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Khovanov homology in thickened surfaces

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KHOVANOV HOMOLOGY IN THICKENED SURFACES

by

Jeffrey Thomas Conley Boerner

<u>An Abstract</u>

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2010

Thesis Supervisors: Professor Charles Frohman
Professor Richard Randell

Mikhail Khovanov developed a bi-graded homology theory for links in S^3. Khovanov's theory came from a Topological quantum field theory (TQFT) and as such has a geometric interpretation, explored by Dror Bar-Natan. Marta Asaeda, Jozef Przytycki and Adam Sikora extended Khovanov's theory to I-bundles using decorated diagrams. Their theory did not suggest an obvious geometric version since it was not associated to a TQFT. In this thesis we develop a geometric version of Asaeda, Przytycki and Sikora's theory for links in thickened surfaces. This version leads to two other distinct theories that we also explore.

Abstract Approved: ______________________________

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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To Nora, Alice, Greta and Hazel

New knowledge is the most valuable commodity on earth. The more truth we have to work with, the richer we become.

Kurt Vonnegut, Breakfast of Champions

ACKNOWLEDGMENTS

I would like to thank Dick Randell for great advice and helpful conversations on Mathematics and life in general. I also want to thank Charlie Frohman for being an unbelievable advisor and friend. Charlie's enthusiasm, knowledge and humor were some of the things I looked forward to every day.

I am also thankful for my fellow graduate students at the University of Iowa. The environment was always supportive and stimulating.

I owe the biggest thanks to my family, as such an undertaking cannot be done alone. Their support was a constant throughout this process and none of it would have been possible without them.

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CHAPTER 1

INTRODUCTION

A knot in Mathematics is very similar to a knot in everyday life. It is a 1-dimensional object (like a string) lying in 3-dimensional space, perhaps wrapping around itself in any number of ways. The one distinction is that Mathematical knots do not have any loose ends. That way if something is knotted, it must stay knotted, no matter how much you move it around (as long as you don't cut it.)

Mathematicians realized knots were objects that could be studied rigorously. The most obvious direction of study was to name all of the simplest knots, starting at the "smallest" and working your way up. Soon it became obvious that the process of distinguishing one knot from another was far from trivial. For this purpose, Knot theorists developed knot invariants. Some invariants can prove quite useful in distinguishing knots from one another.

In my view there are two kinds of invariants. Ones that are easy to define, but hard to determine and ones that are hard to define but relatively easy to determine (assuming enough time and computing power). The easy to define ones are often the result of associating a number to a diagram and then minimizing over all diagrams of the knot. The difficult to define ones are often difficult because they must be invariant under the Reidemeister moves. However to compute them one can look at any diagram of that knot.

The link polynomials, for instance the Jones and Alexander polynomials, have algorithms to determine them and so are relatively easy to compute. In addition, they can sometimes tell you something about the easy to define invariants. For example, the Jones polynomial determines a bound on crossing number and the Alexander polynomial determines a bound on genus.

1.1 Link homologies

Recently these polynomials have been taken one step further with the introduction of link homologies, specifically Heegaard Floer and Khovanov homology. The graded Euler characteristics of these theories yield the Alexander and Jones polynomials, respectively.

Initially all of these quantities are defined for links existing in S^3. It is natural to attempt to extend them to links embedded in other 3-manifolds. The simplest 3-manifolds that are not S^3 or $\mathbb{R}^3$ are probably thickened surfaces, so that is what we concern ourselves with here.

CHAPTER 2

PRELIMINARIES

It will be assumed that the reader has a strong general mathematical background including homological algebra but little or no knowledge of knot theory. In this chapter we will formally introduce knots, links and link diagrams. Then the Kauffman bracket and the Jones polynomial will be defined.

2.1 Knots and diagrams

Definition 2.1. A *knot* is a smooth embedding of S^1 into S^3. An *n-component link* is n copies of S^1 smoothly embedded into S^3. A knot is a 1-component link.

Two knots are equivalent if there is an orientation preserving diffeomorphism from S^3 to itself such that one knot is taken to the other knot.

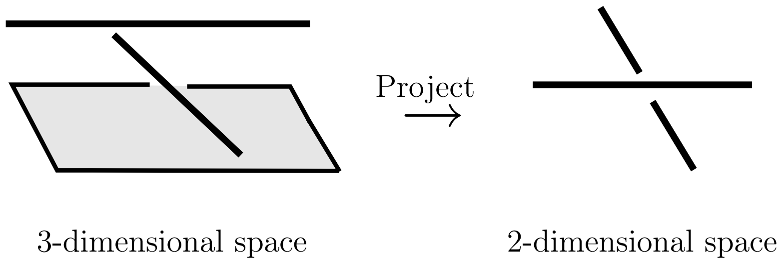
Links are 1-dimensional objects living in 3-dimensional space. To view links in 2-dimensions we use *link diagrams*. A link diagram is a projection of a link to a 2-dimensional plane with additional markings to indicate over and under crossings. Figure 2.1 shows this process.

2.2 Reidemeister moves

To deal with link diagrams Kurt Reidemeister developed three moves shown in Figure 2.2.

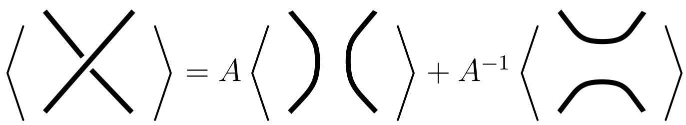
Theorem 2.1 (Reidemeister)**.** *Any two diagrams of the same link are related by some sequence of the three Reidemeister moves.*

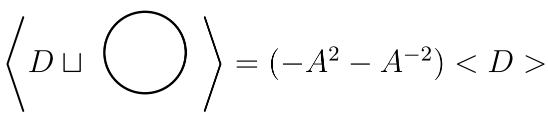
Figure 2.1: Projecting a link to a diagram.

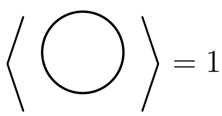


2.3 Kauffman bracket

Let D be a link diagram. The Kauffman bracket of D, denoted $< D >$, is an element of $\mathbb{Z}[A, A^{-1}]$ and is determined by the following three rules:

1. 

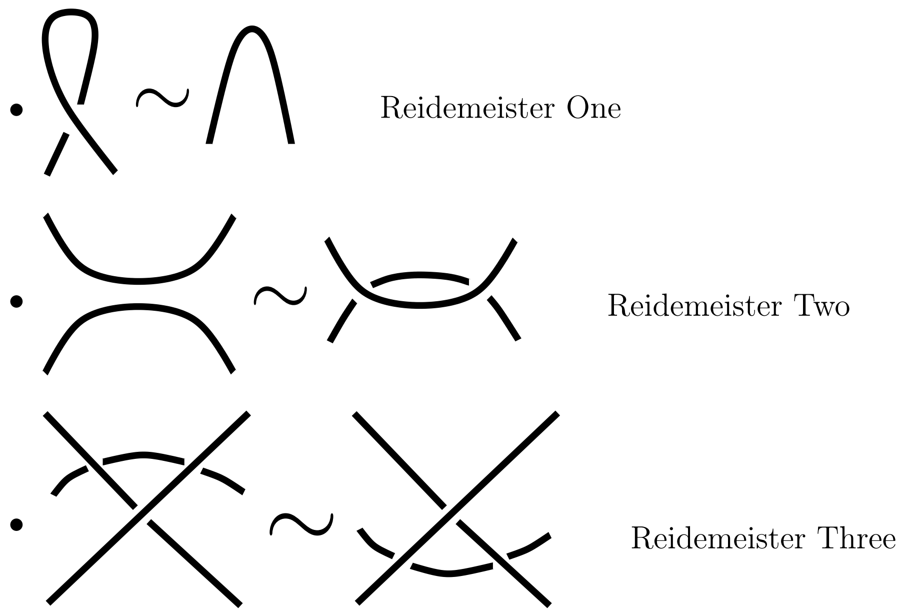
2. 

3. 

After examining the Kauffman bracket rules one can notice that an algorithm for computing the bracket polynomial is to examine every possible smoothing (Kauffman state) and count the number of circles in each state. This procedure is utilized in the Kauffman bracket state sum formula:

$$< D >= \sum_{S} A^{\sigma(S)}(-A^2 - A^{-2})^{\#S-1}, \tag{2.1}$$

Figure 2.2: The Reidemeister moves.



where S is a Kauffman state, $\sigma(S)$ = (number of A smoothings) $-$ (number of A^{-1}) smoothings, and #S is the number of components of S.

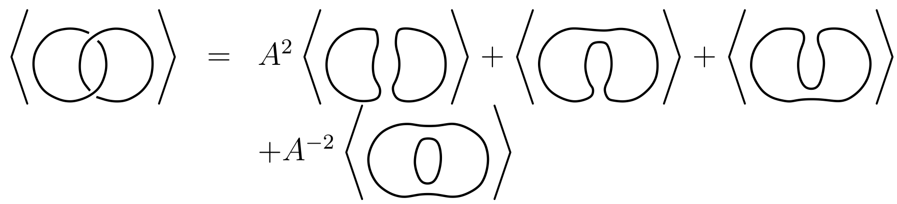
Example 2.1. The Kauffman bracket of the standard diagram of the Hopf link is shown in Figure 2.3.

2.4 Jones polynomial

The Kauffman bracket is not an invariant of links as it does not respect the Reidemeister one move. However, one can adjust it accordingly to make it into the Jones polynomial, which is a link invariant.

Definition 2.2. The *writhe* of a crossing is either $+1$ or -1 as seen in Figure 2.4. The *writhe* of an oriented link diagram, D, is the sum of the writhes of the crossings

Figure 2.3: Computing the Kauffman bracket of the Hopf link.

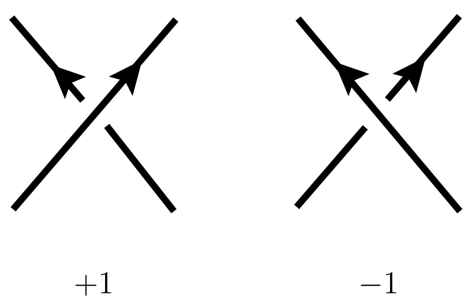


$$= A^2(-A^2 - A^{-2}) + 1 + 1 + A^{-2}(-A^2 - A^{-2})$$

$$= -A^4 - 1 + 2 - 1 - A^{-4} = -A^4 - A^{-4}$$

and is denoted $w(D)$.

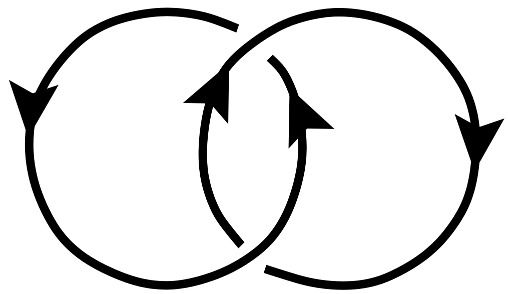
Figure 2.4: The writhe of a crossing.



Example 2.2. Consider the oriented diagram of the Hopf link in Figure 2.5. Note this diagram has writhe of 2.

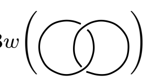
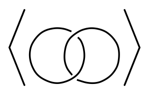
Theorem 2.2 (Kauffman)**.** *Let L be an oriented link and D a diagram of L. Then $V_L(t) = (-A)^{-3w(D)} < D >_{t^{-1/2}=A^2}$ is an invariant of L.*

Figure 2.5: An oriented Hopf link.



$V_L(t)$ is the Jones polynomial of L.

Example 2.3. The Jones polynomial of the Hopf link with the orientation in Figure 2.5 is

$V_{\text{Hopf link}}(t) = (-A)^{-3w}$ () $\Big\langle$  $\Big\rangle = (-A)^{-3*2}(-A^4 - A^{-4})$

$$= (-A)^6(-A^4 - A^{-4}) = A^6(-A^4 - A^{-4}) = -A^10 - A^2$$

$$= -t^{-5/2} - t^{-1/2}$$

2.5 3-manifold definitions

In the subsequent chapters it will be necessary to know a few basic definitions concerning 3-manifolds.

Definition 2.3. Let S be a surface embedded in three-manifold M. S is *compressible* if S contains an essential curve that bounds a disk, D, in M such that $S \cap D = \partial D$. If no such curves exist and S is not a two-sphere, then S is *incompressible*.

Definition 2.4. A surface is *orientable* if it is two-sided. Otherwise it is said to be non-orientable. Equivalently, a surface is non-orientable if it has an embedded Möbius band.

CHAPTER 3

TQFTS AND KHOVANOV HOMOLOGY

3.1 Frobenius algebras and TQFTs

Definition 3.1. A *Frobenius algebra* is a finite dimensional, unital, associative algebra, A, over a field k equipped with a linear functional $\epsilon : A \to k$, such that the kernel of ϵ contains no nonzero left ideal of A.

Definition 3.2. A *(1+1) dimensional Topological Quantum Field Theory* (TQFT) is a functor from the category $2Cob$ to the category of algebras over a field k.

Remark 3.1. It can be shown that the algebra that a circle is sent to under a TQFT is necessarily a Frobenius algebra. Therefore there is a 1-1 correspondence between Frobenius algebras and (1+1) dimensional TQFTs.

3.2 Khovavov homology

In [4] Mikhail Khovanov introduced a homology theory for links in S^3. In order to construct his link homology, Khovanov used the Frobenius extension $(\mathbb{Z}, A, \epsilon, \mu)$, where $A = \mathbb{Z}[x]/(x^2)$ and $\epsilon(x) = 1, \epsilon(1) = 0$. He assigned each circle in a Kauffman state either a 1 or an x. Borrowing the terminology from [7], we will call these decorated Kauffman states *enhanced Kauffman states.*

Khovanov's boundary operator on enhanced Kauffman states consists of changing a positive smoothing to a negative smoothing. When two circles combine as the result of changing a smoothing, the multiplication operation of A is used. When one circle splits into two circles the co-multiplication operation is used.

Table 3.1: The operations in A.

$m(x \otimes x) = 0$	$\Delta(1) = 1 \otimes x + x \otimes 1$
$m(x \otimes 1) = x$	$\Delta(x) = x \otimes x$
$m(1 \otimes 1) = 1$	

The multiplication and co-multiplication of A is shown in Figure 3.1.

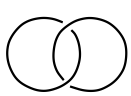
We also have the following grading coming from [7]:

Definition 3.3. Enhanced Kauffman states in Khovavov homology are bi-graded. If D is a link diagram and S is an enhanced Kauffman states then we have,

$$i(S) = \frac{w(D) - \sigma(S)}{2}, j(S) = -\frac{\sigma(S) + 2\tau(S) - 3w(D)}{2},$$

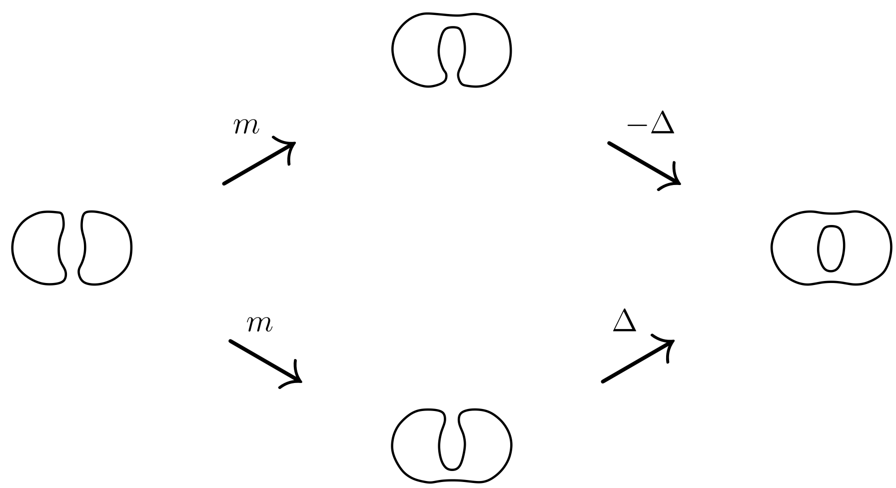
where $w(D)$ is the writhe of the diagram D, $\tau(S)$ is the number of xs - the number of 1s in S.

What is remarkable about Khovanov's homology theory is that the graded Euler characteristic of the homology is the Jones polynomial of the link.

Example 3.1. We will compute the Khovanov homology of the Hopf link. Let $D =$ . Note $w(D) = 2$. The complex of Kauffman states and chain group generators is outlined in Figure 3.1

We will now compute the homology of the Khovanov chain complex of the Hopf link. The computations are shown in Figure 3.2.

Figure 3.1: The complex of Kauffman states of the Hopf link.



$\sigma = 2$	$\sigma = 0$	$\sigma = -2$
$i = 0$	$i = 1$	$i = 2$
$1 \otimes 1 \in C_{0,4}(D)$	$1 \oplus 0 \in C_{1,4}(D)$	$1 \otimes 1 \in C_{2,6}(D)$
$1 \otimes x \in C_{0,2}(D)$	$0 \oplus 1 \in C_{1,4}(D)$	$1 \otimes x \in C_{2,4}(D)$
$x \otimes 1 \in C_{0,2}(D)$	$x \oplus 0 \in C_{1,2}(D)$	$x \otimes 1 \in C_{2,4}(D)$
$x \otimes x \in C_{0,0}(D)$	$0 \oplus x \in C_{1,2}(D)$	$x \otimes x \in C_{2,2}(D)$

Table 3.2: The boundary map of the Hopf link.

$d_0(1 \otimes 1) = m(1 \otimes 1) \oplus m(1 \otimes 1) = 1 \oplus 1$
$d_0(1 \otimes x) = m(1 \otimes x) \oplus m(1 \otimes x) = x \oplus x$
$d_0(x \otimes 1) = m(x \otimes 1) \oplus m(x \otimes 1) = x \oplus x$
$d_0(x \otimes x) = m(x \otimes x) \oplus m(x \otimes x) = 0$
$d_1(1 \oplus 0) = -\Delta(1) \oplus \Delta(0) = -1 \otimes x - x \otimes 1$
$d_1(0 \oplus 1) = -\Delta(0) \oplus \Delta(1) = 1 \otimes x + x \otimes 1$
$d_1(x \oplus 0) = -\Delta(x) \oplus \Delta(0) = -x \otimes x$
$d_1(0 \oplus x) = -\Delta(0) \oplus \Delta(x) = x \otimes x$

Figure 3.2: Khovanov homology of the Hopf link computations.

$$H_0(L) = ker(d_0)/im(d_{-1}) =< x \otimes x, 1 \otimes x - x \otimes 1 >$$

$$H_1(L) = ker(d_1)/im(d_0) =< x \oplus 0 + 0 \oplus x, 1 \oplus 0 + 0 \oplus 1 > / < 1 \oplus 1, x \oplus x >$$

$$\cong < 0 >$$

$$H_2(L) = ker(d_2)/im(d_1) =< 1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x > / < -1 \otimes x - x \otimes 1, x \otimes x >$$

$$=< 1 \otimes 1, 1 \otimes x >$$

In light of these computations the Khovanov homology of the Hopf link is shown in Figure 3.3.

