἡ ΔΕ 5
πρὸς τὴν ΑΒ, ἡ ΑΒ πρὸς ΔΖ· ὥστε μέση ἀνάλογόν
ἐστὶν ἡ ΑΒ τῶν ΕΔ, ΔΖ. καὶ διὰ τοῦτο καὶ αἱ κατα-

γόμεναι ἐπὶ τὴν AB παράλληλοι τῇ ΔE δυνήσονται τὰ
 παρὰ τὴν τρίτην ἀνάλογον παρακείμενα τῶν ΔE , AB ,
 τουτέστι τὴν AN . διὰ δὴ τοῦτο μέση ἀνάλογον γίνε- 10
 ται ἡ ΔE δευτέρα διάμετρος τῶν BA , AN τοῦ εἴδους
 πλευρῶν.

δεῖ δὲ εἰδέναι καὶ τοῦτο διὰ τὸ εὐχρηστον τῶν
 καταγραφῶν· ἐπεὶ γὰρ ἄνισοί εἰσιν αἱ AB , ΔE διά- 15
 μετροι· ἐν μόνῳ γὰρ τῷ κύκλῳ ἴσαι εἰσὶν· δηλον, ὅτι
 ἡ μὲν πρὸς ὀρθὰς

ἀγομένη τῇ ἐλάσσονι

αὐτῶν ὡς ἐνταῦθα ἡ

ΔZ ἄτε τρίτη ἀνά-

λογον οὕσα τῶν ΔE , 20

AB μείζων ἐστὶν ἀμ-

φοῖν, ἡ δὲ πρὸς ὀρ-

θὰς ἀγομένη τῇ μείζονι ὡς ἐνταῦθα ἡ AN διὰ τὸ τρίτην

ἀνάλογον εἶναι τῶν AB , ΔE ἐλάσσων ἐστὶν ἀμφοῖν·

ὥστε καὶ συνεχῶς εἶναι τὰς τέσσαρας ἀνάλογον· ὡς γὰρ 25

ἡ AN πρὸς ΔE , ἡ ΔE πρὸς AB καὶ ἡ AB πρὸς ΔZ .

Εἰς τὸ ιζ'.

Ὁ μὲν Εὐκλείδης ἐν τῷ πεντεκαιδεκάτῳ θεωρήματι

τοῦ τρίτου βιβλίου τῆς στοιχειώσεως ἔδειξεν, ὅτι ἡ

230 πρὸς ὀρθὰς ἀγομένη ἀπ' ἄκρας τῆς διαμέτρου ἐκτός
 τε πίπτει καὶ ἐφάπτεται τοῦ κύκλου, ὁ δὲ Ἀπολλώνιος
 ἐν τούτῳ καθολικόν τι δείκνυσι δυνάμενον ἐφαρμό-
 σαι ταῖς τρισὶ τοῦ κώνου καὶ τῷ κύκλῳ.

τοσοῦτον διαφέρει ὁ κύκλος τῶν τοῦ κώνου το- 5

μῶν, ὅτι ἐπ' ἐκείνου μὲν αἱ τεταγμένως κατηγμέναι
 πρὸς ὀρθὰς ἄγονται τῇ διαμέτρῳ· οὐδὲ γὰρ ἄλλαι
 εὐθείαι παράλληλοι ἑαυταῖς ὑπὸ τῆς διαμέτρου τοῦ
 κύκλου διχοτομοῦνται· ἐπὶ δὲ τῶν τριῶν τομῶν οὐ
 πάντως πρὸς ὀρθὰς ἄγονται, εἰ μὴ ἐπὶ μόνους τοὺς
 ἄξονας. 10

Εἰς τὸ ιη'.

Ἐν τισιν ἀντιγράφοις τὸ θεώρημα τοῦτο ἐπὶ μόνης
 παραβολῆς καὶ ὑπερβολῆς ἐστίν, κάλλιον δὲ καθολι-
 κώτερον ἔχειν τὴν πρότασιν, εἰ μὴ ὅτι τὸ ἐπὶ τῆς
 ἐλλείψεως ἐκείνοις ὡς ἀναμφίβολον παραλέλειπται· ἡ
 γὰρ ΓΔ ἐντὸς οὐσα τῆς τομῆς πεπερασμένης οὔσης
 καὶ αὐτὴ κατ' ἀμφοτέρωθεν τέμνει τὴν τομὴν. 15

δεῖ δὲ ἐπιστῆσαι, ὅτι, κὰν ἡ AZB τέμνη τὴν το-
 μὴν, ἡ αὐτὴ ἀπόδειξις ἀρμόζει. 20

Εἰς τὸ κ'.

Ἀπὸ τούτου τοῦ θεωρήματος ἀρχόμενος ἐφεξῆς ἐν
 πάσι τὰ συμπώματα τῆς παραβολῆς αὐτῇ δείκνυσιν
 ὑπάρχοντα καὶ οὐκ ἄλλη τινί, ὡς ἐπὶ τὸ πολὺ δὲ τῇ
 ὑπερβολῇ καὶ τῇ ἐλλείψει καὶ τῷ κύκλῳ τὰ αὐτὰ δείκ-
 νουσιν ὑπάρχοντα. 25

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ἐπειδὴ δὲ οὐκ ἄχρηστον φαίνεται τοῖς τὰ μηχανι-
 κὰ γράφουσι διὰ τὴν ἀπορίαν τῶν ὀργάνων καὶ
 πολλάκις διὰ συνεχῶν σημείων γράφειν τὰς τοῦ κώ-
 νου τομὰς ἐν ἐπιπέδῳ, διὰ τούτου τοῦ θεωρήματος
 ἔστι πορίσασθαι συνεχῆ σημεία, δι' ὧν γραφήσεται ἡ
 παραβολὴ κανόνος παραθέσει. ἐὰν γὰρ ἐκθῶμαι εὐ- 5

θείαν ὡς τὴν AB καὶ ἐπ' αὐτῆς λάβω συνεχῆ σημεῖα ὡς τὰ E, Z καὶ ἀπ' αὐτῶν πρὸς ὀρθὰς τῇ AB καὶ ποιήσω ὡς τὰς $EG, ZΔ$ λαβὼν ἐπὶ τῆς EG τυχὸν σημεῖον τὸ Γ , εἰ μὲν εὐρυτέραν βουλευθείην ποιῆσαι παραβολήν, πόρρω τοῦ E , εἰ δὲ στενωτέραν, ἐγγύτερον, καὶ ποιήσω, ὡς τὴν AE πρὸς AZ , τὸ ἀπὸ EG πρὸς τὸ ἀπὸ $ZΔ$, τὰ $\Gamma, Δ$ σημεῖα ἐπὶ τῆς τομῆς ἔσται. ὁμοίως δὲ καὶ ἄλλα ληψόμεθα, δι' ὧν γραφῆσεται ἡ παραβολή.

10

Εἰς τὸ κα'.

15

Τὸ θεώρημα σαφῶς ἔκκειται καὶ πῶσιν οὐκ ἔχει δεῖ μέντοι ἐπιστῆσαι, ὅτι ἡ παρ' ἣν δύνανται, τουτέστιν ἡ ὀρθία πλευρά, ἐπὶ τοῦ κύκλου ἴση ἐστὶ τῇ διαμέτρῳ. εἰ γάρ ἐστιν, ὡς τὸ ἀπὸ $ΔE$ πρὸς τὸ ὑπὸ AEB , ἡ ΓA πρὸς AB , ἴσον δὲ τὸ ἀπὸ $ΔE$ τῷ ὑπὸ AEB ἐπὶ τοῦ κύκλου μόνου, ἴση ἄρα καὶ ἡ ΓA τῇ AB .

20

δεῖ δὲ καὶ τοῦτο εἰδέναι, ὅτι αἱ καταγόμεναι ἐν τῇ τοῦ κύκλου περιφερείᾳ πρὸς ὀρθὰς εἰσι πάντως τῇ διαμέτρῳ καὶ ἐπ' εὐθείας γίνονται ταῖς παραλλήλοις τῇ AG .

25

διὰ δὲ τούτου τοῦ θεωρήματος τῷ αὐτῷ τρόπῳ τοῖς ἐπὶ τῆς παραβολῆς εἰρημένοις προσέχοντες γρά-

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φομεν ὑπερβολὴν καὶ ἔλλειψιν κανόνος παραθέσει. ἐκκείσθω γὰρ εὐθεῖα ἡ AB καὶ προσεκβεβλήσθω ἐπ' ἄπειρον ἐπὶ τὸ H , καὶ ἀπὸ τοῦ A ταύτη πρὸς ὀρθὰς ἦχθω ἡ AG , καὶ ἐπεξεύχθω ἡ $BΓ$ καὶ ἐκβεβλήσθω,

καὶ εἰλήφθω τινὰ σημεῖα ἐπὶ τῆς ΑΗ τὰ Ε, Η, καὶ 5
ἀπὸ τῶν Ε, Η τῆ ΑΓ παράλληλοι ἤχθωσαν αἱ ΕΘ,
ΗΚ, καὶ γινέσθω τῷ μὲν ὑπὸ ΑΗΚ ἴσον τὸ ἀπὸ
ΖΗ, τῷ δ' ὑπὸ ΑΕΘ ἴσον τὸ ἀπὸ ΔΕ· διὰ γὰρ τῶν
Α, Δ, Ζ ἤξει ἡ ὑπερβολή. ὁμοίως δὲ κατασκευάσο-
μεν καὶ τὰ ἐπὶ τῆς ἐλλείψεως. 10

Εἰς τὸ κγ'.

Δεῖ ἐπιστήσαι, ὅτι ἐν τῇ προτάσει δύο διαμέτρους
λέγει οὐχ ἀπλῶς τὰς τυχούσας, ἀλλὰ τὰς καλουμένας
συζυγεῖς, ὧν ἑκατέρω παρὰ τεταγμένως κατηγμένην
ἦκται καὶ μέσον λόγον ἔχει τῶν τοῦ εἶδους πλευρῶν 15
τῆς ἐτέρας διαμέτρου, καὶ διὰ τοῦτο δίχα τέμνουσι
τὰς ἀλλήλων παραλλήλους, ὡς δέδεικται ἐν τῷ ιε' θεω-
ρήματι. εἰ γὰρ μὴ οὕτως ληφθῆ, συμβήσεται τὴν
μεταξὺ εὐθείαν τῶν δύο διαμέτρων τῇ ἐτέρῃ αὐτῶν
παράλληλον εἶναι· ὅπερ οὐχ ὑπόκειται. 20

ἐπειδὴ δὲ τὸ Η ἔγγιόν ἐστι τῆς διχοτομίας τῆς
ΑΒ ἤπερ τὸ Θ, καὶ ἐστι τὸ μὲν ὑπὸ ΒΗΑ μετὰ τοῦ
ἀπὸ ΗΜ ἴσον τῷ ἀπὸ ΑΜ, τὸ δὲ ὑπὸ ΑΘΒ μετὰ

236 τοῦ ἀπὸ ΘΜ ἴσον τῷ αὐτῷ, τὸ δὲ ἀπὸ ΘΜ τοῦ ἀπὸ
ΗΜ μείζον, τὸ ἄρα ὑπὸ ΒΗΑ μείζον τοῦ ὑπὸ ΒΘΑ.

Εἰς τὸ κε'.

Ἐν τισι φέρεται καὶ αὕτη ἡ ἀπόδειξις·

εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τὸ Θ, καὶ ἐπε- 5
ξεύχθω ἡ ΖΘ· ἡ ΖΘ ἄρα ἐκβαλλομένη συμπίπτει τῇ
ΔΓ· ὥστε καὶ ἡ ΖΕ. πάλιν δὲ εἰλήφθω, καὶ ἐπε-
ξεύχθω ἡ ΚΖ καὶ ἐκβεβλήσθω· συμπεσεῖται δὲ τῇ ΒΑ

ἐκβαλλομένη· ὥστε καὶ ἡ ΖΗ.

Εἰς τὸ κζ'.

10

Τὸ θεώρημα τοῦτο πώσεις ἔχει πλείους, πρῶτον μὲν, ὅτι ἡ ΕΖ ἢ ἐπὶ τὰ κυρτὰ μέρη τῆς τομῆς λαμβάνεται ὡς ἐνταῦθα ἢ ἐπὶ τὰ κοῖλα, ἔπειτα, ὅτι ἡ ἀπὸ τοῦ Ε παρὰ τεταγμένως κατηγμένην ἔσω μὲν καθ' ἓν σημεῖον συμβάλλει ἀδιαφόρως τῇ διαμέτρῳ ἀπείρῳ οὔσῃ, ἔξω δὲ οὔσα καὶ μάλιστα ἐπὶ τῆς ὑπερβολῆς ἔχει θέσιν ἢ ἐξωτέρω τοῦ Β ἢ ἐπὶ τοῦ Β ἢ μεταξὺ τῶν Α, Β.

15

Εἰς τὸ κζ'.

Ἐν τισιν ἀντιγράφοις τοῦ κζ' θεωρήματος φέρεται τοιαύτη ἀπόδειξις·

20

ἔστω παραβολή, ἥς διάμετρος ἡ ΑΒ, καὶ ταύτην τεμνέτω εὐθεία τις ἡ ΗΔ ἐντὸς τῆς τομῆς. λέγω,

238 ὅτι ἡ ΗΔ ἐκβαλλομένη ἐφ' ἑκάτερα τὰ μέρη συμπεσεῖται τῇ τομῇ.

ἤχθω γάρ τις διὰ τοῦ Α παρατεταγμένως ἡ ΑΕ· ἡ ΑΕ ἄρα ἐκτὸς πεσεῖται τῆς τομῆς.

ἦτοι δὴ ἡ ΗΔ τῇ ΑΕ παράλληλός ἐστιν ἢ οὔ.

5

εἰ μὲν οὖν παράλληλός ἐστιν, αὐτὴ τεταγμένως κατῆκται· ὥστε ἐκβαλλομένη ἐφ' ἑκάτερα, ἐπεὶ δίχα τέμνεται ὑπὸ τῆς διαμέτρου, συμπεσεῖται τῇ τομῇ. μὴ ἔστω δὴ παράλληλος τῇ ΑΕ, ἀλλὰ ἐκβαλλομένη συμπιπέτω τῇ ΑΕ κατὰ τὸ Ε ὡς ἡ ΗΔΕ.

10

ὅτι μὲν οὖν τῇ τομῇ ἐπὶ τὰ ἕτερα μέρη συμπίπτει, ἐφ' ἃ ἐστὶ τὸ Ε, δῆλον· εἰ γὰρ τῇ ΑΕ συμβάλ-

λει, πολὺ πρότερον τεμεί τὴν τομήν.

λέγω, ὅτι καὶ ἐπὶ τὰ ἕτερα μέρη ἐκβαλλομένη συμ-
πίπτει τῇ τομῇ.

