GEOMETRIC DISCRETIZATION SCHEMES AND DIFFERENTIAL COMPLEXES FOR ELASTICITY

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GEOMETRIC DISCRETIZATION SCHEMES AND DIFFERENTIAL COMPLEXES FOR ELASTICITY

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SUMMARY

In this research, we study two different geometric approaches, namely, the discrete exterior calculus and differential complexes, for developing numerical schemes for linear and nonlinear elasticity. Using some ideas from discrete exterior calculus (DEC), we present a geometric discretization scheme for incompressible linearized elasticity. After characterizing the configuration manifold of volume-preserving discrete deformations, we use Hamilton's principle on this configuration manifold. The discrete Euler-Lagrange equations are obtained without using Lagrange multipliers. The main difference between our approach and the mixed finite element formulations is that we simultaneously use three different discrete spaces for the displacement field. We test the efficiency and robustness of this geometric scheme using some numerical examples. In particular, we do not see any volume locking and/or checkerboarding of pressure in our numerical examples. This suggests that our choice of discrete solution spaces is compatible. On the other hand, it has been observed that the linear elastostatics complex can be used to find very efficient numerical schemes. We use some geometric techniques to obtain differential complexes for nonlinear elastostatics. In particular, by introducing stress functions for the Cauchy and the second Piola-Kirchhoff stress tensors, we show that 2D and 3D nonlinear elastostatics admit separate kinematic and kinetic complexes. We show that stress functions corresponding to the first Piola-Kirchhoff stress tensor allow us to write a complex for 3D nonlinear elastostatics that similar to the complex of 3D linear elastostatics contains both the kinematics an kinetics of motion. We study linear and nonlinear compatibility equations for curved ambient spaces and motions of surfaces in $\mathbb{R}^3$. We also study the relationship between the linear elastostatics complex and the de Rham complex. The

geometric approach presented in this research is crucial for understanding connections between linear and nonlinear elastostatics and the Hodge Laplacian, which can enable one to convert numerical schemes of the Hodge Laplacian to those for linear and possibly nonlinear elastostatics.

CHAPTER I

INTRODUCTION

Solving PDEs has always been a challenging task in computational mechanics. Having the correct solution spaces that posses the essential mathematical structure of the solutions of PDEs is crucial for designing stable numerical schemes. For example, there have been many efforts during the past five decades to find a stable mixed finite element method for linear elasticity. However, it was not until recently that Arnold and Winther [12] obtained such mixed finite elements. The main reason for their successful formulation is that they use a proper trial space that respects the correct geometric structure of the solution. Ideas related to differential forms play a crucial role in their derivations. In fact, the above formulation is closely related to the Hodge Laplacian problem that is defined on Riemannian manifolds, i.e. a manifold with a metric. One can consider the celebrated de Rham complex as the smooth structure of the solution spaces of this problem and try to discretize this complex. This was the main idea of the Finite Element Exterior Calculus introduced by Arnold and his coworkers [11]. By a proper discretization of the de Rham complex, they define the problem of the abstract Hodge Laplacian on a Hilbert complex with proper Hilbert spaces as discrete trial spaces and then they introduce an efficient numerical scheme by studying this problem. Efficient numerical schemes for elasticity have many engineering applications. It is known that both linear and nonlinear elasticity have rich geometric structures. This suggests that one may be able to obtain efficient numerical schemes for them by defining discrete analogous of these smooth structures.

1.1 Main Contributions

In this thesis, we study two different geometric approaches, namely, the discrete exteior calculus and differential complexes, for developing numerical schemes for linear and nonlinear elasticity as follows.

Discrete Exterior Calculus. The main challenge of the elasticity problem is that unlike electromagnetism that only requires differential forms one needs to consider various types of tensors for elasticity. We introduce a geometric structure-preserving scheme for linearized incompressible elasticity. First, we show that in the smooth case, the governing equations of incompressible elasticity can be obtained using Hamilton's principle over the space of divergence-free vector fields without using Lagrange multipliers. Then, we develop a discrete theory for linearized elasticity by assuming the domain to be a simplical complex and choosing a discrete displacement field, which is a primal vector field, as our primal unknown. Thus, we do not need to worry about compatibility equations. Then, we use a discrete definition of divergence to specify the space of discrete divergence-free vector fields over a simplical mesh and choose this space as our solution space. Motivated by the Lagrangian structure of the smooth case, we define a discrete Lagrangian and use Hamilton's principle over the discrete solution space without using Lagrange multipliers. We observe that pressure gradient appears in the discrete governing equations. We use the discrete Laplace-Beltrami operator to obtain the discrete pressure – a dual 0-form. This can be thought of as a geometric justification for the known fact that using different function spaces for pressure and displacement is crucial for obtaining robust numerical schemes for incompressible elasticity. Finally, we consider some numerical examples that suggest that our method is free of locking and checkerboarding of pressure.

The approach that we use for imposing the incompressibility constraint is equivalent to the method of Lagrange multipliers for obtaining the mixed formulation of incompressible elasticity. We directly use the space of discrete divergence-free displacements given by the DEC theory. This is the deviation of our approach from the FE method: The definition of the discrete divergence implies a specific linear interpolation for displacements. However, we are free to choose other interpolation methods for the displacement field when we calculate the kinetic and elastic energies. Therefore, unlike the FE method, we simultaneously use three different discrete solution spaces for the displacement field, in general. Similar to the mixed FE formulations, discrete solution spaces for the displacement field should be compatible with each other and with the discrete space of the pressure field to obtain a stable numerical scheme. Our numerical examples suggest that a choice of discrete spaces given by $\mathbb{P}_1$ polynomials over primal meshes for the incompressibility constraint, $\mathbb{P}_1$ polynomials over support volumes for the elastic energy, and $\mathbb{P}_0$ polynomials over primal meshes for the pressure field is a compatible choice.

Differential Complexes. Formulating linear elastostatics complex in terms of differential forms has been crucial in developing stable numerical schemes as this allows one to confine solution spaces to proper subsets of differential forms [10, 8, 12]. Therefore, it may be possible to obtain stable numerical schemes for nonlinear elastostatics by properly expressing it in terms of differential forms. There have been some efforts in the past to rewrite nonlinear elasticity in terms of differential forms [69]. The complexes that Arnold and his coworkers [10, 8, 12] have used are as follows: For 2D linear elastostatics, they consider kinetic complexes consisting of Airy stress functions. For 3D linear elastostatics, they use complexes that contain both the kinematics and kinetics of motion through the linear compatibility equations and Beltrami stress functions. There have been some ideas for defining stress functions for the Cauchy stress tensor for nonlinear elastostatics [103]. However, as far as we know, to this date there have not been similar complexes for nonlinear elastostatics.

We introduce the notion of stress functions for the first and second Piola-Kirchhoff stress tensors. First Piola-Kirchhoff stress functions enable us to derive a complex for 3D nonlinear elastostatics that contains both the kinematics and kinetics of motion, i.e. it contains the *generalized* compatibility equations and first Piola-Kirchhoff stress functions. In $\mathbb{R}^3$, we show that this complex is equivalent to the $\mathbb{R}^3$-valued de Rham complex, and therefore we can express 3D nonlinear elastostatics entirely in terms of differential forms. This implies that 3D nonlinear elastostatics is related to the de Rham complex more directly than 3D linear elastostatics as the nonlinear case is equivalent to a twisted de Rham complex but the linear case is equivalent to a certain restriction of another twisted de Rham complex. We also write kinetic complexes for 3D nonlinear elastostatics in terms of Cauchy and second Piola-Kirchhoff stress functions. For 2D nonlinear elastostatics, the kinematic and the kinetic complexes are separate. This can be considered as a result of the shorter length of de Rham complex for 2-manifolds in comparison with 3-manifolds. We derive the associated 2D kinetic complexes for various types of stress functions. The kinematic complex of 2D case in $\mathbb{R}^2$ is shown to be equivalent to the $\mathbb{R}^2$-valued de Rham complex.

We write the linear elastostatics complex as the linearization of a sequence of differential operators associated with nonlinear elasticity on manifolds with constant sectional curvatures. At first sight, 3-manifolds with constant sectional curvatures may seem to be too abstract and unphysical. However, note that 3D bodies equipped with nontrivial metrics (the Green deformation tensors) are special cases of such manifolds. By using some classical results in differential geometry, we obtain linear and nonlinear compatibility equations for motions in curved ambient spaces and also motions of surfaces in $\mathbb{R}^3$. We should mention that these results are equivalent to the compatibility equations obtained by using other approaches discussed in [103] for curved ambient spaces and [37] for surfaces. We also study the relation between linear elastostatics complex and the de Rham complex. Our results are useful if one wants to derive new numerical schemes for linear elastostatics from the existing numerical schemes of the Hodge Laplacian. For example, it is possible to define discrete differential forms as discrete cochains of simplical complexes [61]. By tracing the above construction, it should be possible to obtain a scheme for linear elastostatics using discrete cochains.

We first discuss our geometric numerical scheme for incompressible elasticity in chapter 2. In chapter 3, we study the complexes of linear and nonlinear elastostatics. Finally, we mention conclusions and future directions in chapter 4.

CHAPTER II

A GEOMETRIC NUMERICAL SCHEME FOR INCOMPRESSIBLE ELASTICITY

Finding robust numerical schemes for solving incompressible elasticity problems has been of great interest due to important applications of incompressible elasticity, e.g. in analyzing biological systems where soft tissue is usually modeled as an incompressible elastic body (see [108] and [112] and references therein). It is well known that numerical methods that are reliable for compressible elasticity severely fail for the case of incompressible problems (see [45, 58, 14] and references therein). Inaccurate results are usually due to the locking phenomenon. Locking, in general, is the loss of accuracy of the solution of a numerical scheme for the approximation of a parameter-dependent problem as the parameter tends to a critical value [15, 100]. For example, locking appears in plate and shell models as the thickness $d \to 0$, analysis of incompressible linear elasticity as Poisson's ratio $\nu \to 1/2$, and heat transfer problems as the ratio of conductivities $\mu \to 0$. A robust numerical method is uniformly convergent for all values of the parameter of the problem. Babuška and Suri [15] gave precise mathematical definitions for locking and robustness and gave some general results on the characterization of these phenomena.

To this date various numerical schemes have been developed for incompressible elasticity. The finite element (FE) method is one of the best numerical methods for compressible elasticity. However, FE results may be inaccurate in the near-incompressible and incompressible regimes. To overcome this difficulty, many different approaches have been proposed in the literature. The standard FE formulation based on displacements using low-order elements exhibits a poor performance for near-incompressible elasticity. It has been observed that higher order elements can avoid locking in near-incompressible linear elasticity [94]. Another approach is to use discontinuous Galerkin FE methods [57, 24, 79, 109, 108]. In these formulations, independent approximations are used on different elements and the continuity across boundaries of elements is weakly enforced. Nonconforming FE methods can also avoid locking for near-incompressible elasticity [46, 106, 35]. The simplicity of the aforementioned methods is due to the fact that they are based on the displacement variational formulation, and therefore, one does not need to include other variables in the formulation. There are some formulations based on the Hu-Washizu variational principle, where the displacement, strain, and stress are considered as independent variables. The method of enhanced assumed strain introduced by [97] and the method of mixed enhanced strain of [70] are both based on the Hu-Washizu variational principle. Another approach that has widely been used for near-incompressible and incompressible elasticity is mixed formulations based on the Hellinger-Reissner variational principle. In the near-incompressible regime, the stress and displacement are both unknowns. For the incompressible regime, the pressure and displacement are the primary unknowns, where the pressure is the Lagrange multiplier of the incompressibility constraint. It was observed that discrete spaces of the displacement and pressure should be compatible [45]. Various mixed formulations have been developed by using different techniques. Brink and Stephan [25] proposed an adaptive coupling of boundary elements and mixed FE method for incompressible elasticity. Cervera et al. [32] developed mixed simplical elements for incompressible elasticity and plasticity. Discontinuous Galerkin methods have also been used in the mixed formulations, see [96, 64] and references therein. It has been observed that reduced/selective integration techniques that are closely related to mixed formulations are useful for incompressible elasticity [65, 80]. In these methods, the inf-sup stability requirement are also enforced for the displacement and implicit pressure interpolant spaces.

Equal-order interpolation with stabilization methods [7] and average nodal pressure elements [21, 75] have also been successfully implemented. Bonet and Burton [21] proposed a linear tetrahedron element that prevents locking by introducing nodal volumes and evaluating nodal pressures in terms of these volumes. Gatica et al. [51] developed a dual-mixed finite element method for incompressible plane elasticity. Hauret et al. [58] introduced a diamond element FE discretization for compressible and incompressible linear and finite elasticity. Using both primal and dual vertices of an arbitrary simplicial mesh for the domain and its barycentric dual mesh, they constructed an associated diamond mesh. They defined interpolation spaces for displacement and pressure supported on the diamond mesh by choosing piecewise linear displacement interpolation on sub-elements and constant pressure interpolation on diamond elements. They proved that the displacement field converges optimally with mesh refinement and also showed that for the problem of linearized incompressible elasticity their scheme satisfies the inf-sup condition, and hence, it is well posed. Alternatively, it is also possible to use element-free Galerkin methods [18]. Vidal et al. [105] introduced a pseudo-divergence-free element-free Galerkin method using a diffuse divergence for near-incompressible elasticity. Similar techniques have been used by other researchers for developing mesh-free methods for mixed formulations and B-bar methods [87, 38, 91].

The incompressibility constraint can be imposed more directly using the stream function formulation [13]. Because the divergence of the displacement is zero in incompressible linear elasticity, there is a scalar-valued function called stream function whose curl gives the displacement. Then, the weak formulation of incompressible linear elasticity can be rewritten as a fourth-order elliptic problem over scalar functions. Auricchio et al. [13] used an isogeometric interpolation base on Non-Uniform Rational B-Splines (NURBS) [66] to obtain a locking-free isogeometric approach for the stream function formulation. The high continuity across the elements is the key advantage of NURBS functions. The solution of the stream function formulation automatically satisfies the incompressibility constraint, i.e. it is divergence-free by construction.

Another interesting idea in the literature (promoted mainly by Arnold and his coworkers [11]) is to use an “elasticity complex”, which is similar in form to the classical de Rham complex. In fact, Eastwood [41] showed that the linear elasticity complex can be constructed from the de Rham complex. Having a complex for a field theory, one then defines a discrete analogue of the continuum complex. In the case of finite element method, this gives the appropriate finite element spaces for different fields (e.g. displacements and stresses in the case of linear elasticity). This has led to the discovery of several stable mixed finite elements for linear elasticity [11].

Ciarlet and Ciarlet [36] proposed a new approach for finding the solution of planar linear elasticity that may be capable of handling near incompressible case as well. They showed that this problem can be alternatively reformulated as minimization of an associated Lagrangian over the strain field. They defined their finite element space over a triangulation of the reference configuration as the space of 2×2 symmetric matrix fields, which are constant over each triangle of the triangulation, has the same values for the degrees of freedom at common edges of any two distinct triangles, and is curl-curl free. This curl-curl free condition plays the role of the compatibility conditions. Thus, this method enables one to directly obtain the strains and stresses as they are considered the primal unknowns. Another numerical scheme for dealing with incompressibility is the finite volume method. Bijelonja et al. [20] developed a finite volume based method for incompressible linear elasticity using the solution of the integral form of the governing equations and the introduction of pressure as an additional variable. Considering several numerical examples, they concluded that such numerical methods are locking free.

Geometric ideas were first introduced in numerical electromagnetism (see [54] and references therein). Here the idea is to use some techniques from differential geometry, algebraic topology, and discrete exterior calculus to write the governing equations and constraints in terms of appropriate geometric entities and then look for the solutions in a proper solution space that satisfies the required constraints. The main advantage of such methods is that by construction they are free of traditional numerical artifacts such as loss of energy or momenta. Hirani et al. [63] used discrete exterior calculus to obtain a numerical method for Darcy flow. They used flux and pressure, which are considered to be differential forms, as the primal unknowns and then the numerical method was derived by using the framework provided by discrete exterior calculus for discretizing differential forms and operators that act on forms. Pavlov et al. [89] proposed a structure-preserving discretization scheme for incompressible fluids. Their main idea is that instead of discretizing spatial velocity, one can discretize push-forward of real-valued functions and the Lie derivative operator. They showed that the space of discrete push-forwards is the space of orthogonal, signed doubly-stochastic matrices and the space of discretized Lie derivatives is the space of antisymmetric null-column matrices. They obtained a discrete in space and continuous in time version of Euler equations using the Lagrange-d'Alembert principle for their discrete Lagrangian and then constructed a fully discrete variational integrator by defining a space-discrete/time-discrete Lagrangian.

The problem with elasticity is that unlike electromagnetism that only requires differential forms one needs to consider various types of tensors for elasticity. There have been recent efforts in the literature in geometrizing discrete elasticity. Chao et al. [33] used geometric ideas to introduce an integrator for nonlinear elasticity. Kanso et al. [69] used bundle-valued differential forms for geometrization of stress. Assuming the existence of some discrete scalar-valued and vector-valued discrete differential forms, Yavari [111] presented a discrete theory with ideas for developing a numerical geometric theory. In this chapter, we introduce a geometric structure-preserving scheme for linearized incompressible elasticity, see also Angoshtari and Yavari [5].

First, we review incompressible linear and nonlinear elasticity, their geometries, and their variational structures. In particular, we derive the governing equations of incompressible finite and linearized elasticity using Hamilton's principle without using Lagrange multipliers by a direct use of the configuration manifold of incompressible elasticity and the Hodge decomposition theorem. Then, we review discrete exterior calculus (DEC) specialized to elasticity applications and study discrete configuration manifold of 2D incompressible linearized elasticity in detail. We write kinetic and elastic energies of a discretized linear elastic body in the language of DEC. Using Hamilton's principle in the discrete configuration manifold of discrete incompressible linearized elasticity then gives the discrete Euler-Lagrange equations. Finally, we consider some numerical examples to demonstrate the efficiency and lack of any volume locking in our geometric scheme.

2.1 Incompressible Elasticity

In this section, we review some basic topics in finite and linearized incompressible elasticity. In particular, we study their variational structure and show that the governing equations of incompressible elasticity can be obtained using the variational principle over the space of volume-preserving deformations.

2.1.1 Incompressible Finite Elasticity

Here, we first review some preliminaries on finite elasticity and then study the variational structure of incompressible finite elasticity, see [82, 117] for more details. Let an m-dimensional Riemannian manifold $(\mathcal{B}, \mathbf{G})$ with local coordinates $\{X^A\}$ be the material manifold for an elastic body, i.e., in this manifold the body is stress free.[1] We assume that ambient space is another Riemannian manifold $(\mathcal{S}, \mathbf{g})$ of dimension $n \geq m$ with local coordinates $\{x^a\}$. Here for the sake of simplicity, we assume that $m = n$.

[1]In general, $(\mathcal{B}, \mathbf{G})$ is the underlying Riemannian manifold of the material manifold, which is where the body is stress-free. See [114, 116, 115] for more details.

We use $(.,.)_G$ and $(.,.)_g$ to denote the inner product using the metrics $\mathbf{G}$ and $\mathbf{g}$, respectively. Motion is a diffeomorphism $\varphi : \mathcal{B}\times\mathbb{R} \to \mathcal{S}$. If we define $\varphi_t := \varphi(\cdot, t) : \mathcal{B} \to \mathcal{S}$, then we note that $\varphi_t(\mathcal{B}) \subset \mathcal{S}$ is a submanifold of $\mathcal{S}$ and hence it inherits the metric structure of $\mathcal{S}$. Let $\varphi_X := \varphi(X, \cdot) : \mathbb{R} \to \mathcal{S}$. Then $\varphi_X(t)$ specifies a curve in $\mathcal{S}$ and so we can define the following vector field covering φ_t:

$$\mathbf{V}(X,t) = \varphi_{X*}\frac{d}{dt} = T_t(\varphi_X)\cdot\frac{d}{dt} = \frac{\partial\varphi_(X,t)}{\partial t}, \tag{1}$$

where the linear mapping $T_t\varphi_X$ is the derivative of φ_X at point t and $\frac{d}{dt}$ is the unit vector in the tangent space of $\mathbb{R}$ at t. This vector field is called material velocity. If we push forward $\mathbf{V}$, we obtain the spatial velocity $\mathbf{v}$, which is a vector field on $\varphi_t(\mathcal{B})$ given by $\mathbf{v}(x,t) = \mathbf{V}(\varphi_t^{-1}(x), t)$. Similarly, one can define the material acceleration $\mathbf{A}$ and the spatial acceleration $\mathbf{a}$ as

$$\mathbf{A}(X,t) = \frac{\partial\mathbf{V}(X,t)}{\partial t}, \qquad \mathbf{a}(x,t) = \mathbf{A}(\varphi_t^{-1}(x),t). \tag{2}$$

A motion φ is called volume preserving if for every nice set $\mathcal{U} \subset \mathcal{B}$ we have

$$\int_{\varphi_t(\mathcal{U})} dv = \int_{\mathcal{U}} dV, \tag{3}$$

where $\mathcal{U}$ is an open set with a piecewise C^1 boundary $\partial\mathcal{U}$ and $dV = \sqrt{\det\mathbf{G}}\ \boldsymbol{d}X^1\wedge\cdots\wedge \boldsymbol{d}X^n$ and $dv = \sqrt{\det\mathbf{g}}\ \boldsymbol{d}x^1\wedge\cdots\wedge\boldsymbol{d}x^n$ are the volume forms of $\mathcal{B}$ and $\mathcal{S}$, respectively. If $\varphi(X,t)$ is a volume-preserving motion then $\operatorname{div}\mathbf{v} = 0$ and its Jacobian $J(X,t) = 1$, where Jacobin is defined as $J = \sqrt{\det\mathbf{g}/\det\mathbf{G}}\det\mathbf{F}$, with $\mathbf{F} = T_X\varphi_t$. Balance of linear momentum reads

$$\rho_0\mathbf{A} = \rho_0\mathbf{B} + \mathrm{Div}\mathbf{P}, \tag{4}$$

where $\rho_0 = \rho_0(X)$ denotes mass density of $\mathcal{B}$, $\mathbf{B}$ is the body force, and $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor. The right Cauchy-Green deformation tensor is defined as $\mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F} = \varphi_t^*\mathbf{g}$. In this work, we consider hyperelastic materials, i.e. we assume existence of a stored energy function $W = W(X, \mathbf{G}, \mathbf{F}, \mathbf{g})$ or $W = W(X, \mathbf{C})$. For such materials we have the following identity

$$\mathbf{P} = \rho_0 \mathbf{g}^\sharp \frac{\partial W}{\partial \mathbf{F}}. \tag{5}$$

In the next section, we study the variational structure of incompressible finite elasticity.

2.1.1.1 Variational Structure of Incompressible Finite Elasticity

Let $\partial_d\mathcal{B}$ denote the subset of $\partial\mathcal{B}$ on which essential boundary conditions are imposed, i.e., $\varphi|_{\partial_d} = \varphi_d$, and let $\partial_\tau\mathcal{B}$ be the portion of $\partial\mathcal{B}$ on which natural boundary conditions $\boldsymbol{\tau} = \langle \mathbf{P}, \mathbf{N}^\flat \rangle$ are imposed, where $\mathbf{N}$ is the outward unit vector field normal to $\partial\mathcal{B}$, $\boldsymbol{\tau}$ is the traction vector, and $\langle , \rangle$ denotes the natural pairing of a vector and a form, i.e., contraction of a covariant index of one tensor with a contravariant index of another tensor. We define the space of configurations of $\mathcal{B}$ to be

$$\mathcal{C} = \{\psi : \mathcal{B} \to \mathcal{S} \mid \psi = \varphi_d \ \text{ on } \ \partial_d\mathcal{B}\}, \tag{6}$$

where φ_d denotes the essential boundary condition on $\partial_d\mathcal{B}$. One can show that $\mathcal{C}$ is a C^∞ infinite-dimensional manifold [44]. A tangent vector to a configuration $\psi \in \mathcal{C}$ is the tangent to a curve $c : (-\epsilon, \epsilon) \to \mathcal{C}$ with $c(0) = \psi$, which is a velocity field $\mathbf{U}$ covering ψ and vanishes on $\partial_d\mathcal{B}$. Therefore, we have

$$T\mathcal{C} = \{(\psi, \mathbf{U}) \,|\, \psi \in \mathcal{C}, \mathbf{U} : \mathcal{B} \to T\psi(\mathcal{B}) \ \& \ \mathbf{U}|_{\partial_d\mathcal{B}} = 0\}. \tag{7}$$

$T\mathcal{C}$ is usually called the space of variations. Let $H^s = W^{s,2}$ be the Sobolev space consisting of all mappings $\xi : \mathcal{B} \to \mathcal{S}$ such that ξ and all its derivatives up to order s

belong to the Hilbert space L^2.[2] Note that by defining an appropriate inner product on H^s, it is possible to show that H^s is a Hilbert space [45]. The configuration space for incompressible elasticity is

$$\mathcal{C}_{vol} = \{\psi \in \mathcal{C} \mid J(\psi) = 1\}. \tag{8}$$

Ebin and Marsden [44] showed that if $\mathcal{C} \subset H^s$ then $\mathcal{C}_{vol}$ is a smooth submanifold of $\mathcal{C}$. The tangent space of $\mathcal{C}_{vol}$ at a configuration $\psi \in \mathcal{C}_{vol}$ is

$$T_\psi \mathcal{C}_{vol} = \left\{ \mathbf{U} \in T_\psi \mathcal{C} \mid \operatorname{div}\left(\mathbf{U} \circ \psi^{-1} \right) = 0 \right\}. \tag{9}$$

For unconstrained finite elasticity, the Lagrangian $L : T\mathcal{C} \to \mathbb{R}$ is defined as

$$L(\varphi, \mathbf{V}) = K - V, \tag{10}$$

where

$$\begin{aligned} K(\mathbf{V}) &= \frac{1}{2} \int_{\mathcal{B}} \rho_0 \, (\mathbf{V}, \mathbf{V})_g dV = \frac{1}{2} \int_{\mathcal{B}} \rho_0 \, \|\mathbf{V}\|_g^2 dV, \\ V(\varphi) &= \int_{\mathcal{B}} \rho_0 \, W(X, \mathbf{F}) dV + \int_{\mathcal{B}} \rho_0 \, \mathcal{V}_{\mathbf{B}}(\varphi) dV \\ &+ \int_{\partial_{\boldsymbol{\tau}} \mathcal{B}} \mathcal{V}_{\boldsymbol{\tau}}(\varphi) dA, \end{aligned} \tag{11}$$

with $D_\varphi \mathcal{V}_{\boldsymbol{\tau}} = -\boldsymbol{\tau}$ and $D_\varphi \mathcal{V}_{\mathbf{B}} = -\mathbf{B}$, where $\mathbf{B}(X,t) = \mathbf{b}(\varphi(X,t),t)$ is the material body force and D_φ denotes derivative with respect to φ. Note that in Euclidian space, one can consider dead loads as $\mathcal{V}_{\boldsymbol{\tau}}(\varphi) = -(\boldsymbol{\tau}, \varphi)_g$ and $\mathcal{V}_{\mathbf{B}}(\varphi) = -(\mathbf{B}, \varphi)_g$. By setting $\delta \int_0^T L dt = 0$ in the time interval $[0, T]$, one obtains the Euler-Lagrange equations for finite elasticity. For unconstrained finite elasticity, the Euler-Lagrange equations are equivalent to the weak form and the strong form of the governing field equations of

[2] Recall that $\xi : \mathcal{B} \to \mathbb{R}^n$ belongs to L^2 if it is square integrable, i.e., $\|\xi\|_{L^2}^2 = \int_{\mathcal{B}} \|\xi\|^2 dV < \infty$.

nonlinear elasticity [82].

For incompressible finite elasticity we require $\delta \int_0^T L dt = 0$ over volume-preserving motions. To solve this problem, one needs to impose the constraint $J = 1$. This can be done by directly imposing the constraint into the Lagrangian using the Lagrange multipliers [86]. An alternative approach, which is more in line with our discretization philosophy, is to consider the Lagrangian (10) on $T\mathcal{C}_{vol}$ instead of $T\mathcal{C}$ as follows [82] (see also [89] for a similar treatment of incompressible perfect fluids). We want to find a curve $\varphi_t \in \mathcal{C}_{vol}$ on the time interval $[0, T]$ with $\varphi_0 = \mathrm{Id}_{\mathcal{B}}$ ($\mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is the identity map) and $\varphi_T = \tilde{\varphi} \in \mathcal{C}_{vol}$ such that

$$I = \delta \int_0^T L(\varphi_t) dt = 0. \tag{12}$$

Let $\varphi_{t,s} \in \mathcal{C}_{vol}$ be a variation field such that $\varphi_{t,0} = \varphi_t$ and $\delta\varphi = \frac{d}{ds}\big|_{s=0} \varphi_{t,s}$ is a vector field in $T\mathcal{C}_{vol}$. We consider proper variations, and therefore, $\varphi_{0,s} = \varphi_0$ and $\varphi_{T,s} = \tilde{\varphi}$. Then, (12) is equivalent to

$$I = \left(\frac{d}{ds} \int_0^T L(\varphi_{t,s}, \dot{\varphi}_{t,s}) dt \right) \bigg|_{s=0} = 0. \tag{13}$$

We have

$$\begin{aligned} L(\varphi_{t,s}, \dot{\varphi}_{t,s}) &= \int_{\mathcal{B}} \left\{ \frac{1}{2} \rho_0 \left(\dot{\varphi}_{t,s}, \dot{\varphi}_{t,s} \right)_g - \rho_0 W \left(X, T_X \varphi_{t,s} \right) - \rho_0 \mathcal{V}_{\mathbf{B}} \left(\varphi_{t,s} \right) \right\} dV \\ &- \int_{\partial_{\boldsymbol{\tau}} \mathcal{B}} \mathcal{V}_{\boldsymbol{\tau}} \left(\varphi_{t,s} \right) dA, \end{aligned} \tag{14}$$

and therefore, using the metric compatibility and symmetry of the Levi-Civita connection [30], we obtain

$$\begin{aligned} I &= \left(\frac{d}{ds}\int_0^T L(\varphi_{t,s},\dot{\varphi}_{t,s})dt\right)\Bigg|_{s=0} = \int_0^T \left\{\frac{d}{ds}L(\varphi_{t,s},\dot{\varphi}_{t,s})\Bigg|_{s=0}\right\}dt \\ &= \int_0^T \left\{\int_{\mathcal{B}}\left[\rho_0(\dot{\varphi}_t,\nabla_{\frac{\partial}{\partial t}}\delta\varphi)_g - \left\langle\rho_0\frac{\partial W}{\partial \mathbf{F}},\nabla(\delta\varphi)\right\rangle + \rho_0\left(\mathbf{B},\delta\varphi\right)_g\right]dV \right. \\ &+ \left.\int_{\partial_{\boldsymbol{\tau}}\mathcal{B}}\left(\boldsymbol{\tau},\delta\varphi\right)_g dA\right\}dt, \end{aligned} \tag{15}$$

where the components of the two-point tensor $\nabla(\delta\varphi)$ read

$$\Big(\nabla(\delta\varphi)\Big)^a{}_A = (\delta\varphi)^a{}_{|A} = \frac{\partial(\delta\varphi)^a}{\partial X^A} + (\delta\varphi)^k\gamma^a_{kl}F^l{}_A, \tag{16}$$

with γ^a_{kl} denoting the Christoffel symbols of the coordinate system $\{x^a\}$ on $\mathcal{S}$ and

$$\left\langle\rho_0\frac{\partial W}{\partial \mathbf{F}},\nabla(\delta\varphi)\right\rangle = \rho_0\left(\frac{\partial W}{\partial \mathbf{F}}\right)_a{}^A\Big(\nabla(\delta\varphi)\Big)^a{}_A. \tag{17}$$

Because of metric compatibility of the Levi-Civita connection, we have

$$(\dot{\varphi}_t,\nabla_{\frac{\partial}{\partial t}}\delta\varphi)_g = \frac{d}{dt}\left(\dot{\varphi}_t,\delta\varphi\right)_g - \left(\nabla_{\frac{\partial}{\partial t}}\dot{\varphi}_t,\delta\varphi\right)_g. \tag{18}$$

As we consider proper variations, substitution of (18) into (15) yields

$$\begin{aligned} I &= -\int_0^T\left\{\int_{\mathcal{B}}\left[\rho_0(\mathbf{A}-\mathbf{B},\delta\varphi)_g + \left\langle\rho_0\frac{\partial W}{\partial \mathbf{F}},\nabla(\delta\varphi)\right\rangle\right]dV\right. \\ &- \left.\int_{\partial_{\boldsymbol{\tau}}\mathcal{B}}\left(\boldsymbol{\tau},\delta\varphi\right)_g dA\right\}dt. \end{aligned} \tag{19}$$

The integrand of I is continuous and T is arbitrary, so setting $I = 0$ results in

$$\int_{\mathcal{B}}\left[\rho_0(\mathbf{A}-\mathbf{B},\delta\varphi)_g + \left\langle\rho_0\frac{\partial W}{\partial \mathbf{F}},\nabla(\delta\varphi)\right\rangle\right]dV - \int_{\partial_{\boldsymbol{\tau}}\mathcal{B}}\left(\boldsymbol{\tau},\delta\varphi\right)_g dA = 0. \tag{20}$$

Equation (20) is called the weak form of the field equations of incompressible finite elasticity. Now observe that

$$\mathrm{Div}\left\langle \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \delta\varphi \right\rangle = \left\langle \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \nabla(\delta\varphi) \right\rangle + \left\langle \mathrm{Div}\left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right), \delta\varphi \right\rangle, \tag{21}$$

where $\mathrm{Div}\left\langle \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \delta\varphi \right\rangle = \left[(\rho_0 \frac{\partial W}{\partial \mathbf{F}})_a{}^A (\delta\varphi)^a \right]_{|A}$ and $\left[\mathrm{Div}(\rho_0 \frac{\partial W}{\partial \mathbf{F}})\right]_a = \left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right)_a{}^A{}_{|A}$. Thus, if $\mathbf{N}$ denotes the unit normal vector field on $\partial\mathcal{B}$, using divergence theorem one concludes that

$$\int_{\partial\mathcal{B}} \left(\left\langle \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \delta\varphi \right\rangle, \mathbf{N} \right)_G dA = \\ \int_{\mathcal{B}} \left\langle \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \nabla(\delta\varphi) \right\rangle \, dV + \int_{\mathcal{B}} \left\langle \mathrm{Div}\left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right), \delta\varphi \right\rangle dV. \tag{22}$$

We assume that $\partial\mathcal{B} = \partial_d\mathcal{B} \cup \partial_{\boldsymbol{\tau}}\mathcal{B}$ and $\partial_d\mathcal{B} \cup \partial_{\boldsymbol{\tau}}\mathcal{B} = \varnothing$, and therefore because $\delta\varphi|_{\partial_d} = 0$, we obtain

$$\int_{\partial\mathcal{B}} \left(\left\langle \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \delta\varphi \right\rangle, \mathbf{N} \right)_G dA = \int_{\partial_{\boldsymbol{\tau}}\mathcal{B}} \left(\left\langle \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \delta\varphi \right\rangle, \mathbf{N} \right)_G dA. \tag{23}$$

Note that we have

$$\left[\mathrm{Div}\left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right)\right]^\sharp = \mathrm{Div}\left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right)^\sharp, \tag{24}$$

with $\left[(\rho_0 \frac{\partial W}{\partial \mathbf{F}})^\sharp\right]^{aA} = g^{ab}\left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right)_b{}^A$. Substituting (22) and (23) into (20) and using (24) yields

$$\int_{\mathcal{B}} \left(\rho_0 \mathbf{A} - \rho_0 \mathbf{B} - \mathrm{Div}\left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right)^\sharp, \delta\varphi \right)_g dV + \\ \int_{\partial_{\boldsymbol{\tau}}\mathcal{B}} \left(\left\langle \left(\rho_0 \frac{\partial W}{\partial \mathbf{F}}\right)^\sharp, \mathbf{N}^\flat \right\rangle - \boldsymbol{\tau}, \delta\varphi \right)_g dA = 0. \tag{25}$$

Therefore, our problem has been reduced to finding $\varphi_t \in \mathcal{C}_{vol}$ that satisfies (25) for every $\delta\varphi \in T\mathcal{C}_{vol}$. Next, we need to use the following lemma.

Lemma 2.1.1. *Let $\boldsymbol{\xi}$ be a vector field on a Riemannian manifold $(\mathcal{M}, \mathbf{g})$. If for every vector field $\mathbf{w} \in \{\mathbf{z} : \mathcal{M} \to T\mathcal{M} | \operatorname{div} \mathbf{z} = 0, \mathbf{z}|_{\partial\mathcal{M}} = 0\}$ we have $\int_{\mathcal{M}} (\boldsymbol{\xi}, \mathbf{w})_g dv = 0$, then there exists a function $p : \mathcal{M} \to \mathbb{R}$ such that $\boldsymbol{\xi} = -\operatorname{div}(p\mathbf{g}^\sharp)$.*

Proof. Let $\Omega^k(\mathcal{M})$ denote the set of k-forms on $\mathcal{M}$ and assume that $\mathcal{M}$ is a compact oriented Riemannian manifold with smooth boundary $\partial\mathcal{M}$. The inner product of k-forms $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^k(\mathcal{M})$ is defined as

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_g = \int_{\mathcal{M}} \boldsymbol{\alpha} \wedge (*\boldsymbol{\beta}) = \int_{\mathcal{M}} \langle \boldsymbol{\alpha}, \boldsymbol{\beta}^\sharp \rangle dv, \tag{26}$$

where $* : \Omega^k(\mathcal{M}) \to \Omega^{n-k}(\mathcal{M})$ is the Hodge star operator. The Hodge decomposition theorem [44, 1] states that $\Omega^k(\mathcal{M})$ has the following orthogonal decomposition

$$\Omega^k(\mathcal{M}) = \mathbf{d}\big(\Omega^{k-1}(\mathcal{M})\big) \oplus \mathfrak{D}_t^k(\mathcal{M}), \tag{27}$$

where

$$\begin{aligned} &\mathbf{d}\big(\Omega^{k-1}(\mathcal{M})\big) = \big\{\boldsymbol{\alpha} \in \Omega^k(\mathcal{M}) \,|\, \exists \boldsymbol{\beta} \in \Omega^{k-1}(\mathcal{M}) \text{ s.t. } \boldsymbol{\alpha} = \mathbf{d}\boldsymbol{\beta}\big\}, \\ &\mathfrak{D}_t^k(\mathcal{M}) = \big\{\boldsymbol{\alpha} \in \Omega_t^k(\mathcal{M}) \,|\, \boldsymbol{\delta}\boldsymbol{\alpha} = 0\big\}, \end{aligned} \tag{28}$$

with $\mathbf{d} : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ and $\boldsymbol{\delta} : \Omega^{k+1}(\mathcal{M}) \to \Omega^k(\mathcal{M})$ denoting the exterior derivative and codifferential operators, respectively, and

$$\Omega_t^k(\mathcal{M}) = \big\{\boldsymbol{\alpha} \in \Omega^k(\mathcal{M}) \,|\, \boldsymbol{\alpha} \text{ is tangent to } \partial\mathcal{M}\big\}. \tag{29}$$

Note that the k-form $\boldsymbol{\alpha} \in \Omega^k(\mathcal{M})$ is tangent to $\partial\mathcal{M}$ if the normal part $\boldsymbol{\alpha}_n = i^*(*\boldsymbol{\alpha})$ is zero, where $i : \partial\mathcal{M} \hookrightarrow \mathcal{M}$ is the inclusion map [1][3]. Thus, if $\boldsymbol{\xi}$ is a vector field on

[3]Note that if $\mathbf{X}$ is a vector field on $\mathcal{M}$, then $\mathbf{X}$ is tangent to $\partial\mathcal{M}$ if and only if $\mathbf{X}^\flat$ is tangent to

$\mathcal{M}$, then the one-form $\boldsymbol{\xi}^\flat$ can be written as

$$\boldsymbol{\xi}^\flat = -\mathbf{d}p + \boldsymbol{\gamma}, \tag{30}$$

where $p : \mathcal{M} \to \mathbb{R}$ is a smooth function, $\boldsymbol{\delta\gamma} = 0$, and $\boldsymbol{\gamma}^\sharp$ is parallel to $\partial\mathcal{M}$, i.e., $\boldsymbol{\gamma}^\sharp(q) \in T_q(\partial\mathcal{M})$ for $q \in \partial\mathcal{M}$. Note that $\mathrm{div}(\boldsymbol{\gamma}^\sharp) = -\boldsymbol{\delta\gamma} = 0$, and therefore $\boldsymbol{\gamma}^\sharp$ is a divergence-free vector field parallel to $\partial\mathcal{M}$. Using (30), we can write the assumption of the lemma as

$$\int_{\mathcal{M}} \left(\boldsymbol{\xi}, \mathbf{w}\right)_g dv = \int_{\mathcal{M}} \left(\boldsymbol{\xi}^\flat, \mathbf{w}^\flat\right)_g dv = \int_{\mathcal{M}} -\left(\mathbf{d}p, \mathbf{w}^\flat\right)_g dv + \int_{\mathcal{M}} \left(\boldsymbol{\gamma}, \mathbf{w}^\flat\right)_g dv = 0, \tag{31}$$

for an arbitrary divergence-free vector field $\mathbf{w}$ that vanishes on $\partial\mathcal{M}$. Therefore, $\mathbf{w}$ is parallel to $\partial\mathcal{M}$ and because the decomposition (27) is orthogonal with respect to the inner product (26), $-(\mathbf{d}p, \mathbf{w}^\flat)_g$ is identically zero and hence (31) is equivalent to $(\boldsymbol{\gamma}, \mathbf{w}^\flat)_g = 0$, which means that $\boldsymbol{\gamma} = 0$ as $\boldsymbol{\gamma}, \mathbf{w}^\flat \in \mathfrak{D}_t^1(\mathcal{M})$. Thus, we obtain

$$\boldsymbol{\xi}^\flat = -\mathbf{d}p. \tag{32}$$

On the other hand, using the identity

$$\frac{\partial g^{ab}}{\partial x^b} = -g^{cb}\gamma^a_{cb} - g^{ac}\gamma^b_{cb}, \tag{33}$$

one can write

$$(pg^{ab})_{|b} = g^{ab}\frac{\partial p}{\partial x^b} + p\left(\frac{\partial g^{ab}}{\partial x^b} + g^{cb}\gamma^a_{cb} + g^{ac}\gamma^b_{cb}\right) = g^{ab}\frac{\partial p}{\partial x^b}, \tag{34}$$

$\partial\mathcal{M}$ [1].

or equivalently

$$\operatorname{div}(p\mathbf{g}^\sharp) = (\mathbf{d}p)^\sharp. \tag{35}$$

Substituting (35) into (32) yields $\boldsymbol{\xi} = -\operatorname{div}(p\mathbf{g}^\sharp)$. This completes the proof. □

Now returning to (25), we note that $\delta\varphi$ is arbitrary and, in particular, it can vanish on the boundary. Hence, the first integral on the left hand side of (25) should vanish. Now by using Lemma 2.1.1, we conclude that

$$\rho_0\mathbf{A} - \rho_0\mathbf{B} - \operatorname{Div}\left(\rho_0\frac{\partial W}{\partial \mathbf{F}}\right)^\sharp = -\operatorname{div}(p\mathbf{g}^\sharp), \tag{36}$$

where the time-dependent function $p : \varphi_t(\mathcal{B}) \times \mathbb{R} \to \mathbb{R}$ is the pressure field. By defining the material pressure field $p_0(X,t) := \varphi_t^* p(X) = p(\varphi_t(X),t)$, and noting that the Jacobian of φ_t is unity, we can use the Piola identity [82] to write

$$\operatorname{div}(p\mathbf{g}^\sharp) = \operatorname{Div}\left(p_0\left(\mathbf{F}^{-1}\right)^\sharp\right), \tag{37}$$

where $(\mathbf{F}^{-1})^\sharp$ is the tensor with components $(\mathbf{F}^{-1})^{aA}$. Substituting (37) into (36) yields

$$\rho_0\mathbf{A} = \rho_0\mathbf{B} + \operatorname{Div}\left[\left(\rho_0\frac{\partial W}{\partial \mathbf{F}}\right)^\sharp - p_0\left(\mathbf{F}^{-1}\right)^\sharp\right], \tag{38}$$

which are the governing equations of incompressible finite elasticity. Substituting (38) back into (25) and using the divergence theorem results in the natural boundary condition

$$\boldsymbol{\tau}|_{\partial_\tau\mathcal{B}} = \left\langle\left(\rho_0\frac{\partial W}{\partial \mathbf{F}}\right)^\sharp - p_0\left(\mathbf{F}^{-1}\right)^\sharp, \mathbf{N}^\flat\right\rangle. \tag{39}$$

Note that p is similar to pressure for perfect fluids and can be considered as the force of the constraint [82]. Also (38) implies that the first Piola-Kirchhoff stress tensor can be written as $\mathbf{P} = \left(\rho_0\frac{\partial W}{\partial \mathbf{F}}\right)^\sharp - p_0\left(\mathbf{F}^{-1}\right)^\sharp$.

2.1.2 Incompressible Linearized Elasticity

In this section we first review some preliminary concepts in linearized elasticity and then study the variational structure of incompressible linearized elasticity formulated on a Riemannian manifold. Here we do not use Lagrange multipliers to enforce the incompressibility constraint. Instead, we confine the displacement field to the set of divergence-free vector fields; motion of an incompressible elastic body extremizes the action function in this space.

Linear elasticity can be considered as the linearization of finite elasticity with respect to a reference motion [82, 118]. The linearized Jacobian about the configuration $\overset{o}{\varphi}$ reads [82]

$$\mathcal{L}_{\overset{o}{\varphi}}(\mathbf{U}) = \overset{o}{J} + \overset{o}{J}\Big[(\operatorname{div}\mathbf{u}) \circ \overset{o}{\varphi}\Big], \tag{40}$$

where $\mathbf{u} = \mathbf{U} \circ \overset{o}{\varphi}{}^{-1}$ and $\overset{o}{J}$ is the Jacobian of $\overset{o}{\varphi}$. For an incompressible motion, we have $J = 1$, and by choosing $\overset{o}{\varphi}$ to be the identity map $\mathrm{Id}_{\mathcal{B}}$, we obtain $1 = J(x) \approx J(\mathrm{Id}_{\mathcal{B}}) + J(\mathrm{Id}_{\mathcal{B}}) \operatorname{div}\mathbf{u}$, which yields

$$\operatorname{Div}\mathbf{U} = \operatorname{div}\mathbf{u} = 0. \tag{41}$$

Note that (41) is the first-order incompressibility condition, i.e., the incompressibility condition is satisfied up to the first order using (41). Let $\mathbf{u} = u^a e_a$ be the displacement field. The linearized strain tensor is $\mathbf{e} = \frac{1}{2}\mathfrak{L}_{\mathbf{u}}\mathbf{g}$ with components $e_{ab} = \frac{1}{2}(u_{a|b} + u_{b|a})$. Balance of linear momentum for an isotropic material reads

$$\rho\ddot{\mathbf{u}} = \rho\mathbf{b} + \operatorname{div}(2\mu\mathbf{e}^\sharp + \lambda(\operatorname{tr}\mathbf{e})\mathbf{g}^\sharp) \qquad \text{in } \mathcal{B}, \tag{42}$$

or in components

$$\rho\ddot{u}^a = \rho b^a + (2\mu\, e^{ab} + \lambda u^c{}_{|c}\, g^{ab})_{|b} \qquad \text{in } \mathcal{B}, \tag{43}$$

where μ and λ are Lamé constants with

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{44}$$

2.1.2.1 Variational Structure of Incompressible Linearized Elasticity

Let us recall that the stored energy per unit volume, $\mathcal{E}$ of an isotropic linear elastic solid can be written as

$$\mathcal{E} = \mu\langle \mathbf{e}^{\sharp}, \mathbf{e}\rangle + \frac{\lambda}{2}(\operatorname{tr}\mathbf{e})^2 = \mu e^{ab}e_{ab} + \lambda(e^a{}_a)^2. \tag{45}$$

Because of metric compatibility of the Levi-Civita connection, we have $(\nabla\mathbf{u})^{\flat} = \nabla\mathbf{u}^{\flat}$, or equivalently, $g_{ac}(u^c{}_{|b}) = u_{a|b}$, where ∇ denotes the Levi-Civita connection of g_{ab}. The incompressibility condition reads $\operatorname{div}\mathbf{u} = u^a{}_{|a} = 0$. Therefore, for an incompressible motion we can write

$$\begin{aligned} e^a{}_a &= \frac{1}{2}g^{ab}\left(u_{a|b} + u_{b|a}\right) = \frac{1}{2}g^{ab}\left(g_{ca}u^c{}_{|b} + g_{cb}u^c{}_{|a}\right) \\ &= \frac{1}{2}\left(\delta^b{}_c u^c{}_{|b} + \delta^a{}_c u^c{}_{|a}\right) = u^a{}_{|a} = 0. \end{aligned} \tag{46}$$

Thus, the stored energy per unit volume of an incompressible linear elastic body reads

$$\mathcal{E} = \mu\langle \mathbf{e}^{\sharp}, \mathbf{e}\rangle = \mu e^{ab}e_{ab}. \tag{47}$$

The Cauchy stress tensor of an isotropic linear elastic body can be written as

$$\boldsymbol{\sigma}^{\flat} = 2\mu\mathbf{e} + \lambda(\operatorname{tr}\mathbf{e})\mathbf{g}. \tag{48}$$

Also we have [82]

$$\operatorname{div}\mathbf{u} = \frac{\operatorname{tr}\boldsymbol{\sigma}}{3k}, \tag{49}$$

where $k = (3\lambda + 2\mu)/3$. If $\nu \to 1/2$, then from (44) we see that $k \to \infty$ and hence $\operatorname{div}\mathbf{u} \to 0$, i.e., when $\nu = 1/2$ any motion is incompressible. However, the converse is not necessarily true because if $\operatorname{tr}\boldsymbol{\sigma} = 0$ then motion is incompressible for any ν. For example, let x and y be the usual Euclidian coordinate system and consider an elastic planar sheet under uniform tension in the x-direction and uniform compression of the same magnitude in the y-direction. Then, trace of stress vanishes and this sheet undergoes an incompressible (linearized) motion regardless of the value of its Poisson's ratio.

Remark. *Note that if $\nu = 1/2$, then the elasticity tensor is neither pointwise stable nor strongly elliptic [82], which leads to coercivity loss in the weak formulation of the problem [45]. This means that the problem is no longer well posed and this is usually called locking. In this case, usually mixed finite element formulations, i.e., approximating displacement and pressure in different finite element spaces, or constrained finite element formulations are used to obtain well-posed weak forms.*

The Lagrangian of a linear incompressible isotropic body is written as $L = K - V$, where

$$\begin{aligned} K &= \frac{1}{2}\int_{\mathcal{B}} \rho(\dot{\mathbf{u}}, \dot{\mathbf{u}})_g dv, \\ V &= \int_{\mathcal{B}} \mu e^{ab} e_{ab} dv - \int_{\mathcal{B}} \rho(\mathbf{b}, \mathbf{u})_g dv - \int_{\partial_{\boldsymbol{\tau}}\mathcal{B}} (\boldsymbol{\tau}, \mathbf{u})_g da, \end{aligned} \tag{50}$$

with $\partial\mathcal{B} = \partial_d\mathcal{B} \cup \partial_\tau\mathcal{B}$, where $\partial_d\mathcal{B}$ and $\partial_\tau\mathcal{B}$ denote the portions of the boundary on which displacements and traction are specified, respectively. Let $\mathcal{B}$ and $\partial\mathcal{B}$ be compact orientable Riemannian manifolds (we assume that the orientation of $\partial\mathcal{B}$ is induced from that of $\mathcal{B}$) and consider the following sets of diffeomorphisms

$$
\begin{aligned}
&\mathcal{D} = \{\psi : \mathcal{B} \to \mathcal{B} | \psi \text{ is diffeomorphism}\}, \\
&\mathcal{D}_{vol} = \{\psi \in \mathcal{D} | \psi \text{ is volume preserving}\}, \\
&\mathcal{D}_q = \{\psi \in \mathcal{D} | \psi(q) = q \text{ for all } q \in \partial\mathcal{B}\}, \\
&\mathcal{D}_{vol,q} = \mathcal{D}_{vol} \cap \mathcal{D}_q,
\end{aligned}
\tag{51}
$$

and the following sets of vector fields

$$
\begin{aligned}
&\mathfrak{V} = \{\mathbf{w} : \mathcal{B} \to T\mathcal{B} | \mathbf{w}(q) \in T_q(\partial\mathcal{B}) \text{ for all } q \in \partial\mathcal{B}\}, \\
&\bar{\mathfrak{U}} = \{\mathbf{w} \in \mathfrak{V} | \operatorname{div} \mathbf{w} = 0\}, \\
&\mathfrak{V}_0 = \{\mathbf{w} \in \mathfrak{V} | \mathbf{w}|_{\partial\mathcal{B}} = 0\}, \\
&\mathfrak{U}_0 = \{\mathbf{w} \in \mathfrak{V} | \operatorname{div} \mathbf{w} = 0, \mathbf{w}|_{\partial\mathcal{B}} = 0\}, \\
&\mathfrak{U} = \{\mathbf{w} : \mathcal{B} \to T\mathcal{B} | \operatorname{div} \mathbf{w} = 0, \mathbf{w}|_{\partial_d\mathcal{B}} = 0\}.
\end{aligned}
\tag{52}
$$

Then $\mathcal{D}$, $\mathcal{D}_{vol}$, $\mathcal{D}_q$, and $\mathcal{D}_{vol,q}$ are infinite-dimensional Lie groups (under composition) with infinite-dimensional Lie algebras $\mathfrak{V}$, $\bar{\mathfrak{U}}$, $\mathfrak{V}_0$, and $\mathfrak{U}_0$, respectively [44]. In fact, $\mathcal{D}_{vol}$ and $\mathcal{D}_q$ are submanifolds (Lie subgroups) of $\mathcal{D}$ and $\mathcal{D}_{vol,q}$ is a submanifold (Lie subgroup) of $\mathcal{D}_q$. Also, we have $\mathfrak{U}_0 \subset \mathfrak{U}$ and $\mathfrak{U}$ is the tangent space of $\mathcal{C}_{vol}$ defined in (8) at the identity map [82], and thus it is an infinite-dimensional manifold and $\mathbf{u}(t)$ is a curve in $\mathfrak{U}$. For an arbitrary $\mathbf{w} \in \mathfrak{U}$, let $\mathbf{h}_s(t) = \mathbf{u}(t) + s\mathbf{w}(t)$ be the variational field for $\mathbf{u}(t)$[4]. The Lagrangian for the variation field reads

$$
\begin{aligned}
L(s) = &\frac{1}{2}\int_{\mathcal{B}} \rho(\dot{\mathbf{u}} + s\dot{\mathbf{w}}, \dot{\mathbf{u}} + s\dot{\mathbf{w}})_g dv \\
&- \int_{\mathcal{B}} \mu\langle \mathbf{e}^\sharp + s\mathbf{e}_w^\sharp, \mathbf{e} + s\mathbf{e}_w\rangle dv + \int_{\mathcal{B}} \rho(\mathbf{b}, \mathbf{u} + s\mathbf{w})_g dv + \int_{\partial_\tau\mathcal{B}} (\boldsymbol{\tau}, \mathbf{u} + s\mathbf{w})_g da,
\end{aligned}
\tag{53}
$$

[4]This choice of the variation field is simpler to work with. The general form of the variation field is a one-parameter family of curves $\mathbf{h}_s(t) = \boldsymbol{\psi}(s,t)$, with $\boldsymbol{\psi}(0,t) = \mathbf{u}(t)$ and $\delta\mathbf{u} = d/ds\,(\boldsymbol{\psi}(s,t))\,|_{s=0}$, which yields the same result as the above choice.

where the components of the tensor $\mathbf{e}_w = \frac{1}{2}\mathfrak{L}_{\mathbf{w}}\mathbf{g}$ are given by

$$(\mathbf{e}_{\mathbf{w}})_{ab} = \frac{1}{2}(w_{a|b} + w_{b|a}). \tag{54}$$

The true motion of a linear incompressible elastic solid satisfies $\delta \int_0^T L dt = 0$ over $\mathfrak{U}$, i.e.

$$\begin{aligned}
&\delta \int_0^T L dt = \frac{d}{ds}\left(\int_0^T L dt\right)\Bigg|_{s=0} \\
&= \int_0^T \left\{ \int_{\mathcal{B}} \left[\rho(\dot{\mathbf{u}}, \dot{\mathbf{w}})_g + \rho(\mathbf{b}, \mathbf{w})_g - 2\mu e^{ab}(\mathbf{e}_{\mathbf{w}})_{ab}\right] dv + \int_{\partial_{\boldsymbol{\tau}}\mathcal{B}} (\boldsymbol{\tau}, \mathbf{w})_g da \right\} dt \\
&= -\int_0^T \left\{ \int_{\mathcal{B}} \left[\rho(\ddot{\mathbf{u}} - \mathbf{b}, \mathbf{w})_g + 2\mu e^{ab}(\mathbf{e}_{\mathbf{w}})_{ab}\right] dv - \int_{\partial_{\boldsymbol{\tau}}\mathcal{B}} (\boldsymbol{\tau}, \mathbf{w})_g da \right\} dt \\
&\qquad + (\dot{\mathbf{u}}, \mathbf{w})_g\Big|_0^T = 0.
\end{aligned} \tag{55}$$

As we consider proper variations, i.e., $\mathbf{w}(0) = \mathbf{w}(T) = 0$, and the integrand of the time integral is continuous, we obtain the following weak form for the linearized incompressible motion

$$\int_{\mathcal{B}} \left[\rho(\ddot{\mathbf{u}} - \mathbf{b}, \mathbf{w})_g + 2\mu e^{ab}(\mathbf{e}_{\mathbf{w}})_{ab}\right] dv - \int_{\partial_{\boldsymbol{\tau}}\mathcal{B}} (\boldsymbol{\tau}, \mathbf{w})_g da = 0. \tag{56}$$

Next, we use the following relation that can be verified by direct substitution:

$$\operatorname{div}\langle 2\mu\mathbf{e}^{\sharp}, \mathbf{w}^{\flat}\rangle = 2\mu\langle \mathbf{e}^{\sharp}, \mathbf{e}_{\mathbf{w}}\rangle + \langle \operatorname{div}(2\mu\mathbf{e}^{\sharp}), \mathbf{w}^{\flat}\rangle. \tag{57}$$

Using (57) and the divergence theorem and because $\mathbf{w}|_{\partial_d\mathcal{B}} = 0$, we obtain

$$\begin{aligned}
&\int_{\partial_{\boldsymbol{\tau}}\mathcal{B}} (\langle 2\mu\mathbf{e}^{\sharp}, \mathbf{w}^{\flat}\rangle, \mathbf{n})_g da = \int_{\partial\mathcal{B}} (\langle 2\mu\mathbf{e}^{\sharp}, \mathbf{w}^{\flat}\rangle, \mathbf{n})_g da \\
&= \int_{\mathcal{B}} 2\mu\langle \mathbf{e}^{\sharp}, \mathbf{e}_{\mathbf{w}}\rangle dv + \int_{\mathcal{B}} (\operatorname{div}(2\mu\mathbf{e}^{\sharp}), \mathbf{w})_g \, dv,
\end{aligned} \tag{58}$$

where $\mathbf{n}$ is the unit outward normal vector field for $\partial\mathcal{B}$. In components this reads

$$\int_{\partial_\tau\mathcal{B}} 2\mu e^{ab} w_b n_a da = \int_{\mathcal{B}} 2\mu e^{ab} (\mathbf{e}_{\mathbf{w}})_{ab} dv + \int_{\mathcal{B}} (2\mu e^{ab})_{|a} w_b dv. \tag{59}$$

Substituting (58) into (56) yields

$$\int_{\mathcal{B}} (\rho\ddot{\mathbf{u}} - \rho\mathbf{b} - \operatorname{div}(2\mu\mathbf{e}^\sharp), \mathbf{w})_g dv + \int_{\partial\mathcal{B}} (\langle 2\mu\mathbf{e}^\sharp, \mathbf{n}^\flat\rangle - \boldsymbol{\tau}, \mathbf{w})_g da = 0. \tag{60}$$

Suppose $\mathbf{w} \in \mathfrak{U}_0 \subset \mathfrak{U}$, then the second term on the left hand side of (60) is identically zero and hence by Lemma 2.1.1, we obtain

$$\rho\ddot{\mathbf{u}} = \rho\mathbf{b} + \operatorname{div}(2\mu\mathbf{e}^\sharp - p\mathbf{g}^\sharp) \quad \text{in } \mathcal{B}, \tag{61}$$

where the time-dependent function $p : \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ is the pressure field. Substituting (61) back into (60) results in

$$-\int_{\mathcal{B}} (\operatorname{div}(p\mathbf{g}^\sharp), \mathbf{w})_g \, dv + \int_{\partial\mathcal{B}} (\langle 2\mu\mathbf{e}^\sharp, \mathbf{n}^\flat\rangle - \boldsymbol{\tau}, \mathbf{w})_g da = 0. \tag{62}$$

Using the relation

$$\operatorname{div}(p\mathbf{g}^\sharp, \mathbf{w})_g = p \operatorname{div} \mathbf{w} + (\operatorname{div}(p\mathbf{g}^\sharp), \mathbf{w})_g, \tag{63}$$

and noting that $\mathbf{w}$ is divergence free, we can use the divergence theorem to write

$$\begin{aligned} \int_{\mathcal{B}} (\operatorname{div}(p\mathbf{g}^\sharp), \mathbf{w})_g \, dv &= \int_{\mathcal{B}} \operatorname{div}(p\mathbf{g}^\sharp, \mathbf{w})_g dv \\ &= \int_{\partial\mathcal{B}} (\langle p\mathbf{g}^\sharp, \mathbf{w}^\flat\rangle, \mathbf{n})_g da = \int_{\partial\mathcal{B}} (\langle p\mathbf{g}^\sharp, \mathbf{n}^\flat\rangle, \mathbf{w})_g da. \end{aligned} \tag{64}$$

Substitution of (64) into (62) yields

$$\int_{\partial\mathcal{B}} (\langle 2\mu\mathbf{e}^\sharp - p\mathbf{g}^\sharp, \mathbf{n}^\flat\rangle - \boldsymbol{\tau}, \mathbf{w})_g da = 0. \tag{65}$$

Since $\mathbf{w}|_{\partial\mathcal{B}}$ is an arbitrary vector field on $\partial\mathcal{B}$ that vanishes on $\partial_d\mathcal{B}$, (65) implies that

$$\boldsymbol{\tau} = \langle 2\mu\mathbf{e}^\sharp - p\mathbf{g}^\sharp, \mathbf{n}^\flat \rangle \qquad \text{on } \partial_{\boldsymbol{\tau}}\mathcal{B}. \tag{66}$$

Equations (61) and (66) are the governing equations and the natural boundary conditions for incompressible linearized elasticity, respectively. In components they read

$$\rho\ddot{u}^a = \rho b^a + (2\mu\, e^{ab} - p\, g^{ab})_{|b} \qquad \text{in } \mathcal{B}, \tag{67}$$

$$\boldsymbol{\tau}^a = 2\mu\, e^{ab} n_b - p\, g^{ab} n_b \qquad \text{on } \partial_{\boldsymbol{\tau}}\mathcal{B}. \tag{68}$$

Remark. *We observed that the case $\nu = 1/2$ corresponds to an incompressible motion, and therefore it satisfies the governing equations of incompressible elasticity. On the other hand, we know that $\nu = 1/2$ should satisfy equation (42) as well. But there is no pressure in the compressible equations and one may wonder how the compressible and incompressible governing equations can be reconciled for this special case. Let $\nu \to 1/2$. Then, from (49) we see that $\operatorname{div}\mathbf{u} \to 0$, and using (48) and (46) we conclude that*

$$\operatorname{tr}\boldsymbol{\sigma} = 2\mu \operatorname{tr}\mathbf{e} + 3\lambda \operatorname{tr}\mathbf{e} \to 3\lambda \operatorname{tr}\mathbf{e}. \tag{69}$$

Equivalently, by assuming that $\operatorname{tr}\boldsymbol{\sigma}$ is bounded and defining $p = -\operatorname{tr}\boldsymbol{\sigma}/3$, we can write

$$\lambda \operatorname{tr}\mathbf{e} \to -p. \tag{70}$$

Substituting (70) into (48) and (42) results in the balance of linear momentum for incompressible linearized elasticity. Therefore, we have shown that balance of linear momentum for compressible and incompressible linearized elasticity are the same for $\nu = 1/2$.

Remark. *The solution space for incompressible fluids is similar to that of incompressible linearized elasticity. Consider the motion of an incompressible fluid in a Riemannian manifold $\mathcal{M}$. The spatial velocity of the fluid,* $\mathbf{v}$*, lies in the set* $\mathfrak{F} = \{\mathbf{w} : \mathcal{M} \to T\mathcal{M} | \operatorname{div} \mathbf{w} = 0$ *and* $\mathbf{w}(q) \in T_q(\partial\mathcal{M}) \ \forall q \in \partial\mathcal{M}\}$*, which is similar to* $\mathfrak{U}$*. The variation of the spatial velocity of an incompressible fluid* $\delta\mathbf{v}$ *satisfies the so-called Lin constraint [83, 89], i.e.,* $\delta\mathbf{v} = \dot{\boldsymbol{\xi}} + [\mathbf{v}, \boldsymbol{\xi}]$*, where* $\boldsymbol{\xi}$ *is an arbitrary divergence-free vector field that vanishes at initial and final times, dot denotes derivative with respect to time, and* $[,]$ *is the usual bracket of vector fields (Lie bracket).*

2.2 Discrete Exterior Calculus

The idea of discrete exterior calculus (DEC) is to define discrete versions of the smooth operators of exterior calculus such that some important theorems, e.g. the generalized Stokes' theorem and the naturality with respect to pull-backs remain valid. However, the convergence of these discrete operators to their smooth counterparts and, in particular, the correct topology to investigate such convergence is still vague and needs more work [61]. For an introductory discussion on the connections between DEC and other structure-preserving schemes such as finite element exterior calculus and mimetic methods, see [60] and references therein. In this section we review some topics from DEC. First, we need to review some concepts from algebraic topology and for this we mainly follow [85].

2.2.1 Primal Meshes

Let $\{v_0, \ldots, v_k\}$ be a geometrically independent set in $\mathbb{R}^N$, i.e., $\{v_1 - v_0, \ldots, v_k - v_0\}$ is a set of linearly independent vectors in $\mathbb{R}^N$. The k-simplex σ^k is defined as

$$\sigma^k = \left\{ x \in \mathbb{R}^N \middle| x = \sum_{i=0}^{k} t_i a_i, \text{ where } 0 \le t_i \le 1, \sum_{i=0}^{k} t_i = 1 \right\}. \tag{71}$$

The numbers t_i are uniquely determined by x and are called barycentric coordinates of the point x of σ with respect to vertices $v_0, \ldots, v_k$. The number k is the dimension of σ^k. Any simplex σ^l spanned by a subset of vertices $\{v_0, \ldots, v_k\}$ is called a face of σ^k and $\sigma^l \prec \sigma^k$ or $\sigma^k \succ \sigma^l$ means that σ^l is a face of σ^k.

A simplical complex K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that (i) every face of a simplex of K is in K and (ii) the intersection of any two simplices is either empty or a face of each of them. The largest dimension of the simplices of K is called the dimension of K. Fig. 1 shows a 2-dimensional simplical complex with $\bullet$ representing its vertices and the solid lines representing its 1-simplices. A subcomplex of K is a subcollection of K that contains all faces of its elements. The collection of all simplices of K of dimension at most p is called the p-skeleton of K and is denoted by $K^{(p)}$. The subset of $\mathbb{R}^N$ that is the union of the simplices of K is denoted by $|K|$ and is called the underlying space or the polytope of K. The topology on $|K|$ is considered to be the usual subspace topology induced from the ambient space $\mathbb{R}^N$. A flat simplical complex K of dimension n in $\mathbb{R}^N$ has all its simplices in the same affine n-space of $\mathbb{R}^N$, i.e., $|K|$ is a subset of an n-dimensional subset of $\mathbb{R}^N$ that has zero curvature. Here we assume that all simplical complexes are flat.

A triangulation of a topological space $\mathcal{X}$ is a simplical complex K with a homeomorphism $\mathfrak{h} : |K| \to \mathcal{X}$. A differentiable manifold always admits a triangulation [84]. Roughly speaking, triangulation of a differential manifold $\mathcal{M} \subset \mathbb{R}^N$ can be considered as a complex $\mathbb{M} = \mathfrak{h}(K)$ that covers $\mathcal{M}$ and its cells, which in general, have curved faces. A triangulation of the polytope $|K|$ is defined to be any simplical complex L such that $|L| = |K|$.

Any ordering of the vertices of σ^k defines an orientation for σ^k. We denote the oriented simplex σ^k by $[v_0, \ldots, v_k]$. Two orderings of a simplex σ^k are equivalent if one is an even permutation of the other. By definition, a zero simplex has only one orientation. One can see that for $1 \le k \le N$, σ^k can have two different orientations.

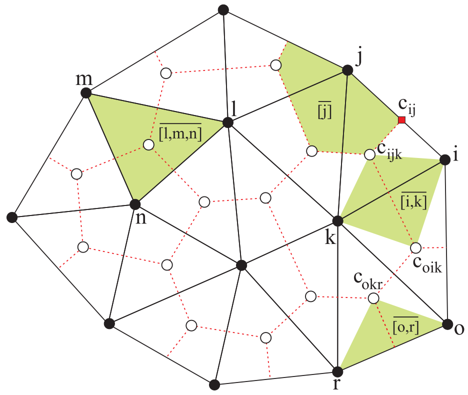


Figure 1: A primal 2-dimensional simplical complex (solid lines) and its dual (dashed lines). The primal vertices are denoted by $\bullet$ and the dual vertices by $\circ$. The circumcenter $c([i,j,k])$ is denoted by c_{ijk}, etc. The highlighted areas denote the support volumes of the corresponding simplices, for example, $\overline{[i,k]}$ is the support volume of the 1-simplex $[i,k]$. Note that the support volume of the primal vertex j coincides with its dual $\overline{[j]}$ and support volume of the primal 2-simplex $[l,m,n]$ coincides with itself.

The equivalence class of the particular ordering is denoted by $(v_0,\ldots,v_k)$. Note that the orientation of σ^k induces an orientation on the $(k-1)$-faces of σ^k defined by $\sigma^{k-1} = (-1)^i[v_0,\ldots,\hat{v}_i,\ldots,v_k]$, where $\hat{v}_i$ means omit the ith vertex. The ordered collection of vectors $(v_1-v_0, v_2-v_0,\ldots,v_k-v_0)$ is called a corner basis at v_0. The span of this basis is called the plane of σ^k and is denoted by $P(\sigma^k)$. The orientation of two oriented simplices σ and τ that have the same dimension can be compared if and only if either they have the same plane or they share a face of dimension $k-1$. If $P(\sigma^k) = P(\tau^k)$, then σ^k and τ^k have the same orientation if and only if their corner basis orient their plane identically. If σ^k and τ^k have a common $k-1$ face, then they have the same orientation if and only if the induced orientation by σ^k on the common face is opposite to that induced by τ^k. If two simplices have the same orientations, we write $\mathrm{sgn}(\sigma^k,\tau^k) = +1$, and if they have opposite orientations we write $\mathrm{sgn}(\sigma^k,\tau^k) = -1$. If two simplices have different dimensions, their orientations can not be compared.

A manifold-like simplical complex K of dimension n is a simplical complex such that $|K|$ is a topological manifold (possibly with boundary) and each simplex of dimension k with $0 \le k \le n-1$ is a face of an n-simplex in the simplical complex. A manifold-like simplical complex of dimension n is called an oriented manifold-like simplical complex if adjacent n-simplices have the same orientations (orient the common $(n-1)$-face oppositely) and simplices of dimensions $n-1$ and lower are oriented individually. In this work, by the primal mesh we mean a manifold-like oriented simplical complex.

2.2.2 Dual Meshes

Dual complexes have an important rule in many computational fields. The two most common dual complexes are circumcentric and barycentric dual complexes. The barycentric dual has the nice property that it can be defined for any simplical complex but the circumcentric dual is well-defined only for well-centered simplical complexes.[5] This means that in problems for which one needs to consider a simplical complex that evolves in time, e.g. finite elasticity, circumcentric dual may not be appropriate. On the other hand, the metric-dependent DEC operators have simpler forms in circumcentric duals [61]. The discretization method that we describe in this paper does not depend on the specific choice of a dual complex as far as a consistent DEC theory is available for that choice of the dual complex. Here we consider circumcentric duals in order to present our method using simpler formulas. Note also that we consider linearized elasticity and hence we are working with a fixed mesh.

The circumcenter of a k-simplex σ^k is the unique point $c(\sigma^k)$ that has the same distance from all the $k+1$ vertices of σ^k. If $c(\sigma^k)$ lies in the interior of σ^k, then σ^k is called a well-centered simplex. A well-centered simplical complex is a simplical complex such that all its simplices (of all dimensions) are well-centered. For example, a planar mesh is well-centered if all of its 2-cells are acute triangles [104]. The

[5]Recently, Hirani et al. [62] introduced the notion of signed duals that allows one to work with meshes that are not well-centered as well.

(circumcenteric) dual complex for an n-dimensional well-centered simplical complex K is a cell complex $\star K$ with cells $\hat{\sigma}$ defined by the duality operator $\star$ as follows: given a k-simplex σ^k in K, the duality operator gives an $(n-k)$-cell of $\star K$ as

$$\hat{\sigma}^{n-k} = \star\sigma^k = \sum_{\sigma^k \prec \sigma^{k+1} \prec \ldots \prec \sigma^n} \epsilon(\sigma^k, \sigma^{k+1}, \ldots, \sigma^n)[c(\sigma^k), c(\sigma^{k+1}), \ldots, c(\sigma^n)], \tag{72}$$

where the coefficients $\epsilon(\sigma^k, \sigma^{k+1}, \ldots, \sigma^n)$ are introduced to glue elements with consistent orientations. Sometimes it is possible to define notions similar to those of a simplical complex for a dual cell. For example, the dual p-skeleton of K is the union of the cells of dimension at most p of $\star K$ and is denoted by $K_{(p)}$. Fig. 1 shows a 2-dimensional simplical complex (solid lines) together with its dual (dashed lines) where $\bullet$ and $\circ$ denote primal and dual vertices, respectively. Note that dual of a primal vertex is a dual 2-cell, dual of a primal 1-simplex is a dual 1-simplex, and dual of a primal 2-simplex is a dual vertex. For example, denoting $c([i,j,k])$ by c_{ijk}, etc., the dual of the primal 1-simplex $[i,k]$ is either $[c_{ijk}, c_{oik}]$ or $[c_{oik}, c_{ijk}]$ depending on the orientation of the primal mesh.

The support volume $\overline{\sigma^k}$ of a k-simplex σ^k in an n-dimensional complex K is the n-dimensional convex hull of the geometric union of σ^k and $\star\sigma^k$, or equivalently

$$\overline{\sigma^k} = \overline{\star\sigma^k} = \mathsf{convex\ hull}(\sigma^k, \star\sigma^k) \cap |K|. \tag{73}$$

Support volumes of various simplices of a 2-dimensional mesh are shown in Fig. 1, where highlighted regions denote the support volumes.

Now we discuss how to orient a dual cell. This is important in the subsequent work. Suppose K is a well-centered n-dimensional primal mesh with the dual $\star K$ and we want to obtain the orientations of the simplices of the dual cell $\hat{\sigma}^k$ that are induced from the orientation of the primal mesh. First we consider the case $1 \leq k \leq n-1$. Let $\sigma^0, \sigma^1, \ldots, \sigma^n$ be primal simplices with $\sigma^0 \prec \sigma^1 \prec \ldots \prec \sigma^n$ and let the orientation of

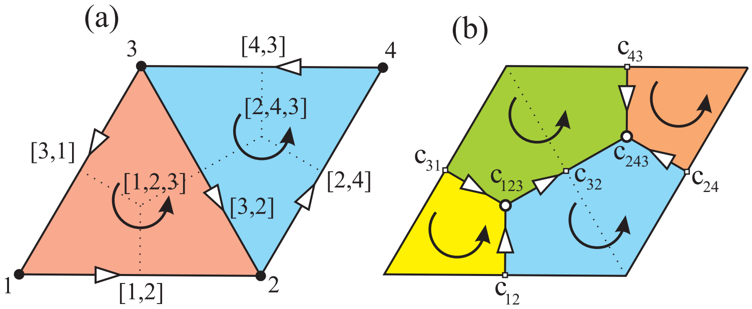


Figure 2: Oriented meshes: (a) primal mesh and (b) associated circumcentric dual mesh. The primal vertices are denoted by $\bullet$ and the dual vertices by $\circ$. The circumcenter $c([1,2,3])$ is denoted by c_{123}, etc.

elementary dual simplices be $s[c(\sigma^k),\dots,c(\sigma^n)]$, where $s=\pm 1$, and the value of s is to be determined. As we mentioned earlier, orientations of σ^k and $[c(\sigma^0),\dots,c(\sigma^k)]$ can be compared as they have the same planes. Similarly, one can compare the orientations of σ^n and $[c(\sigma^0),\dots,c(\sigma^n)]$. Now let us define

$$s = \operatorname{sgn}([c(\sigma^0),\dots,c(\sigma^k)],\sigma^k)\times\operatorname{sgn}([c(\sigma^0),\dots,c(\sigma^n)],\sigma^n). \tag{74}$$

If $k=n$, then the dimension of the dual is 0 and hence it can have only one orientation by definition. For $k=0$ we define $s=\operatorname{sgn}([c(\sigma^0),\dots,c(\sigma^n)],\sigma^n)$. Thus, we note that unlike the primal mesh for which the orientations of the simplices with dimensions less than n are arbitrary, the orientation of none of the simplices of the dual mesh is arbitrary; the dual orientations are induced by the orientation of the primal mesh. The correct orientation of the dual cell is important when we deal with discrete dual forms. In this work by a dual mesh we mean the oriented dual of a well-centered primal mesh. We clarify the previous definitions in the following simple example.

Example 2.2.1. *(Orienting a 2-dimensional Dual Mesh). Consider the 2-dimensional primal mesh and its circumcentric dual shown in Figs. 2(a) and (b), respectively. The primal mesh is oriented and we want to obtain the induced orientation of the dual mesh. By definition, orientation of 2-simplices of the dual mesh are the same as* *those of primal 2-simplices and thus the correct orientation of the dual 2-simplices is counterclockwise. Now consider the dual 1-simplices. As an example, we obtain the orientation of* $\star[3,2]$*, which consists of two elementary 1-simplices with vertices* $\{c_{123}, c_{32}\}$ *and* $\{c_{243}, c_{32}\}$*. The orientation of these elementary simplices is* $s_1[c_{32}, c_{123}]$ *and* $s_2[c_{32}, c_{243}]$*, respectively, where*

$$\begin{aligned} s_1 &= \operatorname{sgn}([2, c_{32}], [3,2]) \times \operatorname{sgn}([2, c_{32}, c_{123}], [1,2,3]) = (-1) \times (+1) = -1, \\ s_2 &= \operatorname{sgn}([2, c_{32}], [3,2]) \times \operatorname{sgn}([2, c_{32}, c_{243}], [2,4,3]) = (-1) \times (-1) = +1. \end{aligned} \quad (75)$$

Thus, the correct orientation of $\star[3,2]$ *is* $[c_{123}, c_{243}]$*. Note that to orient a dual simplex, one needs to orient its elementary duals individually and if the primal mesh is correctly oriented, then these elementary duals will have the same orientations. Similarly, one can obtain the orientations of the other simplices of the dual mesh as is shown in Fig. 2(b). Note that there is an easy rule for orienting dual 1-simplices in* $\mathbb{R}^2$*: consider a 1-dimensional face of a primal 2-simplex. If the orientation induced on that face by the orientation of the 2-simplex is the same as the orientation of that face then the direction of the dual of that face points into the 2-simplex, otherwise it points out of the 2-simplex.*

2.2.3 Discrete Vector Fields

As will be explained in the next section, we choose the displacement vector field as our main unknown in incompressible linearized elasticity. Thus, we need to introduce the concept of discrete vector fields on a flat simplical mesh. Here, there are at least two possibilities: primal and dual vector fields as follows.

A primal discrete vector field $\mathbf{X}$ on an n-dimensional primal mesh K is a map from primal vertices $K^{(0)}$ to $\mathbb{R}^n$. The space of primal vector fields is denoted by $\mathfrak{X}_d(K)$. One can assume that the value of the primal vector field is constant on each of the n-cells of $\star K$. A dual discrete vector field $\mathbf{X}$ on the dual of an n-dimensional

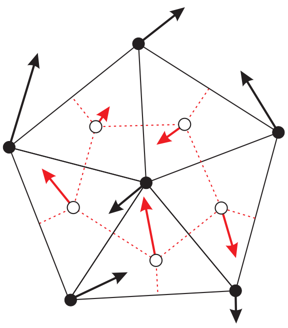


Figure 3: A primal vector field (arrows on primal vertices $\bullet$) and a dual vector field (arrows on dual vertices $\circ$) on a 2-dimensional mesh. The solid and dashed lines denote the primal and dual meshes, respectively.

