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Endomorphisms, composition operators and Cuntz families

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ENDOMORPHISMS, COMPOSITION OPERATORS AND CUNTZ FAMILIES

by

Samuel William Schmidt

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

May 2010

Thesis Supervisor: Professor Paul S. Muhly

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If b is an inner function, then composition with b induces an endomorphism, β, of $L^\infty(\mathbb{T})$ that leaves $H^\infty(\mathbb{T})$ invariant. In this document we investigate the structure of the endomorphisms of $B(L^2(\mathbb{T}))$ and $B(H^2(\mathbb{T}))$ that implement β via the representations of $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ in terms of multiplication operators on $B(L^2(\mathbb{T}))$ and $B(H^2(\mathbb{T}))$. Our analysis, which was inspired by the work of R. Rochberg and J. McDonald, will range from the theory of composition operators on spaces of analytic functions to recent work on Cuntz families of isometries and Hilbert C^*-modules.

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2010 graduation.

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If b is an inner function, then composition with b induces an endomorphism, β, of $L^\infty(\mathbb{T})$ that leaves $H^\infty(\mathbb{T})$ invariant. In this document we investigate the structure of the endomorphisms of $B(L^2(\mathbb{T}))$ and $B(H^2(\mathbb{T}))$ that implement β via the representations of $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ in terms of multiplication operators on $B(L^2(\mathbb{T}))$ and $B(H^2(\mathbb{T}))$. Our analysis, which was inspired by the work of R. Rochberg and J. McDonald, will range from the theory of composition operators on spaces of analytic functions to recent work on Cuntz families of isometries and Hilbert C^*-modules.

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CHAPTER 1
INTRODUCTION

It is the goal of this thesis to link the venerable theory of composition operators on spaces of analytic functions to the representation theory of C^*-algebras. The theory of composition operators is full of equations that involve operators that intertwine various types of representations. In certain situations the equations can be recast in terms of "covariance equations" that are familiar from the theory of C^*-algebras, their endomorphisms and their representations; doing this yields both new theorems and new understanding of known results.

Much of what is done in this thesis was inspired by papers by Richard Rochberg [17] and John McDonald [16]. In [17, Theorem 1], Rochberg performs calculations in order expand functions in the disc algebra with respect to a particular basis. It turns out that from a more contemporary perspective, one can identify in his calculations certain Cuntz families of isometries on Hilbert space. Also, there are connections to certain Hilbert C^*-modules that lie at the heart of what Rochberg was studying. In [16], McDonald built upon Rochberg's work and proved, among other things, that the canonical transfer operator associated to composition with a finite Blaschke product leaves the Hardy space $H^2(\mathbb{T})$ invariant. The work presented here is, in large part, the result of trying to formulate Rochberg's [17, Theorem 1] in the setting of C^*-algebras and endomorphisms, using McDonald's observation on transfer operators [16, Lemma 2].

We denote the unit circle in the complex plane by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and the open unit disc by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The Lebesgue space is defined as $L^2(\mathbb{T}) := \{\xi : \mathbb{T} \to \mathbb{C} : \frac{1}{2\pi}\int_{\mathbb{T}} |\xi(z)|^2 dm(z) < \infty\}$ where m is normalized Lebesgue measure on $\mathbb{T}$. Then $L^2(\mathbb{T})$ is a Hilbert space with inner product $\langle \xi, \eta \rangle := \frac{1}{2\pi}\int_{\mathbb{T}} \xi(z)\overline{\eta(z)}dm(z)$ for ξ and η in $L^2(\mathbb{T})$.

Definition 1.1 (Hardy Space on $\mathbb{T}$)**.** *The **Hardy Space** on the unit circle, denoted $H^2(\mathbb{T})$, is defined as the collection of functions that are square integrable and whose negatively indexed Fourier coefficients are zero. That is,*

$$H^2(\mathbb{T}) := \{\xi \in L^2(\mathbb{T}) : \hat{\xi}(n) := \frac{1}{2\pi}\int_0^{2\pi} \xi(e^{it})e^{-int}dt = 0 \; \forall \; n < 0\}. \tag{1.1}$$

The inner product and norm on this space is inherited from $L^2(\mathbb{T})$.

For more details on the Hardy space see A.18. From this definition it is obvious that $H^2(\mathbb{T})$ is a subspace of $L^2(\mathbb{T})$. We will let $\{e_n\}_{n\in\mathbb{Z}}$ be the orthonormal basis for $L^2(\mathbb{T})$ given by $e_n(z) := z^n$. We denote the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ by P. That is,

$$P(\sum_{n\in\mathbb{Z}} \hat{\xi}(n)e_n) := \sum_{n\in\mathbb{N}} \hat{\xi}(n)e_n$$

takes an element of $L^2(\mathbb{T})$ expressed in terms of its Fourier series and drops the negatively indexed terms.

Definition 1.2. *The **multiplication operator** on $L^2(\mathbb{T})$ determined by a function $\varphi \in L^\infty(\mathbb{T})$ will be denoted $\pi(\varphi)$. It is given by the formula*

$$(\pi(\varphi)f)(z) := \varphi(z)f(z).$$

Definition 1.3. *The **Toeplitz operator** on $H^2(\mathbb{T})$ determined by $\varphi \in L^\infty$ will be denoted by $\tau(\varphi)$. It is defined by the formula $\tau(\varphi) := P\pi(\varphi)P$ restricted to $H^2(\mathbb{T})$.*

The use of the notation π and τ is nonstandard. More commonly, one writes M_f for the multiplication operator determined by f and T_f for the Toeplitz operator determined by f, but for the purposes of this thesis, we have found the standard notation to be a bit awkward. This choice will serve to highlight the symbol and bring it “out of the basement”, so to speak. The map π is a C^*-representation of $L^\infty(\mathbb{T})$ on $L^2(\mathbb{T})$ that is continuous with respect to the weak-$*$ topology on $L^\infty(\mathbb{T})$ and the weak operator topology on $B(L^2(\mathbb{T}))$. Also, τ is a (completely) positive linear map from $L^\infty(\mathbb{T})$ to $B(H^2(\mathbb{T}))$ with similar continuity properties. Both π and τ will be fixed throughout this document.

Definition 1.4. *An **inner function** is an analytic function g, defined on the unit* *disc such that* $|g(z)| \leq 1$ *for all* $z \in \mathbb{D}$ *and such that* $\lim_{r\to 1-} |g(re^{i\theta})| = 1$ *for almost all* $\theta \in [0, 2\pi]$.

Concrete examples of inner functions are Blaschke products. We adopt the following notation. If $w \in \mathbb{D}$, $w \neq 0$ then

$$b_w(z) := \frac{|w|}{w}\frac{w-z}{1-\overline{w}z}; \tag{1.2}$$

if $w = 0$, then $b_0(z) := z$. Given a sequence $\{\alpha_j\}_{j=1}^{\infty}$ in $\mathbb{D}$ with $\sum_{n=1}^{\infty}(1-|\alpha_n|) < \infty$, then the infinite product

$$\prod_{n=1}^{\infty} b_{a_n} \tag{1.3}$$

converges uniformly on compact subsets of $\mathbb{D}$ to an inner function b. We call b the **Blaschke product** with zeros at the points $\{a_n\}_{n=1}^{\infty}$.

Definition 1.5. *If* $\{\alpha_j\}_{j=1}^{N}$ *is a finite collection of (not necessarily distinct) points in* $\mathbb{D}$ *then the* ***finite Blaschke product*** *is defined as*

$$b := \prod_{n=1}^{N} b_{a_n}. \tag{1.4}$$

We call N *the total order of* b.

Sometimes, we write

$$b(z) = z^n \prod_{j=1}^{m} \frac{|\alpha_j|}{\alpha_j}\frac{\alpha_j - z}{1-\overline{\alpha_j}z} \tag{1.5}$$

where m is possibly infinite to emphasize the order n, of the zero of b at 0.

We fix throughout an inner function b which at times will further be assumed to be a finite Blaschke product.

Definition 1.6. *We define* $\beta : L^{\infty}(\mathbb{T}) \to L^{\infty}(\mathbb{T})$ *to be composition by* b*, that is,*

$$\beta(\varphi) := \varphi \circ b$$

It is known that β induces a $*$-endomorphism of $L^{\infty}(\mathbb{T})$ that is continuous with respect to the weak-$*$ topology on $L^{\infty}(\mathbb{T})$. The key difficulty, of course, is to to show that if $\varphi \in L^{\infty}(\mathbb{T})$, then $\beta(\varphi) = \varphi \circ b$ is a bounded measurable function and that if $\varphi_1 = \varphi$ a.e. then $\varphi_1 \circ b = \varphi \circ b$ a.e. To prove this we may assume that φ is real valued since we may always write φ in terms of it's real and imaginary parts, that is $\varphi = \varphi_{re} + i\varphi_{im}$. Now note Ryff's [18, Theorem 2.2] shows that if b is a holomorphic map of $\mathbb{D}$ into $\mathbb{D}$, and if $\varphi \in H^p(\mathbb{T})$, then $(\varphi \circ b)(e^{it})$ exists a.e. and $\lim_{r\to 1-}(\varphi \circ b)(re^{it}) = (\varphi \circ b)(e^{it})$ a.e. So if $\varphi_+ = P\varphi$, then Ryff's theorem proves the result since $\varphi = 2Re(\varphi_+) + \hat{\varphi}(0)$ and $\varphi_+ \in H^2(\mathbb{T})$. When b is a finite Blaschke product this statement is fairly elementary; if b is an arbitrary inner function, it is somewhat more substantial. We give an operator-theoretic proof in Corollary 3.3. When β leaves a subspace of $L^\infty(\mathbb{T})$ invariant, we will continue to use the notation β for its restriction to the subspace for notational convenience. The primary focus of this document is the following problem.

Problem 1.7 (The Central Problem)**.** *Describe all $*$-endomorphisms α of $B(L^2(\mathbb{T}))$ such that*

$$\alpha \circ \pi = \pi \circ \beta \tag{1.6}$$

and describe all $*$-endomorphisms α_+ of $B(H^2(\mathbb{T}))$ such that

$$\alpha_+ \circ \tau = \tau \circ \beta. \tag{1.7}$$

If an endomorphism α of $B(L^2(\mathbb{T}))$ satisfies (1.6), the pair (π, α) is called a *covariant representation* of the pair $(L^\infty(\mathbb{T}), \beta)$. The first part of our problem is thus to identify all endomorphisms α of $B(L^2(\mathbb{T}))$ that yield a covariant representation (π, α) of $(L^\infty(\mathbb{T}), \beta)$. It will become clear that it is natural to consider the related equation (1.7) which is a more or less the "analytic" version of (1.6). As we shall see, it may be interpreted as describing certain covariant representations of the Toeplitz algebra.

Definition 1.8. *The* ***Toeplitz algebra****, denoted $\mathfrak{T}$, is the C^*-algebra generated by all the Toeplitz operators $\tau(\varphi)$, $\varphi \in L^\infty(\mathbb{T})$.*

We write $\mathfrak{T}(C(\mathbb{T}))$ for the C^*-algebra generated by all Toeplitz operators $\tau(\phi)$, where $\phi \in C(\mathbb{T})$. It is well known [5, Chapter 7] that the resulting C^*-algebra is $\mathfrak{T}(C(\mathbb{T})) = \{\tau(\varphi) + k \ : \ \varphi \in C(\mathbb{T}), k \in \mathfrak{K}\}$, we note this as it will become relevant later on.[1]

[1] See A.12 for the definition of $\mathfrak{K}$.

It is not clear *a priori* that *any* endomorphisms satisfying (1.6) or (1.7) exist. They do, however, as we shall show in Theorem 3.4, where Rochberg's work plays a central role. Then, in Corollary 3.6, we show how Rochberg's analysis yields a complete description of all solutions to (1.7). Identifying all solutions to (1.6) is more complicated, and it is here that we must assume that b is a finite Blaschke product. The set of solutions to (1.6) is described in Theorem 5.3 under this restriction.

In solving Problem 1.7 we obtain many new proofs of known results. We do not take any position on the matter of which proofs are simpler or more elementary. Our more modest goal is to separate what can be derived through elementary Hilbert space considerations from what requires more specific function-theoretic analysis. In this respect, we were inspired by the work of Helson and Lowdenslager [9], Halmos and others who cast Hardy space theory in Hilbert space terms and, in particular, showed that Beurling's theorem about invariant subspaces of the shift operator can be proved with elementary Hilbert space methods. Indeed, as we shall see, our main Theorem 3.4 is a straightforward corollary of Beurling's theorem and requires no more technology than Helson and Lowdenslager's approach to that result. This paper, therefore, has something of a didactic component. When we reprove or reinterpret a known result, we call attention to it and give references to alternative approaches.

CHAPTER 2
PRELIMINARIES

2.1 Composition on $L^\infty(\mathbb{T})$

When H is a Hilbert space it is well known that, $B(H)$, the space of bounded operators on H is the dual space of the space of trace class operators on H [1]. The weak-$*$ topology on $B(H)$ or *ultraweak* topology is different from the weak operator topology, but the two coincide on bounded subsets of $B(H)$. It follows that our representation π is continuous with respect to the weak-$*$ topology on $L^\infty(\mathbb{T})$ and either the weak operator topology or the ultraweak topology on $B(L^2(\mathbb{T}))$. To see this consider a net φ_n converging weak-$*$ to φ in $L^\infty(\mathbb{T})$. That is, $\int \varphi_n f dm \to \int \varphi f dm \; \forall f \in L^1(\mathbb{T})$. However, every $f \in L^1(\mathbb{T})$ can be written as $f = g\overline{h}$ where $g, h \in L^2$. Then $\int \varphi_n f dm = \int \varphi_n g\overline{h} dm = \langle \pi(\varphi_n)g, h\rangle$. Therefore $\varphi_n \to \varphi$ w^* if and only if $\langle \pi(\varphi_n)g, h\rangle \to \langle \pi(\varphi)g, h\rangle$ for all $g, h \in L^2(\mathbb{T})$ if and only if $\pi(\varphi_n) \to \pi(\varphi)$ weakly. As we mentioned earlier, when b is a *finite* Blaschke product it is clear that composition with b induces an endomorphism of $L^\infty(\mathbb{T})$. It is well known, as well as proved in this thesis, that finite Blaschke products are N-to-1, local homeomorphisms of the unit circle which are analytic in the open unit disc. In fact, since a finite Blaschke product has poles outside the closed disc $\overline{\mathbb{D}}$, one can find an open set containing $\overline{\mathbb{D}}$ on which b is analytic. We will see that these properties are more than sufficient to satisfy the conditions of lemma 2.1 and that composition with b will be well-defined on $L^\infty(\mathbb{T})$. It is less clear that composition with an arbitrary inner function will induce and endomorphism of $L^\infty(\mathbb{T})$. There are two complications. First, the boundary values of a general inner function b are only defined on a set $F \subseteq \mathbb{T}$ with $m(\mathbb{T} \backslash F) = 0$. Second, an element of $L^\infty(\mathbb{T})$ is an *equivalence class* of measurable functions containing a bounded representative,

[1] See A.11.

where two functions are equivalent if and only if they differ on a null set. Thus we want to know that if we extend a general inner function b arbitrarily on $\mathbb{T}\backslash F$, mapping to $\mathbb{T}$, and if φ and ψ differ at most on a null set, then so do $\varphi \circ b$ and $\psi \circ b$. A little reflection reveals that for this to happen, it is necessary and sufficient that the following assertion be true:

> If b is an inner function whose domain on $\mathbb{T}$ is the measurable set F, then for every null set E of $\mathbb{T}$, $b^{-1}(E)$ is a null set of F.