15

ἔστω γὰρ παρ' ἣν δύνανται ἡ MA , καὶ ἐκβε-
βλήσθω ἐπ' εὐθείας αὐτῇ ἡ AZ · ἡ MA ἄρα τῇ AB
πρὸς ὀρθάς ἐστιν. πεποιήσθω, ὡς τὸ ἀπὸ AE πρὸς
τὸ $AE\Delta$ τρίγωνον, οὕτως ἡ MA πρὸς AZ , καὶ διὰ
τῶν M, Z τῇ AB παράλληλοι ἤχθωσαν αἱ ZK, MN ·
τετραπλεύρου οὖν ὄντος τοῦ $ΛΑΔΗ$ καὶ θέσει οὔσης
τῆς $ΛΑ$ ἤχθω τῇ $ΛΑ$ παράλληλος ἡ $ΓΚΒ$ ἀποτέμ-
νουσα τὸ $ΓΚΗ$ τρίγωνον τῷ $ΛΑΔΗ$ τετραπλεύρῳ
ἴσον, καὶ διὰ τοῦ B τῇ ZAM παράλληλος ἤχθω ἡ
 $ΕΒΝ$. καὶ ἐπεὶ ἐστιν, ὡς τὸ ἀπὸ AE πρὸς τὸ $AE\Delta$
τρίγωνον, ἡ MA πρὸς AZ , ἀλλ' ὡς μὲν τὸ ἀπὸ AE
πρὸς τὸ $AE\Delta$ τρίγωνον, τὸ ἀπὸ $ΓΒ$ πρὸς τὸ $\Delta ΓΒ$
τρίγωνον· παράλληλος γὰρ ἐστιν ἡ AE τῇ $ΓΒ$, καὶ
ἐπιξυγνύουσιν αὐτὰς αἱ $ΓΕ, ΑΒ$ · ὡς δὲ ἡ MA πρὸς

20

25

240 AZ , τὸ $AMNB$ παραλληλόγραμμον πρὸς τὸ $ΑΞ$ παραλ-
ληλόγραμμον, ὡς ἄρα τὸ ἀπὸ $ΓΒ$ πρὸς τὸ $\Gamma\Delta B$ τρί-
γωνον, οὕτως τὸ $AMNB$ παραλληλόγραμμον πρὸς τὸ
 $AZEB$ παραλληλόγραμμον· ἐναλλάξ, ὡς τὸ ἀπὸ $ΓΒ$ πρὸς
τὸ $AMNB$ παραλληλόγραμμον, οὕτως τὸ $\Gamma\Delta B$ τρίγωνον
πρὸς τὸ $AZEB$ παραλληλόγραμμον. ἴσον δὲ ἐστὶ τὸ
 $ZABE$ παραλληλόγραμμον τῷ $\Gamma B\Delta$ τριγώνῳ· ἐπεὶ γὰρ
τὸ $\Gamma ΗΚ$ τρίγωνον τῷ $ΑΛΗΔ$ τετραπλεύρῳ ἐστὶν ἴσον,
κοινὸν δὲ τὸ $ΗΔΒΚ$ τετράπλευρον, τὸ $ΛΑΒΚ$ παραλληλό-
γραμμον τῷ $\Gamma\Delta B$ τριγώνῳ ἐστὶν ἴσον· τὸ δὲ $ΛΑΒΚ$

5

10

παραλληλόγραμμον τῷ $ZAB\Xi$ παραλληλογράμμῳ ἔστιν
 ἴσον· ἐπὶ γὰρ τῆς αὐτῆς βάσεως ἔστι τῆς AB καὶ ἐν
 ταῖς αὐταῖς παραλλήλοις ταῖς AB, ZK . ἴσον ἄρα
 ἔστι τὸ $\Gamma\Delta B$ τρίγωνον τῷ ΞZAB παραλληλογράμμῳ·
 ὥστε καὶ τὸ ἀπὸ ΓB τῷ $AMNB$ παραλληλογράμμῳ 15
 ἔστιν ἴσον. τὸ δὲ $MABN$ παραλληλόγραμμον ἴσον
 ἔστι τῷ ὑπὸ MAB · ἢ γὰρ MA πρὸς ὀρθάς ἔστι τῇ
 AB · τὸ ἄρα ὑπὸ MAB ἴσον ἔστι τῷ ἀπὸ ΓB καὶ
 ἔστιν ἡ MA ὀρθία τοῦ εἴδους πλευρά, ἢ δὲ AB διά- 20
 μετρος, καὶ ἡ ΓB τεταγμένως· παράλληλος γὰρ ἔστι
 τῇ AE · τὸ Γ ἄρα πρὸς τῇ τομῇ ἔστιν. ἢ $\Delta H\Gamma$ ἄρα
 συμβάλλει τῇ τομῇ κατὰ τὸ Γ · ὅπερ ἔδει δεῖξαι.
 σχολία εἰς τὸ προτεθὲν θεώρημα.

[πεποιήσθω δὴ, ὡς τὸ ἀπὸ AE πρὸς τὸ $AE\Delta$
 τρίγωνον, ἢ MA πρὸς AZ] τοῦτο δέδεικται ἐν 25
 σχολίῳ τοῦ ia' θεωρήματος. ἀναγράφας γὰρ τὸ ἀπὸ
 AE καὶ παρὰ τὴν πλευρὰν αὐτοῦ τῷ $AE\Delta$ τριγώνῳ
 ἴσον παραβαλὼν ἔξω τὸ ζητούμενον.

242 **εἰς τὸ αὐτό.**

[τετραπλεύρου ὄντος τοῦ $\Lambda A\Delta H$ ἤχθω τῇ ΛA
 παράλληλος ἢ ΓKB ἀποτέμνουσα τὸ ΓHK τρί-
 γωνον τῷ $\Lambda A\Delta H$ τετραπλεύρῳ ἴσον] τοῦτο δὲ
 ποιήσομεν οὕτως· ἐὰν γὰρ, ὡς ἐν τοῖς στοιχείοις ἐμά- 5
 θομεν, τῷ δοθέντι εὐθυγράμμῳ τῷ $\Lambda A\Delta H$ τετρα-
 πλεύρῳ ἴσον καὶ ἄλλῳ τῷ δοθέντι τῷ $AE\Delta$ τριγώνῳ
 ὅμοιον τὸ αὐτὸ συστησώμεθα τὸ ΣTY , ὥστε ὁμόλογον
 εἶναι τὴν ΣY τῇ $A\Delta$, καὶ ἀπολάβωμεν τῇ μὲν ΣY

ἴσην τὴν ΗΚ, τῇ δὲ ΤΥ ἴσην τὴν ΗΓ, καὶ ἐπιζεύ- 10
 ξωμεν τὴν ΓΚ, ἔσται τὸ ζητούμενον. ἐπεὶ γὰρ ἡ
 πρὸς τῷ Υ γωνία ἴση ἐστὶ τῇ Δ, τουτέστι τῇ Η, διὰ
 τοῦτο ἴσον καὶ ὅμοιον τὸ ΓΗΚ τῷ ΣΤΥ. καὶ ἴση
 ἡ Γ γωνία τῇ Ε, καὶ εἰσιν ἐναλλάξ· παράλληλος ἄρα 15
 ἐστὶν ἡ ΓΚ τῇ ΑΕ.

φανερὸν δὴ, ὅτι, ὅταν ἡ ΑΒ ἄξων ἐστίν, ἡ ΜΑ
 ἐφάπτεται τῆς τομῆς, ὅταν δὲ μὴ ἄξων, τέμνει, εἰ
 πρὸς ὀρθὰς ἄγεται πάντως τῇ διαμέτρῳ.

Εἰς τὸ κη'.

Ὅτι, κὰν ἡ ΓΔ τέμνη τὴν ὑπερβολήν, τὰ αὐτὰ 20
 συμβήσεται, ὥσπερ ἐπὶ τοῦ ὀκτωκαιδεκάτου.

Εἰς τὸ λ'.

[Καὶ ὡς ἄρα ἐπὶ μὲν τῆς ἐλλείψεως συνθέντι,
 ἐπὶ δὲ τῶν ἀντικειμένων ἀνάπαλιν καὶ ἀνα-

244 στρέψαντι] ἐπὶ μὲν οὖν τῆς ἐλλείψεως ἐροῦμεν·
 ἐπειδὴ ἐστίν, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΔΖ, τὸ
 ὑπὸ ΑΗΒ πρὸς τὸ ἀπὸ ΗΕ, ὡς δὲ τὸ ἀπὸ ΔΖ πρὸς
 τὸ ἀπὸ ΖΓ, τὸ ἀπὸ ΕΗ πρὸς τὸ ἀπὸ ΗΓ, δι' ἴσου,
 ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΖΓ, τὸ ὑπὸ ΑΗΒ 5
 πρὸς τὸ ἀπὸ ΗΓ· συνθέντι, ὡς τὸ ὑπὸ ΑΖΒ μετὰ
 τοῦ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΖΓ, τουτέστι τὸ ἀπὸ ΑΓ
 πρὸς τὸ ἀπὸ ΓΖ· ἡ γὰρ ΑΒ τέτμηται εἰς μὲν ἴσα
 κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Ζ· οὕτως τὸ ἀπὸ
 ΓΒ πρὸς τὸ ἀπὸ ΓΗ· καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΑΓ 10
 πρὸς τὸ ἀπὸ ΓΒ, τὸ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΓΗ. ἐπὶ
 δὲ τῶν ἀντικειμένων· ἐπεὶ ἐστίν, ὡς τὸ ὑπὸ ΒΖΑ

πρὸς τὸ ἀπὸ ΖΓ, τὸ ὑπὸ ΑΗΒ πρὸς τὸ ἀπὸ ΓΗ,
 διότι δι' ἴσου, ἀνάπαλιν, ὡς τὸ ἀπὸ ΖΓ πρὸς τὸ ὑπὸ
 ΒΖΑ, τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΑΗΒ· ἀναστρέψαντι, 15
 ὡς τὸ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΓΑ, τὸ ἀπὸ ΗΓ πρὸς
 τὸ ἀπὸ ΓΒ· εὐθεία γάρ τις ἢ ΑΒ τέτμηται δίχα κατὰ
 τὸ Γ, καὶ πρόσκειται ἢ ΖΑ, καὶ τὸ ὑπὸ ΒΖΑ μετὰ
 τοῦ ἀπὸ ΑΓ ἴσον ἐστὶ τῷ ἀπὸ ΓΖ, ὥστε τὸ ἀπὸ ΓΖ
 τοῦ ὑπὸ ΒΖΑ ὑπερέχει τῷ ἀπὸ ΑΓ, καὶ καλῶς εἴρη- 20
 ται τὸ ἀναστρέψαντι.

Εἰς τὸ λα'.

[Διελόντι τὸ ἀπὸ ΓΒ πρὸς τὸ ὑπὸ ΑΗΒ μεί-
 ζονα λόγον ἔχει ἢπερ τὸ ἀπὸ ΓΒ πρὸς τὸ ὑπὸ
 ΑΘΒ] ἐπεὶ γὰρ εὐθεία ἢ ΑΒ τέτμηται δίχα κατὰ τὸ 25
 Γ, καὶ πρόσκειται αὐτῇ ἢ ΒΗ, τὸ ὑπὸ ΑΗΒ μετὰ
 τοῦ ἀπὸ ΓΒ ἴσον ἐστὶ τῷ ἀπὸ ΓΗ· ὥστε τὸ ἀπὸ ΓΗ
 τοῦ ὑπὸ ΑΗΒ ὑπερέχει τῷ ἀπὸ ΓΒ. διὰ δὲ τὴν

246 αὐτὴν αἰτίαν καὶ τὸ ἀπὸ ΓΘ τοῦ ὑπὸ ΑΘΒ ὑπερέχει
 τῷ ἀπὸ ΓΒ· ὥστε ὀρθῶς εἴρηται τὸ διελόντι.

Εἰς τὸ λβ'.

Ἐν τῷ ἑπτακαιδεκάτῳ θεωρήματι ἀπλούστερον 5
 ἔδειξεν, ὅτι ἢ διὰ τῆς κορυφῆς παρὰ τὴν κατηγμένην
 τεταγμένως ἀγομένη ἐφάπτεται, ἐνταῦθα δὲ τὸ ἐν τοῖς
 στοιχείοις ἐπὶ τοῦ κύκλου μόνου δεδειγμένον καθολι-
 κώτερον ἐπὶ πάσης κώνου τομῆς ὑπάρχον ἐπιδείκνυσι.

δεῖ μέντοι ἐπιστήσαι, ὅπερ κάκει ἐδείχθη, ὅτι καμ- 10
 πύλην μὲν ἴσως γραμμὴν οὐδὲν ἄτοπόν ἐστὶν ἐμπί-
 πτειν μεταξὺ τῆς εὐθείας καὶ τῆς τομῆς, εὐθείαν δὲ

ἀμήχανον· τεμεί γὰρ αὕτη τὴν τομὴν καὶ οὐκ ἐφά-
ψεται· δύο γὰρ ἐφαπτομένας εὐθείας κατὰ τοῦ αὐτοῦ
σημείου εἶναι ἀδύνατον.

πολυτρόπως δεδειγμένου τούτου τοῦ θεωρήματος 15
ἐν διαφόροις ἐκδόσεσιν ἡμεῖς τὴν ἀπόδειξιν ἀπλου-
στέραν καὶ σαφεστέραν ἐποιήσαμεν.

Εἰς τὸ λδ'. Δεῖ ἐπιστήσαι, ὅτι ἡ ΓΔ κατηγμένη ἐπὶ τὴν διά-
μετρον ἐπὶ μὲν τῆς ὑπερβολῆς τὰς ΔΒ, ΔΑ ὀρίζουσα 20
τὴν ΒΑ καταλιμπάνει ὀφείλουσαν τμηθῆναι εἰς τὸν
τῶν ΒΔΑ λόγον, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύ-
κλου ἀνάπαλιν τὴν ΒΑ τέμνουσα εἰς ὠρισμένον λόγον
τὸν τῶν ΒΔΑ ἐπιζητεῖν ἡμᾶς ποιεῖ τὸν τῶν ΒΕ,
ΕΑ· οὐδὲν γὰρ δυσχερὲς λόγου δοθέντος ἴσον αὐτῷ
πορίσασθαι.

248 δεῖ μέντοι εἰδέναι, ὅτι καθ' ἐκάστην τομὴν κατα-
γραφαί εἰσι δύο τοῦ Ζ σημείου ἢ ἐσωτέρω τοῦ Γ
λαμβάνομένου ἢ ἐξωτερῶ· ὥστε εἶναι τὰς πάσας πτώ-
σεις ἕξ.