Figure 3.3: The Khovanov homology of the Hopf link.

$$H_{i,j} = \begin{cases} \mathbb{Z} & \text{if } (i,j) = (0,0),(0,2),(2,4),(2,6) \\ 0 & \text{otherwise} \end{cases}$$

Now let's verify that the graded Euler characteristic is actually the Jones polynomial. Khovanov uses a different variable than the one used in the Jones polynomial. He uses q and q is related to t by $q = t^{1/2}$.

Note the graded Euler characteristic of the Hopf link homology is shown in Figure 3.4.

Figure 3.4: The graded Euler characteristic of the Hopf link.

$$\begin{aligned}\chi(H(L)) &= (-1)^0q^0rk(H_{0,0}(L)) + (-1)^0q^2rk(H_{0,2}(L)) + (-1)^2q^6rk(H_{2,6}(L)) \\ &\quad +(-1)^2q^4rk(H_{2,4}(L)) \\ &= 1+q^2+q^6+q^4\end{aligned}$$

Converting this to t yields $1+q^2+q^4+q^6 = 1+t+t^2+t^3$.

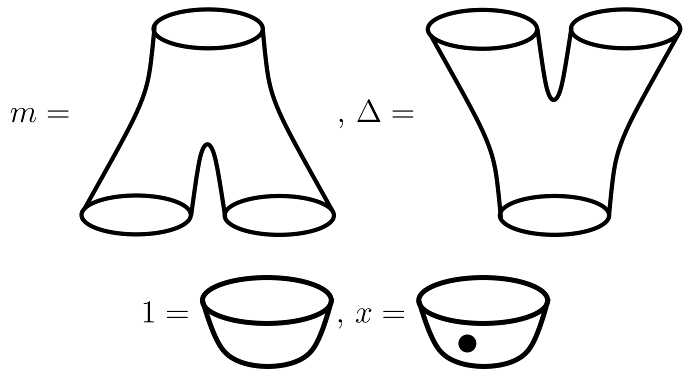
In addition, Khovanov's Jones polynomial is normalized so that the unknot has a value of $(-t-t^{-1})$ instead of 1, thus if we multiply the Jones polynomial of the Hopf link by $(-t-t^{-1})$, we should arrive at the graded Euler characteristic of the chain complex. This is verified in Equation 3.1.

$$V_{\bigcirc\!\!\!\bigcirc}(t) * (-t-t^{-1}) = (-t^{-5/2}-t^{-1/2})(-t^{1/2}-t^{-1/2}) = t^2+t^3+1+t \tag{3.1}$$

3.3 Viro's version

After Khovanov introduced his link homology, Oleg Viro modified it slightly in [7] to produce a theory for framed links. In this theory the x was replaced by a $+$, the 1 by a $-$ and the gradings were $I(S) = \sigma(S)$ and $J(S) = \sigma(S) + 2\tau(S)$. In his

Figure 3.5: Multiplication, co-multiplication, 1 and x in Bar-Natan's theory.



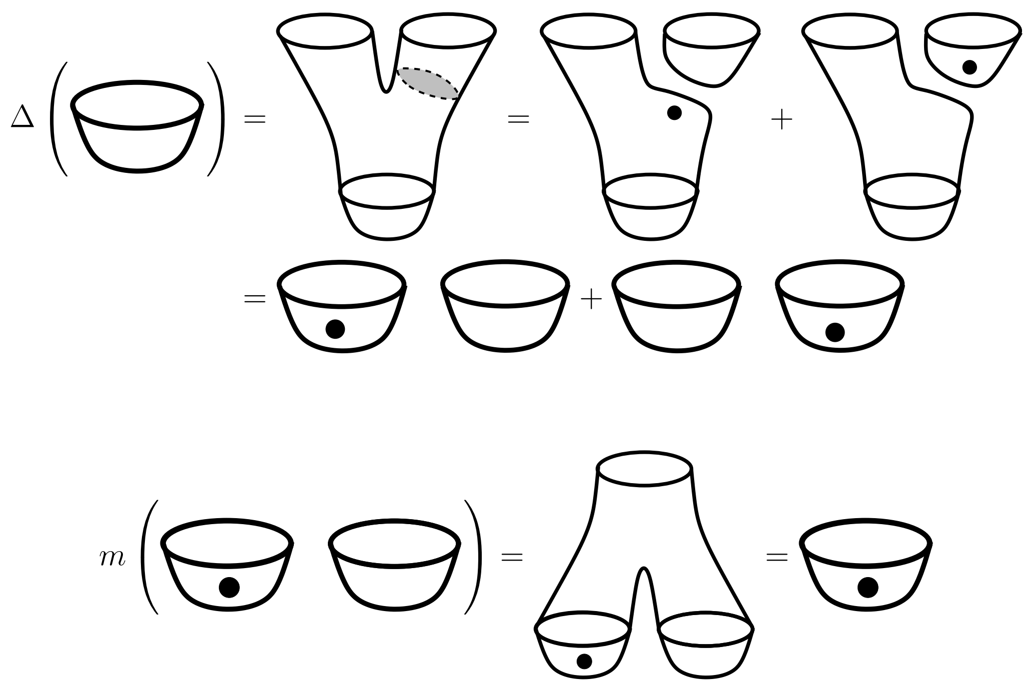
definition of the theory Viro did away with any direct reference to the algebra A.

3.4 Bar-Natan's cobordism theory

In [2] Bar-Natan developed a more geometric version of the link homology Khovanov had defined. Bar-Natan defined the surfaces corresponding to a state of the diagram to be all of the marked cobordisms from the empty diagram to that state. Bar-Natan subjected these surfaces to the relations coming from the TQFT associated to A. By the neck-cutting relation, these surfaces are spanned by disks with at most one dot, where a dot represents multiplication by x. The multiplication, co-multiplication, 1 and x in Bar-Natan's theory can be seen in Figure 3.5.

Note how this formulation corresponds to using the Frobenius extension A in Figure 3.6. As one can see, Bar-Natan's geometric definition yields the same homology as the algebraic approach.

Figure 3.6: An example of co-multiplication and multiplication in Bar-Natan's theory.



CHAPTER 4

KHOVANOV HOMOLOGY IN I-BUNDLES

After Khovanov developed his homology theory for links in S^3, Asaeda, Przytycki and Sikora extended Khovanov's theory to I-bundles over a surface. Their theory will be presented in this chapter. Their approach is by way of Viro's construction of Khovanov homology of framed links. We can recover Khovanov's original construction if the base space of the I-bundle is a disk.

Let M be an I-bundles over a surface F. Therefore, M is either $F \times I$ if F is orientable, or M is the twisted I-bundle over F. We assume that F is not $\mathbb{RP}^2$.

Definition 4.1. Let D be a link diagram. A *Kauffman state* of D is an assignment of a $+$ or $-$ smoothing to each crossing of the diagram. The smoothings can be seen in Figure 4.1. An *Enhanced Kauffman state* is a Kauffman state of D with an assignment of a $+$ or $-$ to each component of the Kauffman state.

Figure 4.1: $+$ and $-$ smoothings of a crossing.

$+$ smoothing $\qquad\qquad$ $-$ smoothing

Definition 4.2. A circle is *trivial* if it bounds a disk in the surface. A circle is *bounding* if it bounds either a disk or a Möbius band.

4.1 Grading of Enhanced Kauffman states and the chain groups

Let S be an enhanced Kauffman state of a diagram D. We define

$I(S) = \#$ of positive smoothings $-$ $\#$ of negative smoothings

$J(S) = I(S) + 2\tau(S)$

$\tau(S) = \#$ of positive trivial components $-$ $\#$ of negative trivial components.

Let $\mathcal{C}(F)$ be the set of all unoriented, unbounding simple closed curves in F, up to homotopy. If the unbounding components of S are $\gamma_1, \ldots, \gamma_n$ and marked by $\epsilon_1, \ldots, \epsilon_n \in \{+1, -1\}$, then

$$\Psi(S) = \sum_i \epsilon_i \gamma_i \in \mathbb{Z}\mathcal{C}(F).$$

Let $C_{ijs}(D)$ be the free abelian group spanned by enhanced states S, such that $i(S) = i, j(S) = j$ and $\Psi(S) = s$.

4.2 The differential

Let v be a crossing of D. We define the incidence number, $[S : S']_v$, to be 1 if the following four conditions are satisfied:

1. the crossing v is marked by $+$ in S and by $-$ in S',

2. S and S' assign the same markers to all the other crossing,

3. the labels of the common circles in S and S' are unchanged,

4. $J(S) = J(S'), \Psi(S) = \Psi(S')$.

Otherwise $[S : S']_v = 0$.

Let $t(S, v)$ denote the number of negative crossings in S coming before v in the ordering.

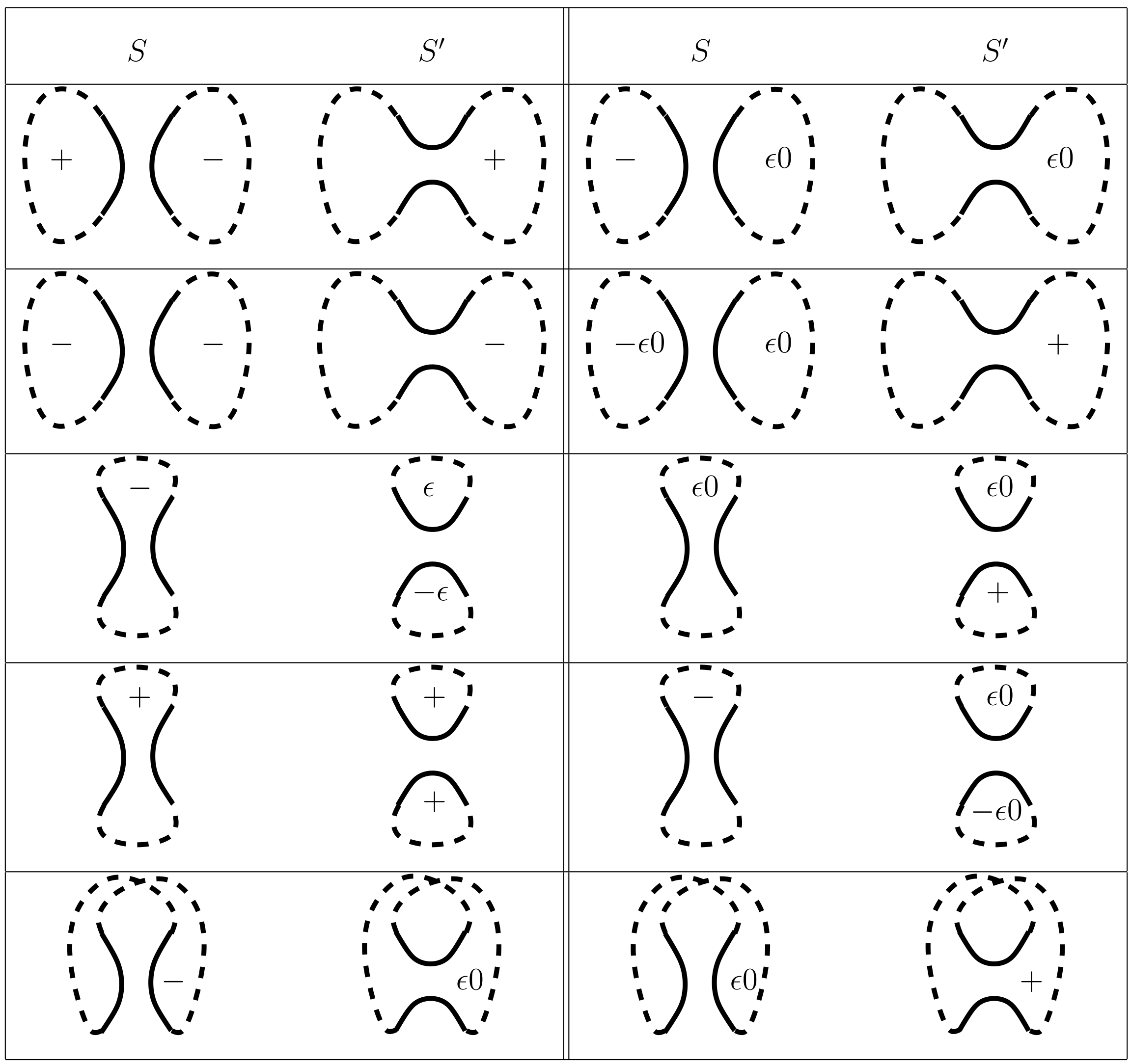
Then the differential $d : C_{i,j,s} \to C_{i-2,j,s}$ is defined by

$$d(S) = \sum_v \sum_{S'} [S : S']_v S' \tag{4.1}$$

Table 4.1 shows how the differential behaves near a crossing. S is the enhanced state smoothed positively at the crossing and S' is an enhanced state incident to S. A 0 indicates that the component is non-trivial and ϵ can be either $+$ or $-$.

Theorem 4.1 (Asaeda-Przytycki-Sikora)**.** *The homology theory constructed in this chapter is an invariant of band links in I-bundles.*

Table 4.1: The differential near a crossing.



CHAPTER 5

FOAMS

As discussed in Chapter 3, Khovavov homology came from a Frobenius extension. Bar-Natan developed a geometric version of the theory by using relations on surfaces coming from the TQFT associated to the Frobenius algebra. Asaeda, Przytycki and Sikora's theory does not have a Frobenius algebra or TQFT associated to it, so it was a natural question as to whether or not it could be realized geometrically. In the next chapter we accomplish that by using Bar-Natan's relations, in addition to some new ones.

After the geometric version of the APS theory is presented, there are two generalizations that are also explored. One that is quite complex, and one that may be a simpler alternative to the APS theory.

All three of these homology theories make use of surfaces embedded in the thickened surfaces, called foams. This chapter will introduce the foams and a few useful results. The subsequent chapters will build on this chapter in different ways to realize the different theories.

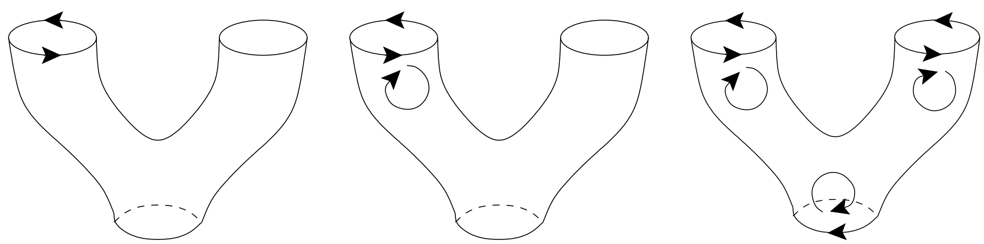
5.1 Definitions

Definition 5.1. Let S be a surface properly embedded in a 3-manifold N. A boundary circle of S is said to be *inessential* if it bounds a disk in N, otherwise it is said to be *essential*.

Definition 5.2. If S is an oriented surface and c is an oriented boundary component of S then the orientation of S is *compatible* with the orientation of c if the boundary orientation of c from S agrees with the orientation of c. Two oriented boundary curves of an orientable connected surface are *compatible* if both curves are compatible with the same orientation on the surface.

Remark 5.1. If S is a connected unoriented orientable surface and c is an oriented boundary component of S then there is exactly one orientation for the other boundary curves to be oriented compatibly with c. This is shown in Figure 5.1.

Figure 5.1: Compatible orientation of boundary curves.



Boundary is oriented | Compatible orientation on surface | Rest of boundary is oriented compatibly

Definition 5.3. A *surface link diagram* is an orientable surface together with a link diagram in the surface. The orientable surface will be referred to as the *underlying surface*.

Definition 5.4. A *state* of a surface link diagram D is a choice of smoothing at each crossing. Therefore a state is represented by a collection of disjoint simple closed curves in the underlying surface of D.

Definition 5.5. Let D be a surface link diagram with underlying surface F whose crossings are ordered. A *pre-foam* with respect to D has the following properties:

- A pre-foam is a compact surface properly embedded in $F \times I$.
- A pre-foam has a state of D as its boundary in the top ($F \times \{0\}$) and essential circles as its boundary in the bottom (F x $\{1\}$). There are no other boundary components.
- Pre-foams may be marked with dots.

Two pre-foams are equivalent if they are isotopic relative to the boundary. The dots on pre-foams are allowed to move freely within components but dots may not switch components.

Definition 5.6. A pre-foam is oriented if every essential curve on every component with boundary in the bottom is oriented compatibly. Inessential curves are not oriented and essential curves on components with no boundary in the bottom are not oriented.

Definition 5.7. Let M_D, (M_D^O) be the free $\mathbb{Z}$-module generated by pre-foams (oriented pre-foams) with respect to the surface link diagram D.

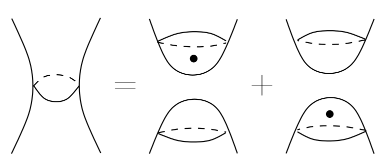
Remark 5.2. In order to continue, local relations need to be defined on M_D and M_D^O. The manner that this is done is to define a submodule of M_D, B, in order that the relations hold in M_D/B. Thus if we want $C = D$, then we have $C - D$ as a generator of B, so then in M_D/B, C is equivalent to D. These are skein relations, so if a relation is $P = P'$ it means that $S = S'$ in M_D/B if there is a 3-ball, A, in the manifold such that S and S' agree outside of A while $S \cap A = P$ and $S' \cap A = P'$.

Also if $z \in \mathbb{Z}$, then the relation $P = z$ means if a surface has P as a subsurface, then the original surface is equal to z times the surface where P is removed in M/B.

Definition 5.8. Let B be the submodule of M_D (M_D^O) generated by the following relations:

1. The neck-cutting relation as seen in Figure 5.2. (NC)

Figure 5.2: The NC relation.