This fact is well known, but exactly who deserves credit for first proving it is unclear to us. The short note by Kametani and Ugaheri [12] proves it in the case that $b(0) = 0$. This implies the general case, as Lebesgue null sets of $\mathbb{T}$ are preserved by conformal maps of the disc, and every inner function b can be written $b = \alpha \circ b_1$ with b_1 an inner function fixing the origin and α a conformal map of the disc. In Corollary 3.3, we will give a proof of this assertion from the abstract Hilbert space perspective. We will need the following lemma. To emphasize the distinction between a measurable function φ and its equivalence class modulo the relation of being equal almost everywhere, we *temporarily* write $[\varphi]$ for the latter.

Lemma 2.1. *Let θ be a Lebesgue measurable function from $\mathbb{T}$ to $\mathbb{T}$. Suppose Θ is defined on trigonometric polynomials p by the formula $\Theta(p) = p \circ \theta$. Then*

1. *Θ has a unique extension to a $*$-homomorphism from $C(\mathbb{T})$ into $L^\infty(\mathbb{T})$, and it is given by the formula $\Theta(\varphi) = [\varphi \circ \theta]$, $\varphi \in C(\mathbb{T})$.*

2. *If Θ is continuous with respect to the weak-$*$ topology of $L^\infty(\mathbb{T})$ restricted to $C(\mathbb{T})$ and the weak-$*$ topology on $L^\infty(\mathbb{T})$, then for each Lebesgue null set E of $\mathbb{T}$, $m(\theta^{-1}(E)) = 0$, and thus Θ extends uniquely to a $*$-endomorphism of $L^\infty(\mathbb{T})$ satisfying $\Theta([\varphi]) = [\varphi \circ \theta]$ for all $[\varphi] \in L^\infty(\mathbb{T})$. The map Θ is completely determined by $[\theta]$.*

Proof. It is clear that Θ, defined in the trigonometric polynomials by $\Theta(p) = p \circ \theta$ satisfies:

- $\Theta(p + q) = \Theta(p) + \Theta(q)$,
- $\Theta(pq) = \Theta(p)\Theta(q)$,
- $\Theta(p^*) = \Theta(p)^*$.

Because the trigonometric polynomials are dense in $C(\mathbb{T})$ and the range of θ is contained in $\mathbb{T}$, Θ extends to a C^*-homomorphism of $C(\mathbb{T})$ into $L^\infty(\mathbb{T})$ without complication. It is sufficient to prove this for p, a trigonometric polynomial. Since θ maps $\mathbb{T}$ to $\mathbb{T}$,

$$\begin{aligned} \|[p \circ \theta]\|_{L^\infty} &\leq \sup_{z \in \mathbb{T}} |p(\theta(z))| \\ &\leq \sup_{z \in \mathbb{T}} |p(z)| \\ &= \|p\|_{C(\mathbb{T})}. \end{aligned}$$

This proves the first assertion.

For the second, fix the Lebesgue null set E, and choose a G_δ set E_0 containing E such that $E_0 \backslash E$ has measure zero. So, if $\{f_n\}_{n \geq 0}$ is a sequence in $C(\mathbb{T})$ such that $f_n \downarrow 1_{E_0}$[2] pointwise, then $[f_n]$ converges to $[1_{E_0}] = [1_E]$ weak-$*$. But also, $f_n \circ \theta \downarrow 1_{E_0} \circ \theta = 1_{\theta^{-1}(E_0)}$ pointwise. Therefore, $[f_n \circ \theta]$ converges to $[1_{E_0} \circ \theta] = [1_{\theta^{-1}(E_0)}]$ weak-$*$. As E is a null set, so is E_0, and the $[f_n]$ converge to 0 weak-$*$. Our hypothesis then implies that the $[\Theta(f_n)] = [f_n \circ \theta]$ converge to 0 weak-$*$, proving that $m(\theta^{-1}(E_0)) = 0$. As $\theta^{-1}(E) \subseteq \theta^{-1}(E_0)$ it follows that $\theta^{-1}(E)$ is also a null set, as desired. Thus, if $\varphi = \psi$ a.e. , then $\varphi \circ \theta = \psi \circ \theta$ a.e. that is, $\Theta([\varphi]) = [\varphi \circ \theta]$ is well defined. $\square$

If b is an inner function that may be defined only on a subset F of $\mathbb{T}$ with $m(\mathbb{T} \backslash F) = 0$, we can extend b to all of $\mathbb{T}$ by setting $b(z) = 1$ for all $z \in \mathbb{T} \backslash F$. Lemma

[2] We write the characteristic function (or indicator function) of a set E as 1_E.

2.1 shows that composition by this extension is equivalent to composition by the original. More precisely, this lemma shows that no matter how we extend b from F to all of $\mathbb{T}$ (so long as the extension maps to $\mathbb{T}$) we obtain a weak$-*$ continuous endomorphism of $L^\infty(\mathbb{T})$ and any two extensions agree. Therefore, generality is not lost in assuming b is an inner function defined on all of $\mathbb{T}$.

2.2 $*$-endomorphisms of $B(H)$

Next, we want to say a few words about $*$-endomorphisms of $B(H)$, where H is a separable Hilbert space. Our discussion largely follows Section 2 of [2]. A *Cuntz family* on H is an N-tuple of isometries $\{S_i\}_{i=1}^N$ on H with mutually orthogonal ranges that together span H; here the number N may be a positive integer or ∞. A Cuntz family $S = \{S_i\}_{i=1}^N$ on H determines a map $\alpha_S : B(H) \to B(H)$ via

$$\alpha_S(T) = \sum_{i=1}^{N} S_i T S_i^*, \qquad T \in B(H). \tag{2.1}$$

(If $N = \infty$, this sum is convergent in the strong operator topology.) The map α_S is readily seen to be a $*$-endomorphism of $B(H)$; multiplicativity is deduced from the fact that a tuple $S = \{S_i\}_{i=1}^N$ is a Cuntz family if and only if the *Cuntz relations*

$$S_i^* S_j = \delta_{ij} I, \qquad 1 \le i, j \le N, \tag{2.2}$$

and

$$\sum_{i=1}^{N} S_i S_i^* = I \tag{2.3}$$

are satisfied. These relations are named after J. Cuntz, who made a penetrating analysis of them in [3].

Theorem 2.2. *Every $*$-endomorphism α of $B(H)$, with H separable, is of the form α_S for some Cuntz family S.*

Proof. We recall the details of the proof of this fact, due to Arveson [2, Proposition 2.1]. Fix a $*$-endomorphism α and define $E = \{S \in B(H) \mid ST = \alpha(T)S, T \in B(H)\}$. It follows from $ST = \alpha(T)S$ that $TS^* = S^*\alpha(T)$ and so for any S_1 and S_2 in E we have

$$S_2^* S_1 T = S_2 \alpha(T) S_1 = T S_2^* S_1.$$

That is, $S_2^* S_1$ commutes with all elements of $B(H)$, and is hence a scalar multiple of the identity. So we can define an inner product $\langle \cdot, \cdot \rangle$ on E by the formula

$$\langle S_1, S_2 \rangle I = S_2^* S_1, \qquad S_1, S_2 \in E,$$

and E with this inner product is a Hilbert space [2, Proposition 2.1]. Now for any orthonormal basis $S = \{S_i\}_{i=1}^N$ for E, the equation

$$S_i^* S_j = \langle S_1, S_2 \rangle I = \delta_{i,j} I$$

follows from the fact that the S_i's are orthonormal. The fact that the S_i's span E implies that,

$$\sum_{i=1}^{N} S_i S_i^* = I$$

since the $S_i S_i^*$ is projection onto the range of S_i. Thus S is a Cuntz family satisfying $\alpha = \alpha_S$. Therefore it is enough to know that E *has* an orthonormal basis - that is, that $E \neq \{0\}$. This follows from the fact that a $*$-endomorphism of $B(H)$, when H is separable, is necessarily ultraweakly continuous[3], and that an ultraweakly continuous unital representation of $B(H)$ is necessarily unitarily equivalent to a multiple of the identity representation of $B(H)$. That multiple is the dimension of E. □

The correspondence between endomorphisms and Cuntz families is not quite one-to-one. Laca observed [13, Proposition 2.2] that if $S = \{S_i\}_{i=1}^N$ and $\tilde{S} = \{\tilde{S}_i\}_{i=1}^N$ are two Cuntz families such that $\alpha_S = \alpha_{\tilde{S}}$, then there is a unitary matrix (u_{ij}) so that $\widetilde{S}_i = \sum_j u_{ij} S_j$, and conversely. The reason is that S and $\tilde{S}$ are both orthonormal bases for the same Hilbert space E. More concretely, we check that the scalars $u_{ij} = S_j^* \widetilde{S}_i$ have the desired properties. Consider,

[3]This non-trivial fact is discussed in detail in [19, Section V.5].

$$\begin{bmatrix} S_1^*\widetilde{S}_1 & S_2^*\widetilde{S}_1 & S_3^*\widetilde{S}_1 & \cdots & S_N^*\widetilde{S}_1 \\ S_1^*\widetilde{S}_2 & S_2^*\widetilde{S}_2 & S_3^*\widetilde{S}_2 & & \\ S_1^*\widetilde{S}_3 & S_2^*\widetilde{S}_3 & S_3^*\widetilde{S}_3 & & \\ \vdots & & & \ddots & \\ S_1^*\widetilde{S_N} & & & & S_N^*\widetilde{S_N} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_N \end{bmatrix} = \begin{bmatrix} \sum_j S_j S_j^* \widetilde{S}_1 \\ \sum_j S_j S_j^* \widetilde{S}_2 \\ \sum_j S_j S_j^* \widetilde{S}_3 \\ \vdots \\ \sum_j S_j S_j^* \widetilde{S_N} \end{bmatrix} = \begin{bmatrix} \widetilde{S}_1 \\ \widetilde{S}_2 \\ \widetilde{S}_3 \\ \vdots \\ \widetilde{S_N} \end{bmatrix}$$

In this computation we note that the u_{ij}'s commute with every element of $B(H)$ and in particular with the $\widetilde{S}_i$'s. This follows from the fact that S is a Cuntz family, which is also why this matrix is unitary as seen by the following computation.

$$\begin{bmatrix} S_1^*\widetilde{S}_1 & S_2^*\widetilde{S}_1 & S_3^*\widetilde{S}_1 & \cdots & S_N^*\widetilde{S}_1 \\ S_1^*\widetilde{S}_2 & S_2^*\widetilde{S}_2 & S_3^*\widetilde{S}_2 & & \\ S_1^*\widetilde{S}_3 & S_2^*\widetilde{S}_3 & S_3^*\widetilde{S}_3 & & \\ \vdots & & & \ddots & \\ S_1^*\widetilde{S_N} & & & & S_N^*\widetilde{S_N} \end{bmatrix}^* \begin{bmatrix} S_1^*\widetilde{S}_1 & S_2^*\widetilde{S}_1 & S_3^*\widetilde{S}_1 & \cdots & S_N^*\widetilde{S}_1 \\ S_1^*\widetilde{S}_2 & S_2^*\widetilde{S}_2 & S_3^*\widetilde{S}_2 & & \\ S_1^*\widetilde{S}_3 & S_2^*\widetilde{S}_3 & S_3^*\widetilde{S}_3 & & \\ \vdots & & & \ddots & \\ S_1^*\widetilde{S_N} & & & & S_N^*\widetilde{S_N} \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{S}_1^{\,*} S_1 & \widetilde{S}_2^{\,*} S_1 & \widetilde{S}_3^{\,*} S_1 & \cdots & \widetilde{S_N}^{\,*} S_1 \\ \widetilde{S}_1^{\,*} S_2 & \widetilde{S}_2^{\,*} S_2 & \widetilde{S}_3^{\,*} S_2 & & \\ \widetilde{S}_1^{\,*} S_3 & \widetilde{S}_2^{\,*} S_3 & \widetilde{S}_3^{\,*} S_3 & & \\ \vdots & & & \ddots & \\ \widetilde{S}_1^{\,*} S_N & & & & \widetilde{S_N}^{\,*} S_N \end{bmatrix} \begin{bmatrix} S_1^*\widetilde{S}_1 & S_2^*\widetilde{S}_1 & S_3^*\widetilde{S}_1 & \cdots & S_N^*\widetilde{S}_1 \\ S_1^*\widetilde{S}_2 & S_2^*\widetilde{S}_2 & S_3^*\widetilde{S}_2 & & \\ S_1^*\widetilde{S}_3 & S_2^*\widetilde{S}_3 & S_3^*\widetilde{S}_3 & & \\ \vdots & & & \ddots & \\ S_1^*\widetilde{S_N} & & & & S_N^*\widetilde{S_N} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_j S_1^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_1 & \sum_j S_2^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_1 & \sum_j S_3^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_1 & \cdots & \sum_j S_N^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_1 \\ \sum_j S_1^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_2 & \sum_j S_2^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_2 & \sum_j S_3^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_2 & & \\ \sum_j S_1^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_3 & \sum_j S_2^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_3 & \sum_j S_3^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_3 & & \\ \vdots & & & \ddots & \\ \sum_j S_1^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_N & & & & \sum_j S_N^*\widetilde{S}_j\widetilde{S}_j^{\,*} S_N \end{bmatrix}$$

$$= \begin{bmatrix} S_1^*S_1 & S_2^*S_1 & S_3^*S_1 & \cdots & S_N^*S_1 \\ S_1^*S_2 & S_2^*S_2 & S_3^*S_2 & & \\ S_1^*S_3 & S_2^*S_3 & S_3^*S_3 & & \\ \vdots & & & \ddots & \\ S_1^*S_N & & & & S_N^*S_N \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & & \\ 0 & 0 & I & & \\ \vdots & & & \ddots & \\ 0 & & & & I \end{bmatrix}$$

Again keeping in mind that the u_{ij}'s commute with each other and that S and $\widetilde{S}$ are both Cuntz families. A similar calculation shows that $(u_{ij})(u_{ij})^* = I_N$, where I_N is the $N \times N$ matrix and I is the identity operator on H. Thus, we have verified that if we want to understand $*$-endomorphisms of $B(H)$ we can equivalently study Cuntz families. Also, unitary equivalent families correspond to unitarily equivalent orthonormal bases.

This transforms our central problem to the following.

Problem 2.3. *To describe the collection of Cuntz families $S = \{S_i\}_{i=1}^N$ on $L^2(\mathbb{T})$ and $R = \{R_i\}_{i=1}^N$ on $H^2(\mathbb{T})$ such that (π, α_S) is a covariant representation of $(L^\infty(\mathbb{T}), \beta)$ and (τ, α_R) is a covariant representation of $(\mathfrak{T}, \beta)$ in the sense of equations* (1.6) *and* (1.7)*:*

$$\sum_{i=1}^N S_i \pi(\varphi) S_i^* = \pi(\beta(\varphi)), \qquad \varphi \in L^\infty(\mathbb{T}), \tag{2.4}$$

and

$$\sum_{i=1}^N R_i \tau(\varphi) R_i^* = \tau(\beta(\varphi)), \qquad \varphi \in L^\infty(\mathbb{T}). \tag{2.5}$$

CHAPTER 3
ROCHBERG'S OBSERVATION

3.1 Cuntz Covariant Representations

As we have noted, our analysis rests on an observation of R. Rochberg in his 1973 paper [17]. A preliminary remark on isometries in abstract Hilbert space will be necessary. For the sake of self containment we state it and provide the elementary proof.

