ADE and Affine ADE Bundles over Complex Surfaces with $p_g = 0$

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Abstract of thesis entitled:

ADE and Affine ADE Bundles over Complex Surfaces with $p_g = 0$

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We study ADE and affine ADE bundles over complex surfaces X with $p_g = 0$.

First, we suppose X admits an ADE singularity. The exceptional locus of this singularity in the minimal resolution Y is an ADE curve of corresponding type. Using this ADE curve and bundle extensions, we construct an ADE bundle over Y which can descend to X. Furthermore, we describe their minuscule representation bundles in terms of configuration of (reducible) (-1)-curves.

Second, we assume X is an elliptic surface with a singular fiber of affine ADE type. Similar to above studies, we construct the affine ADE bundle over X which is trivial on each irreducible component of the affine ADE curve.

Third, when X is the blowup of $\mathbb{P}^2$ at $n \leq 9$ points, there is a canonical E_n bundle over it. We give a detailed study of the relationship between the geometry of X and the deformability of this bundle.

摘要

我们研究了 $p_g = 0$ 的复曲面 X 上的 ADE 向量丛和仿射 ADE 向量丛。

首先，我们假设 X 上有一个 ADE 奇异点。 这个奇异点在极小分解 Y 中的例外轨迹是一条相应形式的 ADE 曲线。 利用这条 ADE 曲线和向量丛的扩张，我们构造了 Y 上的一个 ADE 向量丛，而且这个向量丛可以下降到 X 上。 此外，我们利用 Y 上 (-1)-曲线的组合，描述了他们的极小表示向量丛。

其次，我们假设 X 是一个椭圆曲面，而且 X 上有一个仿射 ADE 形式的奇异纤维。 类似于以前，我们构造了 X 上的一个仿射 ADE 向量丛，而且这个向量丛在这条仿射 ADE 曲线上的每一个不可约成分上都是平凡的。

然后，当 X 是 $\mathbb{P}^2$ 上突起 $n \leq 9$ 个点时， X 上有一个典型的 E_n 向量丛。 我们详细的研究了 X 的几何和这个 E_n 向量丛的可变形性之间的关系。

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Contents

I ADE bundles **9**

1 ADE Lie algebra bundles **10**

1.1 ADE singularities 10

1.2 ADE bundles 12

2 Minuscule representations and (-1)-curves **16**

2.1 Standard representations 16

2.2 Minuscule representations 17

2.3 Configurations of (-1)-curves 17

2.4 Minuscule representations from (-1)-curves 19

2.5 Bundles from (-1)-curves 21

2.6 Outline of Proofs for $\mathfrak{g} \neq E_8$ 22

3 A_n case **24**

3.1 A_n standard representation bundle $\mathfrak{L}_\eta^{(A_n,\mathbb{C}^{n+1})}$ 24

3.2 A_n Lie algebra bundle $\zeta_\eta^{A_n}$ 28

3.3 A_n minuscule representation bundle $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ 28

4 D_n case **30**

4.1 D_n standard representation bundle $\mathfrak{L}_\eta^{(D_n,\mathbb{C}^{2n})}$ 30

4.2 D_n Lie algebra bundle $\zeta_\eta^{D_n}$ 34

4.3 D_n spinor representation bundles $\mathfrak{L}_\eta^{(D_n,\mathcal{S}^\pm)}$ 34

5 E_n case **39**

5.1 E_6 case 39

5.2 E_7 case 42

5.3 E_8 case 44

6 Proof of Theorem 1.2.1 **45**

II Affine ADE bundles **50**

7 Affine ADE Lie algebra bundles **51**

7.1 Affine ADE curves 51

7.2 Affine ADE bundles 53

8 Trivialization of $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ over C_i's after deformations **57**

8.1 Trivializations in loop ADE cases 58

8.2 Trivializations in affine ADE cases 60

8.3 Proof (except the loop E_8 case) 60

8.4 Proof for the loop E_8 case 62

III Deformability **65**

9 E_n-bundle over X_n with $n \leq 9$ **66**

9.1 E_n-bundle over X_n with $n \leq 9$ 66

9.2 Deformability of such $\mathcal{E}_0^{\widehat{E}_8}$ 68

9.3 Negative curves in X_9 70

9.4 Proof of Theorems 9.2.1 and 9.2.2 75

A Minuscule configurations **78**

B Affine Lie algebras **80**

Introduction

It has long been known that there are deep connections between Lie theory and the geometry of surfaces. A famous example is an amazing connection between Lie groups of type E_n and del Pezzo surfaces X of degree $9-n$ for $1 \leq n \leq 8$. The root lattice of E_n can be identified with $K_X^{\perp}$, the orthogonal complement to K_X in $Pic(X)$. Furthermore, all the lines in X form a representation of E_n. Using the configuration of these lines, we can construct an E_n Lie algebra bundle over X [22]. If we restrict it to the anti-canonical curve in X, which is an elliptic curve Σ, then we obtain an isomorphism between the moduli space of degree $9-n$ del Pezzo surfaces which contain Σ and the moduli space of E_n-bundles over Σ. This work is motivated from string/F-theory duality, and it has been studied extensively by Friedman-Morgan-Witten [12][13][14], Donagi [3][4][6][8], Leung-Zhang [21][22][23] and others [7][20][24][25].

In the first part of this thesis, we study the relationships between simply-laced, or ADE, Lie theory and rational double points of surfaces. Suppose

$$\pi : Y \to X$$

is the minimal resolution of a compact complex surface X with a rational double point. Then the dual graph of the exceptional divisor $\sum_{i=1}^{n} C_i$ in Y is an ADE Dynkin diagram. From this we have an ADE root system $\Phi := \{\alpha = \sum a_i[C_i] | \alpha^2 = -2\}$ and we can construct an ADE Lie algebra bundle over Y:

$$\mathcal{E}_0^{\mathfrak{g}} := O_Y^{\oplus n} \oplus \bigoplus_{\alpha \in \Phi} O_Y(\alpha)$$

Even though this bundle can not descend to X, we show that it can be deformed to one which can descend to X provided that $p_g(X) = 0$.

Theorem 0.0.1. (*Proposition 1.2.1, 1.2.2, Theorem 1.2.1 and Lemma 1.2.2*) *Assume Y is the minimal resolution of a surface X with a rational double point p of type $\mathfrak{g}$ and $C = \Sigma_{i=1}^n C_i$ is the exceptional divisor. If $p_g(X) = 0$, then*

(i) given any $(\varphi_{C_i})_{i=1}^n \in \Omega^{0,1}(Y, \bigoplus_{i=1}^n O(C_i))$ with $\overline{\partial}\varphi_{C_i} = 0$ for every i, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$ such that $\overline{\partial}_\varphi := \overline{\partial} + ad(\varphi)$ is a holomorphic structure on $\mathcal{E}_0^{\mathfrak{g}}$. We denote this new holomorphic bundle as $\mathcal{E}_\varphi^{\mathfrak{g}}$.

(ii) Such a $\overline{\partial}_\varphi$ is compatible with the Lie algebra structure.

(iii) $\mathcal{E}_\varphi^{\mathfrak{g}}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(C_i, O_{C_i}(C_i)) \cong \mathbb{C}$.

(iv) There exists $[\varphi_{C_i}] \in H^1(Y, O(C_i))$ such that $[\varphi_{C_i}|_{C_i}] \neq 0$.

(v) Such a $\mathcal{E}_\varphi^{\mathfrak{g}}$ can descend to X if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$ for every i.

Remark 0.0.1. *Infinitesimal deformations of holomorphic bundle structures on $\mathcal{E}_0^{\mathfrak{g}}$ are parametrized by $H^1(Y, End(\mathcal{E}_0^{\mathfrak{g}}))$, and those which also preserve the Lie algebra structure are parametrized by $H^1(Y, ad(\mathcal{E}_0^{\mathfrak{g}})) = H^1(Y, \mathcal{E}_0^{\mathfrak{g}})$, since $\mathfrak{g}$ is semi-simple. If $p_g(X) = q(X) = 0$, e.g. rational surface, then for any $\alpha \in \Phi^-$, $H^1(Y, O(\alpha)) = 0$. Hence $H^1(Y, \mathcal{E}_0^{\mathfrak{g}}) = H^1(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$.*

This generalizes the work of Friedman-Morgan [12], in which they considered E_n bundles over generalized del Pezzo surfaces. In this thesis, we will also describe the minuscule representation bundles of these Lie algebra bundles in terms of (-1)-curves in Y.

Here is an outline of our results in Part I. We first study (-1)-curves in Y which are (possibly reducible) rational curves with self intersection -1. If there exists a (-1)-curve C_0 in X passing through p with minuscule multiplicity C_k (Definition 2.3.3), then (-1)-curves l's in Y with $\pi(l) = C_0$ form the minuscule representation[1] V of $\mathfrak{g}$ corresponding to C_k (Proposition 2.4.1). When V is the

[1]Here V is the lowest weight representation with lowest weight dual to $-C_k$, i.e. V is dual to the highest weight representation with highest weight dual to C_k.

standard representation of $\mathfrak{g}$, the configuration of these (-1)-curves determines a symmetric tensor f on V such that $\mathfrak{g}$ is the space of infinitesimal symmetries of (V, f). We consider the bundle

$$\mathfrak{L}_0^{(\mathfrak{g},V)} := \bigoplus_{\substack{l:(-1)-curve \\ \pi(l)=C_0}} O_Y(l)$$

over Y constructed from these (-1)-curves l's. This bundle can not descend to X as it is not trivial over each C_i.[2]

Theorem 0.0.2. (*Theorem 2.5.1 and 2.5.2*) *For the bundle $\mathfrak{L}_0^{(\mathfrak{g},V)}$ with the corresponding minuscule representation $\rho : \mathfrak{g} \longrightarrow End(V)$,*

(*i*) *there exists $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$ such that $\overline{\partial}_\varphi := \overline{\partial}_0 + \rho(\varphi)$ is a holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)}$. We denote this new holomorphic bundle as $\mathfrak{L}_\varphi^{(\mathfrak{g},V)}$.*

(*ii*) *$\mathfrak{L}_\varphi^{(\mathfrak{g},V)}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i))$.*

(*iii*) *When V is the standard representation of $\mathfrak{g}$, there exists a holomorphic fiberwise symmetric multi-linear form*

$$f : \bigotimes^r \mathfrak{L}_\varphi^{(\mathfrak{g},V)} \longrightarrow O_Y(D)$$

with $r = 0, 2, 3, 4$ when $\mathfrak{g} = A_n, D_n, E_6, E_7$ respectively such that $\mathcal{E}_\varphi^{\mathfrak{g}} \cong aut_0(\mathfrak{L}_\varphi^{(\mathfrak{g},V)}, f)$.

When V is a minuscule representation of $\mathfrak{g}$, there exists a unique holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)} := \bigoplus_l O(l)$ such that the action of $\mathcal{E}_\varphi^{\mathfrak{g}}$ on this bundle is holomorphic and it can descend to X as well.

In the second part of this thesis, we study the relationships between simply-laced affine, or affine ADE, Lie theory and singular fibers of relatively minimal elliptic surfaces. When X is a relatively minimal elliptic surface, Kodaira classified all possible singular fibers (see e.g. [2]) and we call such a curve $C = \cup C_i$

[2]Unless specify otherwise, C_i always refers to an irreducible component of C, i.e. $i \neq 0$.

a Kodaira curve. Its irreducible components C_i's span a sublattice of $Pic\,(X)$ which is isomorphic to the root lattice of an *affine* root system $\Phi_{\widehat{\mathfrak{g}}}$ and therefore we can construct an affine Lie algebra bundle $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ over X.

Theorem 0.0.3. (*Lemma 1.2.1, Proposition 8.1.2 and Theorem 8.2.1*) *Given any complex surface X with $p_g = 0$. If X has a Kodaira curve $C = \cup_{i=0}^r C_i$ of type $\widehat{\mathfrak{g}}$, then*

(i) given any $(\varphi_{C_i})_{i=0}^r \in \Omega^{0,1}(X, \bigoplus_{i=0}^r O(C_i))$ with $\overline{\partial}\varphi_{C_i} = 0$ for every i, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha\in\Phi_{\widehat{\mathfrak{g}}}^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha\in\Phi_{\widehat{\mathfrak{g}}}^+} O(\alpha))$ such that $\overline{\partial}_\varphi := \overline{\partial} + ad(\varphi)$ is a holomorphic structure on $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$. We denote the new bundle as $\mathcal{E}_\varphi^{\widehat{\mathfrak{g}}}$.

(ii) $\overline{\partial}_\varphi$ is compatible with the Lie algebra structure on $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$.

(iii) $\mathcal{E}_\varphi^{\widehat{\mathfrak{g}}}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(C_i, O_{C_i}(C_i)) \cong \mathbb{C}$.

(iv) There exists $[\varphi_{C_i}] \in H^1(X, O(C_i))$ such that $[\varphi_{C_i}|_{C_i}] \neq 0$.

In the third part of this thesis, we explain how the geometry of X_9, a blowup of $\mathbb{P}^2$ at nine points, can be reflected by the deformability of the $\widehat{E}_8$-bundle $\mathcal{E}_0^{\widehat{E}_8}$ over it. Among other things, we obtained the following results.

Theorem 0.0.4. (*Theorem 9.2.1*) *$\mathcal{E}_0^{\widehat{E}_8}$ is totally non-deformable if and only if the nine blowup points in $\mathbb{P}^2$ are in general position.*

Theorem 0.0.5. (*Theorem 9.2.2*) *Suppose $-K_{X_9}$ is nef, then*

(i) X_9 admits an elliptic fibration with a multiple fiber of multiplicity m $(m \geq 1)$ if and only if $\mathcal{E}_0^{\widehat{E}_8}$ is deformable in $(-mK)$-direction but not in $(-m+1)K$-direction.

(ii) X_9 has a (maximal) ADE curve C of type $\mathfrak{g}$ if and only if $\mathcal{E}_0^{\widehat{E}_8}$ is (maximal) $\mathfrak{g}$-deformable.

(iii) X_9 has a (maximal) Kodaira curve C of type $\widehat{\mathfrak{g}}$ if and only if $\mathcal{E}_0^{\widehat{E}_8}$ is (maximal) $\widehat{\mathfrak{g}}$-deformable.

The organization of this thesis is as follows. Section 1 gives the construction of *ADE* Lie algebra bundles over Y directly. In section 2, we review the definition of minuscule representations and construct all minuscule representations using (-1)-curves in Y. Using these, we construct the Lie algebra bundles and minuscule representation bundles which can descend to X in A_n, D_n and E_n $(n \neq 8)$ cases separately in section 3, 4 and 5. The proofs of the main theorems in this thesis are given in section 6.

In part two, section 7 gives the construction of the (affine) ADE Lie algebra bundles directly from (affine) ADE curves. In section 8, we assume $p_g(X) = 0$. We construct deformations of the holomorphic structures on these bundles such that the new bundles are trivial over irreducible components of the curve.

We will consider the E_n-bundle over a blowup of $\mathbb{P}^2$ at $n \leq 9$ points in section 9 and show how the deformability of this bundle can reflect the geometry of the underlying surface.

In the Appendix A, we construct surfaces with ADE curves and a particular (-1)-curve. In the Appendix B, we review the basic construction of affine Lie algebras.

Notations: For a holomorphic bundle $(E_0, \overline{\partial}_0)$ with $E_0 = \oplus_i O(D_i)$, $\overline{\partial}_0$ means the $\overline{\partial}$-operator for the direct sum holomorphic structure. If we construct a new holomorphic structure $\overline{\partial}_\varphi$ on E_0, we denote the resulting bundle as E_φ.

Part I

ADE bundles

Chapter 1

ADE Lie algebra bundles

1.1 ADE singularities

A rational double point p in a surface X can be described locally as a quotient singularity $\mathbb{C}^2/\Gamma$ with Γ a finite subgroup of $SL(2,\mathbb{C})$. It is also called a Kleinian singularity or ADE singularity [2].

Klein [19] determined the structure of the quotient space $\mathbb{C}^2/\Gamma$. For each subgroup Γ, the $\mathbb{C}$-algebra $\mathbb{C}[u,v]^\Gamma$ of Γ-invariant polynomials on $\mathbb{C}^2$ is generated by three fundamental generators x, y, z, satisfying a relation $R(x,y,z) = 0$, where R is a polynomial on $\mathbb{C}^3$. We list these equations below:

$$A_n : x^2 + y^2 + z^{n+1} \quad n \geq 1 \tag{1.1}$$

$$D_n : x^2 + y^{n-1} + yz^2 \quad n \geq 4 \tag{1.2}$$

$$E_6 : x^2 + y^3 + z^4 \tag{1.3}$$

$$E_7 : x^2 + y^3 + yz^3 \tag{1.4}$$

$$E_8 : x^2 + y^3 + z^5 \tag{1.5}$$

They correspond to Γ being a cyclic group, a dihedral group and the groups of the tetrahedron, the octahedron, and the icosahedron respectively.

That means the quotient variety $\mathbb{C}^2/\Gamma$ may be viewed as a hypersurface in $\mathbb{C}^3$

given by the equation $R(x,y,z)=0$:

$$\mathbb{C}^2/\Gamma = \{(x,y,z)\in\mathbb{C}^3 | R(x,y,z)=0\}$$

The hypersurface $\mathbb{C}^2/\Gamma$ has an isolated singularity at the origin, the corresponding singularity is called of type A_n, D_n, E_6, E_7 or E_8 respectively. The reason is if we consider the minimal resolution Y of X, then every irreducible component of the exceptional divisor $C=\sum_{i=1}^n C_i$ is a smooth rational curve with normal bundle $O_{\mathbb{P}^1}(-2)$, i.e. a (-2)-curve, and the dual graph of the exceptional divisor is an ADE Dynkin diagram. The corresponding roots in the Dynkin diagrams are labelled as follows:

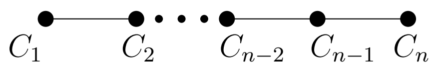


Figure 1. The Dynkin diagram of A_n

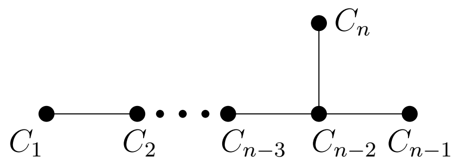


Figure 2. The Dynkin diagram of D_n

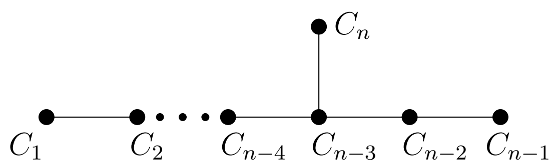


Figure 3. The Dynkin diagram of E_n

There is a natural decomposition

$$H^2(Y,\mathbb{Z}) = H^2(X,\mathbb{Z}) \oplus \Lambda,$$

where $\Lambda = \{\sum a_i[C_i] | a_i \in \mathbb{Z}\}$. The set $\Phi := \{\alpha\in\Lambda | \alpha^2=-2\}$ is a simply-laced (i.e. ADE) root system of a simple Lie algebra $\mathfrak{g}$ and $\Delta = \{[C_i]\}$ is a base of Φ.

For any $\alpha \in \Phi$, there exists a unique divisor $D = \sum a_i C_i$ with $\alpha = [D]$, and we define a line bundle $O(\alpha) := O(D)$ over Y.

1.2 ADE bundles

We define a Lie algebra bundle of type $\mathfrak{g}$ over Y as follows:

$$\mathcal{E}_0^{\mathfrak{g}} := O^{\oplus n} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha).$$

For every open chart U of Y, we take x_α^U to be a nonvanishing holomorphic section of $O_U(\alpha)$ and h_i^U $(i = 1, \cdots, n)$ nonvanishing holomorphic sections of $O_U^{\oplus n}$. Define a Lie algebra structure $[,]$ on $\mathcal{E}_0^{\mathfrak{g}}$ such that $\{x_\alpha^U$'s, h_i^U's$\}$ is the Chevalley basis [17], i.e.

(a) $[h_i^U, h_j^U] = 0$, $1 \leq i, j \leq n$.

(b) $[h_i^U, x_\alpha^U] = \langle \alpha, C_i \rangle x_\alpha^U$, $1 \leq i \leq n$, $\alpha \in \Phi$.

(c) $[x_\alpha^U, x_{-\alpha}^U] = h_\alpha^U$ is a $\mathbb{Z}$-linear combination of h_i^U.

(d) If α, β are independent roots, and $\beta - r\alpha, \cdots, \beta + q\alpha$ is the α-string through β, then $[x_\alpha^U, x_\beta^U] = 0$ if $q = 0$, otherwise $[x_\alpha^U, x_\beta^U] = \pm(r+1)x_{\alpha+\beta}^U$.

Since $\mathfrak{g}$ is simply-laced, all its roots have the same length, we have any α-string through β is of length at most 2. So (d) can be written as $[x_\alpha^U, x_\beta^U] = n_{\alpha,\beta} x_{\alpha+\beta}^U$, where $n_{\alpha,\beta} = \pm 1$ if $\alpha + \beta \in \Phi$, otherwise $n_{\alpha,\beta} = 0$. From the Jacobi identity, we have for any $\alpha, \beta, \gamma \in \Phi$, $n_{\alpha,\beta} n_{\alpha+\beta,\gamma} + n_{\beta,\gamma} n_{\beta+\gamma,\alpha} + n_{\gamma,\alpha} n_{\gamma+\alpha,\beta} = 0$. This Lie algebra structure is compatible with different trivializations of $\mathcal{E}_0^{\mathfrak{g}}$ [22].

By Friedman-Morgan [12], a bundle over Y can descend to X if and only if its restriction to each irreducible component C_i of the exceptional divisor is trivial. But $\mathcal{E}_0^{\mathfrak{g}}|_{C_i}$ is not trivial as $O([C_i])|_{C_i} \cong O_{\mathbb{P}^1}(-2)$. We will construct a new holomorphic structure on $\mathcal{E}_0^{\mathfrak{g}}$, which preserves the Lie algebra structure and therefore the resulting bundle $\mathcal{E}_\varphi^{\mathfrak{g}}$ can descend to X.

As we have fixed a base Δ of Φ, we have a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots.

Definition 1.2.1. *Given any* $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$*, we define* $\overline{\partial}_\varphi : \Omega^{0,0}(Y, \mathcal{E}_0^{\mathfrak{g}}) \longrightarrow \Omega^{0,1}(Y, \mathcal{E}_0^{\mathfrak{g}})$ *by*

$$\overline{\partial}_\varphi := \overline{\partial}_0 + ad(\varphi) := \overline{\partial}_0 + \sum_{\alpha\in\Phi^+} ad(\varphi_\alpha),$$

where $\overline{\partial}_0$ *is the standard holomorphic structure of* $\mathcal{E}_0^{\mathfrak{g}}$*. More explicitly, if we write* $\varphi_\alpha = c_\alpha^U x_\alpha^U$ *locally for some one form* c_α^U*, then* $ad(\varphi_\alpha) = c_\alpha^U ad(x_\alpha^U)$*.*

Proposition 1.2.1. $\overline{\partial}_\varphi$ *is compatible with the Lie algebra structure, i.e.* $\overline{\partial}_\varphi[,] = 0$*.*

Proof. This follows directly from the Jacobi identity. □

For $\overline{\partial}_\varphi$ to define a holomorphic structure, we need

$$0 = \overline{\partial}_\varphi^2 = \sum_{\alpha\in\Phi^+} (\overline{\partial}_0 c_\alpha^U + \sum_{\beta+\gamma=\alpha} (n_{\beta,\gamma} c_\beta^U c_\gamma^U)) ad(x_\alpha^U),$$

that is $\overline{\partial}_0\varphi_\alpha + \sum_{\beta+\gamma=\alpha}(n_{\beta,\gamma}\varphi_\beta\varphi_\gamma) = 0$ for any $\alpha \in \Phi^+$. Explicitly:

$$\begin{cases} \overline{\partial}_0\varphi_{C_i} = 0 & i = 1, 2\cdots, n \\ \overline{\partial}_0\varphi_{C_i+C_j} = n_{C_i,C_j}\varphi_{C_i}\varphi_{C_j} & \text{if } C_i + C_j \in \Phi^+ \\ \vdots & \end{cases}$$

Proposition 1.2.2. *Given any* $(\varphi_{C_i})_{i=1}^n \in \Omega^{0,1}(Y, \bigoplus_{i=1}^n O(C_i))$ *with* $\overline{\partial}_0\varphi_{C_i} = 0$ *for every* i*, it can be extended to* $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$ *such that* $\overline{\partial}_\varphi^2 = 0$*. Namely we have a holomorphic vector bundle* $\mathcal{E}_\varphi^{\mathfrak{g}}$ *over* Y*.*

To prove this proposition, we need the following lemma. For any $\alpha = \sum_{i=1}^n a_i C_i \in \Phi^+$, we define $ht(\alpha) := \sum_{i=1}^n a_i$.