χρήται δὲ καὶ δύο λήμμασιν, ἅπερ ἐξῆς γράψομεν. 5

[μεῖζον ἄρα τὸ ὑπὸ ΑΝΞ τοῦ ὑπὸ ΑΟΞ· ἢ
ΝΟ ἄρα πρὸς ΕΟ μεῖζονα λόγον ἔχει ἢπερ ἢ
ΟΑ πρὸς ΑΝ] ἐπεὶ γὰρ τὸ ὑπὸ ΑΝ, ΝΞ μεῖζόν
ἐστί τοῦ ὑπὸ ΑΟ, ΟΞ, γινέσθω τῷ ὑπὸ ΑΝ, ΝΞ
ἴσον τὸ ὑπὸ τῆς ΑΟ καὶ ἄλλης τινὸς τῆς ΕΠ, ἣτις 10
μεῖζων ἔσται τῆς ΕΟ· ἔστιν ἄρα, ὡς ἢ ΟΑ πρὸς ΑΝ,
ἢ ΝΞ πρὸς ΕΠ· ἢ δὲ ΝΞ πρὸς ΕΟ μεῖζονα λόγον
ἔχει ἢπερ πρὸς τὴν ΕΠ· καὶ ἢ ΟΑ ἄρα πρὸς ΑΝ

ἐλάττωνα λόγον ἔχει ἥπερ ἡ ΝΞ πρὸς ΞΟ.

φανερὸν δὴ καὶ τὸ ἀνάπαλιν, ὅτι, κἂν ἡ ΝΞ πρὸς ΞΟ μείζονα λόγον ἔχη ἥπερ ἡ ΟΑ πρὸς ΑΝ, τὸ ὑπὸ ΕΝ, ΝΑ μείζον ἐστὶ τοῦ ὑπὸ ΑΟ, ΟΞ. 15

γινέσθω γάρ, ὡς ἡ ΟΑ πρὸς ΑΝ, οὕτως ἡ ΝΞ πρὸς μείζονα δηλονότι τῆς ΞΟ ὡς τὴν ΞΠ· τὸ ἄρα ὑπὸ ΕΝ, ΝΑ ἴσον ἐστὶ τῷ ὑπὸ ΑΟ, ΞΠ· ὥστε μείζον ἐστὶ τὸ ὑπὸ ΕΝ, ΝΑ τοῦ ὑπὸ ΑΟ, ΟΞ. (20) 20

εἰς τὸ αὐτό.

[ἀλλ' ὡς μὲν τὸ ὑπὸ ΒΚ, ΑΝ πρὸς τὸ ἀπὸ ΓΕ, τὸ ὑπὸ ΒΔΑ πρὸς τὸ ἀπὸ ΕΔ] ἐπεὶ οὖν διὰ

250 τὸ παραλλήλους εἶναι τὰς ΑΝ, ΕΓ, ΚΒ ἐστὶν, ὡς ἡ ΑΝ πρὸς ΕΓ, ἢ ΑΔ πρὸς ΔΕ, ὡς δὲ ἡ ΕΓ πρὸς ΚΒ, ἢ ΕΔ πρὸς ΔΒ, δι' ἴσου ἄρα, ὡς ἡ ΑΝ πρὸς ΚΒ, ἢ ΑΔ πρὸς ΔΒ· καὶ ὡς ἄρα τὸ ἀπὸ ΑΝ πρὸς τὸ ὑπὸ ΑΝ, ΚΒ, τὸ ἀπὸ ΑΔ πρὸς τὸ ὑπὸ ΑΔΒ. 5

ὡς δὲ τὸ ἀπὸ ΕΓ πρὸς τὸ ἀπὸ ΑΝ, τὸ ἀπὸ ΕΔ πρὸς τὸ ἀπὸ ΔΑ· δι' ἴσου ἄρα, ὡς τὸ ἀπὸ ΕΓ πρὸς τὸ ὑπὸ ΑΝ, ΚΒ, τὸ ἀπὸ ΕΔ πρὸς τὸ ὑπὸ ΑΔΒ· καὶ ἀνάπαλιν, ὡς τὸ ὑπὸ ΚΒ, ΑΝ πρὸς τὸ ἀπὸ ΕΓ, τὸ ὑπὸ ΒΔΑ πρὸς τὸ ἀπὸ ΕΔ. 10

Εἰς τὸ λζ'.

Διὰ τούτων τῶν θεωρημάτων φανερόν, ὅπως ἐστὶ δυνατὸν διὰ τοῦ δοθέντος σημείου ἐπὶ τῆς διαμέτρου καὶ τῆς κορυφῆς τῆς τομῆς ἐφαπτομένην ἀγαγεῖν.

Εἰς τὸ λη'. 15

Ἐν τισιν ἀντιγράφοις τὸ θεώρημα τοῦτο ἐπὶ μόνης

τῆς ὑπερβολῆς εὐρίσκεται δεδειγμένον, καθολικῶς δὲ ἐνταῦθα δέδεικται· τὰ γὰρ αὐτὰ συμβαίνει καὶ ἐπὶ τῶν ἄλλων τομῶν. καὶ τῷ Ἀπολλωνίῳ δὲ δοκεῖ μὴ μόνον τὴν ὑπερβολὴν, ἀλλὰ καὶ τὴν ἔλλειψιν ἔχειν δευτέραν διάμετρον, ὡς πολλάκις αὐτοῦ ἠκούσαμεν ἐν τοῖς προλαβοῦσιν.

καὶ ἐπὶ μὲν τῆς ἐλλείψεως πῶσιν οὐκ ἔχει, ἐπὶ δὲ τῆς ὑπερβολῆς τρεῖς· τὸ γὰρ Ζ σημεῖον, καθ' ὃ συμβάλλει ἡ ἐφαπτομένη τῇ δευτέρᾳ διαμέτρῳ, ἢ κατω-

252 τέρῳ τοῦ Δ ἐστὶν ἢ ἐπὶ τοῦ Δ ἢ ἀνωτέρῳ τοῦ Δ, καὶ διὰ τοῦτο τὸ Θ ὁμοίως αὐτῷ τρεῖς ἔξει τόπους, καὶ προσεκτέον, ὅτι, εἴτε κατωτέρῳ πέσῃ τὸ Ζ τοῦ Δ, καὶ τὸ Θ τοῦ Γ ἔσται κατωτέρῳ, εἴτε τὸ Ζ ἐπὶ τὸ Δ, καὶ τὸ Θ ἐπὶ τὸ Γ, εἴτε ἀνωτέρῳ τὸ Ζ τοῦ Δ, καὶ τὸ Θ τοῦ Γ ἔσται ἀνωτέρῳ.

Εἰς τὸ μα'.

Τὸ θεώρημα τοῦτο ἐπὶ μὲν τῆς ὑπερβολῆς πῶσιν οὐκ ἔχει, ἐπὶ δὲ τῆς ἐλλείψεως, ἐὰν ἡ καταγομένη ἐπὶ τὸ κέντρον πίπτῃ, τὰ δὲ λοιπὰ γένηται τὰ αὐτά, τὸ ἀπὸ τῆς κατηγμένης εἶδος ἴσον ἔσται τῷ ἀπὸ τῆς ἐκ τοῦ κέντρου εἶδει.

ἔστω γὰρ ἔλλειψις, ἥς διάμετρος ἡ ΑΒ, κέντρον τὸ Δ, καὶ κατήχθω τεταγμένως ἡ ΓΔ, καὶ ἀναγεγράφθω ἀπὸ τε τῆς ΓΔ καὶ τῆς ΑΔ εἶδη ἰσογώνια τὰ ΑΖ, ΔΗ, ἐχέτω δὲ ἡ ΔΓ πρὸς ΓΗ τὸν συγκείμενον λόγον ἐκ τε τοῦ ὄν ἔχει ἡ ΑΔ πρὸς ΔΖ καὶ τοῦ ὄν ἔχει ἡ ὀρθία πρὸς τὴν πλαγίαν.

λέγω, ὅτι τὸ AZ ἴσον ἐστὶ τῷ ΔΗ.

ἐπεὶ γὰρ ἐν τῷ ῥητῷ δέδεικται, ὡς τὸ ἀπὸ ΑΔ
 πρὸς τὸ AZ, οὕτως τὸ ὑπὸ ΑΔΒ πρὸς τὸ ΔΗ, φημί,
 ὅτι καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΑΔ πρὸς τὸ ὑπὸ ΑΔΒ,
 οὕτως τὸ AZ πρὸς τὸ ΔΗ. ἴσον δὲ τὸ ἀπὸ ΑΔ τῷ
 ὑπὸ ΑΔΒ· ἴσον ἄρα καὶ τὸ AZ τῷ ΔΗ.

Εἰς τὸ μβ'. 25

Τὸ θεώρημα τοῦτο ἔχει πτώσεις ια, μίαν μὲν, εἰ
 ἐσωτέρω λαμβάνοιτο τὸ Δ τοῦ Γ· δῆλον γάρ, ὅτι καὶ

254 αἱ παράλληλοι ἐσωτέρω πεσοῦνται τῶν ΑΓΘ. ἐτέρας
 δὲ πέντε οὕτως· ἐὰν τὸ Δ ἐξωτέρω ληφθῆ τοῦ Γ, ἢ

μὲν ΔΖ παράλληλος δηλονότι ἐξωτέρω πεσεῖται τῆς
 ΘΓ, ἢ δὲ ΔΕ ἢ μεταξὺ τῶν Α, Β ἢ ἐπὶ τὸ Β ἢ με-
 ταξὺ τῶν Β, Θ ἢ ἐπὶ τὸ Θ ἢ ἐξωτέρω τοῦ Θ· τοῦ

γὰρ Α ἐξωτέρω πεσεῖν αὐτὴν ἀδύνατον, ἐπειδὴ τὸ Δ
 ἐξωτέρω ἐστὶ τοῦ Γ καὶ δηλονότι καὶ ἢ δι' αὐτοῦ
 παράλληλος ἀγομένη τῇ ΑΓ. ἐὰν δὲ τὸ Δ ἐπὶ τὰ

ἕτερα μέρη ληφθῆ τῆς τομῆς, ἢ ἀμφότεραι αἱ παράλ-
 ληλοι μεταξὺ τῶν Θ, Β περατωθήσονται, ἢ ἢ μὲν ΔΖ
 ἐσωτέρω τοῦ Θ, τὸ δὲ Ε ἐπὶ τὸ Θ, ἢ τῆς ΔΖ ὡσαύ-
 τως μενούσης τὸ Ε ἐξωτέρω τοῦ Θ ἐλεύσεται· τοῦ δὲ

Ε πάλιν ἐξωτέρω πίπτοντος τὸ Ζ ἢ ἐπὶ τὸ Θ πεσεῖται,
 ὡς εἶναι τὴν ΓΘΔ μίαν εὐθεῖαν, εἰ καὶ μὴ σώζεται

κυρίως τότε τὸ τῆς παραλλήλου ἰδίωμα, ἢ ἐξωτέρω 15

τοῦ Θ. δεῖ δὲ ἐπὶ τῆς ἀποδείξεως τῶν τελευταίων
 πέντε πτώσεων τὴν ΔΖ ἐκβάλλειν ἕως τῆς τομῆς καὶ
 τῆς ΗΓ παραλλήλου καὶ οὕτως ποιεῖσθαι τὴν ἀπό-

δειξιν.

δυνατὸν δὲ καὶ ἄλλην μίαν καταγραφὴν ἐπινοεῖν
ἐκ τούτων, ὅταν δὴ λαμβανομένου ἑτέρου σημείου αἰ
ἐξ ἀρχῆς εὐθείαι ποιῶσι τὸ λεγόμενον, ἀλλὰ τοῦτο
θεώρημα μᾶλλον ἐστὶν ἢ πῶσις. 20

Εἰς τὸ μγ'.

Ἐν τισι φέρεται ἀπόδειξις τοῦ θεωρήματος τούτου
τοιαύτη. 25

256 ἐπεὶ γὰρ ἴσον ἐστὶ τὸ ὑπὸ ΖΓΔ τῷ ἀπὸ ΓΒ, ἔστιν
ἄρα, ὡς ἡ ΖΓ πρὸς ΓΒ, ἢ ΓΒ πρὸς ΓΔ· καὶ ὡς
ἄρα τὸ ἀπὸ τῆς ΓΖ εἶδος πρὸς τὸ ἀπὸ τῆς ΓΒ εἶδος,
οὕτως ἡ ΖΓ πρὸς τὴν ΓΔ. ἀλλ' ὡς μὲν τὸ ἀπὸ ΖΓ
πρὸς τὸ ἀπὸ ΓΒ, τὸ ΕΖΓ τρίγωνον πρὸς τὸ ΛΓΒ
5 τρίγωνον, ὡς δὲ ἡ ΖΓ πρὸς ΓΔ, τὸ ΕΖΓ τρίγωνον
πρὸς τὸ ΕΓΔ τρίγωνον· ὡς ἄρα τὸ ΕΓΖ τρίγωνον
πρὸς τὸ ΒΛΓ τρίγωνον, τὸ ΕΓΖ πρὸς τὸ ΕΓΔ τρί-
γωνον. ἴσον ἄρα τὸ ΕΓΔ τρίγωνον τῷ ΒΓΛ. καὶ
ὡς ἄρα ἐπὶ μὲν τῆς ὑπερβολῆς ἀναστρέψαντι, ἐπὶ δὲ
10 τῆς ἐλλείψεως ἀνάπαλιν καὶ διελόντι, [ὡς] τὸ ΕΖΓ
τρίγωνον πρὸς τὸ ΕΛΒΖ τετράπλευρον, οὕτως τὸ
ΕΓΖ πρὸς τὸ ΕΔΖ τρίγωνον· ἴσον ἄρα τὸ ΕΔΖ
τρίγωνον τῷ ΕΛΒΖ τετραπλεύρῳ. καὶ ἐπεὶ ἐστὶν,
ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ἀπὸ ΓΒ, τὸ ΕΓΖ πρὸς τὸ
15 ΛΓΒ τρίγωνον, ἐπὶ μὲν τῆς ὑπερβολῆς διελόντι, ἐπὶ
δὲ τῆς ἐλλείψεως ἀνάπαλιν καὶ ἀναστρέψαντι καὶ ἀνά-
παλιν ἐστὶν, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΒΓ, τὸ
ΕΛΒΖ τετράπλευρον πρὸς τὸ ΒΛΓ τρίγωνον. ὁμοίως

δὲ καί, ὡς τὸ ἀπὸ ΓΒ πρὸς τὸ ὑπὸ ΑΚΒ, οὕτως τὸ 20
 ΛΓΒ τρίγωνον πρὸς τὸ ΜΛΒΚ τετράπλευρον· δι'
 ἴσου ἄρα, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ὑπὸ ΑΚΒ, τὸ
 ΕΛΒΖ τετράπλευρον πρὸς τὸ ΛΒΚΜ. ὡς δὲ τὸ
 ὑπὸ ΑΖΒ πρὸς τὸ ὑπὸ ΑΚΒ, τὸ ἀπὸ ΕΖ πρὸς τὸ 25
 ἀπὸ ΗΚ, ὡς δὲ τὸ ἀπὸ ΕΖ πρὸς τὸ ἀπὸ ΗΚ, τὸ
 ΕΔΖ τρίγωνον πρὸς τὸ ΗΘΚ τρίγωνον· καὶ ὡς ἄρα
 τὸ ΕΔΖ πρὸς τὸ ΗΘΚ, τὸ ΕΛΒΖ τετράπλευρον
 πρὸς τὸ ΜΛΒΚ. ἐναλλάξ, ὡς τὸ ΕΔΖ πρὸς τὸ
 ΕΛΒΖ, οὕτως τὸ ΗΘΚ πρὸς τὸ ΜΛΒΚ. ἴσον δὲ
 258 τὸ ΕΔΖ τῷ ΕΛΒΖ ἐδείχθη· ἴσον ἄρα καὶ τὸ ΗΘΚ
 τῷ ΜΛΒΚ τετραπλεύρῳ. τὸ ἄρα ΜΓΚ τρίγωνον
 τοῦ ΗΘΚ διαφέρει τῷ ΛΒΓ.

