ON THE EXISTENCE OF CENTRAL FANS OF CAPILLARY SURFACES

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The following faculty members have examined the final copy of this dissertation for form and content, and recommended that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with a major in Applied Mathematics.

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DEDICATION

I would like to dedicate the dissertation to my lovely wife Samia. In fact, The reason for dedicating this dissertation to her is simply because I wouldn't be interested in completing my graduate studies and doing a PhD degree without her continuous encouragement. Moreover, the endless spiritual support that she provided to me during these years has had a major impact in accomplishing this goal. In 2003 I promised her to pursue my PhD studies if I have the chance to do it, but many circumstances were against achieving the goal. After 10 years, I hereby fulfill my promise.

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ABSTRACT

We prove that under some conditions, the central fans of capillary surfaces exist and are stable. We perturb the contact angle of a capillary surface for a bounded domain which is not necessarily symmetric, that has a central fan, and prove that the central fan will continue to exist after the perturbation. We prove the result for some smooth conditions with sufficient regularity. We provide examples to illustrate the existence and stability of central fans.

TABLE OF CONTENTS

CHAPTER 1

1 INTRODUCTION

1.1 The Mathematical Theory of Capillarity

Capillarity, or capillary action, is the tendency or the ability of a liquid to flow on narrow spaces or move in some certain direction by way of cohesive and adhesive forces.

A simple illustration of this physical phenomenon is liquid inside a thin tube. Capillary action occurs at an interface whenever the cohesive forces between the molecules of the same substance differs from the adhesive forces between molecules that attract between unlike substances. When the adhesive forces is greater than the cohesion forces, the liquid is attracted to the molecules of the wall causing it to wet it, so the level of the surface at the middle will be lower than the level of the liquid at the surrounding area, and this will result in a concave formation of the liquid in a circular tube.

The importance of studying capillarity phenomena lies in the fact that they arise in our lives in different aspects where they control many vital and essential processes in it. Some examples of this phenomena are absorption of a liquid by paper towels, the supply of water from the roots of the tree to the leaves of a tree, a drop of water lying on a flat surface, the surface of the water in a drinking glass, and much more.

The study of capillary surfaces goes back to ancient times, and has been under investigations and studies by some of the greatest scientists. The effects of surface tension have been studied for millennia and the development of “European” science was affected by many disagreements between “Aristotelian” and “Archimedean” theories, including their theories about why objects float in water; these disagreements are illustrated by Galileo Galilei’s famous three-day discussion in 1611 which was recorded in 1612 as Discourse on Floating Bodies.

In the preface of the most important and influential book in the literature on the mathematical theory of capillarity, *Equilibrium Capillary Surfaces* by Robert Finn (Springer-

Verlag, 1986), one finds “Attempts to explain observed phenomena go back at least to Leonardo da Vinci.” In addition, in 1712 articles on capillarity by Brook Taylor and Francis Hauksbee are reproduced prior to the Introduction of Finn’s book. Taylor studied the behavior of a capillary surface near a corner and made some capillarity experiments. In 1805, Thomas Young introduced the notion of mean curvature of a surface H of a surface S, and related it to the pressure change δp across the surface. He also showed that the contact angle depends only on the physical material, not the gravity field or the geometry of the container. In the following year, P.S. Laplace introduced the mean curvature notion and derived a formal analytic expression. Laplace produced the equation of prescribed mean curvature, from which he derived the capillary equation. He found the first explicit formula permitting quantitative prediction for a solution of a capillary problem of a circular capillary tube in a gravity field.

In 1830, Gauss used the principle of virtual work formulated by Johann Bernoulli to unify the work of Young and Laplace by deriving the capillary equation by a variational process, and characterized these surfaces. His approach became the basic foundations for the modern mathematical theory of capillarity, and since that time, a considerable amount of work has appeared in the literature discussing the theory, and the topic became an area of active research. Most of the studies in the eighteen and nineteen centuries were restricted to the case of a capillarity tube that have some symmetric configurations. The problem of finding the shape of a capillary surface has started to attract some prominent mathematicians as Plateau, Monge, Poisson, Raileigh, Neumann, Minkowski, and Poincare [1], [2]. Capillary surfaces and the mathematical theory describing them were important in the history of mathematics (and science) and remain so today. The reasons for this are the type of nonlinearity inherent in the variational principle for which they are minimizers and the contact angle condition which represents their boundary condition at an intersection of solid-liquid, solid-gas and liquid-gas interfaces. It is worth noting that a major impulse to study capillary surfaces today comes from spacecraft and the problem of dealing with liquids, where the shape of the surface is determined by surface tension. In fact, this theory demonstrated to NASA that the proposed design in the mid-1960s of fuel tanks for maneuvering thrusters would, in micro-gravity, have failed to deliver fuel to the thrusters and would probably have resulted in the destruction of manned spacecraft on atmospheric reentry.

The main question of the mathematical theory of capillary is the question of existence, regularity, and behavior of such surfaces. One of the earliest studies of the existence of these surfaces were done by Concus and Finn [3], where they studied the behavior of a capillary surface in a wedge, and they obtained estimates from below and above for the heights of the free surface and obtained necessary and sufficient conditions for the capillary surface to be bounded or not. Serrin [4] studied the variational solutions of quasilinear elliptic differential equations. In 1970, Concus and Finn [5] presented explicit asymptotic form for capillary free surfaces, and several existence results have been obtained by C. Gerhardt, E. Giusti, M. Giaquinta, P. Concus and R. Finn, and others.

In 1973, M. Emmer [6] obtained a variational solution of capillary problem, and Uraltseva obtained a classical solution, and in 1976 E. Giusti [7] proved the existence of variational solution for the mixed boundary value problem with a contact angle boundary condition and Dirichlet boundary condition. Concus and Finn [8] studied capillary free surfaces in the absence of a gravity and in the presence of gravity, and they provided necessary conditions for the existence of solutions of capillary equations. They also studied [9] the singular solutions of capillary equations, and they proved the existence and uniqueness of these solutions. C. Gerhardt [10] studied the existence and regularity of variational solutions of capillary equations, then he extended his studies to the surfaces of prescribed mean curvature [11]. L. Simon and J. Spruck [12] proved that under certain conditions the capillary surface extended continuously to the boundary. Details on Capillary surfaces can be found in [13].

