Belyi pairs and scattering constants

DISSERTATION

zur Erlangung des akademischen Grades

Dr. rer. nat.
im Fach Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

von
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8. September 1980, Berlin

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eingereicht am: 25. Mai 2010
Tag der mündlichen Prüfung: 14. September 2010

Abstract

In this dissertation non-holomorphic Eisenstein series and Dessins d'Enfants are considered. Non holomorphic Eisenstein series are created out of subgroups of the modular group by summing up over all elements modulo the stabilizer of a cusp. The second main object, Dessins d'Enfants, are bipartite graphs that are embedded into topological surfaces.

There is a correspondence between Dessins D'Enfants, Belyi pairs (a non-singular algebraic curve with a map to $\mathbb{P}^1$ ramified above at most three points) and subgroups $\Gamma \subset \Gamma(2)$ of finite index. Therefore Eisenstein series and Dessins d'Enfants are related and a focus of this work is how to use the one to find information about the other.

The main results concerning Dessins d'Enfants in this thesis are investigations of symmetries of Dessins.
We have been able to interpret automorphisms of algebraic curves on the associated Dessin, the subgroups and in particular the sets of cusps. Furthermore, we describe the relation of Dessins for subgroups $\Gamma \subset \Gamma' \subset \Gamma(2)$. Therefore, with help of the Dessins we can decide if two subgroups are contained in each other.
Together with our results on the Dessins for principal congruence subgroups this leads to an implemented algorithm that checks if a subgroup is a congruence subgroup or not.

On the side of Eisenstein series we consider scattering constants, Green's functions and Kronecker limit formulas.
We found symmetries in the scattering matrix for certain groups. For Green's functions we established a trace formula. We showed that Eisenstein series fulfill an identity we call Kronecker limit formula in which they are compared with functions coming from certain modular forms. Then combining the Kronecker limit formula with our trace formula allows us to determine scattering constants for particular subgroups.

Most of the work done in this thesis culminates in the calculation of the scattering constants for the Fermat curves. Concerning this result, one has to note that for nearly all $N \in \mathbb{N}$ the subgroup associated to the N-th Fermat curve is non-congruence.

Zusammenfassung

Diese Dissertation behandelt nicht-holomorphe Eisensteinreihen und Dessins d'Enfants. Nicht-holomorphe Eisensteinreihen entstehen aus Untergruppen der Modulgruppe, indem man über alle Elemente der Gruppe modulo dem Stabilisator einer Spitze aufsummiert. Die zweite Struktur, Dessins d'Enfants, sind bipartite Graphen, die in topologische Flächen eingebettet sind.

Dessins d'Enfants stehen in Korrespondenz zu Belyi-Paaren (einer nicht-singulären algebraischen Kurve mit einem Morphismus nach $\mathbb{P}^1$, der nur über höchstens drei Punkten verzweigt ist) und Untergruppen $\Gamma \subset \Gamma(2)$ von endlichem Index. Deshalb bestehen zwischen Eisensteinreihen und Dessins d'Enfants Verbindungen und ein Schwerpunkt dieser Arbeit ist es, Informationen und Wissen über das eine Objekt in das andere zu übertragen.

Bezüglich Dessins d'Enfants beschäftigen wir uns mit Symmetrien.
Wir waren in der Lage, Automorphismen von algebraischen Kurven im assoziierten Dessin, in der zugehörigen Untergruppe sowie insbesondere auf den Spitzen zu interpretieren. Außerdem beschreiben wir die Zusammenhänge zwischen Dessins für Untergruppen $\Gamma \subset \Gamma' \subset \Gamma(2)$, dadurch können wir für zwei Untergruppen anhand ihres Dessins entscheiden, ob sie in einander enthalten sind. In Kombination mit hier erbrachten Resultaten zu den Hauptkongruenzuntergruppen führt dies zu einem implementierten Algorithmus, der prüft, ob eine Gruppe eine Kongruenzuntergruppe ist oder nicht.

Auf der Seite der Eisensteinreihen untersucht dieser Text Streukonstanten, Greensche Funktionen und Kroneckergrenzformeln.
In der Streumatrix fanden wir Symmetrien (für bestimmte Gruppen). Für Greensche Funktionen wurde eine Spurformel bewiesen. Wir zeigten, dass Eisensteinreihen eine Identität erfüllen, die wir Kroneckergrenzformel nennen. Dabei wird der konstante Term der Eisensteinreihe mit Funktionen verglichen, die von ausgezeichneten Modulformen kommen. Indem wir die Grenzformel mit der Spurformel verbinden, konnten wir Streukonstanten für gewisse Untergruppen ausrechnen.

Die Dissertation gipfelt in der Berechnung der Streukonstanten für die Fermatkurven. Bezüglich dieses Ergebnisses ist zu beachten, dass für die meisten $N \in \mathbb{N}$ die zur N-ten Fermatkurve assoziierte Untergruppe eine Nichtkongruenzuntergruppe ist.

Contents

Introduction

The topics of this dissertation are non-holomorphic Eisenstein series on the one hand and Dessins d'Enfants on the other. These two objects are related and a focus of this work is to find out how to use the one to find information about the other.

Non-holomorphic Eisenstein series are created out of subgroups of the modular group by summing up over all elements modulo the stabilizer of a cusp. Hence, in these series the information of the subgroup is encrypted. They are eigenfunctions of the hyperbolic Laplacian and play a central role in spectral theory. Eisenstein series had first been introduced by H. Maass [Maa49]. In this thesis we will use the definition presented by T. Kubota [Kub73]. In this text the focus lies on one special term of the Eisenstein series, the constant term and the scattering matrices together with scattering constants, which are special values of scattering matrices, that can be derived from them. Scattering matrices for automorphic forms were treated by P. Lax and R. Phillips [LP67] with physical motivation. U. Kühn [Küh05] realized that scattering constants play a role in arithmetic intersection theory. For the Green's functions derived from Eisenstein series used by U. Kühn, we established a trace formula (Proposition 4.2.8). In Proposition 4.3.5 we showed that Eisenstein series fulfill an identity we call Kronecker limit formula. Then combining the Kronecker limit formula with our trace formula allows to determine scattering constants for particular groups (Propositions 4.3.8 and 4.3.10).

We can divide the subgroups of the full modular group $\Gamma(1)$ into groups of two different kinds. On the one hand side are the congruence subgroups, groups such that all matrices whose entries fulfill certain congruence conditions are contained in them. On the other side there are non-congruence subgroups. The scattering matrices for congruence subgroups are essentially known, see D. Hejhal [Hej83] and M. Huxley [Hux84]. Results for non-congruence subgroups had hardly been found. A.B. Venkov [Ven90] worked with cycloidal non-congruence subgroups. He uses the fact, that the scattering matrix changes in a known way, if a subgroup has the same number of cusps than the supergroup. In general, there is a strong relation between the scattering matrix of a group and the ones of subgroups. This has been studied of the thesis of C. Keil [Kei06] and the Diplomarbeit of the author A. Posingies [Pos07]. The relations of scattering matrices have implications to scattering constants. These symmetries of scattering constants have been studied in this thesis (see Propositions 3.2.6, 3.3.3 and Conjecture 3.3.4). One main result is the calculation of the scattering constants for Fermat curves (Theorem 5.4.8); concerning this result, one has to note that for nearly all $N \in \mathbb{N}$ the subgroup associated to the N-th Fermat curve is non-congruence.

The first problem concerning non-congruence subgroups is to find a description of a subgroup with whom we can work. There exist several constructions for non-congruence subgroups (e.g. in [Ven90]). To check that the group received is non-congruence, most

of the constructions make use of known properties concerning the widths of cusps and the index of the subgroup. Another possibility is to take any subgroup and test if it is congruence or not. Congruence test for subgroups given in special forms has been developed e.g. by T. Hsu [Hsu96] as well as by M.-L. Lang, C.H. Lim and S.-P. Tan [LLT95].
In Section 2.4 we present a new algorithm (Algorithm 2.4.2) to test if a subgroup $\Gamma \subset \Gamma(2)$ is congruence. The algorithm is implemented in Maple (in Appendix A.2 together with A.1).

The second main object are Dessins d'Enfants. They are bipartite graphs that are embedded into topological surfaces. The name was given by A. Grothendieck [Gro97] because of their simple appearance. The motivation to study such objects grew after G. Belyi [Bel79] published a result that connects Dessins with algebraic curves defined over number fields. Dessins are a description of covers of $\mathbb{P}^1$, that are only ramified in at most three points. G. Belyi showed that such covers exist for all non-singular algebraic curves defined over a number field. Beside that, Dessins define subgroups of $\Gamma(2)$, since $\Gamma(2) \cong \Pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{3 \text{ points}\})$. Therefore, one can study fundamental properties of such subgroups via Dessins d'Enfants. In the last thirty years, Dessins d'Enfants have been studied by many mathematicians with different aims. A prominent role takes e.g. the study in regard to the action of the absolute Galois group (see e.g. [SL97] and [Sch94]). The main results concerning Dessins d'Enfants in this thesis are investigations of symmetries of Dessins.
We have been able to interpret automorphisms of algebraic curves on the associated Dessin, the subgroups and in particular the sets of cusps (Propositions 2.1.9 and 2.1.11). Furthermore, in Theorem 2.3.1 we describe the relation of Dessins for subgroups $\Gamma \subset \Gamma' \subset \Gamma(2)$. Therefore, with help of the Dessins we can decide wether two subgroups are contained in each other or not.
Together with results on the Dessin for principal congruence subgroups (Theorem 2.2.4) this leads to the algorithm (Algorithm 2.4.2) already mentioned.

In the following, we will present the content of this dissertation in detail.

In Chapter 1 Dessins d'Enfants are treated. We start by defining Dessins and giving some equivalences: Belyi pairs, Belyi permutations, subgroups of the modular group and Dessins are in correspondence to each other. In Section 1.2 we give a sketch of the proof of these equivalences.
In the rest of Chapter 1, we will present some properties and details concerning the Belyi equivalences.
At first, in Section 1.3, we study the relation of Dessins and the Belyi permutations. In particular, we show how to construct σ_∞, the Belyi permutations for the faces of the Dessin, without using σ_0 and σ_1. After that in 1.4, we give an explicit description of the correspondence of cusps of the Belyi pair and the associated subgroup (Theorem 1.4.6). Section 1.5 is devoted to the full Dessin, a variation of Dessin d'Enfants that allows to see more of the structure of the Belyi pair. Finally, Section 1.6 gives a description how to calculate the Dessin for a subgroup $\Gamma \subset \Gamma(2)$. So far, we only considered the inverse problem to get the subgroup out of a Dessin.

After this repetition, in Chapter 2 we come to new results.

In Section 2.1, we study how automorphisms of an algebraic curve (with Belyi map $\boldsymbol{\beta}$) translate to the associated subgroup. We restrict the examination to automorphisms that map the full Dessin, i.e. $\boldsymbol{\beta}^{-1}(\mathbb{R})$, to itself. The result is that such automorphisms induce automorphisms on the associated subgroup given by exchanging and possibly inverting the generators of $\Gamma(2)$ plus a conjugation (Proposition 2.1.9). The conjugation is not uniquely determined but all this automorphisms induce the same action on the cusps (Proposition 2.1.11).
The next part, Section 2.2, deals with the Dessins for principal congruence subgroups $\Gamma(N)$ for even N. See for example Figure 0.1 on page 3 for the Dessin for $\Gamma(6)$. This is a Dessin of genus 1, i.e. opposite sides in the figure have to be identified.

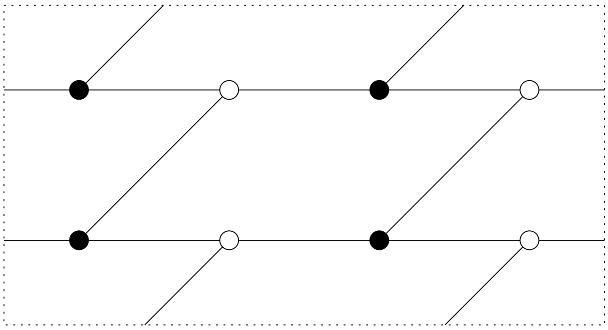


Figure 0.1.: Dessin for $\Gamma(6)$

We have been able to develop an algorithm (in Theorem 2.2.4) that calculates the Belyi permutations for $\Gamma(N)$. A Maple implementation of it can be found in Appendix A.1.
After that, in Section 2.3, we discuss the question how to distinguish Dessins of subgroups $\Gamma \subset \Gamma' \subset \Gamma(2)$. The solution of this problem is that the fact $\Gamma \subset \Gamma'$ translates into the existence of a map from the set the Belyi permutations of Γ act on into the set where the Belyi permutations of Γ' act on (Theorem 2.3.1). With this result we can construct new subgroups of arbitrary index for a given group Γ'. In Algorithm 2.3.4 the construction is explained and we give two examples. Another possible application for that result is to answer the question if a group is a congruence subgroup. We combine the results of the Sections 2.2 and 2.3 in Section 2.4. That yields an implemented algorithm (Algorithm 2.4.2 and Appendix A.2) that decides if a subgroup given by Belyi permutations is congruence.

In the third chapter, we concentrate on Eisenstein series and scattering constants. For each cusp S_j of a finite index subgroup $\Gamma \subset \Gamma(1)$ we have an Eisenstein series $E_j^{\Gamma}(z,s)$ from whose expansion in a cusp S_k we can derive the scattering constant C_{jk}^{Γ}. After the basic definitions and properties in Sections 3.1 and 3.2 we consider sum relations of Eisenstein series and scattering constants. Most of it bases on Proposition 3.1.7 (originally from [Pos07]). A generalized formula for sums of scattering constants is presented in Proposition 3.2.6.

In Section 3.3 we recall the results from Section 2.1 about automorphisms of Dessins. Calculations of some examples imply that automorphisms lead to identities of scattering constants (Conjecture 3.3.4): For an automorphism of a Belyi pair $\alpha : C \to C$ we expect

$$C_{jk}^{\Gamma} = C_{\alpha(j)\alpha(k)}^{\Gamma}$$

for the associated subgroup Γ and the induced morphism on the cusps. We could proof that formula only in a special case, when $\alpha : \Gamma \to \Gamma$ is given by conjugation only (Proposition 3.3.3).

Chapter 4 continues working with Eisenstein series. Here, the focus lies on Green's functions $g_j^{\Gamma}(z)$ derived from Eisenstein series and Kronecker limit formulas for them. By subtracting the pole, we can define (log-log singular) Green's functions for the cusps. They fulfill the following identities for $\Gamma \subset \Gamma' \subset \Gamma(2)$ with cusps S_k (for both groups) and $S_j^{\Gamma'} = \bigcup_{i \in I_j} S_i^{\Gamma}$; the b's denote the widths in Γ, the w's the widths in Γ'

$$\sum_{i \in I_j} \frac{b_i}{w_j} g_i^{\Gamma}(z) = g_j^{\Gamma'}(z),$$

as shown in Proposition 4.2.5 and

$$\mathrm{Tr}_{\Gamma|\Gamma'}\left(g_j^{\Gamma}(\gamma_k z)\right) = g_j^{\Gamma'}(\gamma_k z)$$

(Proposition 4.2.8). This is content of Section 4.2.
In Section 4.3 we show that Eisenstein series fulfill Kronecker limit formulas for distinguished modular forms f_j and $A \in \mathbb{R}$ (Proposition 4.3.5):

$$4\pi \lim_{s \to 1}\left(E_j^{\Gamma}(z,s) - \frac{1}{vol(\Gamma)(s-1)}\right) = -\log ||f_j||^2 + A.$$

With the summation formulas for Green's functions we can establish Kronecker limit formula (Proposition 4.3.10) and then calculate scattering constants (Proposition 4.3.8). In the last section 4.4 the Green's functions (Theorem 4.4.6) and Kronecker limit formulas (Proposition 4.4.12) for the group $\Gamma(2)$ are presented.

The Fermat curves F_N are the topic of Chapter 5. We will study them concerning their Dessins and their scattering constants. In the first section, we give definitions and properties already known for Fermat curves.
Section 5.2 gives the Belyi permutations (Proposition 5.2.2), the identifications between the cusps of the curve F_N and the associated group Γ_N (Proposition 5.2.5) and some figures with the Dessins for several N. Furthermore, we use the results of Chapter 2 to show once more, that for all $N \in \mathbb{N} \setminus \{1, 2, 4, 8\}$ the subgroup Γ_N is a non-congruence subgroup (Proposition 5.2.7).
After this application of Chapter 2 to Fermat curves we apply Chapter 4 to them. In Section 5.3 Kronecker limit formulas for the Fermat curves are established (Theorem 5.3.9). This allows to calculate the scattering constants in Section 5.4: Following Theorem 5.4.8

all scattering constants for the Fermat curves are:

$$\begin{aligned} C_{jj} &= \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left((12N+2)\log(2) + (-3N+6)\log(N)\right)\right) \\ C_{kj} &= \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left(2\log(2) + 6\log(N)\right)\right) \\ C_{lj} &= \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left(2\log(2) + 6\log(N) + 3N\log\left|1-\zeta_N^n\right|\right)\right), \end{aligned}$$

where ζ_N^n is a N-th root of unity determined by the cusps S_j and S_l and

$$C^{\Gamma(1)} = -\frac{6}{\pi}\left(12\zeta'(-1) + \log(4\pi) - 1\right).$$

To complete the discussion of Fermat curves, in Section 5.5 we give the numerical counterpart to the results of of Section 5.4, i.e. an attempt to calculate the scattering constants by approximation.

I would like to thank my supervisor U. Kühn who encouraged me to write this thesis. Without his input and support the work would never have been possible.
Furthermore my thanks goes to the International Graduate School *A*rithmetic and Geometry at Humboldt-Universität zu Berlin, particularly its speaker J. Kramer, for much more than financial support over the last four years, and to the Hausdorff Research Institute for Mathematics for giving me the opportunity to stay in Bonn.

1. Belyi pairs and Dessin d'Enfants

In this first chapter we will introduce most of the objects that are important in this thesis, in particular Belyi pairs, Dessin d'Enfants and subgroups of the modular group, and describe the connections between them.

Section 1.1 will introduce the objects and state the theorem of Belyi which allows to find correspondences between Belyi pairs, Dessin d'Enfants and subgroups of the modular group (Theorem 1.1.8).

In the second Section 1.2 we will give a sketch of the proof of the Belyi equivalences with a construction of a group $\Gamma \subset \Gamma(1)$ out of a Belyi pair.

The remaining sections explain parts of the Belyi correspondences more in detail. The focus lies on the understanding how different properties of the corresponding object have a counterpart in the others and how to calculate the corresponding objects out of each other. In Section 1.3 the connection of Dessins and the Belyi permutation as treated. Section 1.4 has cusps in the center of attention; we can find cusps in the objects of all corresponding sets and can give maps between them. The next section 1.5 varies the definition of Dessins to full Dessins to see more structure in the Dessins. This is needed in Section 1.6, where we calculate Dessins out of information on the subgroup.

1.1. Equivalences of Belyi pairs

We will start this introduction with the theorem of G. Belyi, which awoke interest in the subject with its surprising statement when it was first published in 1979.

Theorem 1.1.1. *Let C be a non-singular algebraic curve defined over a number field. Then there exists a morphism, defined over $\mathbb{Q}$*

$$\boldsymbol{\beta} : C \longrightarrow \mathbb{P}^1$$

with at most three critical values.

Proof: See [Bel79]. □

Definition 1.1.2. *A tuple $(C, \boldsymbol{\beta})$, consisting of a non-singular algebraic curve C and a morphism $\boldsymbol{\beta} : C \to \mathbb{P}^1$ that is ramified in at most 3 points is called a Belyi pair.*

Dessins d'Enfants will be a major object of the following chapters. They are not uniformly defined in literature, therefore we will give the definition we use in this thesis. Good introductions to the topic are [LZ04] and [Wol06].

Definition 1.1.3. *Let M be an oriented compact 2-manifold and D a connected bipartite graph on M which cuts M into simply connected cells, i.e. $M \setminus D$ is a disjoint union of simply connected open sets. Then we call D a Dessin d'Enfants or simply a Dessin on M.*

The actual position of the vertices and edges of the Dessin are not so important, only their relative arrangements. We will consider Dessins up to the following notion of isomorphisms.

Definition 1.1.4. *Two Dessins D on M and D' on M' are isomorphic, if there exists an orientation preserving homeomorphism $u : M \to M'$ such that the restriction of u to D is a graph isomorphism of D and D'.*

Dessins fascinate because they are, via the theorem of Belyi, connected to many other objects. These are, e.g., subgroups of the full modular group.

Definition 1.1.5. *Let*

$$SL_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \,\middle|\, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

be the special linear group. We define $\Gamma(1) := PSL_2(\mathbb{Z})$, its projectivisation, to be the full modular group.

For every $N \in \mathbb{N}$ there is a morphism of groups

$$\rho_N : SL_2(\mathbb{Z}) \longrightarrow SL_2\left(\mathbb{Z}/N\mathbb{Z}\right)$$

given by the reduction modulo N.

The kernel $\Gamma(N) := \ker(\rho_N)$ is called the principal congruence subgroup of level N. We will consider it as a subgroup of $\Gamma(1)$.

A subgroup $\Gamma \subset \Gamma(1)$ such that there exists $N \in \mathbb{N}$ with $\Gamma(N) \subset \Gamma$ is a congruence subgroup. Every other group is called non-congruence subgroup.

The group $\Gamma(2)$ is of particular interest in the following. Thus, we will state some properties.

Lemma 1.1.6. *The group $\Gamma(2)$ is of index six in $\Gamma(1)$. It is described as follows*

$$\Gamma(2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma(1) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \mod 2 \right\} \tag{1.1.6.1}$$

and it is a free group of rank 2 and generated by

$$\gamma_0 := \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix} \right) \qquad \text{and} \qquad \gamma_1 := \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}. \tag{1.1.6.2}$$

Proof: Well known properties that are easy to check. □

There is a connection to permutations as well.

Definition 1.1.7. *A triple of permutations $(\sigma_0, \sigma_1, \sigma_\infty) \in S_n^3$, where $n \in \mathbb{N}$ and S_n is the symmetric group of degree n, with $\sigma_0\sigma_1\sigma_\infty = id$ and such that $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ acts*

transitively we will call a triple of Belyi permutations.
If it is clear in the content which triple is meant, then we will call the permutations it consists of Belyi permutations.

Now, we may state the theorem that shows the equivalences.

Theorem 1.1.8. *The following sets are in 1-1 correspondence (regarded with suitable identifications):*

(i) Belyi pairs $(C, \boldsymbol{\beta})$ *of degree* n*.*

(ii) Pairs (M, Φ)*, where* M *is a compact Riemann surface and* $\Phi : M \longrightarrow \mathbb{P}^1(\mathbb{C})$ *a* n*-sheeted cover with at most three critical values.*

(iii) Dessins d'Enfants with n *edges.*

(iv) Triples of Belyi Permutations in S_n^3*.*

(v) Subgroups $\Gamma \subset \Gamma(2)$ *of index* n*.*

Proof: The theorem is a combination of Theorem I.5.2 in [Bos92] and the equivalences in Theorem 1 in [Bir94]. We will give some details throughout this chapter 1, in particular in Section 1.2. □

1.2. About the proof of the Belyi correspondences

Now, we will discuss very briefly the proof of Theorem 1.1.8. The focus lies on how to associate a subgroup as mentioned in (v) with a Belyi pair from (i). References for the following are [Bos92], [JS96] and [Wol06].

The proof of Theorem 1.1.8 can be divided into 5 steps.

1st step: Belyi pairs correspond to ramified covers of compact Riemann surfaces.

2nd step: Find the Dessin D for a Belyi pair $(C, \boldsymbol{\beta})$.

3rd step: Get the Belyi permutations out of a Dessin.

4th step: Define a subgroup $\Gamma \subset \Gamma(2)$ via Belyi permutation.

5th step: Construct a Belyi pair from a subgroup.

The 1st step: That non-singular projective algebraic curves correspond to compact Riemann surfaces is part of the GAGA principle; actually, it goes back to Riemann. The morphism becomes a ramified cover. We will not distinguish between the descriptions (i) and (ii).

The 2nd step: Let (C, β) be a Belyi pair, seen as a Riemann surface equipped with a ramified cover. We may identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \infty$ and assume the critical values of β to be the set $\{0, 1, \infty\}$. Then the following data define a Dessin D on the Riemann surface defined by C:

$\beta^{-1}(\,]0, 1[\,)$ form the edges of D.

$\beta^{-1}(0)$ is the set of white vertices of the graph D.

$\beta^{-1}(1)$ is the set of black vertices of the graph D.

The 3rd step: Let D be a Dessin with n edges. The edges incident with the vertices of a certain color define a permutation.

To define the Belyi permutations as in Theorem 1.1.8 give numbers to all the edges. Then collect the (numbers of the) edges incident with one vertex into a cycle according to the orientation. The Belyi permutation σ_0 consists of the cycle coming from white vertices. The Belyi permutation σ_1 consists of the cycle coming from black vertices. The third Belyi permutation is given by $\sigma_1^{-1}\sigma_0^{-1}$.

The group $\langle \sigma_0, \sigma_1 \rangle$ acts transitively since the Dessin is connected.

To understand this procedure we will discuss an example below.

Example 1.2.1. *Consider the Dessin in Figure 1.1.*

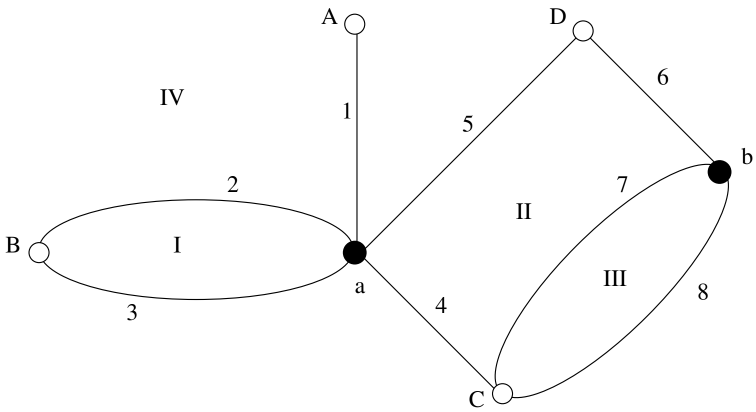


Figure 1.1.: Dessin on the plane

The Dessin consists of 8 edges. Hence, the permutations will be in S_8. At first, we will deal with the black vertices. In the vertex a five edges meet. If we count them according to the standard positive orientation we get the cycle (12345). *For the second black vertex, the vertex b, we get* (678). *To obtain the permutation we have to combine both cycles.*

Now, look at the white vertices. The vertex A is incident with only one edge, the edge 1. *Thus, we would get a cycle consisting of only one element, this corresponds to a fixed point in the permutation for the white vertices. The other white vertices give longer cycles. We get* (23), (487) *and* (56) *for* B, C *and* D, *respectively.*

To assign only one permutation to one kind of vertices we compose the cycles. Since the graph is bipartite, the composition is independent of the order of the cycles. Thereby, we get the permutations σ_0 and σ_1. The last permutation σ_∞ can be calculated with help of the first two.

The resulting Belyi permutations are:

$$\begin{aligned}\sigma_0 &= (23)(487)(56)\\ \sigma_1 &= (12345)(678)\\ \sigma_\infty &= \sigma_1^{-1}\sigma_0^{-1} = (1642)(57)\end{aligned}$$

The 4th step: Given a triple of Belyi permutations $(\sigma_0, \sigma_1, \sigma_\infty)$ fulfilling the properties from Theorem 1.1.8 (iv). The relation $\sigma_0\sigma_1\sigma_\infty = id$ leads to $\langle\sigma_0, \sigma_1, \sigma_\infty\rangle = \langle\sigma_0, \sigma_1\rangle =: G$. Hence, there is a canonical morphism

$$\begin{aligned}\varphi' : \langle p, q\rangle &\longrightarrow G\\ p &\longmapsto \sigma_0\\ q &\longmapsto \sigma_1,\end{aligned}$$

where $\langle p, q\rangle$ is the free group generated by two elements.

Definition 1.2.2. *Since $\Gamma(2)$ is freely generated by the two matrices*

$$\gamma_0 := \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1\end{smallmatrix}\right) \text{ and } \gamma_1 := \left(\begin{smallmatrix} 1 & -2 \\ 2 & -3\end{smallmatrix}\right)$$

it is isomorphic to $\langle p, q\rangle$ and we get a surjection

$$\begin{aligned}\varphi : \Gamma(2) &\longrightarrow G\\ \gamma_0 &\longmapsto \sigma_0\\ \gamma_1 &\longmapsto \sigma_1.\end{aligned} \tag{1.2.2.1}$$

Let $H \subset G$ be the stabilizer subgroup of an edge. Then we define the group $\Gamma \subset \Gamma(2)$ that we were searching for via

$$\Gamma := \varphi^{-1}(H). \tag{1.2.2.2}$$

Steps 2 to 4 describe a kind of "algorithm" how to find a subgroup when a Belyi pair is given. Not every step in this "algorithm" has been determined uniquely. Namely, the edges have been numbered randomly and in the end a stabilizer subgroup has been chosen. How does the resulting group Γ vary by doing other choices?

Remark 1.2.3. The construction of the subgroup shows that a different numbering of the edges does not change the group Γ at all; as long as we take the stabilizer to the same edge (the same edge of the Dessin, not the edge with the same number). The numbers are only mean to describe the set of edges.

Lemma 1.2.4. *In the situation of Definition 1.2.2: The choice of another stabilizer subgroup yields a group Γ' that is conjugated in $\Gamma(2)$ to the original group Γ.*

Proof: Let $H = \mathrm{Stab}_G(a)$ and $H' = \mathrm{Stab}_G(b)$ be two stabilizer subgroups of G, where a and b represent edges. They correspond to Γ and Γ', respectively. Since G acts transitively, the groups H and H' are conjugated in G. Denote by $\sigma \in G$ an element with $\sigma^{-1}H\sigma = H'$. Fix a matrix $\gamma \in \Gamma(2)$ with $\varphi(\gamma) = \sigma$.
Then $\gamma^{-1}\Gamma\gamma = \Gamma'$:
Remark that $\sigma(b) = a$. Now, take $\tau \in \Gamma$, this element satisfies $\varphi(\tau)(a) = a$. When we conjugate it with γ and map the resulting element to G, we have

$$\varphi(\gamma^{-1}\tau\gamma)(b) = \sigma^{-1}\varphi(\tau)\sigma(b) = b.$$

Hence $\gamma^{-1}\tau\gamma \in \Gamma'$ and $\gamma^{-1}\Gamma\gamma \subset \Gamma'$. The other inclusion follows similarly. □

Remark 1.2.5. From Lemma 1.2.4 we can see, that a subgroup $\Gamma \subset \Gamma(2)$ is uniquely determined by giving a Dessin with an edge that shall be stabilized.
Therefore, in this thesis, to work with a particular subgroup, we will often assume that a Dessin together with a marked edge is given.

The 5th step The subgroup of $\Gamma(2)$ defined by this procedure in the steps one to four is indeed closely related to the algebraic curve. To understand this relation, we have to introduce some facts.

Lemma 1.2.6. *We denote by $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ the upper half plane, then the group $\Gamma(1)$ acts on $\mathbb{H}$ via fractional linear transformation*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \frac{az+b}{cz+d} \quad \text{for } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \text{ and } z \in \mathbb{H}.$$

We can add a boundary to $\mathbb{H}$ by joining the upper half plane with $\mathbb{P}^1(\mathbb{Q}) \cong \mathbb{Q} \cup \infty$, the rational projective line, and get $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$. The action of $\Gamma(1)$ on $\mathbb{H}$ can be extended to $\overline{\mathbb{H}}$ by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)((p:q)) = (ap+bq : cp+dq) \quad \text{for } (p:q) \in \mathbb{P}^1(\mathbb{Q}).$$

Proof: See [Miy06]. □

Proposition 1.2.7. *Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup. The quotient $Y(\Gamma) := \Gamma\backslash\mathbb{H}$ admits the structure of a Riemann surface. The surface $Y(\Gamma)$ can be compactified with respect to the action on $\overline{\mathbb{H}}$ to $X(\Gamma) := \overline{Y(\Gamma)}$.*

Proof: See [Shi71]. □

Remark 1.2.8. Let C be a non-singular algebraic curve defined over a number field. The group $\Gamma \subset \Gamma(2)$ defined via the steps 2,3,4 explained above has the property

$$C(\mathbb{C}) \cong X(\Gamma).$$

1.3. The action of the Belyi permutations on a Dessin

We would like to understand better how to imagine the action of the permutations σ_0, σ_1 and σ_∞ in the Dessin.

Since the subgroup of $\Gamma(2)$ associated to a Dessin is given by the preimage of a stabilizer subgroup, see Equation (1.2.2.2), it is enough to concentrate on the action on a single edge, without loss of generality we may give this edge the number 1.

The permutation σ_0 maps an edge to the next edge (according to the orientation) that is incident with the same white vertex, σ_1 maps the edge to the next edge that is incident with the same black vertex and σ_∞ is, because of $\sigma_\infty = \sigma_1^{-1}\sigma_0^{-1}$, the map that maps an edge not to the next but to the second edge that borders the same cell. The permutations are drawn in Figure 1.2 as arrows. There we have $\sigma_0(1) = 7$, $\sigma_1(1) = 6$ and $\sigma_\infty(1) = \sigma_1^{-1}\sigma_0^{-1}(1) = \sigma_1^{-1}(2) = 3$.

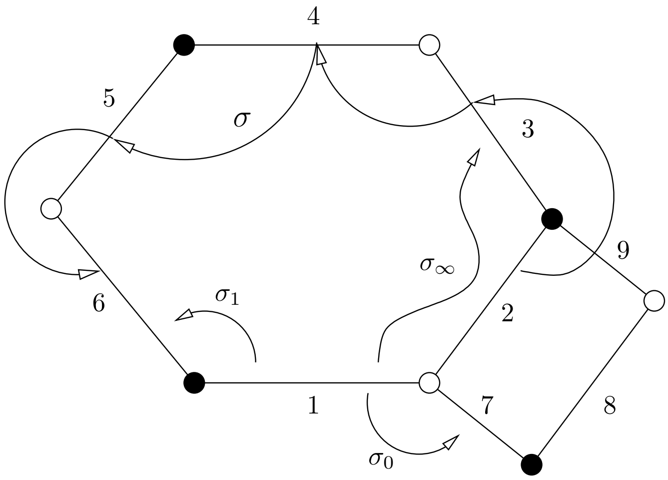


Figure 1.2.: Action of the permutations σ_0, σ_1 and σ_∞

Under φ, in Formula (1.2.2.1), we were looking at the image of a $\gamma \in \Gamma$ in $G = \langle \sigma_0, \sigma_1 \rangle$. For the action of the permutations and movements in the Dessin it helps to stay on the level of words in σ_0 and σ_1. The action of every word in σ_0, σ_1 and σ_∞ on an edge can be visualized as a kind of path in the Dessin. Since we consider the action on edges, the correct graph to find this path in is the corresponding line graph, i.e. the graph where the edges become vertices and that has edges in between two such vertices if the original edges have been adjacent. For our purpose, an intuitive description is sufficient. Let us consider the permutation $\sigma = \sigma_0\sigma_1^{-1}\sigma_0^{-1}\sigma_1^2$. Its action on the edge 2 is

$$\sigma(2) = \sigma_0\sigma_1^{-1}\sigma_0^{-1}\sigma_1^2(2) = \sigma_0\sigma_1^{-1}\sigma_0^{-1}(3) = \sigma_0\sigma_1^{-1}(4) = \sigma_0(5) = 6.$$

In Figure 1.2, we can see the action of σ on the edge 2 in the Dessin. It determines a sequence of edges in the Dessin: $(2, 3, 4, 5, 6)$.

Remark 1.3.1. Let D be a Dessin with permutations σ_0, σ_1 and a distinguished edge e. Then we can imagine elements of $\Gamma(2)$ as paths in the Dessin and switch between the sets

$$\Gamma(2) \quad \longleftrightarrow \quad \{\text{words in } \sigma_0, \sigma_1\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{paths of adjacent edges in the} \\ \text{Dessin starting in } e \end{array} \right\}$$

The first arrow is given by replacing $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ in a word description of a $\gamma \in \Gamma(2)$ by σ_0 and σ_1, respectively.

The second arrow shall not be understood as a well defined map. There is no way to establish a 1-1 correspondence there. We will go from the set on the left to the set on the right and back. But never in a unique way.

To go from words to paths we will stay on the level of understanding that the path for σ in Figure 1.2 gave.

For the direction from paths to word there is a construction. Let $(e_0, e_1, \ldots, e_n)$ be path of edge in a Dessin, i.e. a sequence of edges, where e_j and e_{j+1} are adjacent (for all $j = 0, 1, \ldots, n-1$). Then we can construct a (not unique) word in σ_0 and σ_1, therefore in $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$, out of it: The aim is to get $\sum_{j=1}^{n} \kappa_j$, where the κ_j are powers of σ_0 and σ_1.
Because of the fact that e_j and e_{j+1} (for $j \in \{0, 1, \ldots, n-1\}$) are adjacent, there exists a vertex V connecting them. Let $i_j = 0$ if V is white and $i_j = 1$ if V is black. There is a power $r_j \in \mathbb{Z}$ with $\sigma_{i_j}^{r_j}(e_j) = e_{j+1}$. The permutation $\kappa_{n-j} := \sigma_{i_j}^{r_j}$ will be in the word.

Now, we will discuss the role of the third permutation σ_∞:

In the permutations σ_0 and σ_1 there is one cycle for every vertex consisting of the edges that are incident with that vertex. In σ_0 are the cycles for the white vertices, in σ_1 the ones for the black vertices. The permutation σ_∞ plays the corresponding role for the cells. For each cell we find a cycle in σ_∞ consisting of edges that border that cell. Here, it is not possible that all edges for one cell occur, since each edge has two sides and may border two cells. Thus, in average only every second edge occur in the cycle associated to a cell.

Example 1.3.2. *Go back to Example 1.2.1 on page 10. There we have four cells (do not forget the outer cell, the Dessin has to be regarded on the sphere). The cell I corresponds to the fixed point* 3*, the cell II to the cycle* (57)*, the cell III to the fixed point* 8 *and the cell IV to the cycle* (1642)*.*

In general: Fix an edge. We can determine for which cell it will be in the corresponding cycle. If we give the edges a direction, see them as directed edges from the white vertex to the black one, then all the edges that contribute to the cycle border the cell with the same side. Taken the standard orientation of the plane, it is the right hand side. This principle is illustrated in Figure 1.3. From this description one realizes, that every second bordering edge occurs in the cycle.

Now, it is easy so resolve the unsatisfactory fact that, so far, we could determine σ_∞ only with help of σ_0 and σ_1. Let us read off σ_∞ directly. For a cell choose an edge in the

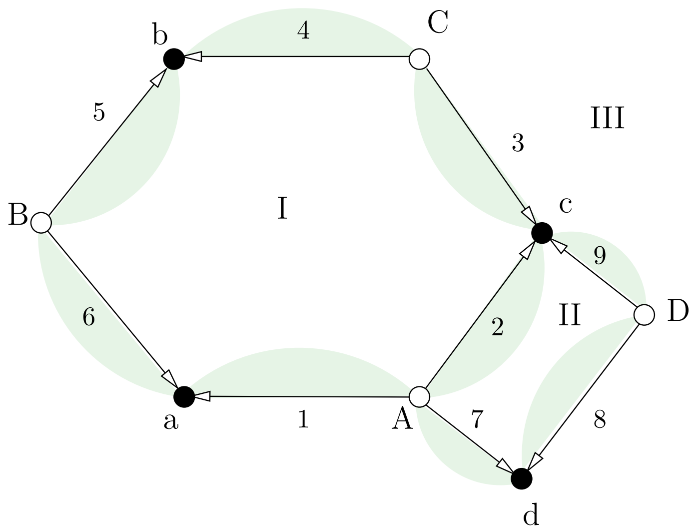


Figure 1.3.: Edges counting for cells

boundary with the property that by turning the edge around its black vertex according to the orientation it is turned into the cell (we will call it property (A)). Starting from such an edge, we count every second edge in the boundary of the cell according to the chosen orientation from the point of view of the center of the cell to get the cycle for that cell. (Note: If there is a vertex with only one edge incident with it, like the edge number 1 in Figure 1.1, both sides of this edge will border the same cell; and both its sides have to be considered.)

The permutation σ_∞ is the product of all such cycles for the cells. Since every second edge bordering a cell has the property (A) and every edge has this property regarding to exactly one cell, the cycles are disjoint and their product unique.

How to see that this is the right permutation?
For each Dessin there are two permutations counting every second edge for cells. The one that has just been described and one, where all edges counts for a cell if they fulfill a property (B), that is that by turning the edge around its white vertex it is turned into the cell. For Figure 1.2 or 1.3 this two permutations are

$$\begin{aligned} &(123)(28)(4976) && \text{with property (A)} \\ &(1538)(246)(79) && \text{with property (B).} \end{aligned}$$

By the description of the action of σ_∞ in Figure 1.2 we can see that the request $\sigma_0\sigma_1\sigma_\infty = id$ is only true for the one with property (A). The permutation with property (B) fulfills the identity $\sigma_\infty\sigma_1\sigma_0 = id$.

Definition 1.3.3. *Given a Dessin. We will call an edge e incident with a cell, if e has the property that by turning the edge around its black vertex according to the orientation it is turned into the cell (property (A)).*

For the map φ (Equation (1.2.2.1)) that maps $\Gamma(2)$ into a permutation group G generated by σ_0 and σ_1 we have $\varphi(\gamma_1^{-1}\gamma_0^{-1}) = \sigma_1^{-1}\sigma_0^{-1} = \sigma_\infty$. This preimage of σ_∞ is

$$\gamma_\infty := \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Similar to G that is generated by two permutation out of $\{\sigma_0, \sigma_1, \sigma_\infty\}$, the group $\Gamma(2)$ is (freely) generated by two matrices out of $\{\gamma_0, \gamma_1, \gamma_\infty\}$.

Remark 1.3.4. We can interpret the Belyi permutations geometrically. A fundamental domain $\mathcal{F}_\Gamma$ for a subgroup $\Gamma \subset \Gamma(2)$ is glued together by $[\Gamma(2) : \Gamma]$ fundamental domains $\mathcal{F}_{\Gamma(2)}$ of $\Gamma(2)$. The Belyi permutation describe how the generators γ_0, γ_1 and γ_∞ act on $\mathcal{F}_\Gamma$. Another way of seeing this is: It is encrypted in the Belyi permutations how the $\mathcal{F}_{\Gamma(2)}$ are glued together to get $\mathcal{F}_\Gamma$; the permutation σ_i shows the order of gluing at the cusp i ($i \in \{0, 1, \infty\}$).

1.4. Details on the map between cusps and ramification points

From the construction of Dessins out of Belyi pairs, as explained in Section 1.1 follows that there is a correspondence between branch points of the Belyi pair and the vertices and cells of the Dessin. Beside that, for the subgroup Γ of $\Gamma(2)$ the so called cusps are related to the vertices and cells of the Dessin. Content of this section is to examine the connection in detail and give an explicit description of the map between the different kinds of cusps.

Definition 1.4.1. *Let $\Gamma \subset \Gamma(1)$. The classes of $\mathbb{P}^1(\mathbb{Q})$ with respect to the action of Γ are called cusps of Γ. We will use the word cusp for a representative of a cusp as well.*

Lemma 1.4.2. *The group $\Gamma(2)$ has three cusps of width 2. They are:*

$$\begin{gathered}\Big\{(p:q) \in \mathbb{P}^1(\mathbb{Q}) | (p,q) = 1, (p,q) \equiv (0,1) \mod 2\Big\} \\ \Big\{(p:q) \in \mathbb{P}^1(\mathbb{Q}) | (p,q) = 1, (p,q) \equiv (1,1) \mod 2\Big\} \\ \Big\{(p:q) \in \mathbb{P}^1(\mathbb{Q}) | (p,q) = 1, (p,q) \equiv (1,0) \mod 2\Big\},\end{gathered} \tag{1.4.2.1}$$

thus a system of representatives is $\{0, 1, \infty\}$. The stabilizer of the three cusps are generated by the matrices

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}, \qquad \gamma_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{1.4.2.2}$$

Proof: These are well known facts that are easy to calculate. □

Remark 1.4.3. The group $\Gamma(1)$ has only one cusp, i.e. for all $S_j, S_k \in \mathbb{P}^1(\mathbb{Q})$ we find a matrix $\gamma_{kj} \in \Gamma(1)$ with $\gamma_{kj}(S_j) = S_k$, in particular, there is $\gamma_j \in \Gamma(1)$ with $\gamma_j(\infty) = S_j$. Similarly, for $S_j \in \mathbb{P}^1(\mathbb{Q})$ we always find a matrix $\gamma_{ji} \in \Gamma(2)$ such that $\gamma_{ji}(i) = S_j$ where $i \in \{0, 1, \infty\}$ is uniquely determined.

Definition 1.4.4. *Let $\Gamma \subset \Gamma(1)$ be of finite index, $S_j \in \mathbb{P}^1(\mathbb{Q})$ and $\Gamma_j := \mathrm{Stab}_\Gamma(S_j)$ its stabilizer in Γ. Take $\gamma_j \in \Gamma(1)$ with $\gamma_j(\infty) = S_j$, then*

$$\gamma_j^{-1}\Gamma_j\gamma_j = \left\{ \left(\begin{smallmatrix} 1 & mb_j \\ 0 & 1 \end{smallmatrix} \right) \middle| m \in \mathbb{Z} \right\},$$

where $b_j \in \mathbb{N}$ and we call b_j the width of the cusp S_j (it is easy to check that the width is well defined).

In the following, we will discuss the connection between cusps of the subgroup $\Gamma \subset \Gamma(2)$, the ramification points of the Belyi pair and the vertices and faces of the Dessin. Theorem 1.1.8 implies a correspondence between them: For every cusp of a subgroup $\Gamma \subset \Gamma(2)$ of width b there exists a vertex or face of the corresponding Dessin of valency $\frac{b}{2}$. (We get the factor $\frac{1}{2}$ since for Dessins the base group is $\Gamma(2)$ where all cusps have width 2.)
But how to find the concrete counterpart?

We will start by illustrating a possibility on an example. Based on the observations there, we will prove the general method.

Example 1.4.5. *Consider again the Dessin D in Figure 1.3 on page 15. We will stabilize edge 1 and get a subgroup $\Gamma \subset \Gamma(2)$.*

With help of the program in the appendix of [Pos07] we can check that Table 1.1 gives non-equivalent cusps with widths.

Cusp	0	1	2	3	4	5	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{5}$	∞
Width	6	4	4	4	4	6	4	8	4	4	6

Table 1.1.: Cusps and widths for Γ

We would like to find the cusps in the Dessin.