ἐπιστήσαι δεῖ ταύτη τῇ δείξει· ὀλίγην γὰρ ἀσάφειαν
 ἔχει ἐν ταῖς ἀναλογίαις τῆς ἐλλείψεως· ἵνα τὰ διὰ 5
 τὴν συντομίαν τοῦ ῥητοῦ ὁμοῦ λεγόμενα διηρημένως
 ποιήσωμεν, οἷον—φησὶ γάρ· ἐπεὶ ἐστίν, ὡς τὸ ἀπὸ
 ΖΓ πρὸς τὸ ἀπὸ ΓΒ, τὸ ΕΓΖ τρίγωνον πρὸς τὸ
 ΛΒΓ, ἀνάπαλιν καὶ ἀναστρέψαντι καὶ ἀνάπαλιν
 —ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ἀπὸ ΓΖ, τὸ ΛΒΓ πρὸς 10
 τὸ ΕΖΓ· ἀναστρέψαντι, ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ
 ΑΖΒ, τουτέστιν ἢ ὑπεροχῇ τοῦ ἀπὸ ΓΒ πρὸς τὸ ἀπὸ
 ΓΖ διὰ τὸ διχοτομίαν εἶναι τὸ Γ τῆς ΑΒ, οὕτως τὸ
 ΛΒΓ τρίγωνον πρὸς τὸ ΛΒΖΕ τετράπλευρον· ἀνά-
 παλιν, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΒΓ, τὸ ΕΛΒΖ 15
 τετράπλευρον πρὸς τὸ ΛΒΓ τρίγωνον.

ἔχει δὲ πτώσεις ἐπὶ μὲν τῆς ὑπερβολῆς $\bar{\iota}\alpha$, ὅσας

είχε και τὸ πρὸ αὐτοῦ ἐπὶ τῆς παραβολῆς, καὶ ἄλλην
 μίαν, ὅταν τὸ ἐπὶ τοῦ H λαμβανόμενον σημεῖον ταύ-
 τὸν ἢ τῷ E · τότε γὰρ συμβαίνει τὸ $E\Delta Z$ τρίγωνον 20
 μετὰ τοῦ $\Lambda B\Gamma$ ἴσον εἶναι τῷ $\Gamma E Z$ · δέδεικται μὲν
 γὰρ τὸ $E\Delta Z$ τρίγωνον ἴσον τῷ $\Lambda B Z E$ τετραπλεύρῳ,
 τὸ δὲ $\Lambda B Z E$ τοῦ $\Gamma Z E$ τριγώνου διαφέρει τῷ $\Lambda B\Gamma$.
 ἐπὶ δὲ τῆς ἐλλείψεως ἢ ταυτόν ἐστὶ τὸ H τῷ E ἢ 25
 ἐσωτέρω λαμβάνεται τοῦ E · καὶ δῆλον, ὅτι ἀμφοτέραι
 αἱ παράλληλοι μεταξὺ πεσοῦνται τῶν Δ, Z , ὡς ἔχει

260 ἐν τῷ ῥητῷ. εἰ δὲ ἐξωτερῶ ληφθῆ τὸ H τοῦ E , καὶ
 ἢ ἀπ' αὐτοῦ τῆ EZ παράλληλος μεταξὺ πέση τῶν $Z,$
 Γ , τὸ Θ σημεῖον ποιῆσει πώσεις πέντε· ἢ γὰρ μεταξὺ
 τῶν Δ, B πίπτει ἢ ἐπὶ τὸ B ἢ μεταξὺ τῶν B, Z ἢ 5
 ἐπὶ τὸ Z ἢ μεταξὺ τῶν Z, Γ . ἐὰν δὲ ἡ διὰ τοῦ H
 τῆ κατηγμένη παράλληλος ἐπὶ τὸ Γ κέντρον πίπτῃ,
 τὸ Θ πάλιν σημεῖον ποιήσῃ ἄλλας πέντε πώσεις
 ὡσαύτως· καὶ δεῖ ἐπὶ τούτῳ σημειώσασθαι, ὅτι τὸ
 ὑπὸ τῶν παραλλήλων ταῖς $E\Delta, EZ$ γιγνόμενον τρί-
 γωνον ἴσον γίνεται τῷ $\Lambda B\Gamma$ τριγώνῳ· ἐπεὶ γὰρ ἐστὶν, 10
 ὡς τὸ ἀπὸ EZ πρὸς τὸ ἀπὸ $H\Gamma$, τὸ $E\Delta Z$ τρίγωνον
 πρὸς τὸ $H\Theta\Gamma$ · ὅμοια γάρ· ὡς δὲ τὸ ἀπὸ EZ πρὸς
 τὸ ἀπὸ $H\Gamma$, τὸ ὑπὸ BZA πρὸς τὸ ὑπὸ $B\Gamma A$, τουτ-
 ἐστὶ τὸ ἀπὸ $B\Gamma$, ὡς ἄρα τὸ $E\Delta Z$ τρίγωνον πρὸς τὸ
 $H\Theta\Gamma$, τὸ ὑπὸ BZA πρὸς τὸ ἀπὸ $B\Gamma$. ὡς δὲ τὸ ὑπὸ 15
 BZA πρὸς τὸ ἀπὸ $B\Gamma$, οὕτως ἐδείχθη ἔχον τὸ $\Lambda B Z E$
 τετράπλευρον πρὸς τὸ $\Lambda B\Gamma$ τρίγωνον· καὶ ὡς ἄρα τὸ
 $E\Delta Z$ τρίγωνον πρὸς τὸ $H\Theta\Gamma$, τὸ $\Lambda B Z E$ τετράπλευ-

ρον πρὸς τὸ $\Lambda\text{B}\Gamma$ τρίγωνον. καὶ ἐναλλάξ. καὶ ἄλλως δὲ
 ταύτας δυνατὸν δεῖξαι λέγοντας, ὅτι ἐπὶ τῶν διπλασίων
 αὐτῶν παραλληλογράμμων ταῦτα δέδεικται ἐν τῷ σχο-
 λίῳ τοῦ μα' θεωρήματος.

20

ἐὰν δὲ ἡ διὰ τοῦ H τῆ EZ παράλληλος ἀγομένη
 μεταξὺ πέση τῶν Γ, A , ἐκβληθήσεται μὲν, ἕως ὅτε ἡ
 ΓE αὐτῆ συμπίσῃ, τὸ δὲ Θ σημεῖον ποιήσῃ πώσεις

262 $\bar{\zeta}$ ἡ γὰρ μεταξὺ τῶν B, Δ ἡ ἐπὶ τὸ B πίπτει ἡ με-
 ταξὺ τῶν B, Z ἡ ἐπὶ τὸ Z ἡ μεταξὺ τῶν Z, Γ ἡ
 ἐπὶ τὸ Γ ἡ μεταξὺ τῶν Γ, A · καὶ ἐπὶ τούτων τῶν
 πώσεων συμβαίνει τὴν διαφορὰν τῶν $\Lambda\text{B}\Gamma, \text{H}\Theta\text{K}$
 τριγῶνων κατωτέρω συνίστασθαι τῆς AB εὐθείας ὑπὸ
 τῆς $\Lambda\Gamma$ ἐκβαλλομένης.

5

ἐὰν δὲ τὸ H ἐπὶ τὰ ἕτερα μέρη ληφθῆ τῆς τομῆς,
 καὶ ἡ ἀπὸ τοῦ H τῆ EZ παράλληλος μεταξὺ πίπτῃ
 τῶν B, Z , ἐκβληθήσεται μὲν διὰ τὴν ἀπόδειξιν, ἕως
 οὗ τέμῃ τὴν $\Lambda\Gamma$, τὸ δὲ Θ σημεῖον ποιήσῃ πώσεις
 $\bar{\zeta}$ ἡ μεταξὺ τῶν B, Z ἡ ἐπὶ τὸ Z πίπτῃ ἡ μεταξὺ
 τῶν Z, Γ ἡ ἐπὶ τὸ Γ ἡ μεταξὺ τῶν Γ, A ἡ ἐπὶ τὸ
 A ἡ ἐξωτέρω τοῦ A . ἐὰν δὲ ἡ ἀπὸ τοῦ H τῆ EZ
 παράλληλος ἐπὶ τὸ Z πίπτῃ, ὥστε μίαν εὐθείαν εἶναι
 τὴν EZH , τὸ Θ σημεῖον ποιήσῃ πώσεις $\bar{\epsilon}$ · ἡ γὰρ
 μεταξὺ τῶν Z, Γ πεσεῖται ἡ ἐπὶ τὸ Γ ἡ μεταξὺ τῶν
 Γ, A ἡ ἐπὶ τὸ A ἡ ἐξωτέρω τοῦ A . ἐὰν δὲ ἡ HK
 μεταξὺ πίπτῃ τῶν Z, Γ , τὸ Θ ποιήσῃ πώσεις $\bar{\epsilon}$ · ἡ
 γὰρ μεταξὺ τῶν Z, Γ πεσεῖται ἡ ἐπὶ τὸ Γ ἡ μεταξὺ
 τῶν Γ, A ἡ ἐπὶ τὸ A ἡ ἐξωτέρω τοῦ A . ἐὰν δὲ ἡ

10

15

20

ΗΚ ἐπὶ τὸ Γ κέντρον πίπτῃ, τὸ Θ σημεῖον ποιήσει
 πώσεις τρεῖς ἢ μεταξὺ πίπτον τῶν Γ, Α ἢ ἐπὶ τὸ
 Α ἢ ἐξωτέρω τοῦ Α· καὶ ἐπὶ τούτων τῶν πώσεων
 συμβήσεται πάλιν τὸ ΗΘΚ τρίγωνον ἴσον γίνεσθαι
 τῷ ΛΒΓ τριγώνῳ. ἐὰν δὲ ἡ ΗΚ μεταξὺ πίπτῃ τῶν
 Γ, Α, τὸ Θ σημεῖον ἢ μεταξὺ τῶν Γ, Α πεσεῖται ἢ
 ἐπὶ τὸ Α ἢ ἐξωτέρω τοῦ Α.

25

συμβαίνει οὖν ἐπὶ τινος ἐλλείψεως τὰς πάσας πώ-
 σεις εἶναι $\overline{μβ}$ καὶ ἐπὶ τῆς τοῦ κύκλου δὲ περιφερείας

264 τοσαύτας, ὡς εἶναι τὰς πάσας πώσεις τούτου τοῦ
 θεωρήματος $\overline{ζς}$.

Εἰς τὸ μδ'.

[Ἐπεὶ οὖν ἀντικείμεναί εἰσιν αἱ ΖΑ, ΒΕ,
 ὧν διάμετρος ἡ ΑΒ, ἡ δὲ διὰ τοῦ κέντρου ἡ
 ΖΓΕ καὶ ἐφαπτόμεναι τῶν τομῶν αἱ ΖΗ, ΔΕ,
 παράλληλός ἐστίν ἡ ΖΗ τῇ ΕΔ] ἐπεὶ γὰρ ὑπερβολή
 ἐστίν ἡ ΑΖ καὶ ἐφαπτομένη ἡ ΖΗ καὶ κατηγμένη ἡ
 ΖΟ, ἴσον ἐστὶ τὸ ὑπὸ ΟΓΗ τῷ ἀπὸ ΓΑ διὰ τὸ λζ'

5

θεώρημα· ὁμοίως δὴ καὶ τὸ ὑπὸ ΕΓΔ τῷ ὑπὸ ΓΒ
 ἐστίν ἴσον. ἔστιν ἄρα, ὡς τὸ ὑπὸ ΟΓΗ πρὸς τὸ ἀπὸ
 ΑΓ, οὕτως τὸ ὑπὸ ΕΓΔ πρὸς τὸ ἀπὸ ΒΓ, καὶ ἐναλ-
 λάξ, ὡς τὸ ὑπὸ ΟΓΗ πρὸς τὸ ὑπὸ ΕΓΔ, τὸ ἀπὸ ΑΓ
 πρὸς τὸ ἀπὸ ΒΓ. ἴσον δὲ τὸ ἀπὸ ΑΓ τῷ ἀπὸ ΒΓ·

10

ἴσον ἄρα καὶ τὸ ὑπὸ ΟΓΗ τῷ ὑπὸ ΕΓΔ. καὶ ἐστίν ἡ
 ΟΓ τῇ ΓΞ ἴση· καὶ ἡ ΗΓ ἄρα τῇ ΓΔ ἐστίν ἴση· ἔστι
 δὲ καὶ ἡ ΖΓ τῇ ΓΕ διὰ τὸ λ'· αἱ ἄρα ΖΓΗ ἴσαι εἰσὶ
 ταῖς ΕΓΔ. καὶ γωνίας ἴσας περιέχουσι τὰς πρὸς τῷ Γ·

15

κατὰ κορυφήν γάρ. ὥστε καὶ ἡ ΖΗ τῆ ΕΔ ἐστὶν ἴση
καὶ ἡ ὑπὸ ΓΖΗ γωνία τῆ ὑπὸ ΓΕΔ. καὶ εἰσὶν ἐναλλάξ· 20
παράλληλος ἄρα ἐστὶν ἡ ΖΗ τῆ ΕΔ.

αὶ πτώσεις αὐτοῦ $\overline{\text{ιβ}}$ εἰσὶν, καθάπερ ἐπὶ τῆς ὑπερ-
βολῆς ἐν τῷ μγ' ἔχει, καὶ ἡ ἀπόδειξις ἡ αὐτή.

Εἰς τὸ με'.

