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Universal deformation rings of modules over self-injective algebras

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UNIVERSAL DEFORMATION RINGS OF MODULES OVER SELF-INJECTIVE ALGEBRAS

by

José Alberto Vélez Marulanda

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

July 2010

Thesis Supervisor: Professor Frauke M. Bleher

ABSTRACT

In this thesis, I apply methods from the representation theory of finite dimensional algebras to the study of versal and universal deformation rings. The main idea is that more sophisticated results from representation theory can be used to arrive at a deeper understanding of deformation rings. Such rings arise naturally in a variety of problems in number theory and group representation theory.

This thesis has two parts. In the first part, Λ is an arbitrary finite dimensional algebra over a field k. If V is a finitely generated Λ-module, I prove that V has a versal deformation ring $R(\Lambda, V)$. Moreover, if Λ is self-injective and the stable endomorphism ring of V is isomorphic to k, then $R(\Lambda, V)$ is universal. If additionally Λ is a Frobenius algebra and Ω denotes the syzygy operator over Λ, I show that the universal deformation rings of V and $\Omega(V)$ are isomorphic. In the second part, I analyze a particular finite dimensional Frobenius algebra Λ over an algebraically closed field k for which all the finitely generated indecomposable modules can be described combinatorially by using certain words in Λ. I use this description to visualize the indecomposable Λ-modules in the stable Auslander-Reiten quiver of Λ and determine all the components of this stable Auslander-Reiten quiver which contain Λ-modules whose endomorphism ring is isomorphic to k. Finally I determine the universal deformation rings of all the modules in these components whose stable endomorphism ring is isomorphic to k.

Abstract Approved: ______________________________
Thesis Supervisor

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Title and Department

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Graduate College
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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

José Alberto Vélez Marulanda

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2010 graduation.

Thesis Committee: ______________________________
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To my mother
Rosa María Marulanda
who is an example of
humility, self-sacrifice, faith
and generosity

An equation for me has no meaning, unless it represents a thought of God.

Srinivasa Ramanujan

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CHAPTER 1

INTRODUCTION

1.1 Motivation

The work in this thesis has to do with applying methods from the representation theory of finite dimensional algebras to the study of versal and universal deformation rings. The main motivation is that more sophisticated results from representation theory can be used to arrive at a deeper understanding of deformation rings.

The first main goal of this thesis is to translate Mazur's deformation theory of Galois representations to deformations of modules for arbitrary finite dimensional algebras over fields. Particular goals are to find sufficient criteria for the existence of a universal deformation ring of a module and for the preservation of this universal deformation ring by the syzygy operator Ω. The second main goal is to apply this deformation theory to a particular finite dimensional algebra and to demonstrate the power of representation theoretic techniques when computing universal deformation rings.

1.2 Overview

In Chapter 2, many of the definitions and well-known results needed for the remainder of this thesis are provided. For example, we define Frobenius and self-injective algebras, projective covers, the syzygy operator, Ext groups, stable endomorphism rings, quivers, almost split sequences, path algebras and stable Auslander-Reiten quivers. We state Schlessinger's criteria for the pro-representability of Artin functors. We mention Morita's Theorem for the equivalence of module categories over algebras and Gabriel's Theorem for the classification of basic algebras over an algebraically closed field.

In Chapter 3, we assume Λ to be a finite dimensional algebra over a field k and V to be a finitely generated Λ-module. We provide the definition of the deformation functor F_V over the category $\hat{\mathcal{C}}$ of complete local commutative Noetherian k-algebras with residue field k. Using similar techniques as in [9] and [3], we prove that V always has a versal deformation ring $R(\Lambda, V)$ and that $R(\Lambda, V)$ is universal provided Λ is self-injective and the stable endomorphism ring of V is isomorphic to k. Then, assuming that Λ is a Frobenius algebra and V has stable endomorphism ring k, we prove that the universal deformation rings of V and $\Omega(V)$ are isomorphic.

In Chapter 4, we assume k to be an algebraically closed field and work with a particular special biserial algebra Λ. We find all the Λ-modules with endomorphism ring k, and describe the components of the stable Auslander-Reiten quiver to which they belong. We determine all the Λ-modules V in these components whose stable endomorphism ring is isomorphic to k, and calculate their universal deformation rings $R(\Lambda, V)$.

CHAPTER 2

BACKGROUND

2.1 Module Theory

2.1.1 Frobenius, Symmetric and Self-injective Algebras

Let k be a field and let Λ be a finite dimensional k-algebra. For every Λ-module M, let $M^* = \mathrm{Hom}_k(M, k)$ be the space of k-linear maps from M to k.

Definition 2.1.1. We say that Λ is a *Frobenius* algebra if there exists a k-linear map $\beta : \Lambda \to k$ such that

(i) $\ker(\beta)$ contains no non-zero left or right ideal.

We say that Λ is a *symmetric* algebra if it is a Frobenius algebra, i.e. it satisfies (i), and moreover

(ii) for all $a, b \in \Lambda$, $\beta(ab) = \beta(ba)$.

We say that Λ is *self-injective* if the left regular Λ-module ${}_\Lambda\Lambda$ is an injective Λ-module.

Proposition 2.1.2. *(i) If Λ is a Frobenius algebra over k, then $(\Lambda_\Lambda)^* \cong {}_\Lambda\Lambda$ as left Λ-modules. In particular Λ is self-injective.*

(i) If Λ is a Frobenius algebra over k, then $({}_\Lambda\Lambda)^* \cong \Lambda_\Lambda$ as right Λ-modules.*

(ii) Suppose Λ is self-injective. Then the following conditions on a finitely generated Λ-module M are equivalent:

(a) M is projective;

(b) M is injective;

(c) M^* is projective;

(d) M^* is injective.

Proof. We will prove (i)*, for the other statements see [2, Prop. 1.6.2]. Assume Λ is a Frobenius k-algebra. Then there exists a k-linear map $\beta : \Lambda \to k$ such that $\ker(\beta)$ does not contain a non-zero left or right ideal. Let $\psi : \Lambda_\Lambda \to \mathrm{Hom}_k({}_\Lambda\Lambda, k)$ be defined by $\psi(a) = \psi_a$ for all $a \in \Lambda$, where $\psi_a(x) = \beta(ax)$ for all $x \in \Lambda$. Let $b \in \Lambda$. Then for all $x \in \Lambda$, $\psi_{ab}(x) = \beta(abx) = \psi_a(bx) = (\psi_a b)(x)$. Thus $\psi(ab) = \psi(a)b$. Therefore ψ is a right Λ-module homomorphism. If $\psi(a) = 0$ then for all $x \in \Lambda$, $\beta(ax) = 0$. This implies that $a\Lambda \subseteq \ker(\beta)$. Thus $a\Lambda = 0$, implying $a = 0$. Hence ψ is injective. Since the k-dimensions of Λ_Λ and $({}_\Lambda\Lambda)^*$ are the same and finite, it follows that ψ is an isomorphism of right Λ-modules. □

Lemma 2.1.3. *Let Λ be a Frobenius k-algebra, let R be a commutative k-algebra and define $R\Lambda = R \otimes_k \Lambda$. If P is a finitely generated projective left $R\Lambda$-module then $\mathrm{Hom}_R(P, R)$ is a finitely generated projective right $R\Lambda$-module.*

Proof. Assume first that $P = R\Lambda$. Since Λ is Frobenius, if follows from Proposition 2.1.2 (i)* that $\Lambda \cong \mathrm{Hom}_k(\Lambda, k)$ as right Λ-modules. By the Change of Rings Theorem (see [5, Thm. 2.38]), the map

$$\hat{\psi} : \quad R \otimes_k \Lambda = R\Lambda \longrightarrow \mathrm{Hom}_R(R\Lambda, R) \qquad (2.1)$$

$$r \otimes a \longmapsto m_r \cdot \psi(a)$$

is an isomorphism of R-modules, where m_r is multiplication by r on R, ψ is the right Λ-module isomorphism in the proof of Lemma 2.1.2, and for all $t \in R$ and

$x \in \Lambda$, $(m_r \cdot \psi(a))(t \otimes x) = m_r(t)\psi_a(x) = rt\beta(ax)$. We now show that $\hat{\psi}$ is also a homomorphism of right $R\Lambda$-modules. Let $s \in R$ and $b \in \Lambda$. Then for all $r \in R$ and $a \in \Lambda$,

$$\begin{aligned}\hat{\psi}((r \otimes a)(s \otimes b)) &= \hat{\psi}(rs \otimes ab) \\ &= m_{rs} \cdot \psi(ab) \\ &= (m_r s) \cdot (\psi(a)b) \\ &= (m_r \cdot \psi(a))(s \otimes b) \\ &= (\hat{\psi}(r \otimes a))(s \otimes b).\end{aligned}$$

Thus

$$R\Lambda \cong \operatorname{Hom}_R(R\Lambda, R)$$

as right $R\Lambda$-modules. Therefore, the result follows when $P = R\Lambda$. Assume now that P is a finitely generated free left $R\Lambda$-module, say $P = (R\Lambda)^n$. Then

$$\operatorname{Hom}_R(P, R) = \operatorname{Hom}_R((R\Lambda)^n, R) \cong (\operatorname{Hom}_R(R\Lambda, R))^n$$

as right $R\Lambda$-modules. Using the first part, we see that $(R\Lambda)^n \cong (\operatorname{Hom}_R(R\Lambda, R))^n$ as right $R\Lambda$-modules. Thus $\operatorname{Hom}_R(P, R)$ is a free right $R\Lambda$-module, hence a projective right $R\Lambda$-module. Finally, assume P to be an arbitrary finitely generated projective left $R\Lambda$-module. Then there exist a finitely generated free left $R\Lambda$-module F and a left $R\Lambda$-module Q so that $F \cong Q \oplus P$. Let $(f_Q, f_P) : F \to Q \oplus P$ be an isomorphism of left $R\Lambda$-modules. Applying $\operatorname{Hom}_R(-, R)$, we obtain an isomorphism of right $R\Lambda$-modules

$$(f_Q, f_P)^* : \operatorname{Hom}_R(Q \oplus P, R) \to \operatorname{Hom}_R(F, R).$$

Since $\operatorname{Hom}_R(Q \oplus P, R) \cong \operatorname{Hom}_R(Q, R) \oplus \operatorname{Hom}_R(P, R)$ as right $R\Lambda$-modules, we obtain

$$\operatorname{Hom}_R(F, R) \cong \operatorname{Hom}_R(Q, R) \oplus \operatorname{Hom}_R(P, R)$$

as right $R\Lambda$-modules. Since $\operatorname{Hom}_R(F, R)$ is a free right $R\Lambda$-module, it follows that $\operatorname{Hom}_R(P, R)$ is a projective right $R\Lambda$-module. □

2.1.2 Projective Covers

Definition 2.1.4. Let R be a commutative Artinian ring and let Λ be a finitely generated R-algebra. Assume M and N to be finitely generated Λ-modules.

(i) The *top* of M, denoted by $\operatorname{top}(M)$, is the quotient $M/\operatorname{rad}(M)$ where $\operatorname{rad}(M)$ is the radical of M, i.e. the intersection of all maximal submodules of M. The *socle* of M, denoted by $\operatorname{soc}(M)$, is the submodule of M generated by all simple submodules of M.

(ii) A surjective Λ-module homomorphism $f : M \to N$ is called an *essential* surjection if a homomorphism $g : X \to M$ is surjective whenever $f \circ g : X \to N$ is surjective.

(iii) A *projective cover* of M is a pair (P, π) where P is a projective Λ-module and $\pi : P \to M$ is an essential surjection of Λ-modules. Equivalently, π induces an isomorphism $P/\operatorname{rad}(P) \cong M/\operatorname{rad}(M)$ (see [1, Prop. I.4.3]).

Theorem 2.1.5. *Let Λ as in Definition 2.1.4. Then every finitely generated Λ-module M has a projective cover (P, π). Moreover, if (P_1, π_1) and (P_2, π_2) are two projective covers of M, then there exists a Λ-module isomorphism $\tau : P_1 \to P_2$ with $\pi_2 \circ \tau = \pi_1$.*

Proof. See [1, Thm. I.4.2]. □

Definition 2.1.6. Let Λ be as in Definition 2.1.4, and let M be a finitely generated Λ-module. The *first syzygy* of M, denoted by $\Omega(M)$, is defined to be the kernel of a projective cover $\pi : P \to M$. By Theorem 2.1.5, it is unique up to isomorphism.

2.1.3 Stable Endomorphism Rings

Definition 2.1.7. Let Λ be as in Definition 2.1.4. Let M and N be finitely generated Λ-modules. We denote by $\operatorname{PHom}_\Lambda(M,N)$ the R-submodule of the R-module $\operatorname{Hom}_\Lambda(M,N)$ consisting of those Λ-module homomorphisms from M to N factoring through a projective Λ-module. Define the R-module of *stable* Λ-module homomorphisms from M to N, denoted by $\underline{\operatorname{Hom}}_\Lambda(M,N)$, to be the quotient

$$\underline{\operatorname{Hom}}_\Lambda(M,N) = \operatorname{Hom}_\Lambda(M,N)/\operatorname{PHom}_\Lambda(M,N).$$

In particular, if $M = N$ then the R-module $\underline{\operatorname{End}}_\Lambda(M) = \underline{\operatorname{Hom}}_\Lambda(M,M)$ is also an R-algebra, called the *stable endomorphism ring* of M.

2.1.4 Ext Groups

Definition 2.1.8. Let Λ be a ring, and let M and N be left Λ-modules.

(i) A *projective resolution* of M is an exact sequence

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \to M \tag{2.2}$$

of Λ-modules such that for all $n \in \mathbb{Z}^+ \cup \{0\}$, P_n is a projective Λ-module and $P_0/\operatorname{Im}(\delta_1) \cong M$ as Λ-modules. Note that projective resolutions always exist.

(ii) Let $\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \to M \to 0$ be a projective resolution of M. By applying $\mathrm{Hom}_\Lambda(-, N)$, we obtain an induced sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(P_0, N) &\xrightarrow{\delta_1^*} \mathrm{Hom}_\Lambda(P_1, N) \\ &\xrightarrow{\delta_2^*} \mathrm{Hom}_\Lambda(P_2, N) \xrightarrow{\delta_3^*} \cdots . \end{aligned} \tag{2.3}$$

Define $\mathrm{Ext}^0_\Lambda(M, N) = \ker(\delta_1^*)$ and for all $n \in \mathbb{Z}^+$,

$$\mathrm{Ext}^n_\Lambda(M, N) = \ker(\delta^*_{n+1}) / \operatorname{Im}(\delta^*_n).$$

Note that $\mathrm{Ext}^0_\Lambda(M, N) \cong \mathrm{Hom}_\Lambda(M, N)$ as abelian groups.

Remark 2.1.9. If R and Λ are as in Definition 2.1.4 and M and N are finitely generated Λ-modules, then $\mathrm{Ext}^n_\Lambda(M, N)$ is an R-module for all $n \geq 0$ and $\mathrm{Ext}^0_\Lambda(M, N) \cong \mathrm{Hom}_\Lambda(M, N)$ as R-modules.

Proposition 2.1.10. *Let Λ be a ring. Suppose that*

$$0 \to M' \to M \to M'' \to 0 \tag{2.4}$$

is a short exact sequence of left Λ-modules. If N is a left Λ-module, then there exists a long exact sequence of abelian groups

$$\begin{aligned} 0 \to &\mathrm{Hom}_\Lambda(N, M') \to \mathrm{Hom}_\Lambda(N, M) \to \mathrm{Hom}_\Lambda(N, M'') \\ &\to \mathrm{Ext}^1_\Lambda(N, M') \to \mathrm{Ext}^1_\Lambda(N, M) \to \mathrm{Ext}^1_\Lambda(N, M'') \to \mathrm{Ext}^2_\Lambda(N, M') \cdots \\ &\to \mathrm{Ext}^n_\Lambda(N, M') \to \mathrm{Ext}^n_\Lambda(N, M) \to \mathrm{Ext}^n_\Lambda(N, M'') \to \mathrm{Ext}^{n+1}_\Lambda(N, M') \to \cdots \end{aligned} \tag{2.5}$$

Proof. See [2, Prop. 2.5.3 (ii)]. ∎

Theorem 2.1.11. *Let Λ be a self-injective finite dimensional k-algebra, where k is a field. Let M and N be finitely generated left Λ-modules, and let S be a simple non-projective Λ-module.*

(i) If M has no projective direct summands, then $M \cong \Omega^{-1}(\Omega(M)) \cong \Omega(\Omega^{-1}(M))$ as Λ-modules.

(ii) $\underline{\mathrm{Hom}}_\Lambda(M, N) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega(M), \Omega(N)) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega^{-1}(M), \Omega^{-1}(N))$ as k-vector spaces.

(iii) For all $i \in \mathbb{Z}^+$,

$$\mathrm{Ext}^i_\Lambda(M, N) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega^i(M), N) \tag{2.6}$$

as k-vector spaces.

(iv) $\underline{\mathrm{End}}_\Lambda(S) \cong \mathrm{End}_\Lambda(S)$.

Proof. See [7, Thm. 2.19] □

2.2 Morita Equivalence

Definition 2.2.1. Let Λ and Λ_0 be finite dimensional k-algebras, where k is a field. Let Λ-mod (resp. Λ_0-mod) be the category of finitely generated Λ-modules (resp. Λ_0-modules). We say that Λ and Λ_0 are *Morita equivalent*, denoted by $\Lambda \sim_M \Lambda_0$, if the categories Λ-mod and Λ_0-mod are equivalent categories.

Definition 2.2.2. Let k be a field and let Λ be a finite dimensional k-algebra. Then Λ is called a *basic algebra* if $\Lambda = P_1 \oplus \cdots \oplus P_n$ as left Λ-modules, where $P_1, \ldots, P_n$ are pairwise non-isomorphic projective indecomposable Λ-modules. If k is algebraically closed, then Λ is basic if and only if all simple Λ-modules are one-dimensional over k.

Theorem 2.2.3. *Let k be an algebraically closed field. Two basic k-algebras are isomorphic if and only if they are Morita equivalent.*

Proof. See [6, Lemma I.2.6]. □

Theorem 2.2.4. *Let k be an algebraically closed field. If Λ is a finite dimensional k-algebra, then there is a unique basic algebra Λ_0 up to isomorphism with $\Lambda \sim_M \Lambda_0$. We call Λ_0 the basic algebra of Λ.*

Proof. See [6, Cor. I.2.7]. □

2.3 Quivers and Path Algebras

2.3.1 Representations of Quivers

Definition 2.3.1. (i) A *quiver* Q is a directed graph $Q = (Q_0, Q_1, s, e)$ where Q_0 is the set of vertices and Q_1 is the set of arrows, and s, e are maps from Q_1 to Q_0 as follows. For any arrow $\alpha \in Q_1$, $s(\alpha)$ is the vertex where α starts and $e(\alpha)$ is the vertex where α ends.

(ii) Suppose Q is a quiver and let k be a field. A *representation* $\mathcal{V}$ of the quiver Q over k is given by (V_i, φ_α) where for any vertex $i \in Q_0$ we have a k-vector space V_i, and for any arrow $i \xrightarrow{\alpha} j$ there is a k-linear transformation $\varphi_\alpha : V_i \to V_j$. Let $\mathcal{V} = (V_i, \varphi_\alpha)$ and $\mathcal{V}' = (V_i', \varphi_\alpha')$ be representations of Q over k. Then a morphism $\eta : \mathcal{V} \to \mathcal{V}'$ is defined to be $\eta = (\eta_i)$, where $\eta_i : V_i \to V_i'$ is a k-linear transformation such that for any arrow $i \xrightarrow{\alpha} j$ there is a commutative diagram

$$\begin{array}{ccc} V_i & \xrightarrow{\varphi_\alpha} & V_j \\ \Big\downarrow{\scriptstyle \eta_i} & & \Big\downarrow{\scriptstyle \eta_j} \\ V_i' & \xrightarrow{\varphi_\alpha'} & V_j' \end{array} \tag{2.7}$$

that is, $\varphi'_\alpha \circ \eta_i = \eta_j \circ \varphi_\alpha$. Denote by $\mathrm{Rep}_k(Q)$ the category of all representations of Q over k.

2.3.2 The Path Algebra of a Quiver

Let Q be a quiver and let k be a field.

Definition 2.3.2. Let $i, j \in Q_0$. A *path* of length $l \geq 1$ from i to j is of the form $(j|\alpha_l, \ldots, \alpha_1|i)$ with arrows α_r satisfying $e(\alpha_r) = s(\alpha_{r+1})$ for all r with $1 \leq r \leq l-1$. We also define for any vertex i of Q a path of length zero (from i to itself), denoted by e_i. The path algebra kQ of Q is defined to be the k-vector space with basis given by the set of all paths in Q. The product of two paths is taken to be the composition if it exists, and zero otherwise. In this way, we obtain an associative k-algebra which has an identity if and only if Q_0 is finite. If Q_0 is finite then the identity is given by $\sum\limits_{e \in Q_0} e$. Note that the path algebra kQ is finite dimensional over k if and only if Q is finite and there is no cyclic path in Q. We denote by J the ideal of kQ generated by all arrows in Q. Then J^n is the ideal of kQ generated by all paths of length greater than or equal to n. We denote by kQ-mod the category of all finitely generated kQ-modules.

2.3.3 Quiver with Relations

Definition 2.3.3. Let Q be a quiver and let k be a field.

(i) Let i and j be vertices of Q. A *relation* θ on Q is an element $\theta = \sum c_w w \in kQ$ where the w are paths between two fixed vertices and $c_w \in k$ for all these w. If $\{\theta_\iota\}_\iota$ is a set of relations on Q then $(Q, \{\theta_\iota\}_\iota)$ is called a *quiver with relations*.

(ii) If $w = (j|\alpha_l, \ldots, \alpha_1|i)$ is a path in Q and $\mathcal{V} = (V_i, \varphi_\alpha)$ is a representation of Q

over k, then w acts on $\mathcal{V}$ via the linear transformation $w(\mathcal{V}) = \varphi_{\alpha_l} \circ \cdots \circ \varphi_{\alpha_1}$. In general, if ρ is a relation on Q, say $\rho = \sum c_n w_n$ where $c_n \in k$ and each w_n is a path, then $\rho(\mathcal{V}) = \sum c_n w_n(\mathcal{V})$.

(iii) Given a quiver with relations $(Q, \{\theta_\iota\}_\iota)$ and a representation $\mathcal{V} = (V_i, \varphi_\alpha)$ of Q over k, then $\mathcal{V}$ is called a *representation* of $(Q, \{\theta_\iota\}_\iota)$ if for all ι we have $\theta_\iota(\mathcal{V}) = 0$. Denote by $\operatorname{Rep}_k(Q, \{\theta_\iota\}_\iota)$ the category of all representations of the quiver with relations $(Q, \{\theta_\iota\}_\iota)$ over k.

Theorem 2.3.4. *Let Q be a quiver, let k be a field and let $(Q, \{\theta_\iota\}_\iota)$ be a quiver with relations.*

(i) The categories $\operatorname{Rep}_k(Q)$ and kQ-mod are equivalent.

(ii) The categories $\operatorname{Rep}_k(Q, \{\theta_\iota\}_\iota)$ and kQ/I-mod are equivalent where I is the ideal of kQ generated by the relations $\{\theta_\iota\}_\iota$.

Proof. See [1, Thm. III.1.5, Prop. III.1.7]. □

Theorem 2.3.5 (Gabriel)**.** *Let k be an algebraically closed field. Any basic finite dimensional k-algebra is of the form kQ/I for a unique quiver Q and some ideal I with $J^n \subseteq I \subseteq J^2$ for some $n \geq 2$, where J is the ideal of kQ generated by all arrows of Q.*

Proof. See [1, Cor. III.1.10]. □

2.4 Almost Split Sequences and Auslander-Reiten Quivers

Let Λ be a finite dimensional algebra over a field k.

Definition 2.4.1. A non-split exact sequence $0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0$ of finitely generated Λ-modules with M and N indecomposable is called an *almost split sequence*, if for any finitely generated Λ-module M' and any Λ-module homomorphism $h : M \to M'$ which is not a split monomorphism, there exists a Λ-module homomorphism $h' : L \to M'$ such that $h' \circ f = h$.

Theorem 2.4.2. *(i) If N is an indecomposable non-projective Λ-module, then there is an almost split sequence ending in N.*

(ii) If M is an indecomposable non-injective Λ-module, then there is an almost split sequence starting in M.

Proof. See [1, Thm. V.1.15] □

Definition 2.4.3. If $0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0$ is an almost split sequence, we define $\tau N = M$ and say that τN is the *Auslander translate* of N. Similarly, we define $\tau^{-1} M = N$. If the k-algebra Λ is symmetric, then $\tau N \cong \Omega^2(N)$ as Λ-modules (see [1, Thm. V.1.15 and Prop. IV.3.8]).

Definition 2.4.4. Suppose M and N are finitely generated Λ-modules. A Λ-module homomorphism $f : M \to N$ is said to be *irreducible* provided f is neither a split monomorphism nor a split epimorphism, and given a factorization $f = g \circ h$ of f, then either g is a split epimorphism or h is a split monomorphism.

Theorem 2.4.5. *Let $0 \to M \xrightarrow{f} L \xrightarrow{g} N \to 0$ be an almost split sequence.*

(i) The irreducible maps starting at M are of the form $f' : M \to L'$ where L' is a non-zero direct summand of L, say $L = L' \oplus L''$ and $f = \begin{pmatrix} f' \\ f'' \end{pmatrix}$ for some $f'' : M \to L''$.

(ii) The irreducible maps ending in N are of the form $g' : L' \to N$ where L' is a non-zero direct summand of L, say $L = L' \oplus L''$, and $g = (g', g'')$ for some $g'' : L'' \to N$.

Proof. See [1, Thm. V.5.3]. □

Definition 2.4.6. Let Λ be a finite dimensional algebra over a field k. The *Auslander-Reiten quiver* of Λ, denoted by $\Gamma(\Lambda)$, is defined to be the quiver whose vertices are the isomorphism classes of indecomposable Λ-modules. In addition, the number of arrows $[M] \to [N]$ in $\Gamma(\Lambda)$ is equal to the k-dimension of the space of irreducible maps from M to N. More precisely, if N is indecomposable and non-projective, then the k-dimension of the space of irreducible maps from M to N is equal to the multiplicity of M as a direct summand of E, where $0 \to \tau N \to E \to N \to 0$ is the almost split sequence ending in N. Similarly, if M is indecomposable and non-injective, then the k-dimension of the space of irreducible maps from M to N is equal to the multiplicity of N as a direct summand of E' where $0 \to M \to E' \to \tau^{-1}M \to 0$ is the almost split sequence starting at M. If N is indecomposable projective, then the k-dimension of the space of irreducible maps from M to N is equal to the multiplicity of M as a direct summand of $\mathrm{rad}(N)$. If M is indecomposable injective, then the k-dimension of the space of irreducible maps from M to N is equal to the multiplicity of N as a direct summand of $M/\operatorname{soc}(M)$. The *stable Auslander-Reiten quiver* $\Gamma_S(\Lambda)$ is obtained from $\Gamma(\Lambda)$ by removing for all projective Λ-modules P and all injective Λ-modules E and all $i \in \mathbb{Z}^+ \cup \{0\}$, $[\tau^{-i}P]$ and $[\tau^i E]$ and all adjacent arrows. In particular, if Λ is self-injective, one removes only the vertices $[P]$ for P projective and the adjacent arrows.

2.5 Special Biserial Algebras

Let k be an algebraically closed field and let Λ be a finite dimensional k-algebra.

Definition 2.5.1. The algebra Λ is *special biserial* provided that its basic algebra kQ/I satisfies the following conditions:

(i) Any vertex of Q is starting point of at most two arrows. Any vertex of Q is end point of at most two arrows.

(ii) Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$. Given an arrow γ, there is at most one arrow β with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$.

The algebra Λ is a *string algebra* if it is special biserial and its basic algebra is of the form kQ/I where I is generated by paths of length greater than or equal to 2.

Remark 2.5.2. Suppose $\Lambda = kQ/I$ is a basic algebra. Let

$$L = \{i \in Q_0 | \Lambda e_i \text{ is injective}\}$$

and define $S = \bigoplus_{i \in L} \mathrm{soc}(\Lambda e_i)$. Then the indecomposable Λ-modules are given by the indecomposable Λ/S-modules together with Λe_i for $i \in L$. Moreover, the Auslander-Reiten quiver $\Gamma(\Lambda/S)$ is obtained from $\Gamma(\Lambda)$ by removing the modules Λe_i with $i \in L$. In the case that Λ is self-injective, $S = \bigoplus_{i \in L} \mathrm{soc}(\Lambda e_i) = \mathrm{soc}(\Lambda)$, and $\Gamma(\Lambda/S)$ is the stable Auslander-Reiten quiver $\Gamma_S(\Lambda)$ (see [6, I.8.11]).

Suppose now that Λ/S is a string algebra. Then it follows that to study the indecomposable non-projective Λ-modules and the stable Auslander-Reiten quiver

$\Gamma_S(\Lambda)$, we may assume that Λ is a string algebra. In particular, this is the case when Λ is special biserial, since then Λ/S is a string algebra (see [6, II.1.3]).

