DIVISORS ON GRAPHS, BINOMIAL AND MONOMIAL IDEALS, AND CELLULAR RESOLUTIONS

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Farbod Shokrieh

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DIVISORS ON GRAPHS, BINOMIAL AND MONOMIAL IDEALS, AND CELLULAR RESOLUTIONS

Approved by:

Dr. Matthew Baker, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. John Etnyre
School of Mathematics
Georgia Institute of Technology

Dr. Anton Leykin
School of Mathematics
Georgia Institute of Technology

Dr. Josephine Yu
School of Mathematics
Georgia Institute of Technology

Dr. David Zureick-Brown
Department of Mathematics and Computer Science
Emory University

Date Approved: June 21, 2013

This thesis is dedicated to my parents and to Helen.

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SUMMARY

We study various binomial and monomial ideals arising from the theory of divisors, orientations, and matroids on graphs. We use ideas from potential theory on graphs and from the theory of Delaunay decompositions for lattices to describe minimal polyhedral cellular free resolutions for these ideals. We show that the resolutions of all these ideals are closely related and that their $\mathbb{Z}$-graded Betti tables coincide. As corollaries, we give conceptual proofs of conjectures and questions posed by Postnikov and Shapiro, by Manjunath and Sturmfels, and by Perkinson, Perlman, and Wilmes. Various other results related to the theory of chip-firing games on graphs – including Merino's proof of Biggs' conjecture and Baker-Shokrieh's characterization of reduced divisors in terms of potential theory – also follow from our general techniques and results.

CHAPTER I

INTRODUCTION

This work is concerned with the development of new connections between the theory of divisors on graphs, potential theory, the theory of lattices, Delaunay decompositions, and commutative algebra.

1.1 Divisors on graphs

Let G be a graph. Let $\mathrm{Div}(G)$ be the free abelian group generated by $V(G)$. An element of $\mathrm{Div}(G)$ is a formal sum of vertices with integer coefficients and is called a *divisor* on G.

We denote by $\mathcal{M}(G)$ the group of integer-valued functions on the vertices. The *Laplacian operator* $\Delta\colon \mathcal{M}(G) \to \mathrm{Div}(G)$ is defined by

$$\Delta(f) = \sum_{v \in V(G)} \sum_{\{v,w\} \in E(G)} (f(v) - f(w))(v) \ .$$

The group of *principal divisors* is defined as the image of the Laplacian operator and is denoted by $\mathrm{Prin}(G)$. Two divisors D_1 and D_2 are called *linearly equivalent* if their difference is a principal divisor. This gives an equivalence relation on the set of divisors. The set of equivalence classes forms a finitely generated abelian group which is called the *Picard group* of G. If G is connected, then the finite (torsion) part of the Picard group has cardinality equal to the number of spanning trees of G. This group has appeared in the literature under many different names; in theoretical physics and in probability theory it was first introduced as the “abelian sandpile group” or “abelian avalanche group” in the context of self-organized critical phenomena [3, 24, 30]. In arithmetic geometry, it appears implicitly in the study of component groups of Néron models of Jacobians of algebraic curves [39, 54]. In algebraic graph theory this group appeared under the name "Jacobian group" or "Picard group" in the study of flows and cuts in graphs [2]. The study of a certain chip-firing game on graphs led to the definition of this group under the name "critical group" [11, 12]. We recommend the recent survey article [38] for a short but more detailed overview of the subject.

The theory of divisors on graphs closely mirrors the theory of divisors on algebraic curves. In fact, Baker and Norine in [5] prove a version of Riemann-Roch theorem in this setting via a combinatorial argument. It was immediately realized (in [31, 45]) that this divisor theory has a natural extension to *metric graphs* (or *abstract tropical curves*). This theory, however, has resisted a more conceptual and cohomological interpretation.

Associated to G there is a canonical ideal which encodes the equivalences of divisors on G. This ideal is already implicitly defined in Dhar's seminal paper [24], but it was first introduced in [22]. Let K be a field and let $\mathbf{R} = K[\mathbf{x}]$ be the polynomial ring in variables $\{x_v : v \in V(G)\}$. The canonical binomial ideal is defined as $\mathbf{I}_G := \langle \mathbf{x}^{D_1} - \mathbf{x}^{D_2} : D_1 \sim D_2 \text{ both nonnegative divisors}\rangle$. A related monomial ideal, which we denote by $\mathbf{M}_G^q$, is a certain initial ideal of $\mathbf{I}_G$ which is defined after fixing a vertex $q \in V(G)$ (see §3.2). This ideal, for the case of complete graphs, was extensively studied in [53]. In [42], Riemann-Roch theory for graphs is linked to Alexander duality (see §11.3) for the ideal $\mathbf{M}_G^q$.

1.2 Minimal free resolutions

Let A be an abelian group and let R be an A-graded polynomial ring over K. Let $\mathfrak{m}$ denote the ideal consisting of all polynomials with zero constant term. We require the A-grading to be "nice", in the sense that a version of Nakayama's lemma holds (see §8.1). For a graded R-module M, a graded free resolution of M is an exact sequence of the form

$$\mathcal{F}\colon 0 \to \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$$

where all F_i's are free R-modules and all differential maps φ_i's are graded. This resolution is called *minimal* if $\varphi_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$ for all $i \geq 0$. The i-th *Betti number* $\beta_i(M)$ of M is the rank of F_i. The i-th *graded Betti number* in degree $\mathsf{j} \in \mathsf{A}$, denoted by $\beta_{i,\mathsf{j}}(M)$, is the rank of the degree j part of F_i. If the grading is "nice" then any finitely generated graded R-module has a minimal free resolution, and the numbers $\beta_{i,\mathsf{j}}(M)$ and $\beta_i(M)$ are independent of the choice of the minimal resolution. These integers encode very subtle numerical information about the module M. Many invariants of M (e.g. its Hilbert series) can be computed using these Betti numbers.

There is a standard way to write down a complex of graded modules from a cell complex $\mathcal{C}$. Namely, one can label 0-dimensional cells of $\mathcal{C}$ by monomials, and then extend the labeling to arbitrary faces by labeling each face F with the least common multiple of the monomial labels on the vertices of F. The resulting labeled cell complex leads to a complex of free graded R-modules

$$\mathcal{F}_{\mathcal{C}} = \bigoplus_{\emptyset \neq F \in \mathcal{C}} R(-\mathbf{m}_F)$$

where $\mathbf{m}_F$ denotes the monomial label of the face F. The differential of $\mathcal{F}_{\mathcal{C}}$ is the homogenized differential of the cell complex $\mathcal{C}$; if $[F]$ denotes the generator of $R(-\mathbf{m}_F)$ we have

$$\partial([F]) = \sum_{\substack{\mathrm{codim(F,F')=1} \\ F' \subset F}} \varepsilon(F, F') \frac{\mathbf{m}_F}{\mathbf{m}_{F'}}\ [F']$$

where $\varepsilon(F, F') \in \{-1, +1\}$ denotes the incidence function indicating the orientation of F' in the boundary of F.

This construction is so general that the resulting complex is expected not to be exact. In the rare case that we do get an exact sequence, the pair $(\mathcal{F}, \partial)$ is called a *cellular free resolution*. If all cells are polyhedral, $(\mathcal{F}, \partial)$ is called a *polyhedral cellular* *free resolution*. If moreover all $\mathbf{m}_F/\mathbf{m}_{F'}$ appearing in the differential maps are non-units in R, then we have a *minimal polyhedral cellular free resolution*.

1.3 Outline and our results

Our first goal is to give a minimal polyhedral cellular free resolution for the ideal $\mathbf{I}_G$. Quite surprisingly, many ideas from potential theory on graphs, from lattices and Delaunay decomposition, and from (a generalized version of) the notion of total unimodularity (developed in §3 and §4) fit together nicely to give a direct and self-contained solution to this problem. This is worked out in §5. Note that as a result we obtain a whole family (as G varies) of ideals with minimal polyhedral cellular free resolution. For complete graphs this is a Scarf complex and for trees this is a Koszul complex.

We then step back and define two more ideals; the *graphic Lawrence ideal* $\mathbf{J}_G$ and one of its initial ideals $\mathbf{O}_G^q$ (defined after fixing a vertex), which we call the *graphic oriented matroid ideal*. These are special classes of more general ideals studied in [8] and [49]. They are intimately related to *graphic hyperplane arrangements* and to *Delaunay decomposition of cut lattices* reviewed in §6. In §7 we take a close look at these ideals, review some general known results, and prove some new results for our special situation.

