BRANCHED COVERS OF CONTACT MANIFOLDS

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to my husband

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Chapter I

INTRODUCTION

This thesis focuses on questions in low-dimemsional topology, contact geometry, and knot theory. We want to understand contact structures via branched covering maps. Contact structures originally arose from areas of physics, but recently they have been seen to have mathematical beauty in their own right and are now being studied by low-dimensional topologists. Topologists are interested in the characteristics, construction, and classification of contact structures. In particular, given known topological constructions and results, one could ask what generalizations can be made to the case of contact manifolds. One such construction is branched covers. In the past 50 years, topologists have proven many amazing results about branched covers and 3-manifolds, and recently much attention has been given to the interaction of these covers with contact structures. Our goal is to better understand branched covers of 3-manifolds and contact manifolds.

A map $p : M \to N$ is called a *branched covering* if there exists a co-dimension 2 subcomplex L such that $p^{-1}(L)$ is a co-dimension 2 subcomplex and $p|_{M-p^{-1}(L)}$ is a covering. We will study here manifolds of dimension 2 or 3. Essentially, a branched covering is a map between manifolds such that away from a set of codimension 2 (called the branch locus) p is a honest covering.

Recall a *contact structure* ξ on an oriented 3-manifold M is a non-integrable plane field in the tangent bundle of M. Branching over knots transverse to the contact structure (i.e. transversal knots) we can pull back the contact structure downstairs to obtain contact structures on the upstairs manifold. Bennequin proved that any link transverse to the standard contact structure in S^3 is transversally isotopic to a

closed braid so often we will think of a transversal link in terms of its braid word [3].

For covers of simply connected spaces, a convenient technique for describing a branched covering map is that of coloring the branch locus, which is defined in Chapter 3. Essentially, a coloring is an assignment to the branch locus of an element of the symmetric group which determines (and is determined by) the covering map. In Chapter 3 we use colorings to prove results on the construction of branched coverings for surfaces and three-manifolds.

The real substance to the subject of branched covers of contact manifolds came in 2002 when Giroux proved the following fundamental theorem: Every contact manifold is a 3-fold simple cover over S^3 branching along some transverse link. The following theorem, proven in Chapter 3, is a strengthening of Giroux's result to a connected branch locus.

Theorem 1.0.1. *Given a contact manifold (M, ξ), there exists a 3-fold simple cover $p : (M, \xi) \to (S^3, \xi_{std})$ whose branch locus is a knot.*

A link L in $(S^3$ is called *universal* if every 3-manifold can be seen as the branched cover over L. Known universal links include the figure-eight knot, Borromean rings, and Whitehead link, see [18]. We call a transversal link L in (S^3, ξ_{std}) *contact universal* if every contact manifold is a branched cover over L. As any such transversal link would also have to be topologically universal, one would want to look at tranversal links that are topologically universal and study lifts of ξ_{std} branching along that link.

Theorem 1.0.2. *Any transversal link that destabilizes is not contact universal.*

Thus for any link which is topologically universal, we must choose a transversal presentation which does not destablize to test for contact universality. This is particularly helpful for the figure-eight knot because Etnyre and Honda showed that the only transversal figure-eight knot which does not destablize is the one described by

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the braid word $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$. We want to determine if every contact manifold can be obtained by branching over this knot.

Harvey, Kawamuro, and Plamenevskaya showed that for any transverse braid $L \subset (S^3, \xi_{std})$ with braid word ω, if for some i, ω contains σ_i^{-1} and not σ_i then every cyclic cover branching along L is an overtwisted manifold. The figure-eight knot, Borromean rings, and Whitehead link all meet this conditions and therefore any cyclic cover branching along any one of these figure-eight knots is overtwisted. We can strengthen their result slightly to the following theorem, which will yield the result that any fully ramified cover branching alone the figure-eight knot, Borromean rings, or Whitehead link will be overtwisted.

Proposition 1.0.3. *If K is transverse knot in (S^3, ξ_{std}) whose braid word contains a σ_i^{-1} and no σ_i for some i then any fully ramified cover branching over K is overtwisted.*

If one of these topologically universal knots is going to be contact universal then a minimal condition would be that tight contact structures can be obtained by branching along the knot. We focus first on the figure-eight knot. One method for determining if a contact structure is overtwisted is the theory of right-veering curves. In 2007 Honda, Kazez, and Matic defined a property of a diffeomorphism called *right-veering*, which indicates whether curves are taken to the right or to the left under the map. If a monodromy for an open book decomposition of a contact manifold takes any curve to the left, then the contact structure is overtwisted. (Open book decompositions of manifolds are discussed in more detail in the next section, but for now imagine any cover of a braid downstairs determines a map, called the monodromy, upstairs.) Using this principle and some nice properties of the figure-eight knot we are able to prove the next theorem.

Theorem 1.0.4. *Every cover of (S^3, ξ_{std}) branching over the figure-eight knot is overtwisted.*

So the figure-eight knot cannot be a contact universal knot as it cannot yield any tight contact structure.

This result is special to the figure-eight knot, and not a property of knots and links whose braid word contains a σ_i^{-1} and no σ_i for some i, as we see with our next result.

Theorem 1.0.5. *Let L in (S^3, ξ_{std}) be the transverse Whitehead link with braid word $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-2}$. There exist covers branching over L that are tight.*

To pin down concrete results about the behavior of branched covers of 3-manifolds, much more needs to be understood about their construction. To do so we will utilize open book decompositions, which are defined below. It is known that every 3-manifold has an open book decomposition. Furthermore, due to the celebrated Giroux correspondence, the study of contact structures up to isomorphism is equivalent to studying open book decompositions up to stabilization. Thus, open books are important because they have immediate applications not only to low-dimensional topology but at the same time to contact geometry.

Given a link K in S^3 (with the standard contact structure if interested in contact manifolds) we want to construct open book decompositions for manifolds obtained by branching along K. Start with the open book decomposition (D^2, id) of S^3. We can consider K as a link braided transversally through the pages. We want to constuct an open book decoposition for the covering manifold. In the case that the cover is cyclic, [17] give an algorithm for doing so, but no algorithm exists for the general case.

Given a general 3-manifold M and open book (Σ, ϕ), covers could be constructed by either branching along a link transverse to the pages or by branching along the binding. Though cyclic covers branched along the binding of the open book decomposition are reasonbly well understood, but almost no work has been done in the non-cyclic case. If branching along a transversal knot, it would be helpful to have a

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method to compute these properties for the covering manifold given the information about the manifold downstairs.

One property of particular interest is overtwistedness. If a contact manifold is overtwisted, any non-branched cover would also be overtwisted. Is the same true of branched covers of overtwisted manifolds? Or, if not always true, would it be true for some class of manifolds? The answer is no.

Theorem 1.0.6. *Given any contact manifold (M, ξ) with ξ overtwisted, there exists a trasversal knot $K \in (M, \xi)$ and integer N such that any n-fold cyclic cover branching along K $(n > N)$ is tight.*

This thesis is organized as follows: Chapter 2 presents basic definitions and theorems in contact geometry. Chapter 3 gives an introduction to branched coverings, including detailed constructions, fundamental theorems, and some new work in topological branched covers. Chapter 4 is devoted to proving our main results.

Chapter II

CONTACT GEOMETRY BACKGROUND

Contact structures have been used in many areas of physics and mathematics in the past twenty years. Some important results whose proofs involve contact structures include proving the Property P Conjecture [21] (which had been outstanding 30 years), giving a surgery characterization of the unknot [25], figure-eight, and trefoil [26], and proving that Heegaard floer homology detects fibered knots [23]. Knots and links in contact structures are also very important, and useful for understanding much about the behavior of the structure and for constructing contact manifolds via surgery and, as we will see in the next chapter, branched covers. One way we study branched covers of manifolds is via open book decompositions. In this chapter we will introduce all of these ideas more carefully and give many examples.

2.1   Contact Structures

This section introduces contact structures and important associated terminology. After giving the basic definitions and examples, we will discuss what is known of their classification.

2.1.1   Basic Definitions and Examples

An oriented 2-plane field ξ on a 3-manifold M is called a *contact structure* if there exists a 1-form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge d\alpha > 0$. Such a ξ is totally non-integrable, and thus there is no embedded surface in M which is tangent to ξ on any open neighborhood. A 3-manifold equipped with a contact structure ξ is called a *contact manifold.*

It will be helpful to establish a few examples we can reference throughout the

paper.
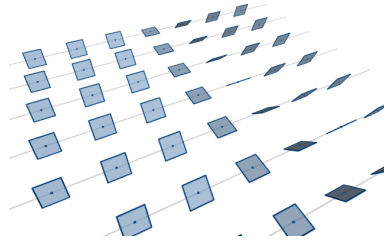
Figure 1: The standard contact structure on $\mathbb{R}^3$ (Picture by Patrick Massot)

Example 2.1.1. Let $M = \mathbb{R}^3$ and $\xi_{std} = \ker(dz - y\,dx)$ where we are using Cartesian coordinates (x, y, z) on $\mathbb{R}^3$. Notice that the plane fields are parallel to the xy-plane when $y = 0$ and moving along any ray perpendicular to the xy-plane the plane field will always be tangent to this ray and rotate by $\pi/2$ in a left handed manner as the ray is traversed. See Figure 1.
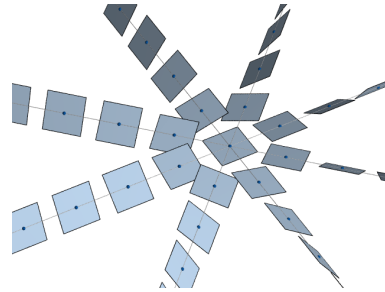
Figure 2: Symmetric Contact Structure on $\mathbb{R}^3$ (Picture by Patrick Massot)

Example 2.1.2. Let $M = \mathbb{R}^3$ and $\xi_{sym} = \ker(dz + r^2 d\theta)$ where we are using cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$. As you move out along any ray perpendicular to the z-axis the contact planes twist clockwise. At the z-axis the contact planes are horizontal. See Figure 2

Example 2.1.3. Let $M = \mathbb{R}^3$ and $\xi_{OT} = \ker(\cos(r)dz + r\sin(r)d\theta)$ where we are using cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$. See Figure 3.
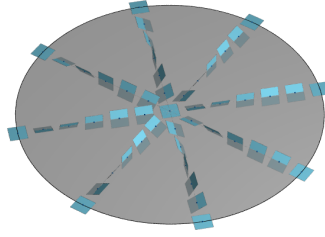
Figure 3: Overtwisted Contact Structure on $\mathbb{R}^3$

Topologists are interested in classification of objects. For example, consider closed orientable surfaces. Every such surface is homeomorphic to a sphere with n holes (i.e. of genus n), and two closed orientable surfaces are homeomorphic if and only if they have the same number of holes. Computationally, two surfaces are homeomorphic if and only if they have the same Euler Characteristic. So we have a classification of closed, orientable surfaces up to homeomorphism and an invariant to determine when two are the same.

Another important set of objects, mentioned above, for which we have a classification is 2-plane fields on closed, oriented 3-manifolds. The theorem is stated below, but first we should give some explanation of the notation. First, Γ is a 2-dimensional invariant of ξ and gives a map Γ_ξ from the group of spin structures on M to a group G in $H_1(M; \mathbb{Z})$. This invariant refines the Euler class because $2\Gamma(\xi, s) = e(\xi)$ where $e(\xi)$ denotes the Euler class. And $\theta(\xi)$ is a rational number which is a 3-dimensional invariant of ξ. For more details on these invariants and a proof of the theorem see [15].

Theorem 2.1.4. *Let ξ_1 and ξ_2 be two 2-plane fields on a closed rational homology 3-sphere. If $e(\xi_0)$ is a torsion class then ξ_1 and ξ_2 are homotopic if and only if, for some choice of spin structure s, $\Gamma(\xi_1, s) = \Gamma(\xi_2, s)$ and $\theta(\xi_1) = \theta(\xi_2)$.*

Remark 2.1.5. There is a similar theorem for general 3-manifolds but the associated invariants are more complicated.

Therefore we have a complete classification of plane fields up to homotopy via invariants that can tell them apart.

With classification being such an important question, it is natural that after defining contact structures, we would immediately ask for a classification. But first we must determine what we want it to mean for two contact structures to be *the same*. The two most commonly used definitions are that they are isotopic through contact structures, and (though less strong) that they are *contactomorphic*. Two contact manifold (M_1, ξ_1) and (M_2, ξ_2) are said to be contactomorphic if there is a diffeomorphism $f : M_1 \to M_2$ with $Tf(\xi_1) = \xi_2$, where $Tf : TM_1 \to TM_2$ denotes the differential of f. Unless otherwise specified, we will always be working up to contactomorphism in this paper.

Theorem 2.1.6. *[16] (Gray's Theorem) Let M be an oriented $(2n+1)$-dimensional manifold and ξ_t, $t \in [0,1]$ a family of contact structures on M that agree off of some compact subset of M. Then there is a family of diffeomorphisms $f_t : M \to M$ such that $(f_t)_* \xi_t = \xi_0$.*

Notice that Gray's theorem tells us that on a compact manifold M, two isotopic contact structures are also contactomorphic: Let ξ, ξ' be isotopic contact structures on a compact manifold M and ξ_t $t \in [0,1]$, the isotopy between them. Gray's theorem gives a diffeomorphism f_t such that $(f_t)_* \xi_t = \xi_0 = \xi$. Thus $(f_1)_* \xi' = (f_1)_* \xi_1 = \xi$ and we see ξ and ξ' are contactomorphic.

While it does not seem reasonable to completely classify contact structures at this point, we would like to find invariants to determine when two contact structures are different. Recall from above that contact structures on closed orientable 3-manifolds are plane fields. Therefore, the invariants of plane fields discussed above give invariants of contact structures and hence can be used to tell when two contact structures are not the same. If the invariants Γ and θ of two contact structures are the same

then we can only conclude that they are homotopic as plane fields but not necessarily through contact structures. Thus the contact structure might or might not be contatomorphic.

We will discuss this through the examples mentioned above. We can find a diffeomorphism of $\mathbb{R}^3$ taking ξ_{std} to ξ_{sym} which makes them contactomorphic. So in some sense they are the same contact structure, but sometimes one is easier to work with than the other. However, they are not contactomorphic to the structure labled OT (see [3] for proof).

Theorem 2.1.7. *[3] (Bennequin) The contact structure ξ_{std} is not contactomorphic to ξ_{OT}.*

Local Model. One important fact to note before we move on is the local model for contact structure. First we state Darboux's theorem in contact geometry.