Remark 3.1. *If V is an isometry on a Hilbert space H, and if D is the subspace $H \ominus VH$ then the subspaces $D, VD, V^2D, \ldots$ are mutually orthogonal.*

Proof. Let $\xi, \eta \in D$ and with out loss of generality let $m, n \in \mathbb{N}$ with $n > M$. Consider,

$$\langle V^n\xi, V^m\eta\rangle = \langle (V^*)^m V^n\xi, \eta\rangle = \langle V^{n-m}\xi, \eta\rangle = 0$$

The above equalities hold because V is an isometry and because η is orthogonal to anything in VH, namely $V^{n-m}\xi$. □

So $(\bigoplus_{k\geq 0} V^kD)^\perp = \bigcap_{j\geq 0} V^jH$. Thus, for an arbitrary Hilbert space H we may write $H = \bigoplus_{k\geq 0} V^kD$ if and only if V is *pure*, that is, $\bigcap_{j\geq 0} V^jH = \{0\}$. We will see that in the case that $H = H^2(\mathbb{T})$ and V is the isometry $\tau(b) = \pi(b)|_{H^2(\mathbb{T})}$ induced by a nonconstant inner function b, that V is pure, and that in fact D is a complete wandering subspace for the unitary $\pi(b)$ in the sense of (3.2) below. This is a minor modification of a point made in [17, Theorem 1].

Lemma 3.2. *Let b be a nonconstant inner function, and let $\mathcal{D} := H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$. Then*

$$H^2(\mathbb{T}) = \bigoplus_{k\geq 0} \pi(b)^k\mathcal{D}, \tag{3.1}$$

and

$$L^2(\mathbb{T}) = \bigoplus_{n\in\mathbb{Z}} \pi(b)^n\mathcal{D}. \tag{3.2}$$

Proof. As noted, in order to prove equation (3.1) it is sufficient to prove that the space $K := \bigcap_{n=0}^{\infty} \pi(b)^n H^2(\mathbb{T})$ is the zero subspace. It is obvious that $\pi(b)$ commutes with $\pi(z)$. That is, the space K is invariant for the unilateral shift $\tau(z) = \pi(z)|_{H^2(\mathbb{T})}$. If $K \neq \{0\}$, then by Beurling's theorem [A.26] there is an inner function θ with $K = \pi(\theta)H^2(\mathbb{T})$. As $\pi(b)K = K$ by definition, we see that $\pi(b)\pi(\theta)H^2(\mathbb{T}) = \pi(\theta)H^2(\mathbb{T})$, and applying $\pi(\theta^{-1})$ to both sides we conclude that $\pi(b)H^2(\mathbb{T}) = H^2(\mathbb{T})$. But b is nonconstant, so by the uniqueness assertion in Beurling's theorem (see [8, Theorem 3]), $\pi(b)H^2(\mathbb{T})$ is a proper subspace of $H^2(\mathbb{T})$. This contradiction shows that $K = \{0\}$, and (3.1) follows.

Since $\pi(b)$ is a unitary on $L^2(\mathbb{T})$, it is immediate from (3.1) that the spaces $\pi(b)^n\mathcal{D}$, $n \in \mathbb{Z}$, are mutually orthogonal. Letting $L = \bigvee_{k\in\mathbb{Z}} \pi(b)^k\mathcal{D}$, it is clear from (3.1) that $L = \bigvee_{k\geq 0} \pi(b)^{-k}H^2(\mathbb{T})$, and thus that L is invariant under $\pi(z)$. By Helson and Lowdenslager's generalization of Beurling's theorem [A.27] (also, see [9, Section 1] or [8, Theorem 3] for the original statement and proof), either there is a unimodular $\theta \in L^\infty(\mathbb{T})$ with $L = \pi(\theta)H^2(\mathbb{T})$ or there is a measurable $E \subseteq \mathbb{T}$ satisfying $L = \pi(1_E)L^2(\mathbb{T})$. In the first case, as clearly $\pi(b)L = L$, we conclude that $\pi(\theta)\pi(b)H^2(\mathbb{T}) = \pi(\theta)H^2(\mathbb{T})$, and applying $\pi(\theta^{-1})$ to both sides we conclude that $\pi(b)H^2(\mathbb{T}) = H^2(\mathbb{T})$, which contradicts the fact that b is not constant. Thus there is $E \subseteq \mathbb{T}$ with $L = \pi(1_E)L^2(\mathbb{T})$. The fact that L contains $H^2(\mathbb{T})$ implies $E = \mathbb{T}$ almost everywhere, so $L = L^2(\mathbb{T})$ as desired. □

Corollary 3.3. *If b is an arbitrary inner function and if β is defined on trigonometric polynomials p by the formula $\beta(p) := p \circ b$, then β extends to a $*$-endomorphism of $L^\infty(\mathbb{T})$ that is continuous with respect to the weak-$*$ topology.*

Proof. Lemma 3.2 implies that $\pi(b)$ is unitarily equivalent to a multiple of the bilateral shift - the multiple being $\dim(\mathcal{D})$. Thus there is a Hilbert space isomorphism W from $L^2(\mathbb{T})$ to $L^2(\mathbb{T}) \otimes \mathcal{D}$ such that $\pi(b) = W^{-1}(\pi(z) \otimes I_{\mathcal{D}})W$. So, for every trigonometric polynomial p,

$$\pi(\beta(p)) = p(\pi(b)) = W^{-1}p(\pi(z) \otimes I_{\mathcal{D}})W.$$

Since π is a homeomorphism with respect to the weak-$*$ topology on $L^\infty(\mathbb{T})$ and the weak operator topology restricted to the range of π, it is evident that b and β satisfy the hypotheses of Lemma 2.1, and the desired result follows. $\square$

Of course, the proof just given recapitulates parts of the well-known theory of the functional calculus for unitary operators.

Theorem 3.4. *Let b be a non-constant inner function and let β be the endomorphism of $L^\infty(\mathbb{T})$ given by $\varphi \mapsto \varphi \circ b$. If $\{v_i\}_{i=1}^N$ is an orthonormal basis for the defect space, $\mathcal{D} = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$, then there is a unique Cuntz family $S = \{S_i\}_{i=1}^N$ on $L^2(\mathbb{T})$ satisfying:*

$$S_i(e_n) = v_i b^n, \qquad 1 \leq i \leq N, \quad n \in \mathbb{Z}. \tag{3.3}$$

and

- *The endomorphism α_S determined by S as in (2.1) satisfies $\alpha_S \circ \pi = \pi \circ \beta$.*
- *Each S_i is reduced by $H^2(\mathbb{T})$, and*
- *if R_i is the restriction of S_i to $H^2(\mathbb{T})$, then $R = \{R_i\}_{i=1}^N$ is a Cuntz family on $H^2(\mathbb{T})$ with the property that $\alpha_R \circ \tau = \tau \circ \beta$.*

The proof of Lemma 3.2 showed that $\mathcal{D} = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$ is nonzero, so it has an orthonormal basis; its dimension N may be finite or infinite. It is well known that N is finite if and only if b is a finite Blaschke product. For if b is a finite Blaschke product, then we will construct an orthonormal basis for $\mathcal{D}$ in Remark 3.7, showing that $\mathcal{D}$ is finite dimensional. If b is not a finite Blascke product then using the structure of inner functions, one can find a sequence, $\{b_i\}_{i=1}^\infty$ of nonconstant inner functions such that b_i divides b_{i+1} and all divide b. It follows from [8, page 12] that

$$\pi(b_i)H^2(\mathbb{T}) \supsetneq \pi(b_{i+1})H^2(\mathbb{T}) \supsetneq \pi(b)H^2(\mathbb{T}) \cdots \qquad \forall i \in \mathbb{N}.$$

Therefore,

$$H^2(\mathbb{T}) \ominus \pi(b_i)H^2(\mathbb{T}) \subsetneq H^2(\mathbb{T}) \ominus \pi(b_{i+1})H^2(\mathbb{T}) \subsetneq \mathcal{D} \qquad \forall i \in \mathbb{N},$$

which shows that the dimension of $\mathcal{D}$ is infinite.

Proof of Theorem 3.4. Lemma 3.2 implies that if v is any unit vector in $\mathcal{D}$, the set $\{vb^n : n \in \mathbb{Z}\}$ is an orthonormal set of vectors in $L^2(\mathbb{T})$. It follows that for any $1 \leq i \leq N$, there is a unique isometry S_i on L^2 satisfying $S_i(e_n) = v_i b^n$ for all $n \in \mathbb{Z}$.

Lemma 3.2 also implies that if v and w are any orthogonal unit vectors in $\mathcal{D}$, the closed linear spans of $\{vb^n : n \in \mathbb{Z}\}$ and $\{wb^n : n \in \mathbb{Z}\}$ are orthogonal. It follows that the isometries in the tuple $S = \{S_i\}_{i=1}^N$ just defined have orthogonal ranges. Let K denote the closed linear span of the ranges of the operators $\{S_i\}_{i=1}^N$. By construction, for all $n \in \mathbb{Z}$ we have $v_i b^n \in K$ for all $1 \leq i \leq N$, and thus $K \supseteq \pi(b)^n \mathcal{D}$ for all $n \in \mathbb{Z}$. By Lemma 3.2 we conclude that $K = L^2(\mathbb{T})$ and S is a Cuntz family of isometries.

Viewing each e_n as an element of $L^\infty(\mathbb{T})$, it is evident that

$$S_i \pi(e_n) = \pi(b^n) S_i = \pi(\beta(e_n)) S_i, \qquad 1 \leq i \leq N, \quad n \in \mathbb{Z}. \tag{3.4}$$

Since this equation is linear in the e_n, we conclude that $S_i \pi(p) = \pi(\beta(p)) S_i$ for every i and every trigonometric polynomial p. Consequently,

$$\pi(\beta(p)) = \pi(\beta(p)) \sum_{i=1}^{N} S_i S_i^* = \sum_{i=1}^{N} S_i \pi(p) S_i^* = \alpha_S(\pi(p))$$

is satisfied for every trigonometric polynomial p. It follows from Corollary 3.3 that equation (2.4) is satisfied for all $\varphi \in L^\infty(\mathbb{T})$.

The fact that $H^2(\mathbb{T})$ is invariant under each S_i is immediate from the definition (3.3). As Lemma 3.2 implies that $\{v_i b^n : 1 \leq i \leq N, n < 0\}$ is an orthonormal basis of $H^2(\mathbb{T})^\perp$, it is also clear from (3.3) that $H^2(\mathbb{T})^\perp$ is invariant under each S_i, so each S_i is reduced by $H^2(\mathbb{T})$. The fact that R is a Cuntz family on $H^2(\mathbb{T})$ satisfying $\alpha_R \circ \tau = \tau \circ \beta$ is then immediate. $\square$

In summary, we have shown how, given an inner function b and the corresponding endomorphism β, to generate a Cuntz family $S = \{S_i\}_{i=1}^N$ satisfying

$$\alpha_S \circ \pi = \pi \circ \beta.$$

Thus, (π, α_S) is a covariant representation of $(L^\infty(\mathbb{T}), \beta)$ implemented by a Cuntz family, i.e. $(\pi,\, \alpha_S)$ is a **Cuntz Covariant Representation**.

3.2 Extending Endomorphisms to the Toeplitz Algebra

Recall that $\mathfrak{T}$ is the C^*-algebra generated by all the Toeplitz operators $\{\tau(\varphi) \mid \varphi \in L^\infty(\mathbb{T})\}$. We shall write $\mathfrak{T}(C(\mathbb{T}))$ for C^*-subalgebra generated by the Toeplitz operators with continuous symbols, i.e., $\mathfrak{T}(C(\mathbb{T}))$ is the C^*-subalgebra of $B(H^2(\mathbb{T}))$ generated by $\{\tau(\varphi) \mid \varphi \in C(\mathbb{T})\}$. As indicated, it is well known that $\mathfrak{T}(C(\mathbb{T})) = \{\tau(\varphi) + k \mid \varphi \in C(\mathbb{T}),\, k \in \mathfrak{K}\}$, where $\mathfrak{K}$ denotes the algebra of compact operators on $H^2(\mathbb{T})$. Details can be found in [5, 7.11 and 7.12].

Corollary 3.5. *If b is an inner function, then the map $\tau(\varphi) \to \tau(\varphi \circ b)$, $\varphi \in L^\infty(\mathbb{T})$, extends to a $*$-endomorphism of $\mathfrak{T}$ that we will continue to denote by β. Further, β leaves $\mathfrak{T}(C(\mathbb{T}))$ invariant if and only if b is a finite Blaschke product. Thus, if ι denotes the identity representation of $\mathfrak{T}$ on $H^2(\mathbb{T})$, then any solution α_+ of equation* (1.7) *(equivalently, any solution $R := \{R_i\}_{i=1}^N$ to equation* (2.5)*) yields a covariant representation (ι, α_+) of $(\mathfrak{T}, \beta)$ and (ι, α_+) preserves $(\mathfrak{T}(C(\mathbb{T})), \beta)$ if and only if b is a finite Blaschke product.*

As elementary as this result seems to be, we do not know how to prove it without recourse to Theorem 3.4.

Proof. The existence of a solution α_+ to equation (1.7) guarantees that the map $\tau(\varphi) \to \tau(\varphi \circ b)$, $\varphi \in L^\infty(\mathbb{T})$, extends to a $*$-endomorphism of $\mathfrak{T}$, because α_+ is a C^*-endomorphism of a larger C^*-algebra, namely $B(H^2(\mathbb{T}))$. Thus Theorem 3.4 shows that composition with b extends to $\mathfrak{T}$. If b is a finite Blaschke product then composition with b leaves $C(\mathbb{T})$ invariant, i.e., β leaves $C(\mathbb{T})$ invariant. Since the solution α_+ to equation (1.7) is of the form α_R where the Cuntz family R is *finite*,

α_+ leaves $\mathfrak{K}$ invariant and, therefore, it leaves $\mathfrak{T}(C(\mathbb{T}))$ invariant when b is a finite Blaschke product. Conversely, if β leaves $\mathfrak{T}(C(\mathbb{T}))$ invariant, then letting $\varphi(z) = z$, we see that $\tau(b) = \alpha_+(\tau(\varphi)) = \tau \circ \beta(\varphi)$ must be of the form $\tau(f) + k$, for some compact operator k and some continuous function f. But then $\tau(b - f) = k$, and so, by [5, 7.15], $b = f$ is continuous, and hence a finite Blaschke product. □

Rochberg's analysis and Laca's result [13, Proposition 2.2] together yield the following.