Roughly speaking, the ideals $\mathbf{J}_G$ and $\mathbf{O}_G^q$ can be thought of as "orientation" variants of the "divisor" ideals $\mathbf{I}_G$ and $\mathbf{M}_G^q$. A powerful technique in the theory of divisors on graphs and chip-firing games is to relate divisors to orientations. Given an orientation, one can form a divisor by reading off the associated indegrees or outdegrees (see, e.g., [15, Theorem 2.3], [5, Theorem 3.3], [36], [48], and [1]). Our next main result shows that, algebraically, there is a good justification for the strength of this method. We show that the relation between the ideals $\mathbf{J}_G$ and $\mathbf{I}_G$ (and similarly $\mathbf{O}_G^q$ and $\mathbf{M}_G^q$) can be understood via *regular sequences*. This is the content of §8 and §9.

These regular sequences allow us to compare many algebraic properties and constructions for the ideals $\mathbf{J}_G$ and $\mathbf{I}_G$ (and similarly $\mathbf{O}_G^q$ and $\mathbf{M}_G^q$). For example, one immediate corollary is to obtain a minimal polyhedral cellular free resolution for the ideal $\mathbf{I}_G$ from a minimal polyhedral cellular free resolution for the ideal $\mathbf{J}_G$. This resolution is essentially equivalent to the one obtained by our potential theoretic considerations (see Remark 10.0.8). We also obtain a minimal polyhedral cellular free resolution for the ideal $\mathbf{M}_G^q$ from a minimal polyhedral cellular free resolution for the ideal $\mathbf{O}_G^q$. It follows that all these resolutions are closely related to Delaunay decompositions of the lattice of integral coboundaries (which we call the *integral cut lattice*) and to the graphic hyperplane arrangement. Moreover, the $\mathbb{Z}$-graded Betti numbers of all these ideals coincide. So $\mathbf{M}_G^q$ and $\mathbf{O}_G^q$ are examples of "nice" initial ideals in the sense of [19], meaning that one can read the Betti numbers of the original ideal from the initial ideal (see [16, 47] for other such examples). Also, we obtain, automatically, an interpretation of the Betti numbers in terms of the number of faces of various dimensions in the graphic hyperplane arrangement, or equivalently, the number of orbits of the Delaunay cells of various dimensions in the cut or principal lattice. These interpretations also imply that Betti numbers can be read from the number of *acyclic partial orientations* of G (see Remark 6.1.3, Example 7.5.5, and Theorem 10.0.5). As a corollary, it follows that the Betti table of all these ideals is independent of the base field K.

For complete graphs, a minimal polyhedral cellular free resolutions for $\mathbf{M}_G^q$ and $\mathbf{I}_G$ was given in [53] and [42], respectively. The case of general graphs was left open in both works. Our work generalizes these constructions to arbitrary graphs, puts their constructions into a larger context, and resolves several questions and conjectures from these papers. We should mention that minimal free resolutions and the Betti numbers for both $\mathbf{M}_G^q$ and $\mathbf{I}_G$ were first established in [48] and independently in [41]. The first Betti number for I_G was computed in [40]. A minimal *cellular* resolution for

$\mathbf{M}_G^q$ was given in [26]. Very recently, the Betti numbers for $\mathbf{M}_G^q$ was also computed in [35].

We also remark that it is possible to directly give a minimal polyhedral cellular free resolution for the ideal $\mathbf{M}_G^q$ by our potential theoretic techniques in §5, but we have chosen to skip the details of this construction here as all the main ideas appear elsewhere in this writing. Moreover, an essentially equivalent (see Remark 5.2.7(ii)) solution for $\mathbf{M}_G^q$ has recently (and independently) appeared in [26], where they leave the solution for $\mathbf{I}_G$ as an open problem.

Our techniques allows us to revisit some of the foundational results on *chip-firing* games and related fields. For example, we remark that our potential theoretic interpretation of Gröbner weights relating $\mathbf{I}_G$ to $\mathbf{M}_G^q$ gives a new proof of the result in [6] interpreting q-reduced divisors as divisors of *minimum total potential* (see Remark 3.3.2). A related problem is to describe the whole Gröbner cone of the initial ideal $\mathbf{M}_G^q$. This was a question of Bernd Sturmfels which we completely answer in §3.4. We show that the rays of the Gröbner cone associated to $\mathbf{M}_G^q$ correspond, in a precise sense, to Green's functions.

The equality of the Betti tables of all of our ideals allows one to prove many numerical facts about one ideal by looking instead at another ideal in this family. We consider a few such examples in §11. One example is the computation of multiplicities. Perhaps the most exciting example of this observation is that we can reprove some important results expressing the h-vectors of $\mathbf{I}_G$ and $\mathbf{M}_G^q$ in terms of the Tutte polynomial. These results were originally proved by Merino in [44] and by Postnikov and Shapiro in [53] using direct combinatorial methods. In our approach, we show that there is a fifth ideal $\mathbf{Mat}_G$, directly related to the cographic matroid of G, with the same Betti table. This observation gives a direct and conceptual proof of the connection with the Tutte polynomial, which is likely to be generalizable.

This work is a first step in understanding the "algebraic geometry" of divisor theory of graphs and related objects. There are many directions that remain open for further explorations. Here we list two examples.

If G is not a tree, the ideal $\mathbf{I}_G$ is not prime and, although it is generated by binomials, it does not define a toric variety. It has been a challenge (certainly to the author) to try to mimic "toric arguments" to obtain results for $\mathbf{I}_G$, with the eventual goal of understanding the divisor theory on graphs at a more conceptual and geometric level. It follows from our work here that the variety associated to $\mathbf{I}_G$ inherits many properties of the toric variety associated to $\mathbf{J}_G$ because it is cut out in $\mathbf{J}_G$ by a regular sequence. We believe this is an important observation and we expect it to lead to some interesting mathematics.

We hope to extend many of our results to more general classes of *matroids*. The most exciting application would be to prove a Merino type result for a more general class of matroids using commutative algebra. This could be a promising approach to Stanley's famous O-sequence conjecture ([57, page 93]).

CHAPTER II

NOTATION AND BACKGROUND

Throughout, we assume $\mathbb{N}$ contains zero. All rings are commutative with 1.

A *graph* means a finite, connected, unweighted multigraph with no loops. As usual, the set of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$. For $A \subseteq V(G)$, we denote by A^c the complement of A in $V(G)$. We set $n = |V(G)|$ and $m = |E(G)|$. For a set of vertices S, the induced subgraph of G with the vertex set S is denoted by $G[S]$.

Let $\mathbb{E}(G)$ denote the set of oriented edges of G; for each edge in $E(G)$ there are two edges e and $\bar{e}$ in $\mathbb{E}(G)$. So we have $|\mathbb{E}(G)| = 2m$. An element e of $\mathbb{E}(G)$ is called an *oriented* edge, and $\bar{e}$ is called the *inverse* of e. We have a map

$$\mathbb{E}(G) \to V(G) \times V(G)$$
$$e \mapsto (e_+, e_-)$$

sending an oriented edge e to its head (or its terminal vertex) e_+ and its tail (or its initial vertex) e_-. Note that $\bar{e}_+ = e_-$ and $\bar{e}_- = e_+$. Given disjoint nonempty subsets A, B of $V(G)$ we define

$$\mathbb{E}(A, B) = \{e \in \mathbb{E}(G) : e_+ \in A, e_- \in B\} \ .$$

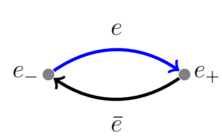

Figure 1: Oriented edges, head, and tail

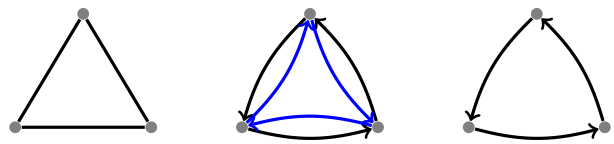

Figure 2: Graph K_3, its oriented edges, and a fixed orientation

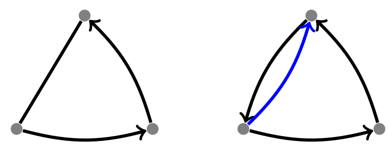

Figure 3: Two equivalent ways to draw a partial orientation

An *orientation* of G is a choice of subset $\mathcal{O} \subset \mathbb{E}(G)$ such that $\mathbb{E}(G)$ is the disjoint union of $\mathcal{O}$ and $\bar{\mathcal{O}} = \{\bar{e} : \ e \in \mathcal{O}\}$. An orientation is called *acyclic* if it contains no directed cycle. A *partial orientation* of G is a choice of subset $\mathcal{P} \subset \mathbb{E}(G)$ that strictly contains an orientation $\mathcal{O}$ of G. For a partial orientation $\mathcal{P}$, the associated (connected) *partition* is the partition of G into totally cyclic subgraphs with edges $\{e, \bar{e} \in \mathcal{P}\}$. A partial orientation is called *acyclic* if the induced orientation on the graph obtained by contracting all its totally cyclic components is acyclic.