2. An unmarked sphere bounding a ball equals zero. (SB)

3. A sphere with a dot bounding a ball equals one. (SD)

4. A component with two dots equals zero. (TD)

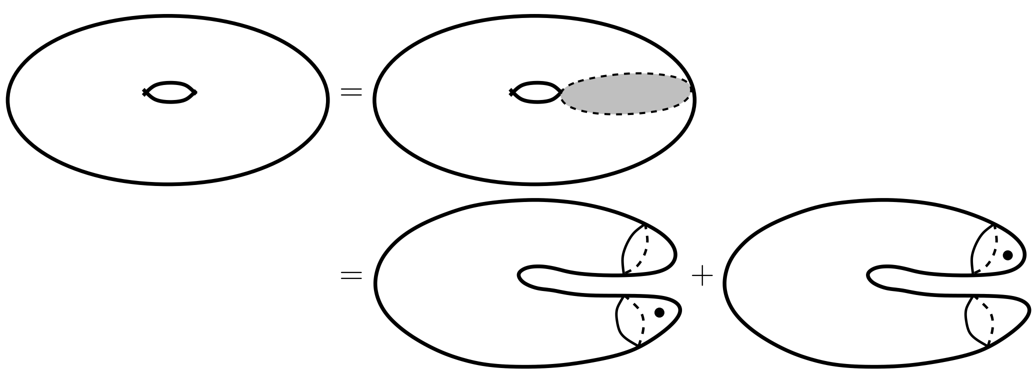
5. A surface with a non-disk, non-sphere incompressible component, that has a dot on that component equals zero. (NDD)

6. An incompressible non-orientable surface equals zero. (NOS)

Remark 5.3. Note that the first four relations are from Bar-Natan's version of Khovanov homology. Thus these relations come from the TQFT associated to the Frobenius extension $\mathbb{Z}[x]/(x^2)$.

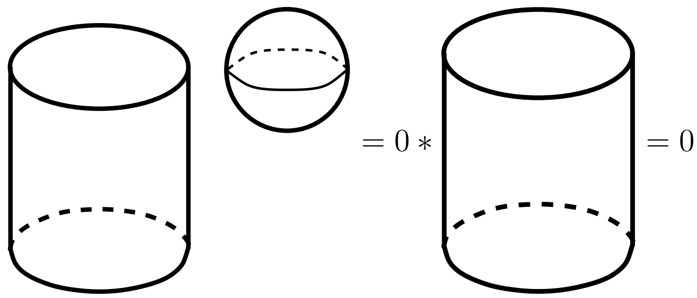
Example 5.1. The local relations can be confusing at first so we will illustrate the effects of the NC and SB relation. Note the NC Relation implies the equality in Figure 5.3 and the SB implies Figure 5.4.

Figure 5.3: The effect of the NC relation.



Definition 5.9. Elements of M_D/B (M_D^O/B) are called *foams* (*oriented foams*) with respect to the diagram D.

Figure 5.4: The effect of the SB relation.



5.2 Gradings

We will define three gradings on foams. The first two are similar to the two gradings from the original Khovanov homology and are defined below.

Definition 5.10. Let S be a state surface.

$I(S) = \{\#$ of positive smoothings in the state corresponding to the top boundary of $S\} - \{\#$ of negative smoothings in the state corresponding to the top boundary of $S\}$

$J(S) = I(S) + 2(2d - \chi(S))$ where d is the number of dots on S.

The third index corresponds to the oriented essential disjoint simple closed curves in the bottom of F x I. Note that given two parallel oriented simple closed curves on a surface it is not difficult to determine if their orientations agree or disagree. To define the third index it is necessary to specify that one of the orientations possible is the positive one and the opposite orientation is the negative one. Thus for each homotopy class of simple closed curve we have chosen a positive orientation and a negative orientation.

Remark 5.4. If F is planar it is often convenient to consider the counter-clockwise orientation to be positive and the clockwise orientation to be negative.

Definition 5.11. Let $\gamma_1, \ldots, \gamma_n$ be a family of disjoint simple closed curves in the bottom of a state surface S. If γ_i and γ_j are parallel then $\gamma_i = \gamma_j$. Then

$K(S) = \sum_{i=1}^{n} k_i \gamma_i$, where

$$k_i = \begin{cases} 1, & \text{if } \gamma_i \text{ is oriented in the positive direction} \\ -1, & \text{if } \gamma_i \text{ is oriented in the negative direction} \end{cases}$$

Theorem 5.5. *The gradings respect the relations that generate B.*

Proof. Consider the fact that any relation that sets something equal to zero respects the grading, since the zero element can have any grading. This means we only need to consider the NC and SD relations.

Note that neither of these relations affect the state of the foam, so the index $I(S)$ is not affected by these relations. Also, neither of these relations affect the curves in the bottom of the foam, so the K index is also not affected by the relations.

To consider the NC relation, first note that when a neck is cut the Euler characteristic goes up by two. Also, each summand adds a dot, thus $2d - \chi(S)$ remains the same.

A sphere has Euler characteristic two, and note if the sphere has a dot, then $2d - \chi(S) = 2 - 2 = 0$, so removing a sphere with a dot does not affect the $J(S)$ index.

□

Lemma 5.6. *$I(S) - 2 = I(b_p(S))$, $J(S) = J(b_p(S))$ and $K(S) = K(b_p(S))$.*

Proof. $I(S)$ has one more positive smoothing and one less negative smoothing than $I(b_p(S))$, so $I(S) - 2 = I(b_p(S))$.

Also, $J(S) = I(S) + 2(2d - \chi(S)) = I(b_p(S)) + 2 + 2(2d - (\chi((b_p(S)) + 1)) = I(b_p(S)) + 2(2d - (\chi((b_p(S)))) = J(b_p(S))$.

The K grading is not affected by placing a bridge, so $K(S) = K(b_p(S))$.

□

5.3 Observations about foams

We will now make some observations about foams that will be useful later.

Lemma 5.7. *A foam with a dotted component that has an essential boundary component is trivial in the quotient.*

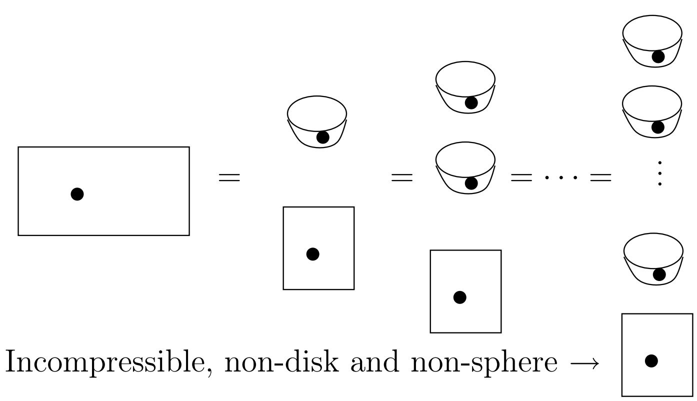
Proof. If the dotted component is incompressible we are done by the NDD relation. If not, then apply the NC relation to this component. If the compressing disk was non-separating, then the result of compressing is a component with two dots, thus it is trivial in the quotient by the TD relation.

If the compressing disk is separating the result of the NC relation is two components. One of them has an essential boundary curve since the NC relation does not affect boundary components. Therefore one of the components is not a disk, but they both have dots.

We are now left with two components, each with dots, and one of them has an essential boundary component.

Note the component that has an essential boundary curve and a dot satisfies the assumptions of the Lemma. Thus we may continue in this manner. Note that each time the NC relation is applied the Euler characteristic goes up by two. Since we are dealing with compact surfaces the Euler characteristic is bounded above, thus the NC relation can only be applied a finite number of times. When the NC relation can no longer be applied there is an incompressible component with a dot that is not a disk or a sphere.

Figure 5.5: Process of compressing to a trivial element.



It can be seen in Figure 5.5 that the original surface is equal to a foam that has an incompressible non-disk and non-sphere component with a dot, which is trivial in the quotient by the NDD relation.

□

Lemma 5.8. *Foams with closed dotted non-sphere components are trivial.*

Proof. If the closed component is incompressible and it has a dot then it is trivial in the quotient, by the NDD relation.

If the closed component is compressible, then the NC relation can be applied. If the compressing disk is non-separating then after the NC relation is applied the new closed component has two dots and is thus trivial in the quotient.

If the compressing disk is separating then we are left with a sum of closed non-sphere surfaces, one of which has a dot in each summand. Note that each summand satisfies the assumptions of this Lemma. Since we are dealing with compact foams and the Euler characteristic is bounded above the process will eventually terminate, as in Lemma 5.7 .

□

Corollary 5.9. *If a foam has a component that is compressible with no inessential boundary curves then it is trivial in the quotient.*

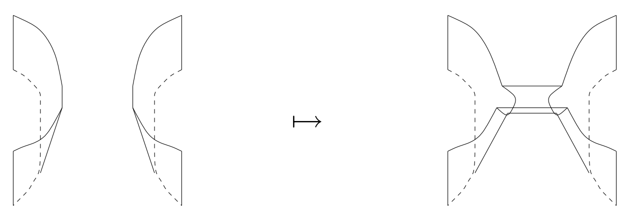
Proof. Since the component is compressible the NC relation can be applied. After the NC relation is applied the only surfaces present are closed surfaces of non-zero genus or surfaces with essential boundary curves. One of these has a dot in each summand and thus the surface is trivial in the quotient by Lemma 5.7.

□

5.4 An operation on foams

Here we define an operation on foams that will lead to the boundary operator in the following chapters. The operation is called *placing a bridge* at a crossing and is shown in Figure 5.6. When a bridge is placed on a foam with oriented boundary the orientation may be affected and we address that below. In fact, orientation sometimes prevents a bridge from being placed. This is called *EO* and is outlined in Definition 5.12. Notationally, $b_p(S)$ will be the foam S, with a bridge placed at the p^{th} crossing.

Figure 5.6: Placing a bridge at the p^{th} crossing.

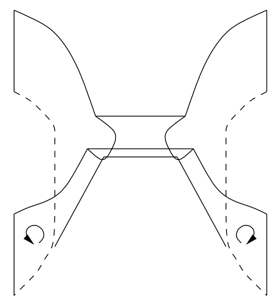


S at p^{th} crossing After bridge is placed

Remark 5.10. Note that the operation of placing a bridge changes the smoothing of one crossing of the associated Kauffman state of the diagram.

Definition 5.12. When a bridge is placed at the p^{th} crossing of a foam S and connects two oriented components, the orientation from one side of the bridge can be slid along the bridge to the other side of the bridge. If the orientation that is slid across the bridge disagrees with existing orientation, then *EO* is said to occur at the p^{th} crossing at S. An example of where EO occurs is shown in Figure 5.7.

Figure 5.7: This is an example of when EO occurs.



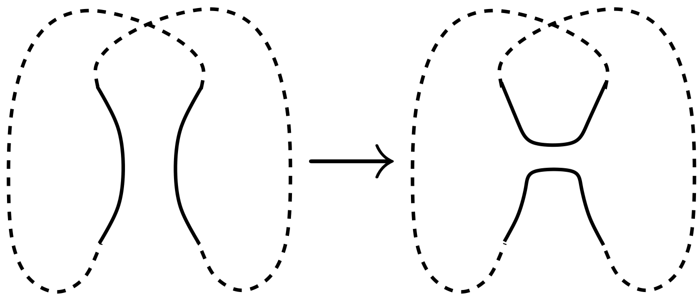
Remark 5.11. If EO does not occur at the p^{th} crossing, then the components are oriented compatibly. Note that $b_p(S)$ is oriented so that all essential curves are oriented compatibly with the existing orientation on the components and inessential components are unoriented.

Lemma 5.12. *If placing a bridge does not affect the number of boundary curves then the resulting foam is trivial in the quotient.*

Proof. Assume placing a bridge turns one boundary curve into one boundary curve. The only way this can happen is if the curves are as they are in Figure 5.8. This is due to the fact that if the upper left connected to the upper right placing a bridge results in two curves, and if the upper left connects to the lower left then we are starting with two curves.

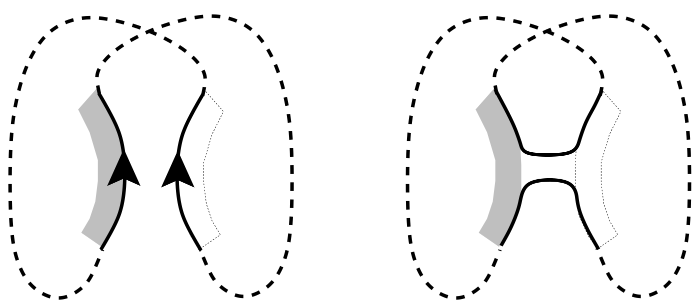
Now note the boundary curve is a circle, so we can place an orientation on it

Figure 5.8: A crossing where the number of boundary curves doesn't change when a bridge is placed.



from the boundary orientation of the surface. Since the original surface is orientable we can color the side to the left of the circle (if we face the direction the arrow is pointing) and leave the other side blank as in Figure 5.9. Thus each side of the component is determined to be dark or blank by the orientation of the curve.

Figure 5.9: The same crossing with one side of the surface shaded.



Surface is shaded based on orientation of curve

A non-orientable surface is created

Now note when a bridge is placed it must connect a dark side to a blank side, thus resulting in a non-orientable surface.

The new surface may or may not be compressible. However, after compressing the component with the boundary curve we are looking at, the surface will still have a Möbius band embedded in it.

□

Theorem 5.13. *b_p is a well-defined map on foams.*

Proof. Note the boundary operator is defined on the original module but it actually operates on a quotient of that module. Thus it needs to be verified that two representations of the same class go to the same class under the boundary operator. Thus assume $[S] = [S']$ in our quotient. Therefore $S - S' \in B$. So we must show that $d(S) - d(S') \in B$. Since d is linear this is equivalent to showing $d(S - S') \in B$. Thus it is sufficient to show that given $b \in B$, that $d(b) \in B$.

We treat each relation generating B below.

- SB, SD and TD

  Note that if a foam has satisfies any of these conditions before placing a bridge it will satisfy them after as well as spheres and dots are not affected by placing a bridge.

- NC

  It must be shown that if foams are related by the neck-cutting relation before applying the boundary operator they are related after applying the boundary operator as well. Placing a bridge does not remove any compressing disks, so the only issue is how orientations are affected. Lemma 5.14 shows if the orientations agree before applying the boundary operator, they agree after applying it as well.

- NDD

  Assume the boundary operator is applied to an incompressible non-disk dotted component (thus this component has Euler characteristic less than or equal to zero). After placing the bridge the new connected component has Euler characteristic less than or equal to negative one and it has a dot. If this new component has an essential boundary component then it is trivial in the quotient by Lemma 5.7.

  Otherwise this component has no essential boundary components and so all boundary components must be inessential. Since it was originally incompressible it must have only one inessential boundary component after placing a bridge. There is a compressing disk present since this component is not a disk, but has an inessential boundary curve. This compressing disk is separating and compressing upon it yields a disk and a closed surface with a dot. This closed surface is not a sphere since there was a compressing disk. The entire foam is now trivial in the quotient by Lemma 5.8.

- NOS

  Assume the boundary operator is applied to a k-foam that has an incompressible non-orientable component. After placing the bridge the component is still non-orientable because a bridge will not change the fact that the component is one-sided. If inessential curves are created, the neck-cutting relation can be applied to separate them from the rest of the component. Notice that this operation still leaves the component non-orientable. If the non-orientable component is compressible the the entire foam is zero by Corollary 5.9, since there are no remaining inessential curves. If the component is now incompressible, then the foam is zero by the NOS relation.

Since we have shown that elements of B remain in B after placing a bridge, we can conclude that b_p is well-defined on foams.

□

Lemma 5.14. *Placing a bridge together with the NC relation is well-defined with regard to orientations of foams, i.e. the operation of placing a bridge and cutting a neck commute.*

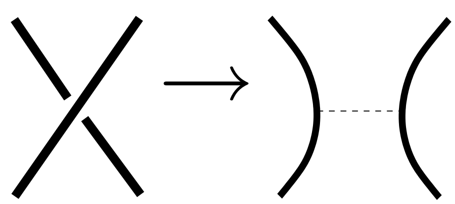
Proof. Note that on the left side of the NC relation boundary orientation from essential boundary circles are forced on other essential boundary circles through the neck. However, on the right side of the relation all essential boundary curves do not need to be compatible with one another since they may no longer lie on the same connected component.

Orientation can only be forced in the left side of the NC relation if each component has an essential boundary component on the right-hand side. Note one component has a dot in each summand, so by Lemma 5.7 both sides are trivial in the quotient. Thus orientations may differ, but the foams are trivial, and therefore equal in the quotient.

□

Definition 5.13. Two boundary curves are said to be *connected* if they are connected by a former crossing smoothed positively as in Figure 5.10.

Figure 5.10: The two curves on the right are said to be connected.



Lemma 5.15. *Let S be an oriented foam and a and b are crossings in the associated diagram. Then $b_a(b_b(S))$ has the same orientation on boundary curves as $b_b(b_a(S))$.*

Proof. The lemma is immediate if the affected components have more than two essential curves. This is due to the fact that at most two essential curves can become inessential when a bridge is placed. On the other hand, if all essential curves are eliminated, then no curves are oriented, so the orientations necessarily agree.

We can then conclude that one of the orderings of crossings has no essential curves on the affected components after one crossing is changed. Without loss of generality assume $b_a(S)$ has no essential curves. Thus $b_a(S)$ has one boundary curve and it is inessential. Applying the NC relation to a curve parallel to the inessential curve yields a disk and a closed surface in a sum with the dots distributed appropriately. Note the closed surface with a dot is not a sphere and thus is trivial in the quotient by Lemma 5.8. Thus the only surface left is a closed surface and a disk with a dot. The boundary curve on the disk cannot become essential, since then the foam would be trivial as it has a dot, by Lemma 5.7. Since only essential curves can be oriented, the boundary curve cannot become oriented either. Therefore $b_b(b_a(S))$ is either trivial or unoriented, and in either case the orientation agrees with that of $b_a(b_b(S))$.

□

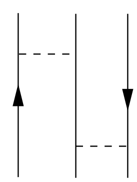
Lemma 5.16. *Let a and b be two crossings of a given diagram and let S be an oriented foam. If $b_a(b_b(S)) = 0$ because of EO, then $b_b(b_a(S))$ is a trivial foam.*

Proof. Assume $b_a(b_b(S)) = 0$ because of EO. This means at least two oriented boundary curves must be present in the foam and these oriented curves can be, at most, two components away from one another (i.e. both connected to the same boundary component.)

First assume the oriented curves are connected. Then assume EO occurs at b. If a bridge is placed at a then the curves connected by b can either continue to have conflicting orientation, or one of them may become a curve on a component with a dot (after applying NC). This is due to the fact that by the orientation rules of b_p, oriented curves can only lose their orientation, not reverse their orientation, and losing orientation only happens when a curve becomes inessential. In either case placing a bridge at b after a results in something trivial in the quotient.

Now assume the oriented curves are both connected to an intermediate curve. Assume without loss of generality this curve is unoriented as in Figure 5.11. (If it is oriented, it cannot be compatible with both curves, since then EO would not occur, so we can reduce to the previous case.) Placing a bridge at either of the crossings orients the previously unoriented curve in such a way that it is not compatible with the other curve. Thus when the second bridge is placed in either order there is EO, so $b_b(b_a(S))$ is a trivial k-foam.

Figure 5.11: Two oriented curves connected to the same unoriented curve.