Lemma 1.2.1. *For any* $\alpha \in \Phi^+$*,* $H^2(Y, O(\alpha)) = 0$*.*

Proof. If $ht(\alpha) = 1$, i.e. $\alpha = C_i$, $H^2(Y, O(C_i)) = 0$ follows from the long exact sequence associated to $0 \to O_Y \to O_Y(C_i) \to O_{C_i}(C_i) \to 0$ and $p_g = 0$.

By induction, suppose the lemma is true for every β with $ht(\beta) = m$. Given any α with $ht(\alpha) = m+1$, by Lemma A in §10.2 of [17], there exists some C_i such that $\alpha \cdot C_i = -1$, i.e. $\beta := \alpha - C_i \in \Phi^+$ with $ht(\beta) = m$. Using the long exact sequence associated to $0 \to O_Y(\beta) \to O_Y(\alpha) \to O_{C_i}(\alpha) \to 0$, $O_{C_i}(\alpha) \cong O_{\mathbb{P}^1}(-1)$ and $H^2(Y, O(\beta)) = 0$ by induction, we have $H^2(Y, O(\alpha)) = 0$. □

Proof. (of Proposition 1.2.2) We solve the equations $\overline{\partial}_0 \varphi_\alpha = \sum_{\beta+\gamma=\alpha} n_{\beta,\gamma}\varphi_\beta\varphi_\gamma$ for $\varphi_\alpha \in \Omega^{0,1}(Y, O(\alpha))$ inductively on $ht(\alpha)$.

For $ht(\alpha) = 2$, i.e. $\alpha = C_i + C_j$ with $C_i \cdot C_j = 1$, since $[\varphi_{C_i}\varphi_{C_j}] \in H^2(Y, O(C_i + C_j)) = 0$, we can find $\varphi_{C_i+C_j}$ satisfying $\overline{\partial}_0 \varphi_{C_i+C_j} = \pm\varphi_{C_i}\varphi_{C_j}$.

Suppose that we have solved the equations for all φ_β's with $ht(\beta) \leq m$. For

$$\overline{\partial}_0 \varphi_\alpha = \sum_{\beta+\gamma=\alpha} n_{\beta,\gamma}\varphi_\beta\varphi_\gamma$$

with $ht(\alpha) = m + 1$, we have $ht(\beta), ht(\gamma) \leq m$. Using $\overline{\partial}_0(\sum_{\beta+\gamma=\alpha} n_{\beta,\gamma}\varphi_\beta\varphi_\gamma) = \sum_{\delta+\lambda+\mu=\alpha}(n_{\delta,\lambda}n_{\delta+\lambda,\mu} + n_{\lambda,\mu}n_{\lambda+\mu,\delta} + n_{\mu,\delta}n_{\mu+\delta,\lambda})\varphi_\delta\varphi_\lambda\varphi_\mu = 0$, $[\sum_{\beta+\gamma=\alpha} n_{\beta,\gamma}\varphi_\beta\varphi_\gamma] \in H^2(Y, O(\alpha)) = 0$, we can solve for φ_α. □

Denote

$$\Psi_Y \triangleq \{\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha)) | \overline{\partial}_\varphi^2 = 0\},$$

and

$$\Psi_X \triangleq \{\varphi \in \Psi_Y | [\varphi_{C_i}|_{C_i}] \neq 0 \text{ for } i = 1, 2, \cdots, n\}.$$

Theorem 1.2.1. *$\mathcal{E}_\varphi^{\mathfrak{g}}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i))$.*

Proof. We will discuss the ADE cases separately in Chapter 3, 4, 5 and the proof will be completed in Chapter 6. □

The next lemma says that given any C_i, there always exists $\varphi_{C_i} \in \Omega^{0,1}(Y, O(C_i))$ such that $0 \neq [\varphi_{C_i}|_{C_i}] \in H^1(Y, O_{C_i}(C_i)) \cong \mathbb{C}$.

Lemma 1.2.2. *For any C_i in Y, the restriction homomorphism $H^1(Y, O_Y(C_i)) \to H^1(Y, O_{C_i}(C_i))$ is surjective.*

Proof. The above restriction homomorphism is part of a long exact sequence induced by $0 \to O_Y \to O_Y(C_i) \to O_{C_i}(C_i) \to 0$. The lemma follows directly from $p_g(Y) = 0$. $\square$

Chapter 2

Minuscule representations and (-1)-curves

2.1 Standard representations

For ADE Lie algebras, $A_n = sl(n+1)$ is the space of tracefree endomorphisms of $\mathbb{C}^{n+1}$ and $D_n = o(2n)$ is the space of infinitesimal automorphisms of $\mathbb{C}^{2n}$ which preserve a non-degenerate quadratic form q on $\mathbb{C}^{2n}$. In fact, E_6 (resp. E_7) is the space of infinitesimal automorphisms of $\mathbb{C}^{27}$ (resp. $\mathbb{C}^{56}$) which preserve a particular cubic form c on $\mathbb{C}^{27}$ (resp. quartic form t on $\mathbb{C}^{56}$) [1]. We call the above representation the *standard representation* of $\mathfrak{g}$, i.e.

$\mathfrak{g}$	standard representation
$A_n = sl(n+1)$	$\mathbb{C}^{n+1}$
$D_n = o(2n)$	$\mathbb{C}^{2n}$
E_6	$\mathbb{C}^{27}$
E_7	$\mathbb{C}^{56}$

Note all these standard representations are the fundamental representations corresponding to the left nodes (i.e. C_1) in the corresponding Dynkin diagrams (Figure 1, 2 and 3) and they are minuscule representations.

2.2 Minuscule representations

Definition 2.2.1. *A minuscule (resp. quasi-minuscule) representation of a semisimple Lie algebra is an irreducible representation such that the Weyl group acts transitively on all the weights (resp. non-zero weights).*

Minuscule representations are always fundamental representations and quasi-minuscule representations are either minuscule or adjoint representations.

$\mathfrak{g}$	Miniscule representations
$A_n = sl\,(n+1)$	$\wedge^k \mathbb{C}^{n+1}$ for $k = 1, 2, \cdots, n$
$D_n = o\,(2n)$	$\mathbb{C}^{2n}, \quad \mathcal{S}^+, \quad \mathcal{S}^-$
E_6	$\mathbb{C}^{27}, \quad \overline{\mathbb{C}^{27}}$
E_7	$\mathbb{C}^{56}$

Note E_8 has no minuscule representation.

2.3 Configurations of (-1)-curves

In this section, we describe (-1)-curves in X and Y.

Definition 2.3.1. *A (-1)-curve in a surface Y is a genus zero (possibly reducible) curve l in Y with $l \cdot l = -1$.*

Remark 2.3.1. *The genus zero condition can be replaced by $l \cdot K_Y = -1$ by the genus formula, where K_Y is the canonical divisor of Y.*

Let C_0 be a curve in X passing through p.

Definition 2.3.2. *(i) C_0 is called a (-1)-curve in X if there exists a (-1)-curve l in Y such that $\pi(l) = C_0$, or equivalently the strict transform of C_0 is a (-1)-curve $\widetilde{C_0}$ in Y. (ii) The multiplicity of C_0 at p is defined to be $\sum_{i=1}^{n} a_i[C_i] \in \Lambda$, where $a_i = \widetilde{C_0} \cdot C_i$.*

Recall from Lie theory, any irreducible representation of a simple Lie algebra is determined by its lowest weight. The fundamental representations[1] are those irreducible representations whose lowest weight is dual to the negative of some simple root. If $C_0 \subset X$ has multiplicity C_k at p whose dual weight determines a minuscule representation V, then we use C_0^k to denote $\widetilde{C_0}$. The construction of such X's and C_0's can be found in the Appendix A.

Definition 2.3.3. *(i) We call C_0 has minuscule multiplicity $C_k \in \Lambda$ at p if C_0 has multiplicity C_k and the dual weight of $-C_k$ determines a minuscule representation V. (ii) In this case, we denote $I^{(\mathfrak{g},V)} = \{l : (-1)\text{-curve in } Y | \pi(l) = C_0\}$.*

If there is no ambiguity, we will simply write $I^{(\mathfrak{g},V)}$ as I. Note that $I \subset C_0^k + \Lambda_{\geq 0}$, where $\Lambda_{\geq 0} = \{\sum a_i[C_i] : a_i \geq 0\}$.

Lemma 2.3.1. *In the above situation, the cardinality of I is given by $|I| = \dim V$.*

Proof. By the genus formula and every $C_i \cong \mathbb{P}^1$ being a (-2)-curve, we have $C_i \cdot K_Y = 0$. Since $C_0^k \cdot K_Y = -1$, each (-1)-curve has the form $l = C_0^k + \sum a_i C_i$ with a_i's non-negative integers. From $l \cdot l = -1$, we can determine $\{a_i\}'$s for l to be a (-1)-curve by direct computations. $\square$

Remark 2.3.2. *The intersection product is negative definite on the sublattice of $Pic(X)$ generated by $C_0^k, C_1, \cdots, C_n$ and we use its negative as an inner product.*

Lemma 2.3.2. *In the above situation, for any $l \in I$, $\alpha \in \Phi$, we have $|l \cdot \alpha| \leq 1$.*

Proof. We claim that for any $v \in C_0^k + \Lambda$, we have $v \cdot v \leq -1$. We prove the claim by direct computations. In $(A_n, \wedge^k \mathbb{C}^{n+1})$ case:

$$\begin{aligned}
&(C_0^k + \textstyle\sum a_i C_i)^2 \\
&= -1 + 2a_k - (a_1^2 + (a_1 - a_2)^2 + \cdots + (a_{k-1} - a_k)^2) - ((a_k - a_{k+1})^2 + \cdots + a_n^2) \\
&\leq -1.
\end{aligned}$$

[1] The usual definition for fundamental representations uses highest weight. But in this thesis, we will use lowest weight for simplicity of notations.

The other cases can be proven similarly.

Since l, $l+\alpha$, $l-\alpha \in C_0^k+\Lambda$ by assumptions, we have $l\cdot l = -1 \geq (l+\alpha)\cdot(l+\alpha)$, hence $l \cdot \alpha \leq 1$. Also $l \cdot l = -1 \geq (l-\alpha)\cdot(l-\alpha)$, hence $l \cdot \alpha \geq -1$. □

Lemma 2.3.3. *In the above situation, for any $l \in I$ which is not C_0^k, there exists C_i such that $l \cdot C_i = -1$.*

Proof. From $l = C_0^k + \sum a_i C_i \neq C_0^k$ $(a_i \geq 0)$, we have $a_k \geq 1$. From $l \cdot l = -1$, we have $(\sum a_i C_i)^2 = -2a_k$. If there does not exist such an i with $l \cdot C_i = -1$, then by Lemma 2.3.2, $l \cdot C_i \geq 0$ for every i, $l \cdot (\sum a_i C_i) \geq 0$. But $l \cdot (\sum a_i C_i) = a_k + (\sum a_i C_i)^2 = -a_k \leq -1$ leads to a contradiction. □

Lemma 2.3.4. *In the above situation, for any $l, l' \in I$, $H^2(Y, O(l-l')) = 0$.*

Proof. Firstly, we prove $H^2(Y, O(C_0^k - l)) = 0$ for any $l = C_0^k + \sum a_i C_i \in I$ inductively on $ht(l) := \sum a_i$. If $ht(l) = 0$, i.e. l is C_0^k, the claim follows from $p_g = 0$. Suppose the claim is true for any $l' \in I$ with $ht(l') \leq m-1$. Then for any $l \in I$ with $ht(l) = m$, by Lemma 2.3.3, there exists i such that $l \cdot C_i = -1$. This implies $(l - C_i) \in I$ with $ht(l-C_i) = m-1$ and therefore $H^2(Y, O(C_0^k - (l - C_i))) = 0$ by induction hypothesis. Using the long exact sequence induced from

$$0 \to O_Y(C_0^k - l) \to O_Y(C_0^k - (l - C_i)) \to O_{C_i}(C_0^k - (l - C_i)) \to 0$$

and $O_{C_i}(C_0^k - (l - C_i)) \cong O_{\mathbb{P}^1}(-1)$ or $O_{\mathbb{P}^1}$, we have the claim.

If $H^2(Y, O(l-l')) \neq 0$, then there exists a section $s \in H^0(Y, K_Y(l'-l))$ by Serre duality. Since there exists a nonzero section $t \in H^0(Y, O(l - C_0^k))$, we have $st \in H^0(Y, K_Y(l' - C_0^k)) \cong H^2(Y, O(C_0^k - l')) = 0$, which is a contradiction. □

2.4 Minuscule representations from (-1)-curves

Recall from the ADE root system Φ, we can recover the corresponding Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. As before, we use $\{x_\alpha\text{'s}, h_i\text{'s}\}$ to denote its Chevalley basis. If

C_0 has minuscule multiplicity C_k, we denote

$$V_0 := \mathbb{C}^I = \bigoplus_{l \in I} \mathbb{C}\langle v_l \rangle,$$

where v_l is the base vector of V_0 generated by l. Then we define a bilinear map $[,] : \mathfrak{g} \otimes V_0 \to V_0$ (possibly up to $\pm$ signs) as follows:

$$[x, v_l] = \begin{cases} \langle x, l \rangle v_l & \text{if } x \in \mathfrak{h} \\ \pm v_{l+\alpha} & \text{if } x = x_\alpha,\ l + \alpha\ \in I \\ 0 & \text{if } x = x_\alpha,\ l + \alpha\ \notin I \end{cases}$$

Proposition 2.4.1. *The signs in the above bilinear map $\mathfrak{g} \otimes V_0 \to V_0$ can be chosen so that it defines an action of $\mathfrak{g}$ on V_0. Moreover, V_0 is isomorphic to the minuscule representation V.*

Proof. For the first part, similar to [27], we use Lemma 2.3.2 to show $[[x, y], v_l] = [x, [y, v_l]] - [y, [x, v_l]]$.

For the second part, since $[x_\alpha, v_{C_0^k}] = 0$ for any $\alpha \in \Phi^-$, $v_{C_0^k}$ is the lowest weight vector of V_0 with weight corresponding to $-C_k$. Also we know the fundamental representation V corresponding to $-C_k$ has the same dimension with V_0 by lemma 2.3.1. Hence V_0 is isomorphic to the minuscule representation V. $\square$

Here we show how to determine the signs. Take any $l \in I$, v_l is a weight vector of the above action. For $x = x_\alpha$ and v_l with weight w, we define $[x, v_l] = n_{\alpha,w} v_{l+\alpha}$, where $n_{\alpha,w} = \pm 1$ if $l + \alpha \in I$, otherwise $n_{\alpha,w} = 0$. By $[[x, y], v_l] = [x, [y, v_l]] - [y, [x, v_l]]$, we have $n_{\alpha,\beta} n_{\alpha+\beta,w} - n_{\beta,w} n_{\alpha,\beta+w} + n_{\alpha,w} n_{\beta,\alpha+w} = 0$.

Remark 2.4.1. *Recall for any $l = C_0^k + \sum a_i C_i \in I$, we define $ht(l) := \sum a_i$. Using this, we can define a filtered structure for I : $I = I_0 \supset I_1 \supset \cdots \supset I_m$, where $m = \max_{l \in I} ht(l)$, $I_i = \{l \in I | ht(l) \leq m - i\}$ and $I_i \backslash I_{i+1} = \{l \in I | ht(l) = m - i\}$. This $ht(l)$ also enables us to define a partial order of I. Say $|I| = N$, we denote $l_N := C_0^k$ since it is the only element with $ht = 0$. Similarly, $l_{N-1} := C_0^k + C_k$. Of course, there are some ambiguity of this ordering, if so, we will just make a choice to order these (-1)-curves.*

2.5 Bundles from (-1)-curves

The geometry of (-1)-curves in Y can be used to construct representation bundles of $\mathcal{E}_{\varphi}^{\mathfrak{g}}$ for every minuscule representation of $\mathfrak{g}$. The proofs of theorems in this section will be given in Chapter 6.

When $C_0 \subset X$ has minuscule multiplicity C_k at p with the corresponding minuscule representation V, we define[2]

$$\mathfrak{L}_0^{(\mathfrak{g},V)} := \bigoplus\nolimits_{l \in I^{(\mathfrak{g},V)}} O(l).$$

$\mathfrak{L}_0^{(\mathfrak{g},V)}$ has a natural filtration $F^{\bullet}$: $\mathfrak{L}_0^{(\mathfrak{g},V)} = F^0\mathfrak{L} \supset F^1\mathfrak{L} \supset \cdots \supset F^m\mathfrak{L}$, induced from the flittered structure on I, namely $F^i\mathfrak{L}_0^{(\mathfrak{g},V)} = \bigoplus_{l \in I_i} O(l)$.

$\mathfrak{L}_0^{(\mathfrak{g},V)}$ can not descend to X as $O_{C_k}(C_0^k) \cong O_{\mathbb{P}^1}(1)$ (because $C_k \cdot C_0^k = 1$ by the definition of the minuscule multiplicity). For any C_i and any $l \in I$, we have $O_{C_i}(l) \cong O_{\mathbb{P}^1}(\pm 1)$ or $O_{\mathbb{P}^1}$ by Lemma 2.3.2. For every fixed C_i, if there is a $l \in I$ such that $O_{C_i}(l) \cong O_{\mathbb{P}^1}(1)$, then $(l + C_i)^2 = -1 = (l + C_i) \cdot K_Y$, i.e. $l + C_i \in I$, also $O_{C_i}(l+C_i) \cong O_{\mathbb{P}^1}(-1)$. That means among the direct summands of $\mathfrak{L}_0^{(\mathfrak{g},V)}|_{C_i}$, $O_{\mathbb{P}^1}(1)$ and $O_{\mathbb{P}^1}(-1)$ occur in pairs, and each pair is given by two (-1)-curves in I whose difference is C_i. This gives us a chance to deform $\mathfrak{L}_0^{(\mathfrak{g},V)}$ to get another bundle which can descend to X.

Theorem 2.5.1. *If there exists a (-1)-curve C_0 in X with minuscule multiplicity C_k at p and $\rho : \mathfrak{g} \longrightarrow End(V)$ is the corresponding representation, then*

$$(\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)} := \bigoplus_{l \in I} O(l),\ \overline{\partial}_{\varphi} := \overline{\partial}_0 + \rho(\varphi))$$

with $\varphi \in \Psi_Y$ is a holomorphic bundle over Y which preserves the filtration on $\mathfrak{L}_0^{(\mathfrak{g},V)}$ and it is a holomorphic representation bundle of $\mathcal{E}_{\varphi}^{\mathfrak{g}}$. Moreover, $\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i))$.

[2]When X is a del Pezzo surface, we use lines in X to construct bundles [FM]. So here we use (-1)-curves in X to construct bundles.

For C_k with $k = 1$, the corresponding minuscule representation V is the standard representation of $\mathfrak{g}$. When $\mathfrak{g} = A_n$, it is simply $sl\,(n+1) = aut_0(V)$. When $\mathfrak{g} = D_n$ (resp. E_6 and E_7), there exists a quadratic (resp. cubic and quartic) form f on V such that $\mathfrak{g} = aut(V, f)$. The next theorem tells us that we can globalize this construction over Y to recover the Lie algebra bundle $\mathcal{E}_\varphi^{\mathfrak{g}}$ over Y. But this does not work for $\mathcal{E}_\varphi^{E_8}$ as E_8 has no standard representation.

Theorem 2.5.2. *Under the same assumptions as in theorem 2.5.1 with $k = 1$, there exists a holomorphic fiberwise symmetric multi-linear form*

$$f : \bigotimes^{r} \mathfrak{L}_\varphi^{(\mathfrak{g},V)} \longrightarrow O_Y(D)$$

with $r = 0, 2, 3, 4$ when $\mathfrak{g} = A_n, D_n, E_6, E_7$ respectively such that $\mathcal{E}_\varphi^{\mathfrak{g}} \cong aut_0(\mathfrak{L}_\varphi^{(\mathfrak{g},V)}, f)$.

It is obvious that $\mathcal{E}_\varphi^{\mathfrak{g}}$ does not depend on the existence of the (-1)-curve C_0, for the minuscule representation bundles, we have the following results.

Theorem 2.5.3. *There exists a divisor B in Y and an integer k, such that the bundle $\mathbb{L}_\varphi^{(\mathfrak{g},V)} := S^k \mathfrak{L}_\varphi^{(\mathfrak{g},V)} \otimes O(-B)$ with $\varphi \in \Psi_X$ can descend to X and does not depend on the existence of C_0.*

2.6 Outline of Proofs for $\mathfrak{g} \neq E_8$

When $\mathfrak{g} \neq E_8$, there exists a natural symmetric tensor f on its standard representation V such that $\mathfrak{g} = aut_0(V, f)$. The set $I^{(\mathfrak{g},\ V)}$ of (-1)-curves has cardinality $N = \dim V$. Given $\eta := (\eta_{i,j})_{N\times N}$ with $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for every $l_i \neq l_j \in I^{(\mathfrak{g},\ V)}$, we consider the operator $\overline{\partial}_\eta := \overline{\partial}_0 + \eta$ on $\mathfrak{L}_0^{(\mathfrak{g},\ V)} := \bigoplus_{l\in I^{(\mathfrak{g},\ V)}} O_Y(l)$. We will look for η which satisfy:

(1) (filtration) $\eta_{i,j} = 0$ for $i > j$ for the partial ordering introduced in §3.4.

(2) (holomorphic structure) $(\overline{\partial}_0 + \eta)^2 = 0$.

(3) (Lie algebra structure) $\overline{\partial}_\eta f = 0$.

(4) (descendent) For every C_k, if $l_i - l_j = C_k$, then $0 \neq [\eta_{i,j}|_{C_k}] \in H^1(Y, O_{C_k}(C_k))$.

Remark 2.6.1. *Property* (2) *implies that we can define a new holomorphic structure on* $\mathfrak{L}_0^{(\mathfrak{g},\ V)}$. *Properties* (1) *and* (3) *require that for any* $\eta_{i,j} \neq 0$, $\eta_{i,j} \in \Omega^{0,1}(Y, O(\alpha))$ *for some* $\alpha \in \Phi^+$. *We will show that if* η *satisfies (1), (2) and (3), then (4) is equivalent to* $\mathfrak{L}_\eta^{(\mathfrak{g},\ V)}$ *being trivial on every* C_k*, i.e.* $\mathfrak{L}_\eta^{(\mathfrak{g},\ V)}$ *can descend to* X.

Denote

$$\Xi_Y^{\mathfrak{g}} \triangleq \{\eta = (\eta_{i,j})_{N\times N} | \eta \text{ satisfies } (1), (2) \text{ and } (3)\},$$

and

$$\Xi_X^{\mathfrak{g}} \triangleq \{\eta \in \Xi_Y^{\mathfrak{g}} | \eta \text{ satisfies } (4)\},$$

then each η in $\Xi_Y^{\mathfrak{g}}$ determines a filtered holomorphic bundle $\mathfrak{L}_\eta^{(\mathfrak{g},\ V)}$ over Y together with a holomorphic tensor f on it. It can descend to X if $\eta \in \Xi_X^{\mathfrak{g}}$.

Since $\mathfrak{g} = aut(V, f)$, for any $\eta \in \Xi_Y^{\mathfrak{g}}$, we have a holomorphic Lie algebra bundle $\zeta_\eta^{\mathfrak{g}} := aut(\mathfrak{L}_\eta^{(\mathfrak{g},\ V)}, f)$ over Y of type $\mathfrak{g}$, and $\mathfrak{L}_\eta^{(\mathfrak{g},\ V)}$ is automatically a representation bundle of $\zeta_\eta^{\mathfrak{g}}$. Furthermore, if $\eta \in \Xi_X^{\mathfrak{g}}$, then $\zeta_\eta^{\mathfrak{g}}$ can descend to X.

For a general minuscule representation of $\mathfrak{g}$, given any $\eta \in \Xi_Y^{\mathfrak{g}}$, we show that there exists a unique holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)}$, such that the action of $\zeta_\eta^{\mathfrak{g}}$ on the new holomorphic bundle $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ is holomorphic. Furthermore, if $\eta \in \Xi_X^{\mathfrak{g}}$, then $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ can descend to X.