The trick is to consider the stabilizer of the cusps and use Remark 1.3.1 to see the stabilizer in the Dessin. Every cusp S_j of Γ has a stabilizer generated by one matrix η_j. We can decompose the generator

$$\eta_j = \gamma_{ij}^{-1}\boldsymbol{\gamma}_i^{b_j/2}\gamma_{ij},$$

where b_j is the cusp width of S_j, $\gamma_{ij} \in \Gamma(2)$ with $\gamma_{ij}(S_j) = S_i$ a matrix that maps S_j to $S_i \in \{0, 1, \infty\}$ and $\boldsymbol{\gamma}_i$ the generator of $\mathrm{Stab}_{\Gamma(2)}(i)$ as given in (1.4.2.2). Section 1.3 tells us how to find a path on the edges of the graph D, when we decompose the stabilizer further into its word in $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$.

The generators of the stabilizers for $S_j = 4$ and $S_l = \frac{2}{5}$ (the most complicated ones in this sample) are

$$\eta_j = (\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1)^{-1}\boldsymbol{\gamma}_0^2\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1, \qquad \eta_l = (\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0)^{-1}\boldsymbol{\gamma}_0^2\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0.$$

The paths that they describe in the Dessin are sketched in Figure 1.4. We may interpret the movement of η_j as going to an edge incident to C, walking once around C and

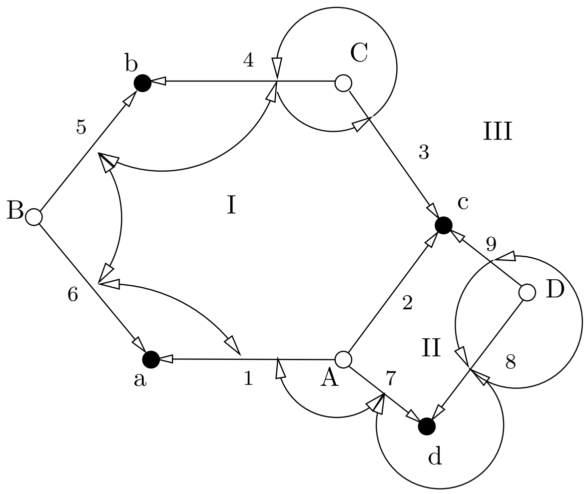


Figure 1.4.: Dessin with stabilizing permutations

going back. A general element of $Stab_\Gamma(S_j)$ is $(\gamma_1\gamma_0\gamma_1)^{-1}(\gamma_0^2)^n\gamma_1\gamma_0\gamma_1$ with $n \in \mathbb{Z}$. Its movement is going to an edge incident to C, walking n times around C and going back (if n is negative, this shall be read as going $|n|$ times backwards around C).

Therefore, is seems to be reasonable to say that C corresponds to the cusp 4. *Accordingly, D corresponds to* $\frac{2}{5}$.

When we decompose all stabilizer for cusps in Table 1.1 and follow the movements that they imply, we get the results from Table 1.2.

Cusp of Γ	Stabilizer	Cusp of D
0	γ_0^3	A
1	γ_1^2	a
2	$\gamma_1^{-1}\gamma_0^2\gamma_1$	B
3	$(\gamma_0\gamma_1)^{-1}\gamma_1^2\gamma_0\gamma_1$	b
4	$(\gamma_1\gamma_0\gamma_1)^{-1}\gamma_0^2\gamma_1\gamma_0\gamma_1$	C
5	$(\gamma_0\gamma_1)^{-2}\gamma_1^2(\gamma_0\gamma_1)^2$	c
$-\frac{1}{2}$	$\gamma_0\gamma_\infty^2\gamma_0^{-1}$	II
$\frac{1}{2}$	$\gamma_0^{-1}\gamma_\infty^4\gamma_0$	III
$\frac{1}{3}$	$\gamma_0^{-1}\gamma_1^2\gamma_0$	d
$\frac{2}{5}$	$(\gamma_1\gamma_0)^{-1}\gamma_0^2\gamma_1\gamma_0$	D
∞	γ_∞^3	I

Table 1.2.: Cuss of the subgroup in D

In the Example 1.4.5, we had 11 vertices and faces of the Dessin, we started with 11 non-equivalent cusps of the subgroup and could, by decomposing the stabilizers, assign to each vertex or face exactly one cusp. We will show that that was no coincidence, it works in general and the assignment is independent of the representative of the cusp and the choice of the generating element of the stabilizer.

Theorem 1.4.6. *Let D be a Dessin with distinguished edge e and $\Gamma \subset \Gamma(2)$ the subgroup defined by stabilizing e. Then there is a bijection*

$$\nu : \{\textit{cusps of } \Gamma\} \longrightarrow \{\textit{vertices and cells of } D\} \tag{1.4.6.1}$$

given via

$$\nu(S) := \begin{cases} \textit{the white vertex incident with } \sigma_{0s}(e) & \textit{if } S \sim_{\Gamma(2)} 0 \\ \textit{the black vertex incident with } \sigma_{1s}(e) & \textit{if } S \sim_{\Gamma(2)} 1 \\ \textit{the cell incident with } \sigma_{\infty s}(e) & \textit{if } S \sim_{\Gamma(2)} \infty \end{cases} \tag{1.4.6.2}$$

where $s \in S$ and σ_{is} ($i \in \{0, 1, \infty\}$) is constructed as follows: Let $i \sim_{\Gamma(2)} s$ be the cusp of $\Gamma(2)$ to which S is equivalent and $\gamma_{is}^{-1}\boldsymbol{\gamma}_i^{b_s/2}\gamma_{is}$ be a generating element for $Stab_\Gamma(s)$, i.e. $\boldsymbol{\gamma}_i$ is one of the generators

$$\boldsymbol{\gamma}_0 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma}_1 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \quad \textit{and} \quad \boldsymbol{\gamma}_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

the matrix $\gamma_{is} \in \Gamma(2)$ with $\gamma_{is}(s) = i$ and b_s the width of S.

Then

$$\sigma_{is} := \varphi(\gamma_{is}),$$

where $\varphi : \Gamma(2) \to \langle \sigma_0, \sigma_1 \rangle$ is the map from Formula (1.2.2.1) that maps the generators of $\Gamma(2)$ to the permutations associated to D.

Before we come to the proof, we make the following

Remark 1.4.7. In Equation (1.4.6.2), the image under ν of a cusp S of $\gamma \subset \Gamma(2)$ is defined via a representative $s \in S$, therefore, by abuse of notation, we will use ν for the underlying map of the rationals as well:

$$\nu : \mathbb{Q} \cup \infty \longrightarrow \{\text{vertices and cells of } D\}$$

Proof: To prove the Theorem 1.4.6, we will show the following properties:

(i) The map ν is well defined on the level of rational number, i.e. for $s \in \overline{\mathbb{Q}}$ the matrix γ_{is} exists and the image is independent of the particular choice.

(ii) The map ν is well defined on the level of cusps, i.e. independent of the choice of the representative in a cusp.

(iii) The map ν is injective.

(iv) The map ν is surjective.

Concerning (i): The group $\Gamma(2)$ has the three cusps $0, 1, \infty$. Hence, for each $s \in \mathbb{Q} \cup \infty$ there exists $\gamma_{is} \in \Gamma(2)$ such that $\gamma_{is}(s) = i$ and $i \in \{0, 1, \infty\}$. Let $\Gamma(2)_0$, $\Gamma(2)_1$ and $\Gamma(2)_\infty$ denote the stabilizer of $0, 1$ and ∞ in $\Gamma(2)$, respectively. The stabilizer of s in Γ can be presented as $\Gamma_s = \gamma_{is}^{-1} (\Gamma(2)_i)^w \gamma_{is}$ were $w = b_s/2$ is half the width of s in Γ, i.e. the cusp width relative to $\Gamma(2)$.

With this description we can interpret elements from Γ_s in the Dessin following the ideas from Section 1.3. Let $\kappa_s = \gamma_{is}^{-1}\boldsymbol{\gamma}_i^{wn}\gamma_{is} \in \Gamma_s$ with $\boldsymbol{\gamma}_i$ as always (it is a generator of $\Gamma(2)_i$) and $n \in \mathbb{Z}$. Now, take the stabilized edge e, with which the group Γ has been defined, and follow its way through the Dessin under the action of the image

$$\sigma_s := \varphi(\kappa_s) = \varphi(\gamma_{is}^{-1})\varphi(\boldsymbol{\gamma}_i^{wn})\varphi(\gamma_{is}) = \sigma_{is}^{-1}\sigma_i^{wn}\sigma_{is}.$$

Now, we repeat the arguments from Example 1.4.5. We can decompose the action into three parts:

- The start is σ_{is}, that is some movement in the Dessin.
- The middle part gives σ_i^{wn}. This is, according to Figure 1.2, a movement around a white or a black vertex or the changing of edges inside of one cell.
- In the end, the inverse of the beginning, σ_{is}, this means going exactly the steps back with which the movement started.

Since $\kappa_s \in \Gamma$, we know that $\sigma_s(e) = e$, thus, if we denote $e' := \sigma_{is}(e)$ we have $\sigma_i^{wn}(e') = e'$. This gives rise to the definition of the map ν in Formula (1.4.6.2).

If we choose a different element $\gamma'_{is} \in \Gamma(2)$ with $\gamma'_{is}(s) = i$, then $\gamma'_{is} = \gamma_i^m\gamma_{is}$ for a $m \in \mathbb{Z}$, thus, the corresponding permutation $\sigma'_{is} := \varphi(\gamma_{is})$ maps e to $\sigma_i^m(e')$. In general, $\sigma_i^m(e') \neq e'$, but, if we remember the actions of the permutations σ_0, σ_1 and σ_∞ as illustrated in Figure 1.2, we still have

$$\sigma_i^m(e') \text{ is incident with the same } \left\{ \begin{array}{ll} \text{white vertex} & (\text{if } i = 0) \\ \text{black vertex} & (\text{if } i = 1) \\ \text{cell} & (\text{if } i = \infty) \end{array} \right\} \text{ than } e'.$$

Hence, the image of a rational number under ν is independent of the choice of γ_{is}.

Concerning (ii): An element $s \in \mathbb{Q} \cup \infty$ in the boundary of the upper half plane is as a cusp characterized by its stabilizer $\Gamma_s \subset \Gamma$. If and only if two elements $s, t \in \mathbb{Q} \cup \infty$ are equivalent under the action of Γ, i.e. they belong to the same cusp, then there exists $\gamma_{ts} \in \Gamma$ with $\gamma_{ts}^{-1}\Gamma_t\gamma_{ts} = \Gamma_s$ (such an element has the property $\gamma_{ts}(s) = t$). In particular, they are equivalent to the same cusp under $\Gamma(2)$. We can describe all elements in the stabilizer of t via an element in the stabilizer of s: $\gamma_{st}^{-1}\gamma_{is}^{-1}\boldsymbol{\gamma}_i^n\gamma_{is}\gamma_{st}$. Since $\gamma_{st} \in \Gamma$, we have $\gamma_{is}\gamma_{st}(e) = \gamma_{is}(e)$ and therefore $\varphi(\gamma_{is}\gamma_{st})(e) = \sigma_{is}\sigma_{st}(e) = \sigma_{is}(e)$. Hence, the images under ν of s and t coincide.

Concerning (iii): Now, consider two elements $s, t \in \mathbb{Q} \cup \infty$ that are mapped to the same, let us say, white vertex. Then, we are in the situation of Figure 1.5: The images (σ_{0s} and σ_{0t}) of the matrices γ_{0s} and γ_{0t} map e to edges e_s and e_t respectively, that

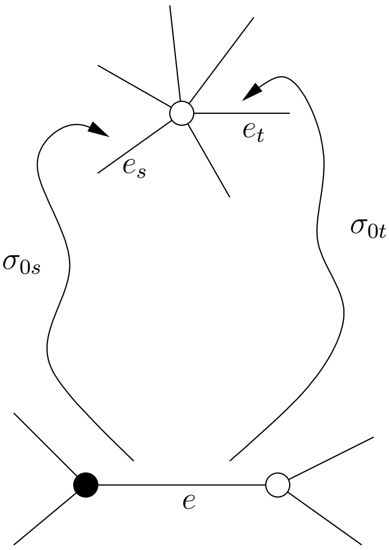


Figure 1.5.: Illustration for equivalent cusps

are incident with the same white vertex. Thus, there is a $n \in \mathbb{Z}$ such that $\sigma_0^n(e_s) = e_t$. The matrix $\gamma_{0t}^{-1}\boldsymbol{\gamma}_0^n\gamma_{0s}$ is in Γ and has the property to map s to t. Hence, s and t are equivalent under Γ and ν is injective.

Concerning (iv): It is not difficult to see that ν is surjective. For each vertex (or cell) E of D we can find a $\sigma_E \in G = \langle \sigma_0, \sigma_1 \rangle$ such that $\sigma_E(e) = e_E$ where e_E is an edge incident with E (since G acts transitively). Take a preimage of $\sigma_E^{-1}\sigma_i^n\sigma_E$ under φ (where n is the number of edges incident with E, and $i \in \{0, 1, \infty\}$ according to E), which exists since φ is surjective. The preimage will have the form $\gamma_E^{-1}\boldsymbol{\gamma}_i^n\gamma_E$ and this element stabilizes $\gamma^{-1}(i) \in \mathbb{Q} \cup \infty$. Therefore, ν is surjective. □

We already discussed that a change of the stabilizer edge yields a subgroup, that is conjugated to the original one. The action on the cusps is similar.

Corollary 1.4.8. *Let D be a Dessin and Γ the associated subgroup with respect to the edge e. If another edge e' is chosen to define the stabilizer we get another subgroup Γ'. As presented in lemma 1.2.4, there is $\gamma \in \Gamma(2)$ such that $\gamma^{-1}\Gamma\gamma = \Gamma'$. Let s be a cusp of Γ. The associated cusp of Γ' is $\gamma^{-1}(s)$.*

Proof: Follows from Theorem 1.4.6 and the proof of Lemma 1.2.4. □

Remark 1.4.9. Let $(C, \boldsymbol{\beta})$ be a Belyi pair, D its Dessin and Γ the corresponding conjugation class of subgroups of $\Gamma(2)$. From the construction of the Dessin as $\boldsymbol{\beta}^{-1}([0,1])$, a correspondence of branch points of $(C, \boldsymbol{\beta})$ is easy to see and the result of Theorem 1.4.6 generalizes to the conjugation class. Therefore we get a 1-1 correspondence

$$\{\text{Branch points of } (C, \boldsymbol{\beta})\} \longleftrightarrow \{\text{Vertices and cells of } D\} \longleftrightarrow \{\text{Cusps of } \Gamma\}.$$

Corollary 1.4.10. *Given a Dessin D and the associated subgroup Γ. A system of representatives for the cusps of Γ is given by a set containing a preimage of every cusps of D under ν (defined by Equation* (1.4.6.2)*).*

Proof: Follows from Theorem 1.4.6. □

Remark 1.4.11. The whole cusp, the full preimage of ν, is normally not computable. But to calculate a single element in a class is no problem, such that a system of representatives is obtainable. It is as well possible to check if two elements of $\mathbb{Q} \cup \infty$ are equivalent under Γ by comparing their images under ν.

Lemma 1.4.12. *Let $(C, \boldsymbol{\beta})$ be a Belyi pair with Dessin D and chosen edge e. Let E be a cusp of D and e' an edge incident with E. Then there exists a permutation $\sigma_{e'e}$ with $\sigma_{e'e}(e) = e'$. Let $\gamma_{e'e} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a preimage of $\sigma_{e'e}$ under φ. Then*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}(i) \in \nu^{-1}(E) \tag{1.4.12.1}$$

with $i = 0$ if E is a white vertex, with $i = 1$ if E is a black vertex, with $i = \infty$ if E is a cell. A different formulation is

$$\left.\begin{array}{ll} \text{if } E \text{ is a white vertex} & -\frac{b}{a} \\ \text{if } E \text{ is a black vertex} & \frac{d-b}{a-c} \\ \text{if } E \text{ is a cell} & -\frac{d}{c} \end{array}\right\} \in \nu^{-1}(E). \tag{1.4.12.2}$$

Proof: We find the permutation $\sigma_{e'e}$ via the explanations of Remark 1.3.1: Take a path of edges in the Dessin and construct the permutation as a word in σ_0 and σ_1. The matrix $\gamma_{e'e}$ is obtained by exchanging σ's by $\boldsymbol{\gamma}$'s in the word. Then, Equation (1.4.12.2) follows easily from Theorem 1.4.6. It is just the explicit description of $\gamma^{-1}(i)$ in part (iv) of the proof. □

1.5. The full Dessin

To get the Dessin from a Belyi pair we took the preimage of $[0, 1]$ as edges and vertices (see Section 1.2, step 2). The Belyi map has three critical values $0, 1, \infty$ and the choice of 0 and 1 was by habit. We could have chosen $[1, \infty]$ or $[\infty, 0]$ as well. In these cases we will get different Dessins, but the information encoded in them would be the same than in the first one.

Let us discuss the three Dessins that are encoded in one Belyi pair and their relations in an example. We take the planar Dessin with 6 edges from Figure 1.6 on page 23.

Imagine the Dessin comes from a Belyi pair, then we can extend the picture to the one showing the preimage of the whole real line. For that, we have to add vertices (the preimages of ∞) in every cell and connect them to the vertices in the boundary of the cells. The result, where the green vertices has been add as preimages of ∞, can be seen in Figure 1.7 on page 24.

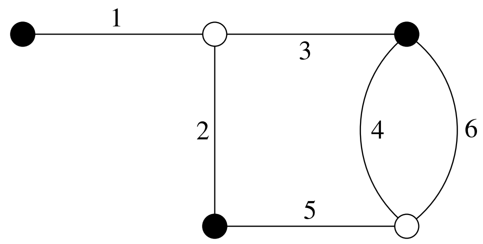


Figure 1.6.: An easy Dessin

We get the other two Dessins by only taking the black and the green or the white and the green vertices into account, respectively. In the first case we get the Dessin for $[1, \infty]$, in the second case the one for $[\infty, 0]$. They are drawn in Figure 1.8.

The edges in all three Dessins (the one from Figure 1.6 and the two from Figure 1.8) are numbered. Use the white vertices to get σ_0 the black vertices to get σ_1, the green vertices to get σ_∞ and use the relation $\sigma_0\sigma_1\sigma_\infty = id$ to calculate the missing one in each case. Then we find out that the permutations in all three cases are the same:

$$\sigma_0 = (123)(456)$$
$$\sigma_1 = (25)(346)$$
$$\sigma_\infty = (162)(35)$$

The easiest way to get this consistent numbering is to see it as a numbering of half spheres in Figure 1.7. For that, realize that the cells, that are all triangles, in the full preimage of $\mathbb{R}$ are preimages of half spheres (a Belyi map maps to $\mathbb{P}^1(\mathbb{C})$ and if we remove the real line two half spheres remain). Now, we color the preimages of the lower half sphere (the triangles were the vertices have the order white-green-black) and give them numbers. In each of the three Dessins exactly one of the three edges that border a cell will be contained. That edge will get the number of the cell.

The coloring and numbering can be seen in figure 1.9.

Sometimes it will be helpful to work with the preimage of the whole real line, the full Dessin associated to a Belyi pair. For that we give a definition, similarly to the one for Dessins (see 1.1.3).

Definition 1.5.1. *Let M be an oriented compact 2-manifold and D a connected tripartite graph on M which cuts M into triangles, i.e. $M \setminus D$ is a disjoint union of open sets bordered by three edges. We call D a full Dessin d'Enfants on M.*

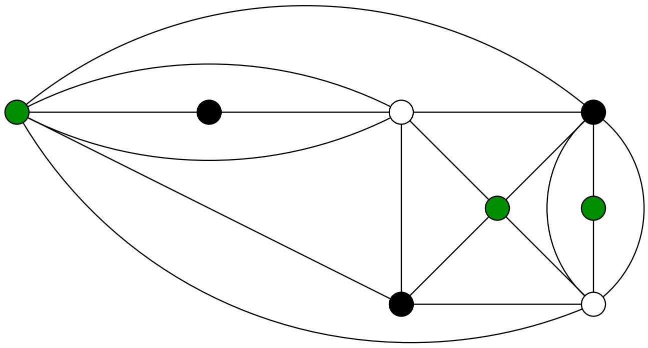


Figure 1.7.: The preimage of $\mathbb{R}$

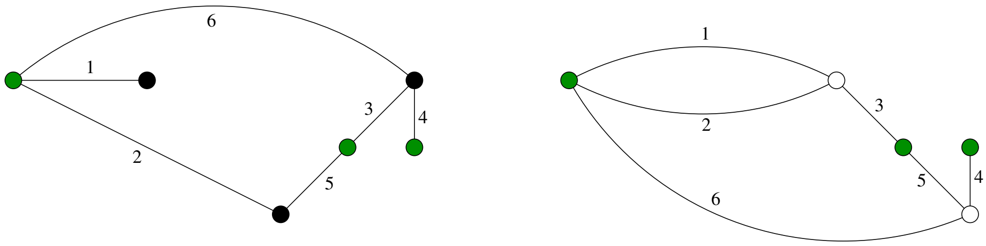


Figure 1.8.: The other two Dessins associated with figure 1.6

Remark 1.5.2. The set of full Dessins M_F and the set of Dessins M_D (both considered up to isomorphisms) can be constructed out of each other.
Let $D \in M_D$ be a Dessin. We can expand it to a full Dessin by adding a vertex in every cell an edges from these vertices to all vertices that border the cell. This yields a full Dessin. Let $D' \in M_F$ be a full Dessin. Then the set of vertices can be divided into three subsets, such that no two vertices in one of these subsets are adjacent. We get a Dessin $D \in M_D$ out of D' by forgetting one of the three subsets and all edge incident with it. To get all Dessins, we have to consider the (up to) three Dessins that we get by forgetting each of the three sets of vertices.

Remark 1.5.3. For a full Dessin D' we have two possibilities to associate three permutations to it.
At first there is the naive way: We give a number to every edge and read off the permutations. The problem with this method is, that we can not easily recover the triple

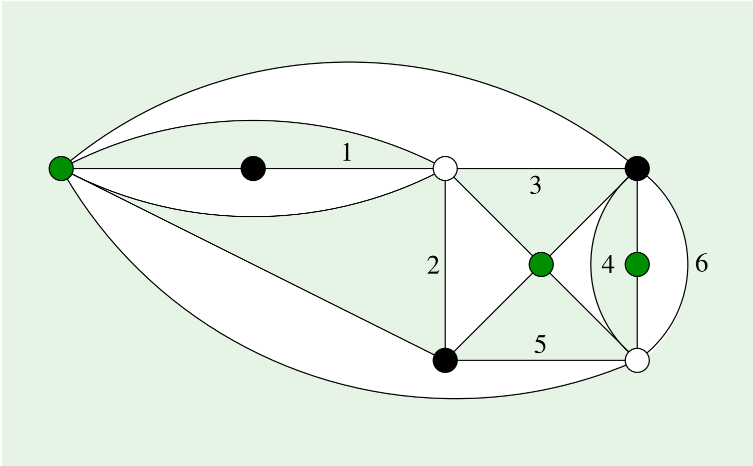


Figure 1.9.: The preimage of $\mathbb{R}$ with colored and numbered half spheres

of Belyi permutations for a (traditional) Dessin D that led to D' according to Remark 1.5.2. Nevertheless, we will need these permutations later (in Lemma 2.1.5) and will call them extended Belyi permutations for the full Dessin D'.
The other possibility is the one already mentioned, where we number every second triangle (when we identify the three classes of vertices with $0, 1, \infty$, then we number the triangles were the vertices have the order $0 - \infty - 1$) and read off the permutations. Following this method we get as permutations the Belyi permutations for the (traditional) Dessin that we get when we forget the class of vertices we identified with ∞.
This second numbering can therefore be used to define the same subgroup $\Gamma \subset \Gamma(2)$ out of the full Dessin. Here we have to stabilize a triangle instead of an edge.

1.6. From a subgroup to the Dessin

In the chapters 1.2 and 1.3 we discussed in detail how to get information about the subgroup from the Dessin or the permutations, now, we would like to do the reverse: Describe the permutations from information of the subgroup $\Gamma \subset \Gamma(2)$.

To do so, we have to understand what is encrypted in the permutations. Every number in the permutations represents a fundamental domain of $\Gamma(2)$ and their arrangement into cycles shows how they are glued together around the cusps. So, we could take a system of representatives for the classes $\Gamma \setminus \Gamma(2)$ and figure out how it glues fundamental domains of $\Gamma(2)$. We will use the connection between the cusps of the subgroup and the Dessin and the tools that we developed in Section 1.4.

In the Belyi permutations the edges were given. An edge of a graph is the pair of the two vertices incident to it. Therefore, all information needed for the graph we would have as well, if we know for every cusp of the Dessin to which cusps it is adjacent and in which order, i.e. for a vertex V of a Dessin we may get a cycle $(V_1, V_2, \ldots, V_n)$ of vertices

(n is the valency of V). Since there is a 1-1 correspondence of vertices and faces of the Dessin to the cusps of the associated subgroup, we can write the same information by using cusps of the subgroup: For a cusp S we will get a cycle $(S_1, S_2, \dots, S_n)$.

In Example 1.6.7 we will see, that the cycles of cusps do not give the Belyi permutations directly and that the consideration of the vertices of the classical Dessin is not enough. Therefore, we will construct the full Dessin as it had been introduced in Section 1.5.

To begin with, we observe that the considerations from Section 1.4 are useful. There, in Theorem 1.4.6, we constructed the bijection between the cusps of the subgroup and the cusps of the Dessin. If we follow the description there, we get information about the relative position of the cusps as well.

Definition 1.6.1. *Let $\Gamma \subset \Gamma(2)$ be a finite index subgroup. We call two cusps S_j and S_k of Γ adjacent, if the corresponding vertices of the full Dessin (see Theorem 1.4.6) are adjacent.*

Lemma 1.6.2. *Let $\Gamma \subset \Gamma(2)$ be a subgroup with cusp S that is equivalent to $i \in \{0, 1, \infty\}$ under the action of $\Gamma(2)$ and $\gamma_S \in \Gamma(2)$ be a matrix with $\gamma_S(i) = S$. Then S corresponds to a vertex in the full Dessin for Γ and the set*

$$\{\gamma_S \boldsymbol{\gamma}_i^{-n}(j)\}_{j \in \{0,1,\infty\} \setminus \{i\}}$$

with $\boldsymbol{\gamma}_i$ the corresponding generator

$$\boldsymbol{\gamma}_0 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma}_1 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \quad \text{or} \quad \boldsymbol{\gamma}_\infty = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$$

of $\Gamma(2)$ and n between 0 and half the width of S is a system of representatives for the vertices that are adjacent to the vertex for S.

Proof: This lemma follows from Theorem 1.4.6 and is quite similar to Lemma 1.4.12. If $\gamma_S(i) = S$, then in the Dessin the word w_S in σ_0 and σ_1 associated to γ_S has the property $w_S^{-1}(e) = e'$, where e is the edge that has to be stabilized to define Γ and e' is an edge incident to the vertex V corresponding to S. The set $\{\sigma_i^n w_S^{-1}(e)\}$ (with n in between 0 and half the width of S, $\sigma_\infty = \sigma_1^{-1}\sigma_0^{-1}$) runs over all edges incident with V or, in the language of full Dessins: The triangle d (a preimage of the lower half sphere) that has to be stabilized to define the subgroup is mapped to $w_S^{-1}(d) = d'$, a preimage of the lower half sphere, which has S as a vertex and $\{\sigma_i^n w_S^{-1}(d)\}$ runs over all such triangles. Hence, $w_S\sigma_i^{-n}(j)$, with $j \in \{0, 1, \infty\} \setminus \{i\}$, are the other vertices of the preimages of the lower half plane that have S as a vertex and $\gamma_S \boldsymbol{\gamma}_i^{-n}(j)$ are cusps of Γ that correspond to these vertices. □

For the classical Dessin we only need the cycle for the cusps above 0 and 1. As an example, we determine the cycles (in cusps of a subgroup) for 0 and 1.

Example 1.6.3. *Let $\Gamma \subset \Gamma(2)$ be a finite index subgroup. Then Γ has the cusps 0 and 1. Let b_0 and b_1 be the widths of 0 and 1, respectively. For these cusps we may take the* *identity for the matrix γ_S from Lemma 1.6.2. Then the cycles are*

$$Z_0 = \left(1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{b_0 - 1}\right)$$

given by $\boldsymbol{\gamma}_0^{-n}(1) = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)^n(1) = \left(\begin{smallmatrix} 1 & 0 \\ 2n & 1 \end{smallmatrix}\right)(1) = \frac{1}{2n+1}$ *for* $n = 0, 1, \ldots, b_0/2 - 1$ *and*

$$Z_1 = \left(0, 2, \frac{4}{3}, \frac{6}{5}, \ldots, \frac{b_1}{b_1 - 1}\right)$$

given by $\boldsymbol{\gamma}_1^{-n}(0) = \left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)^n(0) = \left(\begin{smallmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{smallmatrix}\right)(0) = \frac{2n}{2n-1}$ *for* $n = 0, 1, \ldots, b_1/2 - 1$.

Lemma 1.6.4. *Let $\Gamma \subset \Gamma(2)$ be a finite index subgroup, $\gamma \in \Gamma(2)$ be a matrix, S be a cusp of Γ, that is $\Gamma(2)$-equivalent to $i \in \{0, 1, \infty\}$, S_1 and S_2 be two other cusps that are adjacent to S and following each other, i.e. if $\gamma_S \in \Gamma(2)$ with $\gamma_S(i) = S$ and $\gamma_S\gamma_i^k(j) = S_1$ ($k \in \mathbb{Z}$, $j \in \{0, 1, \infty\} \setminus \{i\}$), then $\gamma_S\gamma_i^{k-1}(j) = S_2$. Then $\gamma(S_1)$ and $\gamma(S_2)$ are adjacent to $\gamma(S)$ and following each other.*

Proof: After the explanations from Lemma 1.6.2, the statement follows easily. In this situation, we have $\gamma_S \in \Gamma(2)$ with $\gamma_S(i) = S$, $\gamma_S\gamma_i^k(j) = S_1$ and $\gamma_S\gamma_i^{k-1}(j) = S_2$ ($k \in \mathbb{Z}$, $j \in \{0, 1\infty\} \setminus \{i\}$). Then $\gamma\gamma_S(i) = \gamma(S)$, $\gamma\gamma_S\gamma_i^k(j) = \gamma(S_1)$ and $\gamma\gamma_S\gamma_i^{k-1}(j) = \gamma(S_2)$, which proves the statement. □

Theorem 1.6.5. *Let $\Gamma \subset \Gamma(2)$ be a finite index subgroup. Then the cycles for the cusps are given as follows:*

A cusp S, which is $\Gamma(2)$-equivalent to 0 and of width b_S, is surrounded by

$$\left(\gamma_{S0}(\infty), \gamma_{S0}(1), \gamma_{S0}\boldsymbol{\gamma}_0^{-1}(\infty), \gamma_{S0}\boldsymbol{\gamma}_0^{-1}(1), \ldots, \gamma_{S0}\boldsymbol{\gamma}_0^{-b_s/2}(\infty), \gamma_{S0}\boldsymbol{\gamma}_0^{-b_s/2}(1)\right) \quad (1.6.5.1)$$

in the given order.

A cusp S, which is $\Gamma(2)$-equivalent to 1 and of width b_S, is surrounded by

$$\left(\gamma_{S1}(0), \gamma_{S1}(\infty), \gamma_{S1}\boldsymbol{\gamma}_1^{-1}(0), \gamma_{S1}\boldsymbol{\gamma}_1^{-1}(\infty), \ldots, \gamma_{S1}\boldsymbol{\gamma}_1^{-b_s/2}(0), \gamma_{S1}\boldsymbol{\gamma}_1^{-b_s/2}(\infty)\right) \quad (1.6.5.2)$$

in the given order.

A cusp S, which is $\Gamma(2)$-equivalent to ∞ and of width b_S, is surrounded by

$$\left(\gamma_{S\infty}(1), \gamma_{S\infty}(0), \gamma_{S\infty}\boldsymbol{\gamma}_\infty^{-1}(1), \gamma_{S\infty}\boldsymbol{\gamma}_\infty^{-1}(0), \ldots, \gamma_{S\infty}\boldsymbol{\gamma}_\infty^{-b_s/2}(1), \gamma_{S\infty}\boldsymbol{\gamma}_\infty^{-b_s/2}(0)\right) \quad (1.6.5.3)$$

in the given order.

Proof: The statement follows mostly from the lemmas 1.6.2 and 1.6.4.
Lemma 1.6.2 gives all edges incident to a cusp S. The second one, Lemma 1.6.4, gives the order for edges that connect a cusp with cusps lying above the same cusp of $\Gamma(2)$. The cycles from the statement then follow from regarding a base triangle of the lower half sphere in Figure 1.10. There, we see which order the edges have. □

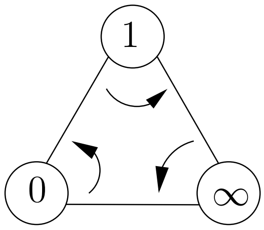


Figure 1.10.: Lower half plane

Remark 1.6.6. We would like to translate the cycles given in Theorem 1.6.5 into the Belyi permutations. As explained in Section 1.5, we get the Belyi permutations by numbering lower half spheres. Hence, the first step toward the Belyi permutations is to rewrite the cycles in lower half spheres. This is done by integrating the cusp S in its cycle from Theorem 1.6.5 at the correct position, i.e. such that the cusps have the order

$$\text{cusp above } 0 \longrightarrow \text{ cusp above } \infty \longrightarrow \text{ cusp above } 1 \longrightarrow \text{ cusp above } 0 \longrightarrow \dots$$

and embracing the three cusps that build a triangle. Formula (1.6.5.1) leads to

$$\left((S,\gamma_{S0}(\infty),\gamma_{S0}(1)),(S,\gamma_{S0}\boldsymbol{\gamma}_0^{-1}(\infty),\gamma_{S0}\boldsymbol{\gamma}_0^{-1}(1)),\dots,(S,\gamma_{S0}\boldsymbol{\gamma}_0^{-b_s/2}(\infty),\gamma_{S0}\boldsymbol{\gamma}_0^{-b_s/2}(1))\right).$$

Now, add all such cycles that corresponds to cusps lying above the same cusp of $\Gamma(2)$ to get the three Belyi permutations and give the triangles numbers. For that, we have to recognize same cusps, i.e. we have to know the classes of cusps or have at least a possibility to reduce all cusps occurring to a system of representatives.
Triangles that occur several times in one permutation are a problem in numbering. These triangles have to have different numbers but we cannot easily distinguish them. Therefore, if that happens, the numbering is no longer straightforward and other information on the subgroup has to be involved in the construction.

Example 1.6.7. *There is a subgroup with the following cusps and widths*

$$0,\ b_0=6,\qquad 1,\ b_1=6,\qquad \infty,\ b_\infty=2,\qquad \frac{1}{2},\ b_{1/2}=2,\qquad \frac{1}{4},\ b_{1/4}=2.$$

This leads to

$$\begin{aligned}
0 &\longrightarrow \left(\infty,1,\frac{1}{2},\frac{1}{3}\sim 1,\frac{1}{4},\frac{1}{5}\sim 1\right)\\
1 &\longrightarrow \left(0,\infty,2\sim 0,\frac{3}{2}\sim\frac{1}{4},\frac{4}{3}\sim 0,\frac{5}{4}\sim\frac{1}{2},\right)\\
\infty &\longrightarrow (1,0)\,,\qquad \frac{1}{2}\longrightarrow\left(\frac{1}{3}\sim 1,0\right),\qquad \frac{1}{4}\longrightarrow\left(\frac{1}{5}\sim 1,0\right)
\end{aligned}$$

and the three Belyi permutation are

$$\sigma_0 = \left((0,\infty,1),\left(0,\frac{1}{2},1\right),\left(0,\frac{1}{4},1\right)\right) = (1,2,3)$$

$$\sigma_1 = \left((0,\infty,1),\left(0,\frac{1}{4},1\right),\left(0,\frac{1}{2},1\right)\right) = (1,3,2)$$

$$\sigma_\infty = \Big((\infty,1,0)\Big)\left(\left(\frac{1}{2},1,0\right)\right)\left(\left(\frac{1}{4},1,0\right)\right) = id.$$

Here, we had no problems to find the Belyi permutations.

With the group with

$$0,\ b_0 = 6, \qquad 1,\ b_1 = 6, \qquad \infty,\ b_\infty = 6$$

this is more difficult. We get

$$\sigma_0 = ((0,\infty,1),(0,\infty,1),(0,\infty,1)) = (1,2,3)$$

$$\sigma_1 = ((0,\infty,1),(0,\infty,1),(0,\infty,1)) =?$$

$$\sigma_\infty = ((\infty,1,0),(\infty,1,0),(\infty,1,0)) =?.$$

The first permutation we can always give. We know that all triangles are distinct and give numbers anyhow. The other Belyi permutation follows only after using $\gamma_0\gamma_1\gamma_\infty = id$ *and we get*

$$\sigma_0 = (1,2,3), \quad \sigma_1 = (1,2,3), \quad \sigma_\infty = (1,3,2).$$

If we only consider the vertices above 0 *and* 1*, like in Example 1.6.3, we would not be able to distinguish these two groups by the cycles. In both examples given, the result would have been, that* 0 *is three times connected to* 1 *and* 1 *three times to* 0*.*

An interesting observation in some example was the form of the permutation group generated by the Belyi permutations for normal subgroups of $\Gamma(2)$.

Proposition 1.6.8. *Let* $\Gamma \subset \Gamma(2)$ *be a subgroup of index* n*. Let* $\sigma_0, \sigma_1 \in S_n$ *be the associated permutations. Then* $\Gamma \lhd \Gamma(2)$ *if and only if* $|\langle\sigma_0,\sigma_1\rangle| = n$*, i.e.* σ_0 *and* σ_1 *generate a transitive subgroup in* S_n *of minimal order.*

Proof: Let $\Gamma \lhd \Gamma(2)$, $G = \langle\sigma_0,\sigma_1\rangle \subset S_n$ and e be the edge that has to be stabilized to define Γ via σ_0 and σ_1. We know that G acts transitively, i.e. $|G| \geq n$.
Now, assume that $|G| > n$. Then there exists $\sigma \in G \setminus \{id\}$, with $\sigma(e) = e$. The matrices in $\Gamma(2)$ that are mapped to σ belong to Γ. Since $\sigma \neq id$, there is a f with $\sigma(f) = f' \neq f$. In G we find as well σ_{fe} with $\sigma_{fe}(e) = f$.
Consider the element $\sigma' := \sigma_{fe}^{-1}\sigma\sigma_{fe}$. In the preimage of σ' under the map φ that defines Γ are matrices of the form $\gamma^1\lambda\gamma$, with $\varphi(\gamma) = \sigma_{fe}$, $\varphi(\lambda) = \sigma$, (i.e. $\lambda \in \Gamma$). Since Γ is normal in $\Gamma(2)$, these kinds of matrices are in Γ. But the element $\sigma_{fe}^{-1}\sigma\sigma_{fe}$ has the property $\sigma_{fe}^{-1}\sigma\sigma_{fe}(e) = \sigma_{fe}^{-1}\sigma(f) = \sigma_{fe}^{-1}(f') \neq e$, therefore no element in the preimage can be in Γ. This is a contradiction to the assumption of the existence of σ.

For the other direction, let $|G| = n$. We have to show that for all $\gamma \in \Gamma$ and $\lambda \in \Gamma(2)$ the matrix $\lambda^{-1}\gamma\lambda \in \Gamma$.
If $|G| = n$, then no permutation in $G \setminus \{id\}$ has fixed points, i.e. $\varphi(\gamma) = id$ ($\forall\ \gamma \in \Gamma$). From this we get the statement by regarding the matrix $\lambda^{-1}\gamma\lambda$ as a movement in the Dessin: The permutation σ_λ will map the fixed edge e to an edge e' that is fixed by $\sigma_\gamma(= id)$ and finally σ_λ^{-1} will map e' back to e, where σ_* is the image of $*$ under φ. Therefore, $\varphi(\lambda^{-1}\gamma\lambda)(e) = e$, hence $\lambda^{-1}\gamma\lambda \in \Gamma$. □

2. Symmetries of Dessins

In this chapter we discuss properties and results concerning Dessins d'Enfants that go beyond the known statements from Chapter 1.

Section 2.1 will explain how automorphisms of Belyi pairs induce automorphisms of the corresponding subgroups. We will come back to the results therein in Chapter 3 Section 3.3.

The three other sections are closely related to each other. At first, in Section 2.2 we calculate the Dessins and the Belyi permutations for $\Gamma(N)$. This is done via an implemented algorithm (in Maple). Then we discuss the Dessins for groups where one is a subgroup of the other, thereby we are able to detect subgroups and to construct them. In the last section 2.4 we combine the results of the two previous sections to develop an algorithm to test if a group given via Belyi permutations is a congruence subgroup. An implementation of this algorithm is given as well.

2.1. Automorphisms on Dessins and Belyi pairs

We consider once more the associated objects of a Belyi pair $(C, \boldsymbol{\beta})$, a Dessin and a finite index subgroup of $\Gamma(2)$. When we take an automorphism of C that respects the Belyi map in the sense of Definition 2.1.1 below, what does it do on the associated subgroup?

In this section, we will see that an automorphism of a Belyi pair induces automorphisms of the corresponding subgroup Γ that act on the cusps of Γ as the original automorphism did on the cusps of the Belyi pair.

At first, we have to introduce the notion of an automorphism of a Belyi pair or a Dessin. To get all the information, we have to consider full Dessins.

Definition 2.1.1. *Let $(C, \boldsymbol{\beta})$ be a Belyi pair and D its full Dessin. An automorphism of $(C, \boldsymbol{\beta})$ is an automorphism α of C with $\alpha(D) = D$ (the colors of the vertices may change).*

Remark 2.1.2. The definition of automorphism of Dessins is not consistent with the notion of isomorphic Dessins in Definition 1.1.4. But here, we will be interested in the induced action of the automorphism on the sets of cusps. The former Definition 1.1.4 would be on the one hand to restrictive by demanding the preservation of the orientation, i.e. forbidding reflections, and on the other hand to open because it involves lots of maps that fix all the cusps.

Remark 2.1.3. Automorphisms of algebraic curves are the isometries of the associated Riemann surfaces. In the case of Belyi pairs and Dessins it is easier to imagine isometries.

Let us see in a classical example of Belyi maps, which possesses a large number of regularities and symmetries, what the automorphisms are.

Example 2.1.4. *Consider the map $\boldsymbol{\beta} : z \mapsto z^n$, for $z \in \mathbb{C}$ and $n \in \mathbb{N}$. This is a Belyi map of degree n for the sphere with the following ramification behavior:*

$$\boldsymbol{\beta}^*(0) = n \cdot 0, \qquad \boldsymbol{\beta}^*(1) = 1 + \zeta_n + \cdots + \zeta_n^{n-1}, \qquad \boldsymbol{\beta}^*(\infty) = n \cdot \infty,$$

where $\zeta_n = e^{2\pi i/n}$, the first primitive n-th root of unity. Hence, the associated Dessin is a n-star, i.e. it consists of a white center incident with n edges that end in black vertices.

From now on, we specialize to the case $n = 5$. The Dessin for $n = 5$ is drawn in Figure 2.1. What symmetries and regularities do we see in the Dessin?

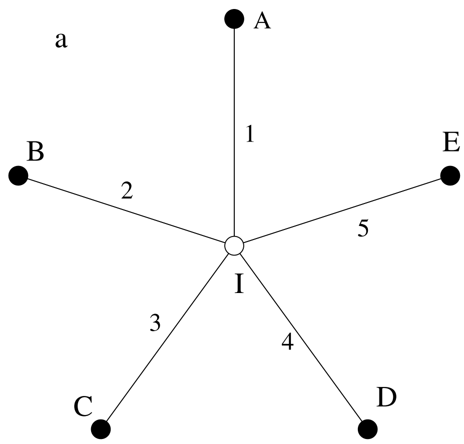


Figure 2.1.: The Dessin for $\boldsymbol{\beta}$ with $n = 5$

Obviously, we can rotate (with an angle of $e^{2\pi i m/5}$, $m \in \{0, 1, 2, 3, 4\}$) and reflect (taking every edge as axis) it.

It is not difficult to find automorphisms of $\mathbb{C} \cup \infty$ that cause these movements: The multiplication with $e^{2\pi i/5}$ will do a rotation, the complex conjugation a reflection. All others are generated by these two maps.

By just considering the standard Dessin, we forget maps are only visible in the full Dessin, the preimage of the whole real line (Figure 2.2). There we can see that the reflection along the equator leaves the full Dessin invariant as well as rotations about an axis though one of the black vertices by $e^{\pi i}$. As a map of $\mathbb{C} \cup \infty$, such a rotation is $z \mapsto \frac{1}{z}$ (the others are combinations of this one with the multiplication with $e^{2\pi i m/5}$) and the reflection at the unit circle is $z \mapsto \frac{1}{\bar{z}}$. Thus, the group A of automorphisms of the Dessin is generated by the three maps

$$\begin{aligned} d: &\quad z \longmapsto z \cdot e^{2\pi i/5} \\ s: &\quad z \longmapsto \bar{z} \\ t: &\quad z \longmapsto \frac{1}{z}. \end{aligned}$$

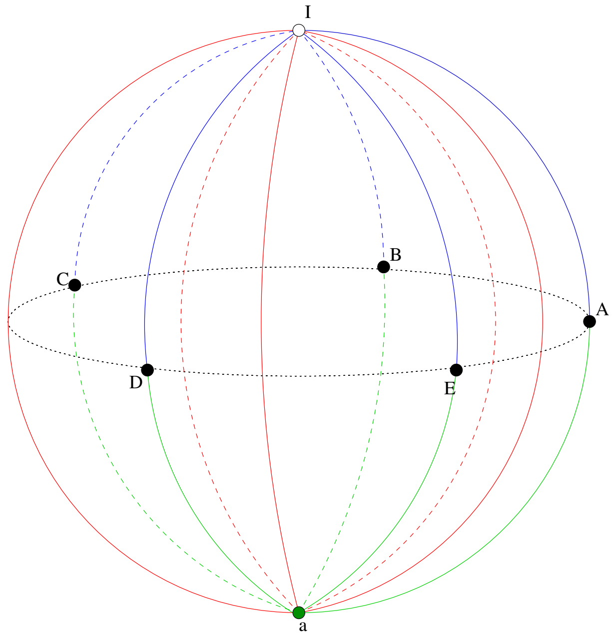


Figure 2.2.: The full Dessin for $\boldsymbol{\beta}$ with $n = 5$

If we numerate the edges of the Dessin, an automorphism of a Belyi pair extends to the associated permutations. Since the automorphism does not change the curve or the Dessin, we expect the action on the permutations to be nearly trivial. Indeed, the Belyi permutations only change slightly under such an automorphism.