2.5.1 String and Band Modules

Let k be an algebraically closed field and let $\Lambda = kQ/I$ be a basic string algebra.

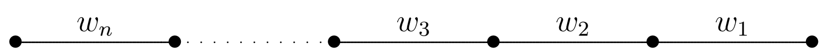
Definition 2.5.3. (i) Given an arrow β of Q, denote by β^{-1} the formal inverse of β. We set $s(\beta^{-1}) = e(\beta)$ and $e(\beta^{-1}) = s(\beta)$, and we write $(\beta^{-1})^{-1} = \beta$. By a *word* of length $n \geq 1$ we mean a sequence $w_n \cdots w_1$, where the w_i are of the form β or β^{-1} with β an arrow, and where $s(w_{i+1}) = e(w_i)$ for $1 \leq i \leq n-1$. We define $(w_n \cdots w_1)^{-1} = {w_1}^{-1} \cdots {w_n}^{-1}$, $s(w_n \cdots w_1) = s(w_1)$ and $e(w_n \cdots w_1) = e(w_n)$. A *rotation* of a word w is a word of the form $w_i \cdots w_1 w_n \cdots w_{i+1}$ for $1 \leq i \leq n$. If v is a vertex of Q, we define an empty word e_v of length zero with $e(e_v) = v = s(e_v)$ and $(e_v)^{-1} = e_v$. On the set of words, we define two equivalence relations: (a) the relation $\sim$ which identifies w with w^{-1}, and (b) the relation $\sim_r$ which identifies a word with its rotations and their inverses.

(ii) A *string* is a representative w of an equivalence class under the relation $\sim$ where either $w = e_v$ for some vertex v of Q, or $w = w_n \cdots w_1$ with $n \geq 1$ and $w_i \neq w_{i+1}^{-1}$ for $i = 1, 2, \ldots, n-1$ and no sub-path of w or its inverse belong to I.

(iii) A *band* is a representative w of an equivalence class under the relation $\sim_r$ where $w = w_n \cdots w_1$ with $n \geq 1$ and $w_{i+1} \neq w_i^{-1}$, $1 \leq i \leq n-1$, $w_n \neq w_1^{-1}$ such that the powers of w are defined, w is not itself a power and w^m has no subword lying in I for any $m \geq 1$.

Definition 2.5.4. For each string or band w, we define an algebra C_w and a functor $\mathrm{G}_w : C_w - \mathrm{mod} \to \Lambda - \mathrm{mod}$. For each C_w-module V we describe $\mathrm{G}_w(V)$ as a representation of a quiver Q_w which defines a Λ-module. On morphisms, the functors are defined in the obvious way.

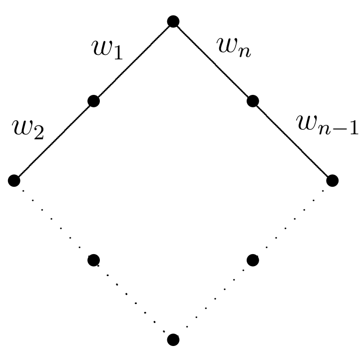
(i) Let w be a string. Then we let $C_w = k$ and take Q_w to be the quiver with underlying graph:



(2.8)

where the edge labeled w_i points to the left if w_i is an arrow and to the right otherwise. For a C_w-module V, we define $\mathrm{G}_w(V)$ to be the representation of Q_w where at each vertex the space is V and for each arrow the map is the identity map.

(ii) Suppose now $w = w_n \cdots w_1$ is a band. Without loss of generality, we may assume that w_n is an arrow. Let $C_w = k[x, x^{-1}]$ and for Q_w take the circular quiver:



(2.9)

where the edge labeled w_i points counterclockwise if w_i is an arrow and clockwise otherwise. Let V be a $k[x, x^{-1}]$-module. We define $\mathrm{G}_w(V)$ to be the representation of Q_w with vector space V at each vertex, such that the linear transformation representing the arrow w_n is x and for every other arrow the map is the identity.

Theorem 2.5.5. *Suppose* Λ *is a string algebra. Then the modules* $\mathrm{G}_w(V)$ *for* w *a string or a band and* V *an indecomposable* C_w*-module form a complete set of representative of indecomposable* Λ*-modules.*

Proof. See [4, Theorem on p. 161]. □

Definition 2.5.6. (i) If C is a string, we write $M(C)$ instead of $\mathrm{G}_C(k)$, and we call $M(C)$ a string module.

(ii) If B is a band, $\lambda \in k^*$, and n is a positive integer, we write $M(B, \lambda, n)$ instead of $\mathrm{G}_B(V_{n,\lambda})$ where $V_{n,\lambda}$ is the $k[x, x^{-1}]$-module which has k-dimension n and on which x acts as the $n \times n$ Jordan block $J_n(\lambda)$. We call $M(B, \lambda, n)$ a band module.

2.5.2 Hooks and Co-hooks

Let k be an algebraically closed field and let $\Lambda = kQ/I$ be a basic string algebra.

Definition 2.5.7. Let C be a string. We say

(i) C *starts on a peak* provided that there is no arrow β with $C\beta$ a string;

(ii) C *starts in a deep* provided that there is no arrow γ with $C\gamma^{-1}$ a string;

(iii) C *ends on a peak* provided that there is no arrow β with $\beta^{-1}C$ a string;

(iv) C *ends in a deep* provided that there is no arrow γ with γC a string.

(v) Assume $C = c_n c_{n-1} \cdots c_1$ with $n \in \mathbb{Z}^+$. We say that C is *directed* if all c_j are arrows, and C is *inverse* if all c_j^{-1} are arrows. We say that C is a *maximal directed string* if C is directed and if for any arrow $\delta \in Q_1$, $\delta C \in I$.

Definition 2.5.8. (i) Assume that C is a string not starting on a peak, say $C\beta$ is a string for some arrow β. Then there is a unique directed string D such that $C\beta D^{-1}$ is a string starting in a deep. We denote $C\beta D^{-1}$ by C_h.

(ii) Assume that C is a string not ending on a peak, say $\beta^{-1}C$ is a string for some arrow β. Then there is a unique directed string D such that $D\beta^{-1}C$ is a string ending in a deep. We denote $D\beta^{-1}C$ by ${}_hC$.

(iii) Assume that C is a string not starting in a deep, say $C\gamma^{-1}$ is a string for some arrow γ. Then there is a unique directed string D such that $C\gamma^{-1}D$ is a string starting on a peak. We denote $C\gamma^{-1}D$ by C_c.

(iv) Assume that C is a string not ending in a deep, say γC is a string for some arrow γ. Then there is a unique directed string D such that $D^{-1}\gamma C$ is a string ending on a peak. We denote $D^{-1}\gamma C$ by ${}_cC$.

Proposition 2.5.9. *The canonical Λ-module homomorphisms $M(C) \to M(C_h)$, $M(C) \to M({}_hC)$ and $M(C_c) \to M(C)$, $M({}_cC) \to M(C)$ are irreducible.*

Proof. See [4, Lemmas on p. 166, p. 168 and p. 169]. □

Definition 2.5.10. We call the following exact sequences "canonical" exact sequences:

(i) Let u be a vertex. For any arrow δ starting at the vertex u, we have an almost split sequnece:

$$0 \to \tau(\Lambda u/\Lambda\delta) \to X \to \Lambda u/\Lambda\delta \to 0 \tag{2.10}$$

where the middle term is indecomposable (see [4, p. 170]). The middle term X is a string module $M(B)$ where B is a string of the form $B = C^{-1}\delta D^{-1}$ and C and D are maximal directed strings.

(ii) Assume that C is a string that neither starts nor ends on a peak. Then ${}_hC$, C_h and ${}_hC_h$ are defined. We have an exact sequence:

$$0 \to M(C) \to M({}_hC) \oplus M(C_h) \to M({}_hC_h) \to 0. \tag{2.11}$$

(iii) If C does not start on a peak but ends on a peak, then $C = {}_cD$ for some string D not starting on a peak. We have an exact sequence:

$$0 \to M(C) \to M(D) \oplus M(C_h) \to M(D_h) \to 0. \tag{2.12}$$

(iv) If C starts on a peak but does not end on a peak, then $C = D_c$ for some string D. We have an exact sequence:

$$0 \to M(C) \to M({}_hC) \oplus M(D) \to M({}_hD) \to 0. \tag{2.13}$$

(v) Finally, suppose C both starts and ends on a peak and suppose $M(C)$ is not injective. Then $C = {}_cD_c$ for some D and we have an exact sequence:

$$0 \to M(C) \to M(D_c) \oplus M({}_cD) \to M(D) \to 0. \tag{2.14}$$

Theorem 2.5.11. *The canonical exact sequences are the almost split sequences containing string modules.*

Proof. See [4, Proposition on p. 172]. □

2.6 Schlessinger's Criteria

Let k be an arbitrary field. Denote by $\hat{\mathcal{C}}$ the category of all complete local commutative Noetherian k-algebras with residue field k. For all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, we fix an isomorphism $R/\mathfrak{m}_R \cong k$, where $\mathfrak{m}_R$ denotes the unique maximal ideal of R. The morphisms of $\hat{\mathcal{C}}$ are continuous k-algebra homomorphisms inducing the identity on k. We denote by $\mathcal{C}$ the full subcategory of $\hat{\mathcal{C}}$ of all Artinian objects in $\hat{\mathcal{C}}$.

Definition 2.6.1. Let $\mathrm{T} : \mathcal{C} \to \mathrm{Sets}$ be a covariant functor such that $\mathrm{T}(k)$ consists of a single element.

(a) A *couple* for T is a pair (A, ζ) where $A \in \mathrm{Ob}(\mathcal{C})$ and $\zeta \in \mathrm{T}(A)$. A *morphism of couples* $u : (A, \zeta) \to (A', \zeta')$ is a morphism $u : A \to A'$ in $\mathcal{C}$ with $\mathrm{T}(u)(\zeta) = \zeta'$.

(b) We can extend T to a functor $\hat{\mathrm{T}} : \hat{\mathcal{C}} \to \mathrm{Sets}$ as follows:

- For all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, $\hat{\mathrm{T}}(R) = \varprojlim_n \mathrm{T}(R/\mathfrak{m}_R^n)$.
- For all $\alpha : R \to R'$ in $\hat{\mathcal{C}}$, $\hat{\mathrm{T}}(\alpha) = \varprojlim_n \mathrm{T}(\alpha_n)$, where for all $n \in \mathbb{Z}^+$, α_n is the induced morphism $\alpha_n : R/\mathfrak{m}_R^n \to R'/\mathfrak{m}_{R'}^n$ in $\mathcal{C}$. A *pro-couple* for T is a pair (B, ζ) where $B \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $\zeta \in \hat{\mathrm{T}}(B)$. A *morphism of pro-couples* $\upsilon : (B, \zeta) \to (B', \zeta')$ is a morphism $\upsilon : B \to B'$ in $\hat{\mathcal{C}}$ with $\hat{\mathrm{T}}(\upsilon)(\zeta) = \zeta'$.

(c) Let (R, ζ) be a pro-couple for T, i.e. $\zeta = \varprojlim_n \zeta_n$ for an inverse system of $\zeta_n \in \mathrm{T}(R/\mathfrak{m}_R^n)$ for $n \in \mathbb{Z}^+$. We say that (R, ζ) *pro-represents* T if the natural transformation $\tau : \mathrm{Hom}_{\hat{\mathcal{C}}}(R, -) \to \mathrm{T}$ is a natural isomorphism, where $\tau =$

$(\tau_A)_{A\in \mathrm{Ob}(\mathcal{C})}$ is defined as follows: For all $A \in \mathrm{Ob}(\mathcal{C})$,

$$\begin{aligned} \tau_A : \quad \mathrm{Hom}_{\hat{\mathcal{C}}}(R, A) &\longrightarrow \mathrm{T}(A) \\ u &\longmapsto \mathrm{T}(u_n)(\zeta_n) \end{aligned} \tag{2.15}$$

where u factors through $u_n : R/\mathfrak{m}_R^n \to A$ for some $n \in \mathbb{Z}^+$, since A is Artinian, and u_n is induced from u. Note that τ_A does not depend on n.

NOTE: This implies that the natural transformation $\hat{\tau} : \mathrm{Hom}_{\hat{\mathcal{C}}}(R, -) \to \hat{\mathrm{T}}$ defined by $\hat{\tau}_B = \varprojlim_n \tau_{B/\mathfrak{m}_B^n}$ is a natural isomorphism. In other words, (R, ζ) represents $\hat{\mathrm{T}}$.

(d) Let $\mathrm{T}' : \mathcal{C} \to \mathrm{Sets}$ be another covariant functor such that $\mathrm{T}'(k)$ consists of a single element. A natural transformation $\eta = (\eta_R)_{R\in \mathrm{Ob}(\mathcal{C})} : \mathrm{T} \to \mathrm{T}'$ is called *smooth* if for every surjection $\alpha : B \to A$ in $\mathcal{C}$, the map

$$\begin{aligned} \mathrm{T}(B) &\longrightarrow \mathrm{T}(A) \times_{\mathrm{T}'(A)} \mathrm{T}'(B) \\ \zeta &\longmapsto (\mathrm{T}(\alpha)(\zeta), \eta_B(\zeta)) \end{aligned} \tag{2.16}$$

is surjective, where $\mathrm{T}(A) \times_{\mathrm{T}'(A)} \mathrm{T}'(B)$ is the pullback of the diagram

$$\begin{array}{ccc} & & \mathrm{T}(A) \\ & & \Big\downarrow \eta_A \\ \mathrm{T}'(B) & \xrightarrow{\mathrm{T}'(\alpha)} & \mathrm{T}'(A) \end{array} \tag{2.17}$$

NOTE: This implies that the natural transformation $\hat{\eta} : \hat{\mathrm{T}} \to \hat{\mathrm{T}}'$ defined by

$$\hat{\eta}_B = \varprojlim_n \eta_{B/\mathfrak{m}_B^n} \tag{2.18}$$

is surjective, in the sense that

$$\hat{\eta}_B : \hat{\mathrm{T}}(B) \longrightarrow \hat{\mathrm{T}}'(B) \tag{2.19}$$

is surjective for all $B \in \mathrm{Ob}(\hat{\mathcal{C}})$.

(e) Let $k[\epsilon]$ with $\epsilon^2 = 0$ be the ring of dual numbers. Then the *tangent space* of T is defined as

$$t_{\mathrm{T}} = \mathrm{T}(k[\epsilon]). \tag{2.20}$$

(f) A pro-couple (R, ζ) of T is called a *pro-representable hull* of T, and a *representable hull* of $\hat{\mathrm{T}}$, if the natural transformation $\tau : \mathrm{Hom}_{\hat{\mathcal{C}}}(R, -) \to \mathrm{T}$ defined in part (c) of Definition 2.6.1 is smooth and $\tau_{k[\epsilon]} : t_R \to t_{\mathrm{T}}$ is bijective, where $t_R = \mathrm{Hom}_{\hat{\mathcal{C}}}(R, k[\epsilon])$.

Theorem 2.6.2 (Schlessinger's Criteria)**.** *Let* $\mathrm{T} : \mathcal{C} \to \mathrm{Sets}$ *be as in Definition 2.6.1. Consider diagrams in* $\mathcal{C}$ *of the form*

$$\begin{array}{ccccc} & & A' \times_A A'' & & \\ & \swarrow^{\beta'} & & \searrow^{\beta''} & \\ A' & & & & A'' \\ & \searrow_{\alpha'} & & \swarrow_{\alpha''} & \\ & & A & & \end{array} \tag{2.21}$$

where $\alpha'' : A'' \to A$ *is a small extension, i.e.* α'' *is surjective and* $\ker(\alpha'') = tA''$ *for some* $t \in A''$ *with* $\mathfrak{m}_{A''} t = 0$*. For each such diagram, consider the natural map of pullbacks*

$$b : \mathrm{T}(A' \times_A A'') \longrightarrow \mathrm{T}(A') \times_{\mathrm{T}(A)} \mathrm{T}(A''). \tag{2.22}$$

Then

(I) T *has a pro-representable hull, and* $\hat{\mathrm{T}}$ *has a representable hull, if and only if* T *has properties* $(H1)$, $(H2)$, $(H3)$ *below:*

(H1) b in (2.22) is always surjective for every diagram (2.21);

(H2) b in (2.22) is bijective if $A = k$ and $A'' = k[\epsilon]$ with $\epsilon^2 = 0$ in (2.21);

(H3) $\dim_k(t_{\mathrm{T}}) < \infty$, where $t_{\mathrm{T}} = \mathrm{T}(k[\epsilon])$ has a natural k-vector space structure using (H2) as follows:

ADDITION.

$$\begin{array}{ccccc} \mathrm{T}(k[\epsilon]) \times_{\mathrm{T}(k)} \mathrm{T}(k[\epsilon]) & \xrightarrow{b^{-1}} & \mathrm{T}(k[\epsilon] \times_k k[\epsilon]) & \xrightarrow{\mathrm{T}(\boxplus)} & \mathrm{T}(k[\epsilon]) \\ \downarrow = & & & & \downarrow = \\ t_{\mathrm{T}} \times t_{\mathrm{T}} & & \dashrightarrow & & t_{\mathrm{T}} \end{array} \tag{2.23}$$

where

$$\begin{array}{rrcl} \boxplus : & k[\epsilon] \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ & (a + b\epsilon, a + c\epsilon) & \longmapsto & a + (b + c)\epsilon \end{array} \tag{2.24}$$

SCALAR MULTIPLICATION. *Let $\lambda \in k$.*

$$\begin{array}{ccccc} \mathrm{T}(k[\epsilon]) \cong \mathrm{T}(k) \times_{\mathrm{T}(k)} \mathrm{T}(k[\epsilon]) & \xrightarrow{b^{-1}} & \mathrm{T}(k \times_k k[\epsilon]) \cong \mathrm{T}(k[\epsilon]) & \xrightarrow{\mathrm{T}(m_\lambda)} & \mathrm{T}(k[\epsilon]) \\ \downarrow = & & & & \downarrow = \\ t_{\mathrm{T}} & & \dashrightarrow & & t_{\mathrm{T}} \end{array} \tag{2.25}$$

where

$$\begin{array}{rrcl} m_\lambda : & k[\epsilon] & \longrightarrow & k[\epsilon] \\ & a + b\epsilon & \longmapsto & a + \lambda b\epsilon. \end{array} \tag{2.26}$$

(II) T is pro-representable, and $\hat{\mathrm{T}}$ is representable, if (H1)-(H3) are satisfied, and T has the additional property (H4):

(H4) b in (2.22) is bijective if $A' = A''$ and $\alpha' = \alpha''$ in (2.21). .

Proof. See [10, Thm. 2.11]. □

CHAPTER 3

DEFORMATION RINGS

3.1 Set Up

Let k be a field and let Λ be a finite dimensional k-algebra. Let V be a fixed left Λ-module of finite k-dimension, and consider the following k-algebra homomorphism:

$$\begin{aligned} \rho: \qquad \Lambda &\longrightarrow \mathrm{End}_k(V) \\ a &\longmapsto (v \mapsto a \cdot v) \end{aligned}$$

Let $\{v_1, \ldots, v_n\}$ be a fixed ordered k-basis of V. Then, relative to this basis, we can identify $\mathrm{End}_k(V)$ with $\mathrm{Mat}_n(k)$. More precisely, for each $a \in \Lambda$ there is a unique matrix $(t_{ij}^a)_{1 \le i,j \le n} \in \mathrm{Mat}_n(k)$ such that

$$a \cdot v_j = t_{1j}^a v_1 + \cdots + t_{nj}^a v_n \tag{3.1}$$

for all $1 \le j \le n$. In other words, $\rho(a) = (t_{ij}^a)$ for all $a \in \Lambda$, and we can identify ρ with the representation

$$\begin{aligned} \rho: \qquad \Lambda &\longrightarrow \mathrm{Mat}_n(k) \\ a &\longmapsto (t_{ij}^a) \end{aligned}$$

corresponding to the Λ-module V. Let k^n be the standard n-dimensional k-vector space with standard k-basis $\{e_1, \ldots, e_n\}$. Then ρ defines a Λ-module structure on k^n by setting $a \cdot x = \rho(a)x$ for all $a \in \Lambda$ and all $x \in k^n$. If we denote k^n with this Λ-structure by ${}_\rho k^n$, then the map $h : {}_\rho k^n \to V$ with $h(e_j) = v_j$ for all $1 \le j \le n$ defines a Λ-module isomorphism. In the following, we will often identify V with ${}_\rho k^n$.

Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian k-algebras with residue field k. In particular, the morphisms in $\hat{\mathcal{C}}$ are continuous k-algebra homomorphisms which induce the identity on k. For each $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, let $\mathfrak{m}_R$ denote its unique maximal ideal, and let $\kappa_R : R \to k$ be the natural surjection of R onto its residue field. We denote by $\mathcal{C}$ the full subcategory of $\hat{\mathcal{C}}$ of all Artinian objects in $\hat{\mathcal{C}}$. For any $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, we denote by $R\Lambda$ the tensor product of k-algebras $R \otimes_k \Lambda$. Note that $R\Lambda$ is an R-algebra. Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, and let M be an $R\Lambda$-module which is a free R-module of rank n. Let $\{m_1, \ldots, m_n\}$ be a fixed ordered R-basis of M. Then, relative to this basis, we can identify $\mathrm{End}_R(M)$ with $\mathrm{Mat}_n(R)$. For each $b \in R\Lambda$, there exists a unique matrix $(r^b_{ij})_{1 \le i,j \le n} \in \mathrm{Mat}_n(R)$ such that

$$b \cdot m_j = r^b_{1j} m_1 + \cdots + r^b_{nj} m_n \tag{3.2}$$

for all $1 \le j \le n$. Thus we get an R-algebra homomorphism

$$\begin{aligned} \widetilde{\pi_M} : \quad R\Lambda &\longrightarrow \mathrm{Mat}_n(R) \\ b &\longmapsto (r^b_{ij}) \end{aligned}$$

corresponding to the $R\Lambda$-module M. Consider the natural injective k-algebra homomorphism

$$\begin{aligned} \iota_R : \quad \Lambda &\longrightarrow R \otimes_k \Lambda = R\Lambda \\ a &\longmapsto 1 \otimes a \end{aligned}$$

Let $\pi_M = \widetilde{\pi_M} \circ \iota_R$. Then $\widetilde{\pi_M}$ is uniquely determined by π_M, since if $b = \sum_{i=1}^{l} s_i \otimes a_i \in R \otimes_k \Lambda = R\Lambda$, then

$$\widetilde{\pi_M}(b) = \sum_{i=1}^{l} s_i \widetilde{\pi_M}(1 \otimes a_i) = \sum_{i=1}^{l} s_i \pi_M(a_i).$$

Let R^n be the standard free R-module with standard R-basis $\{e^R_1, \ldots, e^R_n\}$. Then π_M defines an $R\Lambda$-module structure on R^n by setting

$$b \cdot y = \sum_{i=1}^{l} (s_i \otimes a_i) \cdot y = \sum_{i=1}^{l} s_i \pi_M(a_i) y \tag{3.3}$$

for all $b = \sum_{i=1}^{l} s_i \otimes a_i \in R \otimes_k \Lambda = R\Lambda$ and all $y \in R^n$. If we denote R^n with this $R\Lambda$-module structure by ${}_{\pi_M}R^n$, then the map $H : {}_{\pi_M}R^n \to M$ with $H(e_j^R) = m_j$ for all $1 \le j \le n$ defines an $R\Lambda$-module isomorphism. In the following, we will often identify M with ${}_{\pi_M}R^n$.

Definition 3.1.1. Suppose R, M, $\{m_1, \ldots, m_n\}$ and $\pi_M : \Lambda \to \mathrm{Mat}_n(R)$ are as above. Then we call π_M the *representation of* Λ *corresponding to* M *relative to the* R*-basis* $\{m_1, \ldots, m_n\}$.

3.2 The Deformation Functor

Definition 3.2.1. Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$.

(a) A *lift* of V over R is an $R\Lambda$-module M which is free over R, together with a Λ-module isomorphism $\phi : k \otimes_R M \to V$. We denote such a lift by (M, ϕ).

(b) Two lifts (M, ϕ) and (M', ϕ') of V over R are said to be *isomorphic* if there is an isomorphism $f : M \to M'$ of $R\Lambda$-modules such that the following diagram commutes:

$$\begin{array}{ccc} k \otimes_R M & \xrightarrow{id \otimes f} & k \otimes_R M' \\ & {\scriptstyle \phi} \searrow \quad \swarrow {\scriptstyle \phi'} & \\ & V & \end{array} \tag{3.4}$$

(c) A *deformation* of V over R is an isomorphism class of lifts of V over R. We denote by $\mathrm{Def}_\Lambda(V, R)$ the set of deformations of V over R.

Definition 3.2.2. We define the *deformation functor* for V to be the covariant functor

$$\mathrm{F}_V : \hat{\mathcal{C}} \longrightarrow \mathrm{Sets} \tag{3.5}$$

such that

(a) for all $R \in \hat{\mathcal{C}}$,

$$\mathrm{F}_V(R) = \mathrm{Def}_\Lambda(V, R); \tag{3.6}$$

(b) for each morphism $\alpha : R \to R'$ in $\hat{\mathcal{C}}$, $\mathrm{F}_V(\alpha)$ is the map

$$\begin{aligned} \mathrm{F}_V(\alpha) : \quad \mathrm{Def}_\Lambda(V, R) &\longrightarrow \mathrm{Def}_\Lambda(V, R') \\ [(M, \phi)] &\longmapsto [(M', \phi')] \end{aligned} \tag{3.7}$$

where $M' = R' \otimes_{R,\alpha} M$ and $\phi' : k \otimes_{R'} M' \to V$ is defined to be the composition

$$k \otimes_{R'} M' = k \otimes_{R'} (R' \otimes_{R,\alpha} M) \xrightarrow{\nu^M_{R',\alpha}} k \otimes_R M \xrightarrow{\phi} V. \tag{3.8}$$

Here $\nu^M_{R',\alpha} : k \otimes_{R'} (R' \otimes_{R,\alpha} M) \to k \otimes_R M$ is the natural isomorphism with $\nu^M_{R',\alpha}(\lambda \otimes (r' \otimes m)) = \lambda \kappa_{R'}(r') \otimes m$ for all $\lambda \in k$, $r' \in R'$, $m \in M$. Note that since α induces the identity on k, $\kappa_{R'} \circ \alpha = \kappa_R$.

Our first goal is to rewrite F_V in terms of representations. Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ be arbitrary. Consider the set

$$E(R) = \{\pi : \Lambda \to \mathrm{Mat}_n(R) : \ \pi \text{ is a } k\text{-algebra homomorphism and } \kappa_R \circ \pi = \rho\}, \tag{3.9}$$

where $\kappa_R : \mathrm{Mat}_n(R) \to \mathrm{Mat}_n(k)$ is the surjective ring homomorphism induced by $\kappa_R : R \to k$. Restricting κ_R to $\mathrm{GL}_n(R)$ gives a surjective group homomorphism $\kappa^*_R : \mathrm{GL}_n(R) \to \mathrm{GL}_n(k)$. Let $G(R) = \ker(\kappa^*_R)$. Then $G(R)$ is a group which acts on $E(R)$ by conjugation. If $\pi \in E(R)$, we denote the corresponding element in $E(R)/G(R)$ by $[\pi]$.

Lemma 3.2.3. *Let $\alpha : R \to S$ be a morphism in $\hat{\mathcal{C}}$, and denote the induced ring homomorphism* $\mathrm{Mat}_n(R) \to \mathrm{Mat}_n(S)$ *also by α. Then we have a well-defined map*

$$\begin{aligned} \mathrm{H}(\alpha): \quad E(R)/G(R) &\longrightarrow E(S)/G(S) \\ [\pi] &\longmapsto [\hat{\alpha}(\pi)] \end{aligned} \tag{3.10}$$

where $\hat{\alpha}(\pi) = \alpha \circ \pi$.

Proof. Since $\alpha : R \to S$ is a morphism in $\hat{\mathcal{C}}$, it induces the identity on k. This means that $\kappa_S \circ \alpha = \kappa_R$. Hence for all $\pi \in E(R)$,

$$\kappa_S \circ (\hat{\alpha}(\pi)) = \kappa_S \circ (\alpha \circ \pi) = \kappa_R \circ \pi = \rho,$$

which implies that $\hat{\alpha}(\pi) \in E(S)$. Assume that $[\pi] = [\pi']$ in $E(R)/G(R)$. Then there exists $X \in G(R)$ such that $X\pi X^{-1} = \pi'$. Therefore $\hat{\alpha}(\pi') = \hat{\alpha}(X\pi X^{-1}) = \alpha(X)\hat{\alpha}(\pi)\alpha(X)^{-1}$. Since $\kappa_S(\alpha(X)) = \kappa_R(X) = I_n$, we have that $\alpha(X) \in G(R)$. Thus $[\hat{\alpha}(\pi)] = [\hat{\alpha}(\pi')]$ in $E(S)/G(S)$. □

Remark 3.2.4. In the following, we often write α instead of $\hat{\alpha}$.

