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Isoperimetric profile of algebras

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The dissertation of Michele D'Adderio is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To “my family”.

EPIGRAPH

The secret of freedom lies in educating people,
whereas the secret of tyranny is in keeping them ignorant.
—Maximilien Robespierre

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ABSTRACT OF THE DISSERTATION

Isoperimetric profile of algebras

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Professor Efim Zelmanov, Chair

The main topic of this thesis is the isoperimetric profile of algebras, introduced by Gromov in [21]. This is an asymptotic geometric invariant of algebras which has a strong connection with the property of amenability. We pursue a systematic study of this invariant, which will lead us to new insights into the phenomenon of amenability of algebras. Also, we show the connections of the isoperimetric profile with other invariant of algebras, like the *GK*-dimension, the growth and the lower transcendence degree. We will use our tools to provide answers to some questions on these topics.

Chapter 1

Introduction

Trying to apply geometric ideas to algebra is a leitmotif of the mathematics of the last century. The geometric theory of groups, one of the main byproduct of this philosophy, reached its exploit in 1981 with Gromov's paper [19]. Other than the celebrated theorem on groups of polynomial growth, this work contains the key idea that, once we chose a set of generators, we can view finitely generated groups as metric spaces with the word-length metric, and hence apply to them the methods of the so called "rough" geometry, whose objects of study are metric spaces considered up to quasi-isometries. There has been a tremendous amount of work on this subject, and this field of research is still flourishing.

The ultimate motivation of the work presented in this thesis is to develop a geometric theory of algebras parallel to the one just mentioned for groups.

The cited work of Gromov was motivated by the study of the notion of growth of groups. Research on this important asymptotic geometric invariant in group theory promoted further investigations, leading, among other things, to another breakthrough in the subject: the discovery by Grigorchuk of the famous group that now bears his name.

In the literature, the concepts for algebras traceable to these body of notions that have been mainly considered are the growth of algebras and the related Gelfand-Kirillov dimension. These turn out to be fundamental tools in noncommutative ring theory.

Together with the growth, the most important asymptotic geometric invariant of groups is the isoperimetric profile.

The geometric concept of an isoperimetric profile was first introduced in algebra for groups by Vershik in [40] and Gromov in [20]. Here is the definition given by Gromov in [21], for semigroups:

Definition. Given an infinite semigroup Γ generated by a finite subset S, and given a finite subset Ω of Γ we define the *boundary* of Ω as

$$\partial_S(\Omega) := \bigcup_{s\in S}(s\Omega \setminus \Omega).$$

Then we define the *isoperimetric profile of a semigroup* Γ with respect to S as the function from $\mathbb{N}$ onto itself given by

$$I_\circ(n;\Gamma,S) := \inf_{|\Omega|=n} |\partial_S(\Omega)|$$

for each $n \in \mathbb{N}$, where $|X|$ denotes the cardinality of the set X.

It's well known that the asymptotic behavior of this function is independent of the set of generators S.

For properties of the isoperimetric profile see [13, 14, 21, 33], the survey [34] and references therein.

The notion of the isoperimetric profile of algebras was introduced by Gromov in [21]:

Definition. Let A be a finitely generated algebra over a field K of characteristic zero. Given two subspaces V and W of A we define the *boundary* of W with respect to V by

$$\partial_V(W) := VW/(VW \cap W).$$

If V is a generating finite dimensional subspace of A, we define the *isoperimetric profile of A with respect to V* to be the maximal function I_* such that all finite dimensional subspaces $W \subset A$ satisfy the following *isoperimetric inequality*

$$I_*(|W|;A,V) = I_*(|W|) \leq |\partial_V(W)|,$$

where $|Z|$ denotes the dimension over the base field K of the vector space Z.

Again, the asymptotic behavior of this function does not depend on the generating subspace.

The main topic of this thesis is this invariant of algebras.

The isoperimetric profile is an asymptotically weakly sublinear function, and it has a strick connection with the notion of amenability.

Amenability of groups has been introduced by John von Neumann in 1929 ([41]). Since than, it has been extensively studied, and it turned out to have connections with many areas of mathematics: group theory, representation theory, random walks, C^*-algebras, von Neumann algebras, Banach-Tarski paradox, property T, etc....

In associative algebras this notion was introduced by Elek [10]. It's easy to see that the isoperimetric profile of an algebra is linear if and only if the algebra is nonamenable (in the sense of Elek). In this sense it can be viewed as a measure of the amenability of an algebra.

There have been several works on amenability of algebras. I will mention the more significant ones: Elek in [10] and [11] studied basic properties of the amenability of algebras and amenability of division algebras. Bartholdi in [2] studied the relation between the amenability of a group and the amenability of its group algebra. In [21] Gromov studied in particular the isoperimetric profile of group algebras and its relation with the isoperimetric profile of the underlying group.

Along these lines, in this thesis we pursue a systematic study of the isoperimetric profile of algebras.

In the second chapter we provide some technical and motivational background on growth, amenability and isoperimetric profile in group theory, and on growth of algebras and Gelfand-Kirillov dimension. We tried to make the rest of this thesis independent on this chapter.

In the third chapter we discuss the first basic properties of the isoperimetric profile, its relation with amenability and we review some of the work of Gromov that provides the fundamental computation of the isoperimetric profile of the algebra of polynomials.

In the fourth chapter we study the behavior of the isoperimetric profile under various ring-theoretic constructions. We will also consider briefly the isoperimetric profile of modules. These tools will enable us to provide new results on the amenability of algebras, generalizing most of the results in Elek's [10, 11].

In the fifth chapter we apply our tools to compute the isoperimetric profile of many algebras.

In the sixth chapter we discuss the relation of the isoperimetric profile with other invariants of algebras. In particular we study the relations of the isoperimetric profile with the lower transcendence degree introduced by Zhang in [45], and we derive from this some consequences on amenability of algebras. We study its relation with the growth, answering a question in [21] Section 1.9. We conclude by stating a conjecture.

Chapter 2

Background

In this chapter we provide some technical and motivational background.

2.1 Growth of groups

We will always consider finitely generated groups, unless otherwise stated.

Efremovič ([9]) and Švarc ([38]) in the fifties, and Milnor ([30]) in 1968 started the study of the concept of growth in groups.

Definition. Let Γ be an infinite group generated by a finite subset $S \subset \Gamma$ such that $1 \notin S$ and $S = S^{-1}$. For every $n \in \mathbb{N}$ we set

$$G_\circ(n;\Gamma,S) := |\cup_{r=0}^{n} S^r|,$$

where $S^0 = \{1\}$ and 1 is the identity of Γ. This gives a monotone increasing function from $\mathbb{N}$ into itself, which depends on Γ and S, called the *growth* of Γ with respect to S. Thinking about the Cayley graph associated to this two data as a metric space, with the distance between two points defined as the minimal length of a trail joining them, we can think of $G_\circ(n;\Gamma,S)$ as the volume of the ball centered in 1 of radius n. This justifies the name of this function.

We need another definition:

Definition. Given two functions $f_1, f_2 : \mathbb{N}_+ \to \mathbb{R}_+$ we say that f_1 is *asymptotically faster* then f_2, and we write $f_1 \succeq f_2$, if there exist positive constants C_1 and C_2 such that $f_1(C_1 x) \geq C_2 f_2(x)$ for all $x \in \mathbb{R}_+$. We say f_1 is *asymptotically equivalent* to f_2, and we write $f_1 \sim f_2$, if $f_1 \succeq f_2$ and $f_2 \succeq f_1$. This is clearly an equivalence relation. We will talk about the equivalence class of such a function as its *asymptotic behavior.*

Remark. Notice that given a function $f : \mathbb{N} \to \mathbb{N}$ we can always consider it as a function on $\mathbb{R}_+$, simply defining for $r \in \mathbb{R}_+$, $f(r) := f(\lfloor r \rfloor)$, where $\lfloor r \rfloor$ denotes the maximal integer $\leq r$. We will often do it, without mentioning it explicitly.

Example 2.1.1. It's easy to check that if we have a polynomial $p(x) \in \mathbb{R}[x]$ with $p(r) > 0$ for $r > 0$, and we consider it as a function $r \mapsto p(r)$, we have $p(r) \sim r^{\deg(p)}$.

Lemma 2.1.1. *Let Γ be an infinite group generated by a finite subset $S \subset \Gamma$ such that $1 \notin S$ and $S = S^{-1}$, and let $S' \subset \Gamma$ be another generating subset with these properties. Then $G_\circ(n;\Gamma,S) \sim G_\circ(n;\Gamma,S')$.*

Proof. Clearly $S \subset \cup_{r=0}^{m} S'^m$ for some m, since S' generates Γ. This implies that $\cup_{s=0}^{n} S^s \subset (\cup_{r=0}^{m} S'^r)^n \subset \cup_{j=0}^{mn} S'^j$ for all $n \in \mathbb{N}$. Hence

$$G_\circ(n;\Gamma,S) \leq G_\circ(mn;\Gamma,S'),$$

which gives

$$G_\circ(n;\Gamma,S) \preceq G_\circ(n;\Gamma,S')$$

since m does not depend on n. In the same way it can be shown that

$$G_\circ(n;\Gamma,S') \preceq G_\circ(n;\Gamma,S),$$

giving the result. □

This lemma shows that the asymptotic behavior of $G_\circ(n;\Gamma,S)$ does not depend on S, and hence it's an invariant of the group. We will call it the *growth* of Γ and we will denote it $G_\circ(n;\Gamma)$ or just $G_\circ(n)$ if there is no possibility of confusion. For basic properties of growth of groups and for reference on what we will discuss in this thesis we refer to [8].

Example 2.1.2. It's easy to check that for any positive integer d the growth of the group $\mathbb{Z}^d$ is

$$G_\circ(n;\mathbb{Z}^d) \sim n^d.$$

It's also easy to show that for the free group F_d in $d \geq 2$ free generators

$$G_\circ(n;F_d) \sim e^n.$$

Definition. A group Γ is said to have *(bounded) polynomial growth* if

$$G_\circ(n;\Gamma) \preceq n^a$$

for some $a \in \mathbb{R}$, $a \geq 0$.

Notice that a group has constant growth if and only if it's finite. If a group Γ is infinite, then $G_\circ(n;\Gamma) \geq n$: if for some m we have $\cup_{r=0}^{m} S^r = \cup_{r=0}^{m+1} S^r$, then $\cup_{r=0}^{m} S^r = \cup_{r=0}^{m+p} S^r$ for any $p \geq 1$, which implies that Γ is finite. On the other hand, we always have

$$|\cup_{r=0}^{m} S^r| \leq \sum_{r=0}^{m} |S|^r \leq |S| \cdot |S|^m,$$

showing that we always have

$$G_\circ(n;\Gamma) \preceq e^n.$$

Hence we just observed that the growth of an infinite group always lies between a linear function and an exponential function. Moreover, the growth can be polynomial of any integral exponent. The next two results (which we mention without proofs) show that this is the case for a big class of groups.

Definition. The class EG of groups containing finite and abelian groups, and closed under taking subgroups, quotients, direct limits and extensions is called the class of *elementary groups*.

For example this class contains nilpotent and solvable groups, but not the free group F_2 on two free generators.

Definition. A group is called *virtually nilpotent* if it contains a finite index subgroup which is nilpotent. More generally, it has *virtually* a certain property if there is a finite index subgroup that has that property.

Remark 1. It's easy to see that the growth of a group Γ and the growth of any finite index subgroup of Γ are the same.

Theorem 2.1.2 (Milnor,Wolf,Chou [31, 43, 6])**.** *An elementary group has polynomial growth if and only if it is virtually nilpotent, and it has exponential growth otherwise.*

Theorem 2.1.3 (Guivarc'h, Bass [22, 3])**.** *For a nilpotent group* Γ *we have*

$$G_\circ(n;\Gamma) \sim n^d,$$

where d is its homogeneous dimension.

All we need to know about the homogeneous dimension is that it is a positive integer for which there is a formula in terms of the lower central series of Γ.

It can be shown that if a group Γ contains a subgroup isomorphic to a free group F_2, then its growth is exponential. It's been an open problem for many years (since Milnor's question in 1968) if there exist groups of *intermediate growth*, i.e. with growth asymptotically faster than any polynomial, but asymptotical slower than the exponential. A positive answer to this question has been provided by Grigorchuck [18] in 1983, where he provided an example of such a group.

We state the celebrated Gromov's theorem on groups with polynomial growth:

Theorem 2.1.4 (Gromov [19])**.** *A group has polynomial growth if and only if it is virtually nilpotent.*

It's worthwhile to mention here that this theorem and even more its proof in [19] signed a landmark in geometric group theory.

This theorem can be considered as a "negative" result: there are few groups of polynomial growth, the virtually nilpotent ones (which we understand reasonably well), and for those we have seen that the exponent of the growth is an integer.

We will see that this is very far from the corresponding situation for algebras.

2.2 Growth of algebras and Gelfand-Kirillov dimension

In this section, unless otherwise stated, with the word "algebra" we will always mean a finitely generated infinite dimensional associative algebra with unit 1 *over a* *fixed field K of characteristic* 0.

In order to define the notion of growth of algebras, we need some definition and some notation.

Definition. We define a *subframe* of an algebra A to be a finite dimensional subspace containing the identity and a *frame* to be a subframe which generates the algebra. (cf. [45])

Given a finite dimensional vector space W over K we will denote its dimension over K by

$$|W| := \dim_K W.$$

Given an algebra A and a frame $V \leq A$, for any $n \in \mathbb{N}$ we set

$$G_*(n; A, V) := |V^n|.$$

This gives us a monotone increasing function from $\mathbb{N}$ into itself, which depend on the algebra A and on V. What does not depend on V is its asymptotic behavior.

Lemma 2.2.1. *Given two frames V and V' of an algebra A, we have*

$$G_*(n; A, V) \sim G_*(n; A, V').$$

Proof. Since V and V' generate A, for some r and s we will have

$$V \leq V'^r \quad \text{and} \quad V' \leq V^s,$$

which imply immediately

$$G_*(n; A, V) \leq G_*(rn; A, V') \quad \text{and} \quad G_*(n; A, V') \leq G_*(sn; A, V),$$

giving the result. □

Hence, as for groups, we can define the growth of an algebra A as the asymptotic behavior of $G_*(n; A, V)$ for a frame V, and we can denote it $G_*(n; A)$ or simply $G_*(n)$ if there is no possibility of confusion.

Example 2.2.1. It's easy to see that the growth of the algebra $K[x_1,\dots,x_d]$ of polynomials in d variables is

$$G_*(n;K[x_1,\dots,x_d]) \sim n^d.$$

Also, the growth of the free algebra $K\langle x,y\rangle$ in two generators is

$$G_*(n;K\langle x,y\rangle) \sim e^n.$$

As in the case of groups, the growth of an (infinite dimensional finitely generated) algebra lies between a linear function and an exponential one, and we have just seen examples of polynomial growth with integer exponent.

As in the case of groups, we say that an algebra has *(bounded) polynomial growth* if its growth is asymptotically bounded by a polynomial.

Contrary to the case of groups, algebras of polynomial growth are far from being completely understood. To see this we now introduce the notion of Gelfand-Kirillov dimension, which was introduced by Gelfand and Kirillov [15, 16] in 1966.

Definition. The *Gelfand-Kirillov dimension* (also called *GK*-dimension) of a (not necessarily finitely generated) algebra A is

$$GK\dim(A) := \sup_V \overline{\lim}_n \log_n |V^n|,$$

where the supremum is taken over all subframes of A.

Remark 2. Notice that, while this definition works also for algebras that are not finitely generated, for finitely generated algebra to compute the *GK*-dimension it's enough to choose any fixed frame V, instead of taking the supremum. For this and all the other properties of the *GK*-dimension that we will mention in this thesis we refer to [27].

It's easy to see that if $G_*(n;A) \sim n^r$ then $GK\dim(A) = r$. Also, if the growth of an algebra is faster then any polynomial, then its *GK*-dimension is infinite.

These remarks provide us examples:

$$GK\dim(K[x_1,\dots,x_d]) = d \quad \text{and} \quad GK\dim(K\langle x,y\rangle) = \infty.$$

In the following proposition we list some of the many properties of the *GK*-dimension. Proofs can be found in [27].

Proposition 2.2.2. *Let A, A_1 and A_2 be algebras. Than*

- $GK\dim(B) \leq GK\dim(A)$ *whenever B is a subalgebra or a homomorphic image of A.*
- $GK\dim(A_1 \oplus A_2) = \max\{GK\dim(A_1), GK\dim(A_2)\}$.
- $GK\dim(A_1 \otimes A_2) \leq GK\dim(A_1) + GK\dim(A_2)$.
- *If A is finitely generated and commutative, then* $GK\dim(A) = K\dim(A)$*, where* $K\dim$ *denotes the classical Krull dimension.*
- *If A is a field, then* $GK\dim(A) = \mathrm{tr.deg}_K(A)$*, where* $\mathrm{tr.deg}_K$ *denotes the transcendence degree of A over K.*

Hence the *GK*-dimension provides an extremely useful invariant of algebras, that is a noncommutative generalization of the Krull dimension and of the transcendence degree of commutative algebras.

Proofs of the following two results can be found in [27].

Theorem 2.2.3 (Bergman)**.** *No algebra has GK-dimension strictly between* 1 *and* 2*.*

The next result suggests that algebras of polynomial growth are more complicated than groups of polynomial growth.