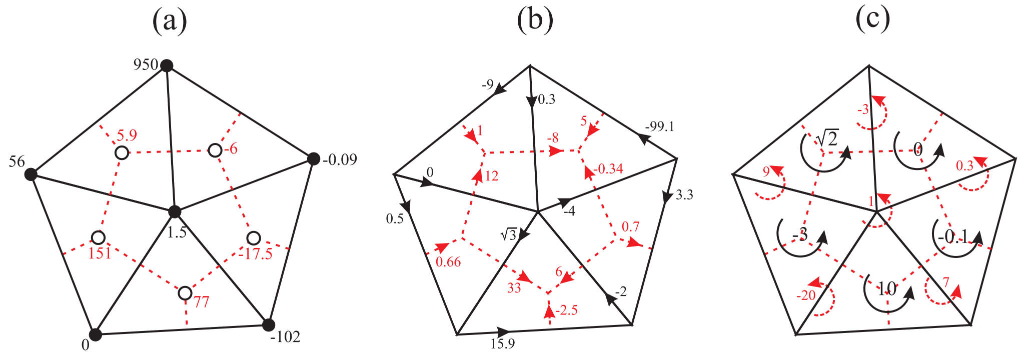


Figure 4: Examples of forms on a 2-dimensional primal mesh (solid lines) and its dual mesh (dashed lines): (a) primal and dual 0-forms, which are real numbers on primal and dual vertices, (b) primal and dual 1-forms, which are real numbers on primal and dual 1-simplices, and (c) primal and dual 2-forms, which are real numbers on primal and dual 2-simplices, respectively.

primal mesh K is a map from dual vertices $K_{(0)}$ to $\mathbb{R}^n$. The space of dual vector fields is denoted by $\mathfrak{X}_d(\star K)$. One can assume that the value of the dual vector field is constant on each of the n-cells of K. In Fig. 3, for a 2-dimensional mesh, primal and dual vectors are denoted by arrows on primal vertices $\bullet$ and dual vertices $\circ$, respectively.

2.2.4 Primal and Dual Discrete Forms

In the smooth case, a k-form on an n-manifold $\mathcal{N}$ is an antisymmetric covariant tensor of order k and the set of k-forms on $\mathcal{N}$ is denoted by $\Omega^k(\mathcal{N})$. Now we define primal and dual discrete k-forms. We need the notion of chains and cochains as follows. A

k-chain on a simplical complex K is a function c_k from the set of oriented k-simplices of K to the integers such that (i) $c_k(-\sigma^k) = -c_k(\sigma^k)$, and (ii) $c(\sigma) = 0$ for all but finitely many oriented k-simplices σ. If we add k-chains by adding their values, we obtain the group of (oriented) k-chains of K, which is denoted by $C_k(K)$. If $k < 0$ or $k > \dim K$, $C_k(K)$ is defined to be the trivial group. For an oriented simplex σ^k, the elementary chain c corresponding to σ^k is the function defined as

$$c(\tau) = \begin{cases} 1, & \tau = \sigma^k, \\ -1, & \tau = -\sigma^k, \\ 0, & \text{otherwise.} \end{cases} \tag{76}$$

In the following the symbol σ^k denotes not only an oriented simplex but also the elementary k-chain c corresponding to σ^k. The meaning is always clear from the context. It can be shown that $C_k(K)$ is free Abelian, i.e., a basis for $C_k(K)$ can be obtained by orienting each k-simplex and using the corresponding elementary chains as a basis. A k-cochain c^k is a homomorphism from the chain group $C_k(K)$ to $\mathbb{R}$. The space of k-cochains is denoted by $C^k(K) = \mathrm{Hom}(C_k(K), \mathbb{R})$. A primal discrete k-form is a k-cochain and the space of discrete k-forms on K is denoted by $\Omega_d^k(K) = C^k(K)$. Similarly, one can define the space of dual discrete k-forms on $\star K$, which is denoted by $\Omega_d^k(\star K)$. For a k-chain $c_k \in C_k(K)$ and a k-form $\boldsymbol{\alpha}^k \in \Omega_d^k(K)$, we denote the value of $\boldsymbol{\alpha}^k$ at c_k by $\langle \boldsymbol{\alpha}^k, c_k \rangle = \boldsymbol{\alpha}^k(c_k)$. Since $C_k(K)$ is a free Abelian group, we have $c_k = \sum_i \hat{c}_k^i \sigma_i^k$, where $\hat{c}_k^i = c_k(\sigma_i^k) \in \mathbb{Z}$, and summation is over all k-simplices of K. The k-form $\boldsymbol{\alpha}^k$ is a linear function of chains, and thus we have

$$\langle \boldsymbol{\alpha}^k, c_k \rangle = \boldsymbol{\alpha}^k \left(\sum_i \hat{c}_k^i \sigma_i^k \right) = \sum_i \hat{c}_k^i \alpha_i, \tag{77}$$

where the coefficients $\alpha_i = \boldsymbol{\alpha}^k(\sigma_i^k) \in \mathbb{R}$ are called the components of the k-form. Thus, one can specify any k-form by a set of real numbers on k-simplices. Similarly, one can define a dual k-form. Fig. 4 shows examples of primal and dual 0, 1, and 2-forms on a 2-dimensional mesh.

2.2.5 Discrete Operators

One of the main goals of this work is to find an appropriate discrete space for displacement field of incompressible linear elasticity, i.e., the discrete space of divergence-free vector fields. Here, we define the discrete divergence using discrete exterior derivative, discrete Hodge star, and discrete flat operator.

Exterior derivative. The discrete exterior derivative is defined as the coboundary operator as follows. A boundary operator $\partial_k : C_k(K) \to C_{k-1}(K)$ is a homomorphism defined on each oriented simplex $\sigma^k = [v_0, \ldots, v_k]$ as

$$\partial_k \sigma^k = \partial_k [v_0, \ldots, v_k] = \sum_{i=0}^{k} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_k]. \tag{78}$$

The coboundary operator $\delta^k : C^k(K) \to C^{k+1}(K)$ is defined as

$$\langle \delta^k c^k, c_{k+1} \rangle = \langle c^k, \partial_{k+1} c_{k+1} \rangle. \tag{79}$$

Using the above definitions, one can show that $\partial_k \circ \partial_{k+1} = 0$ and $\delta^{k+1} \circ \delta^k = 0$. The sequence

$$0 \to C_n(K) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{k+1}} C_k(K) \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_1} C_0(K) \to 0, \tag{80}$$

is called the chain complex induced by the boundary operator. Similarly, the sequence

$$0 \leftarrow C^n(K) \xleftarrow{\delta^{n-1}} \cdots \xleftarrow{\delta^k} C^k(K) \xleftarrow{\delta^{k-1}} \cdots \xleftarrow{\delta^0} C^0(K) \leftarrow 0, \tag{81}$$

is called the cochain complex induced by the coboundary operator. The discrete exterior derivative $\mathbf{d} : \Omega_d^k(K) \to \Omega_d^{k+1}(K)$ is defined to be the coboundary operator δ^k.

It follows that $\mathbf{d}^{k+1} \circ \mathbf{d}^k = 0$. Similarly, one can define discrete exterior derivative over a dual mesh. Let K be an n-dimensional primal mesh. Then, the dual boundary operator $\partial_k : C_k(\star K) \to C_{k-1}(\star K)$ on each oriented dual simplex $\star\sigma^{n-k} = \star[v_0, \ldots, v_{n-k}]$ is defined as

$$\partial_k \star [v_0, \ldots, v_{n-k}] = \sum_{\sigma^{n-k+1} \succ \sigma^{n-k}} s_{\sigma^{n-k+1}} \star \sigma^{n-k+1}, \tag{82}$$

where $s_{\sigma^{n-k+1}} = \pm 1$. If $0 \leq k \leq n-1$, $s_{\sigma^{n-k+1}}$ is chosen such that $s_{\sigma^{n-k+1}}\sigma^{n-k+1}$ induces the same orientation on σ^{n-k} as its original orientation. For $k = n$, s_{σ^1} is chosen such that the orientation of $s_{\sigma^1} \star \sigma^1$ is the same as that induced by $\star\sigma^0$ on its geometric boundary. Note that unlike the primal boundary, the dual boundary of a dual cell is not necessarily the same as its geometric boundary. For example, in Fig. 2 the dual boundary of $\star\sigma_2^0$ is $[c_{24}, c_{243}] + [c_{243}, c_{123}] + [c_{123}, c_{12}]$, which is different from the geometric boundary of $\star\sigma_2^0$.

The dual discrete exterior derivate $\mathbf{d} : \Omega_d^k(\star K) \to \Omega_d^{k+1}(\star K)$ is defined to be the dual coboundary operator defined similar to (79) by using dual boundary operator. There is a major difference between the dual discrete exterior derivative and the primal one as we explain next. Consider primal and dual zero-forms on the planar mesh shown in Fig. 5. Let $\{f^1, \ldots, f^4\}$ and $\{f^{123}, f^{243}\}$ be the values of 0-forms on primal and dual vertices, respectively. The value of $\mathbf{d}f$ on the primal and dual 1-simplices are shown in Fig. 5(a) and (b), respectively. In the continuous case, we have $\mathbf{d}(f + a) = \mathbf{d}f$, where a is a real constant. The same is true for a primal 0-form as the value of $\mathbf{d}f$ is the differences of values at the end points of each primal 1-simplex. But the value of $\mathbf{d}f$ for dual 1-simplices with one end on the boundary is not the difference of the values at their end points and thus, for a dual 0-form we have $\mathbf{d}(f + a) \neq \mathbf{d}f$. As a consequence, the discrete Laplace-Beltrami operator on dual 0-forms is bijective.

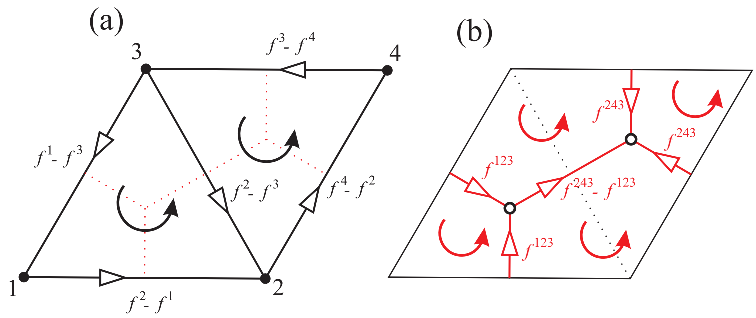


Figure 5: The discrete 1-forms $\mathbf{d}f$ obtained from (a) primal and (b) dual 0-form f. The sets $\{f^1, f^2, f^3, f^4\}$ and $\{f^{123}, f^{243}\}$ are the sets of values of primal and dual 0-forms, respectively. Note that f^{123} is the value of f at $c([1,2,3])$, etc.

Hodge Star. Recall that in the smooth case, the Hodge star operator $* : \Omega^k(\mathcal{N}) \to \Omega^{n-k}(\mathcal{N})$ for a smooth n-manifold $\mathcal{N}$, is uniquely defined by the identity [1]

$$\boldsymbol{\alpha} \wedge *\boldsymbol{\beta} = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}^\sharp \rangle \boldsymbol{\mu}, \tag{83}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are k-forms and $\boldsymbol{\mu}$ is the volume form for $\mathcal{N}$. For an n-dimensional mesh, the discrete Hodge star is defined as follows. Suppose $1 \le k \le n-1$, then the discrete Hodge star operator is a map $* : \Omega_d^k(K) \to \Omega_d^{n-k}(\star K)$ that for a k-simplex σ^k and a discrete k-form $\boldsymbol{\alpha}$, satisfies the identity

$$\frac{1}{|\star\sigma^k|} \langle *\boldsymbol{\alpha}, \star\sigma^k \rangle = \frac{1}{|\sigma^k|} \langle \boldsymbol{\alpha}, \sigma^k \rangle, \tag{84}$$

where $|\sigma^k|$ and $|\star\sigma^k|$ denote the volumes of σ^k and $\star\sigma^k$, respectively. Thus, one can obtain the components of the $(n-k)$-form $*\boldsymbol{\alpha}$ using the above relation, which uniquely determines $*\boldsymbol{\alpha}$. Note that the left and right-hand sides of (84) depend on the orientations of the dual and primal meshes, respectively. But as the primal and dual vertices have only one orientation, by definition, for the cases $k=0$ and $k=n$ one side of (84) will be independent of the orientation while the other side changes sign by changing the orientation. Thus, we need to modify the above definition for these cases. For $k = 0$ we define

$$\frac{1}{s|\star\sigma^0|}\langle *\boldsymbol{\alpha}, \star\sigma^0\rangle = \frac{1}{|\sigma^0|}\langle \boldsymbol{\alpha}, \sigma^0\rangle, \tag{85}$$

where we assume the volume of a primal or dual vertex to be +1, i.e., $|\sigma^0| := 1$, and $s = (-1)^{n-1}\text{sgn}(\partial(\star\sigma^0), \star\sigma^1)$, where an edge $\sigma^1 \succ \sigma^0$ pointing away from σ^0 and $\partial(\star\sigma^0)$ has the orientation induced by $\star\sigma^0$. Thus, if the dual of an outgoing 1-simplex has the same orientation as the orientation induced by $\star\sigma^0$, then $s = (-1)^{n-1}$, otherwise $s = (-1)^n$. Similarly, if $k = n$ we define

$$\frac{1}{|\star\sigma^n|}\langle *\boldsymbol{\alpha}, \star\sigma^n\rangle = \frac{1}{s|\sigma^n|}\langle \boldsymbol{\alpha}, \sigma^n\rangle, \tag{86}$$

where the value of s is determined as follows. Consider the induced orientation of σ^n on an edge $\sigma^{n-1} \prec \sigma^n$. If $\star\sigma^{n-1}$ points away from $\star\sigma^n$, then $s = (-1)^{n-1}$, otherwise $s = (-1)^n$. Note that the above relations can be used to define the discrete Hodge star as a map $* : \Omega_d^k(\star K) \to \Omega_d^{n-k}(K)$.

Flat Operator. One can define different flat operators, for example, a flat operator that associates a dual 1-form to a primal vector field or a primal 1-form to a dual vector field. Here, we need the former case. Note that [61] denotes this type of flat operator by $\flat_{\text{pdd}}$. The discrete flat operator on a primal vector field, $\flat : \mathfrak{X}_d(K) \to \Omega_d^1(\star K)$ is defined by its operation on dual elementary chains: given a primal vector field $\mathbf{X}$ and a primal $(n-1)$-simplex σ^{n-1}, we define

$$\langle \mathbf{X}^\flat, \star\sigma^{n-1}\rangle = \sum_{\sigma^0 \prec \sigma^{n-1}} \mathbf{X}(\sigma^0) \cdot (\star\boldsymbol{\sigma}^{n-1}), \tag{87}$$

where $\star\boldsymbol{\sigma}^{n-1}$ is the vector corresponding to $\star\sigma^{n-1}$, i.e., it has the length $|\star\sigma^{n-1}|$ in the direction of $\star\sigma^{n-1}$, and "$\cdot$" is the usual inner product of $\mathbb{R}^n$. Using this definition of flat operator, the primal discrete divergence theorem holds automatically. Also note that discrete flat operator is neither surjective nor injective, see [61] for more discussions.

Divergence. For vector fields on smooth manifolds the following relation holds [1]

$$\mathrm{div}\mathbf{X} = -\boldsymbol{\delta}\mathbf{X}^\flat = *\mathbf{d} * \mathbf{X}^\flat, \tag{88}$$

where $\boldsymbol{\delta} : \Omega^{k+1}(\mathcal{N}) \to \Omega^k(\mathcal{N})$ is the codifferential operator. Since we have already defined the discrete flat operator, discrete Hodge star, and discrete exterior derivative, we can directly use (88) as the definition of the discrete divergence as follows. Let $\mathbf{X}$ be a primal vector field, then the discrete divergence $\mathrm{div}\mathbf{X}$ is the dual 0-form given by

$$\langle \mathrm{div}\mathbf{X}, \star\sigma^n \rangle = * \, \mathbf{d} * \mathbf{X}^\flat. \tag{89}$$

Then, the following divergence theorem holds on a primal mesh, which can be proved by a direct calculation, see Lemma 6.1.6 of [61].

Divergence Theorem on a Primal Mesh. Let K be an n-dimensional primal mesh and σ^0 be one of its primal vertices. Let $\mathbf{X}$ be a primal vector field on the mesh. Then

$$|\sigma^n| \langle \mathrm{div}\mathbf{X}, \star\sigma^n \rangle = \sum_{\sigma^{n-1} \prec \sigma^n} s_{n-1} |\sigma^{n-1}| \left(\sum_{\sigma^0 \prec \sigma^{n-1}} \mathbf{X}(\sigma^0) \right) \cdot \frac{\star\sigma^{n-1}}{|\star \sigma^{n-1}|}, \tag{90}$$

where $s_{n-1} = +1$ if the orientation of σ^n is such that the dual edges $\star\sigma^{n-1}$ point outwards and $s_{n-1} = -1$ otherwise.

In the next section, we show that the discrete divergence on a planar simply-connected mesh is surjective and use this discrete divergence to characterize the space of discrete displacement fields of incompressible linearized elasticity.

Laplace-Beltrami. The smooth Laplace-Beltrami operator $\Delta : \Omega^0(\mathcal{N}) \to \Omega^0(\mathcal{N})$ is defined as $\Delta = \text{div} \circ \text{grad}$. As the gradient of a smooth function $f : \mathcal{N} \to \mathbb{R}$ is $(\mathbf{d}f)^\sharp$, we can write

$$\Delta f = *\mathbf{d} * \left[(\mathbf{d}f)^\sharp \right]^\flat = *\mathbf{d} * \mathbf{d}f. \tag{91}$$

We already know the definitions of discrete $\mathbf{d}$ and $*$, and hence we can use (91) to define the primal and dual discrete Laplace-Beltrami operators $\Delta : \Omega^0_d(K) \to \Omega^0_d(K)$ and $\Delta : \Omega^0_d(\star K) \to \Omega^0_d(\star K)$, respectively. Obviously, the smooth Δ operator is not injective. The same is true for the primal discrete Δ operator, but as we will show in the sequel, the dual discrete Δ operator is bijective. In §2.4 we use the dual discrete Δ operator to calculate the discrete pressure field from the pressure gradient.

2.2.6 Affine Interpolation

To define the elastic energy, we need to interpolate the discrete displacement field over support volumes. For this we use the so-called $\mathbb{P}_1$ polynomials [45]. Let $\{\mathbf{r}^i \in \mathbb{R}^n\}_{i=1,\dots,n+1}$ be a set a of $n+1$ geometrically-independent points that are the vertices of the n-simplex τ^n and suppose that $\{x^i\}$ is the canonical Euclidean coordinate system for $\mathbb{R}^n$. Let $\{\mathbf{U}^i \in \mathbb{R}^n\}_{i=1,\dots,n+1}$ be a primal vector field on these points, i.e., $\mathbf{U}^i$ is the value of the vector field at the vertex $\mathbf{r}^i$. For $1 \leq i \leq n+1$, let $\lambda_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be the associated barycentric coordinates [45]. Then, the interpolating function $\mathfrak{A} : \tau^n \to \mathbb{R}^n$ is given by $\mathfrak{A}(x^1, \dots, x^n) = \sum_{i=1}^{n+1} \lambda_i(x^1, \dots, x^n)\mathbf{U}^i$, and we have $\mathfrak{A}(\mathbf{r}^i) = \mathbf{U}^i$. Alternatively, it is easier to use the following non-standard form:

$$\mathfrak{A}(x^1, \dots, x^n) = \mathbf{q}^{n+1} + \sum_{i=1}^{n} x^i \mathbf{q}^i, \tag{92}$$

where the constant vectors $\mathbf{q}^i \in \mathbb{R}^n,\ i = 1, \dots, n+1$, are given by

$$\mathbf{q}^i = \sum_{j=1}^{n+1} \mathbb{Q}^{ij} \mathbf{U}^j,\ i = 1, \dots, n+1, \tag{93}$$

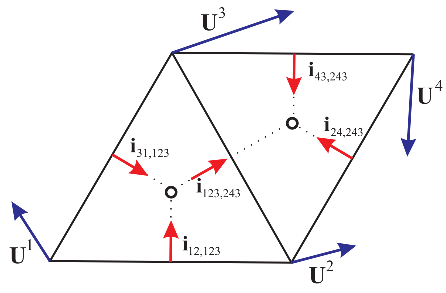


Figure 6: A discrete primal vector field, see Fig. 2 for the numbering of the simplices and orientation of the primal and dual meshes. The vector $\mathbf{i}_{31,123}$ is the unit vector with the same orientation as $[c_{31}, c_{123}]$, etc.

and the diagonal matrices $\mathbb{Q}^{ij} \in \mathbb{R}^{n \times n}$, $i, j = 1, \ldots, n+1$, depend only on $\mathbf{r}^i$'s and are independent of $\mathbf{U}^i$'s.

2.3 Discrete Configuration Manifold of Incompressible Linearized Elasticity

As we mentioned in §2.1.2, in linearized elasticity one needs to find the unknown displacement field, which is a vector field on the reference configuration of the elastic body. Thus, we need to consider a fixed well-centered primal mesh for representing the reference configuration, and therefore linearized elasticity is similar to fluid mechanics in the sense that both need a fixed mesh. Note that choosing such well-centered primal meshes is always possible in $\mathbb{R}^2$ as equilateral triangles fill $\mathbb{R}^2$, and hence one can always approximate planar regions with these well-centered simplices. Generating well-centered meshes is not a straightforward task, in general. See [104] and references therein for further discussions. However, the approach that we develop here can be extended to arbitrary domains by either generating well-centered meshes for that domain or if it is not possible to generate a well-centered mesh, by using another DEC theory that is appropriate for other types of meshes.

We select the displacement field as our primary unknown, which is a primal discrete vector field. Note that by choosing displacement field as our unknown, we do not need to consider compatibility equations. In order to design a structure-preserving scheme, we require that unknown variables remain in the correct space not only when they converge to the final solution, but also during the process of finding the solution. For incompressible elasticity, this means that we need to search in the space of discrete divergence-free vector fields. The configuration space of incompressible elasticity is similar to that of incompressible fluids. Pavlov et al. [89] developed a structure-preserving method for incompressible perfect fluids. In that scheme, they discretized push-forward of real-valued functions and showed that the space of divergence-free vector fields can be described by some orthogonal matrices. However, in order to define their discrete operators, they had to impose a nonholonomic constraint on the orthogonal matrices, which perhaps makes sense for fluids but is not reasonable for elasticity. Here, we propose a different idea for describing the space of discrete divergence-free vector fields. For better understanding the idea, we return to the primal mesh shown in Fig. 2 and calculate the divergence of a discrete primal vector field, see Fig. 6. Let $\mathbf{U}^i$ denote the vector field at vertex i. A straightforward calculation using the definitions of the previous section yields

$$
\begin{aligned}
\langle \mathrm{div}\mathbf{U}, \star[1,2,3] \rangle &= \frac{1}{|[1,2,3]|} \sum_{i=1}^{4} \mathbf{c}_{1i} \cdot \mathbf{U}^i, \\
\langle \mathrm{div}\mathbf{U}, \star[2,4,3] \rangle &= \frac{1}{|[2,4,3]|} \sum_{i=1}^{4} \mathbf{c}_{2i} \cdot \mathbf{U}^i,
\end{aligned}
\tag{94}
$$

where "$\cdot$" denotes the usual inner product and the vectors $\mathbf{c}_{ij} \in \mathbb{R}^2$ are given by

$$
\begin{aligned}
&\mathbf{c}_{11} = |[3,1]|\mathbf{i}_{31,123} + |[1,2]|\mathbf{i}_{12,123}, \\
&\mathbf{c}_{12} = |[1,2]|\mathbf{i}_{12,123} - |[3,2]|\mathbf{i}_{123,243}, \\
&\mathbf{c}_{13} = |[3,1]|\mathbf{i}_{31,123} - |[3,2]|\mathbf{i}_{123,243}, \;\; \mathbf{c}_{14} = \mathbf{0},
\end{aligned}
\tag{95}
$$

and

$$
\begin{aligned}
&\mathbf{c}_{21} = \mathbf{0}, \ \ \mathbf{c}_{22} = |[2,4]|\mathbf{i}_{24,243} + |[3,2]|\mathbf{i}_{123,243}, \\
&\mathbf{c}_{23} = |[4,3]|\mathbf{i}_{43,243} + |[3,2]|\mathbf{i}_{123,243}, \\
&\mathbf{c}_{24} = |[2,4]|\mathbf{i}_{24,243} + |[4,3]|\mathbf{i}_{43,243},
\end{aligned}
\tag{96}
$$

with $\mathbf{i}_{31,123}$ denoting a unit vector with the same orientation as $[c_{31}, c_{123}]$, etc. (see Fig. 6). Now we use (94) to impose $\mathrm{div}\mathbf{U} = 0$, which results in

$$
\mathbb{I}_{2\times 8}\boldsymbol{X}_{8\times 1} = \mathbf{0}, \tag{97}
$$

where

$$
\begin{aligned}
\mathbb{I}_{2\times 8} &= \begin{bmatrix} \mathbf{c}_{11}^{\mathsf{T}} & \mathbf{c}_{12}^{\mathsf{T}} & \mathbf{c}_{13}^{\mathsf{T}} & \mathbf{c}_{14}^{\mathsf{T}} \\ & & & \\ \mathbf{c}_{21}^{\mathsf{T}} & \mathbf{c}_{22}^{\mathsf{T}} & \mathbf{c}_{23}^{\mathsf{T}} & \mathbf{c}_{24}^{\mathsf{T}} \end{bmatrix}, \\
\boldsymbol{X}_{8\times 1} &= \left\{\mathbf{U}^1, \cdots, \mathbf{U}^4\right\}^{\mathsf{T}}.
\end{aligned}
\tag{98}
$$

Note that the matrix $\mathbb{I}$ only depends on the mesh and does not depend on $\mathbf{U}$.

Remark. *The weak form of the incompressibility constraint reads*

$$
\int_{\mathcal{B}} q \, \mathrm{div}\mathbf{U} = 0, \qquad \forall q \in L^2(\mathcal{B}). \tag{99}
$$

Using the notation of [45], consider a Lagrange finite element over the mesh of Fig. 6, where the displacement field is approximated by continuous $\mathbb{P}_1$ polynomials and the pressure field by $\mathbb{P}_0$ polynomials, i.e. the displacement field is continuous and piecewise-linear while the pressure is piecewise-constant over triangles. For the simplex $[1,2,3]$, one can write $|[3,2]|\mathbf{i}_{123,243} = |[1,2]|\mathbf{i}_{12,123} + |[3,1]|\mathbf{i}_{31,123}$. Using this relation and the similar one for $[2,4,3]$, it is straightforward to show that (97) is the same as the discretization of (99) via the Lagrange finite elements.

Now let us consider an n-dimensional primal mesh K_h such that $|K_h| \subset \mathbb{R}^n$ is simply-connected[6] and denote the number of primal and dual vertices with P_h and D_h, where h is the diameter of the primal mesh, i.e., $h = \sup\{\mathsf{diam}(\sigma_i^n)|\sigma_i^n \in K\}$, with $\mathsf{diam}(\sigma_k^n) = \sup\left\{d(x,y)|x,y \in \sigma_k^n\right\}$, and $d(x,y)$ denotes the standard distance between x and y. Now we impose the essential boundary conditions. Suppose S_h denotes the number of those primal vertices that are located on the boundary of K_h and their displacements are specified. Note that these known displacements can be nonzero or even time dependent. The unknown primal displacement field $\mathbf{U}$ is denoted by $\{\mathbf{U}^i\}_{i=1,\ldots,\bar{\mathsf{P}}_h}$, where $\bar{\mathsf{P}}_h = \mathsf{P}_h - \mathsf{S}_h$, and $\mathbf{U}^i$ is the displacement at the vertex i. Imposing the incompressibility constraint using the procedure that resulted in (97) yields

$$\mathbb{I}^h_{\mathsf{D}_h \times n\bar{\mathsf{P}}_h} \boldsymbol{X}_{n\bar{\mathsf{P}}_h \times 1} = \mathbf{u}^h_{\mathsf{D}_h \times 1}, \tag{100}$$

where $\mathbb{I}^h$ is the reduced incompressibility matrix and depends only on the mesh, $\mathbf{u}^h$ depends on the known values of the displacements and also the mesh, and

$$\boldsymbol{X}_{n\bar{\mathsf{P}}_h \times 1} = \left\{\mathbf{U}^1_{n\times 1} \cdots \mathbf{U}^{\bar{\mathsf{P}}_h}_{n\times 1}\right\}^{\mathsf{T}}. \tag{101}$$

Thus, the displacement field is divergence free if and only if the vector $\boldsymbol{X}$ satisfies (100). Note that if the known displacements are all zero, then $\mathbf{u}^h = \mathbf{0}$. There is a systematic way to obtain the reduced incompressibility matrix and $\mathbf{u}^h$, which is a consequence of the discrete divergence theorem, cf. (90). Here we explain the method in $\mathbb{R}^2$, but it is also possible to extend it to higher dimensions.

Let $n = 2$ and consider a subset of a 2-dimensional primal mesh and its dual that

[6] Later, we will also discuss the effect of non-simply-connectedness.

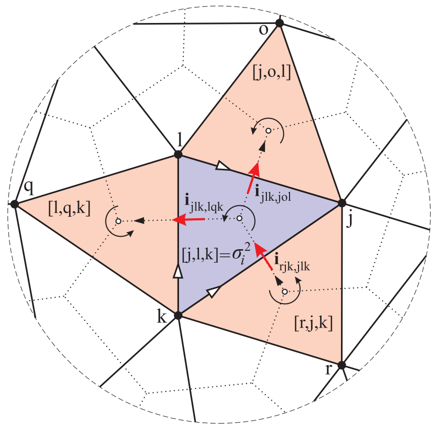


Figure 7: A subset of a primal mesh and its associated dual mesh. The vector $\mathbf{i}_{jlk,lqk}$ is the unit vector with the same orientation as $[c([j,l,k]),c([l,q,k])]$, etc.

are shown in Fig. 7. We define the matrix $\bar{\mathbb{I}}^h_{\mathsf{D}_h\times(2\mathsf{P}_h)}$, the incompressibility matrix, as

$$\bar{\mathbb{I}}^h=\begin{bmatrix}\mathbf{c}_{11}^{\mathsf{T}} & \cdots & \mathbf{c}_{1\mathsf{P}_h}^{\mathsf{T}}\\ \vdots & \ddots & \vdots\\ \mathbf{c}_{\mathsf{D}_h1}^{\mathsf{T}} & \cdots & \mathbf{c}_{\mathsf{D}_h\mathsf{P}_h}^{\mathsf{T}}\end{bmatrix}_{\mathsf{D}_h\times(2\mathsf{P}_h)}, \tag{102}$$

where $\mathbf{c}_{im}\in\mathbb{R}^2$, $i=1,\ldots,\mathsf{D}_h$, $m=1,\cdots,\mathsf{P}_h$, are specified as follows. Note that the number of the rows of $\bar{\mathbb{I}}^h$ is equal to the number of dual vertices (or equivalently primal 2-cells) as the divergence of a primal vector field is a dual zero-form. Now suppose that we order primal vertices and primal 2-cells of the mesh such that the vertices j, l, and k are the jth, lth, and kth primal vertices of the primal mesh, respectively, and $[j,l,k]$ is the ith 2-cell, i.e., $\sigma_i^2=[j,l,k]$, as shown in Fig. 7. Then, in the ith row of $\bar{\mathbb{I}}^h$, $\mathbf{c}_{im}$ is nonzero if and only if the mth primal vertex is a face of the ith primal 2-cell. This means that the only nonzero elements in the ith row corresponding to $\sigma_i^2=[j,l,k]$, are $\mathbf{c}_{ij}$, $\mathbf{c}_{il}$, and $\mathbf{c}_{ik}$. The vector $\mathbf{c}_{ij}$ is given by

$$\mathbf{c}_{ij}=s_k|[k,j]|\mathbf{i}_{rjk,jlk}+s_l|[l,j]|\mathbf{i}_{jlk,jol}, \tag{103}$$

where $\mathbf{i}_{rjk,jlk}$ denotes the unit vector with the same orientation as $[c([r,j,k]),c([j,l,k])]$, etc., and s_k is +1 if the orientation of $[k,j]$ is the same as the orientation induced by $\sigma_i^2=[j,l,k]$, otherwise $s_k=-1$. One can determine s_l similarly. Here we have $s_k=+1$ and $s_l=-1$. Similarly, we obtain

$$\begin{aligned}\mathbf{c}_{il}&=-|[l,j]|\mathbf{i}_{jlk,jol}-|[k,l]|\mathbf{i}_{jlk,lqk},\\ \mathbf{c}_{ik}&=-|[k,l]|\mathbf{i}_{jlk,lqk}+|[k,j]|\mathbf{i}_{rjk,jlk}.\end{aligned}\tag{104}$$

Noting that $\mathbf{i}_{jol,jlk}=-\mathbf{i}_{jlk,jol}$, one can rewrite (103) and (104) as

$$\begin{aligned}\mathbf{c}_{ij}&=|[k,j]|\mathbf{i}_{rjk,jlk}+|[l,j]|\mathbf{i}_{jol,jlk},\\ \mathbf{c}_{il}&=|[l,j]|\mathbf{i}_{jol,jlk}+|[k,l]|\mathbf{i}_{lqk,jlk},\\ \mathbf{c}_{ik}&=|[k,l]|\mathbf{i}_{lqk,jlk}+|[k,j]|\mathbf{i}_{rjk,jlk},\end{aligned}\tag{105}$$

which means that for writing nonzero $\mathbf{c}_{im}$'s for the ith 2-cell, one simply needs to consider unit normal vectors pointing into that cell and then consider all those terms with a plus sign in each nonzero $\mathbf{c}_{im}$'s. Thus, we can write the incompressibility matrix without using the orientation of the primal and dual meshes. The condition $\mathrm{div}\mathbf{U}=0$ is equivalent to

$$\bar{\mathbb{I}}^h_{\mathsf{D}_h\times(2\mathsf{P}_h)}\bar{\boldsymbol{X}}_{(2\mathsf{P}_h)\times 1}=\mathbf{0},\tag{106}$$

where $\bar{\boldsymbol{X}}$ is defined similarly to (101) but contains both known and unknown displacements.

Suppose $|K_h|\subset\mathbb{R}^2$ is simply-connected, i.e. its fundamental group and consequently its first homology group are both trivial. Then, the Euler characteristic of $|K_h|$ reads [78]

$$\begin{aligned}\chi(|K_h|)&=\#(0\text{-simplices})-\#(1\text{-simplices})\\ &\quad+\#(2\text{-simplices})=\mathsf{P}_h-\mathsf{E}_h+\mathsf{D}_h=1,\end{aligned}\tag{107}$$

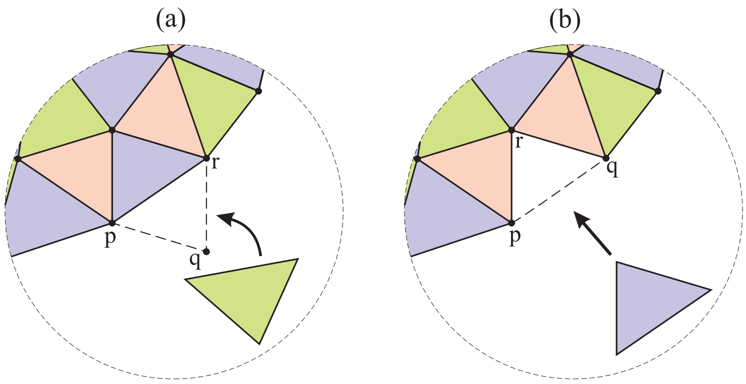


Figure 8: Two possible ways for adding a triangle to a 2-dimensional shellable mesh, either (a) the new triangle introduces a new primal vertex or (b) the new triangle does not introduce any new primal vertices.

where # denotes "number of" and we have used the fact that the number of 2-simplices of the primal mesh is equal to the number of dual vertices. Let P_h^i and P_h^b denote the number of primal vertices that belong to interior and boundary of $|K_h|$, respectively. Then we have $\mathsf{P}_h = \mathsf{P}_h^i + \mathsf{P}_h^b$. Similarly, let E_h^i and E_h^b denote the number of primal 1-simplices that belong to interior and boundary of $|K_h|$, respectively. We have $\mathsf{E}_h = \mathsf{E}_h^i + \mathsf{E}_h^b$. Using the above definitions, one can show that the following relations hold

$$3\mathsf{D}_h = \mathsf{E}_h^b + 2\mathsf{E}_h^i, \qquad \mathsf{P}_h^b = \mathsf{E}_h^b. \tag{108}$$

Using (107) and (108) we obtain

$$\mathsf{D}_h = 2\mathsf{P}_h^i + \mathsf{P}_h^b - 2 = 2\mathsf{P}_h - \mathsf{P}_h^b - 2. \tag{109}$$

Thus, we always have

$$\mathsf{D}_h < 2\mathsf{P}_h. \tag{110}$$

So, if $\bar{\mathbb{I}}^h$ is full-ranked, then $\mathsf{rank}(\bar{\mathbb{I}}^h) = \mathsf{D}_h$. Now, we show that the incompressibility matrix of a planar simply-connected mesh is always full-ranked.

Theorem 2.3.1. *Let K_h be a 2-dimensional well-centered primal mesh such that*

$|K_h|$ *is a simply-connected set. Then, the associated incompressibility matrix* $\bar{\mathbb{I}}^h$ *is full-ranked.*

Proof. Since $|K_h|$ is simply connected, because of (110) we need to show that the rows of $\bar{\mathbb{I}}^h$ are linearly independent. We use induction to complete the proof. Since K_h is shellable[7], one can consider a construction of K_h by starting with one triangle and then adding one triangle at a time such that the resulting simplical complex at each step is homeomorphic to a square. Thus, at each step the mesh has the same topological properties as K_h and, in particular, it is simply connected. Using (103) and (104), we conclude that the incompressibility matrix of a single triangle is full ranked, i.e., there exist non-zero elements in the matrix since edges of a triangle cannot be parallel to each other. Now suppose that in the process of constructing K_h we have a mesh with m triangles, K^m, that has a full-ranked incompressibility matrix $\bar{\mathbb{I}}_m$, i.e., rows of $\bar{\mathbb{I}}_m$ are linearly independent. As Fig. 8 shows, there are two possibilities for adding a new triangle to K^m: (i) the new triangle adds a new primal vertex to the mesh as in Fig. 8(a), and (ii) no new primal vertex is added to the mesh as in Fig. 8(b). In case (i) the incompressibility matrix of the resulting mesh, $\bar{\mathbb{I}}_{m+1}$, is full ranked because it is obtained from $\bar{\mathbb{I}}_m$ by adding a row corresponding to the added triangle and two columns for the displacement of the new vertex q (see Fig. 8(a)). The only nonzero entries in the new columns are placed on the new row and hence the new row is linearly independent from other rows. In case (ii) note that $\bar{\mathbb{I}}_{m+1}$ is obtained from $\bar{\mathbb{I}}_m$ by adding a new row corresponding to the new primal 2-cell, which has the vertices p, q, and r as is shown in Fig. 8(b). The matrix $\bar{\mathbb{I}}_m$ is full ranked and so it has D^m linearly independent columns, where D^m is the number of primal 2-cells of K^m. Thus, because of (110), the number of linearly-dependent columns of

[7] A simplical complex is called regular if it is homeomorphic to the unit cube. A regular simplical complex is shellable if either it consists of a single complex or it is possible to obtain a smaller regular complex by removing one of its simplices. All 2-dimensional regular complexes are shellable. Delaunay triangulation of a regular complex is shellable in any dimension [2].

$\bar{\mathbb{I}}_m$ is equal to $2\mathsf{P}^m - \mathsf{D}^m = \mathsf{P}_m^b + 2$. Due to the structure of $\bar{\mathbb{I}}_m$, one can choose all the independent columns from those that do not correspond to p, q, and r. Suppose that we choose such independent columns. The matrix $\bar{\mathbb{I}}_{m+1}$ is obtained by adding a row to $\bar{\mathbb{I}}_m$ that has zero components except for the ones that correspond to p, q, and r. This means that all the chosen independent columns of $\bar{\mathbb{I}}_m$ still remain independent for $\bar{\mathbb{I}}_{m+1}$ and at least one of the columns that corresponds to p, q, and r becomes independent of other columns. Therefore, $\bar{\mathbb{I}}_{m+1}$ has at least $\mathsf{D}^m + 1$ independent columns and since $\mathsf{D}^{m+1} = \mathsf{D}^m + 1$, we conclude that $\bar{\mathbb{I}}_{m+1}$ has exactly D^{m+1} independent columns and rows, and therefore it is full ranked. This completes the proof. □

Remark. *The assumption of simply connectedness is necessary in the above proof. Note that the incompressibility matrix I is important in our numerical scheme and not $\bar{I}$. The incompressibility matrix is obtained by removing some rows of $\bar{I}$ and it may or may not remain full-ranked even for a simply-connected domain. Here the important thing is that the number of columns of I remains greater than the number of its rows. This guarantees that the nullity of incompressibility matrix is greater than zero and hence the space of divergence-free vector fields would be a nontrivial finite-dimensional set. For both simply-connected and non-simply-connected domains one can obtain the incompressibility matrix with larger number of columns by mesh refinement.*

Also note that the extension of this theorem to 3-dimensional meshes is not straightforward. In particular, a simply-connected mesh in $\mathbb{R}^3$ is not necessarily shellable. In fact, Rudin [93] showed that there exists an unshellable triangulation for a tetrahedron.

Remark. *The above theorem tells us that the discrete primal-dual divergence over a planar simply-connected mesh is surjective because the discrete divergence operator from the space of discrete primal vector fields to the space of discrete dual zero-forms is a linear map defined by the matrix $\bar{\mathbb{I}}$. Thus, $\bar{\mathbb{I}}$ being full-ranked implies that the associated linear map is surjective. This is interesting as the discrete flat operator* *that is used in the definition of divergence is not surjective.*

Now, let us consider (106). In general, using the rank-nullity theorem and (110), we can write

$$\mathsf{nullity}(\bar{\mathbb{I}}^h) = 2\mathsf{P}_h - \mathsf{rank}(\bar{\mathbb{I}}^h) = 2\mathsf{P}_h - \mathsf{D}_h = \mathsf{P}_h^b + 2 > 0, \tag{111}$$

which means that for an arbitrary planar simply-connected mesh, the space of discrete divergence-free primal vector fields is finite-dimensional. In particular if $\{\bar{\mathbf{w}}_i, \ldots, \in \mathbb{R}^{2\mathsf{P}_h}\}$ is a basis for the null space of $\bar{\mathbb{I}}^h$, we then can write

$$\bar{\boldsymbol{X}} = \sum_i \hat{D}_i \bar{\mathbf{w}}_i, \tag{112}$$

where $\hat{D}_i$ are real numbers. Therefore, a displacement field is divergence free if and only if it can be expressed as in (112).

By imposing the essential boundary conditions in (106), we obtain (100), i.e., $\mathbb{I}^h$ is obtained by eliminating those columns of $\bar{\mathbb{I}}^h$ that correspond to the specified displacements. The vector $\mathbf{u}^h$ is obtained by moving terms that include the specified displacements to the right-hand side of (106). If there are "too many" boundary vertices with specified displacements, then the number of rows of $\mathbb{I}^h$ may exceed the number of its columns, and therefore (100) may not admit any solution. This is similar to the continuous case where there may not exist a divergence-free vector field for some choices of boundary conditions. We elucidate this in the following example.

Example 2.3.2. *(Incompressibility matrix for a planar mesh). Consider a mesh consisting of equilateral triangles with unit lengths as shown in Fig. 9. Using (105), we obtain the incompressibility matrix* $\bar{\mathbb{I}}^h_{6\times 14}$ *as*

$$\bar{\mathbb{I}}^h = \left[\mathbb{J}^h \; \mathbb{K}^h\right], \tag{113}$$

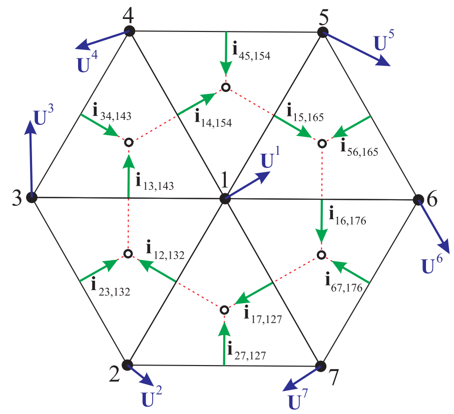


Figure 9: A 2-dimensional primal mesh with its associated dual mesh and the associated unit vectors. The vector $\mathbf{i}_{12,132}$ is the unit vector with the same orientation as $[c([1,2]), c([1,3,2])]$, etc.

where

$$\mathbb{J}^h = \begin{bmatrix} \mathbf{i}_{12,132}+\mathbf{i}_{143,13} & \mathbf{i}_{12,132}+\mathbf{i}_{23,132} & \mathbf{i}_{23,132}+\mathbf{i}_{143,13} & \mathbf{0} \\ \mathbf{i}_{13,143}+\mathbf{i}_{154,14} & \mathbf{0} & \mathbf{i}_{13,143}+\mathbf{i}_{34,143} & \mathbf{i}_{34,143}+\mathbf{i}_{154,14} \\ \mathbf{i}_{14,154}+\mathbf{i}_{165,15} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{14,154}+\mathbf{i}_{45,154} \\ \mathbf{i}_{15,165}+\mathbf{i}_{176,16} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{i}_{16,176}+\mathbf{i}_{127,17} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{i}_{17,127}+\mathbf{i}_{132,12} & \mathbf{i}_{27,127}+\mathbf{i}_{132,12} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{114}$$

and

$$\mathbb{K}^h = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{i}_{45,154}+\mathbf{i}_{165,15} & \mathbf{0} & \mathbf{0} \\ \mathbf{i}_{15,165}+\mathbf{i}_{56,165} & \mathbf{i}_{56,165}+\mathbf{i}_{176,16} & \mathbf{0} \\ \mathbf{0} & \mathbf{i}_{16,176}+\mathbf{i}_{67,176} & \mathbf{i}_{67,176}+\mathbf{i}_{127,17} \\ \mathbf{0} & \mathbf{0} & \mathbf{i}_{17,127}+\mathbf{i}_{27,127} \end{bmatrix}, \tag{115}$$

where $\mathbf{i}$*'s are unit vectors shown in Fig. 9 with* $\mathbf{i}_{143,13} = -\mathbf{i}_{13,143}$*, etc. Also we have*

$$\bar{\boldsymbol{X}}_{14\times1} = \left\{\mathbf{U}^1, \cdots, \mathbf{U}^7\right\}^{\mathsf{T}}. \tag{116}$$

A primal vector field on this mesh is divergence-free if and only if (106) is satisfied. Since $\bar{\mathbb{I}}^h$ *is full ranked in this example we have* $\mathsf{nullity}(\bar{\mathbb{I}}^h) = 14 - 6 = 8$*, so we observe that similar to the continuous case where the set of divergence-free vector fields* $\bar{\mathfrak{U}}$ *defined in (52) is nonempty and, in fact, is infinite-dimensional, the set of discrete divergence-free vector fields on this mesh is also nonempty but, of course, it is finite-dimensional. Now suppose that all the boundary vertices have specified displacements, i.e.,* $\mathbf{U}^i = \widetilde{\mathbf{U}}^i$ *for* $i = 2, \ldots, 7$*. Then, by moving the terms corresponding to the known displacements to the right-hand side of (106), we obtain (100), where* $\boldsymbol{X}_{2\times1} = \mathbf{U}^1$*, with*

$$\mathbb{I}^h_{6\times2} = \begin{bmatrix} \mathbf{i}_{12,132} + \mathbf{i}_{143,13} \\ \mathbf{i}_{13,143} + \mathbf{i}_{154,14} \\ \mathbf{i}_{14,154} + \mathbf{i}_{165,15} \\ \mathbf{i}_{15,165} + \mathbf{i}_{176,67} \\ \mathbf{i}_{67,176} + \mathbf{i}_{17,127} \\ \mathbf{i}_{17,127} + \mathbf{i}_{132,12} \end{bmatrix}, \tag{117}$$

and

$$\mathbf{u}^h_{6\times1} = -\left\{ \begin{matrix} (\mathbf{i}_{12,132} + \mathbf{i}_{23,132}) \cdot \widetilde{\mathbf{U}}^2 + (\mathbf{i}_{23,132} + \mathbf{i}_{143,13}) \cdot \widetilde{\mathbf{U}}^3 \\ (\mathbf{i}_{13,143} + \mathbf{i}_{34,143}) \cdot \widetilde{\mathbf{U}}^3 + (\mathbf{i}_{34,143} + \mathbf{i}_{154,14}) \cdot \widetilde{\mathbf{U}}^4 \\ (\mathbf{i}_{14,154} + \mathbf{i}_{45,154}) \cdot \widetilde{\mathbf{U}}^4 + (\mathbf{i}_{45,154} + \mathbf{i}_{165,15}) \cdot \widetilde{\mathbf{U}}^5 \\ (\mathbf{i}_{15,165} + \mathbf{i}_{56,165}) \cdot \widetilde{\mathbf{U}}^5 + (\mathbf{i}_{56,165} + \mathbf{i}_{176,67}) \cdot \widetilde{\mathbf{U}}^6 \\ (\mathbf{i}_{16,176} + \mathbf{i}_{67,176}) \cdot \widetilde{\mathbf{U}}^6 + (\mathbf{i}_{67,176} + \mathbf{i}_{127,17}) \cdot \widetilde{\mathbf{U}}^7 \\ (\mathbf{i}_{27,127} + \mathbf{i}_{132,12}) \cdot \widetilde{\mathbf{U}}^2 + (\mathbf{i}_{17,127} + \mathbf{i}_{27,127}) \cdot \widetilde{\mathbf{U}}^7 \end{matrix} \right\}. \tag{118}$$

Here it is obvious that (100) cannot admit any solutions for some values of the boundary displacements. In our example, if all the boundary displacements vanish, then $\boldsymbol{X} = \mathbf{0}$ *is the only solution. Thus, we see that contrary to the continuous case where the set of divergence-free vector fields that vanish on the boundary of a manifold (*$\mathfrak{U}_0$ *defined in (52)) is always infinite-dimensional, the corresponding discrete set may only contain the zero vector field.*

In general, if

$$2\mathsf{S}_h < \mathsf{P}_h^b + 2, \tag{119}$$

then from (109) we have

$$\begin{aligned} \mathsf{nullity}(\mathbb{I}^h) &= 2\bar{\mathsf{P}}_h - \mathsf{rank}(\mathbb{I}^h) = 2\mathsf{P}_h - 2\mathsf{S}_h - \mathsf{rank}(\mathbb{I}^h) \\ &\geq 2\mathsf{P}_h - \mathsf{D}_h - 2\mathsf{S}_h = \mathsf{P}_h^b + 2 - 2\mathsf{S}_h > 0, \end{aligned} \tag{120}$$

and, therefore, the space of divergence-free vector fields satisfying the essential boundary conditions would be finite-dimensional. Also note that if the essential boundary conditions are not imposed on all the boundary vertices, then one can satisfy (119) by choosing finer meshes on that part of the boundary with no essential boundary conditions. To summarize, we observe that the dimension of the space of divergence-free vector fields on a planar simply-connected mesh is $2\mathsf{P}_h - \mathsf{D}_h$, but by imposing essential boundary conditions, this space may become empty or may contain only the zero vector field.

Let K_h be an n-dimensional primal mesh, possibly not simply-connected and suppose $\mathsf{nullity}(\mathbb{I}^h) = \mathsf{N}$. Let $\{\mathbf{w}_i \in \mathbb{R}^{n\bar{\mathsf{P}}_h}\}_{i=1}^{\mathsf{N}}$ be a basis for the null space of $\mathbb{I}^h$. Then, from (100) we conclude that if a time-dependent displacement field is divergence-free we have

$$\boldsymbol{X}(t) = \boldsymbol{X}^\circ + \sum_{i=1}^{\mathsf{N}} D_i(t)\mathbf{w}_i, \tag{121}$$

where $\boldsymbol{X}^\circ$ is a solution to the inhomogeneous linear system (100) and D_i's are some real-valued functions of time and $\boldsymbol{X}^\circ$ and $\mathbf{w}_i$'s are time independent. This completely determines the space of time-dependent displacement fields. Note that if the essential boundary conditions are time dependent, then $\bar{\mathbb{I}}^h$ and so $\mathbf{w}_i$'s are still time independent but $\mathbf{u}^h$ becomes time dependent, which implies that $\boldsymbol{X}^\circ$ is time dependent, as well.

Remark. *(Non-simply-connected meshes) If K_h has some holes, then (106) and (108) are still valid but (107) reads*

$$\chi(|K_h|) = \mathsf{P}_h - \mathsf{E}_h + \mathsf{D}_h = 1 - \mathsf{H}, \tag{122}$$

where H denotes the number of holes. Thus, (109) is replaced by

$$\mathsf{D}_h = 2\mathsf{P}_h - \mathsf{P}_h^b - 2 + 2\mathsf{H}, \tag{123}$$

and so (110) is not necessarily valid. Thus, the effect of holes is similar to the effect of essential boundary conditions in the sense that both can cause the number of the rows of the incompressibility matrix exceed the number of its columns. In particular, note that a non-simply-connected domain may have an incompressibility matrix with the number of its rows exceeding the number of columns even without any essential boundary conditions if there are too many holes in the mesh, i.e., if $\mathsf{P}_h^b + 2 < 2\mathsf{H}$. Similar to the problems that have too many nodes with essential boundary conditions, here one can obtain an incompressibility matrix with more columns than rows by refining the mesh.

2.4 Discrete Governing Equations

As we showed in §2.1.1 and §2.1.2, incompressible finite and linear elasticity solutions extremize the action in the space of divergence-free motions. This is the procedure

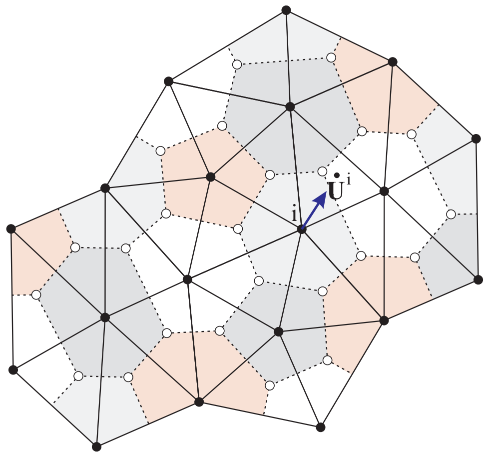


Figure 10: Dual cells that are used for defining the kinetic energy. Primal and dual vertices are denoted by • and ∘, respectively. The solid lines denote the boundary of the primal 2-cells and the colored regions denote the dual of each primal vertex. The material properties, displacements, and velocities are considered to be constant on each dual 2-cell. For example, consider the primal vertex i (σ_i^0). The velocity at the corresponding dual cell is assumed to be equal to the velocity at vertex i, which is denoted by $\dot{\mathbf{U}}^i$.

that we use for obtaining the governing equations in our discrete formulation. In the previous section, we characterized the space of discrete divergence-free motions in (121). Now, we need to write a discrete Lagrangian. We first define discrete kinetic and stored energies in the following.

2.4.1 Kinetic Energy

Let K_h be an n-dimensional mesh. The discrete displacement field is a primal vector field with displacement $\mathbf{U}^i$ at the primal vertex σ_i^0. As the numbers of primal vertices and dual n-cells are equal, we can associate $\mathbf{U}^i$ to $\star\sigma_i^0$, see Fig. 10. This means that we are assuming that the primal mesh is the union of the dual cells and we consider constant displacement and velocity on each dual cell. Suppose we order the primal vertices such that $i = 1, \ldots, \bar{\mathsf{P}}_h$ denote the primal vertices without essential boundary conditions and $i = \bar{\mathsf{P}}_h + 1, \ldots, \mathsf{P}_h$ denote those primal vertices with essential boundary conditions. We can now define the discrete kinetic energy, K^d, as

$$K^d = \frac{1}{2}\sum_{i=1}^{\bar{\mathsf{P}}_h} \rho_i | \star \sigma_i^0 | \ \dot{\mathbf{U}}^i \cdot \dot{\mathbf{U}}^i + \frac{1}{2}\sum_{i=\bar{\mathsf{P}}_h+1}^{\mathsf{P}_h} \rho_i | \star \sigma_i^0 | \ \dot{\mathbf{U}}^i \cdot \dot{\mathbf{U}}^i, \tag{124}$$

where "$\cdot$" denotes the usual dot product and ρ_i is the density on the dual cell $\star\sigma_i^0$, which can have different values on different cells if the elastic body is inhomogeneous. In fact, mass density can be considered as a primal 0-form. Note that time-dependent essential boundary conditions contribute to the kinetic energy through the term

$$K_e^d = \frac{1}{2}\sum_{i=\bar{\mathsf{P}}_h+1}^{\mathsf{P}_h} \rho_i | \star \sigma_i^0 | \ \dot{\mathbf{U}}^i \cdot \dot{\mathbf{U}}^i, \tag{125}$$

but because the variation of K_e^d is zero, it does not contribute to the Euler-Lagrange equations and hence one can safely exclude this term from the following calculations. Using (101) we can rewrite the discrete kinetic energy as

$$K^d = \frac{1}{2}\dot{\boldsymbol{X}}^{\mathsf{T}}\mathbf{M}\dot{\boldsymbol{X}} + K_e^d, \tag{126}$$

where $\mathbf{M} \in \mathbb{R}^{(n\bar{\mathsf{P}}_h)\times(n\bar{\mathsf{P}}_h)}$ is a diagonal square matrix with elements

$$M_{jk} = \begin{cases} \rho_i | \star \sigma_i^0 |, \text{ if } j = k = n(i-1) + s, \\ \qquad\qquad \text{with } 1 \le s \le n, \ 1 \le i \le \bar{\mathsf{P}}_h, \\ 0, \qquad\qquad \text{if } j \ne k. \end{cases} \tag{127}$$

We will use (126) to write the discrete Lagrangian.

2.4.2 Elastic Stored Energy

In this section we define a discrete elastic stored energy. For the sake of clarity, we do not use summation convention throughout this section unless it is explicitly stated otherwise. We order the primal vertices such that $\{\sigma_i^0\}_{i=1}^{\bar{\mathsf{P}}_h}$ and $\{\sigma_i^0\}_{i=\bar{\mathsf{P}}_h+1}^{\mathsf{P}_h}$ denote the

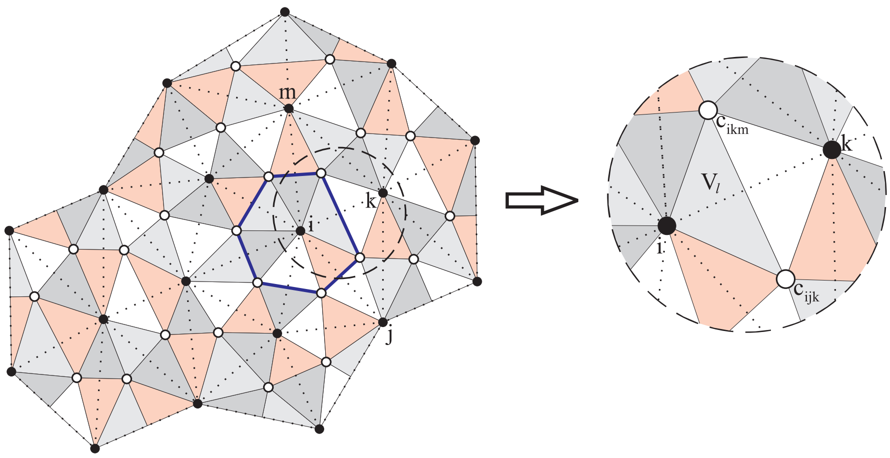


Figure 11: Regions that are used for calculating the elastic stored energy. Primal and dual vertices are denoted by • and ∘, respectively. The dotted lines denote the primal one-simplices. Displacement is interpolated using affine functions in each of the colored triangles which are the intersection of a support volume of a primal 1-simplex with a dual 2-cell. The elastic body is assumed to be homogeneous in each dual 2-cell. The region bounded by the solid lines denotes the dual of the primal vertex i. The stored energy at this dual cell is obtained by summing the internal energy of the corresponding 6 smaller triangles.

primal vertices without and with essential boundary conditions, respectively. We define the discrete elastic stored energy as $E^d = \sum_l \mathcal{E}^l$, where $\mathcal{E}^l$ is the internal energy of a portion of support volumes of 1-simplices that is calculated by interpolation of discrete displacements using an affine interpolation function. To fix ideas, we derive the explicit form of the discrete stored energy in $\mathbb{R}^2$ ($n = 2$). Consider a primal mesh as shown in Fig. 11. Discrete stored energy is written as

$$E^d = \sum_{l}^{2\mathsf{E}_h} \mathcal{E}^l, \tag{128}$$

where E_h is the number of primal 1-simplices and $\mathcal{E}^l$'s are the energies associated to the colored regions. This choice of regions follows naturally from our previous assumption of homogeneous material properties within each dual cell and the fact that we need three vertices for a planar affine interpolation. To obtain the explicit form of $\mathcal{E}^l$, consider the enlarged part of Fig. 11 and suppose that i, j, k, and m are the

ith, jth, kth, and mth primal vertices, respectively. Here we have $V_l = |\overline{[i,k]} \bigcap (\star\sigma_i^0)|$, where $\overline{[i,k]}$ denotes the support volume of $[i,k]$ defined in (73). Assuming summation convention on indices $a, b = 1, \ldots, n$, we define

$$\mathcal{E}^l = \int_{V_l} \mu e^{ab} e_{ab} dv, \tag{129}$$

where strains are calculated considering an affine interpolating function for vertices i, c_{ikm}, and c_{ijk}. Using (92), let

$$u|_{V_l} = \mathbf{q}_l^{n+1} + \sum_{b=1}^{n} x^b \mathbf{q}_l^b, \tag{130}$$

where $\mathbf{q}_l^b \in \mathbb{R}^n$, $b = 1, \ldots, n+1$, are constant vectors associated to V_l that can be written as (see (93))

$$\mathbf{q}_l^b = \mathbb{Q}_l^{bi} \mathbf{U}^i + \mathbb{Q}_l^{bc_{ijk}} \mathbf{U}^{c_{ijk}} + \mathbb{Q}_l^{bc_{ikm}} \mathbf{U}^{c_{ikm}}, \tag{131}$$

with $\mathbf{U}^{c_{ijk}}$ and $\mathbf{U}^{c_{ikm}}$ denoting displacements of c_{ijk} and c_{ikm}, respectively, and $\mathbb{Q}_l^{bi}$, $\mathbb{Q}_l^{bc_{ijk}}$, and $\mathbb{Q}_l^{bc_{ikm}}$ defined in §2.2.6 and are calculated using the vertices i, c_{ijk}, and c_{ikm}. To obtain $\mathbf{U}^{c_{ijk}}$ and $\mathbf{U}^{c_{ikm}}$ we need to interpolate displacements of $\mathfrak{J}_{ijk} = \{i, j, k\}$ and $\mathfrak{J}_{ikm} = \{i, k, m\}$, respectively. Using (92), we can write

$$\mathbf{U}^{c_{ijk}} = \sum_{b \in \mathfrak{J}_{ijk}} \mathbb{Q}_{[i,j,k]}^{3b} \mathbf{U}^b + \sum_{a=1}^{2} \sum_{b \in \mathfrak{J}_{ijk}} x_{c_{ijk}}^a \mathbb{Q}_{[i,j,k]}^{ab} \mathbf{U}^b, \tag{132}$$

$$\mathbf{U}^{c_{ikm}} = \sum_{b \in \mathfrak{J}_{ikm}} \mathbb{Q}_{[i,k,m]}^{3b} \mathbf{U}^b + \sum_{a=1}^{2} \sum_{b \in \mathfrak{J}_{ikm}} x_{c_{ikm}}^a \mathbb{Q}_{[i,k,m]}^{ab} \mathbf{U}^b, \tag{133}$$

where $x_{c_{ijk}}^a$ and $x_{c_{ikm}}^a$ denote the a-coordinate of c_{ijk} and c_{ikm}, respectively, and the index $[i,j,k]$ in $\mathbb{Q}_{[i,j,k]}^{ab}$ emphasizes that this matrix is obtained by interpolation over

$[i, j, k]$. Let $\mathfrak{J}_{ijkm} = \{i, j, k, m\}$, then substituting (132) and (133) into (131) yields

$$\mathbf{q}_l^b = \sum_{a \in \mathfrak{J}_{ijkm}} \mathbb{H}_l^{ba} \mathbf{U}^a, \quad b = 1, 2, 3, \tag{134}$$

where by defining $\mathfrak{S} = \{[i, j, k], [i, k, m]\}$, we can write $\mathbb{H}_l^{ba} \in \mathbb{R}^{2\times 2}$ as

$$\begin{aligned}
\mathbb{H}_l^{bi} &= \mathbb{Q}_l^{bi} + \sum_{\sigma \in \mathfrak{S}} \mathbb{Q}_l^{bc_\sigma} \left(\mathbb{Q}_\sigma^{3i} + \sum_{a=1}^{2} x_{c_\sigma}^a \mathbb{Q}_\sigma^{ai} \right), \\
\mathbb{H}_l^{bj} &= \mathbb{Q}_l^{bc_{ijk}} \left(\mathbb{Q}_{[i,j,k]}^{3j} + \sum_{a=1}^{2} x_{c_{ijk}}^a \mathbb{Q}_{[i,j,k]}^{aj} \right), \\
\mathbb{H}_l^{bk} &= \sum_{\sigma \in \mathfrak{S}} \mathbb{Q}_l^{bc_\sigma} \left(\mathbb{Q}_\sigma^{3k} + \sum_{a=1}^{2} x_{c_\sigma}^a \mathbb{Q}_\sigma^{ak} \right), \\
\mathbb{H}_l^{bm} &= \mathbb{Q}_l^{bc_{ikm}} \left(\mathbb{Q}_{[i,k,m]}^{3m} + \sum_{a=1}^{2} x_{c_{ikm}}^a \mathbb{Q}_{[i,k,m]}^{am} \right).
\end{aligned} \tag{135}$$

Note that $\mathbb{H}_l^{ba}$'s are diagonal matrices and hence symmetric. Because the ambient space is flat, we have

$$u^a{}_{|b} = \frac{\partial u^a}{\partial x^b} = q_l^{b,a}, \tag{136}$$

where $q_l^{b,a}$ denotes the a-component of $\mathbf{q}_l^b$. Also as $g_{ab} = \delta_{ab}$, using summation convention on index c, we can write $u_{a|b} = g_{ca} u^c{}_{|b} = \delta_{ca} q_l^{b,c} = q_l^{b,a}$, and hence

$$e_{ab} = \frac{1}{2} \left(q_l^{b,a} + q_l^{a,b} \right). \tag{137}$$

Note that (with summation convention on indices c and d)

$$e^{ab} = g^{ac} g^{bd} e_{cd} = \delta^{ac} \delta^{bd} e_{cd} = e_{ab}. \tag{138}$$

So using (137) and (138) we obtain

$$\mathcal{E}^l = \frac{1}{4} \mu_l V_l \left(q_l^{b,a} + q_l^{a,b} \right) \left(q_l^{b,a} + q_l^{a,b} \right), \tag{139}$$

where we use summation convention on $a, b = 1, 2$, and μ_l is the value of Lamé constant μ at V_l[8]. Alternatively, we can write (139) as

$$\mathcal{E}^l = \frac{1}{2}\mu_l V_l \left[(\mathbf{q}_l^1)^\mathsf{T} \boldsymbol{J} \mathbf{q}_l^1 + (\mathbf{q}_l^2)^\mathsf{T} \boldsymbol{K} \mathbf{q}_l^2 + (\mathbf{q}_l^1)^\mathsf{T} \boldsymbol{L} \mathbf{q}_l^2 \right], \tag{140}$$

where

$$\boldsymbol{J} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{K} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ \boldsymbol{L} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}. \tag{141}$$

Note that $\boldsymbol{L}$ is asymmetric and as we will see in the sequel, it induces asymmetry in subsequent matrices. Substituting (134) into (140) results in

$$\mathcal{E}^l = \sum_{a,b \in \mathfrak{J}_{ijkm}} (\mathbf{U}^a)^\mathsf{T} \mathbb{S}_l^{ab} \mathbf{U}^b, \tag{142}$$

with the matrices $\mathbb{S}_l^{ab} \in \mathbb{R}^{2\times 2}$ given by

$$\mathbb{S}_l^{ab} = \frac{1}{2}\mu_l V_l \left[(\mathbb{H}_l^{1a})^\mathsf{T} \boldsymbol{J} \mathbb{H}_l^{1b} + (\mathbb{H}_l^{2a})^\mathsf{T} \boldsymbol{K} \mathbb{H}_l^{2b} + (\mathbb{H}_l^{1a})^\mathsf{T} \boldsymbol{L} \mathbb{H}_l^{2b} \right]. \tag{143}$$

Note that $\mathbb{H}_l$'s, $\boldsymbol{J}$, and $\boldsymbol{K}$ are diagonal and so symmetric but $\boldsymbol{L}$ is asymmetric, and therefore $\mathbb{S}_l^{ab} \neq (\mathbb{S}_l^{ba})^\mathsf{T}$, and in particular, $\mathbb{S}_l^{aa}$'s are not symmetric.

Next, we impose the essential boundary conditions and obtain an expression for E^d. First consider the following definitions

$$\begin{aligned} &\mathfrak{N}(\sigma_a^0) = \left\{ \sigma^0 \in K^h \middle| \ \exists \sigma^1 \in K^h \text{ s.t. } \sigma_a^0, \sigma^0 \prec \sigma^1 \ \& \ \sigma_a^0 \neq \sigma^0 \right\}, \\ &\mathfrak{E} = \left\{ \sigma^0 \in K^h \middle| \text{ Essential B.C. is imposed on } \sigma^0 \right\}, \\ &\mathfrak{E}(\sigma_a^0) = \mathfrak{N}(\sigma_a^0) \cap \mathfrak{E}. \end{aligned} \tag{144}$$

[8]Recall that μ_l can be considered as a primal 0-form.

Thus, $\mathfrak{N}(\sigma_a^0)$ is the set of neighbors of σ_a^0 and $\mathfrak{E}(\sigma_a^0)$ is the set of the neighbors of σ_a^0 that have essential boundary conditions. Substituting (142) into (128) yields the following expression for the discrete elastic stored energy

$$E^d = \boldsymbol{X}^\mathsf{T}\mathbf{S}\boldsymbol{X} + \mathbf{s}\cdot\boldsymbol{X} + E_e^d, \tag{145}$$

where $\boldsymbol{X} \in \mathbb{R}^{n\bar{\mathsf{P}}_h}$ is defined in (101), the matrix $\mathbf{S} \in \mathbb{R}^{n\bar{\mathsf{P}}_h\times n\bar{\mathsf{P}}_h}$ can be written as

$$\mathbf{S} = \begin{bmatrix} \mathbb{D}^{11} & \cdots & \mathbb{D}^{1\bar{\mathsf{P}}_h} \\ \vdots & \ddots & \vdots \\ \mathbb{D}^{\bar{\mathsf{P}}_h 1} & \cdots & \mathbb{D}^{\bar{\mathsf{P}}_h\bar{\mathsf{P}}_h} \end{bmatrix}, \tag{146}$$

with

$$\mathbb{D}_{2\times2}^{ab} = \begin{cases} \sum_l \mathbb{S}_l^{ab}, & \text{if } b \in \mathfrak{N}(a), \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{147}$$

The vector $\mathbf{s} \in \mathbb{R}^{n\bar{\mathsf{P}}_h}$ is defined as $\mathbf{s} = \left\{\mathbf{d}^1, \cdots, \mathbf{d}^{\bar{\mathsf{P}}_h}\right\}^\mathsf{T}$, with

$$\mathbf{d}^a = \sum_{b\in\mathfrak{E}(a)} (\mathbf{U}^b)^\mathsf{T}\left[(\mathbb{S}_l^{ab})^\mathsf{T} + \mathbb{S}_l^{ba}\right], \tag{148}$$

and finally, the scalar E_e^d is given by

$$E_e^d = \sum_{a\in\mathfrak{E}}\sum_l (\mathbf{U}^a)^\mathsf{T}\,\mathbb{S}_l^{aa}\,\mathbf{U}^a. \tag{149}$$

The summation on l in (147) denotes summation over all those regions whose elastic energies are affected by the displacements of vertices a and b. Equation (149) has a similar interpretation. The matrix $\mathbf{S}$ is not symmetric, in general, and the vector $\mathbf{s}$ is zero if boundary vertices have zero displacements. Because both $\mathbf{S}$ and $\mathbf{S}^\mathsf{T}$ appear in the Euler-Lagrange equations, the governing equations remain symmetric in the sense that reciprocity holds. Also similar to K_e^d defined in (125), E_e^d does not contribute to the Euler-Lagrange equations either. Finally, note that there are other possible choices for writing a discrete elastic energy. In the next section, we use the discrete kinetic and elastic stored energies to write a discrete Lagrangian and obtain the Euler-Lagrange equations for linearized incompressible elasticity.

2.4.3 Discrete Euler-Lagrange Equations

In this section we use Hamilton's principle in the space of divergence-free displacements to obtain the Euler-Lagrange equations for the unknown $\boldsymbol{X}$. As in the continuous case that was discussed in §2.1.2.1, we do not use Lagrange multipliers to impose the incompressibility constraint. Instead, we confine the solution space to the divergence-free displacements and gradient of pressure appears naturally.

Similar to the Lagrangian in the continuous case, we define the discrete Lagrangian as

$$L^d = K^d - V^d, \tag{150}$$

where $V^d = E^d - B^d - T^d$, with B^d and T^d denoting the work of body forces and tractions at boundary nodes, respectively. We model a body force with a primal vector field. Let $\mathbf{B}^i$ be the body force at vertex i. Then, we have

$$B^d = \sum_{i=1}^{\mathsf{P}_h} m^i \mathbf{B}^i \cdot \mathbf{U}^i = \mathbf{b} \cdot \boldsymbol{X} + B_e^d, \tag{151}$$

where $m^i = \rho_i |\star \sigma_i^0|$, is the mass of the dual cell $\star\sigma_i^0$ and $\mathbf{b} \in \mathbb{R}^{n\bar{\mathsf{P}}_h}$ is defined as $\mathbf{b} = \left\{\mathbf{b}^1, \cdots, \mathbf{b}^{\bar{\mathsf{P}}_h}\right\}^{\mathsf{T}}$, with $\mathbf{b}^i = m^i \mathbf{B}^i$, and

$$B_e^d = \sum_{i=\bar{\mathsf{P}}_h+1}^{\mathsf{P}_h} m^i \mathbf{B}^i \cdot \mathbf{U}^i. \tag{152}$$

Note that similar to the previous section, we order the primal vertices such that

$\{\sigma_i^0\}_{i=1}^{\bar{\mathsf{P}}_h}$ and $\{\sigma_i^0\}_{i=\bar{\mathsf{P}}_h+1}^{\mathsf{P}_h}$ denote the primal vertices without and with essential boundary conditions, respectively. Let us define

$$T^d = \mathbf{t} \cdot \boldsymbol{X}, \tag{153}$$

where the vector $\mathbf{t} \in \mathbb{R}^{n\bar{\mathsf{P}}_h}$ is defined as $\mathbf{t} = \{\mathbf{t}^1, \cdots, \mathbf{t}^{\bar{\mathsf{P}}_h}\}$, with $\mathbf{t}^i = \mathbf{0}$, if the traction at σ_i^0 is zero. Note that we assume that the set of vertices with essential boundary conditions and the set of vertices with natural boundary conditions are disjoint. Therefore, the specified displacements do not contribute to T^d. Substituting (126), (145), (151), and (153) into (150) results in

$$L^d = \frac{1}{2}\dot{\boldsymbol{X}}^{\mathsf{T}}\mathbf{M}\dot{\boldsymbol{X}} - \boldsymbol{X}^{\mathsf{T}}\mathbf{S}\boldsymbol{X} + \mathbf{F} \cdot \boldsymbol{X} + L_e^d, \tag{154}$$

where

$$\mathbf{F} = -\mathbf{s} + \mathbf{b} + \mathbf{t}, \quad L_e^d = K_e^d - E_e^d + B_e^d. \tag{155}$$

Let the variational field of $\boldsymbol{X}$ be a 1-parameter family of divergence-free vector fields $\boldsymbol{X}_\epsilon$ that satisfy the essential boundary conditions and

$$\boldsymbol{X}_0 = \boldsymbol{X}, \qquad \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \boldsymbol{X}_\epsilon = \delta\boldsymbol{X}. \tag{156}$$

Note that $\boldsymbol{X}_\epsilon$ satisfies (100), i.e.

$$\mathbb{I}^h \boldsymbol{X}_\epsilon = \mathbf{u}^h, \tag{157}$$

and therefore

$$\mathbb{I}^h \,\delta\boldsymbol{X} = \mathbf{0}, \tag{158}$$

which means that $\delta \boldsymbol{X} \in \mathsf{Ker}(\mathbb{I}^h)$. Hamilton's principle tells us that

$$\delta \int_{t_1}^{t_2} L^d dt = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{t_1}^{t_2} L_\epsilon^d dt = 0. \tag{159}$$

Using (154), we can write

$$\begin{aligned}\delta \int_{t_1}^{t_2} L^d dt &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{t_1}^{t_2} \left(\frac{1}{2}\dot{\boldsymbol{X}}_\epsilon^\mathsf{T} \mathbf{M} \dot{\boldsymbol{X}}_\epsilon - \boldsymbol{X}_\epsilon^\mathsf{T} \mathbf{S} \boldsymbol{X}_\epsilon + \mathbf{F} \cdot \boldsymbol{X}_\epsilon + L_e^d\right) dt \\ &= \int_{t_1}^{t_2} \left[\dot{\boldsymbol{X}}^\mathsf{T} \mathbf{M} \left(\frac{d}{dt}\delta \boldsymbol{X}\right) - \boldsymbol{X}^\mathsf{T} \left(\mathbf{S} + \mathbf{S}^\mathsf{T}\right) \delta \boldsymbol{X} + \mathbf{F} \cdot \delta \boldsymbol{X}\right] dt \\ &= -\int_{t_1}^{t_2} \left[\mathbf{M} \ddot{\boldsymbol{X}} + \left(\mathbf{S} + \mathbf{S}^\mathsf{T}\right) \boldsymbol{X} - \mathbf{F}\right] \cdot \delta \boldsymbol{X} dt = 0,\end{aligned} \tag{160}$$

where we used symmetry of the matrix $\mathbf{M}$, the integration by parts for simplifying the kinetic energy contribution, and the assumption that $\delta \boldsymbol{X}$ is a proper variation, i.e., $\delta \boldsymbol{X} = \mathbf{0}$ at both t_1 and t_2. Because the integrand of (160) is a continuous function of time and t_1 and t_2 are arbitrary, we obtain

$$\left[\mathbf{M} \ddot{\boldsymbol{X}} + \left(\mathbf{S} + \mathbf{S}^\mathsf{T}\right) \boldsymbol{X} - \mathbf{F}\right] \cdot \delta \boldsymbol{X} = 0. \tag{161}$$

Note that we can write

$$\mathbb{R}^{n \bar{\mathsf{P}}_h} = \mathsf{Ker}(\mathbb{I}^h) \oplus \mathsf{Ker}(\mathbb{I}^h)^\perp, \tag{162}$$

and since $\delta \boldsymbol{X} \in \mathsf{Ker}(\mathbb{I}^h)$, from (161) we conclude that

$$\mathbf{M} \ddot{\boldsymbol{X}} + \left(\mathbf{S} + \mathbf{S}^\mathsf{T}\right) \boldsymbol{X} - \mathbf{F} = \boldsymbol{\Lambda}, \tag{163}$$

where $\boldsymbol{\Lambda} \in \mathsf{Ker}(\mathbb{I}^h)^\perp$.

Remark. *In the smooth case, we observed that confining the variations to the divergence-free vector fields results in the appearance of pressure gradient in the balance of linear momentum. As the vector $\boldsymbol{\Lambda}$ appears in the discrete governing equations through a similar procedure, it is reasonable to expect that this vector should somehow be related*

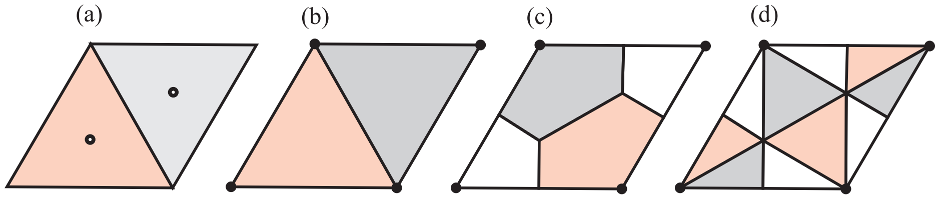


Figure 12: Discrete solution spaces: (a) $\mathbb{P}_0$ over primal meshes for the pressure field, (b) $\mathbb{P}_1$ over primal meshes for the displacement field in the incompressibility constraint, (c) $\mathbb{P}_0$ over dual meshes for the displacement field for approximating the kinetic energy, and (d) $\mathbb{P}_1$ over support volumes for the displacement field for approximating the elastic energy.

to discrete pressure gradient. As a matter of fact, the vector $\mathbf{\Lambda} \in \mathbb{R}^{n\bar{P}_h}$ can be written as

$$\mathbf{\Lambda} = \left\{\mathbf{\Lambda}^1, \cdots, \mathbf{\Lambda}^{\bar{P}_h}\right\}^{\mathsf{T}}, \tag{164}$$

where $\mathbf{\Lambda}^i \in \mathbb{R}^n$ can be thought as the value of the gradient of pressure at the primal vertex σ_i^0. Although we do not conduct a convergence analysis to show that the pressure field that is obtained by this assumption converges to the smooth pressure field, our numerical examples in the next section demonstrate that this assumption is valid. On the other hand, this correspondence suggests that pressure should be a dual zero-form because $\nabla p = (\mathbf{d}p)^\sharp$. This is a geometric justification for the known fact that using different function spaces for displacement and pressure is crucial in incompressible linearized elasticity [45, 58]. Also note that we do not obtain the pressure gradient for vertices with essential boundary conditions.