Definition 3.2.5. For all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, let $\mathrm{H}(R) = E(R)/G(R) \in \mathrm{Ob}(\mathrm{Sets})$. Let $\alpha : R \to R'$ and $\alpha' : R' \to R''$ be morphisms in $\hat{\mathcal{C}}$. Consider also the identity morphism $id_R : R \to R$. By Lemma 3.2.3, we have the following:

(i) $\mathrm{H}(id_R) = id_{\mathrm{H}(R)}$;

(ii) $\mathrm{H}(\alpha' \circ \alpha) = \mathrm{H}(\alpha') \circ \mathrm{H}(\alpha)$.

Thus we obtain a covariant functor

$$\mathrm{H} : \hat{\mathcal{C}} \longrightarrow \mathrm{Sets} \tag{3.11}$$

Lemma 3.2.6. *For all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ there is a bijection*

$$\tau_R : \mathrm{F}_V(R) \longrightarrow \mathrm{H}(R) \tag{3.12}$$

which sends a deformation $[(M, \phi)]$ of V over R to $[\pi_M]$ where π_M is the representation of Λ corresponding to M relative to a suitable R-basis of M.

Proof. Let (M, ϕ) be a lift of V over R. By Nakayama's Lemma we can lift the k-basis $\{v_1, \ldots, v_n\}$ of V to an R-basis of M, say $\{m_1, \ldots, m_n\}$, such that $\phi(1 \otimes m_j) = v_j$ for all $1 \leq j \leq n$. Let $\pi_M : \Lambda \to \mathrm{Mat}_n(R)$ be the representation of Λ corresponding to M relative to $\{m_1, \ldots, m_n\}$ as defined in Definition 3.1.1. We need to prove that $\kappa_R \circ \pi_M = \rho$. We identify V with ${}_\rho k^n$ and M with ${}_{\pi_M} R^n$ as described in Section 3.1. Then $\phi(1 \otimes e_j^R) = e_j$ for all $1 \leq j \leq n$. Thus for all $a \in \Lambda$ and all $1 \leq j \leq n$,

$$\begin{aligned} \rho(a)e_j &= a \cdot \phi(1 \otimes e_j^R) = \phi(1 \otimes ((1 \otimes a) \cdot e_j^R)) = \phi(1 \otimes (\pi_M(a)e_j^R)) \\ &= \kappa_R(\pi_M(a))\phi(1 \otimes e_j^R) = \kappa_R(\pi_M(a))e_j. \end{aligned}$$

Thus $\rho(a) = \kappa_R(\pi_M(a))$ for all $a \in \Lambda$.

Assume now that (M, ϕ) and (M', ϕ') are two isomorphic lifts of V over R, so that there exists an $R\Lambda$-module isomorphism $f : M \to M'$ satisfying $\phi' \circ (id \otimes f) = \phi$. Let $\{m_1, \ldots, m_n\}$ (respectively $\{m'_1, \ldots, m'_n\}$), be an R-basis of M (respectively M') such that for all $1 \leq j \leq n$

$$\phi(1 \otimes m_j) = v_j = \phi'(1 \otimes m'_j). \tag{3.13}$$

Let $\pi_M, \pi_{M'} : \Lambda \to \mathrm{Mat}_n(R)$ be the representations of Λ corresponding to M and M' relative to $\{m_1, \ldots, m_n\}$ and $\{m'_1, \ldots, m'_n\}$, respectively. Since f is an $R\Lambda$-module isomorphism, there exists a unique matrix $T_f \in \mathrm{GL}_n(R)$ such that for all $a \in \Lambda$

$$T_f \pi_M(a) T_f^{-1} = \pi_{M'}(a). \tag{3.14}$$

Identifying V with ${}_\rho k^n$, M with ${}_{\pi_M} R^n$ and M' with ${}_{\pi_{M'}} R^n$, we get for all $1 \leq j \leq n$,

$$\begin{aligned} e_j = \phi(1 \otimes e_j^R) &= \phi'((id \otimes f)(1 \otimes e_j^R)) = \phi'(1 \otimes T_f e_j^R) \\ &= \kappa_R(T_f)\phi'(1 \otimes e_j^R) = \kappa_R(T_f)e_j. \end{aligned}$$

Thus $\kappa_R^*(T_f) = \kappa_R(T_f) = I_n$. Therefore $T_f \in G(R)$, which implies $[\pi_M] = [\pi_{M'}]$ in $E(R)/G(R) = \mathrm{H}(R)$. So we have a well-defined map

$$\begin{aligned} \tau_R : \qquad \mathrm{F}_V(R) &\longrightarrow \mathrm{H}(R) \\ [(M, \phi)] &\longmapsto [\pi_M]. \end{aligned} \tag{3.15}$$

Claim 1: τ_R is surjective for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$.

Proof of Claim 1. Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and $[\pi] \in E(R)/G(R) = \mathrm{H}(R)$. Then $\pi : \Lambda \to \mathrm{Mat}_n(R)$ is a k-algebra homomorphism with $\kappa_R \circ \pi = \rho$. Consider the free R-module R^n, and let $\{e_1^R, \ldots, e_n^R\}$ be its canonical R-basis. Then π defines an $R\Lambda$-module structure on R^n by setting

$$b \cdot y = \sum_{i=1}^{l} (s_i \otimes a_i) y = \sum_{i=1}^{l} s_i \pi(a_i) y \tag{3.16}$$

for all $b = \sum_{i=1}^{l} s_i \otimes a_i \in R \otimes_k \Lambda = R\Lambda$ and for all $y \in R^n$. We denote R^n with this $R\Lambda$-module structure by ${}_\pi R^n$. Define $\phi : k \otimes_R ({}_\pi R^n) \to V$ by $\phi(1 \otimes e_j^R) = v_j$ for all $1 \leq j \leq n$. Then ϕ is an isomorphism of k-vector spaces. We need to prove that ϕ is a Λ-module homomorphism. Let $a \in \Lambda$ and $1 \leq j \leq n$. Then

$$\begin{aligned}\phi(a \cdot (1 \otimes e_j^R)) &= \phi(1 \otimes \pi(a) e_j^R) \\ &= \kappa_R(\pi(a))\phi(1 \otimes e_j^R) \\ &= \rho(a)\phi(1 \otimes e_j^R) \\ &= a \cdot \phi(1 \otimes e_j^R).\end{aligned}$$

Thus ϕ is a Λ-module isomorphism. So $({}_\pi R^n, \phi)$ is a lift of V over R. By definition of τ_R, we have that $\tau_R([({}_\pi R^n, \phi)]) = [\pi_{{}_\pi R^n}]$, where $\pi_{{}_\pi R^n}$ is the representation of Λ corresponding to ${}_\pi R^n$ relative to $\{e_1^R, \ldots, e_n^R\}$ as in Definition 3.1.1. It follows that $\pi_{{}_\pi R^n} = \pi$, and hence $[\pi] = [\pi_{{}_\pi R^n}] = \tau_R([({}_\pi R^n, \phi)])$. Therefore τ_R is surjective for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$. This proves Claim 1.

Claim 2: τ_R is injective for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$.

Proof of Claim 2. Let (M, ϕ) and (M', ϕ') be lifts of V over R, and let $\tau_R([(M, \phi)]) = [\pi_M]$ and $\tau_R([(M', \phi')]) = [\pi_{M'}]$. Let the corresponding R-bases of M and M' be $\{m_1, \ldots, m_m\}$ and $\{m'_1, \ldots, m'_n\}$, respectively, and assume that $[\pi_M] = [\pi_{M'}]$. Then there exists a matrix $U \in G(R)$ such that $U\pi_M(a)U^{-1} = \pi_{M'}(a)$ for all $a \in \Lambda$. Define $f : M \to M'$ by

$$f(m_j) = u_{1j}m'_1 + \cdots + u_{nj}m'_n. \tag{3.17}$$

for all $1 \leq j \leq n$. Since U is invertible, we have that f is an R-module isomorphism. Identifying M with ${}_{\pi_M}R^n$ and M' with ${}_{\pi_{M'}}R^n$, we get for all $b = \sum_{i=1}^{l} s_i \otimes a_i \in R \otimes_k \Lambda = R\Lambda$ and for all $1 \leq j \leq n$,

$$f(b \cdot e_j^R) = U\left(\sum_{i=1}^{l} s_i \pi_M(a_i) e_j^R\right) = \sum_{i=1}^{l} s_i U \pi_M(a_i) e_j^R = \sum_{i=1}^{l} s_i \pi_{M'}(a_i) U e_j^R = b \cdot f(e_j^R).$$

Hence $f : M \to M'$ is an $R\Lambda$-module isomorphism. Now, for all $1 \le j \le n$ we have

$$\begin{aligned}
\phi'((id \otimes f)(1 \otimes e_j^R)) &= \phi(1 \otimes f(e_j^R)) \\
&= \phi(1 \otimes U e_j^R) \\
&= \kappa_R(U)\phi(1 \otimes e_j^R) \\
&= \phi(1 \otimes e_j^R).
\end{aligned}$$

Hence $\phi = \phi' \circ (id \otimes f)$. Therefore the lifts (M, ϕ) and (M', ϕ') are isomorphic. Thus $[(M, \phi)] = [(M', \phi')]$ in $\mathrm{Def}_\Lambda(V, R) = \mathrm{F}_V(R)$. This proves Claim 2. $\square$

Proposition 3.2.7. *Let $\alpha : R \to R'$ be a morphism in $\hat{\mathcal{C}}$. Then we have the following commutative diagram:*

$$\begin{array}{ccc}
\mathrm{H}(R) & \xrightarrow{\mathrm{H}(\alpha)} & \mathrm{H}(R') \\
\Big\downarrow{\scriptstyle \eta_R} & & \Big\downarrow{\scriptstyle \eta_{R'}} \\
\mathrm{F}_V(R) & \xrightarrow{\mathrm{F}(\alpha)} & \mathrm{F}_V(R')
\end{array} \tag{3.18}$$

where $\eta_R = \tau_R^{-1}$ for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and τ_R is the bijection from Lemma 3.2.6. In other words, $\eta_R([\pi]) = [({}_\pi R^n, {}_\pi\phi)]$ where ${}_\pi\phi : k \otimes_R ({}_\pi R^n) \to V$ is defined by ${}_\pi\phi(1 \otimes e_j^R) = v_j$ for all $1 \le j \le n$.

Proof. Let $[\pi] \in \mathrm{H}(R)$. Then $\mathrm{H}(\alpha)([\pi]) = [\hat{\alpha}(\pi)] = [\alpha \circ \pi] \in \mathrm{H}(R')$, and $\eta_{R'}([\alpha \circ \pi]) = [({}_{\alpha\circ\pi}(R')^n, {}_{\alpha\circ\pi}\,\phi')]$. On the other hand, $\eta_R([\pi]) = [({}_\pi R^n, {}_\pi\,\phi)]$. Then

$$\mathrm{F}_V(\alpha)([({}_\pi R^n, {}_\pi\,\phi)] = [(R' \otimes_{R,\alpha} ({}_\pi R^n), \widetilde{{}_\pi\phi})]$$

where $\widetilde{{}_\pi\phi}$ is the composition $k \otimes_{R'} (R' \otimes_{R,\alpha} ({}_\pi R^n)) \xrightarrow{\nu_{R',\alpha}^{{}_\pi R^n}} k \otimes_R ({}_\pi R^n) \xrightarrow{{}_\pi\phi} V$ and $\nu_{R',\alpha}^{{}_\pi R^n}$ is as in (3.8). Let $f : R' \otimes_{R,\alpha} ({}_\pi R^n) \to {}_{\alpha\circ\pi}(R')^n$ be the map defined by $f(r' \otimes y) = r'\alpha(y)$ for all $r' \in R'$ and $y \in R^n$. Then f is an R'-module isomorphism. We need to prove that f is an $R'\Lambda$-module homomorphism. Indeed, let $r' \in R'$, $b' = \sum_{i=1}^{l} s_i' \otimes a_i \in R' \otimes_k \Lambda = R'\Lambda$ and $y \in R^n$. Then

$$\begin{aligned} f(b' \cdot (r' \otimes y)) &= \sum_{i=1}^{l} f(s_i' r' \otimes \pi(a_i)y) \\ &= \sum_{i=1}^{l} s_i' r' \alpha(\pi(a_i)y) \\ &= \sum_{i=1}^{l} s_i' r' \alpha(\pi(a_i))\alpha(y) \\ &= \sum_{i=1}^{l} s_i' \alpha(\pi(a_i)) r' \alpha(y) \\ &= b' \cdot f(r' \otimes y). \end{aligned}$$

Hence f is an $R'\Lambda$-module isomorphism. On the other hand, note that $\{1_{R'} \otimes e_1^R, \ldots, 1_{R'} \otimes e_n^R\}$ is an R'-basis of $R' \otimes_{R,\alpha} ({}_\pi R^n)$ satisfying $\widetilde{{}_\pi\phi}(1 \otimes (1_{R'} \otimes e_j^R)) = v_j$ for all $1 \leq j \leq n$. Also, ${}_{\alpha\circ\pi}\phi((id \otimes f)(1 \otimes (1_{R'} \otimes e_j^R))) = {}_{\alpha\circ\pi}\phi(1 \otimes f(1_{R'} \otimes e_j^R)) = {}_{\alpha\circ\pi}\phi(1 \otimes \alpha(e_j^R)) = {}_{\alpha\circ\pi}\phi(1 \otimes e_j^{R'}) = v_j$ for all $1 \leq j \leq n$. Therefore, $\widetilde{{}_\pi\phi} = {}_{\alpha\circ\pi}\phi \circ (id \otimes f)$, and hence $[({}_{\alpha\circ\pi}(R')^n, {}_{\alpha\circ\pi}\phi')] = [(R' \otimes_{R,\alpha} ({}_\pi R^n), \widetilde{{}_\pi\phi})]$. This implies $\eta_{R'} \circ \mathrm{H}(\alpha) = \mathrm{F}_V(\alpha) \circ \eta_R$. □

Note that for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, η_R is a bijection, hence an isomorphism in the category Sets. Thus we have the following consequence of Proposition 3.2.7.

Corollary 3.2.8. *There is a natural isomorphism* $\eta : \mathrm{H} \to \mathrm{F}_V$*, where for all* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, $\eta_R : \mathrm{H}(R) \to \mathrm{F}_V(R)$ *is as in Proposition 3.2.7.*

3.3 Schlessinger's Criteria (H1), (H2) and (H4)

In this section, we check Schlessinger's criteria (H1), (H2) and (H4) for the functor H. Recall that $\mathcal{C}$ is the full subcategory of $\hat{\mathcal{C}}$ of Artinian objects.

Lemma 3.3.1. *Assume that $\alpha : R \to S$ is a surjection in $\mathcal{C}$. Then α induces a surjection $\alpha : G(R) \to G(S)$.*

Proof. Assume $\alpha : R \to S$ is surjective. Then the induced group homomorphism $\alpha : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ is surjective. Let $Y \in G(S)$, i.e. $Y \in \mathrm{GL}_n(S)$ with $\kappa_S(Y) = I_n$. Then there exists $X \in \mathrm{GL}_n(R)$ with $\alpha(X) = Y$. Since α is a morphism in $\mathcal{C}$, $\kappa_S \circ \alpha = \kappa_R$. Hence $\kappa_R(X) = \kappa_S(\alpha(X)) = \kappa_S(Y) = I_n$, and $X \in G(R)$. □

To verify Schlessinger's criteria, consider pullback diagrams in $\mathcal{C}$ of the form

$$\begin{array}{ccccc} & & R' \times_R R'' & & \\ & \overset{\beta'}{\swarrow} & & \overset{\beta''}{\searrow} & \\ R' & & & & R'' \\ & \underset{\alpha'}{\searrow} & & \underset{\alpha''}{\swarrow} & \\ & & R & & \end{array} \tag{3.19}$$

where α'' is a small extension, i.e. α'' is surjective and $\ker(\alpha'') = tR''$ for some $t \in R''$ with $\mathfrak{m}_{R''}t = 0$. For each such diagram consider the natural map of pullbacks

$$b : \mathrm{H}(R' \times_R R'') \longrightarrow \mathrm{H}(R') \times_{\mathrm{H}(R)} \mathrm{H}(R''). \tag{3.20}$$

Lemma 3.3.2 (H1)**.** *The map b is surjective.*

Proof. Let $([\pi'], [\pi'']) \in \mathrm{H}(R') \times_{\mathrm{H}(R)} \mathrm{H}(R'')$. Then $[\alpha'(\pi')] = [\alpha''(\pi'')]$ in $\mathrm{H}(R) = E(R)/G(R)$, i.e. there exists $X \in G(R)$ with

$$\alpha'(\pi') = X\alpha''(\pi'')X^{-1}. \tag{3.21}$$

Since α'' is surjective, it induces by Lemma 3.3.1 a surjection $\alpha'' : G(R'') \to G(R)$. This means that there exists $X'' \in G(R'')$ with $\alpha''(X'') = X$. Thus $\alpha'(\pi') = \alpha''(X''\pi''X''^{-1})$. Therefore $(\pi', X''\pi''X''^{-1}) \in E(R' \times_R R'')$ and

$$b([(\pi', X''\pi''X''^{-1})]) = ([\pi'], [X''\pi''X''^{-1}]) = ([\pi'], [\pi'']).$$

□

Let $\pi'' \in E(R'')$ with $\alpha''(\pi'') = \pi$. Define

$$G_{\pi''}(R'') = \{X'' \in G(R'') : X''\pi''{X''}^{-1} = \pi''\} \tag{3.22}$$

and

$$G_{\pi}(R) = \{X \in G(R) : X\pi X^{-1} = \pi\}. \tag{3.23}$$

Lemma 3.3.3. *The map b is injective if the map*

$$G_{\pi''}(R'') \longrightarrow G_{\pi}(R) \tag{3.24}$$

induced by α'' is surjective for all $\pi'' \in E(R'')$ with $\alpha''(\pi'') = \pi$.

Proof. Let $\pi''', \widetilde{\pi'''} \in E(R' \times_R R'')$ with

$$([\pi'], [\pi'']) = b([\pi''']) = b([\widetilde{\pi'''}]) = ([\widetilde{\pi'}], [\widetilde{\pi''}]),$$

where $\pi' = \beta'(\pi''')$, $\pi'' = \beta''(\pi''')$, $\widetilde{\pi'} = \beta'(\widetilde{\pi'''})$ and $\widetilde{\pi''} = \beta''(\widetilde{\pi'''})$. Then there exist $X' \in G(R')$ and $X'' \in G(R'')$ with $X'\pi'{X'}^{-1} = \widetilde{\pi'}$ and $X''\pi''{X''}^{-1} = \widetilde{\pi''}$. Let $\overline{X'}$ and $\overline{X''}$ be the images of X' and X'' in $G(R)$ under α' and α'', respectively. Let $\pi = \alpha'(\pi') = \alpha''(\pi'')$ and $\tilde{\pi} = \alpha'(\widetilde{\pi'}) = \alpha''(\widetilde{\pi''})$. Then $\overline{X'}\pi\overline{X'}^{-1} = \tilde{\pi} = \overline{X''}\pi\overline{X''}^{-1}$. Hence

$\overline{X''}^{-1}\overline{X'} \in G_\pi(R)$. By assumption, there exists $Y \in G_{\pi''}(R'')$ with $\overline{Y} = \overline{X''}^{-1}\overline{X'}$. Let $\widetilde{X''} = X''Y$. Then

$$\widetilde{X''}\pi''\widetilde{X''}^{-1} = X''Y\pi''Y^{-1}X''^{-1} = X''\pi''X''^{-1} = \widetilde{\pi''},$$

and

$$\overline{\widetilde{X''}} = \overline{X''}\,\overline{Y} = \overline{X''}\,\overline{X''}^{-1}\overline{X'} = \overline{X'}.$$

Thus X' and $\widetilde{X''}$ define an element X''' in $G(R' \times_R R'')$ and

$$X'''\pi'''X'''^{-1} = \widetilde{\pi'''}.$$

Hence $[\pi'''] = [\widetilde{\pi'''}]$ in $\mathrm{H}(R' \times_R R'')$. □

Lemma 3.3.4 (H2)**.** *The map b is injective if $R = k$ and $R'' = k[\epsilon]$ with $\epsilon^2 = 0$.*

Proof. Let $\pi'' \in E(R'')$. Since $R = k$ and $\alpha''(\pi'') = \rho$, we have $G(R) = G(k) = \{I_n\}$, and thus $G_\rho(R) = G_\rho(k) = \{I_n\}$. Hence the map $G_{\pi''}(R'') \to G_\rho(k)$ is surjective. Thus (H2) follows from Lemma 3.3.3. □

Lemma 3.3.5 (H4)**.** *If $\mathrm{End}_\Lambda(V) \cong k$ then b is injective if $R' = R''$ and $\alpha' = \alpha''$.*

Because of Lemma 3.3.3, to prove Lemma 3.3.5 it suffices to prove the following result:

Lemma 3.3.6. *If $\mathrm{End}_\Lambda(V) \cong k$ then for all $R \in \mathrm{Ob}(\mathcal{C})$ and for all $\pi \in E(R)$, $G_\pi(R)$ consists only of the scalar matrices in $G(R)$.*

Proof. Since R is Artinian, we prove this by induction on the length of R. If $R = k$ then $G(k) = \{I_n\}$, and hence $G_\pi(R) = G_\rho(k) = \{I_n\}$. Now consider a small extension of the form

$$0 \to tR \to R \xrightarrow{\lambda} R_0 \to 0$$

where $R_0 \in \mathrm{Ob}(\mathcal{C})$, $t \in R$ with $\mathfrak{m}_R t = 0$ and $tR \cong k$. By induction assumption, we have for all $\pi \in E(R_0)$ that $G_\pi(R_0)$ consists only of the scalar matrices in $G(R_0)$. Let $\pi \in E(R)$ and $X \in G_\pi(R)$. Then $\kappa_R(X) = I_n$ and $X\pi X^{-1} = \pi$. Let $\lambda(\pi) = \pi_0$ and $\lambda(X) = X_0$. Then $X_0\pi_0 X_0{}^{-1} = \pi_0$, and thus $X_0 \in G_{\pi_0}(R_0)$, which means that X_0 is a scalar matrix. Thus there exists $r_0 \in R_0$ with $X_0 = r_0 \cdot I_n$. Let $r \in R$ with $\lambda(r) = r_0$. Then $X = r \cdot I_n + tB$ for a certain $B \in \mathrm{Mat}_n(R)$. We have

$$r\pi + (tB)\pi = X\pi = \pi X = \pi r + \pi(tB).$$

Thus $(tB)\pi = \pi(tB)$. Hence tB defines an element in $\mathrm{End}_{R\Lambda}(M)$ where M is the $R\Lambda$-module ${}_\pi R^n$. Observe that $\mathrm{Im}(tB) \subseteq tM \cong V$, since $tR \cong k$ and $k \otimes_R M \cong V$. Therefore

$$tB \in \mathrm{Hom}_{R\Lambda}(M, tM) \cong \mathrm{Hom}_{R\Lambda}(M, V).$$

Since $\mathfrak{m}_R$ annihilates V and $M/\mathfrak{m}_R M \cong k \otimes_R M \cong V$, it follows that

$$\mathrm{Hom}_{R\Lambda}(M, V) \cong \mathrm{Hom}_\Lambda(V, V) = \mathrm{End}_\Lambda(V) \cong k.$$

This means that tB is a scalar matrix in tR. Hence there exists $a \in R$ with $tB = taI_n$. Thus $X = (r + ta)I_n$, and X is a scalar matrix. □

Next we want to show that (H4) is also valid in case Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$. Our proof follows the steps used to prove [3, Lemma 2.3]. We need two Lemmas.

Lemma 3.3.7. *Assume that Λ is self-injective. Let $\lambda : R \to R_0$ be a surjective morphism in $\mathcal{C}$. Let M be a finitely generated $R\Lambda$-module which is free over R and* *consider* $M_0 = R_0 \otimes_{R,\lambda} M$. *Assume that* $f_0 \in \mathrm{End}_{R_0\Lambda}(M_0)$ *factors through a projective* $R_0\Lambda$*-module. Then there exists* $f \in \mathrm{End}_{R\Lambda}(M)$ *factoring through a projective* $R\Lambda$*-module such that the following diagram commutes:*

$$\begin{CD} M @>{f}>> M \\ @V{p_M}VV @VV{p_M}V \\ M_0 @>{f_0}>> M_0 \end{CD}$$

where $p_M : M \to M_0$ *is the natural surjection.*

Proof. Since R, R_0 are Artinian, it suffices to prove this Lemma in the case that λ is a small extension in $\mathcal{C}$, i.e. $\lambda : R \to R_0$ is surjective and there exists $t \in R$ such that $\ker(\lambda) = tR \cong k$ and $\mathfrak{m}_R t = 0$. Assume that we have a commutative diagram

$$\begin{array}{ccccc} & & P_0 & & \\ & {\scriptstyle u_0}\nearrow & & \searrow{\scriptstyle v_0} & \\ M_0 & & \xrightarrow{\quad f_0 \quad} & & M_0 \end{array}$$

where P_0 is a projective $R_0\Lambda$-module. Since M is a finitely generated $R\Lambda$-module, we may assume without loss of generality that P_0 is also a finitely generated $R\Lambda$-module. There exist idempotents $\epsilon_1, \dots, \epsilon_l \in R_0\Lambda$ such that

$$P_0 = (R_0\Lambda)\epsilon_1 \oplus \cdots \oplus (R_0\Lambda)\epsilon_l. \tag{3.25}$$

Since $\mathfrak{m}_R \otimes_k \Lambda = \mathrm{rad}(R) \otimes_k \Lambda \subseteq \mathrm{rad}(R\Lambda)$, it follows that $tR \otimes_k \Lambda \subseteq \mathrm{rad}(R\Lambda)$. Moreover, tensoring the short exact sequence $0 \to tR \to R \xrightarrow{\lambda} R_0 \to 0$ of R-modules with Λ over k gives a short exact sequence of R-modules

$$0 \to tR \otimes_k \Lambda \to R \otimes_k \Lambda \xrightarrow{\lambda \otimes id} R_0 \otimes_k \Lambda \to 0.$$

It follows that $(R \otimes_k \Lambda)/(tR \otimes_k \Lambda) \cong R_0 \otimes_k \Lambda = R_0\Lambda$ as R-algebras. Therefore the hypotheses of the Theorem on Lifting Idempotents (see [5, Theorem 6.7]) are satisfied. Hence there exist idempotents $e_1, \ldots, e_l \in R\Lambda$ so that $(\lambda \otimes id)(e_j) = \epsilon_j$ for all $1 \leq j \leq l$. Define

$$P = (R\Lambda)e_1 \oplus \cdots \oplus (R\Lambda)e_l. \tag{3.26}$$

Then P is a projective $R\Lambda$-module satisfying $R_0 \otimes_{R,\lambda} P \cong P_0$. Let $p_P : P \to P_0$ be the natural projection. We have a diagram

$$\begin{array}{ccccc} & & P & & \\ & & \Big\downarrow{\scriptstyle v_0 \circ p_P} & & \\ M & \xrightarrow{p_M} & M_0 & \longrightarrow & 0 \end{array}$$

with exact bottom row. Since P is a projective $R\Lambda$-module, there exists an $R\Lambda$-module homomorphism $v : P \to M$ such that $p_M \circ v = v_0 \circ p_P$. Now consider a projective resolution of the $R\Lambda$-module M

$$\cdots \xrightarrow{\delta_3} Q_2 \xrightarrow{\delta_2} Q_1 \xrightarrow{\delta_1} Q_0 \to M \to 0. \tag{3.27}$$

Applying $\mathrm{Hom}_{R\Lambda}(-, tP)$, we obtain the sequence

$$0 \to \mathrm{Hom}_{R\Lambda}(Q_0, tP) \xrightarrow{\delta_1^*} \mathrm{Hom}_{R\Lambda}(Q_1, tP) \xrightarrow{\delta_2^*} \mathrm{Hom}_{R\Lambda}(Q_2, tP) \xrightarrow{\delta_3^*} \cdots$$

Therefore

$$\mathrm{Ext}^1_{R\Lambda}(M, tP) \cong \ker(\delta_2^*)/\,\mathrm{Im}(\delta_1^*). \tag{3.28}$$

Since all the terms in the projective resolution (3.27) of M are projective as R-modules, tensoring (3.27) with k over R gives a projective resolution of the Λ-module

$k \otimes_R M$

$$\cdots \xrightarrow{id\otimes\delta_3} k \otimes_R Q_2 \xrightarrow{id\otimes\delta_2} k \otimes_R Q_1 \xrightarrow{id\otimes\delta_1} k \otimes_R Q_0 \to k \otimes_R M \to 0. \tag{3.29}$$

Applying $\mathrm{Hom}_\Lambda(-, tP)$, we obtain the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(k \otimes_R Q_0, tP) \xrightarrow{(id\otimes\delta_1)^*} \mathrm{Hom}_\Lambda(k \otimes_R Q_1, tP) \xrightarrow{(id\otimes\delta_2)^*} \\ \mathrm{Hom}_\Lambda(k \otimes_R Q_2, tP) \xrightarrow{(id\otimes\delta_3)^*} \cdots \end{aligned} \tag{3.30}$$

and hence $\mathrm{Ext}^1_\Lambda(k \otimes_R M, tP) \cong \ker((id \otimes \delta_2)^*)/\operatorname{Im}((id \otimes \delta_1)^*)$. Since $\mathfrak{m}_R$ annihilates tP and $Q_i/\mathfrak{m}_R Q_i \cong k \otimes_R Q_i$ for all $i \geq 0$, we obtain for all $i \geq 0$ a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{R\Lambda}(Q_i, tP) & \xrightarrow{\delta^*_{i+1}} & \mathrm{Hom}_{R\Lambda}(Q_{i+1}, tP) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_\Lambda(k \otimes_R Q_i, tP) & \xrightarrow{(id \otimes \delta_{i+1})^*} & \mathrm{Hom}_\Lambda(k \otimes_R Q_{i+1}, tP) \end{array}$$