CHAPTER 2

2 BACKGROUND

2.1 Equations of Prescribed Mean Curvature

Let $a^{ij} = a^{ji}$, $i, j = 1, ..., n$, and consider the second order quasilinear operator Q of the form:

$$Qu = \sum_{i,j=1}^{n} a^{ij}(x, u, Du)D_{ij}u + b(x, u, Du)$$

where $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n, n \geq 2$. Let $\Pi \subseteq \Omega \times \mathbb{R} \times \mathbb{R}^n$, then Q is elliptic in Π if the coefficient matrix $[a^{ij}(x, z, p)] > 0$ for all $(x, z, p) \in \Pi$. That is, there exists minimum eigenvalue $\lambda(x, z, p)$ and maximum eigenvalue $\Lambda(x, z, p)$ both positive and

$$0 < \lambda(x, z, p) \left|\xi\right|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, z, p)\xi_i\xi_j \leq \Lambda(x, z, p) \left|\xi\right|^2$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$, and for each $(x, z, p) \in \Pi$.
If further, we have $\Lambda(x, z, p)/\lambda(x, z, p) \leq \mathrm{M} \in \mathbb{R}$ for all $(x, z, p) \in \Pi$, then we say that Q is uniformly elliptic in Π. If Q is elliptic in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, then we say that Q is elliptic.

The equation of prescribed mean curvature is a quasilinear second order, elliptic equation of the form

$$\sum_{i=1}^{n} D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) \quad = \quad nH(x, u)$$

where H is the prescribed mean curvature of the solution u. If

$$H(x, u) = \kappa u + \lambda$$

then the equation is called a “capillary equation”. If in addition, $H = 0$, then the equation becomes the minimal surface equation.

2.2 Capillary Problem

A capillary surface can be defined as the interface separating two fluids adjacent to each other. Capillary surfaces occur when two fluids are adjacent to each other without mixing, and these surfaces separate the two substances. As a consequence of this, ideal capillary surfaces have no thickness. If gravity is absent then the surface has a constant mean curvature.

The definition of a capillary problem can be stated as the problem of finding a function $f \in C^2(\Omega)$ such that

$$\operatorname{div}(Tf) = \kappa f + \lambda \qquad \text{in } \Omega \tag{2.1}$$

subject to the condition

$$Tf \cdot \nu = \cos\gamma \qquad \text{on } \partial\Omega \ , \tag{2.2}$$

where

$$Tf = \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}.$$

The quantity $\kappa = \frac{\rho \mathrm{g}}{\sigma}$ is a constant, where ρ is the density of the liquid, g is the gravitational acceleration, and σ is the surface tension. The quantity λ is a physical constant that is determined by volume constraints when $\kappa = 0$.

The quantity $\gamma = \gamma(s)$, $0 \leq \gamma(s) \leq \pi$, is a function of position on $\partial\Omega$, which is called the contact angle, and it is the angle at which the the boundary of the supporting surface and the capillary surface intersect. The surface $z = f(x, y)$ describes the shape of the static liquid gas interface in a vertical cylindrical tube of Ω. From the definition of κ, we can see that it is positive if the gravity field is downward, and negative if the gravity field is upward. The case $\kappa = 0$ refers to an absence of gravity; that is $\mathrm{g} = 0$, and the mean curvature H in this case is constant.

Definition of a Corner: Let Ω be a connected domain. Then we say that Ω has a corner at $O = (0,0)$ if and only if $O \in \partial\Omega$, and there exists $\delta > 0, \alpha \in [0,\pi]$, and $\theta_1, \theta_2 \in C^0([0,\delta] :\to C^0((-\pi,\pi))$ such that $\theta_1(r) < \theta_2(r)$ for each $r \in (0,\delta)$, and

$$\lim_{r\downarrow 0}\theta_1(r) = -\alpha,\ \lim_{r\downarrow 0}\theta_2(r) = \alpha,$$

$$\Omega \cap D(\delta) = \{(r\cos(\theta), r\sin(\theta)) : 0 < r < \delta, \theta_1(r) < \theta < \theta_2(r)\}$$

where

$$D(\delta) = \{x \in \mathbb{R}^2 : |x| \leq \delta\}.$$

We write

$$\partial^+\Omega = \{(r\cos(\theta_2(r)), r\sin(\theta_2(r))) : 0 < r \leq \delta\},$$

and

$$\partial^-\Omega = \{(r\cos(\theta_1(r)), r\sin(\theta_1(r))) : 0 < r \leq \delta\}.$$

Let $\Omega \subset R^2$,$O =$(0,0) be a corner. We assume that $\partial\Omega$ is piecewise smooth, and tangent rays to $\partial\Omega$ make an angle of 2α at O and $\theta = \pm\alpha$ are the tangent rays to $\partial\Omega$ at O. If $0 < \alpha < \pi/2$ then we say that the corner is convex, while if $\pi/2 < \alpha < \pi$ then we say that the corner is nonconvex, or re-entrant. If $\alpha = 0$ or π then the region has a cusp.

2.3 Dirichlet Problem

Let Q be an elliptic differential operator. Then, we define the Dirichlet problem as the problem of finding a solution to the equation

$$Qu = 0 \qquad \text{in } \Omega$$

subject to the condition

$$u = \phi \qquad \text{on } \partial\Omega$$

for some $\varphi \in L^1(\partial\Omega)$.

Theorem 2.1. *(Uniqueness Result): Let Q be elliptic operator in the bounded region Ω. Let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. If $Qu = Qv$ in Ω and $u = v$ on $\partial\Omega$ then $u = v$ in Ω.*

Proof. See [14], Theorem (10.2). □

2.4 Variational Solutions

A classical solution of a partial differential equation is a function that satisfies the equation and the boundary condition at every point of the domain. But, because of the lack of regularity, we don't always expect classical solutions to exist. For example, Korevaar [15] provided a solution of a capillary problem that is discontinuous at the corner. Hence, we need a weaker concept of "solution", that is a weak or generalized solution. We can obtain such solution by two approaches, the Perron method and the variational method. This section is devoted to the second type.

Definition of Locally Lipschitz Domains: Let $\Omega \subset \mathbb{R}^n$ be an open connected set, with boundary $\partial\Omega$. The boundary $\partial\Omega$ is called a locally Lipschitz boundary if for each point $x \in \partial\Omega$, there is a neighborhood of x, say U_x such that $\partial\Omega \cap U_x$ is the graph of a Lipschitz continuous function.