Theorem 2.2.4 (Warfield)**.** *For any real number* $r \geq 2$ *there exists an algebra A with* $GK\dim(A) = r$*.*

It is also clear that $GK\dim(A) = 0$ if and only if A is locally finite-dimensional, i.e. every finitely generated subalgebra is finite dimensional, and $GK\dim(A) \geq 1$ if A is not locally finite dimensional. These observations together with the previous theorems show that for any algebra A

$$GK\dim(A) \in \{0, 1\} \cup [2, \infty].$$

Algebras of *GK*-dimension 1 are well understood:

Theorem 2.2.5 (Small, Stafford, Warfield [36])**.** *A finitely generated algebra of GK-dimension* 1 *satisfies a polynomial identity.*

The following example will be relevant in the next chapters.

Example 2.2.2. For $n \in \mathbb{N}$, $n \geq 1$, the *Weyl algebra* $A_n = A_n(K)$ is the ring of polynomial in the $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ with coefficients in K subject to the relations

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0 \quad \text{and} \quad [x_i, y_j] = \delta_{ij},$$

where $[a,b] := ab - ba$, and δ_{ij} is the Kronecker symbol. Observe that

$$A_{n+1} \cong A_n[y_{n+1}][x_{n+1}, \delta] \quad \text{with} \quad \delta = \frac{\partial}{\partial y_{n+1}},$$

where with $B[x;\delta]$ we denote the *Ore extension* of the algebra B with respect to the K-derivation δ of B, i.e. polynomials in x with coefficients in B subject to the relations $[x,b] = \delta(b)$, where $b \in B$.

To compute the GK-dimension of the Weyl algebra we can use the following lemma, which is useful in many computations (see [27]). We could have used it to compute the GK-dimension of the algebra of polynomials.

Lemma 2.2.6. *Let A be an algebra with a derivation δ such that each finite dimensional subspace of A is contained in a δ-stable finitely generated subalgebra of A. Then $GK\dim(A[x;\delta]) = GK\dim(A) + 1$.*

It's easy to check that the hypothesis of the lemma are satisfied for the Weyl algebras. Hence by induction we get

$$GK\dim(A_n) = 2n.$$

While the GK-dimension well behaves under many ring theoretic construction, it does not work as much as well under taking localizations.

If A is an algebra, a *right Ore set* $\Omega \subseteq A$ is a multiplicative closed subset of A which satisfies the *right Ore condition*, i.e. $cA \cap a\Omega \neq \emptyset$ for all $c \in \Omega$ and $a \in A$. If all the elements of Ω are regular, we can consider the ring of right fractions $A\Omega^{-1}$, and identify A with the subset $\{a/1 \mid a \in A\} \subseteq A\Omega^{-1}$.

There are analogous left versions of these notions.

Definition. An algebra A is called an *Ore domain* if the subset $A \setminus \{0\}$ is a (right) Ore set.

Hence given an Ore domain A we can consider its ring of quotient $Q(A)$, i.e. the (right) localization of $Q(A) := A(A \setminus \{0\})^{-1}$. This is going to be a division algebra.

Example 2.2.3. The Weyl algebras are Ore domains (in fact they are notherian), hence we can consider their quotient division algebras, that we will denote $D_n := Q(A_n)$. The study of these division algebras motivated the pioneering work of Gelfand and Kirillov.

A well known theorem of Makar-Limanov (see [29] or [27]) shows that the quotient division algebra D_1 of the Weyl algebra A_1 contains a subalgebra isomorphic to a free algebra on two free generators. By what we already mentioned about GK-dimension, this implies that $GK\dim(D_1) = \infty$, while $GK\dim(A_1) = 2$. This shows that the GK-dimension can blow up under localizations, and this is one of the main motivation to look for another invariant of algebras that behaves better under this fundamental construction. We will see that the isoperimetric profile does work better.

We conclude this section with another example showing that the GK-dimension can increase with localization. It will be relevant later.

Example 2.2.4. Consider in the Weyl algebra A_1 the multiplicative closed set Ω generated by x and y. It can be shown that this is an Ore set, and that the localization $A_1\Omega^{-1}$ has GK-dimension 3. Notice also that this algebra is finitely generated and it's a noetherian domain (see [27], Example 4.11 for details).

2.3 Amenability of groups

Amenability of groups was introduced in 1929 by John von Neumann ([41]) in terms of invariant means. Here we give a more combinatorial definition, which is essentially due to Følner, that is easier to generalize for algebras.

Definition. A group Γ is *amenable* if for every finite subset $S \subset \Gamma$ and every $\varepsilon > 0$ there exists a finite subset $\Omega \subset \Gamma$ such that

$$|\Omega \cup S\Omega| < (1+\varepsilon)|\Omega|.$$

For a good introduction to this subject and for reference to what we are going to state in this section we refer to Wagon's [42].

Example 2.3.1. Finite groups are amenable: choose $\Omega := \Gamma$.

It can be showed that abelian groups are also amenable.

The free group F_2 on two free generators x and y is not amenable: if we choose $S = \{x,y\}$, than for any finite subset $\Omega \subset F_2$ we have

$$|\Omega \cup S\Omega| \geq 2|\Omega|.$$

The following proposition gives us more examples of amenable groups:

Proposition 2.3.1. *If Γ is a group and $G_\circ(n;\Gamma) \precneqq e^n$, than Γ is amenable.*

Proof. Suppose Γ is not amenable. Then there exist a finite $S \subset \Gamma$ and $\varepsilon > 0$ such that for any finite $\Omega \subset \Gamma$

$$|\Omega \cup S\Omega| \geq (1+\varepsilon)|\Omega|.$$

In particular we can take $\Omega = \cup_{r=0}^{m-1} S^r$. Hence we get

$$|\cup_{r=0}^{m} S^r| - |\cup_{r=0}^{m-1} S^r| \geq \varepsilon |\cup_{r=0}^{m-1} S^r|,$$

which is the same as

$$G_\circ(m;\Gamma,S) - G_\circ(m-1;\Gamma,S) \geq \varepsilon G_\circ(m-1;\Gamma,S).$$

This implies that $G_\circ(m;\Gamma)$ grows exponentially. □

We define AG to be the class of amenable groups. We have the following properties:

Theorem 2.3.2. *The class of amenable groups AG is closed under taking subgroups, homomorphic images, extensions and direct limits.*

This together with the previous examples shows for instance that nilpotent and solvable groups are amenable. Also, groups that contain a noncommutative free subgroup are not amenable.

Call NF the class of such groups. We have seen the inclusions

$$EG \subseteq AG \subseteq NF.$$

In 1968 Milnor asked if the first inclusion is an equality. Grigorchuk's group ([18]) provided an example showing that the inclusion is in fact strict: since Grigorchuk's group has intermediate growth, it's amenable, but for the same reason it cannot be elementary by the theorem of Milnor, Wolf and Chou.

In 1929 von Neumann asked if the second inclusion is an equality. In 1981 Ol'shanskii in [35] provided a group which is not amenable and it does not contain noncommutative free subgroups (in fact all the elements have finite order).

It's a result of Tits [39] that equalities holds if we restrict ourselves to consider only group of matrices.

We end this section mentioning that amenability turns out to be important in a number of fields of mathematics, including group theory, representation theory, random walks, C^*-algebras, von Neumann algebras, Banach-Tarski paradox, property T. It has a strict connection with the isoperimetric profile.

2.4 Isoperimetric profile of groups

The notion of an isoperimetric profile in algebra was first introduced in groups by Vershik in [40] and by Gromov in [20].

Definition. Given an infinite semigroup Γ generated by a finite subset S, and given a finite subset Ω of Γ we define the *boundary* of Ω as

$$\partial_S(\Omega) := \bigcup_{s\in S}(s\Omega \setminus \Omega).$$

Then we define the *isoperimetric profile of a semigroup* Γ with respect to S as the function from $\mathbb{N}$ onto itself given by

$$I_\circ(n;\Gamma,S) := \inf_{|\Omega|=n} |\partial_S(\Omega)|$$

for each $n \in \mathbb{N}$, where $|X|$ denotes the cardinality of the set X.

It's well known that the asymptotic behavior of this function is independent of the set of generators S. Hence this provide another asymptotic geometric invariant of groups.

We want to stress here a few properties of this invariant. First of all notice that

$$|\partial_S(\Omega)| = |\bigcup_{s\in S}(s\Omega \setminus \Omega)| \leq |S||\Omega|,$$

hence we always have

$$I_\circ(n;\Gamma) \preceq n,$$

i.e. the isoperimetric profile of a semigroup is a sublinear function. The following proposition shows that the linearity of this invariant is equivalent to the nonamenability of the group.

Proposition 2.4.1. *A group Γ is nonamenable if and only if there exists a finite subset $S \subset \Gamma$ such that*

$$I_\circ(n;\Gamma,S) \sim n.$$

Proof. If Γ is nonamenable than there exist $S \subset \Gamma$ finite and $\varepsilon > 0$ such that for any finite subset $\Omega \subset \Gamma$

$$|\partial_S(\Omega)| = |\Omega \cup S\Omega| - |\Omega| \geq \varepsilon|\Omega|,$$

which implies

$$I_\circ(n;\Gamma,S) \succeq n,$$

that gives

$$I_\circ(n;\Gamma,S) \sim n.$$

If Γ is amenable, then for any finite subset $S \subset \Gamma$ and any $\varepsilon > 0$ we can find a finite subset $\Omega \subset \Gamma$ (depending on ε) such that

$$|\partial_S(\Omega)| = |\Omega \cup S\Omega| - |\Omega| < \varepsilon|\Omega|,$$

and this prevent $I_\circ(n;\Gamma,S) \succeq n$, as we wanted. □

In this sense the isoperimetric profile can be viewed as a measure of the amenability of the group.

Looking at the definition, it becomes immediately clear that to compute the isoperimetric profile, even for easy examples, it's not easy to prove a lower bound. The key result which lies at the hearth of almost any computation of isoperimetric profiles is the following remarkable inequality which is due to Coulhon and Saloff-Coste (cf. [34]). We need some definitions.

Let Γ be an infinite semigroup generated by a finite subset S. Let $B(n) := \cup_{i=0}^{n} S^i$, where $S^0 = \{1\}$ and 1 is the identity element of Γ. Define $\Phi(\lambda) := \min\{n \in \mathbb{N} \mid |B(n)| > \lambda\}$ for $\lambda > 0$. This is the inverse function of the growth of Γ. The following proof is due to Gromov ([21]).

Theorem 2.4.2 (Coulhon, Saloff-Coste)**.** *Let Γ be an infinite semigroup with the cancellation property (i.e. $xz = yz$ implies $x = y$ for any $x, y, z \in \Gamma$) generated by a finite subset S. For any finite non-empty subset Ω of Γ we have*

$$|\partial_S(\Omega)| \geq \frac{|\Omega|}{2\Phi(2|\Omega|)}.$$

Remark 3. A typical example of a semigroup with the cancellation property is a sub-semigroup of a group.

Proof. Observe that if $a, b \in \Gamma$ then

$$|\partial_{ab}(\Omega)| \leq |\partial_a(\Omega)| + |\partial_b(\Omega)|.$$

Hence by induction, if $a_i \in \Gamma$ for $i = 1, \ldots, n$, then

$$|\partial_{a_1 a_2 \cdots a_n}(\Omega)| \leq \sum_i |\partial_{a_i}(\Omega)| \leq n \max_i |\partial_{a_i}(\Omega)|.$$

This immediately implies that for any $m \geq 1$ and $y \in B(m)$

$$|\partial_y(\Omega)| \leq m \max_{s \in S} |\partial_s(\Omega)|.$$

Therefore

$$\begin{aligned} |\partial_S(\Omega)| &\geq \max_{s \in S} |\partial_s(\Omega)| \geq \frac{1}{m} \frac{1}{|B(m)|} \sum_{y \in B(m)} |\partial_y(\Omega)| \\ &= \frac{1}{m} \frac{1}{|B(m)|} \sum_{y \in B(m)} (|\Omega| - |y\Omega \cap \Omega|) \\ &= \frac{1}{m} \frac{1}{|B(m)|} \left(|B(m)||\Omega| - \sum_{y \in B(m)} |y\Omega \cap \Omega| \right). \end{aligned}$$

We need a lemma.

Lemma 2.4.3.
$$\sum_{y\in B(m)} |y\Omega\cap\Omega| \leq |\Omega|^2.$$

Proof. We have

$$\sum_{y\in B(m)} |y\Omega\cap\Omega| = \sum_{y\in B(m)}\sum_{x_1\in\Omega}\sum_{x_2\in\Omega}\chi(yx_1 = x_2),$$

where $\chi(\mathscr{P}) = 1$ if the proposition $\mathscr{P}$ is true, $\chi(\mathscr{P}) = 0$ if $\mathscr{P}$ is false. By cancellation property, the ordered pair $(x_1,x_2)\in\Omega\times\Omega$ uniquely determines y such that $yx_1 = x_2$, hence

$$\sum_{y\in B(m)}\sum_{x_1\in\Omega}\sum_{x_2\in\Omega}\chi(yx_1 = x_2) \leq |\Omega\times\Omega| = |\Omega|^2.$$

□

Using the lemma and choosing $m = \Phi(2|\Omega|)$, we get

$$\begin{aligned} |\partial_S(\Omega)| &\geq \frac{1}{m}\frac{1}{|B(m)|}\left(|B(m)||\Omega| - \sum_{y\in B(m)}|y\Omega\cap\Omega|\right) \\ &\geq \frac{1}{m}\frac{1}{|B(m)|}\left(|B(m)||\Omega| - |\Omega|^2\right) \\ &= \frac{|\Omega|}{m}\left(1 - \frac{|\Omega|}{|B(m)|}\right) \\ &> \frac{|\Omega|}{2m} = \frac{|\Omega|}{2\Phi(2|\Omega|)}, \end{aligned}$$

completing the proof. □

The following results are based on this inequality and the results on the growth of groups that we have mentioned before (see [34]).

Theorem 2.4.4. *Let Γ be a finitely generated group. The following conditions are equivalent:*

- $I_\circ(n;\Gamma) \sim n^{(d-1)/d}$ *where $d \geq 1$ is an integer.*
- *The growth of Γ is polynomial of degree d.*

In a way this theorem tells us that for groups of polynomial growth, the growth and the isoperimetric profile provide the same amount of information. The next result shows that this invariants are not equipollent.

Theorem 2.4.5. *Let Γ have a finite index subgroup which is polycyclic. Then*

- $I_\circ(n;\Gamma) \sim n^{(d-1)/d}$ *if and only if* $G_\circ(n;\Gamma) \sim n^d$.
- $I_\circ(n;\Gamma) \sim n/\log n$ *if and only if* Γ *has exponential growth.*

Hence for example a polycyclic group Γ of exponential growth and the free group F_2 on two generators have the same growth, but $I_\circ(n;\Gamma) \sim n/\log$, while $I_\circ(n;F_2) \sim n$. In general the isoperimetric profile of a group is believed to be a finer invariant than the growth, but there is no proof of this statement.

It's worthwhile to mention that there are many other example of groups that have exponential growth, but non linear isoperimetric profile. For example the wreath product of a non trivial finite group with $\mathbb{Z}^d$ with $d \geq 2$ have an isoperimetric profile asymptotically strictly between $n/\log n$ and n (the rate depending on d).

We don't discuss here the important connection of the isoperimetric profile with random walks (see [34]).

Chapter 3

Preliminaries

In this chapter we give basic definitions and properties of the isoperimetric profile, including its connection with amenability and the fundamental computation of this invariant for the algebra of polynomials, which is due to Gromov.

3.1 The Isoperimetric Profile

Unless otherwise stated, by an *algebra* A we will mean an infinite dimensional associative algebra with unit 1 over a fixed field K of characteristic 0.

Given two subspaces V and W of an algebra A we will denote the quotient space $V/(V \cap W)$ simply by V/W. Also, given a subset S of A and a subspace V of A we define $SV := span_K\{sv | s \in S, v \in V\}$.

In this notation, given a subspace V of A and a subset S of A, the *boundary* of V with respect to S is defined by

$$\partial_S(V) := SV/V.$$

We will denote the dimension over K of a subspace V of A by $|V|$. Also, for any finite set S we denote by $|S|$ its cardinality. Hopefully this will not cause any confusion.

We are interested in the dimension of the boundary, hence we can always assume that 1 (the identity of A) is in S, since

$$\partial_{S\cup\{1\}}(V) = (S \cup \{1\})V/V = (SV+V)/V \cong SV/(SV \cap V) = SV/V = \partial_S(V).$$

Claim. *If $V \subset A$ is a finite dimensional subspace and $a, b \in A$,*

$$|\partial_{ab}(V)| \leq |\partial_a(V)| + |\partial_b(V)|.$$

Proof. We want to estimate $|\partial_{ab}(V)| = |abV/V|$. Let's consider a basis $e_1, e_2, \dots, e_r$ of the subspace $\{v \in V \mid bv \in V\}$ of V, and let's complete it to a basis of V, say $e_1, e_2, \dots, e_r, e_{r+1}, \dots, e_n$. Hence $be_j \notin V$ for $j > r$. When we multiply by a after b we see that $abe_i \in aV$ for $1 \leq i \leq r$, so they will contribute at most $|\partial_a(V)| = |aV/V|$ to $|\partial_{ab}(V)|$. Clearly the other abe_j's ($j > r$) will contribute at most $|\partial_b(V)|$ to $|\partial_{ab}(V)|$. This proves the claim. □

In the same way it can be proved the following more general inequality:

$$|\partial_{ST}(V)| \leq |\partial_S(V)| + |S||\partial_T(V)|, \qquad (\bullet)$$

where S and T are finite subsets of A. Notice also that if S is a finite subset of A and $V = span_K S = KS$, then $\partial_S(W) = \partial_V(W)$ for all subspaces W of A. Hence the same inequality is true if we assume S and T to be finite dimensional subspaces.