Recall that $\mathsf{Ker}(\mathbb{I}^h)^\perp$ is the orthogonal complement of the null space of $\mathbb{I}^h$, which is the row space of $\mathbb{I}^h$, i.e., the space spanned by the rows of $\mathbb{I}^h$. To obtain $\mathbf{\Lambda}$, note that from the rank-nullity theorem, one can write

$$\dim(\mathsf{Ker}(\mathbb{I}^h)^\perp) = n\bar{\mathsf{P}}_h - \mathsf{nullity}(\mathbb{I}^h) = \mathsf{rank}(\mathbb{I}^h) = \mathsf{R}. \tag{165}$$

Let $\{\mathbf{z}^1, \ldots, \mathbf{z}^{\mathsf{R}}\}$ be a basis for $\mathsf{Ker}(\mathbb{I}^h)^\perp$. Then, we have

$$\boldsymbol{\Lambda}(t) = \sum_{i=1}^{\mathsf{R}} \Lambda_i(t)\, \mathbf{z}^i, \tag{166}$$

where the time-dependent functions Λ_i's are unknowns to be determined. Thus, we have the following discrete governing equations for the unknowns $\boldsymbol{X}$ and Λ_i's:

$$\mathbf{M}\ddot{\boldsymbol{X}} + \left(\mathbf{S} + \mathbf{S}^{\mathsf{T}}\right)\boldsymbol{X} - \mathbf{F} = \sum_{i=1}^{\mathsf{R}} \Lambda_i \mathbf{z}^i, \tag{167}$$

$$\mathbb{I}^h \boldsymbol{X} = \mathbf{u}^h. \tag{168}$$

The number of unknowns is $n\bar{\mathsf{P}}_h + \mathsf{R}$ and since $\mathsf{rank}(\mathbb{I}^h) = \mathsf{R}$, we conclude that $\mathbb{I}^h$ has R independent rows (columns) and thus (168) has R independent equations and so the number of independent equations becomes $n\bar{\mathsf{P}}_h + \mathsf{R}$, which is equal to the number of unknowns. Therefore, one can solve (167) and obtain the displacement field and the pressure gradient.

Remark. *The difference between our approach for deriving (167) and (168) and that of the FE method is as follows. As we mentioned in §2.3, (168) is equivalent to a Lagrange finite element approximation, where the pressure field is approximated by $\mathbb{P}_0$ polynomials over the primal mesh and the displacement field is approximated by piecewise linear $\mathbb{P}_1$ polynomials over the primal mesh. On the other hand, we used two different spaces for the displacement field for writing (167), see Fig. 12. For discretizing the kinetic energy, the displacement field is discontinuous and is approximated by $\mathbb{P}_0$ polynomials over the dual mesh. However, for discretizing the elastic energy, it is continuous and is approximated by $\mathbb{P}_1$ polynomials over (part of) the support volumes.*

Remark. *(Incompressible linear elastostatics) For incompressible linearized elastostatics, the displacements and pressures are time independent, and therefore (167)* *and (168) are equivalent to*

$$\mathbb{K}_{(n\bar{\mathsf{P}}_h+\mathsf{D}_h)\times(n\bar{\mathsf{P}}_h+\mathsf{R})}\ \mathbb{X}_{(n\bar{\mathsf{P}}_h+\mathsf{R})\times 1} = \mathbb{F}_{(n\bar{\mathsf{P}}_h+\mathsf{D}_h)\times 1}, \tag{169}$$

where

$$\mathbb{K} = \begin{bmatrix} (\mathbf{S}+\mathbf{S}^{\mathsf{T}})_{n\bar{\mathsf{P}}_h\times n\bar{\mathsf{P}}_h} & -\mathbf{Z}_{n\bar{\mathsf{P}}_h\times \mathsf{R}} \\ \mathbb{I}^h_{\mathsf{D}_h\times n\bar{\mathsf{P}}_h} & \mathbf{0}_{\mathsf{D}_h\times\mathsf{R}} \end{bmatrix}, \tag{170}$$

with the i*th column of* $\mathbf{Z}$ *equal to* $\mathbf{z}^i$ *for* $i = 1,\ldots,\mathsf{R}$. *By defining* $\boldsymbol{\lambda} = \{\Lambda_1,\cdots,\Lambda_{\mathsf{R}}\}^{\mathsf{T}} \in \mathbb{R}^{\mathsf{R}}$*, the vectors* $\mathbb{X}$ *and* $\mathbb{F}$ *can be written as*

$$\mathbb{X} = \left\{\begin{array}{c} \boldsymbol{X} \\ \boldsymbol{\lambda} \end{array}\right\} \in \mathbb{R}^{n\bar{\mathsf{P}}_h+\mathsf{R}}, \qquad \mathbb{F} = \left\{\begin{array}{c} \mathbf{F} \\ \mathbf{u^h} \end{array}\right\} \in \mathbb{R}^{n\bar{\mathsf{P}}_h+\mathsf{D}_h}. \tag{171}$$

2.4.4 Discrete Pressure Field

Equations (167) and (168) enable us to obtain the displacement field of linear incompressible elasticity together with the pressure gradient $\boldsymbol{\Lambda}$. The next step is to calculate stresses. Note that we do not define a notion of discrete stress and, instead, we define stresses on subregions with constant strains, cf. §2.4.2. Therefore, we define the stress to be

$$\sigma^{ab} = 2\mu\, e^{ab} - p\, g^{ab}, \tag{172}$$

on each shaded region of Fig. 11. Using (137) and (138), the stress of the subregion l can be written as

$$\sigma_l^{ab} = \mu_l \left(q_l^{a,b} + q_l^{b,a}\right) - p_l\, \delta^{ab}. \tag{173}$$

Thus, we need to calculate the value of pressure on each dual vertex. The number of unknown pressures is D_h. Let $\{p^1,\ldots,p^{\mathsf{D}_h}\}$ denote the value of unknown pressures at dual vertices. We need to obtain D_h independent equations from the pressure gradient $\mathbf{\Lambda}$ to be able to specify the pressure. One way to do so is to use the discrete Laplace-Beltrami operator. This is explained in the following example.

Example 2.4.1. *Consider the planar mesh of Fig. 9 once again. Let p be a dual 0-form on this mesh representing the pressure and suppose p^{123} denotes $\langle p, \star[1,2,3]\rangle$, etc. We know that Δp is also a dual 0-form. Let us define the vector $\mathbf{p} \in \mathbb{R}^6$ as*

$$\mathbf{p} = \left\{p^{132}, p^{143}, p^{154}, p^{165}, p^{176}, p^{127}\right\}^{\mathsf{T}}. \tag{174}$$

One can use (91) to calculate Δp. On the other hand, one can obtain Δp using the pressure gradient $\mathbf{\Lambda}$. Suppose, for example, that the vertices 2 and 3 have essential boundary conditions. Thus, pressure gradient obtained from (167) lies in $\mathbb{R}^{10}$ and Δp is equal to the divergence of $\mathbf{\Lambda}$. To calculate the discrete pressure field, we need to obtain the discrete pressure gradient $\mathbb{G}_p \in \mathbb{R}^{14}$, which is a primal vector field. To this end, we need to assign a pressure gradient to those vertices that have essential boundary conditions. This can be done by assuming that pressure gradient of these vertices are equal to those of their closest interior primal vertices. This way, using $\mathbf{\Lambda}$, we can obtain the pressure gradient $\mathbb{G}_p \in \mathbb{R}^{14}$. Now we equate the expressions for Δp obtained using the previous two approaches. This gives

$$\mathbb{L}_{6\times 6}\,\mathbf{p} = \bar{\mathbb{I}}_{6\times 14}\mathbb{G}_p, \tag{175}$$

where $\bar{\mathbb{I}}$ *is given in (113) and*

$$\mathbb{L} = \begin{bmatrix} r_{1,2}+r_{1,3}+r_{2,3} & -r_{1,3} & 0 & 0 & 0 & -r_{1,2} \\ -r_{1,3} & r_{1,3}+r_{1,4}+r_{3,4} & -r_{1,4} & 0 & 0 & 0 \\ 0 & -r_{1,4} & r_{1,4}+r_{1,5}+r_{4,5} & -r_{1,5} & 0 & 0 \\ 0 & 0 & -r_{1,5} & r_{1,5}+r_{1,6}+r_{5,6} & -r_{1,6} & 0 \\ 0 & 0 & 0 & -r_{1,6} & r_{1,6}+r_{1,7}+r_{6,7} & -r_{1,7} \\ -r_{1,2} & 0 & 0 & 0 & -r_{1,7} & r_{1,7}+r_{1,2}+r_{2,7} \end{bmatrix}, \tag{176}$$

with

$$r_{i,j} = \frac{|[i,j]|}{|\star[i,j]|}. \tag{177}$$

The right-hand side of (175) *is a discrete analogue of* $\Delta = \operatorname{div} \circ \operatorname{grad}$. *Note that here* $\mathbb{L}$ *is symmetric and invertible. Therefore, we are able to obtain a unique dual* 0*-form* p.

Let us consider a planar mesh K_h. Fig. 7 shows part of this mesh. We define a symmetric matrix $\mathbb{L}^h \in \mathbb{R}^{\mathsf{D}_h \times \mathsf{D}_h}$ as follows. Each row of $\mathbb{L}^h$ corresponds to a dual vertex of K_h. All the elements in each row are zero except the diagonal elements and the ones that correspond to the dual vertices that are joined to the reference dual vertex by a dual 1-simplex. For example, in Fig. 7, nonzero components of the row that corresponds to $\star[j,l,k]$ are those that correspond to $\star[j,l,k]$, $\star[j,o,l]$, $\star[l,q,k]$, and $\star[r,j,k]$. Let us denote these components of $\mathbb{L}^h$ by $\mathbb{L}_{jlk,jlk}$, $\mathbb{L}_{jlk,jol}$, $\mathbb{L}_{jlk,lqk}$, and $\mathbb{L}_{jlk,rjk}$, respectively. Then, we have

$$\begin{aligned} &\mathbb{L}_{jlk,jlk} = r_{l,k} + r_{k,j} + r_{j,l},\ \mathbb{L}_{jlk,jol} = -r_{j,l}, \\ &\mathbb{L}_{jlk,lqk} = -r_{l,k},\ \mathbb{L}_{jlk,rjk} = -r_{k,j}. \end{aligned} \tag{178}$$

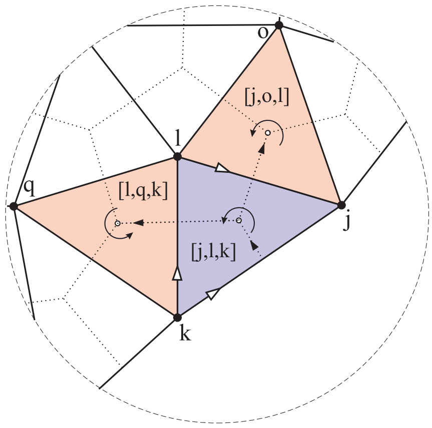


Figure 13: Part of a primal mesh and its associated dual mesh. The 2-simplex $[j, l, k]$ lies on the boundary.

Note that if $[j, l, k]$ is on the boundary as shown in Fig. 13, then the only nonzero components are $\mathbb{L}_{jlk,jlk}$, $\mathbb{L}_{jlk,jol}$, and $\mathbb{L}_{jlk,lqk}$, which are defined as above. Note also that by construction, the matrix $\mathbb{L}^h$ is symmetric. The sum of the components of the row corresponding to an internal primal n-simplex is zero while the same sum for rows that correspond to boundary n-simplices is not zero. As we will see in the following theorem, as a result of this specific structure, the symmetric matrix $\mathbb{L}^h$ is nonsingular and because Δp can be calculated using this matrix, the dual Laplace-Beltrami operator is injective. Note that the primal Δ operator is not injective. The reason for this is that the dual coboundary operator is not the same as the geometric boundary of a dual cell, see (82) and (91), and the corresponding discussions.

Theorem 2.4.2. *Let K_h be a planar well-centered primal mesh such that $|K_h|$ is a simply-connected set. Then, the matrix $\mathbb{L}^h \in \mathbb{R}^{\mathsf{D}_h \times \mathsf{D}_h}$ is nonsingular.*

Proof. The proof of this theorem is similar to that of Theorem 2.3.1. Using the fact that K_h is shellable, one can use induction and the specific structure of the matrix $\mathbb{L}$ to complete the proof. $\square$

Let the vector $\mathbf{p} \in \mathbb{R}^{\mathsf{D}_h}$ denote the pressure p on K_h, i.e., the ith component of $\mathbf{p}$ is $p^i = \langle p, \hat{\sigma}_i^0 \rangle$, where $\hat{\sigma}_i^0$ is the ith dual vertex. Then, one can use $\mathbb{L}^h$ to calculate

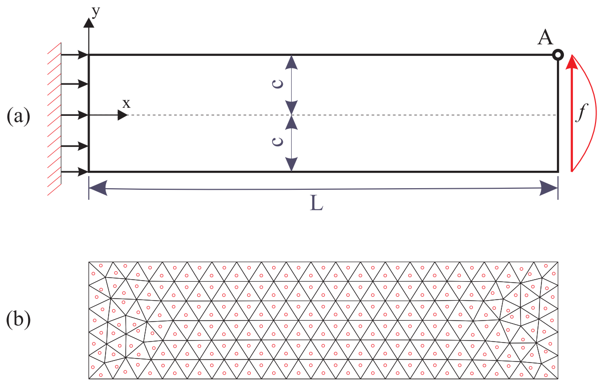


Figure 14: Cantilever beam: (a) Geometry, boundary conditions, and loading, (b) a well-centered primal mesh with $\circ$ denoting the circumcenter of each primal 2-cell.

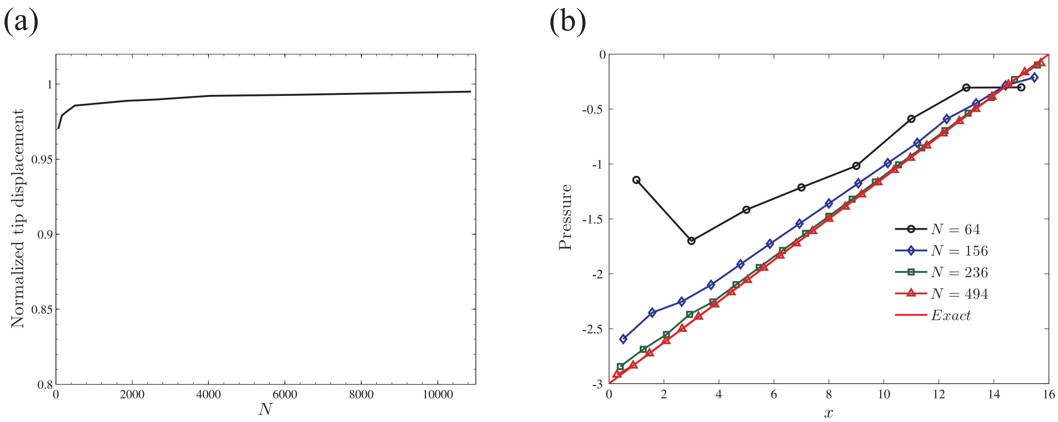


Figure 15: Cantilever beam: (a) Convergence of the normalized displacement of the tip point A (ratio of the numerically-calculated and exact displacements). N is the number of primal 2-cells of the mesh. (b) Pressure of the dual vertices that correspond to the primal 2-cells that are on the bottom of the beam.

Δp. Alternatively, Δp can be calculated using the pressure gradient $\mathbb{G}_p \in \mathbb{R}^{n\mathsf{P}_h}$, which is a vector that has the same components as $\mathbf{\Lambda} \in \mathbb{R}^{n\bar{\mathsf{P}}_h}$ at vertices without essential boundary conditions. Those components that are associated with vertices with essential boundary conditions are chosen to be equal to the pressure gradient of the closest internal primal vertex. For example, suppose in Fig. 13, vertex k has essential boundary conditions and the closest internal vertex to k is l. Then, pressure gradient at vertex k is assumed to be equal to the pressure gradient at l.[9] Then,

[9]If there are more than one closest vertexes, one can associate the average pressure to vertex k.

equating Δp's obtained using the above two approaches we obtain

$$\mathbb{L}^h \mathbf{p} = \bar{\mathbb{I}}^h \mathbb{G}_p, \tag{179}$$

and because $\mathbb{L}^h$ is nonsingular one can solve (179) and obtain the pressure on each dual vertex. However, our numerical experiments show that the direct use of (179) does not yield satisfactory results for pressure and this is not unusual as we have not imposed the natural boundary conditions on each primal 2-cell with such boundary conditions yet. Recall that if τ^a denotes the a-component of the traction $\boldsymbol{\tau}$, then using the summation convention on index b, we can write

$$\tau^a = \sigma^a{}_b \, n^b = (2\mu \, e^a{}_b - p \, g^a{}_b) n^b, \tag{180}$$

where n^b is the b-component of the unit outward normal vector at the natural boundary. Now suppose that in Fig. 13, the 2-cell $[j, l, k]$ lies on the natural boundary. Then, using (173) and (180) we can write

$$\tau^a_{kj} = \left[\mu_{jlk} \left(q^{a,b}_{jlk} + q^{b,a}_{jlk} \right) - p_{jlk} \, \delta^{ab} \right] n^b_{kj}, \tag{181}$$

where we assume summation convention on index b. Using (181) we can determine the pressure at all the 2-cells with natural boundary conditions. Next, we omit the rows that correspond to those 2-cells with natural boundary conditions in (179) and move all the terms containing the known values of the pressure to the right-hand side of the remaining equations. This way, from (179) we obtain the required number of equations to determine pressure at all the dual vertices. Finally, one can use (173) to calculate the stress on each dual subregion.

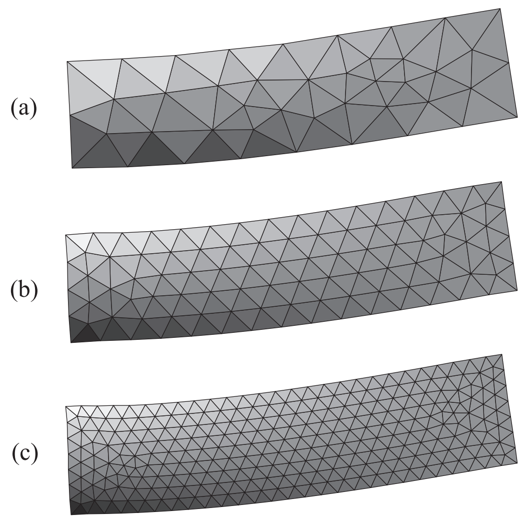


Figure 16: The pressure field for the beam problem for meshes with (a) $N = 64$, (b) $N = 156$, (c) $N = 494$, where N is the number of primal 2-cells of the mesh.

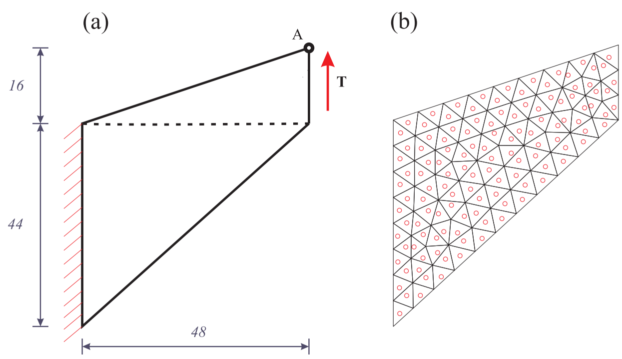


Figure 17: Cook's membrane: (a) Geometry, boundary conditions, and loading, (b) a well-centered mesh with $N = 123$ primal 2-cells with $\circ$ denoting the circumcenter of each primal 2-cell.

2.5 Numerical Examples

To demonstrate the efficiency and robustness of our geometric method, in this section we consider the following two 2-dimensional benchmark problems: a cantilever beam subjected to a parabolic end load and Cook's membrane.

Cantilever beam. As our first example, we consider a planner cantilever beam shown in Fig. 14 that has a closed-form solution for its displacements field [102]. The parabolic load per unit length at the right boundary is given by $f(y) = \frac{F}{2I}(c^2 - y^2)$,

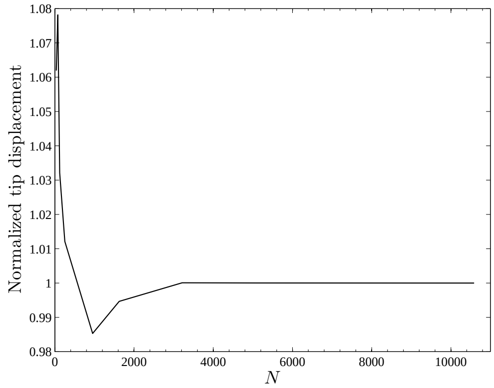


Figure 18: Convergence of the normalized displacement of the tip point A of the Cook's membrane. N is the number of primal 2-cells of the mesh.

where $I = 2c^3/3$. Thus, the total shear load on the right boundary is F. Now, we consider the analytical solution for a beam under this load given by

$$\begin{aligned} u_x &= \frac{(1-\nu^2)Fy}{6EI}\left(3x^2 - 6Lx + \frac{\nu\, y^2}{1-\nu}\right) - \frac{Fy}{6I\mu}\left(y^2 - 3c^2\right), \\ u_y &= \frac{(1-\nu^2)F}{6EI}\left[\frac{3\nu(L-x)y^2}{1-\nu} + 3Lx^2 - x^3\right], \end{aligned} \tag{182}$$

and impose the displacements at $x = 0$. The divergence of the displacement field reads

$$\operatorname{div}\mathbf{u} = \frac{1}{EI}(1+\nu)(1-2\nu)F(x-L)y, \tag{183}$$

and hence, for $\nu = 0.5$ we have $\operatorname{div}\mathbf{u} = \mathbf{0}$. Note that if $\nu = 0.5$, then as we explained in Remark 2.1.2.1, the above displacements also satisfy the equations of incompressible linearized elasticity. For this case, pressure is given by

$$p(x,y) = -\frac{F(x-L)y}{2I}. \tag{184}$$

We assume the following parameters: $L = 16$, $c = 2$, $F = 1$, $E = 10^7$, and $\nu = 0.5$. We show one of the primal meshes with $N = 236$ primal 2-cells in Fig. 14(b). We

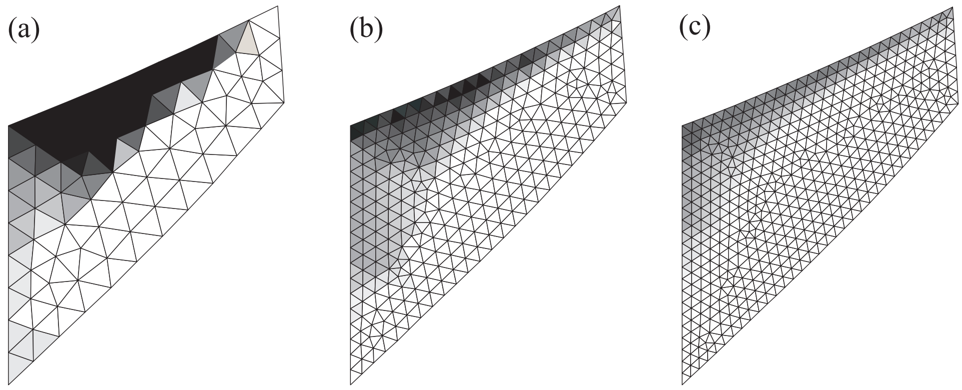


Figure 19: The pressure field for the Cook's membrane for meshes with (a) $N = 123$, (b) $N = 530$, (c) $N = 955$, where N is the number of primal 2-cells of the mesh.

increase the number of primal 2-cells and study the convergence of the solutions. In Fig 15(a), we plot the normalized vertical displacement of the tip point A defined as U_y^A/u_y^A, where U_y^A and u_y^A denote the vertical displacement of point A obtained by our structure-preserving scheme and the exact solution, respectively. We see that the numerical solutions converge to the exact solution. Moreover, we observe a smooth pressure field for the beam. As an example, in Fig. 15(b) we show the variation of pressure at the bottom of the beam ($y = -c$) in the x-direction and again we observe that pressure converges to its exact value. We plot the pressure field over different primal meshes with exaggerated deformations in Fig. 16. The pressure field is free from checkerboarding and becomes smoother and smoother upon mesh refinement.

Cook's membrane. Now we consider the Cook's membrane problem, which is a standard benchmark problem that has been used in the past to investigate the incompressible and near-incompressible solutions under combined bending and shear [58, 87]. Fig. 17 depicts the geometry, boundary conditions, and loading of the problem together with a well-centered mesh with $N = 123$ primal 2-cells. The left boundary is fully clamped and the right boundary is subjected to a distributed shearing load of magnitude $T = 6.25$ per unit length (a total vertical force of 100 is imposed on the right boundary). The material is assumed to be homogeneous with the parameters

$E = 250$ and $\nu = 0.5$. Now we study the variations of the vertical displacement of the tip point A upon mesh refinement. The result is plotted in Fig. 18 that shows the convergence of the normalized displacements by increasing the number of primal 2-cells, N. Note that we use the limit value of the numerically-calculated displacement $U_y^A = 4.2002$ for normalization of displacements in this figure. Finally, we observe that our structure-preserving scheme is free of checkerboarding as is clearly seen in Fig. 19. In this figure, we plot the pressure field over deformed configuration of Cook's membrane. We see that pressure field becomes smoother and smoother upon mesh refinement. Also note that the rate of convergence of the results in our numerical examples is comparable with those of finite element mixed formulations [58, 87].

CHAPTER III

COMPLEXES OF LINEAR AND NONLINEAR ELASTOSTATICS

The main reason that elasticity is harder to discritize compared to electromagnetism is that unlike electromagnetism that deals merely with forms, one has to consider higher order tensors for elasticity. Interestingly, by using some methods from the theory of relativity, Eastwood [41] showed that it is possible to express linear elastostatics in terms of forms. He observed that there is a relation between the linear elastostatics complex and a twisted de Rham complex through a general construction known as the Bernstein-Gelfand-Gelfand (BGG) resolution [19, 28]: One starts from a twisted de Rham complex and constructs another complex called the associated BGG complex that has properties similar to those of the original de Rham complex. Arnold and his coworkers [10, 8, 12] used this important fact to develop stable mixed finite elements formulations for linear elastostatics. One can either directly discretize the linear elastostatics complex [12] or use its relation with the de Rham complex [10]. Motivated by the Eastwood's BGG construction for linear elastostatics, Geymonat and Krasucki [52] deduced a Hodge orthogonal decomposition for symmetric matrix fields in L^2 analogous to the classical Hodge decomposition. This shows that the similarities between the linear elastostatics complex and the de Rham complex also extend to less smooth Sobolev spaces.

The linear elastostatics complex was first introduced by Kröner [74] in connection with linear elastic dislocation theory. As far as we know, Calabi [26] was the first who mathematically studied this complex. He obtained a complex on n-manifolds with constant sectional curvatures (Clifford-Klein spaces) which is equivalent to the linear elastostatics complex in $\mathbb{R}^3$. He calculated the cohomology groups of this complex if the underlying manifold is also compact. Later, Eastwood [41] observed that the linear elastostatics complex is the special case of the Calabi complex and showed that in $\mathbb{R}^3$ the linear elastostatics complex is equivalent to a BGG complex that can be derived from a twisted de Rham complex on the 3-sphere $\mathcal{S}^3$ or the linear projective space $\mathbb{R}P^3$. Of course, his derivation does not imply that the linear elastostatics complex is metric independent as one needs a metric to identify this complex with the BGG complex. In this chapter, we study linear and nonlinear elastostatics complexes, see also [6]. We begin by reviewing some geometric preliminaries and then, we explain the differential operators of linear and nonlinear elastostatics. In particular, we study compatibility equations and introduce various notions of stress functions for nonlinear elastostatics. Finally, we write differential complexes of linear and nonlinear elastostatics. We also discuss the relation between the linear elastostatics complex and the de Rham complex.

3.1 Algebraic and Geometric Preliminaries

For understanding the linear and nonlinear elastostatics complexes and sheaves, various algebraic and geometric notions are required. We review these preliminaries in this section.

3.1.1 Categories and Functors

A category $\mathscr{C}$ is a collection of objects $\mathsf{Ob}(\mathscr{C})$ and for any two objects A and B a set $\mathsf{Mor}(A,B)$ called the set of morphisms of A into B and for any three objects $A,B,C\in\mathsf{Ob}(\mathscr{C})$ a law of composition $\mathbf{c}:\mathsf{Mor}(B,C)\times\mathsf{Mor}(A,B)\to\mathsf{Mor}(A,C)$ such that (i) $\mathsf{Mor}(A,B)\cap\mathsf{Mor}(A',B')=\varnothing$ unless $A=A'$ and $B=B'$ for which $\mathsf{Mor}(A,B)=\mathsf{Mor}(A',B')$, (ii) for each object A there is an identity morphism $\mathrm{Id}_A\in\mathsf{Mor}(A,A)$, and (iii) the law of composition is associative, i.e. if $f\in\mathsf{Mor}(A,B)$, $g\in\mathsf{Mor}(B,C)$, and $h\in\mathsf{Mor}(C,D)$ then $\mathbf{c}(\mathbf{c}(h,g),f)=\mathbf{c}(h,\mathbf{c}(g,f))$, for all $A,B,C,D\in\mathsf{Ob}(\mathscr{C})$ [76].

In general, a morphism $f \in \mathsf{Mor}(A, B)$ is not a map from A into B. But in the cases that we consider in the sequel, f is either a map from set A into set B or a map from a set related to object A into a set related to object B. If the objects A and B have a special structure and their morphisms preserve that structure, then their morphisms are usually called homomorphisms[1] and $\mathsf{Hom}(A, B) := \mathsf{Mor}(A, B)$.

For example, all smooth n-dimensional manifolds together with local diffeomorphisms (i.e. immersions) between them form the category $\mathscr{M}f_n$. Also let V and W be arbitrary finite-dimensional vector spaces. Then all finite-dimensional vector spaces together with linear maps between them (i.e. $\mathsf{Mor}(V, W) = L(V, W)$) is a category. The morphisms of this category are called the homomorphisms of linear spaces or simply homomorphisms. Another important example of homomorphism is the group homomorphism. Let G and H be groups, then $\phi : G \to H$ is a group homomorphism if $\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$ for all $g_1, g_2 \in G$, where the dots denote the group operation of G and H, respectively. We will see other types of homomorphisms in the sequel.

A morphism $f \in \mathsf{Mor}(A, B)$ is called endomorphism if $A = B$ and we define $\mathsf{End}(A) := \mathsf{Mor}(A, A)$. A morphism f is called isomorphism if it is invertible, i.e. there exists $g \in \mathsf{Mor}(B, A)$ such that $\mathbf{c}(f, g) = \mathrm{Id}_B$ and $\mathbf{c}(g, f) = \mathrm{Id}_A$. Invertible morphisms of an object into itself are called automorphisms. The sets of automorphisms of A is denoted by $\mathsf{Aut}(A)$. Let V be an n-dimensional vector space. Then the general linear group of V, $GL(V)$, is identical to $\mathsf{Aut}(V)$. Thus, $GL(V)$ is the set of all invertible linear maps $f : V \to V$. Similarly, $\mathfrak{gl}(V) := \mathsf{End}(V)$. Note that $GL(V)$ and $\mathfrak{gl}(V)$ can be considered as the set of all invertible $n \times n$ matrices and all $n \times n$ matrices, respectively. The set $GL(V)$ is a group using the composition of functions as the group action. The representation of a group G on a finite-dimensional vector space V is a group homomorphism $\phi : G \to GL(V)$. Equivalently, we can consider a

[1] A homomorphism is an algebraic concept, which is different from a homeomorphism, which is a topological concept.

representation of G as a map $\hat{\phi} : G \times V \to V$ such that $\hat{\phi}(g_1 \cdot g_2, v) = \hat{\phi}(g_1, \hat{\phi}(g_2, v))$ for all $g_1, g_2 \in G$ and $v \in V$.

Let $\mathscr{C}$ and $\mathscr{B}$ be categories. A (covariant) functor F of $\mathscr{C}$ into $\mathscr{B}$ is a map that to each $A \in \mathsf{Ob}(\mathscr{C})$ associates an object $F(A) \in \mathsf{Ob}(\mathscr{B})$ and to each $f \in \mathsf{Mor}(A, B)$ associates a morphism $F(f) \in \mathsf{Mor}(F(A), F(B))$ such that (i) $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ for all $A \in \mathsf{Ob}(\mathscr{C})$, and (ii) for all $f \in \mathsf{Mor}(A, B)$ and $g \in \mathsf{Mor}(B, C)$ we have $F(\mathbf{c}_{\mathscr{C}}(g, f)) = \mathbf{c}_{\mathscr{B}}(F(g), F(f))$, where $\mathbf{c}_{\mathscr{C}}$ and $\mathbf{c}_{\mathscr{B}}$ are the composition laws of $\mathscr{C}$ and $\mathscr{B}$, respectively [76]. For contravariant functors, the condition (i) remains the same but $F(f) \in \mathsf{Mor}(F(B), F(A))$ and $F(\mathbf{c}_{\mathscr{C}}(g, f)) = \mathbf{c}_{\mathscr{B}}(F(f), F(g))$.

The product category $\mathscr{C} \times \mathscr{C}'$ is the category with $\mathsf{Ob}(\mathscr{C} \times \mathscr{C}') = \{(A, A') : A \in \mathsf{Ob}(\mathscr{C}), A' \in \mathsf{Ob}(\mathscr{C}')\}$ and $\mathsf{Mor}((A, A'), (B, B')) = \{(f, f') : f \in \mathsf{Mor}(A, B), f' \in \mathsf{Mor}(A', B')\}$. In a similar manner, one can define the product of more than two categories. A bifunctor F is a functor from $\mathscr{C} \times \mathscr{C}'$ into a category $\mathscr{B}$ with $F(A, A') := F((A, A')) \in \mathsf{Ob}(\mathscr{B})$. For morphisms there are four possibilities: co-covariant, contra-covariant, co-contravariant, and contra-contravariant bifunctors. For example, if F is a contra-covariant functor, then $F(f, f') := F((f, f')) \in \mathsf{Mor}(F(B, A'), F(A, B'))$. The space of homomorphisms of vector spaces $L(V, W) = \mathsf{Hom}(V, W)$ is a contra-covariant bifunctor. Let V' and W' be vector spaces, $\boldsymbol{f} \in L(V, V')$, and $\boldsymbol{g} \in L(W, W')$. Then $L(\boldsymbol{f}, \boldsymbol{g}) : L(V', W) \to L(V, W')$, $\boldsymbol{h} \mapsto \boldsymbol{g} \circ \boldsymbol{h} \circ \boldsymbol{f}$. Similar conclusions are valid for group homomorphisms. It is also possible to define multifunctors, i.e. functors from $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$ into $\mathscr{B}$. We use functors for defining natural bundles and also special vector bundles whose sections are tensor fields.

3.1.2 Tensor Product and Exterior Power

Let V, W, and X be vector spaces over $\mathbb{R}$ ($\mathbb{R}$-modules in general). A tensor product of V and W is a vector space $V \otimes_{\mathbb{R}} W$ over $\mathbb{R}$ ($\mathbb{R}$-module in general) together with an $\mathbb{R}$-bilinear map $\vartheta : V \times W \to V \otimes_{\mathbb{R}} W$ such that for every $\mathbb{R}$-bilinear map $\gamma : V \times W \to X$

there exists a unique linear map $\Phi : V \otimes_{\mathbb{R}} W \to X$ such that $\gamma = \Phi \circ \vartheta$, i.e. the diagram

$$\begin{tikzcd} V \otimes_{\mathbb{R}} W \arrow[dr, "\Phi"] & \\ V \times W \arrow[u, "\vartheta"] \arrow[r, "\gamma"] & X \end{tikzcd}$$

commutes [48]. Alternatively, one can define a tensor product as a universal object of a category [76]. If the scaler field is obvious, then usually the above tensor product is written as $V \otimes W$. It is possible to show the existence and uniqueness (up to a unique isomorphism) of tensor products. Thus, one can think of a tensor product to be a family, i.e. although a tensor product of two vector spaces is not unique, one can determine all tensor products by knowing just one of them. Usually, the general properties of a tensor product is important not a specific element of the class of tensor products. So we speak of the tensor product of vector spaces to refer to any element of the class of tensor products. Assume $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Since the map ϑ is usually not important, $\vartheta(\mathbf{v}, \mathbf{w})$ is written as $\mathbf{v} \otimes \mathbf{w}$. We have $\dim(V \otimes W) = \dim V \times \dim W$. In particular, let $\{\mathbf{v}_i\}$ and $\{\mathbf{w}_j\}$ be bases for V and W, respectively. Then $\{\mathbf{v}_i \otimes \mathbf{w}_j\}$ is a basis for $V \otimes W$. It is also possible to define the tensor products of homomorphisms of vector spaces. Let $\boldsymbol{k} \in L(V, W)$ and $\boldsymbol{h} \in L(M, N)$. Then

$$\begin{aligned} \boldsymbol{k} \otimes \boldsymbol{h} : V \otimes M &\longrightarrow W \otimes N, \\ \mathbf{v} \otimes \mathbf{m} &\longmapsto \boldsymbol{k}(\mathbf{v}) \otimes \boldsymbol{h}(\mathbf{m}). \end{aligned} \tag{185}$$

Thus, the tensor product is a co-covariant bifunctor.

Let $V^* = L(V, \mathbb{R})$ be the dual set of V with the basis $\{\mathbf{v}^i\}$, where $\mathbf{v}^i(\mathbf{v}_k) = \delta^i{}_k$. For finite-dimensional vector spaces, we have $V^{**} = (V^*)^* \approx V$, where $\approx$ means isomorphic. This result is due to the existence of the natural isomorphism $\iota_n : V \to V^{**}$ given by

$(\iota_n(\mathbf{v}))(\mathbf{v}^*) = \mathbf{v}^*(\mathbf{v})$ for $\mathbf{v} \in V$ and $\mathbf{v}^* \in V^*$. One can show that

$$L(V, W) \approx W \otimes V^* \approx L^2(W^*, V; \mathbb{R}), \tag{186}$$

where $L^2(W^*, V; \mathbb{R})$ is the set of $\mathbb{R}$-bilinear maps $\boldsymbol{f} : W^* \times V \to \mathbb{R}$. Suppose $\mathbf{v} = v^i \mathbf{v}_i$ and $\mathbf{w} = w^j \mathbf{w}_j$, where we use the summation convention. Then, (186) implies that the tensor product $\mathbf{w}_j \otimes \mathbf{v}^i$ can be considered as the bilinear map $\mathbf{w}_j \otimes \mathbf{v}^i : W^* \times V \to \mathbb{R}$ with

$$\mathbf{w}_j \otimes \mathbf{v}^i(\mathbf{w}^l, \mathbf{v}_k) = \left(\left(\iota_n \left(\mathbf{w}_j \right) \right) \left(\mathbf{w}^l \right) \right) \cdot \mathbf{v}^i(\mathbf{v}_k) = \delta_j{}^l \delta^i{}_k. \tag{187}$$

Note that the same symbol $\mathbf{w}_j \otimes \mathbf{v}^i$ is used to denote both an element of the tensor product and also a bilinear map. This simplifies the notation and introduces no confusion as long as we know the isomorphism that relates the isomorphic sets. Similarly, we may write $\mathbf{v}$ to denote $\iota_n(\mathbf{v}) \in V^{**}$.

Consider an $\mathbb{R}$-bilinear map $\boldsymbol{f} : W^* \times V \to \mathbb{R}$ and let $f^j{}_i = \boldsymbol{f}(\mathbf{w}^j, \mathbf{v}_i)$. Then, in the coordinate systems $\{\mathbf{v}_i\}$ and $\{\mathbf{w}_j\}$, the map $\boldsymbol{f}$ can be expressed as $\boldsymbol{f} = f^j{}_i \, \mathbf{w}_j \otimes \mathbf{v}^i$. Similar results hold for the tensor product of arbitrary finite number of linear spaces over $\mathbb{R}$. In particular, the tensor product is associative and we define

$$\otimes^r W = \underbrace{W \otimes \cdots \otimes W}_{r}. \tag{188}$$

A tensor is an element of a tensor product of linear spaces over $\mathbb{R}$. The main observation is that such elements can be considered as $\mathbb{R}$-valued $\mathbb{R}$-multilinear maps. Thus, one can define tensors to be multilinear maps as well. Geometers use the notation $W \otimes V^*$ but they actually mean $L(V, W)$. It is straightforward to define tensor fields as sections of vector bundles using tensor products.

Another notion closely related to tensor products is exterior powers. Let W and

X be vector spaces over $\mathbb{R}$ as before. An $\mathbb{R}$-multilinear map

$$\gamma : \underbrace{W \times \cdots \times W}_{r} \to X \tag{189}$$

is called alternating if $\gamma(\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_r) = 0$ whenever $\bar{\mathbf{w}}_i = \bar{\mathbf{w}}_j$ for any two indices $i \neq j$. Note that an alternating map is antisymmetric.

An exterior rth power $\Lambda^r_{\mathbb{R}} W$ over $\mathbb{R}$ (or simply $\Lambda^r W$) is a vector space $\Lambda^r W$ over $\mathbb{R}$ with an alternating map

$$\alpha : \underbrace{W \times \cdots \times W}_{r} \to \Lambda^r W$$

such that for every alternating $\mathbb{R}$-multilinear map γ as (189), there exists a unique $\mathbb{R}$-linear map $\Psi : \Lambda^r W \to X$ such that $\gamma = \Psi \circ \alpha$, i.e. the diagram

$$\begin{tikzcd}
\Lambda^r W \arrow[dr, "\Psi"] & \\
\underbrace{W \times \cdots \times W}_{r} \arrow[u, "\alpha"] \arrow[r, "\gamma"'] & X
\end{tikzcd}$$

commutes [48]. Similar to tensor products, exterior powers exist and are unique (up to a unique isomorphism). The notation $\alpha(\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_r) = \bar{\mathbf{w}}_1 \wedge \cdots \wedge \bar{\mathbf{w}}_r$ is used for the image of α. Suppose $n = \dim W$. For $0 \leq r \leq n$ we have $\dim(\Lambda^r W) = \binom{n}{r}$. In particular, let $\{\mathbf{w}_i\}_{i=1}^n$ be a basis for W. Then, the monomials $\mathbf{w}_{i_1} \wedge \cdots \wedge \mathbf{w}_{i_r}$ with $i_1 < \cdots < i_r$ form a basis for $\Lambda^r W$. Let $\mathsf{Alt}^r(W)$ denote the set of alternating $\mathbb{R}$-multilinear maps

$$\mathbf{h} : \underbrace{W \times \cdots \times W}_{r} \to \mathbb{R}.$$

We can write

$$\Lambda^r W^* \approx \mathsf{Alt}^r(W). \tag{190}$$

Let $\mathsf{Alt}^r(W;X)$ denote the set of alternating $\mathbb{R}$-multilinear maps as in (189). Then we have

$$\Lambda^r W^* \otimes X \approx \mathsf{Alt}^r(W;X). \tag{191}$$

Later we will use the relations (190) and (191) for defining forms and vector bundle-valued forms, respectively.

3.1.3 Lie Algebras and Lie Groups

An algebra E is a vector space (a module in general) over a field A (a ring in general) together with an A-bilinear map $E \times E \to E$. A Lie algebra $(\mathfrak{g}, [,])$ is a vector space $\mathfrak{g}$ over $\mathbb{R}$ together with the Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is an antisymmetric $\mathbb{R}$-bilinear mapping that satisfies the Jacobi identity, i.e.

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0, \tag{192}$$

for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{g}$ [27]. Let $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras. Then $\phi : \mathfrak{g} \to \mathfrak{k}$ is a homomorphism of Lie algebras if it is $\mathbb{R}$-linear and compatible with the brackets, i.e. $[\phi(\mathbf{X}), \phi(\mathbf{Y})]_{\mathfrak{k}} = \phi([\mathbf{X}, \mathbf{Y}]_{\mathfrak{h}})$, $\forall \mathbf{X}, \mathbf{Y} \in \mathfrak{g}$.

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a Lie subalgebra, denoted by $\mathfrak{h} \leq \mathfrak{g}$, if it is closed under the Lie bracket, i.e. if $[\mathfrak{h}, \mathfrak{h}] = \{[\mathbf{X}, \mathbf{Y}] : \mathbf{X}, \mathbf{Y} \in \mathfrak{h}\} \subset \mathfrak{h}$. For any subset $A \subset \mathfrak{g}$, the smallest subalgebra of $\mathfrak{g}$ that contains A exists and is called the subalgebra generated by A. A subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is an ideal in $\mathfrak{g}$ and we write $\mathfrak{h} \lhd \mathfrak{g}$ if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Let $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k]$. Then for each $k \in \mathbb{N}$ we have $\mathfrak{g}^k \lhd \mathfrak{g}$ and $\mathfrak{g} \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^k \supset \mathfrak{g}^{k+1} \supset \cdots$, which is called the lower central series of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is called nilpotent if $\mathfrak{g}^k = 0$ for some $k \in \mathbb{N}$. Similarly, let $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(k+1)} := [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$. We have $\mathfrak{g}^{(k)} \lhd \mathfrak{g}$ and $\mathfrak{g} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(k)} \supset \mathfrak{g}^{(k+1)} \supset \cdots$, which is called the derived series of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is called solvable if $\mathfrak{g}^{(k)} = 0$ for some $k \in \mathbb{N}$. Suppose $\mathfrak{h} \leq \mathfrak{g}$. If $\mathfrak{g}$ is nilpotent (solvable) then $\mathfrak{h}$ is also nilpotent (solvable).

A Lie algebra $\mathfrak{g}$ is called semisimple if it has no nonzero solvable ideal and is called simple if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and the only ideals in $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$. The center of $\mathfrak{g}$ is defined as $\mathfrak{z}(\mathfrak{g}) \coloneqq \{\mathbf{X} \in \mathfrak{g} : [\mathbf{X}, \mathbf{Y}] = 0,\ \forall \mathbf{Y} \in \mathfrak{g}\}$. The Lie algebra $\mathfrak{g}$ is called reductive if any solvable ideal of $\mathfrak{g}$ is contained in $\mathfrak{z}(\mathfrak{g})$. The direct sum of Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$, $\mathfrak{h} \oplus \mathfrak{g}$, is the direct sum of vector spaces $\mathfrak{h}$ and $\mathfrak{g}$ endowed with the componentwise bracket. One can show that a finite direct sum of simple Lie algebras is semisimple.

A Lie group $\mathcal{G}$ is a manifold and also a group such that the group multiplication $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a smooth mapping [73, 27]. Let $g, x \in \mathcal{G}$. The multiplication $\mu(g, x)$ is usually denoted by $g \cdot x$. The left translation $\lambda_g : \mathcal{G} \to \mathcal{G},\ x \mapsto g \cdot x$, and the right translation $\rho^g : \mathcal{G} \to \mathcal{G},\ x \mapsto x \cdot g$, are both diffeomorphisms of $\mathcal{G}$ with inverses $\lambda_{g^{-1}}$ and $\rho^{g^{-1}}$, respectively. Let $\mathfrak{X}(\mathcal{G})$ denote the linear space of smooth vector fields of the Lie group $\mathcal{G}$. A smooth vector field $\boldsymbol{\xi} \in \mathfrak{X}(\mathcal{G})$ is called left invariant if $\lambda_g^* \boldsymbol{\xi} = \boldsymbol{\xi}$ for all $g \in \mathcal{G}$, where $\lambda_g^* \boldsymbol{\xi}(x) = (T_{g \cdot x} \lambda_{g^{-1}}) \cdot \boldsymbol{\xi}(g \cdot x)$. If we assume $x = g^{-1}$, then the definition of the left invariant vector fields yields $\boldsymbol{\xi}(x) = T_e \lambda_x \cdot \boldsymbol{\xi}(e)$, where e is the unit element of $\mathcal{G}$. Thus, any left invariant vector field is uniquely determined by $\boldsymbol{\xi}(e) \in T_e\mathcal{G}$. Conversely, given $\mathbf{X} \in T_e\mathcal{G}$, one can obtain a left invariant vector field $\boldsymbol{L}_{\mathbf{X}} \in \mathfrak{X}_L(\mathcal{G})$, where $\mathfrak{X}_L(\mathcal{G}) \subset \mathfrak{X}(\mathcal{G})$ is the space of left-invariant vector fields.

Table 1: Common Lie groups and their Lie algebras.

Lie Group		Lie Algebra
Name	Definition	
General Linear Group	$GL(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$	$\mathfrak{gl}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n}\}$
Positive General Linear Group	$GL^+(\mathbb{R}^n) = \{A \in GL(\mathbb{R}^n) : \det A > 0\}$	$\mathfrak{gl}^+(\mathbb{R}^n) = \mathfrak{gl}(\mathbb{R}^n)$
Special Linear Group	$SL(\mathbb{R}^n) = \{A \in GL(\mathbb{R}^n) : \det A = 1\}$	$\mathfrak{sl}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} : \operatorname{tr} A = 0\}$
Orthogonal Group	$O(\mathbb{R}^n) = \{A \in GL(\mathbb{R}^n) : AA^\top = \mathrm{Id}_{\mathbb{R}^n}\}$	$\mathfrak{o}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} : A + A^\top = 0\}$
Special Orthogonal Group	$SO(\mathbb{R}^n) = \left\{A \in GL(\mathbb{R}^n) : AA^\top = \mathrm{Id}_{\mathbb{R}^n},\ \det A = 1\right\}$	$\mathfrak{so}(\mathbb{R}^n) = \mathfrak{o}(\mathbb{R}^n)$
Euclidean Group	$Euc(\mathbb{R}^n) = \left\{\begin{pmatrix} 1 & 0 \\ \mathbf{v} & A \end{pmatrix} \in GL(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in SO(\mathbb{R}^n)\right\}$	$\mathfrak{euc}(\mathbb{R}^n) = \left\{\begin{pmatrix} 0 & 0 \\ \mathbf{v} & A \end{pmatrix} \in \mathfrak{gl}(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in \mathfrak{so}(\mathbb{R}^n)\right\}$
Positive Affine Group	$Aff^+(\mathbb{R}^n) = \left\{\begin{pmatrix} 1 & 0 \\ \mathbf{v} & A \end{pmatrix} \in GL^+(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in GL^+(\mathbb{R}^n)\right\}$	$\mathfrak{aff}^+(\mathbb{R}^n) = \left\{\begin{pmatrix} 0 & 0 \\ \mathbf{v} & A \end{pmatrix} \in \mathfrak{gl}(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in \mathfrak{gl}(\mathbb{R}^n)\right\}$

This defines a linear isomorphism $\boldsymbol{L} : T_e\mathcal{G} \to \mathfrak{X}_L(\mathcal{G})$, $\mathbf{X} \mapsto \boldsymbol{L}_{\mathbf{X}}$. Note that $\mathfrak{X}(\mathcal{G})$ with the usual bracket of vector fields is a Lie algebra. Also the pullback along a diffeomorphism is compatible with the Lie bracket of vector fields, so we have $\lambda_g^*[\boldsymbol{\xi}, \boldsymbol{\eta}] = [\lambda_g^*\boldsymbol{\xi}, \lambda_g^*\boldsymbol{\eta}]$ for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}(\mathcal{G})$. Therefore, $\mathfrak{X}_L(\mathcal{G})$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$. Note that $\boldsymbol{L}$ endows a Lie bracket to $\mathfrak{g} := T_e\mathcal{G}$ given by $[\mathbf{X}, \mathbf{Y}] := [\boldsymbol{L}_{\mathbf{X}}, \boldsymbol{L}_{\mathbf{Y}}](e)$, for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$. The pair $(\mathfrak{g}, [])$ is called the Lie algebra of the Lie group $\mathcal{G}$. Also note that similar to left invariant vector fields, one can define the notion of right invariant vector fields $\mathfrak{X}_R(\mathcal{G})$. We summarize some of the most common Lie groups and their Lie algebras in Table 1. Each of these groups is a subgroup of the general linear group $GL(\mathbb{R}^n)$ with the composition (or the matrix multiplication) as its group multiplication. The linear space $\mathfrak{gl}(\mathbb{R}^n)$ is equipped with the commutator of matrices as its Lie bracket, i.e. $[A, B] = AB - BA$, $\forall A, B \in \mathbb{R}^{n\times n}$. Let $x \in \mathcal{G}$ and $\boldsymbol{\xi} \in \mathcal{G}$. The flow of the vector field $\boldsymbol{\xi}$ is the mapping $\mathrm{Fl}^{\boldsymbol{\xi}}$ defined as $\mathrm{Fl}^{\boldsymbol{\xi}}(t, x) := c_x(t)$, where $c_x : I_x \subset \mathbb{R} \to \mathcal{G}$ is an integral curve of $\boldsymbol{\xi}$ with I_x an open interval containing 0, i.e. $c_x'(t) := T_t c_x \cdot 1 = \boldsymbol{\xi}(c(t))$ and $c_x(0) = x$. The exponential map $\exp : \mathfrak{g} \to \mathcal{G}$ is a local diffeomorphism given by $\exp(\mathbf{X}) := \mathrm{Fl}^{\boldsymbol{L}_{\mathbf{X}}}(1, e)$.

A subgroup $\mathcal{H}$ of a Lie group $\mathcal{G}$ that is also a submanifold of $\mathcal{G}$ is called a Lie subgroup. A Lie subgroup $\mathcal{H}$ is a closed subset of $\mathcal{G}$, and conversely, any closed subgroup of $\mathcal{G}$ is a Lie subgroup. The Lie algebra $\mathfrak{h}$ of a Lie subgroup $\mathcal{H}$ is a Lie subalgebra of $\mathfrak{g}$. A subgroup $\mathcal{H}$ of $\mathcal{G}$ is called normal if $\mathrm{conj}_g(h) \in \mathcal{H}$, $\forall g \in \mathcal{G}$ and $\forall h \in \mathcal{H}$. A connected Lie subgroup $\mathcal{H} \subset \mathcal{G}$ is normal if and only if the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal.

3.1.3.1 Homogeneous Spaces and Group Actions

Let $\mathcal{H}$ be a Lie subgroup of $\mathcal{G}$. The homogeneous space of $\mathcal{G}$ corresponding to $\mathcal{H}$ is the coset space $\mathcal{G}/\mathcal{H} = \{g \cdot \mathcal{H} : g \in \mathcal{G}\}$, where $g \cdot \mathcal{H} = \{g \cdot h : h \in \mathcal{H}\}$ [73, 27]. The homogeneous space $\mathcal{G}/\mathcal{H}$ is equipped with the quotient topology, i.e. $\mathcal{U} \subset \mathcal{G}/\mathcal{H}$ is open if and only if $p^{-1}(\mathcal{U})$ is open in $\mathcal{G}$, where $p : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ is the natural projection. If $\mathcal{H}$ is a closed subgroup of $\mathcal{G}$ (and so a Lie subgroup), then $\mathcal{G}/\mathcal{H}$ is a smooth manifold with $\dim(\mathcal{G}/\mathcal{H}) = \dim \mathcal{G} - \dim \mathcal{H}$ and the projection p is a submersion, i.e. $T_g p$ is a surjective map for all $g \in \mathcal{G}$.

A left action of a Lie group $\mathcal{G}$ on a manifold $\mathcal{M}$ is a smooth mapping $\ell : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that $\ell(e, x) = x$ and $\ell(g_1, \ell(g_2, x)) = \ell(g_1 \cdot g_2, x)$. Usually, if there is no risk of confusion, $\ell(g, x)$ is denoted by $g \cdot x$. Consider mappings $\ell_g : \mathcal{M} \to \mathcal{M}$ and $\ell^x : \mathcal{G} \to \mathcal{M}$ with $\ell_g(x) = \ell^x(g) = \ell(g, x)$. Then, ℓ_g is a diffeomorphism with the inverse $\ell_{g^{-1}}$. A representation of $\mathcal{G}$ on a vector space V is a left action of $\mathcal{G}$ on V with linear mappings $\ell_g : V \to V$. But a representation $\varrho : \mathfrak{d} \times V \to V$ of a Lie algebra $\mathfrak{d}$ is not a group action since $\varrho_{\mathbf{0}} \coloneqq \varrho(\mathbf{0}, \cdot) \neq \mathrm{Id}_V$. Similarly, a right action is a smooth mapping $r : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ such that $r(x, e) = x$ and $r(r(x, g_1), g_2) = r(x, g_1 \cdot g_2)$. The following notation is usually used for right actions: $r_x(g) = r^g(x) = r(x, g)$. Note that each left action ℓ induces a right action $r_\ell(x, g) = \ell(g^{-1}, x)$. The orbit of a left action through a point $x \in M$ is $\mathcal{G} \cdot x = \{g \cdot x : g \in \mathcal{G}\}$. Two orbits are either disjoint or equal. The manifold $\mathcal{M}$ is the union of disjoint orbits. The set of all orbits is denoted by $\mathcal{M}/\mathcal{G}$. The isotropy subgroup of $\mathcal{G}$ at x or the stabilizer of x is defined as $\mathcal{G}_x = \{g \in \mathcal{G} : \ell(g, x) = x\}$. One can define orbits and isotropy groups for right actions in a similar way. An action is called transitive if it has only one orbit[2]. An action is called effective if only neutral element $e \in \mathcal{G}$ acts as the identity of $\mathcal{M}$. A free action is an action that all of its isotropy subgroups equal $\{e\}$.

The homogeneous space $\mathcal{G}/\mathcal{H}$ can be considered as the orbits of the free right action $r : \mathcal{G} \times \mathcal{H} \to \mathcal{G}$, $(g, h) \mapsto \mu(g, h)$, where μ is the group multiplication of $\mathcal{G}$. The group multiplication of $\mathcal{G}$ induces a smooth transitive left action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{H}$ given by $\ell(g_1, g_2 \cdot \mathcal{H}) \coloneqq (g_1 \cdot g_2) \cdot \mathcal{H}$. For each right action $r : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$, the fundamental

[2]Note that the relation $\dim(\mathcal{M}/\mathcal{G}) = \dim \mathcal{M} - \dim \mathcal{G}$ is not valid for an action of a Lie group $\mathcal{G}$ on a manifold $\mathcal{M}$, in general.

vector field $\boldsymbol{\zeta}_{\mathbf{X}} \in \mathfrak{X}(\mathcal{M})$ is defined as

$$\boldsymbol{\zeta}_{\mathbf{X}}(x) = \frac{d}{dt}\Big|_{t=0} r(x, \exp(t\mathbf{X})) = T_e(r_x) \cdot \mathbf{X}, \tag{193}$$

for all $x \in \mathcal{M}$ and $\mathbf{X} \in \mathfrak{g}$. Thus, we obtain a linear mapping $\boldsymbol{\zeta} : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$, which is a Lie algebra homomorphism, i.e. $\boldsymbol{\zeta}_{[\mathbf{X},\mathbf{Y}]} = [\boldsymbol{\zeta}_{\mathbf{X}}, \boldsymbol{\zeta}_{\mathbf{Y}}]$. Also we have $T_x(r^g) \cdot \boldsymbol{\zeta}_{\mathbf{X}}(x) = \boldsymbol{\zeta}_{\mathrm{Ad}(g^{-1})(\mathbf{X})}(x \cdot g)$. For a left action $\ell : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, we define

$$\boldsymbol{\zeta}_{\mathbf{X}}(x) = \frac{d}{dt}\Big|_{t=0} \ell(\exp(t\mathbf{X}), x) = T_e(\ell^x) \cdot \mathbf{X}. \tag{194}$$

The mapping $\boldsymbol{\zeta}$ is not a Lie algebra homomorphism for left actions since in this case we have $\boldsymbol{\zeta}_{[\mathbf{X},\mathbf{Y}]} = -[\boldsymbol{\zeta}_{\mathbf{X}}, \boldsymbol{\zeta}_{\mathbf{Y}}]$. For left actions, one can show that $T_x(\ell_g) \cdot \boldsymbol{\zeta}_{\mathbf{X}}(x) = \boldsymbol{\zeta}_{\mathrm{Ad}(g)(\mathbf{X})}(g \cdot x)$.

Let $\phi : \mathcal{M} \to \bar{\mathcal{M}}$ be a mapping between manifolds $\mathcal{M}$ and $\bar{\mathcal{M}}$ with left actions $\ell : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ and $\bar{\ell} : \mathcal{G} \times \bar{\mathcal{M}} \to \bar{\mathcal{M}}$, respectively. Then, ϕ is called $\mathcal{G}$-equivariant if we have $\phi(\ell(g, x)) = \bar{\ell}(g, \phi(x))$, $\forall x \in \mathcal{M}$ and $\forall g \in \mathcal{G}$. Similarly, if $\mathcal{M}$ and $\bar{\mathcal{M}}$ have right actions $r : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ and $\bar{r} : \bar{\mathcal{M}} \times \mathcal{G} \to \bar{\mathcal{M}}$, then ϕ is $\mathcal{G}$-equivariant if $\phi(r(x, g)) = \bar{r}(\phi(x), g)$ for all $x \in \mathcal{M}$ and $g \in \mathcal{G}$.

3.1.3.2 Representations of Lie Groups and Lie Algebras

We briefly review some basic concepts of the representation theory that will be used later. For more detailed discussions, see [27, 47, 67]. The representation of a group G on a finite-dimensional vector space V is a group homomorphism $\phi : G \to GL(V)$. Equivalently, we can consider a representation of G as a map $\hat{\phi} : G \times V \to V$ such that $\hat{\phi}(g_1 \cdot g_2, v) = \hat{\phi}(g_1, \hat{\phi}(g_2, v))$, $\forall g_1, g_2 \in G$ and $v \in V$. Let $H \subset G$ be a subgroup and let $W \subset V$ be a subspace. The restriction $\varrho|_H : H \to GL(V)$ is a representation of H on V. A subspace $W \subset V$ is G-invariant under ϱ if $\varrho(g)(W) \subset W$, $\forall g \in G$. A G-invariant subspace W defines the subrepresentation $\varrho_W : G \to GL(W)$, $g \mapsto \varrho(g)$, and the representation $\hat{\varrho} : G \to GL(V/W)$ given by $\hat{\varrho}(g)(\mathbf{v} + W) := \varrho(g)(\mathbf{v}) + W$, $\forall g \in G$. Let $V^* := L(V, \mathbb{R})$ be the dual of V. The pairing between V^* and V is defined as $\langle \mathbf{v}^*, \mathbf{v} \rangle := \mathbf{v}^*(\mathbf{v})$, $\forall \mathbf{v} \in V$ and $\mathbf{v}^* \in V^*$. The dual representation $\varrho^* : G \to GL(V^*)$ is given by $\varrho^*(g) := [\varrho(g^{-1})]^{\mathsf{T}}$ [47]. We have $\langle \varrho^*(g)(\mathbf{v}^*), \varrho(g)(\mathbf{v}) \rangle = \langle \mathbf{v}^*, \mathbf{v} \rangle$. Moreover, ϱ induces a representation on $\otimes^n V$, $\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n \mapsto \varrho(g)(\mathbf{v}_1) \otimes \cdots \otimes \varrho(g)(\mathbf{v}_n)$. Similarly, if $\varepsilon : G \to GL(X)$ is another representation, we obtain the representation $\varrho_{V \otimes X}(g) : V \otimes X \to V \otimes X$, $\mathbf{v} \otimes \mathbf{x} \mapsto \varrho(g)(\mathbf{v}) \otimes \varepsilon(g)(\mathbf{x})$. A (left) G-module (V, ς) is a vector space V together with a representation $\varsigma : G \to GL(V)$ of $\mathcal{G}$ on V. A homomorphism of G-modules from (V, ς) into (V', ς') is a G-equivariant linear mapping $\psi : V \to V'$, i.e. ψ is a linear map and $\psi(\varsigma(g, \mathbf{v})) = \varsigma'(g, \psi(\mathbf{v}))$, $\forall g \in G$ and $\forall \mathbf{v} \in V$.

A representation of a Lie algebra on V is a Lie algebra homomorphism $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$. Alternatively, a representation can be considered as an $\mathbb{R}$-bilinear map $\varrho : \mathfrak{g} \times V \to V$ such that $\forall \mathbf{X}, \mathbf{Y} \in \mathfrak{g}$ and $\forall \mathbf{v} \in V$ we have $\varrho([\mathbf{X}, \mathbf{Y}], \mathbf{v}) = \varrho(\mathbf{X}, \varrho(\mathbf{Y}, \mathbf{v})) - \varrho(\mathbf{Y}, \varrho(\mathbf{X}, \mathbf{v}))$. The adjoint map or the adjoint representation of $\mathfrak{g}$ is defined as $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathbf{X} \mapsto [\mathbf{X}, \cdot]$, and is a representation of $\mathfrak{g}$ on $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra and let $W \subset V$ be a subspace. It is obvious that $\varrho|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{h}$ on W. A subspace W is called $\mathfrak{g}$-invariant under ϱ if $\varrho(\mathbf{X})(W) \subset W$, $\forall \mathbf{X} \in \mathfrak{g}$. A $\mathfrak{g}$-invariant subspace W defines a representation $\varrho_W : \mathfrak{g} \to \mathfrak{gl}(W)$, $\mathbf{X} \mapsto \varrho(\mathbf{X})$, that is called a subrepresentation of $\mathfrak{g}$. Moreover, ϱ induces a representation $\hat{\varrho} : \mathfrak{g} \to GL(V/W)$ given by $\hat{\varrho}(\mathbf{X})(\mathbf{v} + W) := \varrho(\mathbf{X})(\mathbf{v}) + W$, $\forall \mathbf{X} \in \mathfrak{g}$. Suppose $f : V \to X$ is a linear map. The transpose $f^{\mathsf{T}} : X^* \to V^*$ is defined as $f^{\mathsf{T}}(\mathbf{x}^*) = \mathbf{x}^* \circ f$, $\forall \mathbf{x}^* \in X^*$. A representation $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ induces the dual representation $\varrho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ given by $\varrho^*(\mathbf{X}) := (\varrho(-\mathbf{X}))^{\mathsf{T}}$ [47], i.e. $\varrho^*(\mathbf{X})(\mathbf{v}^*) := \mathbf{v}^*(\varrho(-\mathbf{X})) = -\mathbf{v}^*(\varrho(\mathbf{X}))$, $\forall \mathbf{v}^* \in V^*$. A $\mathfrak{g}$-module (V, ϱ) is a vector space V (over the field of $\mathfrak{g}$) together with a representation $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ on V. One can define the category of $\mathfrak{g}$-modules as follows. Let $(V, \varrho : \mathfrak{g} \to \mathfrak{gl}(V))$ and $(V', \varrho' : \mathfrak{g} \to \mathfrak{gl}(V'))$ be two objects. A homomorphism of $\mathfrak{g}$-modules from (V, ϱ) into (V', ϱ') is an $\mathbb{R}$-linear map $\phi : V \to V'$, which is compatible with the action of $\mathfrak{g}$, i.e. $\phi(\varrho(\mathbf{X})(\mathbf{v})) = \varrho'(\mathbf{X})(\phi(\mathbf{v}))$. Such a homomorphism is also called $\mathfrak{g}$-invariant or $\mathfrak{g}$-equivariant.

Suppose V is a finite-dimensional vector space. The group $GL(V)$ is a Lie group with the Lie algebra $\mathfrak{gl}(V)$. Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groups and $\phi : \mathcal{G} \to \mathcal{H}$ be a homomorphism of Lie groups. Then, the tangent map $T_e\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. In particular, if $\varrho : \mathcal{G} \to GL(V)$ is a group representation of $\mathcal{G}$ on V, then $T_e\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation of $\mathfrak{g}$ on V called the infinitesimal representation corresponding to ϱ. The conjugation by g is defined as $\mathrm{conj_g} : \mathcal{G} \to \mathcal{G},\ x \mapsto gxg^{-1}$. Let $\mathrm{Ad}(g) \coloneqq T_e(\mathrm{conj}_g) : \mathfrak{g} \to \mathfrak{g}$. Then the map $\mathrm{Ad} : \mathcal{G} \to GL(\mathfrak{g})$ is a representation of $\mathcal{G}$ on $\mathfrak{g}$ called the adjoint representation of G. Furthermore, the map $\mathrm{ad} \coloneqq T_e(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation of Lie algebra $\mathfrak{g}$ on $\mathfrak{g}$ called the adjoint representation of $\mathfrak{g}$. We have $\mathrm{ad}(\mathbf{X})(\mathbf{Y}) = [\mathbf{X}, \mathbf{Y}]$ for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$. In general, any representation of a Lie group induces a representation on its Lie algebra which is obtained by differentiation [47]. All of the above $\mathfrak{g}$-representations that was constructed from the representation V can be also obtained by differentiating the counterpart $\mathcal{G}$-representation. As an example, consider a representation of $\mathfrak{g}$ that is induced by a representation of $\mathcal{G}$. Then, the dual representation of $\mathfrak{g}$ can also be obtained by differentiating the dual representation of $\mathcal{G}$.

3.1.3.3 The Killing Form

Recall that the trace $\mathrm{tr}(f)$ of a linear mapping $f : V \to V$ is defined as the trace of a matrix representation of f, which does not depend on the specific choice of coordinate systems on V. The Killing form B of the Lie algebra $\mathfrak{g}$ is a symmetric $\mathbb{R}$-bilinear mapping given by $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},\ (\mathbf{X}, \mathbf{Y}) \mapsto \mathrm{tr}(\mathrm{ad}(\mathbf{X}) \circ \mathrm{ad}(\mathbf{Y}))$. The Killing form is a $\mathfrak{g}$-invariant bilinear form in the sense that $B([\mathbf{X}, \mathbf{Y}], \mathbf{Z}) = B(\mathbf{X}, [\mathbf{Y}, \mathbf{Z}])$, $\forall \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{g}$. Moreover, the Killing form is invariant under automorphisms of $\mathfrak{g}$-module $(\mathfrak{g}, \mathrm{ad})$, i.e. $B(\phi(\mathbf{X}), \phi(\mathbf{Y})) = B(\mathbf{X}, \mathbf{Y})$, where $\phi : \mathfrak{g} \to \mathfrak{g}$ is an arbitrary automorphism of $\mathfrak{g}$-module $(\mathfrak{g}, \mathrm{ad})$. The Killing form is called nondegenerate if $\{\mathbf{X} \in \mathfrak{g} : B(\mathbf{X}, \mathbf{Y}) = 0, \forall \mathbf{Y} \in \mathfrak{g}\} = \{\mathbf{0}\}$. A Lie algebra is semisimple if and only if its Killing form is nondegenerate. The Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defines the mapping $\hat{B} : \mathfrak{g} \to \mathfrak{g}^*$ given by $\hat{B}(\mathbf{X}) = B(\mathbf{X}, \cdot)$, where $\mathfrak{g}^* := L(\mathfrak{g}, \mathbb{R})$ is the dual of the $\mathfrak{g}$. For a finite-dimensional vector space $\mathfrak{g}$ one can show that B is nondegenerate if and only if $\hat{B}$ is an isomorphism [76].

3.1.4 Fiber Bundles

Here we review three types of fiber bundles: vector bundles, principal bundles, and associated bundles. More details can be found in [73].

3.1.4.1 Fibered Manifolds

A map $f : \mathcal{M} \to \mathcal{N}$ is called submersion if it is a submersion at each $x \in \mathcal{M}$, i.e. the rank of $T_x f : T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$ is equal to $\dim \mathcal{N}$. The map f is called immersion if $T_x f$ is injective for all x. A triple $(\mathcal{M}, p, \mathcal{N})$, where $p : \mathcal{M} \to \mathcal{N}$ is a surjective submersion, is called a fibered manifold. The manifolds $\mathcal{M}$ and $\mathcal{N}$ are called the total space and the base, respectively. The mapping p is called the projection. The fibered manifold $(\mathcal{M}, p, \mathcal{N})$ is also denoted by $\mathcal{M} \to \mathcal{N}$ or just $\mathcal{M}$ if the projection p is clear. A trivial fibered manifold with a fiber $\mathcal{S}$ is $(\mathcal{M} \times \mathcal{S}, \bar{\pi}_{\mathcal{M}}, \mathcal{M})$, where $\bar{\pi}_{\mathcal{M}} = \mathrm{pr}_1$, i.e. $\bar{\pi}_{\mathcal{M}}$ is the projection on the first coordinate.

A section of $(\mathcal{M}, p, \mathcal{N})$ is a smooth mapping $\boldsymbol{s} : \mathcal{N} \to \mathcal{M}$ such that $p \circ \boldsymbol{s} = \mathrm{Id}_{\mathcal{N}}$. The space of all smooth sections of $(\mathcal{M}, p, \mathcal{N})$ is denoted by $\Gamma(\mathcal{M})$ or $\Gamma(\mathcal{M} \to \mathcal{N})$. Let $x \in \mathcal{N}$ and recall that a subset $\mathcal{A} \subset \mathcal{M}$ is a smooth submanifold of $\mathcal{M}$ if it is a smooth manifold and the inclusion $i : \mathcal{A} \hookrightarrow \mathcal{M}$ is an embedding, i.e. i is an immersion and $i : \mathcal{A} \to i(\mathcal{A})$ is a homeomorphism, where $i(\mathcal{A})$ has the subspace topology. Then, $\mathcal{M}_x := p^{-1}(x)$ is a submanifold of $\mathcal{M}$ called the fiber over x.

Suppose $(\bar{\mathcal{M}}, \bar{p}, \bar{\mathcal{N}})$ is another fibered manifold. Then a morphism from $(\mathcal{M}, p, \mathcal{N})$ into $(\bar{\mathcal{M}}, \bar{p}, \bar{\mathcal{N}})$ is a smooth fiber-respecting map $\phi : \mathcal{M} \to \bar{\mathcal{M}}$, i.e. the smooth map ϕ

transforms each fiber of $\mathcal{M}$ into a subset of a fiber of $\bar{\mathcal{M}}$. The relation $\phi(\mathcal{M}_x) \subset \bar{\mathcal{M}}_{\bar{x}}$ defines a map $\underline{\phi} : \mathcal{N} \to \bar{\mathcal{N}}$ such that $\bar{p} \circ \phi = \underline{\phi} \circ p$, i.e. the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \bar{\mathcal{M}} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \bar{p}} \\ \mathcal{N} & \xrightarrow{\underline{\phi}} & \bar{\mathcal{N}} \end{array}$$

In this case, it is usually said that ϕ covers $\underline{\phi}$. All fibered manifolds together with their morphisms form a category denoted by $\mathscr{F}\mathscr{M}$. As we mentioned earlier, $\mathscr{M}f_m$ is the category of m-dimensional manifolds and their local diffeomorphisms. A natural bundle is a functor of $\mathscr{M}f_m$ into $\mathscr{F}\mathscr{M}$ that to each manifold $\mathcal{M}$ associates a fibered manifold $(F(\mathcal{M}), p_{\mathcal{M}}, \mathcal{M})$ and to each $f : \mathcal{M} \to \mathcal{N}$ associates a fiber-respecting morphism $F(f) : F(\mathcal{M}) \to F(\mathcal{N})$ covering f. Thus, the following diagram commutes.

$$\begin{array}{ccc} F(\mathcal{M}) & \xrightarrow{F(f)} & F(\mathcal{N}) \\ \Big\downarrow{\scriptstyle p_{\mathcal{M}}} & & \Big\downarrow{\scriptstyle p_{\mathcal{N}}} \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array}$$

3.1.4.2 Fiber Bundles

A fiber bundle $(\mathcal{E}, p, \mathcal{M}, \mathcal{S})$ is a fibered manifold $(\mathcal{E}, p, \mathcal{M})$ together with a manifold $\mathcal{S}$ such that each $x \in \mathcal{M}$ has an open neighborhood $U \subset \mathcal{M}$ with $p^{-1}(U)$ being diffeomorphic to $U \times \mathcal{S}$ via a fiber-respecting diffeomorphism ψ such that the following diagram commutes.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & U \times \mathcal{S} \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

The manifolds $\mathcal{E}$, $\mathcal{M}$, and $\mathcal{S}$ are called the total space, the base space, and the standard fiber, respectively. The mapping p is called the projection. The pair (U, ψ) is called a fiber chart or a local trivialization of $\mathcal{E}$. Note that a local trivialization is different from a (local) chart of the manifold $\mathcal{E}$, which is a pair $(\mathcal{U}, u)$, where $\mathcal{U} \subset \mathcal{E}$ is an open set and $u : \mathcal{U} \to u(\mathcal{U}) \subset \mathbb{R}^k$, with $k = \dim \mathcal{E}$, is a diffeomorphism. A trivialization can be considered as an alternative coordinate system induced by the additional bundled structure of $\mathcal{E}$.

A set of charts $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ such that $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $\mathcal{M}$, is called a fiber bundle atlas. Suppose $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $s \in \mathcal{S}$. Then, $\psi_\alpha \circ \psi_\beta^{-1}(x, s) = (x, \psi_{\alpha\beta}(x, s))$, where $\psi : U_{\alpha\beta} \times \mathcal{S} \to \mathcal{S}$ is a diffeomorphism of $\mathcal{S}$ for each x. Alternatively, one can assume that $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(\mathcal{S})$, where $\mathrm{Diff}(\mathcal{S})$ is the group of all diffeomorphisms of $\mathcal{S}$. The mappings $\psi_{\alpha\beta}$ are called the transition functions of the bundle.

3.1.4.3 Vector Bundles

Consider a fiber bundle $(\mathcal{E}, p, \mathcal{M}, \mathcal{S})$ with $\mathcal{S}$ being a finite-dimensional vector space V. In this case, a fiber chart (U, ψ) is called a vector bundle chart (or a local trivialization of $\mathcal{E}$). Moreover, assume that the transition functions of a fiber bundle atlas of $\mathcal{E}$ are fiber linear isomorphisms, i.e. $\psi_{\alpha\beta} : U_{\alpha\beta} \to GL(V)$. Such fiber bundle atlases are called vector bundle atlases. Two different vector bundle atlases are called equivalent if their union is a vector bundle atlas. This defines an equivalence relation. A fiber bundle $(\mathcal{E}, p, \mathcal{M}, V)$ together with an equivalence class of vector bundle atlases is called a vector bundle. If there exists at least one vector bundle atlas, then there also exists an equivalent class of vector bundle atlases.

If $(\mathcal{E}, p, \mathcal{M}, V)$ is a vector bundle, then $\mathcal{E}_x$ for all $x \in \mathcal{M}$ is a vector space since the transition functions are homomorphisms of the vector space V and they induce a unique linear structure on each $\mathcal{E}_x = p^{-1}(x)$. In particular, for each $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{E}_x$ with $\psi_\alpha(\mathbf{u}_i) = (x, \mathbf{v}_i)$ for $i = 1, 2$, and $c_1, c_2 \in \mathbb{R}$, we define $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 := \psi_\alpha^{-1}(x, c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$. Furthermore, $0_x \in \mathcal{E}_x$ is given by $0_x = \psi_\alpha^{-1}(x, 0)$. The zero section $\mathbf{0} : \mathcal{M} \to \mathcal{E}$ is defined as $\mathbf{0}(x) = 0_x$.

The tangent bundle $(T\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M}, \mathbb{R}^m)$ with $m = \dim \mathcal{M}$ and $\pi_{\mathcal{M}}(T_x\mathcal{M}) = x$, is a vector bundle. One can obtain the atlas $\{(\pi_{\mathcal{M}}^{-1}(U_\alpha), Tu_\alpha)\}$ of $T\mathcal{M}$ from an atlas $\{(U_\alpha, u_\alpha)\}$ of $\mathcal{M}$. The chart changing maps for this atlas are given by

$$Tu_\alpha \circ (Tu_\beta)^{-1}(\mathbf{y}, \mathbf{Y}) = (u_{\alpha\beta}(\mathbf{y}), T_{\mathbf{y}}(u_{\alpha\beta}) \cdot \mathbf{Y}), \tag{195}$$

where $\mathbf{y} \in u_\beta(U_\alpha \cap U_\beta)$ and $\mathbf{Y} \in \mathbb{R}^m$. Since chart changing maps are diffeomorphisms, we have $T_{\mathbf{y}} u_{\alpha\beta} \in GL(\mathbb{R}^m)$ and therefore, $\{(U_\alpha, Tu_\alpha)\}$ is a vector bundle atlas for the tangent bundle. As we will see in the sequel, tensors and k-forms are vector bundles as well.

Let $(\mathcal{E}, p, \mathcal{M}, V)$ and $(\mathcal{F}, q, \mathcal{N}, W)$ be vector bundles. Then, a homomorphism of vector bundles from $(\mathcal{E}, p, \mathcal{M}, V)$ into $(\mathcal{F}, q, \mathcal{N}, W)$ is a fiber-respecting fiber-linear smooth mapping $\phi : \mathcal{E} \to \mathcal{F}$ covering $\underline{\phi} : \mathcal{M} \to \mathcal{N}$, i.e. the following diagram commutes.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[d, "p"] & \mathcal{F} \arrow[d, "q"] \\ \mathcal{M} \arrow[r, "\underline{\phi}"] & \mathcal{N} \end{tikzcd}$$

Thus, $\phi_x := \phi|_{\mathcal{E}_x} : \mathcal{E}_x \to \mathcal{F}_{\underline{\phi}(x)}$ is a homomorphism of linear spaces. The smooth vector bundles and their homomorphisms form a category $\mathscr{V}\mathscr{B}$.

Let $(\mathcal{E}, p, \mathcal{M}, V)$ be a vector bundle. Suppose $\{\mathbf{v}_j\}_{j=1}^k$ is a basis for V and (U_α, ψ_α) is a vector bundle chart. Consider the sections $\boldsymbol{s}_j : U_\alpha \to \mathcal{E}$, $x \mapsto \psi_\alpha^{-1}(x, \mathbf{v}_j)$, for $j = 1, \ldots, k$. Then, for each $x \in U_\alpha$, the set $\{\boldsymbol{s}_j(x)\}_{j=1}^k$ is a basis for $\mathcal{E}_x$. The set $\{\boldsymbol{s}_j\}_{j=1}^k$ is called a local frame field for $\mathcal{E}$ over U_α.