Definition of Functions of Bounded Variations: Let Ω be an open bounded domain in $\mathbb{R}^n$. A function $f \in L^1(\Omega)$ is said to be of bounded variation on Ω if $\int_\Omega |Df| < \infty$, where

$$\int_{\Omega} |Df| = \sup\{\int_{\Omega} f \cdot \mathrm{div}(g)dx \ : \ g \in C_0^1(\Omega;\mathbb{R}^n) \ , |g(x)| \leq 1 \text{ for } x \in \Omega\}.$$

The set of all functions of bounded variations on Ω is denoted by $BV(\Omega)$. The integral $\int_\Omega |Dv|$ is called the total variation of u in Ω, and denoted by $V(u,\Omega)$.

Variational Method: Let $\Omega \subset \mathbb{R}^n$ be an open connected set, with $\partial\Omega$ locally Lipschitz. Consider the following energy functional

$$M(v) = \int_{\Omega} \sqrt{1+|\nabla v|^2}dx + \int_{\Omega} \int_0^v H(x,t)dtdx$$

where

$$v = \varphi \text{ on } \partial\Omega$$

for some $\varphi \in L^1(\partial\Omega)$.

G. Williams proved [17] that if Ω is a bounded domain with a locally Lipschitz boundary, $H(x,t)$ increases in t, $H(x,t_0) \in L^n(\Omega)$ for each fixed $t_0 \in R$, $\varphi \in L^1(\partial\Omega)$, and $\psi \in W^{1,\infty}(\Omega)$ such that $\psi \leq \varphi$ on $\partial\Omega$, then we have

$$\inf_{K_1} M(v) = \inf_{K_2}\{M(v) + \int_{\partial\Omega} |v-\varphi| \, dH_{n-1}\}$$

where $K_1 = \{v \in W^{1,1}(\Omega) : v \geq \psi \text{ in } \Omega, \ v = \varphi \text{ on } \partial\Omega\}$, $K_2 = \{v \in BV(\Omega) : v \geq \psi \text{ in } \Omega\}$.

2.5 Behavior of Capillary Surfaces Near Corners

The central question that addresses this theme is the following: How does a generalized solution of a capillary problem behave near the corner ? For a convex corner, and a constant contact angle γ, Paul Concus and R. Finn showed that if $\left|\frac{\pi}{2} - \gamma\right| \leq \alpha$ then f is bounded in Ω. Also for a convex corner, L. Simon [18] proved that if $\left|\frac{\pi}{2} - \gamma\right| < \alpha$ then f is C^1 up to the corner. Tam [19] proved that f and the normal vector are continuous up to the corner in the borderline case $\left|\frac{\pi}{2}-\gamma\right|=\alpha$. It is worth noting that there is a general existence theory that covers the case $\kappa \geq 0$ and an existence theorem for $\kappa < 0$ and $|\kappa|$ small. K. Lancaster and D. Siegel [20] proved that a bounded capillary solution with contact angle $\gamma : \partial\Omega \to (0,\pi)$ which need not be continuous, must be continuous at O if $\left|\gamma_0^+ - \gamma_0^-\right| \leq \pi - 2\alpha$, and $\left|\gamma_0^+ + \gamma_0^- - \pi\right| < 2\alpha$ where $\gamma^+(s)$ and $\gamma^-(s)$ denote γ along the arcs $\partial\Omega^+$ and $\partial\Omega^-$ respectively, where $s=0$ corresponds to the corner O, and $\gamma_0^\pm = \lim_{s\downarrow 0}\gamma^\pm(s)$ where $0<\gamma_0^\pm<\pi$.

For a nonconvex corner, if $\kappa > 0$ then the solution exists over piecewise smooth domains Ω. If $\kappa = 0$, γ is constant, and $\partial^\pm\Omega$ straight boundary segments forming a convex corner, Concus and Finn showed that a bounded solution exists in a neighborhood of O only if $\left|\frac{\pi}{2}-\gamma\right| \leq \alpha$. Korevaar [15] gave examples of capillary surfaces that are discontinuous at the corner, and for any γ, $0<\gamma<\pi/2$.

2.6 Central Fans

Definition of Radial Limits: Suppose Ω has a corner at O and $f \in C^0(\Omega)$. We say that the radial limit of f at O in the direction $\theta \in (-\alpha,\alpha)$ exists if and if the following limit exists

$$Rf(\theta) = \lim_{r\downarrow 0} f(r\cos(\theta), r\sin(\theta))$$

Definition of Fans: We say that the radial limit of f at O has a fan if and only if there exist $\alpha_1, \alpha_2 \in [-\alpha,\alpha]$, $\alpha_1 < \alpha_2$ such that $Rf(\theta) = Rf(\alpha_1)$ for all $\theta \in [\alpha_1,\alpha_2]$. We say that the angular interval $[\alpha_1,\alpha_2]$ is a fan. If $\alpha > \pi/2$ and $\alpha_2 - \alpha_1 = \pi$ we say the radial limit of f at O has a central fan $[\alpha_1,\alpha_2]$.

The main Theorem of the existence of central fans is the following Theorem due to Lancaster and Siegel [20].

Theorem 2.2. *(Existence of central Fans): Let $\Omega \in \mathbb{R}^2$, and $\Omega^* = \Omega \cap B_\delta(O)$ for some $\delta > 0$. Let f be a bounded solution to a capillary equation satisfying the contact angle condition on $\partial^\pm\Omega^* \setminus \{O\}$, discontinuous at O, with*

$$0 < \gamma_0 \le \gamma^{\pm}(s) \le \gamma_1 < \pi.$$

If $\alpha \ge \pi/2$ then $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. If $\alpha < \pi/2$ and there exists constants $\underline{\gamma}^{\pm}, \bar{\gamma}^{\pm}, 0 < \underline{\gamma}^{\pm} \le \bar{\gamma} < \pi$ satisfying $\underline{\gamma}^{+} + \underline{\gamma}^{-} > \pi - 2\alpha$ and $\bar{\gamma}^{+} + \bar{\gamma}^{-} < 2\alpha + \pi$, so that $\underline{\gamma}\pm \le \gamma^{\pm}(s) \le \bar{\gamma}^{\pm}$ for all s, $0 < s < s_0$ for some s_0. then again $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. Furthermore, in either case, $Rf(\theta)$ is continuous function on $[-\alpha, \alpha]$ which behaves in one of the following ways:

(i) There exists α_1, α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$, and Rf is constant function on $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$, and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2]$. Label these cases (I) and case (D), respectively.