Chapter 3

A_n case

We recall that $A_n = sl(n+1,\mathbb{C}) = aut_0(\mathbb{C}^{n+1})$ (where aut_0 means tracefree endomorphisms). The standard representation of A_n is $\mathbb{C}^{n+1}$ and minuscule representations of A_n are $\wedge^k\mathbb{C}^{n+1}$, $k = 1, 2, \cdots, n$.

3.1 A_n standard representation bundle $\mathfrak{L}_\eta^{(A_n,\mathbb{C}^{n+1})}$

We consider a surface X with an A_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1, then $I^{(A_n,\mathbb{C}^{n+1})} = \{C_0^1 + \sum_{i=1}^k C_i | \ 0 \leq k \leq n\}$ has cardinality $n+1$. We order these (-1)-curves: $l_k = C_0^1 + \sum_{i=1}^{n+1-k} C_i$ for $1 \leq k \leq n+1$. For any $l_i \neq l_j \in I$, $l_i \cdot l_j = 0$. Fix any C_i, we have

$$l_k \cdot C_i = \begin{cases} 1, & k = n+2-i \\ -1, & k = n+1-i \\ 0, & \text{otherwise.} \end{cases}$$

Define $\mathfrak{L}_0^{(A_n,\mathbb{C}^{n+1})} := \bigoplus_{l \in I} O(l)$ over Y, for simplicity, we write it as $\mathfrak{L}_0^{A_n}$. $\mathfrak{L}_0^{A_n}$ can not descend to X, since for any C_i,

$$\mathfrak{L}_0^{A_n}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus(n-1)} \oplus O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1).$$

Our aim is to find a new holomorphic structure on $\mathfrak{L}_0^{A_n}$ such that the resulting bundle can descend to X. First, we define $\overline{\partial}_\eta : \Omega^{0,0}(Y, \mathfrak{L}_0^{A_n}) \longrightarrow \Omega^{0,1}(Y, \mathfrak{L}_0^{A_n})$ on

$\mathfrak{L}_0^{A_n} = \bigoplus_{k=1}^{n+1} O(l_k)$ as follows:

$$\overline{\partial}_\eta = \begin{pmatrix} \overline{\partial} & \eta_{1,2} & \cdots & \eta_{1,n+1} \\ 0 & \overline{\partial} & \cdots & \eta_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\partial} \end{pmatrix}$$

where $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for any $j > i$. When $j > i$, $l_i - l_j \in \Lambda$ is a positive root because of $l_i \cdot l_j = 0$ and our ordering of l_k's.

The integrability condition $\overline{\partial}_\eta^2 = 0$ is equivalent to, for $i = 1, 2, \cdots, n$,

$$\begin{cases} \overline{\partial}\eta_{i,i+1} = 0, \\ \overline{\partial}\eta_{i,j} = -\sum_{m=i+1}^{j-1} \eta_{i,m} \cdot \eta_{m,j}, \quad j \geq i+2, \end{cases}$$

Note $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j)) = \Omega^{0,1}(Y, O(\alpha))$ for some $\alpha \in \Phi^+$. From

$$\sum_{m=i+1}^{j-1} [\eta_{i,m} \cdot \eta_{m,j}] \in H^2(Y, O(l_i - l_j)) = 0,$$

we can find $\eta_{i,j}$, such that $\overline{\partial}\eta_{i,j} = -\sum_{m=i+1}^{j-1} \eta_{i,m} \cdot \eta_{m,j}$. That is

Proposition 3.1.1. *Given any $\eta_{i,i+1} \in \Omega^{0,1}(Y, O(l_i - l_{i+1}))$ with $\overline{\partial}\eta_{i,i+1} = 0$ for $i = 1, 2, \cdots n$, there exists $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for every $j > i$ such that $\overline{\partial}_\eta$ defines a holomorphic structure on $\mathfrak{L}_0^{A_n}$, i.e. $\overline{\partial}_\eta^2 = 0$.*

We want to prove that there exists $\eta \in \Xi_Y^{A_n}$ such that $\mathfrak{L}_\eta^{A_n}$ can descend to X, i.e. $\mathfrak{L}_\eta^{A_n}|_{C_i}$ is trivial for every C_i. To prove this, we will construct $n+1$ holomorphic sections of $\mathfrak{L}_\eta^{A_n}|_{C_i}$ which are linearly independent everywhere on C_i. The following lemma will be needed for all the ADE cases.

Lemma 3.1.1. *Consider a vector bundle $(\mathfrak{L} := \bigoplus_{i=1}^N O(l_i), \overline{\partial}_{\mathfrak{L}} = \overline{\partial}_0 + (\eta_{i,j})_{N\times N})$ over Y with $\eta_{i,j} = 0$ whenever $i \geq j$. Suppose C is a smooth (-2)-curve in Y with $H^1(C, O_C(l_i)) = 0$ for every $i = 1, 2, \cdots N$, then for any fixed i and any*

$s_i \in H^0(C, O_C(l_i))$, the following equation for s_1, s_2, $\cdots$ s_{i-1} has a solution,

$$\begin{pmatrix} \overline{\partial} & \eta_{1,2}|_C & \eta_{1,3}|_C & \cdots & \cdots & \eta_{1,N}|_C \\ 0 & \overline{\partial} & \eta_{2,3}|_C & \cdots & \cdots & \eta_{2,N}|_C \\ 0 & 0 & \overline{\partial} & \cdots & \cdots & \eta_{3,N}|_C \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \overline{\partial} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.$$

Proof. The above equation is equivalent to:

$$\overline{\partial} s_i = 0, \tag{1}$$

$$\eta_{i-1,i} s_i + \overline{\partial} s_{i-1} = 0, \tag{2}$$

$$\vdots$$

$$\eta_{1,i} s_i + \cdots + \eta_{1,2} s_2 + \overline{\partial} s_1 = 0. \tag{i}$$

Equation (1) is automatic as $s_i \in H^0(C, O_C(l_i))$. For equation (2), since $\overline{\partial}\eta_{i-1,i} = 0$ and $\overline{\partial} s_i = 0$, we have $[\eta_{i-1,i} s_i] \in H^1(C, O_C(l_{i-1})) = 0$, hence we can find s_{i-1} satisfying $\overline{\partial} s_{i-1} = -\eta_{i-1,i} s_i$.

Inductively, suppose we have found $s_i, \cdots, s_{j-1}$ for the first $(i-j)$ equations, then for the $(i-j+1)$-th equation: $\eta_{j,i} s_i + \cdots + \eta_{j,j+1} s_{j+1} + \overline{\partial} s_j = 0$, we have

$$\eta_{j,i} s_i + \cdots + \eta_{j,j+1} s_{j+1} \in \Omega^{0,1}(C, O_C(l_j)).$$

From $\overline{\partial}_{\mathfrak{L}}^2 = 0$, we have

$$\overline{\partial}\eta_{k,m} = -(\eta_{k,k+1} \cdot \eta_{k+1,m} + \eta_{k,k+2} \cdot \eta_{k+2,m} + \cdots + \eta_{k,m-1} \cdot \eta_{m-1,m}).$$

Then

$$\overline{\partial}(s_m) = -(\eta_{m,m+1} s_{m+1} + \cdots + \eta_{m,i} s_i)$$

implies

$$\overline{\partial}(\eta_{j,i} s_i + \cdots + \eta_{j,j+1} s_{j+1}) = 0$$

Therefore $[\eta_{j,i}s_i+\cdots+\eta_{j,j+1}s_{j+1}]\in H^1(C,O_C(l_j))=0$, hence we can find s_j such that $\overline{\partial}s_j=-(\eta_{j,i}s_i+\cdots+\eta_{j,j+1}s_{j+1})$. □

Let us recall a standard result which says that the only non-trivial extension of $O_{\mathbb{P}^1}(1)$ by $O_{\mathbb{P}^1}(-1)$ is the trivial bundle. We will give an explicit construction of this trivialization as we will need a generalization of it later.

Lemma 3.1.2. *For an exact sequence over $\mathbb{P}^1: 0\to O_{\mathbb{P}^1}(-1)\to E\to O_{\mathbb{P}^1}(1)\to 0$, the bundle E is determined by the extension class $[\varphi]\in Ext^1_{\mathbb{P}^1}(O(1),O(-1))\cong\mathbb{C}$ up to a scalar multiple. If $[\varphi]\neq 0$, E is trivial, namely there exists two holomorphic sections for E which are linearly independent at every point in $\mathbb{P}^1$.*

Proof. With respect to the (topological) splitting $E=O_{\mathbb{P}^1}(-1)\oplus O_{\mathbb{P}^1}(1)$, the holomorphic structure on E is given by

$$\overline{\partial}_E=\begin{pmatrix}\overline{\partial} & \varphi\\ 0 & \overline{\partial}\end{pmatrix}$$

with $\varphi\in Ext^1_{\mathbb{P}^1}(O(1),O(-1)))$. Let t_1,t_2 be a base of $H^0(\mathbb{P}^1,O(1))\cong\mathbb{C}^2$. Since $[\varphi t_i]\in H^1(\mathbb{P}^1,O(-1))=0$, we can find $u_1,u_2\in\Omega^0(\mathbb{P}^1,O(-1))$, such that

$$\begin{pmatrix}\overline{\partial} & \varphi\\ 0 & \overline{\partial}\end{pmatrix}\cdot\begin{pmatrix}u_i\\ t_i\end{pmatrix}=0,$$

i.e. $s_1=(u_1,t_1)^t$ and $s_2=(u_2,t_2)^t$ are two holomorphic sections of E. Explicitly, we can take $s_1=(\frac{1}{1+|z|^2},z)^t$, $s_2=(\frac{-\overline{z}}{1+|z|^2},1)^t$ in the coordinate chart $\mathbb{C}\subset\mathbb{P}^1$. It can be checked that s_1 and s_2 are linearly independent over $\mathbb{P}^1$. □

From the above lemma, we have the following result.

Lemma 3.1.3. *Under the same assumption as in Lemma 3.1.1. Suppose $\mathfrak{L}|_C\cong O_{\mathbb{P}^1}^{\oplus m}\oplus(O_{\mathbb{P}^1}(1)\oplus O_{\mathbb{P}^1}(-1))^{\oplus n}$ with each pair of $O_{\mathbb{P}^1}(\pm 1)$ corresponding to two (-1)-curves l_i and l_{i+1} with $l_i-l_{i+1}=C$. Then $\mathfrak{L}|_C$ is trivial if and only if $[\eta_{i,i+1}|_C]\neq 0$ for every $\eta_{i,i+1}\in\Omega^{0,1}(Y,O(C))$.*

Proof. For simplicity, we assume $m = n = 1$ and $O_C(l_1) \cong O_{\mathbb{P}^1}$, $O_C(l_2) \cong O_{\mathbb{P}^1}(-1)$, $O_C(l_3) \cong O_{\mathbb{P}^1}(1)$ with $l_2 - l_3 = C$. If $[\eta_{2,3}|_C] \neq 0$, by Lemma 3.1.1 and Lemma 3.1.2, there exists two holomorphic sections for $\mathfrak{L}|_C$ which are linearly independent at every point in C: $s_1 = (x_1, u_1, t_1)^t$ and $s_2 = (x_2, u_2, t_2)^t$ with u_1, t_1, u_2, t_2 given in the proof of Lemma 3.1.2. By $H^0(Y, O_C(l_1)) \cong H^0(\mathbb{P}^1, O) \cong \mathbb{C}$, there exists one holomorphic section for $\mathfrak{L}|_C$ which is nowhere zero on C: $s_3 = (x_3, 0, 0)^t$. These s_1, s_2, s_3 give a trivialization of $\mathfrak{L}|_C$. If $[\eta_{2,3}|_C] = 0$, then $\mathfrak{L}_N|_C$ is an extension of $O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1)$ by $O_{\mathbb{P}^1}$ and there is no such nontrivial extension. □

Proposition 3.1.2. *The bundle $\mathfrak{L}_\eta^{A_n}$ over Y with $\eta \in \Xi_Y^{A_n}$ can descend to X if and only if $0 \neq [\eta_{n+1-i,n+2-i}|_{C_i}] \in H^1(Y, O_{C_i}(C_i))$ for every i, i.e. $\eta \in \Xi_X^{A_n}$.*

Proof. Restricting $\mathfrak{L}_0^{A_n}$ to C_i, the corresponding line bundle summands are

$$O_{C_i}(l_k) \cong \begin{cases} O_{\mathbb{P}^1}(1), & k = n+2-i \\ O_{\mathbb{P}^1}(-1), & k = n+1-i \\ O_{\mathbb{P}^1}, & \text{otherwise.} \end{cases}$$

By Lemma 3.1.3 and our assumption, we have the proposition. □

3.2 A_n Lie algebra bundle $\zeta_\eta^{A_n}$

As $A_n = sl(n+1, \mathbb{C}) = aut_0(\mathbb{C}^{n+1})$, $\zeta_\eta^{A_n} := aut_0(\mathfrak{L}_\eta^{A_n})$ ($\eta \in \Xi_X^{A_n}$) is an A_n Lie algebra bundle over Y which can descend to X. This $\zeta_\eta^{A_n}$ does not depend on the existence of C_0. And $\mathfrak{L}_\eta^{A_n}$ is automatically a representation bundle of $\zeta_\eta^{A_n}$.

3.3 A_n minuscule representation bundle $\mathfrak{L}_\eta^{(A_n, \wedge^k \mathbb{C}^{n+1})}$

Consider a surface X with an A_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_k. By Proposition 2.3.1, $I^{(A_n, \wedge^k \mathbb{C}^{n+1})}$ has cardinality $\binom{k}{n+1}$. Define $\mathfrak{L}_0^{(A_n, \wedge^k \mathbb{C}^{n+1})} := \bigoplus_{l \in I} O(l)$ over Y.

Lemma 3.3.1. $\mathfrak{L}_0^{(A_n,\wedge^k\mathbb{C}^{n+1})} = (\wedge^k\mathfrak{L}_0^{A_n})(C_0^k - kC_0^1 - \sum_{j=1}^{k-1}(k-j)C_j)$.

Proof. The bundles on both sides have the same rank, so we only need to check that every line bundle summand in the right-hand side is $O_Y(l)$ for l a (-1)-curve in $I^{(A_n,\wedge^k\mathbb{C}^{n+1})}$. For any k distinct elements l_{i_j} in $I^{(A_n,\mathbb{C}^{n+1})}$, we denote $l = l_{i_1} + l_{i_2} + \cdots + l_{i_k} + C_0^k - (l_1 + l_2 + \cdots l_k)$, then $O_Y(l)$ is a summand in the right-hand side. Since the intersection number of any two distinct (-1)-curves in $I^{(A_n,\mathbb{C}^{n+1})}$ is zero, we have $l^2 = l \cdot K_Y = -1$. i.e. $l \in I^{(A_n,\wedge^k\mathbb{C}^{n+1})}$. $\square$

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{(A_n,\wedge^k\mathbb{C}^{n+1})}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus(\binom{k}{n-1}+\binom{k-2}{n-1})} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus\binom{k-1}{n-1}}.$$

Proposition 3.3.1. *Fix any $\eta \in \Xi_Y^{A_n}$, there exists a unique holomorphic structure on $\mathfrak{L}_0^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ such that the action of $\zeta_\eta^{A_n}$ on the resulting bundle $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ is holomorphic. Furthermore, if $\eta \in \Xi_X^{A_n}$, then $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ can descend to X.*

Proof. As the action of $\zeta_\eta^{A_n}$ on $\mathfrak{L}_\eta^{A_n}$ is holomorphic, $\zeta_\eta^{A_n}$ acts on $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})} := (\wedge^k\mathfrak{L}_\eta^{A_n})(C_0^k - kC_0^1 - \sum_{j=1}^{k-1}(k-j)C_j)$ holomorphically. The last assertion follows from Proposition 3.1.2 and the fact that $O(C_0^k - kC_0^1 - \sum_{j=1}^{k-1}(k-j)C_j)|_{C_i}$ is trivial for every C_i. $\square$

Chapter 4

D_n case

We recall that $D_n = o(2n, \mathbb{C}) = aut(\mathbb{C}^{2n}, q)$ for a non-degenerate quadratic form q on the standard representation $\mathbb{C}^{2n}$. The other minuscule representations are $\mathcal{S}^+$ and $\mathcal{S}^-$ and the adjoint representation is $\wedge^2\mathbb{C}^{2n}$.

4.1 D_n standard representation bundle $\mathfrak{L}_\eta^{(D_n,\mathbb{C}^{2n})}$

We consider a surface X with a D_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1, then $I^{(D_n,\mathbb{C}^{2n})} = I_1 \cup I_2$ with $I_1 = \{C_0^1 + \sum_{i=1}^{k} C_i | 0 \leq k \leq n-1\}$ and $I_2 = \{F - l | l \in I_1\}$, where $F = 2C_0^1 + 2C_1 + \cdots + 2C_{n-2} + C_{n-1} + C_n$. We order these (-1)-curves: $l_k = F - C_0^1 - \sum_{i=1}^{k-1} C_i$ and $l_{2n-k+1} = C_0^1 + \sum_{i=1}^{k-1} C_i$ for $1 \leq k \leq n$.

For any $l_i \neq l_j \in I$, we have $l_i \cdot l_j = 0$ or 1. Given any $l_i \in I$, there exists a unique $l_j \in I$ such that $l_i \cdot l_j = 1$. In this case, $l_i + l_j = F$.

Define $\mathfrak{L}_0^{(D_n,\mathbb{C}^{2n})} := \bigoplus_{l \in I} O(l)$ over Y, for simplicity, we write it as $\mathfrak{L}_0^{D_n}$. If we ignore C_n, then we recover the A_{n-1} case as in the last section. They are related by the following.

Lemma 4.1.1. $\mathfrak{L}_0^{D_n} = \mathfrak{L}_0^{A_{n-1}} \oplus (\mathfrak{L}_0^{A_{n-1}})^*(F)$.

Proof. Since A_{n-1} is a Lie subalgebra of D_n, we can decompose the representation of D_n as sum of irreducible representations of A_{n-1}. By the branching rule, we have $2n = n + n$, that is $\mathbb{C}^{2n} = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ with $\mathbb{C}^{2n}$ and $\mathbb{C}^n$ the standard representations of D_n and A_{n-1} respectively. For $I^{(D_n,\mathbb{C}^{2n})} = I_1 \cup I_2$, I_1 forms the standard representation $\mathbb{C}^n$ of A_{n-1}, and I_2 forms the $(\mathbb{C}^n)^*$. $\square$

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{D_n}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus(2n-4)} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 2}.$$

Similar to $(A_n, \mathbb{C}^{n+1})$ case, we define $\overline{\partial}_\eta : \Omega^{0,0}(Y, \mathfrak{L}_0^{D_n}) \longrightarrow \Omega^{0,1}(Y, \mathfrak{L}_0^{D_n})$ on $\mathfrak{L}_0^{D_n} = \bigoplus_{k=1}^{2n} O(l_k)$ by $\overline{\partial}_\eta := \overline{\partial}_0 + (\eta_{i,j})_{2n\times 2n}$, where $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for any $j > i$, otherwise $\eta_{i,j} = 0$.

By Lemma 2.3.4 and arguments similar to the proof of Proposition 3.1.1 for the A_n case, given any $\eta_{i,i+1}$ with $\overline{\partial}\eta_{i,i+1} = 0$ for every i, there exists $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for every $j > i$ such that $\overline{\partial}_\eta^2 = 0$.

From the configuration of these $2n$ (-1)-curves, we can define a quadratic form q on the vector space $V_0 = \mathbb{C}^I = \bigoplus_{l\in I} \mathbb{C}\langle v_l \rangle$ spanned by these (-1)-curves,

$$q : V_0 \otimes V_0 \longrightarrow \mathbb{C},\ q(v_{l_i}, v_{l_j}) = l_i \cdot l_j.$$

The D_n Lie algebra is the space of infinitesimal automorphism of q, i.e. $D_n = aut(V_0, q)$.

Correspondingly, we have a fiberwise quadratic form q on the bundle $\mathfrak{L}_\eta^{D_n}$:

$$q : \mathfrak{L}_\eta^{D_n} \otimes \mathfrak{L}_\eta^{D_n} \longrightarrow O(F).$$

Proposition 4.1.1. *There exists η with $\overline{\partial}_\eta^2 = 0$ such that $\overline{\partial}_\eta q = 0$.*

Proof. $\overline{\partial}_\eta q = 0$ if and only if $q(\overline{\partial}_\eta s_i, s_j) + q(s_i, \overline{\partial}_\eta s_j) = 0$ for any $s_i \in H^0(Y, O(l_i))$ and $s_j \in H^0(Y, O(l_j))$. From the definition of q, this is equivalent to $\eta_{2n+1-j,i} + \eta_{2n+1-i,j} = 0$, i.e. $\eta_{i,j} = -\eta_{2n+1-j,2n+1-i}$ for any $j > i$. From $l_i + l_{2n+1-i} = l_j + l_{2n+1-j} = F$, we have

$$\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j)) = \Omega^{0,1}(Y, O(l_{2n+1-j} - l_{2n+1-i})) \ni \eta_{2n+1-j,2n+1-i}.$$

We construct η which satisfies $\overline{\partial}_\eta^2 = 0$ with $\eta_{i,j} = -\eta_{2n+1-j,2n+1-i}$ inductively on $j-i$. For $j-i=1$, we can always take $\eta_{i,i+1} = -\eta_{2n-i,2n+1-i}$. Note we have $\eta_{n,n+1} = 0$. For $j-i=2$, we have

$$\overline{\partial}\eta_{i,i+2} = -\eta_{i,i+1}\eta_{i+1,i+2},$$

$$\overline{\partial}\eta_{2n-i-1,2n-i+1} = -\eta_{2n-i-1,2n-i}\eta_{2n-i,2n-i+1} = -\eta_{i+1,i+2}\eta_{i,i+1} = -\overline{\partial}\eta_{i,i+2},$$

so we can take $\eta_{i,i+2} = -\eta_{2n-i-1,2n-i+1}$.

Repeat this process inductively on $j-i$, we can take $\eta_{i,j} = -\eta_{2n+1-j,2n+1-i}$ for any $j > i$. So there exists η satisfying $\overline{\partial}_\eta q = 0$. $\square$

Until now, we have proved $\Xi_Y^{D_n}$ is not empty.

Restricting $\mathfrak{L}_0^{D_n}$ to C_n, the corresponding line bundle summands are:

$$O_{C_n}(l_j) \cong \begin{cases} O_{\mathbb{P}^1}(1), & j = n+1 \text{ or } n+2 \\ O_{\mathbb{P}^1}(-1), & j = n-1 \text{ or } n \\ O_{\mathbb{P}^1}, & \text{otherwise.} \end{cases}$$

The pairs of $O_{\mathbb{P}^1}(\pm 1)$ in $\mathfrak{L}_0^{D_n}|_{C_n}$ are given by $\{l_{n-1}, l_{n+1}\}$ and $\{l_n, l_{n+2}\}$. To construct a trivialization of $\mathfrak{L}_\eta^{D_n}|_{C_n}$, we need the following generalizations of Lemma 3.1.2 and Lemma 3.1.3.