Corollary 3.6. *A Cuntz family $R = \{R_i\}_{i=1}^N$ in $B(H^2(\mathbb{T}))$ satisfies $\alpha_R \circ \tau = \tau \circ \beta$ if and only if there is an orthonormal basis $\{v_i\}_{i=1}^N$ for $\mathcal{D} = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$ so that the R_i may be expressed in terms of it as in Theorem 3.4.*

Proof. Theorem 3.4 asserts that if R is a Cuntz family in $B(H^2(\mathbb{T}))$ of the indicated form, then $\alpha_R \circ \tau = \tau \circ \beta$. For the converse, suppose $R := \{R_i\}_{i=1}^N$ is a Cuntz family in $B(H^2(\mathbb{T}))$ so that $\alpha_R \circ \tau = \tau \circ \beta$. Then, as we saw in Corollary 3.5, α_R leaves the Toeplitz algebra $\mathfrak{T}$ invariant. Choose any orthonormal basis $\{v_i\}_{i=1}^N$ for $\mathcal{D}$ and let $\widetilde{R} = \{\widetilde{R}_i\}_{i=1}^N$ be corresponding Cuntz family on $H^2(\mathbb{T})$ obtained from Theorem 3.4. Then the equation $\alpha_{\widetilde{R}} \circ \tau = \tau \circ \beta$ is also satisfied, by Theorem 3.4. It follows that α_R and $\alpha_{\widetilde{R}}$ agree on $\mathfrak{T}$. Since $\mathfrak{T}$ is ultraweakly dense in $B(H^2(\mathbb{T}))$ (because $\mathfrak{T}$ contains $\mathfrak{K}$) and since α_R and $\alpha_{\widetilde{R}}$ are ultraweakly continuous maps of $B(H^2(\mathbb{T}))$, $\alpha_R = \alpha_{\widetilde{R}}$ on all of $B(H^2(\mathbb{T}))$. Thus by [13, Proposition 2.2], there is a unitary $N \times N$ *scalar* matrix (u_{ij}) such that $R_i = \sum_j u_{ij} \widetilde{R}_j$. But $\{(\sum_j u_{ij} v_j)\}_{i=1}^N$ is also an orthonormal basis of $\mathcal{D}$, and so the R_i's have the desired form. □

3.3 Composition Operators

To identify all the solutions to equation (1.6) in Problem 1.7, we need to restrict attention to finite Blaschke products. For this reason and to get a clearer picture of the Cuntz isometries implementing α and α_+ we emphasize:

From now on, b will denote a **finite** *Blaschke product.*

Remark 3.7. *It was previously remarked that the nonconstant inner function b is a finite Blaschke product if and only if the space $\mathcal{D} = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$ has finite dimension. In this case the elements of $\mathcal{D}$ are rational functions with poles outside the closed unit disc. In fact, writing*

$$b(z) = \prod_{j=1}^{N} b_{\alpha_j}, \tag{3.5}$$

where the α_i are the not-necessarily-distinct zeros of b, one can check that the functions $\{w_i\}_{i=1}^N$ constructed from partial products of b by way of

$$w_j(z) = \frac{(1-|\alpha_j|^2)^{1/2}}{1-\overline{\alpha_j}z} \prod_{k=1}^{j-1} b_{\alpha_k}, \qquad 1 \le j \le N,$$

(the product $\prod_{k=1}^{j-1} b_{\alpha_k}$ is interpreted as 1 when $j=1$), form an orthonormal basis for $\mathcal{D}$ (see [20, p. 305]). We call this the canonical *orthonormal basis for $\mathcal{D}$. Note that the elements of the canonical basis are nonzero on $\mathbb{T}$ and hence invertible elements of $C(\mathbb{T})$.*

In [18, Theorem 1], Ryff shows that if φ is analytic on the disc $\mathbb{D}$ and maps $\mathbb{D}$ into $\mathbb{D}$, then composition with φ induces a bounded operator on all the H^p spaces. The the principal ingredient in his proof is Littlewood's subordination theorem. In [18, Theorem 3], Ryff shows further that composition with φ is an isometry on H^p if and only if φ is an inner function that vanishes at the origin. The following consequence of Theorem 3.4 is a variation on this theme with a very elementary proof.

Corollary 3.8. *Let b be a finite Blaschke product and define Γ_b on trigonometric polynomials p by $\Gamma_b(p) := p \circ b$. Then Γ_b extends in a unique way to a bounded operator on $L^2(\mathbb{T})$ that leaves $H^2(\mathbb{T})$ invariant.*

Moreover, letting Γ_b now denote the extension, the following are equivalent:

1. *Γ_b is an isometry.*

2. *$b(0) = 0$.*

3. *Γ_b is reduced by $H^2(\mathbb{T})$.*

Proof. Fix an element w of the canonical basis for $\mathcal{D} = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$. By Theorem 3.4 there is a unique isometry S on $L^2(\mathbb{T})$ satisfying

$$S(e_n) = wb^n, \qquad n \in \mathbb{Z}. \tag{3.6}$$

As observed in Remark 3.7, w is an invertible element of $C(\mathbb{T})$, so the operator $\pi(w)$ is invertible. The relation (3.6) then implies that for any trigonometric polynomial p we have

$$\pi(w^{-1})S(p) = \Gamma_b(p),$$

so the bounded operator $\pi(w^{-1})S$ is an extension of Γ_b to all of $L^2(\mathbb{T})$. Uniqueness of the extension follows from the density of the trigonometric polynomials in $L^2(\mathbb{T})$. The fact that $\Gamma_b(e_n) = b^n$ is in $H^2(\mathbb{T})$ for every $n \geq 0$ implies that this extension leaves $H^2(\mathbb{T})$ invariant.

If $b(0) = 0$, then $b(z) = zb_1(z)$, where b_1 is in $H^2(\mathbb{T})$. It follows that for any $n > m$ we have $\langle b^n, b^m \rangle = \langle z^{n-m} b_1^{n-m}, 1 \rangle = 0$, so that the family $\{b^n\}_{n \in \mathbb{Z}}$ is orthonormal. Since $\Gamma_b(e_n) = b^n$ for all $n \in \mathbb{Z}$, we conclude that Γ_b is an isometry, as it maps an orthonormal basis to an orthonormal family. Conversely, if Γ_b is an isometry,

$$b(0) = \langle b, e_0 \rangle = \langle \Gamma_b(e_1), \Gamma_b(e_0) \rangle = \langle e_1, e_0 \rangle = 0.$$

This establishes the equivalence of (1) and (2).

It will be useful later to deduce the equivalence of (2) and (3) from the assertion that if a vector $\xi \in L^2(\mathbb{T})$ has the property that the pointwise product ξb is in $H^2(\mathbb{T})$, then $\Gamma_b^* \xi$ is in $H^2(\mathbb{T})$ if and only if $(\xi b)(0) = 0$. To prove this assertion, note that $\Gamma_b^* \xi$ is in $H^2(\mathbb{T})$ if and only if $(\Gamma_b^* \xi, z^{-n}) = 0$ for all $n > 0$, and this is equivalent to

$$0 = \langle \xi, \Gamma_b(z^{-n}) \rangle = \langle \xi, b^{-n} \rangle = \langle \xi b^n, 1 \rangle = \langle (\xi b) b^{n-1}, 1 \rangle = (\xi b)(0) b^{n-1}(0), \qquad n > 0,$$

which is equivalent to $(\xi b)(0) = 0$. It follows from this assertion that $\Gamma_b^* \xi \in H^2(\mathbb{T})$ for all $\xi \in H^2(\mathbb{T})$ if and only if $b(0) = 0$. $\square$

All of our proofs to this point have used only elementary operator theory. To go further, we require more detailed information about finite Blaschke products.

CHAPTER 4
THE MASTER ISOMETRY

4.1 Defining C_b

We have seen that the isometries in Theorem 3.4 are closely related to the composition operator Γ_b. However, unless $b(0) = 0$, Γ_b is not an isometry. For this reason and other considerations we want to replace Γ_b by an isometry, denoted C_b, that has similar properties. This section is devoted to defining and developing the properties of C_b. We will see that C_b has the following properties:

- *every* Cuntz family $S = \{S_i\}_{i=1}^N$ satisfying equation (2.4) can be expressed in terms of C_b.
- C_b is *reduced* by $H^2(\mathbb{T})$.

We refer to C_b as the *master isometry determined by* b (or by the endomorphism β induced by b.) Much of the material below is contained in results already in the literature (see in particular [16] and [7]). But many calculations are done under the additional hypothesis that $b(0) = 0$, which we want specifically to avoid. In the interest of clarity, we present all of the details.

Lemma 4.1. *Define*

$$J_0(z) := \frac{b'(z)z}{Nb(z)}. \tag{4.1}$$

Then the restriction of J_0 to $\mathbb{T}$ is a positive continuous function and, in particular, is bounded away from zero.

Proof. Of course, J_0 is a rational function. What needs proof is that on $\mathbb{T}$, J_0 is positive, non-vanishing and has no poles. If $\alpha_j \neq 0$, then

$$\begin{aligned}
\frac{b_j'(z)}{b_j(z)} &= \frac{\frac{|\alpha_j|}{\alpha_j}\frac{|\alpha_j|^2-1}{(1-\overline{\alpha_j}z)^2}}{\frac{|\alpha_j|}{\alpha_j}\frac{\alpha_j-z}{1-\overline{\alpha_j}}} \\
&= \frac{(|\alpha_j|^2-1)}{(1-\overline{\alpha_j}z)(\alpha_j-z)} \\
&= \frac{(|\alpha_j|^2-1)}{z\overline{z}(1-\overline{\alpha_j}z)(\alpha_j-z)} \\
&= \frac{1}{z}\frac{1-|\alpha_j|^2}{(\overline{\alpha_j}-\overline{z})(\alpha_j-z)} \\
&= \frac{1}{z}\frac{1-|\alpha_j|^2}{\overline{(\alpha_j-z)}(\alpha_j-z)} \\
&= \frac{1}{z}\frac{1-|\alpha_j|^2}{|(\alpha_j-z)|^2}
\end{aligned}$$

That is, $\frac{b_j'(z)}{b_j(z)} = \frac{1}{z}\frac{1-|\alpha_j|^2}{|\alpha_j-z|^2}$, and $\frac{b_j'}{b_j}(z) = \frac{1}{z}$ if $\alpha_j = 0$, $\frac{b_j'}{b_j}(z) = \frac{1}{z}$. In either case, $\frac{zb_j'(z)}{b_j(z)}$ is strictly positive on $\mathbb{T}$. Basic calculus then implies that

$$\begin{aligned}
J_0(z) &:= \frac{b'(z)z}{Nb(z)} \\
&= \frac{\sum_{i=1}^N b_i'(z)\prod_{j\neq i} b_j(z)z}{N\prod_{j=1}^N b_j(z)} \\
&= \sum_{i=1}^N \frac{b_i'(z)\prod_{j\neq i} b_j(z)z}{N\prod_{j=1}^N b_j(z)} \\
&= \frac{1}{N}\sum_{i=1}^N \frac{zb_i'(z)}{b_i(z)}
\end{aligned}$$

and the result follows.

□

Using the argument principal, it is easy to show that b is a local homeomorphism of a neighborhood of $\mathbb{T}$ onto a neighborhood of $\mathbb{T}$. So, the restriction of b to $\mathbb{T}$ is a local homeomorphism of $\mathbb{T}$ to $\mathbb{T}$. We need a bit more information, which is provided by the next lemma. The proof follows [16, Lemma 1] closely.

Lemma 4.2. *There is a homeomorphism* $\theta : [0, 2\pi] \to [\theta(0), \theta(0) + N \cdot 2\pi]$*, which is increasing and*

1. $b(e^{it}) = e^{i\theta(t)}$.

2. *The derivative of* θ *on* $(0, 2\pi)$ *is* $\frac{b'(e^{it})}{b(e^{it})}e^{it} \gneq 0$.

3. *If* $(t_{j-1}, t_j) = \theta^{-1}(\theta(0) + (j-1) \cdot 2\pi, \theta(0) + j \cdot 2\pi)$, $j = 1, 2, \ldots, N$, *and if* $A_j := \{e^{it} \mid t_{j-1} < t < t_j\}$, *then* $\cup_{j=1}^{N} A_j = \mathbb{T}$, *except for a finite set of points, and* b *maps each* A_j *diffeomorphically onto* $\mathbb{T}\backslash\{b(1)\}$.

4. *If* $\sigma_j : \mathbb{T}\backslash\{b(1)\} \to A_j$ *denotes the inverse of the restriction of* b *to* A_j, *then as* s *ranges over* $(\theta(0) + 2\pi(j-1), \theta(0) + 2\pi j)$, e^{is} *ranges over* $\mathbb{T}\backslash\{b(1)\}$ *and*
$$\sigma_j(e^{is}) = e^{i\theta^{-1}(s)}.$$

Proof. Each b_j is analytic in a neighborhood of the closed unit disc and maps $\mathbb{T}$ homeomorphically onto $\mathbb{T}$ in an orientation preserving fashion. If the plane is slit along the ray through the origin and $b_j(1)$, then one can define an analytic branch of $\log z$ in the resulting region. On $\mathbb{T}\backslash\{b_j(1)\}$, $\log b_j(e^{it}) = i\theta_j(t)$ for a smooth function $\theta_j(t)$ defined initially on $(0, 2\pi)$, and mapping to $(\theta_j(0), \theta_j(0) + 2\pi)$. Further, if one differentiates the defining equation for θ_j, one finds that $i\theta_j'(t) = \frac{b_j'(e^{it})}{b_j(e^{it})}e^{it}i$, so θ_j' is strictly positive, as was shown in the preceding lemma. Hence θ_j is strictly increasing. Since $b_j(e^{i0}) = b_j(1) = b_j(e^{i2\pi})$, θ_j extends to a homeomorphism from $[0, 2\pi]$ *onto* $[\theta_j(0), \theta_j(0) + 2\pi]$. If θ is defined on $[0, 2\pi]$ by the formula $\theta(t) := \sum_{j=1}^{N} \theta_j(t)$, then θ is a strictly increasing homeomorphism from $[0, 2\pi]$ onto $[\theta(0), \theta(0) + N \cdot 2\pi]$ such that $b(e^{it}) = e^{i\theta(t)}$. The remaining assertions are now clear. $\square$

Definition 4.3. *The* (canonical) transfer operator *determined by the Blaschke product* b *is defined on measurable functions* ξ *by the formula*

$$\mathcal{L}(\xi)(z) := \frac{1}{N} \sum_{b(w)=z} \xi(w).$$

Of course, an alternate formula for $\mathcal{L}$ is $\mathcal{L}(\xi)(z) = \frac{1}{N} \sum_{j=1}^{N} \xi(\sigma_j(z))$, when $z \in \mathbb{T}\backslash\{b(1)\}$. It is clear that $\mathcal{L}$ carries measurable functions to measurable functions, preserves order, and is unital. Because b is a local homeomorphism, $\mathcal{L}$ carries $C(\mathbb{T})$ into itself. It is not difficult to see that $\mathcal{L}$ is a bounded linear operator on $L^2(\mathbb{T})$. However, we present a proof of this that connects $\mathcal{L}$ with the adjoint of Γ_b. For this purpose, note that by Lemma 4.1, $\pi(J_0)$ is a bounded, positive, invertible operator on $L^2(\mathbb{T})$ with inverse $\pi(J_0^{-1})$.