Theorem 2.1.8. *[6] (Darboux) Suppose ξ_i is a contact structure on the manifold M_i, $i = 0, 1$, of the same dimension. Given any points p_0 and p_1 in M_0 and M_1, respectively, there are neighborhoods N_i of p_i in M_i and a contactomorphism from $(N_0, \xi_0|_{N_0})$ to $(N_1, \xi_1|_{N_1})$. Moreover, if α_i is a contact form for ξ_i near p_i then the contactomorphism can be chosen to pull α_1 back to α_0.*

This says that any point in any contact 3-manifold has a neighborhood that can be identified with the standard contact structure on an open ball in $\mathbb{R}^3$. For this reason, when we are only interested in local behavior, we will often focus on the case of $(\mathbb{R}^3, \xi_{std})$.

2.1.2   Tight and Overtwisted Contact Structures

In Example 2.1.3, look at the following disk:

$$D = \{(r.\theta, z) | z = 0, r \leq \pi\}.$$

The disk D is tangent to ξ_{OT} along the boundary. Any contact structure is called *overtwisted* if such an embedded disk exists, and *tight* otherwise.

Clearly every contact structure is either tight or overtwisted be definition. The usefulless of dividing contact structures into these two classes is not immediately clear, but we will show why this is a helpful definition to have. Recall that we are interested in classification of contact structures, and we have an invariant which can determine if two contact structures are homotopic through plane fields, but none (yet) that can determine if they are contactomorphic or isotopic through contact structures. In 1989 Eliashberg showed that every homotopy class of an oriented 2-plane field contains exactly one overtwisted contact structure and the classification of overtwisted contact structures on a given closed 3-manifold coincides with the homotopy classification of tangent 2-plane fields [7]. Therefore, for overtwisted contact structures, we have a classification and our invarients for 2-plane fields are complete invariants in this class as well. We then need to address tight contact structures.

Notice also that this means that every closed, oriented 3-manifold admits an overtwisted contact structure. Naturally, we might think that every 3-manifold also has a tight contact strucutre. Etnyre and Honda showed that there exists three manifolds that admit no tight contact structure [10]. So in addition to asking for a classification and invariants, we also would simply like to know which 3-manifolds even admit tight contact structures.

2.2 Links, Knots, and Braids in Contact Structures

Knot theory has applications all over mathematics: geometric group theory, algebra, mathematical physics, and many branches of topology. We will be using some of the applications in this paper so we must discuss some of the fundamentals of knots and links in contact manifolds. We will assume that the reader has a basic knowledge of knots and braids in topological manifolds, and for details see [27].

2.2.1 Transverse Knots

There are two types of knots that are studied in contact manifolds: Legendrian and transverse. We will focus on and overview transverse knots, but see [9] for a discussion of Legendrian knots and more details on transverse knots. A *transverse knot* in a contact manifold (M, ξ) is an oriented, embedded S^1 whose tangent vector at every point is transverse to ξ. Two transverse knots are transverse isotopic if there is an isotopy taking one to the other while staying transverse.

In this section we will assume our knots and links are in $(\mathbb{R}^3, \xi_{std})$. Transverse knots are pictured using front projections $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ with $(x, y, z) \mapsto (x, z)$. One can show that the front projection of a transveral knot is an immersed curve and any immursed curve in $\mathbb{R}^2$ is the front projection of a transverse knot if it satisfies two constraints: no vertical tangencies pointing down, and no double points from a positive crossing with both strands pointing down. Both of these are pictured in Figure 4.
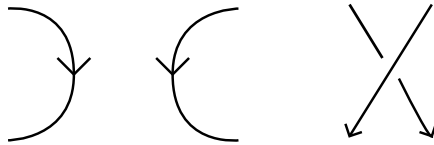
Figure 4: Segments not allowed in projections of transverse knots.

Classical Invariants. Given two transverse knots, we want to be able to tell if they are transversely isotopic. Thus, we would like an invariant we can compute that will determine when two transverse knots are not the same (and hopefully, also determine when they are the same.) Given a transverse knot T, we still have the most basic invariant - the topological knot type $\kappa(T)$. Clearly two transverse knots with different knot types cannot be transverally isotopic. This is a very weak invariant as it cannot distinguish different transverse knots of the same knot type. So we would like to find

such an invariant. For notation, denote the set of all transverse knots that realize a fixed topological knot type K by $T(K)$.

The main invariant for transverse knots is the self-linking number. To define the self-linking number of T we assume it is homologically trivial. Thus there is a surface Σ such that $\partial \Sigma = T$. The contact planes form a trivial two dimensional bundle as any orientable two plane bundle is trivial over a surface with boundary, meaning $\xi|_\Sigma$ is trivial and thus there exists a non-zero vector field V over Σ in ξ. Define a new knot T' by pushing off from T slightly in the direction of V. Now we have two transverse knots, and we compute their linking number $l(T, T')$ and this is precisely the self-linkng number of T, denoted $sl(T)$. Notice that if V were to be a non-zero vector field in $\xi \cap T\Sigma$ along T that we could extend over all of Σ then we could push T to form T' totally "above" T and thus $sl(T)$ would be 0. An alternate way to view the self linking number is to start with a vector field that points out of Σ. Then the self-linking number is the obstruction to extending V over Σ to a non-zero vector field in ξ. If $\Pi(T)$ is the front projection of T, then the $sl(T)$ is the writhe of $\Pi(T)$, (see [9]). There is a formula to compute the self-linking number of T given a braid presentaton as well, which we will see in Section 2.2.2.

Notice that this gives an invariant of transverse knots; i.e. if two transverse knots are transversally isotopic then they must have the same self-linking number. To see this, notice if two transverse knots T_0 and T_1 are transversally isotopic then there exists an isotopy $\phi_t : M \to M$ such that $(\phi_t)_* \xi = \xi$ and $\phi_t(T_0) = T_t$. Now we can use ϕ_t to isotop Σ and V (the surface and non-zero vector field used to compute the self-linking number for T_0) and use their images to compute the self-linking number for T_1. At all points along the isotopy, we can compute the self-linking number of the T_t. But this is an integer that must change continuously as t varies between 0 and 1, and thus cannot change. Therefore, T_0 and T_1 must have the same self-linking

number. However, two transverse knots in the same knot type with the same self-linking number need not be transversely isotopic. For examples, see [8]. A knot type whose transverse knots are classified by their self-linking number is called *transversely simple*.

Stabilizations. Given a transverse knot T, a *stabilization* of T will produce a transverse knot in the same knot type which is not transversely isotopic to T. Drawn as front projections, the move is pictured below. Stabilizing a transverse knot reduces the self-linking number by two.

Figure 5: Transverse Stabilization

2.2.2 Braids

For the majority of this paper we will look at links and knots as braids. Recall a closed *braid* is a knot or link in $\mathbb{R}^3$ that can be parametrized by a map $f : S^1 \to \mathbb{R}^3$ where $s \mapsto (r(s), \theta(s), z(s))$ for which $r(s)$ is not zero and $\theta'(s) > 0$. In the 1920s Alexander showed that every link in $\mathbb{R}^3$ is isotopic to a closed braid by giving an algorithm to braid any link. As we will see, braids are especially useful for constructing three-manifolds.

An open n-strand braid is a picture of n horizontal strands, oriented from left to right and labeled from bottom to top, with positive and negative crossings. A closed braid is associated to an open braid by identifying the beginning and end of the strands. A open braid is obtained from a closed braid (thought of as braided about the z-axis) by isotoping the braid so that all crossings appear below the x-axis and cutting the braid along its intersection with the half-plane $y > 0, x = 0$. An example is pictured in Figure 6.
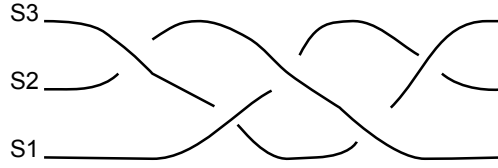
Figure 6: A 3-Braid

We denote the simple n-strand braid with one positive crossing between the ith and $(i+1)^{st}$ strands by σ_i, and similarly σ_i^{-1} if the crossing is negative. A braid can be pictured by concatenation of the braids $\sigma_i^{\pm}$, and thus we call the σ_i the the generators. This list of generators that form the braid is called the *braid word* of the braid. Any braid word uniquely defines the braid, one knot or braid may have many different braid words. For example, in Figure 6, the braid word would be $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}$.

The set of all braids on n stands form a group, called the *braid group*, and is denoted B_n [4]. The generators of the group are σ_i, $i = 1, ..., n-1$, and the group operation is conncatonation [4].

A fixed topological knot K can have many different associated braids. Alexander's theorem does not give a unique braid representation. Markov's Theorem, stated below, gives us a relationship between different braid representations of the same knot.

Theorem 2.2.1. *[22] (Markov's Theorem) Let X ,X′ be closed braid reperesentatives of the same oriented link type K in oriented 3-space. Then there exists a sequence of closed braid representatives of K:*

$$X = X_1 \to X_2 \to \cdots \to X_r = X'$$

taking such that each X_{i+1} is obtained from X_i by either (i) braid isotopy, or (ii) a single stabilization or destabilization.

By braid isotopy, we mean simply an isotopy of a closed braid, through closed braids, in the complement of the braid axis. A braid stabilization is shown for open

braids in Figure 7 and increases the braid index by 1. Going the opposite direction is called a destablilization. Notice stablilzation can be done by adding either a positive or negative crossing.
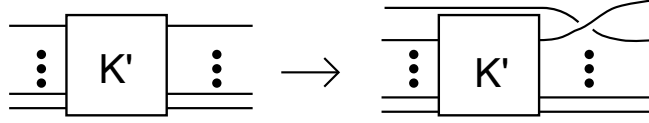
Figure 7: Braid Stablilzation

Because we will soon be focused on transverse knots in contact manifolds, we need to know how transverse knots and links work as braids. Alexander's algorithm shows that all links are isotopic to closed braids, but we need that all transverse links in $(\mathbb{R}^3, \xi_{std})$ are *transversely* isotopic to a closed braid. One might worry that a problem would arise at some point in Alexander's algorithm and a move might be made that was not transverse. Consider $\mathbb{R}^3$ with the symmetric contact structure ξ_{sym} defined in Section 1.1. Then any closed braid about z-axis can be made transverse to the contact planes by "pushing out" radially [3]. As we push out, the planes in ξ_{std} are almost tangent to the planes $\theta = \theta_0$, for all fixed values of θ_0, which clearly our braid will intersect transversally. To see the other direction, we have the following theorem.

Theorem 2.2.2. *[3] (Bennequin) Any transverse knot in $(\mathbb{R}^3, \xi_{sym})$ is transversely isotopic to a closed braid*

Stabilizations of Braids.   Given a braided transverse knot T, there are two braid stablizations that can be done: a positive one and a negative one. Stablization corresponds to adding an additional strand to the braid and adding a positive (or negative) crossing with that strand and the adjacent one at the end of the braid word. That is, if T is a transverse link and ω is a corresponding $(n + 1)$ braid,

then $\omega\sigma_n$ would be a positive stabilization and $\omega\sigma_n^{-1}$ would be the braid word of a negative stabilization. Positive braid stablizations do not change the transverse link, but negative stablilzations correspond to doing a transverse stablilzation [9].

Self-Linking Numbers of Braids. We can also give a formula for the self-linking number $sl(L)$ in terms of a braid representation for L. Given a link L, braid L around the z axis in $\mathbb{R}^3$ with the symmetric contact structure. We then have

$$sl(L) = a(L) - n(L)$$

where $n(L)$ is the number of strands in the braid representing L and $a(L)$ is the algebraic length (sum of exponents on the generators) of the braid [3].

Given two braid words for two transverse knots, how can we tell if they represent the same transverse knot?

Theorem 2.2.3. *(Orevkov and Shevchishin 2003, [24]). Two braids represent the same transverse knot if and only if they are related by positive stabilization and braid isotopy.*

2.3 Open Book Decompositions of Contact Manifolds
2.3.1 Open Book Decompositions

There are a few different ways to construct and visualize 3-manifolds. In this paper we will use open book decompositions. Though they are a great way to visualize 3-manifolds topologically, the real power in open book decompositions comes with the Giroux correspondence. The Giroux correspondence states that given a closed oreinted 3-manifold M there is a 1-1 correspondence between open book decompositions up to positive stablilzation and oriented contact structures on M up to isotopy [14]. We will also see their usefulness in terms of branched covers in the next chapter. But before we can get to all of the applications we must go through the definitions and theory.

Definition 2.3.1. An *open book decomposition* is a pair (B, π) where

1. B is an oriented link in M called the *binding* of the open book and

2. $\pi : (M - B) \to S^1$ is a fibration of the complement of B such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta \subset M$ and $\partial \Sigma_\theta = B$

The surface $\Sigma = \Sigma_\theta$ is called the *page*.

For almost all of this paper, we will use abstact open book decompositions, which are defined below. An abstract open book only determines a manifold up to diffeomorphism. For everything we will do in this paper, diffeomorphism is strong enough, and this way of thinking of open books is more useful for our purposes.

Definition 2.3.2. An (abstract) open book is a pair (Σ, ϕ) where

1. Σ is an oriented compact surface with boundary and

2. $\phi : \Sigma \to \Sigma$ is a diffeomorphism such that ϕ is the identity in a neighborhood of $\partial \Sigma$. The map ϕ is called the *monodromy*.

Given an abstract open book we can construct a 3-manifold M_ϕ by

$$M_\phi = \Sigma_\phi \cup_\psi \left(\coprod_{|\partial \Sigma|} S^1 \times D^2 \right)$$

Above, $|\partial \Sigma|$ is the number of boundary components of Σ. The mapping torus of ϕ is Σ_ϕ and $\cup_\psi$ means that the diffeomorphism ψ is used to identify the boundaries of the two manifolds. (Recall we construct a mapping torus by taking $\Sigma \times [0,1]$ modded out by the equivalence relation $\sim$ where $\sim$ identifies $(\phi(x), 0)$ with $(x, 1)$). For each boundary component b of Σ, $\psi : \partial(S^1 \times D^2) \to b \times S^1 \subset \Sigma_\phi$ is the unique (up to isotopy) diffeomorphism that takes $S^1 \times \{p\}$ to b (where $p \in \partial D^2$) and $\{q\} \times \partial D^2$ to $\{q'\} \times [0,1]/\sim$ where $q' \in \partial \Sigma$.

Let (Σ, ϕ) be an open book decomposition for a manifold M. For notation, let $\Sigma_t = \Sigma \times \{t\}$ in $\Sigma \times [0, 1]$. The following lemma gives the relationship between open book decompositions and abstract open book decompositions. Note, two abstract open books (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are called *equivalent* if there is a diffeomrophism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ \phi_2 = \phi_1 \circ h$.