□

Remark 5.17. Here are two observations that will be helpful later:

1. Bridging an essential boundary curve to itself can produce at most one inessential boundary curve at a time. This is due to the fact that if two inessential boundary curves are created from one curve, then the original curve was also inessential.

2. If a component has an inessential boundary component then either this compo-

nent is a disk or it is compressible. (Just push the disk the curve bounds into $F \times I$ to obtain a compressing disk.)

CHAPTER 6

A GEOMETRIC INTERPRETATION

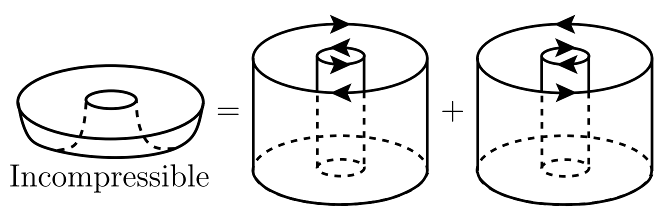
6.1 New Relations

In order to realize the APS theory using foams we need to add in four more relations.

Let $A \subset M_D^O$ be the submodule generated by the following relations:

- A surface with an incompressible component with negative Euler characteristic equals zero. (NEC)
- If the annulus in Figure 6.1 on the left side of the equation is incompressible then we have the relation in Figure 6.1. (UTA)

Figure 6.1: The UTA relation.



- An annulus with its boundary completely in the bottom equals one. (BDA)
- An Incompressible torus equals zero. (IT)

Elements of $M_D^O/(B \cup A)$ are called A-foams with respect to the diagram D.

We need to check that the gradings are well-defined in the quotient. This is addressed in Lemma 6.1.

Lemma 6.1. *The indices $I(S)$, $J(S)$ and $K(S)$ are well-defined for A-foams.*

Proof. By Theorem 5.5, we only need to show that the indices are well-defined for the four additional relations, UTA, BDA, NEC and IT. As before we only need to consider relations that do not send something to zero. Thus we only need to address the UTA and BDA relations.

When considering the UTA relation, note that both sides of the equality have annuli without dots which do not contribute dots or Euler characteristic, thus the $J(S)$ grading on each side of the equality is the same.

We may also remove an annulus without a dot, since an annulus has Euler characteristic zero, so then $2d - \chi(S) = 0$. Thus the BDA relation is well-defined.

Note that a bottom annulus has two essential homotopic curves. Since these curves are in the bottom they must be oriented and they must be oriented in a way that is compatible with each other. Thus they must be oriented in the opposite direction of one other. Therefore in the $K(S)$ index, these curves cancel out in the sum, so the annulus does not contribute to the $K(S)$ index, so we may remove the annulus without affecting the K grading.

For the UTA relation the left side of the equation has no curves in the bottom and in each summand on the right side there are homotopic curves oriented in the opposite direction. Since these curves have opposite orientation, they cancel out in the sum that determines $K(S)$, so neither side of the equality contributes to the $K(S)$

grading.

□

Definition 6.1. Let $C^A_{i,j,s}(D)$ be the submodule of P generated by all A-foams, S, such that $I(S) = i$, $J(S) = j$ and $K(S) = s$.

Theorem 6.2. *The collection of A-foams is generated by foams whose components are vertical annuli and disks.*

Proof. For the purpose of this proof, let $g(S)$ be the genus of S and $b(S)$ be the number of boundary components of S.

By the NC relation, A-foams are generated by incompressible foams. By the NEC relation, all components have non-negative Euler characteristic. Since we are restricted to orientable surfaces and by the fact that $\chi(S) = 2 - 2g(S) - b(S)$, we can conclude that each component must satisfy the condition that $2 - 2g(S) - b(S) \geq 0$. Thus, $2 \geq 2g(S) + b(S)$. So if g(S) = 1, b(S) = 0, so S is a torus. If S is an incompressible torus, then it is zero by the IT relation.

If $g(S) = 0$, then $b(S) = 0$, 1 or 2. Thus S is either a sphere, disk or annulus. Thickened surfaces are irreducible, so sphere can be addressed by relations. Annuli with their boundary in the top are replaced by vertical annuli using the UTA relation. Annuli with their boundary completely in the bottom zero if they have a dot and 1 without a dot, by the BDA relation.

□

6.2 The Boundary Operator

We define

$$d_p^A(S) = \begin{cases} 0, & \text{if } p^{\text{th}} \text{ crossing of } S \text{ is smoothed negatively} \\ 0, & \text{if EO occurs} \\ b_p(S), & \text{otherwise} \end{cases}$$

Then we have the differential $d^A : C^A_{i,j,s}(D) \to C^A_{i-2,j,s}(D)$, defined by

$$d^A(S) = \sum_{p \text{ a crossing of } D} (-1)^{t(S,p)} d_p^A(S), \tag{6.1}$$

where $t(S,p) = |\{\ j$ a crossing of $D :\ j$ is after p in the ordering of crossings and j is smoothed negatively in boundary state of S$\}|$

We need the following two lemmas in order to show that d^A is well-defined on A-foams.

Lemma 6.3. *Let S be a foam. If placing a bridge on S turns two essential boundary curves into one essential boundary curve, or the placing of a bridge on S turns one essential boundary curve into two essential curves then the result of placing this bridge on S is a foam that is trivial in the quotient.*

Proof. By the relations we may assume we are starting with vertical annuli. Thus if two essential boundary curves are turned into one essential boundary curve, we are left with a component with three essential curves and Euler characteristic of negative one. Note a surface that has three boundary components and Euler characteristic negative one is a disk with two holes. Since all of the boundary components are essential, this twice-punctured disk is incompressible. Then since it has negative Euler characteristic it is trivial in the quotient.

□

Lemma 6.4. *If two non-homotopic essential curves are bridged together, then the new foam is trivial in the quotient.*

Proof. If two non-homotopic essential boundary curves are bridged together, then the result is one essential boundary curve since it could only be inessential if the original curves were homotopic. Thus by Lemma 6.3, this new foam is trivial in the quotient.

□

Theorem 6.5. *d^A is well-defined on A-foams.*

Proof. We must address the four additional relations on A-foams.

1. BDA

   A bridge can't be placed in the bottom, thus the annulus will not affect the boundary operator. Therefore the resulting foams will still be equivalent in the quotient as everything else will be affected in the same manner under the boundary operator.

2. NEC

   Assume the boundary operator is applied to a A-foam that has an incompressible component with negative Euler characteristic. When the bridge is placed the new component has Euler characteristic less than or equal to -2. Note that at most one inessential boundary curve can be created when a bridge is placed since we are starting with no inessential curves as the surface is incompressible.

If no inessential curves are created, then all curves are essential. If the component is incompressible, we are done by NEC. If the component is compressible then we are done by Corollary 5.9.

If an inessential curve is created, then apply the NC relation to yield a disk. The total Euler characteristic increases by two as a result, but since one component is disk, the other component has Euler characteristic at most -1. The summand where the non-disk surface has a dot is trivial by Lemma 5.7. Thus we only need to consider the other summand

If the non-disk surface is incompressible, the A-foam is trivial. If the non-disk is compressible then the foam is trivial by Corollary 5.9.

3. UTA

Based on the fact that placing a bridge is well-defined on foams we may assume we are bridging between incompressible components. We do not need to consider if we bridge to an incompressible component with negative Euler characteristic or anything that isn't a disk, but has a dot, as these cases have already been addressed. Thus we may consider that to begin with all components are incompressible annuli without dots or disks with one or no dots.

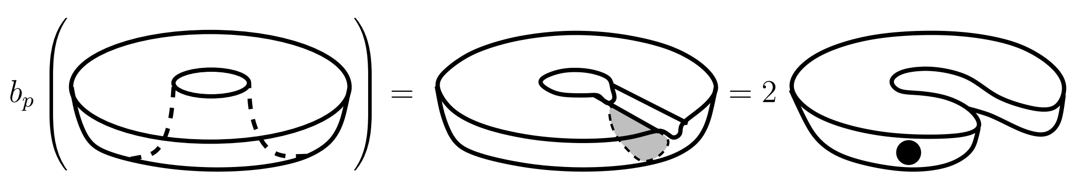
We will begin by looking at all the bridges we can place on the top annulus on the left side of the UTA relation. First look at all the bridges we can place on the outside curve. We can connect to the inside curve or itself. We can also connect to another essential or inessential curve. The essential curve could be part of a vertical or top annulus. The annulus could be next to it, or these annuli could be nested. However, there is only one way they could be nested such that the UTA relation would apply.

It is also necessary to consider all ways to place bridges on the inside curve but this is symmetric to what we are already considering.

(a) The annulus is bridged to itself.

The result of placing a bridge on the left side of the UTA relation in this case is shown in Figure 6.2 and the right side of the UTA relation is shown in Figure 6.3. Both sides results in 2 times a disk with a dot.

Figure 6.2: Annulus bridging to itself, left side.



(b) Nested annuli

The results of placing a bridge on both sides of the UTA relation in this case are shown in Figures 6.4 and 6.5.

(c) Next door annuli

Figure 6.3: Annulus bridging to itself, right side.

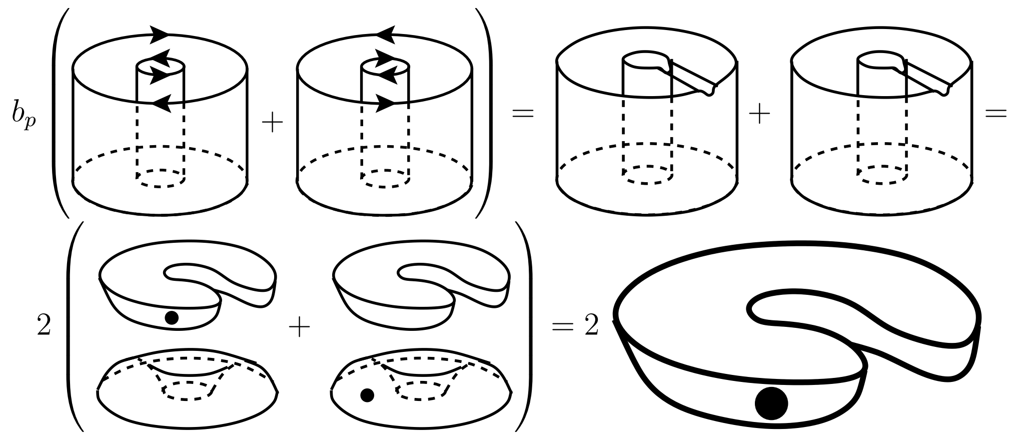
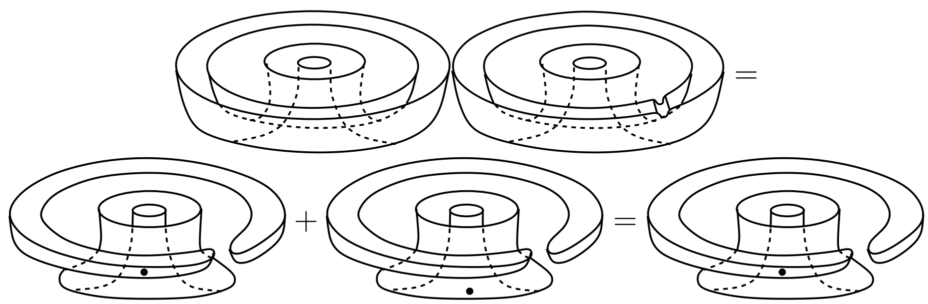


Figure 6.4: Nested annuli, left side.



The results of placing a bridge on both sides of the UTA relation in this case are shown in Figures 6.6 and 6.7.

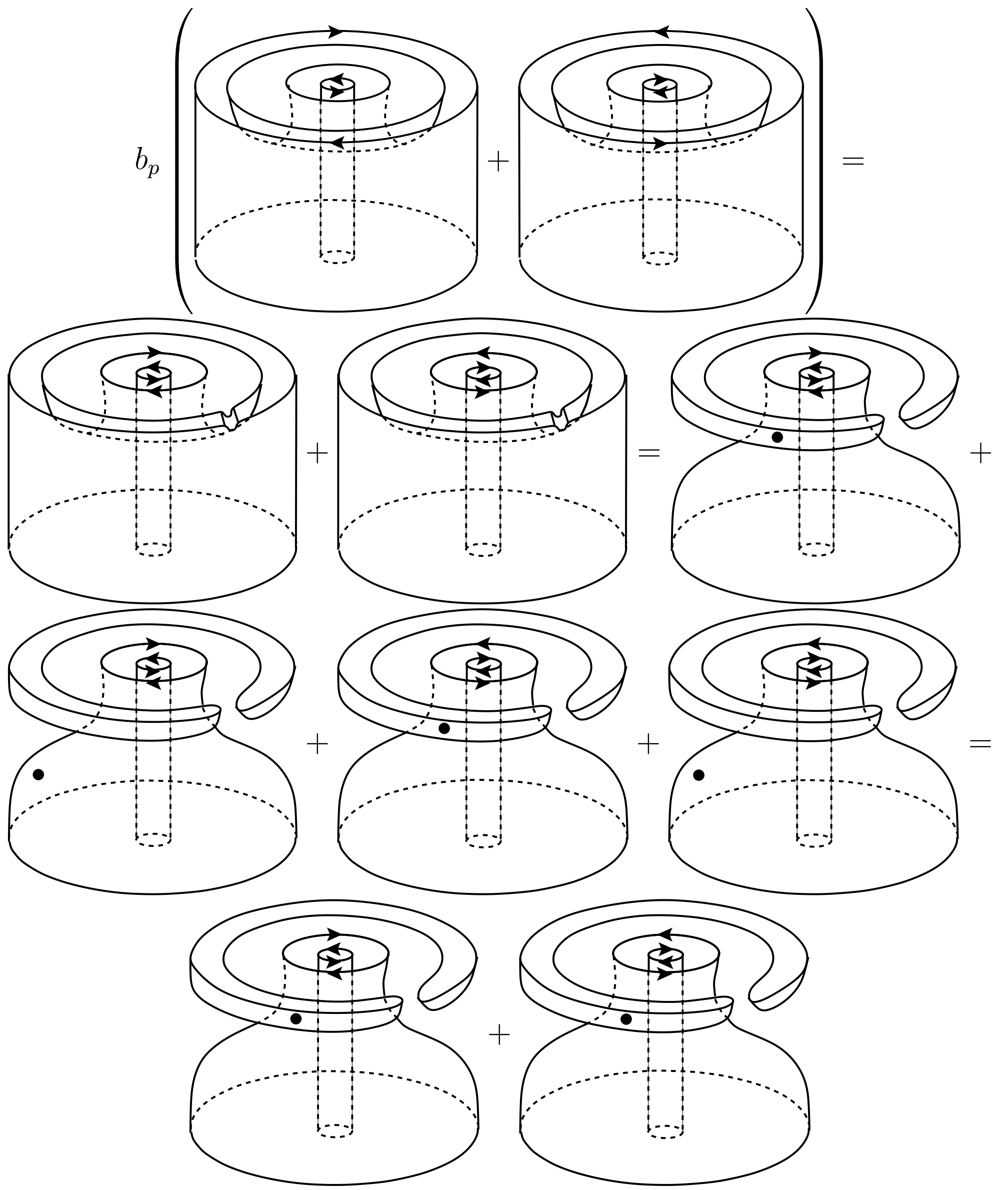
(d) The annulus on the left side of the UTA relation is bridged to disk.

This case is immediate since bridging an annulus to a disk results in an annulus isotopic to the original annulus.

(e) The annulus is bridged to an oriented vertical annulus.

If the annuli are not nested, we get zero on both sides again by Lemma

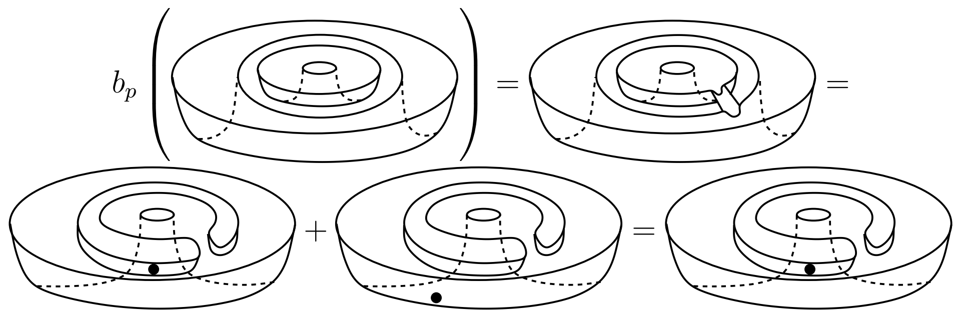
Figure 6.5: Nested annuli, right side.



6.4.

Thus, assume annuli are nested. The left side of the UTA relation is shown in Figure 6.8 and the right side of the UTA relation is shown in Figure 6.9. Upon inspection, it can be seen that the two sides coincide.

Figure 6.6: Next door annuli, left side.



4. IT

   After a bridge is placed on a foam that has an incompressible torus component it still has an incompressible torus component, so after placing a bridge the foam is still in B.

Thus when the boundary operator is applied to elements of B, they remain in B, so by the remarks at the beginning of this proof, the boundary operator is well-defined on the quotient.

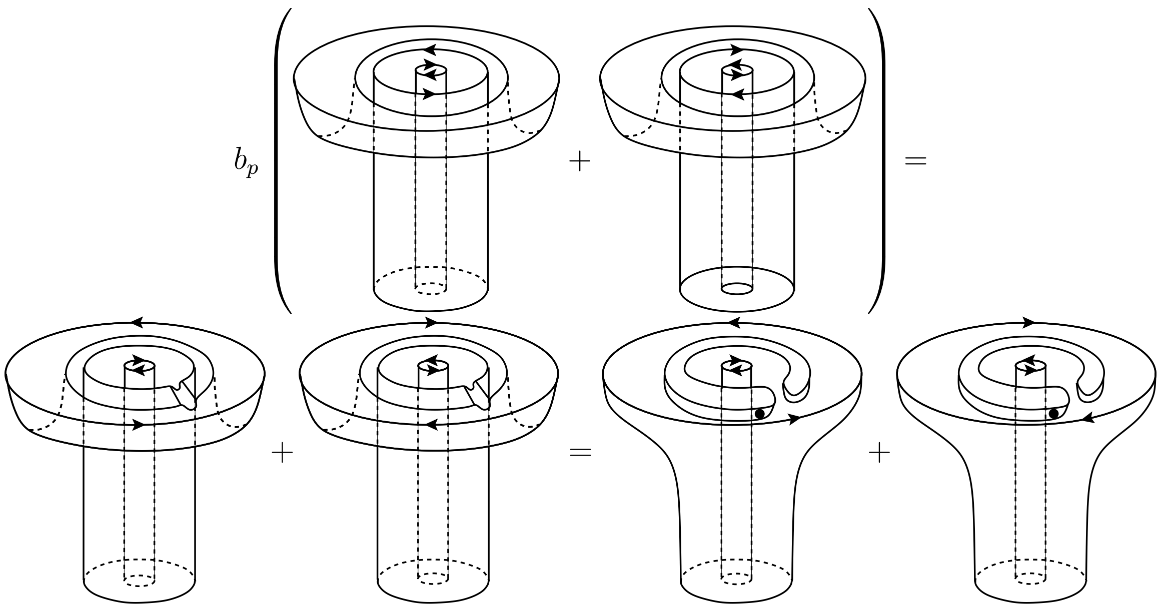
□

6.3 $d^A \circ d^A = 0$

Theorem 6.6. $d^A \circ d^A = 0$

Proof. Note that by how the negative signs are distributed in the definition of the boundary operator all that needs to be shown is that given two crossings a and b and a foam S, that $d_a^A(d_b^A(S)) = d_b^A(d_a^A(S))$. This is the case if EO does not occur since the resulting foams are isotopic and by Lemma 5.15 we know the orientations agree.