Definition. We define a *subframe* of an algebra A to be a finite dimensional subspace containing the identity and a *frame* to be a subframe which generates the algebra. (see [45])

Remark. The previous discussion shows that as long as we are interested in the dimension of the boundary $\partial_V(W)$, instead of taking an arbitrary finite dimensional subspace V of an algebra A, we can take a subframe, without loosing anything.

Convention. In the rest of the paper by a *subspace* we will always mean *finite dimensional subspace*, unless otherwise specified.

Given a subframe V of A, in the Introduction we defined the *isoperimetric profile of A with respect to V* (see [21]) to be the maximal function I_* such that all finite dimensional subspaces $W \subset A$ satisfy the *isoperimetric inequality*

$$I_*(|W|; A, V) = I_*(|W|) \leq |\partial_V(W)|.$$

Notice that for any $n \in \mathbb{N}$

$$I_*(n; A, V) = I_*(n) = \inf |\partial_V(W)|,$$

where the infimum is taken over all subspaces W of A of dimension n.

We are interested in the asymptotic behavior of the function I_*.

Definition. Given two functions $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ we say that f_1 is *asymptotically faster* then f_2, and we write $f_1 \succeq f_2$, if there exist positive constants C_1 and C_2 such that $f_1(C_1 x) \geq C_2 f_2(x)$ for all $x \in \mathbb{R}_+$. We say f_1 is *asymptotically equivalent* to f_2, and we write $f_1 \sim f_2$, if $f_1 \succeq f_2$ and $f_2 \succeq f_1$.

Remark. We can always consider the function $I_*(\,\cdot\,)$ as a function on $\mathbb{R}_+$, simply defining for $r \in \mathbb{R}_+$, $I_*(r) := I_*(\lfloor r \rfloor)$, where $\lfloor r \rfloor$ denotes the maximal integer $\leq r$. We will often do it, without mentioning it explicitly.

Definition. We say that an algebra A *has an isoperimetric profile* if there exists a subframe V of A such that for any other subframe W of A we have

$$I_*(n; A, W) \preceq I_*(n; A, V).$$

Otherwise we say that A has no isoperimetric profile.

In case A has an isoperimetric profile, we will refer to this function, or its asymptotic behavior, as *the isoperimetric profile of* A, and we'll denote it also by $I_*(A)$. If the subframe V of A is such that $I_*(n; A, V)$ is the isoperimetric profile of A we will say that V *measures* the profile of A.

First of all we want to show that an arbitrary finitely generated algebra has an isoperimetric profile. We need the following proposition.

Proposition 3.1.1. *If V and W are two frames of A, then $I_*(\,\cdot\,; A, V) \sim I_*(\,\cdot\,; A, W)$.*

Proof. Clearly $V \subset W^m$ for some $m \in \mathbb{N}$, since W is a generating subspace. Now, given a subspace Z of A, we have

$$|\partial_V(Z)| \leq |\partial_{W^m}(Z)| \leq \left(\sum_{i=0}^{m-1} |W|^i \right) |\partial_W(Z)|,$$

where the second inequality follows by induction on m using (•). This gives

$$I_*(\,\cdot\,; A, V) \preceq I_*(\,\cdot\,; A, W),$$

since $\sum_{i=0}^{m-1} |W|^i$ is a constant which does not depend on Z. The other inequality is proved in the same way. □

Observe that in a finitely generated algebra A, any subframe V is contained in a frame W, and obviously $I_*(n;A,V) \leq I_*(n;A,W)$. This together with the previous proposition shows that A has an isoperimetric profile, and any frame of A measures $I_*(A)$.

We will see later examples of algebras with an isoperimetric profile which are not finitely generated (see Example 4.2.1), and we will give also an example of an algebra which has no isoperimetric profile (see Example 3.3.1).

3.2 Isoperimetric profile and Amenability

In a way, the isoperimetric profile measures the degree of amenability of an algebra.

Definition. We say that an algebra A is *amenable* if for each $\varepsilon > 0$ and any subframe V of A, there exists a subframe W of A with $|VW| \leq (1+\varepsilon)|W|$. This is the so called *Følner condition*.

We will see a lot of examples of amenable algebras in the rest of this work.

Notice that the Følner condition can be restated in the following way using the boundary: for any subspace $V \subset A$ and $\varepsilon > 0$ there exists a subspace $W \subset A$ such that $|\partial_V(W)|/|W| \leq \varepsilon$.

Proposition 3.2.1. *An algebra A is amenable if and only if $I_*(n;A,V) \precnsim n$ for any subframe V of A.*

Proof. First of all observe that we always have $I_*(n;A,V) \preceq n$, since $|\partial_V(W)| \leq |V||W|$ for any fixed subframe V and any subspace W. So if $I_*(n;A,V) \precnsim n$ is not true we have $I_*(n;A,V) \sim n$, and hence for n big enough the value of $I_*(n;A,V)/n$ is bounded away from 0.

If A is amenable, given a subframe V of A, we can find a sequence $\{W_k\}_{k\in\mathbb{N}}$ of subspaces with $|\partial_V(W_k)|/|W_k| \leq 1/k$, and with $|W_k|$ tending to infinity as k tends to infinity. But then

$$I_*(|W_k|;A,V)/|W_k| \leq |\partial_V(W_k)|/|W_k| \leq 1/k,$$

showing that $I_*(n;A,V) \precneqq n$.

If A is not amenable, there is a subframe V and an $\varepsilon > 0$ such that for any subspace W we have $|\partial_V(W)|/|W| > \varepsilon$, i.e. $|\partial_V(W)| > \varepsilon|W|$, and hence $I_*(n;A,V) \succeq n$. So we have $I_*(n;A,V) \sim n$. □

Corollary 3.2.2. *An algebra A is nonamenable if and only if A has isoperimetric profile* $I_*(n;A) \sim n$.

Proof. If A is nonamenable, then by the previous Proposition there exists a subframe $V \subset A$ such that $I_*(n;A,V) \sim n$. Then clearly A has isoperimetric profile, and $I_*(n;A) \sim I_*(n;A,V) \sim n$.

If A is amenable, then by the previous Proposition A cannot have isoperimetric profile $I_*(n;A) \sim n$. □

Corollary 3.2.3. *If all the finitely generated subalgebras of an algebra A are amenable, then A is amenable.*

Proof. If A is not amenable, than there is a subframe V and an $\varepsilon > 0$ such that for any subspace W we have $|\partial_V(W)|/|W| > \varepsilon$, i.e. $|\partial_V(W)| > \varepsilon|W|$. In particular this is true for any subspace W of $K[V]$, where $K[V]$ is the subalgebra of A (finitely) generated by V. Hence $I_*(n;K[V],V) \succeq n$, i.e. $I_*(n;K[V]) \sim n$, which says that $K[V]$ is not amenable. □

Remark 4. The converse of the previous corollary is not true. For example, we will show later in the paper that the algebra $A = K[x,y] \oplus K\langle w,z\rangle$ is amenable, since we'll prove (see Proposition 4.1.2 and Proposition 3.3.4) that $I_*(A) \preceq I_*(K[x,y]) \sim n^{1/2}$. But it's known (cf. [5]) that the finitely generated subalgebra $K\langle w,z\rangle$ (a free algebra of rank 2) is not amenable:

Example 3.2.1. If we call $F_k = K\langle x_1,\dots,x_k\rangle$ the free algebra in $k \geq 2$ noncommuting variables, and $V = K + Kx_1 + \cdots + Kx_k$, clearly for any subspace W we have $|\partial_V(W)| \geq |W|$, which shows that $I_*(n;F_k) \sim n$ for $k \geq 2$.

We have the following Corollary (see also [5])

Corollary 3.2.4. *Free algebras of finite rank* ≥ 2 *are not amenable.*

It can be proved in a similar way the following proposition (see [10] and [45]).

Proposition 3.2.5. *If A is an amenable domain, then A is a right Ore domain.*

Proof. If A is not right Ore, then there are two elements a and b such that $aA + bA = aA \oplus bA$. Hence, if we set $V := K + Ka + Kb$, then $|VW| \geq 2|W|$ for every subspace $W \subset A$. Hence we get $I_*(n; A, V) \sim n$, showing that A is not amenable. $\square$

3.3 Orderable semigroups and the algebra of polynomials

Let Γ be an infinite semigroup generated by a finite subset S. Let $B(n) := \cup_{i=0}^{n} S^i$, where $S^0 = \{1\}$ and 1 is the identity element of Γ. Define $\Phi(\lambda) := \min\{n \in \mathbb{N} \mid |B(n)| > \lambda\}$ for $\lambda > 0$. This is the inverse function of the growth of Γ.

The following result is due to Coulhon and Saloff-Coste. We gave a proof of it in Chapter 2.

Theorem 3.3.1 (Coulhon, Saloff-Coste)**.** *Let Γ be an infinite semigroup with the cancellation property (i.e. $xz = yz$ implies $x = y$ for any $x, y, z \in \Gamma$) generated by a finite subset S. For any finite non-empty subset Ω of Γ we have*

$$|\partial_S(\Omega)| \geq \frac{|\Omega|}{2\Phi(2|\Omega|)}.$$

Remark 5. A typical example of a semigroup with the cancellation property is a subsemigroup of a group.

Corollary 3.3.2. *The free abelian semigroup on $d \in \mathbb{N}$ generators $\mathbb{Z}^d_{\geq 0}$ has isoperimetric profile $I_\circ(n; \mathbb{Z}^d_{\geq 0}) \sim n^{\frac{d-1}{d}}$.*

Proof. The lower bound is given by Theorem 3.3.1, since clearly the growth of $\mathbb{Z}^d_{\geq 0}$ is polynomial of exponent d.

For the upper bound, let S be the set of standard generators of $\mathbb{Z}^d_{\geq 0}$ and consider the cubes $C_n = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d_{\geq 0} \mid x_i \leq n-1 \text{ for all } i\}$. Now $|C_n| = n^d$ and $|\partial_S(C_n)| = dn^{d-1} = d|C_n|^{\frac{d-1}{d}} \sim |C_n|^{\frac{d-1}{d}}$. From this it follows easily the upper bound. $\square$

Given a semigroup Γ generated by a finite subset S, we can consider its semigroup algebra $K\Gamma$. We have the following inequality ([21]):

$$I_*(K\Gamma) \preceq I_\circ(\Gamma).$$

This because given a finite subset Ω of Γ, clearly $|\Omega| = |K\Omega|$, and also

$$|\partial_S(\Omega)| = |\partial_S(K\Omega)|.$$

The other inequality is not always true (see [21]).

Definition. A semigroup Γ is said to be *orderable* if there exists a total order on Γ, which we denote by $<$, such that $x < y$ implies $xz < yz$ and $zx < zy$ for any $x, y, z \in \Gamma$.

We have the following result ([21]).

Theorem 3.3.3 (Gromov)**.** *If Γ is an orderable semigroup generated by a finite set S, then $I_\circ(\Gamma) \sim I_*(K\Gamma)$.*

Proof. By the previous discussion, we only need to prove

$$I_\circ(\Gamma) \preceq I_*(K\Gamma).$$

Consider the elements of $K\Gamma$ as functions on Γ with finite support. Let ν be the function $\nu : K\Gamma \to \Gamma$ defined by $\nu(f) := \min \operatorname{supp}(f)$, where $f \in K\Gamma$ and $\operatorname{supp}(f)$ denote the support of f. Given a subspace V of $K\Gamma$, it's easy to see that $|\nu(V)| = |V|$, and moreover

$$\left| \sum_{i=1}^{k} V_i \right| \geq \left| \bigcup_{i=1}^{k} \nu(V_i) \right|,$$

where V_i is a subspace of $K\Gamma$ for all i's. It's also clear that

$$\nu(\gamma V) = \gamma \nu(V).$$

All these properties imply that

$$|\partial_S(V)| \geq |\partial_S(\nu(V))|.$$

This gives the inequality that we wanted. □

Corollary 3.3.4. *The isoperimetric profile of the algebra $A = K[x_1, \ldots, x_d]$ of polynomials is $I_*(n;A) \sim n^{\frac{d-1}{d}}$.*

Proof. Observe that the semigroup algebra of the semigroup $\mathbb{Z}^d_{\geq 0}$ is isomorphic to the algebra $K[x_1, \ldots, x_d]$. Also, $\mathbb{Z}^d_{\geq 0}$ is clearly orderable (see [32]), hence the result follows from Corollary 3.3.2 and Theorem 3.3.3. □

We can now give an example of an algebra which has no isoperimetric profile.

Example 3.3.1. Consider the algebra $A = K[x_1, x_2, \dots]$ of polynomials in infinitely many variables. For any $d \in \mathbb{N}$, call $W_d = span_K\{x_1, \dots, x_d\}$. We can consider the vector space $V_n^{(d)} = span_K\{x_1^{m_1} \cdots x_d^{m_d} \mid m_i \leq n-1 \text{ for all } i\}$. We have $|V_n^{(d)}| = n^d$ and $|\partial_{W_d}(V_n^{(d)})| = dn^{d-1} = d|V_n^{(d)}|^{\frac{d-1}{d}}$, which easily implies the upper bound

$$I_*(n; A, W_d) \preceq n^{\frac{d-1}{d}}.$$

Now A is a free $K[x_1, \dots, x_d]$-module, hence we can apply Proposition 4.9.1, which we will prove later, to get

$$n^{\frac{d-1}{d}} \sim I_*(n; K[x_1, \dots, x_d], W_d) \preceq I_*(n; A, W_d),$$

giving $I_*(n; A, W_d) \sim n^{\frac{d-1}{d}}$.

Notice that any subspace $W \subset A$ is contained in W_d^m for some d and $m \in \mathbb{N}$. Hence we can apply (•) to see that

$$I_*(n; A, W) \preceq I_*(n; A, W_d) \sim n^{\frac{d-1}{d}}.$$

This shows that A cannot have an isoperimetric profile.

We mention here another interesting result, which is due to Bartholdi ([2]).

Theorem 3.3.5 (Bartholdi)**.** *A group Γ is amenable if and only if its group algebra $F\Gamma$ is amenable for any field F.*

Notice that Theorem 3.3.3 gives more for orderable groups, but it does not say anything on groups which are not orderable, while this theorem holds for any group. We remind also that in general the isoperimetric profile of a group is not equivalent to the isoperimetric profile of its group algebra (see [21]).

Part of the text of chapters 3, 4, 5 and 6 of this thesis is a modified version of "On isoperimetric profiles of algebras", D'Adderio Michele, *J. Algebra*, **322**, 2009.

Chapter 4

Ring theoretic constructions

In this section we study the behavior of the isoperimetric profile under various ring-theoretic constructions. In the process we derive many consequences on the amenability of algebras.

4.1 Subalgebras and homomorphic images

In general, the isoperimetric profile for algebras does not decrease when passing to subalgebras or homomorphic images.

Lemma 4.1.1. *If A and B are two algebras, V is a subframe of A and W is a subframe of B, then $I_*(n;A\oplus B, V+W) \le I_*(n;A,V)$ and $I_*(n;A\oplus B,V+W)\le I_*(n;B,W)$.*

Proof. We identify A and B with their obvious copies in $A\oplus B$. Let V be a subframe of A, W a subframe of B and let $Z\subset A$ be any subspace. We have

$$|\partial_{V+W}(Z)| = |\partial_V(Z)|,$$

where the second boundary is in the algebra A. This proves the first inequality. The second is proved in the same way. □

We have the following consequence.

Proposition 4.1.2. *If A and B are two finitely generated algebras, then $I_*(A\oplus B)\preceq I_*(A)$ and $I_*(A\oplus B)\preceq I_*(B)$.*

Proof. If V is a frame of A and W is a frame of B, than $V+W$ is a frame of $A\oplus B$. By the previous Lemma

$$I_*(A\oplus B)\sim I_*(A\oplus B,V+W)\preceq I_*(A,V)\sim I_*(A),$$

and

$$I_*(A\oplus B)\sim I_*(A\oplus B,V+W)\preceq I_*(B,W)\sim I_*(B),$$

completing the proof. □

Observe that A is a subalgebra of $A\oplus B$, and also A is isomorphic to a homomorphic image of $A\oplus B$. If we now consider a direct sum $A\oplus B$ of two finitely generated algebras with $I_*(A)\precneqq I_*(B)$ (cf. Remark 4), it follows immediately from the previous proposition that we do not have in general inequality for subalgebras and homomorphic images.

From this and what we saw in the previous sections it follows for example that amenability for algebras does not pass to quotients and subalgebras (see also [5]).

In fact this phenomenon can occur also when we deal with domains: given an amenable domain, it's not true that a subdomain must be amenable. In fact it's well known that the Weyl algebra A_1 is amenable (we will see this later), since it has finite GK-dimension, hence by [11] (or even by our results later in this thesis) its quotient division algebra D_1 is still amenable. But it's also known (see [29]) that D_1 contains a subalgebra isomorphic to a free algebra of rank 2, which is nonamenable.

4.2 Localization

The isoperimetric profile behaves well with nice localizations.