Lemma 2.3.3. *[11]*

- *An open book decomposition (B, π) of M gives an abstract open book (Σ_π, ϕ_π) such that $(M_{\phi_\pi}, B_{\phi_\pi})$ is diffeomorphic to (M, B).*

- *An abstract open book determines M_ϕ and an open book (B_ϕ, π) up to diffeomorphism.*

- *Equivalent open books give diffeomorphic 3-manifolds.*

Example 2.3.4. One example we will use often throughout this paper is the open book decomposition (D^2, id) for S^3. There are many other open books for S^3 but we will use this one the most.

Example 2.3.5. Let Σ be the annulus, and ϕ be a right-handed Dehn twist around the core curve. Then $(\Sigma, \phi^k) = L(k, k - 1)$.

Stablilzations of Open Books. It is clear that any abstract open book decomposition determines a 3-manifold. Alexander showed that the other direction holds as well: Every closed oriented 3-manifold has an open book decomposition. But 3-manifolds do not have unique open books; even S^3 has many different associated open books. Given one open book, we might want to get another open book for the same manifold, or tell when two open books determine the same manifold.

Definition 2.3.6. A positive (negative) stabilization of an abstract open book (Σ, ϕ) is the open book (Σ', ϕ')

1. with page $\Sigma' = \Sigma \cup 1$-handle and

2. monodromy $\phi' = \phi \circ \tau_c$ where τ_c is a right- (left-) handed Dehn twist along a curve c in Σ' that intersects the co-core of the 1-handle exactly one time.

Positive or negative stablization of an open book does not change the 3-manifold.

Open Books for Contact Manifolds.

Definition 2.3.7. A contact structure ξ on M is supported by an open book decomposition if ξ can be isotoped through contact structures so that there is a contact 1-form α for ξ such that

1. $d\alpha$ is a positive area form on each page Σ_t of the open book and

2. $\alpha > 0$ on the binding.

The next two theorems show that every open book decomposition supports a contact structure and every oriented contact manifold is supported by an open book decomposition. Finally, we state the celebrated Giroux correspondence which gives the 1-1 relationship between these two structures.

Theorem 2.3.8. *[29] (Thurston-Winkelnkemper) Every open book decomposition (Σ, ϕ) supports a contact structure ξ_ϕ on M_ϕ*

Theorem 2.3.9. *[14] (Giroux) Every oriented contact structure on a closed oriented 3-manifold is supported by an open book decomposition.*

Theorem 2.3.10. *[14] (Giroux) Let M be a closed oriented 3-manifold. Then there is a one-to-one correspondence between the set of oriented contact structures on M up to isotopy and the set of open book decompositions of M up to positive stabilization.*

Example 2.3.11. Consdier the open book we have been using for S^3: (D^2, id). This open book supports the tight contact structure and thus is an open book for (S^3, ξ_{std}). It does not support the overtwisted contact structure.

Knots in Open Books. Given a link K inside a 3-manifold M, there are three natural ways K might appear in an open book decomposition for M: as the binding (the boundary of the page Σ), braided transversely through the pages so that K intersects each Σ the same number of times, or sitting on a page. This paper will primarily deal with the second case, occasionally the first, but we will not use the third here.

2.3.2 Pseudo-Anosov Homeomorphisms

Given an open book decomposition (Σ, ϕ) recall that the monodromy ϕ is a homeomorphism of the surface Σ. Recall a homeomorphism of a closed surface Σ is called pseudo-Anosov if there exists a transverse pair of measured foliations on Σ, F^s (stable) and F^u (unstable), and a real number $\lambda > 1$ such that the foliations are preserved by f and their transverse measures are multiplied by $\frac{1}{\lambda}$ and λ. See [6] for more details on pseudo-Anosov homeomorphisms.

We recall Thurston's classification of surface automorphisms.

Theorem 2.3.12. *Let Σ be an oriented hyperbolic surface with geodesic boundary, and let $h \in Aut(\Sigma, \partial\Sigma)$. Then h is freely isotopic to either*

1. *a pseudo-Anosov homeomorphism ϕ*

2. *a periodic homeomorphism ϕ*

3. *a reducible homeomorphism ϕ that fixes setwise a collection of simple closed geodesic curves.*

In any mapping class there is one such representative ϕ and it is called the *Thurston representative* of h.

2.3.3 Right-veering

Honda, Kazez, and Matic introduced the notion of *right-veering* diffeomorphisms in 2005 [19]. Given a homeomorphism of a surface ϕ, whether it is left-veering, right-veering, or neither can give insight into whether the open book (Σ, ϕ) gives a tight or overtwisted contact structure. To get some intuition, it might help to look at a special case first.

Let S be a compact surface with a nonempty boundary. Choose any oriented properly embedded arc $\alpha : [0, 1] \rightarrow S$ with $\alpha(0), \alpha(1) \in \partial S$ such that α divides S into two regions. Call the region where the boundary orientation induced from the region coincides with the orientation on α the region *to the left* of α and the other *to the right*.

Let $\beta : [0, 1] \rightarrow S$ be another properly embedded arc with $\alpha(0) = \beta(0) \in \partial S$. We say that β is *to the right* of α if, after isotoping β so that it intersects α minimally, there is some $c \in [0, 1]$ such that for all $0 < t < c$, either $\beta(t)$ lies in the region to the right of $\alpha(t)$ or $\beta(t) = \alpha(t)$.

Example 2.3.13. Consider the two pictures in Figure 8, each of which has oriented arcs A and B. The shaded region is to the right of A. On the left, the curve B lies in the region to the left of A and therefore we say B is to the left of A. On the right, the curve B lies in the region to the left of A and in the region to the right of A. But there is a connected subarc of B, containing the initial point, which lied entirely in the region to the left of A, and therefore we say B is to the left of A. Notice if we oriented the curves in the opposite directions, the shaded regions would be to the *left* of A and therefore curve B would be to the right of A in the picture on the left, but B would be to the left of A in the other. When the curves share only one endpoint, orientation is implied to be out of the common endpoint, but when both endpoints are shared it is very important to specify orientation.
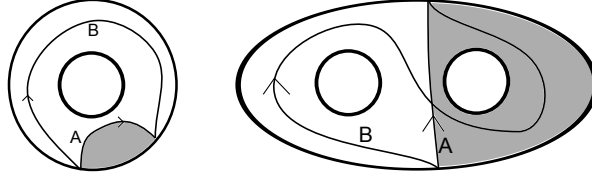
Figure 8: In both pictures the shaded region is to the right of A.

The case above is useful for developing intuition, but it will not happen in general that α divides our surface into two disconnected regions. For example, imagine an annulus with α running between the two boundary componets. So we need a more general notion of when one curve is "to the left" or "to the right" of another.

Once again we start with a curve α whose endpoints lie on the boundary of S. We want to define what it means for another curve β to be to the left or right of α. Let α and β be two non-isotopic curves whose starting points coincide and lie on the boundary of S. If after isotoping the curves to be minimally intersecting, the ordered pair of tangent vectors $\{\dot{\beta}(0), \dot{\alpha}(0)\}$ define a positive orientation on S then we say β is *to the right* of α. If they define a negtive orientation, we say β is *to the left* of α.
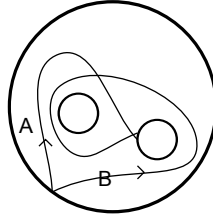
Figure 9: In the figure above B is to the left of A.

Definition 2.3.14. Let $h : S \to S$ be a diffeomorphism that restricts to the identity on ∂S. We say that h is *right-veering* if for every oriented arc $\gamma : [0, 1] \to S$ with $\gamma(0), \gamma(1) \in \partial S$, $h(\gamma)$ is to the right of γ or isotopic to γ. If every $h(\gamma)$ is always to the left of γ then we say h is *left-veering*.

Equivalently, we could define h to be right (left) veering if for every arc $\gamma : [0, 1] \to S$ with $\gamma(0), \gamma(1) \in \partial S$, $h(\gamma)$ is to the right (left) of γ at each endpoint.

This definition has the advantage of not having to worry about the orientation of the arcs. Sometimes we will only be concerned with the behavior at a particular boundary component. Let C be a boundary component of Σ. If for every oriented arc $\gamma : [0,1] \to S$ with $\gamma(0) \in C$, $h(\gamma)$ is to the right (left) of γ, then we say h is right (left)-veering with respect to C.

Example 2.3.15. Let f be a map from the annulus to itself given by a positive Dehn twist around the core curve. Then any arc would be mapped back to itself or to the right. Therefore f is right-veering.

The notion of right-veering and left-veering homeomorphisms is by definition a term describing automorphisms of surfaces. As one might imagine, they were developed for application to open book decomposition, which are presentations of (contact) manifolds involving automorphisms of surfaces. So the first question that should be asked is if there is a relationship between the right or left-veering properties of the monodromy map and the corresponding contact structure.

Theorem 2.3.16. *(Honda-Kazez-Matic) [19] A contact structure (M, ξ) is tight if and only if all of its open book decompositions have right-veering monodromy.*

Notice that an immediate corollary of this theorem is that if even one open book decomposition that supports a contact manifold has a monodromy that is not right-veering then the contact structure is overtwisted. Moreover, because a right-veering monodromy must move *every* arc to the right, we need only find one arc on the page of one open book whose image under the monodromy is to the left.

Perhaps we also need only look at one open book decomposition to determine that a contact structure is tight. One might hope that stabiliation preserves the left-veering or right-veering property, and thus that if one monodromy is right-veering all are right-veering. However, this is far from the case.

Theorem 2.3.17. *[5] (Colin, Honda) Let S be a compact oriented surface with nonempty boundary and h be a diffeomorphism of S which is the identity on ∂S. Then there exists a sequence of positive stabilizations of (S, h) to (S', h') , where $\partial S'$ is connected and h' is right-veering and freely homotopic to a pseudo-Anosov homeomorphism.*

Applying this theorem to open book decompositions, it says that for any contact manifold we can always find a supporting open book that has a connecting binding and a right-veering pseudo-Anosov monodromy. Thus, finding a supporting open book with left-veering monodromy is sufficient to say the contact manifold is overtwisted, but finding a right-veering monodromy is not sufficient to say the structure is tight.

2.3.4   Fractional Dehn Twist Coefficients

We would like introduce the notion of Fractional Dehn Twist Coefficients, as defined in [19]. Let Σ be a surface with geodesic boundary, and $\phi : \Sigma \to \Sigma$ a pseudo-Anosov homeomorphism equipped with stable and unstable laminations. Let C be a boundary component of Σ. Then around C is a semi-open annulus A whose metric completion has geodesic boundary consisting of n infinite geodesics $\lambda_1, ... \lambda_n$. Number the λ_i so that i increases modulo n in the direction consistant with the orientation of C. Let P_i be a semi-infinite geodesic which begins on C, is perpendicular to C, and runs parallel (as it heads away from the boundary) to to λ_i and λ_{i+1} (mod n). Label points (called *prongs*) $x_1, ... x_n$ so that $x_i = P_i \cap C$. (See Figure 10.) The diffeomorphism ϕ rotates the prongs and that there is an integer k such that ϕ maps $x_i \mapsto x_{i+k}$ for all i.

Let h be a diffeomorphism and ϕ as above its pseudo-Anosov represntative. Let $H : \Sigma \times [0, 1] \to \Sigma$ be an isotopy from h to ϕ. Define $\beta : C \times [0, 1] \to C \times [0, 1]$ by sending $(x, t) \mapsto (H(x, t), t)$. Then the arc $\beta(x_i \times [0, 1])$ connects $(x_i, 0)$ and $(x_{i+k}, 1)$ where k is from above. Define the *fractional Dehn twist coefficient* (FDTC) of C to be $c \equiv \frac{k}{n}$ modulo 1, the number of times $\beta(x_i \times [0, 1]$ circles around $C \times [0, 1]$ (in the
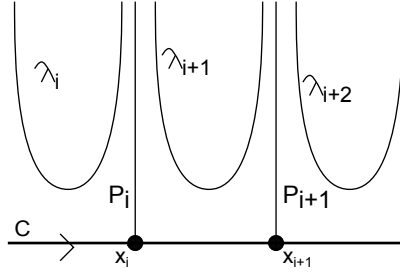
Figure 10: Finding the prongs for a boundary component for a pseudo-Anosov map on a surface

direction of the orientation on C is considered positive). For more details please see [19].

Proposition 2.3.18. *(Honda,Kazez,Matic) If h is isotopic to a pseudo-Anosov homeomorphism, then the following are equivalent:*

1. *h is right-veering with respect to C.*

2. *$c > 0$ for the boundary component C .*

Theorem 2.3.19. *[28] (Roberts) Assume the surface S has one boundary component and h is a diffeomorphism that restricts to the identity on the boundary. If h is isotopic to a pseudo-Anosov homeomorphism and the fractional Dehn twist coefficient of h is c, then $M = (S, h)$ carries a taut foliation transverse to the binding if $c > 1$.*

Eliashberg and Thurston proved that any contact structure close enough to a taut foliation is tight. Honda, Kazez, and Matic showed the contact structure supported by the open book is close to Robert's foliation so it is tight. So now we see the benefit of fractional Dehn twist coefficients. Left-veering curves imply overtwisted contact structures, but right-veering curves tell us nothing. But the theorem above says that a high enough positive fractional Dehn twist can tell us that our contact structure is tight. Computing the FDT coefficients can be difficult though. In particular, for a map isotopic to a pseudo-Anosov homeomorphism, how do we find the laminations? For most cases, the exact fractional Dehn twist cofficient is not important. Knowing

a lower bound, such as $c > 1$ is all we need to say the structure is tight. To that end, Roberts and Kazez gave a method for bounding a fractional Dehn twist coefficient.

Let h be a pseudo-Anosov homeomorphism on a surface S. Let α be an oriented, properly embedded arc which begins on a boundary component C. Isotop the α and $h(\alpha)$, relative to their boundaries, to intersect minimally. Define $i_h(\alpha)$ to be a signed count of the number points, x, in the interiors of α and $h(\alpha)$ with the property that the union of the initial segments of these arcs, up to x, is contained in an annular neighborhood of C. More details can be found in [20].

Theorem 2.3.20. *[20] Suppose h is right-veering at C. Then either*

1. *$c(h) \notin \mathbb{Z}$ and $i_h(\alpha) = \lfloor c(h) \rfloor$ or*

2. *$c(h) \in \mathbb{Z}$ and $i_h(\alpha) \in \{c(h) - 1, c(h)\}$.*

Chapter III

TOPOLOGICAL BRANCHED COVERS

Our overarching goal is to understand 3-manifolds using branched covers. We will see that any 3-manifold can be seen as a cover branching over some knot in S^3. First we need to understand the basics. We will start with the 2-manifold case, then use those results to develop the 3-manifold case. Finally, we will introduce a beautiful and useful theory called *coloring* the branch locus. This method will be fundamental in our main proofs. After presenting the basics, we will discuss some of the history and important results in the field, as well as prove results about constuction of branched covers and improvements on 3-manifold constructions.