Theorem 4.4.

$$\mathcal{L}\pi(J_0)^{-1} = \Gamma_b^* \tag{4.2}$$

Proof. For ξ and η in $L^2(\mathbb{T})$,

$$(\Gamma_b^*\xi, \eta) = (\xi, \Gamma_b\eta) = \int_0^{2\pi} \xi(z)\overline{\eta(b(z))}\, dm(z) = \sum_{j=1}^N \int_{A_j} \xi(z)\overline{\eta(b(z))}\, dm(z).$$

From the first and third assertions of Lemma 4.2,

$$\int_{A_j} \xi(z)\overline{\eta(b(z))}\, dm(z) = \int_{t_{j-1}}^{t_j} \xi(e^{it})\overline{\eta(e^{i\theta(t)})}\, dt.$$

Changes the variable to $s = \theta(t)$, the third and fourth assertions of Lemma 4.2 imply

$$\int_{t_{j-1}}^{t_j} \xi(e^{it})\eta(e^{i\theta(t)})\, dt = \int_{\theta(t_{j-1})}^{\theta(t_j)} \xi(e^{i\theta^{-1}(s)})\eta(e^{is})(\theta^{-1})'(s)\, ds.$$

Calculating $(\theta^{-1})'(s) = (\theta'(t))^{-1} = (\theta'(\theta^{-1}(s)))^{-1}$ and using the second assertion of Lemma 4.2, we deduce

$$\int_{\theta(t_{j-1})}^{\theta(t_j)} \xi(e^{i\theta^{-1}(s)})\eta(e^{is})(\theta^{-1})'(s)\, ds = \int_{\theta(t_j)}^{\theta(t_j)} \xi(e^{i\theta^{-1}(s)})\eta(e^{is})\frac{b(e^{i\theta^{-1}(s)})}{b'(e^{i\theta^{-1}(s)})e^{i\theta^{-1}(s)}}\, ds.$$

But by the fourth statement of Lemma 4.2 $e^{i\theta^{-1}(s)} = \sigma_j(e^{is})$, when $s \in (\theta(0) + 2\pi(j-1), \theta(0) + 2\pi j)$. So the last integral is

$$\int_{\theta(t_{j-1})}^{\theta(t_j)} \xi(\sigma_j(e^{is}))\eta(e^{is})\frac{b(\sigma_j(e^{is}))}{b'(\sigma_j(e^{is}))\sigma_j(e^{is})}\, ds.$$

As e^{is} sweeps out $\mathbb{T}\backslash\{b(1)\}$ as s ranges over each interval $(\theta(t_{j-1}), \theta(t_j)) = (\theta(0) +$

$2\pi(j-1), \theta(0)+2\pi j)$, we conclude that

$$\begin{aligned}
(\Gamma_b^*\xi, \eta) &= \sum_{j=1}^{N} \int_{\theta(0)+2\pi(j-1)}^{\theta(0)+2\pi j} \xi(\sigma_j(e^{is}))\eta(e^{is}) \frac{b(\sigma_j(e^{is}))}{b'(\sigma_j(e^{is}))\sigma_j(e^{is})}\, ds \\
&= \frac{1}{N}\sum_{j=1}^{N} \int_{\mathbb{T}} \xi(\sigma_j(z))\overline{\eta(z)} \frac{Nb(\sigma_j(z))}{b'(\sigma_j(z))\sigma_j(z)}\, dm(z) \\
&= \frac{1}{N}\sum_{j=1}^{N} \int_{\mathbb{T}} \xi(\sigma_j(z))(J_0(\sigma_j(z)))^{-1}\overline{\eta(z)}\, dm(z) \\
&= (\mathcal{L}(\pi(J_0)^{-1}\xi), \eta),
\end{aligned}$$

showing that $\Gamma_b^* = \mathcal{L}\pi(J_0)^{-1}$. □

Notation 4.5.

$$J(z) := \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \ln(J_0(e^{it}))dt\right]$$

Of course J is the unique outer function [A.23] that is positive at 0 and satisfies the equation $|J(z)| = J_0(z)$ for all $z \in \mathbb{T}$. (See [8, Theorem 5] and the surrounding discussion.) Significantly, J does not vanish on $\mathbb{D}$ and J is in $H^\infty(\mathbb{T})$; note that J_0 is not even in $H^2(\mathbb{T})$ except in trivial cases. We will work primarily with $J^{\frac{1}{2}}$, which is $\exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \ln(J_0(e^{it}))dt\right]$. Note that $J^{1/2}$ and $J^{-1/2}$ are both in $H^\infty(\mathbb{T})$.

Lemma 4.6. *For all $\varphi \in L^\infty(\mathbb{T})$,*

$$\mathcal{L}\pi(\varphi)\Gamma_b = \pi(\mathcal{L}(\varphi)).$$

In particular, $\mathcal{L}$ is a left inverse for Γ_b.

Proof. Take $\xi \in L^2(\mathbb{T})$ and $\varphi \in L^\infty(\mathbb{T})$ and calculate:

$$\begin{aligned}
\mathcal{L}(\pi(\varphi)\Gamma_b(\xi))(z) &= \frac{1}{N}\sum_{b(w)=z} (\pi(\varphi)\Gamma_b(\xi))(w) \\
&= \frac{1}{N}\sum_{b(w)=z} \varphi(\omega)\xi(b(w)) \\
&= \frac{1}{N}\sum_{b(w)=z} \varphi(\omega)\xi(z) \\
&= \mathcal{L}(\varphi)(z)\xi(z) \\
&= (\pi(\mathcal{L}(\varphi))\xi)(z).
\end{aligned}$$

The remaining claim is the trivial case when $\varphi = 1$ is constant. □

Lemma 4.7. *Set*

$$C_b := \pi(J^{\frac{1}{2}})\Gamma_b.$$

Then C_b is an isometry on $L^2(\mathbb{T})$ and

$$C_b^* = \mathcal{L}\pi(J^{-\frac{1}{2}}).$$

Further, if $\{S_i\}_{i=1}^N$ is the Cuntz family constructed in Theorem 3.4 using an orthonormal basis $\{v_i\}_{i=1}^N$ for $H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$, then $S_i = \pi(v_i J^{-\frac{1}{2}})C_b$ for all $1 \le i \le N$.

Proof. The key is the relation $\mathcal{L}\pi(J_0^{-1}) = \Gamma_b^*$ from Theorem 4.4. We just compute:

$$C_b^* C_b = \Gamma_b^* \pi(\overline{J^{\frac{1}{2}}})\pi(J^{\frac{1}{2}})\Gamma_b = \Gamma_b^*\pi(|J|)\Gamma_b = \Gamma_b^*\pi(J_0)\Gamma_b = \mathcal{L}\Gamma_b = I.$$

and

$$C_b^* = \Gamma_b^*\pi(\overline{J^{\frac{1}{2}}}) = \mathcal{L}\pi(J_0)^{-1}\pi(\overline{J^{\frac{1}{2}}}) = \mathcal{L}\pi(J^{-\frac{1}{2}}).$$

For the final assertion, simply observe that the definition of S_i (using $\{v_i\}_{i=1}^N$) shows that $S_i = \pi(v_i)\Gamma_b$. As $C_b = \pi(J^{\frac{1}{2}})\Gamma_b$, we conclude

$$S_i \;=\; \pi(v_i)\pi(J^{-\frac{1}{2}})\pi(J^{\frac{1}{2}})\Gamma_b = \pi(v_i J^{-\frac{1}{2}})C_b.$$

□

Proposition 4.8. *$H^2(\mathbb{T})$ reduces C_b and C_b implements $\mathcal{L}$ in the sense that*

$$C_b^*\pi(\varphi)C_b = \pi(\mathcal{L}(\varphi)), \tag{4.3}$$

for all $\varphi \in L^\infty(\mathbb{T})$.

Proof. Since Γ_b and $\pi(J^{\frac{1}{2}})$ leave $H^2(\mathbb{T})$ invariant, so does $C_b = \pi(J^{\frac{1}{2}})\Gamma_b$. On the other hand, $C_b^* = \mathcal{L}\pi(J^{-\frac{1}{2}})$ by Lemma 4.7, so one way to show that $H^2(\mathbb{T})$ reduces C_b is to show that $\mathcal{L}$ leaves $H^2(\mathbb{T})$ invariant. McDonald did this in [16, Lemma 2].

We can also prove this directly: fixing $\eta \in H^2(\mathbb{T})$, we must show that $\mathcal{L}\eta \in H^2(\mathbb{T})$. By Theorem 4.4 we have that $\mathcal{L} = \Gamma_b^*\pi(J_0)$, so it suffices to show that the vector $\xi = \pi(J_0)\eta \in L^2(\mathbb{T})$ is mapped into $H^2(\mathbb{T})$ by Γ_b^*. By the definition (4.1) of

J_0 we have

$$b(z)\xi(z) = b(z)J_0(z)\eta(z) = zb'(z)\eta(z),$$

showing that $b\xi$ is in $H^2(\mathbb{T})$ and that $(b\xi)(0) = 0$. Thus $\Gamma_b^*\xi \in H^2(\mathbb{T})$ by the argument given in the proof of Corollary 3.8.

Equation (4.3) follows from Lemmas 4.7 and 4.6 because

$$C_b^*\pi(\varphi)C_b = \mathcal{L}\pi(J^{-\frac{1}{2}})\pi(\varphi)\pi(J^{\frac{1}{2}})\Gamma_b = \mathcal{L}\pi(\varphi)\Gamma_b = \pi(\mathcal{L}(\varphi)).$$

□

4.2 C_b and Intertwiners

We shall now prove some results about isometries which intertwine π and $\pi\circ\beta$ in a sense which we will make clear.

Theorem 4.9. *Denote the restriction of C_b to $H^2(\mathbb{T})$ by C_{b+}; the following hold.*

1. *If T is a bounded operator on $L^2(\mathbb{T})$, then T satisfies*

$$T\pi(\varphi) = \pi(\beta(\varphi))T, \qquad \varphi \in L^\infty(\mathbb{T}), \tag{4.4}$$

   if and only if $T = \pi(m)C_b$ for some function $m \in L^\infty(\mathbb{T})$.

2. *If T is a bounded operator on $H^2(\mathbb{T})$, then*

$$T\tau(\varphi) = \tau(\beta(\varphi))T, \qquad \varphi \in H^\infty(\mathbb{T}), \tag{4.5}$$

   if and only if $T = \tau(m)C_{b+}$ for some function $m \in H^\infty(\mathbb{T})$.

Further, if $T = \pi(m)C_b$, $m \in L^\infty(\mathbb{T})$, (resp. if $T = \tau(m)C_{b+}$, $m \in H^\infty(\mathbb{T})$) then $\|T\| = (\|\mathcal{L}(|m|^2)\|_\infty)^{\frac{1}{2}}$, and T is an isometry if and only if $\mathcal{L}(|m|^2) = 1$ a.e.

We say that an operator T is an *intertwiner* if it satisfies (1) or (2) because these equations say that T intertwines π and $\pi\circ\beta$ or τ and $\tau\circ\beta$ respectively.

Proof. We begin by proving assertion (1). If $T = \pi(m)C_b$ for some $m \in L^\infty(\mathbb{T})$, a short calculation shows that T satisfies (4.4). The formula (4.3) then implies

$$T^*T = C_b^*\pi(\overline{m})\pi(m)C_b = \pi(\mathcal{L}(|m|^2)),$$

and $\|T\| = (\|\mathcal{L}(|m|^2)\|_\infty)^{\frac{1}{2}}$ follows as π is faithful and the following computation.

$$\begin{aligned}
\|T\|^2 &= \|T^*T\| \\
&= \|\pi(\mathcal{L}(|m|^2)\| \\
&= \sup_{\xi\in H} \frac{\|\pi(\mathcal{L}(|m|^2)\xi\|}{\|\xi\|} \\
&= \sup_{\xi\in H} \frac{\|\mathcal{L}(|m|^2)\xi\|}{\|\xi\|} \\
&= \|\mathcal{L}(|m|^2)\|_\infty
\end{aligned}$$

The fact that T is isometric if and only if $\mathcal{L}(|m|^2) = 1$ a.e. is immediate.

Suppose conversely that T is an operator on $L^2(\mathbb{T})$ satisfying (4.4). Define $m := \pi(J^{-1/2})T(\mathbf{1})$, where $\mathbf{1}$ is the constant function that is identically equal to 1. Note that *a priori* we have that $m \in L^2(\mathbb{T})$, but not necessarily that $m \in L^\infty(\mathbb{T})$. We will address this shortly.

The hypothesis (4.4) and the definition of m imply that if $\varphi \in L^\infty(\mathbb{T})$ then

$$\begin{aligned}
T\varphi &= T\pi(\varphi)\mathbf{1} \\
&= \pi(\beta(\varphi))T(\mathbf{1}) \\
&= \pi(\beta(\varphi))\pi(J^{1/2})m \\
&= \pi(J^{1/2}\varphi \circ b)m \\
&= mC_b(\varphi)
\end{aligned}$$

That is,

$$T\varphi = mC_b(\varphi), \qquad \varphi \in L^\infty(\mathbb{T}) \tag{4.6}$$

More generally for $\varphi \in L^2(\mathbb{T})$, there is a sequence φ_n in $L^\infty(\mathbb{T})$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{T})$. The boundedness of C_b implies that the sequence of vectors $C_b\varphi_n$ is convergent in $L^2(\mathbb{T})$ with limit $C_b\varphi$. The boundedness of T together with (4.6) implies that the sequence $mC_b\varphi_n$ is convergent in $L^2(\mathbb{T})$ with limit $T\varphi$. By passing to a subsequence if necessary, we may assume that $C_b\varphi_n \to C_b\varphi$ pointwise a.e., and $mC_b\varphi_n \to T\varphi$ pointwise a.e, and deduce $T\varphi = mC_b\varphi$. We conclude that

$$T\varphi = mC_b(\varphi), \qquad \varphi \in L^2(\mathbb{T}). \tag{4.7}$$

Fix an orthonormal basis $\{v_i\}_{i=1}^N$ for $H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$ and set $S_i = \pi(v_i J^{-\frac{1}{2}})C_b$. By Lemma 4.7 and Theorem 3.4, $\{S_i\}_{i=1}^N$ is a Cuntz family, so for any $\xi \in L^2(\mathbb{T})$ we have

$$\begin{aligned} m\xi = m\sum_{j=1}^N S_j S_j^*\xi &= m\sum_{j=1}^N \pi(v_j J^{-1/2})C_b S_j^*\xi \\ &= \sum_{j=1}^N v_j J^{-1/2} m C_b S_j^* \xi \\ &= \sum_{j=1}^N v_j J^{-1/2} T S_j^* \xi \qquad \text{by (4.7).} \end{aligned}$$

Thus multiplication by m is the operator $\sum_{j=1}^N \pi(v_j J^{-1/2})TS_j^*$ on $L^2(\mathbb{T})$. As this operator is bounded we deduce that $m \in L^\infty(\mathbb{T})$ as desired.

The proof of assertion (2) is similar. We make the parallel argument while keeping track of the differences. If $T = \tau(m)C_{b+}$ for some $m \in H^\infty(\mathbb{T})$, it is easily seen that (4.5) is satisfied, since $\tau(m)$ and $\tau(\varphi)$ commute when m and φ are in $H^\infty(\mathbb{T})$, and since C_{b+} is the restriction of C_b to a reducing subspace. Furthermore,

$$T^*T = C_{b+}^*\tau(m)^*\tau(m)C_{b+} = PC_b^*P\pi(\overline{m})P\pi(m)PC_bP|_{H^2(\mathbb{T})}.$$

Since $H^2(\mathbb{T})$ is invariant under $\pi(m)$ and reduces C_b, we deduce

$$T^*T = PC_b^*\pi(\overline{m})\pi(m)C_bP|_{H^2(\mathbb{T})} = P\pi(\mathcal{L}(|m|^2))P|_{H^2(\mathbb{T})} = \tau(\mathcal{L}(|m|^2)).$$

Thus $\|T^*T\| = \|\tau(\mathcal{L}(|m|^2))\| = \|\mathcal{L}(|m|^2)\|_\infty$, which proves the formula for the norm of T. Also, it shows that T is an isometry if and only if $\mathcal{L}(|m|^2) = 1$ a.e.

Suppose conversely that T on $H^2(\mathbb{T})$ satisfies equation (4.5) and set $m := \tau(J^{-1/2})T(\mathbf{1})$; we know $m \in H^2(\mathbb{T})$ and wish to deduce that $m \in H^\infty(\mathbb{T})$. The fact that $J^{-\frac{1}{2}} \in H^\infty(\mathbb{T})$ and the properties of C_{b+} show that

$$T\varphi = T\tau(\varphi)\mathbf{1} = \tau(\beta(\varphi))\tau(J^{1/2})m = mC_{b+}(\varphi)$$

for all $\varphi \in H^\infty(\mathbb{T})$ and hence all $\varphi \in H^2(\mathbb{T})$. With $S_i = \pi(v_i J^{-\frac{1}{2}})C_b$ as before, we note that $H^2(\mathbb{T})$ reduces S_i, by Theorem 3.4, and we set $R_i := S_i|_{H^2(\mathbb{T})}$. Theorem 3.4 asserts that $\{R_i\}_{i=1}^N$ is a Cuntz family of isometries on $H^2(\mathbb{T})$. Since $H^2(\mathbb{T})$

reduces C_b, we have for any $\xi \in H^2(\mathbb{T})$

$$\begin{aligned} m\xi &= m\sum_{j=1}^{N} R_j R_j^* \xi \\ &= m\sum_{j=1}^{N} \pi(v_j J^{-1/2}) C_b R_j^* \xi \\ &= \sum_{j=1}^{N} v_j J^{-1/2} m C_{b+} R_j^* \xi \\ &= \sum_{j=1}^{N} \pi(v_j J^{-1/2}) T R_j^* \xi. \end{aligned}$$

As $v_j J^{-1/2} \in H^\infty$ for each j, the conclusion is that multiplication by m is the bounded operator $\sum_{j=1}^{n} \tau(v_j J^{-1/2}) T R_j^*$ on $H^2(\mathbb{T})$. Thus $m \in H^\infty(\mathbb{T})$. $\square$

We have called C_b *the* master isometry. One reason for the use of the definite article is that when one builds the Deaconu-Renault groupoid G determined by b, viewed as a local homeomorphism of $\mathbb{T}$, then C_b appears as the image of a special isometry S in the groupoid C^*-algebra $C^*(G)$ under a representation that gives rise to the Cuntz families we consider here. We have not seen any compelling reason to bring this technology into this note - nevertheless, $C^*(G)$ and S are lying in the background and may prove useful in the future. For further information about the use of groupoids and C^*-algebras generated by local homeomorphisms, see [11].