(ii) There exists $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that

$$-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha$$

with $\alpha_R = \alpha_L + \pi$, and Rf is constant on $[-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha]$ and either increasing on $[\alpha_1, \alpha_L]$ and decreasing on $[\alpha_R, \alpha_2]$ or decreasing on $[\alpha_1, \alpha_L]$ and increasing on $[\alpha_R, \alpha_2]$. Label these case (ID) and case (DI), respectively.

Proof. See [20], Theorem 1. □

Note that the existence of central fans is concluded from (ii) in the re-entrant corner cases. Since the size of the central fan is π, then it cannot exist for domains with convex corners. The fans where Rf is constant on $[-\alpha, \alpha_1]$, and $[\alpha_2, \alpha]$ are called "side fans".

Lancaster and Siegel concluded the following two important corollaries.

Corollary 2.1. *Let f be a bounded solution to (2.1) satisfying (2.2) on $\partial^{\pm}\Omega \setminus O$, with $\lim_{s\downarrow 0}\gamma^{\pm}(s) = \gamma_0$, $0 < \gamma_0 < \pi$. Then for $\alpha \ge \pi/2$, case (ID) cannot occur when $\alpha + \gamma_0 \le 3\pi/2$, and case (DI) cannot occur when $\alpha \le \gamma_0 + \pi/2$. If $\alpha < \pi/2$ and $|\pi/2 - \gamma_0| < \alpha$ or if $\alpha = \pi/2$ then f must be continuous up to O.*

Corollary 2.2. *Let $\Omega \in \mathbb{R}^2$ be a connected open subset that is symmetric about the $x-$axis, $\alpha \geq \pi/2, \gamma(x,-y) = \gamma(x,y)$ for all $x \in \partial\Omega$, and $\lim_{s\downarrow 0} \gamma^{\pm}(s) = \gamma_0$, satisfying*

$$\alpha - \pi/2 \leq \gamma_0 \leq \pi/2 \quad or \quad \pi/2 \leq \gamma_0 \leq 3\pi/2.$$

Let $f(x,y)$ be a bounded solution to (2.1) satisfying (2.2) that is even in y. Then f must be continuous up to O. The condition on the symmetry of f is automatic when $\kappa \geq 0$.

Theorem 2.3. *(Concus-Finn Conjecture): Let $\Omega \in \mathbb{R}^2$ be a connected open subset, with $0 < \alpha < \pi/2$,. Suppose that $\lim_{s\downarrow 0} \gamma^{\pm}(s) = \gamma_0^{\pm}$, where $0 < \gamma_0^-, \gamma_0^+ < \pi$. If $2\alpha + \left|\gamma_0^+ - \gamma_0^-\right| > \pi$ then any solution to (2.1) and (2.2) with $\kappa \geq 0$ has a jump discontinuity at O.*

Proof. See [21]. □

A nonconvex version of the conjecture has also been proved by Lancaster in 2010.

Theorem 2.4. *Let $\Omega \in \mathbb{R}^2$ be a connected open subset, with $\pi/2 \leq \alpha \leq \pi$. Let $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{O\}$ be a bounded solution of (2.1), (2.2). Suppose $\lim_{s\downarrow 0} \gamma^{\pm}(s) = \gamma_0^{\pm}$ where $0 < \gamma_0^-, \gamma_0^+ < \pi$. If*

$$2\alpha + \left|\gamma_0^+ - \gamma_0^-\right| > \pi$$

then f is discontinuous at O whenever

$$\left|\gamma_0^+ - \gamma_0^-\right| > 2\alpha - \pi,$$

or

$$\left|\gamma_0^+ + \gamma_0^- - \pi\right| > 2\pi - 2\alpha.$$

Proof. See [22], Theorem 2.1. □

The main comparison principle that will be used is the following one

Theorem 2.5. *(Concus-Finn Comparison Principle): Let* $\kappa \geq 0$, *and* $N(u) \geq N(v)$ *in* Q, *where* Q *is an elliptic operator and* $N(u) = div(Tu) - \kappa u$. *Assume that* $u \in C^2(\mathbb{R})$ *and* $N(u) = 0$. *Let* $\sum$ *be the boundary of* Ω, *and assume it admits a decomposition* $\sum = \sum_\alpha \cup \sum_\beta \cup \sum_0$ *such that* $\sum_0$ *can be covered, for any* $\epsilon > 0$ *by a countable number of discs* B_{δ_i} *of radius* δ_i *such that* $\sum \delta_i < \epsilon$, *and no regularity hypothesis needed on* $\sum_\alpha$ *or* $\sum_0$. *Suppose that*

$$v \geq u \qquad \text{on } \textstyle\sum_\alpha$$

$$\nu \cdot Tv \geq \nu \cdot Tu \qquad \text{on } \textstyle\sum_\beta$$

Then we have the following:

(i) If $k > 0$ *or if* $\sum_\alpha \neq \varnothing$ *then* $v \geq u$ *in* Ω. *Equality holds at any point if and only if* $v = u$

(ii) If $k = 0$, $\sum_\alpha = \varnothing$ *then* $v(x) = u(x) + constant$ *in* Ω.

Proof. See [13], Theorem 5.1. □

CHAPTER 3

3 TECHNICAL PREPARATORY RESULTS

3.1 Statement of The Problem

Studying the behavior of solutions of capillary boundary value problems near a re-entrant (or nonconvex) corner is a central topic in the geometric analysis of capillary surfaces, since it gives us good insight into the structure of the solution, and may suggest numerical techniques for computing solutions of these partial differential equations. The question of whether central fans exist or not is significantly important, and may lead to the question whether a solution of the equation is continuous at the corner or not.