If A is an algebra, a *right Ore set* $\Omega\subseteq A$ is a multiplicative closed subset of A which satisfies the *right Ore condition*, i.e. $cA\cap a\Omega\neq\emptyset$ for all $c\in\Omega$ and $a\in A$. If all the elements of Ω are regular, we can consider the ring of right fractions $A\Omega^{-1}$, and identify A with the subset $\{a/1\mid a\in A\}\subseteq A\Omega^{-1}$.

There are analogous left versions of these notions.

Notice that we will have slightly different results for the left and the right cases in this section. This depends on the fact that the definition of the boundary is not symmetric.

Lemma 4.2.1. *Let A be an algebra and let Ω be a right Ore set of regular elements in (i) and (ii) and a left Ore set of regular elements in (iii).*

(i) If V is a subframe of A, then

$$I_*(n;A,V) = I_*(n;A\Omega^{-1},V).$$

(ii) If W is a subframe of $A\Omega^{-1}$, then we can find an $m \in \Omega$ such that $Wm \subset A \subset A\Omega^{-1}$. For any such m

$$I_*(n;A\Omega^{-1},W) \leq I_*(n;A,Wm+K).$$

(iii) If W is a subframe of $\Omega^{-1}A$, we can find an $m \in \Omega$ such that $mW \subset A \subset \Omega^{-1}A$. For any such m

$$I_*(n;\Omega^{-1}A,W) \leq I_*(n;A,mW+K).$$

Proof. (i) Let V be a subframe of A. Of course V is also a subframe of $A\Omega^{-1}$. Given any subspace Z of $A\Omega^{-1}$, clearly we can find an element $m \in \Omega$ such that $Zm \subseteq A \subseteq A\Omega^{-1}$. We have

$$|\partial_V(Zm)| = |VZm| - |Zm| = |VZ| - |Z| = |\partial_V(Z)|.$$

Hence

$$I_*(n;A,V) \leq I_*(n;A\Omega^{-1},V),$$

which implies

$$I_*(n;A,V) = I_*(n;A\Omega^{-1},V).$$

(ii) Given now a subframe W of $A\Omega^{-1}$, again we can find an $m \in \Omega$ such that $Wm \subset A \subset A\Omega^{-1}$. If Z is a subspace of A, we have

$$|\partial_W(mZ)| = |WmZ| - |mZ| \leq |WmZ + Z| - |Z| = |\partial_{Wm+K}(Z)|.$$

The above inequality shows that

$$I_*(n;A\Omega^{-1},W) \leq I_*(n;A,Wm+K).$$

(iii) Suppose that W is a subframe of $\Omega^{-1}A$. As before we can find an $m \in \Omega$ such that $mW \subset A \subset \Omega^{-1}A$. If Z is a subspace of A, we have

$$|\partial_W(Z)| = |WZ| - |Z| \leq |mWZ + Z| - |Z| = |\partial_{mW+K}(Z)|.$$

The above inequality gives

$$I_*(n;\Omega^{-1}A,W) \leq I_*(n;A,mW+K).$$

□

The following corollary follows easily from this lemma.

Corollary 4.2.2. *Let A be an algebra and let Ω be a right Ore set of regular elements in (i) and a left Ore set of regular elements in (ii). Then*

(i) A has an isoperimetric profile if and only if $A\Omega^{-1}$ does, and in this case $I_*(A) \sim I_*(A\Omega^{-1})$. Moreover, any subframe of A that measures $I_*(A)$, measures also $I_*(A\Omega^{-1})$, and viceversa if W measures $I_*(A\Omega^{-1})$, then for any $m \in \Omega$ such that $Wm \subset A$, $Wm + K$ measures $I_*(A)$.

(ii) If both A and $\Omega^{-1}A$ have isoperimetric profiles, then $I_*(\Omega^{-1}A) \preceq I_*(A)$.

Remark. In [45], the remark after Proposition 2.1 may suggest that $I_*(A) \preceq I_*(\Omega^{-1}A)$ is not true in general.

We can now give an example of an algebra with an isoperimetric profile, which is not finitely generated.

Example 4.2.1. If $A = K[x_1,\dots,x_d]$ is the algebra of polynomials in d variables, then we already saw that $I_*(A) \sim n^{\frac{d-1}{d}}$. If we denote as usual by $K(x_1,\dots,x_d)$ the quotient field of A, using the previous corollary we have

$$I_*(K(x_1,\dots,x_d)) \sim n^{\frac{d-1}{d}}.$$

Notice that $K(x_1,\dots,x_d)$ is not finitely generated as an algebra.

Another immediate consequence of this corollary is for example that

$$I_*(K[x_1^{\pm 1},\dots,x_d^{\pm 1}]) \sim n^{\frac{d-1}{d}},$$

where $K[x_1^{\pm 1},\dots,x_d^{\pm 1}]$ is the algebra of Laurent polynomials in d variables (see [21]).

The following consequences on the amenability of a localization follow easily from Lemma 4.2.1 and Proposition 3.2.1.

Corollary 4.2.3. *Let A be an algebra and let Ω be a right Ore set of regular elements in (i) and a left Ore set of regular elements in (ii). Then*

(i) A is amenable if and only if $A\Omega^{-1}$ is amenable.

(ii) If A is amenable, then $\Omega^{-1}A$ is amenable.

4.3 Subadditivity

Definition. We say that a function $f:\mathbb{R}_+\to\mathbb{R}_+$ is (*asymptotically*) *subadditive* if there exist positive constants $C_1,C_2>0$ such that for every finite set of positive real numbers $r_1,\dots,r_k$ we have

$$C_2 f(C_1(r_1+\cdots+r_k))\le f(r_1)+\cdots+f(r_2).$$

Example 4.3.1. The function $f(x)=x^\alpha$ for $0\le\alpha\le 1$ is subadditive with constants $C_1=C_2=1$.

For example the isoperimetric profile of an infinite group is subadditive with constants $C_1=C_2=1$ (cf. [21]).

The following lemma motivates our definition of subadditivity.

Lemma 4.3.1. *Given two functions $f,g:\mathbb{R}_+\to\mathbb{R}_+$, if $f\sim g$, then f is subadditive if and only if g is.*

Proof. Suppose that f is subadditive, with constants C_1 and C_2. By assumption there exist positive constants A_1,A_2,B_1 and B_2 such that

$$f(A_1n)\ge A_2g(n)\quad\text{and}\quad g(B_1n)\ge B_2f(n)$$

for all $n\in\mathbb{N}$. Given $r_1,\dots,r_k\in\mathbb{N}$ we have

$$\begin{aligned}A_2C_2B_2g\left(\frac{C_1}{A_1B_1}(r_1+\cdots+r_k)\right) &\le C_2B_2f\left(\frac{C_1}{B_1}(r_1+\cdots+r_k)\right)\\ &\le B_2\left(f\left(\frac{r_1}{B_1}\right)+\cdots+f\left(\frac{r_k}{B_1}\right)\right)\\ &\le g(r_1)+\cdots+g(r_k).\end{aligned}$$

The converse is proved in the same way. □

Subadditivity seems a natural property of the isoperimetric profile. Unfortunately, we are not able to prove it in general for any algebra.

In [21] Gromov states the following result. We give a proof for completeness.

Lemma 4.3.2. *If Γ is an infinite semigroup with the cancellation property generated by a finite set S, then $I_\circ(n;\Gamma,S)$ is subadditive. In particular for $r,s \in \mathbb{N}$*

$$I_\circ(r+s;\Gamma,S) \leq I_\circ(r;\Gamma,S) + I_\circ(s;\Gamma,S).$$

Proof. If X and Y are two finite subset of Γ, then clearly

$$|\partial_S(X\cup Y)| \leq |\partial_S(X)| + |\partial_S(Y)|,$$

since

$$\begin{aligned}
\partial_S(X\cup Y) &= \left(\bigcup_{s\in S} s(X\cup Y)\right)\setminus(X\cup Y)\\
&= \left(\bigcup_{s\in S} s(X\cup Y)\setminus X\right)\cap\left(\bigcup_{s\in S} s(X\cup Y)\setminus Y\right)\\
&\subseteq \left(\bigcup_{s\in S} s(X)\setminus X\right)\cup\left(\bigcup_{s\in S} s(Y)\setminus Y\right)\\
&= \partial_S(X)\cup\partial_S(Y).
\end{aligned}$$

Another obvious observation is that

$$|\partial_S(X)| = |\partial_S(X\gamma)|$$

for any $\gamma \in \Gamma$.

Now, given $r,s \in \mathbb{N}$, let $X \subset \Gamma$ with $|X| = r$ such that $|\partial_S(X)| = I_\circ(r;\Gamma,S)$, and let $Y \subset \Gamma$ with $|Y| = s$ such that $|\partial_S(Y)| = I_\circ(s;\Gamma,S)$.

We claim that we can find an element $\beta \in \Gamma$ such that $X \cap Y\beta = \emptyset$. In fact, for any $x \in X$ and $y \in Y$, by the cancellation property there exists at most one $\gamma \in \Gamma$ such that $x = y\gamma$. Since X and Y are finite and Γ is infinite, there exists a $\beta \in \Gamma$ such that $x \neq y\beta$ for all $x \in X$ and $y \in Y$. Hence $X \cap Y\beta = \emptyset$.

Using the observations we made before, this implies

$$\begin{aligned} |\partial_S(X \cup Y\beta)| &\leq |\partial_S(X)| + |\partial_S(Y\beta)| \\ &= |\partial_S(X)| + |\partial_S(Y)| \\ &= I_\circ(r;\Gamma,S) + I_\circ(s;\Gamma,S). \end{aligned}$$

Since now $|X \cup Y\beta| = r + s$, this gives the result. □

In [21], pag. 8, Gromov claims that clearly

$$I_*(r+s;K\Gamma) \leq I_*(r;K\Gamma) + I_*(s;K\Gamma).$$

We don't see how to prove it for general infinite groups Γ. But the previous two lemmas and Theorem 3.3.3 imply the following corollary:

Corollary 4.3.3. *If Γ is a finitely generated orderable semigroup with the cancellation property then $I_*(K\Gamma)$ is subadditive.*

We now show that the isoperimetric profile of a domain is subadditive. We need the following proposition, which was showed to me by Zelmanov.

Proposition 4.3.4 (Zelmanov)**.** *Let A be a domain over K, and let V and W be finite dimensional subspaces of A, with $|V| = m$ and $|W| = n$. If $V \cap Wa \neq \{0\}$ for all $a \in A \setminus \{0\}$, then A is algebraic of bounded degree.*

To prove this proposition we need the following lemma.

Lemma 4.3.5. *In the hypothesis of the previous proposition, let $\{w_1,\ldots,w_n\}$ be a basis of W. Then for any nonzero element $a \in A$ there exist polynomials $f_1(t),\ldots,f_n(t)$, not all zero and all of degree $\leq m$ such that*

$$w_1 f_1(a) + \cdots + w_n f_n(a) = 0.$$

Proof. Given $0 \neq a \in A$, we have $V \cap W1 \neq \{0\}, V \cap Wa \neq \{0\}, \ldots, V \cap Wa^m \neq \{0\}$.

Hence there are coefficients $\alpha_{ij} \in K$ such that

$$\begin{aligned} 0 &\neq \alpha_{01} w_1 + \cdots + \alpha_{0n} w_n \in V, \\ 0 &\neq \alpha_{11} w_1 a + \cdots + \alpha_{1n} w_n a \in V, \\ &\vdots \\ 0 &\neq \alpha_{m1} w_1 a^m + \cdots + \alpha_{mn} w_n a^m \in V. \end{aligned}$$

Since $|V| = m$, these elements are linearly dependent, hence there exist $\beta_0, \ldots, \beta_m$ not all zero such that

$$\beta_0(\alpha_{01} w_1 + \cdots + \alpha_{0n} w_n) + \beta_1(\alpha_{11} w_1 a + \cdots + \alpha_{1n} w_n a) + \cdots$$
$$\cdots + \beta_m(\alpha_{m1} w_1 a^m + \cdots + \alpha_{mn} w_n a^m) = 0,$$

which implies

$$w_1(\beta_0 \alpha_{01} + \beta_1 \alpha_{11} a + \cdots + \beta_m \alpha_{m1} a^m) + w_2(\beta_0 \alpha_{02} + \beta_1 \alpha_{12} a + \cdots + \beta_m \alpha_{m2} a^m) + \cdots$$
$$\cdots + w_n(\beta_0 \alpha_{0n} + \beta_1 \alpha_{1n} a + \cdots + \beta_m \alpha_{mn} a^m) = 0.$$

We set $f_i(t) := \beta_0 \alpha_{0i} + \beta_1 \alpha_{1i} t + \cdots + \beta_m \alpha_{mi} t^m$ for $i = 1, \ldots, n$. If all the f_i's are zero, then $\beta_i \alpha_{ij} = 0$ for $0 \leq i \leq m$ and $1 \leq j \leq n$. But each row $(\alpha_{i0}, \ldots, \alpha_{in})$ is not the zero vector, because $\sum_j \alpha_{ij} w_j a^i \neq 0$. Hence $\beta_i = 0$ for all i, a contradiction. □

We can now prove the proposition.

Proof. Let $\{w_1, \ldots, w_n\}$ be a basis of W. By the lemma, for $0 \leq i \leq m$ we can find polynomials $f_{i1}, \ldots, f_{in}$, not all zero and of degree $\leq m$ such that

$$\sum_j w_j f_{ij}\left(a^{(m+1)^i}\right) = 0. \tag{*}$$

We have

$$\det \left\| f_{ij}\left(a^{(m+1)^i}\right) \right\| = 0.$$

We got in this way a polynomial of degree bounded by a function of m and n only, satisfied by a. If this is not the zero polynomial, we are done.

Suppose this is not the case. Let $f_{ij}(t) := \alpha_{ij0} + \alpha_{ij1}t + \cdots + \alpha_{ijm}t^m$, and suppose that

$$\det \left\| f_{ij}\left(t^{(m+1)^i}\right) \right\| = 0.$$

Observe that in each row of the matrix $\left\| f_{ij}\left(t^{(m+1)^i}\right) \right\|$ there are at least two nonzero polynomials. In fact we know that they are not all zero. If only one of them is zero, then the equation (*) gives a zero divisor, which doesn't exist by our assumption. Moreover, we can assume that in each row the entries have no common divisors of the form t^k with $k \geq 1$, since otherwise we can factor it out, preserving the relation (*). Hence in particular in each row there is at least one polynomial with nonzero constant term.

Since these rows are linearly dependent, we can take a minimal linearly dependent set of rows, call r the cardinality of this set and call the indices of these rows $j_1, j_2, \ldots, j_r$. By construction all the minors of order r in these rows are zero. Considering these minors modulo $t^{(m+1)^{j_1+1}}$ we can replace the coefficients in the first of our rows by their constant terms, still having the first row non zero and depending on the others. Hence we can find polynomials $b(t), c_2(t), \ldots, c_r(t)$ such that

$$b(t)\alpha_{j_1k0} = \sum_{i=2}^{r} c_i(t) f_{j_ik}\left(t^{(m+1)^{j_i}}\right)$$

for all $k = 1, \ldots, n$. By assumption $b(t) \neq 0$. Observe now that (*) implies

$$\begin{aligned} b(a)\left(\sum_{k=1}^{n} w_k \alpha_{j_1k0}\right) &= \sum_{k=1}^{n} w_k b(a) \alpha_{j_1k0} \\ &= \sum_{k=1}^{n} w_k \sum_{i=2}^{r} c_i(a) f_{j_ik}\left(a^{(m+1)^{j_i}}\right) \\ &= \sum_{i=2}^{r} c_i(a)\left(\sum_{k=1}^{n} w_k f_{j_ik}\left(a^{(m+1)^{j_i}}\right)\right) = 0. \end{aligned}$$

Since $\sum_{k=1}^{n} w_k \alpha_{j_1k0} \neq 0$, we must have $b(a) = 0$. It's now clear that $b(t)$ also has degree bounded by a function of m and n only. This completes the proof. □

The following lemma is crucial.

Lemma 4.3.6. *If A is an (infinite dimensional) division algebra, then given two finite dimensional subspaces V and $W \subset A$ there exists a nonzero element $a \in A$ such that $V \cap Wa = \{0\}$.*

Proof. Suppose the contrary. Then by the previous proposition we know that A is algebraic of bounded degree. Hence by a theorem of Jacobson (see [23]) A is locally finite, i.e. any finitely generated subalgebra of A is finite dimensional. But for any nonzero $a \in A$ we have $v = wa$ for some nonzero $v \in V$ and some nonzero $w \in W$, i.e. $a = w^{-1}v$. Hence a is contained in the subalgebra generated by V and W, which is finite dimensional. This gives a contradiction, since A is not finite dimensional. □

We are now able to prove the main result of this subsection.

Theorem 4.3.7. *If A is a nonamenable domain, then $I_*(A)$ is subadditive. If A is an amenable domain, then $I_*(A,V)$ is subadditive for any subframe V of A.*

Proof. If A is nonamenable, then by Corollary 3.2.2 $I_*(n;A) \sim n$, hence by Lemma 4.3.1 $I_*(A)$ is subadditive.

If A is amenable, then by Proposition 3.2.1 we know that $I_*(A,V) \precnsim n$ for any subframe V of A. In this case, we know that A is a right Ore domain, hence it admits a ring of quotients D, which is of course a division algebra. By Lemma 4.2.1, $I_*(n;A,V) = I_*(n;D,V)$, hence again by Lemma 4.3.1 we reduced the problem to show that D has a subadditive isoperimetric profile.