Figure 6.7: Next door annuli, right side.



By Lemma 5.16 we know that if EO occurs for one order of a and b, then $d_a^A(d_b^A(S))$ and $d_b^A(d_a^A(S))$ are both trivial in the quotient.

Thus the partials always commute, so after adding in the appropriate negative signs, $d^A \circ d^A = 0$. □

6.4 Equating the Homology Theories

Let p be a crossing of the diagram D. Consider the skein triple in F in Figure 6.10.

Now define:

$\alpha_0 : C^A_{i,j,s}(D_\infty) \to C^A_{i-1,j-1,s}(D_p)$ is the the natural embedding as depicted in Figure 6.11.

$\beta : C^A_{i,j,s}(D_p) \to C^A_{i-1,j-1,s}(D_0)$ is the natural projection where foams with a

Figure 6.8: Bridge to vertical annulus, left side.

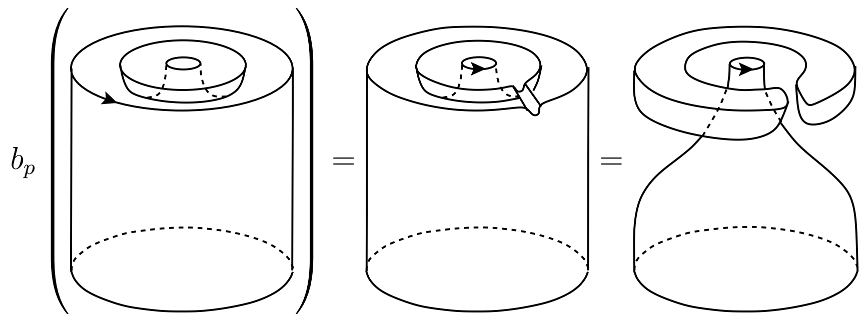
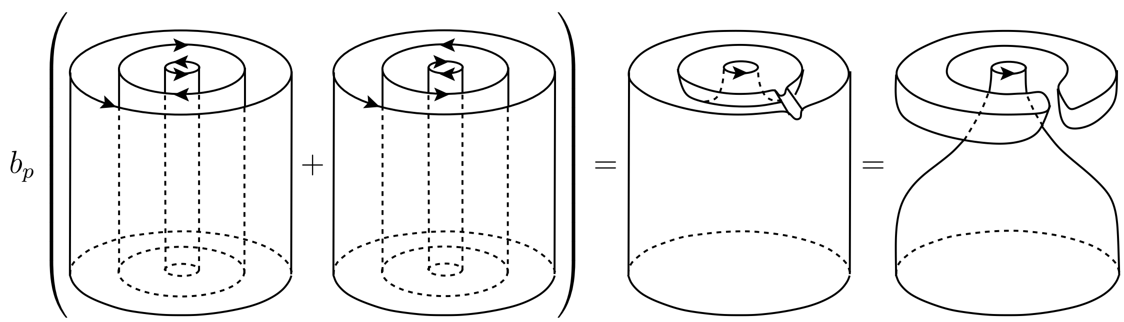


Figure 6.9: Bridge to vertical annulus, right side.



negative smoothing at p are sent to 0 and foams with a positive smoothing are affected as in Figure 6.12.

Let $\alpha : C^A_{i,j,s}(D_\infty) \to C^A_{i-1,j-1,s}(D_p)$ be defined by $\alpha(S) = (-1)^{t'(S)}\alpha_0(S)$ where $t'(S) = |\{$ j a crossing of D : j is before p in the ordering of the crossings of D and j is smoothed negatively in boundary state of S$\}|$

Theorem 6.7 ([7])**.** *α and β are chain maps and the sequence*

$$0 \to C^A_{i+1,j+1,s}(D_\infty) \xrightarrow{\alpha} C^A_{i,j,s}(D_p) \xrightarrow{\beta} C^A_{i-1,j-1,s}(D_0) \to 0$$

is exact.

Figure 6.10: The skein triple.

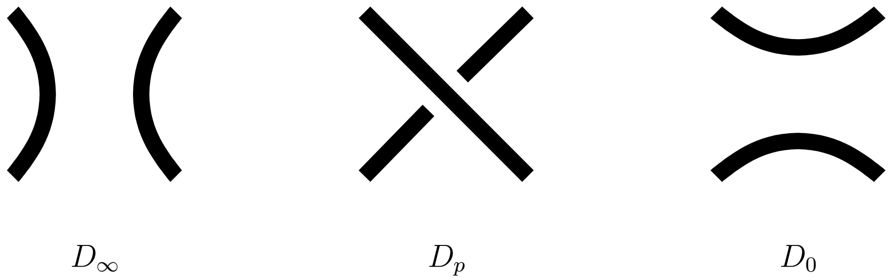
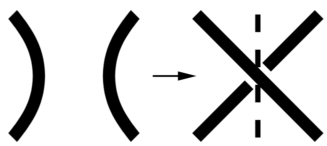


Figure 6.11: The α chain map.

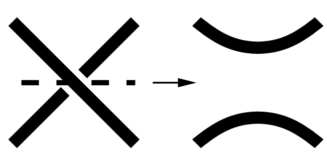


Proof. This proof is adapted from the proof in [1]. First note that $d_p^A(\alpha(S)) = 0$ since $\alpha(S)$ has a negative smoothing at the p-th crossing, so the boundary operator in the p direction is always 0.

Let $\hat{d}_q = (-1)^{t(S,q)} d_q^A$.

Let $S_q = d_q^A(S)$. Note that if $q \neq p$ then $d_q^A(S)$ is topologically the same in

Figure 6.12: The β chain map.



$C^A_{i,j,s}(D_\infty)$ and $C^A_{i,j,s}(D_p)$, thus α_0 and d^A_q commute.

Also note that either $t'(S)+1 = t'(S_q)$ or $t'(S) = t'(S_q)$ depending on whether p or q comes first in the ordering of crossings. Consider that if $t'(S)+1 = t'(S_q)$ then $t(\alpha_0(S), q) + 1 = t(S, q)$ and if $t'(S) = t'(S_q)$, then $t(\alpha_0(S), q) = t(S, q)$. So, in either case $t(\alpha_0(S), q) + t'(S) = t'(S_q) + t(S, q)$ mod 2.

Then note

$$\hat{d}_q(\alpha(S)) = \hat{d}_q((-1)^{t'(S)}\alpha_0(S)) = (-1)^{t'(S)}\hat{d}_q(\alpha_0(S))$$

$$= (-1)^{t'(S)}(-1)^{t(\alpha_0(S),q)}{d^A}_q(\alpha_0(S)) = (-1)^{t'(S)+t(\alpha_0(S),q)}{d^A}_q(\alpha_0(S))$$

$$= (-1)^{t'(S_q)+t(S,q)}\alpha_0({d^A}_q(S)) = (-1)^{t'(S_q)}\alpha_0((-1)^{t(S,q)}{d^A}_q(S)) = \alpha(\hat{d}_q(S))$$

Then we have $d^A(\alpha(S)) = \sum_{q\neq p}\hat{d}_q(\alpha(S)) = \alpha(\sum_{q\neq p}\hat{d}_q(S)) = \alpha(d^A(S))$, so α is a chain map.

Now consider β. Note $\beta(d^A_p(S)) = 0$ since $d^A_p(S)$ is smoothed negatively at p and β sends foams with the negative smoothing at p to 0. Also, for $q \neq p$ $d^A_q(\beta(S)) = \beta(d^A_q(S))$ since the bridge is placed away from p, so the result is the same. Also, β doesn't change the number or placement of negative crossings, so we have $\hat{d}_q\beta = \beta\hat{d}_q$. Then $d^A(\beta(S)) = \beta(d^A(S))$ and thus β is a chain map.

Now the exactness of the sequence is addressed. Since α is an embedding it is 1-1. The image of α is all foams in $C^A_{i,j,s}(D_p)$ that have a state as the top boundary smoothed negatively at p. The kernel of β is precisely these foams. Since β is a projection, it is onto $C^A_{i-1,j-1,s}(D_0)$. Thus the sequence is exact.

□

Let $\bar{C}_{i,j,s}(D)$ be the chain groups defined in [1] and $\bar{H}_{i,j,s}(D)$ be the homology groups defined in [1].

In [1] a circle in F is said to be trivial if it bounds a disk in F and non-trivial otherwise. Thus when referring to circles of an enhanced state coming from [1] these terms will be used. Also note that trivial circles correspond to inessential boundary curves in the top and non-trivial circles correspond to essential circles in the top.

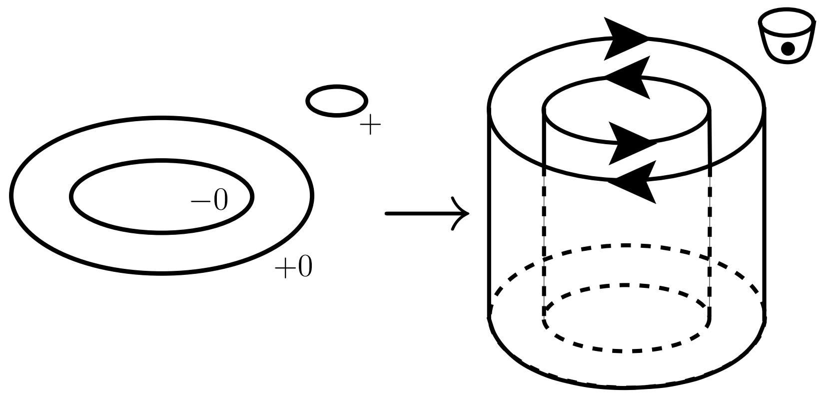
Definition 6.2. $\Phi : \bar{C}_{i,j,s} \to C^A_{i,j,s}$ is defined by taking an enhanced state in $\bar{C}$ and changing each circle as determined in Table 6.1 to get a foam with the same state in C^A.

Table 6.1: The Φ map.

Trivial circle marked with a $+$	$\to$	disk with a dot
Trivial circle marked with a $-$	$\to$	disk without a dot
Nontrivial circle marked with a $+0$	$\to$	vertical annulus with the pos. orient.
Nontrivial circle marked with a -0	$\to$	vertical annulus with the neg. orient.

Example 6.1. Figure 6.13 shows an example of how the Φ map affects an enhanced state from [1].

Figure 6.13: The Φ map.



Lemma 6.8. $\Phi : \bar{C}_{i,j,s} \rightarrow C^A_{i,j,s}$ *is a chain map* $\forall i, j, s$.

Proof. To show Φ is a chain map, we need to show $\Phi\tilde{d^A} = d\Phi$. Where $\tilde{d}$ is the boundary operator coming from [1]. First it will be shown that $\Phi\tilde{d_i} = d^A_i\Phi\ \forall i$.

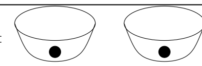
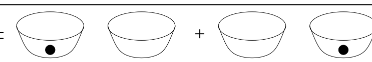
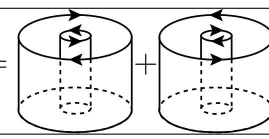
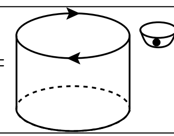
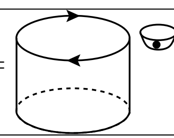
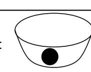
This is treated by cases. In the [1] theory a trivial circle may have a $+$ or a $-$ and a non-trivial circle may have a $+0$ or a -0. Thus a $+$ will refer to a trivial circle marked with a $+$ and a $+0$ will refer to a non-trivial circle marked with a $+0$. The notation is similar for $-$ and -0. If the marking on a circle is not specified then T refers to a trivial circle and N refers to a non-trival circle. Based on how trivial and non-trivial curves may change when a smoothing changes the possible cases are outlined in Table 6.2.

The boundary operator from [1] is determined by how circles change (with respect to being trivial and nontrivial) when a smoothing is switched. Tables 6.3 and 6.4 shows what the partial boundary operator for the theory coming from [1] does in

Table 6.2: The cases that need to be checked.

How curves may change	Possible initial markings	Possible outcomes	Number of possibilities
T → TT or NN	2 choices, + or −	2 results	$2 * 2 = 4$
N → NT or NN	2 choices, +0 or −0	2 results	$2 * 2 = 4$
TT → T	$2 + 1 = 3$ choices, ++, +−, or −−	1 result	3
NN → T or N	$2 + 1 = 3$ choices, +0+0, +0−0, or −0−0	2 results	$3 * 2 = 6$
TN → N	$2 * 2 = 4$ choices, ++0, +−0, −+0 or −−0	1 result	4
		Total	21 cases

Table 6.3: Checking all the cases.

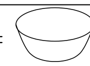
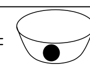
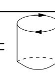
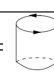
	Circle change	$\Phi\tilde{d}_i$
1.	T $\rightarrow$ TT	$\Phi(\tilde{d}_i(+)) = \Phi(++) =$ 
2.	T $\rightarrow$ NN	$\Phi(\tilde{d}_i(+)) = \Phi(0) = 0$
3.	T $\rightarrow$ TT	$\Phi(\tilde{d}_i(-)) = \Phi((+-) + (-+)) =$ 
4.	T $\rightarrow$ NN	$\Phi(\tilde{d}_i(-)) = \Phi((+0-0) + (-0+0)) =$ 
5.	N $\rightarrow$ NT	$\Phi(\tilde{d}_i(+0)) = \Phi(+0+) =$ 
6.	N $\rightarrow$ NN	$\Phi(\tilde{d}_i(+0)) = \Phi(0) = 0$
7.	N $\rightarrow$ NT	$\Phi(\tilde{d}_i(-0)) = \Phi(-0+) =$ 
8.	N $\rightarrow$ NN	$\Phi(\tilde{d}_i(-0)) = \Phi(0) = 0$
9.	TT $\rightarrow$ T	$\Phi(\tilde{d}_i(++)) = \Phi(0) = 0$
10.	TT $\rightarrow$ T	$\Phi(\tilde{d}_i(+-)) = \Phi(+) =$ 

all of the above cases when one crossing is switched and then Φ is applied.

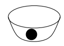
Note that under Φ the associated state isn't affected, thus for example if T $\rightarrow$ TT by changing a smoothing before applying Φ, then after applying Φ the boundary circles behave the same way, and an inessential boundary circle turns into two inessential boundary circles by placing a bridge.

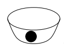
The following 21 items show what $d_i^A\Phi$ is in each of the cases when the bound-

Table 6.4: Checking all the cases, continued.

	Circle change	$\Phi\tilde{d}_i$
11.	TT $\to$ T	$\Phi(\tilde{d}_i(--)) = \Phi(-) =$ 
12.	NN $\to$ T	$\Phi(\tilde{d}_i(+0+0)) = \Phi(0) = 0$
13.	NN $\to$ N	$\Phi(\tilde{d}_i(+0+0)) = \Phi(0) = 0$
14.	NN $\to$ T	$\Phi(\tilde{d}_i(+0-0)) = \Phi(+) =$ 
15.	NN$\to$ N	$\Phi(\tilde{d}_i(+0-0)) = \Phi(0) = 0$
16.	NN $\to$ T	$\Phi(\tilde{d}_i(-0-0)) = \Phi(0) = 0$
17.	NN $\to$ N	$\Phi(\tilde{d}_i(-0-0)) = \Phi(0) = 0$
18.	TN $\to$ N	$\Phi(\tilde{d}_i(++0)) = \Phi(0) = 0$
19.	TN $\to$ N	$\Phi(\tilde{d}_i(+-0)) = \Phi(0) = 0$
20.	TN $\to$ N	$\Phi(\tilde{d}_i(--0)) = \Phi(-0) =$ 
21.	TN $\to$ N	$\Phi(\tilde{d}_i(-+0)) = \Phi(+0) =$ 

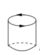
ary circles are affected as in the previous tables.

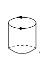
1. Note $\Phi(+) =$ . After a bridge is placed there are two trivial boundary curves in the top. This has Euler characteristic equal to 0, and thus it is a compressible annulus. Compress the annulus to get two disks, each with a dot.

2. $\Phi(+) =$ . When a bridge is placed there are two non-trivial boundary components in the top. This is an incompressible annulus with a dot, so it is trivial in the quotient.

3. $\Phi(-) =$ . After a bridge is placed there are two trivial boundary curves in the top. This is a compressible annulus. Compress the annulus to get disk with dot, disk + disk, disk with dot.

4. $\Phi(-) =$ . After a bridge is placed there are two non-trivial boundary curves in the top. This is an incompressible annulus, so have unoriented annulus = average of oriented annuli.

5. $\Phi(+0) =$ . After a bridge is placed there is a non-trivial boundary curve in the top and a trivial boundary curve in the top. Compress the neck that is near the trivial boundary curve to get an annulus, oriented same way as the original annulus and a disk with a dot.

6. $\Phi(+0) =$ . After a bridge is placed there are two non-trivial boundary curves on the top. One can only compress and separate boundary curves if we have at least 4 non-trivial and we only have three, so we have a surface that is trivial

in the quotient by Lemma 6.3

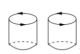
7. Refer to 5.

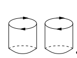
8. Refer to 6.

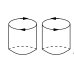
9. $\Phi(++) =$ . After a bridge is placed there is one trivial boundary component. Now we have two dots on the same component, so it is trivial in the quotient.

10. $\Phi(+-) =$ . After a bridge is placed there is one trivial component. These two disks combined to make a disk with a dot.

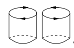
11. $\Phi(--) =$ . After a bridge is placed there is one trivial boundary component. This leaves us with a disk.

12. $\Phi(+0+0) =$ . Placing a bridge would result in a trivial boundary component in the top. Thus the original boundary components must have been parallel. Therefore the bridge falls into the category of (EO) since they are oriented the same way. Thus the result is trivial in the quotient.