In his 1997 Math Reviews Featured Review of [20], Robert Finn wrote: "*The paper is perhaps as important for questions to which it calls attention as for those it answers. Notably, no general conditions are presented under which a central fan will appear.*"

In the 2004 Pacific Journal of Mathematics article "*On a Theorem of Lancaster and Siegel*", Danzhu Shi and Robert Finn [23] proved that a symmetry condition for the domain and contact angle which Lancaster and Siegel assumed is necessary; they did this by making an arbitrarily small, specific perturbation of the geometry of the domain in a particular case and showing that the central fan which had existed in the symmetric case did not exist in the perturbed case. One conclusion which this article might suggest is that central fans are unstable. This dissertation is concerned with investigating the following questions:

1. Can capillary surfaces whose geometry or contact angle is not symmetric about the x-axis have central fans?

2. Are central fans stable with respect to the "right" topology?

These questions may not have complete, nontrivial answers. In my dissertation, I have investigated these questions by positively answer (2) in a manner that establishes examples which positively answer (1). I will prove that these central fans are stable with respect to restricted perturbations of the contact angle. The idea used is to perturb the contact angle of a capillary surface which has a central fan and show that the perturbed capillary surface still has a central fan. The perturbation in question is allowed to break the symmetry of the problem by breaking the symmetry of the contact angle; this provides examples which answer (1) in a positive manner and shows that central fans are stable in a particular topology.

3.2 Lemmas

The following Lemmata and Theorems will be used in Chapter 4 to establish the stability of central fans. From now on, we assume $\kappa > 0$.

Lemma 3.1. *Let Ω be a bounded open domain in $\mathbb{R}^2$ with Lipschitz boundary and let Γ be an open subset of $\partial\Omega$ which is a C^4 curve. Let $\gamma \in L^\infty(\partial\Omega)$ satisfy $\delta \leq \gamma \leq \pi - \delta$ almost all on $\partial\Omega$ for some $\delta > 0$ and $\gamma \in C^{1,\beta}(\Gamma)$ for some $\beta \in (0,1)$. Suppose there exists $f \in C^2(\Omega) \cap L^\infty(\Omega)$ which satisfies*

$$div(Tu) = \kappa u \ \ in \ \ \Omega \tag{3.1}$$

and

$$Tu \cdot \nu = \cos(\gamma) \ \ on \ \ \Gamma, \tag{3.2}$$

where

$$Tf = \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}.$$

Then $f \in C^2(\Omega \cup \Gamma)$.

Proof. This follows from the proof of Theorem 1 of [12], which relies on local arguments (e.g. page 31 of [12]). □

Lemma 3.2. *Let Ω be a bounded open domain in $\mathbb{R}^2$ with Lipschitz boundary and let Γ be an open subset of $\partial\Omega$ which is a $C^{1,\beta}$ curve for some $\beta \in (0,1)$. Let $\phi \in L^\infty(\partial\Omega)$ be in $C^{1,\beta}(\Gamma)$.*

Suppose $g \in C^2(\Omega) \cap L^\infty(\Omega)$ *is the variational solution of*

$$div(Tu) = \kappa u \;\; in \;\; \Omega \qquad \text{and} \qquad u = \phi \;\; on \;\; \partial\Omega;$$

that is, g *minimizes* $J(\cdot)$ *over* $BV(\Omega)$*, where*

$$J(u) = \int\limits_{\Omega} \sqrt{1 + |Du|^2} + \int\limits_{\Omega} \int\limits_{0}^{u} \kappa t dt dx + \int\limits_{\partial\Omega} |u - \phi| \, ds$$

for $u \in BV(\Omega)$*. Set*

$$T = \{(x,t) \in \Gamma \times \mathbb{R} : \min\{\phi(x), g(x)\} \leq t \leq \max\{\phi(x), g(x)\}\}$$

and let G *be the graph of* g *over* Ω*. Then for each* $x_0 \in \Gamma$*, there exists a* $\delta > 0$ *such that* $\{x \in \partial\Omega : \; |x - x_0| \leq \delta\} \subset \Gamma$ *and* $\{(x,t) \in T \cup G : \; |x - x_0| < \delta\}$ *is a* $C^{1,\beta}$ *manifold -with-boundary whose boundary is* $\{(x, \phi(x) : x \in \Gamma, \; |x - x_0| < \delta\}$*.*

Proof. The proof follows from the proof of Theorem 4.2 of [24]. □

Theorem 3.1. *Let* Ω *be a bounded open domain in* $\mathbb{R}^2$ *with Lipschitz boundary and let* Γ *be an open subset of* $\partial\Omega$ *which is a* C^4 *curve or a finite disjoint union of* C^4 *curves. Let* $\gamma \in L^\infty(\partial\Omega)$ *satisfy* $\delta \leq \gamma \leq \pi - \delta$ *almost all on* $\partial\Omega$ *for some* $\delta > 0$ *and* $\gamma \in C^{1,\beta}(\Gamma)$ *for some* $\beta \in (0,1)$*. Suppose there exists* $f \in C^2(\Omega) \cap L^\infty(\Omega)$ *which satisfies (3.1) and (3.2). Let* $\epsilon > 0$*. Define* $g = g_\epsilon \in BV(\Omega)$ *to be the minimizer over* $BV(\Omega)$ *of* $J_\epsilon(\cdot)$*, where*

$$J_\epsilon(u) = \int\limits_{\Omega} \sqrt{1 + |Du|^2} + \int\limits_{\Omega} \int\limits_{0}^{u} k t dt dx + \int\limits_{\partial\Omega} |u - (f + \varepsilon)| \, ds$$

for $u \in BV(\Omega)$*. We have the following:*

(i) $g \in C^2(\Omega)$ *and satisfies (3.1).*

(ii) $g \in C^{1,\beta}(\Omega \cup E)$ *for each compact subset* E *of* Γ *and hence the contact angle*

$$\gamma_g \stackrel{def}{=} \arccos\left(Tg \cdot \mu\right) \in [0, \pi] \tag{3.3}$$

is well-defined and continuous on Γ, *where* μ *denotes the outward unit normal to* $\partial\Omega$.

(iii) Suppose there is a finite set $A = \{x_1, \ldots, x_m\} \subset \partial\Omega$ *such that* $\Gamma = \partial\Omega \setminus A$. *Then* $f \leq g \leq f + \epsilon$ *in* Ω.

(iv) Suppose there is a finite set $A = \{x_1, \ldots, x_m\} \subset \partial\Omega$ *such that* $\Gamma = \partial\Omega \setminus A$. *Then* $\gamma_g < \gamma$ *on* Γ.