Ἐπιστήσαι χρὴ τῷ θεωρήματι τούτῳ πλείους ἔχοντι 25
πτώσεις. ἐπὶ μὲν γὰρ τῆς ὑπερβολῆς ἔχει $\bar{\alpha}$ · τὸ γὰρ

266 ἀντὶ τοῦ Β λαμβανόμενον σημεῖον ἢ ταυτόν ἐστὶ τῷ
Α ἢ ταυτόν τῷ Γ· τότε γὰρ συμβαίνει τὸ ἀπὸ τῆς ΑΘ
τρίγωνον ὅμοιον τῷ ΓΔΛ ταυτόν εἶναι τῷ ἀπο-
τεμνομένῳ τριγώνῳ ὑπὸ τῶν παραλλήλων ταῖς ΔΛΓ.
ἐὰν δὲ μεταξὺ ληφθῇ τὸ Β σημεῖον τῶν Α, Γ, καὶ 5
τὰ Δ, Λ ἀνωτέρω ὡς τῶν περάτων τῆς δευτέρας
διαμέτρου, γίνονται πτώσεις τρεῖς· τὰ γὰρ Ζ, Ε
ἢ ἀνωτέρω τῶν περάτων φέρονται ἢ ἐπ' αὐτὰ ἢ
κατωτέρω. ἐὰν δὲ τὰ Δ, Λ ἐπὶ τὰ πέρατα ὡς τῆς
δευτέρας διαμέτρου, τὰ Ζ, Ε κατωτέρω ἐνεχθήσονται. 10
ὁμοίως δὲ καὶ † ἐὰν ἐξωτερῶ ληφθῇ τοῦ Γ τὸ Β,
[καὶ] ἡ ΘΓ ἐπὶ τὸ Γ ἐκβληθήσεται, συμβαίνει δὲ οὕτως
γίνεσθαι ἄλλας πτώσεις τρεῖς· τοῦ γὰρ Δ σημείου ἢ
ἀνωτέρω φερομένου τοῦ πέρατος τῆς δευτέρας διαμέ-
τρου ἢ ἐπ' αὐτὸ ἢ κατωτέρω καὶ τὸ Ζ ὁμοίως φερό- 15
μενον ποιήσει τὰς τρεῖς πτώσεις. ἐὰν δὲ ἐπὶ τὰ ἕτερα
μέρη τῆς τομῆς ληφθῇ τὸ Β σημεῖον, ἢ μὲν ΓΘ
ἐκβληθήσεται ἐπὶ τὸ Θ διὰ τὴν ἀπόδειξιν, αὶ δὲ ΒΖ,
ΒΕ ποιούσι πτώσεις τρεῖς, ἐπειδὴ τὸ Λ ἐπὶ τὸ πέρασ

φέρεται τῆς δευτέρας διαμέτρου ἢ ἀνωτέρω ἢ κατωτέρω. 20

ἐπὶ δὲ τῆς ἐλλείψεως καὶ τῆς τοῦ κύκλου περιφερείας οὐδὲν ποικίλον ἐροῦμεν, ἀλλὰ ὅσα ἐν τῷ προλαβόντι θεωρήματι ἐλέγχθη· ὡς εἶναι τὰς πώσεις τοῦ θεωρήματος τούτου $\overline{\rho\delta}$.

268 δύναται δὲ τὰ τῆς προτάσεως δείκνυσθαι καὶ ἐπὶ ἀντικειμένων.

Εἰς τὸ μζ'. Τοῦτο τὸ θεώρημα πώσεις ἔχει πλείους, ἃς δεῖξομεν προσέχοντες ταῖς πώσεις τοῦ μβ'.

ὑποδείγματος δὲ χάριν, ἐὰν τὸ Z ἐπὶ τὸ B πίπτοιτο, 5
αὐτόθεν ἐροῦμεν· ἐπεὶ τὸ BΔΛ ἴσον ἐστὶ τῷ ΘΒΔΜ,
κοινὸν ἀφηγήσθω τὸ ΝΜΔΒ· λοιπὸν ἄρα τὸ ΛΝΜ
τῷ ΝΘΒ ἐστὶν ἴσον.

ἐπὶ δὲ τῆς λοιπῆς ἐροῦμεν· ἐπειδὴ τὸ ΛΕΔ τῷ 10
ΘΒΔΜ ἐστὶν ἴσον, τουτέστι τῷ ΚΗΔΜ καὶ τῷ ΗΖΕ,
τουτέστι τῷ ΖΚΝ καὶ τῷ ΝΕΔΜ, κοινὸν ἀφηγήσθω
τὸ ΝΕΔΜ· καὶ λοιπὸν ἄρα τὸ ΛΝΜ τῷ ΚΖΝ ἴσον.

Εἰς τὸ μζ'.

Τοῦτο τὸ θεώρημα ἐπὶ μὲν τῆς ὑπερβολῆς πώσεις ἔχει, ὅσας τὸ πρὸ αὐτοῦ ἐπὶ τῆς παραβολῆς εἶχεν, τὰς

270 δὲ ἀποδείξεις αὐτῶν ποιησόμεθα προσέχοντες ταῖς
πώσεις τοῦ μγ' θεωρήματος, καὶ ἐπὶ τῆς ἐλλείψεως
δὲ τὰς ἀποδείξεις ἐκ τῶν πώσεων τοῦ μγ', οἷον ἐπὶ
τῆς ὑποκειμένης καταγραφῆς τοῦ Η σημείου ἐκτὸς
εἰλημμένου, ἐπειδὴ ἴσον ἐστὶ τὸ ΛΑΓ τρίγωνον τοῖς 5
ΘΗΩ, ΩΓΜ, τουτέστι τοῖς ΟΘΓ, ΟΗΜ τριγώνοις,
τῷ δὲ ΛΑΓ ἴσον ἐστὶ τὸ τε ΞΠΓ τρίγωνον καὶ τὸ

ΛΑΠΞ τετράπλευρον, τουτέστι τὸ ΝΘΠ τρίγωνον
 διὰ τὰ δεδειγμένα ἐν τῷ μγ' θεωρήματι, καὶ τὰ ΞΠΓ,
 ΝΘΠ ἄρα τρίγωνα ἴσα ἐστὶ τοῖς ΟΘΓ, ΟΜΗ τρι- 10
 γώνοις. κοινὸν ἀφηγήσθω τὸ ΘΟΓ τρίγωνον· λοιπὸν
 ἄρα τὸ ΞΟΝ τῷ ΗΟΜ ἴσον ἐστίν. καὶ παράλληλος
 ἢ ΝΞ τῇ ΜΗ· ἴση ἄρα ἢ ΝΟ τῇ ΟΗ.

Εἰς τὸ μη'.

Καὶ τούτου αἱ πώσεις ὡσαύτως ἔχουσι τοῖς προειρη- 15
 μένοις ἐπὶ τοῦ μζ' κατὰ τὴν τῆς ὑπερβολῆς κατα-
 γραφήν.

Εἰς τὸ μθ'.

[Λοιπὸν ἄρα τὸ ΚΛΝ τρίγωνον τῷ ΔΛΠΓ
 παραλληλογράμμῳ ἐστὶν ἴσον. καὶ ἴση ἐστὶν ἢ 20
 ὑπὸ ΔΛΠ γωνία τῇ ὑπὸ ΚΛΝ γωνίᾳ· διπλάσιον
 ἄρα ἐστὶ τὸ ὑπὸ ΚΛΝ τοῦ ὑπὸ ΛΔΓ] ἐκκείσθω
 γὰρ χωρὶς τὸ ΚΛΝ τρίγωνον καὶ τὸ ΔΛΠΓ παραλ-

272 ληλόγραμμον. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΚΛΝ τρίγωνον
 τῷ ΔΠ παραλληλογράμμῳ, ἦχθω διὰ τοῦ Ν τῇ ΛΚ
 παράλληλος ἢ ΝΡ, διὰ δὲ τοῦ Κ τῇ ΛΝ ἢ ΚΡ·
 παραλληλόγραμμον ἄρα ἐστὶ τὸ ΛΡ καὶ διπλάσιον τοῦ
 ΚΛΝ τριγώνου· ὥστε καὶ τοῦ ΔΠ παραλληλογράμμου. 5
 ἐκβεβλήσθωσαν δὴ αἱ ΔΓ, ΛΠ ἐπὶ τὰ Σ, Τ, καὶ
 κείσθω τῇ ΔΓ ἴση ἢ ΓΣ, τῇ δὲ ΛΠ ἢ ΠΤ, καὶ
 ἐπεξεύχθω ἢ ΣΤ· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΔΤ
 διπλάσιον τοῦ ΔΠ· ὥστε ἴσον τὸ ΛΡ τῷ ΛΣ. ἔστι δὲ
 αὐτῷ καὶ ἰσογώνιον διὰ τὸ τὰς πρὸς τῷ Λ γωνίας κατὰ 10
 κορυφήν οὔσας ἴσας εἶναι· τῶν δὲ ἴσων καὶ ἰσογώνιων

παράλληλογράμμων ἀντιπεπόνθασιν αἱ περὶ τὰς ἴσας
γωνίας πλευραί· ἔστιν ἄρα, ὡς ἡ ΚΛ πρὸς ΛΤ,
τουτέστι πρὸς ΔΣ, ἡ ΔΛ πρὸς ΛΝ, καὶ τὸ ὑπὸ ΚΛΝ
ἴσον ἐστὶ τῷ ὑπὸ ΛΔΣ. καὶ ἐπεὶ διπλὴ ἐστὶν ἡ ΔΣ
τῆς ΔΓ, τὸ ὑπὸ ΚΛΝ διπλάσιον ἐστὶ τοῦ ΛΔΓ. 15

ἐὰν δὲ ἡ μὲν ΔΓ τῇ ΛΠ ἐστὶ παράλληλος, ἡ δὲ
ΓΠ τῇ ΛΔ μὴ ἐστὶ παράλληλος, τραπέζιον μὲν δηλον-
ότι ἐστὶ τὸ ΔΓΠΛ, καὶ οὕτως δὲ φημι, ὅτι τὸ ὑπὸ
ΚΛΝ ἴσον ἐστὶ τῷ ὑπὸ ΔΛ καὶ συναμφοτέρου τῆς
ΓΔ, ΛΠ. ἐὰν γὰρ τὸ μὲν ΛΡ ἀναπληρωθῆ, ὡς
προείρηται, ἐκβληθῶσι δὲ καὶ αἱ ΔΓ, ΛΠ, καὶ τεθῆ
τῇ μὲν ΛΠ ἴση ἡ ΓΣ, τῇ δὲ ΔΓ ἡ ΠΤ, καὶ ἐπι-
ζευχθῆ ἡ ΣΤ, παράλληλόγραμμον ἔσται τὸ ΔΤ δι-
πλάσιον τοῦ ΔΠ, καὶ ἡ ἀπόδειξις ἡ αὐτὴ ἀρμόσει. 20
χρησιμεύσει δὲ τοῦτο εἰς τὸ ἐξῆς. 25

Εἰς τὸ ν'.

Αἱ πώσεις τούτου τοῦ θεωρήματος ὡσαύτως ἔχουσι
ταῖς τοῦ μγ', ὁμοίως δὲ καὶ ἐπὶ τοῦ να'.

274 **Εἰς τὸν ἐπίλογον.**

Τὴν ἐκ τῆς γενέσεως διάμετρον λέγει τὴν
γεναμένην ἐν τῷ κώνῳ κοινὴν τομὴν τοῦ τέμνοντος
ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου· ταύτην
δὲ καὶ ἀρχικὴν διάμετρον λέγει. καὶ φησιν, ὅτι πάντα 5
τὰ δεδειγμένα συμπώματα τῶν τομῶν ἐν τοῖς προειρη-
μένοις θεωρήμασιν ὑποθεμένων ἡμῶν τὰς ἀρχικὰς
διαμέτρους συμβαίνειν δύνανται καὶ τῶν ἄλλων πασῶν
διαμέτρων ὑποτιθεμένων.

Εἰς τὸ νδ'.

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[Καὶ ἀνεστάτω ἀπὸ τῆς AB ἐπίπεδον ὀρθὸν πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ ἐν αὐτῷ περὶ τὴν AB γεγράφθω κύκλος ὁ $AEBZ$, ὥστε τὸ τμήμα τῆς διαμέτρου τοῦ κύκλου τὸ ἐν τῷ AEB τμήματι πρὸς τὸ τμήμα τῆς διαμέτρου τὸ ἐν τῷ AZB τμήματι μὴ μείζονα λόγον ἔχειν τοῦ ὄν ἔχει ἡ AB πρὸς $BΓ$] ἔστωσαν δύο εὐθεῖαι αἱ $AB, BΓ$, καὶ δέον ἔστω περὶ τὴν AB κύκλον γράψαι, ὥστε τὴν διάμετρον αὐτοῦ τέμνεσθαι ὑπὸ τῆς AB οὕτως, ὥστε τὸ πρὸς τῷ $Γ$ μέρος αὐτῆς πρὸς τὸ λοιπὸν μὴ μείζονα λόγον ἔχειν τοῦ τῆς AB πρὸς $BΓ$. ὑποκείσθω μὲν νῦν τὸν αὐτόν, καὶ τεμήσθω ἡ AB δίχα κατὰ τὸ $Δ$, καὶ δι' αὐτοῦ πρὸς ὀρθὰς τῇ AB ἤχθω ἡ $EΔZ$, καὶ γεγονέτω, ὡς ἡ AB πρὸς

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276 $BΓ$, ἡ $EΔ$ πρὸς $ΔZ$, καὶ δίχα τεμήσθω ἡ EZ . δῆλον δὴ, ὅτι, εἰ μὲν ἡ AB τῇ $BΓ$ ἐστὶν ἴση καὶ ἡ $EΔ$ τῇ $ΔZ$, διχοτομία ἔσται τῆς EZ τὸ $Δ$, εἰ δὲ ἡ AB τῆς $BΓ$ μείζων καὶ ἡ $EΔ$ τῆς $ΔZ$, ἡ διχοτομία κατωτέρω ἐστὶ τοῦ $Δ$, εἰ δὲ ἡ AB τῆς $BΓ$ ἐλάσσων, ἀνωτέρω.

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ἔστω δὲ νῦν τέως κατωτέρω ὡς τὸ H , καὶ κέντρον τῷ H διαστήματι τῷ HZ κύκλος γεγράφθω· δεῖ δὴ διὰ τῶν A, B σημείων ἤξειν ἢ ἐσωτέρω ἢ ἐξωτέρω. καὶ εἰ μὲν διὰ τῶν A, B σημείων ἔρχοιτο, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· ὑπερπιπτέτω δὲ τὰ A, B , καὶ ἐκβληθεῖσα ἐφ' ἐκάτερα ἡ AB συμπιπτέτω τῇ περιφερείᾳ

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κατὰ τὰ Θ, K , καὶ ἐπεζεύχθωσαν αἱ $Z\Theta, \Theta E, EK, KZ$,
καὶ ἤχθω διὰ τοῦ B τῆ μὲν ZK παράλληλος ἡ MB ,
τῆ δὲ KE ἡ BL , καὶ ἐπεζεύχθωσαν αἱ MA, AL .