It follows that

$$\mathrm{Ext}^1_{R\Lambda}(M, tP) \cong \mathrm{Ext}^1_\Lambda(k \otimes_R M, tP). \tag{3.31}$$

Since $tP \cong k \otimes_R P \cong k \otimes_{R_0} P_0$ is a finitely generated projective Λ-module and Λ is assumed to be self-injective, tP is also an injective Λ-module. Hence $\mathrm{Ext}^1_\Lambda(k \otimes_R M, tP) = 0$, and thus

$$\mathrm{Ext}^1_{R\Lambda}(M, tP) = 0. \tag{3.32}$$

Considering the short exact sequence of $R\Lambda$-modules

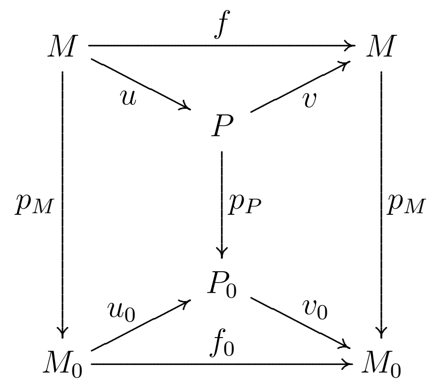
$$0 \to tP \to P \xrightarrow{p_P} P_0 \to 0, \tag{3.33}$$

we obtain the corresponding long exact cohomology sequence

$$0 \to \mathrm{Hom}_{R\Lambda}(M, tP) \to \mathrm{Hom}_{R\Lambda}(M, P) \xrightarrow{(p_P)_*} \tag{3.34}$$

$$\mathrm{Hom}_{R\Lambda}(M, P_0) \to \mathrm{Ext}^1_{R\Lambda}(M, tP) = 0.$$

Thus there exists $u \in \mathrm{Hom}_{R\Lambda}(M, P)$ such that $p_P \circ u = (p_P)_*(u) = u_0 \circ p_M$. Thus, letting $f = v \circ u$, we have a commutative diagram:



□

Lemma 3.3.8. *Assume that Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$. Let $R \in \mathrm{Ob}(\mathcal{C})$, let $\pi \in E(R)$ and let M be the $R\Lambda$-module ${}_\pi R^n$. Then $G_\pi(R)$ consists only of matrices of the form $rI_n + A_f$ where $r \in R$ and $A_f \in \mathrm{Mat}_n(R)$ is the matrix corresponding to a homomorphism $f \in \mathrm{End}_{R\Lambda}(M)$ which factors through a projective $R\Lambda$-module.*

Proof. Since R is Artinian, we prove this by induction on the length of R. If $R = k$ then $G(k) = \{I_n\}$, and hence $G_\pi(R) = G_\rho(k) = \{I_n\}$. Now consider a small extension of the form

$$0 \to tR \to R \xrightarrow{\lambda} R_0 \to 0, \tag{3.35}$$

where $R_0 \in \mathrm{Ob}(\mathcal{C})$, $t \in R$ with $\mathfrak{m}_R t = 0$ and $tR \cong k$. Let $\pi \in E(R)$ and $X \in G_\pi(R)$. Then $\kappa_R(X) = I_n$ and $X\pi X^{-1} = \pi$. Let $\lambda(\pi) = \pi_0$ and $\lambda(X) = X_0$. Then

${X_0}\pi_0{X_0}^{-1} = \pi_0$, and thus $X_0 \in G_{\pi_0}(R_0)$. Let M_0 be the $R_0\Lambda$-module ${}_{\pi_0}(R_0)^n$. By induction, $X_0 = r_0 I_n + A_{f_0}$ for some $r_0 \in R_0$ and some $f_0 \in \mathrm{End}_{R_0\Lambda}(M_0)$ factoring through a projective $R_0\Lambda$-module. Let M be the $R\Lambda$-module ${}_{\pi}R^n$. Then $R_0 \otimes_k M \cong M_0$ as $R_0\Lambda$-modules. By Lemma 3.3.7, we can lift f_0 to an endomorphism $f \in \mathrm{End}_{R\Lambda}(M)$ which factors through a projective $R\Lambda$-module. Let $r \in R$ with $\lambda(r) = r_0$. Then $X = r \cdot I_n + A_f + tB$ for a certain $B \in \mathrm{Mat}_n(R)$. We have

$$r\pi + A_f\pi + (tB)\pi = X\pi = \pi X = \pi r + \pi A_f + \pi(tB).$$

Since $f \in \mathrm{End}_{R\Lambda}({}_{\pi}R^n)$, we have $A_f\pi = \pi A_f$. Thus $(tB)\pi = \pi(tB)$. Hence tB defines an element in $\mathrm{End}_{R\Lambda}(M)$. Observe that $\mathrm{Im}(tB) \subseteq tM \cong V$, since $tR \cong k$ and $k \otimes_R M \cong V$. Therefore

$$tB \in \mathrm{Hom}_{R\Lambda}(M, tM) \cong \mathrm{Hom}_{R\Lambda}(M, V).$$

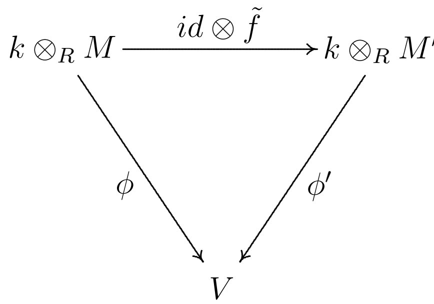
Since $\mathfrak{m}_R$ annihilates V and $M/\mathfrak{m}_R M \cong k \otimes_R M \cong V$, it follows that $\mathrm{Hom}_{R\Lambda}(M, V) \cong \mathrm{End}_\Lambda(V)$. Since $\underline{\mathrm{End}}_\Lambda(V) \cong k \cong tR$, there exists $a \in R$ such that $tB - taI_n$ factors through a projective Λ-module. By Lemma 3.3.7, there exists $g \in \mathrm{End}_{R\Lambda}(M)$ factoring through a projective $R\Lambda$-module such that $tB - taI_n = tA_g$. Thus $X = (r + ta)I_n + A_f + tA_g = (r + ta)I_n + A_{f+tg}$ where $f + tg \in \mathrm{End}_{R\Lambda}(M)$ factors through a projective $R\Lambda$-module. □

Lemma 3.3.9 (H4)**.** *Assume that Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$. Then b is injective if $R' = R''$ and $\alpha' = \alpha''$.*

Proof. By Lemmas 3.3.7 and 3.3.8, $G_{\pi'}(R') \to G_\pi(R)$ is surjective for all $\pi' \in R'$ and $\pi = \alpha'(\pi')$. Thus b is injective by Lemma 3.3.3. □

The following result, which we prove using Lemma 3.3.7, will be useful in Section 3.6.

Lemma 3.3.10. *Assume that* Λ *is self-injective and* $\underline{\mathrm{End}}_\Lambda(V) \cong k$. *Let* $R \in \mathrm{Ob}(\mathcal{C})$ *and* (M, ϕ), (M', ϕ') *be lifts of* V *over* R. *Assume that* $f : M \to M'$ *is an* $R\Lambda$*-module isomorphism. Then* f *induces an isomorphism of* $R\Lambda$*-modules* $\tilde{f} : M \to M'$ *so that the following diagram commutes:*



(3.36)

Proof. Note that the $R\Lambda$-module isomorphism f induces an isomorphism of Λ-modules $id \otimes f : k \otimes_R M \to k \otimes_R M'$. Consider the Λ-module isomorphism $h = \phi^{-1} \circ \phi' \circ (id \otimes f) \in \mathrm{End}_\Lambda(k \otimes_R M)$ and consider $h + \mathrm{PEnd}_\Lambda(k \otimes_R M) \in \underline{\mathrm{End}}_\Lambda(k \otimes_R M)$. Since $k \otimes_R M \cong V$, we have $\underline{\mathrm{End}}_\Lambda(k \otimes_R M) \cong k$. Therefore there exist $c \in k^*$ and a Λ-module homomorphism $g \in \mathrm{End}_\Lambda(k \otimes_R M)$ factoring through a projective Λ-module such that $h = c\, id_{k \otimes_R M} - g$. By Lemma 3.3.7, there exists an $R\Lambda$-module homomorphism $\hat{g} : M \to M$ such that $id \otimes \hat{g} = g$. Let $r \in R$ such that $\kappa_R(r) = c$ and define $\hat{h} = r\, id_M - \hat{g}$. Note that $id \otimes \hat{h} = c\, id_{k \otimes_R M} - g = h$. Since h is a Λ-module isomorphism, it follows by Nakayama's Lemma that $\hat{h}$ is an $R\Lambda$-modules isomorphism.

Define $\tilde{f} = f \circ \hat{h}^{-1} : M \to M'$. Note that $\tilde{f}$ is an $R\Lambda$-module isomorphism. Then

$$\begin{aligned}\phi' \circ (id \otimes \tilde{f}) &= \phi' \circ ((id \otimes f) \circ (id \otimes \hat{h}^{-1})) \\ &= \phi' \circ ((id \otimes f) \circ h^{-1}) \\ &= \phi' \circ ((id \otimes f) \circ (id \otimes f)^{-1} \circ (\phi')^{-1} \circ \phi) \\ &= \phi.\end{aligned}$$

□

3.4 Schlessinger's Criterion (H3)

In this section, we check Schlessinger's criterion (H3) for the functor H. Let $k[\epsilon]$ be the ring of dual numbers with $\epsilon^2 = 0$. Consider the set $\mathrm{H}(k[\epsilon]) = E(k[\epsilon])/G(k[\epsilon])$. Note that for all $[\pi] \in \mathrm{H}(k[\epsilon])$ and for all $a \in \Lambda$ there exists $Y(a) \in \mathrm{Mat}_n(k)$ such that

$$\pi(a) = \rho(a) + \epsilon Y(a). \tag{3.37}$$

By Lemmas 3.3.2 and 3.3.4, we have a bijection

$$b : \mathrm{H}(k[\epsilon] \times_k k[\epsilon]) \longrightarrow \mathrm{H}(k[\epsilon]) \times \mathrm{H}(k[\epsilon]). \tag{3.38}$$

We have shown in the proof of Lemma 3.3.2 that for $[\pi], [\pi'] \in \mathrm{H}(k[\epsilon])$ there exists $X \in G(k[\epsilon])$ depending on π and π' such that

$$b^{-1}([\pi], [\pi']) = [(\pi, X\pi' X^{-1})] \tag{3.39}$$

Now consider the k-algebra homomorphism

$$\begin{aligned}\boxplus : \qquad k[\epsilon] \times_k k[\epsilon] &\longrightarrow k[\epsilon] \\ (r + \epsilon s, r + \epsilon s') &\longmapsto r + \epsilon(s + s')\end{aligned} \tag{3.40}$$

We usually write $(r+\epsilon s)\boxplus(r+\epsilon s')$ instead of $\boxplus((r+\epsilon s, r+\epsilon s'))$. Then $\boxplus$ induces a ring homomorphism $\mathrm{Mat}_n(k[\epsilon]\times_k k[\epsilon])\to \mathrm{Mat}_n(k[\epsilon])$ which we also denote by $\boxplus$. We define an addition on $\mathrm{H}(k[\epsilon])$ as follows. For all $[\pi],[\pi']\in \mathrm{H}(k[\epsilon])$, we set

$$[\pi]+[\pi'] = \mathrm{H}(\boxplus)(b^{-1}([\pi],[\pi'])) = [\hat{\boxplus}(\pi, X\pi' X^{-1})] = [\pi \boxplus X\pi' X^{-1}]. \tag{3.41}$$

Let $\lambda\in k$. Consider the k-algebra homomorphism

$$\begin{aligned} \mu_\lambda : \quad k[\epsilon] &\longrightarrow k[\epsilon] \\ r+\epsilon s &\longmapsto r+\epsilon(\lambda s). \end{aligned} \tag{3.42}$$

We define a scalar multiplication on $\mathrm{H}(k[\epsilon])$ as follows. For all $\lambda\in k$ and all $[\pi]\in \mathrm{H}(k[\epsilon])$, we set

$$\lambda[\pi] = [\mu_\lambda \circ \pi]. \tag{3.43}$$

Then (3.41) and (3.43) define a k-vector space structure on $\mathrm{H}(k[\epsilon])$.

Lemma 3.4.1. *There is a k-linear isomorphism*

$$\mathcal{T} : \mathrm{H}(k[\epsilon]) \to \mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n) \tag{3.44}$$

Proof. We first define the map $\mathcal{T}$. Let $[\pi]\in \mathrm{H}(k[\epsilon])$ and let ${}_\pi(k[\epsilon])^n$ be the corresponding $k[\epsilon]\Lambda$-module. Let $\alpha : k\to k[\epsilon]$ be the injective k-linear map defined by $\alpha(r)=\epsilon r$. Then α defines an injective k-linear map

$$\begin{aligned} \alpha : \quad {}_\rho k^n &\longrightarrow {}_\pi(k[\epsilon])^n \\ x &\longmapsto \epsilon x \end{aligned} \tag{3.45}$$

Moreover, α is a Λ-module homomorphism, since for all $a\in\Lambda$ and all $x\in k^n$,

$$\alpha(a\cdot x) = \alpha(\rho(a)x) = \rho(a)\epsilon x = (\pi(a)-\epsilon Y(a))\epsilon x = \pi(a)\epsilon x = a\cdot\alpha(x)$$

where $\pi(a) = \rho(a) + \epsilon Y(a)$ as in (3.37). Hence ${}_\rho k^n \cong \epsilon\left({}_\pi(k[\epsilon])^n\right)$ as Λ-modules. Let $\beta : k[\epsilon] \to k$ be the surjective k-algebra homomorphism defined by $\beta(r + \epsilon s) = r$. Then β defines a surjective k-linear map

$$\begin{aligned} \beta : {}_\pi(k[\epsilon])^n &\longrightarrow {}_\rho k^n \\ y + \epsilon z &\longmapsto y \end{aligned} \tag{3.46}$$

where $y, z \in k^n$. Moreover, β is a Λ-module homomorphism, since for all $a \in \Lambda$ and all $y, z \in k^n$,

$$\begin{aligned} \beta(a \cdot (y + \epsilon z)) = \beta(\pi(a)(y + \epsilon z)) = \beta((\rho(a) + \epsilon Y(a))y + \epsilon\rho(a)z) = \rho(a)y \\ = a \cdot \beta(y + \epsilon z). \end{aligned}$$

Since $\ker(\beta) = \epsilon\left({}_\pi(k[\epsilon])^n\right)$, we have ${}_\rho k^n \cong {}_\pi(k[\epsilon])^n / \epsilon\left({}_\pi(k[\epsilon])^n\right)$ as Λ-modules. Hence we have a short exact sequence of Λ-modules

$$\zeta_\pi : 0 \to {}_\rho k^n \xrightarrow{\alpha} {}_\pi(k[\epsilon])^n \xrightarrow{\beta} {}_\rho k^n \to 0. \tag{3.47}$$

Let $[\zeta_\pi]$ be the corresponding element in $\mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n)$, and define for all $[\pi] \in \mathrm{H}(k[\epsilon])$, $\mathcal{T}([\pi]) = [\zeta_\pi]$.

Claim 1: $\mathcal{T}$ is well-defined.

Proof of Claim 1. Let $[\pi] = [\pi']$ in $\mathrm{H}(k[\epsilon])$. Then there exists $X \in G(k[\epsilon])$ such that $X\pi X^{-1} = \pi'$. Let ${}_\pi(k[\epsilon])^n$ and ${}_{\pi'}(k[\epsilon])^n$ be the $k[\epsilon]\Lambda$-modules corresponding to π and π', respectively. Define

$$\begin{aligned} g : {}_\pi(k[\epsilon])^n &\longrightarrow {}_{\pi'}(k[\epsilon])^n \\ y &\longmapsto Xy. \end{aligned} \tag{3.48}$$

Then g is a k-linear isomorphism. Moreover, g is a Λ-module isomorphism, since for all $a \in \Lambda$ and all $y \in (k[\epsilon])^n$,

$$g(a \cdot y) = X\pi(a)y = \pi'(a)Xy = a \cdot g(y).$$

We obtain the following diagram of Λ-modules

$$\begin{array}{cccccccccc}
\zeta_\pi : & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{\alpha} & {}_\pi(k[\epsilon])^n & \xrightarrow{\beta} & {}_\rho k^n & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle id} & & \\
\zeta_{\pi'} : & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{\alpha} & {}_{\pi'}(k[\epsilon])^n & \xrightarrow{\beta} & {}_\rho k^n & \longrightarrow & 0
\end{array}$$

Since $X \in G(k[\epsilon])$, there exists $X' \in \mathrm{Mat}_n(k)$ such that $X = I_n + \epsilon X'$. Thus for all $x \in k^n$, $g(\alpha(x)) = \epsilon X x = \epsilon x = \alpha(x)$, and for all $y, z \in k^n$, $\beta(g(y + \epsilon z)) = \beta((I_n + \epsilon X')(y + \epsilon z)) = \beta(y + \epsilon(X'y + z)) = y = \beta(y + \epsilon z)$. Hence the diagram commutes, which implies that $[\zeta_\pi] = [\zeta_{\pi'}]$ in $\mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n)$. Therefore $\mathcal{T}([\pi]) = \mathcal{T}([\pi'])$, which proves Claim 1.

Claim 2: $\mathcal{T}$ is additive.

Proof of Claim 2. Let $([\pi], [\pi']) \in \mathrm{H}(k[\epsilon]) \times \mathrm{H}(k[\epsilon])$. Then by (3.39) and (3.41), there exists $X \in G(k[\epsilon])$ such that

$$[\pi] + [\pi'] = [\pi \boxplus X\pi' X^{-1}]. \tag{3.49}$$

Let $\theta = \pi \boxplus X\pi' X^{-1}$. Then $\mathcal{T}([\theta]) = [\zeta_\theta]$ where ζ_θ is the extension of Λ-modules

$$\zeta_\theta : 0 \to {}_\rho k^n \xrightarrow{\alpha} {}_\theta(k[\epsilon])^n \xrightarrow{\beta} {}_\rho k^n \to 0. \tag{3.50}$$

Let ${}_\pi(k[\epsilon])^n$ and ${}_{X\pi' X^{-1}}(k[\epsilon])^n$ be the $k[\epsilon]\Lambda$-modules corresponding to π and $X\pi' X^{-1}$, respectively. Then $\mathcal{T}([\pi]) = [\zeta_\pi]$ and $\mathcal{T}([\pi']) = [\zeta_{\pi'}]$ where ζ_π and $\zeta_{\pi'}$ are the extensions of Λ-modules

$$\zeta_\pi : 0 \to {}_\rho k^n \xrightarrow{\alpha} {}_\pi(k[\epsilon])^n \xrightarrow{\beta} {}_\rho k^n \to 0 \tag{3.51}$$

and

$$\zeta_{X\pi'X^{-1}} : 0 \to {}_\rho k^n \xrightarrow{\alpha} {}_{X\pi'X^{-1}}(k[\epsilon])^n \xrightarrow{\beta} {}_\rho k^n \to 0. \tag{3.52}$$

We now describe the Baer sum $\zeta_\pi \underset{B}{+} \zeta_{X\pi'X^{-1}}$ of ζ_π and $\zeta_{X\pi'X^{-1}}$. Let

$$M = ({}_\pi(k[\epsilon])^n) \times_{{}_\rho k^n} ({}_{X\pi'X^{-1}}(k[\epsilon])^n) \tag{3.53}$$

be the pullback of

$$\begin{array}{ccc} & & {}_\pi(k[\epsilon])^n \\ & & \downarrow \beta \\ {}_{X\pi'X^{-1}}(k[\epsilon])^n & \xrightarrow{\beta} & {}_\rho k^n \end{array} \tag{3.54}$$

Thus $M = \{(y+\epsilon z, y+\epsilon z') : y, z, z' \in k^n\}$ is a submodule of ${}_\pi(k[\epsilon])^n \times {}_{X\pi'X^{-1}}(k[\epsilon])^n$. Let $B = \{(\epsilon z, -\epsilon z) : z \in k^n\}$. Then B is a Λ-submodule of M, and the Baer sum $\zeta_\pi \underset{B}{+} \zeta_{X\pi'X^{-1}}$ is the extension

$$\zeta_\pi \underset{B}{+} \zeta_{X\pi'X^{-1}} : 0 \to {}_\rho k^n \xrightarrow{\bar\alpha} M/B \xrightarrow{\bar\beta} {}_\rho k^n \to 0$$

where $\bar\alpha(x) = (\epsilon x, 0) + B$ for all $x \in k^n$ and $\bar\beta((y+\epsilon z, y+\epsilon z') + B) = y$ for all $y, z \in k^n$ (see [11, p. 78]). Note that $[\zeta_\pi] + [\zeta_{X\pi'X^{-1}}] = [\zeta_\pi \underset{B}{+} \zeta_{X\pi'X^{-1}}]$ in $\mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n)$. Let

$$\begin{array}{lccc} f : & M & \longrightarrow & {}_\theta(k[\epsilon])^n \\ & (y+\epsilon z, y+\epsilon z') & \longmapsto & (y+\epsilon z) \boxplus (y+\epsilon z') \end{array} \tag{3.55}$$

Then f is a surjective k-linear map. We obtain that f is a Λ-module homomorphism, since for all $a \in \Lambda$ and all $(y + \epsilon z, y + \epsilon z') \in M$,

$$\begin{aligned} f(a \cdot (y + \epsilon z, y + \epsilon z')) &= f(\pi(a)(y + \epsilon z), X\pi'(a)X^{-1}(y + \epsilon z')) \\ &= \pi(a)(y + \epsilon z) \boxplus X\pi'(a)X^{-1}(y + \epsilon z') \\ &= (\pi(a) \boxplus X\pi'(a)X^{-1})((y + \epsilon z) \boxplus (y + \epsilon z')) \\ &= \theta(a)((y + \epsilon z) \boxplus (y + \epsilon z')) \\ &= a \cdot f((y + \epsilon z, y + \epsilon z')). \end{aligned}$$

Moreover, $\ker(f) = B$. Hence f induces a Λ-module isomorphism $\bar{f} : M/B \to {}_{\theta}(k[\epsilon])^n$. We get the following diagram of Λ-modules:

$$\begin{array}{lccccccccc} \zeta_\pi \underset{B}{+} \zeta_{X\pi'X^{-1}} : & 0 & \longrightarrow & {}_{\rho}k^n & \xrightarrow{\bar{\alpha}} & M/B & \xrightarrow{\bar{\beta}} & {}_{\rho}k^n & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle id} & & \\ \zeta_\theta : & 0 & \longrightarrow & {}_{\rho}k^n & \xrightarrow{\alpha} & {}_{\theta}(k[\epsilon])^n & \xrightarrow{\beta} & {}_{\rho}k^n & \longrightarrow & 0 \end{array}$$

For all $x \in k^n$, $\bar{f}(\bar{\alpha}(x)) = \bar{f}((\epsilon x, 0) + B) = \epsilon x = \alpha(x)$ and for all $y, z, z' \in k^n$, $\beta(\bar{f}((y + \epsilon z, y + \epsilon z') + B)) = \beta(y + \epsilon(z + z')) = y = \bar{\beta}((y + \epsilon z, y + \epsilon z') + B)$. Hence the diagram commutes, which implies that $[\zeta_\theta] = [\zeta_\pi \underset{B}{+} \zeta_{X\pi'X^{-1}}]$. Thus

$$\begin{aligned} \mathcal{T}([\pi] + [\pi']) = \mathcal{T}([(\pi \boxplus X\pi'X^{-1})]) = \mathcal{T}([\theta]) = [\zeta_\theta] = [\zeta_\pi \underset{B}{+} \zeta_{X\pi'X^{-1}}] \\ = [\zeta_\pi] + [\zeta_{X\pi'X^{-1}}] = [\zeta_\pi] + [\zeta_{\pi'}] = \mathcal{T}([\pi]) + \mathcal{T}([\pi']). \end{aligned}$$

Claim 3: $\mathcal{T}$ is k-linear.

Proof of Claim 3. Let $\lambda \in k$ and $[\pi] \in \mathrm{H}(k[\epsilon])$. If $\lambda = 0$ then $\mu_\lambda \circ \pi = \rho$ viewed as $\rho : \Lambda \to \mathrm{Mat}_n(k) \subseteq \mathrm{Mat}_n(k[\epsilon])$. Hence

$$\zeta_\rho : 0 \to {}_{\rho}k^n \xrightarrow{\alpha} {}_{\rho}(k[\epsilon])^n \xrightarrow{\beta} {}_{\rho}k^n \to 0 \tag{3.56}$$

splits. Thus $\mathcal{H}(0\cdot[\pi]) = [\zeta_\rho] = [0]$ in this case. Now assume $\lambda \neq 0$. Let ${}_{\mu_\lambda\circ\pi}(k[\epsilon])^n$ be the $k[\epsilon]\Lambda$-module corresponding to $\mu_\lambda \circ \pi$. Let $m_\lambda : k^n \to k^n$ be multiplication by λ. Then we have a pullback diagram:

$$\begin{array}{rcccccccc}
m_\lambda(\zeta_\pi): & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{(\alpha,0)} & P & \xrightarrow{p_2} & {}_\rho k^n & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle m_\lambda} & & \\
\zeta_\pi: & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{\alpha} & {}_\pi(k[\epsilon])^n & \xrightarrow{\beta} & {}_\rho k^n & \longrightarrow & 0
\end{array}$$

where

$$P = \{(\lambda x + \epsilon y, x) | x, y \in k^n\} \subseteq {}_\pi(k[\epsilon])^n \oplus {}_\rho k^n. \tag{3.57}$$

Note that $\lambda[\zeta_\pi] = [m_\lambda(\zeta_\pi)]$ in $\mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n)$. Define

$$\begin{array}{rccc}
g: & P & \longrightarrow & {}_{\mu_\lambda\circ\pi}(k[\epsilon])^n \\
 & (\lambda x + \epsilon y, x) & \longmapsto & x + \epsilon y.
\end{array} \tag{3.58}$$

Then g is a k-linear isomorphism. Moreover, g is a Λ-module homomorphism, since for all $a \in \Lambda$ and all $(\lambda x + \epsilon y, x) \in P$,

$$\begin{aligned}
g(a \cdot (\lambda x + \epsilon y, x)) &= g((\pi(a)(\lambda x + \epsilon y), \rho(a)x)) \\
&= g((\lambda\rho(a)x + \epsilon(\lambda Y(a)x + \rho(a)y), \rho(a)x)) \\
&= \rho(a)x + \epsilon(\lambda Y(a)x + \rho(a)y) = (\rho(a) + \lambda\epsilon Y(a))(x + \epsilon y) \\
&= (\mu_\lambda \circ \pi)(a)(x + \epsilon y) = a \cdot g((\lambda x + \epsilon y, x))
\end{aligned}$$

where $\pi(a) = \rho(a) + \epsilon Y(a)$ as in (3.37). Thus we have the following diagram of Λ-modules

$$\begin{array}{rcccccccc}
m_\lambda(\zeta_\pi): & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{(\alpha,0)} & P & \xrightarrow{p_2} & {}_\rho k^n & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle id} & & \\
\zeta_{\mu_\lambda\circ\pi}: & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{\alpha} & {}_{\mu_\lambda\circ\pi}(k[\epsilon])^n & \xrightarrow{\beta} & {}_\rho k^n & \longrightarrow & 0
\end{array}$$

For all $x \in k^n$, $g((\alpha, 0)(x)) = g((\epsilon x, 0)) = \epsilon x = \alpha(x)$ and for all $x, y \in k^n$, $\beta(g(\lambda x + \epsilon y, x)) = \beta(x + \epsilon y) = x = p_2((\lambda x + \epsilon y, x))$. Hence the diagram commutes, which implies that $[m_\lambda(\zeta_\pi)] = [\zeta_{\mu_\lambda \circ \pi}]$. Therefore

$$\mathcal{T}(\lambda[\pi]) = \mathcal{T}([\mu_\lambda \circ \pi]) = [\zeta_{\mu_\lambda \circ \pi}] = [m_\lambda(\zeta_\pi)] = \lambda[\zeta_\pi] = \lambda \mathcal{T}([\pi]).$$

Claim 4: $\mathcal{T}$ is surjective.