3.1 Ordinary Covering Spaces

Recall a map $p : M \to N$ is called a *covering* if there exists an open cover $\{U_\alpha\}$ of N such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in M, each of which is mapped homeomorphically onto U_α by p. It will be helpful to review some facts from algebraic topology about covering spaces. First, we recall an important classification theorem for covering spaces.

Theorem 3.1.1. *[12] Let X be a CW-complex. The isomorphism classes of connected coverings of X preserving base points are in $1 - 1$ correspondence with the subgroups of $\pi_1(X, x_0)$.*

This relationship is of course that for any covering space $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$, the corresponding subgroup H of $\pi_1(X, x_0)$ is $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ [12].

3.1.1 The Monodromy

Given a connected n-fold covering space $p : \widetilde{X} \to X$ we get a homomorphism

$$m : \pi_1(X, x_0) \to S_n$$

(where S_n is the symmetric group of n letters) as follows: let $x_1, \ldots, x_n$ be any fixed numbering of the points in $p^{-1}(x_0)$. Given any loop $\gamma : S^1 \to X$ based at x_0 let $\widetilde{\gamma}_i$ be the lift of γ to a path beginning at x_i. The other end point of the path will be a point x_k. We define $\sigma_\gamma(i) = k$. Clearly σ_γ is an element of S_n and one can easily check that it is independent of the homotopy class of γ as a based loop. Thus we can define $m([\gamma]) = \sigma_\gamma$ where $[\gamma]$ is the element of $\pi_1(X, x_0)$ that γ defines. Notice that if we labeled the points in another order then we would get another homomorphism that was conjugate to the one above.

So to every connected n-fold covering space we get a conjugacy class of representation called the monodromy of the covering space. Notice that if the covering space is not connected we still get a monodromy representation.

Lemma 3.1.2. *[12] If $p : \widetilde{X} \to X$ is an n-fold covering space then $\widetilde{X}$ is connected if and only if the image of the monodromy acts transitively on $\{1, \ldots, n\}$. More precisely the number of components of $\widetilde{X}$ is precisely the number of equivalence classes of $\{1, \ldots, n\}$ under the action of the image of the monodromy.*

Given a connected manifold X and a homomorphism $m : \pi_1(X, x_0) \to S_n$, choose one representative $i_1, \ldots, i_n$ from each equivalence class of $\{1, \ldots, n\}$ under the action of $\pi_1(X, x_0)$. Let $H_j = \{g \in \pi_1(X, x_0) : m(g)(i_j) = i_j\}$ and $\widetilde{X}_j$ the covering space corresponding to H_j. If $\widetilde{X} = \cup_{j=1}^n \widetilde{X}_j$ then $\widetilde{X} \to X$ is a covering space of X for some labeling of the points $p^{-1}(x_0)$ one may check that the monodromy of p is m.

So to every monodromy representation of $\pi_1(X, x_0)$ into S_n we get an n-fold covering space and it will be connected if and only if the image of the monodromy acts transitively on $1, \ldots, n$.

Example 3.1.3. Consider $p : X \to S^1$. The group $\pi_1(S^1, x_0)$ is generated by one element, call it γ. Let the image of γ under the monodromy be the element $(146)(23)(5)$. Then the cover would be three disjoint copies of S^1, one three-fold, one 2-fold, and one fold.

3.2 Branched Coverings of Manifolds

For the majority of this paper we will be interested in branched coverings. Essentially, a branched covering is a map between manifolds such that away from a set of codimension 2 (called the *branch locus*) p is a honest covering. More precicely we give the following definition.

Definition 3.2.1. A map $p : M \to N$ is called a *branched covering* if there exists a co-dimension 2 complex L such that $p^{-1}(L)$ is a co-dimension 2 complex and $p|_{M - p^{-1}(L)}$ is a covering.

If $p : \widetilde{X} \to X$ is a covering space branched over B then the *coloring* of this is the monodromy map for the ordinary covering space $(\widetilde{X} - p^{-1}(B)) \to (X - B)$.

As we will see, when X is a simply connected space, not only does branched covering give us a coloring, but also any coloring gives us a branched covering.

3.2.1 Surfaces

3.2.1.1 Basic Definitions and Examples

Let M, N be 2-manifolds, and $p : M \to N$ a branched covering. Thus there exists a discrete set $\{x_1, ..., x_k\}$ such that $p^{-1}(\{x_1, ..., x_k\})$ is also discrete and $p|_{M - p^{-1}(x_i)}$ is a covering. The set $\{x_1, ..., x_k\} \subset N$ is called the *branch locus* or *branch points*. Often the term "branch point" is also used to describe a preimage in M of one of the branch points in N.

Remark 3.2.2. For any branch point $x \in M$, there is a neighborhood U containing x such that on U, p looks like $z \mapsto z^m$ for some m. We call m the *branching index* of x.
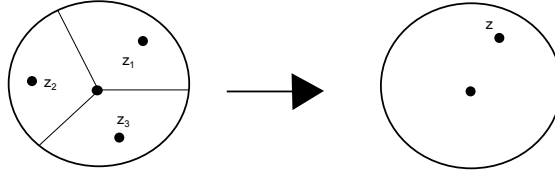
Figure 11: Cyclic Branched Cover over Disk

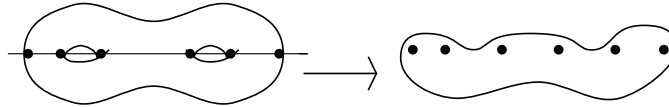**Example 3.2.3.** Let $p : D^2 \to D^2$ by $z \mapsto z^3$, as shown in Figure 11. Notice that every point other than the origin has exactly three preimages, like the point z in the figure. But the origin has one preimage, the origin. Therefore, this is a 3-fold branched covering with branch locus the origin.

Figure 12: Branched Cover of 2-sphere by genus 2 surface.

Example 3.2.4. Let $p : \Sigma_g \to \Sigma_g/\phi = S^2$ where $\phi : \Sigma_g \to \Sigma_g$ is hyperelliptic involution. Figure 12 shows the case for g=2. Notice there would always be $2g + 2$ branch points.

Riemann-Hurwitz Formula.   Recall that if $\Sigma_{g,d}$ is a surface with genus g and d boundary components, then the Euler characteristic of $\Sigma_{g,d}$ is given by the formula

$$\chi(\Sigma) = 2 - 2g - d$$

The Euler characteristic is a tool for idenifying a surface. Recall that any surface is determined up to homeomorphism by the Euker characteristic and number of boundary components. For an n-fold covering map $p : M \to N$, we have the relationship $\chi(M) = n\chi(N)$. The Riemann-Hurwitz formula generlizes this to the case of branch covers.

Theorem 3.2.5. *[27] (Riemann-Hurwitz Formula) Suppose $p : M^2 \to N^2$ is an n-fold branched covering of compact 2-manifolds, $y_1, ..., y_j$ are the preimages of the branch points, and $d_1, ..., d_j$ the corresponding branching indices. Then*

$$\chi(M) = n\chi(N) - \sum_{i=1}^{j}(d_i - 1)$$

It is a standard result of complex analysis that any compact orientable surface M can be seen as some branched cover over the disk (if M has boundary) or the sphere (if M is closed). Restrictions can be placed on either the fold of the cover or the number of branched points without changing the result. In particular, for every closed surface (so M a sphere with g holes) Example 3.2.4 shows there exists a 2-fold cyclic branched covering of M over the sphere with $2g + 2$ branched points. It is also known that there exists a branched covering of M over the sphere with exactly three branched points.

3.2.1.2   Colorings of Branch Sets in Surfaces

Lemma 3.2.6. *Given any surface Σ and finite set of points B, any ordinary finite fold covering space of $\Sigma - B$ extends to a covering space of Σ branched over B.*

Proof. Let Σ be a surface, B a finite set of points on Σ, and $X = \Sigma/B$. Let $\widetilde{X}$ be a covering space of X. Then we have a covering map $p : \widetilde{X} \to X$. We want to extend p to a branched cover $p' : \widetilde{\Sigma} \to \Sigma$. Intuitively, $\widetilde{\Sigma}$ is constructed by filling in the "holes" of $\widetilde{X}$ and near those holes, p' looks like $z \mapsto z^m$ for some m.

Let $b \in B$. We have a disk D_b containing b such that the annulus $A_b = D_b - b$ is contained in the image of p. Because p is a covering, the inverse image under p of A_b must be disjoint annuli. Let A be one of those annuli. For any fixed radius r, we can isotop p on the circle of radius r inside A to be the map $(r, \theta) \mapsto (r, n\theta)$ for some n. Then on a subannuli of A we can isotop further to $(r, \theta) \mapsto (r^n, n\theta) = z^n$. This map clearly can be extended to the disk. $\square$

It is clear then how to color the branch locus for any branched covering over a surface. In general, specifying a coloring of the branch locus of a surface will not determine a unique branched covering space. If the surface downstairs is simply connected then any combinatorial data coloring the branch locus will uniquely determine the covering manifold. Because the surface is simply connected, we can label any point independent of the colorings of the other points in the branch locus: the coloring of each branch point is determined by the preimage of based loops in the fundamental group, and for a simply connected surface, there are no relations between those loops. In particular if the base is D^2 then we have the following simple description.

The branched set B is a collection of points $B = \{x_1, \ldots, x_k\}$. We know the fundamental group of $D^2 - B$ is

$$\pi_1(D^2 - B, x_0) = *^k \mathbb{Z},$$

where x_0 is any base point and $*^k \mathbb{Z}$ means the free product of $\mathbb{Z}$ with itself k times, that is $*^k \mathbb{Z}$ is the free group on k generators. Thus one may specify a monodromy and hence a cover of $D^2 - B$ by choosing k arbitrary elements of S_n.

To make this more explicit we set some notation that will be used throughout the rest of the paper.

Remark 3.2.7. Assume that D^2 is the unit disk in $\mathbb{R}^2$. Let $x_1, \ldots, x_n$ be points on the y-axis contained in D^2 so that their indices increase as one moves up the y-axis. Let x_0 be the point $(-1, 0)$. We can now pick explicit generators of $\pi_1(D^2 - B, x_0)$ as follows. Let s_i be a circle of radius ϵ about x_i where ϵ is chosen so that all the s_i are disjoint. Now let γ_i be the loop that starts at x_0 goes along the straight line towards x_i until it hits s_i, then traverses s_i counterclockwise and finally returns to x_0 along the straight line. Notice that $\gamma_1, \ldots, \gamma_n$ generate $\pi_1(D^2 - B, x_0)$. Thus the generators of $\pi_1(D^2 - B, x_0)$ are in one to one correspondence with the branched locus B.

So one can specify a "coloring" of $D^2 - B$ by labeling the points in B with elements
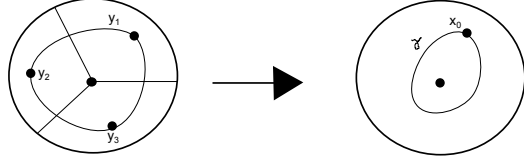
Figure 13: A Branched Covering Example

of S_n and this will uniquely specify a covering space of D^2 branched along B. We denote the label on x_i by c_i.

Example 3.2.8. Let $p : D^2 \to D^2$ by $z \mapsto z^3$ We see that the inverse image of γ takes y_1 to y_2, y_2 to y_3, y_3 to y_1. Therefore we would color the origin (123).

3.2.1.3 Building a Branched Cover From a Coloring

Continuing the notation above let C_i be the horizontal line segment from x_i to the boundary of D^2 with non-negative x-coordinates. We call these the branch line or branch cut associated to x_i.

Remark 3.2.9. Given any loop γ in $D^2 - B$ based at x_0, one may isotop γ to be transverse to the branch cuts. We construct a word in the γ_i and γ_i^{-1} by traversing γ and each time we intersect a branch cut C_i positively we pick up a γ_i and if we intersect it negatively we pick up a γ_i^{-1}. This word gives an element in $\pi_1(D^2 - B, x_0)$ that agrees with $[\gamma]$.

Now given a coloring of B by S_n we build a covering space as follows. Take n copies of $(D^2 - B) \setminus \cup_{i=1}^k C_i$ which we denote by $S_1, \ldots S_n$. We call S_i the ith sheet of the covering. Note that each copy S_i has two copies of C_j in its boundary. We denote them $C_{j,i}^+$ and $C_{j,i}^-$ where the orientation on $C_{j,i}^+$ coming form C_j agrees with the boundary orientation of S_i and $C_{j,i}^-$ is the other copy. Now form the space Σ' from $\cup_{i=1}^n S_i$ by identifying $C_{j,i}^-$ with $C_{j,c_j(i)}^+$.

Lemma 3.2.10. *The surface Σ' is an n-fold covering space of $D^2 - B$. And thus by Lemma 3.2.6 we get a cover $\widetilde{\Sigma}$ of D^2 branched over B.*

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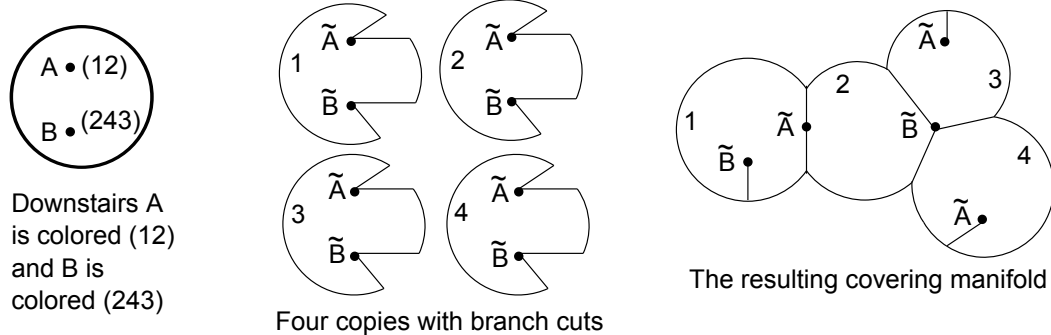
Figure 14: Example of a Construction

Proof. Let $\{U_\alpha\}$ be an open cover of $D^2 - B$ such that for every α, U_α intersects at most one C_j and for any C_j which does intersect U_α, $U_\alpha \cap C_j$ is a connected set. (This condition is not necessary but will make our work simpler.) For any α, if U_α is disjoint from each C_j, then by construction each preimage $p^{-1}(U_\alpha)$ is clearly mapped homeomorphically onto U_α. If U_α intersects some C_j, then C_j divides U_α into two pieces, call them U_α^+ and U_α^- where U_α^+ is above C_j. (Recall that C_j is a horizontal line segment with positive x coordinate so the notion of above means towards the positive y direction.) Then each preimage of U_α contains is cut in two pieces by the preimages of C_j. We form Σ' by identifying $C_{j,i}^-$ with $C_{j,c_j(i)}^+$. Notice that this will identify a preimage of U_α^+ with a preimage of U_α^- on each sheet above. Clearly then this set, which we will call $p^{-1}(U_\alpha)_{j,c(j)}$ is identified homeomorphically with U_α. $\quad\square$

Example 3.2.11. Suppose our disc downstairs had 2 branched points, one colored (12) and the other colored (243). This describes a 4-fold cover, so first we take 4 copies of the disc downstairs. Then we make branched cuts going out from each branched point to the boundary of the disc. The combinatorial data shows how to glue the cuts together. The Figure 3.2.1.3 shows the construction and we see the resulting surface is a disc.