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ἔσσονται δὴ καὶ αὐταὶ παράλληλοι ταῖς $Z\Theta, \Theta E$ διὰ τὸ
ἴσην εἶναι τὴν μὲν $A\Delta$ τῆ ΔB , τὴν δὲ $\Delta\Theta$ τῆ ΔK
καὶ πρὸς ὀρθᾶς εἶναι τὴν $Z\Delta E$ τῆ ΘK . καὶ ἐπεὶ ὀρθή
ἐστὶν ἡ πρὸς τῷ K γωνία, καὶ παράλληλοι αἱ MBL
ταῖς ZKE , ὀρθὴ ἄρα καὶ ἡ πρὸς τῷ B · διὰ τὰ αὐτὰ
δὴ καὶ ἡ πρὸς τῷ A . ὥστε ὁ περὶ τὴν ML κύκλος
γραφόμενος ἤξει διὰ τῶν A, B . γεγράφθω ὡς ὁ
 $MA\Lambda B$. καὶ ἐπεὶ παράλληλός ἐστὶν ἡ MB τῆ ZK ,
ἔστιν, ὡς ἡ $Z\Delta$ πρὸς ΔM , ἡ $K\Delta$ πρὸς ΔB . ὁμοίως
δὴ καὶ, ὡς ἡ $K\Delta$ πρὸς ΔB , ἡ $E\Delta$ πρὸς ΔA . καὶ

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278 ἐναλλάξ, ὡς ἡ $E\Delta$ πρὸς ΔZ , τουτέστιν ἡ AB πρὸς
 $B\Gamma$, ἡ $\Lambda\Delta$ πρὸς ΔM .

ὁμοίως δέ, κὰν ὁ γραφόμενος περὶ τὴν ZE κύκλος
τέμνοι τὴν AB , τὸ αὐτὸ δειχθήσεται.

Εἰς τὸ νε'.

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[Καὶ ἐπὶ τῆς $A\Delta$ γεγράφθω ἡμικύκλιον τὸ
 $AZ\Delta$, καὶ ἤχθω τις εἰς τὸ ἡμικύκλιον παράλ-
ληλος τῆ $A\Theta$ ἡ ZH ποιούσα τὸν τοῦ ἀπὸ ZH
πρὸς τὸ ὑπὸ ΔHA λόγον τὸν αὐτὸν τῷ τῆς ΓA
πρὸς τὴν διπλασίαν τῆς $A\Delta$] ἔστω ἡμικύκλιον τὸ
 $AB\Gamma$ ἐπὶ διαμέτρου τῆς $A\Gamma$, ὁ δὲ δοθεὶς λόγος ὁ τῆς
 EZ πρὸς ZH , καὶ δεόν ἐστω ποιῆσαι τὰ προκείμενα.

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κεῖσθω τῆ EZ ἴση ἡ $Z\Theta$, καὶ τετμήσθω ἡ ΘH
δίχα κατὰ τὸ K , καὶ ἤχθω ἐν τῷ ἡμικυκλίῳ τυχούσα

εὐθεία ἡ ΓΒ ἐν γωνία τῇ ὑπὸ ΑΓΒ, καὶ ἀπὸ τοῦ
 15
 Λ κέντρου ἤχθω ἐπ' αὐτὴν κάθετος ἡ ΛΣ καὶ ἐκβλη-
 θείσα συμβαλλέτω τῇ περιφερεία κατὰ τὸ Ν, καὶ διὰ
 τοῦ Ν τῇ ΓΒ παράλληλος ἤχθω ἡ ΝΜ· ἐφάπεται
 ἄρα τοῦ κύκλου. καὶ πεποιήσθω, ὡς ἡ ΖΘ πρὸς ΘΚ,
 ἡ ΜΞ πρὸς ΞΝ, καὶ κείσθω τῇ ΞΝ ἴση ἡ ΝΟ, καὶ
 20
 ἐπεζεύχθωσαν αἱ ΛΞ, ΛΟ τέμνουσαι τὸ ἡμικύκλιον
 κατὰ τὰ Π, Ρ, καὶ ἐπεζεύχθω ἡ ΠΡΔ.

ἐπεὶ οὖν ἴση ἐστὶν ἡ ΞΝ τῇ ΝΟ, κοινὴ δὲ καὶ
 πρὸς ὀρθᾶς ἡ ΝΛ, ἴση ἄρα καὶ ἡ ΛΟ τῇ ΛΞ. ἔστι
 δὲ καὶ ἡ ΛΠ τῇ ΛΡ· καὶ λοιπὴ ἄρα ἡ ΠΟ τῇ ΡΞ

280 ἐστὶν ἴση. παράλληλος ἄρα ἐστὶν ἡ ΠΡΔ τῇ ΜΟ.
 καὶ ἐστὶν, ὡς ἡ ΖΘ πρὸς ΘΚ, ἡ ΜΞ πρὸς ΝΞ· ὡς
 δὲ ἡ ΘΚ πρὸς ΘΗ, ἡ ΝΞ πρὸς ΞΟ· δι' ἴσου ἄρα,
 ὡς ἡ ΘΖ πρὸς ΘΗ, ἡ ΜΞ πρὸς ΞΟ· ἀνάπαλιν, ὡς
 ἡ ΗΘ πρὸς ΘΖ, ἡ ΟΞ πρὸς ΞΜ· συνθέντι, ὡς ἡ
 5
 ΗΖ πρὸς ΖΘ, τουτέστι πρὸς ΖΕ, ἡ ΟΜ πρὸς ΜΞ,
 τουτέστιν ἡ ΠΔ πρὸς ΔΡ. ὡς δὲ ἡ ΠΔ πρὸς ΔΡ,
 τὸ ὑπὸ ΠΔΡ πρὸς τὸ ἀπὸ ΔΡ, ἴσον δὲ τὸ ὑπὸ ΠΔΡ
 τῷ ὑπὸ ΑΔΓ· ὡς ἄρα ἡ ΗΖ πρὸς ΖΕ, τὸ ὑπὸ ΑΔΓ
 πρὸς τὸ ἀπὸ ΔΡ. ἀνάπαλιν ἄρα, ὡς ἡ ΕΖ πρὸς ΖΗ,
 10
 τὸ ἀπὸ ΔΡ πρὸς τὸ ὑπὸ ΑΔΓ.

Εἰς τὸ νη'.

[Καὶ ἐπὶ τῆς ΑΕ γεγράφθω ἡμικύκλιον τὸ
 ΑΕΖ, καὶ τῇ ΑΔ παράλληλος ἤχθω ἐν αὐτῷ ἡ
 ΖΗ λόγον ποιούσα τὸν τοῦ ἀπὸ ΖΗ πρὸς τὸ
 15
 ὑπὸ ΑΗΕ τὸν τῆς ΓΑ πρὸς τὴν διπλασίαν τῆς

ΑΕ] ἔστω ἡμικύκλιον τὸ ΑΒΓ καὶ ἐν αὐτῷ εὐθεία
 τις ἡ ΑΒ, καὶ κείσθωσαν δύο εὐθεῖαι ἄνισοι αἰ ΔΕ,
 ΕΖ, καὶ ἐκβεβλήσθω ἡ ΕΖ ἐπὶ τὸ Η, καὶ τῇ ΔΕ
 ἴση κείσθω ἡ ΖΗ, καὶ τετμήσθω ὅλη ἡ ΕΗ δίχα
 κατὰ τὸ Θ, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου
 τὸ Κ, καὶ ἀπ' αὐτοῦ κάθετος ἐπὶ τὴν ΑΒ ἤχθω
 καὶ συμβαλλέτω τῇ περιφερείᾳ κατὰ τὸ Λ, καὶ διὰ
 τοῦ Λ τῇ ΑΒ παράλληλος ἤχθω ἡ ΛΜ, καὶ ἐκβλη-

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282 θείσα ἡ ΚΑ συμβαλλέτω τῇ ΛΜ κατὰ τὸ Μ, καὶ
 πεποιήσθω, ὡς ἡ ΘΖ πρὸς ΖΗ, ἡ ΛΜ πρὸς ΜΝ,
 καὶ τῇ ΛΝ ἴση ἔστω ἡ ΛΞ, καὶ ἐπεξεύχθωσαν αἰ
 ΝΚ, ΚΞ καὶ ἐκβεβλήσθωσαν, καὶ ἀναπληρωθεὶς ὁ
 κύκλος τεμνέτω αὐτὰς κατὰ τὰ Π, Ο, καὶ ἐπεξεύχθω
 ἡ ΟΡΠ.

5

ἐπεὶ οὖν ἔστιν, ὡς ἡ ΖΘ πρὸς ΖΗ, ἡ ΛΜ πρὸς
 ΜΝ, συνθέντι, ὡς ἡ ΘΗ πρὸς ΗΖ, ἡ ΛΝ πρὸς
 ΝΜ· ἀνάπαλιν, ὡς ἡ ΖΗ πρὸς ΗΘ, ἡ ΝΜ πρὸς ΝΛ,
 ὡς δὲ ἡ ΖΗ πρὸς ΗΕ, ἡ ΜΝ πρὸς ΝΞ· διελόντι,
 ὡς ἡ ΖΗ πρὸς ΖΕ, ἡ ΝΜ πρὸς ΜΞ. καὶ ἐπεὶ ἴση
 ἔστιν ἡ ΝΛ τῇ ΛΞ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΛΚ,
 ἴση ἄρα καὶ ἡ ΚΝ τῇ ΚΞ. ἔστι δὲ καὶ ἡ ΚΟ τῇ
 ΚΠ ἴση· παράλληλος ἄρα ἡ ΝΞ τῇ ΟΠ. ὁμοιον
 ἄρα τὸ ΚΜΝ τρίγωνον τῷ ΟΚΡ τριγώνῳ καὶ τὸ
 ΚΜΞ τῷ ΠΡΚ. ἔστιν ἄρα, ὡς ἡ ΚΜ πρὸς ΚΡ,
 ἡ ΜΝ πρὸς ΡΟ. ἀλλὰ καί, ὡς αὐτὴ ἡ ΚΜ πρὸς ΚΡ,
 ἡ ΜΞ πρὸς ΠΡ· καὶ ὡς ἄρα ἡ ΝΜ πρὸς ΡΟ, ἡ ΜΞ
 πρὸς ΠΡ· καὶ ἐναλλάξ, ὡς ἡ ΝΜ πρὸς ΜΞ, ἡ ΟΡ

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15

πρὸς ΡΠ. ἀλλ' ὡς μὲν ἡ ΝΜ πρὸς ΜΞ, ἡ ΗΖ 20
 πρὸς ΖΕ, τουτέστιν ἡ ΔΕ πρὸς ΕΖ, ὡς δὲ ἡ ΟΡ
 πρὸς ΡΠ, τὸ ἀπὸ ΟΡ πρὸς τὸ ὑπὸ ΟΡΠ· καὶ ὡς ἄρα
 ἡ ΔΕ πρὸς ΕΖ, τὸ ἀπὸ ΟΡ πρὸς τὸ ὑπὸ ΟΡΠ. ἴσον
 δὲ τὸ ὑπὸ ΟΡΠ τῷ ὑπὸ ΑΡΓ. ὡς ἄρα ἡ ΔΕ πρὸς
 ΕΖ, τὸ ἀπὸ ΟΡ πρὸς τὸ ὑπὸ ΑΡΓ.

- 284 Εἶρηται μὲν ἐν τοῖς μετὰ τὸ ι' θεώρημα σχολίοις
 ὁ σκοπὸς τῶν ιγ' πρώτων θεωρημάτων καὶ ἐν τοῖς
 εἰς τὸ ἐκκαιδέκατον ὁ τῶν ἐξῆς τριῶν, δεῖ δὲ εἰδέναι,
 ὅτι ἐν μὲν τῷ ιζ' φησὶν, ὅτι ἡ διὰ τῆς κορυφῆς
 παρὰ τεταγμένως κατηγμένην ἀγομένη ἐκτὸς πίπτει, 5
 ἐν δὲ τῷ ιη' φησὶν, ὅτι ἡ παρὰλληλος τῇ ὀπωσοῦν
 ἐφαπτομένη ἐντὸς τῆς τομῆς ἀγομένη τεμνεί τὴν τομῆν,
 ἐν τῷ ιθ', ὅτι ἡ ἀπὸ τινος σημείου τῆς διαμέτρου
 παρὰ τεταγμένως κατηγμένην συμπύπτει τῇ τομῇ, ἐν
 τῷ κ' καὶ κα' τὰς καταγομένας ζητεῖ τῶν τομῶν, ὅπως 10
 ἔχουσι πρὸς ἀλλήλας καὶ τὰ τῆς διαμέτρου ὑπ' αὐτῶν
 γινόμενα τμήματα, ἐν τῷ κβ' καὶ κγ' λέγει περὶ τῆς
 εὐθείας τῆς κατὰ δύο σημεία τῇ τομῇ συμπιπτούσης,
 ἐν τῷ κδ' καὶ κε' περὶ τῆς εὐθείας τῆς καθ' ἐν τῇ
 τομῇ συμπιπτούσης, τουτέστιν ἐφαπτομένης, ἐν τῷ κς' 15
 περὶ τῆς ἀγομένης παραλλήλου τῇ διαμέτρῳ τῆς παρα-
 βολῆς καὶ τῆς ὑπερβολῆς, ἐν τῷ κς' περὶ τῆς τεμνούσης
 τὴν διάμετρον τῆς παραβολῆς, ὅτι κατ' ἀμφοτέρω μέρη
 συμπύπτει τῇ τομῇ, ἐν τῷ κη' περὶ τῆς ἀγομένης
 παραλλήλου τῇ ἐφαπτομένη μιᾶς τῶν ἀντικειμένων, 20
 ἐν τῷ κθ' περὶ τῆς διὰ τοῦ κέντρου τῶν ἀντικειμένων