Lemma 2.1.5. *Let $(C, \boldsymbol{\beta})$ be a Belyi pair, D the full Dessin, $(\sigma_0, \sigma_1, \sigma_\infty)$ the triple of the extended associated Belyi permutations, i.e. the permutations that we recive from the full Dessin when we give a number to every edge (see Remark 1.5.3), and α an automorphism of $(C, \boldsymbol{\beta})$. Then $\boldsymbol{\beta}' := \boldsymbol{\beta} \circ \alpha : C \to \mathbb{P}^1$ is a Belyi map and for the set of extended Belyi permutations for the pair $(C, \boldsymbol{\beta}')$ holds*

$$\{\sigma_0', \sigma_1', \sigma_\infty'\} \in \left\{\{\sigma_0, \sigma_1, \sigma_\infty\}, \{\sigma_0^{-1}, \sigma_1^{-1}, \sigma_\infty^{-1}\}\right\}.$$

Proof: Since α is an automorphism, the branch behavior does not change from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$. Hence $\boldsymbol{\beta}'$ is a Belyi map.

The map α maps edges to each other, therefore we can define an action of α on the extended Belyi permutations via the edges. For an extended Belyi permutation

$$\sigma = \left(e_{1_1} e_{1_2} \dots e_{1_{n_1}}\right) \dots \left(e_{r_1} e_{r_2} \dots e_{1_{n_r}}\right)$$

one gets a well defined permutations via

$$\alpha(\sigma) := \left(\alpha(e_{1_1})\alpha(e_{1_2}) \dots \alpha(e_{1_{n_1}})\right) \dots \left(\alpha(e_{r_1})\alpha(e_{r_2}) \dots \alpha(e_{1_{n_r}})\right).$$

Since circles are mapped to circles, we have

$$\alpha_G(\text{the cycle for a cusp } C) = \text{the cycle for } \alpha(C).$$

For the same reasons cusps that lie above the same critical value are mapped to cusps lying above the same critical value, i.e. it is possible that the cusps above 0, 1 and ∞ are interchanged but they can not get mixed, i.e. it holds $\sigma' = \alpha(\sigma)$ for the extended Belyi permutations and the claim from the lemma follows. □

Lemma 2.1.6. *Let $(C, \boldsymbol{\beta})$ be a Belyi pair, $(\sigma_0, \sigma_1, \sigma_\infty)$ the triple of associated Belyi permutations and α an automorphism of $(C, \boldsymbol{\beta})$. Then*

$$\{\sigma_0', \sigma_1', \sigma_\infty'\} \in \Big\{ \{\sigma_0, \sigma_1, \sigma_\infty\}, \{\sigma_0^{-1}, \sigma_1^{-1}, \sigma_\infty^{-1}\}, \{\overline{\sigma}_0, \sigma_1, \sigma_\infty\}, \{\overline{\sigma}_0^{-1}, \sigma_1^{-1}, \sigma_\infty^{-1}\},$$
$$\{\sigma_0, \overline{\sigma}_1, \sigma_\infty\}, \{\sigma_0^{-1}, \overline{\sigma}_1^{-1}, \sigma_\infty^{-1}\}, \{\sigma_0, \sigma_1, \overline{\sigma}_\infty\}, \{\sigma_0^{-1}, \sigma_1^{-1}, \overline{\sigma}_\infty^{-1}\} \Big\},$$

where $(\sigma_0', \sigma_1', \sigma_\infty')$ is the triple of Belyi permutations for $(C, \boldsymbol{\beta} \circ \alpha)$ and $\overline{\sigma}_0 = \sigma_1^{-1}\sigma_\infty^{-1}$, $\overline{\sigma}_1 = \sigma_\infty^{-1}\sigma_0^{-1}$, $\overline{\sigma}_\infty = \sigma_0^{-1}\sigma_1^{-1}$.

Proof: We cannot define the action of α on the Belyi permutations as in Lemma 2.1.5, because if e is an edge of the Dessin D, then $\alpha(e)$ is an edge in the full Dessin and does not necessarily have a number. But to get the Belyi permutations for $(C, \boldsymbol{\beta})$ we had to number the edges of the Dessin D for $(C, \boldsymbol{\beta})$. This induces a numbering of (half of the) triangles in the full Dessin D' (see Section 1.5). Thus, we can define the action of α on the Belyi permutations the following way: The number $\alpha(n)$ of the edge $\alpha(e)$, where e in an edge of the Dessin, is the number of the (unique) numbered triangle that $\alpha(e)$ is a side of. This action is surjective on all given numbers, since α can only do two different things: It can map all numbered triangles to all numbered triangles or no numbered triangle to a numbered triangle. A mixture is not possible, since α is an isometry.
With this action, we get $\alpha(\sigma_i) = \sigma_i'$, where $i \in \{0, 1, \infty\}$ and σ_i' is the Belyi permutation for $(C, \boldsymbol{\beta} \circ \alpha)$.

What permutations can we get?
When the numbered triangles are mapped to the numbered triangles, than it is easy to see that we have only the possibilities from Lemma 2.1.5. In the other case, we get one of the remaining triples for $(\sigma_0', \sigma_1', \sigma_\infty')$, i.e. a triple containing a $\overline{\sigma}_i$. The reason for that is that the Dessin of $(C, \boldsymbol{\beta} \circ \alpha)$ with edges numbered as it has just been explained induces a numbering of triangles in the full Dessin of $(C, \boldsymbol{\beta} \circ \alpha)$ (that is the full Dessin of $(C, \boldsymbol{\beta})$ with possibly changed colors of the edges). This numbering is obtained form the original one by flipping numbers at the edges that are contained in $\alpha(D)$. Therefore, the permutations $\{\sigma_0', \sigma_1', \sigma_\infty'\}$ are obtained out of the full Dessin by changing the numbering via a reflection at the edge $0-1$, the edge $1-\infty$ or the edge $\infty-0$ (and possibly inverting the orientation). The sets

$$\{\overline{\sigma}_0, \sigma_1, \sigma_\infty\}, \{\overline{\sigma}_0^{-1}, \sigma_1^{-1}, \sigma_\infty^{-1}\}, \{\sigma_0, \overline{\sigma}_1, \sigma_\infty\}, \{\sigma_0^{-1}, \overline{\sigma}_1^{-1}, \sigma_\infty^{-1}\},$$
$$\{\sigma_0, \sigma_1, \overline{\sigma}_\infty\}, \{\sigma_0^{-1}, \sigma_1^{-1}, \overline{\sigma}_\infty^{-1}\}$$

are the result of this procedure. □

Definition 2.1.7. *Let $(C, \boldsymbol{\beta})$ be a Belyi pair, $(\sigma_0, \sigma_1, \sigma_\infty)$ its Belyi permutations and α an automorphism of $(C, \boldsymbol{\beta})$, then α acts on $G := \langle \sigma_0, \sigma_1 \rangle$, since it acts on the generators, and we will denote the action by α_G.*

An automorphism of a Belyi pair induces an automorphism of the associated subgroup. To understand the proof better, we start by discussing it in the example started above.

Example 2.1.8. *We take the Belyi pair $(\mathbb{P}^1, \boldsymbol{\beta})$ from Example 2.1.4. The Belyi permutations for the Dessin are*

$$\sigma_0 = (12345), \qquad \sigma_1 = id, \qquad \sigma_\infty = (15432).$$

We denote the permutation group with G, $G = \langle \sigma_0, \sigma_1 \rangle$ and define the associated subgroup Γ by fixing the edge 1:

$$\Gamma := \varphi^{-1} \left(\mathrm{Stab}_G(1) \right), \tag{2.1.8.1}$$

where

$$\begin{aligned} \varphi : \Gamma(2) &\longrightarrow G \\ \boldsymbol{\gamma}_0 &\longmapsto \sigma_0 \\ \boldsymbol{\gamma}_1 &\longmapsto \sigma_1 \end{aligned} \tag{2.1.8.2}$$

At first, take the automorphism $d : z \mapsto z \cdot e^{2\pi i/5}$. The action of this automorphism is easy to understand:

$$d(I) = I, \ d(a) = a, \ d(A) = B, \ d(B) = C, \ d(C) = D, \ d(D) = E, \ d(E) = A$$

(according to the designations from Figure 2.1). The permutations stay the same. Thus, if we like to define a subgroup for the Dessin, taking the action of the automorphism into account, we would leave the the map φ unchanged, since the permutations did not change, but move the edge stabilized from 1 to 2, i.e. we get

$$\Gamma' := \varphi^{-1} \left(\mathrm{Stab}_G(2) \right).$$

The group Γ' is conjugated to Γ (Lemma 1.2.4), following the explanations in Section 1.2, it holds

$$\Gamma' = \boldsymbol{\gamma}_0^{-1} \Gamma \boldsymbol{\gamma}_0.$$

But $\Gamma = \Gamma'$, since G is the cyclic group with 5 elements and $\mathrm{Stab}_G(1) = \mathrm{Stab}_G(2) = id$, therefore the conjugation map with $\boldsymbol{\gamma}_0$ is an automorphism of Γ:

$$\begin{aligned} d_\Gamma : \Gamma &\longrightarrow \Gamma \\ \gamma &\longmapsto \boldsymbol{\gamma}_0^{-1} \gamma \boldsymbol{\gamma}_0 \end{aligned}$$

We will consider a second automorphism, where the construction is a bit different. Take the automorphism $t : z \mapsto \frac{1}{z}$*. Here, we have to decide which one of the black vertices of the Dessin is the real cusp above* 1 *to figure out the action of the automorphism on the cusps. When we set* $A = 1$*, then*

$$t(I) = a,\ t(a) = I,\ t(A) = A,\ t(B) = E,\ t(C) = D,\ t(D) = C,\ t(E) = B.$$

Hence, the fixed edge 1 *stays the same but the permutations change:*

$$\sigma_0' = t_G(\sigma_0) = t_G((12345)) = (15432) = \sigma_\infty, \qquad \sigma_1' = t_G(\sigma_1) = t_G(id) = id = \sigma_1$$

A subgroup Γ' *is defined via* $\Gamma' := \varphi'^{-1}(\mathrm{Stab}_G(1))$ *with*

$$\begin{aligned} \varphi' : \Gamma(2) &\longrightarrow G \\ \gamma_0 &\longmapsto \sigma_0' \\ \gamma_1 &\longmapsto \sigma_1'. \end{aligned}$$

The groups Γ *and* Γ' *are identical and we get the automorphism* t_Γ *of* Γ *by changing each appearance of* $\boldsymbol{\gamma}_0$ *in the word description of an element in* Γ *to* $\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_0^{-1}$ *(that matrix corresponds to* σ_∞*).*

In this example we saw two different types of maps for the automorphism. In the first part, the automorphism was just a conjugation, in the second part, the automorphism was given by changing the generators. In the general case, we obtain the automorphism by a combination of these two methods.

Proposition 2.1.9. *An automorphism* α *of a Belyi pair* $(C, \boldsymbol{\beta})$ *with a marked edge* e *in the Dessin induces an automorphism* α_Γ *of the associated subgroup.*

Proof: Let σ_0 and σ_1 be the Belyi permutations for $(C, \boldsymbol{\beta})$, generating $G = \langle \sigma_0, \sigma_1 \rangle$, then the associated subgroup is $\Gamma = \varphi^{-1}(\mathrm{Stab}_G(e))$, with φ like in Equation (2.1.8.2).

With α we get permutations σ_0' and σ_1' (still generating the same group G) and $\varphi' := \alpha_G \circ \varphi$:

$$\begin{aligned} \varphi' : \quad \Gamma(2) &\longrightarrow G \\ \gamma_0 &\longmapsto \sigma_0' \\ \gamma_1 &\longmapsto \sigma_1' \end{aligned}$$

In φ' the generator of $\Gamma(2)$ are mapped to the images of the original permutations. On the permutations α_G is basically nothing but a kind of permutation, i.e. we can try to get the map α on the subgroup by translating the action on the σ's into action on the generators of $\Gamma(2)$. Generators for $\Gamma(2)$ are two out of

$$\boldsymbol{\gamma}_0 = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right), \quad \boldsymbol{\gamma}_1 = \left(\begin{smallmatrix} 1 & -2 \\ 2 & -3 \end{smallmatrix}\right), \quad \boldsymbol{\gamma}_\infty = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) (= \boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_0^{-1}).$$

The map ψ on $\Gamma(2)$ may be defined via

$$\begin{aligned}\psi: \quad \Gamma(2) &\longrightarrow \Gamma(2)\\ \boldsymbol{\gamma}_0 &\longmapsto \eta_0\\ \boldsymbol{\gamma}_1 &\longmapsto \eta_1\\ \boldsymbol{\gamma}_\infty &\longmapsto \eta_\infty\end{aligned}$$

where η_i ($i = 0, 1, \infty$) is the element of $\Gamma(2)$ that corresponds to σ_i' in the following way: We know that $\sigma_i' \in \{\sigma_0, \sigma_1, \sigma_\infty, \sigma_0^{-1}, \sigma_1^{-1}, \sigma_\infty^{-1}, \overline{\sigma}_0, \overline{\sigma}_1, \overline{\sigma}_\infty, \overline{\sigma}_0^{-1}, \overline{\sigma}_1^{-1}, \overline{\sigma}_\infty^{-1}\}$ and we set

$$\eta_i := \begin{cases} \boldsymbol{\gamma}_j^\epsilon & \text{if} \quad \sigma_i' = \sigma_j^\epsilon \qquad j \in \{0, 1, \infty\} \\ (\boldsymbol{\gamma}_\infty \boldsymbol{\gamma}_1)^{-\epsilon} & \text{if} \quad \sigma_i' = \overline{\sigma}_0^\epsilon \\ (\boldsymbol{\gamma}_0 \boldsymbol{\gamma}_\infty)^{-\epsilon} & \text{if} \quad \sigma_i' = \overline{\sigma}_1^\epsilon \\ (\boldsymbol{\gamma}_1 \boldsymbol{\gamma}_0)^{-\epsilon} & \text{if} \quad \sigma_i' = \overline{\sigma}_\infty^\epsilon \end{cases} \tag{2.1.9.1}$$

A matrix $\gamma \in \Gamma(2)$ is a word, i.e.

$$\gamma = \sum_{n=1}^{r} \boldsymbol{\gamma}_i^\epsilon \ (i \in \{0, 1\}, \epsilon \in \{-1, 1\}) \text{ and } \psi(\boldsymbol{\gamma}) = \sum_{n=1}^{r} \eta_i^\epsilon.$$

The problem is that the map ψ restricted to Γ does not map to Γ: We have the identity $\varphi \circ \psi = \varphi' = \alpha_G \circ \varphi$ and for $\gamma \in \Gamma$ holds $\varphi(\gamma)(e) = e$. In $\alpha_G \circ \varphi$ the edge e will be replaced by $\alpha(e) = e'$, i.e. if e was fixed by a permutation then e' will be fixed now and we have $\varphi \circ \psi(\gamma) = \alpha_G \circ \varphi(\gamma)$. Therefore, a permutation coming from γ stabilizes e' under $\alpha_G \circ \varphi$. To come back to Γ, we have to conjugate $\psi(\Gamma)$ with a matrix μ such that $\varphi(\mu)(e') = e$, because the image of $\mu\psi(\Gamma)\mu^{-1}$ under φ stabilizes e again.

Therefore, we can define an automorphism of Γ.
Here, we give the morphism by defining the images of the standard generators $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ of $\Gamma(2)$. By the unique representation of elements of Γ in words in $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$, the map on Γ is defined:

$$\begin{aligned}\alpha_\Gamma: \quad \Gamma &\longrightarrow \Gamma\\ \boldsymbol{\gamma}_0 &\longmapsto \mu\eta_0\mu^{-1}\\ \boldsymbol{\gamma}_1 &\longmapsto \mu\eta_1\mu^{-1},\end{aligned}$$

such that for words in the generators we have $\alpha_\Gamma(\gamma_i\gamma_j\ldots\gamma_k) = \mu\eta_i\eta_j\ldots\eta_k\mu^{-1}$. □

Remark 2.1.10. The automorphism $\alpha_\Gamma : \Gamma \to \Gamma$ constructed in the proof of Proposition 2.1.9 is not unique, since for the matrix μ there are a lot of possibilities. The different automorphisms can be transferred into each other by conjugation, i.e. if α_Γ and α_Γ' are automorphisms coming from the same α, then there exists $\kappa \in \Gamma(2)$ such that $\alpha_\Gamma'(\gamma) = \kappa\alpha_\Gamma(\gamma)\kappa^{-1}$ $(\forall\, \gamma \in \Gamma)$.

The next proposition will show that, nevertheless, all the automorphisms α_Γ act equally on the set of cusps of Γ and there they mimic the automorphism on the curve.

Proposition 2.1.11. *The automorphisms α_Γ, constructed as in Proposition 2.1.9 act on the set of cusps of Γ in a unique way. The following diagram commutes*

$$\begin{array}{ccc} \{\text{Cusps of } (C, \boldsymbol{\beta})\} & \xrightarrow{\alpha} & \{\text{Cusps of } (C, \boldsymbol{\beta})\} \\ \Big\downarrow \nu & & \Big\downarrow \nu \\ \{\text{Cusps of } \Gamma\} & \xrightarrow{\alpha_\Gamma} & \{\text{Cusps of } \Gamma\}, \end{array}$$

where ν is the map between the cusps of the Belyi pair, which we identify with the cusps of the Dessin, and the cusps of the subgroup defined in Theorem 1.4.6. Hence, α_Γ acts on the cusps of Γ the same way as α on the cusps of $(C, \boldsymbol{\beta})$.

Proof: The statement consists of the following three parts that we have to show:

(i) An automorphism α_Γ induces a well defined action on the set of cusps.

(ii) For two automorphisms α_Γ and α'_Γ coming from the same α the induced action is the same.

(iii) The above diagram commutes.

Concerning (i):
The automorphism acts on the cusps of Γ via an action on the stabilizer: Let S be a cusp of Γ and $\Gamma_S = \mathrm{Stab}_\Gamma(S)$. Then we can write Γ_S in the form $\gamma_{is}^{-1}(\Gamma(2)_i)^w\gamma_{is}$, where i denotes the cusp of $\Gamma(2)$ out of $\{0, 1, \infty\}$ equivalent to S, $\gamma_{is} \in \Gamma(2)$ such that $\gamma_{is}(S) = i$ and w is half the cusps width of S. The image $\alpha_\Gamma\left(\gamma_{is}^{-1}(\Gamma(2)_i)^w\gamma_{is}\right)$ stabilizes a cusp as well (the cusp $\mu\psi(\gamma_{is})^{-1}(j)$, with ψ and μ as in Proposition 2.1.9 and j the cusp of $\Gamma(2)$ that is stabilized by $\psi(\gamma_i)$). The cusp that is stabilized by the images must be of the same width as S, otherwise the inverse of α would create elements in the stabilizer of S that does not exist. Hence, $\alpha_\Gamma(\Gamma_S)$ equals the stabilizer of a cusp.
If two cusps S and T are equivalent under the action of Γ, then there exists a matrix γ_{st} in Γ with $\gamma_{st}(T) = S$, i.e. $\gamma_{st}^{-1}\Gamma_S\gamma_{st} = \Gamma_T$. Thus $\alpha_\Gamma(\Gamma_T) = \mu\psi(\gamma_{st}^{-1})\psi(\Gamma_S)\psi(\gamma_{st})\mu^{-1}$ and we see directly that the images of S and T are Γ equivalent via $\mu\psi(\gamma_{st})\mu^{-1}$.

Therefore, we have an action of α_Γ on the set of cusps of Γ.

Concerning (ii):
Let α_Γ and α'_Γ come from the same α on $(C, \boldsymbol{\beta})$. In Remark 2.1.10 it was mentioned that there is $\kappa \in \Gamma(2)$ such that $\alpha'_\Gamma(\gamma) = \kappa\alpha_\Gamma(\gamma)\kappa^{-1}$ for $\gamma \in \Gamma$. From the description of the automorphisms in Proposition 2.1.9 it follows that $\varphi(\kappa)(e) = e$, i.e. $\kappa \in \Gamma$. Since κ maps a cusp into itself, the action on the cusps is the same for α_Γ and α'_Γ.

Concerning (iii):
Let S be a cusp of $(C, \boldsymbol{\beta})$ and $\boldsymbol{\beta}(S) = i$. In the image $\nu(S)$ are all the $q \in \mathbb{Q} \cup \infty$ that are stabilized by matrices of form $\kappa\gamma_i^w\kappa^{-1}$, where γ_i is the generator of $\mathrm{Stab}_{\Gamma(2)}(i)$, w the

valency of S and $\varphi(\kappa)$ is a permutation that maps the fixed edge e to an edge, that is incident to S.

On the one hand side, we have to show that $\alpha_\Gamma(\kappa\gamma_i^w\kappa^{-1}) = \mu\psi(\kappa)\eta_i^w\psi(\kappa)^{-1}\mu^{-1}$ "stabilizes" the cusp $S' := \alpha(S)$, i.e. we have to show:

(a) $\varphi(\psi(\kappa)^{-1}\mu^{-1})$ maps e to an edge incident to S'.

(b) η_i is a generator of $\mathrm{Stab}_{\Gamma(2)}(\boldsymbol{\beta}'(S'))$.

(c) w is the valency of S'.

The claim **(c)** is the easiest one, since the valencies do not change under automorphisms, S and S' have the same valency.
To **(a)**: It holds $\varphi(\mu^{-1})(e) = e'$. The permutations satisfy $\varphi(\kappa)(e) = e_s$, where e_s is incident to S, and $\varphi(\psi(\kappa)) = \alpha_G(\varphi(\kappa))$ such that it maps $e' = \alpha(e)$ to $\alpha(e_s)$, an edge that is incident to the image $\alpha(S) = S'$.
To **(b)**: The construction of η_i, see Equation (2.1.9.1), gives that the matrix η_i is generator of the stabilizer of cusp of $\Gamma(2)$ and $\varphi(\eta_i) = \alpha_G(\sigma_i)$. The permutation $\alpha_G(\sigma_i)$ contains the cycle for S', hence it corresponds to the cusps of $\Gamma(2)$ that belongs to S'.

On the other hand, it remains to show that every cusp in $\nu(S')$ comes via α_Γ from a cusp in $\nu(S)$, i.e. we have to give a preimage for every matrix $\gamma\gamma_i^w\gamma^{-1}$, that stabilizes a cusps from $\nu(S')$. All such stabilizing matrices have the form $\gamma\gamma_i^w\gamma^{-1}$, where $\gamma \in \Gamma(2)$ and $\boldsymbol{\beta}(S') = i$.
The inverse of ψ^{-1} is known and the matrix $\mu^{-1}\psi^{-1}(\gamma\gamma_i^w\gamma^{-1})\mu$ is the preimage we were searching for. With considerations similar to the ones done before (in **(a)**, **(b)** and **(c)**), one can show that this matrix stabilizes a cusp in $\nu(S)$. □

Let us follow this in the example.

Example 2.1.12. *We take the same Belyi pair as before, the one from Example 2.1.4, and fix the edge* 1*, the one connecting* 0 *with* 1*. The associated subgroup* Γ *has been defined in equations* (2.1.8.1) *and* (2.1.8.2)*, it consists of all elements of* $\Gamma(2)$ *such that in the word description the generator* $\boldsymbol{\gamma}_0$ *occurs a number of times that is* $0 \mod 5$ *(*$\boldsymbol{\gamma}_0^{-1}$ *is counted as* -1*).*

With the explanation from Chapter 1.4, i.e. Lemma 1.4.12, we can get the following identifications of cusps.

Cusps in $(C,\boldsymbol{\beta})$	0	1	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4	∞
D	I	A	B	C	D	E	a
Γ	0	1	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	∞

(For this identification of cusps, we have to move around in the Dessin with the matrices $\boldsymbol{\gamma}_0^n$*,* $n \in \{0, 1, \ldots 4\}$*.) The cusps* 0 *and* ∞ *have width* 10*, the other are of width* 2*. Generators of the stabilizers are*

Cusp	0	1	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	∞
Gens	$\boldsymbol{\gamma}_0^5$	$\boldsymbol{\gamma}_1$	$\boldsymbol{\gamma}_0^{-1}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0$	$\boldsymbol{\gamma}_0^{-2}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0^2$	$\boldsymbol{\gamma}_0^{-3}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0^3$	$\boldsymbol{\gamma}_0^{-4}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0^4$	$\boldsymbol{\gamma}_\infty^5$

Now, we have to consider the image of the stabilizer under a morphism α_Γ and calculate the element in $\mathbb{Q} \cup \infty$ that is fixed by the image. This is done similar to the methods used in Remark 1.3.1, Example 1.4.5 and Lemma 1.4.12.

How does the morphisms d_Γ and t_Γ act on the cusps of Γ?

***Rotation** d_Γ: The map d_Γ is simply given by the conjugation with $\gamma_0 = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right)$. In the next table we see what happens with the cusps under d_Γ.*

old cusp	0	1	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	∞
calculation	$\gamma_0^{-1}(0)$	$\gamma_0^{-1}(1)$	$\gamma_0^{-2}(1)$	$\gamma_0^{-3}(1)$	$\gamma_0^{-4}(1)$	$\gamma_0^{-5}(1)$	$\gamma_0^{-1}(\infty)$
new cusp	0	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{11}$	∞

The cusp $\frac{1}{11}$ is Γ-equivalent to 1 because $\gamma_0^5\left(\frac{1}{11}\right) = 1$ and $\gamma_0^5 \in \Gamma$.

***Map** t_Γ: This map exchanges every γ_0 in the word description of an element in Γ with $(\gamma_0\gamma_1)^{-1}$. Therefore, we get the following table*

old cusp	0	1	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	∞
calculation	γ_∞^5	$id(1)$	$\gamma_0\gamma_1(1)$	$(\gamma_0\gamma_1)^2(1)$	$(\gamma_0\gamma_1)^3(1)$	$(\gamma_0\gamma_1)^4(1)$	γ_0^5
new cusp	∞	1	$-1 \sim \frac{1}{9}$	$-3 \sim \frac{1}{7}$	$-5 \sim \frac{1}{5}$	$-7 \sim \frac{1}{3}$	0

The equivalences in the third row are obtained via

$$\gamma_0^{-4}\left(\gamma_0\gamma_1\right)^{-1}, \gamma_0^{-3}\left(\gamma_0\gamma_1\right)^{-2}, \gamma_0^{-2}\left(\gamma_0\gamma_1\right)^{-3}, \gamma_0^{-1}\left(\gamma_0\gamma_1\right)^{-4}$$

(all four are in Γ), respectively.

We realize that in both cases the action on the cusps of Γ is the same as the one on the cusps of the Belyi pair had been.

2.2. The Dessins and Belyi permutations for $\Gamma(N)$

In this section we will work with congruence subgroups and distinguish Dessins for congruence subgroups. This will not lead to simple formulas but to an implemented algorithm which gives the permutations for some congruence subgroups.

The congruence subgroups known best are $\Gamma(N)$, $\Gamma_0(N)$ and $\Gamma_1(N)$. In all three groups are the elements of $\Gamma(1)$ that fulfill certain congruence properties. For

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \text{ we have } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N,$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N) \text{ we have } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod N$$

and for

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N) \text{ holds } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N.$$

Since all groups that we can describe via Dessins are subgroups of $\Gamma(2)$, neither $\Gamma_0(N)$ nor $\Gamma_1(N)$ will be expressible by a Dessin as it is defined here and explained in Section 1.6. Solely $\Gamma(N)$, the principal congruence subgroup, for even N corresponds to a Dessin.

We will start with stating some properties of $\Gamma(N)$.

Proposition 2.2.1. *Let $N \in \mathbb{N}$ ($N > 2$) and $\Gamma(N)$ be the principal congruence subgroup of level N. Then $\Gamma(N) \lhd \Gamma(1)$, the index is*

$$[\Gamma(1) : \Gamma(N)] = \frac{1}{2} N^3 \prod_{p|N} \left(1 - p^{-2}\right), \tag{2.2.1.1}$$

where we sum over all p that are prime divisor of N. The number of cusps is $\frac{[\Gamma(1):\Gamma(N)]}{N}$, all cusps are of same width N. If N is even (i.e. $\Gamma(N) \subset \Gamma(2)$), then the same number of cusps lie above each cusps $0, 1, \infty$ of $\Gamma(2)$.

Two cusps $(p : q), (s : t) \in \mathbb{P}^1(\mathbb{Q})$ (with $(p, q) = (s, t) = 1$) are equivalent under $\Gamma(N)$ if and only if (as vectors)

$$\pm \begin{pmatrix} p \\ q \end{pmatrix} \equiv \begin{pmatrix} s \\ t \end{pmatrix} \mod N. \tag{2.2.1.2}$$

Proof: Most of the statement can be found in [Shi71].
It is obvious that the cusp at infinity has width N, all others have the same width, because all the stabilizer are conjugated to each other and $\Gamma(N)$ is normal. With all cusps being of same width, is follows directly that we get the same number of cusps above each cusp of $\Gamma(2)$. □

Corollary 2.2.2. *Let $N \in \mathbb{N}$ be even and $\Gamma(N)$ be the principal congruence subgroup of level N. The Dessin that corresponds to $\Gamma(N)$ consists of $\mu := \frac{1}{6} N^2 \prod_{p|N} (1 - p^{-2})$ white and black vertices, respectively, of valency $N/2$ and μ faces bounded by N edges each.*

Proof: Follows from Proposition 2.2.1. □

It would be desirable, to describe the Dessin and the permutations corresponding to the principal congruence subgroup generally, just using N, as we can do it for the Fermat curves (see Section 5.2). Unfortunately, this seems to be impossible. Already in the index of $\Gamma(N)$ the divisors of N play a major roll. It is to anticipate that the influence of the divisors on the Dessin and the permutations will be even greater.

Nevertheless, for one particular even N it is not difficult to calculate the Dessin and the permutations.

Example 2.2.3. *In Figure 2.3, the Dessins for $\Gamma(N)$ for $N = 2, 4, 6$ can be seen. The first two are on the sphere, the third one is on the torus, i.e. opposite sides in the figure have to be identified. The associated Belyi permutations are*

- *two times the identity for $\Gamma(2)$,*

- $\sigma_0 = (12)(34)$ *and* $\sigma_1 = (14)(23)$ *for* $\Gamma(4)$ *and*
- $\sigma_0 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$ *as well as* $\sigma_1 = (1,4,10)(2,7,6)(3,11,9)(5,8,12)$ *for* $\Gamma(6)$.

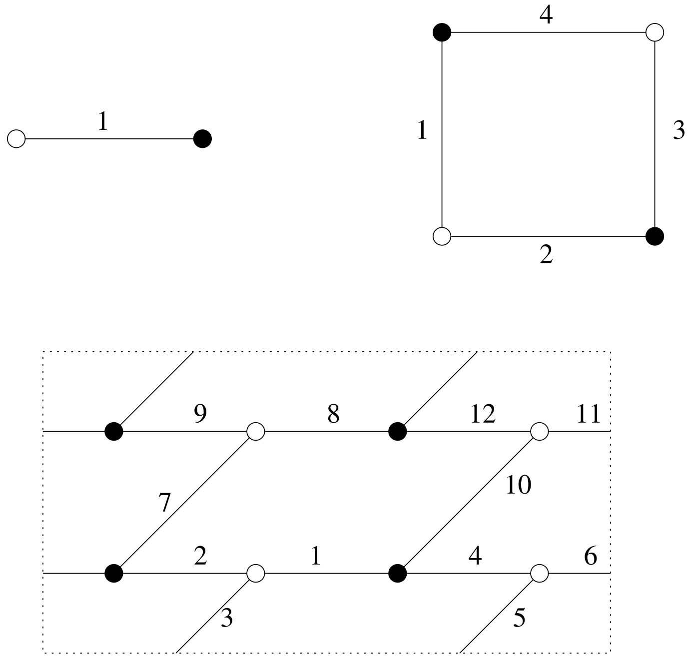


Figure 2.3.: Dessins for $\Gamma(2)$, $\Gamma(4)$ and $\Gamma(6)$

These examples had been calculated by hand. For higher N this is no longer possible. Theorem 1.6.5 gives a possibility to calculate the Belyi permutations with a computer. We apply Theorem 1.6.5 and the explanation from Remark 1.6.6 to principal congruence subgroups.

Theorem 2.2.4. *Let N be an even positive integer. The Belyi permutations for $\Gamma(N)$ are*

$$\sigma_0 = (1,2,\dots,N)(N+1,N+2,\dots,2N)\dots(I-N,I-N+1,\dots,I)$$
$$\sigma_1 = P$$

where $I = [\Gamma(2) : \Gamma(N)]$ is the index in $\Gamma(2)$ and P is the result of the Maple algorithm from Appendix A.1, which implements the following steps:

Algorithm 2.2.5. *(i) Calculate a system $\mathfrak{S}$ of representatives for the cusps of $\Gamma(N)$ lying above 0 and 1.*

(ii) For all $s \in \mathfrak{S}$: build a matrix $\gamma \in \Gamma(2)$ with $\gamma(i) = s$, where $i \in \{0,1\}$ is the cusps of $\Gamma(2)$ to which s is equivalent.

(iii) *Let* Z_i *denote the cycle of cusps for* $i \in \{0, 1\}$. *Calculate* $Z_s := \gamma(Z_i)$ *(component-by-component) for* $s \sim_{\Gamma(2)} i$.

(iv) *Reduce the elements in* Z_s *to cusps in* $\mathfrak{S}$.

(v) *Transcribe the information of the* Z_s *into permutations.*

Proof: The first permutation can be chosen as in the theorem, since Proposition 2.2.1 claims that all cusps are of same width N.

Regarding the algorithm: The calculation bases on Theorem 1.6.5 and Remark 1.6.6, but for the principal congruence subgroups the calculation is easier. It is enough to consider the cusps above 0 and 1 to get a unique description of the edges. The reason is that there are no double edges in $\Gamma(N)$:
In $\Gamma(N)$ all cusps are of same width, e.g. the cycle for a cusp S above $i \in \{0, 1\}$ is just the translate of the cycle for i by a matrix γ_{Si}. The cycles for 0 and 1 had been given in Example 1.6.3. By using the description of a cusp of $\Gamma(N)$ in Proposition 2.2.1 we see that 0 and 1 are connected with N non-equivalent cusps. Because of $\Gamma(N) \lhd \Gamma(2)$ the translates of non-equivalent cusps stay non-equivalent (if $\kappa\gamma(S_1) = \gamma(S_2)$ with $\kappa \in \Gamma(N)$ and $\gamma \in \Gamma(2)$ then $\gamma^{-1}\kappa\gamma(S_1) = S_2$ where $(\gamma^{-1}\kappa\gamma \in \Gamma(N))$. Hence, there are no double edges.

Therefore, the reasons we had to consider the full Dessin in Theorem 1.6.5 and the problems that may occur in the transcription into permutations are inexistent for $\Gamma(N)$.

After this conclusion, we can see that the steps of the algorithm follow Theorem 1.6.5.

That the Maple algorithm in Appendix A.1 actually implements these steps is clear from the commentaries in the appendix and the algorithm itself for all but for the first step; the system of representatives. This is explained in Proposition 2.2.6. □

Proposition 2.2.6. *Let* $N \in \mathbb{N}$ *be even. Every tuple* (a, b), *where* $a \in \{1, 2, \ldots, N\}$ *and* $b \in \{1, 2, \ldots, N/2\}$, *defines a cusp for* $\Gamma(N)$ *if* $(a, b, N) = 1$. *In the case* $b = N/2$ *the nominator* a *has to be in* $\{1, 2, \ldots, N/2\}$. *The cusp is given by* $\frac{a+kN}{b}$ *where* $k \in \mathbb{N}_0$ *is chosen suitable such that* $(a + kN, b) = 1$.
These cusps together with ∞ *form a system of representatives for the cusps of* $\Gamma(N)$.

Proof: The key to understand this statement is the Formula (2.2.1.2) describing the equivalence of cusps for principal congruence subgroups.

We can see that in the set constructed there are no two cusps $\Gamma(N)$-equivalent to each other. Thus, it is enough to show that every cusps is $\Gamma(N)$-equivalent to one of the cusps constructed above.

For $\frac{p}{q} \in \mathbb{Q}$ with $(p, q) = 1$ regard $\binom{p}{q}$ and reduce modulo N: $\binom{p'}{q'} \equiv \binom{p}{q} \mod N$. If q' is bigger than $N/2$, replace $q' := N - q'$ and $p' := N - p'$.
If $q' = 0$, then $\frac{p}{q} \sim \infty$. Otherwise (p', q') is a pair as in the statement. Both elements are is the range allowed. It is left over to show that $(p', q', N) = 1$. If $(p', q') = 1$, then this is done. If $(p', q') > 1$, then at least one of them has to be coprime to N, otherwise $\frac{p}{q} = \pm\frac{p'+k_pN}{q'+k_qN}$ $(k_p, k_q \in \mathbb{Z})$ could not fulfill $(p, q) = 1$. □

Remark 2.2.7. From the description of the cusps for $\Gamma(2)$ in Formula (1.4.2.1) it follows, that the cusps of $\Gamma(N)$, for even N, that lie above 0 and 1 are the ones with an odd denominator.

The Belyi permutation σ_1 calculated with the Maple program from Appendix A.1 for the values $N = 2, 4, 6, 8, 10, 12$ can be found in Table 2.1.

N	Belyi permutation σ_1 for $\Gamma(N)$
2	$id \in S_1$
4	$(14)(23)$
6	$(1,4,10)(2,7,6)(3,11,9)(5,8,12)$
8	$(1,8,24,17)(2,32,21,15)(3,6,22,19)(4,30,23,13)(5,12,18,25)(7,10,20,27)$ $(9,16,26,29)(11,14,28,31)$
10	$(1,10,40,41,26)(2,35,14,39,47)(3,23,18,13,8)(4,28,58,17,33)$ $(5,49,42,57,21)(6,15,55,45,46)(7,31,19,54,38)(9,48,27,22,34)$ $(11,20,60,44,37)(12,32,24,59,53)(16,25,29,43,52)(30,50,36,51,56)$
12	$(1,12,54,66,55,37)(2,48,26,61,14,96)(3,83,78,62,72,34)(4,9,51,63,58,40)$ $(5,45,29,64,17,93)(6,80,75,65,69,31)(7,18,84,56,46,67)$ $(8,90,32,57,20,76)(10,15,81,59,43,70)(11,87,35,60,23,73)$ $(13,24,52,47,88,41)(16,21,49,44,85,38)(19,30,82,89,94,71)$ $(22,27,79,86,91,68)(25,36,50,95,74,39)(28,33,53,92,77,42)$

Table 2.1.: Some Belyi permutations σ_1 for $\Gamma(N)$

2.3. Distinguishing super- and subgroups

The aim of this section is to study how the fact that a finite index subgroup of $\Gamma(2)$ is subgroup of another one is visualized in the Dessin and encrypted in the permutations.

Thereby we will find a method to construct subgroups of a given group and a possibility to check if a group is contained in another.

Given two Dessins D and D' with Belyi permutations $\sigma_0, \sigma_1 \in S_n$, $\sigma_0', \sigma_1' \in Sn'$ and subgroups Γ, Γ', respectively. When do they describe subgroups?
Let us assume $n' < n$. Since n and n' stand for the index in $\Gamma(2)$ the relation possible is $\Gamma \subset \Gamma'$. The first idea is to find a relation of the groups $G := \langle \sigma_0, \sigma_1 \rangle \subset S_n$ and $G' := \langle \sigma_0', \sigma_1' \rangle \subset S_{n'}$, e.g. lift σ_0' and σ_1' to G or reduce the other way around. This does not yield an answer to the question.
The correct approach to the problem is to study maps between the underlying sets and see if there is one compatible with the Belyi permutations. Geometrically that means that we try to cut the fundamental domain of Γ' into fundamental domains of Γ in a way that is compatible with the action of $\Gamma(2)$ on them.

Theorem 2.3.1. *Let Γ and Γ' be finite index subgroups of $\Gamma(2)$ with associated Belyi permutations $\sigma_0, \sigma_1 \in S_n$ and fixed edge e for Γ and $\sigma_0', \sigma_1' \in S_{n'}$ and fixed edge e' for Γ'.*

Then we have: $\Gamma \subset \Gamma'$ *if and only if there exists a map*

$$\mu : \{1, \dots, n\} \longrightarrow \{1, \dots n'\}$$

such that

$$\mu(e) = e' \quad \textit{and} \quad \mu(\sigma_i(a)) = \sigma'_i(\mu(a)) \tag{2.3.1.1}$$

for $i = 0, 1$ *and* $a \in \{1, \dots, n\}$.

Proof: At first, let $\Gamma \subset \Gamma'$ be a subgroup. The map μ can be constructed as follows: We start with $\mu(e) := e'$. For another element $a \in \{1, \dots n\}$ there exists a word w in σ_0 and σ_1 with $w(e) = a$, since $\langle \sigma_0, \sigma_1 \rangle$ acts transitively. To w we may associate a word w' in σ'_0 and σ'_1 by replacing σ_i^ϵ ($i \in \{0, 1\}$, $\epsilon \in \{-1, 1\}$) in w with $(\sigma'_i)^\epsilon$ and we set $\mu(a) := w'(e')$.
It is necessary to show that this definition of μ is well defined. If it is well defined, then it automatically fulfills the required property. The map is not well defined if there are two words w_1 and w_2 with $w_1(e) = w_2(e)$ but $w'_1(e') \neq w'_2(e')$. In this case, there would be an element in Γ given by $w_1^{-1} w_2$ (replace σ_i^ϵ ($i \in \{0, 1\}$, $\epsilon \in \{-1, 1\}$) by γ_i^ϵ) that is not in Γ', which contradicts the assumption.

Now, take the map μ as given. We have to show that every element of Γ is one of Γ' as well.
Each element $\gamma \in \Gamma$ corresponds to a word w (via φ, see (1.2.2.1) on page 11) in the σ_i with $w(e) = e$. Similarly, it corresponds to a word w' via the map φ' which defines Γ'. We have $\gamma \in \Gamma'$ if $w'(e') = e'$.
The w' coming from φ' is nothing else than the w' coming from w via replacing σ_i^ϵ with $(\sigma'_i)^\epsilon$. Inductively, the property $\mu(\sigma_i(a)) = \sigma'_i(\mu(a))$ (for $i = 0, 1$ and $a \in \{1, \dots, n\}$) expands to words such that $\mu(w(a)) = w'(\mu(a))$ and we have

$$w'(e') = w'(\mu(e)) = \mu(w(e)) = \mu(e) = e'.$$

□

Remark 2.3.2. The map μ from Theorem 2.3.1 is, if it exists, unique. The demands from Equation (2.3.1.1) give a construction: Since $G := \langle \sigma_0, \sigma_1 \rangle$ acts transitively, for every edge d, $d = 1, \dots, n$, there is a word w in σ_0 and σ_1 with $w = \prod_{j=1}^{r} \kappa_j^{r_j}(e') = d'$ ($\kappa_j \in \{\sigma_0, \sigma_1\}$, $r_j \in \mathbb{Z}$). The map μ is given via $\mu(d) = \prod_{j=1}^{r} (\kappa'_j)^{r_j}(e)$, where σ_i in the word had been replaced by σ'_i, $i \in \{0, 1\}$.
The word w is not unique. But since this construction uses only the required property from (2.3.1.1), the image must be unique for μ to exists. Therefore a possibility to construct the map or disprove its existence is to start with $e \mapsto e'$ and to apply the Belyi permutations (as illustrated in Figure 2.4) so long until σ_0 and σ_1 had been applied to all $d \in \{1, 2, \dots, n\}$. Either this construction gives rise to a contradiction (one element shall be mapped to different images), then the Belyi permutations do not describe subgroups, or a map μ will be the result, that shows that the groups defined via the Belyi permutations are subgroups.

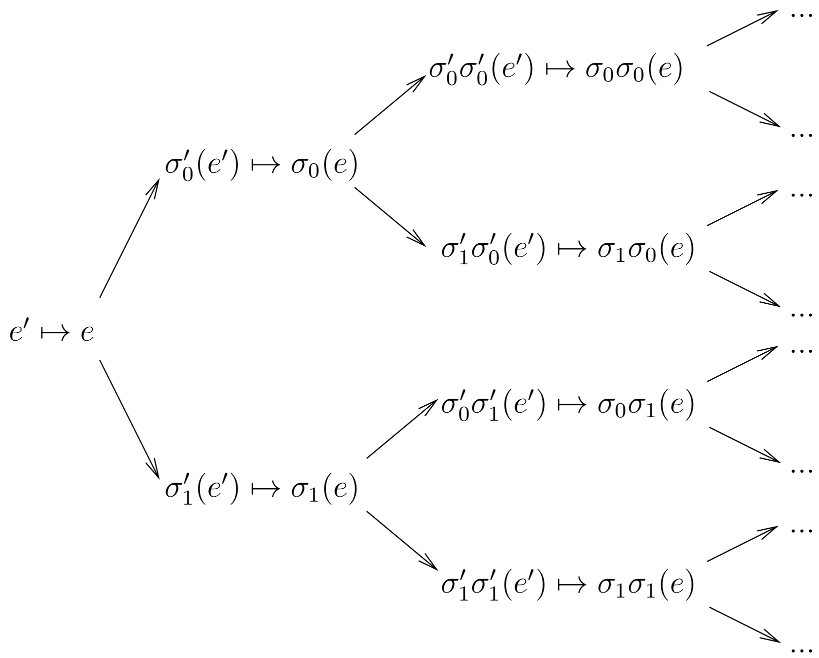


Figure 2.4.: Application of Belyi permutations

Remark 2.3.3. Some properties of the map μ from Theorem 2.3.1: The map is always surjective. All fibers are of same cardinality $\frac{n}{n'}$. Above each cycle of σ_i' lie cycles whose lengths are multiples of the length of the cycle below and they add up to $\frac{n}{n'}$ times the length of the cycle below.

As mentioned in the proof of Theorem 2.3.1, we get the following commutative diagram

$$\begin{array}{ccc} \{1,\dots,n\} & \xrightarrow{w} & \{1,\dots,n\} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ \{1,\dots,n'\} & \xrightarrow{w'} & \{1,\dots,n'\}, \end{array}$$

where w is a word in σ_0 and σ_1 and w' the associated word in σ_0' and σ_1'.

Construction of Subgroups: With Theorem 2.3.1 and the observations in Remark 2.3.3 it is not difficult to construct subgroups for a group out of the couple of permutations σ_0 and σ_1.