It is easy to see that more complicated coverings will get more complex to construct very fast. Even for a simple coloring of points on a disc, it seems necessary to go

through the construction of drawing and gluing to find the covering manifold. This leads us to our first proposition which gives the covering manifold explicitly from the combinatorial data alone when covering over the disc.

Proposition 3.2.12. *Let $p : M \to D^2$ be n-fold cover branching along k points with M a connected 2-manifold. Let $c_1, ..., c_k \in S^n$ be the colorings induced by p. Then for the manifold M,*

1. *The number of boundary components, d, is the number of of cycles in the product $c_k...c_1$ (where any number that does not appear counts as its own cycle).*

2. *For each c_i there is one branch point upstairs for each non-trivial cycle and the branching index of each branch point is the order of the corresponding cycle. From this the genus follows immediately from the Riemann-Hurwitz formula.*

Proof. The branched cover of a disk with k branch points will be some closed oriented surface. The surface is determined by the genus and the number of boundary components.

Using the notation established in Remark 3.2.7 suppose that $\{c_1, \ldots, c_k\}$ is a coloring of the points $B = \{x_1, \ldots, x_k\}$ in D^2 and $p : \Sigma \to D^2$ is the corresponding branched covering. This defines a homomorphism $\pi_1(D^2 - B, x_0) \to S_n$. So we get the homomorphism $\pi(S^1, x_0) \to \pi_1(D^2 - B, x_0) \to S_n$, where the first homomorphism is induced by the inclusion map of ∂D^2 into $D^2 - B$. Since ∂D^2 is homotopic to the work in the generators $\gamma_1 \cdots \gamma_k$ we see that the generator of $\pi_1(S^1, x_0)$, which is $[\partial D^2]$ is mapped to $c_1 \circ \ldots \circ c_k$. Now we see that the covering space of $\partial \Sigma \to \partial D^2$ is the covering map corresponding to $c_1 \circ \ldots \circ c_k$ and thus by Lemma 3.1.2, Σ has the claimed number of boundary components.

Now notice that if s_i is the circle from Remark 3.2.7 then $p : p^{-1}(s_i) \to s_i$ is an ordinary covering of a circle and it is determined by c_i. Thus the number of components of $p^{-1}(s_i)$ is the same as the number of cycles in c_i. Each circle s in

$p^{-1}(s_i)$ surounds exactly one branched point and the ramification index is the degree of the cover $s \to s_i$. □

Example 3.2.13. Let $N = D^2$ with three branch points each colored (1234). Thus we are representing a 4-fold cyclic cover $p : M \to N$ with three branch points and we want to find the covering manifold M. To calculate the number of boundary components of M we compute $(1234)(1234)(1234) = (1432)$ and see there is one cycle so one boundary component. Now we compute the genus by first computing the Euler characteristic. According to the theorem, the number of inverse images of branch points is 3 because there are 3 non-trivial cycles, one for each branch point, and each has branching index 4.

$$\chi(M) = n\chi(N) - 3(d - 1) = 4(1) - 3(3) = 4 - 9 = -5$$

Now, $\chi(M) = 2 - 2g - d$ so $-5 = 2 - 2g - 1$ and therefore the genus is 3. So M is a surface with genus 3 and 1 boundary component. The cut and paste method discussed above involving branch cuts will confirm this the cover is this surface.

Example 3.2.14. Let $N = D^2$ with two branch points, colored $(145)(23)$, and $(15)(43)(2)$. Thus we are representing a 5-fold cover $p : M \to N$ with two branch points and we want to find M. To calculate the number of boundary components of M we compute $(12)(43)(145)(23) = (13)(245)$ and see there are two disjoint cycles so two boundary components. Now we compute the genus by first computing the Euler characteristic. Notice there are three inverse images with index two, one inverse image with index three, and one with index 1.

$$\chi(M) = n\chi(N) - (3(2 - 1) + 1(3 - 1) + 1(1 - 1)) = 5(1) - 3(1) - 2 = 0$$

Now, $\chi(M) = 2 - 2g - d$ so $0 = 2 - 2g - 2$ and therefore the genus is 0. So M is a surface with genus 0 and 2 boundary components - an annulus. Again the cut and

paste method discussed above involving branch cuts will confirm this the cover is this surface.

Corollary 3.2.15. *If p above is a cyclic covering of D^2 branched over k points, then the number of boundary components is $d = \gcd(n, k)$ and the genus is $g = \frac{1}{2}\left(k(n-1) + (2 - n - d)\right)$.*

Proof. First we give the formula for the boundary. We showed above that the number of boundary components is the number of cycles in the product $c_k...c_1$. For an n-fold cover, each $c_1 = (12...n)$. If there are k branch points, then $(c_k...c_1)(j) = (k + j) \bmod(n)$. The order of the cycle containing j in the product $c_k...c_1$ is the number of iterations before j comes back to itself; i.e. $c_k...c_1(j) = (k+j) \bmod(n)$. Then j comes back to itself after $\frac{n}{\gcd(n,k)}$ iterations, meaning each cycle has length $\frac{n}{d}$ and thus the number of total cycles is exactly $\gcd(n, k)$.

And finally the formula for g:

$$\chi(M) = n\chi(N) - \sum_{y_i}(r_i - 1)$$

$$\chi(M) = n \times 1 - k \times (n - 1) = n - kn + k$$

And the genus then is given by the formula $2 - 2g - d = n - kn - k$. Solving for g,

$$g = \frac{1}{2}\left(k(n-1) + 2 - n - d\right)$$

$\square$

3.2.2 3-Manifolds

Before we discuss the generalization of 2-manifold results to 3-manifolds, we will present some basic definitions and constructions.

3.2.2.1 Basic Definitions

Let M, N be 3-manifolds and $p : M \to N$ a branched covering. That is, there exists a one-dimensional complex L such that $p^{-1}(L)$ is a one-dimensional complex and

$p|_{M-p^{-1}(L)}$ is a covering.

Branched covers of surfaces have been well-understood for some time. Low-dimensional topologists sought to determine if branched coverings could be as powerful a tool for studying 3-manifolds as they are for surfaces. The first progress on this question was given by Alexander in the 1920s when he showed that every compact, closed, oriented 3-manifold is some branched cover branching along a 1-complex in S^3 [2]. This result shows that branched covers are not simply a method for constructing some 3-manifolds, but a tool for constructing every three-manifold. Yet this is simply an existence result; the degree of the cover could be arbitrarily large and the complex could be unusably complicated. One would like to know if, as with surfaces, restrictions can be placed on the branch locus or the cover and still construct every 3-manifold. This questions was answered in 1980.

Theorem 3.2.16. *(Hilden-Montesinos) Let M be a compact oriented 3-manifold. Then there exists a 3-fold branched covering $p : M \to S^3$ branching along a knot.*

In Section 3.2 we saw that all surfaces can be constructed by either looking only at covers with three branched points or looking only at 2-fold cyclic covers. Hilden and Montessinos showed we can look only at all 3-fold covers to obtain all 3-manifolds. Could we also look only at covers over one fixed branch locus, or over a finite set of knots and still construct all closed oriented 3-manifolds, mirroring the result for surfaces?

Universal Links. Not only can we restrict to a finite subset of links, but in fact we can restrict to just one link. A link K is called *universal* if every 3-manifold can be obtained as a branched cover branching along K. Thurston showed the existance of universal links, [30] and since then many explicit universal links and knots have been found, including the figure-eight knot, Borromean rings, Whitehead link, and 9_{46} [18, 30]. Thus, to study closed oriented 3-manifolds, we can restrict either to

studying covers over one particular knot, or restrict to 3-fold simple covers and vary the knots.

Though many results have been found involving the existance of branched coverings with certain properties, the actual constructions are often difficult. The next two section will present some of the known methods for visualizing and constructing branched covers, with particular emphasis on the case for three-manifolds. In addition, we will prove some results about their construction for both the case of surfaces and 3-manifolds.

3.2.2.2 Coloring 3-manifolds

Lemma 3.2.17. *Given any 3-manifold M link L in M, any ordinary finite fold covering space of $M - L$ extends to a covering space of M branched over L.*

Near any point on L, we can intersect with a disk transverse to L and reduce this problem to the same argument made in Lemma 3.2.6.

In general branched covers are complicated, but if the base is S^3 then we have the following simple description.

Any link in S^3 can be assumed to miss a fix point in S^3 and thus we can think of links in S^3 as the same a links in $\mathbb{R}^3$. Now for a link L in $\mathbb{R}^3$ we can project it to the xy-plane (and after isotopy we can assume this projection is generic) to get a *diagram* for L. A diagram is an immersed curve in $\mathbb{R}^2$ with only transverse double points and at each double points over and under crossing information is recorded.

Recall the Wirtinger presentation: to each strand in the diagram we have a generator and to each crossing we have a relation. Recall that the generator for each strand is really the meridian to the strand. That is take a base point x_0 with very positive z-coordinate and orient the knot L. Then you get the curve γ_i associated to the ith strand as follows. Let D_i be a small disk that is transverse the the ith strand, intersects it once and does not intersect the other strands. Orient D_i so that

it intersect L positively. Now let γ_i be the straight line form x_0 to the point on ∂D_i closest to x_0, then traverses ∂D_i positively and then returns to x_0 on the straight line. These are the generators for $\pi_1(S^3 - L, x_0)$.
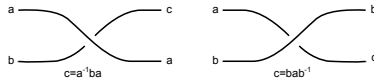
Figure 15: The relations at crossings (reading left to right) where each a, b, c is an element of S^n and points are colored with basepoint above the braid.

So a homomorphism $\pi_1(S^3 - L, x_0) \to S_n$ is determined by specifying an element of S_n for each generator, that is for each strand in the diagram, in such a way that they satisfy the relations at the crossings. We call this a coloring of the diagram and note that it determines a cover of S^3 branched along the link L.

Lemma 3.2.18. *If K is a connected knot, then for any n-fold branched cover along K, each element coloring a strand of K must have the same number of disjoint cycles of order j for all $j \in \mathbb{N}$.*

Proof. Let the strands of K be colored $c_1, ..., c_k$. Start with any one arc, colored c_j. Flow along the knot until you come to the first crossing. Let c_i by the color of the crossing strand. Then the next piece of the knot (the arc reached by flowing under the crossing) is colored either $c_i c_j c_i^{-1}$ or $c_i^{-1} c_j c_i$. Neither changes the cycle structure, and therefore c_j has the same cycle structure as the next piece of the knot. Continue flowing along the knot. Because it is connected, every arc will be crossed and so each piece must have the same cycle structure. $\square$

Coloring Open Braids. When the branch locus is presented as an open braid, in addition to the fact that at any crossing we have a conjugacy relation that must be satisfied, the colorings at the end must match the colors at the start of the braid. Therefore, any one point on the braid can be colored anything, but then you must push through the braid to see what restrictions are placed on the other colorings.

When all the conjugacy relations and the end matching relations are satisfied, we say the braid is *correctly colored*. Note that a correctly colored open braid gives coloring of the respective closed braid.

Admissible Transformations. Suppose we have a branch locus $L \subset S^3$ which is correctly colored and yields a covering manifold M. An *admissible transformation* of the colored diagram is a manipulation to the link and its coloring that does not change the covering manifold.
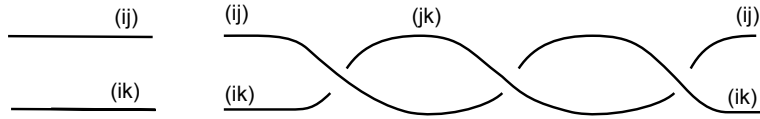
Figure 16: An admissible transformation.

Example 3.2.19. We claim that the transformation shown in Figure 3.2.2.2 is admissible. Let $p : \widetilde{M} \to M$ be a branched cover whose branch locus K in M has a portion colored as shown on the left in figure 3.2.2.2. We can enclose this portion of the branch locus in a ball B and consider $p' : (\widetilde{M} - p^{-1}(B)) \to M - B$. Along $K \cap B$, notice that the intersection with D^2 would give a disk with two branch points, colored (ij) and (ik). The branched cover of a disk with this coloring is again a disk. Therefore, to obtain $\widetilde{M}$ from $(\widetilde{M} - p^{-1}(B))$ we simply insert back the missing 3-balls. Now replace $K \cap B$ with K', the transverse transformation shown on the right in Figure 3.2.2.2. Notice again that the intersection of the strands with D^2 would give a disk with two branch points, colored (for the appropriate association of abc to ijk) (ab) and (bc). Again the cover is a disk, so the cover of B branching along K' is still a 3-sphere. Because there is a unique way insert the copies of the 3-sphere into $(\widetilde{M} - p^{-1}(B))$ upstairs, the cover after transformation yields the same manifold as $\widetilde{M}$. Therefore this transformation is admissible.

3.3 Branched Covers of Contact Structures

How do we take branched covers of contact manifolds? In other words, given a transverse link L in (M, ξ) and branched covering $p : \widetilde{M} \to M$ branching along L, how would we define $\widetilde{\xi}$?

Construct $\widetilde{M}$ as normal. Let $\widetilde{L} = p^{-1}(L)$. Then $p : (\widetilde{M} - \widetilde{L}) \to M - L$ is a true cover, and thus $\xi = p^*(\xi)$ on $(\widetilde{M} - \widetilde{L})$. Then extend to a plane field on all of $\widetilde{M}$ and perturb to make contact [13].