Proof of Claim 4. Let $[\zeta] \in \mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n)$, corresponding to a short exact sequence of Λ-modules

$$\zeta : 0 \to {}_\rho k^n \xrightarrow{\alpha} M \xrightarrow{\beta} {}_\rho k^n \to 0. \tag{3.59}$$

We define a $k[\epsilon]$-module structure on M by

$$(r + \epsilon s) \cdot m = rm + s(\alpha \circ \beta)(m) \tag{3.60}$$

for all $r, s \in k$ and all $m \in M$. Thus M becomes module for $k[\epsilon]\Lambda = k[\epsilon] \otimes_k \Lambda$ via

$$((r + \epsilon s) \otimes a) \cdot m = (r + \epsilon s)(a \cdot m) = r(a \cdot m) + s(a \cdot (\alpha \circ \beta)(m)) \tag{3.61}$$

for all $a \in \Lambda$, where the last equality follows, since α and β are Λ-module homomorphisms. We now prove that in this way M becomes a free $k[\epsilon]$-module. Consider the standard k-basis $\{e_1, \ldots, e_n\}$ of ${}_\rho k^n$. Since β is surjective, there exist $m_1, \ldots, m_n \in M$ such that $\beta(m_j) = e_j$ for all $1 \le j \le n$. Then $m_1, \ldots, m_n, \epsilon m_1, \ldots, \epsilon m_n$ generate M as a k-vector space, since $\beta(m_1), \ldots, \beta(m_n)$ generate $\mathrm{Im}(\beta)$ and $\epsilon m_1 = \alpha(\beta(m_1)), \ldots, \epsilon m_n = \alpha(\beta(m_n))$ generate $\mathrm{Im}(\alpha) = \ker(\beta)$. This implies that $\{m_1, \ldots, m_n, \epsilon m_1, \ldots, \epsilon m_n\}$ is a k-basis of M, since $\dim_k M = 2n$. It follows that $\{m_1, \ldots, m_n\}$ is a $k[\epsilon]$-basis of M. Define $\pi_\zeta : \Lambda \to \mathrm{Mat}_n(k[\epsilon])$ to be equal to the representation π_M of Λ corresponding to M relative to the $k[\epsilon]$-basis $\{m_1, \ldots, m_n\}$, as described in Definition 3.1.1. Identifying M with ${}_{\pi_\zeta}(k[\epsilon])^n$ and $\{m_1, \ldots, m_n\}$ with the standard basis $\{e_1^{k[\epsilon]}, \ldots, e_n^{k[\epsilon]}\}$, we have for all $a \in \Lambda$ and $1 \leq j \leq n$

$$(\kappa_{k[\epsilon]} \circ \pi_\zeta(a))e_j = \beta(\pi_\zeta(a)m_j) = \beta(a \cdot m_j) = a \cdot e_j = \rho(a)e_j.$$

Thus $\kappa_{k[\epsilon]} \circ \pi_\zeta = \rho$. It follows that $[\pi_\zeta] \in \mathrm{H}(k[\epsilon])$ and $\mathcal{T}([\pi_\zeta]) = [\zeta]$. Therefore, $\mathcal{T}$ is surjective, which proves Claim 4.

Claim 5: $\mathcal{T}$ is injective.

Proof of Claim 5. Suppose $\mathcal{T}([\pi]) = \mathcal{T}([\pi'])$. This means that ζ_π and $\zeta_{\pi'}$ are equivalent extensions, i.e. there is a commutative diagram of Λ-modules

$$\begin{array}{ccccccccc}
\zeta_\pi : & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{\alpha} & {}_\pi(k[\epsilon])^n & \xrightarrow{\beta} & {}_\rho k^n & \longrightarrow & 0 \\
& & & \downarrow id & & \downarrow \varphi & & \downarrow id & & \\
\zeta_{\pi'} : & 0 & \longrightarrow & {}_\rho k^n & \xrightarrow{\alpha} & {}_{\pi'}(k[\epsilon])^n & \xrightarrow{\beta} & {}_\rho k^n & \longrightarrow & 0
\end{array}$$

where φ is a Λ-module isomorphism. Moreover, φ is a $k[\epsilon]\Lambda$-module isomorphism since for all $b = \sum_{i=1}^{l} s_i \otimes a_i \in k[\epsilon] \otimes_k \Lambda = k[\epsilon]\Lambda$ and for all $y \in {}_\pi(k[\epsilon])^n$,

$$\begin{aligned}
\varphi(b \cdot y) &= \varphi\left(\sum_{i=1}^{l} s_i \pi(a_i) y\right) \\
&= \sum_{i=1}^{l} s_i \varphi(a_i \cdot y) \\
&= \sum_{i=1}^{l} s_i (a_i \cdot \varphi(y)) \\
&= \sum_{i=1}^{l} s_i \pi'(a_i) \varphi(y) \\
&= b \cdot \varphi(y).
\end{aligned}$$

Hence there exists $X \in \mathrm{GL}_n(k[\epsilon])$ such that $\varphi(y) = Xy$ for all $y \in {}_\pi(k[\epsilon])^n$ and

$X\pi(a) = \pi'(a)X$ for all $a \in \Lambda$. For all $1 \le j \le n$ we have

$$\kappa_{k[\epsilon]}(X)e_j = \beta\left(Xe_j^{k[\epsilon]}\right) = \beta\left(\varphi\left(e_j^{k[\epsilon]}\right)\right) = \beta\left(e_j^{k[\epsilon]}\right) = e_j.$$

Thus $\kappa_{k[\epsilon]}(X) = I_n$, which means $X \in G(k[\epsilon])$. Hence $[\pi] = [\pi']$ in $\mathrm{H}(k[\epsilon])$, and $\mathcal{T}$ is injective. This proves Claim 5, completing the proof of Lemma 3.4.1. □

Corollary 3.4.2 (H3). *The k-vector space $\mathrm{H}(k[\epsilon])$ has finite k-dimension.*

Proof. Let P be a finitely generated projective Λ-module together with a surjective Λ-module homomorphism $f : P \to {}_\rho k^n$. Then, by dimension shifting, $\mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n)$ is a quotient of $\mathrm{Hom}_\Lambda(\ker(f), {}_\rho k^n)$. Since $\dim_k \mathrm{Hom}_\Lambda(\ker(f), {}_\rho k^n) < \infty$, it follows that $\dim_k(\mathrm{Ext}^1_\Lambda({}_\rho k^n, {}_\rho k^n)) < \infty$. Hence the result follows from Lemma 3.4.1. □

3.5 Continuity of the Deformation Functor

Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$. We have that $(\{R/\mathfrak{m}_R^i\}_{i\in\mathbb{Z}^+}, \{\alpha_{ji}\}_{j\ge i})$ is an inverse system where

$$\alpha_{ji} : R/\mathfrak{m}_R^j \longrightarrow R/\mathfrak{m}_R^i \tag{3.62}$$

is the natural surjection for $j \ge i$. Since R is complete, it follows that

$$R \cong \varprojlim_i R/\mathfrak{m}_R^i. \tag{3.63}$$

Note that $(\{\mathrm{Mat}_n(R/\mathfrak{m}_R^i)\}_{i\in\mathbb{Z}^+}, \{\alpha_{ji}\}_{j\ge i})$ is an inverse system where for $j \ge i$,

$$\alpha_{ji} : \mathrm{Mat}_n(R/\mathfrak{m}_R^j) \longrightarrow \mathrm{Mat}_n(R/\mathfrak{m}_R^i) \tag{3.64}$$

is the ring homomorphism induced by $\alpha_{ji} : R/\mathfrak{m}^j \to R/\mathfrak{m}^i$. It follows that there is an isomorphism of k-algebras

$$\varphi : \varprojlim_i \mathrm{Mat}_n(R/\mathfrak{m}_R^i) \to \mathrm{Mat}_n(R). \tag{3.65}$$

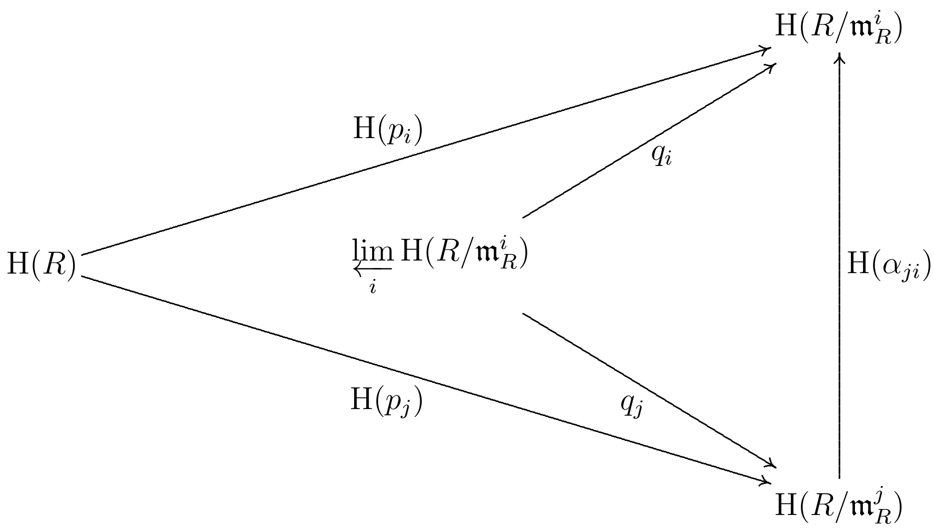
We also have that $(\{\mathrm{H}(R/\mathfrak{m}_R^i)\}_{i\in\mathbb{Z}^+}, \{\mathrm{H}(\alpha_{ji})\}_{j\geq i})$ is an inverse system.

Proposition 3.5.1. *For all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$,*

$$\mathrm{H}(R) \cong \varprojlim_i \mathrm{H}(R/\mathfrak{m}_R^i) \tag{3.66}$$

and this isomorphism is natural with respect to morphisms in $\hat{\mathcal{C}}$.

Proof. Consider the natural projections $p_i : R \to R/\mathfrak{m}_R^i$ for all $i \in \mathbb{Z}^+$. Note that for all $j \geq i$, $\mathrm{H}(\alpha_{ji}) \circ \mathrm{H}(p_j) = \mathrm{H}(\alpha_{ji} \circ p_j) = \mathrm{H}(p_i)$. Thus we have a diagram for $j \geq i$



(3.67)

where for all $s \in \mathbb{Z}^+$, $q_s : \varprojlim_i \mathrm{H}(R/\mathfrak{m}^i) \to \mathrm{H}(R/\mathfrak{m}^s)$ is the natural projection. By the universal property of inverse limits, there is a unique morphism $\iota_R : \mathrm{H}(R) \to \varprojlim_i \mathrm{H}(R/\mathfrak{m}_R^i)$ such that $q_s \circ \iota_r = \mathrm{H}(p_s)$ for all $s \in \mathbb{Z}^+$. By uniqueness of ι_R we have $\iota_R([\pi]) = ([p_i \circ \pi])_i$ for all $[\pi] \in \mathrm{H}(R)$.

Claim 1: ι_R is surjective.

Proof of Claim 1. Let $([\pi_i])_i \in \varprojlim_i \mathrm{H}(R/\mathfrak{m}_R^i)$. Then $[\pi_i] = \mathrm{H}(\alpha_{i+1,i})([\pi_{i+1}]) = [\alpha_{i+1,i} \circ \pi_{i+1}]$ for all $i \in \mathbb{Z}^+$. Hence for all $i \in \mathbb{Z}^+$, there exists $X_{i,i-1} \in G(R/\mathfrak{m}_R^i)$ such that $\pi_i = X_{i,i-1}^{-1}(\alpha_{i+1,i} \circ \pi_{i+1})X_{i,i-1}$. Let $Z_{1,0} = I_n$. We prove by induction that for all $i \in \mathbb{Z}^+$, there exists $Z_{i+1,i} \in G(R/\mathfrak{m}_R^{i+1})$ such that $\alpha_{i+1,i}(Z_{i+1,i}^{-1}\pi_{i+1}Z_{i+1,i}) = Z_{i,i-1}^{-1}\pi_i Z_{i-1,i}$.

We have $\pi_1 = X_{1,0}^{-1}(\alpha_{2,1} \circ \pi_2)X_{1,0}$. Since $\alpha_{2,1}$ is surjective, there exists $Z_{2,1} \in G(R/\mathfrak{m}_R^2)$ with $\alpha_{2,1}(Z_{2,1}) = X_{1,0}$. Thus

$$\alpha_{2,1}(Z_{2,1}^{-1}\pi_2 Z_{2,1}) = \pi_1 = Z_{1,0}^{-1}\pi_1 Z_{1,0}.$$

Since $\alpha_{i+1,i}$ is surjective, there exists $Z_{i+1,i} \in G(R/\mathfrak{m}_R^{i+1})$ with

$$\alpha_{i+1,i}(Z_{i+1,i}) = X_{i,i-1}Z_{i,i-1}.$$

Therefore,

$$\alpha_{i+1,i}(Z_{i+1,i}^{-1}\pi_{i+1}Z_{i+1,i}) = Z_{i,i-1}^{-1}X_{i,i-1}^{-1}(\alpha_{i+1,i} \circ \pi_{i+1})X_{i,i-1}Z_{i,i-1} = Z_{i,i-1}^{-1}\pi_i Z_{i,i-1}.$$

Note that for all $i \in \mathbb{Z}^+$, $Z_{i,i-1} \in G(R/\mathfrak{m}_R^i)$. Hence for all $a \in \Lambda$ we get

$$\overleftarrow{\pi_i}(a) = (Z_{i,i-1}^{-1}\pi_i(a)Z_{i,i-1})_i \in \varprojlim_i(\mathrm{Mat}_n(R/\mathfrak{m}_R^i)). \tag{3.68}$$

Using the isomorphism $\varphi : \varprojlim_i \mathrm{Mat}_n(R/\mathfrak{m}_R^i) \to \mathrm{Mat}_n(R)$ from (3.65), we get a representation of Λ

$$\begin{aligned} \pi : \Lambda &\longrightarrow \mathrm{Mat}_n(R) \\ a &\longmapsto \varphi(\overleftarrow{\pi_i}(a)). \end{aligned} \tag{3.69}$$

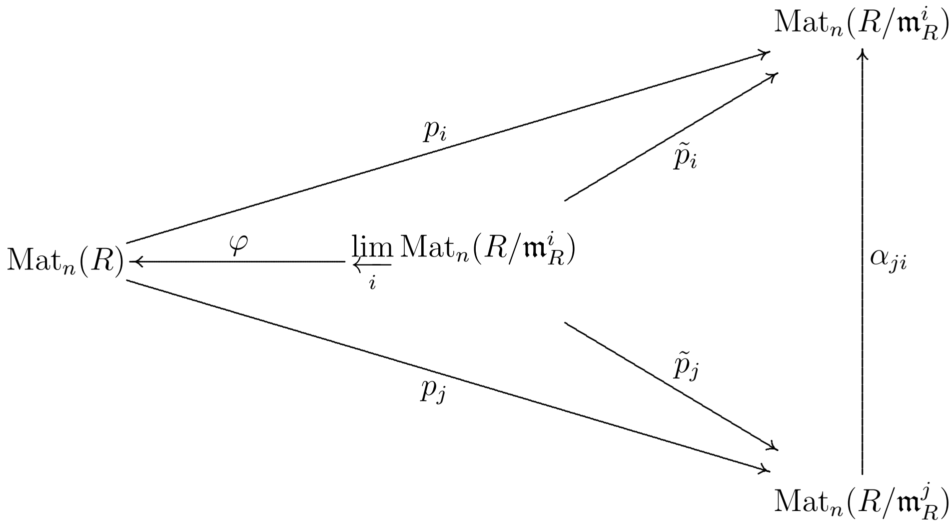
Note that we have a commutative diagram:

$$\begin{array}{ccc} \varprojlim_i \mathrm{Mat}_n(R/\mathfrak{m}_R^i) & \xrightarrow{\ \varphi\ } & \mathrm{Mat}_n(R) \\ \Big\downarrow {\scriptstyle \overleftarrow{\kappa_{R/\mathfrak{m}_R^i}}} & & \Big\downarrow {\scriptstyle \kappa_R} \\ \varprojlim_i \mathrm{Mat}_n(k) & \xrightarrow{\ \overline{\varphi}\ } & \mathrm{Mat}_n(k) \end{array}$$

Hence for all $a \in \Lambda$ we have

$$\kappa_R(\varphi(\overleftarrow{\pi_i}(a))) = \overline{\varphi}\left(\overleftarrow{\kappa_{R/\mathfrak{m}_R^i}}(\overleftarrow{\pi_i}(a))\right) = \overline{\varphi}\left((\kappa_{R/\mathfrak{m}_R^i}(Z_{i,i-1}^{-1}\pi_i(a)Z_{i,i-1}))_i\right) = \overline{\varphi}\left((\rho(a))_i\right)$$
$$= \rho(a).$$

This proves that $[\pi] \in \mathrm{H}(R)$. Note that we have a commutative diagram for all $j \geq i$



(3.70)

Therefore

$$\iota_R([\pi]) = ([p_i \circ \pi])_i = ([p_i \circ \varphi \circ \overleftarrow{\pi_i}])_i = ([\tilde{p}_i \circ \overleftarrow{\pi_i}])_i = ([Z_{i,i-1}^{-1}\pi_i Z_{i,i-1}])_i = ([\pi_i])_i.$$

This proves Claim 1.

Claim 2: ι_R is injective.

Proof of Claim 2. Let $[\pi]$ and $[\pi']$ in $\mathrm{H}(R)$ such that $\iota_R([\pi]) = \iota_R([\pi'])$ in $\varprojlim_i \mathrm{H}(R/\mathfrak{m}_R^i)$. Then for all $i \geq 1$, $[p_i \circ \pi] = [p_i \circ \pi']$ in $\mathrm{H}(R/\mathfrak{m}_R^i)$. This means that for each $i \geq 1$ there exists $X_i \in G(R/\mathfrak{m}_R^i)$ such that $p_i \circ \pi = X_i(p_i \circ \pi')X_i^{-1}$. Therefore for each $i \geq 1$, $S_i = \{X_i \in G(R/\mathfrak{m}_R^i) : p_i \circ \pi = X_i(p_i \circ \pi')X_i^{-1}\}$ is nonempty. On the other hand, for all $a \in \Lambda$, $\pi(a), \pi'(a) \in \mathrm{Mat}_n(R)$. Hence $\alpha_{ji}(p_j(\pi(a))) = p_i(\pi(a))$ and $\alpha_{ji}(p_j(\pi'(a)) = p_i(\pi'(a))$ for all $j \geq i$ and for all $a \in \Lambda$. Let $j \geq i$, $X_j \in S_j$ and $a \in \Lambda$. Then $p_i(\pi(a)) = \alpha_{ji}(X_j)p_i(\pi'(a))\alpha_{ji}(X_j)^{-1}$, which means $\alpha_{ji}(X_j) \in S_i$. Therefore

$(\{S_i\}_{i\in\mathbb{Z}^+}, \{\alpha_{ji}\}_{j\geq i})$ is an inverse system. Since $R/\mathfrak{m}_R^i$ is Artinian for all $i \geq 1$, it follows that $\varprojlim_i S_i$ is not empty. Let $(X_i)_i \in \varprojlim_i S_i$ and define $X = \varphi((X_i)_i)$. Then $X \in G(R)$ and $\pi = X\pi' X^{-1}$. Hence $[\pi] = [\pi']$ in $\mathrm{H}(R)$.

Claim 3: Let $\beta : R \to R'$ be a morphism in $\hat{\mathcal{C}}$. Then

$$\iota_{R'} \circ \mathrm{H}(\beta) = \varprojlim_i \mathrm{H}(\beta_i) \circ \iota_R \tag{3.71}$$

where $\beta_i : R/\mathfrak{m}_R^i \to R'/\mathfrak{m}_{R'}^i$ is the morphism induced by β.

Proof of Claim 3. For each $i \geq 0$, let $p_i : R \to R/\mathfrak{m}_R^i$ and $p_i' : R' \to R'/\mathfrak{m}_{R'}^i$ be the natural projections. Note that $p_i' \circ \beta = \beta_i \circ p_i$. Let $[\pi] \in \mathrm{H}(R)$. Then $(\iota_{R'} \circ \mathrm{H}(\beta))([\pi]) = \iota_{R'}(\mathrm{H}(\beta)([\pi])) = \iota_{R'}([\beta \circ \pi]) = ([p_i' \circ \beta \circ \pi])_i = ([\beta_i \circ p_i \circ \pi])_i = \varprojlim_i \mathrm{H}(\beta_i)(([p_i \circ \pi])_i) = \varprojlim_i \mathrm{H}(\beta_i)(\iota_R([\pi])) = (\varprojlim_i \mathrm{H}(\beta_i) \circ \iota_R)([\pi])$. This proves Claim 3, and hence completes the proof of Proposition 3.5.1. $\square$

Theorem 3.5.2. *Let* $\mathrm{F}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$ *be the deformation functor from Definition 3.2.2. Then* F_V *has a representable hull, in the sense of Definition 2.6.1 (f).*

Proof. This follows from Corollary 3.2.8, Lemmas 3.3.2 and 3.3.4, Corollary 3.4.2, Proposition 3.5.1 and Theorem 2.6.2. $\square$

Theorem 3.5.3. *Let* $\mathrm{F}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$ *be the deformation functor from Definition 3.2.2.*

(a) If $\mathrm{End}_\Lambda(V) \cong k$*, then* F_V *is representable.*

(b) If Λ *is self-injective and* $\underline{\mathrm{End}}_\Lambda(V) \cong k$*, then* F_V *is representable.*

Proof. This follows from Corollary 3.2.8, Lemmas 3.3.2, 3.3.4, 3.3.5 and 3.3.9, Corollary 3.4.2, Proposition 3.5.1 and Theorem 2.6.2. $\square$

Definition 3.5.4. Let $\mathrm{F}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$ be the deformation functor from Definition 3.2.2.

(a) If F_V is representable, as in Theorem 3.5.3, then there exists an object $R(\Lambda, V) \in \mathrm{Ob}(\hat{\mathcal{C}})$ and a lift (U, ϕ_U) of V over $R(\Lambda, V)$ such that for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and all lifts (M, ϕ) of V over R there exists a unique morphism $\alpha : R(\Lambda, V) \to R$ in $\hat{\mathcal{C}}$ with $[(M, \phi)] = \mathrm{F}_V(\alpha)([(U, \phi_U]))$. We call $R(\Lambda, V)$ the *universal deformation ring* of V and $[(U, \phi_U)]$ the *universal deformation* of V over $R(\Lambda, V)$. Note that $R(\Lambda, V)$ is unique up to unique isomorphism in $\hat{\mathcal{C}}$.

(b) In general, as in Theorem 3.5.2, there still exists $R(\Lambda, V) \in \mathrm{Ob}(\hat{\mathcal{C}})$ and a lift (U, ϕ_U) of V over $R(\Lambda, V)$ such that for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and all lifts (M, ϕ) of V over R there exists some (not necessarily unique) morphism $\alpha : R(\Lambda, V) \to R$ in $\hat{\mathcal{C}}$ with $[(M, \phi)] = \mathrm{F}_V(\alpha)([(U, \phi_U)])$. In this case we call $R(\Lambda, V)$ the *versal deformation ring* of V and $[(U, \phi_U)]$ the *versal deformation* of V over $R(\Lambda, V)$. Note that $R(\Lambda, V)$ is unique up to (not necessarily unique) isomorphism in $\hat{\mathcal{C}}$.

3.6 Universal Deformation Rings and the Syzygy Operator Ω

The goal of this section is to prove that if Λ is a Frobenius algebra (see Definition 2.1.1) and $\underline{\mathrm{End}}_\Lambda(V) \cong k$, then $\underline{\mathrm{End}}_\Lambda(\Omega(V)) \cong k$ and $R(\Lambda, V) \cong R(\Lambda, \Omega(V))$, where $R(\Lambda, V)$ and $R(\Lambda, \Omega(V))$ are the universal deformation rings of V and $\Omega(V)$, respectively. Our proof follows the steps used to prove [3, Prop. 2.4 and Cor. 2.5(i),(iii)].

Lemma 3.6.1. *Assume that Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$. Then*

$$\underline{\mathrm{End}}_\Lambda(\Omega(V)) \cong k.$$

In particular, $\Omega(V)$ has a universal deformation ring $R(\Lambda, \Omega(V))$ in $\hat{\mathcal{C}}$.

Proof. Since Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$, it follows from Theorem 2.1.11 (ii) that $\underline{\mathrm{End}}_\Lambda(\Omega(V)) \cong k$. The second statement follows from Theorem 3.5.3 (b). □

Lemma 3.6.2. *Let $(P(V), \epsilon)$ be a projective Λ-module cover of V. Let $R \in \mathrm{Ob}(\mathcal{C})$ be Artinian and let $(P_R(P(V)), \eta)$ be a projective $R\Lambda$-module cover of $P(V)$ viewed as an $R\Lambda$-module. Then for all lifts (M, ϕ) of V over R, there exists an essential surjection $\psi_M : P_R(P(V)) \to M$ of $R\Lambda$-modules such that $\phi \circ p_M \circ \psi_M = \epsilon \circ \eta$, where $p_M : M \to k \otimes_R M$ is the surjective $R\Lambda$-module homomorphism with $p_M(m) = 1 \otimes m$ for all $m \in M$. In particular, $(P_R(P(V)), \psi_M)$ is a projective $R\Lambda$-module cover of M.*

Proof. Since $P_R(P(V))$ is a projective $R\Lambda$-module and $\phi \circ p_M : M \to V$ is a surjective $R\Lambda$-module hommormophism, there exists an $R\Lambda$-module homomorphism $\psi_M : P_R(P(V)) \to M$ with $\phi \circ p_M \circ \psi_M = \epsilon \circ \eta$. Tensoring $P_R(P(V))$ and M with k over R, we obtain $\phi \circ (id \otimes \psi_M) = \epsilon \circ (id \otimes \eta)$. Hence $id \otimes \psi_M$ is surjective, which implies by Nakayama's Lemma that ψ_M is surjective. Let X be an $R\Lambda$-module and let $h : X \to P_R(P(V))$ be an $R\Lambda$-module homomorphism such that $\psi_M \circ h$ is surjective. Then $\epsilon \circ \eta \circ h = \phi \circ p_M \circ \psi_M \circ h$ is surjective. Since ϵ and η are essential, we obtain that h is surjective. In particular, ψ_M is an essential surjection and $(P_R(P(V), \psi_M)$ is a projective $R\Lambda$-module cover of M. □

Let $R \in \mathrm{Ob}(\mathcal{C})$ and let (M, ϕ) be a lift of V over R. Let $(P(V), \epsilon)$, $(P_R(P(V)), \eta)$ and $\psi_M : P_R(P(V)) \to M$ be as in Lemma 3.6.2. Define $\Omega_R(M) = \ker(\psi_M)$. Because $P_R(P(V))$ and M are free over R, it follows that $\Omega_R(M)$ is free over R. Since ϵ is surjective and since $k \otimes_R P_R(P(V))$ is a projective Λ-module, there exists a Λ-module homomorphism $\Phi_M : k \otimes_R P_R(P(V) \to P(V)$ such that $\phi \circ (id \otimes \psi_M) = \epsilon \circ \Phi_M$. Define $\Omega_R(\phi) : k \otimes_R \Omega_R(M) \to \Omega(V)$ to be the restriction of Φ_M to $k \otimes_R \Omega_R(M)$, i.e. we have a commutative diagram of Λ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k \otimes_R \Omega_R M & \longrightarrow & k \otimes_R P_R(P(V)) & \xrightarrow{id \otimes \psi_M} & k \otimes_R M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Omega_R(\phi)} & & \downarrow{\scriptstyle \Phi_M} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \Omega(V) & \hookrightarrow & P(V) & \xrightarrow{\epsilon} & V & \longrightarrow & 0 \end{array}$$

Since $\epsilon \circ \Phi_M = \phi \circ (id \otimes \psi_M)$ is surjective and ϵ is essential, Φ_M is surjective. Hence Φ_M is a Λ-module isomorphism since the k-dimensions of $k \otimes_R P_R(P(V))$ and $P(V)$ are equal and finite. It follows that $\Omega_R(\phi) : k \otimes_R \Omega_R(M) \to \Omega(V)$ is also a Λ-module isomorphism. Therefore, $(\Omega_R(M), \Omega_R(\phi))$ is a lift of $\Omega(V)$ over R.

Suppose now that Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$. Let (M', ϕ') be a lift of V over R which is isomorphic to (M, ϕ). In other words, there exists an $R\Lambda$-module isomorphism $f : M \to M'$ such that $\phi' \circ (id \otimes f) = \phi$. We obtain a commutative diagram of $R\Lambda$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_R(M) & \longrightarrow & P_R(P(V)) & \xrightarrow{\psi_M} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Omega_R(M') & \longrightarrow & P_R(P(V)) & \xrightarrow{\psi_{M'}} & M' & \longrightarrow & 0 \end{array}$$

where λ exists since $P_R(P(V))$ is a projective $R\Lambda$-module and μ is the restriction of λ to $\Omega_R(M)$. Since $\psi_{M'}$ is essential and $f \circ \psi_M$ is surjective, it follows that λ is surjective. Hence λ is an $R\Lambda$-module isomorphism, since $P_R(P(V))$ is a free R-module of finite rank. It follows that μ is also an $R\Lambda$-module isomorphism. By Lemmas 3.3.10 and 3.6.1, μ induces an $R\Lambda$-module isomorphism $\tilde{\mu} : \Omega_R(M) \to \Omega_R(M')$ such that $\Omega_R(\phi') \circ (id \otimes \tilde{\mu}) = \Omega_R(\phi)$. Hence $[(\Omega_R(M), \Omega_R(\phi))] = [(\Omega_R(M'), \Omega_R(\phi'))]$.