One should **not** infer from the use of the definite article that C_b is uniquely determined by the abstract properties that we have shown it has. If V is an isometry on $L^2(\mathbb{T})$ satisfying:

- that implements $\mathcal{L}$, that is $V^*\pi(\varphi)V = \pi(\mathcal{L}(\varphi))$,
- $V\pi(\varphi) = \pi(\beta(\varphi))V$ and
- V is reduced by $H^2(\mathbb{T})$.

Then it can be shown that

- by Theorem 4.9, V must be of the form $V = \pi(m)C_b$ for some $m \in L^\infty(\mathbb{T})$ satisfying $\mathcal{L}(|m|^2) = 1$,
- $|m| = 1$ a.e.
- m is an inner function with the property that $\mathcal{L}(\overline{m})$ is constant.

In general we have been unable to deduce more about m than this. However, we note that m need not be constant.

Example 4.10. *If $b(z) = z^2$ and $m(z) = z^k$ for any odd positive integer k , then $V = \pi(m)C_b$ will satisfy the above properties . However, m is not constant.*

Proof. Let $V = \pi(m)C_b$, with b and m as indicated. Then

$$V^*\pi(\varphi)V = C_b^*\pi(|z^k|^2\varphi)C_b = C_b^*\pi(\varphi)C_b = \pi(\mathcal{L}(\varphi)).$$

Also, $V\pi(\varphi) = \pi(m)C_b\pi(\varphi) = \pi(m)\pi(\beta(\varphi))C_b = \pi(\beta(\varphi))\pi(m)C_b = \pi(\beta(\varphi))V$ and it is obvious that V is reduced by $H^2(\mathbb{T})$. If $b(z) = z^2$ then for any $z = e^{it} \in \mathbb{T}$ the set $\{w : b(w) = z\}$ is equal to $\{e^{it/2}, e^{i(t/2+\pi)}\}$. Further,

$$(\mathcal{L}m)(e^{it}) = \frac{1}{2}(m(e^{it/2}) + m(e^{i(t/2+\pi)}) = \frac{1}{2}(e^{ikt/2} + e^{ikt/2+k\pi}) = 0$$

The other conclusions can be easily verified as well. Yet, z^k is hardly constant. $\square$

It is a matter of particular interest to us to know exactly what the situation is when b is a more general Blaschke product. It remains to be investigated.

CHAPTER 5
HILBERT MODULES AND ORTHONORMAL BASIS

The endomorphism β of $L^\infty(\mathbb{T})$ and the transfer operator $\mathcal{L}$ may be used to endow $L^\infty(\mathbb{T})$ with the structure of a Hilbert C^*-module over $L^\infty(\mathbb{T})$. We will exploit this structure in order to solve Problem 1.7. We do not need much of the general theory about these modules. Rather, we provide enough background so that the formulas we use make sense. Excellent references for the basics of the theory are [14, 15].

Suppose A is a C^*-algebra and that E is a right A-module. Then E is called a *Hilbert C^*-module* over A in case E is endowed with an A-valued sesquilinear form $\langle \cdot, \cdot \rangle : E \times E \to A$ that is subject to the following conditions.

1. $\langle \cdot, \cdot \rangle$ is conjugate linear in the first variable, so $\langle \xi \cdot a, \eta \cdot b \rangle = a^* \langle \xi, \eta \rangle b$.

2. For all $\xi \in E$, $\langle \xi, \xi \rangle$ is a positive element in A that is 0 if and only if $\xi = 0$.

3. E is complete in the norm defined by the formula $\|\xi\|_E := \|\langle \xi, \xi \rangle\|_A^{\frac{1}{2}}$.

Of course, it takes a little argument to prove that $\| \cdot \|_E$ is a norm on E.

Remark 5.1. *Let us note that the notation $\langle \cdot, \cdot \rangle$ has been the inner product in $L^2(\mathbb{T})$, however, we now define it to be the inner product in a correspondence. We change notation for the remainder of this chapter and use the notation $(\cdot, \cdot)$ for the inner product on $L^2(\mathbb{T})$.*

In the application of Hilbert modules that we have in mind, our C^*-algebra A will be unital, and we will denote the unit by $\mathbf{1}$. A vector $v \in E$ is called a *unit vector* if $\langle v, v \rangle = \mathbf{1}$. Note that this says more than simply $\|v\| = 1$. A family $\{v_i\}_{i \in I}$ of vectors in E is called an *orthonormal set* if $\langle v_i, v_j \rangle = \delta_{ij}\mathbf{1}$. Further, if linear combinations of vectors from $\{v_i\}_{i \in I}$ (where the coefficients are from A) are dense in E then we say that $\{v_i\}_{i \in I}$ is an *orthonormal basis* for E. In this event, every vector $\xi \in E$ has the representation

$$\xi = \sum_{i \in I} v_i \cdot \langle v_i, \xi \rangle, \tag{5.1}$$

where the sum converges in the norm of E. In general, a Hilbert C^*-module need not have an orthonormal basis. Also, in general, two orthonormal bases need not have the same cardinal number. Nevertheless, two orthonormal bases $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ are linked by a unitary matrix over A in the usual way:

$$w_j = \sum_{i \in I} v_i \cdot \langle v_i, w_j \rangle = \sum_{i \in I} v_i \cdot u_{ij}.$$

So if the cardinality of I is n and the cardinality of J is m, then $U = (u_{ij})$ is a unitary matrix in $M_{nm}(A)$, i.e., $UU^* = \mathbf{1}_n$ in $M_n(A)$, while $U^*U = \mathbf{1}_m$ in $M_m(A)$. And conversely, any such matrix transforms the orthonormal basis $\{v_i\}_{i \in I}$ for E into an orthonormal basis $\{w_j\}_{j \in J}$ for E via this formula. In our application of these notions, the coefficient algebra A will be commutative so, as is well known, all unitary matrices are square and, therefore, any two orthonormal bases have the same number of elements.

We shall view $L^\infty(\mathbb{T})$ as right module over $L^\infty(\mathbb{T})$ via the formula

$$\xi \cdot a := \xi \beta(a), \qquad a, \xi \in L^\infty(\mathbb{T}), \tag{5.2}$$

where the product on the right hand side is the usual pointwise product in $L^\infty(\mathbb{T})$. Also, we shall use $\mathcal{L}$ to endow $L^\infty(\mathbb{T})$ with the $L^\infty(\mathbb{T})$-valued inner product defined by the formula

$$\langle \xi, \eta \rangle := \mathcal{L}(\bar{\xi}\eta), \qquad \xi, \eta \in L^\infty(\mathbb{T}). \tag{5.3}$$

Using the fact that $\mathcal{L} \circ \beta = \mathrm{id}_{L^\infty(\mathbb{T})}$ (Lemma 4.6), it is straightforward to see that $L^\infty(\mathbb{T})$ is a Hilbert C^*-module over $L^\infty(\mathbb{T})$, which we shall denote by $L^\infty(\mathbb{T})_{\mathcal{L}}$. The only thing that may seem problematic is the fact that $L^\infty(\mathbb{T})_{\mathcal{L}}$ is complete in the norm defined by the inner product. However, a moment's reflection reveals that the norm is equivalent to the $L^\infty(\mathbb{T})$-norm, which is complete. We remark that (5.2) and (5.3) make sense when the functions in $L^\infty(\mathbb{T})$ are restricted to lie in $C(\mathbb{T})$, and $C(\mathbb{T})$ also is a Hilbert module over $C(\mathbb{T})$ in this structure, but we will focus on the

$L^\infty(\mathbb{T})$ case in what follows. ‘

Vectors $\{m_i\}_{i=1}^N$ in $L^\infty(\mathbb{T})_{\mathcal{L}}$ form an orthonormal basis for $L^\infty(\mathbb{T})_{\mathcal{L}}$ if and only if

$$\mathcal{L}(\overline{m_i}m_j) = \langle m_i, m_j\rangle = \delta_{ij}\mathbf{1},$$

where $\mathbf{1}$ is the constant function 1, and

$$f = \sum_{i=1}^N m_i \cdot \langle m_i, f\rangle = \sum_{i=1}^N m_i\beta(\mathcal{L}(\overline{m_i}f)), \qquad f \in L^\infty(\mathbb{T})_{\mathcal{L}}.$$

We have intentionally used N, the order of the Blaschke product b, as the upper limit in these sums because $L^\infty(\mathbb{T})_{\mathcal{L}}$ has an orthonormal basis with N elements, viz. $\{\sqrt{N}1_{A_i}\}_{i=1}^N$, where the A_i's are the arcs in Lemma 4.2, and because any two orthonormal bases for $L^\infty(\mathbb{T})_{\mathcal{L}}$ have the same number of elements, as we noted above.

Remark 5.2. *As a map on $L^\infty(\mathbb{T})$, $\mathbb{E} := \beta\circ\mathcal{L}$ is the conditional expectation onto the range of β. Indeed, $\mathbb{E}$ is a weak-$*$ continuous, positivity preserving, idempotent unital linear map on $L^\infty(\mathbb{T})$. So it is the restriction to $L^\infty(\mathbb{T})$ of an idempotent and contractive linear map on $L^1(\mathbb{T})$ that preserves the constant functions. Hence $\mathbb{E}$ is a conditional expectation by the corollary to [4, Theorem 1]. Of course, the range of $\mathbb{E}$ consists of functions in the range of β by definition. On the other hand, if $f = \beta(g)$ for some function $g \in L^\infty(\mathbb{T})$, then $\mathbb{E}(f) = \beta\circ\mathcal{L}\circ\beta(g) = \beta(g) = f$, since $\mathcal{L}$ is a left inverse for β. Thus, to say that $\{m_i\}_{i=1}^N$ is an orthonormal basis for $L^\infty(\mathbb{T})_{\mathcal{L}}$ is to say that*

$$f = \sum_{i=1}^N m_i\mathbb{E}(\overline{m_i}f), \qquad f \in L^\infty(\mathbb{T}).$$

In light of the discussion in Section 2, the following describes all solutions to (1.6).

Theorem 5.3. *If a Cuntz family $S = \{S_i\}_{i=1}^N$ on $B(L^2(\mathbb{T}))$ gives rise to a covariant representation (π, α_S) of $(L^\infty(\mathbb{T}), \beta)$, then there is an orthonormal basis $\{m_i\}_{i=1}^N$ for $L^\infty(\mathbb{T})_{\mathcal{L}}$ such that*

$$S_i = \pi(m_i)C_b, \qquad 1 \le i \le N. \tag{5.4}$$

Further, if $\{m_i\}_{i=1}^N$ *is any family of functions in* $L^\infty(\mathbb{T})$ *such that the operators* S_i *defined by* (5.4) *form a Cuntz family* S *such that* (π, α_S) *is a covariant representation of* $(L^\infty(\mathbb{T}), \beta)$*, then* $\{m_i\}_{i=1}^N$ *is an orthonormal basis for* $L^\infty(\mathbb{T})_{\mathcal{L}}$*. Conversely, if* $\{m_i\}_{i=1}^N$ *is any orthonormal basis for* $L^\infty(\mathbb{T})_{\mathcal{L}}$ *and* S_i *is defined by* (5.4) *for* $1 \leq i \leq N$*, then* $S = \{S_i\}_{i=1}^N$ *is a Cuntz family and* (π, α_S) *is a covariant representation of* $(L^\infty(\mathbb{T}), \beta)$*.*

Proof. Suppose $\{S_i\}_{i=1}^N$ is a Cuntz family on $L^2(\mathbb{T})$ that satisfies equation (2.4). If both sides of this equation are multiplied on the right by S_j, then one finds from equation (2.2) that $S_j\pi(\cdot) = \pi \circ \beta(\cdot)S_j$ for each j. By Theorem 4.9, for each j there is $m_j \in L^\infty(\mathbb{T})$ satisfying $S_j = \pi(m_j)C_b$ and $\mathbf{1} = \mathcal{L}(|m_j|^2) = \langle m_j, m_j\rangle$. The fact that S satisfies equation (2.2) then yields

$$\delta_{i,j}I_{L^2(\mathbb{T})} = S_i^*S_j = C_b^*\pi(\overline{m_i}m_j)C_b = \pi(\mathcal{L}(\overline{m_i}m_j)) = \pi(\langle m_i, m_j\rangle).$$

Since π is faithful, $\langle m_i, m_j\rangle = \delta_{i,j}\mathbf{1}$, where $\mathbf{1}$ is the constant function 1. Thus, $\{m_i\}_{i=1}^N$ is an orthonormal set in $L^\infty(\mathbb{T})_{\mathcal{L}}$. We now show that the $\{m_i\}_{i=1}^N$ span $L^\infty(\mathbb{T})_{\mathcal{L}}$. If $f \in L^\infty(\mathbb{T})_{\mathcal{L}}$ satisfies $\langle f, m_i\rangle = 0$ for all i, then we have

$$\begin{aligned}
(\pi(f)C_b)^* = C_b^*\pi(\overline{f})\left(\sum_{i=1}^N S_iS_i^*\right) &= C_b^*\pi(\overline{f})\left(\sum_{i=1}^N \pi(m_i)C_bS_i^*\right) \\
&= \sum_{i=1}^N C_b^*\pi(\overline{f})\pi(m_i)C_bS_i^* \\
&= \sum_{i=1}^N \pi(\langle f, m_i\rangle)S_i^* \qquad \text{by (4.3)} \\
&= 0,
\end{aligned}$$

and thus $\pi(f)C_b = 0$, which in turn implies $fJ^{\frac{1}{2}} = \pi(f)J^{\frac{1}{2}} = \pi(f)C_b\mathbf{1} = 0$, and thus $f = 0$. This shows that $\{m_i\}_{i=1}^N$ is an orthonormal basis for $L^\infty(\mathbb{T})_{\mathcal{L}}$.