- ἐκβαλλομένης, ἐν τῷ λ' φησιν, ὅτι διχοτομεῖται ἢ διὰ
τοῦ κέντρου ἐκβαλλομένη τῆς ἐλλείψεως καὶ τῶν ἀντικει-
μένων, ἐν τῷ λα' φησίν, ὅτι ἐπὶ τῆς ὑπερβολῆς ἢ
ἐφαπτομένη τὴν διάμετρον τέμνει μεταξὺ τῆς κορυφῆς 25
καὶ τοῦ κέντρου, ἐν τῷ λβ' καὶ γ' καὶ δ' καὶ ε' καὶ
ζ' περὶ τῶν ἐφαπτομένων ποιεῖται τὸν λόγον, ἐν τῷ
- 286 λζ' περὶ τῶν ἐφαπτομένων καὶ τῶν ἀπὸ τῆς ἀφῆς
κατηγμένων τῆς ἐλλείψεως καὶ τῆς ὑπερβολῆς, ἐν τῷ
λη' περὶ τῶν ἐφαπτομένων τῆς ὑπερβολῆς καὶ τῆς
ἐλλείψεως, ὅπως ἔχουσι πρὸς τὴν δευτέραν διάμετρον,
ἐν τῷ λθ' καὶ μ' περὶ τῶν αὐτῶν ποιεῖται τὸν λόγον 5
τοὺς συγκειμένους ἐκ τούτων λόγους ἐπιζητῶν, ἐν τῷ
μα' περὶ τῶν ἀναγραφομένων παραλληλογράμμων ἀπὸ
τῆς κατηγμένης καὶ τῆς ἐκ τοῦ κέντρου τῆς ὑπερβολῆς
καὶ τῆς ἐλλείψεως, ἐν τῷ μβ' ἐπὶ τῆς παραβολῆς λέγει
ἴσον εἶναι τὸ ὑπὸ τῆς ἐφαπτομένης καὶ τῆς κατηγμένης 10
καταλαμβάνομενον τρίγωνον τῷ ἰσοῦψει αὐτῷ παραλ-
ληλογράμμῳ, ἡμίσειαν δ' ἔχοντι βάσιν, ἐν τῷ μγ'
ἐπὶ τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως ζητεῖ, πῶς
ἔχουσι πρὸς ἄλληλα τὰ ὑπὸ τῶν ἐφαπτομένων καὶ
τῶν κατηγμένων ἀπολαμβάνομενα τρίγωνα, ἐν τῷ 15
μδ' τὸ αὐτὸ ἐν ταῖς ἀντικειμέναις, ἐν τῷ με' τὸ
αὐτὸ ἐπὶ τῆς δευτέρας διαμέτρου τῆς ὑπερβολῆς
καὶ τῆς ἐλλείψεως, ἐν τῷ μζ' περὶ τῶν μετὰ τὴν
ἀρχικὴν διάμετρον τῆς παραβολῆς ἐτέρων, ἐν τῷ μζ'
περὶ τῶν ἐτέρων διαμέτρων τῆς ὑπερβολῆς καὶ τῆς 20
ἐλλείψεως, ἐν τῷ μη' περὶ τῶν ἐτέρων διαμέτρων τῶν

ἀντικειμένων, ἐν τῷ μθ' περὶ τῶν παρ' ἃς δύνανται
αἱ καταγόμεναι ἐπὶ τὰς ἑτέρας διαμέτρους τῆς παρα-
βολῆς, ἐν τῷ ν' περὶ τοῦ αὐτοῦ τῆς ὑπερβολῆς καὶ
τῆς ἑλλείψεως, ἐν τῷ να' περὶ τοῦ αὐτοῦ τῶν ἀντικει-
μένων. ταῦτα εἰπὼν καὶ προσθεὶς τοῖς εἰρημένοις

25

288 ἐπίλογόν τινα ἐν τῷ νβ' καὶ νγ' δεικνύει πρόβλημα,
ὡς δυνατὸν ἐν ἐπιπέδῳ γράψαι τὴν παραβολήν, ἐν
τῷ νδ' καὶ νε' λέγει, πῶς δεῖ γράψαι τὴν ὑπερβολήν,
ἐν τῷ νς' καὶ νζ' καὶ νη', πῶς δεῖ γράψαι τὴν ἑλλειψιν,
ἐν τῷ νθ' λέγει, πῶς δεῖ γράφειν ἀντικειμένας, ἐν
τῷ ξ' περὶ τῶν συζύγων ἀντικειμένων.

5

APPENDIX B
PASSAGES ON COMPOUND RATIO

B.1 From Eutocius' Commentary on Archimedes' *On the Sphere and the Cylinder*

- 120 Ἐπεὶ οὖν ὁ τῆς ΡΛ πρὸς ΛΧ λόγος συνήπται
 ἔκ τε τοῦ, ὃν ἔχει ἡ ΡΛ πρὸς ΛΔ, καὶ ἡ ΛΔ πρὸς
 ΛΧ ὅτι μὲν ἡ σύνθεσις τῶν λόγων λαμβάνεται τῆς
 ΛΔ μέσης λαμβανομένης, ὡς κὰν τῆ Στοιχειώσει ἐλαμ-
 βάνετο, φανερόν· ἐπεὶ δὲ τὸ λεγόμενον ἀδιαρθρώτως 5
 πῶς καὶ οὐχ οὕτως, ὥστε τὴν ἔννοιαν ἀποπληρῶσαι,
 λέλεκται, ὡς ἔστιν εὐρεῖν ἐντυγχάνοντας Πάππῳ τε καὶ
 Θέωνι καὶ Ἀρκαδίῳ ἐν πολλοῖς συντάγμασιν οὐκ ἀπο-
 δεικτικῶς, ἀλλ' ἐπαγωγῆ τὸ λεγόμενον παριστώσιν, οὐ-
 δὲν ἄτοπον πρὸς βραχὺ ἐνδιατρίψαντας τῷ λόγῳ τὸ σα- 10
 φέστερον παραστήσαι.
- φημὶ τοίνυν, ὅτι, ἐὰν δύο ἀριθμῶν ἦτοι μεγεθῶν μέσος
 τις ὄρος ληφθῆ, ὁ τῶν ἐξ ἀρχῆς ληφθέντων ἀριθμῶν
 λόγος σύγκειται ἐκ τοῦ λόγου, ὃν ἔχει ὁ πρῶτος πρὸς τὸν
 μέσον, καὶ τοῦ, ὃν ἔχει ὁ μέσος πρὸς τὸν τρίτον. 15
 ὑπομνηστέον δὴ πρότερον, πῶς ἐλέγετο λόγος ἐκ λό-
 γων συγκείσθαι. ὡς γὰρ ἐν τῇ Στοιχειώσει· ὅταν αἱ τῶν
 λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποι-
 ῶσιν τινα, πηλικότητος δηλονότι λεγομένης τοῦ ἀριθ-
 μοῦ, οὗ παρώνυμός ἐστιν ὁ διδόμενος λόγος, ὡς φασιν 20
 ἄλλοι τε καὶ Νικόμαχος ἐν τῷ πρώτῳ Περὶ μουσικῆς
 καὶ Ἡρόνας ἐν τῷ ὑπομνήματι τῷ εἰς τὴν Ἀριθμητικὴν
 εἰσαγωγῆν, ταῦτόν δὲ εἰπεῖν καὶ τοῦ ἀριθμοῦ τοῦ πολλα-

- πλασιαζομένου ἐπὶ τὸν ἐπόμενον ὄρον τοῦ λόγου καὶ
 ποιούντος τὸν ἡγούμενον. καὶ κυριώτερον μὲν ἐπὶ τῶν 25
 πολλαπλασίων ἢ πηλικότης ἂν λαμβάνοιτο, ἐπὶ δὲ τῶν
 ἐπιμορίων ἢ ἐπιμερῶν οὐκέτι τὴν πηλικότητα δυνατὸν
 λαμβάνεσθαι ἀδαιρέτου μενούσης τῆς μονάδος· ὥστ' ἐπ'
 ἐκείνων διαιρετέον τὴν μονάδα, ὃ εἰ καὶ μὴ κατὰ τὸ προσ-
 ἦγον τῇ ἀριθμητικῇ ἀλλὰ τῇ λογιστικῇ τυγχάνει. διαι- 30
 ρεῖται δὲ ἡ μονὰς κατὰ τὸ μέρος ἢ τὰ μέρη, ἀφ' ὧν ὠνό-
 122 μασται ὁ λόγος, ὥστε εἶναι ὡς ἐν σαφεστέρω τῷ λέγειν
 τοῦ μὲν ἡμιολίου λόγου πηλικότητα πρὸς τὴν μονάδι καὶ
 τὸ ἡμισυ τῆς μονάδος, τοῦ δὲ ἐπιτρίτου πρὸς τὴν μονάδι
 τὸ τρίτον, ὥστε, καθὰ καὶ ἀνωτέρω εἴρηται, τὴν πηλι-
 κότητα τοῦ λόγου ἐπὶ τὸν ἐπόμενον ὄρον πολλαπλασια- 5
 ζομένην ποιεῖν τὸν ἡγούμενον. τοῦ γὰρ ἐννέα πρὸς τὰ
 ἕξ ἡμιολίου πηλικότης οὕσα ἡ μονὰς καὶ τὸ ἡμισυ πολλα-
 πλασιασθεῖσα ἐπὶ τὸν ζ ποιεῖ τὸν θ, καὶ ἐπὶ τῶν ἄλλων
 δὲ τὸ αὐτὸ ἔξεστι κατανοεῖν.
- τούτων δὴ προσαφημισθέντων ἐπανακτέον ἐπὶ τὸ προ- 10
 τεθέν. ἔστωσαν γὰρ οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β,
 μέσος δὲ αὐτῶν εἰλήφθω τις ὁ Γ· δεικτέον δὴ, ὅτι ὁ
 τοῦ Α πρὸς τὸν Β λόγος συνήπται ἐκ τοῦ, ὃν ἔχει ὁ Α
 πρὸς τὸν Γ, καὶ ὁ Γ πρὸς τὸν Β.
- εἰλήφθω γὰρ τοῦ μὲν Α, Γ λόγου πηλικότης ὁ Δ, τοῦ 15
 δὲ Γ, Β ὁ Ε· ὁ ἄρα Γ τὸν Δ πολλαπλασιάσας τὸν Α
 ποιεῖ, ὁ δὲ Β τὸν Ε πολλαπλασιάσας τὸν Γ. ὁ δὲ Δ
 τὸν Ε πολλαπλασιάσας τὸν Ζ ποιεῖτω. λέγω, ὅτι ὁ Ζ
 πηλικότης ἐστὶ τοῦ τοῦ Α πρὸς τὸν Β λόγου, τουτέστιν,

- 20 ὅτι ὁ Z τὸν B πολλαπλασιάσας τὸν A ποιεῖ. ὁ γὰρ B
 τὸν Z πολλαπλασιάσας τὸν H ποιεῖτω. ἐπεὶ οὖν ὁ B τὸν
 μὲν Z πολλαπλασιάσας τὸν H πεποίηκεν, τὸν δὲ E πολλα-
 πλασιάσας τὸν Γ, ἔστιν ἄρα, ὡς ὁ Z πρὸς τὸν E, ὁ H
 πρὸς τὸν Γ. πάλιν, ἐπεὶ ὁ Δ τὸν μὲν E πολλαπλασιάσας
 τὸν Z πεποίηκεν, τὸν δὲ Γ πολλαπλασιάσας τὸν A πε- 25
 ποίηκεν, ἔστιν ἄρα, ὡς ὁ E πρὸς τὸν Γ, ὁ Z πρὸς τὸν
 A. ἐναλλάξ, ὡς ὁ E πρὸς τὸν Z, ὁ Γ πρὸς τὸν A, καὶ
 ἀνάπαλιν, ὡς ὁ Z πρὸς τὸν E, οὕτως ὁ A πρὸς τὸν Γ.
 ἀλλ' ὡς ὁ Z πρὸς τὸν E, ἐδείχθη ὁ H πρὸς τὸν Γ· καὶ
 ὡς ἄρα ὁ H πρὸς τὸν Γ, ὁ A πρὸς τὸν Γ· ἴσος ἄρα ὁ 30
 A τῷ H. ἀλλ' ὁ B τὸν Z πολλαπλασιάσας τὸν H πε-
- 124 ποίηκεν· καὶ ὁ B ἄρα τὸν Z πολλαπλασιάσας τὸν A
 ποιεῖ· ὁ Z ἄρα πληκτικὸς ἐστὶ τοῦ τοῦ A πρὸς τὸν B
 λόγου. καὶ ἔστιν ὁ Z τοῦ Δ ἐπὶ τὸν E πολλαπλασιασ-
 θέντος, τουτέστι τῆς πληκτικότητος τοῦ A, Γ λόγου ἐπὶ
 τὴν πληκτικότητα τοῦ Γ, B λόγου· ὁ ἄρα τοῦ A πρὸς τὸν 5
 B λόγος σύγκειται ἔκ τε τοῦ, ὃν ἔχει ὁ A πρὸς τὸν Γ,
 καὶ ὁ Γ πρὸς τὸν B· ὅπερ ἔδει δεῖξαι.
- ἵνα δὲ καὶ ἐπὶ ὑποδείγματος φανερόν γένηται τὸ εἰ-
 ρημένον, παρεμπιπτέτω τοῦ ιβ καὶ τοῦ β μέσος τις ἀριθ-
 μὸς ὁ δ. λέγω, ὅτι ὁ τοῦ ιβ πρὸς τὸν β λόγος, τουτέστιν 10
 ὁ ἑξαπλάσιος, σύγκειται ἔκ τε τοῦ τριπλασίου τοῦ ιβ
 πρὸς τὰ δ καὶ τοῦ διπλασίου τοῦ δ πρὸς τὰ β.
- ἐὰν γὰρ τὰς πληκτικότητας τῶν λόγων πολλαπλασιά-
 σωμεν ἐπ' ἀλλήλας,
 τουτέστι τὸν γ ἐπὶ τὸν β, γίνεται ὁ
 ζ πληκτικὸς ὢν τοῦ ιβ πρὸς τὰ β λόγου, καὶ ἔστιν ἑξα-

πλάσιος, ὄνπερ καὶ προέκειτο ὑποδείξει. 15

εἰ δὲ καὶ ὁ μέσος παρεμπίπτων μὴ ὑπάρχη τοῦ μὲν
 μείζονος ἐλάττων, τοῦ δὲ ἐλάττονος μείζων, ἀλλ' ἢ τὸ
 ἀνάπαλιν ἢ ἀμφοτέρων μείζων ἢ ἀμφοτέρων ἐλάττων, καὶ
 οὕτως ἢ σύνθεσις ἢ προειρημένη ἀκολουθήσει. τοῦ θ
 καὶ τοῦ ζ μέσος τις παρεμπίπτει ἀμφοτέρων μείζων ὁ 20
 ιβ. λέγω, ὅτι ἐκ τε τοῦ ὑπεπιπίπτου τοῦ θ πρὸς τὸν ιβ
 λόγου καὶ τοῦ διπλασίου τοῦ ιβ πρὸς τὸν ζ σύγκειται
 ὁ ἡμόλιος τοῦ θ πρὸς τὰ ζ.