Let $\mathcal{O}$ be an orientation of G. A vertex q is called a *source* for $\mathcal{O}$ if $q = e_-$ for every $e \in \mathcal{O}$ which is incident to q. Let $\mathcal{P}$ be a partial orientation of G. Let H be the associated connected component containing the vertex q. Then q is called a *source* for $\mathcal{P}$ if H corresponds to a source in the graph obtained by contracting all components of $\mathcal{P}$ (see Example 7.5.5).

For an abelian group A, we let $C^0(G, A)$ denote the set of all A-valued functions on $V(G)$. It is endowed with the bilinear form

$$\langle f_1, f_2 \rangle = \sum_{v \in V(G)} f_1(v) f_2(v) \ .$$

Also, $C^1(G, A)$ will denote the space of all A-valued functions g on $\mathbb{E}(G)$ such that $g(\bar{e}) = -g(e)$ for all $e \in \mathbb{E}(G)$. After fixing an orientation $\mathcal{O} \subset \mathbb{E}(G)$ we have

$C^1(G,A) = C^1_{\mathcal{O}}(G,A) \oplus C^1_{\bar{\mathcal{O}}}(G,A)$, where $C^1_{\mathcal{O}}(G,A)$ denotes the space of all A-valued functions on $\mathcal{O}$. The group $C^1(G,A)$ (and therefore $C^1_{\mathcal{O}}(G,A)$) is endowed with the bilinear form

$$\langle g_1, g_2 \rangle = \sum_{e \in \mathcal{O}} g_1(e) g_2(e) = \frac{1}{2} \sum_{e \in \mathbb{E}(G)} g_1(e) g_2(e) \tag{1}$$

The usual coboundary map $d \colon C^0(G,A) \to C^1(G,A)$ is defined by

$$(df)(e) = f(e_+) - f(e_-) = -(df)(\bar{e}) \ .$$

After fixing an orientation $\mathcal{O} \subset \mathbb{E}(G)$, we also obtain the restricted coboundary map $d_{\mathcal{O}} \colon C^0(G,A) \to C^1_{\mathcal{O}}(G,A)$.

Let R be a commutative ring with 1. We let $C_0(G,R)$ denote the free R-module generated by $V(G)$. Elements of $C_0(G,R)$ are of the form $\sum_{v \in V(G)} a_v(v)$ for $a_v \in R$. It is endowed with a bilinear form induced by $\langle (u),(v) \rangle = \delta_v(u)$ for $u,v \in V(G)$. Here $\delta_v(u)$ denotes the usual Kronecker delta function.

Likewise, we let $C_1(G,R)$ denote the free R-module generated by $\mathbb{E}(G)$. Elements of $C_1(G,R)$ are of the form $\sum_{e \in \mathbb{E}(G)} a_e(e)$ for $a_e \in R$. It is endowed with a bilinear form induced by

$$\langle (e),(e') \rangle = \begin{cases} 1, & \text{if } e' = e \\ -1, & \text{if } e' = \bar{e} \\ 0, & \text{otherwise} \end{cases}$$

for $e, e' \in \mathbb{E}(G)$. The usual boundary map $\partial \colon C_1(G,R) \to C_0(G,R)$ is defined by

$$\partial(e) = (e_+) - (e_-) \ .$$

The bilinear forms defined above provide canonical isomorphisms $C_0(G,R) \cong C^0(G,R)$ and $C_1(G,R) \cong C^1(G,R)$. Then the maps ∂ and d are adjoint with respect to these bilinear forms. We let $e^* \in C^1(G,R)$ denote the image of $(e) \in C_1(G,R)$ under this isomorphism, i.e.

$$e^* := \langle (e), \cdot \rangle \ .$$

The characteristic function of v or $\chi_v = \delta_v \in C^0(G, R)$ is the image of $(v) \in C_0(G, R)$ under the canonical isomorphism.

Let K be a field. Associated to G we define two polynomial rings:

- Let $\mathbf{R} = K[\mathbf{x}]$ denote the polynomial ring in n variables $\{x_v : v \in V(G)\}$.
- Let $\mathbf{S} = K[\mathbf{y}]$ denote the polynomial ring in $2m$ variables $\{y_e : e \in \mathbb{E}(G)\}$ or $\{y_e, y_{\bar{e}} : e \in \mathcal{O}\}$ (for any orientation $\mathcal{O}$).

CHAPTER III

DIVISORS AND POTENTIAL THEORY ON GRAPHS

Following [5], we let $\mathrm{Div}(G)$ be the free abelian group generated by $V(G)$. Equivalently, $\mathrm{Div}(G) = C_0(G, \mathbb{Z})$. An element of $\mathrm{Div}(G)$ is written as $\sum_{v \in V(G)} a_v(v)$ for $a_v \in \mathbb{Z}$ and is called a *divisor* on G. The coefficient a_v in D is denoted by $D(v)$. A divisor D is called *effective* if $D(v) \geq 0$ for all $v \in V(G)$. The set of effective divisors is denoted by $\mathrm{Div}_+(G)$. We write $D \leq E$ if $E - D \in \mathrm{Div}_+(G)$. For $D \in \mathrm{Div}(G)$, let $\deg(D) = \sum_{v \in V(G)} D(v)$. Given disjoint nonempty subsets A, B of $V(G)$ one can assign a divisor $D(A, B) = \sum_{v \in A} |\{w \in B : \{v, w\} \in E(G)\}|\ (v)$.

We denote by $\mathcal{M}(G)$ the group of integer-valued functions on the vertices. Equivalently, $\mathcal{M}(G) = C^0(G, \mathbb{Z})$. For $A \subseteq V(G)$, $\chi_A \in \mathcal{M}(G)$ denotes the $\{0,1\}$-valued characteristic function of A. The *Laplacian operator* $\Delta\colon \mathcal{M}(G) \to \mathrm{Div}(G)$ is defined by

$$\Delta(f) = \sum_{v \in V(G)} \sum_{\{v,w\} \in E(G)} (f(v) - f(w))(v) \ .$$

Remark 3.0.1. With the identification $\mathcal{M}(G) = C^0(G, \mathbb{Z})$ and $\mathrm{Div}(G) = C_0(G, \mathbb{Z})$ and the canonical isomorphism $C_1(G, R) \cong C^1(G, R)$, the operator Δ is identified with $\partial_{\mathcal{O}} d_{\mathcal{O}}\colon C^0(G, \mathbb{Z}) \to C_0(G, \mathbb{Z})$, where $\partial_{\mathcal{O}}$ and $d_{\mathcal{O}}$ denote the usual (restricted) boundary and coboundary maps for an arbitrary orientation $\mathcal{O}$. Somewhat more canonically, $\Delta = \frac{1}{2}\partial d$. It follows that Δ is a self-adjoint operator.

The group of *principal divisors* is defined as the image of the Laplacian operator and is denoted by $\mathrm{Prin}(G)$. It is easy to check that $\mathrm{Prin}(G) \subseteq \mathrm{Div}^0(G)$ where $\mathrm{Div}^0(G)$ denotes the set consisting of divisors of degree zero. The quotient

$\mathrm{Pic}^0(G) = \mathrm{Div}^0(G)/\mathrm{Prin}(G)$ is a finite group whose cardinality is the number of spanning trees of G (see, e.g., [6] and references therein). The full *Picard group* of G is defined as

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\mathrm{Prin}(G)$$

which is isomorphic to $\mathbb{Z} \oplus \mathrm{Pic}^0(G)$. Since principal divisors have degree zero, the map $\deg\colon \mathrm{Div}(G) \to \mathbb{Z}$ descends to a well-defined map $\deg\colon \mathrm{Pic}(G) \to \mathbb{Z}$. Two divisors D_1 and D_2 are called *linearly equivalent* if they become equal in $\mathrm{Pic}(G)$. In this case we write $D_1 \sim D_2$.