Algorithm 2.3.4. *Let $\Gamma \subset \Gamma(2)$ be given via $\sigma_0, \sigma_1 \in S_n$. Then with some choices described below one can construct the Belyi permutations for a subgroup as follows:*

(i) Preparatory choices:

  a) Decide on an index m.

b) *For each cycle* c_{i_j} *in* σ_i *(*$i \in \{0,1\}$*) pick a permutation* $\sigma_{i_j} \in S_m$*. The group generated by all the* σ_{i_j} *must act transitively.*

c) *Mark one position in each cycle* c_{i_j} *in* σ_i*.*

(ii) *For each cycle do the following steps:*

a) *Open the cycle* c_{i_j} *in front of the marked position. We get a sequence* s_{i_j}*.*

b) *Copy* s_{i_j} *via adding n m times such that out of a sequence* $s_{i_j} = (a_1, a_2, \dots, a_r)$ *comes the sequences* $(a_1, a_2, \dots, a_r), (n+a_1, n+a_2, \dots, n+a_r), \dots, ((m-1)n+ a_1, (m-1)n + a_2, \dots, (m-1)n + a_r)$*.*

c) *Merge the sequences to cycles according to the permutation* σ_{i_j}*. In* σ_{i_j} *a cycle* $(b_1, b_2, \dots, b_s)$ *stand for the merging of* $((b_1-1)n + a_1, (b_1-1)n + a_2, \dots, (b_1-1)n + a_r), ((b_2-1)n + a_1, (b_2-1)n + a_2, \dots, (b_2-1)n + a_r), \dots$ *until* $((b_s-1)n + a_1, (b_s-1)n + a_2, \dots, (b_s-1)n + a_r)$ *in this order to a cycle. The composition of these cycles is a permutation* $\sigma'_{i_j} \in S_{mn}$*.*

(iii) *For* $i \in \{0,1\}$ *the new Belyi permutation is*

$$\sigma'_i := \prod_{c_{i_j} \text{ in } \sigma_j} \sigma'_{i_j}.$$

Remark 2.3.5. It is not certain that the Algorithm 2.3.4 yields all possible subgroups of a given group $\Gamma \subset \Gamma(2)$.

Proposition 2.3.6. *Let* $\sigma_0, \sigma_1 \in S_n$ *be such that* $\langle \sigma_0, \sigma_1 \rangle$ *acts transitively. Then it holds: The permutations* $\sigma'_0, \sigma'_1 \in S_{mn}$ *constructed with Algorithm 2.3.4 act transitively and if* Γ *is defined via* σ_0 *and* σ_1 *by fixing the edge e, then the group* Γ' *defined with* σ'_0, σ'_1 *and fixed edge* $e + kn$ *(for a k in* $\{0, 1, \dots, m-1\}$*) is a subgroup of* Γ*.*

Proof: It is easy to realize that all cycles that occur in the construction are disjoint, therefore, the permutations are well defined.
We still have to show that the obtained permutations act transitively and that for every word w' in σ'_0 and σ'_1 that fixes $e + kn$ the corresponding word w in σ_0 and σ_1 fixes e.

Transitivity: Let $G' = \langle \sigma'_0, \sigma'_1 \rangle$ be the subgroup of S_{mn} generated by the two permutations. We will show the claim in two steps. At first, we assume that for every $k \in \{0, \dots m-1\}$ there is a $g_k \in G'$ with $g_k(1) = kn + 1$, and show that then G' acts transitively. Secondly, we will explain, why such elements g_k exist.
Since $G = \langle \sigma_0, \sigma_1 \rangle$ acts transitively, for $l \in \{1, \dots n\}$ there is a word w_l in σ_0 and σ_1 with $w_l(1) = l$; to w_l exists a corresponding word w'_l (replace σ_i^ϵ by $(\sigma'_i)^\epsilon$ for $i \in \{0,1\}$, $\epsilon \in \{-1, 1\}$). Let $w'_l = \prod_{j=1}^r \kappa_j^{r_j}$, where $r_j \in \mathbb{Z}$, $\kappa_j \in \{\sigma'_0, \sigma'_1\}$. We can adjust w'_l such that the sequence $s_0 := 1, s_1 := \kappa_r^{r_r}(s_0), \dots, s_r := \kappa_1^{r_1}(s_{r-1})$ stays in $\{1, \dots n\}$. If $s_{j-1} \in \{1, \dots n\}$ but $s_j \notin \{1, \dots n\}$, then at least one of the two alternatives $\kappa_j^{r'_j}$ or $\kappa_j^{-r'_j}$ (with $r'_j \equiv r_j \mod a$, where a is the length of the cycle of κ_j in which s_{j-1} lies) will force the sequence to stay in $\{1, \dots, n\}$ and will map s_{j-1} to $\kappa_j^{r_j}(s_{j-1})$.

If we apply the modified word not on 1 but on $nk+1$ (for a $k \in \{0, \dots, m-1\}$), the sustained sequence is in $\{nk+1, \dots n(k+1)\}$ and $w'_l(nk+1) = nk+l$.
Hence, the modified words w'_l composed with g_k yield elements with $w'_l \circ g_k(1) = nk+l$. That proves transitivity.
The elements $g_k \in G'$ exist because of the transitivity of the group generated by the permutations σ_{i_j}. This transitivity means that we get a word v in the σ_{i_j} connecting the first n elements with the k-th n elements $(k-1)n+1, \dots, kn$. To modify v to obtain a word in σ'_0 and σ'_1 that actually connects 1 with $(k-1)n+1$ is still some work. Let $v = \prod_{l=1}^{r} \tau_l^{r_l}$, where τ_l is out of $\{\sigma_{i_j}\}$, $r_l \in \mathbb{Z}$. From v we derive $v' = \nu_0 \prod_{l=1}^{r} (\tau'_l)^{hr_l} \nu_l$. Here τ'_l is σ'_i if $\tau_l = \sigma_{i_j}$ and h is the length of the j-th cycle in the corresponding σ_i. The ν_l are permutations as constructed before, permutations that only move in one copy of the n elements. ν_l maps the current image of 1 to the element in the same copy of n that is equivalent ($\mod n$) to the marked spot in the j-th cycle of σ_i where $\tau_l = \sigma_{i_j}$; ν_0 maps the current image of 1 to the 1 in the current copy of n. The permutation v' has the property $v'(1) = (k-1)n+1$, hence, it can be taken for g_k.

Subgroup: The map $\mu : \{1, \dots, mn\} \longrightarrow \{1, \dots, n\}$ is given by $\mu(a) \equiv a \mod n$. Then holds $\mu^{-1}(e) = \{e, n+e, \dots, (k-1)n+e\}$. We have to show that $\mu(\sigma'(a)) = \sigma(\mu(a))$ $(\forall a \in \{1, \dots, mn\})$. But from the construction of multiplying cycles by adding the kn to every element of the cycle and merging these cycles by opening them at the same position, the required commutativity follows directly. □

Example 2.3.7. *Take permutations from* S_5

$$\sigma_0 = (145)(23), \quad \sigma_1 = (1234).$$

Both permutations consist of two cycle (when we count fixed points as well). Let

$$c_{0_1} = (1\underline{4}5),\ c_{0_2} = (\underline{2}3),\ c_{1_1} = (\underline{1}234),\ c_{1_2} = (\underline{5}),$$

where the underlined element is marked. For an index 3 subgroup we may choose the following permutations (which act transitively)

$$\sigma_{0_1} = (132),\ \sigma_{0_2} = (12),\ \sigma_{1_1} = id,\ \sigma_{1_2} = (23).$$

The resulting permutations will be in S_{15}*.*
For σ'_0 *we get: Coming from* c_{0_1} *we obtain* $(4,5,1,14,15,11,9,10,6)$ *via opening the cycle in front of* 4 *tripling the sequence by adding* 5 *and* 10*, respectively, to get* $4,5,1$*;* $9,10,6$ *and* $14,15,11$ *and merging them to one cycle in the order* $1,3,2$*, i.e. at first comes the sequence with numbers between* 1 *and* 5*, then the one with numbers in between* 11 *and* 15 *and finally the sequence for* 6 *to* 10*. From* c_{0_2} *comes* $(2,3,7,8)(12,13)$*. Here we get two cycles, since* c_{0_2} *consists of two. Hence, the permutation is*

$$\sigma'_0 = (1,14,15,11,9,10,6,4,5)(2,3,7,8)(12,13).$$

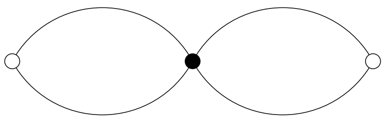
Similarly, we obtain

$$\sigma_1' = (1,2,3,4)(6,7,8,9)(10,15)(11,12,13,14).$$

Example 2.3.8. *We start with the two permutations*

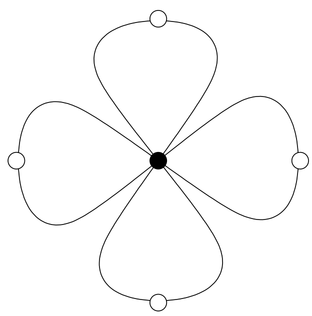
$$\sigma_0 = (12)(34) \quad \textit{and } \sigma_1 = (1234).$$

The associated Dessin is

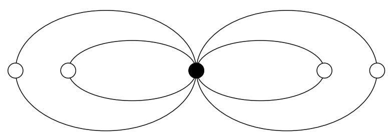


Below, we present 4 *different subgroups of* $\langle \sigma_0, \sigma_1 \rangle$ *of index two with their Dessins. We have to choose permutations from* S_2 *and mark positions for the three cycle* $c_{0_1} = (12)$, $c_{0_2} = (34)$ *and* $c_{1_1} = (1234)$. *In the following table, the first permutation and first marked position will always be the ones for* c_{0_1}, *the second count for* c_{0_2} *and the third ones for* c_{1_1}.

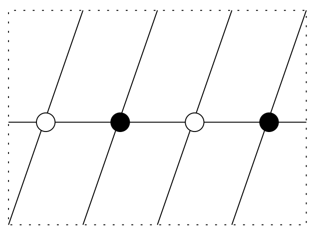
The permutations $id, (12), (12)$ *with marked elements* $1, 3$ *and* 1 *lead to*
$\sigma_0' = (12)(34)(56)(78)$ *and*
$\sigma_1' = (12345678)$.



The permutations $id, (12), (12)$ *with marked elements* $1, 3$ *and* 2 *lead to*
$\sigma_0' = (12)(34)(56)(78)$ *and*
$\sigma_1' = (16785234)$.



The permutations $(12), (12), id$ *with marked elements* $1, 3$ *and* 1 *lead to*
$\sigma_0' = (1256)(3478)$ *and*
$\sigma_1' = (1234)(5678)$.



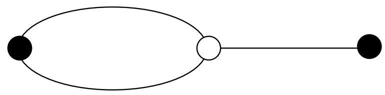
The permutations $(12), (12), (12)$ *with marked elements* $1, 3$ *and* 1 *lead to* $\sigma'_0 = (1256)(3478)$ *and* $\sigma'_1 = (12345678)$.

The associated Dessin has genus 2; it is not helpful to see it drawn.

Example 2.3.9. *This time we look at subgroups of index 3. We start with the two permutations*

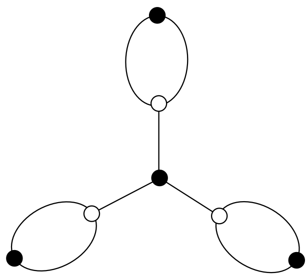
$$\sigma_0 = (123) \quad \textit{and } \sigma_1 = (12).$$

The associated Dessin is

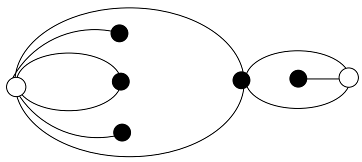


Below, we present 3 genus 0 subgroups of $\langle \sigma_0, \sigma_1 \rangle$ *with their Dessins. We have to choose permutations from* S_3 *and mark positions for the three cycle* $c_{0_1} = (123)$, $c_{1_1} = (12)$ *and* $c_{1_2} = (3)$. *In the following table, the first permutation and first marked position will always be the ones for* c_{0_1}, *the second count for* c_{1_1} *and the third ones for* c_{1_2}.

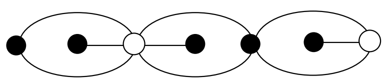
The permutations $id, id, (123)$ *with marked elements* $1, 1$ *and* 3 *lead to* $\sigma'_0 = (123)(456)(789)$ *and* $\sigma'_1 = (12)(369)(45)(78)$.



The permutations $(12), (23), id$ *with marked elements* $1, 1$ *and* 3 *lead to* $\sigma'_0 = (123456)(789)$ *and* $\sigma'_1 = (12)(4578)$.



The permutations $(12), (23), id$ *with marked elements* $2, 1$ *and* 3 *lead to* $\sigma'_0 = (156423)(789)$ *and* $\sigma'_1 = (12)(4578)$.



Other subgroups are defined by

$$\sigma_0' = (123456)(789) \quad and \quad \sigma_1' = (124578)(36),$$
$$\sigma_0' = (123456)(789) \quad and \quad \sigma_1' = (124578)(369),$$
$$\sigma_0' = (156489723) \quad and \quad \sigma_1' = (124578)(396).$$

These three pairs define subgroups of genus 1, 2, 3, *respectively.*

In the Dessins of genus 0 in the examples, one can see quite well that the Dessins for the subgroups arise from the Dessin of the supergroup by gluing several together.

2.4. Detect congruence and non-congruence subgroups

When we combine the results of the last two sections 2.2 and 2.3 we can find an algorithm to decide for a subgroup $\Gamma \subset \Gamma(2)$ if it is a congruence subgroup.

In Definition 1.1.5 a congruence subgroup has been defined as a subgroup which contains $\Gamma(N)$ for a N. A theorem of K. Wohlfahrt reduces the test for a group to be a congruence subgroup to a single N that has to be tested.

Theorem 2.4.1. *Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup. We define N to be the least common multiple of the widths of all cusps of Γ. Then Γ is a congruence subgroup if and only if $\Gamma(N) \subset \Gamma$.*

Proof: See [Woh64]. □

Therefore, it is easy to check if a subgroup given by permutations is a congruence subgroup.

Algorithm 2.4.2. *Let $\Gamma \subset \Gamma(2)$ be a finite index subgroup given by Belyi permutations σ_0 and σ_1 and e an edge that has to be fixed. Then the following steps answer the question for Γ to be a congruence subgroup.*

(i) Find out for which N it has to be checked if $\Gamma(N) \subset \Gamma$.

(ii) Get the Belyi permutations σ_0', σ_1' for $\Gamma(N)$.

(iii) Check, if $\Gamma(N) \subset \Gamma$.

In this form, the Algorithm 2.4.2 is quite trivial. The interesting part is that with the work from Sections 2.2 and 2.3 we can perform all three steps.

Details on (i)**:** The number N is two times the least common multiple of the orders of σ_0, σ_1 and σ_∞. Therefore we calculate $\sigma_\infty = \sigma_1^{-1}\sigma_0^{-1}$ and then N is given by $N = 2 \cdot lcm\,(ord(\sigma_0)ord(\sigma_1)ord(\sigma_\infty))$.

Details on (ii)**:** Here we use the results of Section 2.2. The answer is given in Theorem 2.2.4, where we find

$$\sigma_0 = (1, 2, \ldots, N/2)(N/2+1, N/2+2, \ldots, N) \ldots (I - N/2, I - N/2 + 1, \ldots, I)$$
$$\sigma_1 = P,$$

where
$$I = \frac{1}{12} N^3 \prod_{p|N} \left(1 - p^{-2}\right)$$

and P is the result of the Maple algorithm from Appendix A.1 run for N.

Details on (iii)**:** This is possible with Section 2.3, where we discussed how to detect subgroup via the Belyi permutations. In particular, we will use Theorem 2.3.1, where the relation $\Gamma(N) \subset \Gamma$ is translated into the existence of a map μ between the underlying sets of edges, that is compatible with the Belyi permutations.
We may choose the edge 1 to be fixed for $\Gamma(N)$, since all stabilizer are trivial. We have to find images for all edges a of $\Gamma(N)$ and check the property $\mu(\sigma_i(a)) = \sigma_i'(\mu(a))$ $(i \in \{0, 1\})$ for them.
Following the construction in the Remark 2.3.2 we start by

$$\mu : 1 \mapsto e.$$

From now on, we always take the edge with the smallest number of $\Gamma(N)$ for which we already have an image but where we did not apply the Belyi permutations on and apply them. Thus, the next step is

$$\mu : \sigma_0'(1) \mapsto \sigma_0(e) \quad \text{and} \quad \sigma_1'(1) \mapsto \sigma_1(e).$$

Continuing like this, we define and check the map μ simultaneously: If for an edge a, $\sigma_i'(a)$ $(i \in \{0, 1\})$ already has an image, then we check the compatibility, i.e. we check if the new image coincides with the one already given. Otherwise we define $\mu(\sigma_i'(a)) := \sigma_i(\mu(a))$.
In the case that the procedure does not give a contradiction in the images for all edges, we have successfully defined μ and Γ is a congruence subgroup. Elsewise, a contradiction shows that Γ is non-congruence.

An implementation of the Algorithm 2.4.2 (except the calculation of σ_1' from step (ii)) for Maple can be found in the Appendix A.2. Together with the algorithm in A.1 they form a full test for a subgroup given by Belyi permutations and an edge to be a congruence subgroup.

Remark 2.4.3. For a non-congruence subgroup Γ, an expanded version of Algorithm 2.4.2 can give an element $\gamma \in \Gamma(N)$ with $\gamma \notin \Gamma$, i.e. counterexample for Γ being congruence. Such examples arise from the contradictions that Algorithm 2.4.2 will give. If one keeps track of the Belyi permutations that have to be applied to get the images of the edges, a contradiction in the algorithm are two words in the Belyi permutation with

$$w_{\Gamma(N)} = \prod_{j=1}^{r} \kappa'_j(e') = w'_{\Gamma(N)} = \prod_{j=1}^{r} \lambda'_j(e')$$

but

$$w_{\Gamma} = \prod_{j=1}^{r} \kappa_j(e) \neq w'_{\Gamma} = \prod_{j=1}^{r} \lambda_j(e),$$

where $\kappa'_j, \lambda'_j \in \{\sigma'_0, \sigma'_1\}$ and $\kappa_j, \lambda_j \in \{\sigma_0, \sigma_1\}$ are the corresponding Belyi permutations for Γ. Therefore $(W')^{-1} \circ W \in \Gamma(N)$ but $(W')^{-1} \circ W \notin \Gamma$, with $W \in \Gamma(2)$ and $W' \in \Gamma(2)$ are the matrices given by replacing the Belyi permutations in the words by the corresponding generators of $\Gamma(2)$, i.e. $\sigma_0 \rightsquigarrow \gamma_0$ and $\sigma_1 \rightsquigarrow \gamma_1$.

Remark 2.4.4. Algorithm 2.4.2 only works for subgroups of $\Gamma(2)$. It is possible to express subgroups of $\Gamma(1)$ with permutations, therefore a modified version of Algorithm 2.4.2 will work for general subgroups of $\Gamma(1)$.

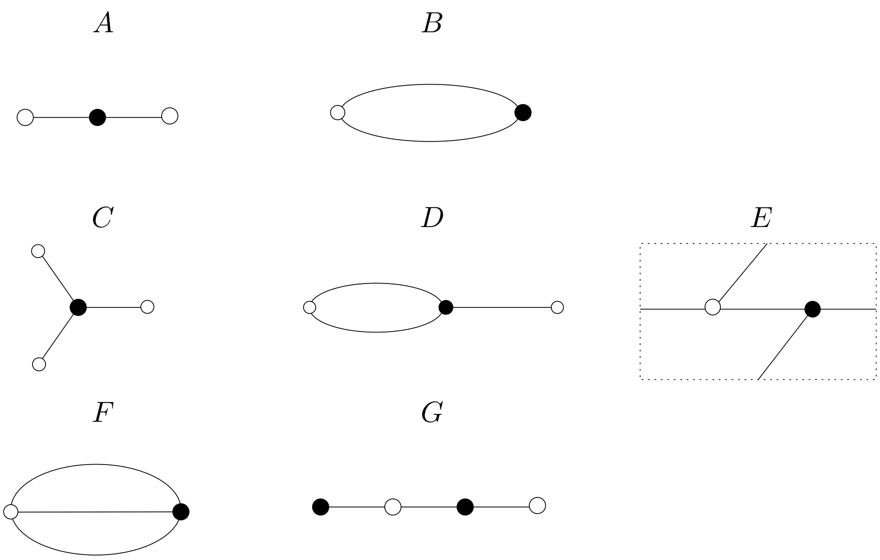


Figure 2.5.: All Dessins with two or three edges

We will use the algorithm to check for all subgroups of $\Gamma(2)$ of index two and three if they are congruence subgroups. We can use Dessins d'Enfants to find all such subgroups. A Dessin with n edges determines a conjugation class of subgroups of index n in $\Gamma(2)$. Since $\Gamma(N) \lhd \Gamma(1)$, either all subgroups in one conjugation class are congruence subgroups or none of them: If $\Gamma(N) \subset \Gamma \subset \Gamma(2)$, then $\Gamma(N) = \lambda^{-1}\Gamma(N)\lambda \subset \lambda^{-1}\Gamma\lambda$ for $\lambda \in \Gamma(2)$. In Figure 2.5 all Dessins (up to coloring) with two or three edges can be seen.

In principal we have to consider for all seven Dessins as well the one were the colors are exchanged, but only in the Dessins *A*, *C* and *D* from Figure 2.5 an exchange of the colors changes the permutations.

In Table 2.3 all subgroups are listed. The first column indicates the Dessin the group is coming from, the second and third the Belyi permutations and in the last column we find the result of the algorithm. A "+" in the last column means that the group defined with help of the two permutation σ_0 and σ_1 is a congruence subgroup. A "−" indicates that it is a non-congruence subgroup. We notice that all three subgroups of index two are congruence but out of the seven subgroups of index three only one is.

Dessin	σ_0	σ_1	congruence?
A	(12)	*id*	+
A	*id*	(12)	+
B	(12)	(12)	+
C	(123)	*id*	−
C	*id*	(123)	−
D	(123)	(12)	−
D	(12)	(123)	−
E	(123)	(123)	+
F	(123)	(132)	−
D	(12)	(13)	−

Table 2.3.: Subgroups of $\Gamma(2)$ of index two and three

3. Eisenstein series and scattering constants

Eisenstein series are real analytic functions which play a central role in spectral theory of the hyperbolic Laplacian. The Eisenstein series here are called non-holomorphic Eisenstein series, because they have a pole. They are created out of subgroups of the modular group by summing up over all elements modulo the stabilizer of a cusp. Hence, in these series the information of the subgroup is encrypted.

In this chapter, we will introduce Eisenstein series, the scattering matrix and scattering constants. We will see that for groups $\Gamma \subset \Gamma'$ the Eisenstein series, scattering constants of Γ' are a weighted sum of some Eisenstein series, scattering constants of Γ, respectively.

In the last section 3.3, we connect scattering constants with the results of Section 2.1 where we treated automorphisms of Belyi pairs and subgroups. We see that automorphisms identifying cusps yield identities of scattering constants.

3.1. Eisenstein series

In this section we will define non-holomorphic Eisenstein series for subgroups of the modular group following [Kub73]. We will state some properties, discuss the Fourier expansion and prove sum relations for the series and their expansions.

For subgroups of the modular group we can define Eisenstein series and other structures coming from them.

Definition 3.1.1. *Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup. Let S_j be a cusp of width b_j with stabilizer Γ_j. We define*

$$\sigma_j := \gamma_j \cdot \begin{pmatrix} \sqrt{b_j} & 0 \\ 0 & 1/\sqrt{b_j} \end{pmatrix} \quad \text{such that} \quad \sigma_j^{-1}\Gamma_j\sigma_j = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle . \tag{3.1.1.1}$$

For each cusp S_j there is a non-holomorphic Eisenstein series $E_j^{\Gamma}(z,s)$, which for $z \in \mathbb{H}$, $s \in \mathbb{C}$ and $\operatorname{Re}(s) > 1$ is defined by the convergent series

$$E_j^{\Gamma}(z,s) = \sum_{\sigma \in \Gamma_j \backslash \Gamma} \operatorname{Im}\left(\sigma_j^{-1}\sigma(z)\right)^s = b_j^{-s} \sum_{\sigma \in \Gamma_j \backslash \Gamma} \operatorname{Im}\left(\gamma_j^{-1}\sigma(z)\right)^s . \tag{3.1.1.2}$$

We state some properties of Eisenstein series.

Proposition 3.1.2. *The function $E_j^{\Gamma}(z,s)$ has a meromorphic continuation to the whole s-plane, with a simple pole in $s = 1$ with residue $3/(\pi \cdot [\Gamma(1) : \Gamma])$.*

Eisenstein series are automorphic forms: For all $\gamma \in \Gamma$ we have $E_j^\Gamma(\gamma(z), s) = E_j^\Gamma(z, s)$. They are eigenforms for the hyperbolic Laplacian Δ:

$$\Delta E_j^\Gamma(\gamma(z), s) = s(s-1)E_j^\Gamma(\gamma(z), s).$$

Proof: See [Kub73] and [Iwa02]. □

For further examinations we need some more information on subgroups of $\Gamma(1)$, i.e. the notion of subcusps and a double coset decomposition of the subgroups.

Definition 3.1.3. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be two subgroups, S_k a cusp of Γ' and $\{S_j\}_{j\in J_k}$ a subset of the cusps of Γ such that $\bigcup_{j\in J_k} S_j = S_k$. Then the S_j ($j \in J_k$) are called the subcusps of S_k with respect to Γ'.*

Remark 3.1.4. If $\Gamma \subset \Gamma(1)$ is of finite index, then there are only finitely many cusps. Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be two finite index subgroups and S_k a cusp of Γ' and $\{S_j\}_{j\in J_k}$ the subcusps of S_k in Γ. For the cusps widths we have

$$\sum_{\{S_j\}_{j\in J_k}} b_j = b_k[\Gamma' : \Gamma].$$

There is a decomposition into double cosets:

Lemma 3.1.5. *Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup, S_j and S_k two cusps of Γ. Denote by Γ_j and Γ_k their stabilizers and by b_j, b_k their widths, respectively. Set*

$$B_j := \gamma_j^{-1}\Gamma_j\gamma_j = \left\langle \left(\begin{smallmatrix} 1 & b_j \\ 0 & 1 \end{smallmatrix}\right) \right\rangle \qquad \text{and}$$
$$B_k := \gamma_k^{-1}\Gamma_k\gamma_k = \left\langle \left(\begin{smallmatrix} 1 & b_k \\ 0 & 1 \end{smallmatrix}\right) \right\rangle,$$

with $\gamma_j, \gamma_k \in \Gamma(1)$ such that $\gamma_j^{-1}(S_j) = \gamma_k^{-1}(S_k) = \infty$.
Then it holds

$$\gamma_j^{-1}\Gamma\gamma_k = \delta_{jk}B_j \cup \bigcup_{c\geq 0}\ \bigcup_{d \, mod \, b_k c} B_j\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)B_k, \tag{3.1.5.1}$$

where the sum is taken over all $c \geq 0$ and $d \mod b_k c$ such that there exists a matrix $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_k$.

Proof: In [Iwa97] is a proof for $B_j := \sigma_j^{-1}\Gamma_j\sigma_j$ using σ_j from Equation (3.1.1.1) instead of using γ_j and $B_j := \gamma_j^{-1}\Gamma_j\gamma_j$, there are some slight adjustments to do for γ_j. That proof can be found in [Pos07]. □

Remark 3.1.6. For each class in the double coset decomposition (Equation (3.1.5.1)) there is exactly one representative $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $0 \leq a < b_j c$ and $0 \leq d < b_k c$. The entries

a and d determine each other uniquely and we get the same decomposition by

$$\gamma_j^{-1}\Gamma\gamma_k = \delta_{jk}B_j \cup \bigcup_{c\geq 0} \bigcup_{a\, mod\, b_j c} B_j\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right)B_k,$$

where the sum is taken over all $c \geq 0$ and $a \mod b_j c$ such that there exists a matrix $\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_k$.

With this knowledge, we continue with properties of Eisenstein series.

Proposition 3.1.7. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups. Let S_k be a cusp of Γ' and $\{S_j\}_{j\in J_k}$ the subcusps of S_k in Γ. The widths will be denoted be b_k and b_j, respectively. Then we have the following relation for Eisenstein series*

$$\sum_{j\in J_k} b_j^s E_j^{\Gamma}(z,s) = b_k^s E_k^{\Gamma'}(z,s). \tag{3.1.7.1}$$

Proof: See [Pos07]. □

Remark 3.1.8. For a finite index subgroup $\Gamma \subset \Gamma(1)$ the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^{b_\infty} \in \Gamma$, where b_∞ is the cusps width in ∞, therefore E_j^{Γ} is b_∞-periodic, i.e. $E_j^{\Gamma}(z+b_\infty, s) = E_\infty^{\Gamma}(z,s)$. Hence, the Eisenstein series admits a Fourier expansion of the form

$$E_j^{\Gamma}(z,s) = \sum_{m\in\mathbb{Z}} a_m(s,y)e^{2\pi i m x/b_\infty}.$$

By mapping another cusps S_k to ∞ via $\gamma_k \in \Gamma(1)$ we get

$$E_j^{\Gamma}(\gamma_k z,s) = \sum_{m\in\mathbb{Z}} a_m(s,y)e^{2\pi i m x/b_k}. \tag{3.1.8.1}$$

This Fourier expansion is called the natural Fourier expansion of E_j^{Γ} in the cusp S_k.

In most of the literature, it is common to normalize this expansion by taking $\sigma_k \in PSL_2(\mathbb{R})$ as in Equation (3.1.1.1) to get a function that is 1-periodic and the expansion has the form

$$E_j^{\Gamma}(\sigma_k z,s) = \sum_{m\in\mathbb{Z}} a_m(s,y)e^{2\pi i m x}. \tag{3.1.8.2}$$

This Fourier expansion is called the normalized Fourier expansion.

Most of the time in this thesis we will work with the natural expansion and simply call it Fourier expansion.

To simplify the notation in the next proposition, we give

Definition 3.1.9. *Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup with cusps S_j and S_k. We set:*

$$r^{\Gamma}_{jk}(c) := \# \left\{ d \mod b_k c \,|\, \exists \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_k \right\} \tag{3.1.9.1}$$

and

$$r^{\Gamma}_{jk,m}(c) := \sum_{\substack{d \bmod b_k c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_k}} e^{2\pi i m d/(c b_k)}. \tag{3.1.9.2}$$

These numbers occur in the Fourier expansion of Eisenstein series.

Proposition 3.1.10. *Eisenstein series admit a Fourier expansion. The natural Fourier expansion of $E^{\Gamma}_j(z,s)$ at the cusp S_k is given by*

$$\begin{aligned} E^{\Gamma}_j(\gamma_k(z), s) &= \delta_{jk}\frac{y^s}{b_j^s} + \frac{1}{b_j^s b_k}\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\left(\sum_{c>0}\frac{1}{c^{2s}} r^{\Gamma}_{jk}(c)\right) y^{1-s} \\ &+ \sum_{m\neq 0}\frac{1}{b_j^s b_k}\left(\sum_{c>0}\frac{1}{c^{2s}} r^{\Gamma}_{jk,m}(c)\right) 2\pi^s\left|\frac{m}{b_k}\right|^{s-1/2}\Gamma(s)^{-1} y^{1/2} K_{s-1/2}(2\pi|m|y/b_k) e^{2\pi i m x/b_k}, \end{aligned} \tag{3.1.10.1}$$

where $z = x + iy$, $\Gamma(\cdot)$ is the Gamma function and $K_*(\cdot)$ the modified Bessel function.

Proof: We will follow the proofs from [Kub73] and [vP05], slightly changed to adjust to the non-normalized situation. We define

$$\tilde{E}_j(z,s) = b_j^s E_j(z,s)$$

and calculate the Fourier expansion for

$$\tilde{E}_j(z,s) = \sum_{\gamma \in \Gamma_j \backslash \Gamma} \operatorname{Im}(\gamma_j^{-1}\gamma(z))^s. \tag{3.1.10.2}$$

With help of the decomposition into double cosets (Lemma 3.1.5) we get

$$\begin{aligned} \tilde{E}_j(\gamma_k(z), s) &= \sum_{\gamma \in \Gamma_j \backslash \Gamma} \operatorname{Im}(\gamma_j^{-1}\gamma\gamma_k(z))^s \\ &= \sum_{\gamma' \in B_j \backslash \gamma_j^{-1}\Gamma\gamma_k} \operatorname{Im}(\gamma'(z))^s \\ &= \delta_{jk}\operatorname{Im}(z)^s + \sum_{c>0}\ \sum_{\substack{d \bmod b_k c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_k}}\ \sum_{m\in\mathbb{Z}} \frac{\operatorname{Im}(z + b_k m)^s}{|c(z + b_k m) + d|^{2s}}. \end{aligned} \tag{3.1.10.3}$$

For the last sum we can use the Poisson summation formula

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} f(t)e^{-2\pi imt}dt$$

and get $(z = x + yi)$:

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{\mathrm{Im}(z+b_k m)^s}{|c(z+b_k m)+d|^{2s}} &= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{\mathrm{Im}(z+b_k t)^s}{|c(z+b_k t)+d|^{2s}} e^{-2\pi imt}dt \\
&= y^s \frac{1}{c^{2s}} \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{1}{|z+b_k t+d/c|^{2s}} e^{-2\pi imt}dt \\
&\overset{(*)}{=} y^s \frac{1}{c^{2s}} \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{1}{|a+iy|^{2s}} e^{-2\pi im(a-x-d/c)/b_k} \frac{1}{b_k} da \\
&= y^s \frac{1}{c^{2s}} \frac{1}{b_k} \sum_{m\in\mathbb{Z}} e^{2\pi im(x-d/c)/b_k} \int_{-\infty}^{\infty} \frac{1}{(a^2+y^2)^s} e^{-2\pi ima/b_k} da,
\end{aligned} \tag{3.1.10.4}$$

where $x + b_k t + d/c$ was substituted with a in $(*)$.

The integrals can be calculated (for the formulas see e.g. [AS64]) and we obtain in the case $m = 0$

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+y^2)^s} da = y^{1-2s}\pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \tag{3.1.10.5}$$

and in the case $m \neq 0$

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+y^2)^s} e^{-2\pi ima/b_k} da = y^{-s+1/2} 2\pi^s |m/b_k|^{s-1/2} \Gamma(s)^{-1} K_{s-1/2}(2\pi|m|y/b_k). \tag{3.1.10.6}$$

If we insert the results (3.1.10.4), (3.1.10.5) and (3.1.10.6) in (3.1.10.3) the proposition follows. □

Remark 3.1.11. The normalized Fourier expansion can be obtained by replacing z with $b_k z$ in Equation (3.1.10.1).

The numbers r_*^Γ (defined in Definition 3.1.9) are the unknown in the Fourier expansion we calculated in Proposition 3.1.10. From the sum relation of the Eisenstein series in Formula (3.1.7.1) follow relations for the r_*^Γ:

Corollary 3.1.12. *Let* $\Gamma \subset \Gamma' \subset \Gamma(1)$ *be finite index subgroups,* S_j *a cusp of* Γ *that we will interpret as one of* Γ' *as well,* S_k *a cusp of* Γ' *and* $\{S_i\}_{i\in I_j}$ *the subcusps of* Γ *such that* $\cup_{i\in I_j} S_i = S_j$*. By* b_* *we denote the widths in* Γ *and with* w_* *the ones in* Γ'*. Then it* *holds*

$$\frac{1}{b_k}\sum_{S_i\subset S_j} r^{\Gamma}_{ik}(c) = \frac{1}{w_k} r^{\Gamma'}_{jk}(c) \qquad (3.1.12.1)$$

and

$$\sum_{S_i\subset S_j} r^{\Gamma}_{ik,m}(c) = \begin{cases} \frac{b_k}{w_k} r^{\Gamma'}_{jk,\frac{w_k}{b_k}m}(c) & \text{if } \frac{b_k}{w_k}|m \\ 0 & \text{elsewise.} \end{cases} \qquad (3.1.12.2)$$

Proof: Follows from Proposition 3.1.7. □

Remark 3.1.13. The number $r^{\Gamma}_{jk}(c)$ is symmetric such that $r^{\Gamma}_{jk}(c) = r^{\Gamma}_{kj}(c)$ (see e.g. [Kub73]). Hence, a formula, similar to (3.1.12.1), where we sum over subcusps of S_k, holds:

$$\frac{1}{b_j}\sum_{S_i\subset S_k} r^{\Gamma}_{ji}(c) = \frac{1}{w_j} r^{\Gamma'}_{jk}(c). \qquad (3.1.13.1)$$

For sums of $r^{\Gamma}_{jk,m}(c)$ there is not such an easy formula that holds generally. But we still have the following

Proposition 3.1.14. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups, S_j a cusp of Γ that we will interpret as one of Γ' as well, S_k a cusp of Γ' and $\{S_i\}_{i\in I_k}$ the subcusps of Γ such that $\cup_{i\in I_k} S_i = S_k$. By b_* we denote the widths in Γ and with w_* the ones in Γ'. For $c \in \mathbb{N}$ and a non-zero integer m holds*

$$\sum_{i\in I_k} r^{\Gamma}_{ji,\frac{b_k}{w_k}m}(c) = \frac{b_j}{w_j} r^{\Gamma'}_{jk,m}(c). \qquad (3.1.14.1)$$

Proof: We have to show that

$$\sum_{i\in I_k} \sum_{\substack{d \mod b_i c \\ \exists\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\gamma_j^{-1}\Gamma\gamma_i}} e^{2\pi i\frac{md}{w_k c}} = \frac{b_j}{w_j} \sum_{\substack{d \mod w_k c \\ \exists\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\gamma_j^{-1}\Gamma'\gamma_k}} e^{2\pi i\frac{md}{w_k c}}. \qquad (3.1.14.2)$$

It is known from Formula (3.1.13.1) that the number of summands on the left hand side is $b := \frac{b_j}{w_j}$ times the number of summands on the right hand side:

$$\sum_{i\in I_k} r^{\Gamma}_{ji}(c) = b\cdot r^{\Gamma'}_{jk}(c).$$

For each couple (c, d) from the left hand side of Equation (3.1.14.2) we get the corresponding normalized representative in the double coset decomposition, see Remark

3.1.6, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_i$ with $0 \leq a < b_i c$. In these representing matrices each a occurs only once:
Assume $\left(\begin{smallmatrix} a & b' \\ c & d' \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_{i'}$ is such another matrix, then $z := \left(\begin{smallmatrix} a & b' \\ c & d' \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ in $\gamma_{i'}^{-1}\Gamma\gamma_i$ is an element in the stabilizer of ∞ and we must have $S_i = S_i'$. Hence, the a in the matrices are pairwise distinct. Let A be the set of a's.

For all S_i a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_i$ lies in $\gamma_j^{-1}\Gamma'\gamma_k$, because $S_i \sim_{\Gamma'} S_k$ (there exists $\gamma_{ik} \in \Gamma'$ with $\gamma_{ik}(S_k) = S_i$ such that $\gamma_i = \gamma_{ik}\gamma_k$ and $\gamma_j^{-1}\Gamma'\gamma_i \subset \gamma_j^{-1}\Gamma'\gamma_i = \gamma_j^{-1}\Gamma'\gamma_{ik}\gamma_k = \gamma_j^{-1}\Gamma'\gamma_k$). Seen as elements in $\gamma_j^{-1}\Gamma'\gamma_k$ we can reduce the entries in $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to the normalized representative in the double coset decomposition for $\gamma_j^{-1}\Gamma'\gamma_k$: $\left(\begin{smallmatrix} a' & * \\ c & d' \end{smallmatrix}\right)$ with $a' \equiv a \mod w_j c$ and $d' \equiv d \mod w_k c$. The entries a' and d' determine each other. In Equation (3.1.14.2) the value of d is only important up to multiples of $w_k c$, i.e. it is enough to calculate the d'.

Thus, we have $b \cdot r_{jk}^{\Gamma'}(c)$ distinct a's in the range 0 to $b_j c$ with the property that there is $\left(\begin{smallmatrix} a' & * \\ c & * \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma'\gamma_k$ where $a' \equiv a \mod w_j c$. Let A' be the set of all possible a':

$$A' = \left\{ a' \mid \exists \left(\begin{smallmatrix} a' & * \\ c & * \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma'\gamma_k, 0 \leq a' < w_j c \right\}.$$

Then $|A'| = r_{jk}^{\Gamma'}(c)$ and it is clear what numbers are in A:

$$A = \{a' + nw_j \mid a' \in A', 0 \leq n < b\}.$$

The reduction $\mod w_j c$ of b different a's will coincide. According to that the reduction $\mod w_k c$ of the corresponding d's will coincide and all possible reductions d' will occur b times. This proves the proposition. □

3.2. Scattering constants

With the help of the constant term of the Fourier expansion the scattering matrix and the scattering constants will be defined. For convenience we will define both in the normalized case; this agrees with the definition in [Kub73] and has the property that scattering matrix and constants are symmetric.

We will define the scattering matrix and constants, give an application and some sum relations.

Definition 3.2.1. *For $\Gamma \subset \Gamma(1)$ of finite index we define the scattering matrix (for the normalized Fourier expansion) to be*

$$\Phi_\Gamma(s) = \left(\pi^{1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \cdot \frac{1}{(b_j b_k)^s} \sum_{c>0} \frac{1}{c^{2s}} r_{jk}(c) \right)_{j,k},$$

where j and k run over all cusps of Γ.

For all pairs j,k we define the (normalized) scattering constant C^{Γ}_{jk} to be the constant term at 1 *of the Dirichlet series $(\Phi_\Gamma)_{jk}(s)$. All entries in the scattering matrix have the same residue $3/(\pi\cdot[\Gamma(1):\Gamma])$ and therefore we set*

$$C^{\Gamma}_{jk} := \lim_{s\to 1}\left(\Phi_\Gamma(s)_{j,k} - \frac{3/(\pi\cdot[\Gamma(1):\Gamma])}{s-1}\right). \tag{3.2.1.1}$$

Remark 3.2.2. If we take the natural Fourier expansion, as in Proposition 3.1.10, as basis to define the (natural) scattering matrix and the (natural) scattering constants, they change slightly:
We get $\frac{1}{b_j^s b_k}$ instead of $\frac{1}{(b_j b_k)^s}$ in the scattering matrix. The residue does not change but the scattering constants. Luckily, the difference is manageable and we have

$$\widetilde{C}^{\Gamma}_{jk} = C^{\Gamma}_{jk} + \frac{3\log(b_k)}{\pi\cdot[\Gamma(1):\Gamma]}, \tag{3.2.2.1}$$

where $\widetilde{C}^{\Gamma}_{jk}$ denotes the scattering constant coming from the constant term in the natural Fourier expansion.

Example 3.2.3. *To give an idea what scattering constants look like, we will give the one for $\Gamma(1)$ that we need later (see e.g. [Küh99]):*

$$C^{\Gamma(1)} := -\frac{6}{\pi}\left(12\zeta'(-1) + \log(4\pi) - 1\right) \tag{3.2.3.1}$$

Example 3.2.4. *Consider the Belyi pair $(E,\boldsymbol{\beta}_E)$ with*

$$E: \quad y^2 = x^3 + 5x + 10$$

and

$$\boldsymbol{\beta}_E: \quad (x,y) \longmapsto \frac{1}{32}\left(y(x-5)+16\right).$$

The curve is referred to as $400H1$ in J. Cremona's tables [Cre97], the rank of its Mordell-Weil group is 1*. The ramification points are all rational points of infinite order, therefore it follows from results of V. Drinfeld [Dri73] and J. Manin [Man72] that the subgroup Γ_E of $\Gamma(2)$ associated to it according to Theorem 1.1.8, is a non-congruence subgroup.*

The point $P=(1,4)$, which lies above 0*, is a generator of the Mordell-Weil group. The points $-P$ and the origin $\mathcal{O}$, the point in ∞, of $E(\mathbb{Q})$ are cusps of $(E,\boldsymbol{\beta}_E)$ as well.*

As scattering constants arose e.g.

$$C^{\Gamma_E}_{\mathcal{O},\mathcal{O}} = \frac{1}{30}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left(14\log(2) + 6\log(5)\right)\right)$$
$$C^{\Gamma_E}_{P,P} = \frac{1}{120}\left(4C^{\Gamma(1)} + \frac{1}{\pi}\left(-131\log(2) + 15\log(5) - 60\,\mathrm{ht}_{NT}(P)\right)\right)$$
$$C^{\Gamma_E}_{P,-P} = \frac{1}{120}\left(4C^{\Gamma(1)} + \frac{1}{\pi}\left(-71\log(2) + 60\,\mathrm{ht}_{NT}(P)\right)\right),$$

where $\mathrm{ht}_{NT}(P)$ *is the Néron-Tate height of the point* P*.*

This is a result from [BKP].
U. Kühn showed in [Küh99] how scattering constants occur in the calculation of Néron-Tate heights. Using his result, we obtained in [BKP] a description for scattering constants for the non-congruence subgroup Γ_E *in terms of* $C^{\Gamma(1)}$*, in logarithms of the places of bad reduction of the curve and Néron-Tate heights.*

Remark 3.2.5. In some special cases the scattering matrices are known. From the description of a scattering matrix one may derive scattering constants.

D. Hejhal [Hej83] and M. Huxley [Hux84] (independently) gave formulas for the scattering matrices of congruence subgroups. They both considered the groups $\Gamma(N), \Gamma_0(N)$ and $\Gamma_1(N)$. Since the scattering matrices of groups and subgroups are strongly related, one can obtain the scattering matrices for all congruence subgroups. Let $(\Gamma(N) \subset)\Gamma$ be a congruence subgroup. The information needed to get the scattering matrix for Γ out of the one for $\Gamma(N)$ is just the cusps, their widths and how they merge into cusps of $\Gamma(N)$. This problem was treated in details in [Kei06].

A.B. Venkov [Ven90] used a special case of this idea: Let $\Gamma \subset \Gamma'$ are subgroups having the same cusps, then the scattering matrices are essentially the same. In this case one can not only go from the subgroup to the supergroup but as well in the opposite direction.

When it is not possible to calculate scattering constants, it is still possible to get some ideas about them by calculating at least some (finitely many) coefficients of the defining series, see Formula (3.1.9.1) and Definition 3.2.1. An algorithm to do it has been presented in [Pos07], to use it, one needs a test for a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to be in the group in question.
An application of this idea can be found in Chapter 5.5.

Using the sum of Eisenstein series in Proposition 3.1.7 we can calculate the sum of scattering constants.

Proposition 3.2.6. *Let* $\Gamma \subset \Gamma' \subset \Gamma(1)$ *be finite index subgroups. Let* S_k^{Γ} *be a cusp of* Γ *that we will interpret as a cusp* $S_k^{\Gamma'}$ *for* Γ' *as well. Let* $S_j^{\Gamma'}$ *be a cusp of* Γ' *and* $\{S_i^{\Gamma}\}_{i\in I_j}$ *the subcusps of* $S_j^{\Gamma'}$ *in* Γ*. The widths will be denoted by* b_k *and* b_i *for* Γ *and by* w_k *and* w_j *for* Γ'*, respectively.*

Then the scattering constants fulfill

$$\sum_{i\in I_j} \frac{b_i}{w_j} C_{ik}^{\Gamma} = C_{jk}^{\Gamma'} - \frac{3}{\pi[\Gamma(1):\Gamma]} \sum_{i\in I} \frac{b_i}{w_j} \log\left(\frac{b_i b_k}{w_j w_k}\right). \tag{3.2.6.1}$$

Proof: The proof is a straight forward generalization of the one of Satz 5.39 from [Pos07] where the group $\Gamma(1)$ has to be replaced by Γ' and the cusps widths by the relative cusps widths $\frac{b_*}{w_*}$. □

In the case of the scattering constants to non-normalized expansions the formulas become easier. This is the first time that we see why one could prefer the non-normalized expansions.