3.1.4.4 The Tangent Bundle of a Vector Bundle

In order to define the notion of connections, we need to study the structure of the tangent bundle $(T\mathcal{E}, \pi_{\mathcal{E}}, \mathcal{E})$ of a vector bundle $(\mathcal{E}, p, \mathcal{M}, V)$. Let $\{(U_\alpha, \psi_\alpha)\}$

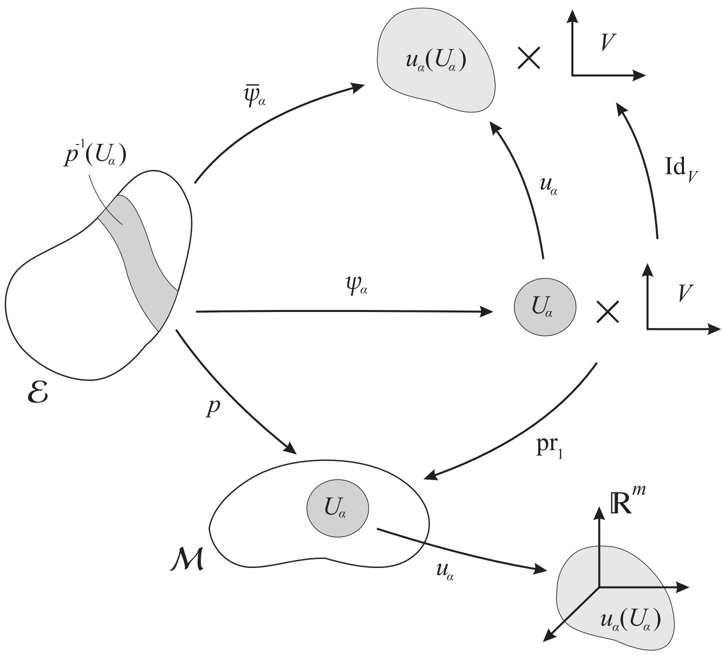


Figure 20: For an arbitrary vector bundle $(\mathcal{E}, p, \mathcal{M}, V)$, any chart (U_α, u_α) of $\mathcal{M}$ such that (U_α, ψ_α) is a vector bundle chart for $\mathcal{E}$, induces the trivialization $(p^{-1}(U_\alpha), \overline{\psi}_\alpha)$ on $\mathcal{E}$.

be a vector bundle atlas for $\mathcal{E}$ such that $\{(U_\alpha, u_\alpha)\}$ is a smooth atlas for an m-dimensional manifold $\mathcal{M}$. As is schematically shown in Fig. 20, one can obtain an atlas $\{(p^{-1}(U_\alpha), \overline{\psi}_\alpha)\}$[3] for $\mathcal{E}$, where

$$\begin{aligned} \overline{\psi}_\alpha : p^{-1}(U_\alpha) &\longrightarrow u_\alpha(U_\alpha) \times V \subset \mathbb{R}^m \times V, \\ z &\longmapsto \left(u_\alpha \circ \mathrm{pr}_1 \circ \psi_\alpha(z), \mathrm{pr}_2 \circ \psi_\alpha(z)\right). \end{aligned} \tag{196}$$

Recall that $\psi_\alpha \circ \psi_\beta^{-1}(x, \mathbf{v}) = (x, \psi_{\alpha\beta}(x) \cdot \mathbf{v})$ for all $x \in U_\alpha \cap U_\beta$ and $\mathbf{v} \in V$. Also for all $\mathbf{y} \in u_\beta(U_\alpha \cap U_\beta)$, we have $u_\alpha \circ u_\beta^{-1}(\mathbf{y}) = u_{\alpha\beta}(\mathbf{y})$ and

$$\overline{\psi}_\alpha \circ (\overline{\psi}_\beta)^{-1}(\mathbf{y}, \mathbf{v}) = \left(u_{\alpha\beta}(\mathbf{y}), \psi_{\alpha\beta}\left(u_\beta^{-1}(\mathbf{y})\right) \cdot \mathbf{v}\right). \tag{197}$$

[3] Strictly speaking, $(p^{-1}(U_\alpha), \overline{\psi}_\alpha)$ is not a chart, since the image of $\overline{\psi}_\alpha$ is not a subset of $\mathbb{R}^l$ for $l = m + \dim V$.

Note that (196) induces an atlas for $T\mathcal{E}$ with charts determined by

$$T\overline{\psi}_\alpha : \pi_{\mathcal{E}}^{-1}\left(p^{-1}\left(U_\alpha\right)\right) \longrightarrow u_\alpha(U_\alpha) \times V \times \mathbb{R}^m \times V. \tag{198}$$

For writing (198), we use the fact that $T_{\mathbf{v}}V \approx V$, i.e. we use V instead of $T_{\mathbf{v}}V$. Let $\{\mathbf{v}_i\}$ be a basis for V, $\mathbf{v} = v^i\mathbf{v}_i \in V$, $\bar{\mathbf{v}} = \bar{v}^i\mathbf{v}_i \in V$, and $f \in C^\infty(V,\mathbb{R})$, where $C^\infty(V,\mathbb{R})$ is the set of smooth functions from V into $\mathbb{R}$. Consider the curve $\mathbf{c}(t) = \mathbf{v} + t\bar{\mathbf{v}}$ in V. Then $\mathbf{c}'(0)f = \frac{d}{dt}\big|_{t=0} f \circ \mathbf{c}(t) = \bar{v}^i\frac{\partial f}{\partial x^i} = \left(\bar{v}^i\frac{\partial}{\partial x^i}\right)f$, where $\frac{\partial}{\partial x^i} \coloneqq \mathbf{c}_i'(0)$ with $\mathbf{c}_i(t) = \mathbf{v} + t\mathbf{v}_i$. Naturally, one can consider the isomorphism $\bar{\iota} : V \to T_{\mathbf{v}}V,\ v^i\mathbf{v}_i \mapsto v^i\frac{\partial}{\partial x^i}$. Whenever we use a vector $\bar{\mathbf{v}} \in V$ as an element of $T_{\mathbf{v}}V$, we mean $\bar{\iota}(\bar{\mathbf{v}})$ that in the base $\{\frac{\partial}{\partial x^i}\}$ has the same components as $\bar{\mathbf{v}}$ in the base $\{\mathbf{v}_i\}$. We have $TV \approx V \times V$.

Using (196), (195), and (197) we obtain the chart changing relations for $T\mathcal{E}$ as follows:

$$\begin{aligned} T\overline{\psi}_\alpha \circ (T\overline{\psi}_\beta)^{-1}(\mathbf{y},\mathbf{v},\mathbf{Y},\mathbf{V}) = \Big(\ & u_{\alpha\beta}(\mathbf{y})\ ,\ \psi_{\alpha\beta}\left(u_\beta^{-1}(\mathbf{y})\right)\cdot\mathbf{v}\ ,\ T_{\mathbf{y}}u_{\alpha\beta}\cdot\mathbf{Y}\ , \\ & \left(T_{\mathbf{y}}(\psi_{\alpha\beta}\circ u_\beta^{-1})\cdot\mathbf{Y}\right)\cdot\mathbf{V} + \psi_{\alpha\beta}\left(u_\beta^{-1}(\mathbf{y})\right)\cdot\mathbf{V}\ \Big), \end{aligned} \tag{199}$$

where $\mathbf{Y} \in \mathbb{R}^m$ and $\mathbf{V} \in V$. For fixed $(\mathbf{y},\mathbf{v})$, (199) is linear in $(\mathbf{Y},\mathbf{V})$, which means that $(T\mathcal{E},\pi_{\mathcal{E}},\mathcal{E},\mathbb{R}^m \times V)$ is a vector bundle with a vector bundle atlas $\{(p^{-1}(U_\alpha),T\overline{\psi}_\alpha)\}$, such that the following diagram commutes.

$$\begin{tikzcd} \pi_{\mathcal{E}}^{-1}(p^{-1}(U_\alpha)) \arrow[rr, "T\overline{\psi}_\alpha"] \arrow[dr, "\pi_{\mathcal{E}}"'] & & p^{-1}(U_\alpha) \times \mathbb{R}^m \times V \arrow[dl, "\mathrm{pr}_1"] \\ & p^{-1}(U_\alpha) & \end{tikzcd}$$

On the other hand, (199) is also linear in $(\mathbf{v},\mathbf{V})$ for fixed $(\mathbf{y},\mathbf{Y})$, which suggests that $(T\mathcal{E},Tp,T\mathcal{M},V\times V)$ is a vector bundle with a vector bundle atlas $\{(\pi_{\mathcal{M}}^{-1}(U_\alpha),T\overline{\psi}_\alpha)\}$, i.e. the following diagram commutes.

$$\begin{tikzcd}
(Tp)^{-1}(\pi_{\mathcal{M}}^{-1}(U_\alpha)) \arrow[rr, "T\overline{\psi}_\alpha"] \arrow[dr, "Tp"'] & & \pi_{\mathcal{M}}^{-1}(U_\alpha) \times V \times V \arrow[dl, "\mathrm{pr}_1"] \\
& \pi_{\mathcal{M}}^{-1}(U_\alpha) &
\end{tikzcd}$$

The vertical space $V\mathcal{E} \subset T\mathcal{E}$ of the vector bundle $(\mathcal{E}, p, \mathcal{M}, V)$ is defined as

$$V\mathcal{E} := \mathsf{Ker}(Tp) = \{Z \in T\mathcal{E} : \; Tp \cdot Z = (p(\pi_{\mathcal{E}}(Z)), 0)\}. \tag{200}$$

For an arbitrary $Z \in V\mathcal{E}$, we have $T\overline{\psi}_\alpha \cdot Z = (\mathbf{y}, \mathbf{v}, 0, \mathbf{V})$. Thus, by using (199), we obtain a trivialization for $V\mathcal{E}$ with transition functions

$$T\overline{\psi}_\alpha \circ (T\overline{\psi}_\beta)^{-1}(\mathbf{y}, \mathbf{v}, 0, \mathbf{V}) = \Big(\, u_{\alpha\beta}(\mathbf{y}) \, , \; \psi_{\alpha\beta}\left(u_\beta^{-1}(\mathbf{y})\right) \cdot \mathbf{v} \, , \; 0 \, , \; \psi_{\alpha\beta}\left(u_\beta^{-1}(\mathbf{y})\right) \cdot \mathbf{V} \, \Big), \tag{201}$$

which are linear in $(\mathbf{v}, \mathbf{V})$ for fixed $\mathbf{y}$. Therefore, $(V\mathcal{E}, \pi_{\mathcal{M}} \circ Tp, \mathcal{M}, V \times V)$ is a vector bundle. Note that tangent vectors to curves that lie in a fix fiber of $\mathcal{E}$ are vertical, i.e. belong to $V\mathcal{E}$.

Let $\mathcal{E}_{\mathcal{M}} \times \mathcal{E} = \{(z, y) \in \mathcal{E} \times \mathcal{E} : \; p(z) = p(y)\}$ and $v_x, w_x \in \mathcal{E}_x$ with $\overline{\psi}_\alpha(v_x) = (u_\alpha(x), \mathbf{v})$ and $\overline{\psi}_\alpha(w_x) = (u_\alpha(x), \mathbf{w})$. The vertical lift $\ell_{\mathcal{E}}^v : \mathcal{E}_{\mathcal{M}} \times \mathcal{E} \to V\mathcal{E}$ is defined as $\ell_{\mathcal{E}}^v(v_x, w_x) = \frac{d}{dt}\big|_{t=0}(v_x + t w_x)$. In a local coordinate chart, we have $(T\overline{\psi}_\alpha) \circ \ell_{\mathcal{E}}^v \left(\; \psi_\alpha^{-1}(x, \mathbf{v}) \; , \; \psi_\alpha^{-1}(x, \mathbf{w}) \; \right) = (u_\alpha(x), \mathbf{v}, 0, \mathbf{w})$. The vertical projection is defined as $\mathrm{pr}_{\mathcal{E}}^v := \mathrm{pr}_2 \circ (\ell_{\mathcal{E}}^v)^{-1} : V\mathcal{E} \to \mathcal{E}$.

3.1.4.5 Tensors and Differential Forms

It is possible to give a description of the size of vector bundles over a fixed base and a fixed standard fiber. In particular, one can construct a set that is in a bijective correspondence with the set of all isomorphism classes of vector bundles over $\mathcal{M}$ with standard fiber V [73]. The main observation is that one can determine a unique class of isomorphic vector bundles by a proper set of transition functions.

Let $\mathscr{V}\mathscr{S}$ be the category of finite-dimensional vector spaces and their homomorphisms. Suppose F is a covariant functor from $\mathscr{V}\mathscr{S}$ into $\mathscr{V}\mathscr{S}$ such that $F : L(V, W) \to L(F(V), F(W))$ is smooth. Then, it is possible to extend F to a covariant functor from $\mathscr{V}\mathscr{B}$ into $\mathscr{V}\mathscr{B}$. Let $(\mathcal{E}, p, \mathcal{M}, V)$ be an object of $\mathscr{V}\mathscr{B}$ with a vector bundle atlas $\{(U_\alpha, \psi_\alpha)\}$ and corresponding transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} \to GL(V)$. Then, $(F(\mathcal{E}), \tilde{p}, \mathcal{M}, F(V))$ is the unique image of $(\mathcal{E}, p, \mathcal{M}, V)$ with an atlas $\{(U_\alpha, \tilde{\psi}_\alpha)\}$ and $\tilde{\psi}_{\alpha\beta} = F(\psi_{\alpha\beta}) : U_{\alpha\beta} \to GL(F(V))$, where $F(\psi_{\alpha\beta})(x) \coloneqq F(\psi_{\alpha\beta}(x))$[4]. Thus, $F(\mathcal{E})$ is just a vector bundle over $\mathcal{M}$ with the standard fiber $F(V)$. Also let $(\underline{\mathcal{E}}, \underline{p}, \mathcal{M}, \underline{V})$ be another object of $\mathscr{V}\mathscr{B}$ and $\boldsymbol{h} : \mathcal{E} \to \underline{\mathcal{E}}$ be a vector bundle homomorphism. We have $F(\boldsymbol{h}) : F(\mathcal{E}) \to F(\underline{\mathcal{E}})$, where $F(\boldsymbol{h})|_{F(\mathcal{E})_x} \coloneqq F(\boldsymbol{h}|_{\mathcal{E}_x})$. If F is a contravariant functor, then one can obtain a covariant functor from $\mathscr{V}\mathscr{B}$ into $\mathscr{V}\mathscr{B}$ by defining $(F(\mathcal{E}), \tilde{p}, \mathcal{M}, F(V))$ to be a vector bundle with $\tilde{\psi}_{\alpha\beta} = F(\psi_{\alpha\beta}^{-1}) : U_{\alpha\beta} \to GL(F(V))$ and using $F(\boldsymbol{h}^{-1})$ instead of $F(\boldsymbol{h})$.

The tensor product $\otimes^r V$ defines a covariant functor such that $F(V) = \otimes^r V$ and $F(\boldsymbol{f}) : \otimes^r V \to \otimes^r W,\ \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_r \mapsto \boldsymbol{f}(\mathbf{v}_1) \otimes \cdots \otimes \boldsymbol{f}(\mathbf{v}_r)$, where $\boldsymbol{f} \in L(V, W)$ and $\mathbf{v}_i \in V$ for $i = 1, \dots, r$. Thus, one can associate the vector bundle $(\otimes^r \mathcal{E}, \tilde{p}, \mathcal{M}, \otimes^r V)$ to $(\mathcal{E}, p, \mathcal{M}, V)$. Similarly, the exterior power is a covariant functor, i.e. $F(V) = \Lambda^r V$ and $F(\boldsymbol{f}) : \Lambda^r V \to \Lambda^r W,\ \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r \mapsto \boldsymbol{f}(\mathbf{v}_1) \wedge \cdots \wedge \boldsymbol{f}(\mathbf{v}_r)$. The duality functor $F(V) = V^*$ is a contravariant functor with $F(\boldsymbol{f}) = \boldsymbol{f}^* : W^* \to V^*,\ \mathbf{w}^* \mapsto \mathbf{w}^* \circ \boldsymbol{f}$.

One can also use multifunctors to construct vector bundles. For example, consider a contra-covariant bifunctor F and let $(\mathcal{E}, p, \mathcal{M}, V)$ and $(\underline{\mathcal{E}}, \underline{p}, \mathcal{M}, \underline{V})$ be vector bundles with transition functions $\psi_{\alpha\beta}$ and $\underline{\psi}_{\alpha\beta}$, respectively. Then, $(F(\mathcal{E}, \underline{\mathcal{E}}), \bar{p}, \mathcal{M}, F(V, \underline{V}))$ is a vector bundle with transition functions $F(\psi_{\alpha\beta}^{-1}, \underline{\psi}_{\alpha\beta})$. In summery, given vector bundles $(\mathcal{E}, p, \mathcal{M}, V)$ and $(\underline{\mathcal{E}}, \underline{p}, \mathcal{M}, \underline{V})$, one can define the following vector bundles with the base $\mathcal{M}$: $\mathcal{E}^*$, $\Lambda^r \mathcal{E}$, $\otimes^r \mathcal{E}$, $\mathcal{E} \otimes \underline{\mathcal{E}}$, $\mathcal{E} \wedge \underline{\mathcal{E}}$, $\mathcal{E} \oplus \underline{\mathcal{E}}$, and $L(\mathcal{E}, \underline{\mathcal{E}})$.

[4]The vector bundle $(F(\mathcal{E}), \tilde{p}, \mathcal{M}, F(V))$ is an element of the unique isomorphic class of vector bundles with the transition functions $\tilde{\psi}_{\alpha\beta}$. Note that $\mathcal{E}_x$ is linear and we have $F(\mathcal{E})_x \coloneqq \tilde{p}^{-1}(x)$ is equal to $F(\mathcal{E}_x)$.

A natural vector bundle is a covariant functor F from $\mathscr{M}f_m$ into $\mathscr{V}\mathscr{B}$, i.e. the functor F associates a vector bundle $(F(\mathcal{M}), p_{\mathcal{M}}, \mathcal{M}, V_{\mathcal{M}})$ to a manifold $\mathcal{M}$ and a vector bundle isomorphism (invertible homomorphism) $F(f) : F(\mathcal{M}) \to F(\bar{\mathcal{M}})$ to each immersion $f : \mathcal{M} \to \bar{\mathcal{M}}$ such that the following diagram commutes.

$$\begin{tikzcd} F(\mathcal{M}) \arrow[r, "F(f)"] \arrow[d, "p_{\mathcal{M}}"] & F(\bar{\mathcal{M}}) \arrow[d, "p_{\bar{\mathcal{M}}}"] \\ \mathcal{M} \arrow[r, "f"] & \bar{\mathcal{M}} \end{tikzcd}$$

For example, $T\mathcal{M}$ is a natural vector bundle with $\mathcal{M} \mapsto (T\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M}, \mathbb{R}^m)$ and $f \mapsto Tf$. Let $T^*\mathcal{M} := (T\mathcal{M})^*$, which is called the cotangent bundle. Then, $T^*\mathcal{M}$ is also a natural vector bundle: $\mathcal{M} \mapsto (T^*\mathcal{M}, \tilde{\pi}_{\mathcal{M}}, \mathcal{M}, L(\mathbb{R}^m, \mathbb{R}))$ and $f \mapsto T^*f$, where $(T^*f)_x := ((T_x f)^{-1})^* : T_x^*\mathcal{M} \to T_{f(x)}^*\bar{\mathcal{M}}$. In general, let F be a covariant functor from $\mathscr{V}\mathscr{S}$ into $\mathscr{V}\mathscr{S}$ as we mentioned earlier. Then, $F(T\mathcal{M})$ defines a natural vector bundle given by $\mathcal{M} \mapsto (F(\mathcal{M}), \bar{p}, \mathcal{M}, F(\mathbb{R}^m))$ and $f \mapsto F(Tf)$. For contravariant functors we have $f \mapsto F((Tf)^{-1})$, where $(Tf)^{-1}|_{f(x)} := (T_x f)^{-1}$.

A tensor field of type $\binom{p}{q}$ is a section of the vector bundle $\otimes^p T\mathcal{M} \otimes \otimes^q T^*\mathcal{M} \to \mathcal{M}$. Let (U, u) be a chart for $\mathcal{M}$. Then, $\{\frac{\partial}{\partial u^i}\}_{i=1}^m$ and $\{du^i\}_{i=1}^m$ are local frame fields for $T\mathcal{M}$ and $T^*\mathcal{M}$ over U, respectively. The corresponding local frame field over U for $\otimes^p T\mathcal{M} \otimes \otimes^q T^*\mathcal{M}$ is

$$\left\{\frac{\partial}{\partial u^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}\right\}_{i_1,\ldots,i_p,j_1,\ldots j_q \in \{1,\ldots,m\}}. \tag{202}$$

Therefore, any $\binom{p}{q}$-tensor field $\boldsymbol{T}$ over U can be written as

$$\boldsymbol{T} = T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} \frac{\partial}{\partial u^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}. \tag{203}$$

Alternatively, using the relation $W \otimes V^* \approx L^2(W^*, V; \mathbb{R})$ (see (186)), one can consider

$\boldsymbol{T}$ as a multilinear function with

$$\boldsymbol{T}\left(du^{i_1}, \ldots, du^{i_p}, \frac{\partial}{\partial u^{j_1}}, \ldots, \frac{\partial}{\partial u^{j_q}}\right) = T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} \in C^\infty(U, \mathbb{R}), \tag{204}$$

with $C^\infty(U, \mathbb{R})$ denoting the set of smooth functions $g : U \subset \mathcal{M} \to \mathbb{R}$. On the other hand, the relation $W \otimes V^* \approx L(V, W)$ yields another interpretation: a $\binom{p}{q}$-tensor field $\boldsymbol{T}$ can be considered as a mapping

$$\boldsymbol{T} : \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{q} \to \Gamma(\otimes^p T\mathcal{M}), \tag{205}$$

which is q-linear over $C^\infty(\mathcal{M}, \mathbb{R})$. To simplify the notation, the same symbols are used to denote tensors in all of the above interpretations, but it should be kept in mind that they are elements of three different sets.

Another type of tensors which is important in nonlinear elasticity is two-point tensors [82]. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. A two-point tensor of type $\binom{p\ r}{q\ s}$ is a section of the vector bundle $\mathcal{T} \to \mathcal{M}$, where the fiber of $\mathcal{T}$ over $x \in \mathcal{M}$ is given by

$$\mathcal{T}_x = \otimes^p T_x\mathcal{M} \otimes \otimes^q T_x^*\mathcal{M} \otimes \otimes^r T_{\varphi(x)}\mathcal{N} \otimes \otimes^s T^*_{\varphi(x)}\mathcal{N}. \tag{206}$$

A differential form or an exterior form of degree k or simply a k-form is a section of the vector bundle $\Lambda^k T^*\mathcal{M} \to \mathcal{M}$. The space of all k-forms is denoted by $\Omega^k(\mathcal{M}) := \Gamma(\Lambda^k T^*\mathcal{M})$. Using (190), one can also consider a k-form $\boldsymbol{\alpha}$ as a mapping

$$\boldsymbol{\alpha} : \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{k} \to C^\infty(\mathcal{M}, \mathbb{R}), \tag{207}$$

which is alternating and k-linear over $C^\infty(\mathcal{M}, \mathbb{R})$. Therefore, a k-form is an alternating $\binom{0}{k}$-tensor. The local frame field for $\Lambda^k T^*\mathcal{M}$ corresponding to the chart (U, u) of

$\mathcal{M}$ is given by

$$\left\{du^{i_1} \wedge \cdots \wedge du^{i_k}\right\}_{\substack{i_1,\ldots,i_k \in \{1,\ldots,m\} \\ i_1 < i_2 < \cdots < i_k}}. \tag{208}$$

Thus, a form $\boldsymbol{\alpha} \in \Omega^k(\mathcal{M})$ over U can be written as

$$\boldsymbol{\alpha} = \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{i_1 \ldots i_k} du^{i_1} \wedge \cdots \wedge du^{i_k} = \frac{1}{k!} \alpha_{i_1 \ldots i_k} du^{i_1} \wedge \cdots \wedge du^{i_k}, \tag{209}$$

where the summation convention is suppressed in the left summation.

Let $\mathcal{V} \to \mathcal{M}$ be a vector bundle. A vector bundle valued k-form is a section of $\Lambda^k T^* \mathcal{M} \otimes \mathcal{V} \to \mathcal{M}$. The space of $\mathcal{V}$-valued k-forms is denoted by $\Omega^k(\mathcal{M}; \mathcal{V})$. From the relation (191), we conclude that a form $\boldsymbol{\beta} \in \Omega^k(\mathcal{M}; \mathcal{V})$ is also an alternating k-linear (over $C^\infty(\mathcal{M}, \mathbb{R})$) mapping

$$\boldsymbol{\beta} : \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{k} \to \Gamma(\mathcal{V}). \tag{210}$$

Suppose V is a finite-dimensional vector space. An alternating k-linear mapping

$$\boldsymbol{\gamma} : \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{k} \to C^\infty(\mathcal{M}, V) \tag{211}$$

is called a V-valued k-form. We have $\boldsymbol{\gamma} \in \Omega^k(\mathcal{M}; V) := \Gamma(\Lambda^k T^* \mathcal{M} \otimes V \to \mathcal{M})$.

3.1.4.6 Principal Bundles

Let $\mathcal{G}$ be a Lie group. A fiber bundle $(\mathcal{P}, p, \mathcal{M}, \mathcal{G})$ is called a principal (fiber) bundle if it has a fiber bundle atlas $\{(U_\alpha, \psi_\alpha)\}$ such that $\psi_\alpha \circ \psi_\beta^{-1}(x, g) = (x, \psi_{\alpha\beta}(x, g)) = (x, \varphi_{\alpha\beta}(x) \cdot g)$, where the dot denotes the group multiplication of $\mathcal{G}$ and the smooth functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathcal{G}$ satisfy the cocycle conditions $\varphi_{\alpha\beta}(x) \cdot \varphi_{\beta\gamma}(x) = \varphi_{\alpha\gamma}(x)$ for $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $\varphi_{\alpha\alpha}(x) = e$, with e being the unit element of $\mathcal{G}$. The bundle atlas $\{(U_\alpha, \psi_\alpha)\}$ is called a principal bundle atlas and $\mathcal{G}$ is called the structure group. The principal bundle $\mathcal{P}$ is also called a principal $\mathcal{G}$-bundle. Suppose $u_x = \psi_\alpha^{-1}(x, g_1) \in \mathcal{P}_x$.

It is possible to define a unique right action $r : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ on $\mathcal{P}$ which is called the principal right action and is given by

$$r(u_x, g_2) = r(\psi_\alpha^{-1}(x, g_1), g_2) := \psi_\alpha^{-1}(x, g_1 \cdot g_2). \tag{212}$$

The principal right action is also written as $u_x \cdot g := r(u_x, g)$ and is free and proper.[5] The orbit of the principal right action through u_x is the fiber $\mathcal{P}_x$. In fact, the mapping $r_{u_x} : \mathcal{G} \to \mathcal{P}$ is a diffeomorphism onto the fiber through u_x. Note that in contrast to the fibers $\mathcal{V}_x$ of a vector bundle $(\mathcal{V}, p, \mathcal{M}, V)$, which inherit the linear structure of the standard fiber V, the fibers $\mathcal{P}_x$ of a principal $\mathcal{G}$-bundle $\mathcal{P}$ are diffeomorphic to $\mathcal{G}$ and do not have a Lie group structure, in general.

One can generalize the previous example to arbitrary homogeneous spaces as follows. Let $\mathcal{G}$ be a Lie group and $\mathcal{H} \subset \mathcal{G}$ be a Lie subgroup. Consider the homogeneous space $\mathcal{G}/\mathcal{H}$ and the natural projection $p : \mathcal{G} \to \mathcal{G}/\mathcal{H}$. Then, $(\mathcal{G}, p, \mathcal{G}/\mathcal{H}, \mathcal{H})$ is a principal fiber bundle. More generally, let $\mathcal{O}$ be a smooth manifold with a smooth, free, proper right action $\mathcal{O} \times \mathcal{G} \to \mathcal{O}$. The set $\mathcal{O}/\mathcal{G}$ with the quotient topology is a smooth manifold with dim $(\mathcal{O}/\mathcal{G}) =$ dim $\mathcal{O}$ − dim $\mathcal{G}$, and $(\mathcal{O}, \bar{p}, \mathcal{O}/\mathcal{G}, \mathcal{G})$ is a principal $\mathcal{G}$-bundle, where $\bar{p} : \mathcal{O} \to \mathcal{O}/\mathcal{G}$ is the natural projection.

Let $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}(\mathcal{P})$. The tangent map of the multiplication $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ of $\mathcal{G}$ makes $T\mathcal{G}$ into a Lie group with the multiplication given by

$$T_{(g_1, g_2)}\mu \cdot (\boldsymbol{\xi}(g_1), \boldsymbol{\eta}(g_2)) = T_{g_1}(\rho^{g_2}) \cdot \boldsymbol{\xi}(g_1) + T_{g_2}(\lambda_{g_1}) \cdot \boldsymbol{\eta}(g_2).$$

As we mentioned earlier, $(g, \mathbf{X}) \mapsto \boldsymbol{L}_{\mathbf{X}}(g)$ defines an isomorphism $\mathcal{G} \times \mathfrak{g} \to T\mathcal{G}$. Let $u_{g_1}, v_{g_2} \in T\mathcal{G}$, where $u_{g_1} = \boldsymbol{L}_{\mathbf{U}}(g_1)$ and $v_{g_2} = \boldsymbol{L}_{\mathbf{V}}(g_2)$. The group multiplication of $T\mathcal{G}$ can be written as $u_{g_1} \cdot v_{g_2} = \boldsymbol{L}_{(\mathrm{Ad}(g_2^{-1})\mathbf{U} + \mathbf{V})}(g_1 \cdot g_2)$. In fact, $T\mathcal{G}$ is isomorphic to the

[5] A right action $r : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ is proper if for any compact subsets $\mathcal{A}, \mathcal{D} \subset \mathcal{P}$, the set $\{g \in \mathcal{G} : (\mathcal{A} \cdot g) \cap \mathcal{D} \neq \varnothing\}$ is also compact.

semidirect product $\mathcal{G} \ltimes \mathfrak{g}$. The fiber bundle $(T\mathcal{P}, Tp, T\mathcal{M}, T\mathcal{G})$ is also a principal fiber bundle with the principal right action $Tr : T\mathcal{P} \times T\mathcal{G} \to T\mathcal{P}$, where $r : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ is the principal right action of $(\mathcal{P}, p, \mathcal{M}, \mathcal{G})$.

Let $(\mathcal{P}, p, \mathcal{M}, \mathcal{G})$ and $(\mathcal{Q}, q, \mathcal{N}, \mathcal{G})$ be principal $\mathcal{G}$-bundles. A fibered manifold morphism $\chi : \mathcal{P} \to \mathcal{Q}$ is called a principal bundle homomorphism if it is $\mathcal{G}$-equivariant with respect to the principal right actions, i.e. if $\chi(u_x \cdot g) = \chi(u_x) \cdot g$, where the dots denote the principal right actions. Principal $\mathcal{G}$-bundles together with their homomorphisms form a category, which is denoted by $\mathscr{PB}(\mathcal{G})$. Let $(\mathcal{P}', p', \mathcal{M}', \mathcal{G}')$ be another principal bundle and $\phi : \mathcal{G} \to \mathcal{G}'$ be a group homomorphism. Then, a fibered manifold morphism $\chi : \mathcal{P} \to \mathcal{P}'$ is called a homomorphism over ϕ of principal bundles if $\chi(u_x \cdot g) = \chi(u_x) \cdot \phi(g)$. The category of principal fiber bundles and their homomorphisms is denoted by $\mathscr{PB}$.

Let $(\mathcal{V}, p, \mathcal{M}, V)$ be a vector bundle. Using the method described in §3.1.4.5, we can define the vector bundle $L(\mathcal{M} \times V, \mathcal{V}) \to \mathcal{M}$ whose fiber over $x \in \mathcal{M}$ is $L(V, \mathcal{V}_x)$. Consider the open subset $\mathcal{GL}(\mathcal{V}) := GL(\mathcal{M} \times V, \mathcal{V}) \subset L(\mathcal{M} \times V, \mathcal{V})$, which is a fiber bundle over $\mathcal{M}$ with fibers $GL(V, \mathcal{V}_x)$, i.e. invertible linear maps from V into $\mathcal{V}_x$. Composition from the right by elements of $GL(V)$ gives a right action of $GL(V)$ on $\mathcal{GL}(\mathcal{V})$. One can show that this right action is the principal right action for the principal $GL(V)$-bundle $\mathcal{GL}(\mathcal{V}) \to \mathcal{M}$. The principal bundle $\mathcal{GL}(\mathcal{V})$ is called the linear frame bundle of $\mathcal{V}$. Note that a local section of $\mathcal{GL}(\mathcal{V})$ specifies a unique local frame field for $\mathcal{V}$. The linear frame bundle of $T\mathcal{M}$ is denoted by $\mathcal{P}^1\mathcal{M}$, i.e. $\mathcal{P}^1\mathcal{M} := \mathcal{GL}(T\mathcal{M})$. Thus, $\mathcal{P}^1\mathcal{M} \to \mathcal{M}$ is a principal $GL(\mathbb{R}^m)$-bundle whose fiber over $x \in \mathcal{M}$ can be considered as the set of all bases of $T_x\mathcal{M}$.

3.1.4.7 Associated Bundles

Suppose $(\mathcal{P}, p, \mathcal{M}, \mathcal{G})$ is a principal bundle and $\mathcal{S}$ is a manifold with a left action $\ell : \mathcal{G} \times \mathcal{S} \to \mathcal{S}$. We can define a right action of $\mathcal{G}$ on $\mathcal{P} \times \mathcal{S}$ as $\tilde{r} : (\mathcal{P} \times \mathcal{S}) \times \mathcal{G} \to$

$\mathcal{P}\times\mathcal{S}$, $((u_x,s),g)\mapsto(u_x\cdot g,g^{-1}\cdot s)$. Let $\mathcal{P}\times_{\mathcal{G}}\mathcal{S}:=(\mathcal{P}\times\mathcal{S})/\mathcal{G}$, i.e. $\mathcal{P}\times_{\mathcal{G}}\mathcal{S}$ is the space of the orbits of $\tilde{r}$. By $[\![u_x,s]\!]\in\mathcal{P}\times_{\mathcal{G}}\mathcal{S}$, we denote the orbit of $(u_x,s)\in\mathcal{P}\times\mathcal{S}$. The space $\mathcal{P}\times_{\mathcal{G}}\mathcal{S}$ is a manifold and the natural projection $\underline{p}:\mathcal{P}\times\mathcal{S}\to\mathcal{P}\times_{\mathcal{G}}\mathcal{S}$ is a surjective submersion. One can show that $(\mathcal{P}\times\mathcal{S},\underline{p},\mathcal{P}\times_{\mathcal{G}}\mathcal{S},\mathcal{G})$ is a principal bundle with the principal right action $\tilde{r}$. Moreover, consider the projection $\tilde{p}:\mathcal{P}\times_{\mathcal{G}}\mathcal{S}\to\mathcal{M}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{P}\times\mathcal{S} \arrow[r, "\underline{p}"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{P}\times_{\mathcal{G}}\mathcal{S} \arrow[d, "\tilde{p}"] \\
\mathcal{P} \arrow[r, "p"] & \mathcal{M}
\end{tikzcd}$$

Then, $(\mathcal{P}\times_{\mathcal{G}}\mathcal{S},\tilde{p},\mathcal{M},\mathcal{S})$ is a fiber bundle, which is called the associated bundle to the principal bundle $\mathcal{P}$ with standard fiber $\mathcal{S}$.[6]

Let $\varrho:\mathcal{G}\to GL(V)$ be a representation of $\mathcal{G}$ on a finite-dimensional vector space V. As we mentioned earlier, ϱ introduces a left action of $\mathcal{G}$ on V. The associated bundle $(\mathcal{P}\times_{\mathcal{G}}V,\tilde{p},\mathcal{M},V)$ is a vector bundle. The linear structure of $\mathcal{P}\times_{\mathcal{G}}V$ is as follows: suppose $c_1,c_2\in\mathbb{R}$, $\mathbf{v}_1,\mathbf{v}_2\in V$, and $u_x\in\mathcal{P}$. Then, we have

$$\begin{aligned}
[\![u_x,c_1\mathbf{v}_1+c_2\mathbf{v}_2]\!] &= (u_x,c_1\mathbf{v}_1+c_2\mathbf{v}_2)\cdot\mathcal{G}=\left\{\big(u_x\cdot g,\varrho(g^{-1})(c_1\mathbf{v}_1+c_2\mathbf{v}_2)\big),\ \forall g\in\mathcal{G}\right\}\\
&= \left\{\big(u_x\cdot g,c_1\varrho(g^{-1})(\mathbf{v}_1)+c_2\varrho(g^{-1})(\mathbf{v}_2)\big),\ \forall g\in\mathcal{G}\right\}\\
&:= c_1[\![u_x,\mathbf{v}_1]\!]+c_2[\![u_x,\mathbf{v}_2]\!]. \qquad (213)
\end{aligned}$$

Let F be a functor as in §3.1.4.5. Then, $\tilde{\varrho}:\mathcal{G}\to GL(F(V))$ is also a representation of $\mathcal{G}$ and the corresponding associated bundle $\mathcal{P}\times_{\mathcal{G}}F(V)\to\mathcal{M}$ is identical to the fiber bundle $F(\mathcal{P}\times_{\mathcal{G}}V)\to\mathcal{M}$.

[6] Actually, an associated bundle is more than a fiber bundle; it is a $\mathcal{G}$-bundle, which means that it has an extra interaction with the structure group of $\mathcal{P}$, see [73] for more details.

3.1.4.8 Jets

Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. We say two smooth mappings $f : \mathcal{M} \to \mathcal{N}$ and $h : \mathcal{M} \to \mathcal{N}$ determine the same jet of order r (or r-jet) at $x \in \mathcal{M}$ if $f(x) = h(x)$ and in some local coordinates around x and $f(x)$, all partial derivatives of f and h up to order r are equal. Then, the chain rule implies that the partial derivatives are equal in all other charts as well. This defines an equivalence relation on the set of smooth mappings, whose classes are called r-jets at x and denoted by $j^r_x f$. The set of all r-jets is denoted by $J^r(\mathcal{M},\mathcal{N})$. The points x and $f(x)$ are called the source and the target of $j^r_x f$, respectively. The space of all r-jets with the source x is denoted by $J^r_x(\mathcal{M},\mathcal{N})$. Similarly, $J^r(\mathcal{M},\mathcal{N})_y$ denotes the space of all r-jets with the target y. We define $J^r_x(\mathcal{M},\mathcal{N})_y := J^r_x(\mathcal{M},\mathcal{N}) \cap J^r(\mathcal{M},\mathcal{N})_y$. For $0 \leq s < r$, we can define the projection map $\pi^r_s : J^r(\mathcal{M},\mathcal{N}) \to J^s(\mathcal{M},\mathcal{N})$ that sends an r-jet to its underlying s-jet. For $s = 0$, we define $\pi^r_0 : J^r(\mathcal{M},\mathcal{N}) \to \mathcal{M} \times \mathcal{N}$. The mapping $j^r f : \mathcal{M} \to J^r(\mathcal{M},\mathcal{N}),\ x \mapsto j^r_x f$, is called the r-jet prolongation of $f : \mathcal{M} \to \mathcal{N}$.

Let $j^r_x f \in J^r_x(\mathcal{M},\mathcal{N})_y$ and $j^r_y g \in J^r_y(\mathcal{N},\mathcal{Q})_z$. Then the composition of r-jets is defined as $(j^r_y g) \circ (j^r_x f) = j^r_x(g \circ f) \in J^r_x(\mathcal{M},\mathcal{Q})_z$. The chain rule implies that this composition is well-defined. An element $X \in J^r_x(\mathcal{M},\mathcal{N})_y$ is called invertible if there exists another element $Y \in J^r_y(\mathcal{N},\mathcal{M})_x$ such that $X \circ Y = j^r_y(\mathrm{Id}_{\mathcal{N}})$, and $Y \circ X = j^r_x(\mathrm{Id}_{\mathcal{M}})$. The existence of an invertible jet implies that $\dim \mathcal{M} = \dim \mathcal{N}$. One can show that $J^r(\mathcal{M},\mathcal{N})$ is a smooth manifold and $(J^r(\mathcal{M},\mathcal{N}), \pi^r_0, \mathcal{M} \times \mathcal{N})$ is a fibered manifold. Moreover, the mappings π^r_s for $0 \leq s < r$ are smooth. Let $\mathscr{M}f$ denote the category of manifolds and smooth mappings and consider a local diffeomorphism $\xi : \mathcal{M} \to \bar{\mathcal{M}}$ and a smooth mapping $\zeta : \mathcal{N} \to \bar{\mathcal{N}}$. The r-jet prolongation bifunctor is defined as $J^r : \mathscr{M}f_m \times \mathscr{M}f \to \mathscr{F}\mathscr{M}$ such that $(\mathcal{M},\mathcal{N}) \mapsto J^r(\mathcal{M},\mathcal{N})$, and $(\xi,\zeta) \mapsto J^r(\xi,\zeta) \in \mathscr{F}\mathscr{M}$, where

$$J^r(\xi,\zeta) : J^r(\mathcal{M},\mathcal{N}) \to J^r(\bar{\mathcal{M}},\bar{\mathcal{N}}),\ X \mapsto j^r_y \zeta \circ X \circ (j^r_x \xi)^{-1}, \tag{214}$$

with x and y being the source and the target of $X \in J^r(\mathcal{M}, \mathcal{N})$.

Let $(\mathcal{E}, p, \mathcal{M})$ be a fibered manifold. The subset of $J^r(\mathcal{M}, \mathcal{E})$ consisting of all r-jets of local sections of $\mathcal{E}$ is called the r-jet prolongation of $\mathcal{E}$ and is denoted by $J^r(\mathcal{E} \to \mathcal{M})$ or $J^r(\mathcal{E})$. It is possible to show that $J^r(\mathcal{E})$ is a smooth submanifold of $J^r(\mathcal{M}, \mathcal{E})$ and $J^r(\mathcal{E}) \to \mathcal{M}$ is a fibered manifold. One can show that $J^r()$ is a functor.[7] The r-jet prolongation of sections of the fibered manifold $\mathcal{E}$ can be considered as the mapping $j^r : \Gamma(\mathcal{E}) \to \Gamma(J^r(\mathcal{E}))$ given by $\boldsymbol{s} \mapsto j^r \boldsymbol{s}$, with $j^r \boldsymbol{s} : \mathcal{M} \to J^r(\mathcal{E}),\ x \mapsto j^r_x \boldsymbol{s}$. An operator $D : \Gamma(\mathcal{E} \to \mathcal{M}) \to \Gamma(\mathcal{E}' \to \mathcal{M})$ is of order $0 \le r \le \infty$ if $j^r_x \boldsymbol{s} = j^r_x \boldsymbol{r}$ implies that $D(\boldsymbol{s})(x) = D(\boldsymbol{r})(x)$. In this case, there exists a fiber-respecting mapping $\tilde{D} : J^r\mathcal{E} \to \mathcal{E}'$ given by $D(\boldsymbol{s})(x) = \tilde{D}(j^r_x \boldsymbol{s})$, that covers $\mathrm{Id}_{\mathcal{M}}$. A differential operator of order r is a smooth mapping $D : \Gamma(\mathcal{E} \to \mathcal{M}) \to \Gamma(\mathcal{E}' \to \mathcal{M})$ of order r. Let $\mathcal{V} \to \mathcal{M}$ and $\mathcal{V}' \to \mathcal{M}$ be vector bundles over a compact manifold. One can show that any linear operator $\Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$ is a linear differential operator [98]. Note that the associated operator $J^r\mathcal{V} \to \mathcal{V}'$ of such a linear operator is a homomorphism of vector bundles.

3.1.5 Derivatives on Vector Bundles

The main goal of this section is to introduce the Maurer-Cartan form. First we need to extend the notion of exterior derivative to bundle-valued forms. We also define Lie derivatives of sections of vector bundles.

Consider vector fields $\boldsymbol{X}, \boldsymbol{Z} \in \mathfrak{X}(\mathcal{M})$. Recall that the Lie bracket $[,] : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ is an $\mathbb{R}$-bilinear mapping, which is antisymmetric and satisfies the Jacobi identity (192). The Lie bracket is not linear over $C^\infty(\mathcal{M}, \mathbb{R})$ since $[f\boldsymbol{X}, \boldsymbol{Z}] = f[\boldsymbol{X}, \boldsymbol{Z}] - (\boldsymbol{Z}f)\boldsymbol{X}$, with $f \in C^\infty(\mathcal{M}, \mathbb{R})$. In a local chart (U, u) of $\mathcal{M}$ let $\boldsymbol{X} = X^i \frac{\partial}{\partial u^i}$ and $\boldsymbol{Z} = Z^i \frac{\partial}{\partial u^i}$. We have

$$[\boldsymbol{X}, \boldsymbol{Z}] = \left(X^i \frac{\partial Z^j}{\partial u^i} - Z^i \frac{\partial X^j}{\partial u^i} \right) \frac{\partial}{\partial u^j}.$$

[7]The functor $J^r()$ is different from the bifunctor $J^r(,)$, which was defined earlier.

The pair $(\mathfrak{X}(\mathcal{M}),[,])$ is a Lie algebra, where $\mathfrak{X}(\mathcal{M})$ is considered to be a vector space over $\mathbb{R}$ and not $C^\infty(\mathcal{M},\mathbb{R})$. Let $\varphi:\mathcal{M}\to\mathcal{N}$ be a local diffeomorphism[8] and $\boldsymbol{Y}_1,\boldsymbol{Y}_2\in\mathfrak{X}(\mathcal{N})$. Then we can write $\varphi^*[\boldsymbol{Y}_1,\boldsymbol{Y}_2]=[\varphi^*\boldsymbol{Y}_1,\varphi^*\boldsymbol{Y}_2]$, i.e. $\varphi^*:\mathfrak{X}(\mathcal{N})\to\mathfrak{X}(\mathcal{M})$ is a Lie algebra homomorphism. One can use natural vector bundles to define the pull-back of sections of vector bundles. Let F be a natural vector bundle and $\boldsymbol{\sigma}\in\Gamma(F(\mathcal{N}))$. The pull-back $\varphi^*\boldsymbol{\sigma}\in\Gamma(F(\mathcal{M}))$ is defined as

$$\varphi^*\boldsymbol{\sigma}=F(\varphi_\ell^{-1})\circ\boldsymbol{\sigma}\circ\varphi, \tag{215}$$

where φ_ℓ^{-1} is the local inverse of φ. In particular, one can use this definition to obtain the pull-back of tensors.

3.1.5.1 Lie Derivative

Consider a vector field $\boldsymbol{X}\in\mathfrak{X}(\mathcal{M})$. The mapping $\mathrm{Fl}_t^{\boldsymbol{X}}:=\mathrm{Fl}^{\boldsymbol{X}}(t,\cdot):\mathcal{M}\to\mathcal{M}$ is a local diffeomorphism. Let $f\in C^\infty(\mathcal{M},\mathbb{R})$. The Lie derivative of f is a smooth mapping $\mathcal{L}_{\boldsymbol{X}}f\in C^\infty(\mathcal{M},\mathbb{R})$ defined as

$$\mathcal{L}_{\boldsymbol{X}}f(x):=\frac{d}{dt}\Big|_{t=0}f\left(\mathrm{Fl}^{\boldsymbol{X}}(t,x)\right),$$

or equivalently

$$\mathcal{L}_{\boldsymbol{X}}f:=\frac{d}{dt}\Big|_{t=0}(\mathrm{Fl}_t^{\boldsymbol{X}})^*f=\frac{d}{dt}\Big|_{t=0}(f\circ\mathrm{Fl}_t^{\boldsymbol{X}}). \tag{216}$$

One can show that $\mathcal{L}_{\boldsymbol{X}}f=\boldsymbol{X}(f)$. Now suppose $\boldsymbol{Y}\in\mathfrak{X}(\mathcal{M})$. The Lie derivative of $\boldsymbol{Y}$ along $\boldsymbol{X}$, $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y}\in\mathfrak{X}(\mathcal{M})$, is given by

$$(\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y})f:=\frac{d}{dt}\Big|_{t=0}\left[\left((\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{Y}\right)f\right]=\frac{d}{dt}\Big|_{t=0}\left[\left(T(\mathrm{Fl}_{-t}^{\boldsymbol{X}})\circ\boldsymbol{Y}\circ\mathrm{Fl}_t^{\boldsymbol{X}}\right)f\right], \tag{217}$$

[8]Note that for a local diffeomorphism $\varphi:\mathcal{M}\to\mathcal{N}$, the pull-back $\varphi^*:\mathfrak{X}(\mathcal{N})\to\mathfrak{X}(\mathcal{M})$ is defined as $(\varphi^*\boldsymbol{Y})(x):=(T_x\varphi)^{-1}\cdot\boldsymbol{Y}(\varphi(x))$, where $x\in\mathcal{M}$ and $\boldsymbol{Y}\in\mathfrak{X}(\mathcal{N})$.

which can be written as

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} := \frac{d}{dt}\Big|_{t=0}(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{Y} = \frac{d}{dt}\Big|_{t=0}\Big(T(\mathrm{Fl}_{-t}^{\boldsymbol{X}})\circ\boldsymbol{Y}\circ\mathrm{Fl}_t^{\boldsymbol{X}}\Big). \tag{218}$$

We have $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} = [\boldsymbol{X},\boldsymbol{Y}]$. Next, we define the Lie derivative of sections of vector bundles. Suppose F is a natural vector bundle. Fixing t, the mapping $\mathrm{Fl}_t^{\boldsymbol{X}} : \mathcal{M}\to\mathcal{M}$ is a local diffeomorphism, and hence one can define the vector bundle isomorphism $F(\mathrm{Fl}_t^{\boldsymbol{X}}) : F(\mathcal{M})\to F(\mathcal{M})$ such that the following diagram commutes.

$$\begin{tikzcd} F(\mathcal{M}) \arrow[r, "F(\mathrm{Fl}_t^{\boldsymbol{X}})"] \arrow[d, "p_{\mathcal{M}}"] & F(\mathcal{M}) \arrow[d, "p_{\mathcal{M}}"] \\ \mathcal{M} \arrow[r, "\mathrm{Fl}_t^{\boldsymbol{X}}"] & \mathcal{M} \end{tikzcd}$$

Here F maps $\mathcal{M}$ to $(F(\mathcal{M}), p_{\mathcal{M}}, \mathcal{M}, V_{\mathcal{M}})$. The pull-back via $\mathrm{Fl}_t^{\boldsymbol{X}}$ of a section $\boldsymbol{s}\in\Gamma(F(\mathcal{M}))$ is defined as

$$(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s} := F(\mathrm{Fl}_{-t}^{\boldsymbol{X}})\circ\boldsymbol{s}\circ\mathrm{Fl}_t^{\boldsymbol{X}}\in\Gamma(F(\mathcal{M})). \tag{219}$$

For a fixed $x\in\mathcal{M}$, $t\mapsto(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s}(x)$ is a curve in the vector space $F(\mathcal{M})_x$, and therefore, the expression $\frac{d}{dt}|_{t=0}\big[(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s}(x)\big]$ is meaningful, i.e. it is the tangent vector to the curve $(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s}(x)$ at $\boldsymbol{s}(x)$. Here we are using the fact that a vector space is isomorphic to its tangent space, and thus we identify the vector space and its tangent space. Using the notation introduced after (198), by $\frac{d}{dt}|_{t=0}\big[(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s}(x)\big]$ we actually mean $(\bar{\imath})^{-1}(\frac{d}{dt}|_{t=0}\big[(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s}(x)\big])$. For sections of fiber bundles, one should consider $\frac{d}{dt}|_{t=0}\big[(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s}(x)\big]$ as an element of the vertical bundle (200), see [27]. The Lie derivative of $\boldsymbol{s}\in\Gamma(F(\mathcal{M})\to\mathcal{M})$ along $\boldsymbol{X}\in\mathfrak{X}(\mathcal{M})$ is given by

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{s} := \frac{d}{dt}\Big|_{t=0}\big[(\mathrm{Fl}_t^{\boldsymbol{X}})^*\boldsymbol{s}\big]. \tag{220}$$

Note that if $F(\mathcal{M}) = T\mathcal{M}$, the definition (220) coincides with (218).

3.1.5.2 Exterior Derivative

Let $\boldsymbol{\alpha} \in \Omega^k(\mathcal{M})$ and $\boldsymbol{X}_0, \dots, \boldsymbol{X}_k \in \mathfrak{X}(\mathcal{M})$. We define $\Omega^0(\mathcal{M}) := C^\infty(\mathcal{M}, \mathbb{R})$ and suppress the summation convention throughout this section unless stated otherwise. The exterior derivative $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ is defined as

$$\begin{aligned} d\boldsymbol{\alpha}(\boldsymbol{X}_0, \dots, \boldsymbol{X}_k) &= \sum_{i=0}^{k} (-1)^i \boldsymbol{X}_i \left(\boldsymbol{\alpha}(\boldsymbol{X}_0, \dots, \widehat{\boldsymbol{X}}_i, \dots, \boldsymbol{X}_k) \right) \\ &+ \sum_{i<j} (-1)^{i+j} \boldsymbol{\alpha}\left([\boldsymbol{X}_i, \boldsymbol{X}_j], \boldsymbol{X}_0, \dots, \widehat{\boldsymbol{X}}_i, \dots, \widehat{\boldsymbol{X}}_j, \dots, \boldsymbol{X}_k \right), \end{aligned} \tag{221}$$

where the hat over a vector field means the omission of that argument. To calculate the first summation, one needs to consider $\boldsymbol{X}_i$ as the mapping $\boldsymbol{X}_i : C^\infty(\mathcal{M}, \mathbb{R}) \to C^\infty(\mathcal{M}, \mathbb{R})$. Let V be a finite-dimensional vector space with a basis $\{\mathbf{v}_i\}$ and suppose $\boldsymbol{\gamma} \in \Omega^k(\mathcal{M}; V)$. One can also use (221) to define $d\boldsymbol{\gamma} \in \Omega^{k+1}(\mathcal{M}; V)$. In this case, we have $\boldsymbol{X}_i : C^\infty(\mathcal{M}, V) \to C^\infty(\mathcal{M}, V)$, $f^j \mathbf{v}_j \mapsto \boldsymbol{X}_i(f^j)\mathbf{v}_j$, where $f^j \in C^\infty(\mathcal{M}, \mathbb{R})$ and we use the summation convention on the index j.

Suppose in a local chart (U, u), $\boldsymbol{\alpha} \in \Omega^k(\mathcal{M})$ is expressed as in (209). One can show that

$$d\boldsymbol{\alpha} = \sum_{i_1 < i_2 < \dots < i_k} d\alpha_{i_1 \dots i_k} \wedge du^{i_1} \wedge \dots \wedge du^{i_k}. \tag{222}$$

Let $\varphi : \mathcal{M} \to \mathcal{N}$ and $\boldsymbol{\eta} \in \Omega^k(\mathcal{N})$. Using the general definition of pull-back given in (215), the pull-back $\varphi^* \boldsymbol{\eta} \in \Omega^k(\mathcal{M})$ is defined as

$$\varphi^* \boldsymbol{\eta}(x)(\boldsymbol{X}_1, \dots, \boldsymbol{X}_k) = \boldsymbol{\eta}(\varphi(x))(T_x\varphi \cdot \boldsymbol{X}_1, \dots, T_x\varphi \cdot \boldsymbol{X}_k), \tag{223}$$

where $x \in \mathcal{M}$. The exterior derivative commutes with the action of local diffeomorphisms, i.e. $\varphi^*(d\boldsymbol{\eta}) = d(\varphi^* \boldsymbol{\eta})$. A differential from $\boldsymbol{\alpha} \in \Omega^k(\mathcal{M})$ is closed if $d\boldsymbol{\alpha} = 0$, and is exact if $\boldsymbol{\alpha} = d\boldsymbol{\beta}$ for some $\boldsymbol{\beta} \in \Omega^{k-1}(\mathcal{M})$. Any exact form is closed, i.e. $d \circ d = 0$. The Poincaré lemma states that a closed differential form is locally exact.

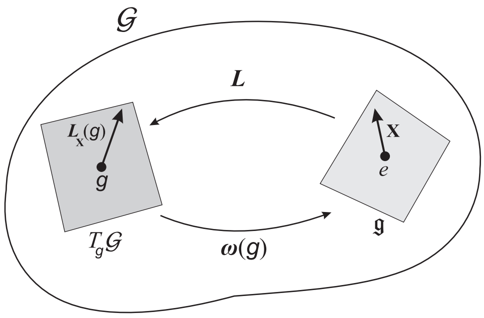


Figure 21: The Maurer-Cartan form $\boldsymbol{\omega} \in \Omega^1(\mathcal{G};\mathfrak{g})$ of a Lie group $\mathcal{G}$ defines the linear isomorphism $\boldsymbol{\omega}(g) : T_g\mathcal{G} \to \mathfrak{g}$.

3.1.5.3 The Maurer-Cartan Form

Using the notion of left invariant vector fields, one can obtain a trivialization for the tangent bundle of a Lie group $\mathcal{G}$. Recall that the mapping $\boldsymbol{L} : \mathfrak{g} \to \mathfrak{X}_L(\mathcal{G}),\ \mathbf{X} \mapsto \boldsymbol{L}_{\mathbf{X}}$, is an isomorphism of vector spaces. The mapping $\mathcal{G} \times \mathfrak{g} \to T\mathcal{G}$ given by $(g, \mathbf{X}) \mapsto \boldsymbol{L}_{\mathbf{X}}(g)$ is a vector bundle isomorphism covering $\mathrm{Id}_{\mathcal{G}}$. The inverse of this mapping can be considered as a $\mathfrak{g}$-valued one-form $\boldsymbol{\omega} \in \Omega^1(\mathcal{G};\mathfrak{g})$ defined as $\boldsymbol{\omega}(g)(\boldsymbol{\xi}) \coloneqq T_g\lambda_{g^{-1}} \cdot \boldsymbol{\xi}(g)$, see Fig. 21. For each $g \in \mathcal{G}$, the mapping $\boldsymbol{\omega}(g) : T_g\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism. This one-form is called the Maurer-Cartan form. We have $\boldsymbol{\omega}(\boldsymbol{L}_{\mathbf{X}}) = \mathbf{X}$, and $\lambda_g^*\boldsymbol{\omega} = \boldsymbol{\omega}$. Also we can write

$$\begin{aligned}
(\rho^g)^*\,\boldsymbol{\omega}(\hat{g})(\boldsymbol{\xi}\,(\hat{g})) &= \boldsymbol{\omega}(\hat{g}\cdot g)\,(T_{\hat{g}}\rho^g \cdot \boldsymbol{\xi}(\hat{g})) = T_{\hat{g}\cdot g}\lambda_{g^{-1}\cdot\hat{g}^{-1}} \circ T_{\hat{g}}\rho^g \cdot \boldsymbol{\xi}(\hat{g}) \\
&= T_g\lambda_{g^{-1}} \circ T_{\hat{g}\cdot g}\lambda_{\hat{g}^{-1}} \circ T_{\hat{g}}\rho^g \cdot \boldsymbol{\xi}(\hat{g}) \\
&= T_g\lambda_{g^{-1}} \circ T_e\rho^g \circ T_{\hat{g}}\lambda_{\hat{g}^{-1}} \circ \boldsymbol{\xi}(\hat{g}) \\
&= T_e\left(\lambda_{g^{-1}} \circ \rho^g\right) \circ T_{\hat{g}}\lambda_{\hat{g}^{-1}} \cdot \boldsymbol{\xi}(\hat{g}) \\
&= \mathrm{Ad}(g^{-1}) \circ \boldsymbol{\omega}(\hat{g})\,(\boldsymbol{\xi}(\hat{g}))\,. \qquad (224)
\end{aligned}$$

Thus, we obtain

$$(\rho^g)^*\,\boldsymbol{\omega} = \mathrm{Ad}(g^{-1}) \circ \boldsymbol{\omega}. \qquad (225)$$

Using the definition (221) and noting that the value of $\boldsymbol{\omega}$ on left invariant vector fields is constant, we can write

$$d\boldsymbol{\omega}(\boldsymbol{L}_{\mathbf{X}}, \boldsymbol{L}_{\mathbf{Y}}) = -\boldsymbol{\omega}([\boldsymbol{L}_{\mathbf{X}}, \boldsymbol{L}_{\mathbf{Y}}]), \ \forall \mathbf{X}, \mathbf{Y} \in \mathfrak{g}. \tag{226}$$

The definition of the Lie bracket of $\mathfrak{g}$ yields $\boldsymbol{\omega}([\boldsymbol{L}_{\mathbf{X}}, \boldsymbol{L}_{\mathbf{Y}}]) = [\mathbf{X}, \mathbf{Y}]$. Thus, (226) can be written as

$$d\boldsymbol{\omega}(\boldsymbol{\xi}, \boldsymbol{\eta}) + [\boldsymbol{\omega}(\boldsymbol{\xi}), \boldsymbol{\omega}(\boldsymbol{\eta})] = 0, \ \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}(\mathcal{G}). \tag{227}$$

This is called the Maurer-Cartan equation. Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\mathcal{M})$ and $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \Omega^1(\mathcal{M}; \mathfrak{v})$, where $\mathfrak{v}$ is a Lie algebra. The bracket $[\boldsymbol{\beta}_1, \boldsymbol{\beta}_2] \in \Omega^2(\mathcal{M}; \mathfrak{v})$ is defined as $[\boldsymbol{\beta}_1, \boldsymbol{\beta}_2](\boldsymbol{X}, \boldsymbol{Y}) := [\boldsymbol{\beta}_1(\boldsymbol{X}), \boldsymbol{\beta}_2(\boldsymbol{Y})] + [\boldsymbol{\beta}_2(\boldsymbol{X}), \boldsymbol{\beta}_1(\boldsymbol{Y})]$, see [95]. In particular, we have $[\boldsymbol{\beta}(\boldsymbol{X}), \boldsymbol{\beta}(\boldsymbol{Y})] = \frac{1}{2}[\boldsymbol{\beta}, \boldsymbol{\beta}](\boldsymbol{X}, \boldsymbol{Y})$. Thus, one can also write the Maurer-Cartan equation as

$$d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}] = 0. \tag{228}$$

3.1.6 Connections

In this section, we discuss various types of connections including general connections, principal connections, induced connections, and linear connections. Let $\mathcal{M}$ be a manifold. Recall that a distribution $\mathcal{D}$ on $\mathcal{M}$ is a subset $\mathcal{D} \subset T\mathcal{M}$ such that for each $x \in \mathcal{M}$, $\mathcal{D}_x := \mathcal{D} \cap T_x\mathcal{M}$ is a vector subspace of the vector space $T_x\mathcal{M}$. Note that $\mathcal{D}$ can be considered as the kernel of a one-form with values in a suitable vector space V, where $\dim V \geq \dim \mathcal{M} - \max_{x \in \mathcal{M}}\{\dim \mathcal{D}_x\}$. A distribution $\mathcal{D}$ is of constant rank k if $\dim \mathcal{D}_x = k$, $\forall x \in \mathcal{M}$. A distribution is smooth if it is the kernel of a smooth one-form. An integral manifold $\mathcal{I} \subset \mathcal{M}$ of $\mathcal{D}$ is a submanifold of $\mathcal{M}$ such that $T_x\mathcal{I} \subset \mathcal{D}_x$ for all $x \in \mathcal{I}$. A maximal integral manifold is an integral manifold such that $T_x\mathcal{I} = \mathcal{D}_x$ for all $x \in \mathcal{I}$. A distribution is integrable if there is a maximal integral manifold through each point $x \in \mathcal{M}$. A distribution $\mathcal{D}$ is called involutive if for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}(\mathcal{M})$ with

$\boldsymbol{\xi}(x), \boldsymbol{\eta}(x) \in \mathcal{D}_x$, $\forall x \in \mathcal{M}$, we have $[\boldsymbol{\xi}, \boldsymbol{\eta}](x) \in \mathcal{D}_x$. The Frobenius theorem states that a smooth constant-rank distribution is integrable if and only if it is involutive.

A vector sub-bundle $(\mathcal{W}, p, \mathcal{M})$ of a vector bundle $(\mathcal{V}, p, \mathcal{M})$ is a vector bundle for which there exists a vector bundle homomorphism $\tau : \mathcal{W} \to \mathcal{V}$ that covers $\mathrm{Id}_{\mathcal{M}}$ and $\tau_x : \mathcal{W}_x \to \mathcal{V}_x$ is a linear embedding. Let $\phi : \mathcal{V} \to \mathcal{V}'$ be a vector bundle homomorphism between $(\mathcal{V}, p, \mathcal{M})$ and $(\mathcal{V}', p', \mathcal{M}')$ such that the rank of $\phi_x := \phi|_{\mathcal{V}_x}$ is constant for all $x \in \mathcal{M}$. Then, $\mathsf{Ker}(\phi)$ is a vector sub-bundle of $\mathcal{V}$, where $(\mathsf{Ker}(\phi))_x := \mathsf{Ker}(\phi_x)$. In particular, suppose $(\mathcal{E}, p, \mathcal{M}, \mathcal{S})$ is a fiber bundle. Since the mapping p is a submersion, the rank of Tp is constant, and hence the vertical bundle $(V\mathcal{E} := \mathsf{Ker}(Tp), \pi_{\mathcal{E}}, \mathcal{E})$ is a vector sub-bundle of $(T\mathcal{E}, \pi_{\mathcal{E}}, \mathcal{E})$. Note that $V\mathcal{E}$ is also a smooth constant-rank distribution of $T\mathcal{E}$. Let $\boldsymbol{\psi}, \boldsymbol{\lambda} \in \mathfrak{X}(\mathcal{E})$ with $\boldsymbol{\psi}(u), \boldsymbol{\lambda}(u) \in V\mathcal{E}$, $\forall u \in \mathcal{E}$. We have $Tp \cdot [\boldsymbol{\psi}, \boldsymbol{\lambda}] = [Tp \cdot \boldsymbol{\psi}, Tp \cdot \boldsymbol{\lambda}] = 0$.[9] This means that $[\boldsymbol{\psi}, \boldsymbol{\lambda}] \in V\mathcal{E}$, i.e. $V\mathcal{E}$ is involutive. Therefore, $V\mathcal{E}$ is integrable as a consequence of the Frobenius theorem.

Let $\boldsymbol{K} \in \Omega^k(\mathcal{M}; T\mathcal{M})$, $\boldsymbol{L} \in \Omega^l(\mathcal{M}; T\mathcal{M})$, and $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\mathcal{M})$. The Frölicher-Nijenhuis bracket $[\boldsymbol{K}, \boldsymbol{L}]$ is defined such that $[\boldsymbol{K}, \boldsymbol{L}] \in \Omega^{k+l}(\mathcal{M}; T\mathcal{M})$. For $k = l = 1$, $[\boldsymbol{K}, \boldsymbol{L}] \in \Omega^2(\mathcal{M}; T\mathcal{M})$ is given by

$$\begin{aligned}
[\boldsymbol{K}, \boldsymbol{L}](\boldsymbol{X}, \boldsymbol{Y}) &= [\boldsymbol{K}(\boldsymbol{X}), \boldsymbol{L}(\boldsymbol{Y})] - [\boldsymbol{K}(\boldsymbol{Y}), \boldsymbol{L}(\boldsymbol{X})] \\
&- \boldsymbol{L}\big([\boldsymbol{K}(\boldsymbol{X}), \boldsymbol{Y}] - [\boldsymbol{K}(\boldsymbol{Y}), \boldsymbol{X}]\big) - \boldsymbol{K}\big([\boldsymbol{L}(\boldsymbol{X}), \boldsymbol{Y}] - [\boldsymbol{L}(\boldsymbol{Y}), \boldsymbol{X}]\big) \\
&+ (\boldsymbol{L} \circ \boldsymbol{K} + \boldsymbol{K} \circ \boldsymbol{L})\,[\boldsymbol{X}, \boldsymbol{Y}]. \qquad (229)
\end{aligned}$$

See [73] for the general definition. Let $\boldsymbol{P} \in \Omega^1(\mathcal{M}; T\mathcal{M})$ be a projection in each fiber of $T\mathcal{M}$, i.e. $\boldsymbol{P} \circ \boldsymbol{P} = \boldsymbol{P}$. The spaces $\mathsf{Ker}(\boldsymbol{P})$ and $\mathsf{Im}(\boldsymbol{P})$ are called the horizontal space and the vertical space of $\boldsymbol{P}$, respectively. If the rank of $\boldsymbol{P}$ is constant, both are

[9] Note that this relation is valid although p is not a local diffeomorphism. For a smooth function f, the Lie brackets of f-related vector fields are f-related, see [73].

vector sub-bundles of $T\mathcal{M}$. Substituting $\boldsymbol{P}$ in (229) yields

$$[\boldsymbol{P},\boldsymbol{P}] = 2\boldsymbol{R} + 2\bar{\boldsymbol{R}}, \tag{230}$$

where $\boldsymbol{R},\bar{\boldsymbol{R}} \in \Omega^2(\mathcal{M};T\mathcal{M})$ are given by

$$\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{P}\big(\,[(\mathrm{Id}_{T\mathcal{M}} - \boldsymbol{P})(\boldsymbol{X}),(\mathrm{Id}_{T\mathcal{M}} - \boldsymbol{P})(\boldsymbol{Y})]\,\big), \tag{231}$$

and

$$\bar{\boldsymbol{R}}(\boldsymbol{X},\boldsymbol{Y}) = (\mathrm{Id}_{T\mathcal{M}} - \boldsymbol{P})\big([\boldsymbol{P}(\boldsymbol{X}),\boldsymbol{P}(\boldsymbol{Y})]\big). \tag{232}$$

The vector-valued form $\boldsymbol{R}$ is called the curvature and $\bar{\boldsymbol{R}}$ is called the co-curvature of $\boldsymbol{P}$. For a constant-rank projection $\boldsymbol{P}$, $\boldsymbol{R}$ and $\bar{\boldsymbol{R}}$ are the obstructions to integrability of $\mathsf{Ker}(\boldsymbol{P})$ and $\mathsf{Im}(\boldsymbol{P})$, respectively. The Bianchi identity reads

$$[\boldsymbol{P},\boldsymbol{R} + \bar{\boldsymbol{R}}] = 0. \tag{233}$$

As we will see later, connections are special cases of vector-valued 1-forms that are also projections in the sense that we mentioned above.

3.1.6.1 General Connections

Let $(\mathcal{E},p,\mathcal{M},\mathcal{S})$ be a fiber bundle and $V\mathcal{E} := \mathsf{Ker}(Tp)$ be the vertical bundle of $\mathcal{E}$. A general connection on $(\mathcal{E},p,\mathcal{M},\mathcal{S})$ is a vector-valued one-form $\boldsymbol{\Phi} \in \Omega^1(\mathcal{E};V\mathcal{E})$ such that $\mathsf{Im}(\boldsymbol{\Phi}) = V\mathcal{E}$ and $\boldsymbol{\Phi}\circ\boldsymbol{\Phi} = \boldsymbol{\Phi}$, i.e. a connection is a projection $T\mathcal{E} \to V\mathcal{E}$. As we mentioned earlier, the vertical bundle $V\mathcal{E}$ is a vector sub-bundle of $T\mathcal{M}$. On the other hand, since $H\mathcal{E} := \mathsf{Ker}(\boldsymbol{\Phi})$ is of constant rank, it is another vector sub-bundle of $T\mathcal{E}$ that is called the horizontal bundle of $\mathcal{E}$ associated to the connection $\boldsymbol{\Phi}$. Elementary linear algebra tells us that $T_\epsilon\mathcal{E} = H_\epsilon\mathcal{E} \oplus V_\epsilon\mathcal{E}$ for $\epsilon \in \mathcal{E}$, i.e. $T\mathcal{E} = H\mathcal{E} \oplus V\mathcal{E}$.

Suppose $\{(W_\alpha,w_\alpha)\}$ is an atlas for $\mathcal{S}$ and $\{(U_\alpha,\psi_\alpha)\}$ is a fiber bundle atlas for $\mathcal{E}$

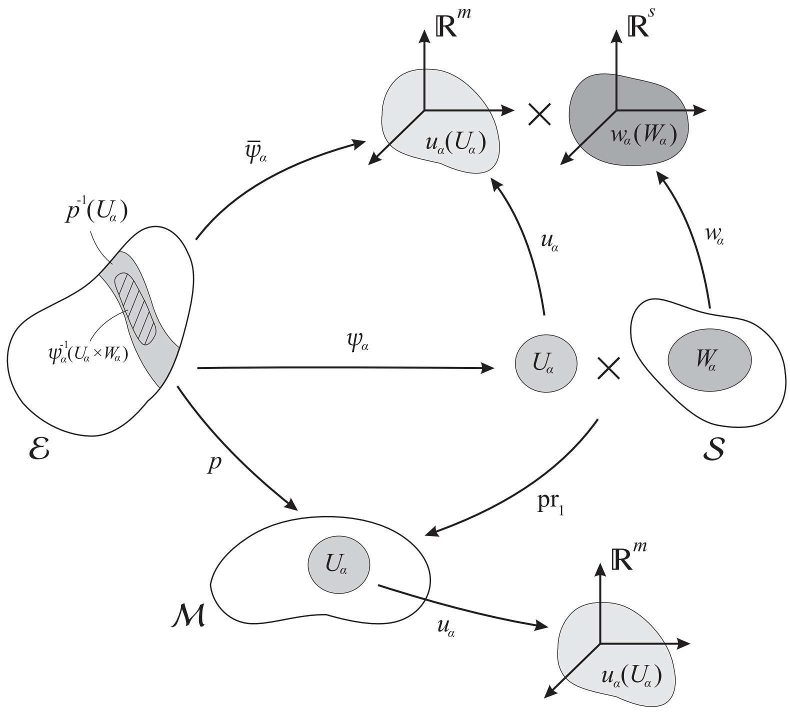


Figure 22: Local charts (W_α, w_α) of $\mathcal{S}$ and (U_α, u_α) of $\mathcal{M}$ such that (U_α, ψ_α) is a fiber bundle chart for a fiber bundle $(\mathcal{E}, p, \mathcal{M}, \mathcal{S})$ induce the chart $(\psi_\alpha^{-1}(U_\alpha \times W_\alpha), \overline{\psi}_\alpha)$ on $\mathcal{E}$.

such that $\{(U_\alpha, u_\alpha)\}$ is an atlas for $\mathcal{M}$. Similar to §3.1.4.4, we can obtain an atlas $\{(\psi_\alpha^{-1}(U_\alpha \times W_\alpha), \overline{\psi}_\alpha)\}$ for $\mathcal{E}$ given by

$$\begin{aligned} \overline{\psi}_\alpha : \psi_\alpha^{-1}(U_\alpha \times W_\alpha) \subset p^{-1}(U_\alpha) &\longrightarrow u_\alpha(U_\alpha) \times w_\alpha(W_\alpha) \subset \mathbb{R}^{m+s}, \\ z &\longmapsto (u_\alpha \circ \mathrm{pr}_1 \circ \psi_\alpha(z), w_\alpha \circ \mathrm{pr}_2 \circ \psi_\alpha(z)), \quad (234) \end{aligned}$$

where $m = \dim \mathcal{M}$ and $s = \dim \mathcal{S}$, see Fig. 22. Using (234), we obtain the atlas $\left\{\left(\pi_{\mathcal{E}}^{-1}\left(\psi_\alpha^{-1}(U_\alpha \times W_\alpha)\right), T\overline{\psi}_\alpha\right)\right\}$ with the charts $T\overline{\psi}_\alpha : \pi_{\mathcal{E}}^{-1}\left(\psi_\alpha^{-1}(U_\alpha \times W_\alpha)\right) \to u_\alpha(U_\alpha) \times w_\alpha(W_\alpha) \times \mathbb{R}^m \times \mathbb{R}^s$ for $(T\mathcal{E}, \pi_{\mathcal{E}}, \mathcal{E})$. Consider a point $\epsilon = \psi_\alpha^{-1}(x, \sigma) \in \psi_\alpha^{-1}(U_\alpha \times W_\alpha)$, with $\overline{\psi}_\alpha(\epsilon) = (\hat{u}^1, \ldots, \hat{u}^m, \hat{w}^1, \ldots, \hat{w}^s)$, and the curves

$$\alpha_i(t) = \left(\overline{\psi}_\alpha\right)^{-1}(\hat{u}^1, \ldots, \hat{u}^i + t, \ldots, \hat{u}^m, \hat{w}^1, \ldots, \hat{w}^s) \text{ for } i = 1, \ldots, m, \qquad (235)$$

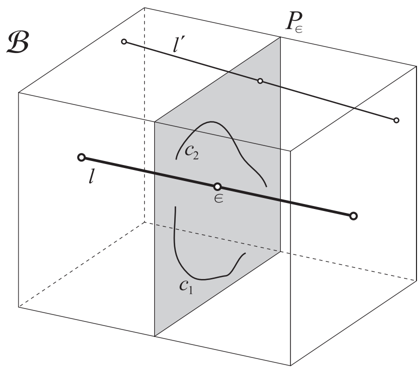


Figure 23: A box $\mathcal{B}$ can be considered as a fiber bundle with the line l as the base space and the "vertical" planes P_ϵ as the fibers, where ϵ is the intersection point of the plane and l. The tangent vectors to curves c_1 and c_2 that remain on a single "vertical" plane belong to $V\mathcal{B}$. Any curve that moves between the fibers such as the line l', has a tangent vector in $H\mathcal{B}$. Note that this is valid for any connection on $\mathcal{B}$.

and

$$\beta_j(t) = \left(\overline{\psi}_\alpha\right)^{-1}(\hat{u}^1, \dots, \hat{u}^m, \hat{w}^1, \dots, \hat{w}^j + t, \dots, \hat{w}^s) \text{ for } j = 1, \dots, s. \tag{236}$$

Note that $p\left(\beta_j(t)\right) = x$, i.e. the curves $\beta_j(t)$ remain in a single fiber of $\mathcal{E}$. We define $\frac{\partial}{\partial \hat{u}^i} := \alpha_i'(0)$ and $\frac{\partial}{\partial \hat{w}^j} := \beta_j'(0)$. The set $\{\frac{\partial}{\partial \hat{u}^1}, \dots, \frac{\partial}{\partial \hat{u}^m}, \frac{\partial}{\partial \hat{w}^1}, \dots, \frac{\partial}{\partial \hat{w}^s}\}$ is a basis for $T_\epsilon \mathcal{E}$. The local expression for $p : \mathcal{E} \to \mathcal{M}$ in local charts $(\psi_\alpha^{-1}(U_\alpha \times W_\alpha), \overline{\psi}_\alpha)$ and (U_α, u_α) is given by $p(\hat{u}^1, \dots, \hat{u}^m, \hat{w}^1, \dots, \hat{w}^s) = (\hat{u}^1, \dots, \hat{u}^m)$, which yields $T_\epsilon p \cdot \frac{\partial}{\partial \hat{u}^i} = \frac{\partial}{\partial u^i}$, and $T_\epsilon p \cdot \frac{\partial}{\partial \hat{w}^j} = 0$. Thus, $\{\frac{\partial}{\partial \hat{w}^1}, \dots, \frac{\partial}{\partial \hat{w}^s}\}$ is a basis for $V_\epsilon \mathcal{E}$. The tangent vectors to any curve that stay in a single fiber of $\mathcal{E}$ belong to $V\mathcal{E}$. A tangent vector $c'(0)$ that belongs to $H\mathcal{E}$ indicates that the curve $c(t)$ does not lay on a single fiber of $\mathcal{E}$, where c is a curve in $\mathcal{E}$, see Fig. 23.

Consider the mapping $(Tp, \pi_{\mathcal{E}}) : T\mathcal{E} \to T\mathcal{M}\,{}_{\mathcal{M}}\!\times \mathcal{E}$, where $T\mathcal{M}\,{}_{\mathcal{M}}\!\times \mathcal{E} := \{(\xi, \epsilon) \in T\mathcal{M} \times \mathcal{E} : \pi_{\mathcal{M}}(\xi) = p(\epsilon)\}$. One can show that $(Tp, \pi_{\mathcal{E}})|_{H\mathcal{E}} : H\mathcal{E} \to T\mathcal{M}\,{}_{\mathcal{M}}\!\times \mathcal{E}$ is a fiber-linear isomorphism. The horizontal lift C associated to the connection $\mathbf{\Phi}$ is defined as

$$C := \left((Tp, \pi_{\mathcal{E}})|_{H\mathcal{E}}\right)^{-1} : T\mathcal{M}\,{}_{\mathcal{M}}\!\times \mathcal{E} \to H\mathcal{E}. \tag{237}$$

The mapping $\chi := \mathrm{Id}_{T\mathcal{E}} - \mathbf{\Phi} = C \circ (Tp, \pi_{\mathcal{E}})$ is called the horizontal projection. The connection $\mathbf{\Phi}$ is also called the vertical projection. Since $V\mathcal{E}$ is integrable, the co-curvature of the connection $\mathbf{\Phi} \in \Omega^1(\mathcal{E}; V\mathcal{E})$ vanishes and therefore, (230) and (231) yield

$$\boldsymbol{R} = \frac{1}{2}[\mathbf{\Phi}, \mathbf{\Phi}] = \mathbf{\Phi}[\chi, \chi] \in \Omega^2(\mathcal{E}; V\mathcal{E}). \tag{238}$$

Let $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}(\mathcal{E})$. Equation (238) states that $\boldsymbol{R}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{2}[\mathbf{\Phi}, \mathbf{\Phi}](\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{\Phi}[\chi(\boldsymbol{\xi}), \chi(\boldsymbol{\eta})]$. Note that as we mentioned earlier, $\boldsymbol{R}$ is an obstruction to integrability of $H\mathcal{E}$. The horizontal lift of $\boldsymbol{X} \in \mathfrak{X}(\mathcal{M})$, $C\boldsymbol{X}$, is defined as $C\boldsymbol{X}(\epsilon) \coloneqq C\left(\boldsymbol{X}\left(p(\epsilon)\right), \epsilon\right) \in H_\epsilon\mathcal{E}$. We have $\boldsymbol{R}(C\boldsymbol{X}, C\boldsymbol{Y}) = [C\boldsymbol{X}, C\boldsymbol{Y}] - C([\boldsymbol{X}, \boldsymbol{Y}])$. Also using (233), the Bianchi identity for a connection $\mathbf{\Phi}$ is given by

$$[\mathbf{\Phi}, \boldsymbol{R}] = 0. \tag{239}$$

Let $(\mathcal{E}, p, \mathcal{M}, \mathcal{S})$ be a fiber bundle and consider a smooth mapping $f : \mathcal{N} \to \mathcal{M}$. The set $\mathcal{N}_{\mathcal{M}} \times \mathcal{E} \coloneqq \{(y, \epsilon) \in \mathcal{N} \times \mathcal{E} : \ f(y) = p(\epsilon)\}$ is a submanifold of $\mathcal{N} \times \mathcal{E}$. The manifold $\mathcal{N}_{\mathcal{M}} \times \mathcal{E}$ is called the pull-back of the fiber bundle $\mathcal{E}$ by f and is denoted by $f^*\mathcal{E}$. We define $f^*p \coloneqq \mathrm{pr}_1 : \mathcal{N}_{\mathcal{M}} \times \mathcal{E} \to \mathcal{N}$, and $p^*f \coloneqq \mathrm{pr}_2 : \mathcal{N}_{\mathcal{M}} \times \mathcal{E} \to \mathcal{E}$. The following diagram commutes.

$$\begin{tikzcd}
f^*\mathcal{E} \arrow[r, "p^*f"] \arrow[d, "f^*p"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{N} \arrow[r, "f"] & \mathcal{M}
\end{tikzcd}$$

One can show that $(f^*\mathcal{E}, f^*p, \mathcal{N}, \mathcal{S})$ is a fiber bundle[10] and p^*f is a fiber-wise diffeomorphism. Note that $(f^*\mathcal{E})_y = (y, \mathcal{E}_{f(y)})$. Let $\mathbf{Z} \in T_{(y,\epsilon)}(f^*\mathcal{E})$ and $\mathbf{\Phi} \in \Omega^1(\mathcal{E}; V\mathcal{E})$ be a connection of $\mathcal{E}$. Then, $f^*\mathbf{\Phi} \in \Omega^1(f^*\mathcal{E}; Vf^*\mathcal{E})$ given by $(f^*\mathbf{\Phi})_{(y,\epsilon)}\mathbf{Z} = T_\epsilon(p^*f)^{-1} \cdot \mathbf{\Phi} \cdot T_{(y,\epsilon)}(p^*f) \cdot \mathbf{Z}$, is a connection on $f^*\mathcal{E}$.