Lemma 4.1.2. *Under the same assumption as in Lemma 3.1.1. Assume $l_{i+1}, l_{i+2}\cdots l_{i+2k}$ satisfy $l_{i+j}\cdot C = -1$ and $l_{i+k+j} = l_{i+j} - C$ for $j = 1, 2, \cdots k$. If $\eta_{i+p,i+q} = 0$ for $2 \le p \le k$, $k+1 \le q \le 2k-1$ and $q - p \le k-1$, i.e. the corresponding submatrix of $\overline{\partial}_{\mathfrak{L}}$ given by $l_{i+1}, l_{i+2}, \cdots l_{i+2k}$ looks like*

$$\left(\begin{array}{c|cccc} & \eta_{i+1,i+k+1} & \eta_{i+1,i+k+2} & \cdots & \eta_{i+1,i+2k} \\ & 0 & \eta_{i+2,i+k+2} & \cdots & \eta_{i+2,i+2k} \\ * & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & \eta_{i+k,i+2k} \\ \hline 0_{k\times k} & & & * & \end{array}\right)$$

with $\eta_{i+1,i+k+1},\ \eta_{i+2,i+k+2}\cdots\eta_{i+k,i+2k}$ *in* $\Omega^{0,1}(Y,O(C))$*. Suppose* $[\eta_{i+1,i+k+1}|_C]$, $[\eta_{i+2,i+k+2}|_C],\cdots,[\eta_{i+k,i+2k}|_C]$ *are nonzero, we can construct* $2k$ *holomorphic sections of* $\mathfrak{L}|_C$ *which are linearly independent at every point in* C*.*

Proof. In order to keep our notations simpler, we assume $k=2$. The above matrix given by $l_{i+1},l_{i+2},l_{i+3},l_{i+4}$ has the form

$$\left(\begin{array}{cc|cc} \overline{\partial} & \eta_{i+1,i+2} & \eta_{i+1,i+3} & * \\ 0 & \overline{\partial} & 0 & \eta_{i+2,i+4} \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times 2}$}} & \overline{\partial} & \eta_{i+3,i+4} \\ & & 0 & \overline{\partial} \end{array}\right).$$

From $H^0(Y,O_C(l_{i+4}))\cong H^0(\mathbb{P}^1,O(1))\cong\mathbb{C}^2$ and $[\eta_{i+2,i+4}|_C]\neq 0$, there exist two holomorphic sections of $\mathfrak{L}|_C$ which are linearly independent at every point in C: $s_1=(y_1,u_1,x_1,t_1)^t$ and $s_2=(y_2,u_2,x_2,t_2)^t$ with u_1,t_1,u_2,t_2 given in Lemma 3.1.2. Similarly, from $H^0(Y,O_C(l_{i+3}))\cong\mathbb{C}^2$ and $[\eta_{i+1,i+3}|_C]\neq 0$, we also have two holomorphic sections of $\mathfrak{L}|_C$ which are linearly independent at every point in C: $s_3=(y_3,0,x_3,0)^t$ and $s_4=(y_4,0,x_4,0)^t$. If there exist a_1,a_2,a_3,a_4 such that $a_1s_1+a_2s_2+a_3s_3+a_4s_4=0$ at some point in C, then we have $a_1t_1+a_2t_2=0$ and $a_1u_1+a_2u_2=0$ at some point, which is impossible by the explicit formulas for u_1,t_1,u_2,t_2 in Lemma 3.1.2. Hence we have the lemma. □

Lemma 4.1.3. *Under the same assumption as in Lemma 3.1.1, we assume* $\mathfrak{L}|_C\cong O_{\mathbb{P}^1}^{\oplus m}\oplus(O_{\mathbb{P}^1}(1)\oplus O_{\mathbb{P}^1}(-1))^{\oplus n}$ *with each pair of* $O_{\mathbb{P}^1}(\pm 1)$ *and the corresponding holomorphic structure as in Lemma 4.1.2. Then* $\mathfrak{L}|_C$ *is trivial if and only if* $[\eta_{i,j}|_C]\neq 0$ *for any* $\eta_{i,j}\in\Omega^{0,1}(Y,O(C))$*.*

Proof. Same arguments as in the proof of Lemma 3.1.3 and Lemma 4.1.2. □

Proposition 4.1.2. *The bundle* $\mathfrak{L}_\eta^{D_n}$ *over* Y *with* $\eta\in\Xi_Y^{D_n}$ *can descend to* X *if and only if for every* C_k *and* $\eta_{i,j}\in\Omega^{0,1}(Y,O(C_k))$, $[\eta_{i,j}|_{C_k}]\neq 0$, *i.e.* $\eta\in\Xi_X^{D_n}$*.*

Proof. Restricting $\mathfrak{L}_0^{D_n}$ to C_i $(1 \le i \le n-1)$, the line bundle summands are

$$O_{C_i}(l_j) \cong \begin{cases} O_{\mathbb{P}^1}(1), & j = i+1 \text{ or } 2n-i \\ O_{\mathbb{P}^1}(-1), & j = i \text{ or } 2n-i+1 \\ O_{\mathbb{P}^1}, & \text{otherwise.} \end{cases}$$

By Lemma 3.1.3, $\mathfrak{L}_\eta^{D_n}|_{C_i}$ is trivial if and only if $[\eta_{i,i+1}|_{C_i}]$, $[\eta_{2n-i,2n+1-i}|_{C_i}]$ are not zeros. For C_n, The pairs of $O_{\mathbb{P}^1}(\pm 1)$ in $\mathfrak{L}_0^{D_n}|_{C_n}$ are given by $\{l_{n-1}, l_{n+1}\}$ and $\{l_n, l_{n+2}\}$. By Lemma 4.1.3 and $\eta_{n,n+1} = 0$ (Proposition 4.1.1), $\mathfrak{L}_\eta^{D_n}|_{C_n}$ is trivial if and only if $[\eta_{n-1,n+1}|_{C_n}]$, $[\eta_{n,n+2}|_{C_n}]$ are not zeros. In fact, this $\mathfrak{L}_\eta^{D_n}$ is just an extension of $\mathfrak{L}_{\eta'}^{A_{n-1}}$ by $(\mathfrak{L}_{\eta'}^{A_{n-1}})^*(F)$ for some $\eta' \in \Xi_X^{A_{n-1}}$ with $\eta' \subset \eta$. $\square$

4.2 D_n Lie algebra bundle $\zeta_\eta^{D_n}$

Note that $\zeta_\eta^{D_n} = aut(\mathfrak{L}_\eta^{D_n}, q)$ is a D_n Lie algebra bundle over Y. In order for $\zeta_\eta^{D_n}$ to descend to X as a Lie algebra bundle, we need to show that $q|_{C_i} : \mathfrak{L}_\eta^{D_n}|_{C_i} \otimes \mathfrak{L}_\eta^{D_n}|_{C_i} \longrightarrow O_{C_i}(F)$ is a constant map for every C_i. This follows from the fact that both $\mathfrak{L}_\eta^{D_n}$ and $O(F)$ are trivial on all C_i's and $\overline{\partial}_\eta q = 0$. From the construction, $\mathfrak{L}_\eta^{D_n}$ is a representation bundle of $\zeta_\eta^{D_n}$.

4.3 D_n spinor representation bundles $\mathfrak{L}_\eta^{(D_n, \mathcal{S}^\pm)}$

We will only deal with $\mathcal{S}^+$, as $\mathcal{S}^-$ case is analogous. Consider a surface X with a D_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_n. By Proposition 2.3.1, $|I^{(D_n, \mathcal{S}^+)}| = 2^{n-1}$. Define $\mathfrak{L}_0^{(D_n, \mathcal{S}^+)} := \bigoplus_{l \in I} O(l)$ over Y.

Lemma 4.3.1. $\mathfrak{L}_0^{(D_n, \mathcal{S}^+)} = \bigoplus_{m=0}^{[\frac{n}{2}]} \wedge^{2m} (\mathfrak{L}_0^{A_{n-1}})^*(mF + C_0^n)$.

Proof. First we check that every line bundle summand in the right-hand side is $O_Y(l)$ for a (-1)-curve l in $I^{(D_n, \mathcal{S}^+)}$. For any $l_i \in I^{(A_{n-1}, \mathbb{C}^n)}$, we have $l_i \cdot C_0^n = 0$, $l_i \cdot F = 0$ and $F \cdot F = 0$, $F \cdot C_0^n = 1$. For any $2m$ distinct elements l_{i_j}'s in

$I^{(A_{n-1},\mathbb{C}^n)}$, we denote $l = -(l_{i_1} + \cdots l_{i_{2m}}) + mF + C_0^n$, then $O_Y(l)$ is a summand in the right-hand side. Since $l^2 = -1$ and $l \cdot K_Y = -1$, $l \in I^{(D_n,\mathcal{S}^+)}$. Also the rank of these two bundles are the same which is $2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2[\frac{n}{2}]}$. Hence we have the lemma. $\square$

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{(D_n,\mathcal{S}^+)}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus 2^{n-2}} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 2^{n-3}}.$$

The D_n Lie algebra bundle $\zeta_0^{D_n}$ has a natural fiberwise action on $\mathfrak{L}_0^{(D_n,\mathcal{S}^+)}$,

$$\rho : \zeta_0^{D_n} \otimes \mathfrak{L}_0^{(D_n,\mathcal{S}^+)} \longrightarrow \mathfrak{L}_0^{(D_n,\mathcal{S}^+)},$$

which can be described easily using the reduction to A_{n-1} (with the node C_n being removed): recall

$$\zeta_0^{D_n} = (\wedge^2 \mathfrak{L}_0^{A_{n-1}}(-F)) \oplus ((\mathfrak{L}_0^{A_{n-1}})^* \otimes \mathfrak{L}_0^{A_{n-1}}) \oplus ((\wedge^2 \mathfrak{L}_0^{A_{n-1}})^*(F)),$$

$$\mathfrak{L}_0^{(D_n,\mathcal{S}^+)} = \bigoplus_{m=0}^{[\frac{n}{2}]} \wedge^{2m} (\mathfrak{L}_0^{A_{n-1}})^*(mF),^1$$

and ρ is given by interior and exterior multiplications for $\wedge^{\cdot} \mathfrak{L}_0^{A_{n-1}}$.

Proposition 4.3.1. *Fix any $\eta \in \Xi_Y^{D_n}$, there exists a unique holomorphic structure on $\mathfrak{L}_0^{(D_n,\mathcal{S}^+)}$ such that the action of $\zeta_\eta^{D_n}$ on the resulting bundle $\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)}$ is holomorphic. Furthermore, if $\eta \in \Xi_X^{D_n}$, then $\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)}$ can descend to X.*

Proof. First, we recall the holomorphic structure on $\zeta_\eta^{D_n}$. In $I^{(D_n,\mathbb{C}^{2n})} = I_1 \cup I_2$ with $I_1 = \{l_i = C_0^1 + \sum_{m=1}^{2n-i} C_m | n+1 \leq i \leq 2n\}$ and $I_2 = \{F - l_i | l_i \in I_1\}$, let s_i, s_i^* and f be local holomorphic sections of $O(l_i), O(F - l_i)$ and $O(-F)$ respectively. By Proposition 4.1.1, we have

$$\overline{\partial}_{\mathfrak{L}_\eta^{D_n}} s_i^* = \sum_{p=1}^{i-1} \eta_{p,i} s_p^*$$

[1]For simplicity, we omit the C_0^n factor.

and

$$\overline{\partial}_{\mathfrak{L}_\eta^{D_n}} s_i = \sum_{p=1}^{n} \eta_{p,2n+1-i} s_p^* - \sum_{p=i+1}^{n} \eta_{i,p} s_p.$$

Back to $\left(\mathfrak{L}_0^{\mathcal{S}^+}\right)^*$, we define $s_{i_1 \cdots i_{2m}} := s_{i_1} \wedge \cdots \wedge s_{i_{2m}} \otimes f^m \in \Gamma(\wedge^{2m} \mathfrak{L}_0^{A_{n-1}}(-mF))$ where $i_j \in \{1, 2, \cdots n\}$ and define $\overline{\partial}_{\left(\mathfrak{L}_\eta^{(D_n, \mathcal{S}^+)}\right)^*}$ as follows:

$$\overline{\partial}_{\mathfrak{L}} s_{i_1 \cdots i_{2m}} = \sum_{p,q} (-1)^{p+q} \eta_{i_p, 2n+1-i_q} s_{i_1 \cdots \widehat{i_p} \cdots \widehat{i_q} \cdots i_{2m}} - \sum_{p} \sum_{k \neq i_p} \eta_{i_p,k} s_{i_1 \cdots i_{p-1} k i_{p+1} \cdots i_{2m}},$$

where $\widehat{i_j}$ means deleting the i_j component. We verify $\overline{\partial}_{\mathfrak{L}}^2 = 0$ by direct computations.

We claim that $\overline{\partial}_{\mathfrak{L}}$ is the unique holomorphic structure such that the action of $\zeta_\eta^{D_n}$ on $\left(\mathfrak{L}_\eta^{(D_n, \mathcal{S}^+)}\right)^*$ is holomorphic, i.e.

$$\overline{\partial}_{\zeta_\eta^{D_n}}(g) \cdot x + g \cdot (\overline{\partial}_{\mathfrak{L}} x) = \overline{\partial}_{\mathfrak{L}}(g \cdot x) \qquad (*)$$

for any $g \in \Gamma(\zeta_\eta^{D_n})$ and $x \in \Gamma\left(\left(\mathfrak{L}_0^{\mathcal{S}^+}\right)^*\right)$.

We prove the above claim by induction on m. When $m = 0$, $x = s_0 \in \Gamma(\wedge^0 \mathfrak{L}_0^{A_{n-1}})$, by direct computations, $(*)$ holds for any $g \in \Gamma(\zeta_\eta^{D_n})$ if and only if $\overline{\partial}_{\mathfrak{L}} s_0 = 0$ and $\overline{\partial}_{\mathfrak{L}} s_{ij} = -\eta_{i,2n+1-j} s_0 - \sum_{p=i+1}^{n} \eta_{i,p} s_{pj} - \sum_{p=j+1}^{n} \eta_{j,p} s_{ip}$ for any $s_{ij} \in \Gamma(\wedge^2 \mathfrak{L}_0^{A_{n-1}})$. When $m = 2$, from the above formula for $\overline{\partial}_{\mathfrak{L}} s_{ij}$, we can get the formula for $\overline{\partial}_{\mathfrak{L}} s_{ijkl}$. Repeat this process inductively, we can get the above formula for $\overline{\partial}_{\mathfrak{L}} s_{i_1 \cdots i_{2m}}$. Hence we have the first part of this proposition.

For the second part, we will rewrite $\overline{\partial}_{\mathfrak{L}}$ in matrix form. Firstly, we have

$$\overline{\partial}_{\mathfrak{L}_\eta^{(D_n, \mathbb{C}^{2n})}} = \left(\begin{array}{c|c} \overline{\partial}_{(\mathfrak{L}_{\eta'}^{A_{n-1}})^*(F)} & B \\ \hline 0 & \overline{\partial}_{\mathfrak{L}_{\eta'}^{A_{n-1}}} \end{array} \right)$$

with $\eta' \subset \eta$ and the upper right block B has the following shape

$$B = \begin{pmatrix} \vdots & \vdots & \ddots \\ \beta & * & \cdots \\ 0 & -\beta & \cdots \end{pmatrix},$$

for $[\beta] \in H^1(Y, O(C_n))$.

In particular, we have an exact sequence of holomorphic bundles:

$$0 \to (\mathfrak{L}_{\eta'}^{A_{n-1}})^*(F) \to \mathfrak{L}_{\eta}^{D_n} \to \mathfrak{L}_{\eta'}^{A_{n-1}} \to 0. \tag{Δ}$$

By tensoring (Δ) with $\mathfrak{L}_{\eta'}^{A_{n-1}}(-F)$, we obtain a bundle S_1 as follows,

$$0 \to O_Y \to S_1 \to \wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F) \to 0,$$

with the induced holomorphic structure given by

$$\overline{\partial}_{S_1} = \left(\begin{array}{c|c} \overline{\partial}_{\wedge^0 \mathfrak{L}_{\eta'}^{A_{n-1}}} & B_1 \\ \hline 0 & \overline{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} \end{array}\right) = \left(\begin{array}{c|c} \overline{\partial}_{\wedge^0 \mathfrak{L}_{\eta'}^{A_{n-1}}} & \pm\beta \quad \cdots \\ \hline 0 & \overline{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} \end{array}\right).$$

The occurrence of $\pm\beta$ in that location is because $l_{n+1} + l_{n+2}$ with $l_{n+1}, l_{n+2} \in I^{(D_n, \mathbb{C}^{2n})}$ is the largest element in $I^{(A_{n-1}, \wedge^2\mathbb{C}^n)}$ and $F - l_{n+1} - l_{n+2} = C_n$ because $F = 2C_0^1 + 2C_1 + \cdots 2C_{n-2} + C_{n-1} + C_n$.

Similarly, we have an extension bundle

$$0 \to \wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F) \to S_2 \to \wedge^4 \mathfrak{L}_{\eta'}^{A_{n-1}}(-2F) \to 0,$$

with

$$\overline{\partial}_{S_2} = \left(\begin{array}{c|c} \overline{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} & B_2 \\ \hline 0 & \overline{\partial}_{\wedge^4 \mathfrak{L}_{\eta'}^{A_{n-1}}(-2F)} \end{array}\right),$$

where

$$B_2 = \begin{pmatrix} \pm\beta & & & \\ 0 & \pm\beta & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \pm\beta \end{pmatrix},$$

for $[\beta] \in H^1(Y, Hom(O(l_i + l_j + l_{n+1} + l_{n+2} - 2F), O(l_i + l_j - F))) = H^1(Y, O(C_n))$ with $i, j \in \{n+3, n+4, \cdots, 2n\}$. And the number of $\pm\beta$'s is $\binom{n-2}{2}$.

Inductively, we obtain $\overline{\partial}_{\left(\mathfrak{L}_{\eta}^{(D_n,\mathcal{S}^+)}\right)^*}$ as above which has the shape that satisfies Lemma 4.1.3:

$$\overline{\partial}_{\left(\mathfrak{L}_{\eta}^{(D_n,\mathcal{S}^+)}\right)^*} = \left(\begin{array}{c|c|c|c} \overline{\partial}_{\wedge^0 \mathfrak{L}_{\eta'}^{A_{n-1}}} & B_1 & \cdots & \cdots \\ \hline 0 & \overline{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} & B_2 & \ddots \\ \hline 0 & 0 & \overline{\partial}_{\wedge^4 \mathfrak{L}_{\eta'}^{A_{n-1}}(-2F)} & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right).$$

The number of $\pm\beta \in \Omega^1(Y, O(C_n))$ in $\overline{\partial}_{\mathfrak{L}}$ is $\binom{n-2}{0} + \binom{n-2}{2} + \cdots + \binom{n-2}{2[\frac{n-2}{2}]} = 2^{n-3}$.

To prove that $\left(\mathfrak{L}_{\eta}^{\mathcal{S}^+}\right)^*$ can descend to X when $\eta \in \Xi_X^{D_n}$, we need to show $\left(\mathfrak{L}_{\eta}^{\mathcal{S}^+}\right)^*|_{C_i}$ is trivial for every C_i. When $i \neq n$, this follows from the fact that $\mathfrak{L}_{\eta'}^{A_{n-1}}$ is trivial (Proposition 3.1.2) and $Ext^1_{\mathbb{P}^1}(O, O) \cong 0$. When $i = n$, this follows from Lemma 4.1.3 and $\beta = \eta_{\mathfrak{n}-1,\mathfrak{n}+1} \in \Omega^{0,1}(Y, O(C_n))$ with $[\eta_{n-1,n+1}|_{C_n}] \neq 0$. $\square$

Chapter 5

E_n case

5.1 E_6 case

We recall that [1] E_6=$aut(\mathbb{C}^{27}, c)$ for a non-degenerate cubic form c on the standard representation $\mathbb{C}^{27}$. The other minuscule representation is $\overline{\mathbb{C}^{27}}$.

We consider a surface X with an E_6 singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1. By Proposition 2.3.1, $I^{(E_6,\mathbb{C}^{27})}$ has cardinality 27. For any two distinct (-1)-curves l_i and l_j in I, we have $l_i \cdot l_j = 0$ or 1.

Define $\mathfrak{L}_0^{(E_6,\mathbb{C}^{27})} := \bigoplus_{l\in I} O(l)$ over Y, for simplicity, we write it as $\mathfrak{L}_0^{E_6}$. If we ignore C_6, then we recover the A_5 case as in section 3.1.

Lemma 5.1.1. *$\mathfrak{L}_0^{E_6} = \mathfrak{L}_0^{A_5} \oplus (\wedge^2 \mathfrak{L}_0^{A_5})^*(H) \oplus (\wedge^5 \mathfrak{L}_0^{A_5})^*(2H)$, where $H = 3C_0^1 + 3C_1 + 3C_2 + 3C_3 + 2C_4 + C_5 + C_6$.*

Proof. E_6 has A_5 as a Lie subalgebra, the branching rule is $27 = 6 + 15 + 6$, i.e. $\mathbb{C}^{27} = \mathbb{C}^6 \oplus \wedge^2(\mathbb{C}^6)^* \oplus \wedge^5(\mathbb{C}^6)^*$. The first 6 (-1)-curves in $I : l_1 = C_0^1$, $l_2 = C_0^1 + C_1, \cdots\ l_6 = C_0^1 + C_1 + C_2 + C_3 + C_4 + C_5$ form the standard representation $\mathbb{C}^6$ of A_5. The next 15 (-1)-curves are given by $H - l_i - l_j$ with $i \neq j \in \{1, 2, \cdots, 6\}$. The remaining 6 (-1)-curves are given by $2H - l_1 - l_2 - \cdots - \widehat{l_i} - \cdots - l_6$. □

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{E_6}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus 15} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 6}.$$

From Lemma 5.1.1, we can easily determine the configuration of these 27 (-1)-curves [29]: Fix any (-1)-curve, there are exactly 10 (-1)-curves intersect it, together with the fixed (-1)-curve, they form 5 triangles. A triple l_i, l_j, l_k is called a triangle if $l_i + l_j + l_k = K'$, where $K' = 3C_0^1 + 4C_1 + 5C_2 + 6C_3 + 4C_4 + 2C_5 + 3C_6$.

From the configuration of these 27 (-1)-curves in Y, we can define a cubic form c on the vector space $V_0 = \mathbb{C}^I = \bigoplus_{l \in I} \mathbb{C}\langle v_l \rangle$ spanned by (-1)-curves,

$$c : V_0 \otimes V_0 \otimes V_0 \longrightarrow \mathbb{C},\ (v_{l_i}, v_{l_j}, v_{l_k}) \mapsto \begin{cases} \pm 1 & \text{if } l_i + l_j + l_k = K' \\ 0 & \text{otherwise.} \end{cases}$$

The signs above can be determined explicitly [1][15] such that $E_6 = aut(V_0, c)$.

Correspondingly, we have a fiberwise cubic form c on the bundle $\mathfrak{L}_\eta^{E_6}$,

$$c : \mathfrak{L}_\eta^{E_6} \otimes \mathfrak{L}_\eta^{E_6} \otimes \mathfrak{L}_\eta^{E_6} \longrightarrow O\,(K').$$

Proposition 5.1.1. *There exists η with $\overline{\partial}_\eta^2 = 0$ such that $\overline{\partial}_\eta c = 0$.*

Proof. Note $\overline{\partial}_\eta c = 0$ if and only if

$$c(\overline{\partial}_\eta s_i, s_j, s_k) + c(s_i, \overline{\partial}_\eta s_j, s_k) + c(s_i, s_j, \overline{\partial}_\eta s_k) = 0 \qquad (*)$$

for any $s_i \in H^0(Y, O(l_i))$, $s_j \in H^0(Y, O(l_j))$ and $s_k \in H^0(Y, O(l_k))$. From the definition of c, if $l_i + l_j + l_k = K'$, then the above equation $(*)$ holds automatically. If $l_i + l_j + l_k \neq K'$, without loss of generality, we assume $l_i \cdot l_j = 0$, then we have the following four cases.

Case (i), if $l_i \cdot l_k = 0$ and $l_j \cdot l_k = 0$, then $(*)$ holds automatically.

Case (ii), if $l_i \cdot l_k = 0$ and $l_j \cdot l_k = 1$, then $(*)$ holds if $\eta_{l_i, K' - l_j - l_k} = 0$.

Case (iii), if $l_i \cdot l_k = 1$ and $l_j \cdot l_k = 0$, then $(*)$ holds if $\eta_{l_j, K' - l_i - l_k} = 0$.

Case (iv), if $l_i \cdot l_k = 1$ and $l_j \cdot l_k = 1$, then $(*)$ holds if $\eta_{l_i, K' - l_j - l_k} \pm \eta_{l_j, K' - l_i - l_k} = 0$, here the sign is determined by the signs of cubic form.

In conclusion, for any $l_i, l_j \in I^{(E_6,\mathbb{C}^{27})}$, if $l_i \cdot l_j \neq 0$, then $\eta_{i,j} = 0$. If $l_i \cdot l_j = 0$, then $l_i - l_j = \alpha$ $(j > i)$ for $\alpha \in \Phi^+$, i.e. $\eta_{i,j} \in \Omega^{0,1}(Y, O(\alpha))$. And for any other $\eta_{p,q} \in \Omega^{0,1}(Y, O(\alpha))$, we have $\eta_{i,j} \pm \eta_{p,q} = 0$. From the signs of the cubic form c, we know that given any positive root α, there exists 6 $\eta_{i,j}$'s in $\Omega^{0,1}(Y, O(\alpha))$, where 3 of them are the same and the other 3 different to the first three by a sign. We use computer to prove we can find such $\eta_{i,j}$'s satisfying $\overline{\partial}_\eta^2 = 0$. □

Until now, we have proved $\Xi_Y^{E_6}$ is not empty.