Corollary 3.2.7. *With the notations from Proposition 3.2.6 we get for the (natural) scattering constant, the scattering constant we get from the natural Fourier expansion, as defined in Remark 3.2.2,*

$$\sum_{i\in I_j} \frac{b_i}{w_j} \widetilde{C}_{ik}^{\Gamma} = \widetilde{C}_{jk}^{\Gamma'} - \frac{3}{\pi[\Gamma(1):\Gamma]} \sum_{i\in I_j} \frac{b_i}{w_j} \log\left(\frac{b_i}{w_j}\right) \tag{3.2.7.1}$$

and

$$\sum_{i\in I_j} \frac{b_i}{w_j} \widetilde{C}_{ki}^{\Gamma} = \widetilde{C}_{kj}^{\Gamma'} - \frac{3}{\pi[\Gamma(1):\Gamma']} \log\left(\frac{b_k}{w_k}\right). \tag{3.2.7.2}$$

Here we get two different formulas, because the scattering constants $\widetilde{C}_{jk}$ are no longer symmetric. Actually, the second Formula (3.2.7.2) is the special case of Formula (3.2.7.1) in which the summands on the right hand side are all the same.

3.3. Application to automorphisms

In Chapter 2.1 we considered the action of certain automorphisms of Belyi pairs on the associated subgroup of the modular group. Now, we may derive identities of scattering constants out if it.

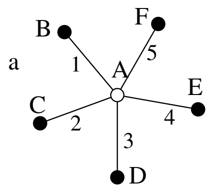
The calculation of some examples indicates that automorphisms and symmetries in Dessins imply identities of scattering constants. Beside the scattering constants for Fermat curves for which the results can be found in Chapter 5.5 the examples that follow on the next pages have been studied.

There, we do not start with a Belyi pair but with a Dessin where we can see that there is an automorphism on the associated subgroup as it has been constructed in Chapter 2.1. In our examples the existance of an automorphism is reflected in the Dessin by regularities and symmetries, i.e. we can turn or mirror the Dessin without really changing it. For the first example (see Example 2.1.4) and the three example of genus 1 an explicit description of the Belyi pairs is known. The Belyi pairs and most of the calculation concerning the example 5)-8) can be found in [Pos07].

Under the point symmetries, in the table, identities of the form $AB = CD$ are listed. Such an identity means that in the subgroup $\Gamma \subset \Gamma(2)$ defined by the Belyi permutations σ_0 and σ_1 by stabilizing the edge with number 1 we expect to have the identity $C^{\Gamma}_{AB} = C^{\Gamma}_{CD}$.
The expectation is based on the calculation of the first 100 coefficients $r^{\Gamma}_{jk}(c)$, i.e. $AB = CD$ means, that $r^{\Gamma}_{AB}(c) = r^{\Gamma}_{CD}(c)$ for $c = 1, 2, \ldots, 100$. The coefficients has been calculated with the algorithm developed in [Pos07] and the Magma implementation given there. Only identities involving exclusively cusps coming from vertices in the Dessins have been considered because for them, the symmetries can be seen in the Dessins.

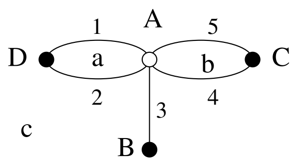
Dessin | **B**elyi permutations and symmetries

1)



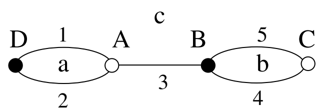
$\sigma_0 = (12345)$ and $\sigma_1 = id$
Symmetries:
$AB = AC = AD = AE = AF$
$BB = CC = DD = EE = FF$
$BC = CD = DE = EF = BF$
$BD = BE = CE = CF = DF$

2)



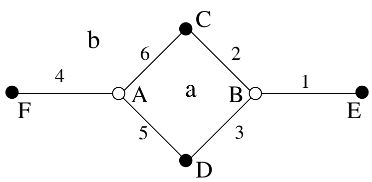
$\sigma_0 = (12345)$ and $\sigma_1 = (12)(45)$
Symmetries:
$DD = CC$
$AD = AC$
$BD = BC$

3)



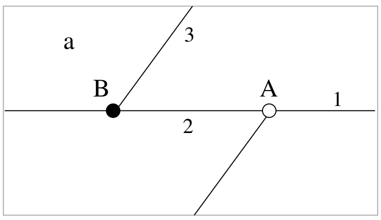
$\sigma_0 = (123)(45)$ and $\sigma_1 = (12)(345)$
Symmetries:
$AA = BB$, $CC = DD$
$AC = BD$, $AD = BC$

4)



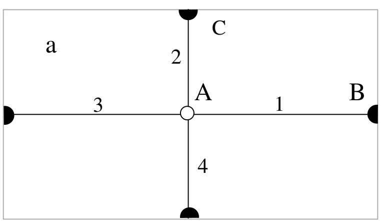
$\sigma_0 = (123)(456)$ and $\sigma_1 = (26)(35)$
Symmetries:
$AA = BB$, $CC = DD$, $EE = FF$
$AC = AD = BC = BD$
$CE = CF = DE = DF$
$AE = BF$, $AF = BE$

5)

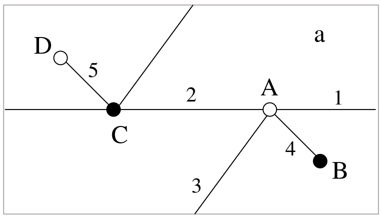


$\sigma_0 = (123)$ and $\sigma_1 = (123)$
Symmetries:
$AA = BB$

6)



$\sigma_0 = (1234)$ and $\sigma_1 = (13)(24)$
Symmetries:
$BB = CC$
$AB = CD$, $AD = BC$

7) 

$\sigma_0 = (1234)$ and $\sigma_1 = (1235)$
Symmetries:
$AA = CC$, $BB = DD$
$AB = CD$, $AD = BC$

8) no figure because of genus two
one white vertex: A
two black vertices: B, C

$\sigma_0 = (123456)$ and $\sigma_1 = (123)(456)$
Symmetries:
$BB = CC$, $AB = AC$

For some of these examples the identities can be proven by properties of Eisenstein series. We have

Lemma 3.3.1. *Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup, S_j and S_k two cusps and $\gamma_{kj} \in \Gamma(1)$ a matrix with $\gamma_{kj}(S_j) = S_k$. Then for the Eisenstein series hold*

$$E_k^{\Gamma}(z,s) = E_j^{\gamma_{kj}^{-1}\Gamma\gamma_{kj}}(\gamma_{kj}^{-1}z, s). \tag{3.3.1.1}$$

Proof: The claim follows from Definition 3.1.1 directly by calculation. Let $\Gamma' = \gamma_{kj}^{-1}\Gamma\gamma_{kj}$ be the conjugated group, Γ_k is the stabilizer of the cusp S_k in Γ and b_k the width of S_k in Γ. Then $\Gamma'_j := \mathrm{Stab}_{\Gamma'}(S_j) = \gamma_{kj}^{-1}\Gamma_k\gamma_{kj}$ and S_j has width b_k in Γ'. We have

$$\begin{aligned}
E_j^{\Gamma'}(\gamma_{kj}^{-1}z, s) &= \frac{1}{b_k^s}\sum_{\gamma\in\Gamma'_j\backslash\Gamma'} \mathrm{Im}\left(\gamma_j^{-1}\gamma\gamma_{kj}^{-1}(z)\right)^s \\
&= \frac{1}{b_k^s}\sum_{\gamma\in\gamma_{kj}^{-1}\Gamma_k\gamma_{kj}\backslash\gamma_{kj}^{-1}\Gamma\gamma_{kj}} \mathrm{Im}\left(\gamma_j^{-1}\gamma\gamma_{kj}^{-1}(z)\right)^s \\
&= \frac{1}{b_k^s}\sum_{\gamma\in\Gamma_k\backslash\Gamma} \mathrm{Im}\left(\gamma_j^{-1}\gamma_{kj}^{-1}\gamma\gamma_{kj}\gamma_{kj}^{-1}(z)\right)^s \\
&= \frac{1}{b_k^s}\sum_{\gamma\in\Gamma_k\backslash\Gamma} \mathrm{Im}\left(\gamma_j^{-1}\gamma_{kj}^{-1}\gamma(z)\right)^s \\
&= E_k^{\Gamma}(z,s),
\end{aligned}$$

since $\gamma_{kj}\gamma_j(\infty) = S_k$. □

Lemma 3.3.1 gives connections of Eisenstein series that we may translate into identities of scattering constants. Thus, from this lemma we derive

Lemma 3.3.2. *Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup, S_j, S_k and S_l cusps and $\gamma_{kj} \in \Gamma(1)$ a matrix with $\gamma_{kj}(S_j) = S_k$. Then for scattering constants hold*

$$C_{jl}^{\Gamma} = C_{kl'}^{\Gamma'}, \tag{3.3.2.1}$$

with $\Gamma' = \gamma_{kj}^{-1}\Gamma\gamma_{kj}$ and l' is the index stands for $S_l' = \gamma_{kj}^{-1}(S_l)$.

Proof: Follows from Lemma 3.3.1. □

When we apply Lemma 3.3.2 on automorphisms on a subgroup $\Gamma \subset \Gamma(2)$ induced by an automorphism of the Belyi pair we get the result that some scattering constants for Γ are equal. The automorphisms in Chapter 2.1 were given by exchanging the generators of $\Gamma(2)$ and a conjugation of the elements (compare with Proposition 2.1.9).

Proposition 3.3.3. *Let $\Gamma \subset \Gamma(2)$ be a finite index subgroup with an automorphism $\alpha : \Gamma \longrightarrow \Gamma$, constructed like in Proposition 2.1.9, that is given by conjugation only. For two cusps S and T we have the identity*

$$C_{ST}^{\Gamma} = C_{\alpha(S)\alpha(T)}^{\Gamma}$$

of scattering constants, where α extends to the cusps according to Proposition 2.1.11.

Proof: Let α be given via $\gamma \mapsto \lambda^{-1}\gamma\lambda$, where $\gamma \in \Gamma$, $\lambda \in \Gamma(1)$. The action on cusps (see Proposition 2.1.11) $\alpha : S \mapsto \alpha(S)$ translates into $\lambda^{-1}(S) = \alpha(S)$.
The scattering constant C_{ST}^{Γ} comes from the Fourier expansion $E_S^{\Gamma}(\gamma_T z, s)$. According to Lemma 3.3.2 and the fact that $\lambda^{-1}\Gamma\lambda = \Gamma$ we have $E_{\lambda(S)}^{\Gamma}(\gamma_T z, s) = E_S^{\Gamma}(\lambda^{-1}\gamma_T z, s)$. We translate the action of λ in the one of α and get

$$\lambda(S) = \alpha^{-1}(S) \quad \text{and} \quad \lambda^{-1}\gamma_T(\infty) = \lambda^{-1}(T) = \alpha(T).$$

Therefore, we may write $E_S^{\Gamma}(\gamma_T z, s) = E_{\alpha(S)}^{\Gamma}(\gamma_{\alpha(T)} z, s)$ from which $C_{ST}^{\Gamma} = C_{\alpha(S)\alpha(T)}^{\Gamma}$ follows. □

The results are promising in direction that they generally show identities of scattering constants if an automorphism identifies the cusps. We risk to state

Conjecture 3.3.4. *Let $\Gamma \subset \Gamma(2)$ be a finite index subgroup together with an automorphism $\alpha : \Gamma \longrightarrow \Gamma$ as in Proposition 2.1.9 and two cusps S and T. Then we have for the scattering constants*

$$C_{ST}^{\Gamma} = C_{\alpha(S)\alpha(T)}^{\Gamma}.$$

*O*n the proof of Conjecture 3.3.4: The hope is that Conjecture 3.3.4 actually follows from a generalization of Proposition 3.3.3.

In Chapter 2.1 was proven, that an automorphism α_Γ of a subgroup $\Gamma \subset \Gamma(2)$ induced by automorphism of a Belyi pair is of the form

$$\begin{aligned} \alpha_\Gamma : \quad & \Gamma \longrightarrow \Gamma \\ & \gamma_0 \longmapsto \mu\eta_0\mu^{-1} \\ & \gamma_1 \longmapsto \mu\eta_1\mu^{-1}, \end{aligned}$$

where $\mu \in \Gamma(2)$ and $\{\eta_0, \eta_1\}$ is a subset of one of the following sets

$$\Big\{\{\gamma_0,\gamma_1,\gamma_\infty\},\{\gamma_0^{-1},\gamma_1^{-1},\gamma_\infty^{-1}\},\{\overline{\gamma}_0,\gamma_1,\gamma_\infty\},\{\overline{\gamma}_0^{-1},\gamma_1^{-1},\gamma_\infty^{-1}\},$$
$$\{\gamma_0,\overline{\gamma}_1,\gamma_\infty\},\{\gamma_0^{-1},\overline{\gamma}_1^{-1},\gamma_\infty^{-1}\},\{\gamma_0,\gamma_1,\overline{\gamma}_\infty\},\{\gamma_0^{-1},\gamma_1^{-1},\overline{\gamma}_\infty^{-1}\}\Big\}, \qquad (3.3.4.1)$$

with

$$\overline{\gamma}_0 = \gamma_1^{-1}\gamma_\infty^{-1}, \quad \overline{\gamma}_1 = \gamma_\infty^{-1}\gamma_0^{-1}, \quad \overline{\gamma}_\infty = \gamma_0^{-1}\gamma_1^{-1}$$

(see Lemma 2.1.6 and Proposition 2.1.9, in particular Formula (2.1.9.1)). Since Lemma 3.3.2 deals with the conjugation by μ we may concentrate on the map given by

$$\tilde{\alpha}_\Gamma : \gamma_i \mapsto \eta_i \qquad (i \in \{0,1\}).$$

The matrix $\eta_\infty = \eta_1^{-1}\eta_0^{-1}$ will be the third matrix from the set out of Formula (3.3.4.1) where η_0 and η_1 had been taken from. This adds up to 42 possiblities for the triple $(\eta_0, \eta_1, \eta_\infty)$: The eight sets from Formula (3.3.4.1) in all possible orders, i.e. times the six permuations in S_3.
It turns out that some of the possible maps for $\tilde{\alpha}_\Gamma$ are given via conjugation over $GL_2(\mathbb{Z})$. Take the matrices

$$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),\ R = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \text{ and } U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

With them we can create half of the maps we considered possible for $\tilde{\alpha}_\Gamma$, i.e. 21 from the 42 triples are given by conjugate the original triple $(\gamma_0, \gamma_1, \gamma_\infty)$ with combinations of R, S and U. In Table 3.2 is illustrated which conjugation yields which triple.

	(12)	(13)	(23)	(123)	(132)	id
$(\gamma_0,\gamma_1,\gamma_\infty)$	–	–	–	R	R^2	id
$(\gamma_0^{-1},\gamma_1^{-1},\gamma_\infty^{-1})$	R^2SU	SU	RSU	–	–	–
$(\overline{\gamma}_0,\gamma_1,\gamma_\infty)$	SR^2	RSR^2	R^2SR^2	–	–	–
$(\overline{\gamma}_0^{-1},\gamma_1^{-1},\gamma_\infty^{-1})$	–	–	–	$RSUS$	R^2SUS	SUS
$(\gamma_0,\overline{\gamma}_1,\gamma_\infty)$	R^2S	S	RS	–	–	–
$(\gamma_0^{-1},\overline{\gamma}_1^{-1},\gamma_\infty^{-1})$	–	–	–	RU	R^2U	U
$(\gamma_0,\gamma_1,\overline{\gamma}_\infty)$	RSR	R^2SR	SR	–	–	–
$(\gamma_0^{-1},\gamma_1^{-1},\overline{\gamma}_\infty^{-1})$	–	–	–	$SUSR$	$RSUSR$	R^2SUSR

Table 3.2.: Conjugations that yield permutated generator triples

To understand the table: Take the entry at second row and the second line for an example. The matrix written there does the following via conjugation

$$(SU)^{-1}\gamma_0 SU = \gamma_\infty^{-1}, \quad (SU)^{-1}\gamma_1 SU = \gamma_1^{-1}, \quad (SU)^{-1}\gamma_\infty SU = \gamma_0^{-1},$$

i.e. it maps $(\gamma_0, \gamma_1, \gamma_\infty)$ to $(\gamma_\infty^{-1}, \gamma_1^{-1}, \gamma_0^{-1})$. This is the same as mapping

$$(\gamma_0,\gamma_1,\gamma_\infty) \overset{\text{2nd line}}{\longmapsto} (\gamma_0^{-1},\gamma_1^{-1},\gamma_\infty^{-1}) \overset{\text{2nd row}}{\longmapsto} (\gamma_\infty^{-1},\gamma_1^{-1},\gamma_0^{-1}).$$

The lemmas 3.3.1 and 3.3.2 had been stated for a conjugation over $\Gamma(1)$ but they hold for $GL_2(\mathbb{Z})$ as well. Therefore, Conjecture 3.3.4 holds for maps α_Γ whose $\tilde{\alpha}_\Gamma$'s are those for which we have a matrix in Table 3.2.

Concerning the other possible $\tilde{\alpha}_\Gamma$: It is easy to check that the triple we found a matrix for in Table 3.2 are exactly those who map $(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_\infty)$ to a triple $(\eta_0, \eta_1, \eta_\infty)$ fulfilling $\eta_0\eta_1\eta_\infty = id$. This is a condition a Belyi triple must fulfill. The hope is that a closer look to the situation in Lemma 2.1.6 shows that the other triple do not occur and the conjecture holds.

Remark 3.3.5. Consider again the examples 1-8.
All identities claimed there follow from Proposition 3.3.3 and the semi proof of Conjecture 3.3.4. Thus, all examples known to the author follows from statements proven.

In the examples 1,6 and 8 we can get the identities via just turning the Dessin where the Belyi permutations stay the same (plus a congugation over $\Gamma(2)$). This turning corresponds to a conjugation with elements in $\Gamma(2)$ and the claim follows from Proposition 3.3.3.

In the other cases we have to reflect or turn the Dessin and the Belyi permutations change (plus possibly a congugation over $\Gamma(2)$). But we will allways get one of the triples for which we found out that it follows from conjugation with combinations of R, S and U, i.e. one of the triples in Table 3.2.

4. Automorphic Green's functions and Kronecker limit formulas

In this chapter we will explore some other properties of Eisenstein series and objects derived from them: Green's functions (in Section 4.2) and Kronecker limit formulas (in Section 4.3). The first section gives the introduction in modular forms needed.

Eisenstein series are real analytic functions which play a central role in spectral theory of the hyperbolic Laplacian.

Kronecker limit formulas show that these functions have a strong relation to holomorphic modular forms which in addition satisfy some integrality condition (see Proposition 4.3.5).

The other feature of Eisenstein series $E_j^{\Gamma}(z,s)$ that we mention in this chapter is that they define Green's functions for the cusps of Γ which, in contrast to randomly chosen Green's functions, allow explicit calculations in Arakelov theory (see [Küh05]).

Both concepts we can only cover slightly and partial.

4.1. Modular forms

For Section 4.3 we need some knowledge of modular forms. Therefore, we will give the basic definitions and properties here.

The main reference for the following is [Miy06].

Definition 4.1.1. *Let $f(z)$ be a function $f : \mathbb{H} \longrightarrow \mathbb{C}$. We define an action of $\Gamma(1)$ on $f(z)$ via*

$$f|_k\gamma(z) = (cz+d)^{-k} f(\gamma z),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ and $k \in \mathbb{Z}$. This action is called the slash operator of weight k or the k-th slash operator.

Definition 4.1.2. *Let $\Gamma \subset \Gamma(1)$ be of finite index, k an integer. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ behaves automorphically of weight k with respect to Γ if*

$$f|_k\gamma(z) = f(z) \qquad \forall \gamma \in \Gamma,$$

i.e. if $f(z)$ fulfills the modular transformation property

$$f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z\right) = (cz+d)^k f(z) \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

Let S_j be a cusp of $\Gamma \subset \Gamma(1)$, a finite index subgroup, and $f(z)$ as in Definition 4.1.2. Then $f|_k\gamma_j(z)$ is b_j-periodic, i.e. $f|_k\gamma_j(z+b_j) = f|_k\gamma_j(z)$, where b_j is the cusp width of S_j. Therefore there exists a function g on $D \setminus \{0\}$ (the unit disc without the center) such that

$$f|_k\gamma_j(z) = g(e^{2\pi i z}) \qquad z \in \mathbb{H}.$$

The function g is meromorphic on $D \setminus \{0\}$, since f is meromorphic. We say that f is meromorphic, is holomorphic in the cusp S_j if g extends meromorphically, holomorphically to 0, respectively.

Definition 4.1.3. *A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called modular function with respect to Γ if it behaves automorphically of weight 0 and is meromorphic in all cusps of Γ; a holomorphic function $f(z)$ is called a modular form (of weight k with respect to Γ) if it behaves automorphically of weight k and is holomorphic in all cusps.*

The set of modular forms of weight k with respect to Γ is denoted by $M_k(\Gamma)$.

Lemma 4.1.4. *Let $\Gamma \subset \Gamma(1)$ be a subgroup. The modular forms $M_*(\Gamma)$ generate a ring, graded by the weight.*
For $f, g \in M_k(\Gamma)$ hold for the sum $f + g \in M_k(\Gamma)$. For $f \in M_k(\Gamma)$, $g \in M_l(\Gamma)$ hold for the product $f \cdot g \in M_{k+l}(\Gamma)$ $(k, l \in \mathbb{N})$.

Proof: See [Miy06]. □

Lemma 4.1.5. *Let $f \in M_k(\Gamma)$ with $\Gamma \subset \Gamma(1)$ and $k \in \mathbb{Z}$. In each cusp S_j of Γ a modular functions or form admits an expansion*

$$f|_k\gamma_j(z) = \sum_{n \geq n_0} a_n q^{n/b_j}, \qquad (n_0 \in \mathbb{Z}, a_{n_0} \neq 0)$$

where $q = e^{2\pi i z}$. This series is called the q-expansion of f in the cusp S_j.
If f is a modular form, then $n_0 \geq 0$.

Proof: See [Miy06]. □

Definition 4.1.6. *The subring of $M_*(\Gamma(1))$ generated by modular forms with integral coefficients will be denoted by $M_*(\Gamma(1))_{\mathbb{Z}}$.*

Lemma 4.1.7. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups and $f \in M_*(\Gamma)$ a modular form. Then f is a root of a normalized polynomial $P_f^{\Gamma|\Gamma'} \in M_*(\Gamma')[X]$:*

$$P_f^{\Gamma|\Gamma'} := \sum_{\gamma \in \Gamma \setminus \Gamma'} (X - f|_k\gamma) \tag{4.1.7.1}$$

Proof: Obviously, looking at the construction, the modular form f is a root of $P_f^{\Gamma|\Gamma'}$. It remains to show that $P_f^{\Gamma|\Gamma'} \in M_*(\Gamma')[X]$, i.e. that all coefficients are modular forms for Γ'. It is sufficient to show that all coefficients fulfill the required transformation rule for Γ', for that it is enough to show

$$\prod_{\gamma \in \Gamma \setminus \Gamma'} (X - f|_k\gamma|_k\sigma(z)) = \prod_{\gamma \in \Gamma \setminus \Gamma'} (X - f|_k\gamma(z))$$

with $\sigma \in \Gamma'$.
It holds $f|_k\gamma|_k\sigma(z) = f|_k(\gamma\sigma)(z)$ and the set $\{\gamma\sigma\}_{\gamma\in\Gamma\setminus\Gamma'}$ is a system of representatives for $\Gamma \setminus \Gamma'$ since $\{\gamma\}_{\gamma\in\Gamma\setminus\Gamma'}$ was one. □

Definition 4.1.8. ***a)*** *We call the polynomial* $P_f^{\Gamma|\Gamma'}$ *(Equation* (4.1.7.1)*) minimal polynomial of* f *with respect to* Γ'*. Its constant term is the norm of* f*, the coefficient of the second highest* X*-power the trace of* f *(with respect to* Γ'*):*

$$\mathrm{Nm}_{\Gamma|\Gamma'}(f) = \prod_{\gamma\in\Gamma\setminus\Gamma'} f|_k\gamma \tag{4.1.8.1}$$

$$\mathrm{Tr}_{\Gamma|\Gamma'}(f) = \sum_{\gamma\in\Gamma\setminus\Gamma'} f|_k\gamma \tag{4.1.8.2}$$

b) *Modular forms* $f \in M_k(\Gamma)$ *with* $P_f^{\Gamma|\Gamma'} \in M_*(\Gamma(1))_{\mathbb{Z}}[X]$ *are called integral modular forms.*

Remark 4.1.9. Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be subgroups. For the norm and the trace we have

$$\begin{aligned} \mathrm{Nm}_{\Gamma|\Gamma'}: &\quad M_k(\Gamma) \longrightarrow M_{[\Gamma':\Gamma]k}(\Gamma') \\ \mathrm{Tr}_{\Gamma|\Gamma'}: &\quad M_k(\Gamma) \longrightarrow M_k(\Gamma'). \end{aligned}$$

Definition 4.1.10. *Let* $f(z) \in M_k(\Gamma)$ *be a modular form for a finite index subgroup* $\Gamma \subset \Gamma(1)$*. Then we define the Petersson norm via*

$$||f(z)||^2 := |f(z)|^2(4\pi\,\mathrm{Im}(z))^k. \tag{4.1.10.1}$$

If $f \in M_k(\Gamma)$*, then* $||f||^2$ *fulfills the modular transformation property for* $k = 0$*.*

Remark 4.1.11. This definition of the Pertersson norm varies the one dominating the literature. We will use this one, because with the factor 4π that is normally omitted, we follow the normalizations presented in [Küh01] and [BBGK07]. Their choice gives the most clear local expansions and most clean final formulas for the arithmetic intersection numbers.

Remark 4.1.12. Let $f(z) \in M_k(\Gamma)$ where $\Gamma \subset \Gamma(1)$ is a subgroup. The two norms $\mathrm{Nm}(f(z))$ and $||f(z)||^2$ commute:

$$\begin{aligned} \mathrm{Nm}\,||f(z)||^2 &= \mathrm{Nm}\left(|f(z)|^2(4\pi\,\mathrm{Im}(z))^k\right) = \mathrm{Nm}\,f(z)\,\mathrm{Nm}\,\overline{f(z)}(\mathrm{Nm}(4\pi\,\mathrm{Im}(z)))^k \\ &= \mathrm{Nm}\,f(z)\overline{\mathrm{Nm}\,f(z)}(\mathrm{Nm}(4\pi\,\mathrm{Im}(z)))^k = |\,\mathrm{Nm}\,f(z)|^2(\mathrm{Nm}(4\pi\,\mathrm{Im}(z)))^k \\ &= ||\,\mathrm{Nm}\,f(z)||^2. \end{aligned}$$

4.2. Automorphic Green's functions for cusps

Arakelov theory is an intersection theory of "compactified" divisors on arithmetic surfaces. Arithmetic surfaces may be seen as considering an algebraic curve C over a number field K together with all reductions above primes. An illustration can be seen in Figure 4.1.

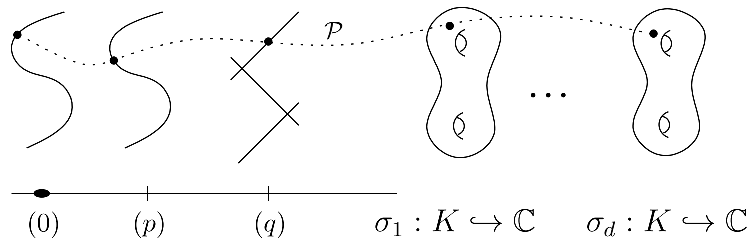


Figure 4.1.: Arithmetic curve

A K-rational point P gives a horizontal divisor $\mathcal{P}$ in this 2-dimensional object. In order to obtain a well-defined pairing of divisors one has to "compactify" them, i.e. one considers pairs $(\mathcal{P}, (g_P^\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$ where g_P^σ is a Green's function for $\sigma(P)$ on the compact Riemann surface $\sigma(C)$. Such a Green's function has a logarithmic singularity in P and, in classical Arakelov theory, is smooth elsewhere. However, it is shown in [Küh01] that one can allow some additional log-log singularities at a finite set of points.

We define a Green's function out of Eisenstein series.

Definition 4.2.1. *For $\Gamma \subset \Gamma(1)$ a finite index subgroup with cusp S_j of width b_j we define*

$$g_j^\Gamma(z) := 4\pi \lim_{s\to 1}\left(E_j(z,s) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma]}\right) - \frac{12\,(\log(4\pi) - \log(b_j))}{[\Gamma(1):\Gamma]}. \qquad (4.2.1.1)$$

Proposition 4.2.2. *The function $g_j^\Gamma(z)$ is a Green's function for S_j. It has* log-log *singularities at all the cusps S_k of Γ and some mild quotient singularities at the elliptic fixed points of Γ.*

Proof: See [Küh05]. □

Remark 4.2.3. Green's functions are only defined up to constants. The normalization in Definition 4.2.1 is motivated by the results from Propositions 4.2.5 and 4.2.8 below.

Lemma 4.2.4. *The Fourier expansion of $g_j^\Gamma(z)$ in the cusp S_k is $(z = x+iy)$*

$$g_j(\gamma_k z) = \delta_{jk} 4\pi \frac{y}{b_k} + 4\pi \left(\widetilde{C}_{jk}^\Gamma - \frac{C^{\Gamma(1)}}{[\Gamma(1):\Gamma]} \right) - \frac{12\left(\log(4\pi y) - \log(b_j)\right)}{[\Gamma(1):\Gamma]} + 4\pi^2 \frac{1}{b_j b_k} \sum_{m\neq 0} e^{-2\pi |m| y/b_k} e^{2\pi i m x/b_k} \sum_{c>0} \frac{1}{c^2} r_{jk,m}^\Gamma(c). \tag{4.2.4.1}$$

Proof: The claim follows from the expansion of Eisenstein series in Equation (3.1.10.1) with help of local expansions around $s=1$ and the identity $K_{1/2}(x) = \sqrt{\pi/(2x)}e^{-x}$. $\square$

Proposition 4.2.5. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be subgroups, S_j a cusp of Γ' and $\{S_i\}_{i\in I_j}$ its subcusps in Γ. The widths in Γ will be denoted by b_i, the width in Γ' with w_j.*
With the chosen normalization the Green's functions $g_i^\Gamma(z)$ have the property

$$\sum_{i\in I_j} \frac{b_i}{w_j} g_i^\Gamma(z) = g_j^{\Gamma'}(z). \tag{4.2.5.1}$$

Proof: To shorten the formulas we set

$$\begin{aligned} K &:= \sum_{i\in I_j} \frac{b_i}{w_j} \left(- \frac{12(\log(4\pi) - \log(b_i))}{[\Gamma(1):\Gamma]} \right) \\ &= - \frac{12\log(4\pi)}{[\Gamma(1):\Gamma']} + \frac{12}{[\Gamma(1):\Gamma]} \sum_{i\in I_j} \frac{b_i}{w_j} \log(b_i). \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{i\in I_j} \frac{b_i}{w_j} g_i^\Gamma(z) &= 4\pi \lim_{s\to 1} \left(\sum_{i\in I_j} \frac{b_i}{w_j} E_i^\Gamma(z,s) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma']} \right) + K \\ &= 4\pi \lim_{s\to 1} \left(\sum_{i\in I_j} \frac{b_i w_j^s}{w_j b_i^s} \frac{b_i^s}{w_j^s} E_i^\Gamma(z,s)) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma']} \right) + K \\ &\overset{(1)}{=} 4\pi \lim_{s\to 1} \left(\sum_{i\in I_j} \left(1 - \log\left(\frac{b_i}{w_j}\right)(s-1)\dots \right) \frac{b_i^s}{w_j^s} E_i^\Gamma(z,s)) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma']} \right) + K \\ &= 4\pi \lim_{s\to 1} \left(\sum_{i\in I_j} - \log\left(\frac{b_i}{w_j}\right)(s-1) \frac{b_i^s}{w_j^s} E_i^\Gamma(z,s) \right) \\ &\quad + 4\pi \lim_{s\to 1} \left(\sum_{i\in I_j} \frac{b_i^s}{w_j^s} E_i^\Gamma(z,s) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma']} \right) + K \end{aligned}$$

$$\begin{aligned}
&\overset{(2)}{=} 4\pi \sum_{i\in I_j} \lim_{s\to 1}\left(-\log\left(\frac{b_i}{w_j}\right)(s-1)\frac{3b_i/w_j}{\pi[\Gamma(1):\Gamma](s-1)}\right)\\
&\quad + 4\pi \lim_{s\to 1}\left(E_j^{\Gamma'}(z,s) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma']}\right) + K\\
&= -\frac{12}{[\Gamma(1):\Gamma]}\sum_{i\in I_j}\frac{b_i}{w_j}\log\left(\frac{b_i}{w_j}\right) + 4\pi\lim_{s\to 1}\left(E_j^{\Gamma'}(z,s) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma']}\right) + K\\
&= 4\pi\lim_{s\to 1}\left(E_j^{\Gamma'}(z,s) - \frac{\Phi_{\Gamma(1)}(s)}{[\Gamma(1):\Gamma']}\right) - \frac{12(\log(4\pi)-\log(w_j))}{[\Gamma(1):\Gamma']}\\
&= g_j^{\Gamma'}(z).
\end{aligned}$$

In (1) the local expansion of $\frac{w_j^s}{b_i^s}$ was used and in (2) the expansion of Eisenstein series at $s=1$ and Proposition 3.1.7. □

Remark 4.2.6. In particular for $\Gamma' = \Gamma(1)$ holds

$$\sum_{S_j \text{ cusp}} b_j g_j^{\Gamma}(z) = g^{\Gamma(1)}(z).$$

For the Green's functions we have a result for the expansions as well. Firstly, we give a definition to state Proposition 4.2.8 more nicely. The slash operator as defined in Definition 4.1.1 is applicable to Green's functions. Similar to Definition 4.1.8 we have

Definition 4.2.7. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups and S_j a cusp of Γ. Then we define the trace of the Green's function g_j with respect to Γ' as*

$$\operatorname{Tr}_{\Gamma|\Gamma'}\left(g_j^{\Gamma}\right) = \sum_{\gamma\in\Gamma\backslash\Gamma'} g_j^{\Gamma}|_0\gamma. \tag{4.2.7.1}$$

Proposition 4.2.8. *Let $g_j(z)$ be the Green's function for the cusp S_j defined via an Eisenstein series for a subgroup $\Gamma \subset \Gamma' \subset \Gamma(1)$ (see Definition 4.2.1). Let S_k be a cusp of Γ and $\gamma_k \in \Gamma(1)$ with $S_k = \gamma_k(\infty)$. Then for the trace we obtain*

$$\operatorname{Tr}_{\Gamma|\Gamma'}\left(g_j^{\Gamma}(\gamma_k z)\right) = g_j^{\Gamma'}(\gamma_k z). \tag{4.2.8.1}$$

Proof: To understand what happens in the sum, we choose a suitable system of representatives. We have

$$\Gamma' = \bigcup_{S_l\sim_{\Gamma'}S_k}\ \bigcup_{0\le n<\frac{b_l}{w_k}} \Gamma\gamma_{lk}\tau_{k,w_k n},$$

where $\gamma_{lk}(S_k) = S_l$ and $\tau_{k,w_k n} = \gamma_k\tau_{w_k n}\gamma_k^{-1} \in \operatorname{Stab}_{\Gamma'}(S_k)$ (the matrix γ_k as in the statement and $\tau_{w_k n} = \left(\begin{smallmatrix} 1 & w_k n \\ 0 & 1 \end{smallmatrix}\right))$. Then

$$\begin{aligned}
\mathrm{Tr}_{\Gamma|\Gamma'}(g_j^{\Gamma}(\gamma_k z)) &= \sum_{\gamma \in \Gamma \backslash \Gamma'} g_j|_0 \gamma(\gamma_k z) \\
&= \sum_{S_l \sim S_k} \sum_{0 \le n < \frac{b_l}{w_k}} g_j(\gamma_{lk}\gamma_k \tau_{w_k n} \gamma_k^{-1} \gamma_k z) \\
&= \sum_{S_l \sim S_k} \sum_{0 \le n < \frac{b_l}{w_k}} g_j(\gamma_l \tau_{w_k n} z) \\
&= \sum_{S_l \sim S_k} \sum_{0 \le n < \frac{b_l}{w_k}} g_j(\gamma_l (z + w_k n))
\end{aligned}$$

with $\gamma_{lk}\gamma_k = \gamma_l$ which fulfills $\gamma_l(\infty) = S_l$.

Remember Formula (4.2.4.1) the Fourier expansion of the Green's function in the cusp S_k. We set

$$\varphi_{jl,m}^{\Gamma} := \frac{1}{b_j b_l} \sum_{c>0} \frac{1}{c^2} \sum_{\substack{d \mod b_l c \\ \exists \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1} \Gamma \gamma_l}} e^{2\pi i m \frac{d}{b_l c}}.$$

We get

$$\begin{aligned}
\mathrm{Tr}_{\Gamma|\Gamma'}(g_j^{\Gamma'}(\gamma_k z)) = &\sum_{S_l \sim S_k} \sum_{0 \le n < \frac{b_l}{w_k}} \left(\delta_{jk} 4\pi \frac{y}{b_j} + 4\pi \left(\widetilde{C}_{jl}^{\Gamma} - \frac{C^{\Gamma(1)}}{[\Gamma(1):\Gamma]} \right) - \right. \\
&\left. \frac{12(\log(4\pi y) - \log(b_j))}{[\Gamma(1):\Gamma]} + 4\pi \sum_{m \neq 0} \pi e^{-2\pi |m| y / b_l} e^{2\pi i m (x + w_k n)/b_l} \varphi_{jl,m}^{\Gamma} \right) \\
= &\, \delta_{jk} 4\pi \frac{y}{w_k} - \frac{12(\log(4\pi y) - \log(b_j))}{[\Gamma(1):\Gamma']} + 4\pi \left(\sum_{S_l \sim S_k} \frac{b_l}{w_k} \widetilde{C}_{jl}^{\Gamma} - \frac{C^{\Gamma(1)}}{[\Gamma(1):\Gamma']} \right) \\
&+ 4\pi \sum_{m \neq 0} \sum_{S_l \sim S_k} \left(\sum_{0 \le n < \frac{b_l}{w_k}} e^{2\pi i m n/(b_l/w_k)} \right) \pi e^{-2\pi |m| y/b_l} e^{2\pi i m x/b_l} \varphi_{jl,m}^{\Gamma} \\
\overset{(1)}{=} &\, \delta_{jk} 4\pi \frac{y}{w_k} - \frac{12(\log(4\pi y) - \log(b_j))}{[\Gamma(1):\Gamma']} \\
&+ 4\pi \left(\widetilde{C}_{jk}^{\Gamma'} - \frac{3}{\pi[\Gamma(1):\Gamma']} \log\left(\frac{b_j}{w_j}\right) - \frac{C^{\Gamma(1)}}{[\Gamma(1):\Gamma']} \right) \\
&+ 4\pi \sum_{m \neq 0} \sum_{S_l \sim S_k} \frac{b_l}{w_k} \pi e^{-2\pi |m| y/w_k} e^{2\pi i m x/w_k} \varphi_{jl,\frac{b_l}{w_k} m}^{\Gamma}
\end{aligned}$$

$$\begin{aligned}&\overset{(2)}{=} \delta_{jk} 4\pi \frac{y}{w_k} - \frac{12(\log(4\pi y) - \log(w_j))}{[\Gamma(1):\Gamma']} + 4\pi \left(\widetilde{C}_{jk}^{\Gamma'} - \frac{C^{\Gamma(1)}}{[\Gamma(1):\Gamma']} \right) \\ &\quad + 4\pi \sum_{m\neq 0} \pi e^{-2\pi|m|y/w_k} e^{2\pi i m x/w_k} \varphi_{jk,m}^{\Gamma'} \\ &= g_j^{\Gamma'}(\gamma_k z).\end{aligned}$$

In (1) the summation formula for non-normalized scattering constants was used

$$\sum_{S_l \sim S_k} \frac{b_l}{w_k} \widetilde{C}_{jl}^{\Gamma} = \widetilde{C}_{jk}^{\Gamma'} - \frac{3}{\pi[\Gamma(1):\Gamma']} \log\left(\frac{b_j}{w_j}\right)$$

and

$$\sum_{0\leq n<k} e^{2\pi i m \frac{n}{k}} = \begin{cases} k & \text{if } k|m \\ 0 & \text{elsewise.} \end{cases}$$

For (2) we have to use Proposition 3.1.14. □

Remark 4.2.9. From Proposition 4.2.8 it follows that for every cusp S_j of $\Gamma \subset \Gamma(1)$ we get for the expansions

$$\mathrm{Tr}_{\Gamma|\Gamma(1)}\, g_j^{\Gamma}(z) = g^{\Gamma(1)}(z).$$

4.3. Kronecker limit formulas and their application on the calculation of scattering constants

Kronecker limit formulas give a connection of special values of eigenfunctions of the hyperbolic Laplacian with distinguished modular forms.

Let $\Gamma \subset \Gamma(1)$ be a subgroup. If we subtract the pole in $s = 1$, the Eisenstein series for $s = 1$ is a well defined automorphic function on $\mathbb{H}$. It essentially comes from the logarithm of the norm of a holomorphic function g, which is very similar to a modular form in the sense of Definition 4.1.3 (such functions are sometimes referred to as modular forms with multiplier systems). Sometimes a distinguished modular form (distinguished i.e. by its integrality) f with same divisor as g exists. Then $\log ||f||^2$ differs from the constant term in $s = 1$ only be a constant A (see 4.3.5).

The constant A that is strongly related to the scattering constant has a meaning in Arakelov theory. It can be used to describe generalized arithmetic intersection numbers (see [Küh99], [Küh05]). More precisely, it gives the arithmetic intersection lying above the archimedean places.

Kronecker limit formulas for Hilbert modular surfaces have been considered by J. Bruinier, J. Burgos Gil and U. Kühn [BBGK07] as an ingredient to calculate the arithmetic self intersection of $\overline{\mathcal{M}}_k$ the bundle of modular forms.

The classical Kronecker limit formula for $\Gamma(1)$ is

Proposition 4.3.1. *For the Eisenstein series $E^{\Gamma(1)}(z,s)$ (Definition 3.1.1) the following formula holds*

$$4\pi \lim_{s\to 1}\left(E^{\Gamma(1)}(z,s)-\frac{3/\pi}{s-1}\right)=-\log||\Delta(z)||^2+24\frac{\zeta'(-1)}{\zeta(-1)}-12\log(4\pi)+24, \quad (4.3.1.1)$$

where $\Delta(z)$ is the well known Delta function (see Remark 4.3.2) and $||\cdot||^2$ the Petersson norm, introduced in Definition 4.1.10.

Proof: This presentation may be found in [Küh99]. □

Remark 4.3.2. We have $\Delta(z)\in M_{12}(\Gamma(1))_{\mathbb{Z}}$ uniquely determined by having a zero in the one cusp of $\Gamma(1)$ and leading coefficient 1. It holds

$$\Delta(z)=q\prod_{n\geq 1}(1-q^n)^{24} \qquad (q=e^{2\pi i z}).$$

Remark 4.3.3. The constant

$$24\frac{\zeta'(-1)}{\zeta(-1)}-12\log(4\pi)+24$$

in Proposition 4.3.1 can be divided in two parts. On the one hand side, there is $12\log(4\pi)$ that comes from the norm $||\cdot||^2$ and on the other hand we have the part coming from the Eisenstein series that is essentially the scattering constant:

$$24\left(\frac{\zeta'(-1)}{\zeta(-1)}-\log(4\pi)+1\right)=4\pi C^{\Gamma(1)} \qquad (4.3.3.1)$$

Now we can see yet another property of the Green's functions defined in Section 4.2; they have the normalization that leads to:

Corollary 4.3.4. *It holds*

$$g^{\Gamma(1)}(z)=-\log||\Delta(z)||^2.$$

We would like to find formulas similar to (4.3.1.1) for other groups.

Proposition 4.3.5. *Let $\Gamma\subset\Gamma(1)$ be a subgroup, S_j a cusp of Γ. Suppose there is a modular form $f_j\in M_k(\Gamma)$ ($k\in\mathbb{N}$), that only vanishes in the cusp S_j. Then there is a constant $A\in\mathbb{R}$ such that*

$$4\pi\lim_{s\to 1}\left(E_j^{\Gamma}(z,s)-\frac{1}{vol(\Gamma)(s-1)}\right)=-\frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}}\log||f_j||^2+A. \qquad (4.3.5.1)$$

Proof: Examine the action of the hyperbolic Laplace operator $\Delta=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ on the functions $4\pi\lim_{s\to 1}\left(E_j^{\Gamma}(z,s)-\frac{1}{vol(\Gamma)(s-1)}\right)$ and $-\log||f_j||^2$. We have on the one hand side (for the expansion in a cusp S_l)

$$\begin{aligned}\Delta\left(4\pi\lim_{s\to 1}\left(E_j^{\Gamma}(\gamma_l z,s)-\frac{1}{vol(\Gamma)(s-1)}\right)\right)\\ &=\Delta\left(4\pi\left(\delta_{jl}\frac{y}{b_l}+\widetilde{C}_{jl}^{\Gamma}-\frac{3\log(y)}{\pi[\Gamma(1):\Gamma]}+\sum_{n\neq 0}a_n(y,1)q^{xn/b_l}\right)\right)\\ &=\frac{12}{[\Gamma(1):\Gamma]},\end{aligned}$$

with $q=e^{2\pi i}$ and $z=x+iy$, where the formula for the expansion can be derived from Formula (4.2.4.1). On the other side,

$$\begin{aligned}\Delta(-\log||f_j||^2)&=\Delta\left(-\log\left(|f_j|^2(4\pi y)^k\right)\right)\\ &=-\Delta\left(\log(f_j)\right)-\Delta(\log(\overline{f_j}))-\Delta\left(\log\left((4\pi y)^k\right)\right)\\ &=0+0+k.\end{aligned}$$

Hence,

$$4\pi\lim_{s\to 1}\left(E_j^{\Gamma}(z,s)-\frac{1}{vol(\Gamma)(s-1)}\right)+\frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}}\log||f_j||^2$$

lies in the kernel of Δ.