Let $\{(U_\alpha, \psi_\alpha)\}$ be a fiber bundle atlas for $\mathcal{E}$ and $\epsilon = \psi_\alpha^{-1}(x, \sigma) \in \mathcal{E}$. We define

[10] One should not confuse the pull-back of a fiber bundle defined here with the pull-back of a section of a fiber bundle given in (215).

$(\psi_\alpha^{-1})^*\mathbf{\Phi} \in \Omega^1(U_\alpha \times \mathcal{S}; U_\alpha \times T\mathcal{S})$ as

$$(\psi_\alpha^{-1})^*\mathbf{\Phi}(\mathbf{X}_x, \mathbf{S}_\sigma) := T_\epsilon\psi_\alpha \cdot \mathbf{\Phi} \cdot T_{(x,\sigma)}\psi_\alpha^{-1} \cdot (\mathbf{X}_x, \mathbf{S}_\sigma) = \mathbf{S}_\sigma - \Gamma^\alpha(\mathbf{X}_x, \sigma), \tag{240}$$

where $\mathbf{X}_x \in T_x\mathcal{M}$, $\mathbf{S}_\sigma \in T_\sigma\mathcal{S}$, and

$$(0_x, \Gamma^\alpha(\mathbf{X}_x, \sigma)) := -T_\epsilon\psi_\alpha \cdot \mathbf{\Phi} \cdot T_{(x,\sigma)}\psi_\alpha^{-1} \cdot (\mathbf{X}_x, 0_\sigma). \tag{241}$$

One can assume $\Gamma^\alpha \in \Omega^1(U_\alpha; \mathfrak{X}(\mathcal{S}))$. The forms Γ^α are called the Christoffel forms of the connection $\mathbf{\Phi}$ with respect to the bundle atlas $\{(U_\alpha, \psi_\alpha)\}$.

Let $(a,b) \subset \mathbb{R}$ be an open interval containing zero and let $c : (a,b) \to \mathcal{M}$ be a smooth curve with $c(0) = x$. For any $\epsilon \in \mathcal{E}_x$, there exists a unique maximal subinterval $(a', b') \subset (a,b)$ containing zero and a unique curve $\tilde{c}_\epsilon : (a', b') \to \mathcal{E}$ such that $\tilde{c}_\epsilon(0) = \epsilon$, $p(\tilde{c}_\epsilon(t)) = c(t)$, and $\mathbf{\Phi}(\tilde{c}_\epsilon'(t)) = 0$, i.e. $\tilde{c}_\epsilon'(t) \in H\mathcal{E}$. The curve $\tilde{c}_\epsilon(t)$ defines the parallel transport of ϵ along $\tilde{c}_\epsilon$.

3.1.6.2 Principal Connections

Let $(\mathcal{P}, p, \mathcal{M}, \mathcal{G})$ be a principal fiber bundle with the principal right action r and let $\mathbf{\Phi} \in \Omega^1(T\mathcal{P}; V\mathcal{P})$ be a general connection. From §3.1.4.6 recall that $(T\mathcal{P}, Tp, T\mathcal{M}, T\mathcal{G})$ is a principal bundle with the principal right action Tr. On the other hand, $(V\mathcal{P}, p \circ \pi_\mathcal{P}, \mathcal{M}, T\mathcal{G})$ is also a fiber bundle with the principal right action $Tr|_{V\mathcal{P}} : V\mathcal{P} \times T\mathcal{G} \to V\mathcal{P}$. A principal connection $\mathbf{\Phi}$ is a general connection that is also $\mathcal{G}$-equivariant for the principal right action r, i.e. if $\mathbf{\Phi} \circ Tr^g = Tr^g \circ \mathbf{\Phi}$, or equivalently $(r^g)^*\mathbf{\Phi} = \mathbf{\Phi}$, $\forall g \in \mathcal{G}$. The vertical bundle $V\mathcal{P} \to \mathcal{P}$ is trivial as $\mathcal{P} \times \mathfrak{g} \to \mathcal{P}$, i.e. $\mathcal{P} \times \mathfrak{g} \approx V\mathcal{P}$ via the vector bundle isomorphism $(z, \mathbf{X}) \mapsto T_e r_z \cdot \mathbf{X} = \boldsymbol{\zeta}_\mathbf{X}(z)$, where $z \in \mathcal{P}$, $\mathbf{X} \in \mathfrak{g}$, and $\boldsymbol{\zeta} : \mathfrak{g} \to \mathfrak{X}(\mathcal{P})$ is the fundamental vector field mapping defined in (193). Define

$\boldsymbol{\gamma} \in \Omega^1(\mathcal{P}; \mathfrak{g})$ such that $\boldsymbol{\gamma}(\boldsymbol{X}(z)) = T_z(r_z)^{-1} \cdot \boldsymbol{\Phi}(\boldsymbol{X}(z))$, or equivalently

$$\boldsymbol{\Phi}(\boldsymbol{X}(z)) = T_e r_z \cdot \boldsymbol{\gamma}(\boldsymbol{X}(z)) = \boldsymbol{\zeta}_{\boldsymbol{\gamma}(\boldsymbol{X}(z))}(z), \tag{242}$$

where $\boldsymbol{X} \in \mathfrak{X}(\mathcal{P})$. The one-form $\boldsymbol{\gamma} \in \Omega^1(\mathcal{P}; \mathfrak{g})$ is called the connection form of the principal connection $\boldsymbol{\Phi}$. Since $\boldsymbol{\zeta}_{\mathbf{Y}}(z) \in V\mathcal{P}$ for all $\mathbf{Y} \in \mathfrak{g}$, we obtain $\boldsymbol{\gamma}(\boldsymbol{\zeta}_{\mathbf{Y}}(z)) = T_e(r_z)^{-1} \cdot \boldsymbol{\Phi}(\boldsymbol{\zeta}_{\mathbf{Y}}(z)) = T_e(r_z)^{-1} \cdot \boldsymbol{\zeta}_{\mathbf{Y}}(z) = \mathbf{Y}$. Moreover, the connection form $\boldsymbol{\gamma}$ is $\mathcal{G}$-equivariant, i.e.

$$\left((r^g)^* \boldsymbol{\gamma}\right)(\boldsymbol{X}(z)) = \boldsymbol{\gamma}(T_z r^g \cdot \boldsymbol{X}(z)) = \mathrm{Ad}(g^{-1})(\boldsymbol{\gamma}(\boldsymbol{X}(z))). \tag{243}$$

Conversely, a one-form $\boldsymbol{\gamma} \in \Omega^1(\mathcal{P}; \mathfrak{g})$ satisfying $\boldsymbol{\gamma}(\boldsymbol{\zeta}_{\mathbf{Y}}) = \mathbf{Y}$, defines a connection $\boldsymbol{\Phi}$ on $\mathcal{P}$ by (242), that is also a principal connection if and only if (243) is satisfied.

The curvature $\boldsymbol{R}$ of a principal connection $\boldsymbol{\gamma}$ has vertical values, and thus, one can define the curvature form $\boldsymbol{\rho} \in \Omega^2(\mathcal{P}; \mathfrak{g})$ such that $\boldsymbol{\rho}(\boldsymbol{X}(z), \boldsymbol{Y}(z)) = -T_z(r_z)^{-1} \cdot \boldsymbol{R}(\boldsymbol{X}(z), \boldsymbol{Y}(z))$, or $\boldsymbol{R}(\boldsymbol{X}(z), \boldsymbol{Y}(z)) = -\boldsymbol{\zeta}_{\boldsymbol{\rho}(\boldsymbol{X}(z), \boldsymbol{Y}(z))}(z)$, where $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\mathcal{P})$. The curvature form $\boldsymbol{\rho}$ is $\mathcal{G}$-equivariant in the sense that $(r^g)^* \boldsymbol{\rho} = \mathrm{Ad}(g^{-1}) \circ \boldsymbol{\rho}$. Also the following Maurer-Cartan formula holds:

$$\boldsymbol{\rho}(\boldsymbol{X}, \boldsymbol{Y}) = d\boldsymbol{\gamma}(\boldsymbol{X}, \boldsymbol{Y}) + [\boldsymbol{\gamma}(\boldsymbol{X}), \boldsymbol{\gamma}(\boldsymbol{Y})]. \tag{244}$$

3.1.6.3 Induced Connections

Let $(\mathcal{P}, p, \mathcal{M}, \mathcal{G})$ be a principal bundle and let $\mathcal{S}$ be a manifold with a left action $\ell : \mathcal{G} \times \mathcal{S} \to \mathcal{S}$. From §3.1.4.6 recall that $(T\mathcal{P}, Tp, T\mathcal{M}, T\mathcal{G})$ is a principal $T\mathcal{G}$-bundle. The mapping $T\ell : T\mathcal{G} \times T\mathcal{S} \to T\mathcal{S}$ induces a left action of $T\mathcal{G}$ on $T\mathcal{S}$. As we mentioned in §3.1.4.7, $(\mathcal{P} \times \mathcal{S}, \underline{p}, \mathcal{P} \times_{\mathcal{G}} \mathcal{S}, \mathcal{G})$ is a principal $\mathcal{G}$-bundle and $(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}, \tilde{p}, \mathcal{M}, \mathcal{S})$ is a fiber bundle. Using $T\underline{p} : T\mathcal{P} \times T\mathcal{S} \to T(\mathcal{P} \times_{\mathcal{G}} \mathcal{S})$, one can show that $T(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}) = T\mathcal{P} \times_{T\mathcal{G}} T\mathcal{S}$. By embedding $\mathcal{P}$ into $T\mathcal{P}$ and $\mathcal{G}$ into $T\mathcal{G}$ as the zero sections, the restriction of $T\underline{p}$

to $\mathcal{P} \times T\mathcal{S}$ implies that $V(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}) = \mathcal{P} \times_{\mathcal{G}} T\mathcal{S}$, where $V(\mathcal{P} \times_{\mathcal{G}} \mathcal{S})$ is the vertical bundle of $\mathcal{P} \times_{\mathcal{G}} \mathcal{S}$. Let $\mathbf{\Phi} \in \Omega^1(\mathcal{P}; V\mathcal{P})$ be a principal connection of $\mathcal{P}$. One can define $\bar{\mathbf{\Phi}} \in \Omega^1(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}; T(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}))$ such that the following diagram commutes.

$$\begin{tikzcd}
T\mathcal{P} \times T\mathcal{S} \arrow[r, "\mathbf{\Phi} \times \mathrm{Id}"] \arrow[d, "T\underline{p}"] & T\mathcal{P} \times T\mathcal{S} \arrow[d, "T\underline{p}"] \\
T\mathcal{P} \times_{T\mathcal{G}} T\mathcal{S} \arrow[r, "\bar{\mathbf{\Phi}}"] & T\mathcal{P} \times_{T\mathcal{G}} T\mathcal{S}
\end{tikzcd}$$

We have $\bar{\mathbf{\Phi}} \circ \bar{\mathbf{\Phi}} = \bar{\mathbf{\Phi}}$ and $\mathsf{Im}(\bar{\mathbf{\Phi}}) = V(\mathcal{P} \times_{\mathcal{G}} \mathcal{S})$. Thus, $\bar{\mathbf{\Phi}}$ is a connection on $\mathcal{P} \times_{\mathcal{G}} \mathcal{S}$, which is called the induced connection.

Let $\phi : \mathcal{G} \times W \to W$ be a representation of the structure group $\mathcal{G}$ on a finite-dimensional vector space W. As mentioned in §3.1.4.7, $(\mathcal{W} = \mathcal{P} \times_{\mathcal{G}} W, \tilde{p}, \mathcal{M}, W)$ is a vector bundle. From §3.1.4.4 recall that $T\mathcal{W} = T\mathcal{P} \times_{T\mathcal{G}} TW \approx T\mathcal{P} \times_{T\mathcal{G}} (W \times W)$ can be considered as the vector bundles $(T\mathcal{W}, \pi_{\mathcal{W}}, \mathcal{W})$ and $(T\mathcal{W}, T\tilde{p}, T\mathcal{M})$. Suppose $\mathbf{\Phi} \in \Omega^1(\mathcal{P}; V\mathcal{P})$ is a principal bundle and $\bar{\mathbf{\Phi}} \in \Omega^1(\mathcal{W}; T\mathcal{W})$ is the induced connection. The following diagram commutes.

$$\begin{tikzcd}
T\mathcal{P} \times TW \arrow[rr, "\mathbf{\Phi} \times \mathrm{Id}"] \arrow[dr, "\pi_{\mathcal{P} \times W}"'] \arrow[ddd, "T\underline{p}"'] & & V\mathcal{P} \times TW \arrow[dl, "\pi_{\mathcal{P} \times W}"] \arrow[ddd, "T\underline{p}"] \\
& \mathcal{P} \times W \arrow[d, "\underline{p}"] & \\
& \mathcal{W} & \\
T\mathcal{W} \arrow[ur, "\pi_{\mathcal{W}}"] \arrow[rr, "\bar{\mathbf{\Phi}}"] \arrow[dr, "T\tilde{p}"'] & & V\mathcal{W} \arrow[ul, "\pi_{\mathcal{W}}"'] \arrow[dl, "T\tilde{p}"] \\
& T\mathcal{M} &
\end{tikzcd}$$

One can show that $\bar{\mathbf{\Phi}}$ is fiber-linear in both vector bundle structures of $T\mathcal{W}$.[11] The induced connection $\bar{\mathbf{\Phi}}$ is called the linear connection of the associated bundle. The

[11]Note that any induced connection $\bar{\mathbf{\Phi}} \in \Omega^1(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}; T(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}))$ is fiber-linear in the vector bundle structure $T(\mathcal{P} \times_{\mathcal{G}} \mathcal{S}) \to \mathcal{P} \times_{\mathcal{G}} \mathcal{S}$.

connector $K : T\mathcal{W} \to \mathcal{W}$ of the linear connection $\bar{\mathbf{\Phi}}$ is defined as $K := \mathrm{pr}^v_{\mathcal{W}} \circ \bar{\mathbf{\Phi}}$, where the vertical projection $\mathrm{pr}^v_{\mathcal{W}} : V\mathcal{W} \to \mathcal{W}$ is defined in §3.1.4.4. One can show K is both $\pi_{\mathcal{W}} - \tilde{p}$–fiber–linear and $T\tilde{p} - \tilde{p}$–fiber–linear, i.e. both $K : (T\mathcal{W}, \pi_{\mathcal{W}}, \mathcal{W}) \to (\mathcal{W}, \tilde{p}, \mathcal{M})$ and $K : (T\mathcal{W}, T\tilde{p}, T\mathcal{M}) \to (\mathcal{W}, \tilde{p}, \mathcal{M})$ are homomorphisms of vector bundles.

3.1.6.4 Linear Connections

Let $(\mathcal{V}, p, \mathcal{M}, V)$ be a vector bundle. A linear connection on $\mathcal{V}$ is a connection $\mathbf{\Psi} \in \Omega^1(T\mathcal{V}; V\mathcal{V})$ such that $\mathbf{\Psi} : T\mathcal{V} \to V\mathcal{V}$ is also $Tp - Tp$–fiber–linear. One can show that $\mathbf{\Psi}$ is the induced connection for a unique principal connection on $\mathcal{P}^1\mathcal{M}$. The connector of $\mathbf{\Psi}$ is given by $K := \mathrm{pr}^v_{\mathcal{V}} \circ \mathbf{\Psi}$. On the other hand, a mapping $K : T\mathcal{V} \to \mathcal{V}$ that is both $\pi_{\mathcal{V}} - p$–fiber–linear and $Tp - p$–fiber–linear such that $K \circ \ell^v_{\mathcal{V}} = \mathrm{pr}_2 : \mathcal{V}_{\mathcal{M}} \times \mathcal{V} \to \mathcal{V}$, specifies a linear connection with the horizontal bundle $H\mathcal{V} = \{\mathbf{X}_u : K(\mathbf{X}_u) = 0_{p(u)}\}$.

Let $\boldsymbol{s} \in \Gamma(\mathcal{V})$ and $\boldsymbol{X} \in \mathfrak{X}(\mathcal{M})$. The covariant derivative of $\boldsymbol{s}$ along $\boldsymbol{X}$ is defined as

$$\nabla_{\boldsymbol{X}} \boldsymbol{s} := K \circ T\boldsymbol{s} \circ \boldsymbol{X} \in \Gamma(\mathcal{V}). \tag{245}$$

The operator $\nabla : \mathfrak{X}(\mathcal{M}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ is also called a linear connection. Suppose $f, g \in C^\infty(\mathcal{M}, \mathbb{R})$, $\boldsymbol{Y} \in \mathfrak{X}(\mathcal{M})$, and $\tilde{\boldsymbol{s}} \in \Gamma(\mathcal{V})$. One can show that

$$\nabla_{f\boldsymbol{X}+g\boldsymbol{Y}} \boldsymbol{s} = f\nabla_{\boldsymbol{X}} \boldsymbol{s} + g\nabla_{\boldsymbol{Y}} \boldsymbol{s}, \tag{246a}$$

$$\nabla_{\boldsymbol{X}}(\boldsymbol{s} + \tilde{\boldsymbol{s}}) = \nabla_{\boldsymbol{X}} \boldsymbol{s} + \nabla_{\boldsymbol{X}} \tilde{\boldsymbol{s}}, \tag{246b}$$

$$\nabla_{\boldsymbol{X}}(f\boldsymbol{s}) = \boldsymbol{X}(f)\boldsymbol{s} + f\nabla_{\boldsymbol{X}} \boldsymbol{s}. \tag{246c}$$

Conversely, an operator $\nabla : \mathfrak{X}(\mathcal{M}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ that satisfies the properties (246) determines a linear connection. The curvature of the covariant derivative or the curvature of the linear connection is defined as

$$\boldsymbol{R}^{\mathcal{V}}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{s} := \nabla_{\boldsymbol{X}} \nabla_{\boldsymbol{Y}} \boldsymbol{s} - \nabla_{\boldsymbol{Y}} \nabla_{\boldsymbol{X}} \boldsymbol{s} - \nabla_{[\boldsymbol{X}, \boldsymbol{Y}]} \boldsymbol{s}. \tag{247}$$

We have $\boldsymbol{R}^{\mathcal{V}} \in \Omega^2(\mathcal{M}; L(\mathcal{V}, \mathcal{V}))$. Let $\boldsymbol{\alpha} \in \Omega^k(\mathcal{M}; \mathcal{V})$. The covariant exterior derivative $d_\nabla : \Omega^k(\mathcal{M}; \mathcal{V}) \to \Omega^{k+1}(\mathcal{M}; \mathcal{V})$ is defined as

$$
\begin{aligned}
(d_\nabla \boldsymbol{\alpha})(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_k) = &\sum_{i=0}^{k} (-1)^i \nabla_{\boldsymbol{X}_i} \left(\boldsymbol{\alpha}(\boldsymbol{X}_0, \ldots, \widehat{\boldsymbol{X}}_i, \ldots \boldsymbol{X}_k) \right) \\
&+ \sum_{i<j} (-1)^{i+j} \boldsymbol{\alpha} \left([\boldsymbol{X}_i, \boldsymbol{X}_j], \boldsymbol{X}_0, \ldots, \widehat{\boldsymbol{X}}_i, \ldots, \widehat{\boldsymbol{X}}_j, \ldots, \boldsymbol{X}_k \right),
\end{aligned}
\tag{248}
$$

where the hat over a vector field means the omission of that argument. In contrast to the exterior derivative (221), d_∇ is not a differential, i.e. $d_\nabla \circ d_\nabla \neq 0$, in general [27]. Straightforward calculations show that the curvature of ∇ is the obstruction for being a differential.

3.2 Differential Operators of Elastostatics

In this section, we study various operators that are required for writing complexes for linear and nonlinear elastostatics. There are three operators in the linear elastostatics complex: (i) The Killing operator that represents linear strains, (ii) the curvature operator that is related to the compatibility equations, and (iii) the Bianchi operator that is related to stress functions. We will obtain the Killing and the curvature operators by linearizing the corresponding operators of nonlinear elasticity. But the Bianchi operator will be written using the Calabi complex. We will show that the Bianchi operator can be identified with the divergence operator in flat ambient spaces. This implies that classical stress functions of linear elastostatics and the one that we introduce here for nonlinear elastostatics are well-defined in flat spaces such as Euclidean space. Note that flatness is an intrinsic notion not an extrinsic one; for example, the cylinders and cones with their standard metrics in $\mathbb{R}^3$ are flat spaces. The Killing and the Bianchi operators are related to the kinematics and the kinetics of motion, respectively. On the other hand, the curvature operator can represent both the kinematics and kinetics of motion. The corresponding kinematic and kinetic complexes are coupled for 3-dimensional manifolds, but they become decoupled for 2-manifolds. We will also derive a sequence of differential operators for linear elastostatics that depends on the projective structures rather than the Riemannian metric. The projective structures are crucial for understanding the relation between the linear elastostatics complex and the de Rham complex. We begin this section by introducing projective structures.

3.2.1 Projective Differential Geometry

Projective structures are closely related to Hilbert's fourth problem. For a complete introduction and brief history of these structures, we refer readers to [42, 27]. Let $\mathcal{M}$ be a manifold with $m = \dim \mathcal{M} \geq 2$. Torsion-free linear connections ∇ and $\hat{\nabla}$ on $T\mathcal{M}$ are called projectively equivalent if and only if there is a one-form $\boldsymbol{\Upsilon} \in \Omega^1(\mathcal{M})$ such that $\hat{\nabla}_{\boldsymbol{X}}\boldsymbol{Y} = \nabla_{\boldsymbol{X}}\boldsymbol{Y} + \boldsymbol{\Upsilon}(\boldsymbol{Y})\boldsymbol{X} + \boldsymbol{\Upsilon}(\boldsymbol{X})\boldsymbol{Y}$, $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\mathcal{M})$. More geometrically, it can be shown that torsion-free connections ∇ and $\hat{\nabla}$ are projectively equivalent if and only if they have the same geodesics up to parametrization. A projective structure $(\mathcal{M}, [\nabla])$ on $\mathcal{M}$ is a projective equivalence class $[\nabla]$ of a torsion-free linear connection ∇ on $T\mathcal{M}$. The subject of Hilbert's fourth problem is to study a metric $\check{\boldsymbol{g}}$ on $\mathbb{R}^n$ such that $\check{\nabla} \in [\nabla]$, where $\check{\nabla}$ is the Levi-Civita connection of $\check{\boldsymbol{g}}$ and ∇ is the standard metric of $\mathbb{R}^n$ [3].

Recall that a linear connection $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ on $T\mathcal{M}$ induces a linear connection on $T^*\mathcal{M}$ that is denoted by the same symbol $\nabla : \mathfrak{X}(\mathcal{M}) \times \Omega^1(\mathcal{M}) \to \Omega^1(\mathcal{M})$ and is given by $\nabla_{\boldsymbol{X}}\boldsymbol{\alpha}(\boldsymbol{Y}_1) = \boldsymbol{X}(\boldsymbol{\alpha}(\boldsymbol{Y}_1)) - \boldsymbol{\alpha}(\nabla_{\boldsymbol{X}}\boldsymbol{Y}_1)$ [30]. More generally, it induces a linear connection on $\otimes^p T\mathcal{M} \otimes \otimes^q T^*\mathcal{M}$, $\Lambda^k T^*\mathcal{M}$, and $S^k T^*\mathcal{M}$. A differential operator in terms of ∇ is called projectively invariant if it remains the same for all $\hat{\nabla} \in [\nabla]$.

The exterior derivative is projectively invariant: For $\boldsymbol{\beta} \in \Omega^k(\mathcal{M})$, one can write

$$\begin{aligned}(d\boldsymbol{\beta})(\boldsymbol{Y}_0, \dots, \boldsymbol{Y}_k) &= \sum_{i=0}^{k} (-1)^i \boldsymbol{Y}_i \left(\boldsymbol{\beta}(\boldsymbol{Y}_0, \dots, \widehat{\boldsymbol{Y}_i}, \dots, \boldsymbol{Y}_k)\right) \\ &+ \sum_{i<j} (-1)^{i+j} \boldsymbol{\beta}([\boldsymbol{Y}_i, \boldsymbol{Y}_j], \boldsymbol{Y}_0, \dots, \widehat{\boldsymbol{Y}_i}, \dots, \widehat{\boldsymbol{Y}_j}, \dots, \boldsymbol{Y}_k) \\ &= \sum_{i=0}^{k} (-1)^i (\nabla_{\boldsymbol{Y}_i} \boldsymbol{\beta})(\boldsymbol{Y}_0, \dots, \widehat{\boldsymbol{Y}_i}, \dots, \boldsymbol{Y}_{k+1}),\end{aligned} \tag{249}$$

where ∇ can be *any* torsion-free connection on $T\mathcal{M}$. Of course, we do not need any connection to define the exterior derivative. The skew-symmetrization in the last term simply cancels out the effect of torsion-free connections. The condition for projective equivalence can be reformulated for other types of tensors as well. For example, let $\boldsymbol{\alpha} \in \Omega^1(\mathcal{M})$. Then, ∇ and $\hat{\nabla}$ are projectively equivalent if and only if $\hat{\nabla}_{\boldsymbol{X}} \boldsymbol{\alpha} = \nabla_{\boldsymbol{X}} \boldsymbol{\alpha} - \boldsymbol{\alpha}(\boldsymbol{X}) \boldsymbol{\Upsilon} - \boldsymbol{\Upsilon}(\boldsymbol{X}) \boldsymbol{\alpha}$. In general, for $\boldsymbol{\beta} \in \Omega^k(\mathcal{M})$, the equivalence condition reads

$$\begin{aligned}\left(\hat{\nabla}_{\boldsymbol{X}} \boldsymbol{\beta}\right)(\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_k) &= (\nabla_{\boldsymbol{X}} \boldsymbol{\beta})(\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_k) - (k+1) \boldsymbol{\Upsilon}(\boldsymbol{X}) \boldsymbol{\beta}(\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_k) \\ &+ (\boldsymbol{\Upsilon} \wedge \boldsymbol{\beta})(\boldsymbol{X}, \boldsymbol{Y}_1, \dots, \boldsymbol{Y}_k).\end{aligned} \tag{250}$$

In order to write invariant differential operators corresponding to the operators of linear elastostatics, we need density bundles.

3.2.1.1 Density Bundles

The linear frame bundle $\mathcal{P}^1\mathcal{M}$ is a principal $GL(\mathbb{R}^m)$-bundle on $\mathcal{M}$, where $\mathcal{P}^1\mathcal{M}_x = GL(\mathbb{R}^m, T_x\mathcal{M})$. For an arbitrary $\alpha \in \mathbb{R}$ consider the representation of $GL(\mathbb{R}^m)$ on $\mathbb{R}$ given by $A \cdot c = |\det A|^{-\alpha} c$. The associated line bundle $\mathcal{P}^1\mathcal{M} \times_{GL(\mathbb{R}^m)} \mathbb{R} \to \mathcal{M}$ is called the bundle of α-densities [27]. On orientable manifolds, the bundle of 1-densities is isomorphic to $\Lambda^m T^*\mathcal{M}$. A Riemannian metric introduces a trivialization for the bundle of α-densities [1]. In particular, an α-density $\boldsymbol{\mu} \in \Gamma(\mathcal{P}^1\mathcal{M} \times_{GL(\mathbb{R}^m)} \mathbb{R})$ can be expressed as $\boldsymbol{\mu} = a\boldsymbol{\mu}_{\alpha,g}$, where $a \in \mathbb{R}$ and $\boldsymbol{\mu}_{\alpha,g}(x)(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m) = |\det[\boldsymbol{g}(x)(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)]|^{\alpha/2}$, with the vector fields $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m \in \mathfrak{X}(\mathcal{M})$ constituting a basis for $T_x\mathcal{M}$ for all $x \in \mathcal{M}$. Let $w \in \mathbb{R}$. The bundle of $(-\frac{w}{m+1})$-densities is called the bundle of projective densities of weight w [42]. We denote this line bundle by $\mathcal{L}^{\langle w \rangle}$ and w is called the projective weight. The equivalence condition in terms of $\boldsymbol{\mu} \in \Gamma(\mathcal{L}^{\langle w \rangle})$ reads $\hat{\nabla}_{\boldsymbol{X}}\boldsymbol{\mu} = \nabla_{\boldsymbol{X}}\boldsymbol{\mu} + w\boldsymbol{\Upsilon}(\boldsymbol{X})\boldsymbol{\mu}$ [42]. In the presence of a Riemannian metric, one can show that $\boldsymbol{\mu}_{\alpha,g}$ is parallel for the Levi-Civita connection, i.e. $\nabla\boldsymbol{\mu}_{\alpha,g} = 0$ [88]. For bundles of projective densities of weight w, we define $\boldsymbol{\mu}^{\langle w \rangle} := \boldsymbol{\mu}_{\alpha,g}$, where $\alpha = -\frac{w}{m+1}$.

3.2.2 The Killing Operator

Let $(\mathcal{B}, \boldsymbol{G})$ and $(\mathcal{S}, \boldsymbol{g})$ be Riemannian manifolds and consider an orientation-preserving embedding $\varphi : \mathcal{B} \to \mathcal{S}$. We equip $T\mathcal{B}$ and $T\mathcal{S}$ with connections $\bar{\nabla}$ and ∇, respectively, that are not necessarily the associated Levi-Civita connections of the metrics. The Green deformation tensor $\boldsymbol{C} \in \Gamma(S^2T^*\mathcal{B})$ is defined by $\boldsymbol{C}(\boldsymbol{X}, \boldsymbol{Y}) := \boldsymbol{G}(\boldsymbol{X}, (T\varphi)^{\mathsf{T}} \circ T\varphi \cdot \boldsymbol{Y}) = (\varphi^*\boldsymbol{g})(\boldsymbol{X}, \boldsymbol{Y})$, where T denotes the transpose with respect to the metrics. The material strain tensor $\boldsymbol{E} \in \Gamma(S^2T^*\mathcal{B})$ is $2\boldsymbol{E} := \boldsymbol{C} - \boldsymbol{G}$. The linearized strain tensor $\boldsymbol{e}(\boldsymbol{U}) \in \Gamma(S^2T^*\mathcal{B})$, $\forall \boldsymbol{U} \in \mathfrak{X}(\varphi_0(\mathcal{B}))$, is the linearization of $\boldsymbol{E}$ with respect to a reference motion $\varphi_0 : \mathcal{B} \to \mathcal{S}$. We have [82, 118]

$$\begin{aligned} 2\boldsymbol{e}(\boldsymbol{U})(\varphi_0^*\boldsymbol{X}, \varphi_0^*\boldsymbol{Y}) = \\ 2\boldsymbol{E}(\varphi_0^*\boldsymbol{X}, \varphi_0^*\boldsymbol{Y}) + \boldsymbol{g}(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\boldsymbol{U}) + \boldsymbol{g}(\nabla_{\boldsymbol{X}}\boldsymbol{U}, \boldsymbol{Y}),\ \forall \boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\varphi_0(\mathcal{B})). \end{aligned} \tag{251}$$

Suppose $\mathcal{B}$ is a connected open subset of $\mathcal{S}$ with $\boldsymbol{G} = \boldsymbol{g}|_{\mathcal{B}}$. Also assume that $\varphi_0 = \mathrm{Id}_{\mathcal{B}}$. Then, we obtain

$$2\boldsymbol{e}(\boldsymbol{U})(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\boldsymbol{U}) + \boldsymbol{g}(\nabla_{\boldsymbol{X}}\boldsymbol{U}, \boldsymbol{Y}), \quad \forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{U} \in \mathfrak{X}(\mathcal{B}). \tag{252}$$

The operator $D_{\mathcal{K}} : \mathfrak{X}(\mathcal{B}) \to \Gamma(S^2T^*\mathcal{B})$, $\boldsymbol{U} \mapsto \boldsymbol{e}$, is metric dependent and is not projectively invariant, in general, where $\Gamma(S^2T^*\mathcal{B})$ is the linear space of sections of the vector bundle $S^2T^*\mathcal{B} \to \mathcal{B}$, i.e. the space of symmetric $\binom{0}{2}$-tensors on $\mathcal{B}$. However, if ∇ is chosen to be the Levi-Civita connection of $\boldsymbol{g}$, $D_{\mathcal{K}}$ induces a projectively invariant operator, i.e. an operator that depends on projective structures on $\mathcal{B}$ rather than its Riemannian metric. Let $(\mathcal{B}, [\nabla])$ be a projective structure on $\mathcal{B}$ arising from the Levi-Civita connection ∇ of $\boldsymbol{g}$. Since ∇ is metric compatible, we can write $2\boldsymbol{e}(\boldsymbol{U})(\boldsymbol{X},\boldsymbol{Y}) = \left(\nabla_{\boldsymbol{X}}\boldsymbol{U}^\flat\right)(\boldsymbol{Y}) + \left(\nabla_{\boldsymbol{Y}}\boldsymbol{U}^\flat\right)(\boldsymbol{X}) =: D_{\mathcal{S}}(\boldsymbol{U}^\flat)(\boldsymbol{X},\boldsymbol{Y})$, where the flat operator $\flat : \mathfrak{X}(\mathcal{B}) \to \Omega^1(\mathcal{B})$ is the natural isomorphism induced by the metric. The operator $D_{\mathcal{S}} : \Omega^1(\mathcal{B}) \to \Gamma(S^2T^*\mathcal{B})$ is not projectively invariant. In the presence of the metric $\boldsymbol{g}$, we can identify $\Omega^k(\mathcal{B})$ and $\Omega^k(\mathcal{B};\mathcal{L}^{\langle w\rangle})$ using the isomorphism $\iota(\boldsymbol{\beta}) = \boldsymbol{\beta} \otimes \boldsymbol{\mu}^{\langle w\rangle}$, where $\boldsymbol{\mu}^{\langle w\rangle}$ was defined in §3.2.1.1. Such an isomorphism also exists for other tensor bundles and is denoted by the same symbol ι. Now, consider the operator $D_1^{\langle w\rangle} : \Omega^1(\mathcal{B};\mathcal{L}^{\langle w\rangle}) \to \Gamma(S^2T^*\mathcal{B} \otimes \mathcal{L}^{\langle w\rangle})$, $D_1^{\langle w\rangle}(\boldsymbol{\beta} \otimes \boldsymbol{\mu}^{\langle w\rangle}) := D_{\mathcal{S}}(\boldsymbol{\beta}) \otimes \boldsymbol{\mu}^{\langle w\rangle}$. Let $\hat{\nabla} \in [\nabla]$ and note that if $\boldsymbol{\upsilon} \in \Omega^1(\mathcal{B};\mathcal{L}^{\langle w\rangle})$, we can write

$$\begin{aligned}\left(\hat{\nabla}_{\boldsymbol{X}}\boldsymbol{\upsilon}\right)(\boldsymbol{Y}) + \left(\hat{\nabla}_{\boldsymbol{Y}}\boldsymbol{\upsilon}\right)(\boldsymbol{X}) &= \left(\nabla_{\boldsymbol{X}}\boldsymbol{\upsilon}\right)(\boldsymbol{Y}) + \left(\nabla_{\boldsymbol{Y}}\boldsymbol{\upsilon}\right)(\boldsymbol{X}) \\ &\quad + (w-2)\big(\boldsymbol{\Upsilon}(\boldsymbol{X})\boldsymbol{\upsilon}(\boldsymbol{Y}) + \boldsymbol{\Upsilon}(\boldsymbol{Y})\boldsymbol{\upsilon}(\boldsymbol{X})\big).\end{aligned} \tag{253}$$

Thus, for $w = 2$, the operator $D_1 := D_1^{\langle 2\rangle}$ is projectively invariant. Since $\boldsymbol{\mu}_{\alpha,g}$ is parallel for the Levi-Civita connection, we conclude that $D_1 \circ \iota = \iota \circ D_{\mathcal{S}}$, i.e. ι becomes a morphism of complexes, and we can replace $D_{\mathcal{S}}$ with D_1.

A motion φ with zero material strain tensor satisfies $\boldsymbol{g}(T\varphi\cdot\boldsymbol{X}, T\varphi\cdot\boldsymbol{Y}) = \boldsymbol{G}(\boldsymbol{X},\boldsymbol{Y})$, i.e. strain-free motions of nonlinear elastostatics are isometries $\mathcal{B} \to \mathcal{S}$. The set of strain-free motions in $\mathbb{R}^n$ with its standard metric is the set of isometries of $\mathbb{R}^n$ and thus, it is in a one-to-one correspondence with the Euclidean group $Euc(\mathbb{R}^n) := \left\{\left(\begin{smallmatrix}1 & 0\\ \mathbf{v} & A\end{smallmatrix}\right) \in GL(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in SO(\mathbb{R}^n)\right\}$, with the special orthogonal group $SO(\mathbb{R}^n) := \{A \in GL(\mathbb{R}^n) : AA^\mathsf{T} = \mathrm{Id}_{\mathbb{R}^n}, \det A = 1\}$. Using the Levi-Civita connection ∇, strain-free displacements of linear elastostatics are infinitesimal isometries, since one can write

$$
\begin{aligned}
2\boldsymbol{e}(\boldsymbol{U})(\boldsymbol{X},\boldsymbol{Y}) &= \boldsymbol{U}(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})) + \boldsymbol{g}([\boldsymbol{X},\boldsymbol{U}],\boldsymbol{Y}) + \boldsymbol{g}(\boldsymbol{X},[\boldsymbol{Y},\boldsymbol{U}]) \\
&= (\mathcal{L}_{\boldsymbol{U}}\boldsymbol{g})(\boldsymbol{X},\boldsymbol{Y}),
\end{aligned}
\tag{254}
$$

where $\mathcal{L}_{\boldsymbol{U}}$ is the Lie derivative in the direction of $\boldsymbol{U}$. Hence, $\boldsymbol{e}(\boldsymbol{U}) = 0$ if and only if $\mathcal{L}_{\boldsymbol{U}}\boldsymbol{g} = 0$, i.e. $\boldsymbol{U}$ is a Killing field. Due to this result and the fact that the operators $D_{\mathcal{K}}$ and $D_{\mathcal{S}}$ are equivalent on a Riemannian manifold, we call both of these operators the Killing operator [42]. For a Killing field $\boldsymbol{U}$, one can show that $\mathrm{Fl}_t^{\boldsymbol{U}} := \mathrm{Fl}^{\boldsymbol{U}}(t,\cdot) : U \subset \mathcal{B} \to \mathcal{B}$ is an isometry for each t [30], where $\mathrm{Fl}^{\boldsymbol{U}}$ is the flow of $\boldsymbol{U}$. Vector fields on $\mathbb{R}^n$ can be considered as mappings $\mathbb{R}^n \to \mathbb{R}^n$. The space of strain-free displacements of linear elastostatics in $\mathbb{R}^n$ with its standard metric is isomorphic to $\mathfrak{euc}(\mathbb{R}^n)$, where $\mathfrak{euc}(\mathbb{R}^n) := \left\{ \left(\begin{smallmatrix} 0 & 0 \\ \mathbf{v} & A \end{smallmatrix}\right) \in \mathfrak{gl}(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in \mathfrak{so}(\mathbb{R}^n) \right\}$, is the Lie algebra of $Euc(\mathbb{R}^n)$, with $\mathfrak{so}(\mathbb{R}^n) := \{A \in \mathbb{R}^{n\times n} : A + A^{\mathsf{T}} = 0\}$, being the Lie algebra of $SO(\mathbb{R}^n)$. Note that the Killing operator does not depend on the curvature of the ambient space.

3.2.3 The Curvature Operator and the Compatibility Equations

Next, we write the second operator in the elastostatics complex. This operator expresses the so-called compatibility equations that address the following problem: Given an arbitrary $\check{\boldsymbol{e}} \in \Gamma(S^2T^*\mathcal{B})$ (or equivalently $\check{\boldsymbol{C}} \in \Gamma(S^2T^*\mathcal{B})$ for nonlinear elasticity), is there any $\boldsymbol{U} \in \mathfrak{X}(\mathcal{B})$ ($\varphi : \mathcal{B} \to \mathcal{S}$) such that $\boldsymbol{e}(\boldsymbol{U}) = \check{\boldsymbol{e}}$ ($\boldsymbol{C}(\varphi) = \check{\boldsymbol{C}}$)? It turns out that the answer depends on the curvature of the ambient space $\mathcal{S}$. Classically, the compatibility equations were expressed for flat ambient spaces. Here, we derive these conditions for non-flat ambient spaces as well. Similar to our treatment of the linear strain, we first obtain the compatibility equations for nonlinear elasticity and then we write linear compatibility equations by linearizing the nonlinear equations.

Consider a motion $\varphi : \mathcal{B} \to \mathcal{S}$ of $(\mathcal{B},\boldsymbol{G})$ in $(\mathcal{S},\boldsymbol{g})$, where $\dim\mathcal{B} = \dim\mathcal{S}$ such that $\varphi(\mathcal{B})$ is an open subset of $\mathcal{S}$. Since φ is a diffeomorphism, it is easy to observe that $\boldsymbol{C} = \varphi^*\boldsymbol{g}$ is symmetric and positive-definite and thus, it is a Riemannian metric for $\mathcal{B}$.

The mapping φ is an isometry between Riemannian manifolds $(\mathcal{B}, \boldsymbol{C})$ and $(\varphi(\mathcal{B}), \boldsymbol{g})$. Hence, the above integrability question is equivalent to: Given a metric $\boldsymbol{C}$ on $\mathcal{B}$, is there any isometry between $(\mathcal{B}, \boldsymbol{C})$ and an open subset of $\mathcal{S}$? Note that we do not a priori know which part of $\mathcal{S}$ would be occupied by $\mathcal{B}$. This suggests that a useful compatibility equation should be expressed only on $\mathcal{B}$. As we will see in the remainder of this section, we need to use the pull-back of some tensors on $\mathcal{S}$. This implies that we have to consider a "homogeneity" assumption for these tensors in the sense that they are constant on $\mathcal{S}$ such that the specific location of $\varphi(\mathcal{B})$ in $\mathcal{S}$ does not matter. In particular, we will express such a homogeneity assumption for the Riemannian curvature. In the following, we need to assume that the connection ∇ of $\mathcal{S}$ is the Levi-Civita connection of $\boldsymbol{g}$. Recall that the curvature of $\mathcal{S}$ is given by $\boldsymbol{R}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})\bar{\boldsymbol{Z}} = \nabla_{\bar{\boldsymbol{X}}}\nabla_{\bar{\boldsymbol{Y}}}\bar{\boldsymbol{Z}} - \nabla_{\bar{\boldsymbol{Y}}}\nabla_{\bar{\boldsymbol{X}}}\bar{\boldsymbol{Z}} - \nabla_{[\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}]}\bar{\boldsymbol{Z}}$, $\forall \bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, \bar{\boldsymbol{Z}} \in \mathfrak{X}(\mathcal{S})$. The Riemannian curvature is given by $\boldsymbol{\mathcal{R}}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{T}}) = \boldsymbol{g}(\boldsymbol{R}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})\bar{\boldsymbol{Z}}, \bar{\boldsymbol{T}})$. Let $\boldsymbol{\Sigma}_x$ be a 2-dimensional subspace of $T_x\mathcal{S}$ and let $\mathbf{X}_1, \mathbf{X}_2 \in \boldsymbol{\Sigma}_x$ be two arbitrary linearly independent vectors. The sectional curvature of $\boldsymbol{\Sigma}_x$ is defined as [30]

$$K(\boldsymbol{\Sigma}_x) = \frac{\boldsymbol{\mathcal{R}}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_2, \mathbf{X}_1)}{(\boldsymbol{g}(\mathbf{X}_1, \mathbf{X}_1)\boldsymbol{g}(\mathbf{X}_2, \mathbf{X}_2))^2 - (\boldsymbol{g}(\mathbf{X}_1, \mathbf{X}_2))^2}. \tag{255}$$

Of course, $K(\boldsymbol{\Sigma}_x)$ is independent of the choice of $\mathbf{X}_1$ and $\mathbf{X}_2$. The linear connection ∇ on $T\mathcal{S}$ induces a linear connection $\varphi^*\nabla$ on $T\mathcal{B}$ given by $(\varphi^*\nabla)_{\boldsymbol{X}}\boldsymbol{Y} = \varphi^*\left(\nabla_{\varphi_*\boldsymbol{X}}\, \varphi_*\boldsymbol{Y}\right)$, $\forall \boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\mathcal{B})$. Using the definition of the Levi-Civita connection ∇ [71], one can write

$$\begin{aligned} \boldsymbol{C}(\boldsymbol{Z}, (\varphi^*\nabla)_{\boldsymbol{X}}\boldsymbol{Y}) &= \boldsymbol{g}(\varphi_*\boldsymbol{Z}, \nabla_{\varphi_*\boldsymbol{X}}\, \varphi_*\boldsymbol{Y}) \\ &= \frac{1}{2}\Big\{\boldsymbol{Y}\left(\boldsymbol{C}(\boldsymbol{X}, \boldsymbol{Z})\right) + \boldsymbol{X}\left(\boldsymbol{C}(\boldsymbol{Z}, \boldsymbol{Y})\right) - \boldsymbol{Z}\left(\boldsymbol{C}(\boldsymbol{X}, \boldsymbol{Y})\right) \\ &\quad - \boldsymbol{C}\left([\boldsymbol{Y}, \boldsymbol{Z}], \boldsymbol{X}\right) - \boldsymbol{C}\left([\boldsymbol{X}, \boldsymbol{Z}], \boldsymbol{Y}\right) - \boldsymbol{C}\left([\boldsymbol{Y}, \boldsymbol{X}], \boldsymbol{Z}\right)\Big\}, \end{aligned} \tag{256}$$

i.e. $\nabla^{\boldsymbol{C}} := \varphi^*\nabla$ is the Levi-Civita connection corresponding to $\boldsymbol{C}$. Moreover, we have

$$(\varphi^*\boldsymbol{R})(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = \varphi^*\left(\boldsymbol{R}(\varphi_*\boldsymbol{X},\varphi_*\boldsymbol{Y})\varphi_*\boldsymbol{Z}\right) = \nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\boldsymbol{Z} - \nabla^{\boldsymbol{C}}_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}, \quad (257)$$

i.e. $\boldsymbol{R}^{\boldsymbol{C}} := \varphi^*\boldsymbol{R}$ is the curvature of $\nabla^{\boldsymbol{C}}$. In other words, if $\varphi : \mathcal{B} \to \mathcal{S}$ is an isometry between $(\mathcal{B},\boldsymbol{C})$ and $(\varphi(\mathcal{B}),\boldsymbol{g})$, then we must have

$$\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) = \boldsymbol{\mathcal{R}}(\varphi_*\boldsymbol{X},\varphi_*\boldsymbol{Y},\varphi_*\boldsymbol{Z},\varphi_*\boldsymbol{T}), \quad (258)$$

where $\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}$ is the Riemannian curvature of $\boldsymbol{C}$. On the other hand, the following result was first proved by Cartan [31, 30]: Suppose $(\mathcal{B},\boldsymbol{C})$ and $(\mathcal{S},\boldsymbol{g})$ are Riemannian manifolds with the same dimension and let $\mathrm{i} : T_X\mathcal{B} \to T_x\mathcal{S}$ be a linear isometry. The exponential maps $\exp^{\boldsymbol{C}}_X : U \subset T_X\mathcal{B} \to \mathcal{U}_X \subset \mathcal{B}$ and $\exp_x : \bar{U} \subset T_x\mathcal{S} \to \bar{\mathcal{U}}_x \subset \mathcal{S}$ are (local) diffeomorphisms and one can define the mapping $\mathrm{f} := \exp_x \circ \mathrm{i} \circ (\exp^{\boldsymbol{C}}_X)^{-1} : \mathcal{U}_X \to \mathcal{S}$. The neighborhood $\mathcal{U}_X$ can be restricted such that $\forall Y \in \mathcal{U}_X$ there is a unique normalized geodesic between X and Y. Let $\mathrm{P}_t : T_X\mathcal{B} \to T_Y\mathcal{B}$ be the parallel transport along this geodesic and consider the mapping $\Psi_X : \bar{\mathrm{P}}_t \circ \mathrm{i} \circ (\mathrm{P}_t)^{-1} : T_Y\mathcal{B} \to T_{\mathrm{f}(Y)}\mathcal{S}$. Then, if we have

$$\begin{aligned}&\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{T}) = \\ &\quad \boldsymbol{\mathcal{R}}(\Psi_X\cdot\mathbf{X},\Psi_X\cdot\mathbf{Y},\Psi_X\cdot\mathbf{Z},\Psi_X\cdot\mathbf{T}),\ \forall Y \in \mathcal{U}_X, \forall \mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{T} \in T_Y\mathcal{B},\end{aligned} \quad (259)$$

the mapping $\mathrm{f} : \mathcal{U}_X \to \mathrm{f}(\mathcal{U}_X)$ is a local isometry at X and $T_X\mathrm{f} = \mathrm{i}$. Hence, if it is possible to choose a linear isometry between the tangent spaces of a point of $(\mathcal{B},\boldsymbol{C})$ and a point of $(\mathcal{S},\boldsymbol{g})$, the curvature condition (259) becomes a sufficient condition for the existence of a local isometry.[12] As we will explain in the following, the above

[12]Let $\varphi : \mathcal{B} \to \mathcal{S}$ be an isometric immersion. Since φ commutes with the parallel transport and preserves curvature, the condition (259) is locally satisfied in neighborhoods of X and $\varphi(X)$ with $\mathrm{i} = T_X\varphi$ [59].

curvature condition can be easily verified for manifolds with a constant sectional curvature. Ambrose [4, 34] proved a global version of the above condition: If $\mathcal{B}$ and $\mathcal{S}$ are complete and simply-connected and a condition similar to (259) is satisfied at a point of $\mathcal{B}$ and a point of $\mathcal{S}$ for a linear isometry, then there exists a global isometric embedding $\mathcal{B} \to \mathcal{S}$.

In this work, we obtain the compatibility equations for two classes of motions: (i) Motions in ambient spaces with constant sectional curvatures, and (ii) motions of hypersurfaces in ambient spaces with constant sectional curvatures.

3.2.3.1 Motions in Ambient Spaces with Constant Sectional Curvatures

Suppose the ambient space $\mathcal{S}$ has a constant sectional curvature $\mathrm{k} \in \mathbb{R}$, i.e. $K(\mathbf{\Sigma}_x) = \mathrm{k}$, $\forall x \in \mathcal{S}$ and $\forall \mathbf{\Sigma}_x \in T_x\mathcal{S}$. Then, it is a well-known fact that if $\mathcal{S}$ is complete[13] and simply-connected, it is isometric to: (i) The n-sphere with radius $1/\sqrt{\mathrm{k}}$, if $\mathrm{k} > 0$, (ii) $\mathbb{R}^n$, if $\mathrm{k} = 0$, and (iii) the hyperbolic space, if $\mathrm{k} < 0$ [71]. In general, it is possible to show that if a Riemannian manifold has constant sectional curvature k, then each $x \in \mathcal{S}$ has a neighborhood that is isometric to an open subset of a sphere if $\mathrm{k} > 0$, $\mathbb{R}^n$ if $\mathrm{k} = 0$, and a hyperbolic space if $\mathrm{k} < 0$ [110]. Such spaces are also called Clifford-Klein spaces [26]. For example, the sectional curvature of a cylinder in $\mathbb{R}^3$ is zero and it is locally isometric to $\mathbb{R}^2$. In fact, the only surfaces of revolution with $\mathrm{k} = 0$ in $\mathbb{R}^3$ are cylinders, planes, and cones. Note that these spaces are flat with respect to their metric induced by the standard metric of the Euclidean space. See [29] for discussions on surfaces of revolution with positive and negative constant sectional curvatures. More general discussions on the classification of Riemannian manifolds with constant sectional curvature can be found in [110]. Since $\mathcal{S}$ has a constant

[13]Hopf-Rinow Theorem [30] states that any two distinct points of a complete, connected Riemannian manifold can be joined by a geodesic. Note that the metric topology on a connected Riemannian manifold coincides with its original topology.

sectional curvature, its curvature can be written as [71]

$$\boldsymbol{R}(\bar{\boldsymbol{X}},\bar{\boldsymbol{Y}})\bar{\boldsymbol{Z}} = \mathrm{k}\left(\boldsymbol{g}(\bar{\boldsymbol{Z}},\bar{\boldsymbol{Y}})\bar{\boldsymbol{X}} - \boldsymbol{g}(\bar{\boldsymbol{Z}},\bar{\boldsymbol{X}})\bar{\boldsymbol{Y}}\right),\ \forall \bar{\boldsymbol{X}},\bar{\boldsymbol{Y}},\bar{\boldsymbol{Z}} \in \mathfrak{X}(\mathcal{S}). \tag{260}$$

The pull-back of (260) along an isometric embedding φ reads

$$\boldsymbol{R}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = \mathrm{k}\boldsymbol{C}(\boldsymbol{Z},\boldsymbol{Y})\boldsymbol{X} - \mathrm{k}\boldsymbol{C}(\boldsymbol{Z},\boldsymbol{X})\boldsymbol{Y},\ \forall \boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z} \in \mathfrak{X}(\mathcal{B}), \tag{261}$$

i.e. $(\mathcal{B},\boldsymbol{C})$ has constant sectional curvature k as well. Therefore, if $\boldsymbol{C} \in \Gamma(S^2T^*\mathcal{B})$ is the Green deformation tensor of a motion, then it must satisfy (261). Conversely, we have the following theorem.

Theorem 3.2.1. *Suppose the manifolds $\mathcal{B}$ and $\mathcal{S}$ have the same dimensions and $(\mathcal{S},\boldsymbol{g})$ has a constant sectional curvature. Let $\boldsymbol{C}$ be a metric on $\mathcal{B}$ with the same constant sectional curvature. Then, for each $X \in \mathcal{B}$, there is a neighborhood $\mathcal{U}_X \subset \mathcal{B}$ of X and an isometry φ_X between $(\mathcal{U}_X,\boldsymbol{C})$ and $(\varphi_X(\mathcal{U}_X),\boldsymbol{g})$. The mapping φ_X is unique up to an isometry of $\mathcal{S}$.*

Proof. Consider arbitrary points $X \in \mathcal{B}$ and $x \in \mathcal{S}$ and let $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$ be arbitrary orthonormal bases for $T_X\mathcal{B}$ and $T_x\mathcal{S}$, respectively. Choose the isometry $\mathrm{i} : T_X\mathcal{B} \to T_x\mathcal{S}$ such that $\mathrm{i}(\mathbf{E}_i) = \mathbf{e}_i$. Then, the condition (259) is satisfied and therefore, there is a local isometry that maps X to x. It is straightforward to conclude that φ_X is unique up to an isometry of $\mathcal{S}$. □

Note that Theorem 3.2.1 implies that there are many local isometries between manifolds with a similar constant sectional curvature. We will study a global version of the above theorem in a future work. The symmetries of the Riemannian curvature determine the number of compatibility equations, i.e. the number of independent equations that we obtain by writing (261) in a local coordinate system. Recall that these symmetries include the first Bianchi identity

$$\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T})+\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{X},\boldsymbol{T})+\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{Z},\boldsymbol{X},\boldsymbol{Y},\boldsymbol{T})=0, \tag{262}$$

and also

$$\begin{aligned}\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) &= -\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{Y},\boldsymbol{X},\boldsymbol{Z},\boldsymbol{T})=-\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{T},\boldsymbol{Z}), && (263)\\ \boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) &= \boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{Z},\boldsymbol{T},\boldsymbol{X},\boldsymbol{Y}). && (264)\end{aligned}$$

For an n-dimensional Riemannian manifold, one can show that the number of independent components of the Riemannian curvature is $n^2(n^2-1)/12$ [99]. For example, for $n=2,3,4$, the number of compatibility equations is $1,6,20$, respectively. Therefore, the number of compatibility equations only depends on the dimension of the ambient space. The symmetries (262) and (263) imply (264), but (263) and (264) do not imply (262), in general. Tensors with the symmetries (263) and (264) belong to $\Gamma(S^2(\Lambda^2T^*\mathcal{B}))$ and have $(n^2-n+2)(n^2-n)/8$ independent components on an n-manifold. Note that for $n=2,3$, (263) and (264) yield (262).

Alternatively, it is also possible to write the compatibility equations in terms of $\boldsymbol{F}:=T\varphi$. Let $\dim\mathcal{B}=\dim\mathcal{S}$. By a $T\mathcal{S}$-valued k-form $\boldsymbol{\alpha}$ over ψ we mean a multilinear mapping that associates an element of $\Lambda^kT^*_X\mathcal{B}\otimes T_{\psi(X)}\mathcal{S}$ to each $X\in\mathcal{B}$, where $\psi:\mathcal{B}\to\mathcal{S}$ is a smooth embedding that we call the underlying embedding of forms. We denote the space of all $T\mathcal{S}$-valued k-forms over ψ by $\Omega^k_\psi(\mathcal{B};T\mathcal{S})$. The connection ∇ of $T\mathcal{S}$ enables us to define the covariant exterior derivative $d^\nabla_k:\Omega^k_\psi(\mathcal{B};T\mathcal{S})\to\Omega^{k+1}_\psi(\mathcal{B};T\mathcal{S})$ by

$$\begin{aligned}(d^\nabla_k\boldsymbol{\alpha})(\boldsymbol{X}_0,\dots,\boldsymbol{X}_k)=&\sum_{i=0}^k(-1)^i\nabla_{\psi_*\boldsymbol{X}_i}\left(\boldsymbol{\alpha}(\boldsymbol{X}_0,\dots,\widehat{\boldsymbol{X}}_i,\dots\boldsymbol{X}_k)\right)\\&+\sum_{i<j}(-1)^{i+j}\boldsymbol{\alpha}\left([\boldsymbol{X}_i,\boldsymbol{X}_j],\boldsymbol{X}_0,\dots,\widehat{\boldsymbol{X}}_i,\dots,\widehat{\boldsymbol{X}}_j,\dots,\boldsymbol{X}_k\right),\end{aligned} \tag{265}$$

where the hat over a vector field implies the omission of that argument. Since $(d_0^{\nabla}\boldsymbol{\alpha})(\boldsymbol{X}) = \nabla_{\psi_*\boldsymbol{X}}\boldsymbol{\alpha}$, we have

$$\begin{aligned}\left(d_1^{\nabla}\circ d_0^{\nabla}(\boldsymbol{\alpha})\right)(\boldsymbol{X}_0,\boldsymbol{X}_1) &= \nabla_{\psi_*\boldsymbol{X}_0}\nabla_{\psi_*\boldsymbol{X}_1}\boldsymbol{\alpha} - \nabla_{\psi_*\boldsymbol{X}_1}\nabla_{\psi_*\boldsymbol{X}_0}\boldsymbol{\alpha} - \nabla_{\psi_*[\boldsymbol{X}_0,\boldsymbol{X}_1]}\boldsymbol{\alpha}\\ &= \boldsymbol{R}(\psi_*\boldsymbol{X}_0,\psi_*\boldsymbol{X}_1)\,\boldsymbol{\alpha}.\end{aligned} \tag{266}$$

Hence, d^{∇} is a differential, i.e. $d^{\nabla}\circ d^{\nabla} = 0$, if and only if the ambient space $\mathcal{S}$ is flat. Let $\varphi:\mathcal{B}\to\mathcal{S}$ be an embedding with the tangent map $\boldsymbol{F} = T\varphi \in \Omega^1_{\varphi}(\mathcal{B};T\mathcal{S})$. One can write

$$\begin{aligned}\left(d_1^{\nabla}\boldsymbol{F}\right)(\boldsymbol{X}_0,\boldsymbol{X}_1) &= \nabla_{\boldsymbol{F}(\boldsymbol{X}_0)}\boldsymbol{F}(\boldsymbol{X}_1) - \nabla_{\boldsymbol{F}(\boldsymbol{X}_1)}\boldsymbol{F}(\boldsymbol{X}_0) - \boldsymbol{F}([\boldsymbol{X}_0,\boldsymbol{X}_1])\\ &= \boldsymbol{T}(\varphi_*\boldsymbol{X}_0,\varphi_*\boldsymbol{X}_1),\end{aligned} \tag{267}$$

where $\boldsymbol{T}$ is the torsion of ∇. If ∇ is torsion-free, then $\boldsymbol{F}$ must satisfy $d^{\nabla}\boldsymbol{F} = 0$. One may want to consider the converse problem as: Given $\boldsymbol{\beta} \in \Omega^1_{\varphi}(\mathcal{B};T\mathcal{S})$, do we have $\boldsymbol{\beta} = T\varphi$? But this is a trivial question since one needs to simply calculate the tangent map of the underlying embedding φ to answer this question. Instead, we define the following *generalized* compatibility problem: Given $\boldsymbol{\beta} \in \Omega^1_{\varphi}(\mathcal{B};T\mathcal{S})$, does $\boldsymbol{\beta}$ belong to the cohomology class of $\boldsymbol{F}$? Equivalently, the generalized compatibility problem can be stated as: Given $\boldsymbol{\beta} \in \Omega^1_{\varphi}(\mathcal{B};T\mathcal{S})$, is there any $\boldsymbol{\alpha} \in \Omega^0_{\varphi}(\mathcal{B};T\mathcal{S})$ such that $\boldsymbol{\beta} - \boldsymbol{F} = d_0^{\nabla}\boldsymbol{\alpha}$? The physical significance of this problem will become clearer in the next section, where this approach allows us to define stress functions for nonlinear elastostatics and obtain a complex for nonlinear elastostatics.

Suppose $\boldsymbol{\beta} - \boldsymbol{F} = d_0^{\nabla}\boldsymbol{\alpha}$, where ∇ is the Levi-Civita connection of the flat manifold $(\mathcal{S},\boldsymbol{g})$. The relations (266) and (267) imply that $d_1^{\nabla}\boldsymbol{\beta} = 0$. If $\mathcal{S} = \mathbb{R}^n$, the vector bundle $T\varphi(\mathcal{B})\to\mathcal{B}$ is the trivial vector bundle $\varphi(\mathcal{B})\times\mathbb{R}^n$. In this case, cohomology groups of the twisted de Rham complex induced by (265) can be computed by the cohomology groups of the de Rham complex [22]. In particular, if $\mathcal{B}$ is contractible, then $d_1^{\nabla}(\boldsymbol{\beta} - \boldsymbol{F}) = d_1^{\nabla}\boldsymbol{\beta} = 0$, implies that there is an $\boldsymbol{\alpha} \in \Omega_{\varphi}^0(\mathcal{B}; T\mathcal{S})$ such that $\boldsymbol{\beta} - \boldsymbol{F} = d_0^{\nabla}\boldsymbol{\alpha}$. Similarly, for non-contractible domains in $\mathbb{R}^n$, one can use the cohomology groups of the de Rham complex to obtain global compatibility equations, see [113]. This result is also locally valid on any flat manifold as we will discuss later.

3.2.3.2 Motions of Hypersurfaces

Suppose $(\mathcal{H}, \widehat{\boldsymbol{g}})$ is a submanifold of a manifold $(\mathcal{S}, \boldsymbol{g})$, where $\widehat{\boldsymbol{g}}$ is induced by $\boldsymbol{g}$. Let $\widehat{\nabla}$ and ∇ be the associated Levi-Civita connections of $\mathcal{H}$ and $\mathcal{S}$, respectively. For any $x \in \mathcal{H}$, we have the decomposition $T_x\mathcal{S} = T_x\mathcal{H} \oplus (T_x\mathcal{H})^{\perp}$, where $(T_x\mathcal{H})^{\perp}$ is the normal complement of $T_x\mathcal{H}$ in $T\mathcal{S}$. Any local vector fields $\boldsymbol{X}$ on $\mathcal{H}$ can be extended to a local vector field $\widetilde{\boldsymbol{X}}$ on $\mathcal{S}$ and we have $\widehat{\nabla}_{\boldsymbol{X}}\boldsymbol{Y} = (\nabla_{\widetilde{\boldsymbol{X}}}\widetilde{\boldsymbol{Y}})^{\mathrm{T}}$, where T denotes the tangent component. The second fundamental form $\boldsymbol{B} \in \Gamma(S^2T^*\mathcal{H} \otimes T\mathcal{H}^{\perp})$ is defined as $\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{Y}) = \nabla_{\widetilde{\boldsymbol{X}}}\widetilde{\boldsymbol{Y}} - \widehat{\nabla}_{\boldsymbol{X}}\boldsymbol{Y}$, with $\widetilde{\boldsymbol{X}}$ and $\widetilde{\boldsymbol{Y}}$ being any local extension of local vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$. Let $\mathfrak{X} \in \Gamma(T\mathcal{H}^{\perp}) =: \mathfrak{X}(\mathcal{H})^{\perp}$. We can associate a linear self-adjoint operator $\mathsf{S}_{\mathfrak{X}} : T\mathcal{H} \to T\mathcal{H}$ to $\boldsymbol{B}$ by $\widehat{\boldsymbol{g}}(\mathsf{S}_{\mathfrak{X}}(\boldsymbol{X}), \boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{Y}), \mathfrak{X})$. The operator S is called the shape operator of $\mathcal{H}$. One can show that $(\nabla_{\boldsymbol{X}}\mathfrak{X})^{\mathrm{T}} = -\mathsf{S}_{\mathfrak{X}}(\boldsymbol{X})$ [30]. On the other hand, we can also define a linear connection $\widehat{\nabla}^{\perp}$ on $T\mathcal{H}^{\perp} \to \mathcal{H}$ by $\widehat{\nabla}^{\perp}_{\boldsymbol{X}}\mathfrak{X} = (\nabla_{\boldsymbol{X}}\mathfrak{X})^{\mathrm{N}}$, where N denotes the normal component. The normal curvature $\widehat{\boldsymbol{R}}^{\perp} : \mathfrak{X}(\mathcal{H}) \times \mathfrak{X}(\mathcal{H}) \times \mathfrak{X}(\mathcal{H})^{\perp} \to \mathfrak{X}(\mathcal{H})^{\perp}$ is the curvature of $\widehat{\nabla}^{\perp}$. Hence, there are two different geometries on $T\mathcal{H}$ and $T\mathcal{H}^{\perp}$. The relation between these geometries is expressed by the Gauss, Ricci, and Codazzi equations as follows. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{T} \in \mathfrak{X}(\mathcal{H})$, and $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{X}(\mathcal{H})^{\perp}$. The following relations hold [30]:

$$\begin{aligned} \boldsymbol{\mathcal{R}}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{T}) = \widehat{\boldsymbol{\mathcal{R}}}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{T}) + \boldsymbol{g}(\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{Z}), \boldsymbol{B}(\boldsymbol{Y}, \boldsymbol{T})) \\ - \boldsymbol{g}(\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{T}), \boldsymbol{B}(\boldsymbol{Y}, \boldsymbol{Z})), \end{aligned} \tag{268}$$

$$\widehat{\boldsymbol{g}}([\mathsf{S}_{\mathfrak{Y}}, \mathsf{S}_{\mathfrak{X}}]\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{R}(\boldsymbol{X}, \boldsymbol{Y})\mathfrak{X}, \mathfrak{Y}) - \boldsymbol{g}(\widehat{\boldsymbol{R}}^{\perp}(\boldsymbol{X}, \boldsymbol{Y})\mathfrak{X}, \mathfrak{Y}), \tag{269}$$

$$\boldsymbol{g}(\boldsymbol{R}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z}, \mathfrak{X}) = \big(\nabla_{\boldsymbol{X}}\mathfrak{B}\big)(\boldsymbol{Y}, \boldsymbol{Z}, \mathfrak{X}) - \big(\nabla_{\boldsymbol{Y}}\mathfrak{B}\big)(\boldsymbol{X}, \boldsymbol{Z}, \mathfrak{X}), \tag{270}$$

where $[\mathsf{S}_{\mathcal{Y}}, \mathsf{S}_{\mathcal{X}}] = \mathsf{S}_{\mathcal{Y}} \circ \mathsf{S}_{\mathcal{X}} - \mathsf{S}_{\mathcal{X}} \circ \mathsf{S}_{\mathcal{Y}}$, $\boldsymbol{\mathcal{B}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{\mathcal{X}}) = \boldsymbol{g}(\boldsymbol{B}(\boldsymbol{X},\boldsymbol{Y}),\boldsymbol{\mathcal{X}})$, and

$$\begin{aligned}\big(\nabla_{\boldsymbol{X}}\boldsymbol{\mathcal{B}}\big)(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{\mathcal{X}}) &= \boldsymbol{X}\big(\boldsymbol{\mathcal{B}}(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{\mathcal{X}})\big) \\ &\quad - \boldsymbol{\mathcal{B}}(\widehat{\nabla}_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{\mathcal{X}}) - \boldsymbol{\mathcal{B}}(\boldsymbol{Y},\widehat{\nabla}_{\boldsymbol{X}}\boldsymbol{Z},\boldsymbol{\mathcal{X}}) - \boldsymbol{\mathcal{B}}(\boldsymbol{Y},\boldsymbol{Z},\widehat{\nabla}^{\perp}_{\boldsymbol{X}}\boldsymbol{\mathcal{X}}).\end{aligned} \tag{271}$$

The equations (268), (269), and (270) are called the Gauss, Ricci, and Codazzi equation, respectively. These equations generalize the compatibility equations of the local theory of surfaces, see [101] for more discussions. To simplify the above equations, we assume that $\mathcal{S}$ has a constant sectional curvature k and $\mathcal{H}$ is a hypersurface, i.e. $\dim\mathcal{S} - \dim\mathcal{H} = 1$. These assumptions are quite natural if we want to study motions of a 2-dimensional surface in $\mathbb{R}^3$. Using (260) and the fact that the second fundamental form of hypersurfaces can be expressed as $\boldsymbol{B}(\boldsymbol{X},\boldsymbol{Z}) = \boldsymbol{g}(\boldsymbol{B}(\boldsymbol{X},\boldsymbol{Z}),\boldsymbol{\mathcal{N}})\boldsymbol{\mathcal{N}}$, where $\boldsymbol{\mathcal{N}}$ is the unit normal vector field, the Gauss equation can be written as

$$\begin{aligned}\widehat{\boldsymbol{\mathcal{R}}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) \quad &+ \quad \widehat{\boldsymbol{g}}(\mathsf{S}_{\boldsymbol{\mathcal{N}}}\boldsymbol{Z},\boldsymbol{X})\widehat{\boldsymbol{g}}(\mathsf{S}_{\boldsymbol{\mathcal{N}}}\boldsymbol{T},\boldsymbol{Y}) - \widehat{\boldsymbol{g}}(\mathsf{S}_{\boldsymbol{\mathcal{N}}}\boldsymbol{T},\boldsymbol{X})\widehat{\boldsymbol{g}}(\mathsf{S}_{\boldsymbol{\mathcal{N}}}\boldsymbol{Z},\boldsymbol{Y}) \\ &+ \quad \mathrm{k}\,\widehat{\boldsymbol{g}}(\boldsymbol{X},\boldsymbol{Z})\,\widehat{\boldsymbol{g}}(\boldsymbol{Y},\boldsymbol{T}) - \mathrm{k}\,\widehat{\boldsymbol{g}}(\boldsymbol{X},\boldsymbol{T})\,\widehat{\boldsymbol{g}}(\boldsymbol{Y},\boldsymbol{Z}) = 0.\end{aligned} \tag{272}$$

Since vector fields in $\mathfrak{X}(\mathcal{H})^{\perp}$ are normal to those in $\mathfrak{X}(\mathcal{H})$, if $\mathcal{S}$ has a constant sectional curvature, we observe that $\boldsymbol{g}(\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}) = 0$. Moreover, for hypersurfaces we have $\boldsymbol{\mathcal{X}} = \mathcal{X}\boldsymbol{\mathcal{N}}$, and $\boldsymbol{\mathcal{Y}} = \mathcal{Y}\boldsymbol{\mathcal{N}}$, with $\mathcal{X},\mathcal{Y} \in \Omega^0(\mathcal{H})$, and since $\boldsymbol{g}(\boldsymbol{\mathcal{N}},\boldsymbol{\mathcal{N}}) = 1$, we conclude that $\widehat{\nabla}^{\perp}_{\boldsymbol{X}}\boldsymbol{\mathcal{N}} = 0$, which implies that $\boldsymbol{g}(\widehat{\boldsymbol{R}}^{\perp}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}) = 0$. Thus, the Ricci equation merely implies that $\mathcal{X}\mathcal{Y}[\mathsf{S}_{\boldsymbol{\mathcal{N}}},\mathsf{S}_{\boldsymbol{\mathcal{N}}}] = 0$, i.e. the Ricci equation becomes vacuous with the above assumptions. Similarly, the Codazzi equation simplifies to read

$$\widehat{\nabla}_{\boldsymbol{X}}\left(\mathsf{S}_{\boldsymbol{\mathcal{N}}}(\boldsymbol{Y})\right) - \widehat{\nabla}_{\boldsymbol{Y}}\left(\mathsf{S}_{\boldsymbol{\mathcal{N}}}(\boldsymbol{X})\right) = \mathsf{S}_{\boldsymbol{\mathcal{N}}}([\boldsymbol{X},\boldsymbol{Y}]). \tag{273}$$

Suppose $\mathcal{B} \subset \mathcal{H}$ is a connected open subset and assume $\mathcal{H}$ and $\mathcal{S}$ are orientable. The eigenvalues λ_i of $\mathsf{S}_{\boldsymbol{\mathcal{N}}}$ are all real and the corresponding eigenvectors constitute an orthonormal basis $\{\widehat{\boldsymbol{e}}_1, \dots, \widehat{\boldsymbol{e}}_n\}$ for $\mathcal{H}$ such that $\{\widehat{\boldsymbol{e}}_1, \dots, \widehat{\boldsymbol{e}}_n, \mathbf{N}\}$ is in the orientation of $\mathcal{S}$. The eigenvalues λ_i are called the principal curvatures of $\mathcal{H}$ and are extrinsic in the sense that they depend on the structure of $\mathcal{H}$ inside $\mathcal{S}$. A direct consequence of the Guass equation is that the products $\lambda_i\lambda_j$, $i \neq j$, are intrinsic if $\mathcal{S}$ has zero sectional curvature, i.e. $\lambda_i\lambda_j$ is merely determined by the induced metric. For the special case of surfaces in $\mathbb{R}^3$, we recover the celebrated Theorem Egregium of Gauss, which states that the Gaussian curvature, i.e. $\lambda_1\lambda_2$, is intrinsic.

Let $\varphi : \mathcal{B} \to \mathcal{S}$ be an orientation-preserving isometric embedding and let $\bar{\boldsymbol{X}} = \varphi_* \boldsymbol{X} \in \mathfrak{X}(\varphi(\mathcal{B}))$. Suppose $\boldsymbol{\theta} \in \Gamma(S^2T^*\mathcal{B})$ is defined as $\boldsymbol{\theta}(\boldsymbol{X}, \boldsymbol{Y}) = \bar{\boldsymbol{g}}(\bar{\mathsf{S}}_{\bar{\mathbf{N}}}\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$, where $\bar{\mathsf{S}}_{\bar{\mathbf{N}}}$ is the shape operator of the hypersurface $\varphi(\mathcal{B}) \subset \mathcal{S}$ with the unit normal vector field $\bar{\mathbf{N}}$ and the induced metric $\bar{\boldsymbol{g}} := \boldsymbol{g}|_{\varphi(\mathcal{B})}$. We call $\boldsymbol{\theta}$ the extrinsic deformation tensor. Let $\boldsymbol{C} = \varphi^*\bar{\boldsymbol{g}}$ be the Green deformation tensor. Of course, equations similar to (272) and (273) are valid for $(\varphi(\mathcal{B}), \bar{\boldsymbol{g}})$ with its shape operator $\bar{\mathsf{S}}_{\bar{\mathbf{N}}}$. The pull-back of the Gauss equation along φ can be written as

$$\begin{aligned} \mathcal{R}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) + \boldsymbol{\theta}(\boldsymbol{X},\boldsymbol{Z})\boldsymbol{\theta}(\boldsymbol{Y},\boldsymbol{T}) - \boldsymbol{\theta}(\boldsymbol{X},\boldsymbol{T})\boldsymbol{\theta}(\boldsymbol{Y},\boldsymbol{Z}) \\ + \mathrm{k}\boldsymbol{C}(\boldsymbol{X},\boldsymbol{Z})\boldsymbol{C}(\boldsymbol{Y},\boldsymbol{T}) - \mathrm{k}\boldsymbol{C}(\boldsymbol{X},\boldsymbol{T})\boldsymbol{C}(\boldsymbol{Y},\boldsymbol{Z}) = 0. \end{aligned} \tag{274}$$

The pull-back of the Codazzi equation simply implies that

$$\left(\nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\boldsymbol{\theta}\right)(\boldsymbol{Y},\boldsymbol{Z}) = \left(\nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\boldsymbol{\theta}\right)(\boldsymbol{X},\boldsymbol{Z}), \tag{275}$$

i.e. $\boldsymbol{\theta} \in \Gamma(S^3T^*\mathcal{B})$. Therefore, if $(\boldsymbol{C}, \boldsymbol{\theta})$ denote the intrinsic and extrinsic deformations of an isometry φ, they must satisfy (274) and (275). The converse of this statement is the compatibility condition for motions of hypersurfaces: Let $(\mathcal{S}, \boldsymbol{g})$ be a Riemannian manifold and $\dim\mathcal{S} - \dim\mathcal{B} = 1$. Given a metric $\boldsymbol{C} \in \Gamma(S^2T^*\mathcal{B})$ and a symmetric tensor $\boldsymbol{\theta} \in \Gamma(S^2T^*\mathcal{B})$, is there an isometric embedding $\varphi : \mathcal{B} \to \mathcal{S}$ such that $\boldsymbol{C} = \varphi_*\bar{\boldsymbol{g}}$ and $\boldsymbol{g}\left(\boldsymbol{B}(\varphi_*\boldsymbol{X}, \varphi_*\boldsymbol{Y}), \mathbf{N}\right) = \boldsymbol{\theta}(\boldsymbol{X}, \boldsymbol{Y})$? Here, $\bar{\boldsymbol{g}}$ and $\boldsymbol{B}$ denote the induced metric and

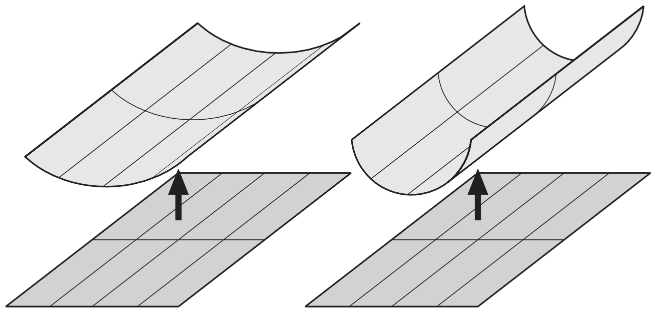


Figure 24: Two isometric embeddings of a plane into $\mathbb{R}^3$: The resulting surfaces are cylinders with different radii but both motions have the same deformation tensor $\boldsymbol{C}$.

the second fundamental form of $\varphi(\mathcal{B})$, respectively. One may wonder why we have to include $\boldsymbol{\theta}$ in the formulation. Roughly speaking, the answer is that we want surfaces with similar deformations to be unique up to isometries of the *ambient* space $\mathcal{S}$. This criterion cannot be satisfied if we only consider $\boldsymbol{C}$. For example, consider isometric deformations of a plane in $\mathbb{R}^3$ into portions of cylinders with different radii as shown in Fig. 24. All these motions have the same intrinsic deformation $\boldsymbol{C}$, but obviously cylinders with different radii are not isometric via isometries of $\mathbb{R}^3$, i.e. cannot be mapped onto each other using rigid motions of $\mathbb{R}^3$. The upshot is expressed in the following theorem [68, 72]:

Theorem 3.2.2. *Let $\mathcal{S} = \mathbb{R}^{n+1}$ and $(\mathcal{B}, \boldsymbol{C})$ be a Riemannian n-manifold with a symmetric tensor $\boldsymbol{\theta} \in \Gamma(S^2T^*\mathcal{B})$ such that (274) and (275) are satisfied. Then, for each $X \in \mathcal{B}$, there is an open neighborhood $\mathcal{U}_X \subset \mathcal{B}$ of X and an isometric embedding $\check{\varphi} : \mathcal{U}_X \to \mathcal{S}$, such that $\check{\varphi}^*(\boldsymbol{g}(\check{\boldsymbol{B}}, \check{\mathbf{N}})) = \boldsymbol{\theta}$, where $\check{\boldsymbol{B}}$ and $\check{\mathbf{N}}$ are the second fundamental form and the unit normal of $\check{\varphi}(\mathcal{U}_X)$, respectively. Moreover, $\check{\varphi}$ is unique up to isometries of $\mathcal{S}$.*

Note that if in addition $\mathcal{B}$ is simply-connected and connected, then under the above assumptions there is a global isometric immersion $\varphi : \mathcal{B} \to \mathcal{S}$, which is unique up to an isometry of $\mathcal{S}$ [72, 101]. Also we should mention that the above compatibility equations are equivalent to those obtained by [37].