13. $\Phi(+0+0) =$ . Placing a bridge results in one non-trivial boundary curve on the top. Thus we have an incompressible pair of pants which is trivial in the quotient.

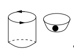
14. $\Phi(+0-0) =$ . After placing a bridge there is one trivial boundary component. Thus the original non-trivial curves were homotopic. Compress upon the disk that is present near the trivial curve on top. This results in a disk on top

with a dot and an annulus on the bottom + disk on top with an annulus with a dot on the bottom which is equivalent to just having a disk with a dot in the quotient.

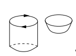
15. $\Phi(+0-0) =$ . After a bridge is placed there is one non-trivial boundary component. As in 13, we have an incompressible pair of pants which is trivial in the quotient.

16. Refer to 12.

17. Refer to 13.

18. $\Phi(++0) =$ . After a bridge is placed there is one non-trivial boundary curve on the top. Note bridging to a disk doesn't change the annulus, except it adds a dot, which makes the foam trivial in the quotient.

19. Refer to 18.

20. $\Phi(--0) =$ . After a bridge is placed there is a non-trivial boundary component on top. Absorbing a disk doesn't change annulus, so we get the same annulus with the same orientation.

21. Refer to 20.

By examining the list and the table, we can see that $d_i\Phi = \Phi\tilde{d}_i$ in each case. Thus note

$$\Phi(\tilde{d}(S)) = \Phi(\sum_i (-1)^{t'(S,i)}\tilde{d}_i(S)) = \sum_i (-1)^{t'(S,i)}\Phi(\tilde{d}_i(S))$$

$$= \sum_i (-1)^{t'(S,i)} d_i^A(\Phi(S)) = d^A(\Phi(S)).$$

Therefore Φ is a chain map, as desired.

□

Theorem 6.9. *Given a link diagram D, $\bar{H}_{i,j,s}(D) \cong H_{i,j,s}(D)$ $\forall i, j, s$ by Φ_*.*

Proof. Let $\bar{I}$, $\bar{J}$, $\bar{K}$ be the indices coming from the [1] theory and let $\bar{S}$ be an enhanced Kauffman state from [1].

Then note clearly $\bar{I}(\bar{S}) = I(\Phi(\bar{S}))$ since the smoothings stay the same under Φ.

In the calculations below it is shown that $\bar{J}(\bar{S}) = J(\Phi(\bar{S}))$.

$$\bar{J}(+) = \bar{I}(+) + 2(\#\text{positive trivial circles} - \#\text{negative trivial circles})$$

$$= I(\Phi(+)) + 2(1-0) = I(S) + 2(2-1)$$

$$= I(\text{a disk with a dot}) + 2(2d - \chi(\text{a disk})) = J(\text{a disk with a dot})$$

$$\bar{J}(-) = \bar{I}(-) + 2(\#\text{positive trivial circles} - \#\text{negative trivial circles})$$

$$= I(\Phi(-)) + 2(0-1) = I(\text{a disk without a dot}) + 2(2d - \chi(\text{a disk}))$$

$$= J(\text{a disk without a dot})$$

$$\bar{J}(+0) = \bar{I}(+0) + 2(\#\text{positive trivial circles} - \#\text{negative trivial circles})$$
$$= I(\Phi(+0)) + 2(0-0) = I(\text{an annulus}) = I(\Phi(+0)) + 2(2d - \chi(\text{an annulus}))$$
$$= J(\text{an annulus with bottom curve oriented in the positive direction})$$

$$\bar{J}(-0) = \bar{I}(-0) + 2(\#\text{positive trivial circles} - \#\text{negative trivial circles})$$
$$= I(\Phi(-0)) + 2(0-0) = I(\text{an annulus}) = I(\Phi(-0)) + 2(2d - \chi(\text{an annulus}))$$
$$= J(\text{an annulus with bottom curve oriented in the negative direction})$$

Then note all non-trivial circles that are present in a smoothing of the diagram appear in the bottom of the foam, so the $\bar{K}$-grading is also preserved under Φ.

The proof will proceed by induction on the number of crossings in the diagram. Assume D has zero crossings. Therefore the boundary maps are all the zero map. Thus the chain groups are also the homology groups. Note Φ is an isomorphism on the chain groups since it takes generators to generators, so it is also an isomorphism on homology in this case.

Let D be a diagram in F, with n crossings and inductively assume Φ_* is an isomorphism for all diagrams with less than n crossings.

Note we have a relation between the short exact sequences coming from the two theories. The diagram commutes since α and $\bar{\alpha}$ are defined identically and the same is true for β and $\bar{\beta}$.

Figure 6.14: The short exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{C}_{i+1,j+1,s}(D_\infty) & \xrightarrow{\bar{\alpha}} & \bar{C}_{i,j,s}(D_p) & \xrightarrow{\bar{\beta}} & \bar{C}_{i-1,j-1,s}(D_0) & \longrightarrow & 0 \\
 & & \downarrow \Phi & \circlearrowright & \downarrow \Phi & \circlearrowright & \downarrow \Phi & & \\
0 & \longrightarrow & C^A_{i+1,j+1,s}(D_\infty) & \xrightarrow{\alpha} & C^A_{i,j,s}(D_p) & \xrightarrow{\beta} & C^A_{i-1,j-1,s}(D_0) & \longrightarrow & 0
\end{array}$$

The short exact sequence induces the long exact sequence shown in Figure 6.15.

Figure 6.15: The long exact sequences.

$$\begin{array}{ccccccccccc}
\ldots\bar{H}_{i+1,j-1,s}(D_0) & \xrightarrow{\partial} & \bar{H}_{i+1,j+1,s}(D_\infty) & \xrightarrow{\bar{\alpha}_*} & \bar{H}_{i,j,s}(D_p) & \xrightarrow{\bar{\beta}_*} & \bar{H}_{i-1,j-1,s}(D_0) & \xrightarrow{\partial} & \bar{H}_{i-1,j+1,s}(D_\infty) & \longrightarrow & \ldots \\
\Phi_* \downarrow & \circlearrowright & \downarrow \Phi_* & \circlearrowright & \downarrow \Phi_* & \circlearrowright & \downarrow \Phi_* & \circlearrowright & \downarrow \Phi_* & & \\
\ldots H_{i+1,j-1,s}(D_0) & \xrightarrow{\partial} & H_{i+1,j+1,s}(D_\infty) & \xrightarrow{\alpha_*} & H_{i,j,s}(D_p) & \xrightarrow{\beta_*} & H_{i-1,j-1,s}(D_0) & \xrightarrow{\partial} & H_{i-1,j+1,s}(D_\infty) & \longrightarrow & \ldots
\end{array}$$

All Φ_*, except the middle one, are isomorphisms by the inductive assumption. Also, the diagram commutes since Φ is a chain map.

Note by the five lemma the middle Φ is an isomorphism. Thus by induction given a link diagram D, $\bar{H}_{i,j,s}(D) \cong H_{i,j,s}(D)\ \forall i, j, s$ by Φ_*.

$\square$

Since Asaeda, Przytycki and Sikora proved invariance for the $\bar{H}(D)$ homology and $\bar{H}(D) \cong H(D)$ by the previous theorem we obtain,

Corollary 6.10. *$H(D)$ is an invariant under Reidemeister moves 2 and 3 and a Reidemeister 1 move shifts the indices in a predictable way.*

CHAPTER 7

K-HOMOLOGY

After constructing the APS theory geometrically there were two directions to pursue. The APS theory allowed disks to become annuli, but not annuli to become twice-punctured disks. One direction would be to allow annuli to become twice-punctured disks and the other direction would be to not let annuli become disks. We will explore both of these directions, starting with the less restrictive one.

The next homology theory presented is actually an infinite family of homology theories. There is one theory for each choice of $k \in \mathbb{N} \cup \{\infty\}$. We need one additional relation for these theories.

7.1 New relations

Definition 7.1. B_k is a submodule of M_D^O generated by the relation:

1. If $k \neq \infty$ then an incompressible component with Euler characteristic less than or equal to $-k$ is zero. (KEC)

Definition 7.2. *k-foams* with respect to the surface link diagram D are elements of $M_D^O/(B \cup B_k)$.

Remark 7.1. Note that the KEC relation sets some foams equivalent to zero, so k-foams respect the grading on foams.

Definition 7.3. Let $C^k_{i,j,s}(D)$ be the free module generated by all k-foams with respect to the diagram D, S, such that $I(S) = i$, $J(S) = j$ and $K(S) = s$.

7.2 The Boundary Operator

In this section we define the boundary operator of the chain complex and show it is well-defined with respect to k-foams.

Definition 7.4. Let p be a crossing of the link diagram D.

$$d_p^k(S) = \begin{cases} 0, & \text{if } p^{\text{th}} \text{ crossing of } S \text{ is smoothed negatively} \\ 0, & \text{if (EO) occurs} \\ d_p^k(S), & \text{else} \end{cases}$$

Then we define the boundary operator $d^k : C^k_{i,j,s}(D) \to C^k_{i-2,j,s}(D)$, by

$$d^k(S) = \sum_{p \text{ a crossing of } D} (-1)^{t(S,p)} d_p^k(S), \tag{7.1}$$

where $t(S,p) = |\{\, j$ a crossing of $D : j$ is after p in the ordering of crossings and j is smoothed negatively in boundary state of S$\}|$

Theorem 7.2. *The boundary operator, d^k, is well-defined on k-foams.*

Proof. In order to show that d^k is well-defined we need to look at the relations that define k-foams and see how d^k affects them. Note the only new relation to consider is KEC.

Assume the boundary operator is applied to a k-foam that has an incompressible component with Euler characteristic less than or equal to $-k$. When the bridge is placed the new component has Euler characteristic less than or equal to $-k-1$. Note that at most one inessential boundary curve can be created when a bridge is placed since we are starting with no inessential curves as the surface is incompressible.

If no inessential curves are created, then all curves are essential. If the component is incompressible, we are done by KEC. If the component is compressible then we are done by Corollary 5.9.

If an inessential curve is created, then apply the NC relation to yield a disk with a dot and a surface of Euler characteristic less than or equal to $-k$.

If the non-disk surface is incompressible, the k-foam is trivial. If the non-disk is compressible then the foam is trivial by Corollary 5.9.

When the boundary operator is applied to elements of $B \cup B_k$, they remain in $B \cup B_k$, so by the remarks at the beginning of this proof, the boundary operator is well-defined on the quotient.

□

7.3 $d^k \circ d^k = 0$

Theorem 7.3. $d^k \circ d^k = 0$

Proof. In order to show $d^k \circ d^k = 0$, we need to show that $d^k_p \circ d^k_q = d^k_q \circ d^k_p \; \forall p, q$ crossings.

Note that by the definition of the boundary operator all that needs to be shown is that given two crossings a and b and a foam S, that $d^k_a(d^k_b(S)) = d^k_b(d^k_a(S))$. This is clearly the case if EO does not occur and if we disregard orientation.

If EO does occur we have shown by Lemma 5.4 that if it occurs for one order of a and b, then $d^k_a(d^k_b(S))$ and $d^k_b(d^k_a(S))$ are both trivial in the quotient. By Lemma 5.15, $d^k_a(d^k_b(S))$ and $d^k_b(d^k_a(S))$ are oriented identically.

Thus the partials always commute, so after adding in the appropriate negative signs, $d^k \circ d^k = 0$.

$\square$

CHAPTER 8

SIMPLE HOMOLOGY

We will describe a simpler method to obtain a homology theory that determines the Kauffman bracket than the one from [1]. It is simpler in the sense that surfaces are not oriented. In addition, the graded Euler characteristic of the chain complex is exactly the Kauffman bracket.

8.1 New relations

Definition 8.1. Let B_S be the submodule of M_D generated by the following relations:

1. An incompressible annulus with both boundary components in the top or bottom equals zero. (TT)

2. A surface with an incompressible component with negative Euler characteristic equals zero. (NEC)

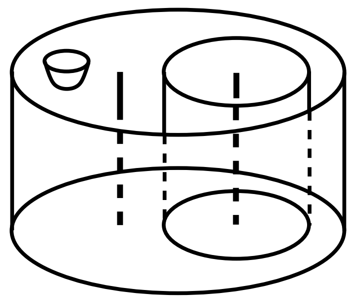
3. An incompressible torus is zero. (IT)

Definition 8.2. The elements of $M_D/(B \cup B_S)$ are called *simple foams* with respect to the link diagram D.

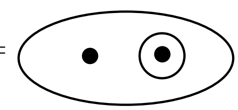
8.2 Grading

Remark 8.1. Since all of the new relations on simple foams set something equal to zero, simple foams continue to respect the I and J gradings of foams. However, since simple foams are not oriented the K grading no longer applies.

Figure 8.1: The simple foam is in the thickened twice punctured disk, where the vertical lines represent the punctures.



Definition 8.3. Let S be a simple foam. $\bar{K}(S)$ is the diagram of the essential curves in the bottom of the foam.

Example 8.1. If S is the simple foam in figure 8.1 then $\bar{K}(S) =$ 

Definition 8.4. Let $C^S_{i,j,s}(D)$ be the chain group generated by simple foams with respect to the diagram D, with the appropriate I, J and $\bar{K}$ grading.

Theorem 8.2. *Simple foams are spanned by simple foams whose components are incompressible vertical annuli or disks.*

Proof. We will proceed similarly to the analogous theorem for A-foams. Therefore we again let $g(S)$ be the genus of S and $b(S)$ be the number of boundary components of S.

By the NC relation, A-foams are generated by incompressible foams. By the NEC relation, all components have non-negative Euler characteristic. Since we are restricted to orientable surfaces and by the fact that $\chi(S) = 2 - 2g(S) - b(S)$, we can conclude that each component must satisfy the condition that $2 - 2g(S) - b(S) \geq 0$.

Thus, $2 \geq 2g(S) + b(S)$. So if g(S) = 1, b(S) = 0, so S is a torus. If S is an incompressible torus, then it is zero by the IT relation.

If $g(S) = 0$, then $b(S) = 0$, 1 or 2. Thus S is either a sphere, disk or annulus. Thickened surfaces are irreducible, so sphere can be addressed by relations. Annuli with their boundary in the top or bottom are zero by the TT relation.

□

8.3 The Boundary operator

We will now define the boundary operator for simple foams.

Definition 8.5. If two distinct essential curves are connected by the crossing p, then EC is said to occur at the p^{th} crossing.

Definition 8.6. $d_p^S : C_{i,j,s}(D) \to C_{i-2,j,s}(D)$

$$d_p^S(S) = \begin{cases} 0, & \text{if } p\text{-th crossing of } S \text{ is smoothed negatively} \\ 0, & \text{if EC occurs at the } p\text{-th crossing of } S \\ d_p^S(S), & \text{else} \end{cases}$$

Then $d^S : C^S_{i,j,s}(D) \to C^S_{i-2,j,s}(D)$ is defined by

$$d^S(S) = \sum_{p \text{ a crossing of } D} (-1)^{t(S,p)} d_p^S(S), \tag{8.1}$$

where $t(S,p) = |\{\, j$ a crossing of D : j is after p in the ordering of crossings and j is smoothed negatively in boundary state of S$\}|$

Theorem 8.3. *The boundary operator, d^S, is well-defined with respect to simple foams.*

Proof. We must show that trivial simple foams remain trivial under the boundary operator. This proof closely follows the proof of Theorem 7.2. In that proof the NC and NDD relation were addressed. This only leaves the TT, NEC and IT relations. We will address these by cases.

- NEC

  Assume the boundary operator is applied to a simple foam with an incompressible negative Euler characteristic component. An incompressible negative Euler characteristic component has only essential boundary curves. EC occurs unless the same essential curve bridges to itself. If the component surface is incompressible the foam is trivial by the NEC relation. If the surface is compressible with no inessential curves, then the foam is trivial by Lemma 5.9.

  If an inessential curve is created then we can remove it from the rest of the component by the NC relation. This leaves a disk and a surface of negative Euler characteristic. If the non-disk component is compressible the foam is trivial by Lemma 5.9. If the non-disk component is incompressible, then the foam is trivial by the NEC relation.

- TT

  Assume the boundary operator is applied to a simple foam with an incompressible top annulus. As before we can assume that the bridge placed on a top annulus bridges the same boundary curve to itself, since if it bridged to another boundary curve, the result would be zero by EC.

If the bridge connects a curve to itself, then there is now either a compressing disk or the surface is incompressible with negative Euler characteristic. The incompressible surface is trivial in the quotient, so we may assume it is compressible. The surface has either one inessential curve or no inessential curves. By Corollary 5.9 we can assume one of the boundary components is inessential. Thus after compression there is a disk and the the rest of the surface. By noting the Euler characteristic what is left is a top annulus which is trivial in the quotient.

- IT

  If a simple foam has an incompressible torus component, then after applying the boundary operator it will continue to have an incompressible torus component, so it remains trivial in the quiotient.

□

8.4 $d^S \circ d^S = 0$

Lemma 8.4. *If $d_a^S(d_b^S(S)) = 0$ by EC, then $d_b^S(d_a^S(S))$ is trivial in the quotient.*

Proof. If $d_a^S(d_b^S(S)) = 0$ by EC then there must be at least two essential boundary curves in the top of the foam and they must either be connected, or both are connected to the same curve.

If they are both connected to the same curve we may assume the intermediate curve is inessential otherwise we are in the other case. When an essential and an inessential curve combine we are left with an essential curve, thus in either order, the result is zero.

If two essential curves are connected to each other we can assume one crossing connects them, while another crossing connects one of the curves to itself. Otherwise, if it connects anywhere else the other crossing still connects two essential curves or an essential and inessential. Either of which results in an essential curve. When a bridge is placed at the self-connecting crossing an inessential curve may be produced along with an essential curve. After applying the NC relation the other crossing either connects an essential curve to an essential or an essential to an inessential on a dotted disk. In either case the result is trivial in the quotient.

□

Theorem 8.5. $d^S \circ d^S = 0$

Proof. The partial boundary operators commute if EC does not occur since the resulting foams are isotopic. Lemma 8.4 shows that if EC occurs, the partials commute then too. Thus with the appropriate negative signs, the partial boundary operators anti-commute, and since d^S is a sum of the partial boundary operators with the negative signs, we have $d^S \circ d^S = 0$.

□

Remark 8.6. Constructing these homology theories via embedded surfaces instead of decorated diagrams seems to lead to simpler proofs of showing that we have a chain complex. Instead, much of the work is shifted to proving that the boundary operator is well-defined.

8.5 Graded Euler characteristic

Definition 8.7. The graded Euler characteristic of a chain complex is

$$\chi(C_{*,*,s}(D)) = \sum_{i,j} A^j (-1)^{\frac{j-i}{2}} rkC_{i,j,s}(D).$$

Theorem 8.7. *The graded Euler characteristic of the chain complex for simple homology is the Kauffman bracket of the link in the surface.*

Proof. We will show that the graded Euler characteristic satisfies the three skein conditions of the Kauffman bracket.