The spectral decomposition of the Laplacian had been studied, see [Iwa02], for functions that are square integrable (the space $\mathfrak{L}(Y_\Gamma)$). To find out, if we can use the result from [Iwa02], we will study the behavior of $4\pi\lim_{s\to 1}\left(E_j^{\Gamma}(z,s)-\frac{1}{vol(\Gamma)(s-1)}\right)+\frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}}\log||f_j||^2$ in the cusps. For that, we will compare the expansions. The expansion of the Eisenstein series in a cusp S_l we have seen above. The expansion $f_j|_{S_l}$ looks like $a_m q^{zm/b_l}\left(1+\sum_n a_n q^{zn/b_l}\right)$, with $q=e^{2\pi i}$, $m\in\left\{0,[\Gamma(1):\Gamma]\cdot\frac{k}{12}\right\}$ and $m=0$ if and only if $j\neq l$, since f_j only vanishes in the cusp S_j and the vanishing order is $[\Gamma(1):\Gamma]\cdot\frac{k}{12}$ (by the theory of modular forms). Therefore, we have ($z=x+iy$)

$$\begin{aligned}\log||f_j|_{S_l}||^2&=\log\left((4\pi y)^k|f_j|_{S_l}|^2\right)\\ &=k\cdot\log(4\pi y)+2\operatorname{Re}\log(f_j|_{S_l})\\ &=k\cdot\log(4\pi y)+2\operatorname{Re}\log\left(a_m q^{zm/b_l}\left(1+\sum_{n>0}a_n q^{zn/b_l}\right)\right)\\ &=k\cdot\log(4\pi y)+2\operatorname{Re}\log(a_m)-\delta_{jl}4\pi\frac{y}{b_l}\cdot[\Gamma(1):\Gamma]\cdot\frac{k}{12}\\ &\quad+2\operatorname{Re}\log\left(1+\sum_{n>0}a_n q^{zn/b_l}\right)\end{aligned}$$

$$= k \cdot \log(4\pi y) + 2\,\mathrm{Re}\log(a_m) - \delta_{jl} 4\pi \frac{y}{b_l} \cdot [\Gamma(1):\Gamma] \cdot \frac{k}{12} + 2\,\mathrm{Re}\left(\sum_{n>0} b_n q^{zn/b_l}\right),$$

where the b_n are suitable such that $\log\left(1 + \sum_{n>0} a_n q^{zn/b_l}\right) = \sum_{n>0} b_n q^{zn/b_l}$.

Now we can see that the value of

$$4\pi \lim_{s\to 1}\left(E_j^{\Gamma}(z,s) - \frac{1}{vol(\Gamma)(s-1)}\right) + \frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}} \log||f_j||^2$$

in S_l is bounded:

$$\lim_{z\to i\infty}\left(4\pi \lim_{s\to 1}\left(E_j^{\Gamma}(\gamma_l z,s) - \frac{1}{vol(\Gamma)(s-1)}\right) + \frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}} \log||f_j|_{S_l}||^2\right)$$

$$= \lim_{z\to i\infty}\left(4\pi\widetilde{C}_{jl}^{\Gamma} + \frac{12\log(4\pi)}{[\Gamma(1):\Gamma]} + 2\,\mathrm{Re}\log(a_m) + \sum_{n\neq 0} a_n e^{-2\pi|n|\frac{y}{b_l}} e^{2\pi i\frac{x}{b_l}}\right.$$

$$\left. +2\,\mathrm{Re}\left(\sum_{n>0} b_n e^{2\pi i n\frac{z}{b_l}}\right)\right)$$

$$= 4\pi\widetilde{C}_{jl}^{\Gamma} + \frac{12\log(4\pi)}{[\Gamma(1):\Gamma]} + 2\,\mathrm{Re}\log(a_m),$$

where S_l was chosen arbitrarily. Therefore, we have

$$4\pi \lim_{s\to 1}\left(E_j^{\Gamma}(z,s) - \frac{1}{vol(\Gamma)(s-1)}\right) + \frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}} \log||f_j||^2 \in \mathfrak{L}(Y_{\Gamma}).$$

The spectral decomposition in [Iwa02] shows that the kernel of the Laplacian are the constant functions. If we look closer at the functions involved here, we see that we get a real number. □

Remark 4.3.6. The constant A from Proposition 4.3.5 in Formula (4.3.5.1) can be calculated by comparison of the Fourier expansions. It is independent of the cusp S_l in which the expansion is taken:
To explain this, we may start with the functions expanded in ∞ to see what happens, if we pass on to another cusp. Changing from ∞ to the cusp S_l means changing z to $\gamma_l z$, where $\gamma_l \in \Gamma(1)$ with $\gamma_l^{-1}\Gamma_l\gamma_l = \left\langle \begin{pmatrix} 1 & b_l \\ 0 & 1 \end{pmatrix} \right\rangle$. Thus, only the parts of the function change that depend on z. This parts coincides for g_j^{Γ} and $-\log||f_j||^2$ such that their difference stays the same.

Corollary 4.3.7. *Let $\Gamma \subset \Gamma(1)$ be a subgroup and S_j a cusp of Γ. Suppose there is a modular form $f_j \in M_k(\Gamma)$ ($k \in \mathbb{N}$), that only vanishes in the cusp S_j. Then there exists*

$A \in \mathbb{R}$ such that

$$g_j^\Gamma(z) = -\frac{1}{[\Gamma(1):\Gamma] \cdot \frac{k}{12}} \log ||f_j||^2 + A. \tag{4.3.7.1}$$

Proof: Follows from Proposition 4.3.5 since $4\pi \lim_{s\to 1} \left(E_j^\Gamma(z,s) - \frac{3/\pi}{[\Gamma(1):\Gamma](s-1)} \right)$ differs from $g_j^\Gamma(z)$ by a constant. □

Proposition 4.3.8. *Let $\Gamma \subset \Gamma(1)$ be a subgroup, S_j and S_l are cusps of Γ. Let $f_j \in M_k(\Gamma)$ ($k \in \mathbb{N}$) be a modular form for Γ, that only vanishes in the cusp S_j and A the constant from corollary 4.3.7 applied on Γ, S_j and f_j.*

Then

$$C_{jl}^\Gamma = \frac{1}{[\Gamma(1):\Gamma]} \left(C^{\Gamma(1)} + \frac{1}{\pi} \left(3\log(b_l/b_j) - \frac{6}{k} \operatorname{Re}\log(a_m) \right) \right) + \frac{1}{4\pi} A \tag{4.3.8.1}$$

where

$$m = \begin{cases} [\Gamma(1):\Gamma] \cdot \frac{k}{12} & \text{if } S_j = S_l \\ 0 & \text{elsewise,} \end{cases}$$

such that a_m is the first non-zero coefficient in $f_j|_{S_l} = \sum_{n \ge 0} a_n q^{n/b_l}$.

Proof: We will get scattering constants by comparing the constant terms in the Kronecker limit formula in Corollar 4.3.7:

$$\text{constant term} \left(g_j^{\Gamma_N}(\gamma_l z) \right) = \text{constant term} \left(-\frac{1}{[\Gamma(1):\Gamma] \cdot \frac{k}{12}} \log ||f_j|_{S_l}(z)||^2 \right) + A \tag{4.3.8.2}$$

For the left hand side, the Green's function, we get

$$\delta_{jl} 4\pi \frac{y}{b_l} + 4\pi \left(\widetilde{C}_{jl}^\Gamma - \frac{C^{\Gamma(1)}}{[\Gamma(1):\Gamma]} \right) - \frac{12\left(\log(4\pi y) - \log(b_j)\right)}{[\Gamma(1):\Gamma]} \tag{4.3.8.3}$$

where the scattering constant $\widetilde{C}_{jl}^{\Gamma_N}$ is the only unknown. On the left hand side, we can calculate the constant term out of the q-expansions of the modular forms. As in the proof of Proposition 4.3.5, we have for a cusp S_j

$$\begin{aligned} \log ||f_j|_{S_l}(z)||^2 = \log ||f_j|_k \gamma_l(z)||^2 &= \log \left(|f_j|_k \gamma_l|^2 (4\pi \operatorname{Im}(z))^k \right) \\ &= k \cdot \log(4\pi \operatorname{Im}(z)) + \log |f_j|_k \gamma_l(z)|^2 \end{aligned}$$

and

$$\begin{aligned}\log|f_j|_k\gamma_l(z)|^2 &= 2\operatorname{Re}\log(f_j|_k\gamma_l(z))\\ &= 2\operatorname{Re}\log\left(\sum_{n\geq 0} a_n q^{n/b_l}\right)\\ &= 2\operatorname{Re}\log\left(a_m q^{m/b_l}\left(1+\frac{a_{m+1}}{a_m}q^{1/b_l}+\dots\right)\right),\end{aligned}$$

where $q = e^{2\pi i z}$ and a_m is the first coefficient that is not zero. This is the one for $m = [\Gamma(1) : \Gamma]\cdot\frac{k}{12}$ if $S_j = S_l$ or $m = 0$ otherwise. From the local expansion of the logarithm $\log(1+x) = \sum_{k\geq 1}(-1)^{k+1}\frac{x^k}{k}$, we see that only $a_m q^{m/b_l}$ makes a contribution to the constant term. We get

$$-\log|f_j|_k\gamma_l(z)|^2 = \delta_{jl}4\pi\frac{y}{b_l}\cdot[\Gamma(1):\Gamma]\cdot\frac{k}{12} - 2\operatorname{Re}\log(a_m) + \text{higher terms}$$

and therefore

$$\begin{aligned}&\text{constant term}\left(-\frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}}\log||f_j|_k\gamma_l(z)||^2\right)\\ &\qquad = -\frac{12}{[\Gamma(1):\Gamma]}\log(4\pi y) + \delta_{jl}4\pi\frac{y}{b_j} - 2\operatorname{Re}\log(a_m)\cdot\frac{1}{[\Gamma(1):\Gamma]\cdot\frac{k}{12}}\end{aligned}$$

Furthermore, we use Remark 3.2.2 which gives

$$\widetilde{C}^{\Gamma}_{jk} = C^{\Gamma}_{jl} + \frac{3\log(b_l)}{\pi[\Gamma(1):\Gamma]}$$

to conclude the statement. □

Proposition 4.3.9. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups and let S_j be a cusp of Γ. Then we have with f_j as in Proposition 4.3.5*

$$\operatorname{Tr}_{\Gamma|\Gamma'} g^{\Gamma}_j(z) = -\frac{1}{[\Gamma(1):\Gamma']\frac{k}{12}}\log \operatorname{Nm}_{\Gamma|\Gamma'}||f_j||^2 + [\Gamma':\Gamma]A, \qquad (4.3.9.1)$$

where $\operatorname{Nm}_{\Gamma|\Gamma'}$ is the norm relative to Γ' and $A \in \mathbb{R}$ the constant from proposition 4.3.5 applied on Γ, S_j and f_j.

Proof: Follows from Corollary 4.3.7 of Proposition 4.3.5 and Remark 4.3.6 since the value of A is independent of the cusps in which we consider the Green's function to be expanded in. □

In Proposition 4.3.8 we used the Kronecker limit formula to get scattering constants. The sum Formula (4.3.9.1) we will use with the aim to calculate the constant A from the Kronecker limit formula (4.3.5.1).

Proposition 4.3.10. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups, let $S_j, S_l \in \mathbb{P}^1(\mathbb{Q})$ be cusps of Γ' of widths w_j and w_k, respectively, and $f_j \in M_k(\Gamma)$ a modular form with the properties of the one in Proposition 4.3.5. Then we have*

$$A_\Gamma = \frac{1}{[\Gamma' : \Gamma]} C_{jl}^{\Gamma'} - 4\pi C^{\Gamma(1)} - 12\log(w_l/w_j) + \frac{24}{k}\operatorname{Re}\log(a_m), \qquad (4.3.10.1)$$

where A_Γ is the constant occurring in the Kronecker limit formula comparing g_j^Γ and f_j (Corollary 4.3.7) and a_m from $\mathrm{Nm}_{\Gamma|\Gamma'} f_j = \sum_{n\geq m} a_n q^{n/w_l}$ with

$$m = \begin{cases} [\Gamma(1) : \Gamma'] \cdot \frac{k}{12} & \text{if } S_j \sim_{\Gamma'} S_l \\ 0 & \text{elsewise.} \end{cases}$$

Proof: The proof is similar to the one for Proposition 4.3.8 using $\mathrm{Tr}_{\Gamma|\Gamma'} g_j^\Gamma = g_j^{\Gamma'}$ as shown in Proposition 4.2.8. □

4.4. Applications to $\Gamma(2)$

The group $\Gamma(2)$ is of particular interest in this work, we introduced it in Chapter 1.1. Here we will calculate the Green's functions, the scattering matrices and Kronecker limit formulas for it.

At first, we will summarize the information we already introduced for $\Gamma(2)$.

Remember: The group $\Gamma(2)$ is of index 6 in $\Gamma(1)$, it is

$$\Gamma(2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod 2 \right\}$$

and it is a free group of rank 2 and generated by

$$\gamma_0 := \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right) \qquad \text{and} \qquad \gamma_1 := \left(\begin{smallmatrix} 1 & -2 \\ 2 & -3 \end{smallmatrix}\right).$$

The group $\Gamma(2)$ has three cusps of width 2. They are:

$$\left\{(p:q) \in \mathbb{P}^1(\mathbb{Q}) | (p,q) = 1, (p,q) \equiv (0,1) \mod 2\right\}$$
$$\left\{(p:q) \in \mathbb{P}^1(\mathbb{Q}) | (p,q) = 1, (p,q) \equiv (1,1) \mod 2\right\}$$
$$\left\{(p:q) \in \mathbb{P}^1(\mathbb{Q}) | (p,q) = 1, (p,q) \equiv (1,0) \mod 2\right\},$$

thus a system of representatives is $\{0, 1, \infty\}$.

To give the explicit formulas later on, here, we present the detailed computation of the Fourier expansion in ∞ of the Green's function (according Definition 4.2.1) for $\Gamma(2)$.

According to the Formula (4.2.4.1) we need the scattering constants and the numbers

$$r^{\Gamma(2)}_{jk,m}(c) := \sum_{\substack{d \mod 2c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma(2)\gamma_k}} e^{\pi i m \frac{d}{c}}.$$

with $j \in \{0, 1, \infty\}$, $k = \infty$, to calculate the expansions of $g_j^{\Gamma(2)}$ in ∞.

We will express the $r^{\Gamma(2)}_{jk,m}(c)$ via the coefficients $r^{\Gamma(1)}_m$ for $\Gamma(1)$.

The coefficients $r^{\Gamma(1)}$ are known.

Proposition 4.4.1. *For* $\Gamma(1)$ *we have*

$$r^{\Gamma(1)}_m(c) = \sum_{d|(c,m)} \mu\left(\frac{c}{d}\right) d,$$

where μ *stands for the Möbius function, defined via*

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree.} \\ 1 & \text{if } n = 1. \\ (-1)^k & \text{otherwise, where } k \text{ is the number of prime factors of } n. \end{cases}$$

Proof: See [vP05] or [Iwa02]. □

Proposition 4.4.2. *For the group* $\Gamma(2)$ *we have*

$$r^{\Gamma(2)}_{\infty\infty,m}(c) = \begin{cases} 0 & \text{if } c \equiv 1 \mod 2 \\ r^{\Gamma(1)}_m(2c) & \text{elsewise} \end{cases} \tag{4.4.2.1}$$

$$r^{\Gamma(2)}_{0\infty,m}(c) = \begin{cases} 0 & \text{if } c \equiv 0 \mod 2 \\ r^{\Gamma(1)}_m(c) & \text{elsewise} \end{cases} \tag{4.4.2.2}$$

$$r^{\Gamma(2)}_{1\infty,m}(c) = \begin{cases} 0 & \text{if } c \equiv 0 \mod 2 \\ r^{\Gamma(1)}_m(2c) & \text{elsewise.} \end{cases} \tag{4.4.2.3}$$

Proof: To get $r^{\Gamma(2)}_{jk}$ for two cusps S_j and S_k we have to know all numbers d, with $0 \le d < 2c$ and such that there exists $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma(2)\gamma_k$.

It is an easy calculation that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ it holds:

$$\begin{aligned} j = k = \infty: &\quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma(2)\gamma_k \iff \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2) \\ j = 0; k = \infty: &\quad \exists \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma(2)\gamma_k \iff (c, d) \equiv (1, 0) \mod 2 \\ j = 1; k = \infty: &\quad \exists \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma(2)\gamma_k \iff (c, d) \equiv (1, 1) \mod 2 \end{aligned}$$

With this information we can express the coefficients $r^{\Gamma(2)}_{jk,m}(c)$ with help of the ones for $\Gamma(1)$. The following formulas are dealing with the cases in which $r^{\Gamma(2)}_{jk,m}(c)$ is not zero, e.g. for $r_{\infty\infty}$ c is even, in the other two cases c is odd.

$$\begin{aligned}
r^{\Gamma(2)}_{\infty\infty,m}(c) &= \sum_{\substack{d \mod 2c \\ (d,c)=1}} e^{\pi i m \frac{d}{c}} \\
&= \sum_{\substack{d \mod 2c \\ (d,2c)=1}} e^{2\pi i m \frac{d}{2c}} \\
&= r^{\Gamma(1)}_{m}(2c) \\
r^{\Gamma(2)}_{0\infty,m}(c) &= \sum_{\substack{d \mod 2c \\ (c,d)=1, d\equiv 0(2)}} e^{\pi i m \frac{d}{c}} \\
&= \sum_{\substack{d \mod c \\ (c,d)=1}} e^{\pi i m \frac{2d}{c}} \\
&= r^{\Gamma(1)}_{m}(c) \\
r^{\Gamma(2)}_{1\infty,m}(c) &= \sum_{\substack{d \mod 2c \\ (c,d)=1, d\equiv 1(2)}} e^{\pi i m \frac{d}{c}} \\
&= \sum_{\substack{d \mod 2c \\ (2c,d)=1}} e^{2\pi i m \frac{d}{2c}} \\
&= r^{\Gamma(1)}_{m}(2c)
\end{aligned}$$

□

Remark 4.4.3. Proposition 4.4.2 with its description of the coefficients of $\Gamma(2)$ via the ones for $\Gamma(1)$ can be used to calculate the scattering constants for $\Gamma(2)$. For the group $\Gamma(2)$ there exist only two different scattering constants, although $\Gamma(2)$ has three cusps. These constants are

$$\begin{aligned}
C_a^{\Gamma(2)} &= -\frac{1}{3\pi}\left(36\zeta'(-1) - 3 + 3\log(4\pi) + 7\log(2)\right) \\
&= \frac{1}{6}C^{\Gamma(1)} - \frac{7}{3\pi}\log(2), && (4.4.3.1) \\
C_b^{\Gamma(2)} &= -\frac{1}{3\pi}\left(36\zeta'(-1) - 3 + 3\log(4\pi) + \log(2)\right) \\
&= \frac{1}{6}C^{\Gamma(1)} - \frac{1}{3\pi}\log(2), && (4.4.3.2)
\end{aligned}$$

where the first case $C_a^{\Gamma(2)}$ is the one with $S_j = S_k$ and in the second case $C_b^{\Gamma(2)}$ we have $S_j \neq S_k$.

For a detailed calculation see Appendix C.

To prepare the calculations of the sum of the r's, here are some identities:

Lemma 4.4.4. *It holds*

(i) $\sum_{c\geq 1}\frac{\mu(2c)}{(2c)^s}=\frac{1}{1-2^s}\cdot\frac{1}{\zeta(s)}$,

(ii) $\sum_{c\geq 1}\frac{\mu(2c-1)}{(2c-1)^s}=\frac{2^s}{2^s-1}\cdot\frac{1}{\zeta(s)}$,

(iii) $\sum_{c\geq 1}\frac{\mu(4c-2)}{(4c-2)^s}=\frac{1}{1-2^s}\cdot\frac{1}{\zeta(s)}$ *and*

(iv) $\sum_{c\geq 1}\frac{\mu(4c)}{(4c)^s}=0$.

Proof: The inverse of the Riemann Zeta function has a description as

$$\frac{1}{\zeta(s)}=\prod_{i\geq 1}\left(1-\frac{1}{p_i^s}\right),$$

where the product is taken over all prime numbers and p_i denotes the i-th prime number. We have

$$\begin{aligned}\sum_{c\geq 1}\frac{\mu(2c)}{(2c)^s}&=\frac{1}{2^s}\frac{\mu(2c)}{(c)^s}\\&=\frac{1}{2^s}\left(-1+\sum_{i>1}\frac{1}{p_i^s}-\sum_{1<i<j}\frac{1}{p_i^sp_j^s}+\sum_{1<i<j<k}\frac{1}{p_i^sp_j^sp_k^s}\cdots\right)\\&=-\frac{1}{2^s}\prod_{i\geq 2}\left(1-\frac{1}{p_i^s}\right)\\&=-\frac{1}{2^s}\left(1-\frac{1}{2^s}\right)^{-1}\frac{1}{\zeta(s)}\\&=\frac{1}{1-2^s}\cdot\frac{1}{\zeta(s)}.\end{aligned}$$

The second and the third formula from the lemma are

$$\begin{aligned}\sum_{c\geq 1}\frac{\mu(2c-1)}{(2c-1)^s}&=\sum_{c\geq 1}\frac{\mu(c)}{c^s}-\sum_{c\geq 1}\frac{\mu(2c)}{(2c)^s}\\&=\frac{1}{\zeta(s)}-\frac{1}{1-2^s}\cdot\frac{1}{\zeta(s)}\\&=\frac{2^s}{2^s-1}\cdot\frac{1}{\zeta(s)}\end{aligned}$$

and

$$\begin{aligned}\sum_{c\geq 1}\frac{\mu(4c-2)}{(4c-2)^s} &= \frac{1}{2^s}\sum_{c\geq 1}\frac{\mu(2)\mu(2c-1)}{(2c-1)^s}\\ &= -\frac{1}{2^s}\sum_{c\geq 1}\frac{\mu(2c-1)}{(2c-1)^s}\\ &= \frac{1}{1-2^s}\cdot\frac{1}{\zeta(s)}.\end{aligned}$$

(Here, the Möbius function is multiplicative, since 2 and $2c-1$ are coprime.)

The fourth claim from the lemma follows directly, since $4c$ is never squarefree and $\mu(4c) = 0$ for all c. □

Proposition 4.4.5. *For* $\Gamma(2)$ *holds*

$$\sum_{c\geq 0}\frac{1}{c^{2s}}r^{\Gamma(2)}_{\infty\infty,m}(c) = \frac{2^{2s}}{\zeta(2s)}\left(\frac{1}{1-2^{2s}}\sum_{\substack{d|m\\ d\equiv 2(4)}} d^{-2s+1} + \sum_{\substack{d|m\\ d\equiv 0(4)}} d^{-2s+1}\right) \tag{4.4.5.1}$$

$$\sum_{c\geq 0}\frac{1}{c^{2s}}r^{\Gamma(2)}_{0\infty,m}(c) = \frac{2^{2s}}{2^{2s}-1}\cdot\frac{1}{\zeta(2s)}\sum_{\substack{d|m\\ d\equiv 1(2)}} d^{-2s+1} \tag{4.4.5.2}$$

$$\sum_{c\geq 0}\frac{1}{c^{2s}}r^{\Gamma(2)}_{1\infty,m}(c) = \frac{2^{2s}}{\zeta(2s)}\left(\frac{1}{1-2^{2s}}\sum_{\substack{d|m\\ d\equiv 1(2)}} d^{-2s+1} + \frac{2^{2s}}{2^{2s}-1}\sum_{\substack{d|m\\ d\equiv 2(4)}} d^{-2s+1}\right). \tag{4.4.5.3}$$

Proof: The claim follows with Proposition 4.4.2 and Lemma 4.4.4. We show the calculation for the first formula to illustrate the procedure.

$$\begin{aligned}\sum_{c\geq 0}\frac{1}{c^{2s}}r^{\Gamma(2)}_{\infty\infty,m}(c) &= \sum_{c\geq 1}\frac{1}{(2c)^{2s}}r^{\Gamma(1)}_{m}(2c)\\ &= \sum_{c\geq 1}\frac{1}{(2c)^{2s}}\sum_{d|(4c,m)}\mu\left(\frac{4c}{d}\right)d\\ &= \sum_{\substack{c\geq 1\\ d|(4c,m)}}\mu\left(\frac{4c}{d}\right)\frac{1}{\left(\frac{4c}{d}\right)^{2s}}d^{-2s+1}2^{2s}\end{aligned}$$

$$= 2^{2s}\left(\sum_{\substack{d|m\\ d\equiv 1(2)}}\sum_{c\geq 1}\mu(4c)\frac{1}{(4c)^{2s}}d^{-2s+1} + \sum_{\substack{d|m\\ d\equiv 2(4)}}\sum_{c\geq 1}\mu(2c)\frac{1}{(2c)^{2s}}d^{-2s+1}\right.$$

$$\left.+ \sum_{\substack{d|m\\ d\equiv 0(4)}}\sum_{c\geq 1}\mu(c)\frac{1}{c^{2s}}d^{-2s+1}\right)$$

$$= \frac{2^{2s}}{\zeta(2s)}\left(\frac{1}{1-2^{2s}}\sum_{\substack{d|m\\ d\equiv 2(4)}} d^{-2s+1} + \sum_{\substack{d|m\\ d\equiv 0(4)}} d^{-2s+1}\right).$$

□

Now, we can come to the Fourier expansion of the Green's functions.

Theorem 4.4.6. *The Green's functions for* $\Gamma(2)$ *have the following expansions*

$$g_\infty^{\Gamma(2)}(\gamma_\infty z) = 2\pi y - \frac{16}{3}\log(2) - 2\log(4\pi y)$$
$$+ 8\sum_{m>0}\left(4\sum_{\substack{d|m\\ d\equiv 0(2)}} d^{-1} - \sum_{d|m} d^{-1}\right)e^{-2\pi m y}\cos(2\pi m x) \qquad (4.4.6.1)$$

$$g_0^{\Gamma(2)}(\gamma_\infty z) = \frac{8}{3}\log(2) - 2\log(4\pi y) + 16\sum_{m>0}\left(\sum_{\substack{d|m\\ d\equiv 1(2)}} d^{-1}\right)e^{-\pi m y}\cos(\pi m x) \qquad (4.4.6.2)$$

$$g_1^{\Gamma(2)}(\gamma_\infty z) = \frac{8}{3}\log(2) - 2\log(4\pi y) + 16\sum_{m>0}\left((-1)^m\sum_{\substack{d|m\\ d\equiv 1(2)}} d^{-1}\right)e^{-\pi m y}\cos(\pi m x). \qquad (4.4.6.3)$$

Proof: The Fourier expansion is given by Lemma 4.2.4 as

$$g_j^{\Gamma(2)}(\gamma_k z) = \delta_{jk}2\pi y + 4\pi\left(\widetilde{C}_{jk}^{\Gamma(2)} - \frac{C^{\Gamma(1)}}{6}\right) - 2\left(\log(4\pi y) - \log(2)\right)$$
$$+ \pi^2\sum_{m\neq 0} e^{-2\pi|m|y/2}e^{2\pi i m x/2}\sum_{c>0}\frac{1}{c^2}r_{jk,m}^{\Gamma}(c)$$

(use $b_j = b_k = 2$, $[\Gamma(1):\Gamma(2)] = 6$).

The scattering constants $C^{\Gamma(1)}$ and $C_{jk}^{\Gamma(2)}$ has been given in Example 3.2.3 and Remark

4.4.3, we conclude with Remark 3.2.2 that

$$\widetilde{C}_a^{\Gamma(2)} = \frac{1}{6}C^{\Gamma(1)} - \frac{11}{6\pi}\log(2) \quad \text{and}$$
$$\widetilde{C}_b^{\Gamma(2)} = \frac{1}{6}C^{\Gamma(1)} + \frac{1}{6\pi}\log(2).$$

In the case of $g_\infty^{\Gamma(2)}$ follows with the results from Proposition 4.4.5 (and $\zeta(2) = \frac{\pi^2}{6}$) that

$$\pi^2 e^{-\pi|m|y} e^{\pi imx} \sum_{c>0} \frac{1}{c^2} r_{\infty\infty,m}^{\Gamma(2)}(c) = \frac{4\pi^2}{\zeta(2)} \left(-\frac{1}{3} \sum_{\substack{d|m \\ d\equiv 2(4)}} d^{-1} + \sum_{\substack{d|m \\ d\equiv 0(4)}} d^{-1} \right) e^{-\pi|m|y} e^{\pi imx}$$
$$= 8 \left(-\sum_{\substack{d|m \\ d\equiv 2(4)}} d^{-1} + 3 \sum_{\substack{d|m \\ d\equiv 0(4)}} d^{-1} \right) e^{-\pi|m|y} e^{\pi imx}.$$

We realize, that all parts but $e^{\pi imx}$ are independent of the sign of m and

$$e^{\pi imx} + e^{-\pi imx} = 2\cos(\pi mx).$$

Furthermore, the coefficient is zero for odd m, for even m all d's in both sum are even and we can divide by m and simplify the congruences. We get

$$\pi^2 \sum_{m\neq 0} e^{-\pi|m|y} e^{\pi imx} \sum_{c>0} \frac{1}{c^2} r_{\infty\infty,m}^{\Gamma(2)}(c) = 4 \sum_{m>0} \left(-\sum_{\substack{d|m \\ d\equiv 1(2)}} d^{-1} + 3 \sum_{\substack{d|m \\ d\equiv 0(2)}} d^{-1} \right) e^{-2\pi|m|y} e^{2\pi imx}$$
$$= 8 \sum_{m>0} \left(4 \sum_{\substack{d|m \\ d\equiv 0(2)}} d^{-1} - \sum_{d|m} d^{-1} \right) e^{-2\pi my} \cos(2\pi mx).$$

The last formula and $\widetilde{C}_a^{\Gamma(2)}$ leads to the first claim in the proposition.

The other two cases are similar. The only thing more difficult is to realize that in the case of the cusp 1, the formula from Proposition 4.4.5 describes an alternating sum. □

Kronecker limit formula for $\Gamma(2)$: For a Kronecker limit formula for $\Gamma(2)$, we need the Green's function that we treated in Section 4.4 and suitable modular forms. Modular forms for $\Gamma(2)$ are known and their behavior in the cusps.

We have

Lemma 4.4.7. *We define (with* $q = e^{2\pi i z}$*):*

$$\theta^2(z) = \prod_{n\geq 1} (1-q^n)^4 \left(1+q^{n-1/2}\right)^8 \tag{4.4.7.1}$$

$$\lambda(z) = -\frac{1}{16} q^{-1/2} \prod_{n\geq 1} \left(\frac{1-q^{n-1/2}}{1+q^n}\right)^8 \tag{4.4.7.2}$$

$$(1-\lambda)(z) = \frac{1}{16} q^{-1/2} \prod_{n\geq 1} \left(\frac{1+q^{n-1/2}}{1+q^n}\right)^8 \tag{4.4.7.3}$$

Then $\theta^2(z)$ *is a modular form for* $\Gamma(2)$ *of weight* 2*, the functions* $\lambda(z)$ *and* $(1-\lambda)(z)$ *are modular functions for the same group. They have the following divisors*

$$\operatorname{div}\theta^2 = 1\cdot 1, \quad \operatorname{div}\lambda = 1\cdot 0 - 1\cdot\infty, \quad \operatorname{div}(1-\lambda) = 1\cdot 1 - 1\cdot\infty.$$

Proof: See [Yan96], or, for more background information, [Miy06] and [EMOT81]. □

Based on that information, we can specify modular forms with exactly one single zero in one cusp. We multiply with $2^{-4/3}$ to simplify the formulas that will appear later on.

Corollary 4.4.8. *The modular forms*

$$G_0(z) := 2^{-4/3}\frac{\lambda(z)}{1-\lambda(z)}\theta^2(z) \tag{4.4.8.1}$$

$$G_1(z) := 2^{-4/3}\theta^2(z) \tag{4.4.8.2}$$

$$G_\infty(z) := 2^{-4/3}\frac{1}{1-\lambda(z)}\theta^2(z) \tag{4.4.8.3}$$

from $M_2(\Gamma(2))$ *have divisors*

$$\operatorname{div} G_0 = 1\cdot 0, \quad \operatorname{div} G_1 = 1\cdot 1, \quad \operatorname{div} G_\infty = 1\cdot\infty.$$

Remark 4.4.9. Concerning the multiplication with a power of 2: We always have to correct the $\log(2)$ terms. This comes from the problem that we consider $X(\Gamma(2))$ as the $\mathbb{P}^1$. But in opposite to $\mathbb{P}^1$ the curve $X(\Gamma(2))$ has bad reduction in 2.

To calculate the logarithm of the Petersson norm, the next lemma is helpful.

Lemma 4.4.10. *It holds*

$$\sum_{n\geq 1}\sum_{k\geq 1}\frac{q^{kn}}{k} = \sum_{n\geq 1} q^n \left(\sum_{d|n} d^{-1}\right)$$

$$= 2\sum_{n}\left(\sum_{\substack{d|n\\ d\equiv 0(2)}} d^{-1}\right) q^{n/2},$$

$$\sum_{n\geq 1}\sum_{k\geq 1}(-1)^{k+1}\frac{q^{kn}}{k} = \sum_{n\geq 1} q^n \frac{1}{2^m}\left(\sum_{\substack{d|n\\ d\equiv 1(2)}} d^{-1}\right)$$

$$= \sum_{n\geq 1} q^n \left(\sum_{\substack{d|n\\ d\equiv 1(2)}} d^{-1} - \sum_{\substack{d|n\\ d\equiv 0(2)}} d^{-1}\right),$$

$$\sum_{n\geq 1}\sum_{k\geq 1}\frac{q^{k(n-1/2)}}{k} = \sum_{n\geq 1} q^{n/2}\frac{1}{2^m}\left(\sum_{\substack{d|n\\ d\equiv 1(2)}} d^{-1}\right),$$

$$\sum_{n\geq 1}\sum_{k\geq 1}(-1)^{k+1}\frac{q^{k(n-1/2)}}{k} = -\sum_{n\geq 1}(-1)^n q^{n/2}\frac{1}{2^m}\left(\sum_{\substack{d|n\\ d\equiv 1(2)}} d^{-1}\right).$$

The number 2^m is always the biggest power of 2 that divides n.

Proof: The proof of these formulas in not complicated. Here a few steps for the second one:

$$\sum_{n}\sum_{k}(-1)^{k+1}\frac{q^{kn}}{k} = \sum_{n\geq 1}\left(\sum_{d|n}(-1)^{d+1}d^{-1}\right) q^n$$

$$= \sum_{n\geq 1}\left(\sum_{\substack{d|n\\ d\equiv 1(2)}} d^{-1}\left(1-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^m}\right)\right)\right) q^n$$

$$= \sum_{n\geq 1}\left(\frac{1}{2^m}\sum_{\substack{d|n\\ d\equiv 1(2)}} d^{-1}\right) q^n$$

or $$= \sum_{n\geq 1} q^n\left(\sum_{\substack{d|n\\ d\equiv 1(2)}} d^{-1} - \sum_{\substack{d|n\\ d\equiv 0(2)}} d^{-1}\right)$$

□

Proposition 4.4.11. *For the modular forms from Corollary 4.4.8 hold:*

$$\log||G_0(z)||^2 = -\frac{8}{3}\log(2) + 2\log(4\pi y) - 16\sum_n \left(\sum_{\substack{d|n \\ d\equiv 1(2)}} d^{-1}\right) e^{-\pi n y}\cos(\pi n x) \tag{4.4.11.1}$$

$$\log||G_1(z)||^2 = -\frac{8}{3}\log(2) + 2\log(4\pi y) - 16\sum_n (-1)^n \left(\sum_{\substack{d|n \\ d\equiv 1(2)}} d^{-1}\right) e^{-\pi n y}\cos(\pi n x) \tag{4.4.11.2}$$

$$\begin{aligned}\log||G_\infty(z)||^2 = {}& \frac{16}{3}\log(2) - 2\pi y + 2\log(4\pi y) \\ & - 8\sum_n \left(4\sum_{\substack{d|n \\ d\equiv 0(2)}} d^{-1} - \sum_{d|n} d^{-1}\right) e^{-2\pi n y}\cos(2\pi n x),\end{aligned} \tag{4.4.11.3}$$

where $z = x + iy$.

Proof: Start with the q-expansions of the modular forms; they are known with the Lemma 4.4.7 and Corollary 4.4.8. With help of

$$\log|z|^2 = 2\operatorname{Re}\log z, \quad \operatorname{Re} q = \cos(2\pi x), \quad \log(1+x) = \sum(-1)^{k+1}\frac{x^k}{k}$$

and Lemma 4.4.10 one can get the required formulas.
Here, we present the last case:

$$\begin{aligned}\log||G_\infty||^2 &= \log\left|\left|2^{-4/3}\theta^2(z)\frac{1}{1-\lambda(z)}\right|\right|^2 \\ &\overset{(1)}{=} \log\left((4\pi y)^2\left|2^{-4/3}\right|^2\left|\theta^2(z)\frac{1}{1-\lambda(z)}\right|^2\right) \\ &= 2\log(4\pi y) - \frac{8}{3}\log(2) + 2\operatorname{Re}\log\left(\theta^2(z)\frac{1}{1-\lambda(z)}\right) \\ &\overset{(2)}{=} 2N^2\log(4\pi y) + 2N^2\operatorname{Re}\log\left(16q^{1/2}\prod_{n\geq 1}(1-q^n)^4(1+q^n)^8\right) \\ &= 2\log(4\pi y) - \frac{8}{3}\log(2) + 2\log(16) - 2\pi y \\ &\quad + 2\operatorname{Re}\log\left(\prod_{n\geq 1}(1-q^n)^4(1+q^n)^8\right)\end{aligned}$$

Equation (1) uses the definition of the Petersson norm 4.1.10 and in (2) the expansion of $\theta^2(z)\frac{1}{1-\lambda(z)}$ from Lemma 4.4.7 has been simplified.
Furthermore

$$\begin{aligned}
&\operatorname{Re}\log\left(\prod_{n\geq 1}(1-q^n)^4(1+q^n)^8\right)\\
&=\operatorname{Re}\left(4\sum_n\log(1-q^n)+8\sum_n\log(1+q^n)\right)\\
&=4\operatorname{Re}\left(\sum_n\sum_k(-1)^{k+1}\frac{(-1)^k q^{kn}}{k}+2\sum_n\sum_k(-1)^{k+1}\frac{q^{kn}}{k}\right)\\
&=4\operatorname{Re}\left(-\sum_{n\geq 1}q^n\left(\sum_{d|n}d^{-1}\right)+2\sum_{n\geq 1}q^n\left(\sum_{\substack{d|n\\ d\equiv 1(2)}}d^{-1}-\sum_{\substack{d|n\\ d\equiv 0(2)}}d^{-1}\right)\right)\\
&=-4\operatorname{Re}\left(\sum_{n\geq 1}q^n\left(4\sum_{\substack{d|n\\ d\equiv 0(2)}}d^{-1}-\sum_{d|n}d^{-1}\right)\right)\\
&=-4\sum_{n\geq 1}\left(4\sum_{\substack{d|n\\ d\equiv 0(2)}}d^{-1}-\sum_{d|n}d^{-1}\right)e^{-2\pi ny}\cos(2\pi nx).
\end{aligned}$$

The combination of the results presented yields the claim.

In the other cases it is helpful to use the following identities that are easy to show: Let n be a natural number, then

$$\sum_{\substack{d|n\\ d\equiv 0(2)}}d^{-1}=\left(1-\frac{1}{2^m}\right)\sum_{\substack{d|n\\ d\equiv 1(2)}}d^{-1}$$

and

$$1-2^{-m}(1-(-1)^n)=(-1)^n,$$

where 2^m the biggest power of 2 that divides n. □

If we compare these results with the expansions of the Green's functions for $\Gamma(2)$, we get limit formulas.

Proposition 4.4.12. *For the group* $\Gamma(2)$ *holds*

$$4\pi \lim_{s\to 1}\left(E_i^{\Gamma(2)}(z,s) - \frac{1}{vol(\Gamma(2))(s-1)}\right) =$$
$$- \log ||G_i(z)||^2 + 4\frac{\zeta'(-1)}{\zeta(-1)} - 2\log(4\pi) + 4 - 2\log(2), \quad (4.4.12.1)$$

where $i \in \{0, 1, \infty\}$ *denotes one of the three cusps of* $\Gamma(2)$*,* $E_i^{\Gamma(2)}(z,s)$ *is a Eisenstein series and* G_i *a modular form Corollary 4.4.8.*

Proof: We have

$$4\pi \lim_{s\to 1}\left(E_i^{\Gamma(2)}(z,s) - \frac{1}{vol(\Gamma(2))(s-1)}\right) = g_i^{\Gamma(2)} + \frac{2}{3}\pi C^{\Gamma(1)} + 2(\log(4\pi) - \log(2)).$$

Now, compare the formulas of the expansion of the $g_i^{\Gamma(2)}$'s from Theorem 4.4.6 with the q-expansions of the modular forms in Proposition 4.4.11, realize that they are equal and get the result by using Formula (4.3.3.1). □

Like in the general case we get a formulation with the Green's functions as well.

Corollary 4.4.13. *For the group* $\Gamma(2)$ *holds*

$$g_i^{\Gamma(2)}(z) = -\log ||G_i(z)||^2, \quad (4.4.13.1)$$

where $i \in \{0, 1, \infty\}$ *denotes one of the three cusps of* $\Gamma(2)$*,* $g_i^{\Gamma(2)}$ *is the Green's function defined in Definition 4.2.1 and* G_i *a modular form Corollary 4.4.8.*

Proof: Follows from Proposition 4.4.12. Compare the formulas form Theorem 4.4.6 and Proposition 4.4.11. □

5. Fermat curves

In this chapter, we want to apply methods and results introduced before. We will take the Fermat curves to do so. There are several reasons why the Fermat curves are in particular suited for that. The first motivation to treat Fermat curves aiming for their scattering constants is that nearly all Fermat curves are associated to non-congruence subgroups (nearly all means all but four).

The other reasons are more pragmatic, they allow us to deal with Fermat curves, i.e. for Fermat curves we have knowledge that is necessary to apply the ideas explained in the first chapters. In particular, we know a Belyi map for these curves that makes them accessible to the theory of Dessins d'Enfants. Additionally, modular forms for the Fermat curves have been studied, i.e. we can try to establish a Kronecker limit formula as explained in Section 4.3 and use it to determine the scattering constants.

We will start with some basic introduction to Fermat curves. After that, we will tackle the problem of Dessins. Here the focus lies on finding the correspondences between the cusps of the Belyi pair and the ones of the subgroup. In the context of the Fermat Dessins we will deal once more with the question if the Fermat curves are congruence subgroups. We will present a proof by giving counter examples that this is not the case.

In Section 5.3 Kronecker limit formulas for the Fermat curves are established.

The Section 5.4 presents the calculation of the scattering constants for Fermat curves. For that we combine the results from Section 5.3 and Section 2.1 with some additional properties of the Fermat curves.

Finally, in Section 5.5 we will explain some calculation that support the other results of this chapter.

5.1. Basics on Fermat curves and the associated subgroups

As a projective curve, the well known Fermat curve is given by

Definition 5.1.1. *Let $N \in \mathbb{N}$. The N-th Fermat curve is given by the equation*

$$F_N : \quad X^N + Y^N = Z^N. \tag{5.1.1.1}$$

For the Fermat curves Belyi morphisms are known.

Lemma 5.1.2. *The map*

$$\begin{aligned} \boldsymbol{\beta}_N : \quad F_N &\longrightarrow \mathbb{P}^1 \\ (X : Y : Z) &\longmapsto (X^N : Z^N) \end{aligned} \tag{5.1.2.1}$$

is a Belyi map for F_N. Its degree is N^2. The ramification points are

$$\begin{aligned} a_j &:= (0 : \zeta^j : 1) \\ b_j &:= (\zeta^j : 0 : 1) \\ c_j &:= (\epsilon\zeta^j : 1 : 0), \end{aligned}$$

where $\zeta = e^{2\pi i/N}$ is the first primitive N-th root of unity, $j \in \{0, \dots N-1\}$ and $\epsilon = e^{\pi i/N}$. Each point has ramification index N.

Proof: Simple calculation. □

Consider the subgroup Γ_N of $\Gamma(2)$ that we will describe in Lemma 5.1.3. This group is a subgroup corresponding to the Belyi pair $(F_N, \boldsymbol{\beta}_N)$ such that

$$\Gamma_N \setminus \mathbb{H} \cong F_N(\mathbb{C}) \setminus \{\text{ramification points}\}.$$

Lemma 5.1.3. *The group Γ_N is the kernel of*

$$\begin{aligned} \Gamma(2) &\longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}, \\ \gamma &\longmapsto (R_0(\gamma), R_1(\gamma)) \mod N, \end{aligned}$$

where $R_i(\gamma)$ denotes the number of generators $\boldsymbol{\gamma}_i$ for $\Gamma(2)$ (see Equation (1.1.6.2) on page 8) that occur in the word description of γ:
Let $\gamma \in \Gamma(2)$ be given via its word in $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$ as $\gamma = \prod_{i=1}^{n} \kappa_i^{r_i}$ with $n \in \mathbb{N}$, $r_i \in \mathbb{Z}$, $\kappa_i \in \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1\}$. Then

$$R_0(\gamma) = \sum_{\kappa_i = \boldsymbol{\gamma}_0} r_i \quad \text{and} \quad R_1(\gamma) = \sum_{\kappa_i = \boldsymbol{\gamma}_1} r_i.$$

Proof: See [MR87] with different generators. □

There is a way to describe R_0 and R_1 explicitly via Dedekind sums.

Remark 5.1.4. In [MR87] the group Γ_N is also described by the kernel of

$$\begin{aligned} \Gamma(2) &\longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}, \\ \gamma &\longmapsto (s(\gamma), t(\gamma)) \mod N. \end{aligned}$$

The numbers $s(\gamma)$ and $t(\gamma)$ are calculated as follows: Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where we can assume c to be non-negative (since $\gamma = -\gamma$), then

$$s(\gamma) = \begin{cases} b/2 & \text{if} \quad c = 0 \\ \frac{a+d}{2c} - 4\left(D(d, c/2) - D(d-c, 2c)\right) & \text{if} \quad c > 0 \end{cases}$$

$$t(\gamma) = \begin{cases} 0 & \text{if} \quad c = 0 \\ 4\left(D(d-c, 2c) - D(d, 2c)\right) & \text{if} \quad c > 0 \end{cases}$$

with

$$D(d,c) = \sum_{0\le j<c} ((j/c))((jd/c))$$

where

$$((x)) = \begin{cases} x-[x]-1/2 & \text{if} \quad x\notin\mathbb{Z} \\ 0 & \text{elsewise.} \end{cases}$$

Some facts about Γ_N.

Lemma 5.1.5. *We have $[\Gamma(1):\Gamma_N]=6N^2$, the group has $3N$ cusps, all of same width $b=2N$. A system of representatives for the cosets $\Gamma_N\setminus\Gamma(2)$ is*

$$\boldsymbol{\gamma}_0^a\boldsymbol{\gamma}_1^b \qquad \textit{with} \qquad a,b\in 0,\ldots,N-1. \tag{5.1.5.1}$$

A system of representatives for the cusps is $S=S_0\cup S_1\cup S_\infty$ with

$$S_0=\{0,2,\ldots,2N-2\}, S_1=\{1,3,\ldots,2N-1\}, S_\infty=\left\{\frac{1}{2},\frac{1}{4},\ldots,\frac{1}{2N}\right\}. \tag{5.1.5.2}$$

The cusps in S_i are $\Gamma(2)$-equivalent to i ($i\in\{0,1,\infty\}$).