3.2.3.3 Linear Compatibility Equations

Now, we linearize the compatibility equations to obtain the second operator of the linear elasticity complex. We first linearize the operator $\Gamma(S^2T^*\mathcal{B}) \to \Gamma(S^2(\Lambda^2T^*\mathcal{B}))$, $\boldsymbol{C} \to \boldsymbol{\mathcal{R}}^{\boldsymbol{C}}$, that is associated to the Riemannian curvature, where $\boldsymbol{C}$ is a Riemannian metric. Let $\boldsymbol{\varepsilon} \in \Gamma(S^2T^*\mathcal{B})$ and consider a curve $t \mapsto \boldsymbol{C} + t\boldsymbol{\varepsilon}$ in $\Gamma(S^2T^*\mathcal{B})$. Note that $\exists \epsilon > 0$ such that for $|t| < \epsilon$ the symmetric tensor $\boldsymbol{C} + t\boldsymbol{\varepsilon}$ is a Riemannian metric on $\mathcal{B}$. The linearization of the above operator is defined as the linear operator $\boldsymbol{\varepsilon} \mapsto \boldsymbol{r}(\boldsymbol{C}, \boldsymbol{\varepsilon}) := \frac{d}{dt}|_{t=0}\boldsymbol{\mathcal{R}}^{\boldsymbol{C}+t\boldsymbol{e}} \in \Gamma(\otimes^4T^*\mathcal{B})$ [50]. One can show that [49]

$$\begin{aligned} 2\boldsymbol{r}(\boldsymbol{C},\boldsymbol{\varepsilon})(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) &= \boldsymbol{L}(\boldsymbol{C},\boldsymbol{\varepsilon})(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) \\ &\quad + \boldsymbol{\varepsilon}(\boldsymbol{R}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z},\boldsymbol{T}) - \boldsymbol{\varepsilon}(\boldsymbol{R}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{T},\boldsymbol{Z}), \end{aligned} \tag{276}$$

where

$$\begin{aligned} &\boldsymbol{L}(\boldsymbol{C},\boldsymbol{\varepsilon})(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) = \\ &\left(\nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\nabla^{\boldsymbol{C}}_{\boldsymbol{Z}}\boldsymbol{\varepsilon}\right)(\boldsymbol{Y},\boldsymbol{T}) + \left(\nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\nabla^{\boldsymbol{C}}_{\boldsymbol{T}}\boldsymbol{\varepsilon}\right)(\boldsymbol{X},\boldsymbol{Z}) - \left(\nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\nabla^{\boldsymbol{C}}_{\boldsymbol{T}}\boldsymbol{\varepsilon}\right)(\boldsymbol{Y},\boldsymbol{Z}) - \left(\nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\nabla^{\boldsymbol{C}}_{\boldsymbol{Z}}\boldsymbol{\varepsilon}\right)(\boldsymbol{X},\boldsymbol{T}) \\ &- \left(\nabla^{\boldsymbol{C}}_{\nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\boldsymbol{Z}}\boldsymbol{\varepsilon}\right)(\boldsymbol{Y},\boldsymbol{T}) - \left(\nabla^{\boldsymbol{C}}_{\nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\boldsymbol{T}}\boldsymbol{\varepsilon}\right)(\boldsymbol{X},\boldsymbol{Z}) + \left(\nabla^{\boldsymbol{C}}_{\nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\boldsymbol{T}}\boldsymbol{\varepsilon}\right)(\boldsymbol{Y},\boldsymbol{Z}) + \left(\nabla^{\boldsymbol{C}}_{\nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\boldsymbol{Z}}\boldsymbol{\varepsilon}\right)(\boldsymbol{X},\boldsymbol{T}). \end{aligned} \tag{277}$$

Our goal is to obtain a necessary and (locally) sufficient condition that guarantees the existence of a displacement field for a given linear strain in an ambient space with constant sectional curvature k. We will study linear compatibility equations for hypersurfaces in a future work. It turns out that by substituting for $\boldsymbol{R}^{\boldsymbol{C}}$ from (261) into (276), one can obtain the desired condition. This is stated in the following theorem due to Calabi [26]:

Theorem 3.2.3 (The Linear Compatibility Equations)**.** *Let* $(\mathcal{S}, \boldsymbol{g})$ *have constant sectional curvature* k *and let* $\mathcal{B} \subset \mathcal{S}$ *with* $\boldsymbol{G} = \boldsymbol{C} = \boldsymbol{g}|_{\mathcal{B}}$*. The linear strain* $\mathbf{e}(\boldsymbol{U})$ *defined* *in (252) satisfies*

$$\begin{aligned}\boldsymbol{I}(\boldsymbol{e})(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) = \boldsymbol{L}(\boldsymbol{g},\boldsymbol{e})(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) + \mathrm{k}\Big\{&\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\boldsymbol{e}(\boldsymbol{X},\boldsymbol{T}) \\ -\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Z})\boldsymbol{e}(\boldsymbol{Y},\boldsymbol{T}) - \boldsymbol{g}(\boldsymbol{Y},\boldsymbol{T})\boldsymbol{e}(\boldsymbol{X},\boldsymbol{Z}) + &\boldsymbol{g}(\boldsymbol{X},\boldsymbol{T})\boldsymbol{e}(\boldsymbol{Y},\boldsymbol{Z})\Big\} = 0.\end{aligned} \tag{278}$$

Conversely, if an arbitrary tensor $\boldsymbol{\varepsilon} \in \Gamma(S^2T^*\mathcal{B})$ *satisfies* $\boldsymbol{I}(\boldsymbol{\varepsilon}) = 0$*, then for each* $X \in \mathcal{B}$*, there is a vector field* $\boldsymbol{U}_X$ *in a neighborhood of* X *such that* $\boldsymbol{\varepsilon} = \boldsymbol{e}(\boldsymbol{U}_X)$*.*

Rather than the direct proof given in [26], another way to justify the above result on the Euclidean space is through the construction of the linear elasticity complex from a twisted de Rham complex that will be explained in the next section. Equivalently, one can obtain (278) by linearizing (267) with respect to $\boldsymbol{F} = T\varphi_0$, where $\varphi_0 = \mathrm{Id}_{\mathcal{B}}$. The equation (278) is called the linear compatibility equation. If we want to refer to the components of $\boldsymbol{I}(\boldsymbol{\varepsilon})$ in a local coordinate system, we call (278) the linear compatibility equations. Note that the tensors $\boldsymbol{r}(\boldsymbol{C},\boldsymbol{\varepsilon})$ and $\boldsymbol{I}(\boldsymbol{\varepsilon})$ inherit the symmetries of the Riemannian curvature, i.e. they satisfy (262) and (263). Consequently, similar to the nonlinear case, the number of independent linear compatibility equations in a n-dimensional ambient space is $n^2(n^2-1)/12$. The tensor $\boldsymbol{I}(\boldsymbol{\varepsilon})(X)$ belongs to a $\big(n^2(n^2-1)/12\big)$-dimensional subspace of the $\big((n^2-n+2)(n^2-n)/8\big)$-dimensional space $S^2(\Lambda^2T^*_X\mathcal{B})$. Let us denote the corresponding tensor bundle by $\mathcal{C}^4\mathcal{B} \to \mathcal{B}$, i.e. $\Gamma(\mathcal{C}^4\mathcal{B})$ is the space of $\binom{0}{4}$-tensors that have the symmetries (262) and (263) of the Riemannian curvature. As we study the relation between the linear elasticity complex and the de Rham complex in the next section, we will observe that if $T^*\mathcal{B}$ is induced by a representation, the representation theory provides some tools to neatly specify tensors with complicated symmetries such as the Riemannian curvature. Let us write the linear compatibility equations in a local coordinate system. To this end, we use normal coordinate systems that facilitate calculations: For any Riemannian manifold $(\mathcal{M},\breve{\boldsymbol{g}})$ and an arbitrary $X \in \mathcal{M}$, there is a local coordinate system $\{X^i\}$ centered at X such that $\nabla_{\partial/\partial X^i}(\partial/\partial X^j) = 0$, at X, where ∇ is the Levi-Civita connection and $\{\partial/\partial X^i\}$ is a local basis for $T\mathcal{M}$ which is orthonormal at X.[14] The coordinate system $\{X^i\}$ is called a normal coordinate system or a geodesic coordinate system at X [71, 88]. The Cartesian coordinate of $\mathbb{R}^n$ gives us a global normal coordinate system for the Euclidean space. Suppose $\{X^i\}$ is a normal coordinate system at an arbitrary $X \in \mathcal{B}$. Also let $\boldsymbol{E}_i := \partial/\partial X^i$ and $\varepsilon_{ij} := \boldsymbol{\varepsilon}(\boldsymbol{E}_i, \boldsymbol{E}_j)$. It is easy to verify that

$$\begin{aligned}(\nabla_{\boldsymbol{E}_i}\nabla_{\boldsymbol{E}_k}\boldsymbol{\varepsilon})(\boldsymbol{E}_j, \boldsymbol{E}_l) &= \\ &\boldsymbol{E}_i\left(\boldsymbol{E}_k\left(\boldsymbol{\varepsilon}(\boldsymbol{E}_j, \boldsymbol{E}_l)\right)\right) - \boldsymbol{E}_i\Big(\boldsymbol{\varepsilon}(\nabla_{\boldsymbol{E}_k}\boldsymbol{E}_j, \boldsymbol{E}_l) + \boldsymbol{\varepsilon}(\boldsymbol{E}_j, \nabla_{\boldsymbol{E}_k}\boldsymbol{E}_l)\Big).\end{aligned} \tag{279}$$

Let $\nabla_{\boldsymbol{E}_i}\boldsymbol{E}_j = \gamma^r_{ij}\boldsymbol{E}_r$, where γ^r_{ij}'s are Christoffel symbols of ∇ and note that $\nabla_{\boldsymbol{E}_i}\boldsymbol{E}_j = \nabla_{\boldsymbol{E}_j}\boldsymbol{E}_i$. Using (279), the linear compatibility equations at X corresponding to the component $\boldsymbol{I}(X)(\boldsymbol{E}_i, \boldsymbol{E}_j, \boldsymbol{E}_k, \boldsymbol{E}_l) := I_{ijkl}(X)$ read

$$\begin{aligned}&\frac{\partial^2\varepsilon_{jl}}{\partial X^i\partial X^k} + \frac{\partial^2\varepsilon_{ik}}{\partial X^j\partial X^l} - \frac{\partial^2\varepsilon_{jk}}{\partial X^i\partial X^l} - \frac{\partial^2\varepsilon_{il}}{\partial X^j\partial X^k} \\ &+ \left(\frac{\partial\gamma^r_{lj}}{\partial X^i} - \frac{\partial\gamma^r_{li}}{\partial X^j}\right)\varepsilon_{rk} + \left(\frac{\partial\gamma^r_{ki}}{\partial X^j} - \frac{\partial\gamma^r_{kj}}{\partial X^i}\right)\varepsilon_{rl} + \mathrm{k}\Big\{\delta_{jk}\varepsilon_{il} - \delta_{ik}\varepsilon_{jl} - \delta_{jl}\varepsilon_{ik} + \delta_{il}\varepsilon_{jk}\Big\} = 0.\end{aligned} \tag{280}$$

For $n = 2$, there is only one compatibility equation corresponding to I_{1212}:

$$\begin{aligned}\frac{\partial^2\varepsilon_{11}}{\partial X^2\partial X^2} - 2\frac{\partial^2\varepsilon_{12}}{\partial X^1\partial X^2} + \frac{\partial^2\varepsilon_{22}}{\partial X^1\partial X^1} + \left(\frac{\partial\gamma^r_{11}}{\partial X^2} - \frac{\partial\gamma^r_{12}}{\partial X^1}\right)\varepsilon_{r2} + \left(\frac{\partial\gamma^r_{22}}{\partial X^1} - \frac{\partial\gamma^r_{21}}{\partial X^2}\right)\varepsilon_{r1} \\ - \mathrm{k}(\varepsilon_{11} + \varepsilon_{22}) = 0.\end{aligned} \tag{281}$$

For $n = 3$, we have 6 compatibility equations corresponding to I_{1212}, I_{1223}, I_{1313}, I_{2113}, I_{2323}, and I_{3123}.

Example 3.2.4 (The Linear Compatibility Equation on 2-Spheres)**.** *Let us calculate the compatibility equation on the* 2*-sphere with radius* $\mathcal{R}$*. As mentioned earlier, we have* $\mathrm{k} = 1/\mathcal{R}^2$*. We choose the spherical coordinate system with* $(X^1, X^2) := (\theta, \phi)$*.*

[14] Let $\{\mathbf{E}_i\}$ be an orthonormal basis for $T_X\mathcal{M}$ and consider a linear isomorphism $u : \mathbb{R}^n \to T_X\mathcal{M}$, $u(\mathbf{e}_i) = \mathbf{E}_i$. Then, the mapping $(\exp_X \circ u)^{-1}$ defines a normal coordinate system [71].

We have $g_{11} = \mathcal{R}^2 \sin^2\phi$, $g_{12} = g_{21} = 0$, and $g_{22} = \mathcal{R}^2$. The nonzero Christoffel symbols are $\gamma_{11}^2 = -\frac{1}{2}\sin 2\phi$, and $\gamma_{12}^1 = \gamma_{21}^1 = \cot\phi$. Note that (θ,ϕ) is an orthogonal coordinate system but it is not a normal coordinate system at any point. Therefore, we must use the general form of the compatibility equations given in (278). Using the relations $\nabla_{\boldsymbol{E}_1}\boldsymbol{E}_1 = \gamma_{11}^2\boldsymbol{E}_2$, $\nabla_{\boldsymbol{E}_1}\boldsymbol{E}_2 = \nabla_{\boldsymbol{E}_2}\boldsymbol{E}_1 = \gamma_{12}^1\boldsymbol{E}_1$, and $\nabla_{\boldsymbol{E}_2}\boldsymbol{E}_2 = 0$, and after lengthy calculations, we obtain the following compatibility equation:

$$\begin{aligned}\frac{\partial^2\varepsilon_{11}}{\partial X^2\partial X^2} - 2\frac{\partial^2\varepsilon_{12}}{\partial X^1\partial X^2} + \frac{\partial^2\varepsilon_{22}}{\partial X^1\partial X^1} - \cot X^2\,\frac{\partial\varepsilon_{11}}{\partial X^2} - \frac{1}{2}\sin 2X^2\,\frac{\partial\varepsilon_{22}}{\partial X^2}&\\ + 2\cot^2X^2\,\varepsilon_{11} = 0.&\end{aligned} \tag{282}$$

Interestingly, the sectional curvature of 2-spheres does not appear in their linear compatibility equations. One should note that ε_{ij}'s are not the conventional components of the linear strain in the spherical coordinate system as the lengths of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are not unity. In fact, we have

$$\varepsilon_{11} = \mathcal{R}^2\sin^2\phi\,\varepsilon_{\theta\theta},\ \ \varepsilon_{12} = \mathcal{R}^2\sin\phi\,\varepsilon_{\theta\phi},\ \ \text{and}\ \varepsilon_{22} = \mathcal{R}^2\varepsilon_{\phi\phi}, \tag{283}$$

where $\varepsilon_{\theta\theta}$, $\varepsilon_{\theta\phi}$, and $\varepsilon_{\phi\phi}$ are the conventional spherical components. Substituting (283) into (282) yields

$$\begin{aligned}\sin^2\phi\,\frac{\partial^2\varepsilon_{\theta\theta}}{\partial\phi^2} - 2\frac{\partial^2\left(\sin\phi\,\varepsilon_{\theta\phi}\right)}{\partial\theta\partial\phi} + \frac{\partial^2\varepsilon_{\phi\phi}}{\partial\theta^2} + \frac{3}{2}\sin 2\phi\,\frac{\partial\varepsilon_{\theta\theta}}{\partial\phi} - \frac{1}{2}\sin 2\phi\,\frac{\partial\varepsilon_{\phi\phi}}{\partial\phi}&\\ + (\sin 2\phi - 1)\varepsilon_{\theta\theta} = 0.&\end{aligned} \tag{284}$$

In summary, we obtained the curvature operator $D_{\mathcal{C}} : \Gamma(S^2T^*\mathcal{B}) \to \Gamma(\mathcal{C}^4\mathcal{B})$, $\boldsymbol{\varepsilon} \mapsto \boldsymbol{I}(\boldsymbol{\varepsilon})$. Similar to the first operator $D_{\mathcal{K}}$, it is possible to obtain a projectively invariant operator D_2 from $D_{\mathcal{C}}$. Consider the operator $D_2^{\langle w\rangle} : \Gamma(S^2T^*\mathcal{B}\otimes\mathcal{L}^{\langle w\rangle}) \to \Gamma(\mathcal{C}^4\mathcal{B}\otimes\mathcal{L}^{\langle w\rangle})$, $\boldsymbol{\varepsilon}\otimes\boldsymbol{\mu}^{\langle w\rangle} \mapsto D_{\mathcal{C}}(\boldsymbol{\varepsilon})\otimes\boldsymbol{\mu}^{\langle w\rangle}$. One concludes that for $w = 2$, the operator $D_2^{\langle w\rangle}$ is projectively invariant [42]. We define $D_2 := D_2^{\langle 2\rangle} : \Gamma(S^2T^*\mathcal{B}\otimes\mathcal{L}^{\langle 2\rangle}) \to \Gamma(\mathcal{C}^4\mathcal{B}\otimes\mathcal{L}^{\langle 2\rangle})$. Similar to

$D_{\mathcal{K}}$, we observe that there is a morphism of complexes ι such that $D_2 \circ \iota = \iota \circ D_{\mathcal{C}}$.

3.2.4 The Bianchi Operator and Stress Functions

Let ∇ be the Levi-Civita connection for $(\mathcal{B}, \boldsymbol{g})$. The operators $D_{\mathcal{S}}$ and $D_{\mathcal{C}}$ defined in the previous sections coincide with the first two operators of the deformation complex in Riemannian geometry obtained by Calabi [26] for manifolds with constant sectional curvatures. For 2-dimensional manifolds, this sequence terminates after $D_{\mathcal{C}}$. However, in general, it behaves similarly to the de Rham complex and terminates after n operators. Let us also write the third operator of this complex as we are interested in linear elasticity in $\mathbb{R}^3$. We call this operator the Bianchi operator as it is closely related to the second Bianchi identity. It is given by $D_{\mathcal{B}} : \Gamma(\mathcal{C}^4\mathcal{B}) \to \Gamma(\mathcal{D}^5\mathcal{B})$,

$$\begin{aligned} &D_{\mathcal{B}}(\boldsymbol{s})(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{T}, \boldsymbol{W}) = \\ &\qquad (\nabla_{\boldsymbol{X}}\boldsymbol{s})(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{T}, \boldsymbol{W}) + (\nabla_{\boldsymbol{Y}}\boldsymbol{s})(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{T}, \boldsymbol{W}) + (\nabla_{\boldsymbol{Z}}\boldsymbol{s})(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{T}, \boldsymbol{W}), \end{aligned} \tag{285}$$

where $\mathcal{D}^5\mathcal{B}$ denotes the space of $\binom{0}{5}$-tensors that have symmetries imposed by $D_{\mathcal{B}}$ and $\mathcal{C}^4\mathcal{B}$: The tensor $\nabla_{\boldsymbol{X}}\boldsymbol{s}$ belongs to $\Gamma(\mathcal{C}^4\mathcal{B})$ and therefore, $D_{\mathcal{B}}(\boldsymbol{s})$ is skew-symmetric in the first three entries and has the symmetries of the Riemannian curvature in the last four entries. For $n = 3$, the bundle $\mathcal{D}^5\mathcal{B}$ has 3-dimensional fibers. The second Bianchi identity implies that $D_{\mathcal{B}}(\boldsymbol{\mathcal{R}}) = 0$, where $\boldsymbol{\mathcal{R}}$ is the Riemannian curvature of $(\mathcal{B}, \boldsymbol{g})$. Since $D_{\mathcal{B}}$ is the operator after $D_{\mathcal{C}}$ in the Calabi complex, the following result holds.

Theorem 3.2.5. *Let the Riemannian manifold $(\mathcal{B}, \boldsymbol{g})$ have a constant sectional curvature. We have $D_{\mathcal{B}}(D_{\mathcal{C}}(\boldsymbol{\varepsilon})) = 0$. Conversely, if $D_{\mathcal{B}}(\boldsymbol{s}) = 0$, then for each $X \in \mathcal{B}$, there is a symmetric tensor $\boldsymbol{\varepsilon}$ in a neighborhood of X such that $\boldsymbol{s} = D_{\mathcal{C}}(\boldsymbol{\varepsilon})$.*

The operators $D_{\mathcal{K}}$ and $D_{\mathcal{C}}$ are related to the kinematics of motion. In contrary to these operators, the physical significance of $D_{\mathcal{B}}$ is not clear at all. For flat manifolds, it is possible to obtain an interesting physical interpretation for $D_{\mathcal{B}}$. We proceed as follows. Since $(\mathcal{B}, \boldsymbol{g})$ is flat, one can choose an orthonormal local coordinate system

$\{X^i\}$ centered at $X \in \mathcal{B}$, i.e. $g_{ij} = \delta_{ij}$ in a neighborhood of X.[15] For such a local coordinate system, it is easy to observe that $\gamma^i_{jk} = 0$. Let $\boldsymbol{h} := D_{\mathbb{B}}(\boldsymbol{s})$. For 2-manifolds, it is straightforward to show that $D_{\mathbb{B}}(\boldsymbol{s}) = 0$, $\forall \boldsymbol{s} \in \Gamma(\mathbb{C}^4\mathcal{B})$, i.e. the Calabi complex terminates after $D_{\mathbb{C}}$ for 2-manifolds. For $n = 3$, the independent components of $\boldsymbol{h}$ are h_{12323}, h_{21313}, and h_{31212}. Using the six independent components of $\boldsymbol{s}$, i.e. s_{1212}, s_{1223}, s_{1313}, s_{2113}, s_{2323}, and s_{3123}, one obtains the following expressions for the components of $\boldsymbol{h}$ in the local coordinate system $\{X^i\}$:

$$\begin{aligned} h_{12323} &= \frac{\partial s_{2323}}{\partial X^1} + \frac{\partial s_{3123}}{\partial X^2} + \frac{\partial s_{1223}}{\partial X^3}, \\ h_{21313} &= \frac{\partial s_{3123}}{\partial X^1} + \frac{\partial s_{1313}}{\partial X^2} + \frac{\partial s_{2113}}{\partial X^3}, \\ h_{31212} &= \frac{\partial s_{1223}}{\partial X^1} + \frac{\partial s_{2113}}{\partial X^2} + \frac{\partial s_{1212}}{\partial X^3}. \end{aligned} \tag{286}$$

For 3-manifolds, the vector bundles $\mathbb{C}^4\mathcal{B}$ and $S^2T\mathcal{B}$ have the same dimension, but there is no global isomorphism between them, in general. The orthonormal coordinate system $\{X^i\}$ enables us to obtain a local isomorphism between corresponding tensors given by

$$s_{2323} \mapsto \sigma^{11},\ s_{3123} \mapsto \sigma^{12},\ s_{1223} \mapsto \sigma^{13},\ s_{1313} \mapsto \sigma^{22},\ s_{2113} \mapsto \sigma^{23},\ \text{and } s_{1212} \mapsto \sigma^{33}. \tag{287}$$

Thus, we can locally identify $\Gamma(\mathbb{C}^4\mathcal{B})$ and $\Gamma(S^2T\mathcal{B})$ via an isomorphism $\tilde{\imath}_X$. If one can obtain an orthonormal coordinate system covering $\mathcal{B}$, this identification is also valid globally. The Cartesian coordinate system provides such a global identification for the Euclidean space with its standard metric. Recall that the divergence of $\binom{2}{0}$-tensor $\boldsymbol{\sigma} \in \Gamma(S^2T\mathcal{B})$ is a $\binom{1}{0}$-tensor given by $(\operatorname{div} \boldsymbol{\sigma})(\boldsymbol{\alpha}) = \operatorname{tr}(\nabla\boldsymbol{\sigma}(\boldsymbol{\alpha}))$, $\forall \boldsymbol{\alpha} \in \Omega^0(\mathcal{B})$. Let $\boldsymbol{\sigma} = \tilde{\imath}_X(\boldsymbol{s})$. Using (286), it is easy to verify that $D_{\mathbb{B}}(\boldsymbol{s}) = 0$, if and only if $\operatorname{div} \boldsymbol{\sigma} = 0$.

[15] Note that normal coordinate systems are orthonormal only at X, in general. The normal coordinate system explained in Footnote 14 is also orthonormal in a neighborhood of X for flat manifolds [82].

Note that to obtain this result, the underlying coordinate system of (287) is assumed to be orthonormal, i.e. $g_{ij} = \delta_{ij}$. Hence, we have proved the following Lemma.

Lemma 3.2.6. *Let $(\mathcal{B}, \boldsymbol{g})$ be a flat 3-manifold. Then, for each $X \in \mathcal{B}$, there exists a neighborhood $\mathcal{V}_X \subset \mathcal{B}$ of X and an isomorphism $\tilde{\imath}_X : \Gamma(\mathcal{C}^4\mathcal{V}_X) \to \Gamma(S^2T\mathcal{V}_X)$ such that $D_{\mathcal{B}}(\boldsymbol{s}) = 0$, if and only if $\operatorname{div}(\tilde{\imath}_X(\boldsymbol{s})) = 0$. In an orthonormal coordinate system centered at X, the expression of $\tilde{\imath}_X$ is given in (287). We denote this isomorphism by $\tilde{\imath}$ if it can be defined globally on $\mathcal{B}$.*

Now, we are ready to give a physical interpretation for $D_{\mathcal{B}}$: In the absence of external body forces, the governing equation of linear elastostatics reads $\operatorname{div} \boldsymbol{\sigma} = 0$, where $\boldsymbol{\sigma} \in \Gamma(S^2T\mathcal{B})$ is the stress tensor on the body $\mathcal{B}$ [82]. Using Theorem 3.2.5 and Lemma 3.2.6, we can prove the existence of the so-called Beltrami stress functions as follows.

Corollary 3.2.7 (Beltrami Stress Functions in Linear Elastostatics)**.** *Let $(\mathcal{B}, \boldsymbol{g})$ be a flat 3-manifold and let $\{X^i\}$ be an orthonormal coordinate system for $\mathcal{B}$ in a neighborhood of an arbitrary point $X \in \mathcal{B}$. If the stress tensor $\boldsymbol{\sigma}$ satisfies $\operatorname{div} \boldsymbol{\sigma} = 0$ on $\mathcal{B}$, there is a tensor $\boldsymbol{\Phi} \in \Gamma(S^2T^*\mathcal{U}_X)$ in a neighborhood $\mathcal{U}_X \subset \mathcal{B}$ of X covered by $\{X^i\}$ such that $\boldsymbol{\sigma}|_{\mathcal{U}_X} = \tilde{\imath}_X(D_{\mathcal{C}}(\boldsymbol{\Phi}))$. The tensor $\boldsymbol{\Phi}$ is called a Beltrami stress function for $(\mathcal{B}, \boldsymbol{g})$ and in the local coordinate system $\{X^i\}$, we have*

$$\left(\tilde{\imath}_X^{-1}(\boldsymbol{\sigma})\right)_{ijkl} = \frac{\partial^2 \Phi_{jl}}{\partial X^i \partial X^k} + \frac{\partial^2 \Phi_{ik}}{\partial X^j \partial X^l} - \frac{\partial^2 \Phi_{jk}}{\partial X^i \partial X^l} - \frac{\partial^2 \Phi_{il}}{\partial X^j \partial X^k}. \tag{288}$$

Conversely, if $(\mathcal{B}, \boldsymbol{g})$ admits a stress function in a neighborhood $\mathcal{U}_X$ of X, we have $(\operatorname{div} \boldsymbol{\sigma})|_{\mathcal{U}_X} = 0$.

The global version of Corollary 3.2.7 is also valid if $\mathcal{B}$ is contractible, i.e. without any holes, and is covered by an orthonormal coordinate system $\{X^i\}$. Since $\mathbb{R}^3$ with its standard metric has a global orthonormal coordinate system, Corollary 3.2.7 is globally valid if $\mathcal{B} \subset \mathbb{R}^3$ is a contractible open subset. For non-contractible bodies, a global result can be obtained using cohomology groups of the Calabi complex. Alternatively, it is also possible to obtain a global result in $\mathbb{R}^3$ using other methods, see [56, 92] and references therein for more discussions. Therefore, we observe that the operators $D_{\mathfrak{K}}$ and $D_{\mathfrak{B}}$ correspond to the kinematics and kinetics of motion, respectively, while depending on the position of $D_{\mathfrak{C}}$ in the short subcomplexes, $D_{\mathfrak{C}}$ can correspond to both the kinematics and the kinetics of motion. If the components Φ_{ij} vanish for $i \neq j$, $\mathbf{\Phi}$ is called a Maxwell stress function and if they vanish for $i = j$, $\mathbf{\Phi}$ is called a Morera stress function [56, 55, 77, 103]. If the only nonzero component is Φ_{33}, $\mathbf{\Phi}$ is called an Airy stress function. For Airy stress functions, we have plane stresses, i.e. $\sigma^{13} = \sigma^{23} = \sigma^{33} = 0$. The converse is also true in $\mathbb{R}^2$: There is a local Airy stress function for a body $\mathcal{B} \subset \mathbb{R}^2$ that satisfies div $\boldsymbol{\sigma} = 0$ [107]. The upshot is the following.

Corollary 3.2.8 (Airy Stress Functions in 2D-Linear Elastostatics)**.** *Let $(\mathcal{B}, \boldsymbol{g})$ be a flat 2-manifold with an orthonormal coordinate system $\{X^i\}$. If the stress tensor $\boldsymbol{\sigma} \in \Gamma(S^2 T\mathcal{B})$ satisfies* div $\boldsymbol{\sigma} = 0$ *on $\mathcal{B}$, then for each $X \in \mathcal{B}$ there is a neighborhood $\mathcal{U}_X \subset \mathcal{B}$ of X and a function $\psi \in \Omega^0(\mathcal{B})$ such that $\boldsymbol{\sigma}|_{\mathcal{U}_X} = D_{\mathcal{A}}(\psi)$. The function ψ is called an Airy stress function for $(\mathcal{B}, \boldsymbol{g})$ and in the local coordinate system $\{X^i\}$, we have*

$$\sigma^{11} = \frac{\partial^2 \psi}{\partial X^2 \partial X^2}, \ \sigma^{12} = -\frac{\partial^2 \psi}{\partial X^1 \partial X^2}, \ \sigma^{22} = \frac{\partial^2 \psi}{\partial X^1 \partial X^1}. \tag{289}$$

Conversely, if $\boldsymbol{\sigma}|_{\mathcal{U}_X} = D_{\mathcal{A}}(\psi)$, for a function $\psi \in \Omega^0(\mathcal{B})$, we have $(\mathrm{div}\, \boldsymbol{\sigma})|_{\mathcal{U}_X} = 0$.

Therefore, as we will discuss later, we also have a kinetic complex for linear elastostatics in $\mathbb{R}^2$, which is not connected to the kinematic complex as in $\mathbb{R}^3$. For 3-manifolds, it is straightforward to check that the operator $D_3 : \Gamma(\mathfrak{C}^4\mathcal{B} \otimes \mathcal{L}^{\langle 2 \rangle}) \to \Gamma(\mathfrak{D}^5\mathcal{B} \otimes \mathcal{L}^{\langle 2 \rangle})$ given by $D_3(\boldsymbol{s} \otimes \boldsymbol{\mu}^{\langle 2 \rangle}) = D_{\mathfrak{B}}(\boldsymbol{s}) \otimes \boldsymbol{\mu}^{\langle 2 \rangle}$ is projectively invariant [42] and there is a morphism ι such that $D_3 \circ \iota = \iota \circ D_{\mathfrak{B}}$.

Theorem 3.2.5 also guarantees the existence of stress functions for nonlinear elastostatics. We have various notions for stress in nonlinear elasticity, and consequently, one can obtain various stress functions for each of these stresses. Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a motion of $(\mathcal{B}, \boldsymbol{G})$ in a flat ambient space $(\mathcal{S}, \boldsymbol{g})$. In the absence of body forces, the elastostatics equilibrium in terms of Cauchy stress tensor $\boldsymbol{\sigma} \in \Gamma(S^2 T\varphi(\mathcal{B}))$ reads $\operatorname{div} \boldsymbol{\sigma} = 0$ [82]. By replacing $(\mathcal{B}, \boldsymbol{g})$ with $(\varphi(\mathcal{B}), \boldsymbol{g})$ in Corollary 3.2.7, we can directly conclude the existence of local stress functions $\boldsymbol{\Phi} \in \Gamma(S^2 T\varphi(\mathcal{B}))$ for the Cauchy stress tensor. We call these stress functions Cauchy stress functions. Now, let $\boldsymbol{C} = \varphi^* \boldsymbol{g}$ be the Green deformation tensor of φ. Clearly, $(\mathcal{B}, \boldsymbol{C})$ is a flat Riemannian manifold. The governing equation of nonlinear elastostatics without body forces can be written as $\operatorname{div}^{\boldsymbol{C}} \boldsymbol{S} = 0$, where $\boldsymbol{S} \in \Gamma(S^2 T\mathcal{B})$ is the second Piola-Kirchhoff stress tensor and $\operatorname{div}^{\boldsymbol{C}}$ is the divergence with respect to the Levi-Civita connection $\nabla^{\boldsymbol{C}}$ of the metric $\boldsymbol{C}$ [82]. Corollary 3.2.9 extends the notion of stress functions to nonlinear elastostatics in terms of the second Piola-Kirchhoff stress tensor as follows.

Corollary 3.2.9 (Second Piola-Kirchhoff Stress Functions in Nonlinear Elastostatics)**.** *Let $\{X^i\}$ be an orthonormal local coordinate system for a flat 3-manifold $(\mathcal{B}, \boldsymbol{C})$, where $\boldsymbol{C}$ is the Green deformation tensor. If the second Piola-Kirchhoff stress tensor satisfies $\operatorname{div}^{\boldsymbol{C}} \boldsymbol{S} = 0$, on $\mathcal{B}$, there exists a tensor $\boldsymbol{\Psi} \in \Gamma(S^2 T^* \mathcal{U}_X)$ in a neighborhood $\mathcal{U}_X \subset \mathcal{B}$ of X covered by $\{X^i\}$ such that $\boldsymbol{S}|_{\mathcal{U}_X} = \tilde{\imath}_X(D^{\boldsymbol{C}}_{\mathfrak{C}}(\boldsymbol{\Psi}))$, where $D^{\boldsymbol{C}}_{\mathfrak{C}}(\boldsymbol{\Psi}) := \boldsymbol{L}(\boldsymbol{C}, \boldsymbol{\Psi})$, and $\boldsymbol{L}$ is defined in (277). We call the tensor $\boldsymbol{\Psi}$ a second Piola-Kirchhoff stress function for $(\mathcal{B}, \boldsymbol{C})$. The components $(\tilde{\imath}_X^{-1}(\boldsymbol{S}))_{ijkl}$ in $\{X^i\}$ are similar to (288). Conversely, if $(\mathcal{B}, \boldsymbol{C})$ admits a second Piola-Kirchhoff stress function in a neighborhood $\mathcal{U}_X \subset \mathcal{B}$ of X, we have $(\operatorname{div}^{\boldsymbol{C}} \boldsymbol{S})|_{\mathcal{U}_X} = 0$.*

In Corollary 3.2.9, note that $\{X^i\}$ must be orthonormal with respect to $\boldsymbol{C}$ not $\boldsymbol{G}$. For $\mathcal{S} = \mathbb{R}^3$ with its standard metric, the motion φ always provides such an orthonormal coordinate system on $\mathcal{B}$ globally. We will study a global version of Corollary 3.2.9 on non-contractible domains in a future work. Similar to the compatibility equations, it is also possible to use the covariant exterior derivative defined in (265). Let $(\mathcal{B}, \boldsymbol{G})$ and $(\mathcal{S}, \boldsymbol{g})$ be flat Riemannian manifolds with the same dimensions that admit global orthonormal coordinate systems $\{X^I\}$ and $\{x^i\}$, respectively. Note that the flatness of $\mathcal{B}$ is required as we need a global orthonormal coordinate system on $\mathcal{B}$. Let $\boldsymbol{E}_I := \partial/\partial X^I$, and $\boldsymbol{e}_i := \partial/\partial x^i$. We have $\boldsymbol{G}(\boldsymbol{E}_I, \boldsymbol{E}_J) = \delta_{IJ}$, and $\boldsymbol{g}(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$. Recall that a two-point tensor of type $\binom{p\ l}{q\ m}$ over an embedding $\varphi : \mathcal{B} \to \mathcal{S}$ is a section of a vector bundle over $\mathcal{B}$ with the fiber $\otimes^p T_X\mathcal{B} \otimes \otimes^q T_X^*\mathcal{B} \otimes \otimes^l T_{\varphi(X)}\mathcal{S} \otimes \otimes^m T_{\varphi(X)}^*\mathcal{S}$ over $X \in \mathcal{B}$ [82]. Let $\boldsymbol{\alpha} \in \Omega_\varphi^1(\mathcal{B}; T\mathcal{S})$ and assume ∇ is the Levi-Civita connection of $\boldsymbol{g}$. Using (265), and the Jacobi identity for brackets, we can write

$$\begin{aligned}\big(d_2^\nabla \circ d_1^\nabla(\boldsymbol{\alpha})\big)(\boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{X}_2) = &\\ \boldsymbol{R}(\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{X}}_1)\boldsymbol{\alpha}(\boldsymbol{X}_2) - \boldsymbol{R}(\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{X}}_2)\boldsymbol{\alpha}(\boldsymbol{X}_1) + \boldsymbol{R}(\bar{\boldsymbol{X}}_1, \bar{\boldsymbol{X}}_2)\boldsymbol{\alpha}(\boldsymbol{X}_0),&\end{aligned} \tag{290}$$

where $\bar{\boldsymbol{X}}_i := \varphi_* \boldsymbol{X}_i$. Hence, if $(\mathcal{S}, \boldsymbol{g})$ is flat, then $d_2^\nabla \circ d_1^\nabla = 0$. For 3-manifolds $\mathcal{B}$ and $\mathcal{S}$, the fibers of $T^*\mathcal{B} \otimes T\varphi(\mathcal{B})$, $\Lambda^2 T^*\mathcal{B} \otimes T\varphi(\mathcal{B})$, and $\Lambda^3 T^*\mathcal{B} \otimes T\varphi(\mathcal{B})$ are 9-, 9-, and 3-dimensional, respectively. The independent components of $\binom{01}{20}$-tensor $\boldsymbol{\beta} \in \Omega_\varphi^2(\mathcal{B}; T\mathcal{S})$ are ${\beta_{12}}^i$, ${\beta_{13}}^i$, and ${\beta_{23}}^i$, for $i = 1, 2, 3$, where ${\beta_{IJ}}^i := \boldsymbol{g}(\boldsymbol{\beta}(\boldsymbol{E}_I, \boldsymbol{E}_J), \boldsymbol{e}_i)$. We have

$$\left(d_2^\nabla \boldsymbol{\beta}\right)(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3) = \left(\frac{\partial {\beta_{23}}^i}{\partial X^1} - \frac{\partial {\beta_{13}}^i}{\partial X^2} + \frac{\partial {\beta_{12}}^i}{\partial X^3}\right)\boldsymbol{e}_i. \tag{291}$$

Let $\boldsymbol{P} \in \Gamma(T\mathcal{B} \otimes T\varphi(\mathcal{B}))$. The divergence of $\binom{11}{00}$-tensor $\boldsymbol{P}$ is a $\binom{01}{00}$-tensor given by $(\text{div}\boldsymbol{P})(\bar{\boldsymbol{\alpha}}) := \text{tr}(\tilde{\nabla}\boldsymbol{P}(\bar{\boldsymbol{\alpha}}))$, $\forall \bar{\boldsymbol{\alpha}} \in \Omega^1(\varphi(\mathcal{B}))$, where $\tilde{\nabla}$ is the Levi-Civita connection of $(\mathcal{B}, \boldsymbol{G})$ [82]. The coordinate systems $\{X^I\}$ and $\{x^i\}$ enable us to define an isomorphism $\hat{\imath} : \Omega_\varphi^2(\mathcal{B}; T\mathcal{S}) \to \Gamma(T\mathcal{B} \otimes T\varphi(\mathcal{B}))$ defined by

$${\beta_{23}}^i \mapsto P^{1i},\ {\beta_{13}}^i \mapsto -P^{2i},\ {\beta_{12}}^i \mapsto P^{3i},\ i = 1, 2, 3. \tag{292}$$

We can readily verify that $d_2^\nabla \boldsymbol{\beta} = 0$, if and only if $\text{div}(\hat{\imath}(\boldsymbol{\beta})) = 0$. Therefore, we have proved the following lemma.

Lemma 3.2.10. *Let* $(\mathcal{B}, \boldsymbol{G})$ *and* $(\mathcal{S}, \boldsymbol{g})$ *be flat Riemannian* 3*-manifolds that admit global orthonormal coordinate systems* $\{X^I\}$ *and* $\{x^i\}$*, respectively. Then, for any embedding* $\varphi : \mathcal{B} \to \mathcal{S}$*, there is an isomorphism* $\hat{\imath} : \Omega^2_\varphi(\mathcal{B}; T\mathcal{S}) \to \Gamma(T\mathcal{B} \otimes T\varphi(\mathcal{B}))$ *such that* $\forall \boldsymbol{\beta} \in \Omega^2_\varphi(\mathcal{B}; T\mathcal{S})$*, we have* $d^\nabla_2 \boldsymbol{\beta} = 0$*, if and only if* $\mathrm{div}(\hat{\imath}(\boldsymbol{\beta})) = 0$*. In the orthonormal coordinates* $\{X^I\}$ *and* $\{x^i\}$*, the expression of* $\hat{\imath}$ *reads as in (292).*

Now, we can define the first Piola-Kirchhoff stress functions as follows: The governing equation of nonlinear elastostatics can be written as $\mathrm{div}\boldsymbol{P} = 0$, where $\varphi : \mathcal{B} \to \mathcal{S}$ is a motion and $\boldsymbol{P} \in \Gamma(T\mathcal{B} \otimes T\varphi(\mathcal{B}))$ is the first Piola-Kirchhoff stress tensor associated to φ [82]. The orthonormal coordinate system $\{x^i\}$ trivializes $T\mathcal{S}$ and therefore, any $\boldsymbol{\beta} \in \Omega^k_\varphi(\mathcal{B}; T\mathcal{S})$ can be written as $\sum_i \boldsymbol{\omega}_i \otimes \boldsymbol{e}_i$, where $\boldsymbol{\omega}_i \in \Omega^k(\mathcal{B})$. Since $d^\nabla(\boldsymbol{\omega}_i \otimes \boldsymbol{e}_i) = (d\boldsymbol{\omega}_i) \otimes \boldsymbol{e}_i$, we conclude that the cohomology group $H^*_\varphi(\mathcal{B}, T\mathcal{S})$ induced by d^∇ is the same as $H^*(\mathcal{B}, \mathbb{R}^3)$ [22]. In particular, if $\mathcal{B}$ is contractible, $H^*_\varphi(\mathcal{B}, T\mathcal{S})$ is trivial. Using this result, it is straightforward to prove the following theorem:

Theorem 3.2.11 (First Piola-Kirchhoff Stress Functions in Nonlinear Elastostatics)**.** *Suppose* $(\mathcal{B}, \boldsymbol{G})$ *and* $(\mathcal{S}, \boldsymbol{g})$ *are flat Riemannian 3-manifolds with global orthonormal coordinate systems* $\{X^I\}$ *and* $\{x^i\}$*, respectively. If the first Piola-Kirchhoff stress tensor* $\boldsymbol{P}$ *corresponding to a motion* $\varphi : \mathcal{B} \to \mathcal{S}$ *satisfies* $\mathrm{div}\boldsymbol{P} = 0$*, then at each* $X \in \mathcal{B}$*, there is a neighborhood* $\mathcal{U}_X$ *of* X *and a tensor* $\Xi \in \Omega^1_\varphi(\mathcal{B}; T\mathcal{S})$ *such that* $\boldsymbol{P}|_{\mathcal{U}_X} = \hat{\imath}(d^\nabla_1(\Xi))$*. We call* Ξ *a first Piola-Kirchhoff stress function for the motion* φ*. In coordinate systems* $\{X^I\}$ *and* $\{x^i\}$*, we have*

$$\left(\hat{\imath}^{-1}(\boldsymbol{P})\right)_{IJ}{}^{i} = \frac{\partial \Xi_J{}^i}{\partial X^I} - \frac{\partial \Xi_I{}^i}{\partial X^J}. \tag{293}$$

Conversely, if $\varphi : \mathcal{B} \to \mathcal{S}$ *admits a first Piola-Kirchhoff stress function* Ξ *in a neighborhood* $\mathcal{U}_X$*, i.e.* $\boldsymbol{P}|_{\mathcal{U}_X} = \hat{\imath}(d^\nabla_1(\Xi))$*, then* $(\mathrm{div}\boldsymbol{P})|_{\mathcal{U}_X} = 0$*.*

The above theorem is also globally valid if $\mathcal{B}$ is contractible. If flat manifolds $\mathcal{B}$ and $\mathcal{S}$ do not admit global orthonormal coordinates, one may restate the above theorem as Corollary 3.2.9. Although we derived nonlinear stress functions separately, one can obtain a relation between them using the relation between the Cauchy and the first and the second Piola-Kirchhoff stress tensors. Also, one should note that the expressions of the isomorphisms $\tilde{\imath}$ and $\hat{\imath}$ in non-orthonormal coordinate systems are not the same as the canonical relations (287) and (292). Stress functions can be defined for nonlinear elastostatics on flat 2-manifolds as well. In particular, it is straightforward to define the Cauchy stress function and the second Piola-Kirchhoff stress functions for flat 2-manifolds using Corollary 3.2.8. Regarding Theorem 3.2.11, suppose the only non-vanishing components of $\boldsymbol{\Xi}$ are $\Xi_3{}^1$ and $\Xi_3{}^2$ and they only depend on X^1 and X^2. Then, the independent components of $\boldsymbol{\xi} = d_1^{\nabla}\boldsymbol{\Xi}$ are

$$\xi_{13}{}^1 = \frac{\partial \Xi_3{}^1}{\partial X^1},\ \xi_{13}{}^2 = \frac{\partial \Xi_3{}^2}{\partial X^1},\ \xi_{23}{}^1 = \frac{\partial \Xi_3{}^1}{\partial X^2},\ \xi_{23}{}^2 = \frac{\partial \Xi_3{}^2}{\partial X^2}. \tag{294}$$

Suppose the associated first Piola-Kirchhoff stress tensor of a motion $\varphi : \mathcal{B} \to \mathcal{S}$ of the 2-manifold $\mathcal{B}$ satisfies $\boldsymbol{P} = D_{\mathcal{SF}}(\boldsymbol{\upsilon})$, where the homomorphism $D_{\mathcal{SF}} : \Omega^0_{\varphi}(\mathcal{B}; T\mathcal{S}) \to \Gamma(T\mathcal{B} \otimes T\varphi(\mathcal{B}))$ in the orthonormal coordinate systems $\{X^I\}$ and $\{x^1\}$ is expressed as

$$P^{11} = \frac{\partial \upsilon^1}{\partial X^2},\ P^{12} = \frac{\partial \upsilon^2}{\partial X^2},\ P^{21} = -\frac{\partial \upsilon^1}{\partial X^1},\ P^{22} = -\frac{\partial \upsilon^2}{\partial X^1}. \tag{295}$$

Note that $D_{\mathcal{SF}}$ is defined by using (292) and (294). Then, one can show that $\mathrm{div}\boldsymbol{P} = 0$. The tensor $\boldsymbol{\upsilon} \in \Omega^0_{\varphi}(\mathcal{B}; T\mathcal{S})$ is called a first Piola-Kirchhoff stress function for the motion φ of the 2-manifold $\mathcal{B}$.

3.3 Complexes in Linear and Nonlinear Elastostatics

We have already derived the differential operators of linear and nonlinear elastostatics. We are now ready to write the associated differential complexes. Let us first introduce resolutions of sheaves, which are suitable for expressing local results. We refer the reader to [23] for further details. Note that sheaves are not required for understanding the complexes of elastostatics and one can skip §3.3.1.

3.3.1 Resolutions of Sheaves

Let $\mathcal{X}$ be a topological space and consider the category $\mathrm{Op}(\mathcal{X})$ with open subsets of $\mathcal{X}$ as its objects. For open sets $U, V \subset \mathcal{X}$, $\mathsf{Mor}(U,V)$ contains only the inclusion map $i_{U,V} : U \hookrightarrow V$ if $U \subset V$, and $\mathsf{Mor}(U,V) = \varnothing$ if otherwise. Recall that a presheaf A of Abelian groups on $\mathcal{X}$ is a contravariant functor from $\mathrm{Op}(\mathcal{X})$ to the category of Abelian groups [23]. Thus, $A(U)$ is an Abelian group and the morphism $r_{U,V} := A(i_{U,V}) : A(V) \to A(U)$, is a homomorphism of Abelian groups called the restriction. A sheaf of Abelian groups $(\mathscr{A}, \pi, \mathcal{X})$ on $\mathcal{X}$ is a topological space $\mathscr{A}$ and a local homeomorphism $\pi : \mathscr{A} \to \mathcal{X}$ called projection. For each $x \in \mathcal{X}$, the stalk of $\mathscr{A}$ at x is defined as $\mathscr{A}_x := \pi^{-1}(x)$, which is an Abelian group with a continuous group operation [23]. Note that unlike vector bundles, the projection of a sheaf is also a local homeomorphism and not merely a surjective map. Let G be an Abelian group. A constant sheaf on $\mathcal{X}$ with stalk G is the sheaf $\mathcal{X} \times G$, where G is equipped with the discrete topology, i.e. any subset of G is an open set. The constant sheaf $\mathcal{X} \times G$ is also denoted by G. Sections of a sheaf $\mathscr{A}$ are defined similar to sections of fibered manifolds. The set of sections of $\mathscr{A}$ on U is denoted by $\mathscr{A}(U)$. If $U = \mathcal{X}$, then $\mathscr{A}(X)$ is also denoted by $\Gamma(\mathscr{A})$.

Let $\mathcal{M}$ be a manifold and consider the space of k-forms $\Omega^k(U)$ on an open subset $U \subset \mathcal{M}$. Then, Ω^k defines a presheaf on $\mathcal{M}$, where $r_{U,V}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}|_U$, with $\boldsymbol{\alpha} \in \Omega^k(V)$ and $U \subset V$. The presheaf Ω^k defines a sheaf $\Omega^k_{\mathcal{M}}$ as follows. Let $\mathcal{S}^x = \{\boldsymbol{\alpha} \in \Omega^k(U) : U \subset \mathcal{M}, x \in U\}$. One can define an equivalence relation on $\mathcal{S}^x$: suppose $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{S}^x$, with $\boldsymbol{\alpha} \in \Omega^k(U)$ and $\boldsymbol{\beta} \in \Omega^k(U')$. Then $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if there is a neighborhood $W \subset U \cap U'$ of x such that $\boldsymbol{\alpha}|_W = \boldsymbol{\beta}|_W$. The equivalence classes $\mathcal{S}^x/\sim$ is called the germs of Ω^k at x and is denoted by $(\Omega^k_{\mathcal{M}})_x$. The germ of $\boldsymbol{\alpha}$ at x, $[\boldsymbol{\alpha}]_x$, is the equivalence class of $\boldsymbol{\alpha}$ in $(\Omega^k_{\mathcal{M}})_x$. Let $\Omega^k_{\mathcal{M}} = \bigsqcup_{x\in\mathcal{M}}(\Omega^k_{\mathcal{M}})_x$, i.e. $\Omega^k_{\mathcal{M}}$ is the disjoint union of $(\Omega^k_{\mathcal{M}})_x$. For a fixed $\boldsymbol{\alpha} \in \Omega^k(U)$, the set $\alpha = \{[\boldsymbol{\alpha}]_x : x \in U\}$, is assumed to be an open set in $\Omega^k_{\mathcal{M}}$ and the topology of $\Omega^k_{\mathcal{M}}$ is taken to be the topology generated by these open sets. Note that $\Omega^k_{\mathcal{M}}$ is an $\Omega^0_{\mathcal{M}}$-module, i.e. $(\Omega^k_{\mathcal{M}})_x$ is an $(\Omega^0_{\mathcal{M}})_x$-module, and we have $[\phi]_x[\boldsymbol{\alpha}]_x + [\psi]_x[\boldsymbol{\beta}]_x = [\phi|_W\boldsymbol{\alpha}|_W + \psi|_W\boldsymbol{\beta}|_W]_x$, where $\phi \in \Omega^0(U_1)$, $\psi \in \Omega^0(U_2)$, $\boldsymbol{\alpha} \in \Omega^k(U_3)$, $\boldsymbol{\beta} \in \Omega^k(U_4)$, and $W \subset \cap_{i=1}^4 U_i$ is a neighborhood of x. One can show that $(\Omega^k_{\mathcal{M}}, \pi, \mathcal{M})$ is a sheaf, where $\pi : \Omega^k_{\mathcal{M}} \to \mathcal{M}$, $[\boldsymbol{\alpha}]_x \mapsto x$. Using a similar construction, one can define the sheaf of germs of an arbitrary presheaf [23]. The sheaf $\Omega^k_{\mathcal{M}}$ defines a presheaf $U \mapsto \Omega^k_{\mathcal{M}}(U)$. There is a natural mapping $\theta_U : \Omega^k(U) \to \Omega^k_{\mathcal{M}}(U)$, $\boldsymbol{\alpha} \mapsto (x \mapsto [\boldsymbol{\alpha}]_x)$. One can show that θ_U defines an isomorphism of presheaves [23], i.e. a natural transformation of functors, and therefore, we can identify $\Omega^k(U)$ and $\Omega^k_{\mathcal{M}}(U)$. But note that there is no one-to-one correspondence between $\Lambda^k T^*\mathcal{M}$ and $\Omega^k_{\mathcal{M}}$. Similar to vector bundles, one can consider algebraic constructions on sheaves. As an example, the tensor product of sheaves $\mathscr{A}$ and $\mathscr{B}$ on $\mathfrak{X}$ is a sheaf $\mathscr{A} \otimes \mathscr{B}$ on $\mathfrak{X}$ defined to be the sheaf of germs of the presheaf $U \mapsto \mathscr{A}(U) \otimes \mathscr{B}(U)$. One can show that $(\mathscr{A} \otimes \mathscr{B})_x = \mathscr{A}_x \otimes \mathscr{B}_x$, $\forall x \in \mathfrak{X}$. Moreover, $\mathscr{A}(U) \otimes \mathscr{B}(U) \approx A(U) \otimes B(U) = A \otimes B(U)$, and hence $\mathscr{A} \otimes \mathscr{B}$ can be considered as the sheaf of germs of the presheaf $U \mapsto A \otimes B(U)$. This identification is also valid for other functors such as $\oplus$, Λ^k, and S^k [23]. We denote the sheaf of germs of local sections of $T\mathcal{M}$ and $T^*\mathcal{M}$ by $\mathscr{T}\mathcal{M}$ and $\mathscr{T}^*\mathcal{M} = (\mathscr{T}\mathcal{M})^*$, respectively. We have $\Omega^k_{\mathcal{M}} = \Lambda^k \mathscr{T}^*\mathcal{M}$.

Suppose $\mathscr{A}$ and $\mathscr{B}$ are sheaves over $\mathfrak{X}$. A homomorphism of sheaves $h : \mathscr{A} \to \mathscr{B}$ is a stalk-preserving mapping covering $\mathrm{Id}_{\mathfrak{X}}$, which is a stalk-wise homomorphism, i.e. the restriction $h_x : \mathscr{A}_x \to \mathscr{B}_x$ is a homomorphism for all $x \in \mathfrak{X}$. A sheaf $\mathscr{C}$ is a subsheaf of $\mathscr{A}$ if it is an open subset of $\mathscr{A}$ and $\mathscr{C}_x = \mathscr{C} \cap \mathscr{A}_x$ is a subgroup of $\mathscr{A}_x$ for all $x \in \mathfrak{X}$. One can show that $\ker h$ and $\operatorname{im} h$ are subsheaves of $\mathscr{A}$ and $\mathscr{B}$, respectively. A sequence of sheaves is exact if the image of each operator is equal to the kernel of the next one. A sequence of sheaves $\cdots \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to \cdots$ induces a sequence of presheaves $\cdots \to \mathscr{A}(U) \to \mathscr{B}(U) \to \mathscr{C}(U) \to \cdots$, $\forall U \subset \mathfrak{X}$. An exact sequence of sheaves does not necessarily induce an exact sequence of presheaves [23]. A resolution of a sheaf $\mathscr{A}$ is a sequence $\{\mathscr{L}^k\}_{k=0}^{\infty}$ of sheaves together with homomorphisms $h^k : \mathscr{L}^k \to \mathscr{L}^{k+1}$ with $h^{k+1} \circ h^k = 0$, and an augmentation homomorphism $\varepsilon : \mathscr{A} \to \mathscr{L}^0$, such that the sequence

$$0 \longrightarrow \mathscr{A} \xrightarrow{\ \varepsilon\ } \mathscr{L}^0 \xrightarrow{\ h^0\ } \mathscr{L}^1 \xrightarrow{\ h^1\ } \mathscr{L}^2 \xrightarrow{\ h^2\ } \cdots \tag{296}$$

is exact. Let $\mathcal{M}$ be a manifold and consider the exterior derivative $d^k : \Omega^k(U) \to \Omega^{k+1}(U)$ for $U \subset \mathcal{M}$. Note that d is only an $\mathbb{R}$-module homomorphism and not an Ω^0-module homomorphism. The exterior derivative induces the homomorphisms $d^k : \Omega^k_{\mathcal{M}} \to \Omega^{k+1}_{\mathcal{M}}$, $[\boldsymbol{\alpha}]_x \mapsto [d^k\boldsymbol{\alpha}]_x$. Clearly, we have $d^{k+1} \circ d^k = 0$. For the constant sheaf $\mathcal{M} \times \mathbb{R}$, or simply $\mathbb{R}$, one can define an augmentation $\varepsilon : \mathbb{R} \to \Omega^0_{\mathcal{M}}$, $(x, c) \mapsto [c]_x$, where $[c]_x$ is the germ of the constant function $f(x) = c$, $\forall x \in \mathcal{M}$. Thus, we obtain the following sequence:

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ \varepsilon\ } \Omega^0_{\mathcal{M}} \xrightarrow{\ d^0\ } \Omega^1_{\mathcal{M}} \xrightarrow{\ d^1\ } \Omega^2_{\mathcal{M}} \xrightarrow{\ d^2\ } \cdots \tag{297}$$

Let $d^k[\boldsymbol{\alpha}]_x = [d^k\boldsymbol{\alpha}]_x = [0]_x$, i.e. $\boldsymbol{\alpha}$ is closed on an open subset $U \subset \mathcal{M}$. The Poincaré lemma [22], states that there is a neighborhood $W \subset U$ of x and $\boldsymbol{\eta} \in \Omega^{k-1}(W)$ such that $\boldsymbol{\alpha}|_W = d^{k-1}\boldsymbol{\eta}$, i.e. $[\boldsymbol{\alpha}]_x = [d^{k-1}\boldsymbol{\eta}]_x = d^{k-1}[\boldsymbol{\eta}]_x$. Therefore, the above sequence is a resolution of the constant sheaf $\mathbb{R}$ on a manifold $\mathcal{M}$, regardless of topological properties of $\mathcal{M}$. The restriction of (297) to any open subset of $\mathcal{M}$ is still exact. Of course, the induced sequence on $\Omega^k(\mathcal{M})$ is not exact in general and its cohomology depends on some topological properties of $\mathcal{M}$. Let V be a finite-dimensional vector space. The twisted exterior derivative $d^k : \Omega^k(\mathcal{M}; V) \to \Omega^{k+1}(\mathcal{M}; V)$ induces a resolution of constant sheaf V given by $0 \to V \to \Omega^0_{\mathcal{M};V} \to \Omega^1_{\mathcal{M};V} \to \cdots$, where $\Omega^k_{\mathcal{M};V}$ is the sheaf of V-valued k-forms.

3.3.2 Linear Elastostatics Complexes

Let $(\mathcal{B}, \boldsymbol{g})$ be a Riemannian 3-manifold with a constant sectional curvature and the Levi-Civita connection ∇. The linear elastostatics complex is induced by the Calabi complex [26]. Suppose $\mathcal{K}(\mathcal{B})$ is the space of Killing vector fields on $\mathcal{B}$. By using the operators defined in the previous section, the Calabi complex is written in the first row of the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(\mathcal{B}) \arrow[r, "\varsigma"] \arrow[d, phantom] & \Omega^1(\mathcal{B}) \arrow[r, "D_{\mathcal{S}}"] \arrow[d, "\iota"] & \Gamma(S^2T^*\mathcal{B}) \arrow[r, "D_{\mathcal{C}}"] \arrow[d, "\iota"] & \Gamma(\mathcal{C}^4\mathcal{B}) \arrow[r, "D_{\mathcal{B}}"] \arrow[d, "\iota"] & \Gamma(\mathcal{D}^5\mathcal{B}) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \mathcal{K}(\mathcal{B}) \arrow[r, "\hat{\varsigma}"] & \Omega^1(\mathcal{B}; \mathcal{L}^{\langle 2\rangle}) \arrow[r, "D_1"] & \Gamma(S^2T^*\mathcal{B} \otimes \mathcal{L}^{\langle 2\rangle}) \arrow[r, "D_2"] & \Gamma(\mathcal{C}^4\mathcal{B} \otimes \mathcal{L}^{\langle 2\rangle}) \arrow[r, "D_3"] & \Gamma(\mathcal{D}^5\mathcal{B} \otimes \mathcal{L}^{\langle 2\rangle}) \arrow[r] & 0
\end{tikzcd} \tag{298}$$

In this diagram, we have $\varsigma(\boldsymbol{K}) = \boldsymbol{K}^\flat$, and $\hat{\varsigma}(\boldsymbol{K}) = \boldsymbol{K}^\flat \otimes \boldsymbol{\mu}^{\langle 2\rangle}$. We observed that the Riemannian metric $\boldsymbol{g}$ allows us to define isomorphisms ι and thus, the Calabi complex can be identified with the second row of the above diagram, which we call the Eastwood complex [42, 41]. Consequently, Cohomology groups of the Calabi and Eastwood complexes are the same. The Eastwood complex depends on the projective structure $[\nabla]$ on $\mathcal{B}$, where $[\nabla]$ is the projective equivalence class of the Levi-Civita connection ∇. In the next section, we will show that if $\mathcal{B}$ is an open subset of $\mathbb{R}^3$, the Eastwood complex is induced by a certain twisted de Rham complex. Of course, this result does not imply that the Calabi complex is metric independent, as we need metric to identify these complexes. Let $\mathcal{B} \subset \mathbb{R}^3$ be an open subset equipped with the standard metric of $\mathbb{R}^3$. Then, $\mathcal{K}(\mathcal{B})$ is isomorphic to $\mathfrak{euc}(\mathbb{R}^3)$. By using Corollary 3.2.7, we can also define Beltrami stress functions for $\mathcal{B} \subset \mathbb{R}^3$. Consequently, we obtain the following diagram.

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{euc}(\mathbb{R}^3) \arrow[r] & \Omega^1(\mathcal{B}) \arrow[r, "D_{\mathbb{S}}"] & \Gamma(S^2T^*\mathcal{B}) \arrow[r, "D_{\mathbb{C}}"] & \Gamma(\mathbb{C}^4\mathcal{B}) \arrow[r, "D_{\mathbb{B}}"] \arrow[d, "\tilde{\iota}"] & \Gamma(\mathbb{D}^5\mathcal{B}) \arrow[r] \arrow[d, "\underline{\iota}"] & 0 \\
 & & & & \Gamma(S^2T\mathcal{B}) \arrow[r, "\mathrm{div}"] & \mathfrak{X}(\mathcal{B}) \arrow[r] & 0
\end{tikzcd}
\tag{299}
$$

Here, we have $\mathrm{div} \circ \tilde{\iota} = \underline{\iota} \circ D_{\mathbb{B}}$, where in the Cartesian coordinate system, the isomorphism $\underline{\iota} : \Gamma(\mathbb{D}^5\mathcal{B}) \to \mathfrak{X}(\mathcal{B})$, $\boldsymbol{h} \mapsto \boldsymbol{Z}$, is given by $h_{12323} \mapsto Z^1$, $h_{21313} \mapsto Z^2$, and $h_{31212} \mapsto Z^3$. The linear elastostatics complex for $\mathcal{B} \subset \mathbb{R}^3$ reads

$$
0 \longrightarrow \mathfrak{euc}(\mathbb{R}^3) \longrightarrow \Omega^1(\mathcal{B}) \xrightarrow{D_{\mathbb{S}}} \Gamma(S^2T^*\mathcal{B}) \xrightarrow{\widetilde{D}_{\mathbb{C}}} \Gamma(S^2T\mathcal{B}) \xrightarrow{\mathrm{div}} \mathfrak{X}(\mathcal{B}) \longrightarrow 0,
\tag{300}
$$

where $\widetilde{D}_{\mathbb{C}} := \tilde{\iota} \circ D_{\mathbb{C}}$. The linear elastostatics complex and the Calabi complex have the same cohomology groups. In particular, (300) is exact on contractible bodies. Since $\Lambda^2\mathbb{R}^4 \approx \mathfrak{euc}(\mathbb{R}^3)$, the complex (300) induces the following resolution of the constant sheaf $\mathcal{B} \times \Lambda^2\mathbb{R}^4$:

$$
0 \longrightarrow \Lambda^2\mathbb{R}^4 \longrightarrow \Omega^1_{\mathcal{B}} \xrightarrow{D_{\mathbb{S}}} S^2\mathscr{T}^*\mathcal{B} \xrightarrow{\widetilde{D}_{\mathbb{C}}} S^2\mathscr{T}\mathcal{B} \xrightarrow{\mathrm{div}} \mathscr{T}\mathcal{B} \longrightarrow 0.
\tag{301}
$$

The complex (300) is the complex that Arnold et al. [8] used for developing a numerical scheme. They rewrote this complex on less smooth spaces and then, they directly discretized the resulting complex. However, since the symmetry of strain and stress tensors is strictly imposed in (300), the resulting discrete scheme is very complicated and requires large numbers of degrees of freedom for each cell of a mesh [10, 11]. Alternatively, one can develop numerical schemes that are based on a mixed formulation that weakly imposes the symmetry of the stress tensor [10]. To this end, Arnold et al. [9] introduced the Arnold-Falk-Winther elastostatics complex as follows:

$$0 \longrightarrow \mathfrak{euc}(\mathbb{R}^3) \longrightarrow \begin{matrix} \mathfrak{X}(\mathcal{B}) \\ \oplus \\ \Omega^1(\mathcal{B}) \end{matrix} \longrightarrow \Gamma(\otimes^2 T^*\mathcal{B}) \longrightarrow \Gamma(\otimes^2 T\mathcal{B}) \longrightarrow \begin{matrix} \mathfrak{X}(\mathcal{B}) \\ \oplus \\ \Omega^1(\mathcal{B}) \end{matrix} \longrightarrow 0. \tag{302}$$

We will study this complex in the next section, where we will show that the complex (302) can be constructed from the same twisted de Rham complex that induces (300). Arnold et al. [9] derived (302) by an equivalent construction.

For 2-manifolds with constant sectional curvatures, the Calabi complex terminates after $D_{\mathbb{B}}$ and therefore, the Calabi and the Eastwood complex for 2-manifolds are as follows.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(\mathcal{B}) \arrow[r, "\varsigma"] & \Omega^1(\mathcal{B}) \arrow[r, "D_{\mathbb{S}}"] \arrow[d, "\iota"] & \Gamma(S^2T^*\mathcal{B}) \arrow[r, "D_{\mathbb{C}}"] \arrow[d, "\iota"] & \Gamma(\mathbb{C}^4\mathcal{B}) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \mathcal{K}(\mathcal{B}) \arrow[r, "\hat{\varsigma}"] & \Omega^1(\mathcal{B}; \mathcal{L}^{\langle 2 \rangle}) \arrow[r, "D_1"] & \Gamma(S^2T^*\mathcal{B} \otimes \mathcal{L}^{\langle 2 \rangle}) \arrow[r, "D_2"] & \Gamma(\mathbb{C}^4\mathcal{B} \otimes \mathcal{L}^{\langle 2 \rangle}) \arrow[r] & 0
\end{tikzcd} \tag{303}$$

Thus, the linear elastostatics complex (300) for an open subset $\mathcal{B} \subset \mathbb{R}^2$ reads

$$0 \longrightarrow \mathfrak{euc}(\mathbb{R}^2) \longrightarrow \Omega^1(\mathcal{B}) \xrightarrow{D_{\mathbb{S}}} \Gamma(S^2T^*\mathcal{B}) \xrightarrow{D_{\mathbb{C}}} \Gamma(\mathbb{C}^4\mathcal{B}) \longrightarrow 0. \tag{304}$$

We call (304) the kinematic complex of 2D linear elastostatics, since it only deals with the kinematics of motion. We also obtain a resolution of constant sheaf $\mathcal{B} \times \Lambda^2\mathbb{R}^3$:

$$0 \longrightarrow \Lambda^2\mathbb{R}^3 \longrightarrow \Omega^1_{\mathcal{B}} \xrightarrow{D_{\mathbb{S}}} S^2\mathscr{T}^*\mathcal{B} \xrightarrow{D_{\mathbb{C}}} \mathscr{C}^4\mathcal{B} \longrightarrow 0, \tag{305}$$

where $\mathscr{C}^4\mathcal{B}$ is the sheaf of germs of local sections of $\mathbb{C}^4\mathcal{B}$. On the other hand, Corollary 3.2.8 implies that we also have the kinetic complex for 2D linear elastostatics:

$$0 \longrightarrow \mathfrak{euc}(\mathbb{R}^2) \xrightarrow{a} \Omega^0(\mathcal{B}) \xrightarrow{D_{\mathcal{A}}} \Gamma(S^2T^*\mathcal{B}) \xrightarrow{\text{div}} \mathfrak{X}(\mathcal{B}) \longrightarrow 0, \tag{306}$$

where the augmentation mapping $a : \mathfrak{euc}(\mathbb{R}^2) \to \Omega^0(\mathcal{B})$ in the Cartesian coordinate $\{X^i\}$ has the expression

$$\begin{bmatrix} 0 & 0 & 0 \\ c_1 & 0 & c_3 \\ c_2 & -c_3 & 0 \end{bmatrix} \xmapsto{a} \big((X^1, X^2) \mapsto c_1 X^1 + c_2 X^2 + c_3\big). \tag{307}$$

The kinetic complex (306) can be considered as a restriction of the kinetic part of the 3D linear elastostatics complex in the following sense: An open subset $\mathcal{B} \subset \mathbb{R}^2$ can be extended to the open subset $\underline{\mathcal{B}} := \mathcal{B} \times (-\epsilon, \epsilon) \subset \mathbb{R}^3$, where $\epsilon > 0$ is an arbitrary real number. Accordingly, a stress tensor $\boldsymbol{\sigma} \in \Gamma(S^2T^*\mathcal{B})$ induces the stress tensor $\underline{\boldsymbol{\sigma}} \in \Gamma^2(S^2T^*\underline{\mathcal{B}})$ defined as $(\underline{\sigma}^{11}, \underline{\sigma}^{12}, \underline{\sigma}^{22}) = (\sigma^{11}, \sigma^{12}, \sigma^{22})$, and $(\underline{\sigma}^{13}, \underline{\sigma}^{23}, \underline{\sigma}^{33}) = (0,0,0)$. Clearly, div $\boldsymbol{\sigma} = 0$, if and only if div $\underline{\boldsymbol{\sigma}} = 0$. In this case, an Airy stress function ψ for $\boldsymbol{\sigma}$ induces a Beltrami stress function $\boldsymbol{\Phi}$ for $\underline{\boldsymbol{\sigma}}$, where the only nonvanishing component of $\boldsymbol{\Phi}$ is $\Phi_{33} = \psi$. Arnold et al. [12] used the kinetic complex (306) to derive the first stable numerical scheme for the mixed formulation of 2D linear elastostatics.