Therefore, if Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$, we obtain for each $R \in \mathrm{Ob}(\mathcal{C})$ a well-defined map

$$g_{\Omega,R} : \quad \mathrm{F}_V(R) \longrightarrow \mathrm{F}_{\Omega(V)}(R) \qquad [(M,\phi)] \longmapsto [(\Omega_R(M), \Omega_R(\phi))] \tag{3.72}$$

Lemma 3.6.3. *Assume Λ is self-injective and $\underline{\mathrm{End}}_\Lambda(V) \cong k$. For any morphism $\theta : R \to R'$ in $\mathcal{C}$ we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{F}_V(R) & \xrightarrow{g_{\Omega,R}} & \mathrm{F}_{\Omega(V)}(R) \\ \downarrow{\scriptstyle \mathrm{F}_V(\theta)} & & \downarrow{\scriptstyle \mathrm{F}_{\Omega(V)}(\theta)} \\ \mathrm{F}_V(R') & \xrightarrow{g_{\Omega,R'}} & \mathrm{F}_{\Omega(V)}(R') \end{array} \tag{3.73}$$

Proof. Let $[(M,\phi)] \in \mathrm{F}_V(R)$. Then $g_{\Omega,R}([(M,\phi)]) = [(\Omega_R(M), \Omega_R(\phi))] \in \mathrm{F}_{\Omega(V)}(R)$. Thus $\mathrm{F}_{\Omega(V)}(\theta)(g_{\Omega,R}([(M,\phi)])) = [(R' \otimes_{R,\theta} \Omega_R(M), \Omega_R(\phi) \circ \nu_{R',\theta}^{\Omega_R(M)})]$ where $\nu_{R',\theta}^{\Omega_R(M)}$ is defined as in Definition 3.2.2 (b). On the other hand

$$g_{\Omega,R'}(\mathrm{F}_V(\theta)([(M,\phi)]) = g_{\Omega,R'}([(R' \otimes_{R,\theta} M, \phi \circ \nu_{R'}^M)]) = [(\Omega_{R'}(R' \otimes_{R,\theta} M), \Omega_{R'}(\phi \circ \nu_{R'}^M))].$$

Using the notation from Lemma 3.6.2 and (3.72), we obtain a commutative diagram of $R'\Lambda$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R'\otimes_{R,\theta}\Omega_R(M) & \longrightarrow & R'\otimes_{R,\theta} P_R(P(V)) & \xrightarrow{id\otimes\psi_M} & R'\otimes_{R,\theta} M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \lambda & & \downarrow id & & \\ 0 & \longrightarrow & \Omega_{R'}(R'\otimes_{R,\theta} M) & \longrightarrow & P_{R'}(P(V)) & \xrightarrow{\psi_{R'\otimes_{R,\theta}M}} & R'\otimes_{R,\theta} M & \longrightarrow & 0 \end{array}$$

where λ exists since $R'\otimes_{R,\theta} P_R(P(V))$ is a projective $R'\Lambda$-module and f is the restriction of λ to $R'\otimes_{R,\theta}\Omega_R(M)$. Since $\psi_{R'\otimes_{R,\theta}M}$ is essential and $id\otimes\psi_M$ is surjective, λ must be surjective. Hence λ is an $R'\Lambda$-module isomorphism since $R'\otimes_{R,\theta} P_R(P(V))$ and $P_{R'}(P(V))$ are free R'-modules of the same finite rank. It follows that f is also an $R'\Lambda$-module isomorphism. By Lemmas 3.3.10 and 3.6.1, f induces an $R'\Lambda$-module isomorphism $\tilde{f} : R'\otimes_{R,\theta}(\Omega_R(M)) \to \Omega_{R'}(R'\otimes_{R,\theta} M)$ such that $(\Omega_R(\phi)\circ\nu_{R',\theta}^{\Omega_R(M)})\circ(id\otimes\tilde{f}) = \Omega_{R'}(\phi\circ\nu_{R',\theta}^{M})$ where $\nu_{R',\theta}^{\Omega_R(M)}$ is as above. Thus $[(R'\otimes_{R,\theta}\Omega_R(M), \Omega_R(\phi)\circ\nu_{R',\theta}^{\Omega_R(M)})] = [(\Omega_{R'}(R'\otimes_{R,\theta} M), \Omega_{R'}(\phi\circ\nu_{R',\theta}^{M}))]$ in $\mathrm{F}_{\Omega(V)}(R')$. □

Lemma 3.6.4. *Assume that Λ is self-injective. Let $R\in\mathrm{Ob}(\mathcal{C})$ and let (U,ϕ) be a lift of $\Omega(V)$ over R. Let $(P(V),\epsilon)$ and $(P_R(P(V)),\eta)$ be as in Lemma 3.6.2, so that $\Omega(V) = \ker(\epsilon)$. Then there exists an injective $R\Lambda$-module homomorphism $\varphi : U\to P_R(P(V))$ such that $\eta\circ\varphi = \phi\circ p_U$, where $p_U : U\to k\otimes_R U$ is the $R\Lambda$-module homomorphism with $p_U(x) = 1\otimes x$ for all $x\in U$.*

Proof. Since R is Artinian, we prove this by induction on the length of R. If $R = k$ then $\eta : P_R(P(V))\to P(V)$ is an $R\Lambda$-module isomorphism. Hence we can let $\varphi = \eta^{-1}\circ\phi\circ p_U$. Now consider a small extension

$$0\to tR\to R\xrightarrow{\lambda} R_0\to 0$$

where $R_0 \in \mathrm{Ob}(\mathcal{C})$, $t \in R$ with $\mathfrak{m}_R t = 0$ and $tR \cong k$. Let $P_{R_0}(P(V)) = R_0 \otimes_{R,\lambda} P_R(P(V))$ and let $\eta_0 : P_{R_0}(P(V)) \to P(V)$ be defined by $\eta_0(r_0 \otimes x) = \kappa_{R_0}(r_0)\eta(x)$ for all $r_0 \in R_0$ and $x \in P_R(P(V))$. Then $(P_{R_0}(P(V)), \eta_0)$ is a projective $R_0\Lambda$-module cover of $P(V)$ viewed as an $R_0\Lambda$-module. Let $U_0 = R_0 \otimes_{R,\lambda} U$ and $\phi_0 = \phi \circ \nu^U_{R_0,\lambda}$, where $\nu^U_{R_0,\lambda}$ is defined as in Definition 3.2.2 (b). Then (U_0, ϕ_0) is a lift of $\Omega(V)$ over R_0. By induction assumption there exists an injective $R_0\Lambda$-module homomorphism $\varphi_0 : U_0 \to P_{R_0}(P(V))$ such that $\eta_0 \circ \varphi_0 = \phi_0 \circ p_{U_0}$. Let $\pi_{P,R_0} : P_R(P(V)) \to R_0 \otimes_{R,\lambda} P_R(P(V))$ and $\pi_{U,R_0} : U \to R_0 \otimes_{R,\lambda} U$ be the natural surjections. Consider a projective resolution of the $R\Lambda$-module U

$$\cdots \xrightarrow{\delta_3} Q_2 \xrightarrow{\delta_2} Q_1 \xrightarrow{\delta_1} Q_0 \to U \to 0. \tag{3.74}$$

Applying $\mathrm{Hom}_{R\Lambda}(-, tP_R(P(V)))$, we obtain the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{R\Lambda}(Q_0, tP_R(P(V))) &\xrightarrow{\delta_1^*} \mathrm{Hom}_{R\Lambda}(Q_1, tP_R(P(V))) \\ &\xrightarrow{\delta_2^*} \mathrm{Hom}_{R\Lambda}(Q_2, tP_R(P(V))) \xrightarrow{\delta_3^*} \cdots \end{aligned}$$

Then $\mathrm{Ext}^1_{R\Lambda}(U, tP_R(P(V)) = \ker(\delta_2^*)/\,\mathrm{Im}(\delta_1^*)$. Since all the terms in the projective resolution (3.74) of U are projective as $R\Lambda$-modules, tensoring (3.74) with k over R gives a projective resolution of the Λ-module $k \otimes_R U$

$$\cdots \xrightarrow{id\otimes\delta_3} k \otimes_R Q_2 \xrightarrow{id\otimes\delta_2} k \otimes_R Q_1 \xrightarrow{id\otimes\delta_1} k \otimes_R Q_0 \to k \otimes_R U \to 0. \tag{3.75}$$

Applying $\mathrm{Hom}_\Lambda(-, tP_R(P(V)))$, we obtain the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(k \otimes_R Q_0, tP_R(P(V))) &\xrightarrow{(id\otimes\delta_1)^*} \mathrm{Hom}_\Lambda(k \otimes_R Q_1, tP_R(P(V))) \\ &\xrightarrow{(id\otimes\delta_2)^*} \mathrm{Hom}_\Lambda(k \otimes_R Q_2, tP_R(P(V))) \xrightarrow{(id\otimes\delta_3)^*} \cdots \end{aligned}$$

Hence $\operatorname{Ext}^1_\Lambda(k\otimes_R U, tP_R(P(V))) = \ker((id\otimes\delta_2)^*)/\operatorname{Im}((id\otimes\delta_1)^*)$. Since $\mathfrak{m}_R$ annihilates $tP_R(P(V))$ and $Q_i/\mathfrak{m}_R Q_i \cong k\otimes_R Q_i$ for all $i \geq 0$, we obtain for all $i \geq 0$ a commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_{R\Lambda}(Q_i, tP_R(P(V))) & \xrightarrow{\delta^*_{i+1}} & \operatorname{Hom}_{R\Lambda}(Q_{i+1}, tP_R(P(V))) \\
\downarrow \cong & & \downarrow \cong \\
\operatorname{Hom}_\Lambda(k\otimes_R Q_i, tP_R(P(V))) & \xrightarrow{(id\otimes\delta_{i+1})^*} & \operatorname{Hom}_\Lambda(k\otimes_R Q_{i+1}, tP_R(P(V)))
\end{array}$$

Hence $\operatorname{Ext}^1_{R\Lambda}(U, tP_R(P(V))) \cong \operatorname{Ext}^1_\Lambda(k\otimes_R U, tP_R(P(V)))$. Since $tP_R(P(V)) \cong k\otimes_R P_R(P(V)) \cong P(V)$ is a finitely generated projective Λ-module and Λ is assumed to be self-injective, $tP_R(P(V))$ is also an injective Λ-module. Thus $\operatorname{Ext}^1_{R\Lambda}(U, tP_R(P(V))) = 0$. Considering the short exact sequence of $R\Lambda$-modules

$$0 \to tP_R(P(V)) \to P_R(P(V)) \xrightarrow{\pi_{P,R_0}} P_{R_0}(P(V)) \to 0, \tag{3.76}$$

the long exact cohomology sequence corresponding to $\operatorname{Hom}_{R\Lambda}(U,-)$ gives a short exact sequence

$$0 \to \operatorname{Hom}_{R\Lambda}(U, tP_R(P(V))) \to \operatorname{Hom}_{R\Lambda}(U, P_R(P(V)) \xrightarrow{(\pi_{P,R_0})_*} \operatorname{Hom}_{R\Lambda}(U, P_{R_0}(P(V))) \to 0.$$

Thus there exists an $R\Lambda$-module homomorphism $\varphi : U \to P_R(P(V))$ such that $\pi_{P,R_0}\circ\varphi = (\pi_{P,R_0})_*(\varphi) = \varphi_0\circ\pi_{U,R_0}$. Hence $\eta\circ\varphi = \eta_0\circ\pi_{P,R_0}\circ\varphi = \eta_0\circ\varphi_0\circ\pi_{U,R_0} = \phi_0\circ p_{U_0}\circ\pi_{U,R_0} = \phi\circ p_U$. Since φ induces an injective homomorphism modulo $\mathfrak{m}_R$ and since U and $P_R(P(V))$ are free R-modules of finite rank, it follows by Nakayama's Lemma that φ is injective. □

Lemma 3.6.5. *Assume that* Λ *is self-injective and* $\underline{\mathrm{End}}_\Lambda(V) \cong k$. *Then for all* $R \in \mathrm{Ob}(\mathcal{C})$, $g_{\Omega,R}$ *is surjective.*

Proof. Let $R \in \mathrm{Ob}(\mathcal{C})$ and let (U, ϕ) be a lift of V over R. Let $(P(V), \epsilon)$ and $(P_R(P(V)), \eta)$ be as in Lemma 3.6.2, so that $\Omega(V) = \ker(\epsilon)$. Let $\varphi : U \to P_R(P(V))$ be the injective $R\Lambda$-module homomorphism from Lemma 3.6.4. We obtain a commutative diagram of $R\Lambda$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\varphi} & P_R(P(V)) & \xrightarrow{\pi} & \mathrm{Coker}(\varphi) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi \circ p_u} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \xi} & & \\ 0 & \longrightarrow & \Omega(V) & \hookrightarrow & P(V) & \xrightarrow{\epsilon} & V & \longrightarrow & 0 \end{array} \tag{3.77}$$

where π is the natural projection and ξ is the $R\Lambda$-module homomorphism induced by η. Since U and $P_R(P(V))$ are free R-modules of finite rank, it follows that $\mathrm{Coker}(\varphi)$ is also free over R. Tensoring the top row of (3.77) with k over R, we obtain a commutative diagram of Λ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k \otimes_R U & \xrightarrow{id \otimes \varphi} & k \otimes_R P_R(P(V)) & \xrightarrow{id \otimes \pi} & k \otimes_R \mathrm{Coker}(\varphi) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\eta}} & & \downarrow{\scriptstyle \zeta} & & \\ 0 & \longrightarrow & \Omega(V) & \hookrightarrow & P(V) & \xrightarrow{\epsilon} & V & \longrightarrow & 0 \end{array} \tag{3.78}$$

where $\bar{\eta}$ (respectively ζ) is induced by η (respectively ξ). Since $\bar{\eta}$ is a Λ-module isomorphism, it follows that ζ is a Λ-module isomorphism. Hence $(\mathrm{Coker}(\varphi), \zeta)$ is a lift of V over R. Since $(P_R(P(V)), \pi)$ is a projective $R\Lambda$-module cover of $\mathrm{Coker}(\varphi)$, it follows from (3.78) and from the definition of $g_{\Omega,R}$ in (3.72) that $g_{\Omega,R}([(\mathrm{Coker}(\varphi),\zeta)]) = [(U,\phi)]$. $\square$

Lemma 3.6.6. *Assume that Λ is a Frobenius algebra and $\underline{\mathrm{End}}_\Lambda(V) \cong k$. Then $g_{\Omega,R}$ is injective for all $R \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Let $[(L_1,\phi_1)],[(L_2,\phi_2)] \in \mathrm{F}_V(R)$ such that

$$[(\Omega_R(L_1),\Omega_R(\phi_1))] = g_{\Omega,R}([(L_1,\phi_1)] = g_{\Omega,R}([(L_2,\phi_2)] = [(\Omega_R(L_2),\Omega_R(\phi_2))].$$

Then there exists an $R\Lambda$-module isomorphism $f : \Omega_R(L_1) \to \Omega_R(L_2)$ with $\Omega_R(\phi_2) \circ (id \otimes f) = \Omega_R(\phi_1)$. Using the notation from before, we obtain for $i \in \{1,2\}$ a short exact sequences of left $R\Lambda$-modules

$$0 \to \Omega_R(L_1) \xrightarrow{\iota_i} P_R(P(V)) \xrightarrow{\pi_i} L_i \to 0 \tag{3.79}$$

where $\pi_i = \psi_{M_i}$ for $i \in \{1,2\}$ and ι_1 is inclusion and ι_2 is induced by f. If M is a left $R\Lambda$-module, we denote by M^* the right $R\Lambda$-module $\mathrm{Hom}_R(M,R)$. Applying $\mathrm{Hom}_R(-,R)$ to (3.79) and using that $\Omega_R(L_1)$ and L_i are free R-modules for $i \in \{1,2\}$, we obtain a short exact sequence of right $R\Lambda$-modules

$$0 \to L_i^* \to P_R^*(P(V)) \to \Omega_R^*(L_1) \to 0 \tag{3.80}$$

for $i \in \{1,2\}$. By Lemma 2.1.3, $P_R^*(P(V))$ is a projective right $R\Lambda$-module. It follows by Schanuel's Lemma (see [5, Thm. 2.24]) that $P_R^*(P(V)) \oplus L_1^* \cong P_R^*(P(V)) \oplus L_2^*$ as right $R\Lambda$-modules. By the Krull-Schmidt-Azumaya Theorem (see [5, Thm 6.12]), we have $L_1^* \cong L_2^*$ as right $R\Lambda$-modules. Therefore $(L_1^*)^* \cong (L_2^*)^*$ as left $R\Lambda$-modules. Let $i \in \{1,2\}$. Then the natural R-module isomorphism $h_i : L_i \to (L_i^*)^*$ given by

$h_i(l)(\psi) = \psi(l)$ for all $l \in L_i$ and $\psi \in L_i^*$ is an $R\Lambda$-module isomorphism, since for all $b \in R\Lambda$

$$h_i(b \cdot l)(\psi) = \psi(b \cdot l) = (\psi \cdot b)(l) = h_i(l)(\psi \cdot b)$$
$$= (b \cdot h_i(l))(\psi).$$

Hence $L_1 \cong L_2$ as left $R\Lambda$-modules. Let $\zeta : L_1 \to L_2$ be an $R\Lambda$-module isomorphism. By Lemma 3.3.10, ζ induces an $R\Lambda$-module isomorphism $\tilde{\zeta} : L_1 \to L_2$ such that $\phi_2 \circ (id \otimes \tilde{\zeta}) = \phi_1$. Hence $[(L_1, \phi_1)] = [(L_2, \phi_2)]$. □

Theorem 3.6.7. *Assume that* Λ *is a Frobenius algebra and* $\underline{\mathrm{End}}_\Lambda(V) \cong k$*. The syzygy operator* Ω *induces a natural isomorphism* g_Ω *between the restrictions of the deformation functors* F_V *and* $\mathrm{F}_{\Omega(V)}$ *to the full subcategory* $\mathcal{C}$ *of Artinian objects* R *in* $\hat{\mathcal{C}}$*. In particular,* $R(\Lambda, V) \cong R(\Lambda, \Omega(V))$*.*

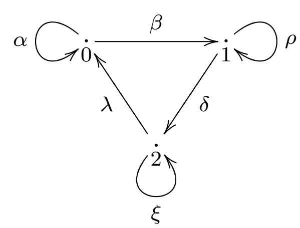
Proof. The first part of the statement is a direct consequence of Lemmas 3.6.3, 3.6.5 and 3.6.6. The second part of the statement follows from Corollary 3.2.8 and Proposition 3.5.1. □

CHAPTER 4

A PARTICULAR EXAMPLE

4.1 Set Up

In this chapter we assume k to be an algebraically closed field and Λ to be the basic algebra kQ/I where Q is the quiver



(4.1)

and I is the ideal of the path algebra kQ generated by the set of relations

$$\{\beta\alpha, \lambda\xi, \alpha\lambda, \rho\beta, \delta\rho, \xi\delta, \alpha^2 - \lambda\delta\beta, \rho^2 - \beta\lambda\delta, \xi^2 - \delta\beta\lambda\}. \tag{4.2}$$

Remark 4.1.1. The algebra Λ is of dihedral type as introduced by K. Erdmann, but not isomorphic to a block of a group algebra (see [6, Lemma IX.5.4]). In particular Λ is a symmetric k-algebra. Notice also that Λ is a special biserial algebra according to Definition 2.5.1. There are three simple Λ-modules up to isomorphism corresponding to the vertices in Q_0, which we denote by S_0, S_1 and S_2. Their projective covers P_0, P_1 and P_2, respectively, can be described using the following diagrams:

$$P_0 = \begin{array}{ccc} & 0 & \\ & & 1 \\ 0 & & \\ & & 2 \\ & 0 & \end{array}, \quad P_1 = \begin{array}{ccc} & 1 & \\ & & 2 \\ 1 & & \\ & & 0 \\ & 1 & \end{array} \quad \text{and} \quad P_2 = \begin{array}{ccc} & 2 & \\ & & 0 \\ 2 & & \\ & & 1 \\ & 2 & \end{array}$$

The stable Auslander-Reiten quiver of Λ has

- infinitely many components of type $\mathbb{Z}A_\infty^\infty$, consisting entirely of string modules,

- two 3-tubes, one corresponding to the maximal directed strings α, ρ, ξ and one corresponding to the maximal directed strings $\delta\beta$, $\lambda\delta$, $\beta\lambda$, and
- infinitely many 1-tubes, consisting entirely of band modules.

4.2 Canonical Homomorphisms

Definition 4.2.1. Let S and T be strings for Λ. Suppose C is a substring of both S and T such that the following conditions (I) and (II) are satisfied.

(I) $S \sim BCD$, where B is a substring which is either of length zero or $B = B'\tau$ for an arrow τ, and D is a substring which is either of length zero or $D = \varphi^{-1}D'$ for an arrow φ. In other words

$$S \sim B' \overset{\tau}{\leftarrow} C \overset{\varphi}{\rightarrow} D'. \tag{4.3}$$

(II) $T \sim ECF$, where E is a substring which is either of length zero or $E = E'\epsilon^{-1}$ for an arrow ϵ, and F is a substring which is either of length zero or $F = \mu F'$ for an arrow μ. In other words

$$T \sim E' \overset{\epsilon}{\rightarrow} C \overset{\mu}{\leftarrow} F'. \tag{4.4}$$

Then there exists a canonical Λ-module homomorphism

$$\alpha_C : M(S) \twoheadrightarrow M(C) \hookrightarrow M(T). \tag{4.5}$$

Theorem 4.2.2. *Each $f \in \mathrm{Hom}_\Lambda(M(S), M(T))$ can be written uniquely as a k-linear combination of canonical Λ-module homomorphisms as in (4.5). In particular, if $M(S) = M(T)$, the canonical endomorphisms generate* $\mathrm{End}_\Lambda(M(S))$.

Proof. See [8]. □

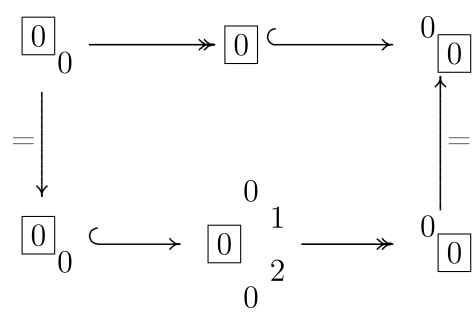
Lemma 4.2.3. *Assume S is a string in Λ. If $S \in \{\beta, \delta, \lambda, \delta\beta, \lambda\delta, \beta\lambda, \dot{0}, \dot{1}, \dot{2}\}$ then* $\mathrm{End}_\Lambda(M(S)) \cong k$.

Proof. Let $S \in \{\beta, \delta, \lambda, \delta\beta, \lambda\delta, \beta\lambda, \dot{0}, \dot{1}, \dot{2}\}$. Since the unique substring C of S satisfying (I) and (II) above is S itself, the only canonical endomorphism α_C of $M(S)$ is the identity homomorphism. By Theorem 4.2.2, $\dim_k \mathrm{End}_\Lambda(M(S)) = 1$. □

4.3 Stable Endomorphism Rings

In this section, we determine all string Λ-modules with stable endomorphism ring k which lie in the same component as a string Λ-module whose endomorphism ring is k. We first look at the string modules corresponding to maximal directed strings of length 1.

Consider the string ${}_{a_1}0 \xrightarrow{\alpha} {}_{a_2}0$. There are two substrings of α satisfying (I) and (II) above, namely, the substring $\dot{0}$ with corresponding canonical endomorphism $a_1 \to a_2$, and α itself which induces the identity endomorphism of $M(\alpha)$. By Theorem 4.2.2, $\dim_k \mathrm{End}_\Lambda(M(\alpha)) = 2$. Looking at the commutative diagram



we see that the canonical map $a_1 \to a_2$ factors through the projective Λ-module P_0. Since $M(\alpha)$ is not projective, this implies

$$\dim_k \mathrm{PEnd}_\Lambda(M(\alpha)) = 1,$$

which means $\dim_k \underline{\mathrm{End}}_\Lambda(M(\alpha)) = 1$. Arguing similarly with each of the strings ρ and ξ, we obtain the following result.

Lemma 4.3.1. *If $S \in \{\alpha, \rho, \xi\}$, then $\dim_k \mathrm{End}_\Lambda(M(S)) = 2$ and $\underline{\mathrm{End}}_\Lambda(M(S)) \cong k$ as k-algebras.*

We now determine all string Λ-modules whose endomorphism ring is isomorphic to k.

Proposition 4.3.2. *The only string Λ-modules with endomorphism ring k are the Λ-modules corresponding to the strings $\dot{0}, \dot{1}, \dot{2}, \beta, \delta, \lambda, \delta\beta, \lambda\delta, \beta\lambda$.*

Proof. Let S be a string which does not represent any of the strings in the Proposition. If the length of S is 1 then S is one of the strings α, ρ or ξ. Hence $\dim_k \underline{\mathrm{End}}_\Lambda(M(S)) = 2$ by Lemma 4.3.1. Assume now that the length of S is greater than 1. Then S contains a maximal directed substring. If S contains a maximal substring ${}_{a_1}i \xrightarrow{\zeta_i} {}_{a_2}i$ of length 1, where $(\zeta_0, \zeta_1, \zeta_2) = (\alpha, \rho, \xi)$, then $S \sim D\alpha D'$ for suitable strings D and D'. So we have at least two canonical endomorphisms of $M(S)$, namely, one given by $a_1 \to a_2$ and another given by the identity. Hence $\dim_k \mathrm{End}_\Lambda(M(S)) \geq 2$. Now suppose that S contains a maximal substring $i \xleftarrow{\gamma_1} j \xleftarrow{\gamma_2} l$ of length 2, where $\gamma_1\gamma_2 \in \{\delta\beta, \lambda\delta, \beta\lambda\}$. Since $S \neq \gamma_1\gamma_2$, either $S \sim D\gamma_1\gamma_2\zeta_l^{-1}D'$ or $S \sim D\zeta_i^{-1}\gamma_1\gamma_2 D'$ for suitable strings D and D'. In other words, S contains a maximal directed substring of length 1. Therefore, it follows as above that $\dim_k \mathrm{End}_\Lambda(M(S)) \geq 2$. $\square$

4.3.1 Stable Endomorphism Rings of the Λ-modules in the Components of the Stable Auslander-Reiten Quiver Containing Simple Λ-modules

Definition 4.3.3. Let σ be the automorphism of Q which induces the permutation $(0, 1, 2)$ on the vertices and the permutation $(\beta, \delta, \lambda)(\alpha, \rho, \xi)$ on the arrows. The σ induces a k-algebra automorphism of $\Lambda = kQ/I$ of order 3, which we also denote by σ.

Proposition 4.3.4. *Let $v \in \{0, 1, 2\}$. The Λ-modules in the component of the stable Auslander-Reiten quiver containing S_v which have stable endomorphism ring k are precisely those of the form:*

(i) $\Omega^n(S_v) = \Omega^n(M(\dot{v}))$,

(ii) $\Omega^n(M(\dot{v}_h))$, *and*

(iii) $\Omega^n(M(\dot{v}_{hh}))$,

for all $n \in \mathbb{Z}$.