Proof: All but the system of cusps follows from lemmas 1.4.2, 5.1.2 and 5.1.3.

To show, that the set from Equation (5.1.5.2) is really a system of representatives, it suffices to show, that the $3N$ elements proposed are not equivalent under Γ_N. If there are equivalent cusps in this set, then they are in the same set out of S_0, S_1 and S_∞.

Let S and S' be two of the proposed cusps $\Gamma(2)$-equivalent to the same cusps i ($i\in\{0,1,\infty\}$). Then, for a $\gamma_{SS'}$ with $\gamma_{SS'}(S')=S$, we can write

$$\kappa_{SS'}{=}\gamma_S\circ\tau_m\circ\gamma_{S'}^{-1}=\gamma_{S\,i}\circ\gamma_i\circ\tau_m\circ\gamma_i^{-1}\circ\gamma_{S'\,i}^{-1}. \tag{5.1.5.3}$$

where $\gamma_S(\infty)=S$, $\tau_m=\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)$ and γ's with double index as above. Attention: Here we have to take care to distinguish γ_i and $\boldsymbol{\gamma}_i$. In the next table on the left hand side only γ_i occurs, a matrix with $\gamma_i(\infty)=i$, on the right hand only $\boldsymbol{\gamma}_i$, one of the generators of $\Gamma(2)$ with $\langle\boldsymbol{\gamma}_i\rangle=Stab_{\Gamma(2)}(i)$.
If m is even, we get as a word in the generators (j,l even, k odd):

$$\begin{array}{rclcl}
\gamma_{j,0} & = & \left(\begin{smallmatrix}1-2j&-j\\-2&1\end{smallmatrix}\right) & = & \boldsymbol{\gamma}_1^{-1}(\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0)^{1-j/2}\\
\gamma_{k,1} & = & \left(\begin{smallmatrix}1-2k&-k-1\\-2&-1\end{smallmatrix}\right) & = & (\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1)^{(1-k)/2}\boldsymbol{\gamma}_1\\
\gamma_{1/l,\infty} & = & \left(\begin{smallmatrix}1&0\\l&1\end{smallmatrix}\right) & = & \boldsymbol{\gamma}_0^{-l/2}\\
\gamma_0\circ\tau_m\circ\gamma_0^{-1} & = & \left(\begin{smallmatrix}1&0\\-m&1\end{smallmatrix}\right) & = & \boldsymbol{\gamma}_0^{m/2}\\
\gamma_1\circ\tau_m\circ\gamma_1^{-1} & = & \left(\begin{smallmatrix}1-m&m\\-m&1+m\end{smallmatrix}\right) & = & \boldsymbol{\gamma}_1^{m/2}\\
\gamma_\infty\circ\tau_m\circ\gamma_\infty^{-1} & = & \left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right) & = & (\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1)^{-m/2}
\end{array}$$

After these preparations, we can determine when the matrix in Formula (5.1.5.3) is in

Γ_N. Let j and j' be two integer even cusps, then the generator $\boldsymbol{\gamma}_1$ appears

$$-1+\left(1-\frac{j}{2}\right)+1+\left(\frac{j'}{2}-1\right)=\frac{j'-j}{2}$$

times in Formula (5.1.5.3). Hence, the matrix is in Γ_N if and only if $j'-j \mid 2N$. For odd integer cusps k and k' we get the condition $k'-k \mid 2N$ and for $\frac{1}{l}$ and $\frac{1}{l'}$ $(l \in 2\mathbb{Z})$ $-m \mid 2N$ and $l'-l \mid 2N$ have to hold for the matrix to be in Γ_N.
By this, the statement follows. □

5.2. The Dessins for Fermat curves

We would like to know the Dessins for the Fermat curves, together with the information which cusp of the curve corresponds to which cusp of the group Γ_N.

Proposition 5.2.1. *The abstract graph associated to the full Dessin of the N-th Fermat curve is the complete tripartite graph with three times N vertices $K_{N,N,N}$.*

Proof: The statement can be found in [LZ04].

For the sequel it is helpful to see a full proof:
It is already known that each cusp has ramification index N. Hence, to show the proposition it is enough to find a path on the curve from each cusp to every cusp lying above the other critical values that is mapped to $\mathbb{R}$. The paths in the last column of the following table will do. In there, ζ_1 and ζ_2 denote two N-th roots of unity and $\sqrt[N]{x}$ is always the positive real root, if x is positive or N odd. If N is even and x negative then $\sqrt[N]{x}=\epsilon\sqrt[N]{|x|}$, where $\epsilon^N=-1$.

Cusps above	explicit description	path $(\lambda \in [0,1])$	
$0 \longleftrightarrow 1$	$(0:\zeta_1:1) \longleftrightarrow (\zeta_2:0:1)$	$\left(\zeta_2\sqrt[N]{\lambda}:\zeta_1\sqrt[N]{1-\lambda}:1\right)$	
$0 \longleftrightarrow \infty$	$(0:1:\zeta_1) \longleftrightarrow (\epsilon\zeta_2:1:0)$	$\left(\zeta_2\sqrt[N]{\lambda-1}:1:\zeta_1\sqrt[N]{\lambda}\right)$	(5.2.1.1)
$1 \longleftrightarrow \infty$	$(1:0:\zeta_1) \longleftrightarrow (1:\epsilon\zeta_2:0)$	$\left(1:\zeta_2\sqrt[N]{\lambda-1}:\zeta_1\sqrt[N]{\lambda}\right)$	

□

The genus of the Fermat curve grows polynomial in N, but when we cut the Dessin D_N open along its edges until it is flat, we get a N-petaled flower, where each petal has $2N$ vertices (all vertices of the Dessin) and the vertices of one type (white or black) surround a cell always in the same order.
For $N=3$ the genus is 1, i.e. the Dessin has to be drawn on the torus. In Figure 5.1 on page 101 the Dessin D_3 can be seen on the torus and in Figure 5.2 cut open (edges with same number has to be identified). The Dessin for $N=5$ can be found in Figure 5.3 on page 102. Again, the Dessin has to be glued at the edges with same numbers, such that the color of the vertices is respected. In Figure 5.4 on page 103 we see an attempt to draw the Dessin for a general N. Here the edges are numbered by Arabic numbers:

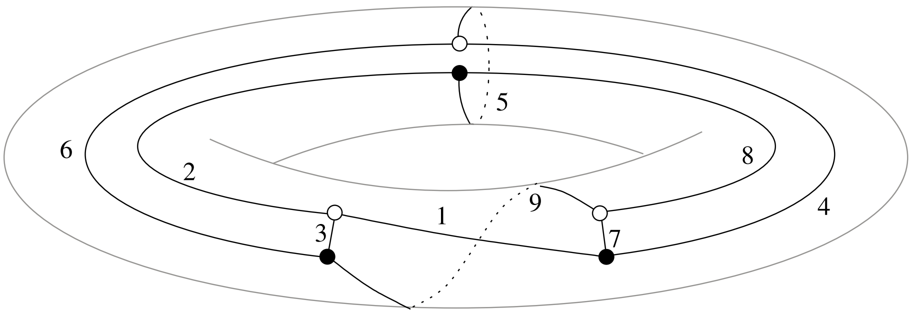


Figure 5.1.: The Dessin D_3 for the Fermat curve $X^3 + Y^3 = Z^3$ on the torus

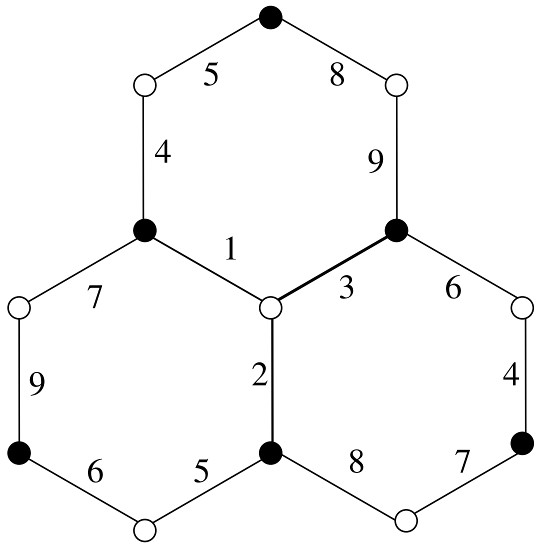


Figure 5.2.: The Dessin D_3 for the Fermat curve $X^3 + Y^3 = Z^3$ flat

$1, 2, \dots n^2$. The white vertices by capital letters $A, B, \dots X$, the black vertices by roman numbers $I, II, \dots, N$, where N denotes the $N \in \mathbb{N}$ from the curve F_N and $IN = N - 1$. Finally the faces are numbered by small letters $a, b, \dots, x$.

In the figures 5.1 to 5.4 the arrangement of the edges can be seen. Formally, we describe the arrangement with help of the Belyi permutations.

Proposition 5.2.2. *The Belyi permutations for the Belyi pair (F_N, β_N) for the N-th Fermat curves are*

$$\sigma_0^{\Gamma_N} = (1, 2, \dots, N)(N + 1, N + 2, \dots, 2N) \dots (N(N - 1) + 1, \dots, N^2) \qquad (5.2.2.1)$$

$$\sigma_1^{\Gamma_N} = (1, N + 1, \dots, N(N - 1) + 1)(2, N + 2, \dots, N(N - 1) + 2) \dots (N, \dots, N^2). \qquad (5.2.2.2)$$

Proof: We know from Proposition 5.2.1 that the abstract graph is $K_{N,N,N}$. The genus of

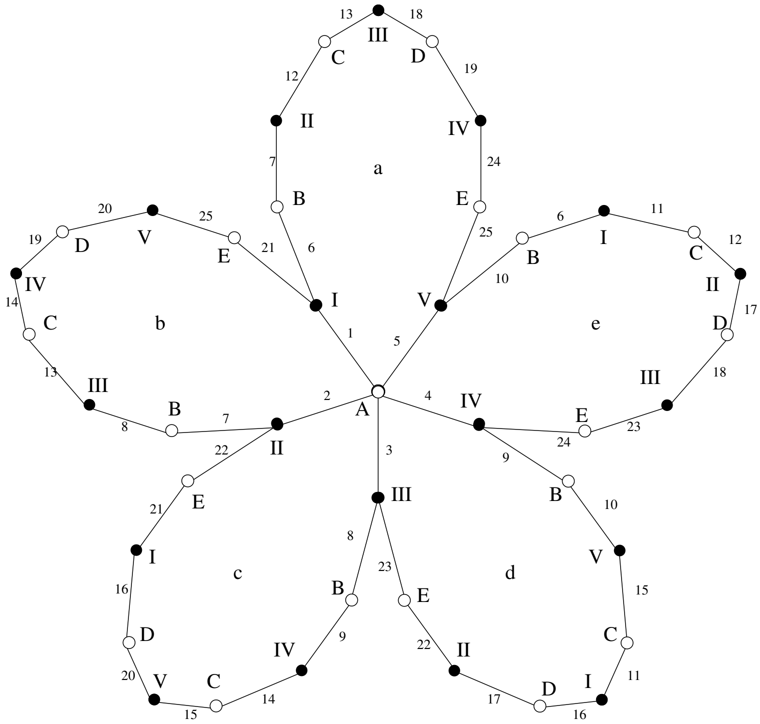


Figure 5.3.: The Dessin D_5 for the Fermat curve $X^5 + Y^5 = Z^5$

the Fermat curve for N is $\frac{1}{2}(N-1)(N-2)$. This follows easily with Riemann-Hurwitz. In [RY70] it was shown, that there is a unique embedding of $K_{N,N,N}$ in a surface of genus $\frac{1}{2}(N-1)(N-2)$. From the description of the embedding in [RY70] the statement follows. □

Corollary 5.2.3. *The three Dessins associated to the pair* $(F_N, \boldsymbol{\beta}_N)$ *where we consider the preimages of* $[0,1]$*,* $[1,\infty]$ *and* $[\infty,0]$*, respectively, are identical.*

Proof: Follows from Proposition 5.2.2. □

Remark 5.2.4. That the permutations given in equations (5.2.2.1) and (5.2.2.2) define Γ_N (following Definition 1.2.2 with any edge) can be seen without the Dessin by using the description of Γ_N in Lemma 5.1.3.

What is still missing is an identification of the different kinds of cusps with each other. That, we will do the most elementary way possible by just calculating the branch behavior.

Proposition 5.2.5. *If we regard the Dessin* D_N *labeled as in Figure 5.4 then we can identify the vertices and cell with the cusps of* Γ_N *and* $(F_N, \boldsymbol{\beta}_N)$ *as indicated in Table 5.1.*

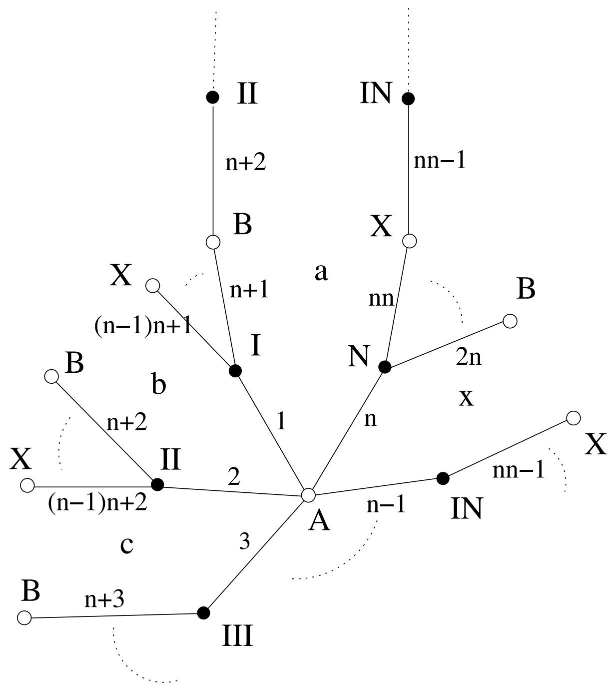


Figure 5.4.: The Dessin for the Fermat curves F_N

Proof: As mentioned before, the assignment is not unique. Here we took the edge between A and I, the edge 1, as the one we stabilize.

We start with the identification of the cusps of the Dessin with the ones of the subgroup. Following the explanations from Chapter 1.4, in particular Lemma 1.4.12, we need permutations which map the edge 1 to edges incident with all vertices and cells. For the vertices we get such permutations via

$$(\gamma_0\gamma_1)^j = \begin{pmatrix} 1 & -2j \\ 0 & 1 \end{pmatrix} \qquad j \in \{0, 1, \ldots, N-1\}$$

(that is: walking once around the cell denoted by a); for the cells via

$$\gamma_0^j = \begin{pmatrix} 1 & 0 \\ -2j & 1 \end{pmatrix} \qquad j \in \{0, 1, \ldots, N-1\}$$

(that is: walking once around the white vertex A). Then we take the preimages of 0, 1 and ∞ as indicated in Lemma 1.4.12 to get the columns for Γ_N in the table.

Now we discuss the identification of cusps of the curve with the ones of the Dessin. Thereby, we have problems with the symmetries of the Dessin. The first choices, the fixing of two cusps lying over different critical values, have to be done randomly, then all other identifications will be set. For convenience, we may identify the white vertex A with $(0:1:1)$ and the black vertex I with $(1:0:1)$.

To establish further identification we calculate the preimage of two elements of the

D_N	F_N	Γ_N
A	$(0:1:1)$	0
B	$(0:\zeta_N:1)$	2
$\vdots$	$\vdots$	$\vdots$
X	$(0:\zeta_N^{N-1}:1)$	$2N-2$
I	$(1:0:1)$	1
II	$(\zeta_N:0:1)$	3
$\vdots$	$\vdots$	$\vdots$
N	$(\zeta_N^{N-1}:0:1)$	$2N-1$
a	$(\epsilon\zeta^{N-1}:1:0)$	∞
b	$(\epsilon:1:0)$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$
x	$(\epsilon\zeta_N^{N-2}:1:0)$	$\frac{1}{2N-2}$

Table 5.1.: Assignments: Cusps of $D_N \longleftrightarrow F_N \longleftrightarrow \Gamma_N$

homotopy group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$: Look at $p_0(\lambda) = \frac{1}{2}e^{2\pi i N\lambda}$ and $p_\infty(\lambda) = 2e^{2\pi i N\lambda}$ ($\lambda \in [0,1]$). The first one is a path going N times around the critical value 0, the second one we can see as a path that goes N times around the critical value ∞ but in the other direction. Their lifts are closed paths on $F_N(\mathbb{C})$. When we know in which order a lift of p_0 and p_∞ intersect the full Dessin, i.e. intersect with the paths on $F_N(\mathbb{C})$ mentioned in (5.2.1.1), we will get the identifications.

There are N lifts of the two paths. We take the ones that starts in $(\sqrt[N]{1/2} : \sqrt[N]{1/2} : 1)$ and $(\epsilon\sqrt[N]{2} : \sqrt[N]{3} : 1)$ (real positive roots), respectively. In these cases the paths have the form $(e^{2\pi i\lambda}\sqrt[N]{1/2} : \ldots : 1)$ and $(e^{2\pi i(\lambda+1/2)}\sqrt[N]{2} : \ldots : 1)$ ($\lambda \in [0,1]$). In the center of p_0 is $(0:1:1)$, in the center of p_∞ the cusp $(\epsilon:1:0)$.

The path p_0 intersects the full Dessin at

$$(\zeta_N^m \sqrt[N]{1/2} : \ldots : 1) = (e^{2\pi i m/N}\sqrt[N]{1/2} : \ldots : 1) \quad \text{and}$$
$$(\zeta_N^m \sqrt[N]{-1/2} : \ldots : 1) = (e^{2\pi i(m+1/2)/N}\sqrt[N]{1/2} : \ldots : 1) \qquad m \in \{0,1,\ldots N-1\}$$

where in the first line we see the intersection with the edges connecting $(0:1:1)$ with cusps lying above 1, i.e. of the form $(\zeta_N^m:0:1)$ and in the second line are the intersections with edges connecting $(0:1:1)$ with cusps lying above ∞, i.e. of the form $(\epsilon\zeta_N^m:1:0)$. From that we conclude that the order of the cusps around $(0:1:1)$ is

$$(1:0:1),(\epsilon:1:0),(\zeta_N:0:1),(\epsilon\zeta_N:1:0),\ldots,(\zeta_N^{N-1}:0:1),(\epsilon\zeta_N^{N-1}:1:0).$$

The path p_∞ intersects the full Dessin at

$$(\zeta_N^m \sqrt[N]{-2} : \ldots : 1) = (e^{2\pi i(m+1/2)/N} \sqrt[N]{2} : \ldots : 1) \quad \text{and}$$
$$(\zeta_N^m \sqrt[N]{2} : \ldots : 1) = (e^{2\pi i m/N} \sqrt[N]{2} : \ldots : 1) \qquad m \in \{0, 1, \ldots N-1\}$$

where in the first line we see the intersection with the edges connecting $(\epsilon : 1 : 0)$ with cusps lying above 0, i.e. of the form $(0 : 1 : \zeta_N^{-m})$ and in the second line are the intersections with edges connecting $(0 : 1 : 1)$ with cusps lying above 1, i.e. of the form $(\zeta_N^m : 0 : 1)$. From that we conclude that the order of the cusps around $(\epsilon : 1 : 0)$ is

$$(0:1:1), (1:0:1), (0:\zeta_N^{N-1}:1), (\zeta_N^{N-1}:0:1), \ldots, (0:\zeta_N:1), (\zeta_N:0:1).$$

(Remember: As a path around $(\epsilon : 1 : 0)$ it is going backwards with growing λ.) □

The Fermat curves as non-congruence subgroups: Most of the subgroups that are associated with the Fermat curves are non-congruence subgroups. In fact, it is long known that the following proposition holds:

Proposition 5.2.6. *The group* Γ_N *is congruence if and only if* $N = 1, 2, 4, 8$.

Proof: This goes back to R. Fricke and F. Klein [FK65]. A proof can be found in [PS91]. □

The understanding of the Dessins for the Fermat curves and the congruence subgroups (see Section 2.2) can be used to construct particular elements that are in $\Gamma(2N)$ but not in Γ_N. Following the Theorem of Wohlfahrt 2.4.1 and realizing that all cusps of $(F_N, \boldsymbol{\beta}_N)$ are of width $2N$, such an element shows that Γ_N is not a congruence subgroup.

To get such elements, we try to construct the map μ from Theorem 2.3.1 that can be used to compare the action of the permutation groups associated with two subgroups. If the groups are not contained in each other the construction produces multiple images from which we can construct elements that are in the smaller group but not in the bigger one (see proof of Theorem 2.3.1).

Proposition 5.2.7. *In the principal congruence subgroups* $\Gamma(N)$ *for even* N *there are the following matrices:*

(i) Let $N \in \mathbb{N}$ *be such that there is a prime* $p \geq 3$ *with* $p|N$*. Then, with* $\gamma := \boldsymbol{\gamma}_0^{(1-p)/2}\boldsymbol{\gamma}_1^{N/p}\boldsymbol{\gamma}_0^{(p-1)/2}\boldsymbol{\gamma}_\infty^{-N/p}$ *we have* $\gamma \in \Gamma(2N)$ *but* $\gamma \notin \Gamma_N$*.*

(ii) Let $N = 2^t$*,* $t \geq 5$*, then with* $\gamma := \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1^6\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0^6\boldsymbol{\gamma}_1$ *we have* $\gamma^{2^{t-4}} \in \Gamma(2^t)$ *but* $\gamma^{2^{t-4}} \notin \Gamma_{2^{t-1}}$*.*

Proof: Concerning *(*i): In the first case, where N is not a power of two, the counter example comes from the fact that there are cusps S in $\Gamma(2N)$ with $S + k \sim_{\Gamma(2N)} S$ for a $k \in \mathbb{N}$, $k < 2N$. We may take the cusp $\frac{1}{p}$, $p \geq 3$, prime with $p|N$. We have $\frac{1}{p} + \frac{2N}{p} \sim_{\Gamma(2N)} \frac{1}{p}$. The number $\frac{2N}{p}$ is an even integer, i.e. $\frac{1}{p} + \frac{2N}{p} = \boldsymbol{\gamma}_\infty^{N/p}\left(\frac{1}{p}\right)$. We have $\gamma_0^{(1-p)/2}(1) = \frac{1}{p}$, i.e. $\gamma_\infty^{N/p}\gamma_0^{(1-p)/2}(1) \sim_{\Gamma(2N)} \frac{1}{p}$. Now, we use the constructions from Remark 1.3.1 and Theorem 1.4.6 and interpret the matrices as permutations and movements in the Dessin for $\Gamma(2N)$. In this setting, both $\sigma_0^{(p-1)/2}$ and $\sigma_0^{(p-1)/2}\sigma_\infty^{-N/p}$ map a chosen edge to edges that are incident to the same black vertex ($\frac{1}{p}$ lies above 1). Hence, there is a $k \in \mathbb{Z}$ such that the application of σ_1^k turns the second edge on the first one. Finally, $\sigma_0^{(1-p)/2}$ closes the circle and leads to an element of $\Gamma(2N)$. It turns out that k is $\frac{N}{p}$. This can be shown e.g. be calculating

$$\gamma = \gamma_0^{(1-p)/2}\gamma_1^{N/p}\gamma_0^{(p-1)/2}\gamma_\infty^{-N/p} = \begin{pmatrix} 1-2N & 4N^2/p \\ -2Np & 4N^2+2n+1 \end{pmatrix}.$$

In γ the generator γ_0 appears $-N/p$ times, this is a number not divisible by N, thus, $\gamma \notin \Gamma_N$.

Concerning (ii): The method applied to the first case breaks down when N is a power of two, i.e. $N = 2^t$ with $t \in \mathbb{N}$. Then, all cusps with the property $S + k \sim_{\Gamma(2N)} S$ for $k \in \mathbb{N}$, $k < 2N$, have even denominator, i.e. they lie above ∞, and an equivalent to the matrix considered in the first case would be e.g. $\gamma_0^{-N/2}\gamma_\infty\gamma_0^{N/2}\gamma_\infty^{-1}$ which is always in Γ_N.
Hence, compared to the first part, the solution for $N = 2^t$, $t \geq 5$, we give here is ad hoc. We take a counterexample for Γ_{16} and lift it to higher N.
The matrix $\gamma := \gamma_0\gamma_1^6\gamma_0\gamma_1\gamma_0^6\gamma_1$ is not in Γ_{16} since both generators appear 8-times, that is not divisible by 16 as demanded for elements in Γ_{16} according to Lemma 5.1.3. That $\gamma \in \Gamma(32)$ holds is easy to check by regarding $\gamma = \left(\begin{smallmatrix} -351 & 736 \\ 320 & -671 \end{smallmatrix}\right)$ modulo 32. Now we look at powers of γ. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ it holds that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{smallmatrix}\right)$. Since $\left\lfloor -\frac{351}{32} \right\rfloor$ and $\left\lfloor -\frac{671}{32} \right\rfloor$ are both odd, we can easily check that the congruences are correct for 64 and inductively complete the argument. The number of generators doubles each time, i.e. there is never the right amount of generators. $\square$

5.3. Modular forms and Kronecker limit formulas for Fermat curves

Our aim is to establish a Kronecker limit formula for the Fermat curves. For that, we need modular forms on F_N. The spaces of modular forms for Fermat curves had been examined by D. Rohrlich [Roh77] and T. Yang [Yan96].

T. Yang gave a basis for the modular forms of weight one. He was using a method, e.g. explained in [Miy06], in which one takes a single modular form and multiplies this one with modular functions in such a way that the result still is a modular form.

T. Yang does not work with $\Gamma(2)$ but with a group Δ that is the free subgroup of $\Gamma(2)$ generated be $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. It holds $\Gamma(2) = \{\pm\}\Delta$. Important is that $-id \notin \Delta$, therefore modular forms for Δ of odd weight exist. Modular functions of Δ and modular forms of even weight are modular functions and modular forms, respectively, for $\Gamma(2)$

as well. The forms we will be working with are of weight 2, such that we can continue working with subgroups of $\Gamma(2)$. (Yang's work is the reason for the notation θ^2 although no θ occurs.)

Remember Lemma 4.4.7 from Chapter 4.3 where modular forms and functions for $\Gamma(2)$ had been introduced: The modular form θ^2, that is of weight 2, and the modular functions λ and $1-\lambda$. Since $\Gamma_N \subset \Gamma(2)$ is a subgroup, all three are modular form and functions, respectively, for Γ_N. D. Rohrlich showed that N-th roots of λ and $1-\lambda$ are modular functions for Γ_N.

Lemma 5.3.1. *The N-th roots*

$$x := \sqrt[N]{\lambda} \quad y := \sqrt[N]{1-\lambda}$$

are modular functions for Γ_N.

Proof: See [Roh77]. □

Lemma 5.3.2. *Remember the cusps of $(F_N, \boldsymbol{\beta}_N)$ in Lemma 5.1.2. We have the following modular functions and forms in $M_*(\Gamma_N)$ with divisors as stated.*

$$\begin{aligned}
\operatorname{div}\, \theta^2 &= \sum_{j=0}^{N-1} N b_j \\
\operatorname{div}\, x &= \sum_j a_j - \sum_j c_j \\
\operatorname{div}\, y &= \sum_j b_j - \sum_j c_j \\
\operatorname{div}\, (x - \zeta^j) &= N b_j - \sum_j c_j \\
\operatorname{div}\, (y - \zeta^j) &= N a_j - \sum_j c_j \\
\operatorname{div}\, (x - \epsilon \zeta^j y) &= N c_j - \sum_j c_j
\end{aligned}$$

The ϵ denotes a N-th root of -1.

Proof: See [Roh77] and [Yan96]. □

Now, we can construct modular forms with special zeros.

Lemma 5.3.3. *For $j = 0, \dots, N-1$ we define*

$$f_{a_j} := \frac{(y-\zeta^j)^N}{y^N}\theta^2 \tag{5.3.3.1}$$

$$f_{b_j} := \frac{(x-\zeta^j)^N}{y^N}\theta^2 \tag{5.3.3.2}$$

$$f_{c_j} := \frac{(x-\epsilon\zeta^j y)^N}{y^N}\theta^2, \tag{5.3.3.3}$$

where $\zeta = e^{2\pi i/N}$ and $\epsilon = e^{\pi i/N}$.
All these are modular forms for Γ_N of weight 2 and

$$\operatorname{div} f_{i_j} = N^2 i_j,$$

where $i_j \in \{a_j, b_j, c_j\}$ stands for a cusp of Γ_N (see Lemma 5.1.2).

Proof: Follows easily from Lemma 5.3.2. □

We will calculate minimal polynomials (see Lemma 4.1.7) for the modular forms from Lemma 5.3.3 relative to $\Gamma(2)$, later on, this will simplify the calculation of the norms.

In Equation (5.1.5.1) representatives for $\Gamma_N \setminus \Gamma(2)$ had been given. We need the transformational behavior of the modular forms and functions under the representatives. The modular form $\theta^2(\cdot)$ is a modular form for $\Gamma(2)$, i.e. its behavior is known. For the modular functions we will start with the action of the slash operator under the generators of $\Gamma(2)$.

Lemma 5.3.4. *Take the generators from Lemma 1.4.2. Then*

$$\begin{array}{rclcrcl} x|_0\gamma_0 & = & \zeta x & \quad & x|_0\gamma_1 & = & x \\ y|_0\gamma_0 & = & y & & y|_0\gamma_1 & = & \zeta y. \end{array}$$

Proof: These transformations have been calculated in [Yan96], we adjusted his results to the different generators considered here. □

Now, it is easy to give the transformations for the representatives for the cosets $\Gamma_N \setminus \Gamma(2)$ and the minimal polynomials.

Lemma 5.3.5. *Denote by $P_{i_j}^{\Gamma_N|\Gamma(2)}$ the minimal polynomial for the modular form f_{i_j}*

(see Lemma 5.3.3) of Γ_N*, then we have*

$$P_{a_j}^{\Gamma_N|\Gamma(2)} = \prod_{k=0,\dots N-1}\left(X - \theta^2(z)\frac{(y(z)-\zeta^k)^N}{y(z)^N}\right)^N$$
$$P_{b_j}^{\Gamma_N|\Gamma(2)} = \prod_{k=0,\dots N-1}\left(X - \theta^2(z)\frac{(x(z)-\zeta^k)^N}{y(z)^N}\right)^N$$
$$P_{c_j}^{\Gamma_N|\Gamma(2)} = \prod_{k=0,\dots N-1}\left(X - \theta^2(z)\frac{(x(z)-\epsilon\zeta^k y(z))^N}{y(z)^N}\right)^N$$

for $j = 0, \dots, N-1$*.*

Proof: We present only the first case in detail, the other follow similarly. To simplify notations let $j(Mz) := cz + d$ with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then we have

$$\begin{aligned}
P_{a_j}^{\Gamma_N|\Gamma(2)} &= \prod_{\gamma\in\Gamma_N\backslash\Gamma(2)}\left(X - f_{a_j}|_2\gamma(z)\right)\\
&\overset{(1)}{=} \prod_{a,b=0,\dots N-1}\left(X - \frac{(y-\zeta^j)^p}{y^N}\theta^2|_2\gamma_0^a\gamma_1^b(z)\right)\\
&= \prod_{a,b=0,\dots N-1}\left(X - j((\gamma_0^a\gamma_1^b)z)^{-2}\theta^2(\gamma_0^a\gamma_1^b z)\frac{(y(\gamma_0^a\gamma_1^b z)-\zeta^j)^N}{y(\gamma_0^a\gamma_1^b z)^N}\right)\\
&\overset{(2)}{=} \prod_{a,b=0,\dots N-1}\left(X - \theta^2(z)\frac{(\zeta^b y(z)-\zeta^j)^N}{y(z)^N}\right)\\
&= \prod_{b=0,\dots N-1}\left(X - \theta^2(z)\frac{\zeta^{bN}(y(z)-\zeta^{j-b})^N}{y(z)^N}\right)^N\\
&\overset{(3)}{=} \prod_{k=0,\dots N-1}\left(X - \theta^2(z)\frac{(y(z)-\zeta^k)^N}{y(z)^p}\right)^N,
\end{aligned}$$

in (1) we entered the system of representatives presented in Equation (5.1.5.2), in (2) Lemma 5.3.4, the transformational behavior, was used, in (3) we used $\zeta^N = 1$ and renumbered. □

Proposition 5.3.6. *With the notations from Lemma 5.3.3 we discover that there are only three different norms for the modular forms from that lemma. They are for all*

$j \in \{0, 1, \ldots, N-1\}$

$$\mathrm{Nm}_{\Gamma_N|\Gamma(2)} f_{a_j} = (-1)^{N^2} \theta^{2N^2}(z) \left(\frac{\lambda(z)}{1-\lambda(z)} \right)^{N^2} \tag{5.3.6.1}$$

$$\mathrm{Nm}_{\Gamma_N|\Gamma(2)} f_{b_j} = (-1)^{N^2} \theta^{2N^2}(z) \tag{5.3.6.2}$$

$$\mathrm{Nm}_{\Gamma_N|\Gamma(2)} f_{c_j} = \theta^{2N^2}(z) \left(\frac{1}{1-\lambda(z)} \right)^{N^2}. \tag{5.3.6.3}$$

Proof: The constant coefficient of the minimal polynomials are the norms. The claim follows from Lemma 5.3.5; in the first case, we have for example

$$\begin{aligned}
\mathrm{Nm}_{\Gamma_N|\Gamma(2)} f_{a_j} &= \text{ constant coefficient of the minimal polynomial} \\
&= \prod_{l=0,\ldots N-1} \left(\theta^2(z) \frac{(y(z)-\zeta^l)^N}{y(z)^N} \right)^N \\
&= \theta^{2N}(z) \frac{1}{y(z)^{N^2}} \left(\prod_{l=0,\ldots N-1} (y(z)-\zeta^l) \right)^{N^2} \\
&= \theta^{2N^2}(z) \frac{(y^N(z)-1)^{N^2}}{y(z)^{N^3}} \\
&= (-1)^{N^2} \theta^{2N^2}(z) \left(\frac{\lambda(z)}{(1-\lambda)(z)} \right)^{N^2}.
\end{aligned}$$

□

Remark 5.3.7. Up to a factor, the norms from Proposition 5.3.6 are powers of the modular form in Corollary 4.4.8, i.e. powers of the modular forms with exactly one single zero in one cusp for $\Gamma(2)$. When we replace the modular forms f_{i_j} by

$$\tilde{f}_{i_j} := 2^{-4/3} f_{i_j} \quad (i_j \in \{a_0, \ldots, a_{N-1}, b_0, \ldots, b_{N-1}, c_0, \ldots, c_{N-1}\}),$$

an alternation that does not change the vanishing behavior in the cusps, we will get (up to sign) the N^2-th power of the modular forms for $\Gamma(2)$. In this case, the expansion of $\log || \mathrm{Nm}_{\Gamma_N|\Gamma(2)} \tilde{f}_{i_j} ||^2$ is easily derived from the one for $\Gamma(2)$. It is just N^2 times the corresponding expansion for $\Gamma(2)$ in Proposition 4.4.11.

Out of the Kronecker limit formula for $\Gamma(2)$ (Proposition 4.4.12) we get a limit formula for the trace.

Lemma 5.3.8. *Let $\Gamma_N \subset \Gamma(2)$ be the subgroup associated to the N-th Fermat curve, S_k a cusp of Γ_N and $\tilde{f}_k$ the modular form for the cusp S_k according to Lemma 5.3.3 and*

Remark 5.3.7. It holds:

$$\mathrm{Tr}_{\Gamma_N|\Gamma(2)}\left(-\log||\tilde{f}_k(z)||^2\right) = N^2 \cdot \mathrm{Tr}_{\Gamma_N|\Gamma(2)}\, g_k^{\Gamma_N}(z) \tag{5.3.8.1}$$

Proof: On the right hand side, we have

$$\mathrm{Tr}_{\Gamma_N|\Gamma(2)}\left(\log||\tilde{f}_k(z)||^2\right) = \log \mathrm{Nm}_{\Gamma_N|\Gamma(2)}\,||\tilde{f}_k||^2 = \log||\,\mathrm{Nm}_{\Gamma_N|\Gamma(2)}\,\tilde{f}_k||^2.$$

In Remark 5.3.7 we realized, that $\mathrm{Nm}_{\Gamma_N|\Gamma(2)}\,\tilde{f}_k = G_k^{N^2}$, where G_k is a the modular form for $\Gamma(2)$ for which $\log||G_k||^2$ has been calculated before (Proposition 4.4.11). Hence, we have

$$\mathrm{Tr}_{\Gamma_N|\Gamma(2)}\left(\log||\tilde{f}_k(z)||^2\right) = \frac{1}{N^2}\log||G_k||^2.$$

On the left hand side gives us Proposition 4.2.8

$$\mathrm{Tr}_{\Gamma_N|\Gamma(2)}\, g_k^{\Gamma_N}(z) = g_k^{\Gamma(2)}(z).$$

Then the proposition follows from the Kronecker limit formula for $\Gamma(2)$, see Proposition 4.4.12. □

The limit formula for the trace gives us individual limit formulas.

Theorem 5.3.9. *Let S_j be a cusp of Γ_N, the subgroup associated to $(F_N, \boldsymbol{\beta}_N)$, and let $\tilde{f}_j \in M_2(\Gamma_N)$ be the corresponding modular defined in Lemma 5.3.3 and Remark 5.3.7. We have*

$$4\pi \lim_{s\to 1}\left(E_j^{\Gamma_N}(z,s) - \frac{1}{vol(\Gamma_N)(s-1)}\right) = \\ -\frac{1}{N^2}\log||\tilde{f}_j(z)||^2 + \frac{1}{N^2}\left(4\frac{\zeta'(-1)}{\zeta(-1)} + 4 - 2\log(4\pi) - 2\log(2N)\right). \tag{5.3.9.1}$$

Proof: That there is an identity

$$4\pi \lim_{s\to 1}\left(E_j^{\Gamma_N}(z,s) - \frac{1}{vol(\Gamma_N)(s-1)}\right) = -\frac{1}{N^2}\log||\tilde{f}_j(z)||^2 + A$$

with a constant $A \in \mathbb{R}$ states Proposition 4.3.5 when we remember that $\tilde{f}_j$ only has a zero of order N^2 in S_j. We have according to Definition 4.2.1

$$4\pi \lim_{s\to 1}\left(E_j^{\Gamma_N}(z,s) - \frac{1}{vol(\Gamma_N)(s-1)}\right) = g_j^{\Gamma_N}(z) + \frac{2\pi}{3N^2}C^{\Gamma(1)} + \frac{2}{N^2}\left(\log(4\pi) - \log(2N)\right).$$

Therefore we can use Lemma 5.3.8 together with Proposition 4.3.9 to calculate A. The contribution from the sum formula in Lemma 5.3.8 is zero. Hence, only the difference to the Green's function is of importance. With

$$24\left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1\right) = 4\pi C^{\Gamma(1)}$$

we derive the statement. □

Corollary 5.3.10. *Let S_k be a cusp of Γ_N and $\tilde{f}_k$ be the corresponding modular form from Lemma 5.3.3. We have*

$$-\log ||\tilde{f}_k(z)||^2 = N^2 g_k^{\Gamma_N}(z). \tag{5.3.10.1}$$

Proof: The statement follows from Theorem 5.3.9 with the normalization of the Green's functions. □

5.4. Scattering constants for Fermat curves

Here, we will finally calculate the scattering constants for the Fermat curves. To do that, we combine the methods from Section 4.3 and Section 3.3. In Section 3.3 it was claimed (and only partial shown) that certain automorphisms of curves yield identities of scattering constants. Therefore, we can reduce the number of constants that we are searching for. With the Kronecker limit formula from Theorem 5.3.9, we can, according to Section 4.3, calculate scattering constants, if we know the modular forms involved well enough. When we look at the expansion in ∞, this is the case.

At first, we consider automorphisms.

Remark 5.4.1. There are two different kinds of automorphisms of F_N. We may interchange the variables (with changing signs), scale them or combine these two means. Let us call morphisms given by exchanges to be of type A and morphisms given by scaling to be of type B. Automorphisms of type A are e.g.

$$(X : Y : Z) \longmapsto (\epsilon X : Z : Y) \qquad \text{or} \qquad (X : Y : Z) \longmapsto (Z : \epsilon Y : X)$$

(where $\epsilon^N = -1$). Automorphisms of type B are given by

$$\begin{array}{rcll} F_N & \longrightarrow & F_N & \\ (X : Y : Z) & \longmapsto & (\zeta^m X : \zeta^n Y : Z) & \qquad m, n \in \{0, 1, \ldots, N-1\}, \end{array} \tag{5.4.1.1}$$

where ζ is a primitive N-th root of unity.

How does these different automorphisms act on the group Γ_N? If some of them act only by conjugation, we can apply Proposition 3.3.3 and follow identities of scattering constants. In the other cases, we get identities only conjecturally.

Automorphisms of type A interchange the cusps above the ramification points, i.e. they interchange generators in elements in Γ_N and the associated automorphisms of Γ_N is not given by conjugation over $\Gamma(2)$. Morphisms of type B, on the contrary, conserve the assignment cusps to ramification points, they preserve the orientation and they are given by conjugation only. These automorphisms lead to the following identities.

Proposition 5.4.2. *Let Γ_N be the group associated to the N-th Fermat curve. Then it holds that*

$$C^{\Gamma_N}_{jk} = C^{\Gamma_N}_{j'k'}$$

where $S_j, S_k, S_{j'}, S_{k'}$ are cusps such that $\boldsymbol{\beta}_N(S_j) = \boldsymbol{\beta}_N(S_{j'})$, $\boldsymbol{\beta}_N(S_k) = \boldsymbol{\beta}_N(S_{k'})$ and

(i) if $\boldsymbol{\beta}_N(S_j) \neq \boldsymbol{\beta}_N(S_k)$, then the cusps $S_{j'}$ and $S_{k'}$ can be chosen arbitrarily above the correct ramification point.

(ii) if $\boldsymbol{\beta}_N(S_j) = \boldsymbol{\beta}_N(S_k)$, then the cusp $S_{k'}$ depends on $S_{j'}$ in the following way: If we normalize the cusps as in Lemma 5.1.2, then S_j and $S_{j'}$ differ by the roots of unity chosen, e.g. in S_j occurs ζ^j and in $S_{j'}$ the root $\zeta^l \cdot \zeta^j$. The difference in the roots for S_k and $S_{k'}$ must be ζ^l as well.

Proof: As explained above, automorphisms of type B act on the group Γ_N by conjugation. Therefore we may apply Proposition 3.3.3 (on page 68), that claims $C^{\Gamma}_{ST} = C^{\Gamma}_{\alpha(S)\alpha(T)}$ for an automorphism $\alpha : \Gamma \to \Gamma$ and two cusps S and T.

We have to understand where we can map a couple (S_j, S_k) of cusps under a morphism of type B. For that look at Formula (5.4.1.1) and write down the action on the cusps. Let the automorphism be given by m and n (both in the range $0, 1, \ldots, N-1$), then we have

$$\begin{aligned} (0 : \zeta^l : 1) &\longmapsto (0 : \zeta^{l+n} : 1) \\ (\zeta^l : 0 : 1) &\longmapsto (\zeta^{l+m} : 0 : 1) \qquad\qquad l \in \{0, 1, \ldots, N-1\} \\ (\epsilon\zeta^l : 1 : 0) &\longmapsto (\epsilon\zeta^{l+m-n} : 1 : 0). \end{aligned}$$

We realize that in (i) we can choose the images of S_j and S_k independently from each other. In (ii) this is not the case. The image S_j determines the one of S_k in the way described in the proposition. □

Conjecture 3.3.4 applied on automorphisms of type A implies identities of scattering constants for the Fermat curves even when we change to cusps lying above another ramification point. Because of their well known and easy structure, we can show results of that type directly. We start with a

Lemma 5.4.3. *Let $\Gamma_N \subset \Gamma(1)$ be the subgroup associated to the N-th Fermat curve ($N \in \mathbb{N}$). Then $\Gamma_N \lhd \Gamma(1)$.*

Proof: Remember the facts that $\Gamma(2) \lhd \Gamma(1)$ (Proposition 2.2.1) and that in Γ_N are all elements from $\Gamma(2)$ in which each generator occurs a number of times that is divisible by N (Lemma 5.1.3).

Let $\gamma \in \Gamma_N$ be a matrix with word $\gamma = \prod_{i=1}^n \kappa_i^{\epsilon_i}$, where $n \in \mathbb{N}$, $\epsilon_i \in \{-1, 1\}$ and $\kappa_i \in \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1\}$, and $\lambda \in \Gamma(1)$. Then

$$\lambda^{-1}\gamma\lambda = \prod_{i=1}^n \lambda^{-1}\kappa_i^{\epsilon_i}\lambda = \prod_{i=1}^n \left(\lambda^{-1}\kappa_i\lambda\right)^{\epsilon_i}$$

and since $\lambda^{-1}\kappa_i\lambda \in \Gamma(2)$, i.e. it is decomposable into a word in $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_1$, the number of each generator in $\lambda^{-1}\gamma\lambda$ still is divisible by N. $\square$

Proposition 5.4.4. *Let $\Gamma_N \subset \Gamma(1)$ be the subgroup associated to the N-th Fermat curve ($N \in \mathbb{N}$). For the scattering constants hold*

$$C^{\Gamma_N}_{1/l,\infty} = C^{\Gamma_N}_{2N-l,0} = C^{\Gamma_N}_{2N-l+1,1} \qquad l \in \{2,4,\dots,2N\}$$
$$C^{\Gamma_N}_{k,\infty} = \begin{cases} C^{\Gamma_N}_{1/(2N-k),0} = C^{\Gamma_N}_{1/k,1} & k \in \{2,4,\dots,2N\} \\ C^{\Gamma_N}_{2N-k,0} = C^{\Gamma_N}_{2N-k+1,1} & k \in \{1,3,\dots,2N-1\} \end{cases},$$

where the index is the cusp.

Proof: We will show much more than the statement claims. To create the scattering constants, we count classes in a double coset decomposition. In this proof, we will not only show that the number of classes is the same but that the classes themselves are equal.

Let $\gamma_{k,\infty} \in \Gamma(1)$ with $k \in \mathbb{Q} \cup \infty$ be a matrix with $\gamma_{k,\infty}(\infty) = k$. We will show that from $\gamma \in \gamma^{-1}_{k,\infty}\Gamma_N$ follows $\gamma \in \gamma^{-1}_{k',\infty}\Gamma_N\gamma_{0,\infty}$ and $\gamma \in \gamma^{-1}_{k'',\infty}\Gamma_N\gamma_{1,\infty}$ with k' and k'' as in the proposition.