3.3.1 Generalizing Topological Results

Topologically, we know from the previous section that we can see any three manifold as the branched cover over S^3 and that we can restrict to looking only at 3-fold covers or looking only at covers over a fixed knot. We want to try to generalize these results to the contact manifolds. Can every contact manifold be seen as a cover over (S^3, ξ_{std})? If so, what restrictions can be placed on the fold of the cover or the branch locus without changing the answer? Giroux gave the following answer [14].

Theorem 3.3.1. *(Giroux) Every contact manifold can be seens as a 3-fold simple cover over some transverse link in (S^3, ξ_{std}).*

Thus every contact manifold is a branched cover over (S^3, ξ_{std}), and (as with the result of Hilden and Montesinos) we can restrict to looking at 3-fold covers. This is not yet a full generalization, as it seems we may need to allow for multiple-component links. We would like to have the same result as in the topological world and restrict to covers over knots. Our next theorem does this.

Theorem 3.3.2. *Given a contact manifold (M, ξ), there exists a 3-fold simple cover $p : (M, \xi) \to (S^3, \xi_{std})$ whose branch locus is a knot.*

Proof. Let L be the branch locus for a cover coming from Theorem 3.3.1, presesnted as a braid. Color the braid according to p. Take a $D^2 \times I$ containing a section of

L with no crossings. For each adjacent pair of strands, if they do not belong to the same connected component of the link, perform the admissible transformation shown in Figure 3.2.2.2. Note that because Theorem 3.3.1 gives us a simple, 3-fold cover, each strand will either be colored (12), (23), or (13).

This move connects the two previously separate components, does not change the fact that the cover is 3-fold or simple, and does not change the manifold upstairs. All that remains is to check that it does not change the contact structure. Notice that the cover of any $D^2 \times t$ is also a disk, so the cover of $D^2 \times I$ is still a ball. This ball, in the cover, is away from the binding and thus the contact structure on it is tight. There is a unique tight contact structure on a ball so the contact structure remains unchained under this transformation. □

Contact Universal Links. Giroux's theorem guarantees that any contact 3-manifold can be obtained via some 3-fold simple branched cover, and the strengthening guarantees the branch locus can be a knot. As with topological 3-manifold, we would like to also be able to obtain any contact 3-manifold by branching over some fixed knot or link.

Definition 3.3.3. A transverse knot K is called contact universal if every contact manifold (M, ξ) can be realized as some cover $p : M \to S^3$ with branch locus K.
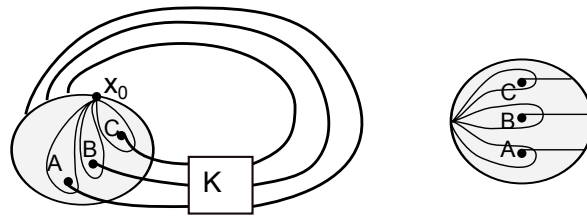
3.4 Covers of Open Book Decompositions

Given an open book decomposition (Σ, ϕ) for a manifold M and a knot inside K we want to see how to take covers over M branching along K in terms of the open book. We can consider the case where the knot is transverse to the pages, or is the binding. Specifically, we want to look at two cases: cyclic covers branched over the binding and general covers over S^3 where K is braided through the pages.

Start with covers over S^3. Let (D^2, id) be the open book decomposition of S^3.

Recall the notation $\Sigma_t = \Sigma \times \{t\}$ in $\Sigma \times [0, 1]$. (Each Σ_t is a D^2.) Let K be a knot braided transversely through the sheets. Each Σ_0 intersects K at k distinct points. Let $L_t = K \cap \Sigma_t$. On Σ_0, label the points of L_0 as $x_1, ...x_k$ so that doing a half twist σ_i would correspond to interchanding points x_i and x_{i+1}.

In any open book decomposition, the monodromy tells how to glue Σ_0 to Σ_1. Let ϕ be the composition of half-Dehn twists that trace out K. Because any map of the disk is isotopic to the identity, the open book (D^2, ϕ) will also give S^3. It can be helpful for intuition to define a continuous family of maps Φ_t to trace out the knot as follows: $\Phi_0 = id$, $\Phi_1 = \phi$, $\Phi_t : \Sigma_0 \to \Sigma_t$ so that $\Phi_t(L_0) = L_t$. We will think of ϕ as the monodromy downstairs. Notice, that though ϕ is isotopic to the identity on D^2, on $D^2 - L$ it is not.

Figure 17: Open Book Decomposition

Next let $p : M \to S^3$ give a covering of S^3. Color the knot as determined by the map. Then each point of L_0 inherits a corresponding color, as shown in figure 3.4. To construct the cover, we need to make our branch cuts along Σ_0. To keep notation and orientation consistant, we will make the branch cuts so that traversing the boundary in the positive direction crosses the branch cuts in the order $c_1, ..., c_k$.

Notice now that we commutative diagram as seen below. It is important to note that this composition is continuous; in particular that given a curve γ downstairs and a lift $\widetilde{\gamma}$ upstairs, $p(\widetilde{\phi}(\widetilde{\gamma})) = \phi(\gamma)$. By definition of the lift, for a curve γ upstairs, $p(\widetilde{\phi}(\gamma)) = \phi(p(\gamma))$.

$$\begin{CD} \widetilde{\Sigma} @>\widetilde{\phi}>> \widetilde{\Sigma} \\ @VpVV @VVpV \\ \Sigma @>\phi>> \Sigma \end{CD}$$

Now we know how to calculate the page $\widetilde{\Sigma}$ of the open book for M - use the same construction as for surfaces. However, the monodromy is significantly more difficult to calculate. In the case for cyclic covers, we can compute the monodromy, though not in terms of the essential curves upstairs. In [17] is given a formula for the monodromy of cylic covers in terms of specific curves they describe upstairs. No such formula exists for a general cover.

Branching Over the Binding   Now suppose K is the binding and we want to do an n-fold cyclic cover. This case is actually very simple. If p is a branched cover over (Σ, ϕ) with branch locus K then any cyclic cover branched over the binding would be (Σ, ϕ^n).

Lifting Open Books of Contact Manifolds   Finally we want to consider using open book decompositions to look at contact manifolds as covers over (S^3, ξ_{std}). The open book decomposition (D^2, id) supports the contact structure. We take covers as discussed previously, and the open book constructed determines a supported contact structure on the covering manifold, as stated more formally below.

Theorem 3.4.1. *Let K be a knot braided transversely through the pages of the open book decomposition (D^2, id), which supports (S^3, ξ_{std}). Let (M, ξ) be the covering contact manifold obtained by branching over K. The open book constructed as described above supports the contact manifold (M, ξ).*

Chapter IV

BRANCHED COVERS OVER TRANSVERSE KNOTS

Our goal is to understand the properties of covering contact structures from the contact manifold downstairs and the combinatorial data of the branch locus. More specifically, given a specific map (via combinatorial data) we want to know what is the covering contact manifold or be able to determine properties of the covering contact structure. Given a specific contact manifold, we would like to understand all possible covers of that manifold over a fixed branch locus.

We will examine both of these problems, starting with what is already known. The second problem will lay the groundwork for a very interesting area of study - finding a contact unversal knot.

4.1   Branching Over a Contact Manifold

Given a contact manifold what can we say about the cover? We certainly know that branched covers of tight manifolds need not stay tight. For example, the 2-fold cyclic branched cover in S^3 with the standard contact structure branched over the figure-eight knot is overtwisted [17]. But once a structure is overtwisted, do all its branched covers stay overtwisted? If not, how rare is it for a cover or an overtwisted manifold to be tight?

Theorem 4.1.1. *Given any 3-manifold M with any overtwisted contact structure, there exists some transverse knot inside M such that some cyclic cover branching over M is tight.*

This is a somewhat surprising result, given the strong contrast to what happens for true covering maps; all non-branched covers of overtwisted manifolds stay overtwisted.

Proof. Let (M, ξ) be any contact manifold. Let (Σ, f) be an open book for ξ.

Recall Theorem 2.3.17 says we can positively stabilize (Σ, f) to get (Σ', f') with connectd boundary and f' a right-veering pseudo-Anosov diffeomorphism. For simplicity in notation, (Σ, f) will now refer to the stabilized open book. Our new open book has connected binding B and monodromy f which is right-veering and freely isotopic to a pseudo-Anosov homeomorphism.

Let c be the fractional Dehn twist of f on $\partial\Sigma$. Because f is right-veering, Proposition 2.3.18 tells us that $c > 0$. Let $\Phi : (\widetilde{M}, \widetilde{\xi}) \to (M, \xi)$ be the n-fold cyclic map branching over B.

Lemma 4.1.2. *$(\widetilde{M}, \widetilde{\xi})$ has open book (Σ, f^n), and f^n is isotopic to pseudo-Anosov homeomorphism.*

Proof. That (Σ, f^n) is an open book for the covering manifold is immediate: The covering manifold is constructed by cutting M along Σ, taking n copies, and glueing them together. Thus clearly the page upstairs is still Σ, and the monodromy is f^n. We still need to show that f^n is also isotopic to a pseudo-Anosov homeomorphism.

Let Ψ be the pseudo-Anosov homeomorphism isotopic to f. Then we have an isotopy $\Phi : \Sigma \times [0, 1] \to \Sigma$ such that $\Phi(x, 0) = f(x)$ and $\Phi(x, 1) = \Psi(x)$. We need an isotopy $\widetilde{\Phi} : \Sigma \times [0, 1] \to \Sigma$ from f^n to Ψ^n. We first define a series of functions g_k as follows. First let $g_2(x, t) = \Phi(\Phi(x, t), t)$. Then, for any $k \in \mathbb{N}$, $k > 2$ $g_k(x, t) = \Phi(g_{k-1}(x, t), t)$. Then define $\widetilde{\Phi} = g_n(x, t)$. Notice that for $n = 2$, $\widetilde{\Phi}(x, 0)$ $= \Phi(\Phi(x, 0), 0) = \Phi(f(x), 0) = f^2(x)$ and $\widetilde{\Phi}(x, 1) = \Phi(\Phi(x, 1), 1) = \Phi(\Psi(x), 1) = \Psi^2(x)$, similarly for larger n. Continuity of this isotopy is immediate from the continuity of Φ. Therefore f^n is isotopic to pseudo-Anosov homeomorphism Ψ^n $\qquad\square$

Downstairs, we have one connected boundary component with a fixed number of prongs $x_0, \dots x_{k-1}$ given by Ψ and fractional Dehn twist coefficient $c \equiv \frac{m}{k}$ (mod 1), as described in Section 2.3.4. Because the branching locus is the boundary and we are

taking a cyclic cover, we have still one boundary component upstairs with the same number of prongs $x_0, ... x_{k-1}$ for Ψ^n. And Ψ moves x_1 to x_{m+1}, therefore Ψ^n would take x_1 to $x_{n \cdot c}$ (mod c). Therefore the fractional Dehn twist Ψ^n is $n \cdot c$. Because we know $c > 0$, for large enough n, $n \cdot c > 1$. By Theorem 2.3.19 this means large enough n will yield a tight contact structure upsairs. □

4.2 Covers over (S^3, ξ_{std})

Given a link L in (S^3, ξ_{std}), what possible contact structures can be seen as covers branching over L? Given a particular coloring, what can we say about its cover?

The goal of this section is to be able to give conditions on a knot or conditions on its lift that will guarantee the covering manifold is tight or overtwisted. We will do this by finding arcs who move to the left or to the right under the monodromy upstairs and showing that the monodromy must be left or right veering.

As before we take our standard open book decomposition for (S^3, ξ_{std}): (D^2, id). Let K be a transverse knot or link braided through the pages. On Σ_0, let γ denote an arc that begins and ends on the boundary of the disk and encloses exactly one branch point, say x_i, and its branch cut. (When it is important which branch point is enclosed, we will use the notation γ_i.) Orient γ so that x_i is in the region to the right of γ as described in Section 2.3.3.

Let p be a branched covering with base (S^3, ξ_{std}), branch locus K, and covering manifold (M, ξ). The lift $\widetilde{\gamma}$ of any γ will have n components if p is a n-fold cover. Let $\widetilde{\gamma}_i$ denote be the piece of $\widetilde{\gamma}$ that has its endpoints on the ith sheet upstairs.

Notice that determining if $\phi(\gamma)$ is to the left or to the right of γ on $D^2 - L_0$ is simply a matter of looking at the image of γ under the half twists that correspond to the braid word. (Of course, on D^2 they will be isotopic to each other.) However, determining if $\widetilde{\phi}(\widetilde{\gamma})$ is to the left or to the right of $\widetilde{\gamma}$ on $\widetilde{\Sigma}$ is a significantly more complex probem because the exact ramifications of the branch points can result in

a huge change in behavior between the two curves once they are isotoped to be minimally self-intersecting.

Thus we want to know not only the lifts of the curves $\phi(\gamma)$ but what it looks like when it is isotoped to be as simple as possible. For this we introduce the following two definitions.

Definition 4.2.1. For any $\phi(\gamma)$ on D^2 as described above, we define the *branching word* of $\phi(\gamma)$ to be a word with letters $x_i^\pm$ which gives the order in which $\phi(\gamma)$ passes through the branch cuts. For any connected component of $\widetilde{\phi(\gamma)}$, we use the same definition.

Notice that for any curve γ, every component of $\widetilde{\phi(\gamma)}$ will have the same branching word as $\phi(\gamma)$. This is immediate to see as $\phi(\gamma)$ lies in the complement of the branching set and thus p gives a true covering on the preimages of $\phi(\gamma)$.

Definition 4.2.2. For any component of $\widetilde{\phi(\gamma)}$, its *reduced branching word* is the branching word of an isotopic copy of $\widetilde{\phi(\gamma)}$ which minimally intersects the branch cuts.

Notice we could define the reduced branching word for $\phi(\gamma)$ as well but because these curves are on D^2, $\phi(\gamma)$ would always be isotopic to γ and therefore would always have empty reduced branching word.

Sometimes we might want to keep track of the sheets a curve passes through by adding some additional notation to the branching word. For each sheet passed, we subindex with the sheet the curve is currently on and the one it is passing to. We demonstrate with an example. Let D^2 downstairs have two branch points A and B, colored $(123)(567)$ and $(157)(346)$ respectively. Let γ enclose the branch point at A and α be the lift of γ under a branched cover that begins on the fourth sheet. Suppose $\widetilde{\phi}(\alpha)$ has branching word $B^{-1}A^{-1}B^{-1}ABA$. Then, because α starts on the fourth sheet, we could track its progress through the branch cuts by adding notaton

for the sheets as follows: $B_{43}^{-1}A_{32}^{-1}B_{22}^{-1}A_{23}B_{34}A_{44}$. We call this the *detailed* branching word.