Proof. Since σ in Definition 4.3.3 is a k-algebra automorphism of Λ, it suffices to prove Proposition 4.3.4 in the case when $v = 0$. Consider the simple Λ-module S_0 corresponding to the string $\dot{0}$ and consider the diagonal in the component of the stable Auslander-Reiten quiver containing S_0

$$\begin{array}{ccccc} \dot{0} & & & & \\ & \searrow & & & \\ & & \dot{0}_h & & \\ & & & \searrow & \\ & & & & \dot{0}_{hh} \\ & & & & \searrow \\ & & & & \ddots \\ & & & & \searrow \\ & & & & \dot{0}_{hh\cdots h} \end{array}$$

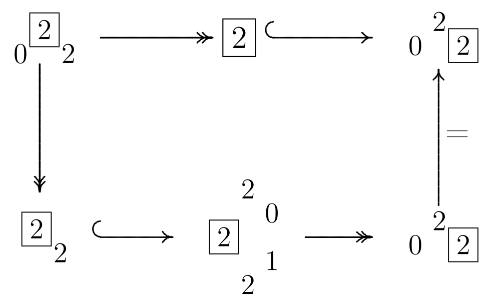
Note that

$$\dot{0}_h = \begin{smallmatrix} & 2 & \\ 0 & & 2 \end{smallmatrix}, \quad \dot{0}_{hh} = \begin{smallmatrix} & 2 & & 1 & \\ 0 & & 2 & & 1 \end{smallmatrix} \quad \text{and} \quad \dot{0}_{hhh} = \begin{smallmatrix} & 2 & & 1 & & 0 & \\ 0 & & 2 & & 1 & & 0 \end{smallmatrix}.$$

Consider first the string

$$\dot{0}_h = \begin{smallmatrix} & {}^{c_1}2 & \\ 0 & & {}_{c_2}2 \end{smallmatrix}.$$

There are two canonical endomorphisms of $M(\dot{0}_h)$, namely, one given by $c_1 \to c_2$ and another given by the identity map. Considering the following commutative diagram



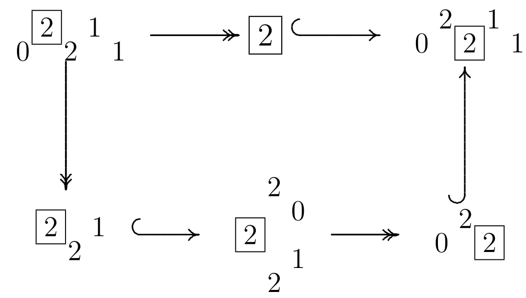
we see that the canonical endomorphism of $M(\dot{0}_h)$ corresponding to $c_1 \to c_2$ factors through the projective Λ-module P_2. Since $\dim_k \operatorname{End}_\Lambda(M(\dot{0}_h)) = 2$, it follows that $\dim_k \underline{\operatorname{End}}_\Lambda(M(\dot{0}_h)) = 1$. Hence $\underline{\operatorname{End}}_\Lambda(M(\dot{0}_h)) \cong k$ as k-algebras. Now consider the Λ-module corresponding to the string

$$\dot{0}_{hh} = \begin{smallmatrix} & {}^{c_1}2 & & {}^{b_1}1 & \\ 0 & & {}_{c_2}2 & & {}_{b_2}1 \end{smallmatrix}.$$

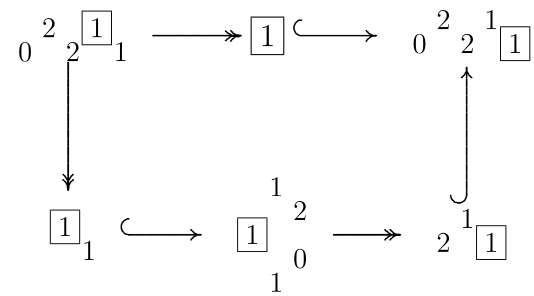
We see that there are three canonical endomorphisms of the Λ-module $M(\dot{0}_{hh})$, namely, those corresponding to the maps $c_1 \to c_2$, respectively $b_1 \to b_2$, respectively the identity. Therefore

$$\dim_k \operatorname{End}_\Lambda(M(\dot{0}_{hh})) = 3.$$

Consider the commutative diagrams



and



We see that the Λ-module endomorphisms of $M(\dot{0}_{hh})$ corresponding to $c_1 \to c_2$, respectively $b_1 \to b_2$, factor through P_2, respectively P_1. Therefore

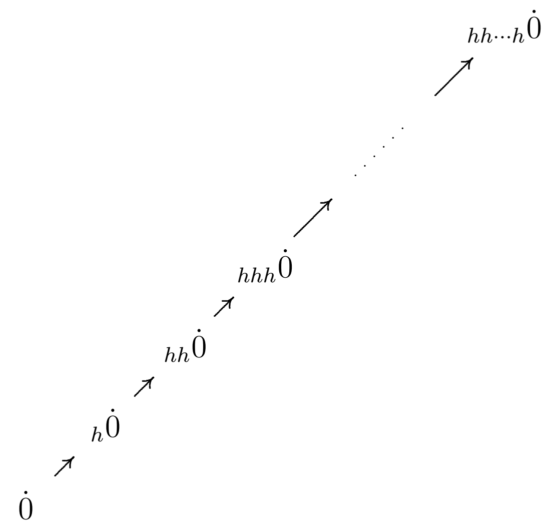
$$\dim_k \underline{\mathrm{End}}_\Lambda(M(\dot{0}_{hh})) = 1.$$

Hence $\underline{\mathrm{End}}_\Lambda(M(\dot{0}_{hh})) \cong k$ as k-algebras. However, if we consider the Λ-module corresponding to the string

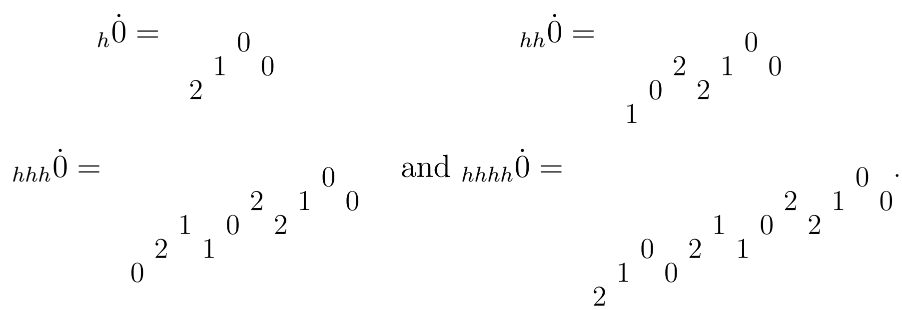
$$\dot{0}_{hhh} = \begin{array}{ccccccc} & 2 & & 1 & & {}_{a_1}0 & \\ {}_{a_0}0 & & 2 & & 1 & & 0 \end{array}$$

we see that the Λ-module endomorphism of $M(\dot{0}_{hhh})$ corresponding to the map $a_1 \to a_0$ does not factor through a projective Λ-module. Let $S_n = \dot{0}_{\underbrace{hhh\cdots h}_{n\text{ times}}}$. Since we always have the endomorphism of $M(S_n)$ corresponding to the map $a_1 \to a_0$ when $n \geq 3$, we obtain $\dim_k \underline{\mathrm{End}}_\Lambda(M(S_n)) \geq 2$ for $n \geq 3$.

Consider now the diagonal



We see that



Note that $\Omega(M({}_h\dot{0})) \cong S_0$, $\Omega(M({}_{hh}\dot{0})) \cong M(\dot{0}_h)$, $\Omega(M({}_{hhh}\dot{0})) \cong M(\dot{0}_{hh})$ and $\Omega(M({}_{hhhh}\dot{0})) \cong M(\dot{0}_{hhh})$. In general, for all $n \geq 1$

$$\Omega(M(\underbrace{hhhh\cdots h}_{n \text{ times}}\dot{0})) \cong M(\dot{0}\underbrace{hhh\cdots h}_{n-1 \text{ times}}).$$

By Theorem 2.1.11, $\underline{\mathrm{End}}_\Lambda(M(\underbrace{hhhh\cdots h}_{n \text{ times}}\dot{0})) \cong k$ if and only if $n \in \{1, 2, 3\}$. Since every Λ-module in the component of the stable Auslander-Reiten quiver containing S_0 lies in the Ω^2-orbit of a module in either one of the diagonals above, this completes the proof of Proposition 4.3.4. □

4.3.2 Stable Endomorphism Rings of the Λ-modules in the 3-Tubes

Proposition 4.3.5. *The Λ-modules in the 3-tubes which have stable endomorphism ring k are precisely those of the form:*

(i) $\Omega^n(M(\alpha))$,

(ii) $\Omega^n(M(\alpha_h))$, *and*

(iii) $\Omega^n(M(\alpha_{hh}))$,

for all $n \in \mathbb{Z}$.

Proof. Since the syzygy functor Ω induces a graph isomorphism between the two 3-tubes, we can by Theorem 2.1.11 restrict ourselves to the Λ-modules in the 3-tube containing $M(\alpha)$. Consider the diagonal

$$\begin{array}{ccccccccc} &&&&&&&& \alpha_{hh\cdots h} \\ &&&&&&& \nearrow & \\ &&&&&& \cdot^{\cdot^{\cdot}} && \\ &&&&& \nearrow &&& \\ &&&& \alpha_{hhh} &&&& \\ &&& \nearrow &&&&& \\ && \alpha_{hh} &&&&&& \\ & \nearrow &&&&&&& \\ \alpha_h &&&&&&&& \\ \nearrow &&&&&&&& \\ \alpha &&&&&&&& \end{array}$$

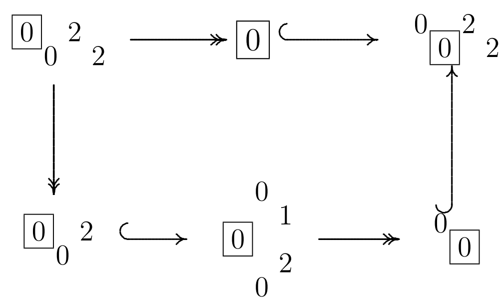
We can describe the Λ-modules $M(\alpha_h)$, $M(\alpha_{hh})$ and $M(\alpha_{hhh})$ by

$$\alpha_h = \begin{array}{cccc} 0 & & 2 & \\ & 0 & & 2 \end{array}, \quad \alpha_{hh} = \begin{array}{cccccc} 0 & & 2 & & 1 & \\ & 0 & & 2 & & 1 \end{array}, \quad \alpha_{hhh} = \begin{array}{cccccccc} 0 & & 2 & & 1 & & 0 & \\ & 0 & & 2 & & 1 & & 0 \end{array}.$$

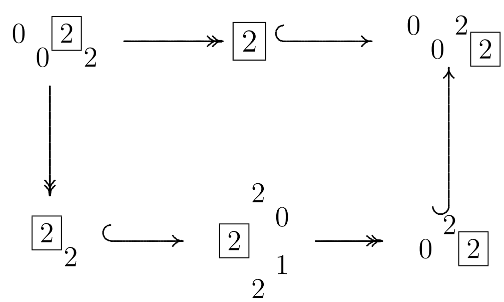
By Lemma 4.3.1, $\underline{\mathrm{End}}_\Lambda(M(\alpha)) \cong k$. Consider now the string

$$\alpha_h = \begin{array}{cccc} {}_{a_1}0 & & {}_{c_1}2 & \\ & {}_{a_2}0 & & {}_{c_2}2 \end{array}.$$

We have three canonical endomorphisms of $M(\alpha_h)$ corresponding to the maps $a_1 \to a_2$, respectively $c_1 \to c_2$, respectively the identity. Hence $\dim_k \operatorname{End}_\Lambda(M(\alpha_h)) = 3$. Consider the commutative diagrams



and

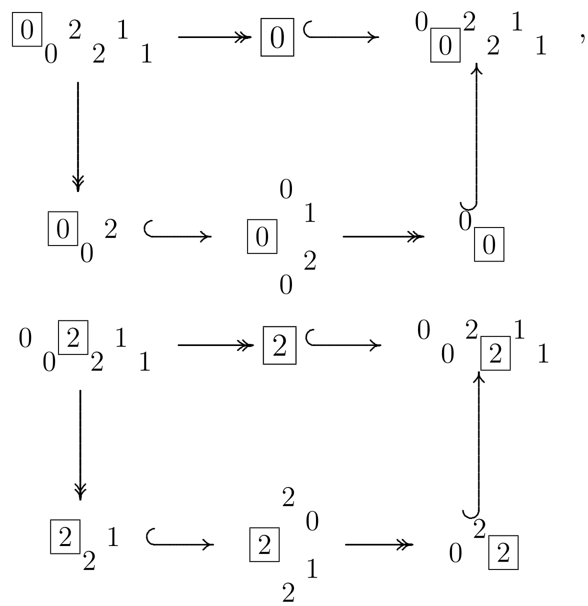


We see that the Λ-module endomorphisms corresponding to $a_1 \to a_2$, respectively $c_1 \to c_2$, factor through P_0, respectively P_2. Hence $\dim_k \underline{\operatorname{End}}_\Lambda(M(\alpha_h)) = 1$.

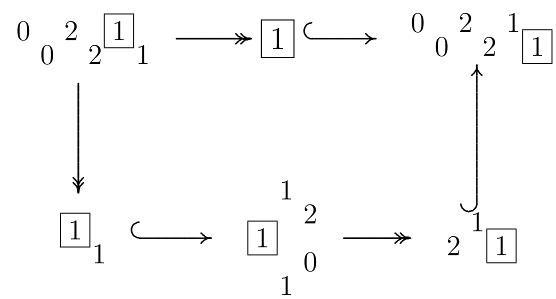
Consider next

$$\alpha_{hh} = \begin{array}{cccccc} {}_{a_1}0 & & {}_{c_1}2 & & {}_{b_1}1 & \\ & {}_{a_2}0 & & {}_{c_2}2 & & {}_{b_2}1 \end{array} .$$

We have four canonical endomorphisms of $M(\alpha_{hh})$, namely those corresponding to the maps $a_1 \to a_2$, respectively $b_1 \to b_2$, respectively $c_1 \to c_2$, respectively the identity map. Therefore $\dim_k \operatorname{End}_\Lambda(M(\alpha_{hh})) = 4$. Consider the commutative diagrams



and



we conclude that $\dim_k \underline{\mathrm{End}}_\Lambda(M(\alpha_{hh})) = 1$.

Consider now the string

$$\alpha_{hhh} = \begin{matrix} {}_{a_3}0 & & 2 & & 1 & & {}_{a_1}0 & \\ & {}_{a_2}0 & & 2 & & 1 & & {}_{a_0}0 \end{matrix} .$$

Note that the canonical endomorphism of $M(\alpha_{hhh})$ corresponding to $(a_1, a_0) \to (a_3, a_2)$ does not factor through a projective Λ-module. Therefore

$$\dim_k \underline{\mathrm{End}}_\Lambda(M(\alpha_{hhh})) \geq 2.$$

Let $S_n = \alpha_{\underbrace{hhh\cdots h}_{n \text{ times}}}$. Since we always have the endomorphism of $M(S_n)$ corresponding to the map $(a_1, a_0) \to (a_3, a_2)$ when $n \geq 3$, we obtain $\dim_k \underline{\mathrm{End}}_\Lambda(M(S_n)) \geq 2$ for

$n \geq 3$. Because every Λ-module in the 3-tube containing α lies in the Ω^2-orbit of a module in the above diagonal, this completes the proof Proposition 4.3.5. □

4.4 Ext groups

In this section, we determine the Ext groups for the Λ-modules found in Propositions 4.3.4 and 4.3.5.

Lemma 4.4.1. *For all $v \in \{0,1,2\}$, $\mathrm{Ext}^1_\Lambda(S_v, S_v) \cong k$ as k-vector spaces.*

Proof. Since σ in Definition 4.3.3 is a k-algebra automorphism of Λ, it suffices to prove Lemma 4.4.1 in the case $v = 0$. Note that $\Omega(S_0)$ can be described by the string

$$S = \begin{array}{cccc} 1 & & & \\ & 2 & & 0 \\ & & 0 & \end{array}.$$

There is only one canonical homomorphism in $\mathrm{Hom}_\Lambda(\Omega(S_0), S_0)$, namely

$$\tau_0 : \quad \begin{array}{ccc} 1 & & \\ & 2 & \boxed{0} \\ & & 0 \end{array} \twoheadrightarrow \boxed{0} \xrightarrow{=} \boxed{0}$$

Since τ_0 does not factor through a projective Λ-module, we obtain

$$\dim_k \underline{\mathrm{Hom}}_\Lambda(\Omega(S_0), S_0) = 1.$$

By Theorem 2.1.11, this means $\mathrm{Ext}^1_\Lambda(S_0, S_0) \cong k$ as k-vector spaces. □

Lemma 4.4.2. *For all $v \in \{0,1,2\}$, $\mathrm{Ext}^1_\Lambda(M(\dot{v}_h), M(\dot{v}_h)) = 0$.*

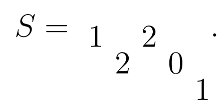
Proof. Using the k-algebra automorphism σ of Λ from Definition 4.3.3, it suffices to prove Lemma 4.4.2 in the case when $v = 0$. Note that $\Omega(M(\dot{0}_h))$ can be described by the string

$$S = \begin{array}{cc} 1 & \\ & 2 \end{array}.$$

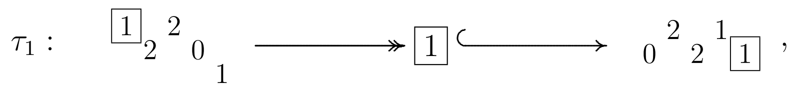
Since $\mathrm{Hom}_\Lambda(\Omega(M(\dot{0}_h)), M(\dot{0}_h)) = 0$, we obtain $\mathrm{Ext}^1_\Lambda(M(\dot{0}_h), M(\dot{0}_h)) = 0$. □

Lemma 4.4.3. *For all* $v \in \{0,1,2\}$, $\mathrm{Ext}^1_\Lambda(M(\dot{v}_{hh}), M(\dot{v}_{hh})) \cong k$ *as* k*-vector spaces.*

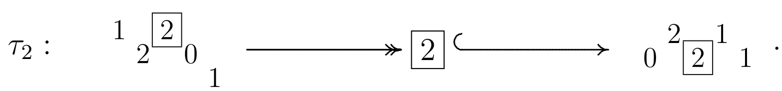
Proof. Using the k-algebra automorphism σ of Λ from Definition 4.3.3, it suffices to prove Lemma 4.4.3 in the case when $v = 0$. Note that $\Omega(M(\dot{0}_{hh}))$ can be described by the string



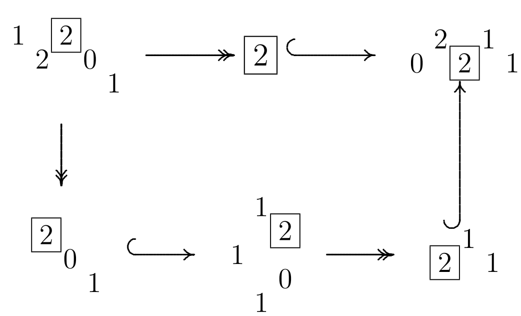
There are only two canonical homomorphisms in $\mathrm{Hom}_\Lambda(\Omega(M(\dot{0}_{hh})), M(\dot{0}_{hh}))$, say τ_1 and τ_2, where



and



Note that τ_1 does not factor through a projective Λ-module. However, looking at the commutative diagram



we see that τ_2 factors through the projective Λ-module P_1. Thus

$$\dim_k \underline{\mathrm{Hom}}_\Lambda(\Omega(M(\dot{0}_{hh})), M(\dot{0}_{hh})) = 1,$$

which implies by Theorem 2.1.11 that $\mathrm{Ext}^1_\Lambda(M(\dot{0}_{hh}), M(\dot{0}_{hh})) \cong k$ as k-vector spaces.

$\square$

The following result follows from Lemmas 4.4.1, 4.4.2 and 4.4.3 together with Theorem 2.1.11.

Proposition 4.4.4. *For all $v \in \{0, 1, 2\}$ and all $n \in \mathbb{Z}$ we have:*

(i) $\operatorname{Ext}^1_\Lambda(\Omega^n(M(\dot{v})), \Omega^n(M(\dot{v}))) \cong k$,

(ii) $\operatorname{Ext}^1_\Lambda(\Omega^n(M(\dot{v}_h)), \Omega^n(M(\dot{v}_h))) = 0$, *and*

(iii) $\operatorname{Ext}^1_\Lambda(\Omega^n(M(\dot{v}_{hh})), \Omega^n(M(\dot{v}_{hh}))) \cong k$.

We next consider the Λ-modules from Proposition 4.3.5.

Lemma 4.4.5. *We have* $\operatorname{Ext}^1_\Lambda(M(\alpha), M(\alpha)) = 0$.

Proof. Note that we can describe $\Omega(M(\alpha))$ by the string

$$S = \begin{smallmatrix} 1 & & \\ & 2 & \\ & & 0 \end{smallmatrix}.$$

Thus $\operatorname{Hom}_\Lambda(\Omega(M(\alpha)), M(\alpha)) = 0$, which implies by Theorem 2.1.11 that

$$\operatorname{Ext}^1_\Lambda(M(\alpha), M(\alpha)) = 0.$$

□

Lemma 4.4.6. *We have* $\operatorname{Ext}^1_\Lambda(M(\alpha_h), M(\alpha_h)) = 0$.

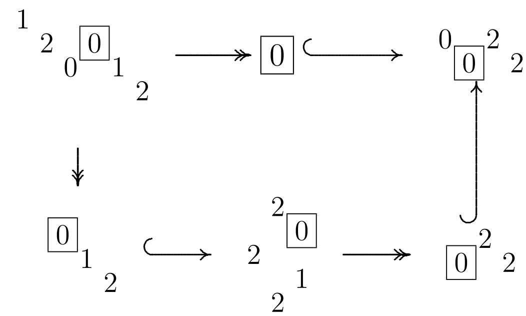
Proof. Note that we can describe $\Omega(M(\alpha_h))$ by the string

$$S = \begin{smallmatrix} 1 & & & & & \\ & 2 & & 0 & & \\ & & 0 & & 1 & \\ & & & & & 2 \end{smallmatrix}.$$

There is only one canonical homomorphism in $\operatorname{Hom}_\Lambda(\Omega(M(\alpha_h)), M(\alpha_h))$, namely

$$\tau_0 : \quad \begin{smallmatrix} 1 & & & & & \\ & 2 & & \boxed{0} & & \\ & & 0 & & 1 & \\ & & & & & 2 \end{smallmatrix} \twoheadrightarrow \boxed{0} \hookrightarrow \begin{smallmatrix} 0 & & 2 & \\ & \boxed{0} & & 2 \end{smallmatrix}$$

Looking at the commutative diagram



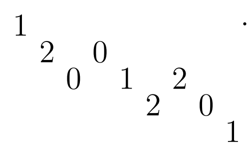
we see that τ_0 factors through the projective Λ-module P_2. Thus

$$\operatorname{Ext}^1_\Lambda(M(\alpha_h), M(\alpha_h)) \cong \underline{\operatorname{Hom}}_\Lambda(\Omega(M(\alpha_h)), M(\alpha_h)) = 0.$$

$\square$

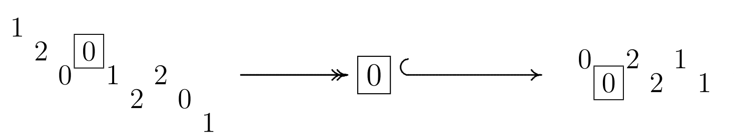
Lemma 4.4.7. *We have* $\operatorname{Ext}^1_\Lambda(M(\alpha_{hh}), M(\alpha_{hh})) \cong k$ *as* k*-vector spaces.*

Proof. Note that we can describe $\Omega(M(\alpha_{hh}))$ by the string

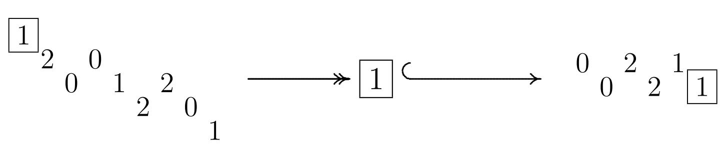


There are three canonical homomorphisms in $\operatorname{Hom}_\Lambda(\Omega(M(\alpha_{hh}), M(\alpha_{hh}))$, namely

τ_0 :

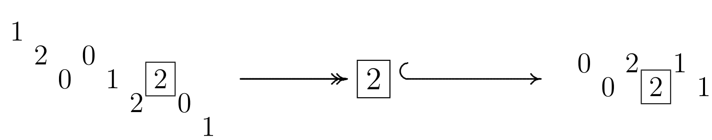


τ_1 :

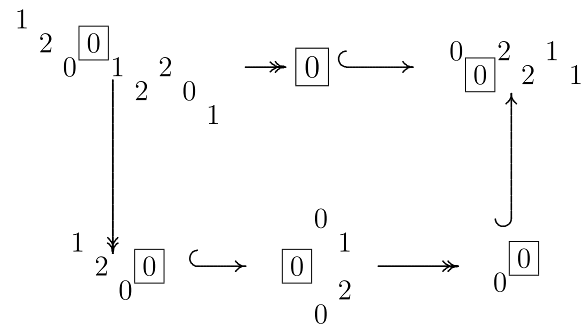


and

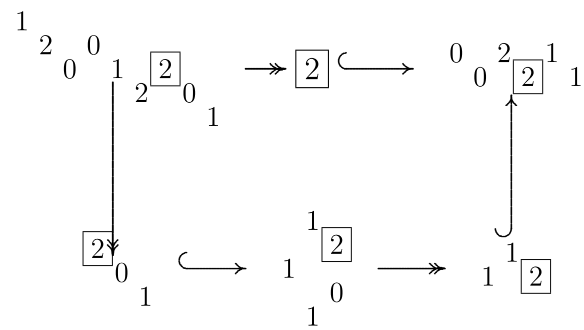
τ_2 :



Looking at the commutative diagrams



and



we see that τ_0 and τ_2 factor through P_0 and P_1, respectively. However τ_1 does not factor through a projective Λ-module. Thus $\dim_k \underline{\mathrm{Hom}}_\Lambda(\Omega(M(\alpha_{hh})), M(\alpha_{hh})) = 1$, which implies $\mathrm{Ext}^1_\Lambda(M(\alpha_{hh}), M(\alpha_{hh})) \cong k$. □

The following result follows from Lemmas 4.4.5, 4.4.6 and 4.4.7 together with Theorem 2.1.11.

Proposition 4.4.8. *For any $n \in \mathbb{Z}$ we have:*

(i) $\mathrm{Ext}^1_\Lambda(\Omega^n(M(\alpha)), \Omega^n(M(\alpha))) = 0$,

(ii) $\mathrm{Ext}^1_\Lambda(\Omega^n(M(\alpha_h)), \Omega^n(M(\alpha_h))) = 0$, *and*

(iii) $\mathrm{Ext}^1_\Lambda(\Omega^n(M(\alpha_{hh})), \Omega^n(M(\alpha_{hh}))) \cong k$.

4.5 Universal Deformation Rings

Let Λ be our basic string algebra from Section 4.1. Since Λ is a symmetric k-algebra, Λ is in particular Frobenius and self-injective. If V is a Λ-module with finite dimension over k and $\underline{\mathrm{End}}_\Lambda(V) \cong k$ then by Theorem 3.5.3 there exists a universal deformation ring $R(\Lambda, V) \in \mathrm{Ob}(\hat{\mathcal{C}})$ of V. Furthermore, by Theorem 3.6.7, $R(\Lambda, \Omega^n(V)) \cong R(\Lambda, V)$ for all $n \in \mathbb{Z}$. In this section we find the universal deformation rings of the Λ-modules V with $\underline{\mathrm{End}}_\Lambda(V) \cong k$ found in Section 4.3.

Definition 4.5.1. Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$. The *cotangent space* t_R^* is defined to be the k-vector space $\mathfrak{m}_R/\mathfrak{m}_R^2$.