Let $r,s \in \mathbb{N}$, and consider two subspaces $W,Z \subset D$ with $|W| = r$ and $|Z| = s$. By the previous lemma, we can find an element $a \in D$ such that $W \cap Za = \{0\}$. If now V is any subframe of D, we have

$$\begin{aligned} |\partial_V(W \oplus Za)| &= |V(W \oplus Za)| - |W \oplus Za| \leq |VW| + |VZa| - |W| - |Za| \\ &= |VW| + |VZ| - |W| - |Z| = |\partial_V(W)| + |\partial_V(Z)|, \end{aligned}$$

which gives the subadditivity of $I_*(n;D,V)$. □

Question 1. Is the isoperimetric profile with respect to some subframe of an algebra always subadditive?

4.4 Free left modules over subalgebras

We now study algebras which are a free left module over some subalgebra.

The proof of the following proposition is a modification of the proof of Theorem 2.4 in [45].

Proposition 4.4.1. *Suppose that $B \subset A$ is a subalgebra and A is a free left B-module. If V is a subframe of B and $I_*(B,V)$ is subadditive, then $I_*(B,V) \preceq I_*(A,V)$.*

Proof. We have $A = \bigoplus_i Ba_i$ where $a_i \in A$. Given any subspace W of A we can find $a_1, \ldots, a_n$ such that $W \subset \bigoplus_{i=1}^n Ba_i$. We can choose a basis of W of the form

$$\{w_i^1 a_1 + y_i^1\}_{i=1}^{p_1} \cup \{w_i^2 a_2 + y_i^2\}_{i=1}^{p_2} \cup \cdots \cup \{w_i^n a_n + y_i^n\}_{i=1}^{p_n}$$

where $w_i^j \in B$ and $y_i^j \in \bigoplus_{k>j} Ba_k$, such that for each j, $\{w_i^j\}_{i=1}^{p_j}$ are linearly independent. Notice that $\{w_i^j a_j + y_i^j\}_{i=1}^{p_j}$ corresponds to a basis of $(W \cup \bigoplus_{k \geq j} Ba_k)/(W \cup \bigoplus_{k>j} Ba_k)$. Let W_j' denote the subspace generated by $\{w_i^j\}_{i=1}^{p_j}$ and let W_j denote the subspace generated by $\{w_i^j a_j + y_i^j\}_{i=1}^{p_j}$. Then

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

and hence

$$|W| = \sum_j |W_j| = \sum_j |W_j'|.$$

Let V be a subframe of B. We have

$$VW_1 = \left\{ xa_1 + y \mid x \in VW_1' \text{ and } y \in \bigoplus_{i=2}^n Ba_i \right\}.$$

Since

$$\sum_{i=2}^n VW_i \subset \bigoplus_{i=2}^n Ba_i \quad \text{and} \quad \left(\bigoplus_{i=2}^n Ba_i \right) \cap Ba_1 = 0,$$

we have

$$\left| \sum_{i=1}^n VW_i \right| \geq |VW_1'| + \left| \sum_{i=2}^n VW_i \right|.$$

By induction on n we have

$$\left| \sum_{i=1}^n VW_i \right| \geq \sum_{i=1}^n |VW_i'|.$$

Using the hypothesis, this implies

$$\begin{aligned} |\partial_V(W)| &= |VW| - |W| = \left|\sum_{i=1}^{n} VW_i\right| - \sum_{i=1}^{n} |W_i| \\ &\geq \sum_{i=1}^{n} |VW_i'| - \sum_{i=1}^{n} |W_i'| = \sum_{i=1}^{n} |\partial_V(W_i')| \\ &\geq \sum_{i=1}^{n} I_*(|W_i'|;B,V) \geq C_2 I_*(C_1 \sum_{i=1}^{n} |W_i'|;B,V) = C_2 I_*(C_1|W|;B,V), \end{aligned}$$

where C_1 and C_2 are two positive constants. Therefore

$$I_*(B,V) \preceq I_*(A,V).$$

□

The following corollaries are immediate consequences of the proposition.

Corollary 4.4.2. *Suppose that $B \subset A$ is a subalgebra and A is a free left B-module. If both A and B have isoperimetric profiles, and $I_*(B)$ is subadditive, then $I_*(B) \preceq I_*(A)$.*

Corollary 4.4.3. *Suppose that $B \subset A$ is a subalgebra and A is a free left B-module. If A is amenable and $I_*(B)$ is subadditive, then B is amenable.*

Let's derive another easy consequence from the previous proposition, which generalizes a result in [11].

Proposition 4.4.4. *If B is a nonamenable division subalgebra of A, then A is nonamenable. If B is an amenable division subalgebra of A, then $I_*(B,V) \preceq I_*(A,V)$ for any subframe V of B. In particular, if both A and B have isoperimetric profiles, then $I_*(B) \preceq I_*(A)$.*

Proof. If B is a nonamenable division subalgebra, then A is a free left B-module. By Theorem 4.3.7, $I_*(B,V)$ is subadditive for any subframe V that measures $I_*(n;B) \sim n$, hence by Proposition 4.4.1

$$n \sim I_*(n;B,V) \preceq I_*(n;A,V)$$

for any subframe V of B that measures $I_*(B)$. Hence $I_*(n;A,V) \sim n$, and so A is nonamenable by Corollary 3.2.2.

If B is an amenable division subalgebra, A is again a free left B-module. By Theorem 4.3.7, $I_*(B,V)$ is subadditive for any subframe V, hence the result follows again from Proposition 4.4.1. □

We are now able to prove the following

Theorem 4.4.5. *Let $B \subset A$ be domains. If both B and A are right Ore, then $I_*(B,V) \preceq I_*(A,V)$ for all subframes V of B.*

Proof. If we call S and D the right quotient division algebras of B and A respectively, by Lemma 4.2.1, if V is a subframe of B we have $I_*(n;B,V) = I_*(n;S,V)$ and $I_*(n;A,V) = I_*(n;D,V)$. Since $I_*(S,V)$ is also subadditive, we can apply Proposition 4.4.4 to $S \subset D$ to get $I_*(S,V) \preceq I_*(D,V)$. Now again by Lemma 4.2.1, $I_*(B,V) \preceq I_*(A,V)$. □

Corollary 4.4.6. *If $B \subset A$ are domains, B is right Ore and both A and B have isoperimetric profiles, then $I_*(B) \preceq I_*(A)$.*

Proof. If A is nonamenable, $I_*(n;A) \sim n$ and there is nothing to prove. Otherwise, the result follows from the previous theorem. □

Remark 6. Notice that the hypothesis on B of being right Ore cannot be dropped. For example we already observed that the quotient division algebra of the Weyl algebra A_1 is amenable, but it contains a subalgebra isomorphic to a free algebra in two variables (see [29]). We show now that this is the only case that can occur.

By a theorem of Jategaonkar ([24]), a domain which is not Ore must contain a subalgebra isomorphic to a noncommutative free algebra. This and the previous proposition imply the following corollary.

Corollary 4.4.7. *If A is an amenable domain, then for any subdomain B of A we have $I_*(B,V) \preceq I_*(A,V)$ for all subframes V of B if and only if A does not contain a subalgebra isomorphic to a noncommutative free algebra.*

4.5 Finite modules over subalgebras

Suppose that B is a subalgebra of an algebra A. Assume that A is a finite right B-module, i.e. $A = WB$, where W is a subframe of A. We want to compare the isoperimetric profiles of A and B.

The following proposition generalizes some of the results in [11].

Proposition 4.5.1. *Let A be an algebra.*

(1) Let B be a subalgebra of A such that A is a finite free right B-module. If B is amenable, then A is also amenable. If both A and B have isoperimetric profiles, then $I_*(A) \preceq I_*(B)$. If moreover B has a subadditive isoperimetric profile and A is also a free left B-module, then $I_*(A) \sim I_*(B)$.

(2) Let B be a division subalgebra of A and let A be a finite right B-module. If B is amenable, then A is also amenable. If both A and B have isoperimetric profiles, then $I_*(A) \sim I_*(B)$.

(3) Let B be a finite dimensional algebra and A an algebra. If A is amenable, then $A \otimes B$ is also amenable. If both A and $A \otimes B$ have isoperimetric profiles, then $I_*(A \otimes B) \preceq I_*(A)$. If moreover A has a subadditive isoperimetric profile, then $I_*(A) \sim I_*(A \otimes B)$.

(4) Let $M_n(A)$ be the algebra of $n \times n$ matrices over A. If A is amenable, then $M_n(A)$ is also amenable. If both A and $M_n(A)$ have isoperimetric profiles, then $I_*(M_n(A)) \preceq I_*(A)$. If moreover A has a subadditive isoperimetric profile, then $I_*(A) \sim I_*(M_n(A))$.

(5) Let G be a finite group and $A * G$ a skew group ring. If A is amenable, then also $A * G$ is amenable. If both A and $A * G$ have isoperimetric profiles, then $I_*(A * G) \preceq I_*(A)$. If moreover A has subadditive isoperimetric profile, then $I_*(A) \sim I_*(A * G)$.

First we need a lemma.

Lemma 4.5.2. *Let $V, W, Z \subset A$ be subspaces of A. Then*

$$\left|\frac{ZVW}{ZW}\right| \leq |Z| \left|\frac{VW}{W}\right|$$

Proof. Let $v_1, \dots, v_m$ be a basis of V and $w_1, \dots, w_n$ a basis of W. Then the products $v_i w_j$ span VW. Clearly at most $|VW/W| = |\partial_V(W)|$ of these products are not in W. For each of them, multiplying on the left by elements of Z, we get at most $|Z|$ products which do not fall into ZW. This proves the result. □

The previous proposition follows from the following lemma together with Propositions 4.4.1, 4.5.3 and 4.3.1.

Lemma 4.5.3. *Let B be a subalgebra of an algebra A, let V be a subframe of A and let A be a finite right B-module.*

(i) If there exist positive constants C_1 and C_2 such that

$$C_1 I_*(C_2 n; A, V) \leq I_*(n+r; A, V)$$

for any $n, r \in \mathbb{N}$, then $I_*(A,V) \preceq I_*(B,V_1)$ for some subframe V_1 of B.

(ii) If A is also free as right B-module, then $I_*(A,V) \preceq I_*(B,V_1)$ for some subframe V_1 of B.

Proof. Since A is a finite right B-module, there exists a subframe W of A such that $A = WB$. It's clear that given the subframe V of A there exists a subframe V_1 of B such that $VW \subseteq WV_1$. For any subspace Z of B, using the previous lemma, we get

$$\begin{aligned} I_*(|WZ|; A, V) &\leq |\partial_V(WZ)| = \left|\frac{VWZ}{WZ}\right| \leq \left|\frac{WV_1 Z}{WZ}\right| \\ &\leq |W|\left|\frac{V_1 Z}{Z}\right| = |W||\partial_{V_1}(Z)|. \end{aligned}$$

Now the hypothesis in (i) gives

$$C_1 I_*(C_2|Z|; A, V) \leq I_*(|WZ|; A, V) \leq |W||\partial_{V_1}(Z)|,$$

which implies $I_*(A,V) \preceq I_*(B,V_1)$.

In (ii), if $A = \oplus_{i=1}^k w_i B$, we choose W to be the span of $\{1 = w_1, w_2, \ldots, w_k\}$. Then

$$I_*(|W||Z|; A, V) = I_*(|WZ|; A, V) \leq |W||\partial_{V_1}(Z)|,$$

which again gives $I_*(A,V) \preceq I_*(B,V_1)$. □

Remark. Notice that the hypothesis in (i) of this lemma is a generalization of the property of being weakly monotone increasing. All the isoperimetric profiles we know satisfy this property.

Question 2. Is it true that $I_*(A)$ satisfies the property in (i) for any algebra A? Is it true if A is a domain?

We are now able to prove the following corollary (cf. [45], Corollary 3.3).

Corollary 4.5.4. *Let $B \subset A$ be prime right Goldie algebras with isoperimetric profiles, and suppose that $I_*(B)$ is subadditive. Then $I_*(B) \preceq I_*(A)$. If moreover A is a finite right B-module and B is artinian, then $I_*(A) \sim I_*(B)$.*

Proof. By Goldie's Theorem, A has a right quotient ring which is a simple artinian algebra. Hence by Corollary 4.2.2 we may assume that A is a simple artinian ring $M_n(A')$ for some division algebra A'. By Proposition 3.1.16 in [28], the quotient ring Q of B embeds into $M_k(A')$ for some $k \leq n$. Therefore by Corollary 4.2.2 and Proposition 4.5.1, (4), we may assume that B is a division algebra. Whence the first statement follows from Proposition 4.4.4.

If B is artinian and A is finite as B-module, then A is artinian. Therefore the second statement follows from Lemma 4.5.3 and Proposition 4.5.1, (4). □

4.6 Tensor products

In this section we study the behavior of the isoperimetric profile with respect to tensor products.

Proposition 4.6.1. *Let A and B be two K-algebras, and let V_A and V_B be two subframes of A and B respectively. If $V := V_A \otimes 1 + 1 \otimes V_B$, then*

$$I_*(nm; A \otimes_K B, V) \leq mI_*(n; A, V_A) + nI_*(m; B, V_B).$$

Proof. Given any two subspaces $W \subset A$ and $Z \subset B$, we have

$$\begin{aligned} I_*(|W||Z|; A \otimes_K B, V) &\leq |\partial_V(W \otimes Z)| = \left| \frac{V_A W \otimes Z + W \otimes V_B Z}{W \otimes Z} \right| \\ &\leq \left| \frac{V_A W \otimes Z}{W \otimes Z} \right| + \left| \frac{W \otimes V_B Z}{W \otimes Z} \right| \\ &= |Z||\partial_{V_A}(W)| + |W||\partial_{V_B}(Z)|, \end{aligned}$$

which gives the result. □

Corollary 4.6.2. *Let A and B be two K-algebras, let V_A and V_B be two subframes of A and B respectively, and let $V := V_A \otimes 1 + 1 \otimes V_B$. If $I_*(n;A,V_A) \preceq n^{1-1/r}$ and $I_*(n;B,V_B) \preceq n^{1-1/s}$ for some real numbers $s \geq r \geq 1$, then*

$$I_*(n;A \otimes_K B, V) \preceq n^{1-\frac{1}{r+s}}.$$

Proof. Given $t \in \mathbb{R}$, $0 < t < 1$ the previous proposition implies

$$\begin{aligned} I_*(n;A \otimes_K B, V) &\preceq n^t I_*(n^{1-t};A,V_A) + n^{1-t} I_*(n^t;B,V_B) \\ &\preceq n^{t+(1-t)(1-1/r)} + n^{t+(1-t)(1-1/s)}. \end{aligned}$$

Substituting $t = r/(r+s)$ we get

$$I_*(n;A \otimes_K B, V) \preceq n^{\frac{r}{r+s}+\frac{s}{r+s}\frac{r-1}{r}} + n^{\frac{r}{r+s}+\frac{s}{r+s}\frac{s-1}{s}} \preceq n^{\frac{r+s-1}{r+s}},$$

since $s \geq r$, hence both the exponents in the sum above are less or equal then the exponent $(r+s-1)/(r+s)$. □

We have also the following immediate consequence of Proposition 4.4.1.

Proposition 4.6.3. *If A and B are two K-algebras, V is a subframe of A and $I_*(A,V)$ is subadditive, then*

$$I_*(A,V) \preceq I_*(A \otimes_K B, V \otimes 1).$$

The relation given in Proposition 4.6.1 looks a bit strange. A more natural relation holds for Følner functions, as we will see later in this work.

4.7 Filtered and Graded Algebras

In this section we consider a filtration on A, i.e. a sequence of subspaces A_i of A

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A, \quad \bigcup_{n=0}^{\infty} A_n = A,$$

with the property that $A_i A_j \subset A_{i+j}$ for all $i, j \geq 0$. We assume also that $A_0 = K$ and that A_1 generates A.

Given a filtered algebra, we can consider its associated graded algebra

$$gr(A) := \bigoplus_{i\geq 0} A_i/A_{i-1},$$

where we agree that $A_{-1} = \{0\}$. This is an algebra with the multiplication derived by the rule

$$[x + A_{i-1}] \cdot [y + A_{j-1}] = [xy + A_{i+j-1}].$$

For any subframe $V \subset A_1$, we can view V also as a subframe of $gr(A)$ via the identification $V \equiv (V \cap A_0)/A_{-1} \oplus V/A_0 = K \oplus V/K$.

Theorem 4.7.1. *If A is an algebra with a filtration given as above, and $gr(A)$ is a domain, then $I_*(gr(A),V) \preceq I_*(A,V)$ for any subframe $V \subset A_1$.*

Proof. Given a subspace W of A we define $W_i = W \cap A_i$ and $gr(W) = \bigoplus_{i\geq 0} W_i/W_{i-1}$. Observe that $gr(W)$ is a finite dimensional subspace of $gr(A)$.

The first remark is that $|W| = |gr(W)|$: this can be seen looking at a basis for W_i and completing it to a basis of W_{i+1} (if $W_i \neq W_{i+1}$, otherwise look at the next index) for each i. These basis elements clearly give a basis for $gr(W)$.

Now we want to compare $|\partial_V(W)|$ and $|\partial_V(gr(W))|$. The remark we need is that for any finite dimensional subspace W of A, and any element $a \in A_1$ we have

$$|a\,gr(W)| = |gr(aW)|,$$

where $a\,gr(W)$ is a short notation for $[a + A_0]gr(W)$.