3.3.3 Nonlinear Elastostatics Complexes

Let $(\mathcal{B}, \boldsymbol{G})$ and $(\mathcal{S}, \boldsymbol{g})$ be Riemannian 3-manifolds and let $C(\mathcal{B}, \mathcal{S})$ denote the space of smooth embeddings $\varphi : \mathcal{B} \to \mathcal{S}$. We can either use the Green deformation $\boldsymbol{C} = \varphi^*\boldsymbol{g}$, or the deformation gradient $\boldsymbol{F} = T\varphi$ to write a sequence of differential operators for nonlinear elastostatics. By using $\boldsymbol{C}$, we will obtain two separate short sequences representing the kinematics and kinetics of motion. Let $(\mathcal{S}, \boldsymbol{g})$ have constant sectional curvature k. We have the kinematic short sequence

$$C(\mathcal{B}, \mathcal{S}) \xrightarrow{D_{\mathcal{M}}} \Gamma_{\mathcal{M}}(S^2T^*\mathcal{B}) \xrightarrow{D_{\mathcal{R}}} \Gamma(\mathcal{C}^4\mathcal{B}), \tag{308}$$

where $\Gamma_{\mathcal{M}}(S^2T^*\mathcal{B})$ is the space of Riemannian metrics on $\mathcal{B}$, $D_{\mathcal{M}}(\varphi) := \varphi^*\boldsymbol{g}$, and the tensor $D_{\mathcal{R}}(\boldsymbol{C})$, $\forall \boldsymbol{C} \in \Gamma_{\mathcal{M}}(S^2T^*\mathcal{B})$, is a $\binom{0}{4}$-tensor given by

$$\begin{aligned}&\big(D_{\mathcal{R}}(\boldsymbol{C})\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) = \\ &\qquad\qquad \boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{T}) - \mathrm{k}\boldsymbol{C}(\boldsymbol{Z},\boldsymbol{Y})\boldsymbol{C}(\boldsymbol{X},\boldsymbol{T}) + \mathrm{k}\boldsymbol{C}(\boldsymbol{Z},\boldsymbol{X})\boldsymbol{C}(\boldsymbol{Y},\boldsymbol{T}).\end{aligned} \tag{309}$$

The compatibility equation (261) implies that $D_{\mathcal{R}} \circ D_{\mathcal{M}} = 0$. Note that $D_{\mathcal{R}}$ and $D_{\mathcal{M}}$ are not linear homomorphisms. If $\mathcal{B} \subset \mathcal{S} = \mathbb{R}^3$, and the Cartesian coordinate system of $\mathbb{R}^3$ is used for both $\mathcal{B}$ and $\mathcal{S}$, one can define the displacement vector field $\boldsymbol{U} \in \mathfrak{X}(\mathcal{B})$ by $\boldsymbol{U}(X) = \varphi(X) - X$, $\forall X \in \mathcal{B}$. Let $\mathbb{P}_{\varphi} : T\mathcal{B} \to T\mathcal{S}$ be the parallel transport in $\mathbb{R}^3$ with respect to φ given by $\mathbb{P}_{\varphi}(X,\mathbf{Y}) = (\varphi(X),\mathbf{Y})$, $\mathbf{Y} \in T_X\mathcal{B}$. It is straightforward to show that $T\varphi = \mathbb{P}_{\varphi} \circ (\mathrm{Id} + \nabla\boldsymbol{U})$. Using the fact that $\mathbb{P}_{\varphi}^{\mathsf{T}} = \mathbb{P}_{\varphi}^{-1}$, one can show that $\boldsymbol{C}^{\sharp} = \mathrm{Id} + \nabla\boldsymbol{U} + \nabla^{\mathsf{T}}\boldsymbol{U} + \nabla^{\mathsf{T}}\boldsymbol{U} \circ \nabla\boldsymbol{U}$ [82], where $\nabla^{\mathsf{T}}\boldsymbol{U} := (\nabla\boldsymbol{U})^{\mathsf{T}}$, and $\boldsymbol{G}(\boldsymbol{C}^{\sharp}(\boldsymbol{X}),\boldsymbol{Y}) = \boldsymbol{C}(\boldsymbol{X},\boldsymbol{Y})$. This defines a mapping $\bar{D}_{\mathcal{M}}(\boldsymbol{U}) = \boldsymbol{C}$. Now, the sequence (308) can be rewritten as

$$\mathfrak{X}(\mathcal{B}) \xrightarrow{\bar{D}_{\mathcal{M}}} \Gamma_{\mathcal{M}}(S^2T^*\mathcal{B}) \xrightarrow{D_{\mathcal{R}}} \Gamma(\mathcal{C}^4\mathcal{B}). \tag{310}$$

Theorem 3.2.1 states that the corresponding sequence of sheaves is an exact sequence, i.e. the sequence (308) and (310) are locally exact. If $(\mathcal{B},\boldsymbol{C})$ and $(\mathcal{S},\boldsymbol{g})$ are flat 3-manifolds, then we can use Corollary 3.2.9 to define a kinetic complex in terms of $\boldsymbol{C}$. Suppose the isomorphism $\tilde{\imath}$ defined in Lemma 3.2.6 is globally defined on $(\mathcal{B},\boldsymbol{C})$. Then, the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & \ker D^{\boldsymbol{C}}_{\mathcal{C}} \arrow[r, hook] & \Gamma(S^2T^*\mathcal{B}) \arrow[r, "D^{\boldsymbol{C}}_{\mathcal{C}}"] & \Gamma(\mathcal{C}^4\mathcal{B}) \arrow[r, "D^{\boldsymbol{C}}_{\mathcal{B}}"] \arrow[d, "\tilde{\imath}"] & \Gamma(\mathcal{D}^5\mathcal{B}) \arrow[r] \arrow[d, "\underline{\iota}"] & 0 \\ & & & \Gamma(S^2T\mathcal{B}) \arrow[r, "\operatorname{div}^{\boldsymbol{C}}"] & \mathfrak{X}(\mathcal{B}) \arrow[r] & 0 \end{tikzcd} \tag{311}$$

In this diagram, $\ker D^{\boldsymbol{C}}_{\mathcal{C}}$ is the kernel of $D^{\boldsymbol{C}}_{\mathcal{C}}$ and $D^{\boldsymbol{C}}_{\mathcal{B}}$ is defined similar to $D_{\mathcal{B}}$ but by using $\nabla^{\boldsymbol{C}}$. The isomorphism $\underline{\iota}$ in the normal coordinate system $\{X^I\}$ reads $h_{12323} \mapsto Z^1$, $h_{21313} \mapsto Z^2$, and $h_{31212} \mapsto Z^3$. Accordingly, one obtains the following kinetic complex for nonlinear elastostatics:

$$0 \longrightarrow \ker D^{\boldsymbol{C}}_s \hookrightarrow \Gamma(S^2T^*\mathcal{B}) \xrightarrow{D^{\boldsymbol{C}}_s} \Gamma(S^2T\mathcal{B}) \xrightarrow{\operatorname{div}^{\boldsymbol{C}}} \mathfrak{X}(\mathcal{B}) \longrightarrow 0, \tag{312}$$

where $D^{\boldsymbol{C}}_s := \tilde{\iota} \circ D^{\boldsymbol{C}}_{\mathcal{C}}$. Corollary 3.2.9 implies that (312) induces an exact sequence of sheaves and is an exact complex on contractible bodies. Note that (308) and (312) cannot be joined, since $D^{\boldsymbol{C}}_s$ and $D_{\mathcal{R}}$ are not the same. In fact, $D^{\boldsymbol{C}}_s$ is the linearization of $D_{\mathcal{R}}$. By using Cauchy stress functions, we obtain another kinetic complex for nonlinear elastostatics:

$$0 \longrightarrow \ker \widehat{D}_{\mathcal{C}} \hookrightarrow \Gamma(S^2T^*\varphi(\mathcal{B})) \xrightarrow{\widehat{D}_{\mathcal{C}}} \Gamma(S^2T\varphi(\mathcal{B})) \xrightarrow{\operatorname{div}} \mathfrak{X}(\varphi(\mathcal{B})) \longrightarrow 0, \tag{313}$$

where $\widehat{D}_{\mathcal{C}}$ in an orthonormal coordinate system $\{x^i\}$ on $(\varphi(\mathcal{B}), \boldsymbol{g})$ has the same expression as the operator $\widetilde{D}_{\mathcal{C}}$ introduced in (300) in an orthonormal coordinate system on $(\mathcal{B}, \boldsymbol{g})$. In contrary to using $\boldsymbol{C}$, using $\boldsymbol{F}$ leads to a complex that contains both the kinematics and kinetics of motion. Suppose $(\mathcal{B}, \boldsymbol{G})$ and $(\mathcal{S}, \boldsymbol{g})$ are flat Riemannian 3-manifolds with global orthonormal coordinate systems $\{X^I\}$ and $\{x^i\}$, respectively. Let ∇ be the Levi-Civita connection of $\boldsymbol{g}$ and consider an embedding $\varphi : \mathcal{B} \to \mathcal{S}$ that represents a motion of $\mathcal{B}$ in $\mathcal{S}$. We call $\boldsymbol{Y} \in \Omega^0_\varphi(\mathcal{B}; T\mathcal{S})$ a uniform vector field of $\mathcal{S}$ on $\mathcal{B}$ covering φ if $\nabla \boldsymbol{Y} = 0$. Let $\mathcal{U}_\varphi(\mathcal{B}, \mathcal{S})$ denote the space of uniform vector fields of $\mathcal{S}$ on $\mathcal{B}$ covering φ. The first row of the following diagram is a twisted de Rham complex that is associated to the motion φ:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{U}_\varphi(\mathcal{B},\mathcal{S}) \arrow[r, hook] & \Omega^0_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "d_0^\nabla"] & \Omega^1_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "d_1^\nabla"] & \Omega^2_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "d_2^\nabla"] \arrow[d, "\hat{\imath}"] & \Omega^3_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r] \arrow[d, "\check{\imath}"] & 0 \\ & & & & \Gamma(T\mathcal{B}\otimes T\varphi(\mathcal{B})) \arrow[r, "\mathrm{div}"] & \Omega^0_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r] & 0 \end{tikzcd} \tag{314}$$

Of course, this twisted de Rham complex is an exact complex if $\mathcal{B}$ is contractible. In the above diagram, the isomorphism $\hat{\imath}$ is defined in (292) and the expression of the isomorphism $\check{\imath} : \Omega^3_\varphi(\mathcal{B};T\mathcal{S}) \to \Omega^0_\varphi(\mathcal{B};T\mathcal{S})$ in the orthonormal coordinates $\{X^I\}$ and $\{x^i\}$ is ${\xi_{123}}^i \mapsto \alpha^i$. Note that similar to (292), the given canonical form of $\check{\imath}$ is only valid in orthonormal coordinate systems. It is easy to check that the above diagram commutes, i.e. $\mathrm{div} \circ \hat{\imath} = \check{\imath} \circ d_2^\nabla$. Thus, we obtain the following complex for nonlinear elastostatics corresponding to a motion φ:

$$0 \longrightarrow \mathfrak{U}_\varphi(\mathcal{B},\mathcal{S}) \hookrightarrow \Omega^0_\varphi(\mathcal{B};T\mathcal{S}) \xrightarrow{d_0^\nabla} \Omega^1_\varphi(\mathcal{B};T\mathcal{S}) \xrightarrow{D_f^{\mathfrak{T}}} \Gamma(T\mathcal{B}\otimes T\varphi(\mathcal{B})) \xrightarrow{\mathrm{div}} \Omega^0_\varphi(\mathcal{B};T\mathcal{S}) \longrightarrow 0, \tag{315}$$

where $D_f^{\mathfrak{T}} := \hat{\imath} \circ d_1^\nabla$. The cohomology groups of the complex (315) are the same as those of the twisted complex (314). Recall the generalized compatibility problem introduced in §3.2.3.1. Then, the space $\mathfrak{U}_\varphi(\mathcal{B},\mathcal{S})$ can be considered as the space of translations in $\mathcal{S}$. The space $\Omega^1_\varphi(\mathcal{B};T\mathcal{S})$ denotes both the space of deformation gradients $\boldsymbol{F}$ and the space of first Piola-Kirchhoff stress functions. Similarly, $\Gamma(T\mathcal{B}\otimes T\varphi(\mathcal{B}))$ represents two different physical spaces, namely, the space of covariant exterior derivatives of deformation gradients and the space of first Piola-Kirchhoff stresses. Let $\mathscr{U}_\varphi$ denote the sheaf of germs of locally uniform vector fields of $\mathcal{S}$ on $\mathcal{B}$ covering φ. Then, the complex (314) induces the following resolution of $\mathscr{U}_{\mathcal{B}}$:

$$0 \longrightarrow \mathscr{U}_\varphi \hookrightarrow \Omega^0_\varphi \xrightarrow{d_0^\nabla} \Omega^1_\varphi \xrightarrow{d_1^\nabla} \Omega^2_\varphi \xrightarrow{d_2^\nabla} \Omega^3_\varphi \longrightarrow 0, \tag{316}$$

where Ω^k_φ is the sheaf of germs of local $T\mathcal{S}$-valued k-forms on $\mathcal{B}$ over φ. Note that (316) is exact on any flat 3-manifold regardless of its topological properties. Similarly, one can write a sequence of sheaves for (315). In particular, if $\mathcal{S} = \mathbb{R}^3$, then we obtain a resolution of the constant sheaf $\mathbb{R}^3$:

$$0 \longrightarrow \mathbb{R}^3 \xrightarrow{\mathrm{a}} \Omega^0_\varphi \xrightarrow{d_0^\nabla} \Omega^1_\varphi \xrightarrow{D_f^{\mathcal{T}}} \mathscr{T}\mathcal{B} \otimes \mathscr{T}\varphi(\mathcal{B}) \xrightarrow{\mathrm{div}} \Omega^0_\varphi \longrightarrow 0, \tag{317}$$

where $\mathscr{T}\mathcal{B} \otimes \mathscr{T}\varphi(\mathcal{B})$ is the sheaf of germs of local sections of $T\mathcal{B} \otimes T\varphi(\mathcal{B})$. The augmentation homomorphism of the above resolution is defined as $\mathrm{a} : (X, \mathbf{v}) \mapsto [\mathbf{v}]_X$, and therefore, each vector $\mathbf{v} \in \mathbb{R}^3$ can be considered as representing the translation by constant vector $\mathbf{v}$. For an open subset $\mathcal{B} \subset \mathcal{S} = \mathbb{R}^3$, one can further simplify (314) and (315). Let the coordinate systems $\{X^I\}$ and $\{x^i\}$ on $\mathcal{B}$ and $\mathcal{S}$ be the Cartesian coordinate system $\{X^i\}$ with the basis vectors $\{\mathbf{E}_i\}$. Let $\boldsymbol{\beta} \in \Omega^k_\varphi(\mathcal{B}; T\mathcal{S})$, where $\boldsymbol{\beta}(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) = \boldsymbol{\beta}^i(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)(\partial/\partial x^i) \in \mathfrak{X}(\varphi(\mathcal{B}))$, $\forall \boldsymbol{X}_j \in \mathfrak{X}(\mathcal{B})$. For any embedding $\varphi : \mathcal{B} \to \mathcal{S} = \mathbb{R}^3$, one can define isomorphisms $\varsigma_k : \Omega^k_\varphi(\mathcal{B}; T\mathcal{S}) \to \Omega^k(\mathcal{B}; \mathbb{R}^3)$, that in the Cartesian coordinate system have the expression $\beta_{I_1 \cdots I_k}{}^j \mapsto \beta_{i_1 \cdots i_k}{}^j$. For $\boldsymbol{\beta} \in \Omega^1_\varphi(\mathcal{B}; T\mathcal{S})$, we have

$$\begin{aligned} \left(d_1^\nabla \boldsymbol{\beta}\right)(\boldsymbol{X}, \boldsymbol{Y}) &= \nabla_{\varphi_* \boldsymbol{X}} \left(\boldsymbol{\beta}^i(\boldsymbol{Y})\mathbf{E}_i\right) - \nabla_{\varphi_* \boldsymbol{Y}} \left(\boldsymbol{\beta}^i(\boldsymbol{X})\mathbf{E}_i\right) - \boldsymbol{\beta}^i([\boldsymbol{X}, \boldsymbol{Y}])\mathbf{E}_i \\ &= \Big(\boldsymbol{X}\left(\boldsymbol{\beta}^i(\boldsymbol{Y})\right) - \boldsymbol{Y}\left(\boldsymbol{\beta}^i(\boldsymbol{X})\right) - \boldsymbol{\beta}^i([\boldsymbol{X}, \boldsymbol{Y}])\Big)\mathbf{E}_i \\ &= \left(\varsigma_2^{-1} \circ d_1 \circ \varsigma_1(\boldsymbol{\beta})\right)(\boldsymbol{X}, \boldsymbol{Y}), \end{aligned} \tag{318}$$

where $d_k : \Omega^k(\mathcal{B}; \mathbb{R}^3) \to \Omega^{k+1}(\mathcal{B}; \mathbb{R}^3)$ is the usual twisted exterior derivative defined as $d_k(\boldsymbol{\omega} \otimes \mathbf{V}) = d_k(\boldsymbol{\omega}) \otimes \mathbf{V}$, $\forall \boldsymbol{\omega} \in \Omega^k(\mathcal{B})$ and $\forall \mathbf{V} \in \mathbb{R}^3$ [22]. Hence, we have $\varsigma_2 \circ d_1^\nabla = d_1 \circ \varsigma_1$. Similarly, one can show that $\varsigma_{k+1} \circ d_k^\nabla = d_k \circ \varsigma_k$, and therefore, for any embedding

$\varphi : \mathcal{B} \to \mathbb{R}^3$, the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r] & \Omega^0_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "d_0^\nabla"] \arrow[d, "\varsigma_0"] & \Omega^1_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "d_1^\nabla"] \arrow[d, "\varsigma_1"] & \Omega^2_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "d_2^\nabla"] \arrow[d, "\varsigma_2"] & \Omega^3_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r] \arrow[d, "\varsigma_3"] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r] & \Omega^0(\mathcal{B};\mathbb{R}^3) \arrow[r, "d_0"] & \Omega^1(\mathcal{B};\mathbb{R}^3) \arrow[r, "d_1"] & \Omega^2(\mathcal{B};\mathbb{R}^3) \arrow[r, "d_2"] \arrow[d, "\hat{\varsigma}"] & \Omega^3(\mathcal{B};\mathbb{R}^3) \arrow[r] \arrow[d, "\check{\varsigma}"] & 0 \\
 & & & & \Gamma(T\mathcal{B}\otimes T\mathcal{B}) \arrow[r, "\mathrm{div}"] & \mathfrak{X}(\mathcal{B}) \arrow[r] & 0
\end{tikzcd} \tag{319}$$

In this diagram, the expression of the isomorphisms $\hat{\varsigma} : \Omega^2(\mathcal{B};\mathbb{R}^3) \to \Gamma(T\mathcal{B}\otimes T\mathcal{B})$, $\boldsymbol{\beta} \mapsto \boldsymbol{\varpi}$, and $\check{\varsigma} : \Omega^3(\mathcal{B};\mathbb{R}^3) \to \mathfrak{X}(\mathcal{B})$, $\boldsymbol{\xi} \mapsto \boldsymbol{X}$, in the Cartesian coordinate system are ${\beta_{23}}^i \mapsto \varpi^{1i}$, ${\beta_{13}}^i \mapsto -\varpi^{2i}$, ${\beta_{12}}^i \mapsto \varpi^{3i}$, and $\xi^i_{123} \mapsto X^i$, $i = 1,2,3$, respectively. Consequently, the 3D nonlinear elastostatics complex (315) in $\mathbb{R}^3$ simplifies to the first row of the following diagram, where $D_{fp} \coloneqq \hat{\varsigma} \circ d_1$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r] & \Omega^0(\mathcal{B};\mathbb{R}^3) \arrow[r, "d_0"] \arrow[d, "\mathrm{i}_0"] & \Omega^1(\mathcal{B};\mathbb{R}^3) \arrow[r, "D_{fp}"] \arrow[d, "\mathrm{i}_1"] & \Gamma(T\mathcal{B}\otimes T\mathcal{B}) \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{i}_2"] & \mathfrak{X}(\mathcal{B}) \arrow[r] \arrow[d, "\mathrm{i}_3"] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r] & \Omega^0_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "d_0^\nabla"] & \Omega^1_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r, "D_f^{\mathcal{T}}"] & \Gamma(T\mathcal{B}\otimes T\varphi(\mathcal{B})) \arrow[r, "\mathrm{div}"] & \Omega^0_\varphi(\mathcal{B};T\mathcal{S}) \arrow[r] & 0
\end{tikzcd} \tag{320}$$

This diagram commutes for any embedding $\varphi : \mathcal{B} \to \mathbb{R}^3$ and the isomorphisms $\mathrm{i}_0 : \alpha^i(X)\mathbf{E}_i \mapsto \alpha^i\ \partial/\partial x^i(\varphi(X))$, $\mathrm{i}_1 : dX^i(X) \otimes \mathbf{E}_j \mapsto dX^i(X) \otimes (\partial/\partial X^j)(\varphi(X))$, $\mathrm{i}_2 : (\partial/\partial X^i) \otimes (\partial/\partial X^j)(X) \mapsto \big(\partial/\partial X^i\ (X)\big) \otimes \big(\partial/\partial x^j\ (\varphi(X))\big)$, $\mathrm{i}_4 : (\partial/\partial X^i)(X) \mapsto (\partial/\partial x^i)(\varphi(X))$, where the coordinate system $\{x^i\}$ on $\mathcal{S}$ is the Cartesian coordinate $\{X^i\}$ on $\mathcal{B}$. The physical interpretation of the complex (320) is as follows: A vector $\mathbf{V} \in \mathbb{R}^3$ is augmented in $\Omega^0(\mathcal{B};\mathbb{R}^3)$ as the uniform translation by $\mathbf{V}$, i.e. $\mathbf{V} \mapsto (X \mapsto \mathbf{V})$. An element $\boldsymbol{U} \in \Omega^0(\mathcal{B};\mathbb{R}^3)$ is considered as a displacement field with $d_0\boldsymbol{U} = T\varphi - \mathrm{Id}_{\mathcal{B}}$, where $\varphi_X = X + \boldsymbol{U}(X)$. Suppose $\mathcal{B}$ is contractible. Given $\boldsymbol{\beta} \in \Omega^1(\mathcal{B};\mathbb{R}^3)$, the condition $d_1\boldsymbol{\beta} = d_1(\boldsymbol{\beta} - \mathrm{Id}_{\mathcal{B}}) = 0$, is the necessary an sufficient condition for the existence of a displacement field $\boldsymbol{U} \in \Omega^0(\mathcal{B};\mathbb{R}^3)$ such that $T\varphi = \boldsymbol{\beta}$, with $\varphi(X) = X + \boldsymbol{U}(X)$. On the other hand, $\boldsymbol{\xi} \in \Omega^1(\mathcal{B};\mathbb{R}^3)$ can be considered as a first Piola-Kirchhoff stress function with $D_{fp}(\boldsymbol{\xi})$ representing the corresponding first Piola-Kirchhoff stress tensor. In fact, $\mathrm{i}_2 \circ D_{fp}(\boldsymbol{\xi})$ is the corresponding stress tensor, but in $\mathbb{R}^3$ we can identify this with $D_{fp}(\boldsymbol{\xi})$. Given a first-Piola Kirchhoff stress tensor $\boldsymbol{\varpi} \in \Gamma(T\mathcal{B} \otimes T\mathcal{B})$, the condition $\operatorname{div} \boldsymbol{\varpi} = 0$, is the necessary and the sufficient condition for the existence of a first Piola-Kirchhoff stress function for $\boldsymbol{\varpi}$. In summary, the linear structure of $\mathbb{R}^3$ allows us to remove the explicit dependence of the complex (315) on φ and obtain the 3D nonlinear elastostatics complex (320).

Finally, let us also mention the complexes for 2D nonlinear elastostatics. The main difference between 2D and 3D cases is that 2D case does not admit a complex that contains both the kinematics and kinetics of motion. For 2-manifolds, the sequences (308) and (310) are still valid. Using Corollary 3.2.8, the kinetic complex in terms of the second Piola-Kirchhoff stress tensor reads

$$0 \longrightarrow \ker D_{\mathcal{A}}^{\boldsymbol{C}} \hookrightarrow \Omega^0(\mathcal{B}) \xrightarrow{D_{\mathcal{A}}^{\boldsymbol{C}}} \Gamma(S^2 T\mathcal{B}) \xrightarrow{\operatorname{div}^{\boldsymbol{C}}} \mathfrak{X}(\mathcal{B}) \longrightarrow 0, \tag{321}$$

where the expression of $D_{\mathcal{A}}^{\boldsymbol{C}}$ in an orthonormal coordinate system of $(\mathcal{B}, \boldsymbol{C})$ is given in (289). In terms of Cauchy stress functions, we obtain the kinetic complex

$$0 \longrightarrow \ker D_{\mathcal{A}} \hookrightarrow \Omega^0(\varphi(\mathcal{B})) \xrightarrow{D_{\mathcal{A}}} \Gamma(S^2 T\varphi(\mathcal{B})) \xrightarrow{\operatorname{div}} \mathfrak{X}(\varphi(\mathcal{B})) \longrightarrow 0, \tag{322}$$

with $D_{\mathcal{A}}$ being defined in (289). Note that for $\mathcal{B} \subset \mathbb{R}^2$, $\ker D_{\mathcal{A}}^{\boldsymbol{C}}$ and $\ker D_{\mathcal{A}}$ can be replaced with $\mathfrak{euc}(\mathbb{R}^2)$ with augmentation mappings similar to (307). For flat 2-manifolds, the complexes (314) and (315) terminate after d_2^{∇} and div, respectively. In particular, we obtain the following kinetic complex in $\mathbb{R}^2$:

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow \Omega^0(\mathcal{B}; \mathbb{R}^2) \xrightarrow{d_0} \Omega^1(\mathcal{B}; \mathbb{R}^2) \xrightarrow{D_{fp}} \Gamma(T\mathcal{B} \otimes T\mathcal{B}) \longrightarrow 0, \tag{323}$$

with similar physical interpretation as the kinematic part of (320). On the other hand, our discussion at the end of §3.2.4 enables us to write the following kinetic complex in terms of the first Piola-Kirchhoff stress tensor:

$$0 \longrightarrow \ker D_{\mathcal{SF}} \hookrightarrow \Omega^0_\varphi(\varphi(\mathcal{B}); T\mathcal{S}) \xrightarrow{D_{\mathcal{SF}}} \Gamma(T\mathcal{B} \otimes T\varphi(\mathcal{B})) \xrightarrow{\text{div}} \Omega^0_\varphi(\mathcal{B}; T\mathcal{S}) \longrightarrow 0. \tag{324}$$

For an open subset $\mathcal{B} \subset \mathbb{R}^2$, this complex simplifies to

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow \Omega^0(\mathcal{B}; \mathbb{R}^2) \xrightarrow{\widehat{D}_{\mathcal{SF}}} \Gamma(T\mathcal{B} \otimes T\mathcal{B}) \xrightarrow{\text{div}} \mathfrak{X}(\mathcal{B}) \longrightarrow 0, \tag{325}$$

where the components of $\boldsymbol{P} = \widehat{D}_{\mathcal{SF}}(\boldsymbol{v})$ in the Cartesian coordinate $\{X^i\}$ is given in (295). Note that $\mathbb{R}^2$ is augmented in $\Omega^0(\mathcal{B}; \mathbb{R}^2)$ as the space of constant functions.

3.4 Linear Elastostatics Complexes and Homogeneous Spaces

In this section, we explain how the linear elastostatics complex arises as a BGG resolution. To this end, we require various notions from differential geometry and the representation theory of Lie groups and Lie algebras. The methods that will be explained in the remainder are first developed for studying the celebrated Minkowski space in the theory of relativity [17, 90]. The application of these methods to the linear elastostatics complex is due to Eastwood [41, 40, 42]. We first mention the required preliminaries. Then, we explain that the linear elastostatics complex is a BGG complex that can be constructed from a twisted de Rham complex. We will also explain the derivation of complexes (302) that Arnold et al. [9] introduced for weakly imposing the symmetry of the stress tensor.

3.4.1 Semisimple Lie Algebras

Let $\mathcal{G}$ be a Lie group with a complex semisimple Lie algebra $\mathfrak{g}$. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal Abelian subalgebra that consists of semisimple elements [27], i.e. a maximal subalgebra such that $[\mathbf{X},\mathbf{Y}] = 0$, $\forall \mathbf{X},\mathbf{Y} \in \mathfrak{h}$ and the linear mapping $\mathrm{ad}(\mathbf{X})(\cdot) = [\mathbf{X},\cdot] : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable, i.e. if there is a basis for $\mathfrak{g}$ consisting of eigenvectors of $\mathrm{ad}(\mathbf{X})$. Cartan subalgebras of semisimple Lie algebras are unique in the sense that any two Cartan subalgebras of $\mathfrak{g}$ are conjugate [47] and thus, one can fix one of them. The zero eigenspace of $\mathrm{ad}(\mathbf{X})$ is $\mathfrak{h}$. A root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is a nonzero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha = \{\mathbf{X} \in \mathfrak{g} : \mathrm{ad}(\mathbf{Y})(\mathbf{X}) = \alpha(\mathbf{Y})\mathbf{X},\ \forall \mathbf{Y} \in \mathfrak{h}\} \neq \{0\}$. Subspaces $\mathfrak{g}_\alpha$ are called root spaces and all are 1-dimensional. The space of all roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ is denoted by $\Delta(\mathfrak{g},\mathfrak{h})$ for which we have the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha. \tag{326}$$

There is a set of simple roots of $\mathfrak{g}$, which is defined to be a subset $\mathcal{S} \subset \Delta(\mathfrak{g},\mathfrak{h})$ such that every root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ can be expressed as a linear combination of elements of $\mathcal{S}$ with all non-negative or non-positive coefficients. The subsets $\mathcal{S}$ are unique up to a conjugation. One can show that $\mathcal{S}$ is a basis for $\mathfrak{h}^*$ and induces an ordering on $\mathfrak{h}^*$ [17]: let $\xi,\eta \in \mathfrak{h}^*$. Then $\xi \geq \eta \iff \xi - \eta = \sum_i a_i\alpha_i$, where $\alpha_i \in \mathcal{S}$ and $a_i \geq 0$. The set of positive roots with respect to $\mathcal{S}$ is $\Delta^+(\mathfrak{g},\mathfrak{h},\mathcal{S}) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) : \alpha > 0\}$, where $\alpha > 0 \iff \alpha = \sum_i a_i\alpha_i$, with $a_i \geq 0$ and at least one $a_i \neq 0$. The Killing form of $\mathfrak{g}$ is nondegenrate on $\mathfrak{h}$ and for each $\xi \in \mathfrak{h}^*$ there is a unique $\mathbf{H}_\xi \in \mathfrak{h}$ such that $\xi(\mathbf{X}) = B(\mathbf{H}_\xi,\mathbf{X})$, $\forall \mathbf{X} \in \mathfrak{h}$. One can also define a nondegenerate complex bilinear form on $\mathfrak{h}^*$ by $\langle\xi,\eta\rangle = B(\mathbf{H}_\xi,\mathbf{H}_\eta)$. The real span of $\Delta(\mathfrak{g},\mathfrak{h})$ is denoted by $\mathfrak{h}^*_{\mathbb{R}}$ and $\mathfrak{h}^*$ is the complexification of $\mathfrak{h}^*_{\mathbb{R}}$ [27]. Moreover, the real dual of $\mathfrak{h}^*_{\mathbb{R}}$, $\mathfrak{h}_{\mathbb{R}}$, is the same as the real span of the elements $\mathbf{H}_\alpha$, $\forall\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$.

As an example, consider the semisimple complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^n)$. Let E_{ij} denote a matrix with zero elements except for the element in the jth column of the ith row which equals to one. It can be shown that the set of $n \times n$ complex trace-free diagonal matrices is a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}$ [27]. Therefore, the rank of $\mathfrak{g}$, which is the dimension of $\mathfrak{h}$, is $n-1$. Let $e_i : \mathfrak{h} \to \mathbb{C}$ be a mapping that returns the ith diagonal element. We have $\Delta(\mathfrak{g},\mathfrak{h}) = \{e_i - e_j : 1 \le i,j \le n, i \ne j\} \subset \mathfrak{h}^*$. The real subspace $\mathfrak{h}_{\mathbb{R}}$ is the space of $n \times n$ real trace-free diagonal matrices. The 1-dimensional root space $\mathfrak{g}_{e_i - e_j}$ is the subspace spanned by E_{ij} and we have the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \ne j} \mathfrak{g}_{e_i - e_j}$. A set of simple roots is $\mathcal{S} = \{\alpha_1, \ldots, \alpha_{n-1}\}$, where $\alpha_i = e_i - e_{i+1}$, and the corresponding positive roots are $\Delta^+(\mathfrak{g},\mathfrak{h},\mathcal{S}) = \{e_i - e_j : 1 \le i < j \le n\}$. Simple roots can be denoted by Dynkin diagrams.[16] The Dynkin diagram of $\mathfrak{sl}(\mathbb{C}^n)$ has $n-1$ vertices: $\overset{\alpha_1}{\bullet}\!-\!\overset{\alpha_2}{\bullet}\!-\!\overset{\alpha_3}{\bullet}\ \cdots\ \overset{\alpha_{n-2}}{\bullet}\!-\!\overset{\alpha_{n-1}}{\bullet}$. Each vertex of the Dynkin diagram denotes a simple root. Borel subalgebras are the maximal solvable subalgebras of $\mathfrak{g}$. All Borel subalgebras are conjugate to the standard Borel subalgebra defined as

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h},\mathcal{S})} \mathfrak{g}_\alpha. \tag{327}$$

The standard Borel subalgebra of $\mathfrak{sl}(\mathbb{C}^n)$ is the set of upper triangular matrices. A parabolic subalgebra of $\mathfrak{g}$ is a subalgebra containing a Borel subalgebra. Up to a conjugation, all parabolic subalgebras have the following standard form [17]. Suppose $\mathcal{S}_{\mathfrak{p}}$ is a subset of $\mathcal{S}$ and let $\Delta(\mathfrak{l},\mathfrak{h}) = \mathrm{span}(\mathcal{S}_{\mathfrak{p}}) \cap \Delta(\mathfrak{g},\mathfrak{h})$, and $\Delta(\mathfrak{u},\mathfrak{h}) = \Delta^+(\mathfrak{g},\mathfrak{h},\mathcal{S}) \backslash \Delta(\mathfrak{l},\mathfrak{h})$. A standard parabolic subalgebra of $\mathfrak{g}$ is defined as $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, where

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l},\mathfrak{h})} \mathfrak{g}_\alpha, \text{ and } \mathfrak{u} = \bigoplus_{\alpha \in \Delta(\mathfrak{u},\mathfrak{h})} \mathfrak{g}_\alpha. \tag{328}$$

The decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is called a Levi decomposition of $\mathfrak{p}$. We also have $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{p}$, where the subalgebra $\mathfrak{u}_-$ is given by

$$\mathfrak{u}_- = \bigoplus_{\alpha \in \Delta(\mathfrak{u},\mathfrak{h})} \mathfrak{g}_{-\alpha}. \tag{329}$$

[16]Dynkin diagrams are used to determine the Cartan matrix of $\mathfrak{g}$ that uniquely specifies $\mathfrak{g}$, see [47].

Given the Dynkin diagram of $\mathfrak{g}$, one can denote a standard parabolic subalgebra by crossing through all nodes that correspond to the simple roots of $\mathfrak{g}$ in $\mathcal{S}\backslash\mathcal{S}_{\mathfrak{p}}$ [17]. For example, for $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^4)$, one can write

$$\times\!\!-\!\!\bullet\!\!-\!\!\bullet = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \in \mathfrak{sl}(\mathbb{C}^4) \right\}, \tag{330}$$

and we have

$$\begin{aligned} &\mathfrak{u}_- = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}(\mathbb{C}^4) \right\}, \quad \mathfrak{l} = \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \in \mathfrak{sl}(\mathbb{C}^4) \right\}, \\ &\mathfrak{u} = \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}(\mathbb{C}^4) \right\}. \end{aligned} \tag{331}$$

Clearly, we have the decomposition $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{l} \oplus \mathfrak{u}$. The celebrated Minkowski space of the theory of relativity correspond to the parabolic subalgebra $\bullet\!\!-\!\!\times\!\!-\!\!\bullet$.

Let $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a complex semisimple Lie algebra $\mathfrak{g}$ over a complex finite-dimensional vector space V. One can show that $\varrho(\mathbf{H}) : V \to V$, $\forall \mathbf{H} \in \mathfrak{h}$, is diagonalizable [27]. A weight of $\mathfrak{g}$ is any element $\eta \in \mathfrak{h}^*$. A weight of V is defined to be an element $\lambda \in \mathfrak{h}^*$ such that $V_\lambda = \{\mathbf{v} \in V : \mathbf{H} \cdot \mathbf{v} = \lambda(\mathbf{H})\mathbf{v}, \forall \mathbf{H} \in \mathfrak{h}\} \neq \{0\}$. The set of all weights of V is denoted by $\Delta(V)$. Note that we have the decomposition

$$V = \bigoplus_{\lambda \in \Delta(V)} V_\lambda. \tag{332}$$

Using $\mathcal{S} = \{\alpha_i\}$, one obtains another basis $\{\lambda_i\}$ for $\mathfrak{h}^*$ by requiring $2\langle\lambda_i, \alpha_j\rangle/\langle\alpha_j, \alpha_j\rangle = \delta_{ij}$. Thus, any $\eta \in \mathfrak{h}^*$ can be written as $\eta = \sum_i c_i^\eta \lambda_i$, where $c_i^\eta = 2\langle\eta, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$. In particular, any weight $\lambda \in \Delta(V)$ can be written as $\lambda = \sum_i c_i^\lambda \lambda_i$. One can use the Dynkin diagram of $\mathfrak{g}$ to denote weights of V by writing the coefficient c_i^λ on the node α_i. For example, consider $\mathfrak{sl}(\mathbb{C}^4)$ with $\lambda_i = \sum_i e_i$, $i = 1, 2, 3$ [27]. Then, for example, we have

$$\lambda = \overset{-1}{\bullet}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{-1}{\bullet} = -e_1 + 2(e_1 + e_2) - (e_1 + e_2 + e_3) = \alpha_2. \tag{333}$$

Since $\Delta(V)$ has finitely many weights, using the ordering that $\mathcal{S}$ induces on $\Delta(V) \subset \mathfrak{h}^*$, we can define highest (lowest) weight λ of $\Delta(V)$ as a weight λ with $\lambda \geq \lambda'$ ($\lambda \leq \lambda'$), $\forall \lambda' \in \Delta(V)$. A weight $\lambda = \sum_i c_i^\lambda \lambda_i$ is called dominant if all $c_i^\lambda \geq 0$, and is called integral if all $c_i^\lambda \in \mathbb{Z}$. There is a one-to-one correspondence between finite-dimensional irreducible $\mathfrak{g}$-modules and dominant integral weights [17, 27]: Highest weight of such a representation is dominant and integral, and conversely for any dominant integral weight $\lambda \in \mathfrak{h}^*$, there is a unique (up to an isomorphism) irreducible $\mathfrak{g}$-module with highest weight λ. If an irreducible representation of $\mathfrak{g}$ on V has highest weight λ, then the induced dual representation has lowest weight $-\lambda$. We assume that the Dynkin diagram of an integral dominant weight λ also denotes the representation with lowest weight $-\lambda$.

It is also possible to show that finite-dimensional irreducible representations of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ are in one-to-one correspondence with dominant integral weights for $\mathfrak{p}$ [17], i.e. weights with non-negative integers on the nodes that correspond to all $\alpha_i \in \mathcal{S}_{\mathfrak{p}}$. Similar to weights for $\mathfrak{g}$, by the Dynkin diagram of λ, which is a dominant integral weight for $\mathfrak{p}$, we also denote the representation of $\mathfrak{p}$ with lowest weight $-\lambda$.[17] For example, the Dynkin diagram $\overset{2}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$ denotes the irreducible representation of $\mathfrak{sl}(\mathbb{C}^4)$ with lowest weight $-(3e_1 + e_2)$ and the Dynkin diagram $\overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$ is the irreducible representation of $\mathfrak{p} \subset \mathfrak{sl}(\mathbb{C}^4)$ given in (330), with lowest weight α_1. Let $\lambda = \sum_{i=1}^{n} c_i^\lambda \lambda_i$ be a dominant integral weight for $\mathfrak{g}$. Also let V_i denote the irreducible $\mathfrak{g}$-module corresponding to λ_i. One can show that the irreducible representation of $\mathfrak{g}$ corresponding to λ is a subrepresentation of the induced representation $S^{c_1^\lambda} V_1 \otimes \cdots \otimes S^{c_n^\lambda} V_n$ [47]. For $\mathfrak{sl}(\mathbb{C}^n)$, we have $\lambda_i = e_1 + \cdots + e_i$, $i = 1, \ldots, n-1$. Consider the standard representation of $\mathfrak{sl}(\mathbb{C}^n)$ on $\mathbb{C}^n$. The fundamental irreducible representation V_i is the induced irreducible representation $\Lambda^i \mathbb{C}^n$ [27]. As we discuss in the next section, irreducible representations of $\mathfrak{sl}(\mathbb{C}^n)$ can be denoted by Young diagrams.

For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, the root reflection $s_\alpha : \mathfrak{h}_\mathbb{R}^* \to \mathfrak{h}_\mathbb{R}^*$ is given by

$$s_\alpha(\xi) = \xi - \frac{2\langle \xi, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha. \tag{334}$$

These reflections are orthogonal mappings with determinant -1 and $s_\alpha(\alpha) = -\alpha$ [27]. The Weyl group of $\mathfrak{g}$, $W_\mathfrak{g}$, is the subgroup of the orthogonal group $O(\mathfrak{h}_\mathbb{R}^*)$ generated by all the reflections s_α, $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ [27], i.e. the group $W_\mathfrak{g}$ consists of mappings $\mathfrak{h}_\mathbb{R}^* \to \mathfrak{h}_\mathbb{R}^*$, with determinant 1 or -1, that are obtained by multiplication (composition) of mappings s_α. Note that (334) defines a left action $W_\mathfrak{g} \times \mathfrak{h}_\mathbb{R}^* \to \mathfrak{h}_\mathbb{R}^*$. It can be shown that $W_\mathfrak{g}$ is generated by $\{s_\alpha : \alpha \in \mathcal{S}\}$, where $\mathcal{S}$ is the set of simple roots. A root reflection s_α with $\alpha \in \mathcal{S}$ is called a simple reflection. For any $\mathrm{w} \in W_\mathfrak{g}$, there is a minimal integer $\ell(\mathrm{w})$, called the length of w, such that w can be written as a composition of $\ell(\mathrm{w})$ simple reflections [17]. Such an expression is not unique in general and is called a reduced expression for w. For example, let $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^n)$. We

[17] Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of Lie groups $\mathcal{G}$ and $\mathcal{P} \subset \mathcal{G}$, respectively. An irreducible representation of $\mathfrak{p}$ is induced by an irreducible representation of $\mathcal{P}$ if and only if the corresponding integral dominant weight for $\mathfrak{p}$ is also integral for $\mathfrak{g}$ [17], i.e. all coefficients of the corresponding Dynkin diagram are integers.

have $\mathfrak{h}^*_{\mathbb{R}} = \{\sum_{k=1}^n a_k e_k : \sum_{k=1}^n a_k = 0\}$. Then, one can show that

$$s_{e_i - e_j}\left(\sum_{k=1}^n a_k e_k\right) = \sum_{\substack{k=1 \\ k\neq i,j}}^n a_k e_k + a_i e_j + a_j e_i, \tag{335}$$

i.e. the reflection $s_{e_i-e_j}$ exchanges the coefficients of e_i and e_j. Thus, $W_{\mathfrak{g}}$ is the permutation group $\mathfrak{S}_n$. Let $(i\cdots k)$ denote $s_{\alpha_i}\cdots s_{\alpha_k}$. The reduced expressions for elements of $W_{\mathfrak{sl}(\mathbb{C}^3)}$ read $\{\mathrm{Id}, (1), (2), (12), (21), (121)\}$. Usually directed diagrams are used for Weyl groups. Let $\mathrm{w}, \mathrm{w}' \in W_{\mathfrak{g}}$. We write $\mathrm{w} \to \mathrm{w}'$ if $\ell(\mathrm{w}') = \ell(\mathrm{w}) + 1$, and $\exists \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ such that $\mathrm{w}' = s_\alpha \mathrm{w}$ [17]. The directed graph of $W_{\mathfrak{sl}(\mathbb{C}^3)}$ is as follows.

$$\begin{tikzcd}
 & (1) \arrow[r] \arrow[dr] & (12) \arrow[dr] & \\
\mathrm{Id} \arrow[ur] \arrow[dr] & & & (121) \\
 & (2) \arrow[r] \arrow[ur] & (21) \arrow[ur] &
\end{tikzcd}$$

Using (334), one can schematically show the action of simple reflections $s_{\alpha_i}(\lambda)$ by the Dynkin diagram of λ. For example, consider $\mathfrak{sl}(\mathbb{C}^5)$. We have $s_{\alpha_3}(\overset{a}{\bullet}-\overset{b}{\bullet}-\overset{c}{\bullet}-\overset{d}{\bullet}) = \overset{a}{\bullet}-\overset{b+c}{\bullet}-\overset{-c}{\bullet}-\overset{c+d}{\bullet}$, i.e. for calculating $s_{\alpha_i}(\lambda)$, the coefficient c_i^λ is replaced by $-c_i^\lambda$ and c_i^λ is added to adjacent coefficients c_{i-1}^λ and c_{i+1}^λ.[18] Let ρ be the sum of all λ_i's, i.e. $\rho = \sum_i \lambda_i$. The Weyl group $W_{\mathfrak{g}}$ has a one-to-one correspondence with the orbit of ρ under the left action of $W_{\mathfrak{g}}$ on $\mathfrak{h}^*_{\mathbb{R}}$. The affine Weyl action on weights is defined as $\mathrm{w}.\lambda = \mathrm{w}(\lambda + \rho) - \rho$, $\forall \mathrm{w} \in W_{\mathfrak{g}}$. A weight λ is called singular if there is an element $\mathrm{w} \in W_{\mathfrak{g}}$ with $w \neq \mathrm{Id}$, such that $\mathrm{w}.\lambda = \lambda$ [17]. Otherwise, λ is called non-singular. If λ is singular, there are some $\mathrm{w} \in W_{\mathfrak{g}}$ and $\alpha_i \in \mathcal{S}$ such that $c_i^{\mathrm{w}(\lambda+\rho)} = 0$, i.e. the Dynkin diagram of $\mathrm{w}(\lambda + \rho)$ has a zero coefficient.

For a standard parabolic subgroup $\mathfrak{p} \subset \mathfrak{g}$, one can define the Hasse diagram $W^{\mathfrak{p}}$, which is the subset of $W_{\mathfrak{g}}$ whose action sends a dominant weight for $\mathfrak{g}$ to a dominant

[18] This role is valid only for $\mathfrak{sl}(\mathbb{C}^n)$ and needs some modifications for other types of Lie algebras, see [17].

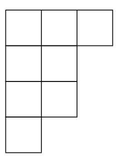
weight for $\mathfrak{p}$ [17]. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the Levi decomposition of $\mathfrak{p}$ as was explained earlier. For an element $\mathrm{w} \in W_{\mathfrak{g}}$, we define $\Delta(\mathrm{w}) = \{\alpha \in \Delta^{+}(\mathfrak{g}, \mathfrak{h}, \mathcal{S}) : \mathrm{w}^{-1}(\alpha) \in -\Delta^{+}(\mathfrak{g}, \mathfrak{h}, \mathcal{S})\}$, where we have $|\Delta(\mathrm{w})| = \ell(\mathrm{w})$ [67]. One can show that $W^{\mathfrak{p}} = \{\mathrm{w} \in W_{\mathfrak{g}} : \Delta(\mathrm{w}) \subset \Delta(\mathfrak{u}, \mathfrak{h})\}$ [17]. Let $\rho^{\mathfrak{p}} = \sum_i \lambda_i$, $\forall i$ such that $\alpha_i \in \mathcal{S} \backslash \mathcal{S}_{\mathfrak{p}}$, i.e. the Dynkin diagram of $\rho^{\mathfrak{p}}$ has 1's on crossed nodes and 0's on all other nodes. The Hasse diagram of $\mathfrak{p}$ is in a one-to-one correspondence with the orbit of $\rho^{\mathfrak{p}}$ under the action of $W_{\mathfrak{g}}$. The Hasse diagram of $\mathfrak{p} \subset \mathfrak{sl}(\mathbb{C}^{n+1})$, with $\mathfrak{p} = \times\!\!-\!\!\bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet$, is given in the following diagram [17].

$$\mathrm{Id} \longrightarrow (1) \longrightarrow (12) \longrightarrow (123) \longrightarrow \cdots \longrightarrow (123\ldots n) \tag{336}$$

3.4.2 Irreducible Representations of $SL(\mathbb{C}^n)$

For obtaining the BGG resolution of linear elasticity, one needs to determine a sequence of projections on some associated vector bundles that correspond to irreducible representations of $SL(\mathbb{C}^n)$. The Dynkin diagram notation for weights is appropriate for calculation of BGG resolutions but it does not specify the irreducible representation of a given weight. In this section, we review Young diagrams which are standard tools for specifying irreducible representations of $GL(\mathbb{C}^n)$ and $SL(\mathbb{C}^n)$. For more details, see [47, 17, 90].

Let $\mathfrak{S}_d$ be the symmetric group of permutations of integers $\{1, \ldots, d\}$ and let $\sigma = \{a_1 \cdots a_k\} \in \mathfrak{S}_d$ denote a k-cycle. Recall that a partition $\eta = (\eta_1, \ldots, \eta_l)$ of an integer d is a set of integers such that $d = \sum_{i=1}^{l} \eta_i$, with $\eta_1 \geq \cdots \geq \eta_l \geq 1$. The number of irreducible representations of $\mathfrak{S}_d$ is equal to the number of partitions of d [47]. Partitions are usually shown by Young diagrams. For example, the Young diagram of the partition $(3, 2, 2, 1)$ of 8 is given by:



By associating integers $1,\dots,d$ to a Young diagram such that each row and each column is in increasing order, we obtain a Young tableau. The canonical Young tableau for the above example is:

1	2	3
4	5	
6	7	
8		

The conjugate of a Young diagram (tableau) is obtained by interchanging rows and columns. Thus, the conjugate partition of $(3,2,2,1)$ is $(4,3,1)$ and the conjugate of the above tableau is as follows.

1	4	6	8
2	5	7	
3			

Given a finite group G, the group ring or group algebra $\mathbb{C}G$ is defined to be the vector space spanned by mappings $\mathcal{E}_g : G \to \mathbb{C}$, $\forall g \in G$, where $\mathcal{E}_g(x)$ is 1 if $x = g$, and 0 otherwise, i.e. the underlying vector space of $\mathbb{C}G$ is the set of all $\mathbb{C}$-valued functions on G. The multiplication $\mathbb{C}G \times \mathbb{C}G \to \mathbb{C}G$ is defined by $\mathcal{E}_{g_1} \cdot \mathcal{E}_{g_2} = \mathcal{E}_{g_1 \cdot g_2}$, or equivalently $f \cdot h(x) = \sum_{g_1 \cdot g_2 = x} f(g_1)h(g_2)$, $\forall f,h \in \mathbb{C}G$ [47]. A representation of $\mathbb{C}G$ is an algebra homomorphism $\hat{\varrho} : \mathbb{C}G \to \mathrm{End}(V)$. A representation $\varrho : G \to GL(V)$ of G induces a representation of $\mathbb{C}G$ by defining $\hat{\varrho}(\mathcal{E}_g) := \varrho(g)$. Consider a Young tableau on a partition $\eta = (\eta_1,\dots,\eta_k)$ of an integer d. Let the subgroups $P_\eta, Q_\eta \subset \mathfrak{S}_d$ be $P_\eta = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves each row}\}$, and $Q_\eta = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves each column}\}$. The elements $a_\eta, b_\eta \in \mathbb{C}\mathfrak{S}_d$ are defined as $a_\eta = \sum_{\sigma \in P_\eta} \mathcal{E}_\sigma$, and $b_\eta = \sum_{\sigma \in Q_\eta} \mathrm{sgn}(\sigma)\mathcal{E}_\sigma$, where $\mathrm{sgn}(\sigma)$ is 1 (-1) if σ is an even (odd) permutation. The Young symmetrizer c_η[19] of the partition η is defined to be [47]

$$c_\eta = a_\eta \cdot b_\eta \in \mathbb{C}\mathfrak{S}_d. \tag{337}$$

[19]As we will see in the sequel, each symmetrizer specifies an irreducible representation. Representations that correspond to different tableaux of a partition are isomorphic [47]; this is the reason for specifying c_η merely with η.

For a vector space V, we have a right action $\otimes^d V \times \mathfrak{S}_d \to \otimes^d V$ given by

$$(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_d) \cdot \sigma = \mathbf{v}_{\sigma(1)} \otimes \cdots \otimes \mathbf{v}_{\sigma(d)}. \tag{338}$$

This right action defines a mapping $\hat{r} : \mathbb{C}\mathfrak{S}_d \to \mathrm{End}(\otimes^d V)$ by $(\sum_i z_i \mathcal{E}_{\sigma_i}, \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_d) \mapsto \sum_i z_i ((\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_d) \cdot \sigma_i)$, where $z_i \in \mathbb{C}$. The image $\hat{r}(c_\eta, \otimes^d V) \subset \otimes^d V$ is denoted by $\mathbb{S}_\eta V$, where the functor $\mathbb{S}_\eta$ is called the Schur functor corresponding to η. Let $\{\mathbf{e}_i\}_{i=1}^n$ be a basis for V. Recall that the corresponding bases for $\binom{n+d-1}{d}$-dimensional space $S^d V$ and $\binom{n}{d}$-dimensional space $\Lambda^d V$ are given by $\{\mathbf{e}_{i_1} \odot \cdots \odot \mathbf{e}_{i_d}\}_{1 \le i_1 \le \cdots \le i_d \le n}$ and $\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_d}\}_{1 \le i_1 < \cdots < i_d \le n}$, respectively, where we have embeddings $S^d V \to \otimes^d V$ and $\Lambda^d V \to \otimes^d V$ defined by $\mathbf{v_1} \odot \cdots \odot \mathbf{v}_d \mapsto \sum_{\sigma \in \mathfrak{S}_d} \mathbf{v}_{\sigma(1)} \otimes \cdots \otimes \mathbf{v}_{\sigma(d)}$, and $\mathbf{v_1} \wedge \cdots \wedge \mathbf{v}_d \mapsto \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma) \mathbf{v}_{\sigma(1)} \otimes \cdots \otimes \mathbf{v}_{\sigma(d)}$, respectively. For the partitions (d) and $(1, \ldots, 1)$ of d, we have $\mathbb{S}_{(d)} V = S^d V$ and $\mathbb{S}_{(1,\ldots,1)} V = \Lambda^d V$. In general, for a partition $\eta = (\eta_1, \ldots, \eta_k)$, one can show that [47]

$$\dim(\mathbb{S}_\eta V) = \begin{cases} 0, & \text{if } k > \dim V = n, \\ \prod\limits_{1 \le i < j \le n} \frac{\eta_i - \eta_j + j - i}{j - i}, & \text{if } k \le n, \end{cases} \tag{339}$$

where for case $k \le n$, it is assumed that $\eta_i = 0$, for $k < i \le n$. Consider the standard action of $GL(\mathbb{C}^n)$ on $V = \mathbb{C}^n$, i.e. $A \cdot \mathbf{v} = A\mathbf{v}$, for $A \in GL(\mathbb{C}^n)$, and $\mathbf{v} \in \mathbb{C}^n$. This action induces a representation on $\otimes^d \mathbb{C}^n$ by $A \cdot (\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_d) = (A\mathbf{v}_1) \otimes \cdots \otimes (A\mathbf{v}_d)$. Then, given a partition η for d, the subspace $\mathbb{S}_\eta \mathbb{C}^n \subset \otimes^d \mathbb{C}^n$ is $GL(\mathbb{C}^n)$-invariant under this representation. In fact, $\mathbb{S}_\eta \mathbb{C}^n$ is an irreducible representation of $GL(\mathbb{C}^n)$ [47]. Since any irreducible representation of $GL(\mathbb{C}^n)$ restricts to an irreducible representation of $SL(\mathbb{C}^n)$, and therefore, an irreducible representation of $\mathfrak{sl}(\mathbb{C}^n)$, we observe that $\mathbb{S}_\eta \mathbb{C}^n$ is also an irreducible representation of $SL(\mathbb{C}^n)$ and $\mathfrak{sl}(\mathbb{C}^n)$. Let λ be a dominant integral weight of $\mathfrak{sl}(\mathbb{C}^n)$ given by

$$\lambda = \sum_{i=1}^{n-1} c_i \lambda_i = \overset{c_1}{\bullet}\!\!-\!\!\overset{c_2}{\bullet}\!\!-\!\!\overset{c_3}{\bullet} \;\cdot\;\cdot\;\cdot\; \overset{c_{n-2}}{\bullet}\!\!-\!\!\overset{c_{n-1}}{\bullet}. \tag{340}$$

The corresponding irreducible representation of $\mathfrak{sl}(\mathbb{C}^n)$ with the highest weight λ is the representation $\mathbb{S}_\eta \mathbb{C}^n \subset \otimes^d \mathbb{C}^n$, where the partition η is given by [47]

$$\eta = \left(\sum_{i=1}^{n-1} c_i, \sum_{i=2}^{n-1} c_i, \ldots, c_{n-2} + c_{n-1}, c_{n-1}, 0 \right), \tag{341}$$

and

$$d = \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} c_i. \tag{342}$$

Note that the partition η is actually obtained from the right side of (341) by removing zero entries from the right end. Moreover, the irreducible representation of $(\eta_1, \ldots, \eta_{n-1}, 0)$ is isomorphic to that of $(\eta_1 + b, \eta_2 + b, \ldots, \eta_{n-1} + b, b)$, where $b \in \mathbb{N}$ is an arbitrary constant [47]. By using (339) and (341), we can write

$$\dim(\mathbb{S}_\eta \mathbb{C}^n) = \prod_{1 \le i < j \le n} \frac{c_i + \cdots + c_{j-1} + j - i}{j - i}. \tag{343}$$

Let us calculate the irreducible representations of some weights that will be used in the sequel. Let $n = 4$. Then, the weights $\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$, $\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}$, and $\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}$ correspond to the representations $\mathbb{C}^4$, $\Lambda^2\mathbb{C}^4$, and $\Lambda^3\mathbb{C}^4$, respectively. The weight $\overset{d}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$ is the highest weight of $S^d\mathbb{C}^4$. Finally, consider the less trivial case $\overset{0}{\bullet}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet}$. Clearly, it corresponds to the canonical Young Tableau $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ for the partition $(2,2,0,0) = (2,2)$ of $d = 4$. We have $P_{(2,2)} = \{\mathrm{Id}, \{12\}, \{34\}, \{12\}\{34\}\}$, and $Q_{(2,2)} = \{\mathrm{Id}, \{13\}, \{24\}, \{13\}\{24\}\}$, which implies that

$$\begin{aligned} c_{(2,2)} = {} & \mathcal{E}_{\mathrm{Id}} + \mathcal{E}_{\{12\}} + \mathcal{E}_{\{34\}} - \mathcal{E}_{\{13\}} - \mathcal{E}_{\{24\}} + \mathcal{E}_{\{13\}\{24\}} + \mathcal{E}_{\{12\}\{34\}} + \mathcal{E}_{\{14\}\{23\}} \\ & - \mathcal{E}_{\{132\}} - \mathcal{E}_{\{124\}} - \mathcal{E}_{\{143\}} - \mathcal{E}_{\{234\}} + \mathcal{E}_{\{1324\}} + \mathcal{E}_{\{1423\}} - \mathcal{E}_{\{1432\}} - \mathcal{E}_{\{1234\}}. \end{aligned} \tag{344}$$

Consider the embedding $S^2(\Lambda^2\mathbb{C}^4) \to \otimes^4\mathbb{C}^4$ given by

$$
\begin{aligned}
(\mathbf{v}_1 \wedge \mathbf{v}_3) \odot (\mathbf{v}_2 \wedge \mathbf{v}_4) \;\;\mapsto\;\; & \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_4 + \mathbf{v}_2 \otimes \mathbf{v}_1 \otimes \mathbf{v}_4 \otimes \mathbf{v}_3 - \mathbf{v}_1 \otimes \mathbf{v}_4 \otimes \mathbf{v}_3 \otimes \mathbf{v}_2 \\
- \;\; & \mathbf{v}_4 \otimes \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 - \mathbf{v}_3 \otimes \mathbf{v}_2 \otimes \mathbf{v}_1 \otimes \mathbf{v}_4 - \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_4 \otimes \mathbf{v}_1 \\
+ \;\; & \mathbf{v}_3 \otimes \mathbf{v}_4 \otimes \mathbf{v}_1 \otimes \mathbf{v}_2 + \mathbf{v}_4 \otimes \mathbf{v}_3 \otimes \mathbf{v}_2 \otimes \mathbf{v}_1, \qquad (345)
\end{aligned}
$$

i.e. $S^2(\Lambda^2\mathbb{C}^4)$ is assumed to be the space of vectors that are alternating for exchanging first with third entries and second with fourth entries and are symmetric for mutual interchanging of first and third entries with second and fourth ones. Note that the alternating entries correspond to the columns of the Young tableau while the symmetric entries correspond to rows. The 20-dimensional subspace $\mathbb{S}_{(2,2)}\mathbb{C}^4$ of the 21-dimensional space $S^2(\Lambda^2\mathbb{C}^4)$ is the span of all vector of the form

$$
(\mathbf{v}_1 \wedge \mathbf{v}_3) \odot (\mathbf{v}_2 \wedge \mathbf{v}_4) + (\mathbf{v}_2 \wedge \mathbf{v}_3) \odot (\mathbf{v}_1 \wedge \mathbf{v}_4). \qquad (346)
$$

Note that the elements of $\mathbb{S}_{(2,2)}\mathbb{C}^4$ has the symmetries of the Riemannian curvature given in (262) and (263). In general, the weight (340) is the highest weight of a subrepresentation of $V' = S^{c_1}\mathbb{C}^n \otimes S^{c_2}(\Lambda^2\mathbb{C}^n) \otimes \cdots \otimes S^{c_{n-1}}(\Lambda^{n-1}\mathbb{C}^n)$, where V' is assumed to be the span of all elements of $\otimes^d\mathbb{C}^n$ that are alternating for permutations of columns and are symmetric for mutual permutations of columns with the same length. As an example, consider the weight $\overset{2}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{1}{\bullet}$ with the following canonical Young tableau.

$$
\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 \\
\hline
6 & 7 & 8 \\
\cline{1-3}
9 \\
\cline{1-1}
\end{array}
$$

We have $V' = S^2\mathbb{C}^4 \otimes S^2(\Lambda^2\mathbb{C}^4) \otimes \Lambda^3\mathbb{C}^4$, where V' is the span of all elements of $\otimes^9\mathbb{C}^4$ that are alternating for permutations $3 \Leftrightarrow 8$, $2 \Leftrightarrow 7$, and $1 \Leftrightarrow 6 \Leftrightarrow 9$, and are symmetric for $4 \Leftrightarrow 5$ and the mutual permutation $2|7 \Leftrightarrow 3|8$, where the numbers are the position of entries in $\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_9$, see (345).

As mentioned in the previous section, the Dynkin diagram of a weight λ is used to denote a representation that its dual has the highest weight λ. For Young diagrams over $\mathbb{C}^n$, the following result holds [39, 17]: Consider a partition $\eta = (\eta_1, \dots, \eta_n)$, with $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n \geq 0$. Then, the Dynkin diagram of the representation $\mathbb{S}_\eta \mathbb{C}^n$ of $SL(\mathbb{C}^n)$ is $\overset{\eta_{n-1}-\eta_n}{\bullet}\!\!-\!\!\!-\!\!\overset{\eta_{n-2}-\eta_{n-1}}{\bullet}\ \ldots\ \overset{\eta_2-\eta_3}{\bullet}\!\!-\!\!\!-\!\!\overset{\eta_1-\eta_2}{\bullet}$. The Dynkin diagrams of some representations are given in the following:

$$\begin{aligned}
\mathbb{C}^4 \approx \Lambda^3(\mathbb{C}^4)^* &= \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}, & (\mathbb{C}^4)^* \approx \Lambda^3\mathbb{C}^4 &= \overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}, \\
\Lambda^2\mathbb{C}^4 \approx \Lambda^2(\mathbb{C}^4)^* &= \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}, & S^d(\mathbb{C}^4)^* &= \overset{d}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}, \\
S^d\mathbb{C}^4 &= \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{d}{\bullet}, & \mathbb{S}_{(2,2)}(\mathbb{C}^4) \approx \mathbb{S}_{(2,2)}(\mathbb{C}^4)^* &= \overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}.
\end{aligned} \tag{347}$$

Note that as isomorphisms of representations, we have $(\Lambda^k V)^* \approx \Lambda^k V^*$, $(S^k V)^* \approx S^k V^*$, $\left(\otimes^k V\right)^* \approx \otimes^k V^*$, and more generally $(\mathbb{S}_\eta V)^* \approx \mathbb{S}_\eta(V^*)$, where all of these representations are induced by the representation V. Moreover, for the standard representation of $SL(V)$ on n-dimensional space V, we have the isomorphism of representations $\Lambda^k V \approx \Lambda^{n-k} V^*$ [47]. Also note that for defining $\mathbb{S}_\eta V$, we only need the action of $\mathfrak{S}_d$ on $\otimes^d V$. Thus, we can safely define $\mathbb{S}_\eta V^*$ but it should be kept in the mind that in the above examples, this representation is induced by the standard action of $SL(V)$ on V and not the standard action of $SL(V^*)$.

3.4.3 Parabolic Geometries

The linear elastostatics complex is equivalent to a complex on a flat parabolic geometry on the 3-sphere. The main goal of this section is to introduce the parabolic geometries. First, we have to define Klein and Cartan geometries. A complete study of Klein and Cartan geometries is available in [95]. Roughly speaking, the Cartan geometries generalize the Klein geometries in the same way that a Riemannian geometry generalizes the Euclidean geometry, i.e. the curved Cartan geometries are locally similar to the flat Klein geometries. On the other hand, Cartan geometries generalize the Riemannian geometry very similar to the Klein geometries that generalize non-Euclidean geometries.

3.4.3.1 Klein Geometries

A Klein geometry is a pair $(\mathcal{G}, \mathcal{H})$, where $\mathcal{G}$ is a Lie group and $\mathcal{H} \subset \mathcal{G}$ is a closed subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{H}$ is connected. The Lie group $\mathcal{G}$ is called the principal group of the geometry and the homogeneous space $\mathcal{G}/\mathcal{H}$ is called the space of the Klein geometry, or by abuse of notation, the Klein geometry. From §3.1.3.1 and §3.1.4.6, we know that $\mathcal{G}/\mathcal{H}$ is a smooth manifold and $(\mathcal{G}, p, \mathcal{G}/\mathcal{H}, \mathcal{H})$ is a principal bundle with the principal right action $r = \mu : \mathcal{G} \times \mathcal{H} \to \mathcal{G}$, where μ is the multiplication of $\mathcal{G}$. The Maurer-Cartan form $\boldsymbol{\omega} \in \Omega^1(\mathcal{G}; \mathfrak{g})$ is a linear isomorphism on each fiber of $\mathcal{G} \to \mathcal{G}/\mathcal{H}$. By restricting (225) to $\mathcal{H}$, we obtain $(\rho^h)^* \boldsymbol{\omega} = \mathrm{Ad}(h^{-1}) \circ \boldsymbol{\omega}$, $\forall h \in \mathcal{H}$. The principal right action of $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ can be written as $r_g = \lambda_g : \mathcal{H} \to \mathcal{G}$. By using (193), we obtain

$$\boldsymbol{\omega}\left(\zeta_{\mathbf{X}}(g)\right) = \boldsymbol{\omega}(T_e r_g \cdot \mathbf{X}) = T_g \lambda_{g^{-1}} \circ T_e \lambda_g \cdot \mathbf{X} = \mathbf{X}, \ \forall g \in \mathcal{G} \text{ and } \forall \mathbf{X} \in \mathfrak{h}. \qquad (348)$$

In general, $\boldsymbol{\omega} \notin \Omega^1(\mathcal{G}; \mathfrak{h})$ and thus, it is not a principal connection for $\mathcal{G} \to \mathcal{G}/\mathcal{H}$. Recall that we have a smooth transitive left action $\ell : \mathcal{G} \times \mathcal{G}/\mathcal{H} \to \mathcal{G}/\mathcal{H}$ given by $g_1 \cdot (g_2 \cdot \mathcal{H}) = (g_1 \cdot g_2) \cdot \mathcal{H}$. A Klein geometry is called effective if ℓ is effective. The kernel $\mathcal{K} \subset \mathcal{G}$ of the Klein geometry is the set of all elements $g \in \mathcal{G}$ such that $\ell_g = \mathrm{Id}_{\mathcal{G}/\mathcal{H}}$. One can show that $\mathcal{K}$ is the maximal normal subgroup of $\mathcal{G}$ that is contained in $\mathcal{H}$. Also $\mathcal{K}$ is a Lie subgroup and its Lie algebra $\mathfrak{k}$ is the maximal ideal in $\mathfrak{g}$ that is contained in $\mathfrak{h}$. By restricting the adjoint action of $\mathcal{G}$ to $\mathcal{H}$, one obtains the mapping $\mathrm{Ad}|_{\mathcal{H}} : \mathcal{H} \to GL(\mathfrak{g})$. On the other hand, as a subgroup of $\mathcal{G}$, the adjoint action of $\mathcal{H}$ is obtained by restricting $\mathcal{G}$ to $\mathcal{H}$ and $\mathfrak{g}$ to $\mathfrak{h}$. Therefore, $\mathfrak{h}$ is an $\mathcal{H}$-invariant subspace of $\mathfrak{g}$, i.e. $\mathrm{Ad}|_{\mathcal{H}}(h)(\mathbf{X}) \in \mathfrak{h}$, $\forall h \in \mathcal{H}$ and $\mathbf{X} \in \mathfrak{h}$. The Klein geometry $(\mathcal{G}, \mathcal{H})$ is called reductive if there is an $\mathcal{H}$-invariant Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$ that is complementary to

$\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ and $\mathrm{Ad}|_{\mathcal{H}}(h)(\mathbf{Y}) \in \mathfrak{n}$, $\forall h \in \mathcal{H}$ and $\mathbf{Y} \in \mathfrak{n}$. The Klein geometry is called split if there exists a Lie subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$, which is complementary to $\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ as a vector space.

A homogeneous vector bundle over the homogeneous space $\mathcal{G}/\mathcal{H}$ is a vector bundle $\pi : \mathcal{V} \to \mathcal{G}/\mathcal{H}$ together with a left action $\tilde{\ell} : \mathcal{G} \times \mathcal{V} \to \mathcal{V}$ such that π is $\mathcal{G}$-equivariant, i.e. $\pi(\tilde{\ell}(g,z)) = \ell(g,\pi(z))$, $\forall g \in \mathcal{G}$ and $\forall z \in \mathcal{V}$, and also $\tilde{\ell}_g : \mathcal{V} \to \mathcal{V}$ is a vector bundle homomorphism for all $g \in \mathcal{G}$, i.e. $\tilde{\ell}_g$ is fiber-linear. Similarly, one can define a homogeneous principal bundle $\pi : \mathcal{P} \to \mathcal{G}/\mathcal{H}$ over $\mathcal{G}/\mathcal{H}$. Here it is required that $\tilde{\ell}_g : \mathcal{P} \to \mathcal{P}$ is a homomorphism of principal bundles, i.e. equivariant with respect to the principal right action of the structure group of $\mathcal{P}$. A homorphism of homogeneous vector bundles (homogeneous principal bundles) is a homomorphism of vector bundles (principal bundles) covering $\mathrm{Id}_{\mathcal{G}/\mathcal{H}}$ that is also $\mathcal{G}$-equivariant.

3.4.3.2 Cartan Geometries

As mentioned earlier, Cartan geometries are "curved" spaces that are locally similar to flat Klein geometries. Cartan geometries are used in important physical theories such as physical gauge theories, where particles (fermions) are considered to be functions on a principal bundle and forces (bosons) are modeled as connections on that principal bundle, see [81] and references therein. There are two seemingly different ways for defining Cartan geometries: the base and the bundle definitions. These two definitions are equivalent if the underlying Klein geometry, i.e. the homogeneous model, of a Cartan geometry is effective [95]. Here we mention the principal bundle definition, see [27, 95] for more details.

Let $\mathcal{H} \subset \mathcal{G}$ be a Lie subgroup of a Lie group $\mathcal{G}$ and let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}$. A Cartan geometry of type $(\mathcal{G},\mathcal{H})$ on a smooth manifold $\mathcal{M}$ is a principal $\mathcal{H}$-bundle $(\mathcal{P},\tilde{p},\mathcal{M},\mathcal{H})$ together with a $\mathfrak{g}$-valued one-form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P};\mathfrak{g})$ called a Cartan connection. The Cartan connection has the following properties:

(i) the linear mapping $\boldsymbol{\omega}(z) : T_z\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $z \in \mathcal{P}$;

(ii) it is $\mathcal{H}$-equivariant[20], i.e. $(r^h)^*\boldsymbol{\omega} = \mathrm{Ad}(h^{-1}) \circ \boldsymbol{\omega}$, $\forall h \in \mathcal{H}$;

(iii) it reproduces the generators of fundamental vector fields, i.e. $\boldsymbol{\omega}\left(\boldsymbol{\zeta}_{\mathbf{X}}(z)\right) = \mathbf{X}$, $\forall \mathbf{X} \in \mathfrak{h}$ and $\forall z \in \mathcal{P}$.

Recall that as mentioned in §3.1.6.2, for the principal bundle $(\mathcal{P}, \tilde{p}, \mathcal{M}, \mathcal{H})$, we have $V\mathcal{P} \approx \mathcal{P} \times \mathfrak{h}$. The property (i) implies that $T\mathcal{P} \approx \mathcal{P} \times \mathfrak{g}$. Note that a Cartan connection $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}; \mathfrak{g})$ is not a principal connection form, since $\boldsymbol{\omega} \notin \Omega^1(\mathcal{P}; \mathfrak{h})$, in general. The homogeneous model for Cartan geometries of type $(\mathcal{G}, \mathcal{H})$ is the principal $\mathcal{H}$-bundle $(\mathcal{G}, p, \mathcal{G}/\mathcal{H}, \mathcal{H})$ together with the Maurer-Cartan form $\bar{\boldsymbol{\omega}} \in \Omega^1(\mathcal{G}; \mathfrak{g})$, where $p : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ is the natural projection. Thus, the homogeneous model for Cartan geometries of type $(\mathcal{G}, \mathcal{H})$ is the Klein geometry $(\mathcal{G}, \mathcal{H})$. The curvature form $\boldsymbol{K} \in \Omega^2(\mathcal{P}; \mathfrak{g})$ of the Cartan geometry $(\mathcal{P} \to \mathcal{M}, \boldsymbol{\omega})$ is defined as

$$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := d\boldsymbol{\omega}(\boldsymbol{\xi}, \boldsymbol{\eta}) + [\boldsymbol{\omega}(\boldsymbol{\xi}), \boldsymbol{\omega}(\boldsymbol{\eta})] = d\boldsymbol{\omega}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}](\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{349}$$

where $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}(\mathcal{P})$. The Maurer-Cartan form $\bar{\boldsymbol{\omega}} \in \Omega^1(\mathcal{G}; \mathfrak{g})$ is a Cartan connection for the Cartan geometry $(\mathcal{G}, p, \mathcal{G}/\mathcal{H}, \mathcal{H})$ on $\mathcal{G}/\mathcal{H}$ and the Maurer-Cartan equation (227) implies that the curvature form of this geometry is zero, i.e. the homogeneous model of a Cartan geometry is flat. Let $\varpi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the natural projection. The form $\boldsymbol{\varrho} := \varpi(\boldsymbol{K}) \in \Omega^2(\mathcal{P}; \mathfrak{g}/\mathfrak{h})$ is called the torsion of the Cartan geometry $(\mathcal{P} \to \mathcal{M}, \boldsymbol{\omega})$. The geometry is called torsion-free if $\boldsymbol{K} \in \Omega^2(\mathcal{P}; \mathfrak{h})$.

Let $\mathcal{G} = Euc(\mathbb{R}^n)$ be the group of rigid motions of $\mathbb{R}^n$ and let $\mathcal{H} \subset \mathcal{G}$ denote the

[20] The right action of $\mathcal{H}$ on $T\mathcal{P}$ is given by $\mathbf{Z}_z \cdot h := T_z r^h \cdot \mathbf{Z}_z$, where $\mathbf{Z}_z \in T_z\mathcal{P}$ and r is the principal right action of $\mathcal{P} \to \mathcal{M}$. Also the adjoint representation $\mathrm{Ad}|_{\mathcal{H}} : \mathcal{H} \times \mathfrak{g} \to \mathfrak{g}$ is a left action of $\mathcal{H}$ on $\mathfrak{g}$, and thus $\mathrm{Ad}(h^{-1}) : \mathfrak{g} \to \mathfrak{g}$, $\forall h \in \mathcal{H}$, defines a right action of $\mathcal{H}$ on $\mathfrak{g}$.

subgroup fixing the origin. By using Table 1, we can write

$$\mathcal{G} = \left\{ \begin{pmatrix} 1 & 0 \\ \mathbf{v} & A \end{pmatrix} \in GL(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in SO(\mathbb{R}^n) \right\}, \tag{350}$$

$$\mathcal{H} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in GL(\mathbb{R}^{n+1}) : A \in SO(\mathbb{R}^n) \right\} \approx SO(\mathbb{R}^n), \tag{351}$$

with the Lie algebras

$$\mathfrak{g} = \mathfrak{euc}(\mathbb{R}^n) = \left\{ \begin{pmatrix} 0 & 0 \\ \mathbf{v} & A \end{pmatrix} \in \mathfrak{gl}(\mathbb{R}^{n+1}) : \mathbf{v} \in \mathbb{R}^n, A \in \mathfrak{so}(\mathbb{R}^n) \right\}, \tag{352}$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{gl}(\mathbb{R}^{n+1}) : A \in \mathfrak{so}(\mathbb{R}^n) \right\} \approx \mathfrak{so}(\mathbb{R}^n). \tag{353}$$

The pair $(\mathcal{G}, \mathcal{H})$ defined above is called the Euclidean model of dimension n. A Euclidean geometry on manifold $\mathcal{M}$ is a Cartan geometry of type $(\mathcal{G}, \mathcal{H})$, where $(\mathcal{G}, \mathcal{H})$ is the Euclidean model. Note that this is the definition of the oriented Euclidean geometry; the unoriented Euclidean geometry is obtained by replacing $SO(\mathbb{R}^n)$ with $O(\mathbb{R}^n)$ in (350) and (351). One can show that a torsion-free Euclidean geometry on $\mathcal{M}$ determines a Riemannian metric on $\mathcal{M}$ up to a constant scale factor and thus, Riemannian geometries on $\mathcal{M}$, i.e. $\mathcal{M}$ together with a Riemannian metric, are equivalent (up to scale) to torsion-free Euclidean geometries on $\mathcal{M}$ [95]. This result shows how Cartan geometries generalize Riemannian geometries.

3.4.3.3 Parabolic Geometries

Let $\mathfrak{g}$ be a semisimple Lie algebra and suppose $k \in \mathbb{N}$. A $|k|$-grading on $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ into a direct sum of subspaces such that: (i) $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we assume $\mathfrak{g}_i = \{0\}$, for $|i| > k$; (ii) the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated (as a Lie subalgebra) by $\mathfrak{g}_{-1}$; and (iii) $\mathfrak{g}_{-k} \neq \{0\}$ and $\mathfrak{g}_k \neq \{0\}$. A Lie algebra

$\mathfrak{g}$ with such a decomposition is called a $|k|$-graded semisimple Lie algebra or simply a $|k|$-graded Lie algebra [27]. From the above definition, we conclude that $\mathfrak{g}_0$, $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$, and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ are Lie subalgebras of $\mathfrak{g}$. Recall the definition of a parabolic subalgebra $\mathfrak{p}$ of a complex semisimple Lie algebra $\mathfrak{g}$ given in §3.4.1. If $\mathfrak{g}$ is a $|k|$-graded semisimple Lie algebra, then $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ is a parabolic subalgebra of $\mathfrak{g}$. Conversely, for any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, one can obtain a $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ [27]. The complexification of a real $|k|$-graded Lie algebra $\mathfrak{g}$ is a complex $|k|$-graded Lie algebra and thus, $\mathfrak{p} \subset \mathfrak{g}$ is a real from of a parabolic subalgebra.

Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form of a $|k|$-graded semisimple Lie algebra $\mathfrak{g}$ and consider the corresponding induced map $\hat{B} : \mathfrak{g} \to \mathfrak{g}^*$. Since $\mathfrak{g}$ is semisimple, B is nondegenerate and therefore, $\hat{B}$ is a linear isomorphism. On the other hand, the restriction $B_{i,j} := B|_{\mathfrak{g}_i \times \mathfrak{g}_j}$, where $i + j = 0$, is also nondegenerate[21] [27]. This implies that if $i + j = 0$, then $\dim \mathfrak{g}_i = \dim \mathfrak{g}_j$. In particular, $B_{0,0}$ and $B_{i,-i}$ are nondegenerate and induce the isomorphisms $\mathfrak{g}_0 \approx \mathfrak{g}_0^*$ and $\mathfrak{g}_i \approx \mathfrak{g}_{-i}^*$. The adjoint representation of $\mathfrak{g}$ induces the $\mathfrak{p}$-modules $\mathfrak{g}$, $\mathfrak{p}$, and $\mathfrak{p}_+$, and hence, the $\mathfrak{p}$-modules $\mathfrak{g}/\mathfrak{p}$ and $(\mathfrak{g}/\mathfrak{p})^*$. Using the Killing form of $\mathfrak{g}$, one concludes that $\mathfrak{p}_+ \approx (\mathfrak{g}/\mathfrak{p})^*$ as $\mathfrak{p}$-modules.

Let $\mathcal{G}$ be a Lie group with the Lie algebra $\mathfrak{g}$ that is a $|k|$-graded semisimple Lie algebra. The Lie subgroups $\mathcal{G}_0$ and $\mathcal{P}$[22] are defined as $\mathcal{G}_0 = \{g \in \mathcal{G} : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i, \forall i = -k, \ldots, k\}$, and $\mathcal{P} = \{g \in \mathcal{G} : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k, \forall i \leq k\}$. We have $\mathcal{G}_0 \subset \mathcal{P} \subset \mathcal{G}$, and the Lie algebras of $\mathcal{G}_0$ and $\mathcal{P}$ are $\mathfrak{g}_0$ and $\mathfrak{p}$, respectively. If $\mathfrak{g}$ is simple, then $\mathcal{P}$ would be to the normalizer $N_{\mathcal{G}}(\mathfrak{p})$ of $\mathfrak{p}$ in $\mathcal{G}$, i.e. $\mathcal{P} = \{g \in \mathcal{G} : \mathrm{Ad}(g)(\mathfrak{p}) \subset \mathfrak{p}\}$. By definition, the subspaces $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$, for $i = -k, \ldots, k$, are $\mathcal{P}$-invariant for the

[21]Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ and let $f : V \times W \to \mathbb{R}$ be a bilinear map. The mapping f is called left nondegenerate and right nondegenerate if $\{\mathbf{v} \in V : f(\mathbf{v}, \mathbf{w}) = 0, \forall \mathbf{w} \in W\} = \{\mathbf{0}\}$, and $\{\mathbf{w} \in W : f(\mathbf{v}, \mathbf{w}) = 0, \forall \mathbf{v} \in V\} = \{\mathbf{0}\}$, respectively. The mapping f is called nondegenerate if it is both left and right nondegenerate. If $\dim V = \dim W$, then f is nondegenerate if it is left (or right) nondegenerate.

[22]We use $\mathcal{P}$ instead of $\mathcal{P}$ to emphasize that it is a Lie subgroup and not a principal bundle, in general.

adjoint action $\mathrm{Ad} : \mathcal{P} \to GL(\mathfrak{g})$. For $j > i$, the quotient $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k/(\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k)$ is isomorphic to $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{j-1}$ as a vector space. In particular, we obtain a left action of $\mathcal{P}$ on $\mathfrak{g}_- \approx \mathfrak{g}/\mathfrak{p}$ induced by the adjoint action. Note that above arguments implies that $\mathcal{P}$ corresponding to $\mathfrak{p}$ exists, but it is not necessarily unique. As an example, let $\mathcal{G} = SL(\mathbb{R}^n)$ and consider the real subalgebra that corresponds to the parabolic subalgebra (330). The parabolic subgroup defined above is the stabilizer of the first axis. This disconnected subgroup is composed of two connected components and one can also choose the connected component containing the identity, i.e. the stabilizer of the ray in the positive direction of the first axis, as the parabolic subgroup corresponding to $\mathfrak{p}$.

Let $\mathcal{G}$ be a Lie group with the $|k|$-graded Lie algebra $\mathfrak{g}$ and suppose $\mathcal{P} \subset \mathcal{G}$ is the Lie subgroup that was introduced earlier. A parabolic geometry of type $(\mathcal{G}, \mathcal{P})$ on a manifold $\mathcal{M}$ is a Cartan geometry $(\mathcal{P} \to \mathcal{M}, \boldsymbol{\omega})$ of type $(\mathcal{G}, \mathcal{P})$ on $\mathcal{M}$. Note that $\dim \mathcal{M} = \dim(\mathcal{G}/\mathcal{P})$ is a necessary condition for the existence of such a geometry. Using the Cartan connection $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}; \mathfrak{g})$, for each $\mathbf{X} \in \mathfrak{g}$ the constant vector field $\varsigma_{\mathbf{X}} \in \mathfrak{X}(\mathcal{P})$ is defined as $\varsigma_{\mathbf{X}}(z) := \boldsymbol{\omega}^{-1}(z)(\mathbf{X}) \in T_z\mathcal{P}$, $\forall z \in \mathcal{P}$. The curvature form $\boldsymbol{K} \in \Omega^2(\mathcal{P}; \mathfrak{g})$ that was introduced in (349) is $\mathcal{P}$-equivariant, i.e. $(r^{\mathrm{p}})^* \boldsymbol{K} = \mathrm{Ad}(\mathrm{p}^{-1}) \circ \boldsymbol{K}$, $\forall \mathrm{p} \in \mathcal{P}$. A differential form (with values in $\mathbb{R}$, a vector space, or a vector bundle) on a fibered manifold is called horizontal if it vanishes for one vertical vector field argument. Using the properties of $\boldsymbol{\omega}$, one can show that $\boldsymbol{K}$ is horizontal [27]. Since $\boldsymbol{\omega}$ trivializes $T\mathcal{P}$, i.e. $T\mathcal{P} \approx \mathcal{P} \times \mathfrak{g}$ via the isomorphism $\boldsymbol{\omega}$, the curvature form can also be expressed by the curvature function $\kappa : \mathcal{P} \to \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ defined as $\kappa(z)(\mathbf{X}, \mathbf{Y}) := \boldsymbol{K}(\varsigma_{\mathbf{X}}(z), \varsigma_{\mathbf{Y}}(z))$, $\forall \mathbf{X}, \mathbf{Y} \in \mathfrak{g}$. Then, the definition of curvature form yields

$$\kappa(z_x)(\mathbf{X}, \mathbf{Y}) = [\mathbf{X}, \mathbf{Y}] - \boldsymbol{\omega}_{z_x}([\boldsymbol{\omega}_{z_x}^{-1}(\mathbf{X}), \boldsymbol{\omega}_{z_x}^{-1}(\mathbf{Y})]). \tag{354}$$

We have $\varsigma_{\mathbf{X}} = \boldsymbol{\zeta}_{\mathbf{X}}$, $\forall \mathbf{X} \in \mathfrak{p}$, that means $\varsigma_{\mathbf{X}}$ is a vertical vector field due to the fact that $V\mathcal{P} \approx \mathcal{P} \times \mathfrak{p}$ via fundamental vector fields.

The homogeneous model $(\mathcal{G}, p, \mathcal{G}/\mathcal{P}, \mathcal{P})$ of parabolic geometries of type $(\mathcal{G}, \mathcal{P})$ is a parabolic geometry and its constant vector fields are the left-invariant vector fields on $\mathcal{G}$. Parabolic geometries of type $(\mathcal{G}, \mathcal{P})$ form a category $\mathscr{P}_{(\mathcal{G},\mathcal{P})}$: a morphism from $(\mathcal{P} \to \mathcal{M}, \boldsymbol{\omega})$ to $(\mathcal{P}' \to \mathcal{M}', \boldsymbol{\omega}')$ is defined to be a homomorphism $\Phi : \mathcal{P} \to \mathcal{P}'$ of principal $\mathcal{P}$-bundles that covers a local diffeomorphism $\underline{\Phi} : \mathcal{M} \to \mathcal{M}'$ such that $\boldsymbol{\omega} = \Phi^* \boldsymbol{\omega}'$.[23] Note that any principal $\mathcal{P}$-homomorphism $\Phi : \mathcal{P} \to \mathcal{P}'$ that covers a local diffeomorphism is a morphism $(\mathcal{P} \to \mathcal{M}, \Phi^*\boldsymbol{\omega}') \to (\mathcal{P}' \to \mathcal{M}', \boldsymbol{\omega}')$. A parabolic geometry is flat if $\kappa = 0$, and is torsion-free if $\kappa(z)(\mathbf{Z}_1, \mathbf{Z}_1) \in \mathfrak{p}$, $\forall z \in \mathcal{P}$ and $\forall \mathbf{Z}_1, \mathbf{Z}_2 \in \mathfrak{g}_-$. The sets of flat parabolic geometries and torsion-free parabolic geometries and their morphisms are subcategories of $\mathscr{P}_{(\mathcal{G},\mathcal{P})}$. In the sequel, we will see that linear elasticity can be considered as a flat parabolic geometry on $\mathcal{G}/\mathcal{P}$, where $\mathcal{G} = SL(\mathbb{R}^{n+1})$ and $\mathcal{P}$ is a proper parabolic subgroup.