Proof. (i) Notice that the existence of g follows from Theorem 5 of [11], or Theorem 2.1 of [7]. The interior regularity of g follows from Theorem 3.1 of [7] (see also [11], page 174; [28], Theorem 3). The fact that g satisfies (3.1) is standard (e.g. [11], page 174).
(ii) The boundary regularity of g follows from Lemma 3.2.
(iii) Notice that $f, g \in C^2(\Omega) \cap C^0(\Omega \cup \Gamma)$. Set $M = \{x \in \Omega : f(x) > g(x)\}$. On $\partial M \cap \Gamma$, $g < f + \epsilon$ and so (ii) implies that $\gamma_g = 0$ on $\partial M \cap \Gamma$. Thus $f = g$ on $\Omega \cap \partial M$ and $\gamma_g = 0$ almost everywhere on $\partial\Omega \cap \partial M$ and so the General Comparison Principle (Theorem 2.5) implies $f \leq g$ in M; hence $M = \varnothing$.

Now let $\tau > 0$ and set $N = \{x \in \Omega : g(x) > f(x) + \epsilon + \tau\}$. Then $g = f + \epsilon + \tau$ on $\Omega \cap \partial N$ and $g > f + \epsilon$ on $\partial N \cap \Gamma$ and so (ii) implies $\gamma_g = \pi$ almost everywhere on $\partial\Omega \cap \partial N$. The General Comparison Principle then implies $g \leq f + \epsilon + \tau$ and so $N = \varnothing$. Therefore $g \leq f + \epsilon + \tau$ in Ω for each $\tau > 0$ and so $g \leq f + \epsilon$ in Ω.
(iv) Suppose first $x \in \Gamma$ and there is a sequence $\{y_j\}$ in Γ such that $x = \lim_{j\to\infty} y_j$ and $g\left(y_j\right) < f\left(y_j\right) + \epsilon$ for each j. Then (ii) implies $\gamma_g\left(y_j\right) = 0$ for each j and so $\gamma_g(x) = 0$. Since $\gamma \in (0, \pi)$, we see that $\gamma_g(x) = 0 < \gamma(x)$.

Suppose next that $x \in \Gamma$ and $g \geq f + \epsilon$ in $P \cap \Gamma$, where P is a neighborhood of x in $\mathbb{R}^2$. From (iii), we see that $g = f + \epsilon$ in $P \cap \Gamma$. If $\gamma_g(x) > \gamma(x)$, then $g\left(x - t\mu(x)\right) > f\left(x - t\mu(x)\right) + \epsilon$

for $t > 0$ small and this contradicts (iii). (Recall that $\mu(x)$ is the exterior unit normal to $\partial\Omega$ at x.) Thus $\gamma_g \leq \gamma$ on Γ.

Finally, suppose $x \in \Gamma$, $\gamma_g(x) = \gamma(x)$ and $g = f + \epsilon$ in $P \cap \Gamma$, where P is a neighborhood of x in $\mathbb{R}^2$. Since $g \leq f + \epsilon$ in Ω and $\gamma_g(x) = \gamma(x)$, the tangent plane Π_g to $z = g$ at $(x, g(x))$ and the tangent plane Π to $z = f + \epsilon$ at $(x, g(x)) = (x, f(x) + \epsilon)$ must coincide. Now the mean curvature H_g of $z = g$ at $(x, g(x))$ is $\kappa g(x)/2$ and the mean curvature H_f of $z = f + \epsilon$ at $(x, g(x))$ is $\kappa f(x)/2 = (\kappa g(x) - \kappa\epsilon)/2$. Since $g = f + \epsilon$ in $P \cap \Gamma$, the (signed) curvature of the curve $z = f(x - t\mu(x)) + \epsilon$ must be strictly less than the (signed) curvature of the curve $z = g(x - t\mu(x))$ for $t > 0$ small and so $g(x - t\mu(x)) > f(x - t\mu(x)) + \epsilon$ for $t > 0$ small, in contradiction to (iii). □

Example 3.1. *(Estimation of The Perturbation of The Contact Angle).*

In this example, we find a lower bound of the gap difference between γ and γ_g. Consider a cylindrical tube made of silver, and with radius 1 cm, with distilled water in it. The contact angle of the water in a silver container is approximately equal to $\pi/2$, so the capillary surface is horizontal. Let f be that surface, and consider g in Theorem 3.1, and assume that $g = \varepsilon$ on $\partial\Omega$. Raise f by ε units up, and consider the lower hemisphere of radius $R \geq 1$, say $p(r)$, such that $p(1) = \varepsilon$. The equation of p is

$$p(r) = \varepsilon + \sqrt{R^2 - 1} - \sqrt{R^2 - r^2}.$$

Notice that $p(0) = \varepsilon + \sqrt{R^2 - 1} - R$, and since $p(0) < p(r) < 1$, we can set

$$\frac{2}{\kappa R} \leq \varepsilon + \sqrt{R^2 - 1} - R, \tag{3.4}$$

that is

$$\frac{2}{\kappa R} \leq p(0) \leq p(r) \leq p(1) = \varepsilon,$$

or in other words

$$-\kappa\varepsilon = N(f+\varepsilon) \le \frac{2}{R} - \kappa p \le N(g) = 0.$$

By Concus-Finn comparison principle, we get $g \le p \le f+\varepsilon$ and so $\gamma_g < \gamma_p < \frac{\pi}{2}$. Hence, we obtain

$$\frac{\pi}{2} - \gamma_g \ge \frac{\pi}{2} - \gamma_p. \tag{3.5}$$

Calculations show that

$$\cos\gamma_p = \frac{1}{R},$$

Letting $R = 2$, then from (3.4) we choose $\varepsilon = 2 - \sqrt{3} + 1/\kappa$. We obtain $\gamma_p = \pi/3$, i.e. from (3.5) we conclude

$$\gamma - \gamma_g \ge \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

This example shows that a lower estimation can be found for the gap difference between the contact angles of f and g.