Proposition 5.1.2. *The bundle $\mathfrak{L}_\eta^{E_6}$ over Y with $\eta \in \Xi_Y^{E_6}$ can descend to X if and only if for every C_k and $\eta_{i,j} \in \Omega^{0,1}(Y, O(C_k))$, $[\eta_{i,j}|_{C_k}] \neq 0$, i.e. $\eta \in \Xi_X^{E_6}$.*

Proof. From Lemma 5.1.1, Proposition 5.1.1 and the order of $I^{(E_6,\mathbb{C}^{27})}$, for $\eta \in \Xi_Y^{E_6}$, $\mathfrak{L}_\eta^{E_6}$ can be constructed from $\mathfrak{L}_{\eta'}^{A_5}$ for some $\eta' \in \Xi_Y^{A_5}$ with $\eta' \subset \eta$. Under the (non-holomorphic) direct sum decomposition $\mathfrak{L}_0^{E_6} = \mathfrak{L}_0^{A_5} \oplus (\wedge^2 \mathfrak{L}_0^{A_5})^*(H) \oplus (\wedge^5 \mathfrak{L}_0^{A_5})^*(2H)$, $\overline{\partial}_\eta$ for $\mathfrak{L}_\eta^{E_6}$ has the following block decomposition:

$$\left(\begin{array}{c|c|c}
 & \begin{matrix} \vdots & \vdots & \vdots & \ddots \\ \pm\beta & * & * & \cdots \\ 0 & \pm\beta & * & \cdots \\ 0 & 0 & \pm\beta & \cdots \end{matrix} & \\
\overline{\partial}_{(\wedge^5 \mathfrak{L}_{\eta'}^{A_5})^*(2H)} & & * \\
\hline
 & & \begin{matrix} \vdots & \vdots & \vdots & \ddots \\ \pm\beta & * & * & \cdots \\ 0 & \pm\beta & * & \cdots \\ 0 & 0 & \pm\beta & \cdots \end{matrix} \\
0 & \overline{\partial}_{(\wedge^2 \mathfrak{L}_{\eta'}^{A_5})^*(H)} & \\
\hline
0 & 0 & \overline{\partial}_{\mathfrak{L}_{\eta'}^{A_5}}
\end{array}\right).$$

Here $\pm\beta \in \Omega^{0,1}(Y, O(C_6))$, it is because the corresponding two (-1)-curves l and l' satisfying $l - l' = C_6$. The signs of β can be determined by $\overline{\partial}_\eta c = 0$.

From above, we know that $\mathfrak{L}_\eta^{E_6}|_{C_k}$ $(k \neq 6)$ is trivial if and only if $\mathfrak{L}_{\eta'}^{A_5}|_{C_k}$ $(k \neq 6)$ is trivial. From Proposition 3.1.2, we have the theorem for $k \neq 6$. For C_6, from Lemma 4.1.3, $\mathfrak{L}_\eta^{E_6}|_{C_6}$ is trivial if and only if these $\pm\beta$'s satisfy $[\beta|_{C_6}] \neq 0$. □

Note that $\zeta_\eta^{E_6} = aut(\mathfrak{L}_\eta^{E_6}, c)$ is an E_6 Lie algebra bundle over Y. In order for $\zeta_\eta^{E_6}$ to descend to X as a Lie algebra bundle, we need to show that $c|_{C_i}$: $\mathfrak{L}_\eta^{E_6}|_{C_i} \otimes \mathfrak{L}_\eta^{E_6}|_{C_i} \otimes \mathfrak{L}_\eta^{E_6}|_{C_i} \longrightarrow O_{C_i}(K')$ is a constant map for every C_i. This follows from the fact that both $\mathfrak{L}_\eta^{E_6}$ and $O(K')$ are trivial on all C_i's and $\overline{\partial}_\eta c = 0$. From the construction, $\mathfrak{L}_\eta^{E_6}$ is a representation bundle of $\zeta_\eta^{E_6}$.

The only other minuscule representation $\overline{\mathbb{C}^{27}}$ of E_6 is the dual of the standard representation $\mathbb{C}^{27}$, therefore $\mathfrak{L}_\eta^{(E_6,\overline{\mathbb{C}^{27}})} = \left(\mathfrak{L}_\eta^{(E_6,\mathbb{C}^{27})}\right)^*$.

5.2 E_7 case

We recall that [1] E_7=$aut(\mathbb{C}^{56}, t)$ for a non-degenerate quartic form t on the standard representation $\mathbb{C}^{56}$. There is no other minuscule representation of E_7.

We consider a surface X with an E_7 singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1. By Proposition 2.3.1, $I^{(E_7,\mathbb{C}^{56})}$ has cardinality 56. For any two distinct (-1)-curves l_i and l_j in I, we have $l_i \cdot l_j = 0, 1$ or 2.

Define $\mathfrak{L}_0^{(E_7,\mathbb{C}^{56})} := \bigoplus_{l\in I} O(l)$ over Y, for simplicity, we write it as $\mathfrak{L}_0^{E_7}$. If we ignore C_7, we recover the A_6 case as in section 3.1.

Lemma 5.2.1. $\mathfrak{L}_0^{(E_7,\mathbb{C}^{56})} = \mathfrak{L}_0^{A_6} \oplus (\wedge^2 \mathfrak{L}_0^{A_6})^*(H) \oplus (\wedge^5 \mathfrak{L}_0^{A_6})^*(2H) \oplus (\wedge^6 \mathfrak{L}_0^{A_6})^*(3H)$, *where* $H = 3C_0^1 + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 2C_5 + C_6 + C_7$.

Proof. Similar to E_6 case. □

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{E_7}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus 32} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 12}$$

The configuration of these 56 (-1)-curves is as follows: Fix any (-1)-curve, there are exactly 27 (-1)-curves intersect it once, 1 (-1)-curve intersects it twice. If $l_i + l_j + l_p + l_q = 2K'$ with $K' = 2C_0^1 + 3C_1 + 4C_2 + 5C_3 + 6C_4 + 4C_5 + 2C_6 + 3C_7$, the four (-1)-curves l_i, l_j, l_p and l_q will form a quadrangle.

From this configuration, we can define a quartic form t on the vector space $V_0 = \mathbb{C}^I = \bigoplus_{l\in I}\mathbb{C}\langle v_l\rangle$ spanned by all the (-1)-curves,

$$t : V_0 \otimes V_0 \otimes V_0 \otimes V_0 \longrightarrow \mathbb{C},\ (v_{l_i}, v_{l_j}, v_{l_p}, v_{l_q}) \mapsto \begin{cases} \pm 1 & \text{if } l_i + l_j + l_p + l_q = 2K' \\ 0 & \text{otherwise.} \end{cases}$$

The signs above can be determined explicitly [1] such that $E_7 = aut(V_0, t)$.

Correspondingly, we have a fiberwise quartic form t on the bundle $\mathfrak{L}_\eta^{E_7}$,

$$t : \mathfrak{L}_\eta^{E_7} \otimes \mathfrak{L}_\eta^{E_7} \otimes \mathfrak{L}_\eta^{E_7} \otimes \mathfrak{L}_\eta^{E_7} \longrightarrow O\,(2K').$$

Proposition 5.2.1. *There exists η with $\overline{\partial}_\eta^2 = 0$ such that $\overline{\partial}_\eta t = 0$.*

Proof. Similar to E_6 case, but even more calculations involved. We will omit the calculations here and only list the conditions for $\overline{\partial}_\eta t = 0$. From $\overline{\partial}_\eta t = 0$ we have when $l_i \cdot l_j \neq 0$, $\eta_{i,j} = 0$. That means all the nonzero $\eta_{i,j}$'s are corresponding to $l_i \cdot l_j = 0$, then $l_i - l_j = \alpha$ for some root α, i.e. $\eta_{i,j} \in \Omega^{0,1}(Y, O(\alpha))$. Conversely, given any positive root α, there exists 12 $\eta_{i,j}$'s in $\Omega^{0,1}(Y, O(\alpha))$, where 6 of them are the same and the other 6 different to the first 6 by a sign. We use computer to prove we can find such $\eta_{i,j}$'s satisfying $\overline{\partial}_\eta^2 = 0$. □

Until now, we have proved $\Xi_Y^{E_7}$ is not empty.

Proposition 5.2.2. *The bundle $\mathfrak{L}_\eta^{E_7}$ over Y with $\eta \in \Xi_Y^{E_7}$ can descend to X if and only if for every C_k and $\eta_{i,j} \in \Omega^{0,1}(Y, O(C_k))$, $[\eta_{i,j}|_{C_k}] \neq 0$, i.e. $\eta \in \Xi_X^{E_7}$.*

Proof. Similar to E_6 case (Proposition 5.1.2). □

Note that $\zeta_\eta^{E_7} = aut(\mathfrak{L}_\eta^{E_7}, t)$ is an E_7 Lie algebra bundle over Y. In order for $\zeta_\eta^{E_7}$ to descend to X as a Lie algebra bundle, we need to show that $t|_{C_i}$: $\mathfrak{L}_\eta^{E_7}|_{C_i} \otimes \mathfrak{L}_\eta^{E_7}|_{C_i} \otimes \mathfrak{L}_\eta^{E_7}|_{C_i} \otimes \mathfrak{L}_\eta^{E_7}|_{C_i} \longrightarrow O_{C_i}\,(2K')$ is a constant map for every C_i. This follows from the fact that both $\mathfrak{L}_\eta^{E_7}$ and $O(2K')$ are trivial on all C_i's and $\overline{\partial}_\eta t = 0$. It is obvious that $\mathfrak{L}_\eta^{E_7}$ is a representation bundle of $\zeta_\eta^{E_7}$.

5.3 E_8 case

Though E_8 has no minuscule representation, the fundamental representation corresponding to C_1 is the adjoint representation of E_8.

We consider a surface X with an E_8 singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1. By direct computations, $|I| = 240$. In this case, $l \in I$ if and only if $l - K' \in \Phi$, where $K' = C_0^1 + 2C_1 + 3C_2 + 4C_3 + 5C_4 + 6C_5 + 4C_6 + 2C_7 + 3C_8$. So $\mathcal{E}_0^{E_8}$ defined in section 1.2 can be written as follows:

$$\mathcal{E}_0^{E_8} := O^{\oplus 8} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha) = (O(K')^{\oplus 8} \oplus \bigoplus_{l \in I} O(l))(-K').$$

We will prove that $(\mathcal{E}_\varphi^{E_8}, \overline{\partial}_\varphi)$ with $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Psi_X$ descends to X in chapter 6.

Chapter 6

Proof of Theorem 1.2.1

In the above three chapters, we have constructed and studied the Lie algebra bundles and minuscule representation bundles in A_n, D_n and E_n $(n \neq 8)$ cases separately. We will prove the holomorphic structures on these bundles can be expressed by forms in the positive root classes and the representation actions.

Proof. (of Theorem 2.5.1 and 2.5.2) Recall that when $\rho : g \longrightarrow End(V)$ is the standard representation, $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ $(\eta \in \Xi_Y^{\mathfrak{g}})$ admits a holomorphic fiberwise symmetric multi-linear form f. And $\overline{\partial}_\eta f = 0$ implies that $\eta_{i,j} = 0$ unless $l_i - l_j = \alpha (j > i)$ for some $\alpha \in \Phi^+$. Thus $\eta_{i,j} = \varphi_\alpha \in \Omega^{0,1}(Y, O(\alpha))$. Furthermore, if $\eta_{i,j}$ and $\eta_{i',j'}$ are in $\Omega^{0,1}(Y, O(\alpha))$, then they are the same up to sign. Thus we can write $\eta_{i,j} = n_{\alpha,w_i}\varphi_\alpha$, where n_{α,w_i}'s are as in Chapter 2, since ρ preserves f. Namely, $\overline{\partial}_\eta = \overline{\partial}_0 + \sum_{\alpha\in\Phi^+} c_\alpha \rho(x_\alpha) = \overline{\partial}_0 + \sum_{\alpha\in\Phi^+} \rho(\varphi_\alpha)$ with $\varphi_\alpha = c_\alpha x_\alpha$.

The holomorphic structure on the bundle $\zeta_\eta^{\mathfrak{g}} := aut(\mathfrak{L}_\eta^{(\mathfrak{g},V)}, f)$ is $\overline{\partial}_\eta = \overline{\partial}_0 + \sum_{\alpha\in\Phi^+} c_\alpha ad(x_\alpha)$, which is the same as $\overline{\partial}_\varphi$ for $\mathcal{E}_\varphi^{\mathfrak{g}}$ in Chapter 1, i.e. $\zeta_\eta^{\mathfrak{g}} = \mathcal{E}_\varphi^{\mathfrak{g}}$.

The only minuscule representations $(\mathfrak{g}, V)$ besides standard representations are $(A_n, \wedge^k C^{n+1})$, $(D_n, S^\pm)$ and $(E_6, \overline{\mathbb{C}^{27}})$. We denote corresponding actions as ρ as usual. In each case, for $\mathcal{E}_\varphi^{\mathfrak{g}}$ to act holomorphically on the corresponding vector bundle, the holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)}$ can only be $\overline{\partial}_\varphi$.

The filtration of $\mathfrak{L}_0^{(\mathfrak{g},V)}$ gives one on $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$, since it is constructed from extensions using elements in $I_i \backslash I_{i+1}$ (section 2.4).

We note that all the above Lie algebra bundles and representation bundles over Y can descend to X if and only if $0 \neq [\varphi_{C_i}|_{C_i}] \in H^1(Y, O_{C_i}(C_i))$ for all C_i's, i.e. $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Psi_X$. $\square$

From the above arguments, Theorem **1.2.1** holds true for ADE except E_8 case.

Proof. (of Theorem 1.2.1) It remains to prove the E_8 case.

$$\mathcal{E}_0^{E_8} := O^{\oplus 8} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha) = (O(K')^{\oplus 8} \oplus \bigoplus_{l \in I} O(l))(-K').$$

We want to show that the bundle $(\mathcal{E}_\varphi^{E_8}, \overline{\partial}_\varphi)$ with $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Psi_X$ can descend to X, i.e. $\mathcal{E}_\varphi^{E_8}|_{C_i}$ is trivial for $i = 1, 2, \cdots 8$. Note $O(K')|_{C_i}$ is trivial for every i, but $O(l)|_{C_i}$ can be $O_{\mathbb{P}^1}(\pm 2)$, hence Lemma 3.1.1 is not sufficient. However, if we ignore C_8 (resp. C_7) in Y, then we recover the A_7 case (resp. D_7 case). Our approach is to reduce the problem of trivializing $\mathcal{E}_\varphi^{E_8}|_{C_i}$ to one for a representation bundle of A_7 (resp. D_7).

Step one, as A_7 is a Lie subalgebra of E_8, the adjoint representation of E_8 decomposes as a sum of irreducible representations of A_7. The branching rule is $248 = 8 + 28 + 56 + 64 + 56 + 28 + 8$, correspondingly, we have the following decomposition of $\mathcal{E}_0^{E_8}$ over Y,

$$\begin{aligned}\mathcal{E}_0^{E_8} = \mathfrak{L}_0^{A_7}(-K') \oplus \wedge^2(\mathfrak{L}_0^{A_7})^*(H - K') \oplus \wedge^5(\mathfrak{L}_0^{A_7})^*(2H - K') \oplus \\ \mathfrak{L}_0^{A_7} \otimes (\mathfrak{L}_0^{A_7})^* \oplus \wedge^3(\mathfrak{L}_0^{A_7})^*(H) \oplus \wedge^6(\mathfrak{L}_0^{A_7})^*(2H) \oplus (\mathfrak{L}_0^{A_7})^*(K'),\end{aligned}$$

where $H = 3C_0^1 + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 2C_6 + C_7 + C_8$ and $K' = C_0^1 + 2C_1 + 3C_2 + 4C_3 + 5C_4 + 6C_5 + 4C_6 + 2C_7 + 3C_8$.

Step two, instead of $\mathfrak{L}_0^{A_7}$, we use $\mathfrak{L}_\varphi^{A_7}$ which is trivial on C_i for $i \neq 8$. We consider the bundle

$$\mathcal{E}'^{E_8} = \mathfrak{L}_{\varphi}^{A_7}(-K') \oplus \wedge^2(\mathfrak{L}_{\varphi}^{A_7})^*(H-K') \oplus \wedge^5(\mathfrak{L}_{\varphi}^{A_7})^*(2H-K') \oplus$$
$$\mathfrak{L}_{\varphi}^{A_7} \otimes (\mathfrak{L}_{\varphi}^{A_7})^* \oplus \wedge^3(\mathfrak{L}_{\varphi}^{A_7})^*(H) \oplus \wedge^6(\mathfrak{L}_{\varphi}^{A_7})^*(2H) \oplus (\mathfrak{L}_{\varphi}^{A_7})^*(K').$$

We have $\overline{\partial}_{\mathcal{E}'^{E_8}} = \overline{\partial}_0 + \sum_{\alpha\in\Phi^+_{A_7}} ad(\varphi_\alpha)$. Since $O(K')$ and $O(H)$ are both trivial on C_i for $i \neq 8$, $\mathcal{E}'^{E_8}$ is trivial on C_i for $i \neq 8$.

Step three, we compare $\mathcal{E}'^{E_8}$ with $\mathcal{E}_{\varphi}^{E_8}$. Topologically they are the same. Holomorphically,

$$\overline{\partial}_{\mathcal{E}_{\varphi}^{E_8}} = \overline{\partial}_0 + \sum_{\alpha\in\Phi^+_{E_8}} ad(\varphi_\alpha) = \overline{\partial}_{\mathcal{E}'^{E_8}} + \sum_{\alpha\in\Phi^+_{E_8}\backslash\Phi^+_{A_7}} ad(\varphi_\alpha).$$

If we write the holomorphic structure of $\mathcal{E}_{\varphi}^{E_8}$ as a 248×248 matrix, then φ_α with $\alpha \in \Phi^+_{E_8}\backslash\Phi^+_{A_7}$ must appear at those positions (β,γ) with $\beta-\gamma=\alpha$, where β has at least one more C_8 than γ. That means, after taking extensions between the summands of $\mathcal{E}'^{E_8}$, we can get $\mathcal{E}_{\varphi}^{E_8}$. Since $\mathcal{E}'^{E_8}$ is trivial on C_i for $i \neq 8$ and $Ext^1_{\mathbb{P}^1}(O,O) \cong 0$, we have $\mathcal{E}_{\varphi}^{E_8}$ trivial on C_i for $i \neq 8$.

Similarly, if we consider the reduction of E_8 to D_7, from the branching rule $248 = 14+64+1+91+64+14$, we have the following decomposition of $\mathcal{E}_0^{E_8}$,

$$\mathcal{E}_0^{E_8} = \mathfrak{L}_0^{D_7}(-K') \oplus \mathfrak{L}_0^{(D_7,\mathcal{S}^+)}(C_7-C_0^6) \oplus O \oplus \mathcal{E}_0^{D_7} \oplus (\mathfrak{L}_0^{(D_7,\mathcal{S}^+)})^*(C_0^6-C_7) \oplus (\mathfrak{L}_0^{D_7})^*(K').$$

Instead of $\mathfrak{L}_0^{D_7}$, we consider $\mathfrak{L}_{\varphi}^{D_7}$. Similar to the reduction to A_7 case as above, we will get for $(\mathcal{E}_{\varphi}^{E_8}, \overline{\partial}_{\varphi})$, if we take $[\varphi_{C_i}|_{C_i}] \neq 0$, then $\mathcal{E}_{\varphi}^{E_8}$ is trivial on C_i for $i \neq 7$. Hence we have proved Theorem 1.2.1 for type E_8. $\square$

Proof. (of Theorem 2.5.3) We only need to find a divisor B in Y such that (i) B is a combination of C_i's and $\widetilde{C_0}$ with the coefficient of $\widetilde{C_0}$ not zero, and (ii) $O(B)$ can descend to X. Then if we take k to be the coefficient of $\widetilde{C_0}$ in B, $\mathbb{L}_{\varphi}^{(\mathfrak{g},V)} := S^k\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)} \otimes O(-B)$ with $\varphi \in \Psi_X$ can descend to X and does not depend on the existence of C_0.

$(A_n,\, C^{n+1})$ case, $B = (n+1)\widetilde{C_0} + nC_1 + (n-1)C_2 + \cdots + C_n$.

$(A_n, \wedge^k C^{n+1})$ case, $B = (n+1)\widetilde{C_0} + (n-k+1)C_1 + \cdots (k-1)(n-k-1)C_{k-1} + k(n-k)C_{k+1} + \cdots kC_n$.

$(D_n,\, C^{2n})$ case, $B = F = 2\widetilde{C_0} + 2C_1 + \cdots + 2C_{n-2} + C_{n-1} + C_n$.

(D_n, S^+) case, $B = 4\widetilde{C_0} + 2C_1 + 4C_2 + \cdots + 2(n-2)C_{n-2} + (n-2)C_{n-1} + nC_n$.

$(E_6,\, C^{27})$ case, $B = 3\widetilde{C_0} + 4C_1 + 5C_2 + 6C_3 + 4C_4 + 2C_5 + 3C_6$.

$(E_7,\, C^{56})$ case, $B = 2\widetilde{C_0} + 3C_1 + 4C_2 + 5C_3 + 6C_4 + 4C_5 + 2C_6 + 3C_7$. □

Remark 6.0.1. *We can determine Chern classes of the Lie algebra bundles and minuscule representation bundles. For any minuscule representation bundle* $\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)}$,

$$c_1(\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)}) = \sum_{l \in I^{(\mathfrak{g},V)}} [l] \in H^2(Y, \mathbb{Z}).$$

For any Lie algebra bundle $\mathcal{E}_{\varphi}^{\mathfrak{g}}$, *we have*

$$c_1(\mathcal{E}_{\varphi}^{\mathfrak{g}}) = 0$$

and

$$c_2(\mathcal{E}_{\varphi}^{\mathfrak{g}}) = \sum_{\alpha \neq \beta \in \Phi} c_1(O(\alpha))c_1(O(\beta)) = \sum_{\alpha \in \Phi^+} c_1(O(\alpha))c_1(O(-\alpha)) = \dim(\mathfrak{g}) - rank(\mathfrak{g}).$$

In particular, the bundles we defined above are not trivial.

Remark 6.0.2. *There are choices in the construction of our Lie algebra bundles and minuscule representation bundles, we will see that these bundles are not unique. Take* $\mathfrak{L}_{\varphi}^{A_2}$ $(\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+_{A_2}} \in \Psi_X)$ *as an example. The holomorphic structure on* $\mathfrak{L}_{\varphi}^{A_2}$ *is as follows:*

$$\overline{\partial}_{\varphi} = \begin{pmatrix} \overline{\partial} & \varphi_{C_2} & \varphi_{C_1+C_2} \\ 0 & \overline{\partial} & \varphi_{C_1} \\ 0 & 0 & \overline{\partial} \end{pmatrix}$$

with $[\varphi_{C_1}|_{C_1}] \neq 0$ *and* $[\varphi_{C_2}|_{C_2}] \neq 0$. *We replace* $\varphi_{C_1+C_2}$ *by* $\varphi_{C_1+C_2} + \psi$, *where* $\psi \in H^1(Y, O(C_1 + C_2)) \neq 0$. *If* $[\psi] \neq 0$, *then* $\overline{\partial}_{\varphi+\psi}$ *is not isomorphic to* $\overline{\partial}_{\varphi}$.

Remark 6.0.3. *Our $\mathfrak{g}$-bundle $\mathcal{E}_\eta^{\mathfrak{g}}$ over Y is given by $aut(\mathfrak{L}_\eta^{(\mathfrak{g},V)}, f)$ with f : $\bigotimes^r \mathfrak{L}_\eta^{(\mathfrak{g},V)} \longrightarrow O_Y(D)$. If $O(D) = O(rD')$ for some divisor D', then*

$$f : \bigotimes^r \mathfrak{L}_\eta^{(\mathfrak{g},V)}(-D') \longrightarrow O_Y.$$

And $Aut(\mathfrak{L}_\eta^{(\mathfrak{g},V)}(-D'), f)$ is a Lie group bundle over Y lifting $\mathcal{E}_\eta^{\mathfrak{g}}$. In general, we only have a $G \times \mathbb{Z}_r$-bundle, or so-called conformal G-bundle in [12].