ἢ γὰρ πηλικότης τοῦ θ πρὸς τὸν ιβ λόγου ἐστὶ τρία
 τέταρτα, τουτέστιν ἡμισυ καὶ τέταρτον, ἢ δὲ πηλικότης 25
 τοῦ ιβ πρὸς τὸν ζ ἐστὶν ὁ β. ἐὰν οὖν πολλαπλασιάσωμεν
 τὸν β ἐπὶ τὸ ἡμισυ καὶ τέταρτον, γίνεται μονὰς α καὶ
 ἡμισυ, ἥτις πηλικότης ἐστὶ τοῦ ἡμιολίου λόγου, ὃν ἔχει
 καὶ ὁ θ πρὸς τὸν ζ. ὁμοίως δέ, κὰν τοῦ θ καὶ ζ μέσος
 ἐμπέση ὁ δ, ἐκ τοῦ θ πρὸς δ διπλασιεπιτετάρτου καὶ τοῦ 30
 δ πρὸς ζ ὑψημιολίου σύγκειται ὁ ἡμόλιος λόγος. πάλιν
 γὰρ τὴν πηλικότητα τοῦ διπλασιεπιτετάρτου τὰ β δ' ἐπὶ
 τὴν πηλικότητα τοῦ ὑψημιολίου, τουτέστι τὰ δύο τρίτα,

126 πολλαπλασιάσαντες ἔξομεν τὸ ἐν ἡμισυ πηλικότητα τοῦ
 ἡμιολίου, ὡς εἴρηται, λόγου. καὶ ἐπὶ πάντων δὲ ὁμοίως
 ὁ αὐτὸς ἀρμόσει λόγος.

συμφανὲς δὲ ἐκ τῶν εἰρημένων, ὡς, ἐὰν δύο δοθέν-
 των ἀριθμῶν ἦτοι μεγεθῶν κὰν μὴ εἷς μέσος, πλείους 5
 δέ, παρεμπίπτωσιν ὄροι, ὁ τῶν ἄκρων λόγος σύγκειται
 ἐκ πάντων τῶν λόγων, ὧν ἔχουσιν οἱ κατὰ τὸ ἐξῆς κεί-
 μενοι ὄροι ἀρχόμενοι ἀπὸ πρώτου καὶ λήγοντες εἰς τὸν

ἔσχατον τῇ κατὰ τοὺς ἔχομένους τάξει.

δύο γὰρ ὄντων ὄρων τῶν A, B παρεμπιπτέωσαν
 πλείους ἐνὸς οἱ Γ, Δ . λέγω, ὅτι ὁ τοῦ A πρὸς τὸν B
 λόγος σύγκειται ἔκ τε τοῦ, ὃν ἔχει ὁ A πρὸς τὸν Γ , καὶ
 ὁ Γ πρὸς τὸν Δ , καὶ ὁ Δ πρὸς τὸν B . 10

ἐπεὶ γὰρ ὁ τοῦ A πρὸς τὸν B σύγκειται ἔκ τε τοῦ, ὃν
 ἔχει ὁ A πρὸς τὸν Δ , καὶ ὁ Δ πρὸς τὸν B , ὡς ἀνωτέρω 15
 εἴρηται, ὁ δὲ τοῦ A πρὸς τὸν Δ λόγος σύγκειται ἔκ τε
 τοῦ, ὃν ἔχει ὁ A πρὸς τὸν Γ , καὶ ὁ Γ πρὸς τὸν Δ , ὁ
 ἄρα τοῦ A πρὸς τὸν B λόγος συνήπται ἔκ τε τοῦ, ὃν
 ἔχει ὁ A πρὸς τὸν Γ , καὶ ὁ Γ πρὸς τὸν Δ , καὶ ὁ Δ πρὸς
 τὸν B . ὁμοίως δὲ καὶ ἐπὶ τῶν λοιπῶν δειχθήσεται.

B.2 From Eutocius' Commentary on Apollonius' Conics

218 Τὸ δὲ ἀπὸ τῆς $B\Gamma$ πρὸς τὸ ὑπὸ $BA\Gamma$ λόγον
 ἔχει τὸν συγκείμενον ἐκ τοῦ ὃν ἔχει ἡ $B\Gamma$ πρὸς
 ΓA καὶ ἡ $B\Gamma$ πρὸς BA . δέδεικται μὲν ἐν τῷ ἔκτῳ
 βιβλίῳ τῆς στοιχειώσεως ἐν τῷ εἰκοστῷ τρίτῳ θεωρή-
 ματι, ὅτι τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα 5
 λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν. ἐπεὶ δὲ
 ἐπακτικώτερον μᾶλλον καὶ οὐ κατὰ τὸν ἀναγκαῖον
 τρόπον ὑπὸ τῶν ὑπομνηματιστῶν ἐλέγετο, ἐζητήσαμεν
 αὐτὸ καὶ γέγραπται ἐν τοῖς ἐκδεδομένοις ἡμῖν εἰς τὸ
 τέταρτον θεώρημα τοῦ δευτέρου βιβλίου τῶν Ἀρχιμή- 10
 δους περὶ σφαίρας καὶ κυλίνδρου καὶ ἐν τοῖς σχολίοις τοῦ
 πρώτου βιβλίου τῆς Πτολεμαίου συντάξεως. οὐ χεῖρον
 δὲ καὶ ἐνταῦθα τοῦτο γραφῆναι διὰ τὸ μὴ πάντως τοὺς
 ἀναγινώσκοντας κάκείνοις ἐντυγχάνειν, καὶ ὅτι σχεδὸν

τὸ ὅλον σύνταγμα τῶν κωνικῶν κέχρηται αὐτῷ. 15

λόγος ἐκ λόγων συγκείσθαι λέγεται, ὅταν αἱ τῶν
 λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποι-
 ῶσί τινα, πηλικότητος δηλονότι λεγομένης τοῦ ἀριθ-
 μού, οὗ παρώνυμός ἐστίν ὁ λόγος. ἐπὶ μὲν οὖν τῶν
 πολλαπλασίων δυνατὸν ἐστὶν ἀριθμὸν ὀλόκληρον εἶναι 20

τὴν πηλικότητα, ἐπὶ δὲ τῶν λοιπῶν σχέσεων ἀνάγκη
 τὴν πηλικότητα ἀριθμὸν εἶναι καὶ μόριον ἢ μόρια, εἰ μὴ
 ἄρα τις ἐθέλοι καὶ ἀρρήτους εἶναι σχέσεις, οἷαί εἰσιν
 αἱ κατὰ τὰ ἄλογα μεγέθη. ἐπὶ πασῶν δὲ τῶν σχέσεων
 δῆλον, ὅτι αὐτὴ ἡ πηλικότης πολλαπλασιαζομένη ἐπὶ 25
 τὸν ἐπόμενον ὄρον τοῦ λόγου ποιεῖ τὸν ἡγούμενον.

ἔστω τοίνυν λόγος ὁ τοῦ Α πρὸς τὸν Β, καὶ εἰ-

220 λήφθω τις αὐτῶν μέσος, ὡς ἔτυχεν, ὁ Γ, καὶ ἔστω
 τοῦ Α, Γ λόγου πηλικότης ὁ Δ, τοῦ δὲ Γ, Β ὁ Ε,
 καὶ ὁ Δ τὸν Ε πολλαπλασιάσας τὸν Ζ ποιεῖτω. λέγω,
 ὅτι τοῦ λόγου τῶν Α, Β πηλικότης ἐστὶν ὁ Ζ, τουτ-
 ἐστὶν ὅτι ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α ποιεῖ. 5

ὁ δὲ Ζ τὸν Β πολλαπλασιάσας τὸν Η ποιεῖτω. ἐπεὶ
 οὖν ὁ Δ τὸν μὲν Ε πολλαπλασιάσας τὸν Ζ πεποίηκεν,
 τὸν δὲ Γ πολλαπλασιάσας τὸν Α πεποίηκεν, ἔστιν
 ἄρα, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α. πάλιν
 ἐπεὶ ὁ Β τὸν Ε πολλαπλασιάσας τὸν Γ πεποίηκεν, 10
 τὸν δὲ Ζ πολλαπλασιάσας τὸν Η πεποίηκεν, ἔστιν
 ἄρα, ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Γ πρὸς τὸν Η. ἐναλλάξ,
 ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Η. ἦν δέ, ὡς
 ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α· ἴσος ἄρα ὁ Η

τῷ Α. ὥστε ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α
 πεποίηκεν. 15

μὴ ταραπτόμενος δὲ τοὺς ἐντυγχάνοντας τὸ διὰ τῶν
 ἀριθμητικῶν δεδειχθαι τοῦτο· οἷ τε γὰρ παλαιοὶ κέ-
 χρηται ταῖς τοιαύταις ἀποδείξεσι μαθηματικαῖς μᾶλλον
 οὔσαις ἢ ἀριθμητικαῖς διὰ τὰς ἀναλογίας, καὶ ὅτι 20
 τὸ ζητούμενον ἀριθμητικόν ἐστίν. λόγοι γὰρ καὶ
 πηλικότητες λόγων καὶ πολλαπλασιασμοὶ τοῖς ἀριθμοῖς
 πρῶτως ὑπάρχουσι καὶ δι' αὐτῶν τοῖς μεγέθεσι, κατὰ
 τὸν εἰπόντα· ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν
 ἀδελφά.

B.3 From Theon's Commentary on Ptolemy's *Syntaxis*

532 **Λήμμα.**

Εἷς λόγος ἐκ δύο λόγων ἢ καὶ πλειόνων συγκείσθαι λέγεται, ὅταν αἱ
 533 τῶν λόγων πηλικότητες πολλαπλασιασθεῖσαι ποιῶσιν τινὰ πηλικότητα
 λόγου.

Ἐχέτω γὰρ τὸ ΑΒ πρὸς τὸ ΓΔ λόγον δεδομένον, καὶ τὸ ΓΔ πρὸς τὸ
 534 ΕΖ λόγον· λέγω ὅτι ὁ τοῦ ΑΒ πρὸς ΕΖ λόγος σύγκειται ἔκ τε τοῦ ΑΒ
 πρὸς ΓΔ καὶ τοῦ ΓΔ πρὸς ΕΖ, τουτέστιν ὅτι ἐὰν ἡ τοῦ ΑΒ πρὸς τὸ ΓΔ
 λόγου πηλικότης πολλαπλασιασθῇ ἐπὶ τὴν τοῦ ΓΔ πρὸς ΕΖ, τοῦ λόγου
 πηλικότητα ποιῆ τὴν τοῦ ΑΒ πρὸς τὴν ΕΖ.

Ἐστω γὰρ πρότερον τὸ μὲν ΑΒ τοῦ ΔΓ μείζον, τὸ δὲ ΓΔ τοῦ ΕΖ.

καὶ

ἔστω τὸ μὲν ΑΒ τοῦ ΓΔ διπλάσιον, τὸ δὲ ΓΔ τοῦ ΕΖ τριπλάσιον. ἐπεὶ
 οὖν τὸ μὲν ΓΔ τοῦ ΕΖ τριπλάσιόν ἐστιν, τὸ δὲ ΑΒ τοῦ ΓΔ διπλάσιον,
 τὸ

ἄρα ΑΒ τοῦ ΕΖ ἐστὶν ἑξαπλάσιον, ἐπεὶ καὶ ἐὰν τὸ τριπλάσιόν τινος

διπλα-
σιάσωμεν, γίνεται αὐτοῦ ἕξαπλάσιον. τοῦτο γάρ ἐστιν κυρίως σύνθεσις.

Ἦ οὕτως. ἐπεὶ τὸ AB τοῦ ΓΔ ἐστὶν διπλάσιον, διηγήσθω τὸ AB εἰς
τὰ τῷ ΓΔ ἴσα τὰ AH, HB. καὶ ἐπεὶ τὸ ΓΔ τοῦ EZ ἐστὶν τριπλάσιον, διὰ
τὰ αὐτὰ δὴ καὶ τὸ HB τοῦ EZ ἐστὶν τριπλάσιον. ὅλον ἄρα τὸ AB τοῦ

EZ

ἐστὶν ἕξαπλάσιον. ὁ ἄρα τοῦ AB πρὸς τὸ EZ λόγος συνήκται διὰ τοῦ

ΓΔ

μέσου ὅρου συγκείμενος ἔκ τε τοῦ AB πρὸς ΓΔ λόγου καὶ τοῦ ΓΔ πρὸς
EZ.

Ἄλλ' ὁμοίως δὲ καὶ ἐὰν ἔλαττον ἢ ἑκατέρου τῶν AB, EZ τὸ ΓΔ, τὸ
αὐτὸ

συναχθήσεται. ἔστω γὰρ πάλιν τὸ μὲν AB τοῦ ΓΔ τριπλάσιον, τὸ δὲ

ΓΔ

ἡμισυ τοῦ EZ. καὶ ἐπεὶ τὸ ΓΔ ἡμισύ ἐστιν τοῦ EZ, τοῦ δὲ ΓΔ τριπλάσιον
τὸ AB, τὸ AB ἄρα ἡμόλιόν ἐστιν τοῦ EZ· ἐὰν γὰρ τὸ ἡμισύ τινος

τριπλα-

σιάσωμεν, ἕξει αὐτὸ ἅπαξ καὶ ἡμισάκις.

Ἦ καὶ οὕτως. ἐπεὶ τὸ μὲν AB τοῦ ΓΔ ἐστὶν τριπλάσιον τὸ δὲ ΓΔ τοῦ
EZ ἡμισυ, οἷον ἐστὶν τὸ AB ἴσων τῷ ΓΔ τριῶν τοιούτων ἐστὶν τὸ EZ
δύο.

ὥστε ἡμόλιον ἔσται τὸ AB τοῦ EZ. ὁ ἄρα τοῦ AB πρὸς τὸ EZ λόγος
συνήκται διὰ τοῦ ΓΔ μέσου ὅρου συγκείμενος ἔκ τε τοῦ AB πρὸς ΓΔ
λόγου καὶ τοῦ ΓΔ πρὸς EZ.

Ἀλλὰ δὴ πάλιν ἔστω τὸ ΓΔ ἑκατέρου τῶν AB, EZ μείζον· καὶ ἔστω

535 τὸ μὲν AB τοῦ ΓΔ ἡμισυ μέρος, τὸ δὲ ΓΔ τοῦ EZ ἐπίτριτον. ἐπεὶ οὖν

οίων ἐστὶν τὸ AB δύο τοιούτων τὸ $\Gamma\Delta$ τεσσάρων, οίων δὲ τὸ $\Gamma\Delta$ τεσσάρων τοιούτων τὸ EZ τριῶν, καὶ οίων ἄρα τὸ AB δύο τοιούτων τὸ EZ τριῶν.

συνήκται ἄρα πάλιν ὁ τοῦ AB πρὸς EZ λόγος διὰ τοῦ $\Gamma\Delta$ μέσου ὄρου ὁ

τῶν δύο πρὸς τὰ τρία τοῦ τε ὑποδιπλασίου καὶ τοῦ ἐπιτρίτου, ὁ ὑψημό- 5
λιος. ὁμοίως δὴ καὶ ἐπὶ πλειόνων καὶ ἐπὶ τῶν λοιπῶν πτώσεων.

Καὶ δῆλον ὡς ὅτι ἐὰν ἀπὸ τοῦ συγκειμένου λόγου εἰς ὅποιοσούν τῶν συντιθέντων ἀφαιρεθῆ, ἐνὸς τῶν ἄκρων ἀφανισθέντος, ὁ λοιπὸς τῶν συντιθέντων καταλειφθήσεται.

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