We have

$$a\,gr(W) = a\bigoplus_{i\geq 0} W_i/W_{i-1} = \bigoplus_{i\geq 0} \frac{aW_i}{aW_i \cap A_i},$$

and

$$gr(aW) = \bigoplus_{i\geq 0} \frac{aW \cap A_i}{aW \cap A_{i-1}},$$

hence we want to show that

$$\frac{aW \cap A_{i+1}}{aW \cap A_i} = \frac{aW_i}{aW_i \cap A_i}.$$

Clearly $aW_i = a(W \cap A_i) \subseteq aW \cap A_{i+1}$. If the other inclusion is false, then there exists $x \in W \setminus A_i$ with $ax \in A_{i+1}$. So $x \in A_{i+p} \setminus A_i$ for some $p \geq 1$, with $ax \in A_{i+1}$. This gives a zero divisor in $gr(A)$, which is a contradiction. Hence $aW_i = aW \cap A_{i+1}$.

Similarly $aW_i \cap A_i = aW \cap A_i$. In fact, it's obvious that $aW_i \cap A_i \subseteq aW \cap A_i$. If the other inclusion is false then there exists $x \in W \setminus A_i$ with $ax \in A_i$. So $x \in A_{i+p} \setminus A_i$ for some $p \geq 1$, with $ax \in A_i$. Again, this gives a zero divisor in $gr(A)$. Hence $aW_i \cap A_i = aW \cap A_i$.

This proves the equality we wanted, giving $|gr(aW)| = |a\,gr(W)|$.

Let's now choose a basis $1 = a_1, a_2, \ldots, a_r$ of V. We have

$$|\partial_V(gr(W))| = \left|\frac{V\,gr(W)}{gr(W)}\right| = \left|\sum_j a_j\,gr(W)\right| - |gr(W)| =$$

$$\begin{aligned}
&= \left|\bigoplus_{i\geq 0}\sum_j \frac{a_j(W\cap A_i)}{a_j(W\cap A_i)\cap A_i}\right| - |gr(W)| = \\
&= \sum_i\left|\sum_j \frac{a_j(W\cap A_i)}{a_j(W\cap A_i)\cap A_i}\right| - |gr(W)| = \sum_i\left|\sum_j \frac{a_jW\cap A_{i+1}}{a_jW\cap A_i}\right| - |gr(W)| = \\
&= \sum_i\left|\sum_j \frac{a_jW\cap A_{i+1}}{(\sum_j a_j)W\cap A_i}\right| - |gr(W)| \leq \sum_i\left|\frac{(\sum_j a_j)W\cap A_{i+1}}{(\sum_j a_j)W\cap A_i}\right| - |gr(W)| = \\
&= \left|\bigoplus_i \frac{(\sum_j a_j)W\cap A_{i+1}}{(\sum_j a_j)W\cap A_i}\right| - |gr(W)| = |gr(VW)| - |gr(W)| = |VW| - |W| = \\
&= |\partial_V(W)|.
\end{aligned}$$

This gives $I_*(gr(A),V) \preceq I_*(A,V)$. □

In [45] (see also [44]) Zhang considers a more general setting.

Definition ([45])**.** Let A and B two K-algebras and let ν be a map from A to B. We call ν a *valuation* from A to B if the following conditions hold:

(v1) $\nu(ta) = t\nu(a)$ for all $a \in A$ and $t \in K$;

(v2) $\nu(a) \neq 0$ for all nonzero $a \in A$;

(v3) for any $a, b \in A$, either $\nu(a)\nu(b) = \nu(ab)$ or $\nu(a)\nu(b) = 0$;

(v4) for any subspace W of A $|\nu(W)| = |W|$.

The main example of a valuation is the leading-term map of a Γ-filtered algebra, where Γ is any ordered semigroup. Let A be an algebra with a filtration $\{A_\gamma \mid \gamma \in \Gamma\}$ of A, which satisfies the following conditions:

(f0) $K \subset A_e$ where e is the unit of Γ;

(f1) $A_\alpha \subset A_\beta$ for all $\alpha < \beta$ in Γ;

(f2) $A_\alpha A_\beta \subset A_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$;

(f3) $A = \cup_{\gamma\in\Gamma}(A_\gamma - A_{<\gamma})$, where $A_{<\gamma} = \cup_{\alpha<\gamma} A_\alpha$;

(f4) $1 \in A_e - A_{<e}$ (and hence $K \subset A_e - A_{<e}$).

Then we define the associated graded algebra to be $gr(A) := \oplus_{\gamma\in\Gamma} A_\gamma / A_{<\gamma}$ with the multiplication determined by $(a + A_{<\alpha})(b + A_{<\beta}) = ab + A_{<\alpha\beta}$. Notice that this is the definition we gave before with $\Gamma = \mathbb{N}$.

We define a map $\nu : A \to gr(A)$ by $\nu(a) = a + A_{<\gamma}$ for all $a \in A_\gamma - A_{<\gamma}$. This ν is called the *leading-term map* of A and it is easy to see that it satisfies (v1,2,3,4) (see [44], Section 6). If also

(f5) $gr(A)$ is a Γ-graded domain,

then $\nu(a)\nu(b) = 0$ will not happen in (v3).

Theorem 4.7.2 (compare to [45], Theorem 4.3)**.** *If A and B are two K-algebras, and ν is a valuation from A to B, then*

$$I_*(B, \nu(V)) \preceq I_*(A, V).$$

Proof. If $W \subset A$, using Lemma 4.1, (3) in [45], we have

$$\begin{aligned} |\partial_V(W)| &= |VW| - |W| = |\nu(VW)| - |\nu(W)| \\ &\geq |\nu(V)\nu(W)| - |\nu(Z)| = |\partial_{\nu(V)}(\nu(W))|, \end{aligned}$$

which gives $I_*(A, V) \succeq I_*(B, \nu(V))$, as we wanted. □

If Γ is an ordered semigroup, B is a Γ-filtered graded K-algebra with the associated graded algebra $gr(B)$ and A is a K-algebra, then $A \otimes_K B$ is Γ-filtered, and its associated graded is isomorphic to $A \otimes_K gr(B)$. Here is another immediate consequence of Theorem 4.7.2:

Corollary 4.7.3. *If Γ is an ordered semigroup, A and B are two finitely generated K-algebras and B is Γ-filtered, then*

$$I_*(A \otimes_K gr(B)) \preceq I_*(A \otimes_K B).$$

4.8 Ore extensions

In this section we study how the isoperimetric profile behaves with Ore extensions.

Definition. Let A be an algebra, and let σ be an endomorphism of A. A linear map δ is a σ-derivation if

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$$

for all $a, b \in A$. We can introduce a ring structure on the free A-module $\oplus_{i\geq 0} x^i A$ by defining

$$ax = x\sigma(a) + \delta(a)$$

for all $a \in A$. The resulting ring will be called an *Ore extension* of A, and it will be denoted by $A[x, \sigma, \delta]$.

Proposition 4.8.1. *Let A be an algebra, σ an automorphism of A and δ a σ-derivation. If $I_*(A, V)$ is subadditive for some subframe $V \subset A$, then*

$$I_*(A, V) \preceq I_*(A[x, \sigma, \delta], V + Vx).$$

Proof. There is a natural filtration of $A[x, \sigma, \delta]$ determined by the degree of x, such that the associated graded algebra is isomorphic to $A[x, \sigma]$. Hence there is a valuation ν from $A[x, \sigma, \delta]$ to $A[x, \sigma]$, which by Theorem 4.7.2 gives

$$I_*(A[x, \sigma, \delta], W) \succeq I_*(A[x, \sigma], W),$$

for any graded subframe $W = \oplus_{i=0}^{m} W_i x^i$.

Hence it's enough to show that $I_*(A[x, \sigma], V + Vx) \succeq I_*(A, V)$, where V is a subframe of A. First observe that the leading-term map of $A[x, \sigma]$ is a valuation from $A[x, \sigma]$ to itself. Again by Theorem 4.7.2 it follows that it's enough to consider only the graded subspaces of $A[x, \sigma]$.

Let V be a subframe of A. Given a graded subspace $Z \subset A[x, \sigma]$, we have $Z =$

$\oplus_{i=0}^{n} Z_i x^i$, where $Z_i \subset A$ for all i. Since $ax = x\sigma(a)$ for all $a \in A$, we get

$$\begin{aligned}
|\partial_{V+Vx}(Z)| &= \left|\sum_{i=0}^{n} VZ_i x^i + VxZ_i x^i\right| - |Z| \\
&= \left|\sum_{i=0}^{n+1} (VZ_i + VZ_{i-1}^{\sigma^{-1}})x^i\right| - \sum_{i=1}^{n} |Z_i| \\
&= \sum_{i=0}^{n+1} \left|VZ_i + VZ_{i-1}^{\sigma^{-1}}\right| - \sum_{i=1}^{n} |Z_i| \\
&\geq \sum_{i=0}^{n} |VZ_i| - \sum_{i=1}^{n} |Z_i| = \sum_{i=0}^{n} |\partial_V(Z_i)| \\
&\geq \sum_{i=1}^{n} I_*(|Z_i|; A, V) \geq C_2 I_*(C_1(\sum_{i=1}^{n} |Z_i|); A, V) = C_2 I_*(C_1|Z|; A, V),
\end{aligned}$$

where by convention $Z_{-1} = Z_{n+1} = \{0\}$, and C_1 and C_2 are the two positive constants coming from the subadditivity assumption. This shows that

$$I_*(A[x,\sigma], V + VX) \succeq I_*(A, V),$$

completing the proof. □

The following corollary follows from the previous proposition and Theorem 4.3.7.

Corollary 4.8.2. *Let A be a domain, σ an automorphism of A and δ a σ-derivation. If A is amenable, then for any subframe $V \subset A$,*

$$I_*(A, V) \preceq I_*(A[x, \sigma, \delta], V + Vx).$$

If A is nonamenable, then $A[x, \sigma, \delta]$ is nonamenable.

Remark 7. Notice that in the proof of the previous proposition we used the following obvious inequality

$$\sum_{i=0}^{n+1} \left|VZ_i + VZ_{i-1}^{\sigma^{-1}}\right| \geq \sum_{i=0}^{n} |VZ_i|.$$

This inequality doesn't appear to be optimal and it's reasonable to expect a better one.

In this direction, in [45], Theorem 5.2, Zhang essentially proves the following

Proposition 4.8.3. *Let A be an algebra, $V \subset A$ a subframe, σ an automorphism of A and δ a σ-derivation. If $I_*(A,V) \succeq n^{\frac{d-1}{d}}$ for some $d \in \mathbb{R}$, $d \geq 1$, then*

$$I_*(A[x,\sigma,\delta],V+Vx) \succeq n^{\frac{d}{d+1}}.$$

This proposition gives for example a lower bound for the isoperimetric profile of iterated Ore extensions, starting from a finitely generated algebra A with $I_*(A) \succeq n^{\frac{d-1}{d}}$, for some $d \geq 1$.

We have also these two easy corollaries.

Corollary 4.8.4. *Let A be a finitely generated algebra and σ an automorphism of A, such that σ^m is an inner automorphism for some $m \in \mathbb{N}$. Then*

$$I_*(n;A[x,\sigma]) \preceq I_*(n;A \otimes_K K[x]).$$

Proof. If σ^m is the inner automorphism given by the conjugation by the invertible element $u \in A$, then $A[x,\sigma]$ is a finite free module over $A[x^m,\sigma] \cong A[u^{-1}x] \cong A \otimes_K K[x]$. The result now follows from Lemma 4.5.3. □

There is also an analogous version of this corollary with the algebra of Laurent skew polynomials $A[x,x^{-1},\sigma]$.

Corollary 4.8.5. *Let A be a finitely generated algebra and σ an automorphism of A, such that σ^m is an inner automorphism for some $m \in \mathbb{N}$. If $I_*(n;A) \sim n^{\frac{d-1}{d}}$ then*

$$I_*(n;A[x,\sigma]) \sim n^{\frac{d}{d+1}}.$$

Proof. It follows from the previous corollary, Corollary 4.6.2 and Proposition 4.8.3. □

4.9 Modules and ideals

If V is a frame of a K-algebra A and M is a left A-module, then we can define the isoperimetric profile of the A-module M as

$$I_*(n;M,V) := \inf|\partial_V(W)| = \inf|VW/W|$$

where the infimum is taken over all n-dimensional subspaces W of M. As for algebras, the asymptotic behavior of this function does not depend on the generating subspace V, hence we can talk about *the isoperimetric profile of the module M* and we will denote it by $I_*(M)$. We observe some properties of this isoperimetric profile.

Proposition 4.9.1. *Let A be an algebra, $V \subset A$ a subframe of A and $M = {}_AM$ a left A-module.*

(i) If $IM = 0$ for some ideal I of A, then $I_*({}_AM, V) \sim I_*({}_{A/I}M, \overline{V})$, where $\overline{V}$ is the image of V in A/I.

(ii) If N is an A-submodule of M, then $I_*(M,V) \preceq I_*(N,V)$.

(iii) If M is a left A-module, then $I_*({}_AM,V) \preceq I_*(A,V)$.

Proof. The first property follows directly from the definitions.

For (ii), given a subspace $W \subset N$, the boundary $\partial_V(W)$ is the same as if we regard W as a subspace of N or of M, hence $I_*(M,V) \preceq I_*(N,V)$.

Now by (ii), $I_*(M,V) \preceq I_*(Am,V)$ for all $m \in M$. Hence we can assume that $M = Am$ for some $m \in M$. By (i) we can also assume that M is faithful. In this case, given a finite dimensional subspace W of A we will have $|Wm| = |W|$. Then clearly $|\partial_V(Wm)| \leq |\partial_V(W)|$. This gives the inequality we wanted. □

Consider now a frame V of an algebra A, and an infinite dimensional ideal J in A. Now J is a left A-module, hence

$$I_*(J) \preceq I_*(A).$$

But also J is an A-submodule of A, hence $I_*(A) \preceq I_*(J)$. Therefore $I_*(A) \sim I_*(J)$ as A-modules.

Remark 8. Notice that the isoperimetric profile of an ideal J of an algebra A as an A-module is a priori different from the isoperimetric profile of J as a subalgebra of A.

Part of the text of chapters 3, 4, 5 and 6 of this thesis is a modified version of "On isoperimetric profiles of algebras", D'Adderio Michele, *J. Algebra*, **322**, 2009.

Chapter 5

Computations of isoperimetric profiles of various algebras

The aim of this section is to prove the following theorem:

Theorem 5.0.2. *The isoperimetric profile of the following algebras is of the form* $n^{\frac{d-1}{d}}$ *where* d *is the GK-dimension of the algebra:*

- *finitely generated algebras of GK-dimension* 1*,*
- *finitely generated commutative domains,*
- *finitely generated prime PI algebras,*
- *universal enveloping algebras of finite dimensional Lie algebras,*
- *Weyl algebras,*
- *quantum skew polynomial algebras,*
- *quantum matrix algebras,*
- *quantum groups* $GL_{q,p_{ij}}(d)$*,*
- *quantum Weyl algebras,*
- *quantum groups* $\mathscr{U}(\mathfrak{sl}_2)$ *and* $\mathscr{U}'(\mathfrak{sl}_2)$.

5.1 Algebras of GK-dimension 1

For finitely generated algebras of GK-dimension 1 the isoperimetric profile is constant.

Proposition 5.1.1. *If A is a finitely generated algebra of GK-dimension 1, then $I_*(A)$ is constant.*

Proof. Let A be a finitely generated algebra of GK-dimension 1. G. Bergman proved (see [27], Theorem 2.5) that for an algebra to have GK-dimension 1 is equivalent to have *linear growth*, i.e. if V is a frame for A, then for all $n \in \mathbb{N}$

$$|V^{n+1}| - |V^n| \leq C,$$

where C is a positive constant. This inequality can also be written as

$$|\partial_V(V^n)| \leq C.$$

Since the growth is linear, this proves that the isoperimetric profile $I_*(A)$ is constant. □

Remark 9. The converse of this proposition is not true.

A cheap example is given by the algebra

$$A = K[x] \oplus K[y,z].$$

We know by Proposition 4.1.2 that $I_*(A) \preceq I_*(K[x])$, and we know by Proposition 3.3.4 that $I_*(K[x])$ is constant. However, $GK \dim A = 2$.

There is a more interesting example (cf. [12], Example 4). Consider the algebra $A = K\langle x,y\rangle/J$, where J is the ideal generated by all monomials in x and y containing at least 2 y's. Clearly $V = K + Kx + Ky$ is a frame of the infinite dimensional algebra A. Observe that the numbers $a_n := |V^n|$ satisfy the relation $a_n = a_{n-1} + n$, with initial conditions $a_1 = 3$ and $a_2 = 5$. Hence A has quadratic growth, and $GK \dim A = 2$. On the other hand, if we put $W_n := span_K\{y, xy, x^2y, \ldots, x^{n-1}y\}$, we have $|W_n| = n$, and

$$|\partial_V(W_n)| = 1$$

for all $n \in \mathbb{N}$. This shows that $I_*(A)$ is constant.

Notice that both of these examples are not domains.

Question 3. Is it true that if a prime noetherian algebra has constant isoperimetric profile, then it has GK-dimension 1?

Notice that the noetherianity assumption can't be dropped: the following example is due to Jason Bell.