Part II

Affine ADE bundles

Chapter 7

Affine ADE Lie algebra bundles

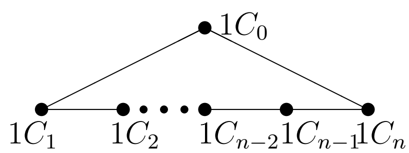
7.1 Affine ADE curves

Definition 7.1.1. *A curve $C = \cup C_i$ in a surface X is called an ADE (resp. affine ADE) curve of type $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) if each C_i is a smooth (-2)-curve in X and the dual graph of C is a Dynkin diagram of the corresponding type.*

It is known that C is an ADE curve if and only if C can be contracted to a rational double point. In this case, the intersection matrix $(C_i \cdot C_j) < 0$ [2].

If C is an affine ADE curve, then the intersection matrix $(C_i \cdot C_j) \leq 0$ and there exists unique n_i's up to overall scalings such that $F := \sum n_i C_i$ satisfies $F \cdot F = 0$. Dynkin diagrams of affine ADE types are drawn as follows and the corresponding n_iC_i's are labelled in the pictures. ADE Dynkin diagrams can be obtained by removing the node corresponding to C_0.

$\hat{A}_n$:



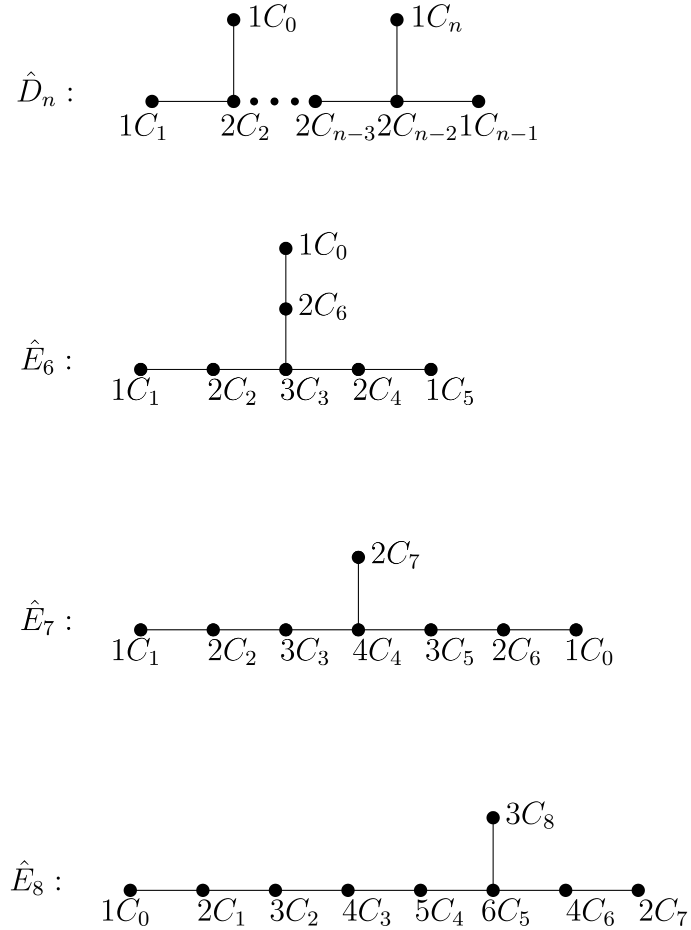


Figure 4. Dynkin diagrams of affine ADE types

Remark 7.1.1. *We will also call a nodal or cuspidal rational curve with trivial normal bundle an* $\widehat{A}_0$ *curve.*

Remark 7.1.2. *By Kodaira's classification of fibers of relative minimal elliptic surfaces [2], every singular fiber is an affine* ADE *curve unless it is rational with a cusp, tacnode or triplepoint (corresponding to type* II *or* $III(\widehat{A}_1)$ *or* $VI(\widehat{A}_2)$ *in Kodaira's notations), which can also be regarded as a degenerated affine* ADE *curve of type* $\widehat{A}_0$*,* $\widehat{A}_1$ *or* $\widehat{A}_2$ *respectively. In this thesis, we will not distinguish affine* ADE *curves from their degenerated forms since they have the same intersection matrices. We also call the affine* ADE *curves as Kodaira curves.*

Definition 7.1.2. *A bundle E is called an ADE (resp. affine ADE) bundle of type $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) if E has a fiberwise Lie algebra structure of the corresponding type.*

In the following section, we will recall an explicit construction of the loop Lie algebra $L\mathfrak{g}$-bundles and the affine Lie algebra $\widehat{\mathfrak{g}}$-bundles from affine ADE curves in X.

7.2 Affine ADE bundles

Suppose $C = \cup_{i=0}^r C_i$ is an affine ADE curve of type $\widehat{\mathfrak{g}}$ in X, we will construct the corresponding affine ADE bundle $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ of type $\widehat{\mathfrak{g}}$ over X as follows.

First, we choose an extended root of $\widehat{\mathfrak{g}}$, say C_0, then $\mathfrak{g}$ is corresponding to the Dynkin diagram consists of those C_i with $i \neq 0$, i.e. $\Phi := \{\alpha = [\sum_{i\neq 0} a_i C_i] \in H^2(X,\mathbb{Z}) | \alpha^2 = -2\}$ is the root system of $\mathfrak{g}$. As above, we have a $\mathfrak{g}$-bundle $\mathcal{E}_0^{(\mathfrak{g},\Phi)} = O^{\oplus r} \oplus \bigoplus_{\alpha\in\Phi} O(\alpha)$. We define

$$\mathcal{E}_0^{(L\mathfrak{g},\Phi)} := \bigoplus_{n\in\mathbb{Z}} (\mathcal{E}_0^{(\mathfrak{g},\Phi)} \otimes O(nF)) \text{ and } \mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)} := \bigoplus_{n\in\mathbb{Z}} (\mathcal{E}_0^{(\mathfrak{g},\Phi)} \otimes O(nF)) \oplus O.$$

We know $\Phi_{\widehat{\mathfrak{g}}} := \{\alpha + nF | \alpha \in \Phi, n \in \mathbb{Z}\} \cup \{nF | n \in \mathbb{Z}, n \neq 0\}$ is an affine root system and it decomposes into union of positive and negative roots, i.e. $\Phi_{\widehat{\mathfrak{g}}} = \Phi_{\widehat{\mathfrak{g}}}^+ \cup \Phi_{\widehat{\mathfrak{g}}}^-$, where $\Phi_{\widehat{\mathfrak{g}}}^+ = \{\sum a_i C_i \in \Phi_{\widehat{\mathfrak{g}}} | a_i \geq 0 \text{ for all } i\} = \{\alpha + nF | \alpha \in \Phi^+, n \in \mathbb{Z}_{\geq 0}\} \cup \{\alpha + nF | \alpha \in \Phi^-, n \in \mathbb{Z}_{\geq 1}\} \cup \{nF | n \in \mathbb{Z}_{\geq 1}\}$ and $\Phi_{\widehat{\mathfrak{g}}}^- = -\Phi_{\widehat{\mathfrak{g}}}^+$.

To describe the Lie algebra structures, we proceed as before, for every open chart U of X, we take a local basis e_i^U of $\mathcal{E}_0^{(\mathfrak{g},\Phi)}|_U$ (e_i^U is just h_j^U or x_α^U as above), e_{nF}^U of $O(nF)|_U$, e_c^U of $O|_U$, compatible with the tensor product, for example, $e_{nF}^U \otimes e_{mF}^U = e_{(n+m)F}^U$. Then define

$$[e_i^U e_{nF}^U, e_j^U e_{mF}^U]_{L\mathfrak{g},\Phi} := [e_i^U, e_j^U]_\Phi e_{(n+m)F}^U, \tag{7.1}$$

$$[e_i^U e_{nF}^U + \lambda e_c^U, e_j^U e_{mF}^U + \mu e_c^U]_{\widehat{\mathfrak{g}},\Phi} := [e_i^U, e_j^U]_\Phi e_{(n+m)F}^U + n\delta_{n+m,0} k(e_i^U, e_j^U) e_c^U. \tag{7.2}$$

Here $[\ ,\ ]_\Phi$ is the Lie bracket on $\mathcal{E}_0^{(\mathfrak{g},\Phi)}$ and $k(x,y) = Tr(adx\ ady)$ is the Killing form on $\mathfrak{g}$.

Lemma 7.2.1. (1) (*resp.* (2)) *defines a fiberwise loop* (*resp. affine*) *Lie algebra structure which is compatible with any trivialization of* $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ (*resp.* $\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}$).

Proof. See Proposition 23 of [21]. □

From the above lemma, we have the following result.

Proposition 7.2.1. *If C is an affine ADE curve of type $\widehat{\mathfrak{g}}$ in X, then* $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ (*resp.* $\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}$) *is a loop* (*resp. affine*) *Lie algebra bundle of type $L\mathfrak{g}$* (*resp.* $\widehat{\mathfrak{g}}$) *over X.*

Note any C_i with $n_i = 1$ can be chosen as the extended root (Appendix B).

Proposition 7.2.2. *The loop Lie algebra bundle* $(\mathcal{E}_0^{(L\mathfrak{g},\Phi)}, [\ ,\ ]_{L\mathfrak{g},\Phi})$ *does not depend on the choice of the extended root.*

Proof. Suppose C_k $(k \neq 0)$ is another root with $n_k = 1$, we denote $\Psi = \{\beta = [\sum_{i\neq k} b_i C_i] \in H^2(X,\mathbb{Z}) | \beta^2 = -2\}$, then Ψ is a root system of $\mathfrak{g}$. As before, we construct the Lie algebra bundle $\mathcal{E}_0^{(\mathfrak{g},\Psi)}$ and $\mathcal{E}_0^{(L\mathfrak{g},\Psi)}$ from Ψ.

We denote $\alpha_0 := \sum_{i\neq 0} n_i C_i = F - C_0$, the longest root in Φ. For any $\alpha = \sum_{i\neq 0} a_i(\alpha) C_i \in \Phi$, $a_k(\alpha)$ can only be 0, ± 1. Hence there is a bijection between Φ and Ψ given by $\alpha \mapsto \beta = \alpha - a_k(\alpha)F$. Then from the definitions of $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ and $\mathcal{E}_0^{(L\mathfrak{g},\Psi)}$, we know they are the same as holomorphic vector bundles.

We compare the Lie brackets on them. We choose a local basis of $\mathcal{E}_0^{(L\mathfrak{g},\Psi)}$ compatible with those of $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ and define $[,]_{L\mathfrak{g},\Psi}$ similarly as $[,]_{L\mathfrak{g},\Phi}$, i.e.

(i) when $\beta = \alpha \in \Phi \cap \Psi$, we take $x_\beta = x_\alpha$;

(ii) when $\beta = \alpha + F \in \Psi^+ \backslash \Phi$, we take $x_\beta = x_\alpha e_F$;

(iii) when $\beta = \alpha - F \in \Psi^- \backslash \Phi$, we take $x_\beta = x_\alpha e_{-F}$;

(iv) take h_i $(i \neq 0, k)$ as before, take $h_0 = -h_{\alpha_0}$ as we want $[x_{C_0}, x_{-C_0}]_{L\mathfrak{g},\Psi} = [x_{-\alpha_0+F}, x_{\alpha_0-F}]_{L\mathfrak{g},\Phi}$.

It is obvious $[\ ,\ ]_{L\mathfrak{g},\Psi} = [\ ,\ ]_{L\mathfrak{g},\Phi}$ on $\mathcal{E}_0^{(L\mathfrak{g},\Psi)} \cong \mathcal{E}_0^{(L\mathfrak{g},\Phi)}$. $\square$

For the affine case, we recall that the Killing form of $\mathfrak{g}$ is the symmetric bilinear map $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $k(x, y) = Tr(adx\ ady)$. It is ad-invariant, that is for $x, y, z \in \mathfrak{g}$, $k([x, y], z) = k(x, [y, z])$.

Lemma 7.2.2. *For any simple simply-laced Lie algebra $\mathfrak{g}$ with a Chavelly basis $\{x_\alpha, \alpha \in \Phi; h_i, 1 \leq i \leq r\}$ and $m^*(\mathfrak{g})$ the dual Coxeter number of $\mathfrak{g}$, we have*

(i) $k(h_i, x_\alpha) = 0$ for any i and α;

(ii) $k(x_\alpha, x_\beta) = 0$ for any $\alpha + \beta \neq 0$;

(iii) $k(h_i, h_j) = 2m^*(\mathfrak{g})\langle C_i, C_j\rangle$;

(iv) $k(x_\alpha, x_{-\alpha}) = 2m^*(\mathfrak{g})$ for any α.

Proof. Directly from the Killing form k being ad-invariant or see [28]. $\square$

Proposition 7.2.3. *The affine Lie algebra bundle $(\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}, [\ ,\ ]_{\widehat{\mathfrak{g}},\Phi})$ does not depend on the choice of the extended root.*

Proof. Follow the notations in Proposition 7.2.2, but we will take $h_0 = -h_{\alpha_0} + 2m^*(\mathfrak{g})e_c$. We will check that $[\ ,\ ]_{\widehat{\mathfrak{g}},\Psi} = [\ ,\ ]_{\widehat{\mathfrak{g}},\Phi}$ on $\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Psi)} = \mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}$:

(a) when $\beta_1 = \alpha_1 + F, \beta_2 = \alpha_2 + F \in \Psi^+\backslash\Phi$, $\alpha_1, \alpha_2 \in \Phi^-\backslash\Psi$ we have

$$[h_{\beta_1}e_{nF}, h_{\beta_2}e_{mF}]_{\widehat{\mathfrak{g}},\Psi} = n\delta_{n+m,0}k(h_{\beta_1}, h_{\beta_2})e_c,$$

which is the same with

$$[h_{-\alpha_1}e_{nF}, h_{-\alpha_2}e_{mF}]_{\widehat{\mathfrak{g}},\Phi} = n\delta_{n+m,0}k(h_{\alpha_1}, h_{\alpha_2})e_c,$$

since $k(h_{\beta_1}, h_{\beta_2}) = 2m^*(\mathfrak{g})\langle\beta_1, \beta_2\rangle = 2m^*(\mathfrak{g})\langle F - \alpha_1, F - \alpha_2\rangle = k(h_{\alpha_1}, h_{\alpha_2})$.

(b) For $[h_ie_{nF}, x_\alpha e_{mF}]_{\widehat{\mathfrak{g}},\Phi}$, automatically from $k(h_i, x_\alpha) = 0$ and loop case.

(c) When $\beta = \alpha + F \in \Psi^+\backslash\Phi$, $\alpha \in \Phi^-\backslash\Psi$,

$$[x_\beta e_{nF}, x_{-\beta}e_{mF}]_{\widehat{\mathfrak{g}},\Psi} = h_\beta e_{(n+m)F} + n\delta_{n+m,0}k(x_\beta, x_{-\beta})e_c,$$

which is the same with

$$[x_{-\alpha}e_{(n+1)F}, x_\alpha e_{(m-1)F}]_{\widehat{\mathfrak{g}},\Phi} = -h_\alpha e_{(n+m)F} + (n+1)\delta_{n+m,0}k(x_\alpha, x_{-\alpha})e_c,$$

by considering $m+n=0$ and $m+n \neq 0$ separately.

(d) For $[x_{\alpha_1}e_{nF}, x_{\alpha_2}e_{mF}]_{\widehat{\mathfrak{g}},\Phi}$ with $\alpha_1+\alpha_2 \neq 0$, automatically from $k(x_{\alpha_1}, x_{\alpha_2}) = 0$ and loop case. □

For simplicity, we will omit Φ in $(\mathfrak{g},\Phi)$, $(L\mathfrak{g},\Phi)$ and $(\widehat{\mathfrak{g}},\Phi)$ when there is no confusion.

Chapter 8

Trivialization of $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ over C_i's after deformations

If $C = \cup C_i$ is an affine ADE curve in X, then the corresponding $F = \sum n_i C_i$ satisfies $F \cdot F = 0$, i.e. $O_F(F)$ is a topologically trivial bundle. If $O_F(F)$ is trivial holomorphically and $q(X) = 0$, then from the long exact sequence of cohomologies induced by $0 \to O_X \to O_X(F) \to O_F(F) \to 0$, we know $H^0(X, O_X(F)) \cong \mathbb{C}^2$. Hence F is a fiber of an elliptic fibration on X.

Suppose X is an elliptic surface, i.e. there is a smooth curve B and a surjective morphism $\pi : X \to B$ whose generic fiber F_b ($b \in B$) is an elliptic curve. Assume π is singular at $b_0 \in B$ and $F_{b_0} = \sum n_i C_i$ is a singular fiber of type $\widehat{\mathfrak{g}}$. Hence, we have a $\widehat{\mathfrak{g}}$-bundle $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ over X. The restriction of $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ to any fiber F_b, other than F_{b_0}, is trivial because $F_b \cap C_i = \varnothing$ for any i. However, $\mathcal{E}_0^{\widehat{\mathfrak{g}}}|_{F_{b_0}}$ is not trivial, for instance $O(-C_i)|_{C_i} \cong O_{\mathbb{P}^1}(2)$. Nevertheless, we will show that after deformations of holomorphic structures, $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ will become trivial on every irreducible component of F_{b_0}.

8.1 Trivializations in loop ADE cases

In part I, we showed how to take successive extensions to make the $\mathfrak{g}$-bundle $\mathcal{E}_0^{\mathfrak{g}}$ trivial on every component C_i of the ADE curve $C = \cup_{i=1}^{r} C_i$. We will use the similar method in loop ADE case.

Definition 8.1.1. *Given any* $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+_{\widehat{\mathfrak{g}}}} \in \Omega^{0,1}(X, \bigoplus_{\alpha \in \Phi^+_{\widehat{\mathfrak{g}}}} O(\alpha))$*, we define* $\overline{\partial}_{(\varphi,\Phi)} : \Omega^{0,0}(X, \mathcal{E}_0^{L\mathfrak{g}}) \longrightarrow \Omega^{0,1}(X, \mathcal{E}_0^{L\mathfrak{g}})$ *by* $\overline{\partial}_{(\varphi,\Phi)} := \overline{\partial}_0 + ad(\varphi)$.

More explicitly, similarly as explained in section 1.2, we have

$$\begin{aligned} \overline{\partial}_{(\varphi,\Phi)} \ :\ &= \overline{\partial}_0 + \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\alpha \in \Phi^+} (c_{\alpha+nF} e_{nF} ad(x_\alpha) + c_{-\alpha+(n+1)F} e_{(n+1)F} ad(x_{-\alpha})) \\ &+ \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{i=1}^{r} c^i_{(n+1)F} e_{(n+1)F} ad(h_i), \end{aligned}$$

Proposition 8.1.1. $\overline{\partial}_{(\varphi,\Phi)}$ *is compatible with the Lie algebra structure on* $\mathcal{E}_0^{L\mathfrak{g}}$.

Proof. $\overline{\partial}_{(\varphi,\Phi)}[\ ,\ ]_{L\mathfrak{g},\Phi} = 0$ follows directly from the Jacobi identity. □

For $\overline{\partial}_{(\varphi,\Phi)}$ to define a holomorphic structure, we need $\overline{\partial}^2_{(\varphi,\Phi)} = 0$, which is equivalent to the following equations:

$$\begin{cases} \overline{\partial}_0 \varphi^i_{nF} = \sum_{p+q=n} \sum_{\alpha \in \Phi^+} \pm a_i(h_\alpha) \varphi_{\alpha+pF} \varphi_{-\alpha+qF}, \\ \\ \overline{\partial}_0 \varphi_{\alpha+nF} = \sum_{p+q=n} (\sum_{\alpha_1+\alpha_2=\alpha} \pm \varphi_{\alpha_1+pF} \varphi_{\alpha_2+qF} + \sum_{i=1}^{r} \langle \alpha, C_i \rangle \varphi_{\alpha+pF} \varphi^i_{qF}), \\ \\ \overline{\partial}_0 \varphi_{-\alpha+nF} = \sum_{p+q=n} (\sum_{\alpha_2-\alpha_1=\alpha} \pm \varphi_{\alpha_1+pF} \varphi_{-\alpha_2+qF} + \sum_{i=1}^{r} \langle -\alpha, C_i \rangle \varphi_{-\alpha+pF} \varphi^i_{qF}), \end{cases}$$

where $a_i(h_\alpha)$ is the coefficient of h_i in h_α.

Proposition 8.1.2. *Given any* $(\varphi_{C_i})_{i=0}^{r} \in \Omega^{0,1}(X, \bigoplus_{i=0}^{r} O(C_i))$ *with* $\overline{\partial} \varphi_{C_i} = 0$ *for every* i*, it can be extended to* $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+_{\widehat{\mathfrak{g}}}} \in \Omega^{0,1}(X, \bigoplus_{\alpha \in \Phi^+_{\widehat{\mathfrak{g}}}} O(\alpha))$ *satisfying* $\overline{\partial}^2_\varphi = 0$*. Namely we have a holomorphic* $L\mathfrak{g}$*-bundle* $\mathcal{E}_\varphi^{L\mathfrak{g}}$ *over* X.

In order to prove this proposition, we need the following lemma.

Lemma 8.1.1. *If $p_g(X) = 0$, then for any $\alpha \in \Phi^+$, $n \in \mathbb{Z}_{\geq 0}$, $H^2(X, O(nF))$, $H^2(X, O(\alpha + nF))$ and $H^2(X, O(-\alpha + (n+1)F))$ are zeros.*

Proof. Since F is an effective divisor and $H^0(X, K_X) = 0$, we have for any $n \geq 0$, $H^0(X, K_X(-nF)) = 0$. This is equivalent to $H^2(X, O(nF)) = 0$ by Serre duality. Similarly, $H^2(X, O(\alpha+nF)) = 0$ follows from $H^0(X, K_X(-\alpha)) \cong H^2(X, O(\alpha)) = 0$ (Lemma 1.2.1). The proof of $H^2(X, O(-\alpha + (n+1)F)) = 0$ uses the fact that $F - \alpha$ is an effective divisor for any $\alpha \in \Phi^+$. $\square$

Proof. (of Proposition 8.1.2): the equation $\overline{\partial}^2_{(\varphi,\Phi)} = 0$ can be rewritten as follows:

$$\begin{cases} \overline{\partial}_0 \varphi_{C_i} = 0 \text{ for } i = 1, 2 \cdots, r, \\ \overline{\partial}_0 \varphi_\alpha = \sum_{\alpha_1+\alpha_2=\alpha} (\pm \varphi_{\alpha_1} \varphi_{\alpha_2}), \\ \overline{\partial}_0 \varphi_{-\alpha_0+F} = \overline{\partial}_0 \varphi_{C_0} = 0, \\ \overline{\partial}_0 \varphi_{-\alpha+F} = \sum_{\alpha_2-\alpha_1=\alpha} (\pm \varphi_{\alpha_1} \varphi_{-\alpha_2+F}), \\ \overline{\partial}_0 \varphi^i_F = \sum_{\alpha \in \Phi^+} (\pm a_i(h_\alpha) \varphi_\alpha \varphi_{-\alpha+F}), \\ \vdots \end{cases}$$

where $\alpha_0 = F - C_0$ is the longest root in Φ.

Firstly, we can solve for all the φ_α's, $\alpha \in \Phi^+$ from $H^2(X, O(\alpha)) = 0$ (Proposition 1.2.2). Secondly, we get all the $\varphi_{-\alpha+F}$'s, $\alpha \in \Phi^+$ from $H^2(X, O(-\alpha + F)) = 0$. Thirdly, since we have all the φ_α's and $\varphi_{-\alpha+F}$'s, we can solve for all the φ^i_F's for $1 \leq i \leq r$ from $H^2(X, O(F)) = 0$. Do this process for $\varphi_{\alpha+nF}$, $\varphi_{-\alpha+(n+1)F}$ and $\varphi^i_{(n+1)F}$ inductively on n. $\square$

By Lemma 1.2.1, there always exists $\varphi_{C_i} \in \Omega^{0,1}(X,\ O(C_i))$ such that $0 \neq [\varphi_{C_i}|_{C_i}] \in H^1(X,\ O_{C_i}(C_i)) \cong \mathbb{C}$ for each $i = 0, 1, \cdots r$.