3.1 Divisors and potential theory

For $p, q \in V(G)$ let the *Green's function* $j_q(p, \cdot)$ denote the unique ($\mathbb{Q}$-valued) solution to the Laplace equation $\Delta f = (p) - (q)$ satisfying $f(q) = 0$. If we think of graph G as an electrical network (in which each edge is a resistor having unit resistance) then $j_q(p, v)$ denotes the electric potential at v if one unit of current enters the network at p and exits at q, with q grounded (i.e., zero potential). It is easy to check that $j_q(p, q) = 0$, $j_q(p, v) = j_q(v, p)$, and $0 \le j_q(p, v) \le j_q(p, p)$ (see [4, 18]). [6, Construction 3.1] explains how to compute these functions using basic linear algebra.

There exists a positive definite, symmetric bilinear form

$$\langle \cdot\,, \cdot \rangle_{\mathrm{en}}\colon\ \mathrm{Div}^0(G) \times \mathrm{Div}^0(G) \to \mathbb{Q}$$

$$\langle D_1, D_2 \rangle_{\mathrm{en}} = \sum_{u,v \in V(G)} D_1(u) j_q(u, v) D_2(v)$$

which is a canonical (i.e. independent of the choice of q) pairing on $\mathrm{Div}^0(G)$ (see [6, 55]). It is called the *energy pairing* on $\mathrm{Div}^0(G)$.

Let $\mathbf{1}$ denote the all-1's divisor. For $D \in \mathrm{Div}(G)$ and $q \in V(G)$, following [6], the *total potential functional* is defined as

$$b_q(D) = \langle \mathbf{1} - n(q), D - \deg(D)(q) \rangle_{\text{en}}$$
$$= \sum_v \sum_p j_q(p,v) D(v) \ .$$

3.2 Divisors and commutative algebra

Any effective divisor D gives rise to a monomial

$$\mathbf{x}^D := \prod_{v \in V(G)} x_v^{D(v)} \in \mathbf{R} \ .$$

Associated to every graph G there is a canonical ideal in $\mathbf{R}$ which encodes the linear equivalences of divisors on G:

$$\mathbf{I}_G := \langle \mathbf{x}^{D_1} - \mathbf{x}^{D_2} : \ D_1 \sim D_2 \text{ both effective divisors} \rangle$$
$$= \operatorname{span}_K \{ \mathbf{x}^{D_1} - \mathbf{x}^{D_2} : \ D_1 \sim D_2 \text{ both effective divisors} \}$$

which was first introduced in [22]. This ideal is graded by both $\operatorname{Pic}(G)$ and $\mathbb{Z}$.

Remark 3.2.1. It is shown in §8.1 (and in [48]) that, although $\operatorname{Pic}(G)$ has torsion elements, it provides a "nice" grading in the sense that Nakayama's lemma holds with respect to this grading and the concept of $\operatorname{Pic}(G)$-graded minimal free resolution makes sense in this context.

Once we fix a vertex q, there is a natural term order that gives rise to a particularly nice Gröbner basis for $\mathbf{I}_G$. This term order was also introduced in [22]. Consider a total ordering of the set of variables $\{x_v : v \in V(G)\}$ compatible with the distances of vertices from q in G:

$$\operatorname{dist}(w,q) < \operatorname{dist}(v,q) \ \Rightarrow \ x_w < x_v \ . \qquad (2)$$

Here, the distance between two vertices in a graph is the number of edges in a shortest path connecting them. This ordering can be thought of as an ordering on the vertices induced by running the breadth-first search (BFS) algorithm starting at the root vertex q. The term order $<_q$ will denote the graded reverse lexicographic ordering (grevlex) on $\mathbf{R}$ induced by the total ordering on the variables given in (2).

The initial ideal $\mathbf{M}_G^q := \mathrm{in}_{<_q}(\mathbf{I}_G)$ for $(\mathbf{I}_G, <_q)$ is canonically defined (up to the choice of the distinguished vertex q). This ideal is extensively studied in [53], where it is denoted by M_G. This ideal is naturally equipped with $\mathrm{Div}(G)$ (fine) and $\mathbb{Z}$ (coarse) gradings.

One of the main results of [22] is the following theorem – see also [48, Section 5] where this result is reproved and generalized to higher syzygy modules.

Theorem 3.2.2. *A Gröbner basis of* $(\mathbf{I}_G, <_q)$ *is*

$$\left\{\mathbf{x}^{D(A^c,A)} - \mathbf{x}^{D(A,A^c)} : A \subsetneq V(G), q \in A\right\} \ .$$

Moreover,

(i) $\mathrm{LM}(\mathbf{x}^{D(A^c,A)} - \mathbf{x}^{D(A,A^c)}) = \mathbf{x}^{D(A^c,A)}$.

(ii) It suffices to consider only those subsets A of $V(G)$ such that both $G[A]$ and $G[A^c]$ are connected. In this case we obtain a minimal Gröbner basis *of* $(\mathbf{I}_G, <_q)$.

As we will see, the minimal Gröbner basis described in part (ii) is also a minimal generating set (see also [48]).

3.3 Potential theory and Gröbner weight functionals for $\mathbf{I}_G$

Let $\vartheta \in C^0(G, \mathbb{R})$ and think of it as a linear functional $\vartheta\colon \mathrm{Div}(G) \to \mathbb{R}$. For $f = \sum c_i \mathbf{x}^{D_i} \in \mathbf{R}$ the ϑ-degree of f, denoted by $\deg_\vartheta(f)$, is the maximum value of $\vartheta(D_i)$. The ϑ-initial form of f is the sum of all terms $c_i\mathbf{x}^{D_i}$ such that $\vartheta(D_i)$ is maximum. For an ideal $I \subset \mathbf{R}$, the ϑ-initial ideal $\mathrm{in}_\vartheta(I)$ is the ideal generated by all ϑ-initial forms.

Fix a term order $<$ for $\mathbf{R}$. The functional ϑ is said to *represent* $<$ *for* I if $\mathrm{in}_\vartheta(I) = \mathrm{in}_<(I)$. It is known that for any term order $<$ and any ideal I, there is a *non-negative* and *integer-valued* functional representing $<$ for I ([58, Proposition 1.11]).

In our situation there is a nice and direct interaction between Gröbner theory and potential theory.

Lemma 3.3.1. $b_q \colon \mathrm{Div}(G) \to \mathbb{Q}$ *is a non-negative rational-valued functional representing* $<_q$ *for* $\mathbf{I}_G$.

Proof. For $D \in \mathrm{Div}(G)$ we know $b_q(D) = \sum_{v,p} j_q(p,v)D(v)$, so the non-negativity and rationality follows immediately. By Theorem 3.2.2, it suffices (see [58, proof of Proposition 1.11]) to check that for any $A \subsetneq V(G)$ with $q \in A$, we have

$$b_q(D(A^c, A)) > b_q(D(A, A^c)) \ .$$

But $D(A, A^c) - D(A^c, A) = \Delta(\chi_A)$, where χ_A denotes the $\{0,1\}$-valued characteristic function of A. The Laplacian operator Δ is self-adjoint (see Remark 3.0.1), which means

$$\sum_v f(v)\Delta(g)(v) = \sum_v g(v)\Delta(f)(v)$$

for all $f, g \in \mathcal{M}(G)$. Therefore for all $f \in \mathcal{M}(G)$ we have

$$\begin{aligned} \sum_v j_q(p,v)\Delta(f)(v) &= \sum_v f(v)\Delta(j_q(p,\cdot))(v) \\ &= \sum_v f(v)(\delta_p(v) - \delta_q(v)) \qquad (3) \\ &= f(p) - f(q) \ . \end{aligned}$$

Therefore we have

$$\begin{aligned} b_q(D(A, A^c) - D(A^c, A)) &= b_q(\Delta(\chi_A)) \\ &= \sum_{v,p} j_q(p,v)\Delta(\chi_A)(v) \\ &= \sum_p (\chi_A(p) - \chi_A(q)) \ . \end{aligned}$$

The result now follows, because for any set $A \subsetneq V(G)$ with $q \in A$, we have $\chi_A(q) = 1$, and there exists a vertex $p \in A^c$ with $\chi_A(p) = 0$. $\square$

Remark 3.3.2. q-reduced divisors (or G-parking functions with respect to q) can be defined as the normal forms of $\mathbf{R}/\mathbf{I}_G$ with respect to the Gröbner basis described in Theorem 3.2.2. It easily follows from Lemma 3.3.1 that a q-reduced divisor is precisely the unique (in each equivalence class) minimizer of the b_q functional. See [6] for a precise statement and a different proof of this fact.

Definition 3.3.3. We let ϑ_q denote the non-negative, integral functional associated to b_q (i.e. obtained from b_q by clearing the denominators). Clearly, ϑ_q will also represent $<_q$ for $\mathbf{I}_G$.