Example 5.1.1 (J. Bell). Consider the algebra A over K with generators x and y and relations x^2, $xy^m x$ for m not a power of 2, and for each $r \geq 2$, $xy^{2^{m_1}}xy^{2^{m_2}}x\cdot\cdots\cdot xy^{2^{m_r}}x$ whenever $\sum_{i=1}^{r} m_i < r2^r$. This ring has $GK\,\mathrm{dim}$ 2 and is prime. Let $V = K + Kx + Ky$, and for $k \geq m+1$ let $W_k = span_K\{y^i x : 2^k + 1 \leq i < 2^{k+1}\}$. Then $xW_k = (0)$ and $yW_k + W_k = W_k + Ky^{2^{k+1}}x$. Hence $|VW_k/W_k| = 1$ and $|W_k| = 2^k$. This easily implies that the isoperimetric profile of A is constant.

5.2 Commutative Domains

We compute the isoperimetric profile of finitely generated commutative domains.

Proposition 5.2.1. *Let A be a finitely generated commutative domain over K, and let $d = GK\,\mathrm{dim} A$. Then $I_*(n;A) \sim n^{\frac{d-1}{d}}$.*

Proof. By the Noether's normalization theorem the ring A is a finitely generated module over a subring B isomorphic to $K[x_1,\ldots,x_d]$.

Theorem 4.4.5 implies that

$$n^{\frac{d-1}{d}} \sim I_*(B) \preceq I_*(A).$$

Considering now the quotient fields $Q \subset S$ of B and A respectively, we have that S is a finite dimensional vector space over Q, hence using Lemma 4.5.3 and Corollary 4.2.2 we have

$$I_*(A) \preceq I_*(B),$$

which gives the result. □

5.3 *PI* algebras

We compute the isoperimetric profile of finitely generated prime *PI* algebras.

Proposition 5.3.1. *If A is a finitely generated prime PI algebra, then $I_*(A) \sim n^{\frac{d-1}{d}}$, where $d = GK\dim A$.*

Proof. A theorem of Berele says that a finitely generated *PI* algebra has finite *GK*-dimension (see [27], 10.7).

Suppose that A is a finitely generated prime *PI* algebra, and consider its quotient algebra Q, which is known to be a full matrix algebra over a division algebra D, which is a finite module over its center F. Clearly $d = GK\dim F$, hence the result follows from Proposition 4.5.1 (2). □

We have also the following

Corollary 5.3.2. *If A is a finitely generated semiprime PI algebra, then $I_*(A) \preceq n^{\frac{d-1}{d}}$, where $d = GK\dim A$.*

Proof. The proof of this corollary goes like the one of the previous proposition. In this case Q is a direct sum of full matrix algebras over division algebras, which are finitely generated over their centers. Hence the same argument we used before together with Proposition 4.1.2 and well known properties of the *GK*-dimension gives the result. □

Notice that in the semiprime case we have a direct sum of subalgebras, hence Proposition 4.1.2 shows that in general we don't have the equivalence.

5.4 Universal enveloping algebras

We compute the isoperimetric profile of universal enveloping algebras of finite dimensional Lie algebras.

Proposition 5.4.1. *The isoperimetric profile of the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ is $I_*(n;\mathscr{U}(\mathfrak{g})) \sim n^{\frac{d-1}{d}}$, where $d = \dim\mathfrak{g}$.*

Proof. Theorem 4.7.1 applies to the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. Since $gr(\mathscr{U}(\mathfrak{g}))$ (with respect to the natural filtration) is isomorphic to the algebra of polynomials in $d = \dim \mathfrak{g}$ variables, we have the lower bound

$$I_*(n;\mathscr{U}(\mathfrak{g})) \succeq I_*(n;gr(\mathscr{U}(\mathfrak{g}))) \sim n^{\frac{d-1}{d}}.$$

Now consider a basis $e_1, e_2, \ldots, e_d$ of $\mathfrak{g}$, fix the order $e_1 < e_2 < \cdots < e_d$ and consider the lexicographical order on the monomials in the e_i's in $\mathscr{U}(\mathfrak{g})$. For any $n \in \mathbb{N}$ consider the subspace $V_n = span_K\{e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} \mid \text{for all } i\ 0 \leq m_i \leq n-1\}$. If we call $\mathscr{U}_1 = span_K\{1, e_1, \ldots, e_d\}$, it follows from the definition of $\mathscr{U}(\mathfrak{g})$ and the PBW theorem that a basis of the boundary $\partial_{\mathscr{U}_1}(V_n)$ is given by the classes of the monomials $e_1^{k_1} e_2^{k_2} \cdots e_d^{k_d}$ such that exactly one of the k_i's is equal to n and all the other are smaller then n. Now $|V_n| = n^d$ and $|\partial_{\mathscr{U}_1}(V_n)| = dn^{d-1} = d|V_n|^{\frac{d-1}{d}}$. From this follows easily the upper bound we needed. □

We want to derive also some consequences in the infinite dimensional case.

Proposition 5.4.2. *If $A = \mathscr{U}(\mathfrak{g})$ is the universal enveloping algebra of an infinite dimensional Lie algebra $\mathfrak{g}$, then for any $0 < \alpha < 1$ there exists a subframe $V \subset \mathscr{U}_1$ such that*

$$I_*(n;\mathscr{U}(\mathfrak{g}), V) \succneqq n^{\alpha}.$$

Proof. A basis of $\mathscr{U}_1$ is given by a basis of $\mathfrak{g}$ and 1. Now $gr(\mathscr{U}(\mathfrak{g}))$ is isomorphic to the polynomial algebra $K[x_1, x_2, \ldots]$ on infinitely many variables, where each variable x_i corresponds to a basis element of $\mathfrak{g}$.

Suppose first that $V = V_d \subset \mathscr{U}_1$, where a basis for V_d is given by the basis elements of $\mathscr{U}_1$ corresponding to $1, x_1, \ldots, x_d$. Then by Theorem 4.7.1

$$I_*(n, gr(A), V) \preceq I_*(n; A, V).$$

But by Proposition 4.4.1, since we can see $gr(A) \cong K[x_1, x_2, \ldots]$ as a free $K[V] \equiv K[x_1, \ldots, x_d]$-module, it follows that $I_*(n, gr(A), V) \succeq I_*(n; K[x_1, \ldots, x_d], V) \sim n^{\frac{d-1}{d}}$. It's easy to see by considering the cubes in the $x_1, \ldots, x_d$ as usual (and it follows also from Proposition 4.9.1) that $I_*(n, gr(A), V) \preceq n^{\frac{d-1}{d}}$, and hence $I_*(n, gr(A), V) \sim n^{\frac{d-1}{d}}$. From this the result easily follows. □

This proposition implies for example that for a finitely generated infinite dimensional Lie algebra (e.g. affine Kac-Moody algebras), its universal enveloping algebras has an isoperimetric profile faster then any polynomial in n of degree $\alpha < 1$.

5.5 Weyl algebras

Consider now the Weyl algebra $A_d = A_d(K)$, i.e. the algebra $K\langle x_1, \dots, x_d, y_1, \dots, y_d\rangle$ subject to the relations

$$[x_i, x_j] = 0 = [y_i, y_j] \quad \text{and} \quad [x_i, y_j] = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker symbol. It is well known that A_d is a domain.

Proposition 5.5.1. *The isoperimetric profile of the Weyl algebra A_d is*

$$I_*(n; A_d) \sim n^{\frac{2d-1}{2d}}.$$

Proof. The lower bound $n^{\frac{2d-1}{2d}} \preceq I_*(n; A_d)$ is given by Theorem 4.7.1, since $gr(A_d)$ (with respect to the filtration determined by total degree) is isomorphic to the algebra of polynomials $K[x_1, \dots, x_d, y_1, \dots, y_d]$.

Now for any $n \in \mathbb{N}$ consider the subspace $V_n = span_K\{x_1^{m_1} \cdots x_d^{m_d} y_1^{m_{d+1}} \cdots y_d^{m_{2d}} \mid$ for all i $0 \le m_i \le n-1\}$. It's easy to see that a basis for A_d is given by the monomials of the form $x_1^{m_1} \cdots x_d^{m_d} y_1^{m_{d+1}} \cdots y_d^{m_{2d}}$. Calling $V = span_K\{x_1, \dots, x_d, y_1, \dots, y_d\}$, it's clear that a basis for $\partial_V(V_n)$ is given by the classes of the monomials $x_1^{k_1} \cdots x_d^{k_d} y_1^{k_{d+1}} \cdots y_d^{k_{2d}}$ such that exactly one of the k_i's is equal to n and all the other are smaller then n. Now $|V_n| = n^{2d}$ and $|\partial_V(V_n)| = 2dn^{2d-1} = 2d|V_n|^{\frac{2d-1}{2d}}$. From this it follows easily the upper bound we needed. □

5.6 Quantized algebras

In this subsection we compute the isoperimetric profile of some quantized algebras related to quantum groups.

We start with quantum skew polynomial algebras. Let $\{p_{ij} \mid 1 \le i < j \le d\}$ be a set of nonzero scalars in K. The *quantum skew polynomial algebra* $K_{p_{ij}}[x_1, \dots, x_d]$

is generated by the variables $x_1, \dots, x_d$ subject to the relations $x_j x_i = p_{ij} x_i x_j$ for all $i < j$. The set of ordered monomials $\{x_1^{l_1} \cdots x_d^{l_d} \mid (l_1, \dots, l_d) \in \mathbb{N}^d\}$ is a basis over K of $K_{p_{ij}}[x_1, \dots, x_d]$. In [44], Example 7.1, Zhang gives a valuation from $K_{p_{ij}}[x_1, \dots, x_d]$ to $K[x_1, \dots, x_d]$, hence by Theorem 4.7.2 we have

$$I_*(n; K_{p_{ij}}[x_1, \dots, x_d]) \succeq I_*(n; K[x_1, \dots, x_d]) \sim n^{\frac{d-1}{d}}.$$

Consider now the subspaces $V_n := span_K\{x_1^{m_1} \cdots x_d^{m_d} \mid \text{ for all } i \;\; 0 \leq m_i \leq n-1\}$ corresponding to the cubes in $\mathbb{Z}^d_{\geq 0}$, and let $V = span_K\{1, x_1, \dots, x_d\}$. Clearly $|V_n| = n^d$, and from the defining relations it follows that $|\partial_V(V_n)| = dn^{d-1} = d|V_n|^{\frac{d-1}{d}}$ (see the proof of Corollary 5.5.1). From this it follows easily the upper bound

$$I_*(n; K_{p_{ij}}[x_1, \dots, x_d]) \preceq I_*(n; K[x_1, \dots, x_d]) \sim n^{\frac{d-1}{d}},$$

giving

$$I_*(n; K_{p_{ij}}[x_1, \dots, x_d]) \sim n^{\frac{d-1}{d}}.$$

The following definition is in [44], Section 7.

Definition. Consider the lexicographical order on $\mathbb{Z}^d$ with $\deg(e_i) < \deg(e_j)$ for $i < j$, where e_i is the vector with 1 in the i-th position, and 0 elsewhere. An algebra A is called a *filtered skew polynomial algebra in d variables* if there is a set of generators $\{x_1, \dots, x_d\}$ of A such that the following three conditions hold.

(q1) The set of monomials $\{x_1^{l_1} \cdots x_d^{l_d} \mid (l_1, \dots, l_d) \in \mathbb{N}^d\}$ is a basis over K of A. We define $\deg(x_1^{l_1} \cdots x_d^{l_d}) = (l_1, \dots, l_d)$ and $F_{(l_1, \dots, l_d)}$ to be the set of all linear combinations of monomials of degree $\leq (l_1, \dots, l_d)$.

(q2) $\{F_{(l_1, \dots, l_d)} \mid (l_1, \dots, l_d) \in \mathbb{N}^d\}$ is a filtration of A.

(q3) The associated graded algebra $gr(A)$ is isomorphic to a quantum skew polynomial algebra.

For example it's easy to see that the Weyl algebras are filtered skew polynomial algebras.

The following proposition is an immediate consequence of Theorem 4.7.1 and what we have shown before.

Proposition 5.6.1. *If A is a filtered skew polynomial algebra in d variables, then*

$$I_*(n;A) \succeq n^{\frac{d-1}{d}}.$$

Now we want to consider the quantum matrix algebras $M_{q,p_{ij}}(d)$ and the quantum groups $GL_{q,p_{ij}}(d)$. See [1] for details on these algebras.

Given a set of nonzero scalars $\{q\} \cup \{p_{ij} \mid 1 \leq i < j \leq d\}$, the *quantum matrix algebra* $M_{q,p_{ij}}(d)$ is generated by $\{x_{ij} \mid 1 \leq i, j \leq d\}$ subject to the relations (7.4.1) of [44, p. 2885]. It's easy to show (cf. [44], Example 7.4) that $M_{q,p_{ij}}(d)$ is a filtered skew polynomial algebra on d^2 variables, hence by Proposition 5.6.1

$$I_*(n;M_{q,p_{ij}}(d)) \succeq n^{\frac{d^2-1}{d^2}}.$$

To prove the other inequality, for each $n \in \mathbb{N}$ we define the subspace

$$V_n := span_K\{x_{11}^{m_{11}} x_{12}^{m_{12}} \cdots x_{1d}^{m_{1d}} x_{21}^{m_{21}} \cdots x_{2d}^{m_{2d}} \cdots x_{dd}^{m_{dd}} \mid \text{for all } i \text{ and } j \;\; 0 \leq m_{ij} \leq n-1\},$$

and we put $V := K + span_K\{x_{ij} \mid 1 \leq i, j \leq d\}$. Using the defining relations it's easy to show that $VV_n \subset V_{n+1}$. This would imply that

$$\begin{aligned} |\partial_V(V_n)| &= |VV_n| - |V_n| \leq |V_{n+1}| - |V_n| \\ &= (n+1)^{d^2} - n^{d^2} \sim n^{d^2-1} = |V_n|^{\frac{d^2-1}{d^2}}. \end{aligned}$$

As usual, from this it follows easily the upper bound

$$I_*(n;M_{q,p_{ij}}(d)) \preceq n^{\frac{d^2-1}{d^2}},$$

which gives

$$I_*(n;M_{q,p_{ij}}(d)) \sim n^{\frac{d^2-1}{d^2}}.$$

The *quantum group* $GL_{q,p_{ij}}(d)$ is defined to be the localization $M_{q,p_{ij}}(d)[D^{-1}]$, where D is the quantum determinant of $M_{q,p_{ij}}(d)$, and $M_{q,p_{ij}}(d)[D^{-1}]$ indicates the right localization with respect to the subset $\{D^n \mid n \in \mathbb{N}\}$. Hence by Corollary 4.2.2 we have

$$I_*(n;GL_{q,p_{ij}}(d)) \sim I_*(n;M_{q,p_{ij}}(d)) \sim n^{\frac{d^2-1}{d^2}}.$$

Consider now the quantum Weyl algebra $A_d(q, p_{ij})$ (see [17] for details on this algebras).

Given a set of nonzero scalars $\{q\} \cup \{p_{ij} \mid 1 \leq i < j \leq d\}$, the *quantum Weyl algebra* $A_d(q, p_{ij})$ is generated by $\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ subject to the relations given in [44], Example 7.5. It's easy to see (cf. [44], Example 7.5) that defining $\deg(x_i) = d+1-i$ and $\deg(y_i) = 2d+1-i$, $A_d(q, p_{ij})$ is a filtered skew polynomial algebra in $2d$ variables. Hence by Proposition 5.6.1 we have

$$I_*(n; A_d(q, p_{ij})) \succeq n^{\frac{2d-1}{2d}}.$$

To prove the other inequality, for each $n \in \mathbb{N}$ we define the subspace

$$V_n := span_K\{x_1^{m_1} \cdots x_d^{m_d} y_1^{n_1} \cdots y_d^{n_d} \mid \text{for all } i \text{ and } j \ \ 0 \leq m_i, n_j \leq n-1\},$$

and we put $V := K + span_K\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$. Again we can show that $VV_n \subset V_{n+1}$, from which it follows easily the upper bound

$$I_*(n; A_d(q, p_{ij})) \preceq n^{\frac{2d-1}{2d}},$$

which gives

$$I_*(n; A_d(q, p_{ij})) \sim n^{\frac{2d-1}{2d}}.$$

Consider now the *quantum group* $\mathscr{U}(\mathfrak{sl}_2)$ (see [25]). This is an algebra isomorphic to an algebra generated by $\{e, f', h\}$ subject to the relations (7.6.2) of [44], pag. 2887.

It's easy to see that it is a filtered skew polynomial algebra in three variables, setting $\deg(h) = (1,0,0)$, $\deg(e) = (0,1,0)$ and $\deg(f') = (0,0,1)$ (cf. [44], Example 7.6). This by Proposition (5.6.1) gives the lower bound

$$I_*(n; \mathscr{U}(\mathfrak{sl}_2)) \succeq n^{\frac{2}{3}}.$$

Now consider for each $n \in \mathbb{N}$ the subspace

$$V_n := span_K\{h^{m_1} e^{m_2} f'^{m_3} \mid 0 \leq m_1 \leq 2(n-1) \text{ and } 0 \leq m_i \leq n-1 \text{ for } i = 2,3\},$$

and let $V = span_K\{1, h, e, f'\}$. We can show that $VV_n \subseteq V_{n+1}$, from which it follows easily the upper bound

$$I_*(n; \mathscr{U}(\mathfrak{sl}_2)) \preceq n^{\frac{2}{3}},$$

which gives

$$I_*(n;\mathscr{U}(\mathfrak{sl}_2)) \sim n^{\frac{2}{3}}.$$

There is also another version of the quantum universal enveloping algebra $\mathscr{U}(\mathfrak{sl}_2)$, say $\mathscr{U}'(\mathfrak{sl}_2)$, which was studied in [26]. Given $q \in K \setminus \{0\}$, the quantum universal enveloping algebra $\mathscr{U}'(\mathfrak{sl}_2)$ is generated by $\{e, f, h\}$ subject to the relations

$$\begin{aligned} qhe - eh &= 2e, \\ hf - qfh &= -2f, \\ ef - qfe &= h + \frac{1-q}{4}h^2. \end{aligned} \tag{5.1}$$

Defining $\deg(h) = (1,0,0)$, $\deg(e) = (0,1,0)$ and $\deg(f) = (0,0,1)$, $\mathscr{U}'(\mathfrak{sl}_2)$ is a filtered skew polynomial algebra in three variables (cf. [44], Example 7.6). This by Proposition 5.6.1 gives the lower bound

$$I_*(n;\mathscr{U}'(\mathfrak{sl}_2)) \succeq n^{\frac{2}{3}}.$$

For the upper bound we can use the same subspaces V_n (where of course we replace f' with f).