Theorem 3.2. *Let Ω be a bounded open domain in $\mathbb{R}^2$ with Lipschitz boundary and let Γ be an open subset of $\partial\Omega$ which is a C^4 curve or a finite disjoint union of C^4 curves. Suppose there is a finite set $A = \{x_1, \dots, x_m\} \subset \partial\Omega$ such that $\Gamma = \partial\Omega \setminus A$. Let f, $\epsilon > 0$ and $g = g_\epsilon$ be as in the previous Theorem. Let $\sigma \in C^{1,\beta}(\Gamma) \cap L^\infty(\partial\Omega)$ satisfy*

$$\gamma_g(x) \le \sigma(x) \le \gamma(x) \ \textit{ for almost all } \ x \in \partial\Omega. \tag{3.6}$$

Then the variational solution h of (3.1)-(3.2) with γ replaced by σ in (3.2) satisfies

$$f \le h \le g \le f + \varepsilon \ \textit{ in } \Omega; \tag{3.7}$$

here h is the minimizer over $BV(\Omega)$ of $K(\cdot)$, where

$$K(u) = \int\limits_{\Omega} \sqrt{1 + |Du|^2} + \int\limits_{\Omega} \int\limits_{0}^{u} \kappa t dt dx - \int\limits_{\partial\Omega} \cos(\sigma) u \; ds$$

for $u \in BV(\Omega)$.

Proof. The variational solution h of (3.1)-(3.2) exists by Theorem 7.9 of [13]). From (3.6) we obtain

$$\cos\gamma(x) \leq \cos\sigma(x) \leq \cos\gamma_g(x),$$

that is

$$T(f) \cdot \nu \leq T(h) \cdot \nu \leq T(g) \cdot \nu \tag{3.8}$$

Using (3.8), we apply Concus-Finn comparison to $u = f, v = h$ and $u = h, v = g$ in both examples, so we conclude that

$$f \leq h \leq g \leq f + \varepsilon \text{ in } \Omega. \tag{3.9}$$

The theorem is therefore proved. $\square$

CHAPTER 4

4 THE MAIN RESULT

4.1 Stability of Central Fans

Theorem 4.1. *Let $\Omega \in \mathbb{R}^2$ be an open, connected, bounded Lipschitz domain, that is symmetric about the $x-$axis with a boundary $\partial\Omega \in C^4$ except at a corner O with an openning angle 2α, with $\alpha > \pi/2$. Suppose $\gamma : \partial\Omega \to (0, \pi)$ is a piecewise $C^{1,\beta}$ map, $\gamma(x, -y) = \gamma(x, y)$, and*

$$f \in C^2(\Omega) \cap C^{1,\beta}(\bar{\Omega} \setminus \{O\})$$

is a solution of (3.1) and (3.2).

Let $\gamma^+(s)$ and $\gamma^-(s)$ denote γ along the arcs $\partial\Omega^+$ and $\partial\Omega^-$ respectively, where $s = 0$ corresponds to the corner O and

$$\gamma_0 = \lim_{s \downarrow 0} \gamma^{\pm}(s), \text{ where } 0 < \gamma_0^{\pm} < \pi.$$

If $\gamma_0 < \alpha - \pi/2$ then f has a central fan at O.

Proof. Since $\gamma_0 < \alpha - \pi/2$ then we get

$$|2\gamma - \pi| > 2\pi - 2\alpha.$$

By Theorem 2.4, this implies that f is discontinuous at O. Since f is even in y, the radial limits of f cannot behave as (I) or (D) of Theorem 2.2, therefore it behaves as (ID) or (DI) of Theorem 2.2, in which we conclude that there exists a central fan at O. □

Theorem 4.2. *Let $\Omega, f,$ and γ be as in the previous Theorem. Suppose there exists a central fan at the corner $O = (0, 0)$, and assume that we have the following case*

$$Rf(\alpha) > Rf(0) \text{ and } Rf(-\alpha) > Rf(0).$$

Then, there exists $\delta > 0$ such that for every $\varepsilon \in (0, \delta]$, the solution h to the capillary problem

$$div(Tu) = \kappa u \ \ in \ \ \Omega \tag{4.1}$$

subject to the condition

$$Tu \cdot \nu = cos\sigma \ \ on \ \ \partial\Omega \tag{4.2}$$

where

$$\gamma_g(x) \leq \sigma(x) \leq \gamma(x),$$

and $g \in BV(\Omega)$ minimize

$$J(u) = \int\limits_{\Omega} \sqrt{1 + |Du|^2} + \int\limits_{\Omega} \int\limits_{0}^{u} ktdtdx + \int\limits_{\partial\Omega} |u - (f + \varepsilon)| \, ds$$

for $u \in BV(\Omega)$, has a central fan.

Proof. Choose δ to be

$$\frac{1}{3}\min\{Rf(\alpha) - Rf(0), \, Rf(-\alpha) - Rf(0)\},$$

and let $\varepsilon \in (0, \delta]$. Then by (3.9) we get

$$Rf(\theta) \leq Rh(\theta) \leq Rf(\theta) + \varepsilon \ \text{ for } \theta \in [-\alpha, \alpha]. \tag{4.3}$$

In particular, we have the following

$$0 \leq Rh(\alpha) - Rf(\alpha) \leq \varepsilon,$$

$$0 \leq Rh(0) - Rf(0) \leq \varepsilon,$$

and

$$0 \leq Rh(-\alpha) - Rf(-\alpha) \leq \varepsilon.$$

We obtain

$$\begin{aligned} Rh(\alpha) - Rh(0) &\geq (Rf(\alpha) - \varepsilon) - (Rf(0) + \varepsilon) \\ &= (Rf(\alpha) - Rf(0)) - 2\varepsilon \geq \varepsilon. \end{aligned}$$

and

$$\begin{aligned} Rh(-\alpha) - Rh(0) &\geq (Rf(-\alpha) - \varepsilon) - (Rf(0) + \varepsilon) \\ &= Rf(-\alpha) - Rf(0) - 2\varepsilon \geq \varepsilon. \end{aligned}$$

Thus, h has a central fan. We conclude that under a small perturbation of the capillary surface having a central fan, the resultant surface is capillary will preserve the central fan. In other words, central fans are stable with respect to small changes in contact angles. □

4.2 Examples

Example 4.1. *(Stability of a Central Fan I).*

Consider Figure 1 of [23]. Let $\partial\Omega$ and γ be as in Theorem 4.1. Let $\alpha = 7\pi/8$, and consider a bounded function $f(x, y)$ that is even in y, and is a bounded solution to a capillary problem with a contact angle $\gamma \leq \pi/4$. Notice that in this case, we have

$$\gamma < \alpha - \pi/2$$

Since Ω is symmetric, and then cases (I) and (D) cannot hold. By Theorem 2.5, a central fan exists. Since $\alpha + \gamma \leq 3\pi/2$ then according to corollary 2.1 case (ID) cannot occur. So, we conclude that the radial limit behaves as (DI). Consider g as in Theorem 4.2, and perturb the angle γ as we did in the Theorem, so that $\gamma_g \leq \sigma \leq \gamma$, and consider the solution h to (4.1) and (4.2). Then, by Theorem 4.2 we conclude that h also has a central fan. This implies that radial limit of h has a central fan, and behaves as (DI), exactly as that of f.