Theorem 8.1.1. *For any given i, the holomorphic $L\mathfrak{g}$-bundle $\mathcal{E}^{L\mathfrak{g}}_\varphi$ over X is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$.*

Proof. The proof will be given in section 8.3 and 8.4. In section 8.3, we deal with all the loop ADE cases except loop E_8 case which will be analyzed in section 8.4. $\square$

8.2 Trivializations in affine ADE cases

Follow the notations in section 8.1, we define $\overline{\partial}_{(\varphi,\Phi)} := \overline{\partial}_0 + ad(\varphi)$ on $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$, note the adjoint action here is defined using the affine Lie bracket.

Proposition 8.2.1. *$\overline{\partial}_{(\varphi,\Phi)}$ is compatible with the Lie algebra structure on $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$.*

Proof. $\overline{\partial}_{(\varphi,\Phi)}[\ ,\ ]_{\widehat{\mathfrak{g}},\Phi} = 0$ follows directly from the Jacobi identity and the Killing from being invariant under the adjoint action. $\square$

It is easy to see that $\overline{\partial}^2_{(\varphi,\Phi)} = 0$ in the affine case is equivalent to $\overline{\partial}^2_{(\varphi,\Phi)} = 0$ in the loop case. Hence we have a new holomorphic structure $\overline{\partial}_{(\varphi,\Phi)}$ on $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$.

Theorem 8.2.1. *For any given i, the holomorphic $\widehat{\mathfrak{g}}$-bundle $\mathcal{E}_\varphi^{\widehat{\mathfrak{g}}}$ over X is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$.*

Proof. This follows from Theorem 8.1.1, $0 \to O \to E_\varphi^{\widehat{\mathfrak{g}}} \to E_\varphi^{L\mathfrak{g}} \to 0$ and $Ext^1_{\mathbb{P}^1}(O,O) = 0$. $\square$

8.3 Proof (except the loop E_8 case)

In this section, we use the symmetry of the affine ADE Dynkin diagram (except $\widehat{E}_8$) to show that $\mathcal{E}_\varphi^{L\mathfrak{g}}$ is trivial on C_i if an only if $[\varphi_{C_i}|_{C_i}] \neq 0$.

Recall, topologically, $\mathcal{E}_\varphi^{L\mathfrak{g}}$ is $\mathcal{E}_0^{L\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}}(\mathcal{E}_0^{(\mathfrak{g},\Phi)} \otimes O(nF))$, but with a holomorphic structure $\overline{\partial}_{(\varphi,\Phi)}$ of the following upper triangular block shape:

$$\overline{\partial}_\varphi = \left(\begin{array}{c|c|c|c|c} \ddots & \ddots & \ddots & \ddots & \ddots \\ \hline \ddots & \overline{\partial}_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)}\otimes O((n+1)F)} & * & * & \ddots \\ \hline \ddots & 0 & \overline{\partial}_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)}\otimes O(nF)} & * & \ddots \\ \hline \ddots & 0 & 0 & \overline{\partial}_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)}\otimes O((n-1)F)} & \ddots \\ \hline \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right).$$

i.e. $\mathcal{E}_\varphi^{L\mathfrak{g}}$ is constructed from successive extensions of $\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)} \otimes O(nF)$'s.

Note $\overline{\partial}_{(\varphi,\Phi)}|_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)}} = \overline{\partial}_0 + \sum_{\alpha\in\Phi^+} ad(\varphi_\alpha)$. By Theorem 1.2.1, for every $i \neq 0$, $\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$. We also know $O(F)|_{C_i}$ is trivial for every i because $F \cdot C_i = 0$. Thus, when $i \neq 0$, $\mathcal{E}_\varphi^{L\mathfrak{g}}|_{C_i}$ is constructed from successive extensions of trivial vector bundles over $C_i \cong \mathbb{P}^1$. This implies that $\mathcal{E}_\varphi^{L\mathfrak{g}}|_{C_i}$ is trivial if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$ as $Ext^1_{\mathbb{P}^1}(O, O) = 0$.

Now we consider $i = 0$. Since $\widehat{\mathfrak{g}} \neq \widehat{E}_8$, the affine Dynkin diagram always admits a diagram automorphism, that means we can write $\mathcal{E}_0^{L\mathfrak{g}}$ as $\bigoplus_{n\in\mathbb{Z}}(\mathcal{E}_0^{(\mathfrak{g},\Psi)} \otimes O(nF))$ (see Proposition 7.2.2). Suppose the extended root corresponding to Ψ is C_k, and the longest root in Ψ is β_0.

We will rewrite the holomorphic structure $\overline{\partial}_{(\varphi,\Phi)}$ in terms of the Ψ root system. Note $\overline{\partial}_{(\varphi,\Phi)}$ is determined by the loop Lie algebra structure which is independent of the choice of the extended root. We choose a local base of $\mathcal{E}_0^{(\mathfrak{g},\Psi)}$ as in Proposition 7.2.2 and define $\overline{\partial}_{(\psi,\Psi)}$ to be the same with $\overline{\partial}_{(\varphi,\Phi)}$, then obviously $\psi_D = \varphi_D$ when $D \neq nF$.

Because $(\mathcal{E}_\varphi^{(L\mathfrak{g},\Phi)}, \overline{\partial}_{(\varphi,\Phi)}) = (\mathcal{E}_\psi^{(L\mathfrak{g},\Psi)}, \overline{\partial}_{(\psi,\Psi)})$ as a holomorphic vector bundle, similar to the arguments in $(\mathcal{E}_\varphi^{(L\mathfrak{g},\Phi)}, \overline{\partial}_{(\varphi,\Phi)})$ case, we have when $i \neq k$, $\mathcal{E}_\varphi^{L\mathfrak{g}}$ is trivial on C_i if and only if $[\psi_{C_i}|_{C_i}] \neq 0$. Note $\psi_{C_0} = \varphi_{-\alpha_0+F} = \varphi_{C_0}$. So we have Theorem 8.1.1 when $\mathfrak{g} \neq E_8$.

8.4 Proof for the loop E_8 case

Similar to the above section, we have when $i = 1, 2, \cdots 8$, $\mathcal{E}_\varphi^{LE_8}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$. The question is what about C_0?

We recall $\mathcal{E}_0^{E_8} := O^{\oplus 8} \oplus \bigoplus_{\alpha\in\Phi} O(\alpha)$. For any $\alpha \in \Phi$, we write $a_1(\alpha)$ as the coefficient of C_1 in α, then $O(\alpha)|_{C_0} \cong O_{\mathbb{P}^1}(a_1(\alpha))$. Among Φ^+, there are 63 roots with $a_1(\alpha) = 0$, corresponding to the positive roots of the Lie sub-algebra E_7; 56 roots with $a_1(\alpha) = 1$, corresponding to weights of the standard representation of E_7; 1 root with $a_1(\alpha) = 2$, which is just the longest root $\alpha_0 = F - C_0$. We denote $\mathcal{E}_0^{E_7} \triangleq O^{\oplus 7} \oplus \bigoplus_{\alpha\in\Phi, a_1(\alpha)=0} O(\alpha)$, $V_0^+ \triangleq \bigoplus_{\alpha\in\Phi,a_1(\alpha)=1} O(\alpha)$ and $V_0^- \triangleq \bigoplus_{\alpha\in\Phi,a_1(\alpha)=-1} O(\alpha)$, then

$$\mathcal{E}_0^{E_8} = \mathcal{E}_0^{E_7} \oplus O \oplus V_0^+ \oplus V_0^- \oplus O(\alpha_0) \oplus O(-\alpha_0).$$

When $O(\alpha)$ is a summand of V_0^+, i.e. $O(\alpha)|_{C_0} \cong O_{\mathbb{P}^1}(1)$, we have $O(\alpha + C_0)|_{C_0} \cong O_{\mathbb{P}^1}(-1)$ and $\alpha + C_0 = F - (\alpha_0 - \alpha)$ with $(\alpha_0 - \alpha) \in \Phi^+$, that is $O(\alpha + C_0)$ is a summand of $V_0^-(F)$. Since $F = \alpha_0 + C_0$ satisfies $F \cdot F = 0$, we have $O(F)|_{C_0} \cong O_{\mathbb{P}^1}$, $O(\alpha_0)|_{C_0} \cong O_{\mathbb{P}^1}(2)$ and $O(2F - \alpha_0)|_{C_0} \cong O_{\mathbb{P}^1}(-2)$.

For the loop E_8-bundle, we have

$$\begin{aligned}
\mathcal{E}_0^{LE_8} &= \bigoplus_{n\in\mathbb{Z}} (\mathcal{E}_0^{E_8} \otimes O(nF)) \\
&= \bigoplus_{n\in\mathbb{Z}} ((\mathcal{E}_0^{E_7} \oplus O \oplus V_0^+ \oplus V_0^- \oplus O(\alpha_0) \oplus O(-\alpha_0)) \otimes O(nF)) \\
&= \bigoplus_{n\in\mathbb{Z}} ((\mathcal{E}_0^{E_7} \oplus O \oplus V_0^+ \oplus V_0^-(F) \oplus O(\alpha_0 - F) \oplus O(F - \alpha_0)) \otimes O(nF)).
\end{aligned}$$

We denote $L_0^{248} \triangleq \mathcal{E}_0^{E_7} \oplus O \oplus V_0^+ \oplus V_0^-(F) \oplus O(\alpha_0 - F) \oplus O(F - \alpha_0)$. From definition of $\overline{\partial}_\varphi$, $\mathcal{E}_\varphi^{LE_8}$ is built from successive extensions of $L_\varphi^{248} \otimes O(nF)$'s, i.e.

$$\overline{\partial}_\varphi = \left(\begin{array}{c|c|c|c}
\ddots & \ddots & \ddots & \ddots \\ \hline
\ddots & \overline{\partial}_{L_\varphi^{248}\otimes O((n+1)F)} & * & \ddots \\ \hline
\ddots & 0 & \overline{\partial}_{L_\varphi^{248}\otimes O(nF)} & \ddots \\ \hline
\ddots & \ddots & \ddots & \ddots
\end{array}\right).$$

So if we can prove $[\varphi_{C_0}|_{C_0}] \neq 0$ implies $(L_\varphi^{248}, \overline{\partial}_\varphi|_{L_\varphi^{248}})$ is trivial over C_0, then $(\mathcal{E}_\varphi^{LE_8}, \overline{\partial}_\varphi)$ is also trivial over C_0 because of $Ext^1_{\mathbb{P}^1}(O,O) = 0$. Note

$$L_0^{248}|_{C_0} \cong O_{\mathbb{P}^1}^{\oplus 133} \oplus O_{\mathbb{P}^1} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 56} \oplus O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(-2).$$

In this decomposition, any of the 56 pairs of $\{O_{\mathbb{P}^1}(-1), O_{\mathbb{P}^1}(1)\}$ is the restriction of $\{O(\alpha), O(\alpha + C_0) = O(F - (\alpha_0 - \alpha))\}$ to C_0 for some α with $a_1(\alpha) = 1$ and the triple $\{O_{\mathbb{P}^1}(2), O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(-2)\}$ is the restriction of $\{O(-C_0), O, O(C_0)\}$ to C_0. We will show that the restriction of $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ to C_0 gives a non-trivial extension for each of these pairs $\{O_{\mathbb{P}^1}(-1), O_{\mathbb{P}^1}(1)\}$'s and the triple $\{O_{\mathbb{P}^1}(-2), O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(2)\}$.

In order to write $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ in matrix form, we need to decompose $\mathcal{E}_0^{E_7}$ into positive parts and non-positive parts, i.e. we denote $\mathcal{E}_0^{(E_7,+)} := \bigoplus_{\alpha \in \Phi^+, a_1(\alpha)=0} O(\alpha)$ and $\mathcal{E}_0^{(E_7,-)} := O^{\oplus 7} \oplus \bigoplus_{\alpha \in \Phi^-, a_1(\alpha)=0} O(\alpha)$. Then $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ can be written as follows: ($\overline{\partial}_\varphi|_{L_\varphi^{248}}$ is a upper triangle matrix since $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ maps any line bundle summand to other more "positive" line bundle summands, i.e. $\overline{\partial}_\varphi : O(D) \to O(D')$ is nonzero only if $D' - D \geq 0$)

$$\overline{\partial}_\varphi|_{L_\varphi^{248}} = \left(\begin{array}{c|c|c|c|c|c|c}
\overline{\partial}_{V_\varphi^-(F)} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ \hline
0 & \overline{\partial}_{O(F-\alpha_0)} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ \hline
0 & 0 & \overline{\partial}_{V_\varphi^+} & A_{34} & A_{35} & A_{36} & A_{37} \\ \hline
0 & 0 & 0 & \overline{\partial}_{\mathcal{E}_\varphi^{(E_7,+)}} & A_{45} & A_{46} & A_{47} \\ \hline
0 & 0 & 0 & 0 & \overline{\partial}_O & A_{56} & A_{57} \\ \hline
0 & 0 & 0 & 0 & 0 & \overline{\partial}_{\mathcal{E}_\varphi^{(E_7,-)}} & A_{67} \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \overline{\partial}_{O(\alpha_0-F)}
\end{array}\right).$$

Now we restrict this to C_0, the 56 pairs $\{O_{\mathbb{P}^1}(-1), O_{\mathbb{P}^1}(1)\}$'s are in $V_0^-(F)|_{C_0} \oplus V_0^+|_{C_0}$. Since $A_{23} = (0, 0, \cdots, 0)_{56\times 1}$ and

$$A_{13} = \begin{pmatrix} \pm\varphi_{C_0} & * & \cdots & * \\ 0 & \pm\varphi_{C_0} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm\varphi_{C_0} \end{pmatrix}_{56\times 56},$$

if $[\varphi_{C_0}|_{C_0}] \neq 0$, then we have a trivialization of the 56 pairs $\{O_{\mathbb{P}^1}(-1), O_{\mathbb{P}^1}(1)\}$'s over C_0 by Lemma 3.1.3 in section 3.1.

For the triple $\{O_{\mathbb{P}^1}(-2), O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(2)\}$, we review the trivialization of A_1 Lie algebra bundle. In A_1 case, we have an A_1-bundle $\mathcal{E}_{\varphi}^{A_1}$, which topologically is $\mathcal{E}_0^{A_1} = O \oplus O(C) \oplus O(-C)$, but with a holomorphic structure as follows:

$$\overline{\partial}_{\varphi} = \left(\begin{array}{c|c|c} \overline{\partial}_0 & \pm\varphi_C & 0 \\ \hline 0 & \overline{\partial}_0 & \pm\varphi_C \\ \hline 0 & 0 & \overline{\partial}_0 \end{array} \right),$$

where $\varphi_C \in H^{0,1}(X, O(C))$. From Part I, we know if $[\varphi_C|_C] \neq 0$, then $\mathcal{E}_{\varphi}^{A_1}$ is trivial on C. Back to our case, the triple $\{O_{\mathbb{P}^1}(-2), O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(2)\}$ has the corresponding submatrices $A_{25} = (\varphi_{C_0})_{1\times 1}$, $A_{57} = (\varphi_{C_0})_{1\times 1}$ and $A_{27} = (0)_{1\times 1}$. Since A_{23}, A_{24}, A_{26}, A_{47} and A_{67} are all zero matrices, from the trivialization of A_1 Lie algebra bundle, we know if $[\varphi_{C_0}|_{C_0}] \neq 0$, then we have a trivialization of the triple $\{O_{\mathbb{P}^1}(-2), O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(2)\}$ over C_0.

Hence if $[\varphi_{C_0}|_{C_0}] \neq 0$, then $(L_{\varphi}^{248}, \overline{\partial}_{\varphi}|_{L_{\varphi}^{248}})$ is trivial on C_0, which implies $(\mathcal{E}_{\varphi}^{LE_8}, \overline{\partial}_{\varphi})$ is also trivial on C_0. Hence, we have Theorem 8.1.1 for LE_8 case.

Part III

Deformability

Chapter 9

E_n-bundle over X_n with $n \leq 9$

When $X = X_n$ is a blowup of $\mathbb{P}^2$ at n points $x_1, \cdots, x_n$ with $n \leq 9$, there is a canonical (affine) Lie algebra bundle $\mathcal{E}_0^{E_n}$ over it, where E_9 is the affine E_8. In this chapter, we will give a detail study of the relationship between the geometry of X_n and the deformability of $\mathcal{E}_0^{E_n}$.

9.1 E_n-bundle over X_n with $n \leq 9$

The Picard group $Pic(X_n) \cong H^2(X_n, \mathbb{Z})$ is a rank $n+1$ lattice with generators $h, l_1, \cdots, l_n$, where h is the class of lines in $\mathbb{P}^2$ and l_i is the exceptional class of the blow-up at x_i. So $h^2 = 1 = -l_i^2$ and $h \cdot l_i = 0 = l_i \cdot l_j$, $i \neq j$. Thus $H^2(X_n, \mathbb{Z}) \cong \mathbb{Z}^{1,n}$. The canonical class is $K_{X_n} = -3h + l_1 + \cdots + l_n$. Denote

$$\Phi_n := \{\alpha \in H^2(X_n, \mathbb{Z}) | \alpha^2 = -2, \alpha \cdot K = 0\}.$$

Then Φ_n is a root system of type E_n when $n \leq 8$ and Φ_9 is an affine real root system of $\widehat{E}_8$ (also denoted as E_9). More explicitly, $\Phi_{\widehat{E}_8} := \Phi_9 \cup \{mK_{X_9} | m \neq 0, m \in \mathbb{Z}\}$ forms a root system of (untwisted) affine E_8-type (that is, $\widehat{E}_8$-type) with $\Phi_{\widehat{E}_8}^{re} := \Phi_9$ the set of real roots and $\Phi_{\widehat{E}_8}^{im} := \{mK_{X_9} | m \neq 0, m \in \mathbb{Z}\}$ the set of imaginary roots (see [16] or [21]). We have an $\widehat{E}_8$-bundle $\mathcal{E}_0^{\widehat{E}_8}$ over X_9:

$$\mathcal{E}_0^{\widehat{E}_8} = O^{\oplus 9} \oplus \bigoplus_{\alpha \in \Phi^{re}_{\widehat{E}_8}} O(\alpha) \bigoplus_{\beta \in \Phi^{im}_{\widehat{E}_8}} O(\beta)$$

The Lie algebra structure on $\mathcal{E}_0^{\widehat{E}_8}$ is explained in [21]. When $n \leq 8$, $\mathcal{E}_0^{E_n} = O^{\oplus n} \oplus \bigoplus_{\alpha \in \Phi_n} O(\alpha)$ is an E_n-bundle over X_n.

Suppose $C = \cup C_i$ is an (affine) ADE curve of type $\mathfrak{g}$ in X_n, then C_i's generates a subroot system Φ inside Φ_n since $C_i \cdot K = 0$ for every i. Therefore the corresponding bundle $\mathcal{E}_0^{\mathfrak{g}}$ is a Lie algebra subbundle of $\mathcal{E}_0^{E_n}$.

Suppose $\mathcal{E}_0^{\mathfrak{g}}$ is a $\mathfrak{g}$-bundle over a surface X corresponding to a root system $\Lambda_{\mathfrak{g}} \subset Pic(X)$ of type $\mathfrak{g}$.

Definition 9.1.1. *A Lie algebra sub-bundle $\mathcal{F}$ of $\mathcal{E}_0^{\mathfrak{g}}$ is called strict if there exists a sub-root lattice Λ of $\Lambda_{\mathfrak{g}}$ such that $\mathcal{F}$ is a direct sum of line bundles corresponding to the roots in Λ.*

In order to describe $\mathcal{E}_0^{\widehat{E}_8}$ as a central extension of a loop Lie algebra bundle over X_9, we pick any smooth (-1)-curve l in X_9, then we have

$$\mathcal{E}_0^{\widehat{E}_8} \cong \mathcal{E}_0^{E_8} \otimes (\bigoplus_{n \in \mathbb{Z}} O(nK_{X_9})) \oplus O,$$

where $\mathcal{E}_0^{E_8}$ is the pull-back of the E_8-bundle over X_8 via $\pi : X_9 \to X_8$, the blow down map of l. The next proposition describes the converse.

Proposition 9.1.1. *When $\mathcal{E}_0^{\widehat{E}_8}$ is a central extension of a loop E_8-sub-bundle over X for some strict E_8-bundle $\mathcal{F}_0^{E_8}$ over X_9, i.e.*

$$\mathcal{E}_0^{\widehat{E}_8} \cong \mathcal{F}_0^{E_8} \otimes (\bigoplus_{n \in \mathbb{Z}} O(nK_{X_9})) \oplus O,$$

as a Lie algebra bundle isomorphism, then there is a unique (possibly reducible) (-1)-curve l in X such that $\mathcal{F}_0^{E_8}$ is constructed from those $\alpha \in \Lambda^{re}$ satisfying $\alpha \cdot l = 0$.

Proof. Denote $\Delta_{E_8} = \{\alpha_1, \cdots, \alpha_8\}$ as a root base of the corresponding E_8 Lie algebra from $\mathcal{F}_0^{E_8}$, we need to find a unique (-1)-curve l in X such that $l \cdot \alpha_i = 0$ for any α_i in Δ_{E_8}. Since $\{\pm 1\} \times W(\widehat{E}_8)$ acts on the set of all root bases of $\widehat{E}_8$ simply transitively [18] and $W(\widehat{E}_8)$ acts on the set of (-1)-curves [21], we only need to find l for one particular root base of any E_8 in $\widehat{E}_8$ and show that such a l is unique. For example, if we take $\alpha_1 = h - l_1 - l_2 - l_3, \alpha_k = l_{k-1} - l_k$ for $k = 2, \cdots 8$, then we can take $l = l_9$ and by the condition that $l \cdot \alpha_i = 0$, $l^2 = -1 = l \cdot K$, we know such a l is unique. □

9.2 Deformability of such $\mathcal{E}_0^{\widehat{E}_8}$

In this section, we will describe relationships between the geometry of X_9 and the deformability of $\mathcal{E}_0^{\widehat{E}_8}$. Similar results for X_n and $\mathcal{E}_0^{E_n}$ with $n \leq 8$ can be easily deduced from this case.

Recall when $Pic(X)$ contains a lattice Λ isomorphic to a root lattice $\Lambda_{\mathfrak{g}}$, then we have a $\mathfrak{g}$-bundle $\mathcal{E}$ over X ([8][13][22][23][21]).

$$\mathcal{E} := O^{\oplus r} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha).$$

Infinitesimal deformations of holomorphic structures on $\mathcal{E}$ are parametrized by $H^1(X, End(\mathcal{E}))$, and those which also preserve the Lie algebra structure are parametrized by $H^1(X, ad(\mathcal{E})) = H^1(X, \mathcal{E})$ since $\mathfrak{g}$ is simple. Hence we introduce the following definitions.

Definition 9.2.1. *(i) $\mathcal{E}$ is called fully deformable if there exists a base $\Delta \subset \Phi$ such that $H^1(X, O(\alpha)) \neq 0$ for any $\alpha \in \Delta$.*

(ii) $\mathcal{E}$ is called $\mathfrak{h}$-deformable if there exists a strict $\mathfrak{h}$ Lie algebra sub-bundle $\mathcal{E}^{\mathfrak{h}} \subseteq \mathcal{E}$ which is fully deformable.

(iii) $\mathcal{E}$ is called deformable in α-direction for $\alpha \in \Phi$ if $H^1(X, O(\alpha)) \neq 0$.

(iv) $\mathcal{E}$ is called totally non-deformable if $H^1(X, O(\alpha)) = 0$ for any $\alpha \in \Phi$.

Recall the holomorphic structure $\overline{\partial}_\varphi$ or $\overline{\partial}_{(\varphi,\Phi)}$ defined as before on $\mathcal{E}$ admits a filtration determined by the height of the roots (if the root base $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$, then for any $\alpha \in \Phi$, we have $\alpha = \sum a_i \alpha_i$ and the height of α is defined to be $ht(\alpha) := \sum a_i$).

Remark 9.2.1. *When $\mathcal{E}$ is fully deformable and if for every simple root $\alpha \in \Delta$, $O(\alpha) = O(C_\alpha)$ for some smooth irreducible curve C_α, then $C = \cup_{\alpha\in\Delta} C_\alpha$ is an ADE or affine ADE curve in X. In this case, we can show that $H^2(X, O(\alpha)) = 0$ for any $\alpha \in \Phi$ and the $\mathfrak{g}$ or $\widehat{\mathfrak{g}}$ bundle $\mathcal{E}$ admits a deformation into a filtrated bundle which is trivial on every C_α. When $\mathcal{E}$ is totally non-deformable, $\overline{\partial}_\varphi$ can only be $\overline{\partial}_0$.*

The main results of this section are the followings.