3.4 Gröbner cone of M_G^q

A modification of the proof of Lemma 3.3.1 shows that the rays of the Gröbner cone associated to $\mathbf{M}_G^q$, in a precise sense, correspond to Green's functions.

The weight functional $\eta \in C^0(G, \mathbb{R})$ defined by $\eta(D) = \sum_{v \in V(G)} \eta(v)(v)$ is in the Gröbner cone if and only if for any set $B \neq \emptyset$ with $q \notin B$ we have

$$\eta(\Delta(\chi_B)) = \sum_{v \in V(G)} \eta(v)\Delta(\chi_B)(v) = \sum_{v \in V(G)} \chi_B(v)\Delta(\eta)(v) > 0 \ . \tag{4}$$

In particular, for each vertex $p \neq q$, setting $B = \{p\}$ we must have:

$$\gamma_p := \Delta(\eta)(p) > 0 \ . \tag{5}$$

This condition is also sufficient because for all $B \neq \emptyset$ with $q \notin B$ we have

$$\eta(\Delta(\chi_B)) = \sum_{v \in V(G)} \chi_B(v)\gamma_v = \sum_{v \in B} \gamma_v \ .$$

It follows that $\eta \in \mathcal{M}(G)$ is a solution to $\Delta(\eta) = \gamma$ for the degree zero divisor $\gamma := \sum_{p \in V(G)} \gamma_p(p)$. From the definition of the Green's function $j_q(p, v)$, and the fact that the Laplacian operator has a 1-dimensional zero-eigenspace generated by the all-1 function $\mathbf{1}$, we obtain:

$$\eta = \sum_{p \in V(G)} \gamma_p j_q(p, \cdot) + k \cdot \mathbf{1} \tag{6}$$

for some constant $k \in \mathbb{R}$. We summarize these observations in the following theorem.

Theorem 3.4.1. *The weight functional $\eta \in C^0(G, \mathbb{R})$ represents $<_q$ for $\mathbf{I}_G$ if and only if there exist $k \in \mathbb{R}$ and real numbers $\gamma_p > 0$ (for $p \in V(G)$) such that*

$$\eta = \sum_{p \in V(G)} \gamma_p j_q(p, \cdot) + k \cdot \mathbf{1} \ .$$

In other words η, up to constant functions, is in the interior of the cone generated by the vectors $(j_q(p, v))_{v \in V(G)}$ for various $p \in V(G)$. Note that these vectors are independent because the matrix $(j_q(p, v))_{p, v \in V(G) \setminus \{q\}}$ is invertible (see [6, Construction 3.1]). The question of describing this Gröbner cone was asked by Bernd Sturmfels.

CHAPTER IV

LATTICES, DELAUNAY DECOMPOSITIONS, TOTAL UNIMODULARITY, AND INFINITE ARRANGEMENTS

4.1 Lattices and Delaunay decompositions

Let Λ be a free $\mathbb{Z}$-module (abelian group), endowed with a positive definite symmetric bilinear pairing $\beta\colon \Lambda \times \Lambda \to \mathbb{Z}$. The pair (Λ, β) (or just Λ, when β is understood) is called a *free bilinear form space over* $\mathbb{Z}$ or, more concisely, an *abstract* $\mathbb{Z}$*-lattice.*

Let (Λ, β) be an abstract $\mathbb{Z}$-lattice. We let $\Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R}$. The bilinear pairing β naturally extends to a bilinear pairing $\beta_{\mathbb{R}}$ on $\Lambda_{\mathbb{R}}$ by $\beta_{\mathbb{R}}(a \otimes \mathbf{u}, b \otimes \mathbf{v}) = ab\ \beta(\mathbf{u}, \mathbf{v})$.

The dual $\mathbb{Z}$-module $\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ is contained (via extension of scalars) in the dual real vector space $\Lambda^\vee_{\mathbb{R}} := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\Lambda_{\mathbb{R}}, \mathbb{R}) = \Lambda^\vee \otimes \mathbb{R}$. The *non-degeneracy* of β is the statement that the homomorphism

$$\Psi\colon \Lambda \to \Lambda^\vee$$
$$\mathbf{v} \mapsto \beta(\mathbf{v}, \cdot)$$

is injective. Clearly every positive definite bilinear pairing is automatically non-degenerate. Therefore the natural extension $\Psi_{\mathbb{R}}\colon \Lambda_{\mathbb{R}} \to \Lambda^\vee_{\mathbb{R}}$ is also injective (e.g., because $\mathbb{R}$ is a flat $\mathbb{R}$-module). Since these vector spaces have the same dimension, it follows that $\Psi_{\mathbb{R}}$ is indeed an isomorphism. In other words, in the language of bilinear forms, $\beta_{\mathbb{R}}$ is a *perfect pairing*[1] on $\Lambda_{\mathbb{R}}$. So, in this situation, any $\varphi \in \Lambda^\vee_{\mathbb{R}}$ is of the form $\varphi(\cdot) = \beta_{\mathbb{R}}(\mathbf{a}, \cdot)$ for some $\mathbf{a} \in \Lambda_{\mathbb{R}}$.

Let $d\colon \Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \to \mathbb{R}$ be any distance function on $\Lambda_{\mathbb{R}}$. The *Delaunay decomposition* of $\Lambda_{\mathbb{R}}$ with respect to the lattice Λ and the distance function d (not necessarily induced

[1] A perfect pairing is sometimes called a *unimodular pairing* in the literature. We will avoid this terminology to avoid confusion.

by the bilinear form) is defined as the collection of cells

$$A_{\mathbf{p}} = \text{conv.hull}\{\mathbf{s} \in \Lambda : d(\mathbf{p}, \mathbf{s}) \text{ is minimal}\} .$$

as $\mathbf{p}$ varies in $\Lambda_{\mathbb{R}}$. It is a classical fact (essentially due to Voronoi and Delaunay) that the collection of Delaunay cells $\{A_{\mathbf{p}}\}$ gives a locally finite, cellular decomposition (face to face tiling) of $\Lambda_{\mathbb{R}}$ which is invariant under the action of Λ (see, e.g., [20]).

4.2 Total unimodularity

Consider a (not necessarily minimal) finite set $\{\varphi_i\}_{i\in I}$ of generators for the free $\mathbb{Z}$-module $\Lambda^\vee$. Extension of scalars gives an inclusion $\Lambda^\vee \hookrightarrow \Lambda^\vee_{\mathbb{R}}$. Clearly, for any subset $J \subseteq I$ such that $\{\varphi_i\}_{i\in J}$ generates $\Lambda^\vee$ as a $\mathbb{Z}$-module, we have $\{\varphi_i\}_{i\in J}$ spans $\Lambda^\vee_{\mathbb{R}}$ as a real vector space (here we have identified $\varphi_i \otimes 1$ with φ_i). The converse is, of course, not true in general.

Definition 4.2.1. Let (Λ, β) be an abstract $\mathbb{Z}$-lattice. A finite set $\{\varphi_i\}_{i\in I}$ of generators for $\Lambda^\vee$ is called *totally unimodular* if for any subset $J \subseteq I$ such that the collection $\{\varphi_i\}_{i\in J}$ spans $\Lambda^\vee_{\mathbb{R}}$ as a real vector space, the collection $\{\varphi_i\}_{i\in J}$ generates $\Lambda^\vee$ as a $\mathbb{Z}$-module.

Example 4.2.2. Let $\Lambda = \mathbb{Z}^2$, generated by $\mathbf{e}_1$ and $\mathbf{e}_2$, endowed with the obvious bilinear pairing induced by $\langle \mathbf{e}_i, \mathbf{e}_j\rangle = \delta_i(j)$. Let $\mathbf{e}_i^* \in (\mathbb{Z}^2)^\vee$ denote the dual basis element $\mathbf{e}_i^*(\mathbf{e}_j) = \delta_i(j)$. Then

- $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_1^* + \mathbf{e}_2^*\}$ generates $\Lambda^\vee$ and is totally unimodular.
- $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_1^* + 2\mathbf{e}_2^*\}$ generates $\Lambda^\vee$ but is not totally unimodular.

  The subcollection $\{\mathbf{e}_1^*, \mathbf{e}_1^* + 2\mathbf{e}_2^*\}$ spans $(\mathbb{R}^2)^\vee$ as a real vector space, but it does not generate $(\mathbb{Z}^2)^\vee$. For example, $\mathbf{e}_2^*$ will not be in the $\mathbb{Z}$-module it generates.