The procedure is that we complete $\gamma_{k,\infty}$ by σ_k to an element of Γ_N and use the fact that $\Gamma_N \lhd \Gamma(1)$. Then we have for k a cusp of Γ_N

$$\gamma^{-1}_{k,\infty}\Gamma_N = \sigma_k^{-1}\sigma_k\gamma^{-1}_{k,\infty}\Gamma_N = \sigma_k^{-1}\Gamma_N = \sigma_k^{-1}\gamma^{-1}_{0,\infty}\Gamma_N\gamma_{0,\infty} = \gamma^{-1}_{k^*,\infty}\Gamma_N\gamma_{0,\infty}$$

and

$$\gamma^{-1}_{k,\infty}\Gamma_N = \sigma_k^{-1}\sigma_k\gamma^{-1}_{k,\infty}\Gamma_N = \sigma_k^{-1}\Gamma_N = \sigma_k^{-1}\gamma^{-1}_{1,\infty}\Gamma_N\gamma_{1,\infty} = \gamma^{-1}_{k^{**},\infty}\Gamma_N\gamma_{1,\infty},$$

where $k^* = \gamma_{0,\infty}\sigma_k(\infty)$ and $k^{**} = \gamma_{1,\infty}\sigma_k(\infty)$.

With

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{and} \quad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

we can choose

$$\gamma_{1/l,\infty} = S^{-1}T^{-l}S \qquad l \in \{2,4,\dots,2N\} \tag{5.4.4.1}$$
$$\gamma_{k,\infty} = T^kS \qquad k \in \{1,2,\dots,2N\}. \tag{5.4.4.2}$$

Then $\sigma_{1/l} := S^{-1}T^{2N-l}S$ and $\sigma_k := T^kS$ (l and k as above) fulfill the requirement to yield a matrix in Γ_N:

$$\sigma_{1/l}\gamma^{-1}_{1/l,\infty} = S^{-1}T^{2N-l}SS^{-1}T^lS = S^{-1}T^{2N}S = S^{-1}\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)^NS \qquad l \in \{2,4,\dots,2N\}$$
$$\sigma_k\gamma^{-1}_{k,\infty} = T^kSS^{-1}T^{-k} = id \qquad k \in \{1,2,\dots,2N\}.$$

Because of $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) = \boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_0^{-1}$ and lemmas 5.1.3 as well as 5.4.3 these matrices are in Γ_N.

Left over to finish the argument is to calculate k^* and k^{**} and ensure that they coincide with the cusps in the claim. Look at

$$\gamma_{0,\infty}S^{-1}T^{2N-l}S(\infty) = T^{2N-l}S(\infty) = 2N-l \tag{5.4.4.3}$$
$$\gamma_{1,\infty}S^{-1}T^{2N-l}S(\infty) = T^{2N-l+1}S(\infty) = 2N-l+1 \tag{5.4.4.4}$$
$$\gamma_{0,\infty}T^{k}S(\infty) = ST^{k}S(\infty) = -\frac{1}{k} \tag{5.4.4.5}$$
$$\gamma_{1,\infty}T^{k}S(\infty) = TST^{k}S(\infty) = \frac{k-1}{k}. \tag{5.4.4.6}$$

Formulas (5.4.4.3) and (5.4.4.4) lead to the first part of the statement. For the second part we divide into cases.

Equation (5.4.4.5) with even k:
Here we have $-\frac{1}{k} \sim_{\Gamma_N} \frac{1}{2N-k}$ via $\left(\begin{smallmatrix} 1 & 0 \\ 2N & 1 \end{smallmatrix}\right) = \boldsymbol{\gamma}_0^{-N}$.

Equation (5.4.4.5) with odd k:
Here we have $-\frac{1}{k} \sim_{\Gamma_N} 2N-k$ via $(\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1)^{(k+1-2N)/2}\boldsymbol{\gamma}_1^{(k-1)/2}\boldsymbol{\gamma}_0^{(k-1)/2}$.

Equation (5.4.4.6) with even k:
Here we have $\frac{k-1}{k} \sim_{\Gamma_N} \frac{1}{k}$ via $\boldsymbol{\gamma}_0^{-k/2}(\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1)^{(2N-k)/2}\boldsymbol{\gamma}_1^{-k/2}$.

Equation (5.4.4.6) with odd k:
Here we have $\frac{k-1}{k} \sim_{\Gamma_N} 2N-k+1$ via $(\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1)^{(-2N+k-1)/2}\boldsymbol{\gamma}_0^{(1-k)/2}\boldsymbol{\gamma}_1^{(1-k)k/2}$.

This discussion finishes the proof. □

Remark 5.4.5. In propositions 5.4.2 and 5.4.4 only identities of scattering constants are mentioned. But these follow in both cases from identities of Eisenstein series.
For Proposition 5.4.2, where the automorphisms were given by conjugation with matrices from $\Gamma(2)$, these identities can be found in Lemma 3.3.1, therefore we will not state them here.
In the proof of Proposition 5.4.4 it was shown that certain decompositions into double cosets of Γ_N are equal, hence, not only the scattering constants are equal but the expansions of Eisenstein series where they came from as well, i.e. we have

$$E_{1/l}^{\Gamma_N}(z,s) = E_{2N-l}^{\Gamma_N}(\gamma_{0,\infty}z,s) = E_{2N-l+1}^{\Gamma_N}(\gamma_{1,\infty}z,s) \qquad l \in \{2,4,\dots,2N\}$$
$$E_{k}^{\Gamma_N}(z,s) = \begin{cases} E_{1/(2N-k)}^{\Gamma_N}(\gamma_{0,\infty}z,s) = E_{1/(2N-k)}^{\Gamma_N}(\gamma_{1,\infty}z,s) & k \in \{0,2,\dots,2N-2\} \\ E_{2N-k}^{\Gamma_N}(\gamma_{0,\infty}z,s) = E_{k+1}^{\Gamma_N}(\gamma_{1,\infty}z,s) & k \in \{1,3,\dots,2N-1\} \end{cases}.$$

Remark 5.4.6. From the propositions 5.4.2 and 5.4.4 it follows that Conjecture 3.3.4 holds for the Fermat curves.

So far, we only got the information that certain scattering constants are equal. Now we use the Kronecker limit formula to actually calculate scattering constants for the Fermat curves in the cases where one cusp is ∞.

Proposition 5.4.7. *Scattering constants for Γ_N, the subgroup associated to the Belyi pair $(F_N, \boldsymbol{\beta}_N)$ (equations (5.1.1.1) and (5.1.2.1)), are*

$$C_{j\infty} = \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left(2\log(2) + 6\log(N)\right)\right) \tag{5.4.7.1}$$

$$C_{k\infty} = \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left(2\log(2) + 6\log(N) + 3N\log|1-\zeta_N^k|\right)\right) \tag{5.4.7.2}$$

$$C_{\infty\infty} = \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left((12N+2)\log(2) + (-3N+6)\log(N)\right)\right), \tag{5.4.7.3}$$

where $\boldsymbol{\beta}_N(S_j) \neq \infty$, $\boldsymbol{\beta}_N(S_k) = \infty$ but $S_k \neq \infty$. The number ζ_N^k is the N-th root of unity that gives the cusp S_k in Lemma 5.1.2.

Proof: This is an application of Proposition 4.3.8.

We will get scattering constants by comparing the constant terms in the Kronecker limit formula in Theorem 5.3.9:

$$\text{constant term}\left(g_k^{\Gamma_N}(z)|_\infty\right) = \frac{1}{N^2}\cdot\text{constant term}\left(-\log||\tilde{f}_k(z)||^2\right) \tag{5.4.7.4}$$

For the left hand side, the Green's function, we get

$$\delta_{jk}4\pi\frac{y}{b_j} + 4\pi\left(\widetilde{C}_{jk}^{\Gamma_N} - \frac{C^{\Gamma(1)}}{[\Gamma(1):\Gamma_N]}\right) - \frac{12\left(\log(4\pi y) - \log(b_j)\right)}{[\Gamma(1):\Gamma_N]} = \delta_{jk}\frac{2\pi y}{N} + 4\pi\left(\widetilde{C}_{jk}^{\Gamma_N} - \frac{C^{\Gamma(1)}}{6N^2}\right) - \frac{2\left(\log(4\pi y) - \log(2N)\right)}{N^2}$$

where the scattering constant $\widetilde{C}_{jk}^{\Gamma_N}$ is the only unknown. On the left hand side, we can calculate the constant term out of the q-expansions of the modular forms. We have for a cusp S_j

$$\begin{aligned}\log||\tilde{f}_j(z)||^2 &= \log\left(|\tilde{f}_j|^2(4\pi\,\mathrm{Im}(z))^2\right)\\ &= 2\log(4\pi\,\mathrm{Im}(z)) + \log|2^{-4/3}f_j(z)|^2\\ &= 2\log(4\pi\,\mathrm{Im}(z)) - \frac{8}{3}\log(2) + \log|f_j(z)|^2\end{aligned}$$

and

$$\begin{aligned}\log|f_j(z)|^2 &= 2\,\mathrm{Re}\log(f_j(z))\\ &= 2\,\mathrm{Re}\log\left(\sum_{n\geq 0} a_n q^{n/(2N)}\right)\\ &= 2\,\mathrm{Re}\log\left(a_m q^{m/(2N)}\left(1 + \frac{a_{m+1}}{a_m}q^{1/(2N)} + \dots\right)\right),\end{aligned}$$

where $q = e^{2\pi i z}$ and a_m is the first coefficient that is not zero. From the local expansion of the logarithm $\log(1+x) = \sum_{k\geq 1}(-1)^{k+1}\frac{x^k}{k}$ we see that only $a_m q^m$ makes a contribution to the constant term.

We calculate $a_m q^m$ for the modular forms f_{c_j} from Lemma 5.3.3. It is

$$\begin{aligned}
f_{c_j} &= \frac{(x-\epsilon\zeta^j y)^N}{y^N}\theta^2 \\
&= \left(\left(-\frac{1}{16}q^{-1/2}\prod_n\left(\frac{1-q^{n-1/2}}{1+q^n}\right)^8\right)^{1/N} - \epsilon\zeta^j\left(\frac{1}{16}q^{-1/2}\prod_n\left(\frac{1+q^{n-1/2}}{1+q^n}\right)^8\right)^{1/N}\right)^N \\
&\quad \cdot 16q^{1/2}\prod_n(1-q^n)^4(1+q^n)^8 \\
&= \left(\epsilon\prod_n\left(1-q^{n-1/2}\right)^{8/N} - \epsilon\zeta^j\prod_n\left(1+q^{n-1/2}\right)^{8/N}\right)^N\prod_n(1-q^n)^4 \\
&= \left(\epsilon\left(1-\frac{8}{N}q^{1/2}\dots\right) - \epsilon\zeta^j\left(1+\frac{8}{N}q^{1/2}+\dots\right)\right)^N\prod_n(1-q^n)^4 \\
&= \left(\left(\epsilon-\epsilon\zeta^j\right) - \frac{8}{N}\epsilon(1+\zeta^j)q^{1/2}+\dots\right)^N\prod_n(1-q^n)^4.
\end{aligned}$$

We have to distinguish two cases

(i) $\zeta^j = 1$: Here, the first part vanishes and we get

$$f_{c_0} = \left(-\frac{8}{N}\cdot 2\epsilon q^{1/2}\right)^N(1+*q^*+\dots).$$

For the logarithm then holds

$$\frac{1}{N^2}\log|f_{c_0}(z)|^2 = -\frac{2}{N}\pi y + \frac{2}{N}(\log(16)-\log(N)) + \text{ higher terms.} \tag{5.4.7.5}$$

(ii) $\zeta^j \neq 1$: Here, we get

$$f_{c_j} = -(1-\zeta^j)^N(1+*q^*+\dots)$$

and

$$\frac{1}{N^2}\log|f_{c_j}(z)|^2 = \frac{2}{N}\log\left|1-\zeta^j\right| + \text{ higher terms.} \tag{5.4.7.6}$$

In a similar way we get that in the other cases, where $S_j \in \{a_0,\dots a_{N-1}, b_0\dots, b_{N-1}\}$, the coefficient a_0 equals 1 or -1 and since $\operatorname{Re}\log(\pm 1) = 0$:

$$\frac{1}{N^2}\log|f_j(z)|^2 = 0 + \text{higher terms.} \tag{5.4.7.7}$$

Let K_j be the constant term of $\frac{1}{N^2}\log|f_j|^2$. Then Formula (5.4.7.4) gives for the scattering constant

$$\widetilde{C}^{\Gamma_N}_{j\infty} = -\delta_{j\infty}\frac{y}{2N} + \frac{1}{6N^2}C^{\Gamma(1)} + \frac{1}{2\pi N^2}\left(\frac{1}{3}\log(2) - \log(N)\right) - \frac{1}{4\pi}K_j.$$

Now, we have to insert the results for K_j (from equations (5.4.7.5), (5.4.7.6) and (5.4.7.7)) and remember Remark 3.2.2 i.e.

$$C^{\Gamma_N}_{jk} = \widetilde{C}^{\Gamma_N}_{jk} - \frac{3\log(2N)}{6N^2\pi}$$

to get the results. □

The combination of the last three propositions give all scattering constants for the Fermat curves.

Theorem 5.4.8. *Scattering constants for* Γ_N*, the subgroup associated to the Belyi pair* $(F_N, \boldsymbol{\beta}_N)$ *(equations* (5.1.1.1) *and* (5.1.2.1)*), are:*
If both cusps are the same, then

$$C_{jj} = \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left((12N+2)\log(2) + (-3N+6)\log(N)\right)\right). \qquad (5.4.8.1)$$

If $S_j \neq S_k$ *and* $\boldsymbol{\beta}_N(S_j) \neq \boldsymbol{\beta}_N(S_k)$*, then*

$$C_{kj} = \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left(2\log(2) + 6\log(N)\right)\right). \qquad (5.4.8.2)$$

If $S_j \neq S_l$ *but* $\boldsymbol{\beta}_N(S_j) = \boldsymbol{\beta}_N(S_l)$*, then*

$$C_{lj} = \frac{1}{6N^2}\left(C^{\Gamma(1)} - \frac{1}{\pi}\left(2\log(2) + 6\log(N) + 3N\log|1-\zeta_N^{l-j}|\right)\right). \qquad (5.4.8.3)$$

The number $\zeta_N^{l-j} = \zeta_N^l\left(\zeta_N^j\right)^{-1}$ *is given by the* N*-th roots of unity that determine the cusps* S_l *and* S_j *in Lemma 5.1.2.*

Proof: The theorem follows from the propositions 5.4.2, 5.4.4 and 5.4.7.

The numbers come from Proposition 5.4.7, where the cusp S_j were ∞. Proposition 5.4.4 generalizes to scattering constants with second cusp 0 and 1, i.e. to cusps lying above the other ramification points. Finally, Proposition 5.4.2 gives us the possibility to change from one cusps to all others lying above the same ramification point.

Formulas (5.4.8.2) and (5.4.8.1) follow quite easily, for Formula (5.4.8.3) one really has to keep track of the shifting of cusps and realize that for the part $|1-\zeta_N^k|$ in Formula (5.4.7.2) holds $|1-\zeta_N^k| = |1-\zeta_N^{-k}|$. □

Remark 5.4.9. There is an alternative method to determine the scattering constants for the Fermat curves by means of Arakelov theory. U. Kühn [Küh05] showed how scattering constants give arithmetic intersection numbers in the infinite places. Together with the intersection numbers in the finite places that Ch. Curilla [Cur10] calculated in his thesis, we could get the scattering constants without using Kronecker limit formulas and the symmetries of the Fermat curves by studying arithmetic intersection numbers in the cusps.

5.5. Numerical results for the scattering constants

We were able to perform some calculations concerning the scattering matrices and constants for the Fermat curves. In them we can see some of the results of the previous sections, i.e. the structure of the scattering matrix. Furthermore, we are able to approximate the value of the scattering constants.

With the description of Γ_N in Remark 5.1.4 we can calculate the coefficients in the scattering matrix and therefore derive a numerical value for the scattering constants from them.

In Appendix B an algorithm is presented with which the first coefficients for

$$N = 3, 4, 5, 6, 7, 8, 9, 11, 13$$

had been calculated. In Table 5.3 on page 121 the first 35 coefficients for $N = 5, 6, 7$ can be seen. To understand the table the notation has to be explained.
The coefficient we calculated is $r_{jk}^{\Gamma_N}(c)$, where the index stands for two cusps S_j and S_k, as it has been introduced in Definition 3.1.9. In the table we omit the (c) to safe space. In Table 5.3 the double index had been replaced by a single one denoting

$$\begin{aligned}
a &:= jj \\
b &:= jk, \text{ where } \boldsymbol{\beta}_N(S_j) \neq \boldsymbol{\beta}_N(S_k) \\
c &:= jl, \text{ where } \boldsymbol{\beta}_N(S_j) = \boldsymbol{\beta}_N(S_k) \text{ and } \gamma_{\text{next}}(S_j) = S_l \\
d &:= jl, \text{ where } \boldsymbol{\beta}_N(S_j) = \boldsymbol{\beta}_N(S_k) \text{ and } \gamma_{\text{next}}^2(S_j) = S_l \\
e &:= jl, \text{ where } \boldsymbol{\beta}_N(S_j) = \boldsymbol{\beta}_N(S_k) \text{ and } \gamma_{\text{next}}^3(S_j) = S_l,
\end{aligned}$$

where $\gamma_{\text{next}} = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, if $\boldsymbol{\beta}_N(S_j) \in \{0, 1\}$ or $\gamma_{\text{next}} = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, if $\boldsymbol{\beta}_N(S_j) = \infty$. These are all possible combinations since $r_{jl}^{\Gamma_N}(c) = r_{lj}^{\Gamma_N}(c)$.

Remark 5.5.1. The number of different coefficient functions coincide with the number of different scattering constants according to Theorem 5.4.8 (Formula (5.4.8.3) does not change when ζ^k is replaced by ζ^{-k}, $k \in \mathbb{Z}$).

Since the series $\sum_{c>0} \frac{1}{c^2} r_{jk}^{\Gamma_N}(c)$ that occurs in the scattering constant diverges, it is not possible to approximate the scattering constants directly with only finitely many coefficients $r_{jk}^{\Gamma_N}(c)$.
In [Pos07] it is explained how one can use test functions with same residue and known scattering constant to approximate scattering constants when finitely many coefficients

are given. Here we take

$$\xi(c) := \frac{2}{3}\phi(c), \tag{5.5.1.1}$$

where $\phi(\cdot)$ is Euler's totient function. Then we get

$$\pi^{1/2}\frac{\Gamma(1/2)}{\Gamma(1)} \cdot \frac{1}{4N^2}\sum_{c=1}^{n}\frac{1}{c^2}\left(r_{jk}^{\Gamma_N}(c) - \frac{2}{3}\xi(c)\right) \approx C_{jk}^{\Gamma_N} - C^{\xi}. \tag{5.5.1.2}$$

With

$$C^{\xi} = \lim_{s\to 1}\left(\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \frac{1}{(2N)^{2s}}\sum_{c=1}^{\infty}\frac{1}{c^{2s}}\xi(c)\right) = \frac{1}{150}C^{\Gamma(1)} - \frac{1}{25\pi}\log(10)$$

the scattering constant $C_{jk}^{\Gamma_N}$ is the only unknown in Equation (5.5.1.2) and can be calculated.

In Table 5.2 the exact and the numerical value of the scattering constants for $N = 5$ can be seen. To determine the numerical value the first 500 coefficients have been used.

Cusps	exact: C_{ex}	numerical: C_{num}	$C_{ex} - C_{num}$	C_{ex}/C_{num}
a	-0,0546771653	-0,0547239002	0,0000467349	0,9991459877
b	-0,0176529251	-0,0176580848	0,0000051597	0,9997078009
c	-0,0228017010	-0,0227691133	-0,0000325877	1,0014312227
d	-0,0381191491	-0,0381025710	-0,0000165781	1,0004350919

Table 5.2.: Comparison of the exact and the numerical scattering constants

	$N=5$				$N=6$					$N=7$				
c	$r_a^{\Gamma_5}$	$r_b^{\Gamma_5}$	$r_c^{\Gamma_5}$	$r_d^{\Gamma_5}$	$r_a^{\Gamma_6}$	$r_b^{\Gamma_6}$	$r_c^{\Gamma_6}$	$r_d^{\Gamma_6}$	$r_e^{\Gamma_6}$	$r_a^{\Gamma_7}$	$r_b^{\Gamma_7}$	$r_c^{\Gamma_7}$	$r_d^{\Gamma_7}$	$r_e^{\Gamma_7}$
1	1	0	0	0	1	0	0	0	0	1	0	0	0	0
2	0	0	5	0	0	0	6	0	0	0	0	7	0	0
3	2	0	0	0	2	0	0	0	0	2	0	0	0	0
4	0	0	0	10	0	0	0	12	0	0	0	0	14	0
5	4	0	0	0	4	0	0	0	0	4	0	0	0	0
6	0	0	0	10	0	0	0	0	24	0	0	0	0	14
7	6	0	0	0	6	0	0	0	0	6	0	0	0	0
8	0	20	10	0	0	24	0	12	0	0	28	0	0	14
9	6	0	0	0	6	0	0	0	0	6	0	0	0	0
10	0	20	0	10	0	0	12	0	24	0	0	0	14	14
11	10	0	0	0	10	0	0	0	0	10	0	0	0	0
12	0	0	10	10	0	24	0	12	0	0	0	14	14	0
13	12	0	0	0	12	0	0	0	0	12	0	0	0	0
14	0	0	20	10	0	0	36	0	0	0	28	28	0	0
15	8	0	0	0	8	0	0	0	0	8	0	0	0	0
16	0	20	20	10	0	24	0	36	0	0	28	14	0	28
17	16	0	0	0	16	0	0	0	0	16	0	0	0	0
18	0	0	30	0	0	0	24	0	24	0	0	28	14	0
19	18	0	0	0	18	0	0	0	0	18	0	0	0	0
20	0	20	0	30	0	0	0	48	0	0	0	0	42	14
21	12	0	0	0	12	0	0	0	0	12	0	0	0	0
22	0	40	10	20	0	0	36	0	48	0	0	0	28	42
23	22	0	0	0	22	0	0	0	0	22	0	0	0	0
24	0	40	10	10	0	72	0	12	0	0	56	0	14	14
25	20	0	0	0	20	0	0	0	0	20	0	0	0	0
26	0	20	0	50	0	0	24	0	96	0	0	14	14	56
27	18	0	0	0	18	0	0	0	0	18	0	0	0	0
28	0	0	30	30	0	48	0	48	0	0	28	28	42	0
29	28	0	0	0	28	0	0	0	0	28	0	0	0	0
30	0	20	30	0	0	0	36	0	24	0	0	56	0	0
31	30	0	0	0	30	0	0	0	0	30	0	0	0	0
32	0	60	50	0	0	72	0	60	0	0	84	0	14	56
33	20	0	0	0	20	0	0	0	0	20	0	0	0	0
34	0	0	50	30	0	0	96	0	0	0	56	70	0	14
35	24	0	0	0	24	0	0	0	0	24	0	0	0	0

Table 5.3.: Coefficients for the scattering matrices for Γ_5, Γ_6 and Γ_7

A. Maple algorithms

A.1. Algorithm to calculate the Belyi permutations for $\Gamma(N)$

This is an implementation for the computer algebra program Maple (version 12) of the algorithm that was presented in Section 2.2. Given a positive even integer N, the program calculates the Belyi permutations associated to the group $\Gamma(N)$. The program works but it is very slow, there is certainly room for improvement.

More precisely, the program gives the permutation σ_1 associated to the cusps above 1 under the assumption that the edges are numbered such that σ_0, the permutation for the cusps above 0, is

$$(1, 2, \ldots, N/2)(N/2+1, N/2+2, \ldots, N)\ldots \\ ((r-1)N/2+1, (r-1)N/2+2, \ldots, [\Gamma(2):\Gamma(N)]),$$

where $r = \frac{2[\Gamma(2):\Gamma(N)]}{N}$ (this is always possible).

Most of the code is self-explanatory. That the part that calculates the cusps of $\Gamma(N)$ (at least two thirds of them) works is due to Proposition 2.2.6. The most crucial point, the actual calculation of the cycles, is done by transforming to cycles for the cusps 0 and 1 that are both known. That this process works was explained in Section 1.6 (Lemma 1.6.2 to Theorem 1.6.5). The cycle for 1 used here does not, at first glance, coincide with the one calculated in Example 1.6.3, but one can easily show that they are equivalent under $\Gamma(N)$.

```
# The following program calculates the permutations associated
# to a principal congruence subgroup. Before starting this program,
# N has to be defined as an even positive integer.

# Calculation of the cusps to be considered,
# i.e. the cusps above 0 and 1.
C:={};
for i from 0 to N-1 do C:=C union {i} end do:
for i from 3 by 2 to N/2-1 do
for j to N do
    if (gcd(j,i)=1) then C:=C union {j/i} elif (gcd(gcd(j,i),N)=1) then
    n:=nops(C): k:=j:
    while (n=nops(C)) do
    if (gcd(i, k)=1) then C:=C union {k/i} end if: k:=k+N:
    end do:
```

```
    end if:
end do: end do:
# N/2 must be dealt with separately, because the identity -a/b=a/b
# leads to less cusps.
if (N/2 mod 2 = 1) then
    for j to N/2 do
    if (gcd(j,i)=1) then C:=C union {j/i} elif (gcd(gcd(j,i),N)=1) then
    n:=nops(C): k:=j:
    while (n=nops(C)) do
    if (gcd(i, k)=1) then C:=C union {k/i} end if: k:=k+N:
    end do:
    end if:
    end do:
end if:

# The following procedure reduces a cusp, lying above 0 or 1,
# to one from the system of representatives.
inRep:=proc(a)
a1:=numer(a) mod N:
a2:=denom(a) mod N:
if a2>(N/2) then a2:=-(a2-N): a1:=-a1: end if:
if a1<0 then a1:=a1+N end if:
if gcd(a1,a2)>1 then
    while gcd(a1,a2)>1 do a1:=a1+N end do:
end if:
# If the denominator is N/2 we have to reduce further.
if (a2=N/2 and a1>N/2) then a1:=N-a1 end if:
return(a1/a2):
end proc:

# This procedure constructs a matrix that maps 0 to a given cusp a/b.
# A matrix that maps 1 to a/b is this one composed with [1,-1,0,1].
neuSp:=proc(a,b)
if b=1 then return [1,a,0,b] end if:
for i from 0 by 2 to b-1 do
if evalb(a*i mod b=1) then return [-floor(a*i/b),a,-i,b]
elif evalb(a*i mod b=b-1) then return [floor(a*i/b)+1,a,i,b]
end if:
end do:
end proc:

# The cycles for the cusps 0 and 1 are
# starting point for the calculation.
# For Gamma(N), Z0 and Z1 are given as below.
```

```
Z0:=[seq(1/(2*i-1), i=1..N/2-1),-1]:
Z1:=[seq(Z0[i]+1, i=1..N/2)]:

# Here, the cycles for all cusps are computed
# using Z0, Z1 and the matrices from neuSp.
Zg:={}: Zu:={};
for a in C do
mat:=neuSp(numer(a), denom(a)):
if evalb(numer(a) mod 2 =0) then
Zg:=Zg union
 {[a,[seq(inRep((mat[1]*Z0[i]+mat[2])/(mat[3]*Z0[i]+mat[4])),
   i=1..N/2)]]}:
else
 # The matrix neuSp has to be composed with [1,-1,0,1].
Zu:=Zu union
 {[a,[seq(inRep((mat[1]*Z1[i]+mat[2]-mat[1])/
   (mat[3]*Z1[i]+mat[4]-mat[3])), i=1..N/2)]]}:
end if:
end do:

# The information is rewritten into permutations.
# The first permutation is chosen to be sigma0=(1,2,3,...)...(...end).
# Only sigma1 will be given.

# The edges from sigma0 get numbers.
kanten:=Array([seq(0, i=1..nops(Zg)*N/2)]): k:=0:
num:=[seq(i, i=1..nops(Zg)*N/2)]:
for a in Zg do
for i from 1 to N/2 do
k:=k+1:
kanten[k]:=[a[1], a[2][i]]:
end do:
end do:

# sigma1 is written by giving the edges the same numbers as in sigma0.
# P will be the permutation (as an array).
P:=Array([seq(0, i=1..nops(Zg))]): j:=0:
for a in Zu do
j:=j+1:
Z:=Array([seq(0, i=1..N/2)]):
for i from 1 to N/2 do
k:=1:
while not ([a[2][i],a[1]]=kanten[num[k]]) do k:=k+1 end do:
Z[i]:=num[k]:
```

```
num:=subsop(k=NULL, num):
end do:
P[j]:=Z:
end do:
```

A.2. Algorithm to test if a subgroup is congruence

The following algorithm will answer the question if a subgroup $\Gamma \subset \Gamma(2)$ defined by two Belyi permutations is a congruence subgroup. The input is two Belyi permutations and an edge to fix. Before the program can run, these three data have to be given: The permutations `s0` and `s1` as a product of disjoint cycles as nested arrays and the edge `e` as an integer.

The first part provides basic data, the degree `M` of the symmetric group from which `s0` and `s1` come and the `N` for which $\Gamma(N)$ must be a subgroup of Γ for Γ to be a congruence subgroup following Theorem 2.4.1.

```
# Given two permutations in form [[a1,a2,...an],...,[x1,x2,...xl]],
# s0 and s1, and an edge e that shall be stabilized.

# They generate group of order M:
M:=max(s0,s1):

# Calculation of the N for which Gamma(N) could be subgroup.
with(group):
si:=mulperms(s1,s0):
N:=2*lcm( seq(nops(s0[i]), i=1..nops(s0)),
   seq(nops(s1[i]), i=1..nops(s1)), seq(nops(si[i]), i=1..nops(si)) ):

# To apply the permutations, change their appearance.
# The next proc rewrites a permutation s into the form
# [s(1), s(2), ...], for that s and n (number of elements
# that s acts on) must be given.
nperm:=proc(s,n)
  erg:=Array([seq(i, i=1..n)]):
if type(s, list) then
  for i from 1 to nops(s) do
  for j from 1 to nops(s[i])-1 do
  erg[s[i][j]]:=s[i][j+1]:
  end do:
  erg[s[i][nops(s[i])]]:=s[i][1]:
  end do:
elif type(s, Array) then
  for i from 1 to ArrayNumElems(s) do
```

```
  for j from 1 to ArrayNumElems(s[i])-1 do
  erg[s[i][j]]:=s[i][j+1]:
  end do:
  erg[s[i][ArrayNumElems(s[i])]]:=s[i][1]:
  end do:
end if:
  return erg
end proc:
```

Now, the information still missing to find out, if Γ is a congruence subgroup, is the permutations for $\Gamma(N)$. The permutation (here called) `con1`, which describes the edges around the cusps $\Gamma(2)$-equivalent to 1, can be calculated with Algorithm A.1. Take care that `con1` is a list or an Array with only one dimension. It may happen that Maple regards `con1` as a two dimensional Array, hence as a matrix, then the procedure `nperm` will give the wrong permutation.

The other permutation `con0`, as it is expected to look like in Algorithm A.1, will be defined below in this algorithm.

At the end, the Boolean variable `sbgrp` will tell us if the group is a subgroup. In `B` the images of the map μ can be seen, there `B[a]` is $\mu(a)$. The algorithm terminates since `con0` and `con1` generate a group that acts transitively.

```
# Permutation con1 for cusps above 1 in Gamma(N) with another program.
# Permutation con0 for the cusps equivalent to 0 in Gamma(N), here:
# In cycle presentation we take (1,2,3..N/2)..(...ind-1, ind).
ds:={seq(ifactors(N)[2][i][1], i=1..nops(ifactors(N)[2]))}:
ind:=N^3/12*mul(1-p^(-2), p in ds):
con0:=[]: for i from 0 to 2*ind/N-1 do
  con0:=[op(con0), seq(i*N/2+j,j=2..N/2),i*N/2+1]
end do:

# construct map  a:{ind} -> {M}.
# Starting from 1 -> e we have to calculate for all of {ind} the values
# a(conj(k))=sj(a(k)) for j=0,1.
# If this calculation yields consistent images,
# the group is a congruence subgroup.
B:=Array([seq(0, i=1..ind)]):    # For the images of {ind}.
U:= [seq(i, i=1..ind)]:          # The preimages.
S0:=nperm(s0, M):
S1:=nperm(s1, M):
Con1:=nperm(con1, ind):
Con0:=con0:
sbgrp:=true:
B[1]:=e:                # initial condition.
```

```
while sbgrp and nops(U)>0 do
i:=1:
while B[U[i]]=0 do i:=i+1 end do:
if  B[Con0[U[i]]]=0 then B[Con0[U[i]]]:=S0[B[U[i]]]
    elif not B[Con0[U[i]]]=S0[B[U[i]]] then sbgrp:=false
end if:
if  B[Con1[U[i]]]=0 then B[Con1[U[i]]]:=S1[B[U[i]]]
    elif not B[Con1[U[i]]]=S1[B[U[i]]] then sbgrp:=false
end if:
U:=subsop(i=NULL,U):
end do:
```

B. Java algorithm to calculate coefficients in the scattering matrix for Fermat curves

The algorithm here and its implementation base on the work done in [Pos07] where one can find more details and explanations. Here, we give a rough idea of the algorithm and a Java code that implements it.

Table B.1.: Calculation of the c-th coefficients for S_j and S_k

```
coefficient = 0
for d from 0 to c − 1 do
  if gcd(c, d) = 1 then
    Create matrix (a b; c d) with entries a, b < c
    for m from 0 to b_j − 1 do
      for n from 0 to b_k − 1 do
        Calculate M = γ_j (1 m; 0 1)(a b; c d)(1 n; 0 1) γ_k^{−1}.
        if M ∈ Γ(2) then
          Find s(M) and t(M).
          if s(M) ≡ t(M) ≡ 0 mod N then
            coefficient = coefficient + 1
          end if
        end if
      end for
    end for
  end if
end for
return coefficient
```

In the algorithm in Table B.1 on page 129 the scheme to calculate the c-th ($c \in \mathbb{N}$) coefficient for given cusps is explained, i.e. it shows how to calculate for cusps S_j and

S_k

$$r_{jk}^{\Gamma_N}(c) = \#\left\{d \mod 2Nc \mid \exists\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma_N\gamma_k\right\}.$$

It is based on the decomposition into double cosets for a group $\Gamma \subset \Gamma(1)$, see Lemma 3.1.5, and the description of the group Γ_N, the subgroup associated to the N-th Fermat curve in Remark 5.1.4.

The rest of this appendix is filled with the code for a Java implementation of Algorithm B.1. All data must be inserted in the main procedure: The N for the curve, the coefficients that shall be calculated (via the minimal and the maximal c) and the cusps (via matrices γ_j and γ_k^{-1}).

```
/*
Calculates coefficients in the scattering matrix for Fermat curves.
*/
import java.io.*;

class Fermat
{
        // gcd
        public static int gcd(int m, int n)
        {
        if (m < n) {
        int t = m;
        m = n;
        n = t;     }
        int r = m % n;
        if (r == 0) { return n; }
        else { return gcd(n, r); }
        }

        // floor function
        public static int gauss(double d)
        {
        int i=0;
        i=(int) d;
        if (i > d) i=i-1;
        return i;
        }

        // calculates upper row for matrix in Gamma(1)
        public static int[] aundb(int m, int n)
        {
        int a=0;
        while  ((a*n % m) != 1) a=a+1;
```

```
int[] ab= {a,(a*n-1)/m};
return ab;
}

// calculates Dirichlet sum
public static double dede(int m, int n)
{
double d=0;
long ml=m;
long nl=n;
for(long j=0; j<nl; j++)
{
double s=0;
double t=0;
if ((1.*j*ml/nl-j*ml/nl)!=0.)
s=(1.*j*ml/nl-j*ml/nl-0.5);
t=1.*j/nl-0.5;
d=d+t*s;
}
return d;
}

// calculates s(M) and t(M) for a given matrix M in Gamma(2)
public static int[] st (ZweiZweiMatrix M)
{
int[] zeug=new int[2];
double d1=0;
double d2=0;
double d3=0;
if (M.c()==0)
    {
    zeug[0]=M.b()/2;
    zeug[1]=0;
    }
else
{
if (M.c()<0) M=new ZweiZweiMatrix
(-M.a(),-M.b(),-M.c(),-M.d());
d1=dede(M.d()-M.c(),2*M.c());
d2=1.*((M.a()+M.d()))/(2.*M.c())-4*(dede(M.d(),M.c()/2)-d1);
d3=4*(d1-dede(M.d(),2*M.c()));
zeug[0]=Math.round(Math.round(d2));
zeug[1]=Math.round(Math.round(d3));
}
```

```
        return zeug;
        }

    public static void main(String[] params) throws IOException
    {
    // input data
    int N=5;  // N
    int cmin=1;  // minimal c, where calculation starts
    int cmax=50; // maximal c, where calculation ends
    int[] koeffs=new int[cmax];
    int bj=10; // widths of the cusps
    int bk=10;

    // cusps j and k,
    // via matrices with Gj(infinity)=j and Gk(k)=infinity.
    ZweiZweiMatrix Gj= new ZweiZweiMatrix(1,0,0,1);
    ZweiZweiMatrix Gk= new ZweiZweiMatrix(0,1,-1,0);

    // array for coefficients
    for(int z=0; z<cmax; z++) koeffs[z]=0;

    // the case d=0, occurs only with c=1.
    ZweiZweiMatrix moeg= new ZweiZweiMatrix(0,-1,1,0);
        for(int m=0; m<bj; m++)
        {
        ZweiZweiMatrix Mm= new ZweiZweiMatrix(1,m,0,1);
        for(int n=0; n<bk; n++)
              {
              ZweiZweiMatrix Mn= new ZweiZweiMatrix(1,n,0,1);
              ZweiZweiMatrix weiter= new ZweiZweiMatrix(1,0,0,1);
              weiter=Gj.mal(Mm.mal(moeg.mal(Mn.mal(Gk))));
              if
              ((Math.abs(weiter.a() % 2) ==1) &&
               (Math.abs(weiter.b() % 2) ==0) &&
               (Math.abs(weiter.c() % 2) ==0) &&
               (Math.abs(weiter.d() % 2) ==1))
              {
              if((Math.abs(st(weiter)[0]) % N==0) &&
                 (Math.abs(st(weiter)[1]) % N==0))
              {
              koeffs[0]=koeffs[0]+1;
              }}}
        }
```

```
    // the case d>0.
    for (int c=cmin; c<=cmax; c++)
    {
    for (int d=1; d<c; d++)
         {
         if (gcd(c,d)==1)
         {
         moeg= new ZweiZweiMatrix(aundb(c,d)[0], aundb(c,d)[1], c, d);
         for(int m=0; m<bj; m++)
         {
         ZweiZweiMatrix Mm= new ZweiZweiMatrix(1,m,0,1);
         for(int n=0; n<bk; n++)
              {
              ZweiZweiMatrix Mn= new ZweiZweiMatrix(1,n,0,1);
              ZweiZweiMatrix weiter= new ZweiZweiMatrix(1,0,0,1);
              weiter=Gj.mal(Mm.mal(moeg.mal(Mn.mal(Gk))));
              if
              ((Math.abs(weiter.a() % 2) ==1) &&
               (Math.abs(weiter.b() % 2) ==0) &&
               (Math.abs(weiter.c() % 2) ==0) &&
               (Math.abs(weiter.d() % 2) ==1))
              {
              if((Math.abs(st(weiter)[0]) % N==0) &&
                 (Math.abs(st(weiter)[1]) % N==0))
              {
              koeffs[c-1]=koeffs[c-1]+1;
              }}}
         }}}
         System.out.println(koeffs[c-1]);
    }}
}
```

To run the program one needs the program `ZweiZweiMatrix.java`, whose code is given below.

```
import java.io.*;
import java.util.*;

class ZweiZweiMatrix implements Serializable
    {
        public static final ZweiZweiMatrix S =
         new ZweiZweiMatrix(new int[][]{{0,-1},{1,0}});
        public static final ZweiZweiMatrix T =
         new ZweiZweiMatrix(new int[][]{{1,1},{0,1}});
```

```
public static final ZweiZweiMatrix T1 =
 new ZweiZweiMatrix(new int[][]{{1,-1},{0,1}});
public static final ZweiZweiMatrix Id =
 new ZweiZweiMatrix(new int[][]{{1,0},{0,1}});
public static ZweiZweiMatrix verschieb( int n)
 { return new ZweiZweiMatrix(1, n, 0, 1); }

private final int[][] werte;
public ZweiZweiMatrix (int[][] werte)
{    this.werte = werte;    }

public ZweiZweiMatrix(int a, int b, int c, int d)
{    this(new int[][]{{a,b},{c,d}});    }

public String toString()
{    return "[" + werte[0][0] + ", " + werte[0][1] + ", "
            + werte[1][0] + ", " + werte[1][1] + "]";
}

public boolean equals(Object o)
{ if (! (o instanceof ZweiZweiMatrix))
       return false;
return Arrays.deepEquals(((ZweiZweiMatrix)o).werte, this.werte);
}

public int hashCode()
{    return Arrays.deepHashCode(this.werte);   }

public boolean inGamma2()
{
return ((werte[0][0] % 2==1) && (werte[0][1] % 2==0) &&
        (werte[1][0] % 2==0) && (werte[1][1] % 2==1));
}

public ZweiZweiMatrix mal (ZweiZweiMatrix andere)
{
return new ZweiZweiMatrix(
            this.a()*andere.a()+this.b()*andere.c(),
            this.a()*andere.b()+this.b()*andere.d(),
            this.c()*andere.a()+this.d()*andere.c(),
            this.c()*andere.b()+this.d()*andere.d());
}

public int a() { return werte[0][0]; }
```

```
    public int b() { return werte[0][1]; }
    public int c() { return werte[1][0]; }
    public int d() { return werte[1][1]; }
}
```

C. Calculation of the scattering constants for $\Gamma(2)$

This appendix is copied essentially from [Pos07].

The scattering matrix for $\Gamma(1)$ is known.

Proposition C.0.1. *For* $\Gamma(1)$ *holds*

$$\Phi_{\Gamma(1)}(s) = \left(\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \sum_{c>0} \frac{1}{c^{2s}}\phi(c)\right) = \left(\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \frac{\zeta(2s-1)}{\zeta(2s)}\right),$$

where $\phi(\cdot)$ *is Euler's totient function.*

Proof: See e.g. [Iwa97]. □

Proposition C.0.2. *The coefficients* $r_{jk}^{\Gamma(2)}(c)$ *for* $c \in \mathbb{N}$ *and* $j,k \in \{0,1,\infty\}$ *are:*

$$\textit{If } j = k \textit{ then } r_{jk}^{\Gamma(2)}(c) = \begin{cases} 2\phi(c), & \textit{if } c \equiv 0 \mod 2; \\ 0, & \textit{if } c \equiv 1 \mod 2. \end{cases}$$

$$\textit{If } j \neq k \textit{ then } r_{jk}^{\Gamma(2)}(c) = \begin{cases} \phi(c), & \textit{if } c \equiv 1 \mod 2; \\ 0, & \textit{if } c \equiv 0 \mod 2. \end{cases}$$

Proof: This is a generalization of Proposition 4.4.2 and follows similarly. For the description via $\phi(\cdot)$ use Proposition C.0.1. □

Now, we will express the entries of the scattering matrix $\Phi^{\Gamma(2)}$ for $\Gamma(2)$, see Definition 3.2.1, via L-series to calculate the scattering constants.

Definition C.0.3. *With*

$$\chi_2(n) := \begin{cases} 1 & \textit{if } (n,2) = 1 \\ 0 & \textit{if } (n,2) = 2 \end{cases} \qquad \textit{for } n \in \mathbb{Z}$$

we define

$$L(\chi_2, s) := \sum_{c>0} \chi_2(c)\frac{1}{c^s}$$

(for $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$*).*

Remark C.0.4. It holds that

$$L(\chi_2, 2s) = \sum_{c>0} \chi_2(c) \frac{1}{c^{2s}} = \sum_{c>0} \frac{1}{(2c-1)^{2s}} = 2^{-2s}(-1 + 2^{2s})\zeta(2s)$$

and

$$L(\chi_2, 2s-1) = 2^{-2s}(-2 + 2^{2s})\zeta(2s-1).$$

Therefore, the quotient is

$$\frac{L(\chi_2, 2s-1)}{L(\chi_2, 2s)} = \frac{(-2 + 2^{2s})\zeta(2s-1)}{(-1 + 2^{2s})\zeta(2s)}. \tag{C.0.4.1}$$

Proposition C.0.5. *For* $\Gamma(2)$ *and cusps* $j \neq k$ *holds*

$$\sum_{c>0} \frac{1}{c^{2s}} r_{jk}^{\Gamma(2)}(c) = \sum_{c>1} \chi_2(c)\phi(c) \frac{1}{c^{2s}} = \frac{L(\chi_2, 2s-1)}{L(\chi_2, 2s)} \tag{C.0.5.1}$$

and in the case $j = k$

$$\sum_{c>0} \frac{1}{c^{2s}} r_{jk}^{\Gamma(2)}(c) = \sum_{c>1} \chi_2(c+1) \cdot 2 \cdot \phi(c) \frac{1}{c^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)} - \frac{L(\chi_2, 2s-1)}{L(\chi_2, 2s)}. \tag{C.0.5.2}$$

Proof: The first identity in the equations (C.0.5.1) and (C.0.5.2) holds because of Proposition C.0.2. The second identity follows by expanding the L-series and comparing the results. □

With Proposition C.0.5, Definition 3.2.1 and Equation (C.0.4.1) we can calculate the scattering constants for Γ(2) using a computer algebra system, i.e. Maple. We get for non-equivalent cusps $j \neq k$

$$C_{jk}^{\Gamma(2)} = \lim_{s \to 1} \left(\frac{\pi^{1/2}}{4^s} \cdot \frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \frac{L(\chi_2, 2s-1)}{L(\chi_2, 2s)} - \frac{3}{6\pi(s-1)} \right) \tag{C.0.5.3}$$

$$= -\frac{1}{3\pi} \left(7\log(2) + 3\log(\pi) + 36\zeta'(-1) - 3\right) \tag{C.0.5.4}$$

$$\approx 0,07097687113 \tag{C.0.5.5}$$

and for equivalent cusps

$$C_{jj}^{\Gamma(2)} = \lim_{s\to 1}\left(\frac{2\pi^{1/2}}{4^s}\cdot\frac{\Gamma(s-1/2)}{\Gamma(s)}\cdot\left(\frac{\zeta(2s-1)}{\zeta(2s)}-\frac{L(\chi_2,2s-1)}{L(\chi_2,2s)}\right)-\frac{3}{6\pi(s-1)}\right) \tag{C.0.5.6}$$

$$= -\frac{1}{3\pi}\left(13\log(2)+3\log(\pi)+36\zeta'(-1)-3\right) \tag{C.0.5.7}$$

$$\approx -0,3702943296. \tag{C.0.5.8}$$

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe. Die Promotionsordnung ist mir bekannt und weder trage ich bereits einen Doktortitel noch habe ich mich anderwärts um einen beworben.

Berlin, den 25.05.2010 Anna Elisabeth Posingies