Theorem 9.2.1. *$\mathcal{E}_0^{\widehat{E}_8}$ over X_9 is totally non-deformable if and only if the nine blowup points in $\mathbb{P}^2$ are in general position.*

Let us recall some facts about elliptic fibrations on X_9 [30][32]. Any elliptic fibration on X_9 must be relatively minimal, i.e. there is no (-1)-curves in any of its fibrations, as there is no elliptic fibration on X_8, this is because the Euler characteristic of any elliptic surface is a multiple of 12 [10] and also $\chi(X_9) = 12$. There is at most one multiple fiber [11], say of multiplicity m. This happens precisely when there exists an irreducible pencil of degree $3m$ in $\mathbb{P}^2$ with 9 base points, each of multiplicity m and X_9 is the blow up of $\mathbb{P}^2$ at these 9 points. We can characterize the existence of such an elliptic fibration on X_9 in terms of deformability of $\mathcal{E}_0^{\widehat{E}_8}$ along imaginary root directions. For instance, X_9 with $-K_{X_9}$ nef admits an elliptic fibration (without multiple fiber) if and only if $\mathcal{E}_0^{\widehat{E}_8}$ is deformable in $(-mK)$-direction for some $m \in \mathbb{N}$ (with $m = 1$). Deformability of $\mathcal{E}_0^{\widehat{E}_8}$ can also detect the existence of ADE or Kodaira curves in X.

Theorem 9.2.2. *Suppose $-K_{X_9}$ is nef, then* *(i) X_9 admits an elliptic fibration with a multiple fiber of multiplicity m ($m \geq 1$) if and only if $\mathcal{E}_0^{\widehat{E}_8}$ is deformable in $(-mK)$-direction but not in $(-m+1)K$-direction.*

(ii) X_9 has an (maximal) ADE curve C of type $\mathfrak{g}$ if and only if $\mathcal{E}_0^{\widehat{E}_8}$ is (maximal) $\mathfrak{g}$-deformable.

(iii) X_9 has a (maximal) Kodaira curve C of type $\widehat{\mathfrak{g}}$ if and only if $\mathcal{E}_0^{\widehat{E}_8}$ is (maximal) $\widehat{\mathfrak{g}}$-deformable.

Here we say an ADE or Kodaira curve C is maximal if it is not proper contained in another ADE or Kodaira curve. We say $\mathcal{E}_0^{\widehat{E}_8}$ is maximal $\mathfrak{g}$ (or $\widehat{\mathfrak{g}}$) deformable if there does not exist another fully deformable (affine) Lie algebra sub-bundle of $\mathcal{E}_0^{\widehat{E}_8}$ containing this $\mathfrak{g}$ (or $\widehat{\mathfrak{g}}$) bundle.

9.3 Negative curves in X_9

In this section, we study negative rational curves in X_9. We can get corresponding results for X_n with $n \leq 8$ from this $n = 9$ case.

A divisor D in X is called a $(-m)$-class if $D \cdot D = -m$ and $D \cdot K = m - 2$. An effective $(-m)$-class is called a $(-m)$-curve. Note when $D = \sum n_i C_i$ is a $(-m)$-curve, we will also denote the corresponding curve $\cup C_i$ as D.

Use the notations in the above section, every effective divisor $D = ah - \sum_{i=1}^{9} a_i l_i \in Pic(X_9)$ must have $a = D \cdot h \geq 0$. It is well-known that all (-1)-classes are effective, and there are infinite number of them in X_9. There are also infinite number of (-2)-classes, but whether they are effective or not depends on the positions of the 9 blow-up points.

Definition 9.3.1. *Let $x_1, \cdots, x_n$ be n distinct points in $\mathbb{P}^2$. These n points are said to be non-special with respect to Cremona transformations if for any Cremona transformation T with centers within x_i's, the points $y_1, \cdots, y_n$ corresponding to*

x_i's under T are distinct points such that no three points among $y_1, \cdots, y_n$ are collinear.

Definition 9.3.2. *([21]) Let $x_1, \cdots, x_9$ be 9 points in $\mathbb{P}^2$, we say they are in general position if they satisfy the following three conditions:*

(i) they are distinct points in $\mathbb{P}^2$;

(ii) they are non-special with respect to Cremona transformations;

(iii) there is a unique cubic curve passing through all of them.

The conditions (i) and (ii) mean that any 8 of these 9 points are in general position. That is, no lines pass through three of them, no conics pass through six of them, and no cubic curves pass through eight of them with one of the eight points being a double point.

If the 9 blowing up points are in general position, then there is no effective (-2)-class in X_9 [21]. In general, there are at most finite number of $(-m)$-curves with $m \geq 3$.

Lemma 9.3.1. *Let $D = ah - \sum_{i=1}^{9} a_i l_i$ be a $(-m)$-curve in X_9 with $m \geq 3$, then*

(i) $m \leq 9$;

(ii) $0 \leq a \leq 3$;

(iii) $-1 \leq a_i \leq 2$ for all i, and there exists some j with $a_j = 1$;

(iv) there are finite number of such curves.

Proof. (i) Since D is a $(-m)$-curve, $D \cdot D = -m$ and $D \cdot K = m - 2$, i.e.

$$\sum a_i^2 = a^2 + m \text{ and } \sum a_i = 3a + m - 2.$$

From the above two equations, we have

$$(3a + m - 2)^2 = (\sum a_i)^2 \leq 9(\sum a_i^2) = 9(a^2 + m).$$

Thus, $a \leq \frac{-m^2+13m-4}{6(m-2)}$, also $a \geq 0$ since D is effective, hence $m \leq 12$.

When $m \geq 10$, we must have $a = 0$, that means $\sum a_i^2 = m$ and $\sum a_i = m-2$, hence $\sum a_i^2 - \sum a_i = 2$, which implies every a_i satisfies $|a_i| \leq 1$ and there exists exactly one a_i with $a_i = -1$. But we also have $\sum a_i = m - 2 \geq 8$, which is impossible since we only have nine a_i's.

(ii) When $m \geq 4$, $a \leq \frac{-m^2+13m-4}{6(m-2)} \leq \frac{8}{3} < 3$. When $m = 3$, $a \leq \frac{-m^2+13m-4}{6(m-2)} = \frac{13}{3} < 5$. Hence we only need to prove there is no (-3)-curve with $a = 4$.

Suppose not, then there exists a_i's such that $\sum a_i^2 = 19$ and $\sum a_i = 13$. From $\sum a_i^2 - \sum a_i = 6$, we know $-2 \leq a_i \leq 3$. If there is any a_i with $a_i = 3$, then the other a_i's can only be 0 or 1, but we have $\sum a_i = 13$ and there is only nine a_i's, which is impossible. Hence $-2 \leq a_i \leq 2$, from $\sum a_i^2 - \sum a_i = 6$, we can have at most three a_i's equal to 2, which is also impossible since $\sum a_i = 13$.

(iii) From $\sum a_i^2 = a^2 + m$, $\sum a_i = 3a + m - 2$ and $0 \leq a \leq 3$, we have

$$\sum a_i = 3a + m - 2 \geq a^2 + m - 2 = \sum a_i^2 - 2.$$

Hence $-1 \leq a_i \leq 2$. And there are three cases:

Case 1, one a_i equal to 2, the others equal to 0 or 1;

Case 2, one a_i equal to -1, the others equal to 0 or 1;

Case 3, all a_i's are equal to 0 or 1.

By $\sum a_i = 3a + m - 2 \geq 1$, we know in case 2 and case 3, there must exist some a_i with $a_i = 1$. In case 1, if there is no a_i with $a_i = 1$, then $D = ah - 2l_j$. From $\sum a_i^2 = a^2 + m$, $\sum a_i = 3a + m - 2$, we have $a = 0$, $m = 4$, hence $D = -2l_j$, which is not an effective divisor.

(iv) It is obvious from the above results. $\square$

From this lemma, we can easily obtain the following as a corollary.

Corollary 9.3.1. *If there exists a $(-m)$-curve in X_9 with $m \geq 3$, then there also exists a $(-m+1)$-curve in X_9.*

Proof. If $D \in |ah - \sum a_i l_i|$ is a $(-m)$-curve in X_9 with $m \geq 3$, then there exists

j with $a_j = 1$ by (iii) of Lemma 9.3.1 . It is easy to check that $D + l_j$ is a $(-m+1)$-curve in X_9. $\square$

If the 9 blowing up points are in general position, then there is no (-2)-curve in X_9, as a consequence, there is also no $(-m)$-curve in X_9 with $m \geq 3$. The following result shows that this happens exactly when X_9 is almost Fano. We include a proof here as we could not find it in the literatures.

Lemma 9.3.2. *X_9 has no $(-m)$-curve with $m \geq 3$ if and only if $-K_{X_9}$ is nef.*

Proof. If $-K$ is nef, then from $C \cdot K^{-1} = 2 - m \geq 0$ for any $(-m)$-curve C, we know $m \leq 2$.

Conversely, assume X_9 has no $(-m)$-curve with $m \geq 3$. Since X_9 is a blowup of $\mathbb{P}^2$ at nine points $\{x_i\}_{i=1}^9$, we have an effective anti-canonical divisor D. Recall when $D \cdot \Sigma < 0$ for any irreducible curve Σ in X, Σ must be a component of D. So if D is an irreducible curve or a Kodaira curve, then D is nef. We denote the image of D in $\mathbb{P}^2$ as C, which is a cubic curve passing through these 9 blowing up points.

(i) If C is smooth, then we are done as $D \cong C$ and therefore irreducible.

(ii) If C is reduced and irreducible, then it must be a nodal or cuspidal cubic. If $\{x_i\}_{i=1}^9 \cap \text{sing}(C) = \varnothing$ ($sing(C)$ means the set of singular points on C), then $D \cong C$ and we are done. Otherwise, say $x_1 \in \text{sing}(C)$ and we write the strict and proper transformations of C in $Bl_{x_1}(\mathbb{P}^2)$ as C_1 and $C_1 + E$ respectively. Then the remaining x_i's must have exactly 1 point (resp. 7 points) lying on E (resp. C_1) in order to avoid having $(-m)$-curve with $m \geq 3$. Thus D is a Kodaira curve of type $\widehat{A}_1$ or $III(\widehat{A}_1)$ for C being a nodal or cuspidal respectively.

(iii) If C is reduced and reducible, then $C = B \cup H_0$ or $H_1 \cup H_2 \cup H_3$ with B and H_j's are conic and distinct lines in $\mathbb{P}^2$. As before, we must have exactly 6 x_i's on B and 3 x_i's on each H_j and none on sing(C). Thus $D \cong C$ is a Kodaira curve of type $\widehat{A}_1$, $\widehat{A}_2$, $III(\widehat{A}_1)$ or $VI(\widehat{A}_2)$.

(iv) If C is non-reduced, $C = 3H$, D must have a $(-m)$-curve with $m \geq 3$.

Hence D is an irreducible curve or a Kodaira curve, and we are done. □

In the following two lemmas, we will use Lemma 2.21 in [2] to give a criteria of a curve in X_n being an ADE or affine ADE curve. Lemma 2.21 can be reformulated as follows: if $C = \cup_{i=1}^{r} C_i$ is a connected curve in a surface X satisfying: (i) $C_i^2 = -2$ and $C_i \cdot K_X = 0$ for any i; (ii) $C_i \cdot C_j \leq 1$ for any $i \neq j$; (iii) $(C_i \cdot C_j)_{r \times r} \leq 0$. Then when $(C_i \cdot C_j)_{r\times r} < 0$, C is an ADE curve, otherwise, it is an affine ADE curve.

Lemma 9.3.3. *Suppose $-K_{X_n}$ $(n \leq 8)$ is nef. Let $C = \cup C_i$ be a connected curve in X_n. If $C \cdot K_{X_n} = 0$, then C is an ADE curve.*

Proof. Since $-K_{X_n}$ is nef, $C \cdot K_{X_n} = 0$ implies $C_i \cdot K_{X_n} = 0$ for each i, i.e. $[C_i] \in \langle K \rangle^{\perp} \cong \Lambda_{E_n}$. We have $C_i^2 < 0$ and $(C_i + C_j)^2 < 0$ for any i and j. Together with the genus formula, we have $C_i^2 = -2$ and $C_i \cdot C_j \leq 1$ for $i \neq j$. By Lemma 2.21 in [2], we know C is an ADE curve. □

For $n = 9$ case, we have the following lemma.

Lemma 9.3.4. *Suppose $-K_{X_9}$ is nef. Let $C = \cup C_i$ be a connected curve in X_9. If $C \cdot K_{X_9} = 0$ and $C_i + K_{X_9}$ is not effective for each i, then C is a smooth elliptic curve, an ADE curve or an affine ADE curve.*

Proof. Since $-K_{X_9}$ is nef, $C \cdot K_{X_9} = 0$ implies $C_i \cdot K_{X_9} = 0$ for each i, i.e. $[C_i] \in \langle K_{X_9} \rangle^{\perp} \cong \Lambda_{E_9}$. We have $C_i^2 \leq 0$ and $(C_i + C_j)^2 \leq 0$ for any i and j. Moreover, for any effective divisor $D \in \langle K_{X_9} \rangle^{\perp}$, if $D^2 = 0$, then $D \in |mK_{X_9}|$ for some non-zero integer m. From $C_i^2 \leq 0$ and genus formula, we have $C_i^2 = -2$ or 0.

If there exists C_i such that $C_i^2 = 0$, then $C_i \in |mK|$ for some non-zero integer m. Since $C_i + K_{X_9}$ is not effective, we know $m = -1$, i.e. $C_i \in |-K|$. If C is not irreducible, then there exists C_j which intersects C_i, which is impossible. So $C = C_i \in |-K|$ is an elliptic curve or an affine A_0 curve by Lemma 9.3.2.

If $C_i^2 = -2$ for any i, then $C_i \cdot C_j \leq 2$ for any $i \neq j$. If there exist C_i and C_j such that $C_i \cdot C_j = 2$, then $(C_i + C_j)^2 = 0$, $C_i + C_j \in |mK|$ for some integer m. Hence $C = C_i \cup C_j$ is an affine A_1 curve, this is because if C_k is another irreducible component of C and assume it intersects with C_i, then it must be an irreducible component of C_j, which contradicts to C_j being irreducible. Otherwise, we will have $C_i^2 = -2$ for each i and $C_i \cdot C_j \leq 1$ for $i \neq j$. By Lemma 2.21 of [2], we know C is an ADE or affine ADE curve. □

9.4 Proof of Theorems 9.2.1 and 9.2.2

Proof. (of Theorem 9.2.1) If the nine blowup points in $\mathbb{P}^2$ are in general position, then for any $\alpha \in \Phi_9$, we have $h^0(X, O(\alpha)) = 0$ [21]. Since $K \cdot K = 0$, we also have $K - \alpha \in \Phi_9$ and therefore $h^2(X, O(\alpha)) = 0$ by Serre duality. However the Riemann-Roch formula gives $\chi(X, O(\alpha)) = 1 + \frac{\alpha^2 - \alpha K}{2} = 0$ and therefore $h^1(X, O(\alpha)) = 0$. For the imaginary roots mK's, from Lemma 4 and Proposition 11 in [21], we have $h^0(X, O(mK)) = 0$ and $h^0(X, O(-mK)) = 1$ for $m \geq 1$. By Serre duality and Riemann-Roch formula, we have $h^1(X, O(mK)) = 0$ for any imaginary root mK. Hence $\mathcal{E}_0^{\widehat{E}_8}$ is totally non-deformable.

Conversely, if $\mathcal{E}_0^{\widehat{E}_8}$ is totally non-deformable, then X has no (possibly reducible) (-2)-curve, hence no $(-n)$-curve with $n \geq 2$. By Proposition 10 in [31], this implies the nine blowup points are non-special with respect to Cremona transformations. Also from $h^1(X, O(mK)) = 0$ for any imaginary root mK, we get $h^0(X, O(-K)) = 1$, we have a unique cubic curve in $\mathbb{P}^2$ passing through all of the blow-up points. Hence, the nine blow-up points in $\mathbb{P}^2$ are in general position. □

Proof. (of Theorem 9.2.2) (i)We have $h^1(X, O(-mK)) = h^0(X, O(-mK)) - 1$

for any m by Riemann-Roch formula. So $\mathcal{E}_0^{\widehat{E}_8}$ is deformable in $(-mK)$-direction if and only if $h^0(X, O(-mK)) = 2$.

Let $F_0 \in |-K|$, then by Proposition 2.2 of [5], X admits an elliptic fibration with a multiple fiber of multiplicity m if and only if $O_{F_0}(F_0)$ is of order m in $Pic(F_0)$. But $O_{F_0}(mF_0) \cong O_{F_0}$ if and only if $h^0(O_{F_0}(mF_0)) = 1$ as $O_{F_0}(mF_0)$ is topologically trivial. By the exact sequence

$$0 \longrightarrow O_X \longrightarrow O_X(mF_0) \longrightarrow O_{F_0}(mF_0) \longrightarrow 0$$

together with $h^1(X, O_X) = 0$, we know $h^0(O_X(mF_0)) = 1 + h^0(O_{F_0}(mF_0))$. So $m = \min\{n : h^0(O_{F_0}(nF_0)) = 1\} = \min\{n : h^0(X, O(-nK)) = 2\}$.

(ii) If X has an ADE curve C of type $\mathfrak{g}$, we can use it to construct a fully deformable $\mathfrak{g}$-subbundle of $\mathcal{E}_0^{\widehat{E}_8}$. When C is maximal, then this $\mathfrak{g}$-subbundle is not contained in any other fully deformable Lie algebra subbundle of $\mathcal{E}_0^{\widehat{E}_8}$.

Conversely, if $\mathcal{E}_0^{\widehat{E}_8}$ is maximal $\mathfrak{g}$-deformable, then we can find a base $\Delta \subset \Phi_{\widehat{E}_8}$ of $\mathfrak{g}$ such that $h^1(X, O(\alpha)) \neq 0$ for every $\alpha \in \Delta$. Since $\chi(O(\alpha)) = 1 + \frac{\alpha^2 - \alpha \cdot K}{2} = 0$, we must have $h^0(O(\alpha)) \neq 0$ or $h^2(O(\alpha)) = h^0(O(K - \alpha)) \neq 0$, that is either α or $K - \alpha$ is effective. Hence, there must exist some integers m's such that $\alpha + mK$ is effective because $-K$ is effective, we denote the largest such m as m_α.

We claim that for every $\alpha \in \Delta$, $C_\alpha \in |\alpha + m_\alpha K|$ is an irreducible (-2)-curve. If so, then $C = \cup_{\alpha \in \Delta} C_\alpha$ is a maximal ADE curve of type $\mathfrak{g}$. If there exists reducible C_α, we write $C_\alpha = \cup D_i$. Then each D_i is perpendicular to K as $-K$ is nef and $C_\alpha \cdot K = 0$. Since $C_\alpha + K$ is not effective, every $D_i + K$ is also not effective and $D_i \notin |-K|$. Hence $D_i^2 = -2$ for any i as $D_i^2 = 0$ will imply $D_i \in |-K|$. We know C_α is connected, this is because if C_α is not connected, then one of its connected component must have self-intersection zero from $C_\alpha^2 = -2$, which contradicts to $C_\alpha + K$ is not effective. Hence $C = \cup_{\alpha \in \Delta} C_\alpha$ is an (affine) ADE curve by Lemma 9.3.4. It is obvious that this curve strictly contains a $\mathfrak{g}$-curve, which contradicts to $\mathcal{E}_0^{\widehat{E}_8}$ being maximal $\mathfrak{g}$-deformable.

(iii) The proof is similar to (ii). $\square$

Remark 9.4.1. *If X_9 admits an elliptic fibration, then we can find m such that $h^1(X_9, O(-mK)) \neq 0$. Conversely, if $h^1(X_9, O(-mK)) \neq 0$, we need to add the condition of $-K$ being nef to show that X admits an elliptic fibration. To see this, we take $x_1, \cdots, x_5$ to be 5 points on a line $l \subset \mathbb{P}^2$, and another 4 generic points (not on l) $x_6, \cdots, x_9$ in $\mathbb{P}^2$. Then we have an one parameter family of conics C_t's passing through these 4 points. If we blow up $\mathbb{P}^2$ at these 9 points and denote the strict transforms of l and C_t with same notations, then $l^2 = -4$, $C_t^2 = 0$. Moreover $C_t + l \in |-K|$ and $h^0(X_9, O(-K)) = 2$. But $-K$ is not nef as $(-K) \cdot l = -2$, which implies that X_9 is not elliptic.*

From the above, we can easily deduce similar results for the E_n-bundle $\mathcal{E}_0^{E_n}$ over X_n when $n \leq 8$, namely

(i) $\mathcal{E}_0^{E_n}$ is totally non-deformable if and only if the n blowup points in $\mathbb{P}^2$ are in general position.

(ii) When $-K_{X_n}$ nef, $\mathcal{E}_0^{E_n}$ is maximal $\mathfrak{g}$-deformable if and only if X_n has a maximal $\mathfrak{g}$ curve.

Appendix A

Minuscule configurations

We now construct examples of surface with an ADE singularity p of type $\mathfrak{g}$ and a (-1)-curve C_0 passing through p with minuscule multiplicity C_k. We call its minimal resolution a surface with minuscule configuration of type $(\mathfrak{g}, V)$, where V is the fundamental representation corresponding to $-C_k$.

First we consider the standard representation $V \simeq \mathbb{C}^{n+1}$ of $A_n = sl\,(n+1)$. When we blowup a point on any surface, the exceptional curve is a (-1)-curve E. If we blowup a point on E, the strict transform of E becomes a (-2)-curve. By repeating this process $n+1$ times, we obtain a chain of (-2)-curves with a (-1)-curve attached to the last one. Namely we have a surface with a minuscule configuration of type $(A_n, \mathbb{C}^{n+1})$.

Suppose that D is a smooth rational curve on a surface with $D^2 = 0$. By blowing up a point on D, we obtain a surface with a chain of two (-1)-curves. If we blowup their intersection point and iterative blowing up points in exceptional curves, then we obtain a surface with minuscule configuration of type $(D_n, \mathbb{C}^{2n})$.

Given a surface together with a smooth rational curve C with $C^2 = 1$ on it. We could obtain every minuscule configuration by the following process. If we blow up three points on C, then the strict transform of C is an (-2)-curve. By the previous construction of iterated blowups of points in these three exceptional curves E_i's, we could obtain many minuscule configurations. Let us denote the number of iterated blowups of the exceptional curve E_i as m_i with $i \in \{1, 2, 3\}$. Then we can obtain minuscule configuration of type $(\mathfrak{g}, V)$ by taking suitable m_i's as follows.

minuscule configuration of type $(\mathfrak{g}, V)$	(m_1, m_2, m_3)
$\left(A_n, \Lambda^k \mathbb{C}^{n+1}\right)$ for any k	$(k-1, 0, n-k)$
$(D_n, \mathbb{C}^{2n})$, (D_n, S^+) and (D_n, S^-)	$(n-3, 1, 1)$
$(E_6, 27)$, $\left(E_6, \overline{27}\right)$	$(2, 1, 2)$
$(E_7, 56)$	$(3, 1, 2)$

Note that we could obtain such a configuration for every adjoint representation of E_n this way. We remark that surfaces in this last construction are necessarily rational surfaces because of the existence of C with $C^2 = 1$.

Appendix B

Affine Lie algebras

In this appendix, we recall some results on affine Lie algebras [18][21]. If $(\mathfrak{g}, [,])$ is a finite dimensional simple Lie algebra, then the corresponding loop Lie algebra is $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with the Lie bracket defined by $[a \otimes t^n, b \otimes t^m]_{L\mathfrak{g}} = [a, b] \otimes t^{m+n}$, where $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$.

The corresponding untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ is constructed as a central extension of $L\mathfrak{g}$, with one-dimensional center $\mathbb{C}c$, i.e. $\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$. The Lie bracket on $\widehat{\mathfrak{g}}$ is defined by the formula $[a \otimes t^n + \lambda c, b \otimes t^m + \mu c]_{L\mathfrak{g}} = [a, b] \otimes t^{m+n} + n\delta_{n+m,0} k(a, b)c$, where $\lambda, \mu \in \mathbb{C}$ and k is the Killing form on $\mathfrak{g}$.

We can obtain the affine Dynkin diagram of $\widehat{\mathfrak{g}}$ from the Dynkin diagram of $\mathfrak{g}$ by adding one node to it, corresponding to the extended root and labelling as C_0. But in the affine ADE except affine E_8 case, from the symmetry of the affine Dynkin diagrams, we have different choices of labelling the extended root.

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