Example 4.2.3. The primary examples of total unimodularity and the most well-known examples arise from totally unimodular matrices or, more generally, weakly unimodular matrices. An $r \times m$ ($r \leq m$) integer matrix $A = (a_{ij})$ is called *weakly unimodular* if every $r \times r$ square submatrix of A has determinant in the set $\{-1, 0, 1\}$. If every square submatrix of A has determinant in the set $\{-1, 0, 1\}$, then A is called a *totally unimodular* matrix. Any totally unimodular matrix is weakly unimodular. A weakly unimodular matrix which contains the identity matrix of size r is automatically totally unimodular.

Let A be a weakly unimodular matrix. Let Λ denote the row space $\text{Image}(A^T) \hookrightarrow \mathbb{Z}^m$ with the bilinear pairing induced by the natural bilinear pairing on $\mathbb{Z}^m$. For $1 \leq j \leq m$ let $\varphi_j \in \Lambda^\vee$ denote the restriction of $\mathbf{e}_j^* \in (\mathbb{Z}^m)^\vee$ to Λ. Concretely, if we denote the i-th row ($1 \leq i \leq r$) of A by $\mathbf{v}_i$, then each φ_j is defined by $\varphi_j(\mathbf{v}_i) = a_{ij}$. By Cramer's rule, the collection $\{\varphi_1, \ldots, \varphi_m\}$ is totally unimodular precisely because A is weakly unimodular.

4.3 Infinite hyperplane arrangements

Consider a finite collection $\{\varphi_i\}_{i \in I} \subset \Lambda_{\mathbb{R}}^\vee$ spanning $\Lambda_{\mathbb{R}}^\vee$ as a vector space over $\mathbb{R}$. For each $\mathbf{p} \in \Lambda_{\mathbb{R}}$ we denote by $C_{\mathbf{p}}$ the polyhedron in $\Lambda_{\mathbb{R}}$ defined by

$$C_{\mathbf{p}} = \{\mathbf{s} \in \Lambda_{\mathbb{R}} : \lfloor \varphi_i(\mathbf{p}) \rfloor \leq \varphi_i(\mathbf{s}) \leq \lceil \varphi_i(\mathbf{p}) \rceil \text{ for all } i \in I\} \ .$$

As usual, $\lfloor x \rfloor$ denotes the largest integer $n \leq x$, and $\lceil x \rceil$ denotes the smallest integer $n \geq x$. Clearly $C_{\mathbf{s}} = C_{\mathbf{p}}$ for all $\mathbf{s} \in \text{rel.int}(C_{\mathbf{p}})$. We denote by $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i \in I})$ the collection of all polyhedra $C_{\mathbf{p}}$ for $\mathbf{p} \in \Lambda_{\mathbb{R}}$.

The following result is well known for the case of totally unimodular matrices (Example 4.2.3) (see, e.g., [29, 51]). We give a proof suited for our general setting.

Theorem 4.3.1. *Fix a finite collection $\{\varphi_i\}_{i \in I} \subset \Lambda_{\mathbb{R}}^\vee$ which spans $\Lambda_{\mathbb{R}}^\vee$ as a vector space over $\mathbb{R}$.*

(i) $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i\in I})$ *is a polyhedral cell decomposition of* $\Lambda_{\mathbb{R}}$ *by bounded convex polyhedra. This cell decomposition is invariant under the translation by*

$$\{\mathbf{s} \in \Lambda_{\mathbb{R}} : \varphi_i(\mathbf{s}) \in \mathbb{Z} \textit{ for all } i \in I\}$$

which is contained in the set of 0*-dimensional polyhedra in* $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i\in I})$*.*

(ii) If, further, $\{\varphi_i\}_{i\in I} \subset \Lambda^\vee$ *and it generates* $\Lambda^\vee$*, then the polyhedral decomposition* $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i\in I})$ *is invariant under the translation action by elements of* Λ *which is contained in the set of* 0*-dimensional polyhedra in* $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i\in I})$*.*

(iii) If, further, $\{\varphi_i\}_{i\in I}$ *is totally unimodular, then* Λ *coincides with the set of* 0*-dimensional polyhedra in* $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i\in I})$*. Moreover,* $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i\in I})$ *coincides with the Delaunay decomposition of* $\Lambda_{\mathbb{R}}$ *with respect to the lattice* Λ *and the metric induced by*

$$\|\mathbf{p}\|^2 = \sum_{i\in I} |\varphi_i(\mathbf{p})|^2 \ . \tag{7}$$

Proof. Consider the map

$$\Phi\colon \Lambda_{\mathbb{R}} \longrightarrow \mathbb{R}^I$$
$$\mathbf{p} \mapsto (\varphi_i(\mathbf{p}))_{i\in I} \ .$$

Since $\{\varphi_i\}_{i\in I}$ spans $\Lambda_{\mathbb{R}}^\vee$ we know Φ is injective. Let $\{\varepsilon_i\}_{i\in I}$ denote the standard basis of $\mathbb{R}^I$, and let $\{\varepsilon_i^*\}_{i\in I}$ denote the dual basis of $(\mathbb{R}^I)^\vee$. Then $\mathcal{H}(\mathbb{R}^I, \{\varepsilon_i^*\}_{i\in I})$ is clearly the Delaunay decomposition of $\mathbb{R}^I$ with respect to the lattice $\mathbb{Z}^I$ with its standard pairing (induced by $\langle\varepsilon_i, \varepsilon_j\rangle = \delta_i(j)$).

The decomposition $\mathcal{H}(\Lambda_{\mathbb{R}}, \{\varphi_i\}_{i\in I})$ is the decomposition of $\Lambda_{\mathbb{R}}$ induced by Φ from this Delaunay decomposition of $\mathbb{R}^I$. It consists of $\Phi^{-1}(C)$ for various cells C in the Delaunay decomposition of $\mathbb{R}^I$ with $\Phi^{-1}(\operatorname{rel.int}(C)) \neq \emptyset$.

(i) immediately follows from the above considerations.

For (ii) note that, since $\{\varphi_i\}_{i\in I}$ generates $\Lambda^\vee$, we have $\Lambda = \Phi^{-1}(\mathbb{Z}^I)$.

For (iii), let $A = \Phi^{-1}(C)$ for cells C in the Delaunay decomposition of $\mathbb{R}^I$ with $\Phi^{-1}(\mathrm{rel.\,int}(C)) \neq \emptyset$. By the total unimodularity assumption, A is 0-dimensional if and only if $A = \{\mathbf{s}\}$ for some $\mathbf{s} \in \Lambda$. Let B be a cell in the Delaunay decomposition of $\Lambda_{\mathbb{R}}$. By definition this means there exists some $\mathbf{p}_0 \in \Lambda_{\mathbb{R}}$ such that

$$A = A_{\mathbf{p}_0} = \text{conv.hull}\{\mathbf{s} \in \Lambda : \|\mathbf{p}_0 - \mathbf{s}\| \text{ is minimal}\} \ .$$

Consider $\Phi(\mathbf{p}_0) \in \mathbb{R}^I$, and let B' denote the corresponding Delaunay cell in $\mathbb{R}^I$, i.e.

$$B' = B'_{\Phi(\mathbf{p}_0)} = \text{conv.hull}\{\mathbf{a} \in \mathbb{Z}^I : \langle \Phi(\mathbf{p}_0) - \mathbf{a}, \Phi(\mathbf{p}_0) - \mathbf{a} \rangle \text{ is minimal}\} \ .$$

B is obviously contained in $\Phi^{-1}(B')$. However the convex polyhedron $\Phi^{-1}(B')$ is the convex hull of its 0-dimensional faces. Therefore $B = \Phi^{-1}(B')$. □

Remark 4.3.2.

(i) Under the total unimodularity assumption, by Theorem 4.3.1(iii), we obtain a finite polyhedral cell decomposition of the quotient torus $\Lambda_{\mathbb{R}}/\Lambda$. This cell decomposition is essential in the study of our binomial ideals.

(ii) If the totally unimodular collection is coming from a weakly unimodular matrix as in Example 4.2.3, then the norm in (7) coincides with the standard norm induced by the bilinear form $\beta_{\mathbb{R}}$. This is because the φ_j's are precisely the restriction of the $\mathbf{e}_j^*$'s to $\Lambda_{\mathbb{R}}$.

CHAPTER V

POTENTIAL THEORY AND THE CELLULAR FREE RESOLUTION OF $\mathbf{I}_G$

Here we use potential theory and the energy pairing to give a self-contained and direct solution to the problem of finding a minimal polyhedral cellular free resolution of the ideal $\mathbf{I}_G$.