1. $\chi(C^S_{*,*,s}(D \cup \bigcirc)) = \sum_{i,j} A^j (-1)^{\frac{j-i}{2}} rkC^S_{i,j,s}(D \cup \bigcirc)$

$$= \sum_{i,j} A^j (-1)^{\frac{j-i}{2}} (rkC^S_{i,j-2,s}(D) + rkC^S_{i,j+2,s}(D))$$

$$= \sum_{i,j} A^2 A^{j-2} (-1)^{\frac{j-2-i+2}{2}} (rkC^S_{i,j-2,s}(D))$$

$$+ \sum_{i,j} A^{-2} A^{j+2} (-1)^{\frac{j+2-i-2}{2}} rkC^S_{i,j+2,s}(D))$$

$$= \sum_{i,j} A^2 A^{j-2} (-1)^{\frac{j-2-i}{2}+1} (rkC^S_{i,j-2,s}(D)$$

$$+ \sum_{i,j} A^{-2} A^{j+2} (-1)^{\frac{j+2-i}{2}-1} rkC^S_{i,j+2,s}(D))$$

$$= -A^2 \sum_{i,j} A^{j-2} (-1)^{\frac{j-2-i}{2}} (rkC^S_{i,j-2,s}(D)$$

$$-A^{-2} \sum_{i,j} A^{j+2} (-1)^{\frac{j+2-i}{2}} rkC^S_{i,j+2,s}(D))$$

$$= (-A^2 - A^{-2}) \chi(C^S_{*,*,s}(D))$$

2. $\chi(C^S_{*,*,s}(D)) = \sum_{i,j} A^j(-1)^{\frac{j-i}{2}} rkC^S_{i,j,s}(D)$

$$= \sum_{i,j} A^j(-1)^{\frac{j-i}{2}}(rkC^S_{i-1,j-1,s}(D_+) + rkC^S_{i+1,j+1,s}(D_-))$$

$$= \sum_{i,j} A^j(-1)^{\frac{j-i}{2}}(rkC^S_{i-1,j-1,s}(D_+)$$

$$+ \sum_{i,j} A^j(-1)^{\frac{j-i}{2}} rkC^S_{i+1,j+1,s}(D_-))$$

$$= \sum_{i,j} A * A^{j-1}(-1)^{\frac{j-1-(i-1)}{2}}(rkC^S_{i-1,j-1,s}(D_+)$$

$$+ \sum_{i,j} A^{-1}A^{j+1}(-1)^{\frac{j+1-(i+1)}{2}} rkC^S_{i+1,j+1,s}(D_-))$$

$$= A\sum_{i,j} A^{j-1}(-1)^{\frac{j-1-(i-1)}{2}}(rkC^S_{i-1,j-1,s}(D_+)$$

$$+A^{-1}\sum_{i,j} A^{j+1}(-1)^{\frac{j+1-(i+1)}{2}} rkC^S_{i+1,j+1,s}(D_-))$$

$$= A\chi(C^S_{*,*,s}(D_+)) + A^{-1}\chi(C^S_{*,*,s}(D_-))$$

3. Consider D, a diagram of a link with no crossings. After removing all inessential curve, the only possible non-trivial foams are ones with essential curves connected to essential curves in the bottom by vertical annuli. Note the graded Euler characteristic is the representation of the link in the Kauffman bracket for surfaces.

□

8.6 Decorated diagram version of simple theory

In this section we will realize this theory via decorated diagrams instead of foams to make it simpler to compute. The symbol a will denote a non-trivial curve and trivial curves will be marked with either a + or −. The multiplication and co-multiplication for this theory is shown in Table 8.1.

Table 8.1: Multiplication table for simple homology.

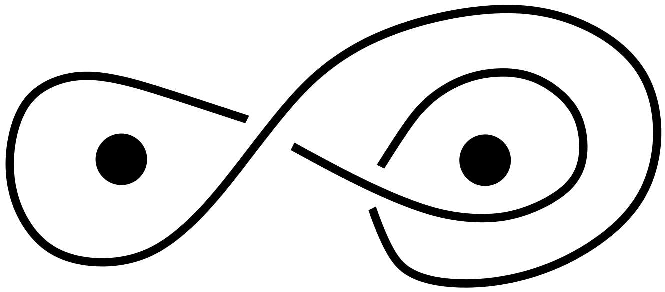
$m(+\otimes +) = 0$	$\Delta(-) = \begin{cases} -\otimes + + \otimes - & \text{two inessential} \\ 0 & \text{two essential} \end{cases}$
$m(a \otimes a) = 0$	
$m(a \otimes +) = 0$	$\Delta(+) = \begin{cases} +\otimes + & \text{two inessential} \\ 0 & \text{two essential} \end{cases}$
$m(a \otimes -) = a$	
$m(+\otimes -) = +$	$\Delta(a) = \begin{cases} a \otimes + & \text{one inessential} \\ 0 & \text{two essential} \end{cases}$
$m(-\otimes -) = -$	

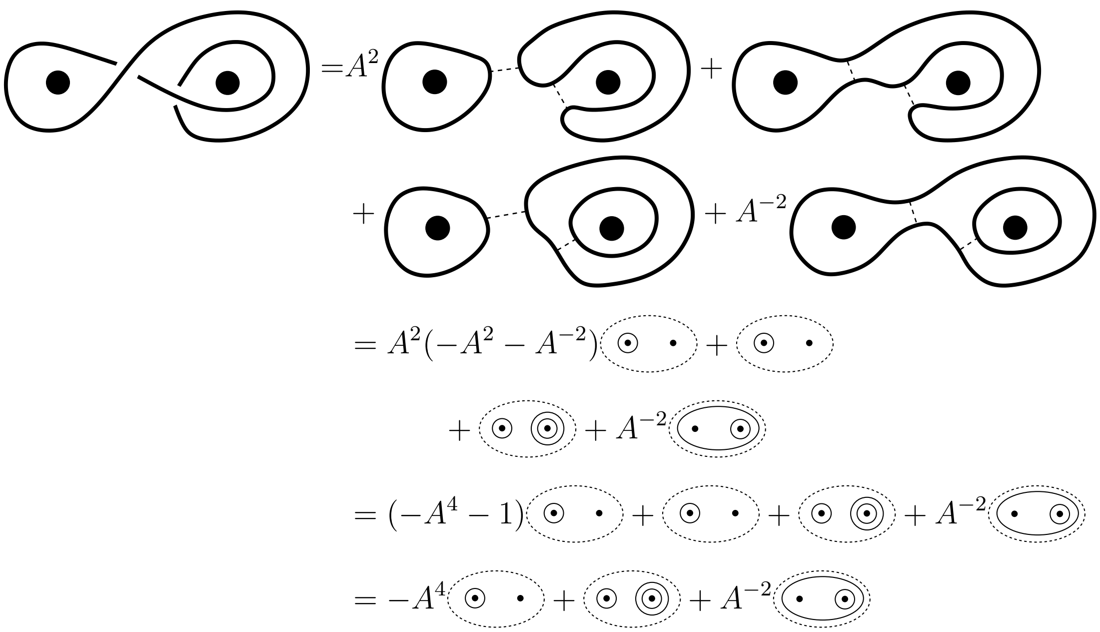
Definition 8.8. If we use decorated diagrams instead of embedded surfaces, Euler characteristic doesn't apply, so here is a revised definition of the J grading.

$$\begin{aligned} J(S) &= I(S) + 2(2(\text{number of } +\text{s}) - \text{number of trivials}) \\ &= I(S) + 2(2(\text{number of } +\text{s}) - (\text{number of } +\text{s} + \text{number of } -\text{s})) \\ &= I(S) + 2(\text{number of } +\text{s} - \text{number of } -\text{s}) \end{aligned}$$

Example 8.2. Now we will compute the simple homology of the link L in the twice-punctured disk. First we compute the Kauffman bracket of L, in Figure 8.2. The complex of smoothings is shown in Figure 8.5.

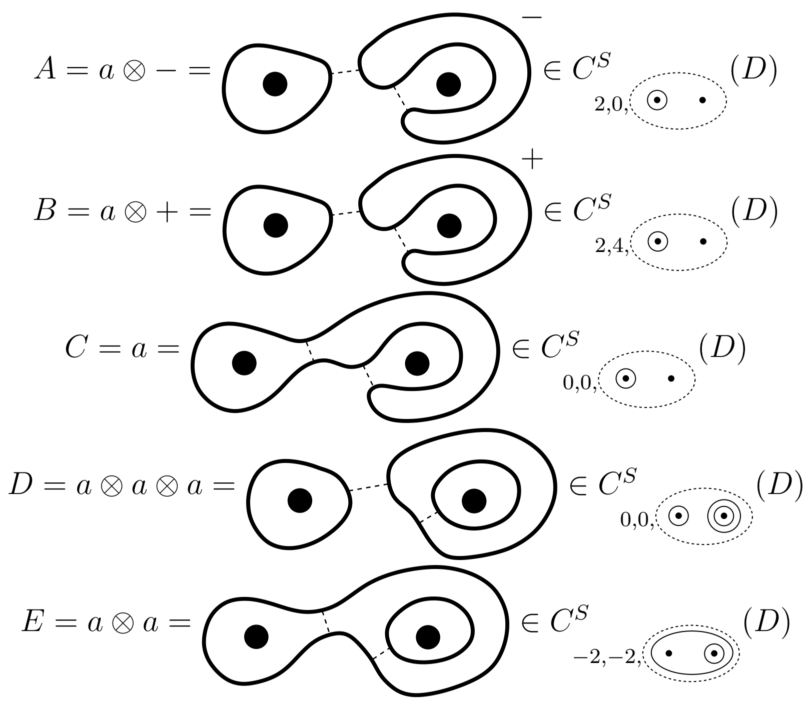
Figure 8.2: Kauffman bracket of L.





The elements of the chain groups are named in Figure 8.3. Note the multiplication can be done without examining the diagram, but when doing the co-

Figure 8.3: Elements of the chain groups.



multiplication one has to examine the diagram.

$d(A) = m(a \otimes -) \oplus a \otimes \Delta(-) = a \oplus 0$, so $d(A) = C$

$d(B) = m(a \otimes +) \oplus a \otimes \Delta(-) = 0 \oplus 0$

$d(C) = \Delta(a) = 0$, since this a curve splits into two a curves.

$d(D) = m(a \otimes a) \otimes a = 0 \otimes a = 0$, since this a curve also splits into two a curves.

Figure 8.4: Homology computations.

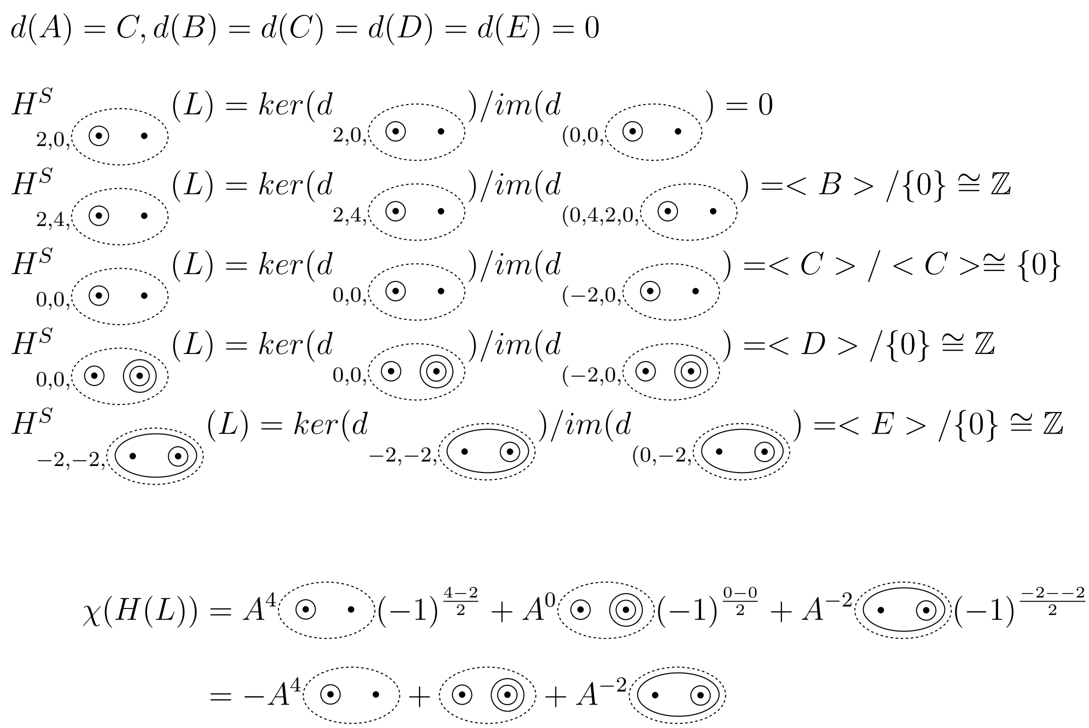
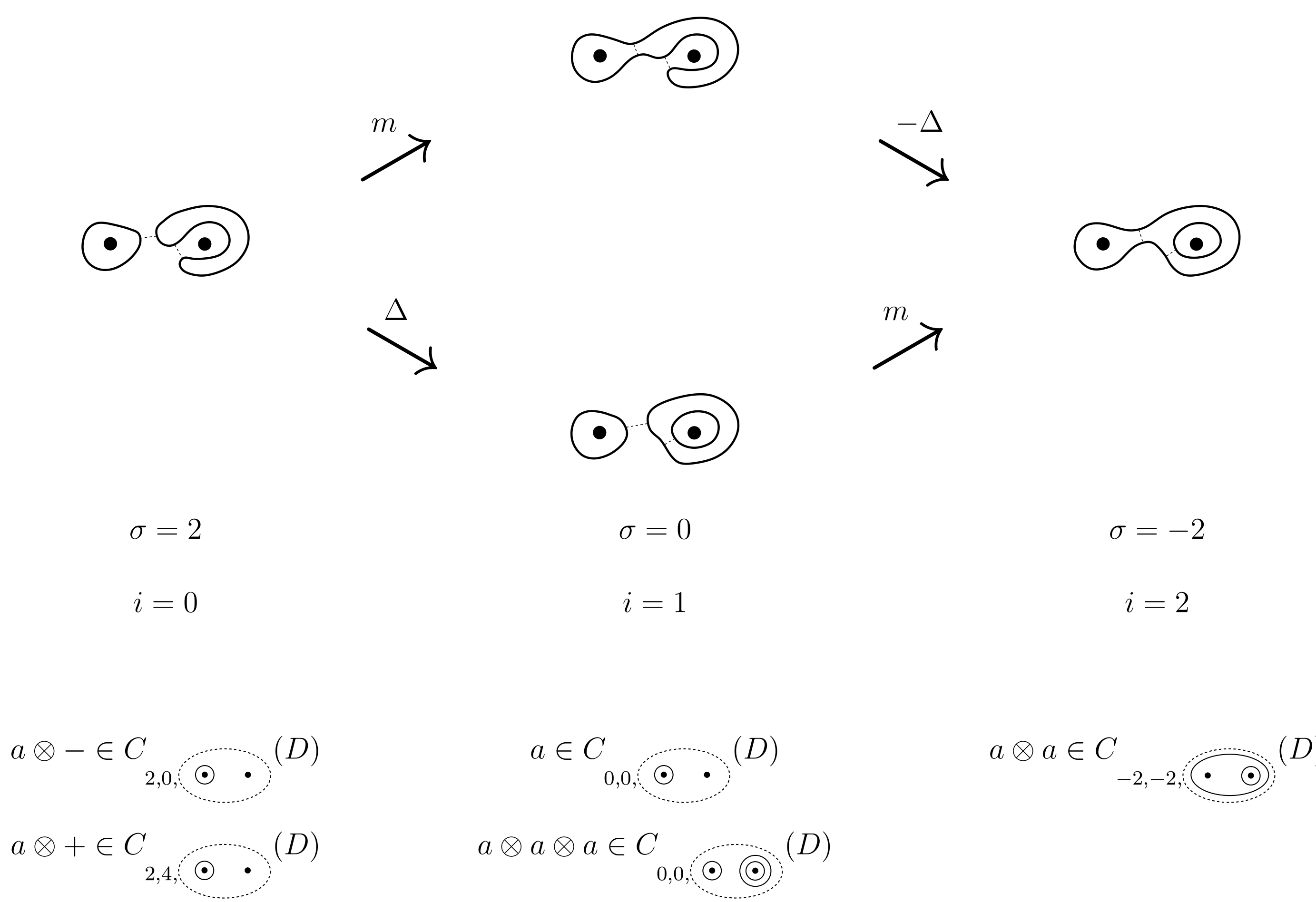


Figure 8.5: The complex of Kauffman states of L.



CHAPTER 9

INVARIANCE

We will prove that k-homology and simple homology are invariants of framed links in thickened surfaces simultaneously. Note that all theorems and statements in this section can be applied to either theory. The strategy to prove invariance will be the one employed by Bar-Natan in [3] involving acyclic sub-complexes and will follow his proof closely. Thus we shall use Theorem 9.1.

Theorem 9.1. *[3] If C' is acyclic, then $H(C/C') \cong H(C)$, that is C and C/C' produce the same homology.*

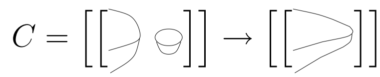
Remark 9.2. The disk without a dot acts like a unit as does the circle marked with a v_- in [3].

9.1 Reidemeister I

Note the complexes C/C' and D are identical apart from a shift in degrees. Then by Theorem 9.1, we have that C and D produce the same homology, up to degree.

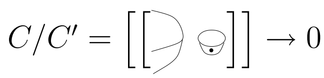
To see how the degrees differ note that foams in C/C' have an extra disk with a dot and an extra crossing that is smoothed positively. Thus if S is a foam in D and S' is the corresponding foam in C/C' then $I(S) + 1 = I(S')$, $d(S) + 1 = d(S')$ and $\chi(S) + 1 = \chi(S')$, thus $J(S') = I(S') + 2(2d(S') - \chi(S')) = I(S) + 1 + 2(2(d(S) + 1) - (\chi(S) + 1)) = I(S) + 1 + 2(2d(S) + 2 - 1 - \chi(S)) = I(S) + 2(2d(S) - \chi(S)) + 3$. So $I(S) + 1 = I(S')$ and $J(S) + 3 = J(S')$. Thus we know exactly how a positive twist

Figure 9.1: The complexes associated with the Reidemeister one move.

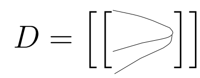


(a) The complex of complexes, C, coming from a positive twist.

(b) The acyclic complex C', where the disk never has a dot.



(c) The complex C/C', where the disk always has a dot.