For the converse assertion, suppose $\{m_i\}_{i=1}^N$ is any orthonormal basis for

$L^\infty(\mathbb{T})_\mathcal{L}$, and set $S_i := \pi(m_i)C_b$. Then from (4.3) we deduce

$$\begin{aligned} S_i^* S_j &= C_b^* \pi(\overline{m_i} m_j) C_b \\ &= \pi(\langle m_i, m_j \rangle) \\ &= \delta_{i,j}\pi(\mathbf{1}) = \delta_{i,j} I_{L^2(\mathbb{T})}. \end{aligned}$$

So the relations (2.2) are satisfied. To verify the Cuntz identity (2.3), note first that equation (2.2) shows that the sum $\sum_{i=1}^N S_i S_i^*$ is a projection. To show that $\sum_{i=1}^N S_i S_i^* = I$, it suffices to show that $\sum_{i=1}^N S_i S_i^*$ acts as the identity operator on a dense subset of $L^2(\mathbb{T})$. So fix $f \in L^\infty(\mathbb{T})$ and observe that we may write

$$\sum_{i=1}^N S_i S_i^* f = \sum_{i=1}^N S_i S_i^* \pi(f) 1 = \sum_{i=1}^N S_i S_i^* \pi(f) \Gamma_b 1 = \sum_{i=1}^N S_i S_i^* \pi(f J^{-\frac{1}{2}}) C_b 1. \tag{5.5}$$

Since $S_i = \pi(m_i)C_b$, the last sum in (5.5) is

$$\sum_{i=1}^N \pi(m_i) C_b C_b^* \pi(\overline{m_i}) \pi(f J^{-\frac{1}{2}}) C_b 1 = \sum_{i=1}^N \pi(m_i) C_b \pi(\mathcal{L}(\overline{m_i} f J^{-\frac{1}{2}}))1,$$

where we have used (4.3). But by Theorem 4.9 the right hand side of this equation is

$$\sum_{i=1}^N \pi(m_i)\pi(\beta(\mathcal{L}(\overline{m_i} f J^{-\frac{1}{2}})))C_b 1 = \pi\left(\sum_{i=1}^N m_i \langle m_i, f J^{-\frac{1}{2}}\rangle\right) C_b 1 = \pi(f J^{-\frac{1}{2}}) C_b 1,$$

because $\{m_i\}_{i=1}^N$ is an orthonormal basis for $L^\infty(\mathbb{T})_\mathcal{L}$, by hypothesis. As $C_b 1 = \pi(J^{\frac{1}{2}})\Gamma_b 1 = \pi(J^{\frac{1}{2}})1$ it follows that $\pi(f J^{-\frac{1}{2}})C_b 1 = f$, and thus $\sum_{i=1}^N S_i S_i^* f = f$. We conclude that $S = \{S_i\}_{i=1}^N$ is a Cuntz family.

To see that this family implements β, simply note that

$$\pi(\beta(\varphi)) = \pi(\beta(\varphi)) \sum_{i=1}^N S_i S_i^* = \sum_{i=1}^N S_i \pi(\varphi) S_i^*$$

since the S_i satisfy equation (4.4). □

Corollary 5.4. *If $\{v_i\}_{i=1}^N$ is an orthonormal basis for the Hilbert* space $H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$, *then the functions $\{v_i J^{-\frac{1}{2}}\}_{i=1}^N$ form an orthonormal basis for the Hilbert* module $L^\infty(\mathbb{T})_\mathcal{L}$.

Proof. By Lemma 4.7, the Cuntz isometries coming from $\{v_i\}_{i=1}^N$ via Theorem 3.4 have the form $\pi(v_i J^{-\frac{1}{2}})C_b$. Therefore by Theorem 5.3 the functions $\{v_i J^{-\frac{1}{2}}\}_{i=1}^N$ form an orthonormal basis for $L^\infty(\mathbb{T})_\mathcal{L}$. □

Corollary 5.5. *If $S^{(1)}$ and $S^{(2)}$ are two Cuntz families in $B(L^2(\mathbb{T}))$ satisfying*

$$\alpha_{S^{(i)}} \circ \pi = \pi \circ \beta, \qquad i = 1, 2,$$

then there is a unitary matrix (u_{ij}) in $M_N(L^\infty(\mathbb{T}))$ such that

$$S_j^{(2)} = \sum_{i=1}^{N} S_i^{(1)} \pi(u_{ij}), \tag{5.6}$$

$j = 1, 2, \cdots, N$. Conversely, if $S^{(1)}$ and $S^{(2)}$ are Cuntz families on $L^2(\mathbb{T})$ that are linked by equation (5.6), then $\alpha_{S^{(1)}}$ implements β if and only if $\alpha_{S^{(2)}}$ implements β. Further, $\alpha_{S^{(1)}} = \alpha_{S^{(2)}}$ on $B(L^2(\mathbb{T}))$ if and only if (u_{ij}) is a unitary matrix of constant functions.

Proof. By Theorem 5.3, we may suppose there are orthonormal bases $\{m_i^{(1)}\}_{i=1}^N$ and $\{m_i^{(2)}\}_{i=1}^N$ for $L^\infty(\mathbb{T})_{\mathcal{L}}$ that define $S^{(1)}$ and $S^{(2)}$. In this event, there is a unitary matrix (u_{ij}) in $M_N(L^\infty(\mathbb{T}))$ so that

$$m_j^{(2)} = \sum_{i=1}^{N} m_i^{(1)} \cdot u_{ij}.$$

But then we may use (5.4) to derive (5.6) as follows:

$$\begin{aligned} S_j^{(2)} &= \pi(m_j^{(2)})C_b = \sum_{i=1}^{N} \pi(m_i^{(1)})\pi(\beta(u_{ij}))C_b = \sum_{i=1}^{N} \pi(m_i^{(1)})C_b\pi(u_{ij}) \\ &= \sum_{i=1}^{N} S_j^{(1)}\pi(u_{ij}). \end{aligned}$$

The same equation proves the converse assertion and the last assertion follows from Laca's Proposition 2.2 in [13]. $\square$

We conclude with a new look at Rochberg's [17, Theorem 1] and related work of McDonald [16]. Because of the complex conjugates that appear in the formula for the inner product on $L^\infty(\mathbb{T})_{\mathcal{L}}$, it is somewhat surprising that $\langle m_i, f\rangle \in H^\infty(\mathbb{T})$ whenever $f \in H^\infty(\mathbb{T})$ and m_i comes from an orthonormal basis for $H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$.

Theorem 5.6. *Let $\{v_i\}_{i=1}^N$ be an orthonormal basis for $\mathcal{D} = H^2(\mathbb{T}) \ominus \pi(b)H^2(\mathbb{T})$ and let $m_i = v_i J^{-\frac{1}{2}}$, so that $\{m_i\}_{i=1}^N$ is an orthonormal basis for $L^\infty(\mathbb{T})_{\mathcal{L}}$ by Corollary 5.4. Then a function $f \in L^\infty(\mathbb{T})$ lies in $H^\infty(\mathbb{T})$ if and only if $\langle m_i, f\rangle$ lies in $H^\infty(\mathbb{T})$ for all i. Further, f lies in the disc algebra $A(\mathbb{D})$ if and only if $\langle m_i, f\rangle$ lies in $A(\mathbb{D})$*

for all i.

Proof. By Remark 3.7 we know the functions m_i are in the disc algebra. It is thus immediate from

$$f = \sum_{i=1}^{n} m_i \cdot \langle m_i, f \rangle, \qquad f \in L^\infty(\mathbb{T}), \tag{5.7}$$

and the fact that β preserves both $H^\infty(\mathbb{T})$ and $A(\mathbb{D})$ that if the coefficients $\langle m_i, f\rangle$ all lie in $H^\infty(\mathbb{T})$ or $A(\mathbb{D})$ then f will also.

Conversely, fix $f \in H^\infty(\mathbb{T})$ and any $v \in \mathcal{D}$. We must show that $\langle vJ^{-1/2}, f\rangle$ is in $H^\infty(\mathbb{T})$. Note that $\langle vJ^{-1/2}, f\rangle$ is in L^∞ so it suffices to show that this function is in $H^2(\mathbb{T})$. To this end, fix a positive integer k, and compute

$$\begin{aligned}
(\langle vJ^{-1/2}, f\rangle, e_{-k}) &= (\mathcal{L}(\overline{vJ^{-1/2}}f), e_{-k}) \\
&= (\Gamma_b^*(J_0\overline{vJ^{-1/2}}f), e_{-k}) && \text{by Theorem 4.4} \\
&= (J_0\overline{vJ^{-1/2}}f, b^{-k}) \\
&= (J^{1/2}\overline{v}f, b^{-k}) && \text{as } J_0 = |J| = J^{1/2}\overline{J^{1/2}} \\
&= (J^{1/2}fb^k, v).
\end{aligned}$$

Since $J^{1/2}f \in H^\infty$ and $k > 0$ the function $J^{1/2}fb^k$ is in $\pi(b)H^2(\mathbb{T})$, so as $v \in \mathcal{D}$ we conclude $(J^{1/2}fb^k, v) = 0$. As $k > 0$ was arbitrary, $\langle vJ^{-1/2}, f\rangle$ is in $H^2(\mathbb{T})$, as desired. If f is further assumed to be in $A(\mathbb{D})$, as $\mathcal{L}$ maps $C(\mathbb{T})$ into itself we conclude $\langle vJ^{-1/2}, f\rangle \in C(\mathbb{T}) \cap H^2(\mathbb{T}) = A(\mathbb{D})$. $\square$

In our notation, Rochberg's Theorem 1 in [17] asserts that if $\{v_i\}_{i=1}^N$ is the *canonical* orthonormal basis (in the sense following Remark 3.7) for $\mathcal{D}$, then for any $f \in A(\mathbb{D})$, there are uniquely determined $f_1, f_2, \cdots, f_N \in A(\mathbb{D})$ satisfying

$$f(z) = \sum_{i=1}^{N} v_i(z)\beta(f_i)(z), \qquad z \in \overline{\mathbb{D}}, \tag{5.8}$$

and that moreover for each $1 \leq i \leq N$ the linear map $f \to f_i$ thus determined on $A(\mathbb{D})$ is continuous in the norm of $A(\mathbb{D})$.

We recover this theorem by applying Theorem 5.6 to the canonical basis

$\{v_i\}_{i=1}^N$ and the function $J^{-1/2}f \in A(\mathbb{D})$: it asserts that (5.8) holds with the functions $f_i = \langle m_i, J^{-1/2}f\rangle \in A(\mathbb{D})$. The norm continuity of the f_i in f is immediate from this formula. Our Theorem 5.6 provides a slightly stronger uniqueness statement: if $f \in A(\mathbb{D})$, assuming only that the f_i are in $L^\infty(\mathbb{T})$, multiplying both sides of (5.8) by $J^{-1/2}$, applying $\langle m_j, -\rangle$, and using the fact that $\{m_i\}_{i=1}^N$ is an orthonormal basis for $L^\infty(\mathbb{T})_{\mathcal{L}}$, one finds that f_j must be given by the formula above.

Rochberg [17] and McDonald [16] establish more information about the f_i using the special structure of the canonical orthonormal basis of $\mathbb{D}$. Our analysis does not seem to contribute anything new to their refinements. On the other hand, our results are explicitly independent of the choice of basis and connect to the structure of the Hilbert module $L^\infty(\mathbb{T})_{\mathcal{L}}$.

Remark 5.7. *The reader may have noticed that if $m \in L^\infty(\mathbb{T})$ and if $T = \pi(m)C_b$, then from the calculations in Theorem 4.9, the norm of T is the norm of m calculated in $L^\infty(\mathbb{T})_{\mathcal{L}}$. This is not an accident. The Hilbert module $L^\infty(\mathbb{T})_{\mathcal{L}}$ becomes a* left *module over $L^\infty(\mathbb{T})$ via the formula $a \cdot \xi := a\xi$, $a \in L^\infty(\mathbb{T})$, $\xi \in L^\infty(\mathbb{T})_{\mathcal{L}}$. This makes $L^\infty(\mathbb{T})_{\mathcal{L}}$ what is known as a C^**-correspondence *or* Hilbert bimodule *over $L^\infty(\mathbb{T})$. Further, if $\psi : L^\infty(\mathbb{T})_{\mathcal{L}} \to B(L^2(\mathbb{T})$ is defined by the formula*

$$\psi(m) = \pi(m)C_b, \qquad m \in L^\infty(\mathbb{T})_{\mathcal{L}},$$

then the pair (π, ψ) turns out to be what is known as a Cuntz-Pimsner covariant representation *of the pair $(L^\infty, L^\infty(\mathbb{T})_{\mathcal{L}})$. This means, in particular, that $\psi(m)^*\psi(m) = \pi(\langle m, m\rangle)$, as we noted in Theorem 4.9. Further, the pair (π, ψ) extends to a C^*-representation of the so-called* Cuntz-Pimsner algebra *of $L^\infty(\mathbb{T})_{\mathcal{L}}$, $\mathcal{O}(L^\infty(\mathbb{T})_{\mathcal{L}})$. We have not made use of this here, however it seems worthy of further investigation. See [6] for further information about Cuntz-Pimsner algebras and their representations.*

APPENDIX
BASICS

In this appendix we offer relevant definitions and review some classical function theoretic results following [10] as our primary resource. We do this so that the casual reader may reference any terminology with which he or she may be unfamiliar. It also serves the purpose of making this document more self-contained. Some of the following appears in the main body of this thesis; we will not worry about repeating anything in favor of continuity for the reader.

A.1 Hilbert Space

We begin with some very basic definitions working toward the definition of Hilbert space.

Definition A.1 (complete normed linear space)**.** *Let $\mathbb{X}$ be a complex vector space. A* ***norm*** *on $\mathbb{X}$ is a non-negative real valued function $\|\cdot\| : \mathbb{X} \to \mathbb{R}_+$ satisfying:*

- $\|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = 0$*;*
- $\|x + y\| \leq \|x\| + \|y\|$*; [Triangle Inequality]*
- $\|\lambda x\| = \lambda \|x\|$.

for $x, y \in \mathbb{X}$ and $\lambda \in \mathbb{C}$. We say that $\mathbb{X}$ is a ***complex normed linear space*** *if it is a complex vector space equipped with a norm.*

One immediately obtains a metric on $\mathbb{X}$ by $\rho(x, y) := \|x - y\|$.

Definition A.2 (Banach space)**.** *If $\mathbb{X}$ is a complex normed linear space that is complete in ρ, then $\mathbb{X}$ is called a* ***Banach space****.*

We are working toward the definition of **Hilbert space**. We do so since we intend to investigate certain algebras by representing them on Hilbert space and studying associated operators.

Definition A.3 (inner product space)**.** *Let H be a complex vector space. An **inner product** on H is a function, $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}$ satisfying:*

- $\langle \xi_1 + \xi_2, \eta\rangle = \langle \xi_1, \eta\rangle + \langle \xi_2, \eta\rangle$*;*
- $\langle \lambda\xi, \eta\rangle = \lambda\langle \xi, \eta\rangle$*;*
- $\langle \eta, \xi\rangle = \overline{\langle \xi, \eta\rangle}$*;*
- $\langle \xi, \xi\rangle \geq 0$ *and* $\langle \xi, \xi\rangle = 0$ *if and only if* $\xi = 0$.

*Such a space H, together with an inner product on H, is called an **inner product space**.*

Given an inner product space, one has the **Cauchy-Schwartz inequality**: $|\langle \xi, \eta\rangle|^2 \leq \langle \xi, \xi\rangle\langle \eta, \eta\rangle$. Also, it is immediate that a every inner product induces a norm on H by $\|\xi\| := \langle \xi, \xi\rangle^{1/2}$ and so, ever inner product space is a normed space.

Definition A.4 (Hilbert space)**.** *A **Hilbert space** is an inner product space H, that is complete with respect to the induced norm.*

Definition A.5. *An **orthonormal basis** for H is a family $\{e_i\}_{i\in I}$ of vectors such that*

- $\langle e_i, e_j\rangle = \delta_{i,j}$*, and*
- *the span of $\{e_i\}_{i\in I}$ is dense in H.*

For our purposes we will usually assume that H is separable, meaning it has a countable orthonormal basis. Now that we have established the definition of Hilbert space let us introduce some examples.

Definition A.6 (Lebesgue spaces)**.** *For $0 < p < \infty$ the L^p **space** on the circle, $\mathbb{T}$, denoted $L^p(\mathbb{T})$ is defined to be all $\xi : \mathbb{T} \to \mathbb{C}$ such that $\int_{\mathbb{T}} |\xi|^p dm < \infty$ where m is Lebesgue measure.*

When $p = 2$, the space $L^2(\mathbb{T})$ is a Hilbert Space with inner product given by $\langle \xi, \eta \rangle := \frac{1}{2\pi} \int_{\mathbb{T}} \xi \overline{\eta} dm$. For reasons we will not get into here, the cases when $p = 1, 2, \infty$ are of the most interest in operator theory.