3.4.4 Associated Representations of Homogeneous Bundles and Invariant Differential Operators

Let $\mathcal{G}$ be a Lie group with a parabolic Lie subgroup $\mathcal{P} \subset \mathcal{G}$, i.e. $\mathcal{G}$ has a semisimple $|k|$-graded Lie algebra $\mathfrak{g}$ and the Lie algebra of $\mathcal{P}$ is $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$, and recall that $(\mathcal{G}, p, \mathcal{G}/\mathcal{P}, \mathcal{P})$ is a principal bundle. Consider the left actions $\underline{\ell} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(g, \tilde{g}) \mapsto g \cdot \tilde{g}$, and $\bar{\ell} : \mathcal{G} \times (\mathcal{G}/\mathcal{P}) \to \mathcal{G}/\mathcal{P}$, $(g, \tilde{g} \cdot \mathcal{P}) \mapsto (g \cdot \tilde{g}) \cdot \mathcal{P}$. Since $p(g \cdot \tilde{g}) = g \cdot p(\tilde{g})$, and $\underline{\ell}_g(\tilde{g} \cdot \mathsf{p}) = \underline{\ell}_g(\tilde{g}) \cdot \mathsf{p}$, $\forall \mathsf{p} \in \mathcal{P}$, the principal bundle $\mathcal{G} \to \mathcal{G}/\mathcal{P}$ is a homogeneous principal bundle. Let $\varrho : \mathcal{P} \times V \to V$ be a representation of $\mathcal{P}$ on V and consider the vector bundle $(\mathcal{G} \times_{\mathcal{P}} V, \tilde{p}, \mathcal{G}/\mathcal{P}, V)$. The projection $\tilde{p}$ is $\mathcal{G}$-equivariant for the left actions of $\mathcal{G}$ on $\mathcal{G} \times_{\mathcal{P}} V$ and $\mathcal{G}/\mathcal{P}$, where $\tilde{\ell} : \mathcal{G} \times (\mathcal{G} \times_{\mathcal{P}} V) \to \mathcal{G} \times_{\mathcal{P}} V$, $g \cdot [\![\tilde{g}, \mathbf{v}]\!] \coloneqq [\![g \cdot \tilde{g}, \mathbf{v}]\!]$. Moreover, the mapping $\tilde{\ell}_g : \mathcal{G} \times_{\mathcal{P}} V \to \mathcal{G} \times_{\mathcal{P}} V$ is a vector bundle homomorphism for all $g \in \mathcal{G}$. Thus, the vector bundle $(\mathcal{G} \times_{\mathcal{P}} V, \tilde{p}, \mathcal{G}/\mathcal{P}, V)$ is a homogeneous vector bundle. On the other

[23] If $\underline{\Phi}$ is not a local diffeomorphism, then $\Phi^*\boldsymbol{\omega}'$ cannot be a Cartan connection since $\Phi^*\boldsymbol{\omega}'(z)$ is not an isomorphism.

hand, suppose $(\mathcal{V}, \pi, \mathcal{G}/\mathcal{P}, V)$ is a homogeneous vector bundle. Let $o := e \cdot \mathcal{P} \in \mathcal{G}/\mathcal{P}$, i.e. o is the orbit that passes through the unit element e. Let $\mathcal{V}_o = \pi^{-1}(o) \approx V$ be the fiber of $\mathcal{V}$ over o. We have $\pi(g \cdot \boldsymbol{\nu}) = g \cdot \pi(\boldsymbol{\nu}) = g \cdot \mathcal{P}$, $\forall g \in \mathcal{G}$ and $\boldsymbol{\nu} \in \mathcal{V}_o$. In particular, $\pi(\mathsf{p} \cdot \boldsymbol{\nu}) = o$, $\forall \mathsf{p} \in \mathcal{P}$, and therefore, the left action of $\mathcal{G}$ on $\mathcal{V}$ induces a left action of $\mathcal{P}$ on $\mathcal{V}_o$, which is a representation of $\mathcal{P}$ on $\mathcal{V}_o$. One can show that $\Phi : \mathcal{G} \times_{\mathcal{P}} \mathcal{V}_o \to \mathcal{V}$, $[\![g, \boldsymbol{\nu}]\!] \mapsto g \cdot \boldsymbol{\nu}$, is an isomorphism of homogeneous vector bundles [27], i.e. Φ covers $\mathrm{Id}_{\mathcal{G}/\mathcal{P}}$ and is fiber-linear and $\mathcal{G}$-equivariant. Also the $\mathcal{P}$-modules V and $(\mathcal{G} \times_{\mathcal{P}} V)_o$ are isomorphic. A $\mathcal{P}$-modules homomorphism $f : V \to W$ induces the homomorphism of homogeneous vector bundles $\hat{f} : \mathcal{G} \times_{\mathcal{P}} V \to \mathcal{G} \times_{\mathcal{P}} W$, $[\![g, \mathbf{v}]\!] \mapsto [\![g, f(\mathbf{v})]\!]$. If V and W are isomorphic $\mathcal{P}$-modules, then $\mathcal{G} \times_{\mathcal{P}} V \approx \mathcal{G} \times_{\mathcal{P}} W$ as homogeneous vector bundles and conversely, if $\mathcal{V}$ and $\mathcal{W}$ are isomorphic homogeneous bundles, then $\mathcal{V}_o \approx \mathcal{W}_o$ as $\mathcal{P}$-modules. Thus, there is a bijection (up to isomorphisms) between finite-dimensional representations of $\mathcal{P}$ and homogeneous vector bundles over $\mathcal{G}/\mathcal{P}$ [27].

The space of smooth sections of $\mathcal{G} \times_{\mathcal{P}} V$ can be identified with the space $C(\mathcal{G}, V)^{\mathcal{P}}$ of smooth $\mathcal{P}$-equivariant functions $\mathcal{G} \to V$ as follows. A mapping $s \in C(\mathcal{G}, V)^{\mathcal{P}}$ induces a smooth section $\boldsymbol{s} \in \Gamma(\mathcal{G} \times_{\mathcal{P}} V)$ given by $\boldsymbol{s}(g \cdot \mathcal{P}) := [\![g, s(g)]\!]$. Conversely, a section $\boldsymbol{s} \in \Gamma(\mathcal{G} \times_{\mathcal{P}} V)$, $g \cdot \mathcal{P} \mapsto [\![g, \mathbf{v}_{g \cdot \mathcal{P}}]\!]$, induces a $\mathcal{P}$-equivariant mapping $s : \mathcal{P} \to V$, $g \mapsto \mathbf{v}_{g \cdot \mathcal{P}}$. This defines a bijection between $C(\mathcal{G}, V)^{\mathcal{P}}$ and $\Gamma(\mathcal{G} \times_{\mathcal{P}} V)$ that allows us to identify these spaces.

Consider the tangent bundle $(T(\mathcal{G}/\mathcal{P}), \pi_{\mathcal{G}/\mathcal{P}}, \mathcal{G}/\mathcal{P})$. The left action of $\mathcal{G}$ on $T(\mathcal{G}/\mathcal{P})$ is given by $T\bar{\ell}_g : T(\mathcal{G}/\mathcal{P}) \to T(\mathcal{G}/\mathcal{P})$. Since $\pi_{\mathcal{G}/\mathcal{P}}(T_x \bar{\ell}_g \cdot \mathbf{z}_x) = g \cdot x = g \cdot \pi_{\mathcal{G}/\mathcal{P}}(\mathbf{z}_x)$, $\forall x \in \mathcal{G}/\mathcal{P}$ and $\forall \mathbf{z}_x \in T_x(\mathcal{G}/\mathcal{P})$, $T(\mathcal{G}/\mathcal{P})$ is a homogeneous vector bundle. A representation of $\mathcal{P}$ that corresponds to the homogeneous bundle $T(\mathcal{G}/\mathcal{P})$ is given by $T_o \bar{\ell}_{\mathsf{p}} : T_o(\mathcal{G}/\mathcal{P}) \to T_o(\mathcal{G}/\mathcal{P})$. Since $p : \mathcal{G} \to \mathcal{G}/\mathcal{P}$ is a surjective submersion, the mapping $T_e p : \mathfrak{g} \to T_o(\mathcal{G}/\mathcal{P})$ is surjective and $\ker T_e p = \mathfrak{p}$. Let $\lambda_{\tilde{g}}(g) = g \cdot \tilde{g}$, and $\rho^{\tilde{g}}(g) = g \cdot \tilde{g}$. We have $\tilde{g} \cdot p(g) = \tilde{g} \cdot (g \cdot \mathcal{P}) = p(\lambda_{\tilde{g}}(g))$, $\forall g, \tilde{g} \in \mathcal{G}$, and therefore $T_o \bar{\ell}_{\mathsf{p}} \circ T_e p = T_{\mathsf{p}} p \circ T_e \lambda_{\mathsf{p}}$, $\forall \mathsf{p} \in \mathcal{P}$. Note that $p(g) = p(g \cdot \mathsf{p}^{-1})$, $\forall \mathsf{p} \in \mathcal{P}$, that implies that $T_{\mathsf{p}} p = T_e p \circ T_{\mathsf{p}} \rho^{\mathsf{p}^{-1}}$. Using these results, one can write $T_o\bar{\ell}_{\mathsf{p}} \circ T_e p = \left(T_e p \circ T_{\mathsf{p}}\rho^{\mathsf{p}^{-1}}\right) \circ T_e\lambda_{\mathsf{p}} = T_e p \circ \left(T_{\mathsf{p}}\rho^{\mathsf{p}^{-1}} \circ T_e\lambda_{\mathsf{p}}\right) = T_e p \circ \mathrm{Ad}(\mathsf{p})$, $\forall \mathsf{p} \in \mathcal{P}$. Let $\mathrm{pr} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ be the natural projection. The mapping $\hat{\iota} : \mathfrak{g}/\mathfrak{p} \to T_o(\mathcal{G}/\mathcal{P})$, $\mathbf{X} + \mathfrak{p} \mapsto T_e p \cdot \mathbf{X}$, is a linear isomorphism since $\mathrm{null}\,\hat{\iota} = 0$, and $\dim(\mathfrak{g}/\mathfrak{p}) = \dim T_o(\mathcal{G}/\mathcal{P})$. Let $\widehat{\mathrm{Ad}} : \mathcal{P} \to GL(\mathfrak{g}/\mathfrak{p})$ be the representation of $\mathcal{P}$ induced by the adjoint representation of $\mathcal{G}$. Since $\widehat{\mathrm{Ad}}(\mathsf{p}) \circ \mathrm{pr} = \mathrm{pr} \circ \mathrm{Ad}(\mathsf{p})$, $\forall \mathsf{p} \in \mathcal{P}$, and using the fact that $\hat{\iota} \circ \mathrm{pr} = T_e p$, one obtains $T_o\bar{\ell}_{\mathsf{p}} \circ \hat{\iota} \circ \mathrm{pr} = \hat{\iota} \circ \mathrm{pr} \circ \mathrm{Ad}(\mathsf{p}) = \hat{\iota} \circ \widehat{\mathrm{Ad}}(\mathsf{p}) \circ \mathrm{pr}$. Then, the surjectivity of pr implies that $\widehat{\mathrm{Ad}}(\mathsf{p}) = \hat{\iota}^{-1} \circ T_o\bar{\ell}_{\mathsf{p}} \circ \hat{\iota}$, i.e. $\hat{\iota}$ is also a homorphism of $\mathcal{P}$-modules $(\mathfrak{g}/\mathfrak{p}, \widehat{\mathrm{Ad}})$ and $(T_o(\mathcal{G}/\mathcal{P}), T_o\bar{\ell}_{\mathsf{p}})$. Hence, the homogeneous vector bundle $T(\mathcal{G}/\mathcal{P}) \to \mathcal{G}/\mathcal{P}$ corresponds to the representation $\widehat{\mathrm{Ad}} : \mathcal{P} \to GL(\mathfrak{g}/\mathfrak{p})$. Equivalently, $T(\mathcal{G}/\mathcal{P})$ corresponds to the representation of $\mathcal{P}$ on $\mathfrak{g}_- \approx \mathfrak{g}/\mathfrak{p}$, where the representation of $\mathfrak{g}_-$ is induced by that of $\mathfrak{g}/\mathfrak{p}$. These relations can be represented as in the following diagram.

$$
\begin{tikzcd}
\text{Homogeneous Bundles} & \mathcal{P}\text{-modules} \\
T(\mathcal{G}/\mathcal{P}) \arrow[r] \arrow[d, Leftrightarrow, "\approx"] \arrow[dd, Leftrightarrow, bend right=60, "\approx"'] & T_o(\mathcal{G}/\mathcal{P}) \arrow[d, Leftrightarrow, "\approx"] \arrow[dl] \\
\mathcal{G} \times_{\mathcal{P}} T_o(\mathcal{G}/\mathcal{P}) \arrow[d, Leftrightarrow, "\approx"] & \mathfrak{g}/\mathfrak{p} \approx \mathfrak{g}_- \arrow[dl] \\
\mathcal{G} \times_{\mathcal{P}} (\mathfrak{g}/\mathfrak{p}) &
\end{tikzcd}
$$

The equivalence between $\mathcal{P}$-representations and homogeneous vector bundles on $\mathcal{G}/\mathcal{P}$ that was mentioned earlier, is compatible with constructions in the following sense [27]. Let $\mathcal{V} \to \mathcal{G}/\mathcal{P}$ and $\mathcal{W} \to \mathcal{G}/\mathcal{P}$ correspond to $\mathcal{P}$-modules V and W, respectively. Then, the homogeneous vector bundles $\mathcal{V}^*$, $\mathcal{V} \oplus \mathcal{W}$, and $\mathcal{V} \otimes \mathcal{W}$ correspond to $\mathcal{P}$-modules V, $V \oplus W$, and $V \otimes W$, respectively. Recall that the Killing form induces a $\mathcal{P}$-module isomorphism $\mathfrak{p}_+ \approx (\mathfrak{g}/\mathfrak{p})^*$. The cotangent bundle $T^*(\mathcal{G}/\mathcal{P})$ correspond to the $\mathcal{P}$-module $(\mathfrak{g}/\mathfrak{p})^*$ or $\mathfrak{p}_+$. More generally, the tensor bundle $\otimes^m T(\mathcal{G}/\mathcal{P}) \otimes \otimes^n T^*(\mathcal{G}/\mathcal{P})$ and $\mathbb{S}_\eta T^*(\mathcal{G}/\mathcal{P})$[24] are homogeneous vector bundles that correspond to $\mathcal{P}$-modules

[24]Note that the left action of $\mathcal{G}$ on $\bigotimes^d V$ induced by the $\mathcal{G}$-module V commutes with the right action of $\mathfrak{S}_d$, i.e. $[g \cdot (\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_d)] \cdot \sigma = g \cdot [(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_d) \cdot \sigma]$, and thus, $\mathbb{S}_\eta V$ is $\mathcal{G}$-invariant. This implies that $\mathbb{S}_\eta \mathcal{V}$ is a homogeneous vector bundle that corresponds to the $\mathcal{P}$-module $(\mathbb{S}_\eta \mathcal{V})_o = \mathbb{S}_\eta \mathcal{V}_o$.

$\otimes^m(\mathfrak{g}/\mathfrak{p}) \otimes \otimes^n(\mathfrak{g}/\mathfrak{p})^*$ and $\mathbb{S}_\eta(\mathfrak{g}/\mathfrak{p})^*$, respectively, where $\mathbb{S}_\eta$ is the Schur functor for a partition η.

Given a Cartan subalgebra $\mathfrak{h}$ and a set of simple roots $\mathcal{S}$ for $\mathfrak{g}$, let $\mathcal{S}_\mathfrak{p} = \mathcal{S}\backslash\{\alpha\}$, where $\alpha \in \mathcal{S}$. Then, the decomposition $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{l} \oplus \mathfrak{u}$ introduced in §3.4.1 is a $|1|$-grading on $\mathfrak{g}$, where $\mathfrak{g}_{-1} = \mathfrak{u}_-$, $\mathfrak{g}_0 = \mathfrak{l}$, and $\mathfrak{g}_1 = \mathfrak{u}$ [17]. The $\mathfrak{p}$-module $\mathfrak{g}/\mathfrak{p}$ is irreducible if and only if $\mathfrak{u}_-$ is Abelian. Thus, the parabolic subalgebra $\mathfrak{p} = \times\!\!-\!\!\bullet\!\!-\!\!\bullet$ defined in (330) introduces a $|1|$-grading on $\mathfrak{sl}(\mathbb{C}^4)$ and the representation $\mathfrak{g}/\mathfrak{p}$ with the Dynkin diagram $\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}$ is irreducible. In general, the Dynkin diagram of the irreducible $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^n)$ is $\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\ \cdots\ \overset{0}{\bullet}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\ \cdots\ \overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}$, where cross can be on any node [17]. Note that as was mentioned earlier, since the $\mathfrak{p}$-module $\mathfrak{g}/\mathfrak{p}$ is integral for $\mathfrak{g}$, it integrates to an irreducible $\mathcal{P}$-module as well.

Let $\Gamma(\mathcal{V})$ be the space of smooth sections of a homogeneous vector bundle $\mathcal{V} \to \mathcal{G}/\mathcal{P}$. One can define a left action $\mathcal{G} \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ given by $(g \cdot \boldsymbol{s})(x) = g \cdot (\boldsymbol{s}\,(g^{-1} \cdot x))$, $\forall g \in \mathcal{G}$ and $\forall x \in \mathcal{G}/\mathcal{P}$. Suppose $\mathcal{W} \to \mathcal{G}/\mathcal{P}$ is another homogeneous vector bundle and $D : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ is a linear differential operator of order r. The linear differential operator D is called an invariant linear differential operator if it is also $\mathcal{G}$-equivariant with respect to the above left action, i.e. $D(g \cdot \boldsymbol{s}) = g \cdot D(\boldsymbol{s})$, $\forall \boldsymbol{s} \in \Gamma(\mathcal{V})$ and $\forall g \in \mathcal{G}$. Suppose $\boldsymbol{\alpha} \in \Gamma(\mathcal{V}|_U)$ is a local section of $\mathcal{V}$, where $U \subset \mathcal{G}/\mathcal{P}$ is an open subset. Note that the domain of $g \cdot \boldsymbol{\alpha}$ is $g \cdot U$. Let $\mathscr{O}(\mathcal{V})$ denote the sheaf of germs of local sections of $\mathcal{V}$. We can define the left action $\mathcal{G} \times \mathscr{O}(\mathcal{V}) \to \mathscr{O}(\mathcal{V})$, $(g, [\boldsymbol{\alpha}]_x) \mapsto [g \cdot \boldsymbol{\alpha}]_{g \cdot x}$. An isomorphism of homogeneous vector bundles $\iota : \mathcal{V} \to \mathcal{W}$ induces $\mathcal{G}$-equivariant isomorphism $\mathscr{O}(\mathcal{V}) \to \mathscr{O}(\mathcal{W})$ defined by $[\boldsymbol{\alpha}]_x \mapsto [\iota \circ \boldsymbol{\alpha}]_x$. The sheaf of germs of local sections of $\mathcal{G} \times_\mathcal{P} V \to \mathcal{G}/\mathcal{P}$ is denoted by $\mathscr{G}(V)$. Clearly, if V and V' are isomorphic $\mathcal{P}$-modules, then $\mathscr{G}(V) \approx \mathscr{G}(V')$. We do not distinguish between isomorphic classes of homogeneous vector bundles, i.e. $\mathscr{G}(V)$ also denotes $\mathscr{O}(\mathcal{V})$ if $\mathcal{V} \approx \mathcal{G} \times_\mathcal{P} V$. Any integral weight of $\mathfrak{g}$ that is dominant for a parabolic subalgebra $\mathfrak{p}$ corresponds to an irreducible representation of $\mathcal{P}$. For such irreducible representations, $\mathscr{G}(V)$ is also denoted by

Dynkin diagrams. For example, let $\mathfrak{p} = \times\!\!-\!\!\bullet\!\!-\!\!\bullet$. Then $\mathscr{T}(\mathcal{G}/\mathcal{P}) = \mathscr{G}(\mathfrak{g}/\mathfrak{p}) = \mathscr{G}\big(\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}\big)$, and $\Omega^0_{\mathcal{G}/\mathcal{P}} = \mathscr{G}(\mathbb{C}) = \mathscr{G}\big(\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\big)$, where $\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$ is the trivial representation $\mathbb{C}$, i.e. $\mathsf{p}\cdot z = z$, $\forall \mathsf{p} \in \mathcal{P}$ and $\forall z \in \mathbb{C}$.

3.4.5 The Linear Elastostatics Complex as a BGG Resolution

We are ready now to explain how the linear elasticity complex arises as a BGG resolution. Let $\mathcal{G} = SL(\mathbb{R}^{n+1})$. Since $SL(\mathbb{R}^{n+1})$ is a real form of $SL(\mathbb{C}^{n+1})$, one can use notations for $SL(\mathbb{C}^{n+1})$ and its parabolic subgroup $\mathcal{P}$ to also denote $SL(\mathbb{R}^{n+1})$ and a real form of $\mathcal{P}$ [42, 27]. In particular, Dynkin diagrams can be used for denoting irreducible representations of $SL(\mathbb{R}^{n+1})$ and induced irreducible homogeneous vector bundles. Let $\mathfrak{p} = \times\!\!-\!\!\bullet\!\!-\!\!\bullet\cdots\bullet\!\!-\!\!\bullet \subset \mathfrak{g} = \mathfrak{sl}(\mathbb{R}^{n+1})$. As was mentioned earlier, this choice induces a $|1|$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$
\begin{aligned}
\mathfrak{g}_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ \mathbf{X} & 0 \end{pmatrix} \in \mathfrak{sl}(\mathbb{R}^{n+1}) : \mathbf{X} \in \mathbb{R}^n \right\}, \\
\mathfrak{g}_{0} &= \left\{ \begin{pmatrix} -\operatorname{tr} A & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{sl}(\mathbb{R}^{n+1}) : A \in \mathfrak{gl}(\mathbb{R}^n) \right\}, \\
\mathfrak{g}_{1} &= \left\{ \begin{pmatrix} 0 & \mathbf{Z}^{\mathsf{T}} \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(\mathbb{R}^{n+1}) : \mathbf{Z} \in \mathbb{R}^n \right\}.
\end{aligned} \tag{355}
$$

Let $\{X^i\}$ be the Cartesian coordinate of $\mathbb{R}^{n+1}$ and $\mathbf{X} = (X^1, \ldots, X^{n+1}) \in \mathbb{R}^{n+1}$. Consider the standard action of $\mathcal{G}$ on $\mathbb{R}^{n+1}$. There exists two choices for the parabolic subgroup corresponding to $\mathfrak{p}$. The parabolic subgroup can be either the stabilizer of the line coincide with the X^1-axis, i.e.

$$
\mathcal{P}' = \left\{ \begin{pmatrix} \det B^{-1} & \mathbf{Z}^{\mathsf{T}} \\ 0 & B \end{pmatrix} \in SL(\mathbb{R}^{n+1}) : B \in GL(\mathbb{R}^n), \mathbf{Z} \in \mathbb{R}^n \right\}, \tag{356}
$$

or the stabilizer of the ray in the positive direction of X^1-axis, i.e.

$$\mathcal{P} = \left\{ \begin{pmatrix} \det B^{-1} & \mathbf{Z}^\mathsf{T} \\ 0 & B \end{pmatrix} \in SL(\mathbb{R}^{n+1}) : B \in GL^+(\mathbb{R}^n), \mathbf{Z} \in \mathbb{R}^n \right\}, \tag{357}$$

where $GL^+(\mathbb{R}^n) = \{A \in GL(\mathbb{R}^n) : \det A > 0\}$. The set $\mathcal{P}'$ is not connected; the two connected components are determined by the sign of $\det B$ [27]. The subgroup $\mathcal{P}$ is the connected component of $\mathcal{P}'$ containing the identity. The homogeneous space $\mathcal{G}/\mathcal{P}'$ is the real projective space $\mathbb{R}P^n$, which is an n-dimensional compact manifold that is orientable if and only if n is odd [30]. This nonorientablity causes some technical issues in the calculations and therefore, it is easier to work with $\mathcal{P}$ instead of $\mathcal{P}'$ [53, 42]. The homogeneous space $\mathcal{G}/\mathcal{P}'$ is a special case of flag manifolds, which are homogeneous spaces corresponding to parabolic subgroups of $SL(\mathbb{C}^{n+1})$ [17, 39]. Projective spaces and Grassmannians are examples of flag manifolds.

Consider an element $Q = \begin{pmatrix} Q_{11} & \bar{\mathbf{Z}}^\mathsf{T} \\ \mathbf{W} & \underline{Q} \end{pmatrix} \in \mathcal{G}$, where $Q_{11} \in \mathbb{R}$. The principal right action of $\mathcal{P}$ on $\mathcal{G}$ is given by $Q \cdot \begin{pmatrix} \det B^{-1} & \mathbf{Z}^\mathsf{T} \\ 0 & B \end{pmatrix} = \begin{pmatrix} (\det B^{-1}) Q_{11} & Q_{11}\mathbf{Z}^\mathsf{T} + \bar{\mathbf{Z}}^\mathsf{T} B \\ (\det B^{-1})\mathbf{W} & \mathbf{W}\mathbf{Z}^\mathsf{T} + \underline{Q} B \end{pmatrix}$. Since $\det B > 0$, the orbit $\mathcal{G}_Q$ that passes through Q is determined by $\mathbf{q} = \begin{Bmatrix} Q_{11} \\ \mathbf{W} \end{Bmatrix} \in \mathbb{R}^{n+1}$, i.e. the orbit $\mathcal{G}_Q$ is determined by the ray emanating from origin in the direction of $\mathbf{q}$. Let $\mathcal{S}^n$ be the unit n-sphere. We have the deffiomorphism $\mathcal{J} : \mathcal{G}/\mathcal{P} \to \mathcal{S}^n$, $Q \cdot \mathcal{P} \mapsto \mathbf{q}/\|\mathbf{q}\|$, where $\|\cdot\|$ is the standard norm of $\mathbb{R}^{n+1}$ [27]. Thus, $\mathcal{G}/\mathcal{P}$ is an n-dimensional orientable manifold. We know that $(\mathcal{G}, p, \mathcal{G}/\mathcal{P}, \mathcal{P})$ is a homogeneous principal $\mathcal{P}$-bundle. Let $\hat{p} := \mathcal{J} \circ p$. Since $\mathcal{P}$ acts freely on $\mathcal{G}$ and the orbits of this action coincide with $\hat{p}^{-1}(\mathbf{X})$, Lemma 10.3 of [73] implies that $(\mathcal{G}, \hat{p}, \mathcal{S}^n, \mathcal{P})$ is a principal $\mathcal{P}$-bundle. The action $\mathcal{G} \times \mathcal{G}/\mathcal{P} \to \mathcal{G}/\mathcal{P}$ induces the action $\mathcal{G} \times \mathcal{S}^n \to \mathcal{S}^n$, $g \cdot \mathbf{X} = \mathcal{J}(g \cdot \mathcal{J}^{-1}(\mathbf{X}))$, which makes the diffeomorphism $\mathcal{J}$ $\mathcal{G}$-equivariant. Thus, for all practical purposes, we can identify $\mathcal{G}/\mathcal{P}$ with $\mathcal{S}^n$ and consider $\mathcal{G} \to \mathcal{S}^n$ as a homogeneous principal bundle. In such a case, $\mathcal{S}^n$ is called the projective n-sphere [40] and n-manifolds with parabolic geometries of type $(\mathcal{G}, \mathcal{P})$ are called oriented projective manifolds of dimension n [27]. Note that the action of $\mathcal{G}$

on $\mathcal{S}^n$ is induced by the restriction of the standard action of $\mathcal{G}$ on $\mathbb{R}^{n+1}$ to $\mathcal{S}^n$. The representation $\mathcal{P} \times \mathbb{R}^n \to \mathbb{R}^n$ corresponding to $T\mathcal{S}^n$[25] is $\hat{B} \cdot \mathbf{W} = (\det B)B\mathbf{W}$, with the Dynkin diagram notation $\overset{1}{\times}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{1}{\bullet}$. The representation of $T^*\mathcal{S}^n$ is $\hat{B} \cdot \mathbf{W} = (\det B^{-\mathsf{T}})B^{-\mathsf{T}}\mathbf{W}$. Let $\mathcal{L}^{\langle w \rangle} \to \mathcal{G}/\mathcal{P}$ be the bundle of project densities of weight w on $\mathcal{G}/\mathcal{P}$ that was introduced in §3.2.1.1. The line bundle $\mathcal{L}^{\langle w \rangle} \to \mathcal{G}/\mathcal{P}$ is a homogeneous vector bundle that corresponds to an irreducible representation of $\mathcal{P}$ with the Dynkin diagram $\overset{w}{\times}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ [16, 53].

Let $\eta = (\eta_1, \dots, \eta_n)$, $\eta_1 \geq \cdots \geq \eta_n \geq 0$, be a partition of $v = \sum_{i=1}^n \eta_i$. The vector bundle $\mathbb{S}_\eta^{\langle w \rangle} T^*\mathcal{S}^n := (\mathbb{S}_\eta(T^*\mathcal{S}^n)) \otimes \mathcal{L}^{\langle w \rangle} \to \mathcal{G}/\mathcal{P}$[26] is an irreducible homogeneous vector bundle that corresponds to an irreducible representation with the Dynkin diagram $\overset{w-v-\eta_1}{\times}\!-\!\overset{\eta_1-\eta_2}{\bullet}\!-\!\overset{\eta_2-\eta_3}{\bullet} \cdots -\!\overset{\eta_{n-1}-\eta_n}{\bullet}$ [42]. The numbers v and w are called the valency and the projective weight of the tensor bundle $\mathbb{S}_\eta^{\langle w \rangle} T^*\mathcal{S}^n$, respectively. One can show that $\mathbb{S}_{(\eta_1,\dots,\eta_n)}^{\langle w \rangle} T^*\mathcal{S}^n \approx \mathbb{S}_{(\eta_1-\eta_n,\dots,\eta_{n-1}-\eta_n,0)}^{\langle w-\eta_n(n+1) \rangle} T^*\mathcal{S}^n$. As an example, if $n = 3$, we have $\mathcal{G} \times_{\mathcal{P}} \left(\overset{c_1}{\times}\!-\!\overset{c_2}{\bullet}\!-\!\overset{c_3}{\bullet}\right) \approx \mathbb{S}_{(c_2+c_3,c_3)}^{\langle c_1+2c_2+3c_3 \rangle} T^*\mathcal{S}^3$ [41]. Let $\mathscr{L}^{\langle w \rangle}$ be the sheaf of germs of local sections of $\mathcal{L}^{\langle w \rangle}$. Using the notation introduced in §3.4.4, we can write $\mathscr{G}\left(\overset{w-v-\eta_1}{\times}\!-\!\overset{\eta_1-\eta_2}{\bullet}\!-\!\overset{\eta_2-\eta_3}{\bullet} \cdots -\!\overset{\eta_{n-1}-\eta_n}{\bullet}\right) = (\mathbb{S}_\eta(\mathscr{T}^*\mathcal{S}^n)) \otimes \mathscr{L}^{\langle w \rangle} := \mathbb{S}_\eta^{\langle w \rangle} \mathscr{T}^*\mathcal{S}^n$.

The homogeneous vector bundle $\mathcal{S}^n \times \mathbb{R}$ with the $\mathcal{G}$-action $g \cdot (x, c) = (g \cdot x, c)$, corresponds to the trivial representation of $\mathcal{P}$ on $\mathbb{R}$. We can identify the sets $\Omega^0(\mathcal{S}^n)$ and $\Gamma(\mathcal{S}^n \times \mathbb{R})$.[27] More generally, let V be a $\mathcal{G}$-module with a basis $\{\mathbf{e}_j\}$ and consider the homogeneous vector bundle $\Lambda^k T^*\mathcal{S}^n \otimes (\mathcal{S}^n \times V)$. Then, we have $\Omega^k(\mathcal{S}^n; V) = \Gamma(\Lambda^k T^*\mathcal{S}^n \otimes (\mathcal{S}^n \times V))$. Note that $\Omega^0(\mathcal{S}^n; V) = \Gamma(\mathcal{S}^n \times V)$, where the right action of $\mathcal{S}^n \times V$ is $g \cdot (x, \mathbf{v}) = (g \cdot x, g \cdot \mathbf{v})$. Suppose $\mathbf{a} \in T^*_{g^{-1}\cdot x}\mathcal{S}^n$. The left action $\mathcal{G} \times T^*\mathcal{S}^n \to T^*\mathcal{S}^n$ is given by $g \cdot \mathbf{a} = \ell^*_{g^{-1}}\mathbf{a}$, where $\ell^*_{g^{-1}}$ is the pull-back with respect to $\ell_{g^{-1}}$, i.e.

[25]Note that the homogeneous bundles $(T(\mathcal{G}/\mathcal{P}), \pi_{\mathcal{G}/\mathcal{P}}, \mathcal{G}/\mathcal{P})$ and $(T\mathcal{S}^n, \mathcal{J}^{-1} \circ \pi_{\mathcal{S}^n}, \mathcal{G}/\mathcal{P})$ are isomorphic.

[26]On an orientable n-dimensional homogeneous space, the line bundle $\mathcal{L}^{\langle -n-1 \rangle}$ that is isomorphic to the bundle of 1-densities, is also isomorphic to $\Lambda^n T^*(\mathcal{G}/\mathcal{P})$ and therefore, it is technically easier to work with orientable spaces [16, 53].

[27]Note that $\mathcal{S}^n \times \mathbb{R}$ with the trivial $\mathcal{G}$-action and $\mathcal{L}^{\langle w \rangle}$ have isomorphic underlying sets but their $\mathcal{G}$-actions are not isomorphic, in general, and thus, they are different homogeneous bundles.

$(g \cdot \mathbf{a})(\mathbf{X}) = \mathbf{a}(T_x\ell_{g^{-1}} \cdot \mathbf{X})$, $\forall \mathbf{X} \in T_x\mathcal{S}^n$. Consequently, the action of $\mathcal{G}$ on $\Omega^k(\mathcal{S}^n)$ introduced in §3.4.4, reads $g \cdot \boldsymbol{\eta} = \ell^*_{g^{-1}}\boldsymbol{\eta}$, $\boldsymbol{\eta} \in \Omega^k(\mathcal{S}^n)$. A section $\boldsymbol{\alpha} \in \Omega^k(\mathcal{S}^n; V)$ can be written as $\boldsymbol{\alpha} = \sum_j \boldsymbol{\alpha}_j \otimes \mathbf{e}_j$, where $\boldsymbol{\alpha}_j \in \Omega^k(\mathcal{S}^n)$. The twisted exterior derivative $d : \Omega^k(\mathcal{S}^n; V) \to \Omega^{k+1}(\mathcal{S}^n; V)$ is defined as $d(\boldsymbol{\eta} \otimes \mathbf{v}) = (d\boldsymbol{\eta}) \otimes \mathbf{v}$ [22]. Using the facts that $\ell^*_{g^{-1}}(\boldsymbol{\eta} \wedge \boldsymbol{\xi}) = \ell^*_{g^{-1}}\boldsymbol{\eta} \wedge \ell^*_{g^{-1}}\boldsymbol{\xi}$, $\boldsymbol{\xi} \in \Omega^l(\mathcal{S}^n)$, and the naturallity of the exterior derivative, we can write

$$\begin{aligned} d(g \cdot \boldsymbol{\alpha}) &= \sum_j d\big(g \cdot (\boldsymbol{\alpha}_j \otimes \mathbf{e}_j)\big) = \sum_j d\big((g \cdot \boldsymbol{\alpha}_j) \otimes (g \cdot \mathbf{e}_j)\big) = \sum_j d(\ell^*_{g^{-1}}\boldsymbol{\alpha}_j) \otimes (g \cdot \mathbf{e}_j) \\ &= \sum_j \ell^*_{g^{-1}}(d\boldsymbol{\alpha}_j) \otimes (g \cdot \mathbf{e}_j) = g \cdot \Big(\sum_j (d\boldsymbol{\alpha}_j) \otimes \mathbf{e}_j\Big) = g \cdot d\boldsymbol{\alpha}. \end{aligned} \tag{358}$$

Therefore, we proved the following:

Theorem 3.4.1. *Let V be a $\mathcal{G}$-module. Then, the twisted exterior derivative $d : \Omega^k(\mathcal{S}^n; V) \to \Omega^{k+1}(\mathcal{S}^n; V)$ is a $\mathcal{G}$-invariant differential operator of order 1.*

Let λ be a dominant integral weight of $\mathfrak{g}$ and recall the definitions of the affine Weyl action $\mathrm{w}.\lambda$ and the Hasse diagram $W^{\mathfrak{p}}$ introduced in §3.4.1. Suppose $E(\lambda)$ and $E_{\mathfrak{p}}(\lambda)$ denote irreducible representations of $\mathfrak{g}$ and $\mathfrak{p}$ with the lowest weight $-\lambda$, respectively, and let $\mathscr{G}_{\mathfrak{p}}(\lambda)$ be the sheaf of germs of local sections of $\mathcal{G} \times_{\mathcal{P}} E\mathfrak{p}(\lambda)$. The following theorem holds.

Theorem 3.4.2 (Baston and Eastwood [17])**.** *Let λ be a dominant integral weight for $\mathfrak{g}$. There exists an exact resolution of constant sheaf $E(\lambda)$ given by $0 \longrightarrow E(\lambda) \longrightarrow \mathscr{D}^\bullet(\lambda)$, that is called a BGG resolution on $\mathcal{G}/\mathcal{P}$, where*

$$\mathscr{D}^k(\lambda) = \bigoplus_{\mathrm{w} \in W^{\mathfrak{p}},\, \ell(\mathrm{w}) = k} \mathscr{G}_{\mathfrak{p}}(\mathrm{w}.\lambda). \tag{359}$$

This theorem is valid for any semisimple Lie group $\mathcal{G}$ with a parabolic subgroup $\mathcal{P}$. Using the Hasse diagram (336), we can write BGG resolutions on $\mathcal{G}/\mathcal{P} \cong \mathcal{S}^n$. For example, the de Rham resolution (297) of the constant sheaf $\mathcal{S}^n \times \mathbb{R}$ reads

$$
\begin{aligned}
0 \longrightarrow\ & \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet} \longrightarrow \mathscr{G}\Big(\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\Big) \longrightarrow \mathscr{G}\Big(\overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\Big) \longrightarrow \\
& \mathscr{G}\Big(\overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\Big) \longrightarrow \cdots \longrightarrow \mathscr{G}\Big(\overset{-n}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\Big) \longrightarrow \mathscr{G}\Big(\overset{-n-1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\Big) \longrightarrow 0.
\end{aligned}
\tag{360}
$$

The general case for $n = 2$ is

$$
0 \longrightarrow \overset{c_1}{\bullet}\!-\!\overset{c_2}{\bullet} \longrightarrow \mathscr{G}\Big(\overset{c_1}{\times}\!-\!\overset{c_2}{\bullet}\Big) \xrightarrow{D^{(c_1+1)}} \mathscr{G}\Big(\overset{-c_1-2}{\times}\!-\!\overset{c_1+c_2+1}{\bullet}\Big) \xrightarrow{D^{(c_2+1)}} \mathscr{G}\Big(\overset{-c_1-c_2-3}{\times}\!-\!\overset{c_1}{\bullet}\Big) \longrightarrow 0, \tag{361}
$$

where $D^{(i)}$ is an ith-order differential operator. Similarly, the general case for $n = 3$ reads

$$
\begin{aligned}
0 \longrightarrow\ & \overset{c_1}{\bullet}\!-\!\overset{c_2}{\bullet}\!-\!\overset{c_3}{\bullet} \longrightarrow \mathscr{G}\Big(\overset{c_1}{\times}\!-\!\overset{c_2}{\bullet}\!-\!\overset{c_3}{\bullet}\Big) \xrightarrow{D^{(c_1+1)}} \mathscr{G}\Big(\overset{-c_1-2}{\times}\!-\!\overset{c_1+c_2+1}{\bullet}\!-\!\overset{c_3}{\bullet}\Big) \xrightarrow{D^{(c_2+1)}} \\
& \mathscr{G}\Big(\overset{-c_1-c_2-3}{\times}\!-\!\overset{c_1}{\bullet}\!-\!\overset{c_2+c_3+1}{\bullet}\Big) \xrightarrow{D^{(c_3+1)}} \mathscr{G}\Big(\overset{-c_1-c_2-c_3-4}{\times}\!-\!\overset{c_1}{\bullet}\!-\!\overset{c_2}{\bullet}\Big) \longrightarrow 0.
\end{aligned}
\tag{362}
$$

Differential operators of BGG resolutions are $\mathcal{G}$-invariant linear differential operators. Consequently, one can use the classification of invariant operators for explicitly writing these operators [42]. Alternatively, as we will discuss later, it is also possible to derive these operators from proper vector-valued de Rham complexes. The later approach can be extended to define BGG sequences on curved parabolic geometries [28]. Consider the Riemannian manifold $(\mathcal{S}^3, \widetilde{\boldsymbol{g}})$, where $\widetilde{\boldsymbol{g}}$ is the round metric of the 3-sphere, i.e. $\widetilde{\boldsymbol{g}}$ is induced by the standard metric of $\mathbb{R}^4$. This manifold has a constant sectional curvature and the great circles of $\mathcal{S}^3$ are the geodesics of the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{\boldsymbol{g}}$. Thus, we have the Calabi complex (298) on $\mathcal{S}^3$. Due to the round metric, we also have the Eastwood complex introduced in the second row of (298). On the other hand, one can show that the Eastwood complex on $\mathcal{S}^3$ is equivalent to the BGG complex associated to $\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$, which is the irreducible representation $\Lambda^2\mathbb{R}^4$

of $SL(\mathbb{R}^4)$ [41]. Hence, the Eastwood complex on $\mathcal{S}^3$ is

$$0 \longrightarrow \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} \longrightarrow \mathscr{G}\big(\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\big) \longrightarrow \mathscr{G}\big(\overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\big) \longrightarrow \mathscr{G}\big(\overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\big) \longrightarrow \mathscr{G}\big(\overset{-5}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\big) \longrightarrow 0. \tag{363}$$

Equivalently, using the notation mentioned earlier, one can write (363) as

$$\begin{aligned} 0 \longrightarrow \Lambda^2\mathbb{R}^4 \longrightarrow \mathbb{S}^{\langle 2\rangle}_{(1)}\mathscr{T}^*\mathcal{S}^3 \longrightarrow \mathbb{S}^{\langle 2\rangle}_{(2)}\mathscr{T}^*\mathcal{S}^3 \longrightarrow \mathbb{S}^{\langle 2\rangle}_{(2,2)}\mathscr{T}^*\mathcal{S}^3 \longrightarrow & \\ \mathbb{S}^{\langle 2\rangle}_{(2,2,1)}\mathscr{T}^*\mathcal{S}^3 \approx \mathbb{S}^{\langle -2\rangle}_{(1,1)}\mathscr{T}^*\mathcal{S}^3 \longrightarrow 0. & \end{aligned} \tag{364}$$

Note that the differential operators of the Eastwood complex on $\mathcal{S}^3$ are projectively invariant and also $\mathcal{G}$-invariant for the action of $\mathcal{G} = SL(\mathbb{R}^4)$. This is the consequence of the fact that projective structures on the homogeneous space $\mathcal{G}/\mathcal{P} \cong \mathcal{S}^n$ are equivalent to parabolic geometries of type $(\mathcal{G},\mathcal{P})$ [27]. In particular, the projective structure arising from the round metric of $\mathbb{R}^3$ is equivalent to the flat homogeneous space of parabolic geometries of type $(\mathcal{G};\mathcal{P})$, i.e. the principal $\mathcal{P}$-bundle $\mathcal{G} \to \mathcal{G}/\mathcal{P} \approx \mathcal{S}^3$ together with the Maurer-Cartan form $\boldsymbol{\omega} \in \Omega^1(\mathcal{G};\mathfrak{g})$ of $\mathcal{G}$ [42]. The Maurer-Cartan form induces a linear connection on irreducible homogeneous vector bundles. These vector bundles are also called tractor bundles. Let $\zeta : \mathbb{R}^3 \to \mathcal{S}^3$ be the central projection of $\mathbb{R}^3$ to a hemisphere of $\mathcal{S}^3$ given by $(X^1, X^2, X^3) \mapsto (1, X^1, X^2, X^3)/\sqrt{1 + \sum_{i=1}^3 (X^i)^2}$. Let $\mathcal{B} \subset \mathbb{R}^3$ be an open subset and let $\widetilde{\mathcal{B}} := \zeta(\mathcal{B}) \subset \mathcal{S}^3$. The Riemannian manifold $(\mathcal{B}, \zeta^*\widetilde{\boldsymbol{g}})$ has a constant sectional curvature. From (256) recall that $\nabla^{\zeta^*\widetilde{\boldsymbol{g}}} := \zeta^*\widetilde{\nabla}$, where $\nabla^{\zeta^*\widetilde{\boldsymbol{g}}}$ is the Levi-Civita connection of $\zeta^*\widetilde{\boldsymbol{g}}$. Suppose ∇ is the Levi-Civita connection of the standard metric $\boldsymbol{g}$ of $\mathbb{R}^3$. A metric $\hat{\boldsymbol{g}}$ on $\mathcal{B}$ is called projectively flat if and only if $\nabla^{\hat{\boldsymbol{g}}} \in [\nabla]$. Thus, the geodesics of projectively flat manifolds in $\mathbb{R}^3$ are lines up to parameterizations. One can show that projectively flat metrics have constant sectional curvatures [43]. Since ζ preserves geodesics, i.e. the images of lines are great circles of $\mathcal{S}^3$, ζ is a morphism of projective structures $(\mathcal{B}, [\nabla])$ and $(\widetilde{\mathcal{B}}, [\widetilde{\nabla}])$, i.e. $\zeta^*\widetilde{\nabla} \in [\nabla]$. This implies that $\zeta^*\widetilde{\boldsymbol{g}}$ is projectively flat. We have the Eastwood complex for $(\mathcal{B}, \zeta^*\widetilde{\boldsymbol{g}})$ as it has a constant sectional curvature. The projective invariance implies that the Eastwood complexes of $(\mathcal{B}, \boldsymbol{g})$ and $(\mathcal{B}, \zeta^*\widetilde{\boldsymbol{g}})$ coincides [40, 42]. In particular, the curvature operator depends on a combination of the certain part of the Riemannian curvature called Schouten tensor. This combination vanishes for projectively flat metrics. The Calabi complexes of $(\mathcal{B}, \boldsymbol{g})$ and $(\mathcal{B}, \zeta^*\widetilde{\boldsymbol{g}})$ are not the same. In particular, the density $\boldsymbol{\mu}_{\alpha,g}$ is parallel for ∇ and not for $\nabla^{\zeta^*\widetilde{\boldsymbol{g}}}$. Alternatively, one can consider the Eastwood complex of $(\mathcal{B}, \zeta^*\widetilde{\boldsymbol{g}})$ as the local expression of the Eastwood complex of $(\widetilde{\mathcal{B}}, \widetilde{\boldsymbol{g}})$ in the local coordinate system introduced by ζ. In summary, we observed that the linear elastostatics resolution (301) is equivalent to a BGG resolution on $\mathcal{S}^3$ in a proper local coordinate system. Similarly, the 2D kinematic elastostatics resolution (305) is equivalent to a BGG resolution corresponding to the irreducible representation $\Lambda^2\mathbb{R}^3 = \overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}$ of $\mathcal{G} = SL(\mathbb{R}^3)$.

3.4.6 The twisted de Rham Complex of Linear Elastostatics

Let $\mathcal{B} \subset \mathbb{R}^3$ be an open subset equipped with the standard metric of $\mathbb{R}^3$. In the previous section, we showed that the elastostatics complex in $\mathbb{R}^3$ is equivalent to the expression of the Eastwood complex of $\mathcal{S}^3$ in the central projection coordinate system. The BGG resolution on $\mathcal{S}^3$ corresponding to the representation $\Lambda^2\mathbb{R}^4$ can be constructed from the $\Lambda^2\mathbb{R}^4$-valued de Rham complex [42]. Consequently, by expressing this construction in the central projection coordinate system, one obtains a similar construction for the Eastwood complex and therefore, the elastostatics complex in $\mathbb{R}^3$. The upshot is the following diagram first derived by Eastwood [41].

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & & \\
 & & 0 \arrow[d] & 0 \arrow[d] & \Gamma(\otimes^2 T\mathcal{B}) \arrow[d, equal] & 0 \arrow[d] & \\
 & & \mathfrak{X}(\mathcal{B}) \arrow[d, "\mu_0"] & \Gamma(T^*\mathcal{B} \otimes T\mathcal{B}) \arrow[d, "\mu_1"] & \Gamma(S^2 T\mathcal{B}) \oplus \Omega^1(\mathcal{B}) \arrow[d, "\mu_2"] & \mathfrak{X}(\mathcal{B}) \arrow[d, "\mu_3"] & \\
0 \arrow[r] & \Lambda^2\mathbb{R}^4 \arrow[r, "\alpha"] & \Omega^0(\mathcal{B}; \Lambda^2\mathbb{R}^4) \arrow[r, "d_0"] \arrow[d, "\eta_0"] & \Omega^1(\mathcal{B}; \Lambda^2\mathbb{R}^4) \arrow[r, "d_1"] \arrow[d, "\eta_1"] & \Omega^2(\mathcal{B}; \Lambda^2\mathbb{R}^4) \arrow[r, "d_2"] \arrow[d, "\eta_2"] & \Omega^3(\mathcal{B}; \Lambda^2\mathbb{R}^4) \arrow[r] \arrow[d, "\eta_3"] & 0 \\
 & & \Omega^1(\mathcal{B}) \arrow[d] & \Gamma(\otimes^2 T^*\mathcal{B}) \arrow[d, equal] & \Gamma(T^*\mathcal{B} \otimes T\mathcal{B}) \arrow[d] & \Omega^1(\mathcal{B}) \arrow[d] & \\
 & & 0 & \mathfrak{X}(\mathcal{B}) \oplus \Gamma(S^2 T^*\mathcal{B}) \arrow[d] & 0 & 0 & \\
 & & & 0 & & &
\end{tikzcd}
\tag{365}
$$

The augmentation mapping $\alpha : \Lambda^2\mathbb{R}^4 \to \Omega^0(\mathcal{B}; \Lambda^2\mathbb{R}^4)$ sends a vector $\mathbf{v} \in \Lambda^2\mathbb{R}^4$ to the constant function $\boldsymbol{\beta}(X) = \mathbf{v}$, $\forall X \in \mathcal{B}$. The $\Lambda^2\mathbb{R}^4$-valued exterior derivates $d_i : \Omega^i(\mathcal{B}; \Lambda^2\mathbb{R}^4) \to \Omega^{i+1}(\mathcal{B}; \Lambda^2\mathbb{R}^4)$ are given by $d_i(\boldsymbol{\alpha} \otimes \mathbf{v}) = (d_i\boldsymbol{\alpha}) \otimes \mathbf{v}$, $\boldsymbol{\alpha} \in \Omega^i(\mathcal{B})$. The components of the homomorphisms μ_k and η_k in the Cartesian coordinate $\{X^i\}$ with the standard orthonormal basis $\{\mathbf{E}_i\}$ is as follows. The 6-dimensional space $\Lambda^2\mathbb{R}^4$ has a basis $\{\mathbf{v}_l\}$ with

$$
\mathbf{v}_1 = \mathbf{E}_1 \wedge \mathbf{E}_2,\ \mathbf{v}_2 = \mathbf{E}_1 \wedge \mathbf{E}_3,\ \mathbf{v}_3 = \mathbf{E}_1 \wedge \mathbf{E}_4,\ \mathbf{v}_4 = \mathbf{E}_2 \wedge \mathbf{E}_3,\ \mathbf{v}_5 = \mathbf{E}_2 \wedge \mathbf{E}_4,\ \mathbf{v}_6 = \mathbf{E}_3 \wedge \mathbf{E}_4. \tag{366}
$$

Suppose δ_{ij} is the Kronecker delta and ϵ_{ijk} is the alternating symbol with $\epsilon_{123} = 1$. Let $\boldsymbol{\tau}^1 := dX^1 \wedge dX^2$, $\boldsymbol{\tau}^2 := dX^1 \wedge dX^3$, $\boldsymbol{\tau}^3 := dX^2 \wedge dX^3$, and $\boldsymbol{w} := dX^1 \wedge dX^2 \wedge dX^3$. We have the isomorphisms $\Omega^0(\mathcal{B}; \Lambda^2\mathbb{R}^4) \approx \Omega^1(\mathcal{B}) \oplus \mathfrak{X}(\mathcal{B})$, $\Omega^1(\mathcal{B}; \Lambda^2\mathbb{R}^4) \approx \Gamma(\otimes^2 T^*\mathcal{B}) \oplus \Gamma(T^*\mathcal{B} \otimes T\mathcal{B})$, $\Omega^2(\mathcal{B}; \Lambda^2\mathbb{R}^4) \approx \Gamma(T^*\mathcal{B} \otimes T\mathcal{B}) \oplus \Gamma(\otimes^2 T\mathcal{B})$, and $\Omega^3(\mathcal{B}; \Lambda^2\mathbb{R}^4) \approx \Omega^1(\mathcal{B}) \oplus \mathfrak{X}(\mathcal{B})$. Using these identifications, the expression of the isomorphisms μ_i and η_i can be written as

$$
\mu_0 : \mathfrak{X}(\mathcal{B}) \to \Omega^0(\mathcal{B}; \Lambda^2\mathbb{R}^4),\ Y^q \frac{\partial}{\partial X^q} \mapsto \left(\epsilon_{pqr} X^q Y^r \, dX^p, Y^q \frac{\partial}{\partial X^q} \right), \tag{367}
$$

$$\eta_0 : \Omega^0(\mathcal{B};\Lambda^2\mathbb{R}^4) \to \Omega^1(\mathcal{B}),\ \left(\theta_p\, dX^p, Y^q\frac{\partial}{\partial X^q}\right) \mapsto (\theta_p - \epsilon_{pqr}X^qY^r)\, dX^p, \tag{368}$$

$$\mu_1 : \Gamma(T^*\mathcal{B}\otimes T\mathcal{B}) \to \Omega^1(\mathcal{B};\Lambda^2\mathbb{R}^4),$$

$$\beta_i{}^q\, dX^i \otimes \frac{\partial}{\partial X^q} \mapsto \left(\epsilon_{pqr}X^q\beta_i{}^r\, dX^i\otimes dX^p, \beta_i{}^q dX^i\otimes\frac{\partial}{\partial X^q}\right), \tag{369}$$

$$\eta_1 : \Omega^1(\mathcal{B};\Lambda^2\mathbb{R}^4) \to \Gamma(\otimes^2 T^*\mathcal{B}),$$

$$\left(\nu_{ip} dX^i\otimes dX^p, \beta_i{}^q dX^i\otimes\frac{\partial}{\partial X^q}\right) \mapsto (\nu_{ip} - \epsilon_{pqr}X^q\beta_i{}^r)\, dX^i\otimes dX^p, \tag{370}$$

$$\mu_2 : \Gamma(\otimes^2 T\mathcal{B}) \to \Omega^2(\mathcal{B};\Lambda^2\mathbb{R}^4),$$

$$\omega^{iq}\frac{\partial}{\partial X^i}\otimes\frac{\partial}{\partial X^q} \mapsto \left(\epsilon_{kqr}X^q\omega^{jr}\delta_{ji}\delta^{kp} dX^i\otimes\frac{\partial}{\partial X^p}, \omega^{iq}\frac{\partial}{\partial X^i}\otimes\frac{\partial}{\partial X^q}\right), \tag{371}$$

$$\eta_2 : \Omega^2(\mathcal{B};\Lambda^2\mathbb{R}^4) \to \Gamma(T^*\mathcal{B}\otimes T\mathcal{B}),$$

$$\left(\xi_i{}^p dX^i\otimes\frac{\partial}{\partial X^p}, \omega^{iq}\frac{\partial}{\partial X^i}\otimes\frac{\partial}{\partial X^q}\right) \mapsto \left(\xi_i{}^p - \epsilon_{kqr}X^q\omega^{jr}\delta_{ji}\delta^{kp}\right) dX^i\otimes\frac{\partial}{\partial X^p}, \tag{372}$$

$$\mu_3 : \mathfrak{X}(\mathcal{B}) \to \Omega^3(\mathcal{B};\Lambda^2\mathbb{R}^4),\ Y^q\frac{\partial}{\partial X^q} \mapsto \left(\epsilon_{pqr}X^qY^r\, dX^p, Y^q\frac{\partial}{\partial X^q}\right), \tag{373}$$

$$\eta_3 : \Omega^3(\mathcal{B};\Lambda^2\mathbb{R}^4) \to \Omega^1(\mathcal{B}),\ \left(\theta_p\, dX^p, Y^q\frac{\partial}{\partial X^q}\right) \mapsto (\theta_p - \epsilon_{pqr}X^qY^r)\, dX^p, \tag{374}$$

where the repetition of an index implies the summation on that index and the values of all indices are 1,2,3. It is straightforward to check that the columns of (365) are exact complexes. Let $\boldsymbol{\beta} \in \Gamma(\otimes^2 T^*\mathcal{B})$. By defining $\beta_{ij} = A_{ij} + S_{ij}$, where $A_{ij} := (\beta_{ij} - \beta_{ji})/2$, and $S_{ij} := (\beta_{ij} + \beta_{ji})/2$, we obtain the decomposition $\Gamma(\otimes^2 T^*\mathcal{B}) \approx \Gamma(S^2T^*\mathcal{B}) \oplus \Gamma(\Lambda^2T^*\mathcal{B})$. Since $\Gamma(\Lambda^2T^*\mathcal{B}) \approx \mathfrak{X}(\mathcal{B})$, we conclude that $\Gamma(\otimes^2T^*\mathcal{B}) \approx \Gamma(S^2T^*\mathcal{B}) \oplus \mathfrak{X}(\mathcal{B})$. Similarly, one can write $\Gamma(\otimes^2T\mathcal{B}) \approx \Gamma(S^2T\mathcal{B}) \oplus \Omega^1(\mathcal{B})$. The compositions $\eta_i \circ d_{i-1} \circ \mu_{i-1}$, for $i = 1, 2, 3$, are algebraic, i.e. involve no differentiation. By removing the isomorphisms corresponding to these compositions, one obtains the 3D elastostatics complex (300). This is equivalent to eliminating $\mathfrak{X}(\mathcal{B})$, $\Gamma(T^*\mathcal{B}\otimes T\mathcal{B})$, and $\Omega^1(\mathcal{B})$ from (365). See [41] for more details. Alternatively, if one only removes the composition

$\eta_2 \circ d_1 \circ \mu_1$ from (365), the resulting complex reads

$$0 \longrightarrow \Lambda^2\mathbb{R}^4 \longrightarrow \Omega^0(\mathcal{B};\Lambda^2\mathbb{R}^4) \longrightarrow \Gamma(\otimes^2 T^*\mathcal{B}) \longrightarrow \Gamma(\otimes^2 T\mathcal{B}) \longrightarrow \Omega^3(\mathcal{B};\Lambda^2\mathbb{R}^4) \longrightarrow 0. \tag{375}$$

This complex is the Arnold-Falk-Winther elastostatics complex introduced in (302). More details on the construction of this sequence can be found in [9] and [42]. Note that the Arnold-Falk-Winther elastostatics complex is obtained from the $\Lambda^2\mathbb{R}^4$-valued de Rham complex and the 3D elastostatics complex (300) can be considered as a proper restriction of this complex. However, this complex does not have a direct physical interpretation.

CHAPTER IV

CONCLUDING REMARKS

We presented a discrete geometric structure-preserving numerical scheme for incompressible linearized elasticity. We proved that the governing equations of finite and linearized incompressible elasticity can be obtained using Hamilton's principle and Hodge decomposition theorem without using Lagrange multipliers. We used ideas from algebraic topology, exterior calculus, and discrete exterior calculus to develop a discrete geometric theory for linearized elasticity. We considered the discrete displacement field as our primary unknown and characterized the space of divergence-free discrete displacements as the solution space. Note that instead of heuristically defining the discrete displacement field and its divergence as a discretization of a smooth vector field and smooth divergence operator, we assume the discrete displacement field to be a discrete primal vector field and use the definition of discrete divergence using DEC techniques. Therefore, we preserve the geometric structure of the smooth problem by considering discrete quantities that have the same geometric structure as their smooth counterparts. This guarantees that the method remains free of numerical artifacts as we remain in the correct discrete space unlike the standard finite element and finite difference schemes.

Motivated by the Lagrangian structure of the smooth case, we defined a discrete Lagrangian and used Hamilton's principle in the space of discrete divergence-free displacement fields to obtain the governing equations of the discrete theory. We observed that the discrete gradient of pressure appears naturally in the governing equations. We used the discrete Laplace-Beltrami operator to determine the pressure field, which is assumed to be a dual 0-form. We then considered some numerical examples and observed that our discretization scheme is free of numerical artifacts, e.g. checkerboarding of pressure. Based on the rate of convergence of the results of the numerical examples, our method is comparable with finite element mixed formulations [58, 87]. We observed that by choosing the displacement field to be a primal vector field, pressure is a dual 0-form; this geometrically justifies the known fact that using different function spaces for displacement and pressure is helpful in the incompressible regime. Also note that our method can be used for analyzing multiply-connected bodies as well.

The smooth weak form of the incompressible elasticity is well-posed. However, it is a well-known fact that the discretization of the weak form is well-posed if and only if the discrete spaces for the displacement and pressure fields are compatible [45]. For example, a $\mathbb{P}_0/\mathbb{P}_1$ Lagrange finite element approximation for the displacement/pressure field is not well-posed. This low order approximation is not well-posed even when using the diamond element approach [58]. Although we do not present any proof, our numerical results suggest that the discrete weak form is well-posed for our choices of the discrete solution spaces. This also suggests that choices that are naturally imposed by the geometry of a problem can be nontrivial and hard to see using other approaches.

The structure of linearized elasticity is similar to that of perfect fluids in the sense that both need a fixed mesh. However, finite elasticity requires the material description of motion. This means that one needs the time evolution of the initial simplical complex of the reference configuration. However, this evolving mesh would not remain a simplical complex, in general, and hence the extension of this work to the case of finite elasticity is not straightforward. Also the convergence issues are not considered in this work. Applications to fluid mechanics and finite elasticity and studying convergence issues are open problems that will be studied later.

We also derived various complexes for nonlinear elastostatics using some geometric methods and expressed nonlinear elastostatics in terms of differential forms. By introducing stress functions for the Cauchy and the second Piola-Kirchhoff stress tensors, we showed that 2D and 3D nonlinear elastostatics admit separate kinematic and kinetic complexes. On the other hand, we showed that stress functions corresponding to the first Piola-Kirchhoff stress tensor allow us to write a complex for 3D nonlinear elastostatics that similar to the complex of 3D linear elastostatics contains both the kinematics an kinetics of motion. We studied linear and nonlinear compatibility equations for curved ambient spaces and motions of surfaces in $\mathbb{R}^3$. We also studied the relationship between the linear elastostatics complex and the de Rham complex.

Our derivations have important consequences as follows. There are standard methods for calculating cohomology groups of the de Rham complex. The relations between the linear and nonlinear elastostatics complexes with the de Rham complex enable us to calculate cohomology groups of the elastostatics complexes. In particular, we can study the compatibility equations and stress functions on non-contractible bodies. On the other hand, the nonlinear elastostatics complex may allow us to introduce a stable numerical scheme for nonlinear elastostatics. To this end, first we need to study a weak formulation of nonlinear elastostatics in terms of stress functions. Developing an exact theory for moving shells is another extension of this work. By exact theory we mean a 2D theory for shells which is not obtained by approximating the 3D theory of elasticity. Our discussions in this work suggest that for developing such an exact theory, in addition to the Green deformation strain tensor, we also need to consider other strain tensors. These extensions will be the subject of our future research.

Bibliography

[1] Abraham, R., Marsden, J. E., and Ratiu, T., *Manifolds, Tensor Analysis, and Applications*. Springer-Verlag, New York, 1988.

[2] Alfeld, P., "Upper and lower bounds on the dimension of multivariate spline spaces," *SIAM J. Numer. Anal.*, vol. 33, pp. 571–588, 1996.

[3] Álvarez Paiva, J. C., "Hilbert's fourth problem in two dimensions," in *Mass Selecta: teaching and learning advanced undergraduate mathematics*, pp. 165–183, Amer. Math. Soc., Providence, RI, 2003.

[4] Ambrose, W., "Parallel translation of Riemannian curvature," *Ann. of Math.*, vol. 64, pp. 337–363, 1956.

[5] Angoshtari, A. and Yavari, A., "A geometric structure-preserving discretization scheme for incompressible linearized elasticity," *Comput. Methods Appl. Mech. Engrg.*, vol. 259, pp. 130–153, 2013.

[6] Angoshtari, A. and Yavari, A., "Linear and nonlinear elastostatics complexes," *submitted*, 2013.

[7] Areias, P. and Matous, K., "Stabilized four-node tetrahedron with nonlocal pressure for modeling hyperelastic materials," *J. Numer. Methods Eng.*, vol. 76, pp. 1185–1201, 2008.

[8] Arnold, D. N., Awanou, G., and Winther, R., "Finite elements for symmetric tensors in three dimensions," *Math. Comput.*, vol. 77, pp. 1229–1251, 2008.

[9] Arnold, D. N., Falk, R. S., and Winther, R., "Finite element exterior calculus, homological techniques, and applications," *Acta Numerica*, vol. 15, pp. 1–155, 2006.

[10] Arnold, D. N., Falk, R. S., and Winther, R., "Mixed finite element methods for linear elasticity with weakly imposed symmetry," *Math. Comput.*, vol. 76, pp. 1699–1723, 2007.

[11] Arnold, D. N., Falk, R. S., and Winther, R., "Finite element exterior calculus: from Hodge theory to numerical stability," *Bul. Am. Math. Soc.*, vol. 47, pp. 281–354, 2010.

[12] Arnold, D. N. and Winther, R., "Mixed finite elements for elasticity," *Numer. Math.*, vol. 92, pp. 401–419, 2002.

[13] Auricchio, F., da Veiga Beirao, L., Buffam, A., Lovadina, C., and Reali, A., "A fully locking-free isogeometric approach for plane linear elasticity problems: a stream function formulation," *Comput. Methods Appl. Mech. Engrg.*, vol. 197, pp. 160–172, 2007.

[14] Auricchio, F., da Veiga Beirao, L., Lovadina, C., and Reali, A., "The importance of the exact satisfaction of the incompressibility constraint in nonlinear elasticity: mixed FEMs versus NURBS-based approximations," *Comput. Methods Appl. Mech. Engrg.*, vol. 199, pp. 314–323, 2010.

[15] Babuška, I. and Suri, M., "On locking and robustness in the finite element method," *SIAM J. Numer. Anal.*, vol. 29, pp. 1261–1293, 1992.

[16] Bailey, T. N., Eastwood, M. G., and Gover, A. R., "Thomas's structure bundle for conformal, projective and related structures," *Rocky Mtn. J. Math.*, vol. 24, pp. 1191–1217, 1994.

[17] Baston, R. J. and Eastwood, M. G., *The Penrose Transformation: its Interaction with Representation Theory.* Oxford University Press, 1989.

[18] Belytschko, T., Lu, Y. Y., and Gu, L., "Element-free Galerkin methods," *Int. J. Numer. Methods Eng.*, vol. 37, pp. 229–256, 1994.

[19] Bernstein, I. N., Gelfand, I. M., and Gelfand, S. I., "Differential operators on the base affine space and a study of $\mathfrak{g}$-modules," in *Lie Groups and their Representations* (Gelfand, I. M., ed.), pp. 21–64, Adam Hilger, New York, 1975.

[20] Bijelonja, I., Demirdzic, I., and Muzaferija, S., "A finite volume method for incompressible linear elasticity," *Comput. Methods Appl. Mech. Engrg.*, vol. 195, pp. 6378–6390, 2006.

[21] Bonet, J. and Burton, A. J., "A simple average nodal pressure tetrahedral element for incompressible and nearly incompressible dynamic explicit applications," *Comm. Num. Meth. Eng.*, vol. 14, pp. 437–449, 1998.

[22] Bott, R. and Tu, L. W., *Differential Forms in Algebraic Topology.* Springer-Verlag, New York, 2010.

[23] Bredon, G. E., *Sheaf Theory.* Springer-Verlog, New York, 1997.

[24] Bridgeman, L. J. and Wihler, T. P., "Stability and a posteriori error analysis of discontinuous Galerkin methods for linearized elasticity," *Comput. Meth. Appl. Mech. Engrg.*, vol. 200, pp. 1543–1557, 2011.

[25] Brink, U. and Stephan, E. P., "Adaptive coupling of boundary elements and mixed finite elements for incompressible elasticity," *Num. Meth. Part. Dif. Eq.*, vol. 17, pp. 79–92, 2001.

[26] Calabi, E., "On compact Riemannian manifolds with constant curvature I," in *Differential Geometry*, pp. 155–180, Proc. Symp. Pure Math. vol. III, Amer. Math. Soc., 1961.

[27] Čap, A. and Slovák, J., *Parabolic Geometries I: Background and General Theory.* American Mathematical Society, Providence, RI, 2009.

[28] Čap, A., Slovák, J., and Souček, V., "Bernstein-Gelfand-Gelfand sequences," *Ann. of Math.*, vol. 154, pp. 97–113, 2001.

[29] Carmo, M., *Differential Geometry of Curves and Surfaces.* Prentice-Hall, New Jersey, 1976.

[30] Carmo, M., *Riemannian Geometry.* Birkhäuser, Boston, 1992.

[31] Cartan, E., *Leçons sur la Geométrie des Espaces de Riemann.* Gauthier-Villars, Paris, 1951.

[32] Cervera, M., Chiumenti, M., Valverde, Q., and Agelet de Saracibar, C., "Mixed linear/linear simplicial elements for incompressible elasticity and plasticity," *Comput. Methods Appl. Mech. Engrg.*, vol. 192, pp. 5249–5263, 2003.

[33] Chao, I., Pinkall, U., Sanan, P., and Schröder, P., "A simple geometric model for elastic deformations," *ACM Trans. on Graph.*, vol. 29, pp. 38:1–38:6, 2010.

[34] Cheeger, J. and Ebin, G., *Comparison Theorems in Riemannian Geometry.* American Mathematical Society, Providence, RI, 2008.

[35] Chen, S., Ren, G., and Mao, S., "Second-order locking-free nonconforming elements for planar linear elasticity," *J. Comp. Appl. Math.*, vol. 233, pp. 2534–2548, 2010.

[36] Ciarlet, P. G. and Ciarlet, P., "Direct computation of stresses in planar linearized elasticity," *Math. Mod. Meth. App. Sci.*, vol. 19, pp. 1043–1064, 2009.

[37] Ciarlet, P. G., Gratie, L., and Mardare, C., "A new approach to the fundamental theorem of surface theory," *Arch. Rat. Mech. Anal.*, vol. 188, pp. 457–473, 2008.

[38] Dolbow, J. and Belytschko, T., "Volumetric locking in the element free Galerkin method," *Int. J. Numer. Methods Eng.*, vol. 46, pp. 925–942, 1999.

[39] Eastwood, M. G., "The generalized Penrose-Ward transform," *Math. Proc. Cambridge Philos. Soc.*, vol. 97, pp. 165–187, 1985.

[40] Eastwood, M. G., "Variations on the de Rham complex," *Notices Amer. Math.Soc.*, vol. 46, pp. 1368–1376, 1999.

[41] Eastwood, M. G., "A complex from linear elasticity," pp. 23–29, Rend. Circ. Mat. Palermo, Serie II, Suppl. 63, 2000.

[42] Eastwood, M. G., "Notes on projective differential geometry," in *Symmetries and overdetermined systems of partial differential equations*, pp. 41–60, IMA Vol. Math. Appl., 144, Springer, New York, 2008.

[43] Eastwood, M. G. and Matveev, V., "Metric connections in projective differential geometry," in *Symmetries and overdetermined systems of partial differential equations*, pp. 339–350, IMA Vol. Math. Appl., 144, Springer, New York, 2008.

[44] Ebin, D. G. and Marsden, J. E., "Groups of diffeomorphisms and the motion of an incompressible fluid," *Ann. of Math.*, vol. 92, pp. 102–163, 1970.

[45] Ern, A. and Guermond, J., *Theory and Practice of Finite Elements.* Springer-Verlag, New York, 2004.

[46] Falk, R. S., "Nonconforming finite element methods for the equations of linear elasticity," *Math. Comp.*, vol. 57, pp. 529–550, 1991.

[47] Fulton, W. and Harris, J., *Representation Theory: A First Course.* Springer-Verlog, New York, 1991.

[48] Garrett, P. B., *Abstract Algebra.* Chapman & Hall/CRC, 2008.

[49] Gasqui, J. and Goldschmidt, H., "Déformations infinitésimales des espaces Riemanniens localement symétriques. I," *Adv. in Math.*, vol. 48, pp. 205–285, 1983.

[50] Gasqui, J. and Goldschmidt, H., *Radon Transforms and the Rigidity of the Grassmannians.* Princeton University Press, Princeton, 2004.

[51] Gatica, G. N., Gatica, L. F., and Stephan, E. P., "A dual-mixed finite element method for nonlinear incompressible elasticity with mixed boundary conditions," *Comput. Methods Appl. Mech. Engrg.*, vol. 196, pp. 3348–3369, 2007.

[52] Geymonat, G. and Krasucki, F., "Hodge decomposition for symmetric matrix fields and the elasticity complex in Lipschitz domains," *Commun. Pure Appl. Anal.*, vol. 8, pp. 295–309, 2009.

[53] Gover, A. R., "Invariants on projective space," *J. Amer. Math. Soc.*, vol. 7, pp. 145–158, 1994.

[54] Gross, P. and Kotiuga, P. R., *Electromagnetic Theory and Computation: A Topological Approach.* Cambridge University Press, Cambridge, 2004.

[55] Gurtin, M. E., "On Helmboltz's theorem and the completeness of the Papkovicb-Neuber stress functions for infinite domains," *Arch. Rat. Mech. Anal.*, vol. 9, pp. 225–233, 1962.

[56] Gurtin, M. E., "A generalization of the Beltrami stress functions in continuum mechanics," *Arch. Rat. Mech. Anal.*, vol. 13, pp. 321–329, 1963.

[57] Hansbo, P. and Larson, M. G., "Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method," *Comput. Meth. Appl. Mech. Engrg.*, vol. 191, pp. 1895–1908, 2002.

[58] Hauret, P., Kuhl, E., and Ortiz, M., "Diamond elements: A finite element/discrete-mechanics approximation scheme with guaranteed optimal convergence in incompressible elasticity," *Int. J. Numer. Meth. Engng*, vol. 72, pp. 253–294, 2007.

[59] Hebda, J. J., "Parallel translation of curvature along geodesics," *Trans. Amer. Math. Soc.*, vol. 299, pp. 559–572, 1987.

[60] Hiemstra, R. R., Huijsmans, R. H. M., and Gerritsma, M. I., "High order gradient, curl and divergence conforming spaces, with an application to compatible NURBS-based Isogeometric Analysis," *E-print arXiv:1209.1793*, 2012.

[61] Hirani, A. N., *Discrete Exterior Calculus.* PhD thesis, California Institute of Technology, 2003.

[62] Hirani, A. N., Kalyanaraman, K., and VanderZee, E. B., "Delaunay Hodge star," *Computer-Aided Design*, vol. 45, pp. 540–544, 2013.

[63] Hirani, A. N., Nakshatrala, K. B., and Chaudhry, J. H., "Numerical method for Darcy flow derived using Discrete Exterior Calculus," *E-print arXiv:0810.3434v3*, 2008.

[64] Houston, P., Schotzau, D., and Wihler, T. P., "An hp-adaptive mixed discontinuous Galerkin FEM for nearly incompressible linear elasticity," *Comput. Methods Appl. Mech. Engrg.*, vol. 195, pp. 3224–3246, 2006.

[65] Hughes, T. J. R., *The Finite Element Method.* Prentice-Hall, Englewood Cliffs, NJ, 2000.

[66] Hughes, T., Cottrell, J., and Bazilevs, Y., "Isogeometric analysis: CAD, finite elements, NURBS, exact geometry, and mesh refinement," *Comput. Methods Appl. Mech. Engrg.*, vol. 194, pp. 4135–4195, 2005.

[67] Humphreys, J. E., *Introduction to Lie Algebras and Representation Theory.* Springer-Verlog, New York, 1972.

[68] Ivey, T. A. and Landsberg, J. M., *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems.* American Mathematical Society, Providence, RI, 2003.

[69] Kanso, E., Arroyo, M., Tong, Y., Yavari, A., Marsden, J. E., and Desbrun, M., "On the geometric character of force in continuum mechanics," *ZAMP*, vol. 58, pp. 843–856, 2007.

[70] Kasper, E. and Taylor, R., "A mixed-enhanced strain method. Part I: Geometrically linear problems," *Comput. Struct.*, vol. 75, pp. 237–250, 2000.

[71] Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry*, vol. 1. Interscience Publishers, New York, 1963.

[72] Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry*, vol. 2. Interscience Publishers, New York, 1969.

[73] Kolář, I., Michor, P. W., and Slovák, J., *Natural Operations in Differential Geometry.* Springer-Verlog, Berlin, 1993.

[74] Kröner, E., "Allgemeine kontinuumstheorie der versetzungen und eigenspannungen," *Archive for Rational Mechanics and Analysis*, vol. 4, pp. 273–334, 1959.

[75] Krysl, P. and Zhu, B., "Locking-free continuum displacement finite elements with nodal integration," *Int. J. Numer. Methods Eng.*, vol. 76, pp. 1020–1043, 2008.

[76] Lang, S., *Algebra.* Springer-Verlog, New York, 2002.

[77] Langhaar, H. L. and Stippes, M., "Three-dimensional stress function," *J. Franklin Inst.*, vol. 258, pp. 371–382, 1954.

[78] Lee, J. M., *Introduction to Topological Manifolds.* Springer-Verlog, New York, 2011.

[79] Liu, R., Wheeler, M. F., and Dawson, C., "A three-dimensional nodal-based implementation of a family of discontinuous Galerkin methods for elasticity problems," *Comput. Struct.*, vol. 87, pp. 141–150, 2009.

[80] Malkus, D. S. and Hughes, T. J. R., "Mixed finite element methods - Reduced and selective integration techniques: A unification of concepts," *Comput. Methods Appl. Mech. Eng.*, vol. 15, pp. 63–81, 1978.

[81] Manin, Y. I., "Strings," *The Mathematical Intelligencer*, vol. 11, pp. 59–65, 1989.

[82] Marsden, J. E. and Hughes, T., *Mathematical Foundations of Elasticity.* Dover Publications, New York, 1994.

[83] Marsden, J. E. and Scheurle, J., "The reduced Euler-Lagrange equations," *Fields Inst. Comm.*, vol. 1, pp. 139–164, 1993.

[84] Munkres, J., *Elementry Differential Topology.* Annals of Mathematics Studies 54, Princeton University Press, 1966.

[85] Munkres, J., *Elements of Algebric Topology.* Springer, New York/Addison-Wesley, Menlo Park CA, 1984.

[86] Ogden, R. W., *Non-linear Elastic Deformations.* Dover Publications, New York, 1997.

[87] Ortiz, A., Puso, M. A., and Sukumar, N., "Maximum-entropy meshfree method for compressible and near-incompressible elasticity," *Comput. Methods Appl. Mech.*, vol. 199, pp. 1859–1871, 2010.

[88] Papapetrou, A., *Lectures on General Relativity.* D. Reidel Publishing Company, Dordrecht, 1974.

[89] Pavlov, D., Mullen, P., Tong, Y., Kanso, E., Marsden, J. E., and Desbrun, M., "Structure-preserving discretization of incompressible fluids," *Physica D*, vol. 240, pp. 443–458, 2011.

[90] Penrose, R. and Rindler, W., *Spinors and Space-time*, vol. I: Two-spinor calculus and relativistic fields. Cambridge University Press, 1984.

[91] Preisig, M., "Locking-free numerical methods for nearly incompressible elasticity and incompressible flow on moving domains," *Comput. Methods Appl. Mech. Engrg.*, vol. 213, pp. 255–265, 2012.

[92] Rieder, G., "Topologische fragen in der theorie der spannungsfunktionen," *Abh. Braunschweig. Wiss. Ges.*, vol. 7, pp. 4–65, 1960.

[93] Rudin, M. E., "An unshellable triangulation of a tetrahedron," *Bull. Amer. Math. Soc.*, vol. 64, pp. 90–91, 1958.

[94] Scott, L. and M.Vogelius, "Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials," *Math. Model. Numer. Anal.*, vol. 19, pp. 111–143, 1985.

[95] Sharpe, R. W., *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program.* Springer-Verlog, New York, 1997.

[96] Shen, Y. and Lew, A. J., "A family of discontinuous Galerkin mixed methods for nearly and perfectly incompressible elasticity," *ESAIM: Math. Model. Num. Anal.*, vol. 46, pp. 1003–1028, 2012.

[97] Simo, J. and Rifai, M., "A class of assumed strain method and the methods of incompatible modes," *Int. J. Num. Methods Eng.*, vol. 29, pp. 1595–1638, 1990.

[98] Slovák, J., *Natural Operators on Conformal Manifolds.* Research Lecture Notes, University of Vienna, 1993.

[99] Srivastava, S. K., *General Relativity And Cosmology.* Prentice-Hall Of India Pvt. Limited, New Delhi, 2008.

[100] Suri, M., "Analytical and computational assessment of locking in the hp finite element method," *Comput. Meth. Appl. Mech. Engrg.*, vol. 133, pp. 347–371, 1996.

[101] Tenenblat, K., "On isometric immersions of Riemannian manifolds," *Boletim da Soc. Bras. de Mat.*, vol. 2, pp. 23–36, 1971.

[102] Timoshenko, S. P. and Goodier, J. N., *Theory of Elasticity.* McGraw-Hill, New York, 1970.

[103] Truesdell, C., "Invariant and complete stress functions for general continua," *Arch. Rat. Mech. Anal.*, vol. 4, pp. 1–27, 1959.

[104] VanderZee, E., Hirani, A. N., Guoy, D., and Ramos, E. A., "Well-centered triangulation," *SIAM J. Sci. Comput.*, vol. 31, pp. 4497–4523, 2010.

[105] Vidal, Y., Villon, P., and Huerta, A., "Locking in the incompressible limit: pseudo-divergence-free element free Galerkin," *Commun. Num. Methods Eng.*, vol. 19, pp. 725–735, 2003.

[106] Wang, L. and Qi, H., "A locking-free scheme of nonconforming rectangular finite element for the planar elasticity," *J. Comp. Math.*, vol. 22, pp. 641–650, 2004.

[107] Wang, Y., *Preconditioning for the mixed formulation of linear plane elasticity.* PhD thesis, Texas A&M University, 2004.

[108] Whiteley, J. P., "Discountinous Galerkin finite element methods for incompressible non-linear elasticity," *Comput. Methods Appl. Mech. Engrg.*, vol. 198, pp. 3464–3478, 2009.

[109] Wihler, T. P., "Locking-free DGFEM for elasticity problems in polygons," *IMA J. Num. Anal.*, vol. 24, pp. 45–75, 2004.

[110] Wolf, J. A., *Spaces of Constant Curvature.* American Mathematical Society, Providence, RI, 2011.

[111] Yavari, A., "On geometric discretization of elasticity," *J. Math. Phys.*, vol. 49:022901, 2008.

[112] Yavari, A., "A geometric theory of growth mechanics," *J. Non. Sci.*, vol. 20, pp. 781–830, 2010.

[113] Yavari, A., "Compatibility equations of nonlinear elasticity for non-simply-connected bodies," *Arch. Rat. Mech. Anal.*, vol. DOI: 10.1007/s00205-013-0621-0, 2013.

[114] Yavari, A. and Goriely, A., "Riemann-Cartan geometry of nonlinear dislocation mechanics," *Arch. Rat. Mech. Anal.*, vol. 205, pp. 59–118, 2012.

[115] Yavari, A. and Goriely, A., "Weyl geometry and the nonlinear mechanics of distributed point defects," *Proceedings of the Royal Society A*, vol. 468, pp. 3902–3922, 2012.

[116] Yavari, A. and Goriely, A., "Riemann-Cartan geometry of nonlinear disclination mechanics," *Math. Mech. Solids*, vol. 18, pp. 91–102, 2013.

[117] Yavari, A., Marsden, J. E., and Ortiz, M., "On the spatial and material covariant balance laws in elasticity," *J. Math. Phys.*, vol. 47, pp. 85–112, 2006.

[118] Yavari, A. and Ozakin, A., "Covariance in linearized elasticity," *J. Appl. Math. Phys. (ZAMP)*, vol. 59, pp. 1081–1110, 2008.