We summarize the computations of this section in the following

Proposition 5.6.2. *With the notations we explained in this subsection,*

(1) $I_*(n;K_{p_{ij}}[x_1,\dots,x_d]) \sim n^{\frac{d-1}{d}}$;

(2) $I_*(n;M_{q,p_{ij}}(d)) \sim n^{\frac{d^2-1}{d^2}}$;

(3) $I_*(n;GL_{q,p_{ij}}(d)) \sim n^{\frac{d^2-1}{d^2}}$;

(4) $I_*(n;A_d(q,p_{ij})) \sim n^{\frac{2d-1}{2d}}$;

(5) $I_*(n;\mathscr{U}(\mathfrak{sl}_2)) \sim n^{\frac{2}{3}}$;

(6) $I_*(n;\mathscr{U}'(\mathfrak{sl}_2)) \sim n^{\frac{2}{3}}$.

All together the computations that we performed in this section give a proof of Theorem 5.0.2.

Part of the text of chapters 3, 4, 5 and 6 of this thesis is a modified version of "On isoperimetric profiles of algebras", D'Adderio Michele, *J. Algebra*, **322**, 2009.

Chapter 6

Relations with other invariants

In this section we compare the isoperimetric profile to some other invariants for infinite dimensional algebras.

6.1 I_* and the Følner function

Given an amenable algebra A and a subframe V of A, we define the *Følner function $F_*(n;A,V)$ with respect to* V (cf. [21]) to be the minimal dimension of a subspace W of A such that

$$|\partial_V(W)| \leq \frac{|W|}{n}.$$

Notice that this function is not defined for a nonamenable algebra.

As we did for the isoperimetric profile, we say that an algebra A has Følner function if there exists a subframe V of A such that

$$F_*(A,W) \preceq F_*(A,V)$$

for any subframe W of A. We denote this function and its asymptotic equivalence class by $F_*(A)$, and we say that a subframe V measures $F_*(A)$ if $F_*(A) \sim F_*(A,V)$.

It can be proved in the same way as we did for the isoperimetric profile that a finitely generated algebra A has Følner function, and its asymptotic behavior is measured by any frame V of A.

Notice that if n is in the image of $F_*(A,V)$, then

$$I_*(n) = I_*(|W|) \le |\partial_V(W)| \le \frac{|W|}{F_*^{-1}(|W|)}$$

for a suitable subspace W of dimension n. This would suggest the inequality

$$I_*(n) \preceq \frac{n}{F_*^{-1}(n)},$$

where $F_*^{-1}(n) := \sup\{k \mid F_*(k) \le n\}$.

Question 4. Is this inequality always true? Is it true for domains? Is it true for semigroups?

Of course there is the analogous definition for semigroups: in this case the Følner function is denoted by $F_\circ$ (cf. [21]).

In [21] there are various proofs of the lower bound for the Følner function of $\mathbb{Z}^d_{\ge 0}$, the upper bound being clear considering the cubes:

$$F_\circ(n;\mathbb{Z}^d_{\ge 0}) \sim n^d.$$

Notice that in this particular case $I_\circ(n) \sim n/F_\circ^{-1}(n)$.

Question 5. Are these two functions always equivalent? Is it true for algebras? Is it true for domains?

The equivalence $I_*(n) \sim n/F_*^{-1}(n)$ is correct at least in the case of polynomial algebras. In fact, using the fact that the Følner functions of an orderable semigroup and its semigroup algebra are asymptotically equivalent (see [21], Section 3), we have

$$F_*(n;K[x_1,\dots,x_d]) \sim n^d.$$

Sometimes the Følner function is easier to handle than the isoperimetric profile (see [13]). For example the Følner function of the tensor products has an easier relation with the Følner functions of the factors.

Proposition 6.1.1. *Given A and B two K-algebras, if V_A and V_B are two subframes of A and B respectively, and $V := V_A \otimes 1 + 1 \otimes V_B$, we have*

$$F_*\left(\frac{mn}{m+n};A\otimes_K B,V\right) \le F_*(m;A,V_A)F_*(n;B,V_B).$$

Proof. We use the proof of Proposition 4.6.1: we keep the same notation we used there, but this time we choose suitable subspaces $W \subset A$ and $Z \subset B$ for which $|\partial_{V_A}(W)| \leq |W|/m$ and $|\partial_{V_B}(Z)| \leq |Z|/n$. We get

$$\begin{aligned} |\partial_V(W \otimes Z)| &\leq |Z||\partial_{V_A}(W)| + |W||\partial_{V_B}(Z)| \\ &\leq \frac{|W||Z|}{m} + \frac{|W||Z|}{n} = \frac{m+n}{mn}|W||Z|, \end{aligned}$$

which gives the result. □

Putting $m = n$ in the proposition we get the following

Corollary 6.1.2. *In the same notation of the previous proposition,*

$$F_*(n; A \otimes_K B, V) \preceq F_*(n; A, V_A) F_*(n; B, V_B).$$

6.2 I_* and the lower transcendence degree

In [45] J. J. Zhang introduced the notion of the lower transcendence degree of an algebra.

Definition. If for every subframe $V \subset A$ there is a subspace $W \subset A$ such that

$$|\partial_V(W)| = 0,$$

then we define the *lower transcendence degree* of A to be 0 and we write $\mathrm{Ld}(A) = 0$. Otherwise there is a subframe V such that for every subspace W

$$|\partial_V(W)| \geq 1.$$

In this case the *lower transcendence degree* of A is defined to be

$$\mathrm{Ld}(A) := \sup_V \sup\{d \in \mathbb{R}_{\geq 0} \mid \exists\, C > 0 \,:\, |\partial_V(W)| \geq C|W|^{1-\frac{1}{d}} \text{ for all } W\},$$

where V ranges over all subframes of A. Hence $\mathrm{Ld}(A)$ is a nonnegative real number or infinity.

Observe that in the definition of the lower transcendence degree we can use the inequality $I_*(|W|;A,V) \succeq |W|^{1-\frac{1}{d}}$ instead of $|\partial_V(W)| \geq C|W|^{1-\frac{1}{d}}$. In the case of a finitely generated algebra, since we already showed that the asymptotic behavior of the isoperimetric profile does not depend on the frame, we can drop the first supremum in the definition and we can take simply some fixed frame V.

It's now clear from the definitions that if two algebras A and B satisfy $I_*(A) \sim I_*(B)$, then $\mathrm{Ld}(A) = \mathrm{Ld}(B)$. The converse is not always true:

Remark 10. In general we do not have the inequality

$$n^{1-\frac{1}{\mathrm{Ld}(A)}} \preceq I_*(n). \tag{6.1}$$

For example in the case $I_*(n) \sim n^{\alpha}/\log n$ for some $0 < \alpha \leq 1$, we would have

$$n^{\beta} \precneqq I_*(n)$$

for any $\beta < \alpha$, but

$$n^{\gamma} \succneqq I_*(n)$$

for any $\gamma \geq \alpha$. For example, $I_*(n) \sim n/\log n$ ($\alpha = 1$) is the isoperimetric profile of the group algebra of a finitely generated polycyclic group of exponential growth (see [33]). Hence $(\mathrm{Ld}(A)-1)/\mathrm{Ld}(A) = \alpha$ in this case, which shows that the inequality is not true.

From this remark we see that if we have for example two algebras A and B with $I_*(n;A) \sim n/\log n$ and $I_*(n;B) \sim n$ (e.g. the group algebra of a finitely generated polycyclic group of exponential growth and a free algebra of rank two), then clearly $I_*(A) \nsim I_*(B)$, but $\mathrm{Ld}(A) = \mathrm{Ld}(B) = \infty$. All this shows that the isoperimetric profile is finer than the lower transcendence degree as an invariant for algebras.

With these observations and all the tools that we have developed we see that this invariant behaves well with localizations. This makes it a good (probably the best known) "trancendence degree" of division algebras infinite dimensional over their centers, whose absence have been a major obstacle to the study of such rings.

The following proposition follows directly from the definitions

Proposition 6.2.1. *If $d = \mathrm{Ld}(A)$, then $n^{\frac{s-1}{s}} \precneqq I_*(n;A,V)$ for any $s \lneqq d$ and some particular subframe $V \subset A$. Moreover, $I_*(n;A,W) \not\succeq n^{\frac{t-1}{t}}$ for any $t > d$ and any subframe $W \subset A$.*

In [45], Proposition 1.4, Zhang proves that for any algebra A,

$$\mathrm{Ld} A \leq \mathrm{Tdeg} A \leq GK\dim A,$$

where Tdeg$A$ is the Gelfand-Kirillov transcendence degree (see [45] for the definition).

This together with Proposition 6.2.1 implies the following theorem, which generalizes a result in [11].

Theorem 6.2.2. *If all the finitely generated subalgebras of an algebra A have finite lower transcendence degree, then A is amenable.*

An example of a finitely generated amenable division algebra with infinite GK-transcendence degree is given in [11]. Theorem 6.2.2 together with previous results in this paper allows us to provide new examples of this sort.

An easy example is the field $F := K(x_1, x_2, \dots)$ of rational functions in infinitely many variables.

Even more interesting examples come from universal enveloping algebras of infinite dimensional Lie algebras with subexponential growth, for example affine Kac-Moody algebras. In fact by [37] these algebras have subexponential growth, and so they are amenable (see [10]). But from Proposition 5.4.2 it follows that they have infinite lower transcendence degree. Since they are domains, we can consider their quotient division algebras to provide examples of division algebras.

In [45, p. 181], Zhang asked if is it true that for any orderable semigroup Γ the semigroup algebra $K\Gamma$ is Ld-stable, i.e. $\mathrm{Ld} K\Gamma = GK\dim K\Gamma$. We conclude the subsection giving a positive answer:

Proposition 6.2.3. *The group algebra $K\Gamma$ of an ordered semigroup Γ is* Ld-*stable.*

Proof. By a theorem of Gromov (see [21], Section 3) we know that $I_\circ(\Gamma, S) \sim I_*(K\Gamma, S)$ for any finite subset $S \subset \Gamma$. Observe that $d := GK\dim K\Gamma$ is the degree of growth of the semigroup Γ, which may be of course infinity. Now by the Couhlon-Saloff-Coste inequality (Theorem 3.3.1) we have

$$I_*(n;\Gamma) \succeq n^{\frac{d-1}{d}},$$

in case d is finite, or

$$I_*(n;\Gamma) \succeq n/\Phi(n),$$

where Φ is the inverse function of the growth of Γ, if d is infinity. In the last case Φ is slower then any positive power of n, hence in both cases

$$\mathrm{Ld} K\Gamma \geq GK \dim K\Gamma.$$

Since the other inequality is always true, this completes the proof. □

6.3 I_* and the growth

The Weyl algebra A_1 and its quotient division algebra D_1 give an example that shows that the isoperimetric profile is not a finer invariant then the GK-dimension. Another example is in [27], Example 4.10, where the algebra $\mathscr{U}(\mathfrak{g})$ and some its localization have different GK-dimensions, but they have the same isoperimetric profiles.

We may ask for an analogue of the Coulhon-Saloff-Coste inequality (Theorem 3.3.1) for algebras. In Remark 9 we considered the algebra $A = K\langle x,y\rangle/J$, where J is the ideal generated by all monomials in x and y containing at least 2 y's. We already showed that this algebra has constant isoperimetric profile, but it has GK-dimension 2. This example shows that we don't have in general an analogue for algebras of the Coulhon-Saloff-Coste inequality. A cheaper example of this type is the algebra $K[x] \oplus K\langle y,z\rangle$, which we also considered in the Remark 9. Both these examples are not domains.

An example of a prime algebra is Example 5.1.1. An example of a domain is given by the quotient division algebra D_1 of the Weyl algebra A_1.

In [21], Section 1.9, Gromov asks if there is a bound on the growth of a domain by its Følner function. Keeping in mind Questions 4 and 5, this bound would correspond to the Coulhon-Saloff-Coste inequality for the isoperimetric profile. The algebra D_1 answers this question in the negative, since in this case clearly the Følner function $F_*(n)$ of D_1 is asymptotically bounded by n^2, but D_1 grows exponentially. Of course D_1 is not finitely generated.

A finitely generated example is given by the localization $A_1\Omega^{-1}$ of the multiplicative closed subset Ω (of the Weyl algebra A_1) generated by x and y. This is a finitely generated noetherian domain with GK-dimension 3 but with lower transcendence degree 2 (see Example 4.11 in [27] for details).

6.4 A conjecture

Let A be a domain, and let $D \subset A$ be a subalgebra of A which is a division algebra (over our base field K). We can see A as a (right) algebra over D. In this case we will use the notation A_D. In a natural way we can define all our invariants over a D instead over K: first of all, given a subset $S \subset A$, we define

$$|S|_D := \text{ right span of } S \text{ over } D.$$

Hence, given any subframe $V \subset A$, and a subspace $W \subset A$, we can define

$$|\partial_V(W)|_D := |VW|_D - |W|_D.$$

With this we can clearly define the notions of isoperimetric profile, Følner function and lower transcendence degree of A over D, and we denote them $I_*(n;A_D)$, $F_*(n;A_D)$ and $\mathrm{Ld}_D(A)$ respectively.

We formulate our conjecture:

Conjecture 1. *Let A be a domain, and let $D \subset A$ be a subalgebra of A which is a division algebra. Then*

$$F_*(n;A) \succeq F_*(n;D) \cdot F_*(n;A_D).$$

To justify our conjecture, we show how this conjecture implies a well known conjecture of Zhang (see [45]).

Lemma 6.4.1. *For simplicity, let's assume that A is finitely generated. Then*

$$\mathrm{Ld}(A) = \sup\{d \in \mathbb{R}, d \geq 1 \mid \exists c > 0 \,:\, F_*(n;A) \geq c \cdot n^d\}.$$

Proof. Suppose that $I_*(n;A) \succeq n^{1-\frac{1}{d}}$ for some $d \in \mathbb{R}$, $d \geq 1$. Hence there exists $b > 0$ such that for all $N \in \mathbb{N}$

$$I_*(N) \geq b \cdot N^{1-\frac{1}{d}}.$$

Also, assume that $I_*(N;A) \leq N/n$; therefore

$$b \cdot N^{1-\frac{1}{d}} \leq I_*(N) \leq N/n,$$

hence

$$N \geq b^d \cdot n^d,$$

i.e. $F_*(n;A) \geq c \cdot n^d$ for $c = b^d$.

Suppose now that for some $d \in \mathbb{R}$, $d \geq 1$, we have a $c > 0$ such that $F_*(n;A) \geq c \cdot n^d$ for all $n \in \mathbb{N}$. By contradiction, if we assume that $I_*(n;A) \not\succeq n^{1-\frac{1}{d}}$, then for every $r \in \mathbb{N}$ we can find an N_r such that

$$I_*(N_r;A) \lneq \frac{1}{r} \cdot (N_r)^{1-\frac{1}{d}} = \frac{N_r}{r \cdot (N_r)^{1/d}}.$$

Hence, by the definition of Følner function,

$$F_*(r \cdot (N_r)^{1/d};A) \leq N_r.$$

This implies

$$N_r \geq c(r \cdot (N_r)^{1/d})^d = cr^d N_r.$$

For r big enough this gives a contradiction. □

The same argument works for the case of Ld_D.

The lemma immediately implies that our conjecture implies the following conjecture of Zhang (see [45]):

Conjecture 2 (Zhang)**.** *Let A be a domain, and let $D \subset A$ be a subalgebra of A which is a division algebra. Then*

$$\mathrm{Ld} A \geq \mathrm{Ld} D + \mathrm{Ld}_D A.$$

This conjecture has many interesting consequences in noncommutative ring theory, especially its part related to noncommutative projective algebraic geometry (see [45] for details). Aside from a new proof of the Artin-Stafford gap (proved by Smoktunowicz), it would imply the following well known conjectures due to Lance Small and Michael Artin.

Conjecture 3 (Small)**.** *Let A be an Ore domain which is not locally PI and F a commutative subalgebra of the quotient division algebra of A. Then $GK\dim F \leq GK\dim A - 1$.*

This conjecture has been recently proved by Jason Bell in [4].

Conjecture 4 (Artin)**.** *Let D be a division algebra over an algebraically closed field k with $GK\dim D > 1$. Then* $\mathrm{Ld}D \geq 2$.

Part of the text of chapters 3, 4, 5 and 6 of this thesis is a modified version of "On isoperimetric profiles of algebras", D'Adderio Michele, *J. Algebra*, **322**, 2009.

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