Definition A.7 (B(H)). *Let H be a Hilbert Space. The continuous linear operators on H is denoted **B(H)**.*

Definition A.8 (dual space). *If V is a vector space then the **dual space** of V is defined to be $V^* := \{f : V \to \mathbb{C} : f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)\}$, where $\alpha, \beta \in \mathbb{C}$ and $x, y \in V$. That is, f is linear.*

Also, let V_* denote the predual of V, that is, $(V_*)^* = V$.

Theorem A.9 (Riesz representation theorem). *Given $f \in H^*$, there is a unique vector $\eta_f \in H$ so that $f(x) = \langle x, \eta_f \rangle$ for all $x \in H$.*

Definition A.10 (Trace Class Operator). *An operator $A \in B(H)$ is said to be **Trace Class** if the sum*

$$\sum_{i=1}^{\infty} \langle (A^*A)^{1/2} e_i, e_i \rangle < \infty \tag{A.9}$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for (separable) H.

In this case, the sum $\sum_{i=1}^{\infty} \langle Ae_i, e_i \rangle$ is absolutely convergent and independent of the choice of $\{e_i\}$. It is called the **trace** of A and is denoted $Tr(A)$.

Remark A.11. *The sum A.9 defines a norm on the trace class operators C_1 and forms a two sided ideal in $B(H)$. For $T \in C_1$, define $\phi_T(A) := Tr(TA)$. Then $\phi_T \in B(H)_*$ and the assignment $T \mapsto \phi_T$ is an isometric isomorphism from C_1 onto $B(H)_*$.*

Definition A.12. *An operator T on a Hilbert space H is called **compact** if the image of the closed unit ball, $\mathbb{B}_1$, has compact closure. That is, $\overline{T(\mathbb{B}_1)}$ is compact. The set of all compact operators is denoted $\mathfrak{K}$.*

Remark A.13. *For a infinite dimensional Hilbert space, the following inclusions are proper: $\mathfrak{F} \subsetneq C_1 \subsetneq \mathfrak{K}$, where $\mathfrak{F}$ are the **finite rank** operators.*

A.2 Topologies on B(H)

There are a number of topologies on $B(H)$ which are relevant in this document. We recall some here and describe conditions for an operator to be be continuous or a net to converge with respect to the given topology.

Definition A.14. $\|T\|_{op} := Sup\{\frac{\|Tx\|}{\|x\|} : x \neq 0\}$ *defines a norm on* $B(H)$. *The induced topology is called the **norm topology** or **operator norm topology***

An operator T is continuous with respect to the norm topology if given $\epsilon > 0$ there exists a $\delta > 0$ so that $\|x - y\| < \delta$ implies $\|T(x - y)\| < \epsilon$.

Definition A.15. *The weak−∗ topology or ultraweak topology on* $B(H)$ *is the topology inherited from the predual* $B_*(H)$ *of* $B(H)$ *(the trace class operators on H). It is the weakest topology such that all elements of the predual are continuous when considered as functions on B(H).*

A net (or sequence) $T_n \subset B(H)$ is convergent to T in the weak−∗ topology if it converges pointwise: $Tr(AT_n) \to Tr(AT)$ for all $A \in C_1$. In this case, one writes $T_n \xrightarrow{w^*} T$.

Definition A.16. *The **weak operator topology** is the weakest topology on the set of bounded operators on* H *such that* $T \mapsto \langle Tx, y\rangle$ *is continuous for any vectors* $x, y \in H$.

A net (or sequence) $\{T_n\} \subset B(H)$ converges to $T \in B(H)$ in the weak operator topology if $\langle T_n x, y\rangle \to \langle Tx, y\rangle \qquad \forall x, y \in H$.

Definition A.17. *The **strong operator topology** on* $B(H)$ *is the weakest topology such that the map* $T \mapsto \|Tx\|$ *is continuous where* $T \in B(H)$ $x \in H$

A net (or sequence) $T_n \subset B(H)$ converges to $T \in B(H)$ in the strong operator topology if $\lim_{n\to\infty} \|T_n x - Tx\| = 0 \qquad \forall x \in H$.

A.3 Function Theoretic Results

Definition A.18 (Hardy Space on $\mathbb{T}$)**.** *For $1 \leq p \leq \infty$, the **Hardy Space** on the unit circle, denoted $H^p(\mathbb{T})$, is defined to be*

$$H^p(\mathbb{T}) := \{\xi \in L^p(\mathbb{T}) : \hat{\xi}(n) = 0 \qquad \forall\, n < 0\}, \tag{A.10}$$

where $\hat{\xi}(n) := \frac{1}{2\pi}\int_0 2\pi\xi(e^{it})e^{-int}dt$.

Historically, Hardy spaces were defined as spaces of analytic functions on the unit disc $\mathbb{D}$.

Definition A.19 (Hardy Space on $\mathbb{D}$)**.** *For $0 < p < \infty$, $H^p(\mathbb{D})$ is defined to be $\{\xi : \mathbb{D} \to \mathbb{C} : \xi$ is holomorphic and $M_p(\xi) := \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |\xi(re^{it})|^p dt$ is finite. $\}$.*

Remark A.20. *Several remarks are in order.*

1. *If $1 \leq p \leq \infty$, then $\|\xi\|_p := (M_p(\xi))^{1/p}$ is a norm and $H^p(\mathbb{D})$ is complete in this norm. If $0 < p < 1$, $M_p(\cdot)^{1/p}$ is not a norm. Nevertheless, the function $(\xi, \eta) \mapsto M_p(\xi - \eta)$ is a complete metric on $H^p(\mathbb{T})$.*

2. *(Fatou's Theorem) If $\xi \in H^p(\mathbb{D})$, then the limit $\lim_{r\to 1-}\xi(re^{i\theta})$ exists for almost all $\theta \in [0, 2\pi]$ and the resulting function $\tilde{\xi}$ on $\mathbb{T}$ is in $L^p(\mathbb{T})$. Further, when $1 \leq p \leq \infty$ $\|\xi\|_{H^p(\mathbb{D})} = \|\tilde{\xi}\|_{H^p(\mathbb{T})}$ and for $z \in \mathbb{D}$ we have the Poisson integral formulation*
$$\xi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2}\tilde{\xi}(e^{i\theta})d\theta. \tag{A.11}$$
Although it is true that $\tilde{\xi}$ exists, when $\xi \in H^p(\mathbb{D})$ and $0 < p < 1$, and $\|\xi\|_{H^p(\mathbb{D})} = \|\tilde{\xi}\|_{L^p(\mathbb{T})}$, formula A.11 does not really make sense because $\tilde{\xi}$ need not be integrable when $0 < p < 1$.

3. *Because of 2), given $\xi \in H^p(\mathbb{D})$, $1 \leq p \leq \infty$, we always extend ξ to $\mathbb{T}$ setting $\xi(e^{i\theta}) = \tilde{\xi}(e^{i\theta})$, wherever $\tilde{\xi}$ is defined, and setting $\xi(e^{i\theta}) = 0$ otherwise.*

We proceed toward some classical function theoretic results. First, we define some classes of functions.

Definition A.21. *An* ***inner function****, g, is an analytic function on the unit disc such that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$ and $|g(z)| = 1$ for almost all $z \in \mathbb{T}$.*

Recall the following notation from Chapter 1. If $w \in \mathbb{D}$, $w \neq 0$ then

$$b_w(z) := \frac{|w|}{w}\frac{w - z}{1 - \overline{w}z}; \tag{A.12}$$

if $w = 0$, then $b_0(z) := z$. Given a sequence $\{\alpha_j\}_{j=1}^{\infty}$ in $\mathbb{D}$ with $\sum_{n=1}^{\infty}(1 - |\alpha_n|) < \infty$, then the infinite product

$$\prod_{n=1}^{\infty} b_{a_n} \tag{A.13}$$

converges uniformly on compact subsets of $\mathbb{D}$ to an inner function b. We call b the **Blaschke product** with zeros at the points $\{a_n\}_{n=1}^{\infty}$.

Definition A.22. *If $\{\alpha_j\}_{j=1}^{N}$ is a finite collection of (not necessarily distinct) points in $\mathbb{D}$ then the* ***finite Blaschke product*** *is defined as*

$$b := \prod_{n=1}^{N} b_{a_n}. \tag{A.14}$$

We call N the total order of b.

Sometimes, we write

$$b(z) = z^n \prod_{j=1}^{m} \frac{|\alpha_j|}{\alpha_j}\frac{\alpha_j - z}{1 - \overline{\alpha_j}z} \tag{A.15}$$

where m is possibly infinite to emphasize the order n, of the zero of b at 0.

Though some of the results in this document have been proved in the greater generality we will often insist that N is finite. In this case we have some additional properties. Finite Blaschke products map the closed unit disc, $\overline{\mathbb{D}}$ onto itself. In fact are analytic in a neighborhood of $\overline{\mathbb{D}}$ and so maps the circle $\mathbb{T}$ onto $\mathbb{T}$. As verified in this thesis an elementary computation based on the argument principal shows that they are, in fact, local homeomorphisms of an annular neighborhood of the circle of degree N. Using the maximum modulus theorem, one easily shows that finite Blaschke products completely classify functions with these properties up to a constant of modulus 1. Continuity on $\mathbb{T}$ puts these functions in $L^2(\mathbb{T})$ and analyticity gives us that $\hat{b}(-n) = 0$ for n in $\mathbb{N}$, where $\hat{b}(n)$ is the n^{th} Fourier coefficient of b.

Definition A.23. *An **outer function**, F, is an analytic function on the unit disc of the form*

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} k(t)dt\right] \tag{A.16}$$

where k is a real-valued integrable function on the circle and λ is a complex number of modulus one.

The following theorem [10, p.62] offers an equivalent formulation of outer functions.

Theorem A.24. *Let F be a non-zero function in $H^1(\mathbb{T})$ then the following are equivalent.*

1. *F is an outer function.*
2. *If f is any function in $H^1(\mathbb{T})$ such that $|f| = |F|$ almost everywhere on $\mathbb{T}$, then $|f(z)| \leq |F(z)|$ at every point z in $\mathbb{D}$.*
3. *$\log |F(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |F(e^{it})| dt$.*

One reason inner functions, finite Blaschke products and outer functions are of interest to us is because of the following factorization theorems.

Theorem A.25. *Let f be a non-zero function in $H^1(\mathbb{T})$. Then f can be written in the form $f = gF$ where g is an inner function and F is an outer function. Moreover, this factorization is unique up to a constant of modulus 1, and F is in $H^1(\mathbb{T})$.*

The following theorems involve the bilateral shift, U on $L^2(\mathbb{T})$ which is given by multiplication by z. It is a fundamental tool in this thesis.

Theorem A.26 (Beurling's Theorem)**.** *Let S be a nonzero closed subspace of $H^2(\mathbb{T})$. Then S is invariant under multiplication by z if and only if $S = FH^2(\mathbb{T})$ where F is an inner function.*

As we have noted, Helson and Lowdenslager used a variational method from Hilbert space in order to prove this theorem and other results in classical function theory and harmonic analysis. Their methods prove a stronger version of Theorem

A.26 which we need. Proof of a variant of this theorem is in [9].

Theorem A.27 (Helson and Lowdenslager)**.** *Let S be a nonzero closed subspace of $L^2(\mathbb{T})$. Then S is invariant under multiplication by z if and only if:*

- $S = FH^2(\mathbb{T})$ *where F is an inner function, or*
- $S = \mathbf{1}_E L^2(\mathbb{T})$ *where E is a measurable subset of $\mathbb{T}$.*

A.4 C^*-algebras

Definition A.28 (Banach algebra)**.** *A **Banach algebra** is an associative algebra A over the real or complex numbers which is also a Banach space. The algebra multiplication and the Banach space norm are required to be related by the following inequality:*

$$\forall x, y \in A, \|x\,y\| \ \leq \|x\|\,\|y\|$$

Definition A.29 (C^*-algebra)**.** *A Banach algebra A over the complex numbers is a C^***-algebra** if it is endowed with an involution $*$ satisfying the following identities:*

- $(x+y)^* = x^* + y^*$ *and* $(xy)^* = y^*x^*$,
- $(x+y)^* = x^* + y^*$, *and* $(xy)^* = y^*x^*$,
- $(\lambda x)^* = \overline{\lambda} x^*$
- $(x^*)^* = x$

and finally, the C^*-identity, $\|a^*a\| = \|a\|^2$

It is a well known result [1] due to a construction by Gelfand, Naimark and Segal that if A is a C^*-algebra, then there is a Hilbert space H on which A can be represented. That is, there is a map $\pi : A \to B(H)$ such that if $x, y \in A$ and $\lambda \in \mathbb{C}$ then:

- $\pi(x+y) = \pi(x) + \pi(y)$,

- $\pi(xy) = \pi(x)\pi(y)$,
- $\pi(\lambda x) = \lambda\pi(x)$ and
- $\pi(x^*) = \pi(x)^*$

This completes the basic background material for this thesis.

REFERENCES

[1] William Arveson. *An invitation to C^*-algebras.* Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 39.

[2] William Arveson. Continuous analogues of Fock space. *Mem. Amer. Math. Soc.*, 80(409):iv+66, 1989.

[3] Joachim Cuntz. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.*, 57(2):173–185, 1977.

[4] R. G. Douglas. Contractive projections on an $\mathfrak{L}_1$ space. *Pacific J. Math.*, 15:443–462, 1965.

[5] Ronald G. Douglas. *Banach algebra techniques in operator theory*, volume 179 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[6] Neal J. Fowler, Paul S. Muhly, and Iain Raeburn. Representations of Cuntz-Pimsner algebras. *Indiana Univ. Math. J.*, 52(3):569–605, 2003.

[7] Hiroyasu Hamada and Yasuo Watatani. Toeplitz-composition C^*-algebras for certain finite Blaschke products, September 2008.

[8] Henry Helson. *Lectures on invariant subspaces.* Academic Press, New York, 1964.

[9] Henry Helson and David Lowdenslager. Invariant subspaces. In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, pages 251–262. Jerusalem Academic Press, Jerusalem, 1961.

[10] Kenneth Hoffman. *Banach spaces of analytic functions.* Dover Publications Inc., New York, 1988. Reprint of the 1962 original.

[11] Marius Ionescu and Paul S. Muhly. Groupoid methods in wavelet analysis. In *Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey*, volume 449 of *Contemp. Math.*, pages 193–208. Amer. Math. Soc., Providence, RI, 2008.

[12] Shunji Kametani and Tadasi Ugaheri. A remark on Kawakami's extension of Löwner's lemma. *Proc. Imp. Acad. Tokyo*, 18:14–15, 1942.

[13] M. Laca. Endomorphisms of $B(H)$ and Cuntz algebras. *J. Operator Theory*, 30(1):85–108, 1993.

[14] E. C. Lance. *Hilbert C^*-modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[15] V. M. Manuilov and E. V. Troitsky. *Hilbert C^*-modules*, volume 226 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2005. Translated from the 2001 Russian original by the authors.

[16] John N. McDonald. Adjoints of a class of composition operators. *Proc. Amer. Math. Soc.*, 131(2):601–606 (electronic), 2003.

[17] Richard Rochberg. Linear maps of the disk algebra. *Pacific Journal of Mathematics*, 44(1), 1973.

[18] J Ryff. Subordinate H^p functions. *Duke Math*, 33:347–354, 1966.

[19] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[20] J. L. Walsh. *Interpolation and approximation by rational functions in the complex domain*. Fourth edition. American Mathematical Society Colloquium Publications, Vol. XX. American Mathematical Society, Providence, R.I., 1965.