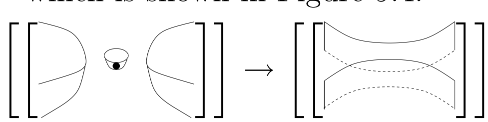


(d) The complex D, where the positive twist has been removed.

affects the homology.

9.2 Reidemeister II

Let C be the complex in Figure 9.2 coming from one side of the Reidemeister two move. Note C has a subcomplex C', depicted in Figure 9.3, where the disk never has a dot. C' is acyclic since the only non-trivial map has no kernel and is onto. Now consider the complex C/C' which is shown in Figure 9.4.

Now we define τ :  on C/C' as in Figure 9.5. Note $\tau = m \circ \Delta^{-1}$.

C/C' has a subcomplex C'', which is shown in Figure 9.6. The second term in the complex C'' consists of elements of the form $(\beta, \tau(\beta))$.

C'' is acyclic since the non-zero map is onto the first factor and by construction of τ (since Δ is a bijection) is onto the second factor as well.

Figure 9.2: The complex coming from one side of the Reidemeister two move.

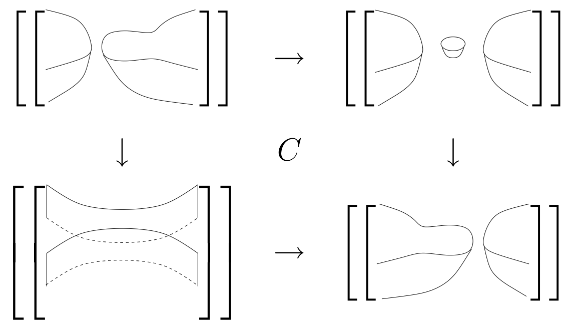
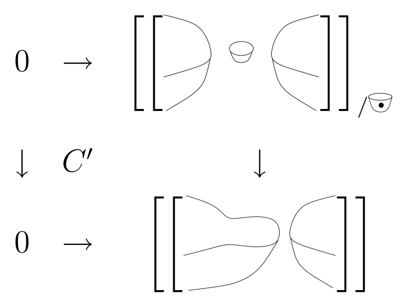
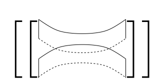


Figure 9.3: The acyclic subcomplex C'.



Modding out by C'' results in the complex with the disk being identified with elements in the complex without the disk, so we have

$(C/C')/C''$ is $0 \to$  $\to 0.$

Note this is the same complex that appears if a Reidemeister II move was performed on the original diagram. By Theorem 9.1, the homology of the original complex is equivalent to the homology produced by $(C/C')/C''$ and thus it is un-

Figure 9.4: The complex C/C'.

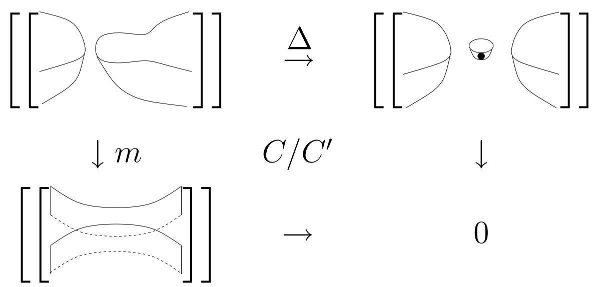
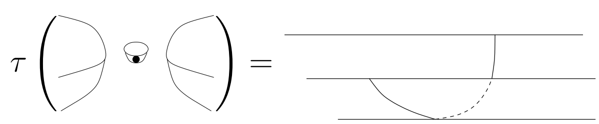


Figure 9.5: The definition of τ.



changed by a Reidemeister II move.

9.3 Reidemeister III

Consider the two cubes that come from the Reidemeister three move, C_A and C_B, as shown in figure 9.7.

Notice the two subcomplexes, C'_A and C'_B in Figure 9.8, where the disk never has a dot, are acyclic as in the Reidemeister two case. Now consider the complexes C_A/C'_A and C_B/C'_B in Figure 9.9, where the disk always has a dot.

Let τ be the same map as defined in the Reidemeister two case. Then we can consider the acyclic subcomplexes of the previous cubes, $C''_A \subset C_A/C'_A$ and $C''_B \subset$

Figure 9.6: The complex C''

.

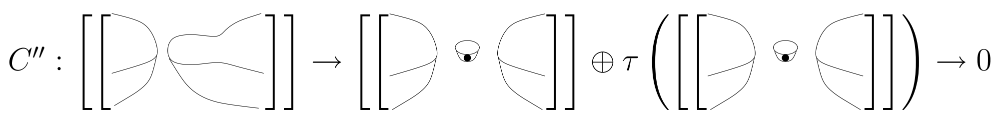
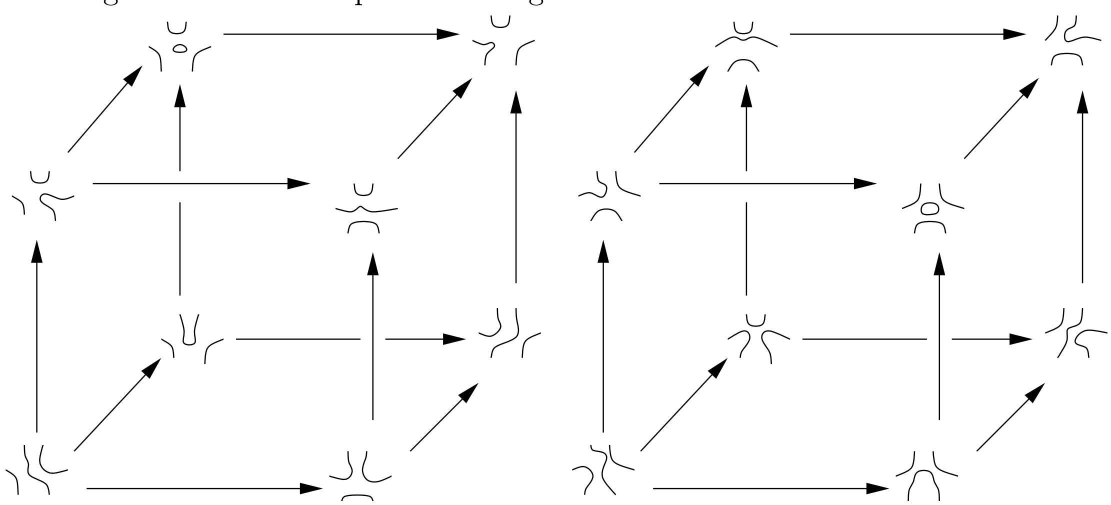


Figure 9.7: The complexes coming from the Reidemeister three move.



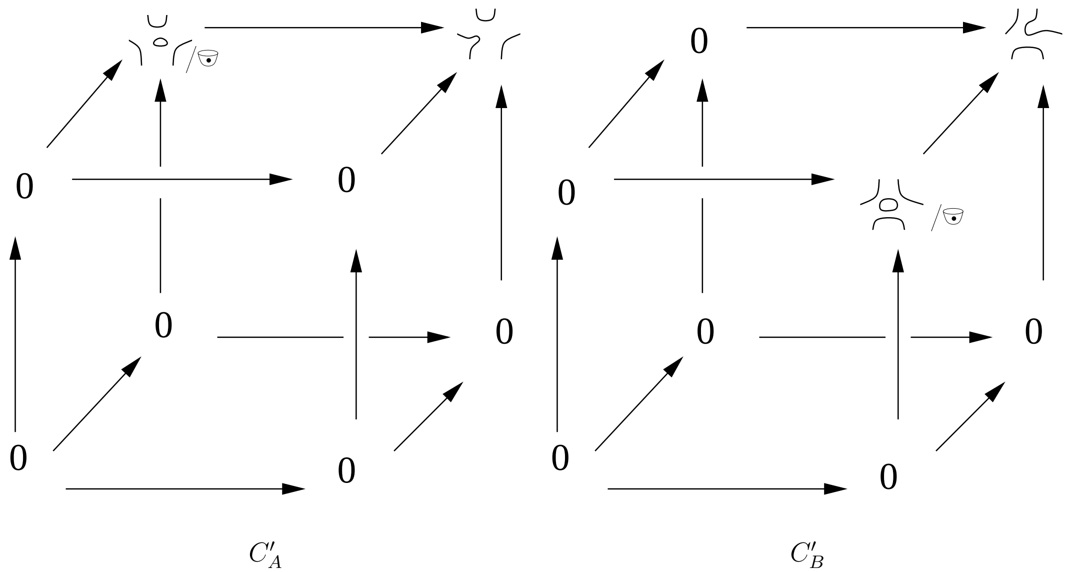
C_A C_B

C_B/C'_B as shown in Figure 9.10. The complexes are acylic because Δ is an isomophism in this complex so the only non-zero map is one-to-one and onto.

Finally we arrive at $(C_A/C'_A)/C''_A$ and $(C_B/C'_B)/C''_B$ in Figure 9.11. Note that when we mod out by C''_A and C''_B we essentially set $\beta = \tau(\beta)$.

We will define Υ as in [3], Υ sends the bottom layer of $(C_A/C'_A)/C''_A$ to the bottom layer of $(C_B/C'_B)/C''_B$. Υ does almost the same to the top of the cubes, except

Figure 9.8: The acyclic subcomplexes C'_A and C'_B.

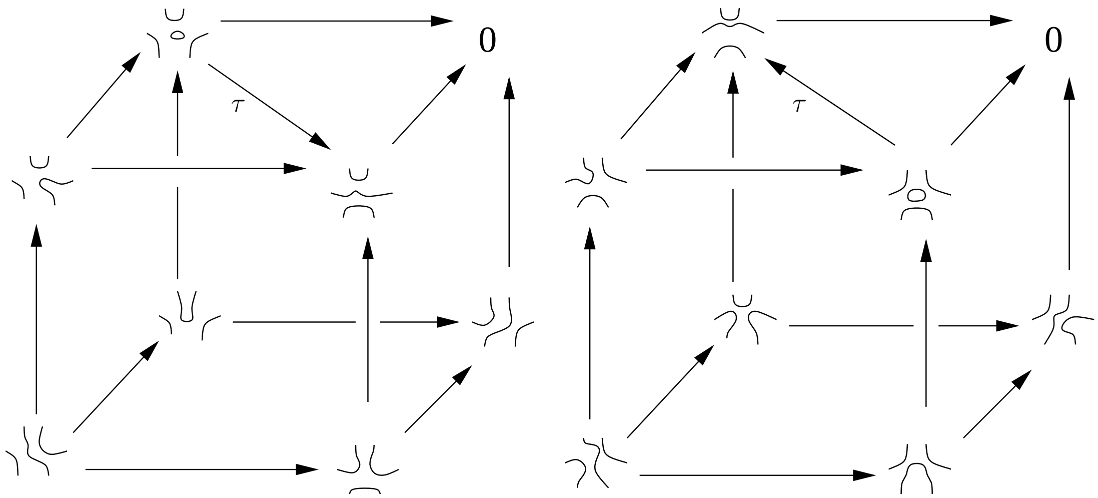


that the top layers are reversed so that the foams with the closed component locally are mapped to each other. We must now verify that $d \circ \Upsilon = \Upsilon \circ d$ so that Υ is a chain map. Since the tops and bottoms are immediately isomorphic, the only thing left to show is that $\tau^A \circ d^A_{1,0,*} = d^B_{1,0,*}$ and $\tau^B \circ d^B_{0,1,*} = d^A_{0,1,*}$. The fact that $\tau^A \circ d^A_{1,0,*} = d^B_{1,0,*}$ is shown in Figure 9.12.

Thus we have $\tau^A \circ d^A_{1,0,*} = d^B_{1,0,*}$. Showing that $\tau^B \circ d^B_{0,1,*} = d^A_{0,1,*}$ is done in the exact same manner, thus we have that Υ is a chain map.

Since the diagram commutes and both maps are isomorphisms the chain complexes are isomorphic. These chain complexes produce the same homology as C_A and C_B respectively since they are quotients of C_A and C_B by acyclic complexes. Thus C_A and C_B produce the same homology and the theory is invariant under the third Reidemeister move. This proves Theorem 9.3.

Figure 9.9: The complexes C_A/C'_A and C_B/C'_B.



C_A/C'_A C_B/C'_B

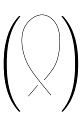
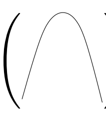
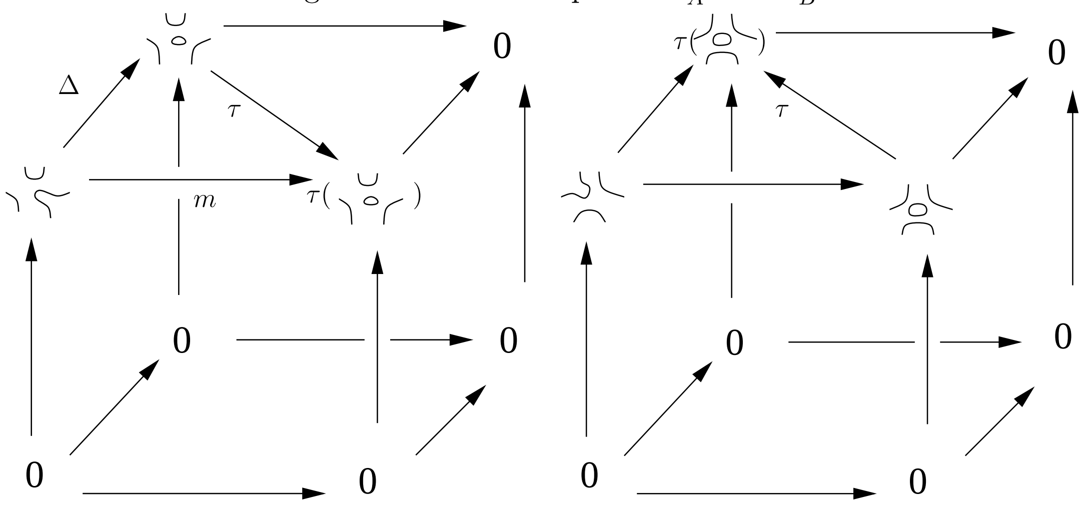
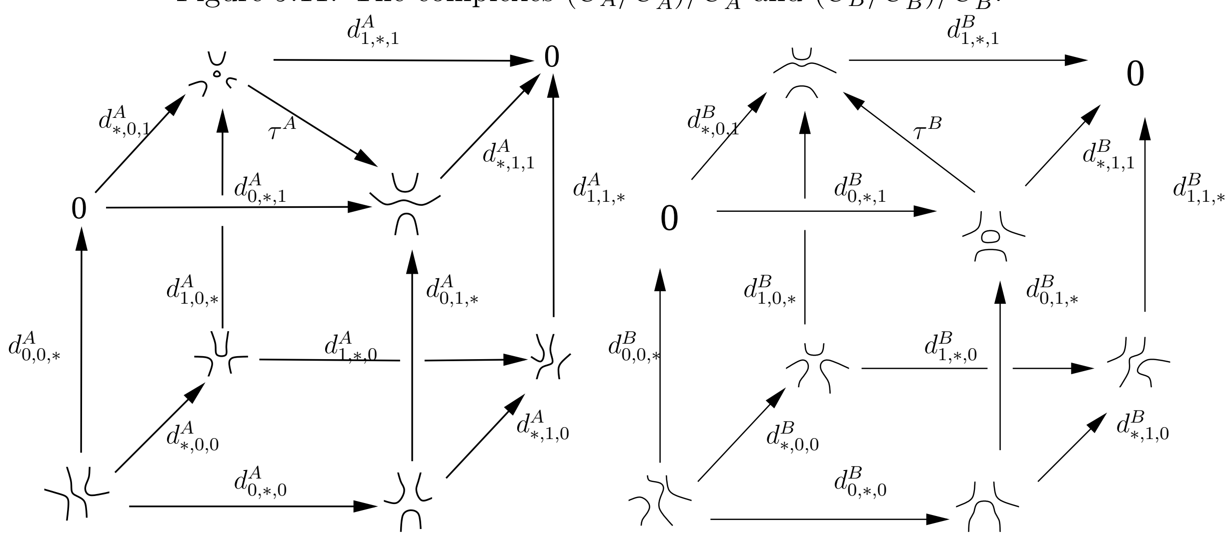
Theorem 9.3. *Both homology theories that have been defined are invariants of framed links and* $H_{i,j,s}$ $\Bigg($  $\Bigg) = H_{i-1,j-3,s}$ $\Bigg($  $\Bigg)$.

Figure 9.10: The complexes C''_A and C''_B.



$C''_A \subset C_A/C'_A$ $\qquad$ $C''_B \subset C_B/C'_B$

Figure 9.11: The complexes $(C_A/C'_A)/C''_A$ and $(C_B/C'_B)/C''_B$.



$(C_A/C'_A)/C''_A$ $\qquad$ $(C_B/C'_B)/C''_B$

Figure 9.12: Showing $\tau^A \circ d^A_{1,0,*} = d^B_{1,0,*}$.

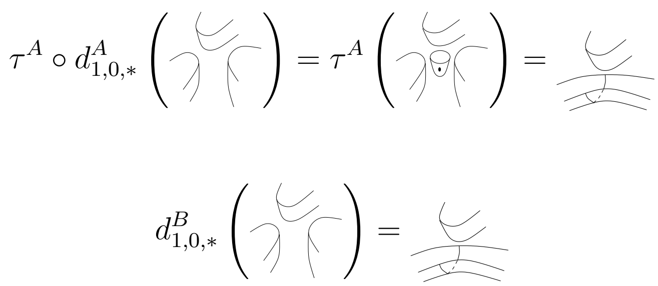


Figure 9.13: Υ is a chain map.

$$
\begin{array}{ccc}
\text{Bottom}_A & \rightarrow & \text{Top}_A \\
{}^{\Upsilon}\downarrow\cong\uparrow & \circlearrowright & \downarrow\cong\uparrow{}^{\Upsilon} \\
\text{Bottom}_B & \rightarrow & \text{Top}_B
\end{array}
$$

REFERENCES

[1] Marta M. Asaeda, Jozef H. Przytycki, and Adam S. Sikora. Categorification of the Kauffman bracket skein module of I-bundles over surfaces. *ALGEBR.GEOM.TOPOL.*, 4:1177, 2004.

[2] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *ALGEBRAIC AND GEOMETRIC TOPOLOGY*, 2:337, 2002.

[3] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *GEOM.TOPOL.*, 9:1443, 2005.

[4] Mikhail Khovanov. A categorification of the Jones polynomial. *DUKE MATH.J*, 3:359, 2000.

[5] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*. Cambridge University Press, 2003.

[6] W. B. Raymond Lickorish. *An Introduction to Knot Theory*. Springer-Verlag, 1997.

[7] Oleg Viro. Remarks on definition of Khovanov homology, 2002.

[8] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *The Annals of Mathematics, Second Series*, 87:56, 2000.