Now with this notation we can obtain the reduced branching word for $\widetilde{\phi}(\alpha)$ in two steps. First, remove any letters where no branching occurs. In our example above we would be left with $B_{43}^{-1}A_{32}^{-1}A_{23}B_{34}$. Next, remove any adjacent letters that cancel each other out, keeping in mind that for transpositions any letter is its own inverse. Here we would be left with $B_{43}^{-1}B_{34}$, and then the empty word. Therefore, without drawing it out, we know that $\widetilde{\phi}(\alpha)$ is isotopic to α. To give one more example, if for the same colorings we had chosen α to be the component that began on the third sheet, then the branching word would be $B_{36}^{-1}A_{65}^{-1}B_{51}^{-1}A_{12}B_{22}A_{23}$ so the reduced branching word would be $B_{36}^{-1}A_{65}^{-1}B_{51}^{-1}A_{12}A_{23}$.

When it becomes important to clearly distinguish between the branched point and the coloring of the branched point, we will use the following convention for notation. Capital letters will signify the specific points, and lower case letters will signify the colorings of those points as determined by the branched covering.

Before we use these definitions to prove any results, we will first see many examples of the branchng word and reduced branching word.
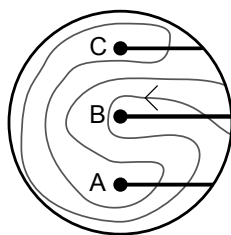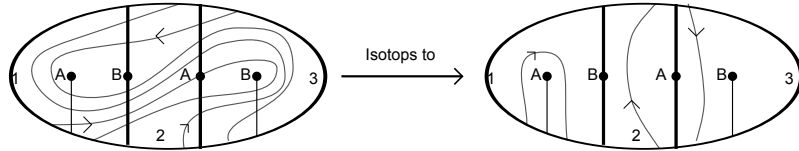

Figure 18: Finding the Branching Word

Example 4.2.3. The branching word of the curve seen in Figure 18 would be (reading left to right) $BA^{-1}C^{-1}A$.

Example 4.2.4. The branching word of γ and of and each component of $\widetilde{\gamma}$ as seen in Figure 19 would both be $ABAB^{-1}A^{-1}B^{-1}$.
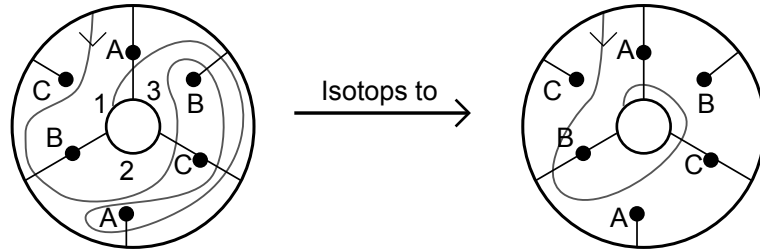
51
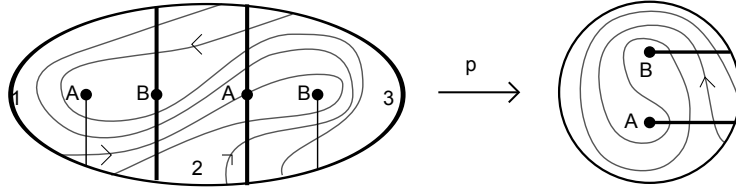
Figure 19: Finding the Branching Word

Figure 20: The Reduced Branching Word

Example 4.2.5. The reduced branching word of any component of $\widetilde{\gamma}$ from Figures 19 and 20 would both be the empty word.

Figure 21: The Reduced Branching Word

Example 4.2.6. The branching word of γ in Figure 21 is $BC^{-1}B^{-1}C^{-1}ACBA^{-1}$. The reduced branching word is $BC^{-1}A^{-1}$.

Before we get to our main proofs we need a lemma.

Lemma 4.2.7. *Let* (D^2, ϕ) *be the open book for* (S^3, ξ_{std})*,* K *a transverse link braided through the pages and* ϕ *the map induced by the braid word of* K*. The page* D^2 *intersects* K *at* k *distinct points. Let* $x_1, ..., x_k$ *be the branch points, colored by* $c_1...c_k$*. Let* p *be a branched covering map over* (S^3, ξ_{std}) *with branch locus* K *and* γ *on* D^2

be a curve that encloses exactly one branch point x_i and is disjoint from the branch cuts so that x_i lies in the region to the right of γ. If for some component α of $\widetilde{\gamma}$, the reduced branching word of $\widetilde{\phi}(\widetilde{\alpha}$ does not start with x_i then $\widetilde{\phi}$ is left-veering.

Proof. Let α be a component of $\widetilde{\gamma}$ such that the reduced branching word of $\widetilde{\phi}(\widetilde{\alpha}$ does not start with x_i. Because γ encloses x_i downstairs, α will locally (restricting to the sheet on which it starts) enclose one preimage of x_i. Isotop $\widetilde{\phi}(\alpha)$ to minimally intersect the branch cuts. Because α has reduced branching word that does not begin with x_i, we know that after isotoping to intersect the branch cuts minimally, $\widetilde{\phi}(\alpha)$ crosses a branch cut other than x_i first. Let $t \in [0,1]$ be such that $\widetilde{\phi}(\alpha(t))$ is the point where $\widetilde{\phi}(\alpha)$ first crosses the branch cut. Then the subarc of $\widetilde{\phi}(\alpha)$ connecting $\widetilde{\phi}(\alpha(0))$ to $\widetilde{\phi}(\alpha(t))$ can be isotoped to never cross α. This subarc would thus lie in the region to the left of α, and therefore $\widetilde{\phi}(\alpha)$ is to the left of α. Therefore $\widetilde{\phi}$ is left-veering. $\square$

Theorem 4.2.8. *Given any transverse knot that destabilizes, every cover branching over that knot will be overtwisted.*

Proof. Let $p : (M,\xi) \rightarrow (S^3, \xi_{std})$ be a branched covering, and S^3 presented as the open book (D^2, id) with all the same notation as above. Let K be a stablization of another transverse knot K', braided through the pages. Assume K' is a j-braid whose braid work σ is written in terms of $\sigma_1, ..., \sigma_{j-1}$. Then K is a $j+1$-braid whose braid word is $\sigma\sigma_j^{-1}$. (See Figure 22).
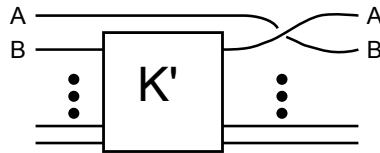

Figure 22: K' inside the stabilized knot K

Color K as determined by the branched covering. Let A, B represet the strands that would be twisted by σ_j, with B the strand that is also included in K'. We will

use a and b to denote the respective colorings. Because the braid for K must match up at each end, but there is only one crossing between thee strands, we see that $a = b$. (See Figure 22).

Let γ on D^2 be an arc with endpoints on the boundary that encloses the branch cut of A and passes through no other branch cuts. Because γ encloses only A, and no twist in σ (the braid word for K') involves the $j + 1$ strand, $\phi(\gamma)$ involves only one twist: it will start at the same points as γ but enclose B after curving to the left (see Figure 23.
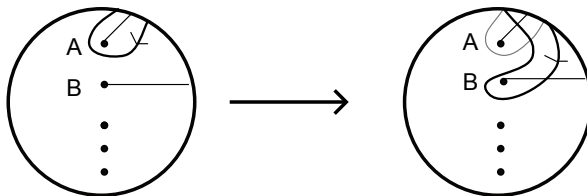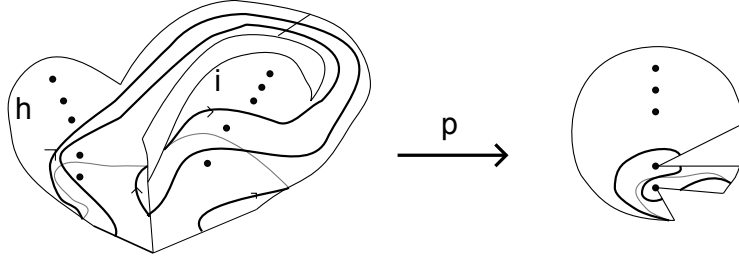

Figure 23: $\phi(\gamma)$

Choose any i such that B branches on the ith sheet. Let α be the component of $\widetilde{\gamma}$ which is contained on the ith sheet. We claim that $\widetilde{\phi}(\alpha)$ is to the left of α, and thus $\widetilde{\phi}$ is left-veering. Recall from Chapter 2 that $\widetilde{\phi}(\alpha)$ is isotopic to the component of $\widetilde{\phi(\gamma)}$ which begins and ends on the ith sheet. By Lemma 4.2.7, we know that if the reduced branching word of $\widetilde{\phi}(\alpha)$ does not begin with x_i then $\widetilde{\phi}$ is left-veering.

Lift $\phi(\gamma)$. We chose i so that it is not fixed by B, and therefore the coloring of B is an element of S_n which sending some number, call it h, to i . Then on the ith sheet, $\widetilde{\phi}(\alpha_i)$ moves through the branch cut at B to the hth sheet (because γ hits the branch cut at B in a negative direction), and then hits the branch cut at A in a positive direction. But we proved $b = a$, meaning A also sends h to i, and therefore $\widetilde{\phi}(\alpha_i)$ then returns to the ith sheet, as shown in Figure 24.

The curve $\widetilde{\phi}(\alpha_i)$ passes first through the branch cut at B. And because it never passes back through that same cut before it returns to the boundary on the ith sheet,

Figure 24: The section of $\widetilde{\phi(\gamma)}$ containing the ith and h^{th} sheets

we know it cannot be isotoped away from the branch cut at B and thus has branching word $B^{-1}A$, which clearly cannot reduced [lemma to cite here]. Therefore, by Lemma 4.2.7, $\widetilde{\phi}$ is left-veering. From Chapter 2 we know that $(\widetilde{\Sigma}, \widetilde{\phi})$ gives the open book decomposition supporting (M, ξ) and therefore (M, ξ) is overtwisted.

$\square$

Notice the main ideas of the proof that covers of stabilized knots are overtwisted boiled down to two main points: First, that we had a σ_j^{-1} to move arcs to the left and no σ_j to move them back; and second that there was a curve that passed to a new sheet at every branch cut, preventing the $\widetilde{\phi}(\alpha_i)$ to be pulled away from the branch cut it crosses in the region to the left. Our next result is a generalization of this method.

Proposition 4.2.9. *If K is transverse knot in (S^3, ξ_{std}) whose braid word contains a σ_i^{-1} and no σ_i for some i then any fully ramified cover branching over K is overtwisted.*

Remark 4.2.10. This was proven in the cyclic case by [17].

Proof. Letting all notation be the same as before, we see we need to show exactly the things mentioned above: that for some curve γ which encloses some point A on a page of the open book downstairs there is a component α_i of its lift so that $\widetilde{\phi}(\alpha_i)$ is to the left of α_i.

Let K be a transverse knot in (S^3, ξ_{std}) whose braid word contains a σ_i^{-1} and no σ_i for some i. Without loss, assume that the braid word starts with σ_i^{-1} (cut the

braid accordingly). Choose γ to be an oriented arc on the D^2 page downstairs that encloses x_{i+1} (from here on called A) so that A lies in the region to the right of γ.

Claim. *The curve $\phi(\gamma)$ is to the left of γ on $D^2 - L_0$.*

Proof. We can conjugate the braid word (cut at the appropriate position) so that the first twist applied to γ is σ_i^{-1}. Notice $\sigma_i^{-1}(\gamma)$ is to the left of γ on $D^2 - L_0$. Any $\sigma_k^{\pm}$ for $k > i$ or $k < i - 1$ would not move $\sigma_i^{-1}(\gamma)$. The only twist which could affect $\sigma_i^{-1}(\gamma)$ are $\sigma_i^{\pm}$ and $\sigma_{i-1}^{\pm}$. By assumption, no σ_i^{+1} occurs in the braid word. Twists σ_i^{-1} and $\sigma_{i-1}^{\pm}$ would keep $\sigma_i^{-1}(\gamma)$ to the left of γ. Continuing down the braid word, any $\sigma_i^{\pm}(\gamma)$ for $k > i$ would still have no effect on the initial behavior of the curve, and any $\sigma_k^{\pm}$ for $k < i - 1$ or σ_{i-1} would keep the image of γ to the left of γ. Therefore $\phi(\gamma)$ will end up to the left of γ on $D^2 - L_0$. □

So downstairs we know that $\phi(\gamma)$ is to the left of γ (on $D^2 - L_0$). Thus, for any lift $\tilde{\gamma}$ the initial tangent vector of $\tilde{\phi}(\tilde{\gamma})$ at the initial point will be to the left of $\tilde{\gamma}$. This means that whatever the branching word is for $\tilde{\phi}(\tilde{\gamma})$, it will not begin with A. Because p is a fully ramified branched cover, $\tilde{\phi}(\tilde{\gamma})$ will branch at each sheet and thus will have the same reduced branching word as branching word. Therefore the reduced branching word will not begin with A, and thus by Lemma 4.2.7 $\tilde{\phi}$ is not right-veering and so the covering contact structure is overtwisted. □

Corollary 4.2.11. *If K a transverse link in (S^3, ξ_{std}) which is the figure-eight knot with braid word $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$, the Borromean rings with braid word $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$, or the Whitehead with braid word $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-2}$ then and branched cover branching over K which is fully ramified yields an overtwisted contact structure upstairs.*

4.3 Contact Universal Knots

4.3.1 The Figure-Eight Knot

Because we have a classification of transverse knots with knot type the figure-eight knot and know the only one that does not destabilize is that with maximal self-linking number (-3), notice an immediate corollary of Theorem 4.2.8 is that a figure-eight knot with any other self-linking number cannot be contact universal. So what about the figure-eight knot with $sl = -3$ - could it be contact universal?

Theorem 4.3.1. *Every cover of S^3 branching over the figure-eight knot is overtwisted.*

Proof. The figure-eight (with sl=-3) is a 3-braid so L_0 is a set of three points $x_1 = C, x_2 = B, x_3 = A$. As in the previous section, let γ be a curve that encloses the branch cut at A. The monodromy ϕ will take γ and perform the half twists $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$. Figure 25 shows γ and $\phi(\gamma)$.
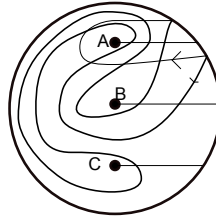
Figure 25: Image of γ under ϕ

We need to show that there is a component α_i of $\widetilde{\gamma}$ such that $\widetilde{\phi}(\alpha_i)$ is to the left of α_i. Recall, from Lemma 4.2.7 we know it suffices to show that that $\widetilde{\phi}(\alpha_i)$ has reduced branching word that does not begin with A. Notice that the branching word for any component α_i is $B^{-1}AC^{-1}A^{-1}BA$. Let a, b, c be the elements of S_n that color the respective branch points A, B, C.