Lemma 4.5.2. *Let $\theta : R \to S$ be a morphism in $\hat{\mathcal{C}}$. Let $\theta^* : t_R^* \to t_S^*$ be the induced map of cotangent spaces. Then θ is surjective if and only if θ^* is surjective.*

Proof. See [10, Lemma 1.1]. □

4.5.1 Universal Deformation Rings of the Λ-modules in the Components of the Stable Auslander-Reiten Quiver Containing Simple Λ-modules

Proposition 4.5.3. *For $v \in \{0, 1, 2\}$, $R(\Lambda, S_v) \cong k[[t]]/(t^2)$*

Proof. We prove this for $v = 0$; the cases when $v = 1$ and $v = 2$ are similar. By Proposition 4.4.4 (i), $\mathrm{Ext}^1_\Lambda(S_0, S_0) \cong k$. Therefore $R(\Lambda, S_0)$ is isomorphic to a quotient algebra of the ring of formal power series $k[[t]]$. Consider $R_0 = k[[t]]/(t^2)$ and let $M(\alpha)$ be the string Λ-module corresponding to the string $0 \xrightarrow{\alpha} 0$. We have a short exact sequence of Λ-modules

$$0 \to S_0 \xrightarrow{\iota} M(\alpha) \xrightarrow{\tau} S_0 \to 0$$

where ι is the canonical Λ-module monomorphism

$$\boxed{0} \hookrightarrow \begin{smallmatrix} 0 \\ \boxed{0} \end{smallmatrix}$$

and τ is the canonical Λ-module epimorphism

$$\boxed{0}_0 \longrightarrow\!\!\!\!\!\rightarrow \boxed{0}$$

The Λ-module $M(\alpha)$ is naturally an R_0-module by letting t act as $\iota \circ \tau$, i.e. for all $m \in M(\alpha)$, $tm = \iota(\tau(m))$. Let $\{\bar{z}_0\}$ be a k-basis of $M(\alpha)/\operatorname{rad}(M(\alpha)) \cong S_0$. Lift $\bar{z}_0$ to an element $z_0 \in M(\alpha)$. Then $tz_0 = \iota(\tau(z_0))$ is not zero and thus gives a k-basis of $\operatorname{rad}(M(\alpha))$. Hence $\{z_0\}$ is an R_0-basis of $M(\alpha)$ implying that $M(\alpha)$ is a free R_0-module of rank 1. Note that $R_0/tR_0 \cong k$, which means we have a short exact sequence of R_0-modules

$$0 \to tR_0 \to R_0 \to k \to 0.$$

Hence, tensoring this sequence with $M(\alpha)$ over R_0, we obtain a short exact sequence

$$0 \to tR_0 \otimes_{R_0} M(\alpha) \to R_0 \otimes_{R_0} M(\alpha) \to k \otimes_{R_0} M(\alpha) \to 0$$

of $R_0\Lambda$-modules. Since $R_0 \otimes_{R_0} M(\alpha) \cong M(\alpha)$, we have $tR_0 \otimes_{R_0} M(\alpha) \cong tM(\alpha)$. Therefore, $S_0 \cong M(\alpha)/tM(\alpha) \cong k \otimes_{R_0} M(\alpha)$ as Λ-modules. If $\zeta : k \otimes_{R_0} M(\alpha) \to S_0$ is a Λ-module isomorphism, then $(M(\alpha), \zeta)$ is a lift of S_0 over R_0. By Theorem 3.5.3, there exists a unique morphism $\theta : R(\Lambda, S_0) \to R_0$ in $\hat{\mathcal{C}}$ such that $\mathrm{F}_{S_0}(\theta)([(U, \phi)]) = [(M(\alpha), \zeta)]$ where $[(U, \phi)]$ is the universal deformation of S_0 over $R(\Lambda, S_0)$. Note that since $(M(\alpha), \zeta)$ is not the trivial lift of S_0 over R_0, θ is a surjection. We want to show that θ is an isomorphism. Suppose that θ is not an isomorphism. Then there exists a surjective morphism $\theta_1 : R(\Lambda, S_0) \to R_1$ in $\hat{\mathcal{C}}$, where $R_1 = k[[t]]/(t^3)$, such that $\theta = \pi \circ \theta_1$, where π is the natural projection

$$\pi : R_1 = k[[t]]/(t^3) \to R_0 = k[[t]]/(t^2).$$

Let $M_1 = R_1 \otimes_{R(\Lambda,S_0),\theta_1} U$. Note that $M_1/tM_1 \cong S_0$ as Λ-modules. Since $R(\Lambda, S_0)$ is the universal deformation ring of S_0, $R_0 \otimes_{R(\Lambda,S_0),\theta} U \cong M(\alpha)$. Therefore,

$$M(\alpha) \cong R_0 \otimes_{R(\Lambda,S_0),\theta} U \cong R_0 \otimes_{R_1,\pi} (R_1 \otimes_{R(\Lambda,S_0),\theta_1} U) \cong R_0 \otimes_{R_1,\pi} M_1.$$

Note that since $\ker(\pi) = (t^2)/(t^3)$, we have $R_0 \otimes_{R_1,\pi} M_1 \cong M_1/t^2M_1$. Hence $M(\alpha) \cong M_1/t^2M_1$ as $R_1\Lambda$-modules. Consider the $R_1\Lambda$-module homomorphism $g : M_1 \to t^2M_1$ defined by $g(x) = t^2x$ for all $x \in M_1$. Since M_1 is free over R_1, it follows that $\ker(g) = \{x \in M_1 : t^2x = 0\} = \{x \in M_1 : x = ty \text{ for some } y \in M_1\} = tM_1$. Clearly, g is onto. Thus $M_1/tM_1 \cong t^2M_1$, which implies that $S_0 \cong t^2M_1$. Hence we get a short exact sequence of $R_1\Lambda$-modules

$$0 \to S_0 \to M_1 \to M(\alpha) \to 0. \tag{4.6}$$

Note that the sequence (4.6) does not split as a sequence of $R_1\Lambda$-modules. We now show that (4.6) does not split as a sequence of Λ-modules. Suppose that $M_1 \cong S_0 \oplus M(\alpha)$ as Λ-modules, and let $\begin{pmatrix} z \\ m \end{pmatrix} \in S_0 \oplus M(\alpha) \cong M_1$. Then t acts on $\begin{pmatrix} z \\ m \end{pmatrix}$ as a matrix $U_t = \begin{pmatrix} 0 & \epsilon \\ 0 & \iota \circ \tau \end{pmatrix}$ where $\epsilon : M(\alpha) \to S_0$ is a surjective Λ-module homomorphism and $\iota \circ \tau$ is the action of t on $M(\alpha)$. Since τ generates $\mathrm{Hom}_\Lambda(M(S), S_0)$ as a k-vector space, there exists $c \in k^*$ so that $\epsilon = c\tau$. Since $t^2M_1 \cong S_0$ is nonzero, there must exist a nonzero element $\begin{pmatrix} z \\ m \end{pmatrix} \in M_1$ with $(U_t)^2 \begin{pmatrix} z \\ m \end{pmatrix} \neq 0$. Since $\ker(c\tau) = tM(\alpha)$, it follows that

$$(U_t)^2 \begin{pmatrix} z \\ m \end{pmatrix} = \begin{pmatrix} (c\tau \circ \iota \circ \tau)(m) \\ (\iota \circ \tau)^2(m) \end{pmatrix} = \begin{pmatrix} (c\tau)(tm) \\ t^2m \end{pmatrix} = 0$$

which is a contradiction. Therefore, (4.6) does not split as a sequence of Λ-modules. However,

$$\mathrm{Ext}^1_\Lambda(M(\alpha), S_0) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega(M(\alpha)), S_0) \cong \underline{\mathrm{Hom}}_\Lambda(M(\lambda\delta), S_0) = 0,$$

which implies that (4.6) has to split. Since this is a contradiction, it follows that $\theta : R(\Lambda, S_0) \to k[[t]]/(t^2)$ is an isomorphism. $\square$

Proposition 4.5.4. *For* $v \in \{0, 1, 2\}$, $R(\Lambda, M(\dot{v}_h)) \cong k$.

Proof. Since $\mathrm{Ext}^1_\Lambda(M(\dot{v}_h), M(\dot{v}_h)) = 0$ by Proposition 4.4.4 (ii) and since $R(\Lambda, M(\dot{v}_h))$ is a k-algebra, it follows that $R(\Lambda, M(\dot{v}_h)) \cong k$. $\square$

Proposition 4.5.5. *For* $v \in \{0, 1, 2\}$, $R(\Lambda, M(\dot{v}_{hh})) \cong k[[t]]$.

Proof. We prove this for $v = 0$; the cases when $v = 1$ and $v = 2$ are similar. Let $V = M(\dot{0}_{hh})$. By Proposition 4.4.4 (iii), $\mathrm{Ext}^1_\Lambda(V, V) \cong k$, which implies that $R(\Lambda, V)$ is isomorphic to a quotient algebra of $k[[t]]$. Let $\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{b}_5\}$ be a k-basis of $V = M(\dot{0}_{hh})$ corresponding to the vertices of the linear quiver used to define $M(\dot{0}_{hh})$ as follows

$$\begin{array}{ccccc} & {}_{\bar{b}_4}2 & & {}_{\bar{b}_2}1 & \\ {}_{\bar{b}_5}0 & & {}_{\bar{b}_3}2 & & {}_{\bar{b}_1}1 \end{array} . \tag{4.7}$$

For all $i \geq 1$, let T_i be the string $T_i = (\dot{0}_{hh}\beta)^{i-1}\dot{0}_{hh}$, which can be visualized as follows

$$\begin{array}{cccccccccccccccccccc}
 & & & & & & & & & & & & & & & & 2 & & 1 & \\
 & & & & & & & & & & & 2 & & 1 & & {}_{c_1}0 & & 2 & & 1 \\
 & & & & & & & & & & {}_{c_2}0 & & 2 & & 1 & & & & & \\
 & & & & & & & & & \iddots & & & & & & & & & & \\
 & & & & & & 2 & & 1 & & & & & & & & & & & \\
 & 2 & & 1 & & {}_{c_{i-1}}0 & & 2 & & 1 & & & & & & & & & & \\
{}_{c_i}0 & & 2 & & 1 & & & & & & & & & & & & & & &
\end{array}$$

and let $R_i = k[[t]]/(t^i)$. Note that if $i = 1$, then $M(T_1) = V$ and $R_1 = k$. Let $i \geq 2$ be fixed, and consider the endomorphism τ_i of the string Λ-module $M(T_i)$ induced by the canonical homomorphism

$$M(T_i) \xrightarrow{\pi_{i,i-1}} M(T_{i-1}) \xhookrightarrow{\iota_{i-1,i}} M(T_i) \tag{4.8}$$

which sends the basis vector c_j of $M(T_i)$ to c_{j+1} for $1 \leq j \leq i-1$ and c_i to zero. Note that $\ker(\tau_i)$ is the unique Λ-submodule of $M(T_i)$ which is isomorphic to V and the image of τ_i is isomorphic to $M(T_{i-1})$. Moreover, the image of τ_i^{i-1} is isomorphic to V and τ_i^i is the zero endomorphism of $M(T_i)$. The Λ-module $M(T_i)$ is naturally an R_i-module by letting t act on $M(T_i)$ as τ_i, i.e. for all $m \in M(T_i)$, $tm = \tau_i(m)$. Let $\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{b}_5\}$ be a k-basis of $M(T_i)/tM(T_i) \cong V$. Lift these elements to elements b_1, b_2, b_3, b_4, b_5 of $M(T_i)$. It follows that b_1, b_2, b_3, b_4, b_5 are linearly independent over k and $\{tb_1, \ldots, tb_5, \ldots, t^{i-1}b_1, \ldots, t^{i-1}b_5\}$ is a k-basis of $tM(T_i) \cong M(T_{i-1})$. Thus $\{b_1, b_2, b_3, b_4, b_5\}$ is an R_i-basis of $M(T_i)$, which means that $M(T_i)$ is free over R_i. By tensoring the short exact sequence

$$0 \to tR_i \to R_i \to k \to 0$$

with $M(T_i)$ over R_i, we obtain a short exact sequence of $R_i\Lambda$-modules

$$0 \to tM(T_i) \to M(T_i) \to k \otimes_{R_i} M(T_i) \to 0.$$

Since $V \cong M(T_i)/tM(T_i)$ as Λ-modules, there exists a Λ-module isomorphism $\zeta_i : k \otimes_{R_i} M(T_i) \to V$. Therefore $(M(T_i), \zeta_i)$ is a lift of V over R_i. For all $i \geq 1$, the $R_i\Lambda$-module $M(T_i)$ is a $k[[t]]\Lambda$-module via the natural projections $p_i : k[[t]] \to R_i$. Moreover, the $k[[t]]\Lambda$-modules $M(T_i)$ form an inverse system $(\{M(T_i)\}_{i\in\mathbb{Z}^+}, \{\pi_{ji}\}_{j\geq i})$ where $\pi_{ji} : M(T_j) \to M(T_i)$ is the composition $\pi_{ji} = \pi_{i+1,i} \circ \cdots \circ \pi_{j+1,j}$ where $\pi_{n,n-1} : M(T_n) \to M(T_{n-1})$ is the canonical homomorphism from (4.8) for $i+1 \leq n \leq j+1$. Let $N = \varprojlim_i M(T_i)$. Then N is a $k[[t]]\Lambda$-module, where t acts on N as $\overleftarrow{\tau_i}$. In particular, $N/tN \cong V$. Let $\{\bar{b}_1, \ldots, \bar{b}_5\}$ be a k-basis of $N/tN \cong V$. By Nakayama's Lemma, we can lift these elements to elements $b_1, \ldots, b_5 \in N$ such that $\{b_1, \ldots, b_5\}$ is a generating set of N. Then $S = \{b_1, \ldots, b_5, tb_1, \ldots, tb_5, t^2b_1, \ldots, t^2b_5, \ldots\}$ generates N as a k-vector space. If S' is an arbitrary finite subset of S, then there exists $i \geq 1$ such that $S' \subseteq \{b_1, \ldots, b_5, tb_1, \ldots, tb_5, \ldots, t^{i-1}b_1, \ldots, t^{i-1}b_5\}$ with the latter being a k-basis of $M(T_i)$. This implies that S' is in particular a k-linear independent subset of S. Hence S is a k-basis of N. Since $\{tb_1, \ldots, tb_5, t^2b_1, \ldots, t^2b_5, \ldots\}$ is a k-basis of tN, it follows that $\{b_1, \ldots, b_5\}$ is a $k[[t]]$-basis of N. Thus N is a free $k[[t]]$-module of rank 5. Consider the short exact sequence

$$0 \to (t) \to k[[t]] \to k \to 0.$$

Tensoring this sequence with N over $k[[t]]$, we obtain a short exact sequence of $k[[t]]\Lambda$-modules

$$0 \to tN \to N \to k \otimes_{k[[t]]} N \to 0.$$

It follows that $N/tN \cong k \otimes_{k[[t]]} N$. Hence there exists an isomorphism of Λ-modules $\zeta : k \otimes_{k[[t]]} N \to V$, since $N/tN \cong V$ as Λ-modules. Therefore (N, ζ) is a lift of V over $k[[t]]$. Let $R(\Lambda, V)$ and $[(U, \phi)]$ be the universal deformation ring and universal deformation of V, respectively. Then there exists a unique morphism $\theta : R(\Lambda, V) \to k[[t]]$ in $\hat{\mathcal{C}}$ such that $\mathrm{F}_V(\theta)([(U, \phi)] = [(N, \zeta)]$. In particular, $N \cong k[[t]] \otimes_{R(\Lambda,V),\theta} U$ as $k[[t]]\Lambda$-modules. We want to show that θ is an isomorphism. Note that $R(\Lambda, V) \cong k[[t]]/I$ for some ideal I of $k[[t]]$ since $\mathrm{Ext}^1_\Lambda(V, V) \cong k$ as k-vector spaces. Hence it is enough to prove that θ is surjective. Consider the lift $(M(T_2), \zeta_2)$ of V over $R_2 = k[[t]]/(t^2)$. Note that $N/t^2N \cong M(T_2)$. There exists a unique morphism $\theta' : R(\Lambda, V) \to R_2$ such that $\mathrm{F}_V(\theta')([(U, \phi)]) = [(M(T_2), \zeta_2)]$. Since $(M(T_2), \zeta_2)$ is not the trivial lift, θ' is surjective. Consider the natural projection $p_2 : k[[t]] \to R_2$, and let

(U_2, ϕ_2) be the lift of V over R_2 corresponding to the morphism $p_2 \circ \theta : R(\Lambda, V) \to R_2$. Then

$$\begin{aligned} U_2 \cong R_2 \otimes_{R(\Lambda,V),p_2\circ\theta} U &\cong R_2 \otimes_{k[[t]],p_2} (k[[t]] \otimes_{R(\Lambda,V),\theta} U) \\ &\cong R_2 \otimes_{k[[t]],p_2} N \\ &\cong N/t^2 N \\ &\cong M(T_2). \end{aligned}$$

By Lemma 3.3.10 this implies that $(U_2, \phi_2) \cong (M(T_2), \zeta_2)$ as lifts of V over R_2, i.e. $[(U_2, \phi_2)] = [(M(T_2), \zeta_2)]$. By uniqueness of θ', it follows that $\theta' = p_2 \circ \theta$. Let θ^*, $(p_2)^*$ and $(\theta')^*$ be the induced maps on the cotangent spaces. Note that $(\theta')^* = (p_2)^* \circ \theta^*$. Assume that θ is not surjective. By Lemma 4.5.2, this implies that θ^* cannot be surjective. Since $\dim_k t^*_{k[[t]]} = 1$, it follows that $\dim_k \operatorname{Im} \theta^* = 0$, which implies that $\theta^* = 0$. Hence $(\theta')^* = (p_2)^* \circ \theta^* = 0$. On the other hand, since θ' is surjective, it follows from Lemma 4.5.2 that $(\theta')^*$ is surjective. In particular, $(\theta')^* \neq 0$ which is a contradiction. Thus θ is surjective, and hence an isomorphism. □

4.5.2 Universal Deformation Rings of the Λ-modules in the 3-Tubes

Proposition 4.5.6. *We have $R(\Lambda, M(\alpha)) \cong k \cong R(\Lambda, M(\alpha_h))$ and $R(\Lambda, M(\alpha_{hh})) \cong k[[t]]$.*

Proof. By Proposition 4.4.8 (i)-(ii), $\operatorname{Ext}^1_\Lambda(M(\alpha), M(\alpha)) = 0 = \operatorname{Ext}^1_\Lambda(M(\alpha_h), M(\alpha_h))$, which implies $R(\Lambda, M(\alpha)) \cong k \cong R(\Lambda, M(\alpha_h))$. Let $V = M(\alpha_{hh})$. By Proposition 4.4.8 (iii), $\operatorname{Ext}^1_\Lambda(V, V) \cong k$, which implies that $R(\Lambda, V)$ is isomorphic to a quotient algebra of $k[[t]]$. Let $\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{b}_5, \bar{b}_6\}$ be a k-basis of $V = M(\alpha_{hh})$ corresponding to the vertices of the linear quiver used to define $M(\alpha_{hh})$ as follows

$$\begin{array}{cccccc} {}_{\bar{b}_6}0 & & {}_{\bar{b}_4}2 & & {}_{\bar{b}_2}1 & \\ & {}_{\bar{b}_5}0 & & {}_{\bar{b}_3}2 & & {}_{\bar{b}_1}1 \end{array} \tag{4.9}$$

For all $i \geq 1$, let T_i be the string $T_i = (\alpha_{hh}\beta)^{i-1}\alpha_{hh}$, which can be visualized as follows

$$\begin{array}{cccccccccccccccccc} {}_{c_i}0 & & 2 & & 1 & & {}_{c_{i-1}}0 & & 2 & & 1 & & \cdots {}_{c_1}0 & & 2 & & 1 & \\ & 0 & & 2 & & 1 & & 0 & & 2 & & 1 & & 0 & & 2 & & 1 \end{array}$$

and let $R_i = k[[t]]/(t^i)$. Note that if $i = 1$, then $M(T_1) = V$ and $R_1 = k$. Let $i \geq 2$ be fixed, and consider the endomorphism τ_i of the string Λ-module $M(T_i)$ induced by the canonical homomorphism

$$M(T_i) \xrightarrow{\pi_{i,i-1}} M(T_{i-1}) \xhookrightarrow{\iota_{i-1,i}} M(T_i) \tag{4.10}$$

which sends the basis vector c_j of $M(T_i)$ to c_{j+1} for $1 \leq j \leq i-1$ and c_i to zero. As in the proof of Proposition 4.5.5, it follows that $M(T_i)$ is an R_i-module by letting t act on $M(T_i)$ as τ_i. Let $\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{b}_5, \bar{b}_6\}$ be a k-basis of $M(T_i)/tM(T_i) \cong V$. Lift these elements to elements $b_1, b_2, b_3, b_4, b_5, b_6$ of $M(T_i)$. As in the proof of Proposition 4.5.5, it follows that $\{b_1, b_2, b_3, b_4, b_5, b_6\}$ is an R_i-basis of $M(T_i)$, which means that $M(T_i)$ is free over R_i. By tensoring the short exact sequence

$$0 \to tR_i \to R_i \to k \to 0$$

with $M(T_i)$ over R_i we obtain a short exact sequence of $R_i\Lambda$-modules

$$0 \to tM(T_i) \to M(T_i) \to k \otimes_{R_i} M(T_i) \to 0.$$

Since $V \cong M(T_i)/tM(T_i)$ as Λ-modules, there exists a Λ-module isomorphism $\zeta_i : k \otimes_{R_i} M(T_i) \to V$. Therefore $(M(T_i), \zeta_i)$ is a lift of V over R_i. For all $i \geq 1$, the

$R_i\Lambda$-module $M(T_i)$ is a $k[[t]]\Lambda$-module via the natural projections $p_i : k[[t]] \to R_i$. Moreover, the $k[[t]]\Lambda$-modules $M(T_i)$ form an inverse system $(\{M(T_i)\}_{i\in\mathbb{Z}^+}, \{\pi_{ji}\}_{j\geq i})$ where $\pi_{ji} : M(T_j) \to M(T_i)$ is the composition $\pi_{ji} = \pi_{i+1,i} \circ \cdots \circ \pi_{j+1,j}$ where $\pi_{n,n-1} : M(T_n) \to M(T_{n-1})$ is the canonical homomorphism from (4.10) for $i+1 \leq n \leq j+1$. Let $N = \varprojlim_i M(T_i)$. Then N is a $k[[t]]\Lambda$-module, where t acts on N as $\overleftarrow{\tau_i}$. In particular, $N/tN \cong V$. Let $\{\bar{b}_1, \ldots, \bar{b}_5, \bar{b}_6\}$ be a k-basis of $N/tN \cong V$. By Nakayama's Lemma, we can lift these elements to elements $b_1, \ldots, b_5, b_6 \in N$ such that $\{b_1, \ldots, b_5, b_6\}$ is a generating set of N. As in the proof of Proposition 4.5.5, it follows that $\{b_1, \ldots, b_5, b_6\}$ is a $k[[t]]$-basis of N. Thus N is a free $k[[t]]$-module of rank 6. Consider the short exact sequence

$$0 \to (t) \to k[[t]] \to k \to 0.$$

Tensoring this sequence with N over $k[[t]]$, we obtain a short exact sequence of $k[[t]]\Lambda$-modules

$$0 \to tN \to N \to k \otimes_{k[[t]]} N \to 0.$$

It follows that $N/tN \cong k \otimes_{k[[t]]} N$. Hence there exists an isomorphism of Λ-modules $\zeta : k \otimes_{k[[t]]} N \to V$, since $N/tN \cong V$ as Λ-modules. Therefore (N, ζ) is a lift of V over $k[[t]]$. Let $R(\Lambda, V)$ and $[(U, \phi)]$ be the universal deformation ring and universal deformation of V, respectively. Then there exists a unique morphism $\theta : R(\Lambda, V) \to k[[t]]$ in $\hat{\mathcal{C}}$ such that $\mathrm{F}_V(\theta)([(U, \phi)] = [(N, \zeta)]$. In particular, $N \cong k[[t]] \otimes_{R(\Lambda,V),\theta} U$ as $k[[t]]\Lambda$-modules. We want to show that θ is an isomorphism. Note that $R(\Lambda, V) \cong k[[t]]/I$ for some ideal I of $k[[t]]$ since $\mathrm{Ext}^1_\Lambda(V, V) \cong k$ as k-vector spaces. Hence it is enough to prove that θ is surjective. Consider the lift $(M(T_2), \zeta_2)$ of V over

$R_2 = k[[t]]/(t^2)$. Note that $N/t^2N \cong M(T_2)$. There exists a unique morphism $\theta' : R(\Lambda, V) \to R_2$ such that $\mathrm{F}_V(\theta')([(U,\phi)]) = [(M(T_2), \zeta_2)]$. Since $(M(T_2), \zeta_2)$ is not the trivial lift, θ' is surjective. Hence we can use the same arguments as in the proof of Proposition 4.5.5 to show that θ is surjective, and hence an isomorphism. □

The following result follows from Propositions 4.5.3, 4.5.4, 4.5.5 and 4.5.6 together with Theorem 3.6.7.

Theorem 4.5.7. *For $v \in \{0, 1, 2\}$ and for all $n \in \mathbb{Z}$ we have:*

(i) $R(\Lambda, \Omega^n(M(\dot{v}))) \cong k[[t]]/(t^2)$,

(ii) $R(\Lambda, \Omega^n(M(\dot{v}_h))) \cong k$,

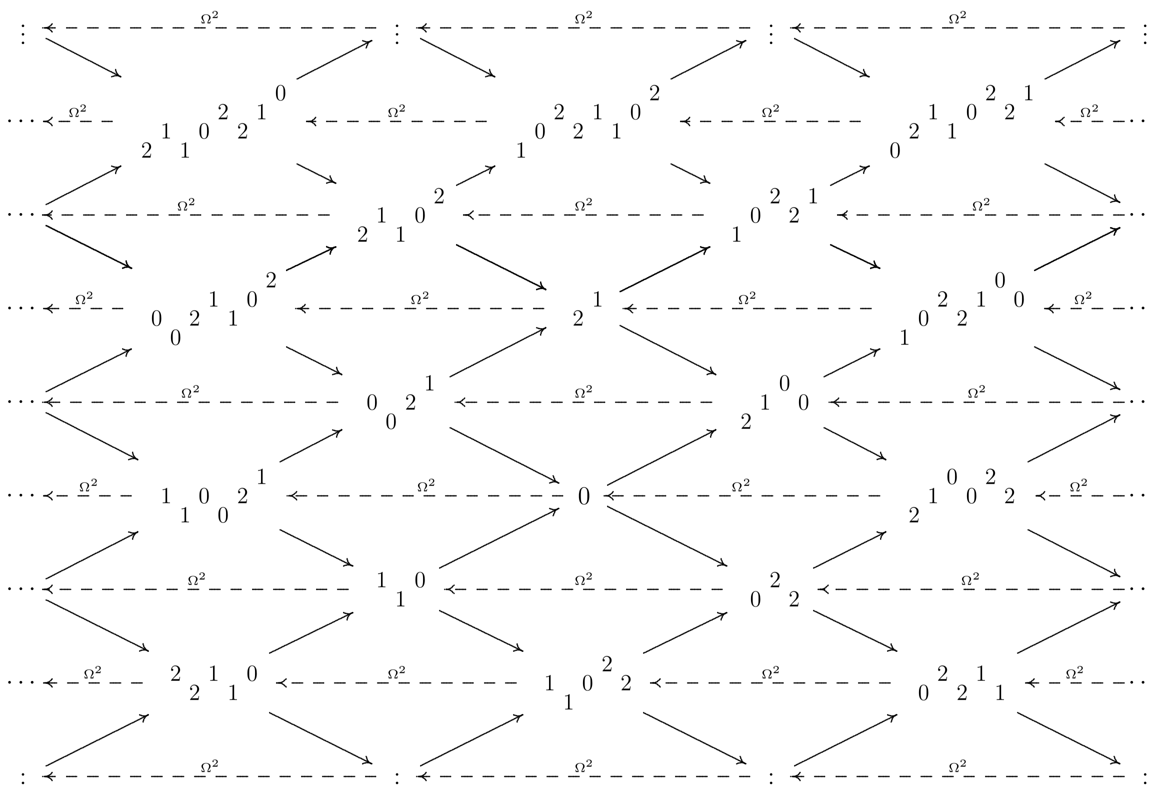
(iii) $R(\Lambda, \Omega^n(M(\dot{v}_{hh}))) \cong k[[t]]$,

(iv) $R(\Lambda, \Omega^n(M(\alpha))) \cong k$,

(v) $R(\Lambda, \Omega^n(M(\alpha_h))) \cong k$, *and*

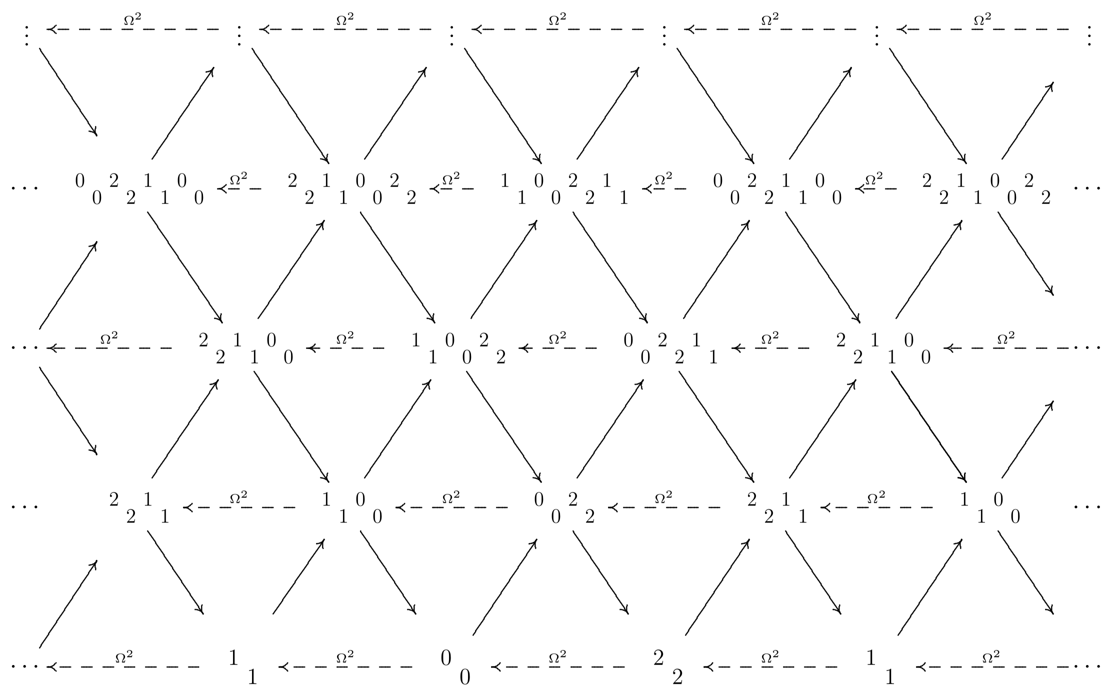
(vi) $R(\Lambda, \Omega^n(M(\alpha_{hh}))) \cong k[[t]]$.

The figures on the following two pages illustrate Theorem 4.5.7.



$\underline{\mathrm{End}}_\Lambda(M)$	$\mathrm{Ext}^1_\Lambda(M,M)$	$R(\Lambda,M)$
$\not\cong k$	–	–
$\not\cong k$	–	–
$\cong k$	k	$k[[t]]$
$\cong k$	0	k
$\cong k$	k	$k[[t]]/(t^2)$
$\cong k$	k	$k[[t]]/(t^2)$
$\cong k$	0	k
$\cong k$	k	$k[[t]]$
$\not\cong k$	–	–

Figure 4.1: Ext groups and universal deformation rings of the Λ-modules with stable endomorphism ring k in the component of the stable Auslander-Reiten quiver containing S_0



$\underline{\mathrm{End}}_\Lambda(M)$	$\mathrm{Ext}^1_\Lambda(M,M)$	$R(\Lambda,M)$
$\ncong k$	–	–
$\ncong k$	–	–
$\cong k$	k	$k[[t]]$
$\cong k$	0	k
$\cong k$	0	k

Figure 4.2: Ext groups and universal deformation rings of the Λ-modules with stable endomorphism ring k in the 3-tube containing $M(\alpha)$

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