5.1 Minimal cellular free resolutions

Let S be a polynomial ring in r variables. Let $\mathcal{C}$ be a regular cell complex. If we *label* the vertices (0-dimensional cells) by monomials in S, we may extend the labeling to arbitrary faces by labeling an arbitrary face F with the *least common multiple* of the monomial labels on the vertices of F. In this way we obtain a *labeled cell complex*, which leads to a complex of free $\mathbb{Z}^r$-graded S-modules

$$\mathcal{F}_{\mathcal{C}} = \bigoplus_{\emptyset \neq F \in \mathcal{C}} S(-\mathbf{m}_F) \tag{8}$$

where $\mathbf{m}_F$ denotes the monomial label of the face F. The homological degree of $S(-\mathbf{m}_F)$ is $\dim(F)$. Let $[F]$ denote the generator of $S(-\mathbf{m}_F)$. The differential of $\mathcal{F}_{\mathcal{C}}$ is the homogenized differential of the cell complex $\mathcal{C}$:

$$\partial([F]) = \sum_{\substack{\mathrm{codim(F,F')=1} \\ F' \subset F}} \varepsilon(F, F') \frac{\mathbf{m}_F}{\mathbf{m}_{F'}} \, [F']$$

where $\varepsilon(F, F') \in \{-1, +1\}$ denotes the incidence function indicating the orientation of F' in the boundary of F (see [43, IX.5] or [17, Section 6.2]). Note that the length of $(\mathcal{F}, \partial)$ is the dimension of $\mathcal{C}$.

It is shown in [9, Proposition 1.2] that the complex $(\mathcal{F}, \partial)$ is exact if and only if every subcomplex $\mathcal{C}_{\leq \mathbf{m}}$ (i.e. the subcomplex of $\mathcal{C}$ consisting of all cells whose labels divide the monomial $\mathbf{m}$) is acyclic over K (i.e. its homology with K coefficients is only in degree 0). In this case $(\mathcal{F}, \partial)$ is called a *cellular free resolution*. If all cells are polyhedral it is called a *polyhedral cellular free resolution*. It is a *minimal* cellular free resolution if all $\mathbf{m}_F/\mathbf{m}_{F'}$ appearing in the differential maps are non-units. See [9] for more details.

5.2 Principal lattice with the energy pairing

Recall the $\mathbb{Z}$-module $\mathrm{Prin}(G)$ is defined as the image of the Laplacian operator $\Delta\colon \mathcal{M}(G) \to \mathrm{Div}(G)$. We have introduced two different canonical bilinear forms on this group. One is the bilinear form induced from the bilinear form on $C_0(G,\mathbb{Z}) = \mathrm{Div}(G)$ defined in §2. The bilinear form that is most relevant in this section is the one induced from the energy pairing defined in §3.1.

Definition 5.2.1. By a *principal lattice* we will mean the pair $(\mathrm{Prin}(G), \langle\cdot,\cdot\rangle_{\mathrm{en}})$ where

$$\langle\cdot,\cdot\rangle_{\mathrm{en}}\colon\ \mathrm{Prin}(G) \times \mathrm{Prin}(G) \to \mathbb{Z}$$

is the restriction of the energy pairing to $\mathrm{Prin}(G) \subseteq \mathrm{Div}^0(G)$.

Remark 5.2.2. It is easy to see (using (3)) that if $D \in \mathrm{Prin}(G)$ then for all $E \in \mathrm{Div}^0(G)$ we have $\langle E, D\rangle_{\mathrm{en}} \in \mathbb{Z}$ and therefore

(i) The restriction of the energy pairing to $\mathrm{Prin}(G)$ is $\mathbb{Z}$-valued.

(ii) The energy pairing descends to a well-defined pairing on $\mathrm{Pic}^0(G)$, which is shown to be non-degenerate in [55].

The principal lattice is an abstract $\mathbb{Z}$-lattice in the sense of §4.1. Its ambient vector space $\mathrm{Prin}(G)_{\mathbb{R}} = \mathrm{Prin}(G) \otimes \mathbb{R}$ coincides with $\mathrm{Div}^0_{\mathbb{R}}(G) = \mathrm{Div}^0(G) \otimes \mathbb{R} \subset C_1(G,\mathbb{R})$.

Our next goal is to find a nice collection of functionals for this lattice. For each $e \in \mathbb{E}(G)$ we define the functional $\zeta_e \in \mathrm{Div}^0_{\mathbb{R}}(G)^\vee$ by

$$\zeta_e(\cdot) = \langle \partial(e), \cdot\rangle_{\mathrm{en}}\ .$$

Lemma 5.2.3.

(i) Any $D \in \mathrm{Div}^0_{\mathbb{R}}(G)$ *is of the form* $D = \Delta(f)$ *for some* $f \in C^1(G, \mathbb{R})$.

(ii) For $D = \Delta(f) \in \mathrm{Div}^0_{\mathbb{R}}(G)$ *we have* $\zeta_e(D) = (df)(e)$.

Proof. (i) This follows from the fact that the kernel of Δ consists of constant functions.

(ii) We have, using (3)

$$\begin{aligned}\langle \partial(e), D\rangle_{\mathrm{en}} &= \langle \partial(e), \Delta(f)\rangle_{\mathrm{en}} \\ &= \sum_{u,v \in V(G)} (\delta_{e_+}(u) - \delta_{e_-}(u)) j_q(u,v)\Delta(f)(v) \\ &= \sum_{u \in V(G)} (\delta_{e_+}(u) - \delta_{e_-}(u)) \sum_{v \in V(G)} j_q(u,v)\Delta(f)(v) \\ &= \sum_{u \in V(G)} (\delta_{e_+}(u) - \delta_{e_-}(u))(f(u) - f(q)) \\ &= f(e_+) - f(e_-)\ .\end{aligned}$$

□

Proposition 5.2.4.

(i) $\{\zeta_e\}_{e \in \mathbb{E}(G)} \subset \mathrm{Prin}(G)^\vee$.

(ii) $\{\zeta_e\}_{e \in \mathbb{E}(G)}$ *generates* $\mathrm{Prin}(G)^\vee$.

(iii) $\{\zeta_e\}_{e \in \mathbb{E}(G)}$ *is totally unimodular for the principal lattice.*

Proof. (i) We need to show that $\zeta_e(D) \in \mathbb{Z}$ for all $D \in \mathrm{Prin}(G)$. Let $D = \Delta(f)$ for $f \in \mathcal{M}(G)$. Then by Lemma 5.2.3(ii) $\zeta_e(D) = (df)(e)$ which is an integer because f is integer-valued.

(ii) Let ζ be an arbitrary element of $\mathrm{Prin}(G)^\vee$. We need to show that $\zeta = \sum_{e \in \mathbb{E}(G)} a_e \zeta_e$ for some integers a_e. Since $\zeta \in \mathrm{Div}^0_{\mathbb{R}}(G)^\vee$ and $\langle \cdot, \cdot \rangle_{\mathrm{en}}$ is positive definite (and therefore non-degenerate), we must have $\zeta(\cdot) = \langle \mathbf{a}, \cdot \rangle_{\mathrm{en}}$ for some $\mathbf{a} \in \mathrm{Div}^0_{\mathbb{R}}(G)$ (see §4.1). For all $p \in V(G)\backslash\{q\}$ we have (see (3))

$$\begin{aligned} \langle \mathbf{a}, \Delta(\chi_p) \rangle_{\mathrm{en}} &= \sum_{u,v \in V(G)} \mathbf{a}(u) j_q(u,v) \Delta(\chi_p)(v) \\ &= \sum_{u \in V(G)} \mathbf{a}(u) \sum_{v \in V(G)} j_q(u,v) \Delta(\chi_p)(v) \\ &= \sum_{u \in V(G)} \mathbf{a}(u) (\chi_p(u) - \chi_p(q)) \\ &= \mathbf{a}(p) \ . \end{aligned} \tag{9}$$

Since $\zeta \in \mathrm{Prin}(G)^{\vee}$ we must have $\mathbf{a}(p) = \langle \mathbf{a}, \Delta(\chi_p) \rangle_{\mathrm{en}} \in \mathbb{Z}$ for all $p \in V(G)\backslash\{q\}$. Since $\mathbf{a}(q) = -\sum_{p \neq q} \mathbf{a}(p)$ we obtain $\mathbf{a} \in \mathrm{Div}^0(G)$. Let

$$\mathbf{a} = \sum_{p \in V(G)} \mathbf{a}(p)(p) = \sum_{p \neq q} \mathbf{a}(p)((p) - (q)) \ . \tag{10}$$

Since G is connected, for each $p \neq q$ there is a directed path from q to p consisting of some oriented edges $\{e^{(i)}\}_{1 \leq i \leq \ell}$ such that $e^{(1)}_{-} = q$, $e^{(\ell)}_{+} = p$, and $e^{(i)}_{+} = e^{(i+1)}_{-}$ for $1 \leq i \leq \ell - 1$. We may write

$$(p) - (q) = \sum_{i=1}^{\ell} (e^{(i)}_{+} - e^{(i)}_{-}) = \sum_{i=1}^{\ell} \partial(e^{(i)}) \ .$$

Substituting this in (10), we conclude that $\mathbf{a} = \sum_{e \in \mathbb{E}(G)} a_e \partial(e)$ for some integers a_e. Therefore $\zeta = \sum_{e \in \mathbb{E}(G)} a_e \zeta_e$ as we want.