Example 4.2. *(Stability of a Central fan II).*

Let a square Γ be at the middle of a disc C, and consider $\Omega = C \setminus \Gamma$. Clearly, there are four corners with $\alpha = 3\pi/4$. Let $\kappa \geq 0$, and let $f(x, y)$ be the solution to the capillary problem in Ω, that is even with y, and with a contact angle $\gamma < \pi/4$. Then by Theorem 2.4, the function f is discontinuous at O. Also, from corollary 2.1 the case (ID) cannot occur. Also, (I) and (D) cannot occur because of the symmetry, therefore the radial limits of f behave as (DI). Again, perturb γ as we did in the previous example, and consider the solution h to (4.1) and (4.2). Theorem 4.2 shows that the radial limits of h behave as (DI).

CHAPTER 5

5 CONCLUSION

5.1 Applications of The Results

Chemists, engineers, physicists and others who study surface chemistry, capillarity and related topics have developed empirical methods of predicting and explaining capillary effects. The mathematical theory of capillarity was developed in 1805-6 by Young and Laplace and placed on a firmer theoretical foundation in 1830 by Gauss, before physicists, chemical engineers and others developed and began using sophisticated experimental techniques to investigate surface tension, surfactants, wetting and dewetting of surfaces, the relationship between surface roughness and the wetting phenomenon, etc.

The principal conclusion of this work is the proof that in specific configurations, the existence of central fans of radial limits of nonparametric capillary surfaces is a stable mathematical phenomenon. I wish to discuss the relationship of this work to the following items.

(1) The validity of the Young-Laplace-Gauss theory has been challenged over time. Is this theory appropriate for its applications? In particular, does this theory yield conclusions which can be tested experimentally? The answer to this last question is "yes". In a series of experiments proposed by Paul Concus (UC Berkeley) and Robert Finn (Stanford) and conducted in NASA drop towers, during space shuttle missions and on the MIR space station, the predictions of the mathematical theory were tested and were found to be correct; the symmetry breaking of a symmetric capillary problem occurred as predicted by the Young-Laplace-Gauss theory, for example, during an experiment on the MIR station. Since the results here require positive gravity, they represent predictions of the Young-Laplace-Gauss theory which do not require space travel in order to be tested. Chemical engineers, physicists, etc. often apply a variety of empirical rules (e.g. "advancing" contact angles, "receding" contact angles) and may question the Young-Laplace-Gauss theory. The conclusions obtained

here may lead to experiments which, like the various NASA experiments, may confirm that the Young-Laplace-Gauss theory offers a useful mathematical description of the macroscopic behavior of fluid surfaces.

(2) Central fans are important because they represent a type of boundary behavior which was completely unexpected by the experts. For example, when, in 1973, Paul Concus computed a nonparametric minimal surface in an L-shaped (i.e. L-Shaped Tromino) domain, he correctly assumed the radial limits at the reentrant corner existed but incorrectly predicted the behavior of these radial limits (i.e. a central fan existed at the reentrant corner but his a priori assumptions about the behavior of these radial limits prevented him from realizing this). The central fan question of finding necessary and sufficient conditions for the existence of central fans may be extremely difficult, but initial steps like the one here which examines the stability of central fans are important for understanding the question and the obstacles to its solution.

(3) What are applications of the work here? Our knowledge of industrial application of this work is limited. One place where my results might have application is in the process of "dip coating" certain types of capacitors. The DuPont company creates certain types of capacitors by dipping a block of material into a solution and allowing the solution to wet the bottom and portion of the sides of the block. The area and shape of the coated region can influence the electrical and magnetic properties of the capacitor. If the horizontal dimensions of the block are equal (i.e. the projection of the block is a square), then the symmetry conditions in Corollary 2.2 are satisfied (provided the container in which the block sits satisfy these conditions), and the capillary surface will either be continuous at the corner or have a central fan. If this symmetry is broken slightly, the results here suggest that the top of the coating (i.e. trace of the capillary surface) may continue to be continuous or "nearly" continuous, which might be advantageous to DuPont. (See [26]).

(4) In example 4.1, and 4.2, we perturb the contact angle of a capillary surface with a central fan $\gamma < \pi/4$, to retain the central fan after perturbation. In fact, if $\gamma = \pi/4$, then $\gamma = \alpha - \pi/2$, and according to corollary (2.2), we conclude that the function f is continuous at the corner, so a small perturbation causes the continuity to vanish and central fans would replace it. On the other hand, we found that central fans resist small perturbations. The results in this study, as illustrated by the examples, are evidence that reducing the contact angle with a small change may not affect the central fan, and they will continue to exit. It turns out that central fans can be more stable than continuity at the nonconvex corner. Maximizing the capillarity of fluids (and hence reducing their contact angles) is widely used in industry. Battery manufacturers seek to maximize the capillarity of the electrolytes to maintain the contact with the electrodes. Since these results show that the central fans of the liquid is stable over the nonconvex corners, I hope this will help to improve their products.

(5) It is a well known fact that raising temperature of the liquid reduces it's surface tension. This is because the temperature causes the kinetic energy of the molecules to increase, therefore causing the cohesive forces between them to reduce. Thus, if a liquid is in a capillary tube, then adhesive forces between molecules of the liquid and the molecules of the walls will increase, and thus more liquid molecules will be attracted to the wall, and this will result in reducing the contact angle. I hope battery manufacturers and other manufacturers dealing with electric circuits can take advantage of my results to improve the resistance efficiency of their products that are exposed to overheating and are influenced by temperature, and to help them in extending the end of life of their products.

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