Lemma 4.3.2. *If for some i, $b^{-1}(i) \neq i$ and $c(a(b^{-1}))(i) \neq a(b(i))$ then the reduced branching word for $\widetilde{\phi}(\alpha_i)$ does not begin with A or A^{-1}.*

Proof. The branching word for any component α_i is $B^{-1}AC^{-1}A^{-1}BA$. Suppose for some i, $\widetilde{\phi}(\alpha_i)$ branches the first time is crosses the branch cut at B (i.e. $b^{-1}(i) \neq i$) and when it crosses the branch cut at C (i.e. $c(a(b^{-1}))(i) \neq a(b(i))$). Say B takes j to i. Then the detailed branching word for $\widetilde{\phi}(\alpha_i)$ would be $B_{ji}^{-1}A_{ik}C_{kl}^{-1}A_{lm}^{-1}B_{mx}A_{xj}$. Where j is distinct from i and k is distinct from l. To remove the first term, B_{ji}^{-1}, it would have to be canceled out by a B_{ij}. But between the B_{ji}^{-1} and the only B^+ term is the C_{kl}^{-1}, which will not be removed in the first step because k is distinct from l and cannot be cancelled out later because of the absense of the C^+ term. Therefore at no point in the reduction algorithm will B_{ji}^{-1} be adjacent to B_{mx}, and thus they cannot cancel. This means the reduced branching word must begin with the B_{ji}^{-1} term. $\square$

Notice that by Lemma 4.2.7, if such a component as specified in the above lemma did exist, then the covering contact structure would be overtwisted.

Lemma 4.3.3. *Given any coloring of the figure-eight knot, there exists i, such that $b^{-1}(i) \neq i$ and $c(a(b^{-1}))(i) \neq a(b(i))$.*

Proof. Suppose no such i existed. For each component $\widetilde{\phi}(\alpha_i)$, the branching word is $B^{-1}AC^{-1}A^{-1}BA$. Choose any i such that the coloring of B includes branching at i. Then we assume that for any such $\widetilde{\phi}(\alpha_i)$, no branching occurs as it passes through the branch cut at C, leaving us with a detailed branching word of $B_{ji}^{-1}A_{ik}C_{kk}^{-1}A_{km}^{-1}B_{ml}A_{lj}$. Then the resulting reduced branching word, after step 1 removed the C term, would be $B_{ji}^{-1}A_{ik}A_{km}^{-1}B_{ml}A_{lj}$. Immediately we see that $m = i$ because A takes i to k and A^{-1} takes k to m. The partially reduced branching word can then be written as $B_{ji}^{-1}A_{ik}A_{ki}^{-1}B_{il}A_{lj}$. This allows us to cancel the A and A^{-1}: $B_{ji}^{-1}B_{il}A_{lj}$. Applyig the same logic again we see that $l = j$, modifying the word to $B_{ji}^{-1}B_{ij}A_{jj}$. Now we can remove the A term and cancel the B and B^{-1} terms, giving us an empty reduced branching word.

We just showed that if B sends i to j, but no branching occurs when $\widetilde{\phi}(\alpha_i)$ passes

through the branch cut at C, then the coloring of A does not include branching at j. Therefore, no number present in the coloring of B is present in the coloring of A.
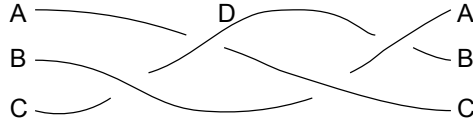
Figure 26: Coloring the Figure-Eight

Figure 26 shows the braid of the figure-eight, cut at $\Sigma_0 = \Sigma_1$. Let D be the only arc of the braid that is alienated from Σ_0 and Σ_1 by crossings. Because the coloring is correct, we immediately get the following relationships:

$$D = A^{-1}BA$$

$$C = DAD^{-1}$$

$$A = C^{-1}BC$$

If A and B have no numbers in common then the first relationship above tell us that $D=B$. Therefore D and A have no numbers in common, which means the second relationship tells us that $C = A$. Then C and B have no numbers in common, so the third relationship tells us that $A = B$, which is a contradiction.

Therefore, no coloring of the figure-eight has A and B without numbers in common, which brings us to a contradiction. Therefore our assumption, that there is no component of $\widetilde{\phi}(\widetilde{\gamma})$ which does not branch both the first time it crosses B and when it crosses at C must be false. $\qquad\square$

So finally, we know there exists i such that $b^{-1}(i) \neq i$ and $c(a(b^{-1}))(i) \neq a(b(i))$, and therefore by Lemma 4.3.2 there exists a component of $\widetilde{\gamma}$ whose image has reduced branching word that does not begin with A, and therefore by Lemma 4.2.7 $\widetilde{\phi}$ is left veering therefore any cover branching over the figure-eight is overtwisted. $\qquad\square$

4.3.2 The Whitehead Link

One might hope that any transverse braid whose braid word contains σ_i^{-1} and no σ_i for some i would have every cover overtwisted, as was the case with the figure-eight and with destabilized links. This would help to rule out the Whitehead link as a contact universal link. Yet this is not the case. Below is one counterexamples.

Theorem 4.3.4. *Let L in (S^3, ξ_{std}) be the transverse Whitehead link with braid word $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-2}$. There exist covers branching over L that are tight.*

Proof. Let K be the transverse Whitehead link with braid word $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-2}$ in (S^3, ξ_{std}) with open book (D^2, ϕ). Let A, B, C denote the branch points on $D^2 \times \{0\}$, reading top to bottom (i.e. A, B would be twisted by σ_2, B, C would be twisted by σ_1). Let $p : (M, \xi) \to (S^3, \xi_{std})$ a 9-fold branched covering branching over K given by colorings a, b, $c = (123)$, (145267389), and (123) respectively. Then by Proposition 3.2.12 we know that the covering manifold will be a genus 1 surface with 3 boundary components.

Let γ be an arc that begins and ends on the boundary of D^2, enclosing the point A such that A lies in the region to the right of γ. Let β do the same for the point C. $\phi(\gamma)$ has branching word $B^{-1}A^{-1}BAC^{-1}A^{-1}B^{-1}ABA$. The curve $\phi(\beta)$ has branching word $CA^{-1}BAC^{-1}A^{-1}B^{-1}A^{-1}BACA^{-1}B^{-1}A$.

Each of the curves γ, β, $\phi(\gamma)$, and $\phi(\beta)$ has nine preimages. Denote the preimages of these curves with a subscript noting which sheet in the preimage the curve begins and ends on. For examplle, $\widetilde{\phi(\beta)}_2$ would be the preimage of $\phi(\beta)$ which begins on the sheet labeled with a 2. We want to focus on particular preimages of $\widetilde{\phi(\gamma)}$ and $\widetilde{\phi(\beta)}$, namely the preimages surrounding the ramfied preimages of A and C. The branch point A downstairs has only one preimage which is ramified, where sheets 1, 2, and 3 connect, likewise for C. For these desired preimages ($i = 1, 2, 3$), the detailed branching words are below.

$$\widetilde{\phi(\gamma)}_1\colon\ B_{1,9}^{-1}A_{9,9}^{-1}B_{9,1}A_{1,2}C_{2,1}^{-1}A_{1,3}^{-1}B_{3,7}^{-1}A_{7,7}B_{7,3}A_{3,1}$$

$$\widetilde{\phi(\gamma)}_2\colon\ B_{2,5}^{-1}A_{5,5}^{-1}B_{5,2}A_{2,3}C_{3,2}^{-1}A_{2,1}^{-1}B_{1,9}^{-1}A_{9,9}B_{9,1}A_{1,2}$$

$$\widetilde{\phi(\gamma)}_3\colon\ B_{3,7}^{-1}A_{7,7}^{-1}B_{7,3}A_{3,1}C_{1,3}^{-1}A_{3,2}^{-1}B_{2,5}^{-1}A_{5,5}B_{5,2}A_{2,3}$$

$$\widetilde{\phi(\beta)}_1\colon\ C_{12}A_{2,1}^{-1}B_{14}A_{4,4}C_{4,4}^{-1}A_{4,4}^{-1}B_{4,1}^{-1}A_{1,3}^{-1}B_{3,8}A_{8,8}C_{8,8}A_{8,8}^{-1}B_{8,3}^{-1}A_{3,1}$$

$$\widetilde{\phi(\beta)}_2\colon\ C_{2,3}A_{3,2}^{-1}B_{26}A_{6,6}C_{6,6}^{-1}A_{6,6}^{-1}B_{6,2}^{-1}A_{2,1}^{-1}B_{1,4}A_{4,4}C_{4,4}A_{4,4}^{-1}B_{4,1}^{-1}A_{12}$$

$$\widetilde{\phi(\beta)}_3\colon\ C_{3,1}A_{1,3}^{-1}B_{38}A_{8,8}C_{8,8}^{-1}A_{8,8}^{-1}B_{8,3}^{-1}A_{3,2}^{-1}B_{2,6}A_{6,6}C_{6,6}A_{6,6}^{-1}B_{6,2}^{-1}A_{2,3}$$

Using the algebraic method described in the previous section, we achieve the reduced branching words given below.

$$\widetilde{\phi(\gamma)}_1\colon\ A_{1,2}C_{2,1}^{-1}$$

$$\widetilde{\phi(\gamma)}_2\colon\ A_{2,3}C_{3,2}^{-1}$$

$$\widetilde{\phi(\gamma)}_3\colon\ A_{3,1}C_{1,3}^{-1}$$

$$\widetilde{\phi(\beta)}_1\colon\ C_{12}A_{21}^{-1}$$

$$\widetilde{\phi(\beta)}_2\colon\ C_{23}A_{32}^{-1}$$

$$\widetilde{\phi(\beta)}_3\colon\ C_{31}A_{13}^{-1}$$

Therefore the preimages $\widetilde{\phi(\gamma)}_i$ and $\widetilde{\phi(\beta)}_i$ can be isotoped to the curves show in Figure 27. The curves γ_i and $\widetilde{\phi(\gamma)}_i$ are shown, paired by color. (We should also note that all of the algebraic work above can be confirmed using the cut and paste method combined with brute force isotoping of curves, but we will not include those calculations here.)

Claim. *The open book constructed above destabilizes to (T, id) where T is the punctured torus.*

Recall that a positive stabilization of an abstract open book (Σ, ϕ) is the open book with page $\Sigma' = \Sigma \cup 1\text{-handle}$ and monodromy $\phi' = \phi \circ \tau_c$ where τ_c is a right-handed Dehn twist along a curve c in Σ' that intersects the co-core of the 1-handle
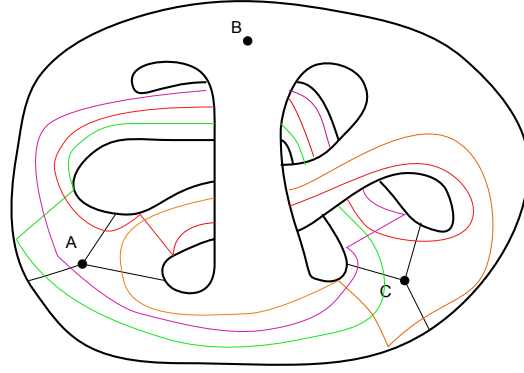
Figure 27: The preimages of particular arcs after isotoping to be minimally intersecting.

exactly one time. We want to show that our open book is a positive stabilization of (T, id) by destabilizing our open book along particular curves.

Choose c to be the bolded blue curve shown in the image on the left in Figure 28. Define $\Phi := D_c^{-1} \circ \widetilde{\phi}$ (D denotes a positive Dehn twist). The picture on the right in Figure 28 shows $\Phi(\widetilde{\gamma}_i)$ and $\Phi(\widetilde{\beta}_i)$ for the preimages from Figure 27 under Φ as well as the original $\widetilde{\gamma}_i$ and $\widetilde{\beta}_i$ for reference.
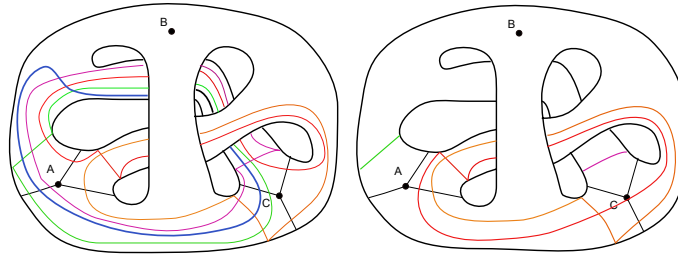

Figure 28: Images of curves under Φ.

Notice that Φ fixes the green curve. Therefore we can cut along it to destabilize our surface. On the new surface choose d to be the bolded blue curve pictured in Figure 29. Let $\Phi' = D_d^{-1} \circ \Phi$. As before the images of our curves under Φ' are below.
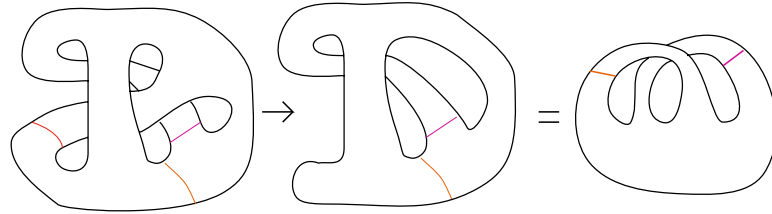
We see that the red curve is fixed, so we cut our surface along it to destabilize.

The two core curves are both fixed under Φ', giving us a monodromy isotopic to the identity. We then have shown that we can destabilize Σ to the punctured torus with identity monodromy.
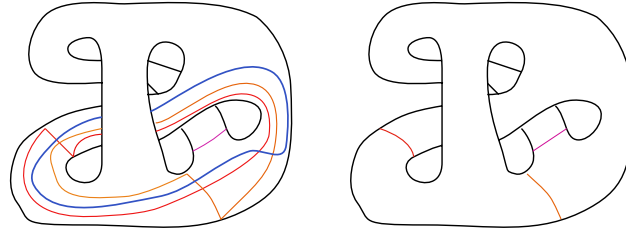
Figure 29: Images under Φ'.

Figure 30: Destabilized Open Book.

From [11] we know this open book yields the manifold $(S^1 \times S^2)\#(S^1 \times S^2)$ and from [14] and [1] we know that when supported with this open book it is Stein fillable and thus tight.

Therefore, we have constructed a cover over the transverse Whitehead link with braid word $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-2}$ which is tight.

$\square$

This tells us that the Whitehead link has the potential to be contact universal, but whether or not it is remains an open question.

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