(iii) Assume $J \subseteq \mathbb{E}(G)$ is such that the collection $\{\zeta_e\}_{e \in J}$ spans $\mathrm{Div}^0_{\mathbb{R}}(G)^{\vee}$ as a real vector space. We need to show that $\{\zeta_e\}_{e \in J}$ also generates $\mathrm{Prin}(G)^{\vee}$ as a $\mathbb{Z}$-module. Let ζ be an arbitrary element of $\mathrm{Prin}(G)^{\vee}$. Then $\zeta = \sum_{e \in J} b_e \zeta_e$ for some $b_e \in \mathbb{R}$ because $\{\zeta_e\}_{e \in J}$ spans $\mathrm{Div}^0_{\mathbb{R}}(G)^{\vee}$. In other words

$$\zeta(\cdot) = \langle \mathbf{b}, \cdot \rangle_{\mathrm{en}} \quad \text{with} \quad \mathbf{b} = \sum_{e \in J} b_e \partial(e)$$

for some $b_e \in \mathbb{R}$. We need to show that $b_e \in \mathbb{Z}$ for all $e \in J$. A computation identical to (9) shows that we have $\mathbf{b} \in \mathrm{Div}^0(G)$. It is a well-known classical fact (due to Poincaré) that the incidence matrix of G is totally unimodular (see, e.g., [10, Proposition 5.3] and §6.2). So $\sum_{e\in J} b_e \partial(e) \in \mathrm{Div}^0(G)$ will automatically imply that all b_e's must be integers. □

Remark 5.2.5. It also follows from the proof of Proposition 5.2.4(ii) that

(i) $\mathrm{Prin}(G)^\vee \cong \mathrm{Div}^0(G)$ and a canonical isomorphism is furnished by the energy pairing.

(ii) $C_1(G,\mathbb{Z}) \xrightarrow{\partial} C_0(G,\mathbb{Z}) \xrightarrow{\deg} \mathbb{Z} \to 0$ is an exact sequece. This statement, when $\mathbb{Z}$ is replaced with $\mathbb{R}$ is classical (see, e.g., [11, Proposition 12.1 and Proposition 28.1]).

We are now ready to apply the results in §4.3 to this setting.

Theorem 5.2.6. *Let* $\mathcal{H}(\mathrm{Div}^0_{\mathbb{R}}(G), \{\zeta_e\}_{e\in\mathbb{E}(G)}) = \{C_{\mathbf{a}}\}$ *be the collection of all polyhedra*

$$C_{\mathbf{a}} = \{\mathbf{b} \in \mathrm{Div}^0_{\mathbb{R}}(G) : \lfloor \zeta_e(\mathbf{a}) \rfloor \le \zeta_e(\mathbf{b}) \le \lceil \zeta_e(\mathbf{a}) \rceil \textit{ for all } e \in \mathbb{E}(G)\} \ . \tag{11}$$

as $\mathbf{a}$ *varies in* $\mathrm{Div}^0_{\mathbb{R}}(G)$*. Then*

(i) $\{C_{\mathbf{a}}\}$ *is a polyhedral cell decomposition of* $\mathrm{Div}^0_{\mathbb{R}}(G)$ *by bounded convex polyhedra.*

(ii) The cell decomposition $\{C_{\mathbf{a}}\}$ *is invariant under the translation by the lattice* $\mathrm{Prin}(G)$*.*

(iii) The set of 0*-dimensional cells in* $\{C_{\mathbf{a}}\}$ *coincides with* $\mathrm{Prin}(G)$*.*

(iv) $\{C_{\mathbf{a}}\}$ *is the same as the Delaunay cell decomposition of* $\mathrm{Div}^0_{\mathbb{R}}(G)$ *with respect to the lattice* $\mathrm{Prin}(G)$ *and the metric induced by the norm*

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p}\rangle_{\mathrm{en}}} = \sqrt{\mathcal{E}(\mathbf{p})} \ . \tag{12}$$

(v) $\{C_{\mathbf{a}}\}$ *descends to a finite polyhedral cell decomposition of* $\mathrm{Div}^0_{\mathbb{R}}(G)/\mathrm{Prin}(G)$*.*

Proof. This result follows from Proposition 5.2.4, Theorem 4.3.1, and Remark 4.3.2(i). We only need to show that the norm defined in (12) is compatible with the one considered in (7). By Lemma 5.2.3(i) any $\mathbf{p} \in \mathrm{Div}^0_{\mathbb{R}}(G)$ is of the form $\Delta(f)$ for some $f \in C^1(G, \mathbb{R})$. By (3), Lemma 5.2.3(ii), and Remark 3.0.1 we have

$$\begin{aligned}
\mathcal{E}(\mathbf{p}) &= \langle \Delta(f), \Delta(f) \rangle_{\mathrm{en}} \\
&= \sum_{v \in V(G)} f(u) \Delta(f)(u) \\
&= \frac{1}{2} \sum_{v \in V(G)} f(u)(\partial df)(u) \\
&= \frac{1}{2} \sum_{e \in \mathbb{E}(G)} (df)(e)(df)(e) \\
&= \frac{1}{2} \sum_{e \in \mathbb{E}(G)} |\zeta_e(\mathbf{p})|^2 \ .
\end{aligned}$$

So the norm defined in (12) is proportional to the norm defined in (7) and they induce the same Delaunay cell decomposition. □

The Delaunay cell decomposition $\{C_{\mathbf{a}}\}$ of Theorem 5.2.6 will be denoted by $\mathrm{Del}(\mathrm{Prin}(G))$. The induced finite cell decomposition of the torus $\mathrm{Div}^0_{\mathbb{R}}(G)/\operatorname{Prin}(G)$ will be denoted by $\mathrm{Del}(\mathrm{Prin}(G))/\operatorname{Prin}(G)$.

Remark 5.2.7.

(i) Since $\zeta_{\bar{e}} = -\zeta_e$ for all $e \in \mathbb{E}(G)$ we could alternatively define $C_{\mathbf{a}}$ in (11) as

$$\{\mathbf{b} \in \mathrm{Div}^0_{\mathbb{R}}(G) : \zeta_e(\mathbf{b}) \leq \lceil \zeta_e(\mathbf{a}) \rceil \text{ for all } e \in \mathbb{E}(G)\} \ .$$

It follows that open cells in this cell complex correspond precisely to equivalence classes of points, where $\mathbf{a} \sim \mathbf{b}$ if and only if $\lceil \zeta_e(\mathbf{a}) \rceil = \lceil \zeta_e(\mathbf{b}) \rceil$ for all $e \in \mathbb{E}(G)$.

(ii) By Lemma 5.2.3(ii) the local picture at the origin is the image of the graphic arrangement defined in §6.1 under the map Δ.

(iii) The cell complexes $\mathrm{Del}(\mathrm{Prin}(G))$ and $\mathrm{Del}(\mathrm{Prin}(G))/\mathrm{Prin}(G)$ are related to the cell complexes $\mathrm{Del}(L(G))$ and $\mathrm{Del}(L(G))/L(G)$ (defined in §6.2) by the (restricted) boundary map (see Remark 5.3.5 and Remark 10.0.8). The finite cell complexes $\mathrm{Del}(L(G))/L(G)$ and $\mathrm{Del}(\mathrm{Prin}(G))/\mathrm{Prin}(G)$ have the same f-vector (i.e. the same number of i-dimensional faces for all i).

The following lemma will be used in the proof of Theorem 5.3.2.

Lemma 5.2.8. *Fix a divisor $E \in \mathrm{Div}(G)$. The subcomplex of $\mathrm{Del}(\mathrm{Prin}(G))$ on the lattice points $P(E) = \{D \in \mathrm{Prin}(G) : D \leq E\}$ is a polyhedral subdivision of a contractible space.*

Proof. $P(E)$ is precisely the set of lattice points inside the closed convex polytope $Q(E) = \{\mathbf{a} \in \mathrm{Div}^0_{\mathbb{R}}(G) : \mathbf{a} \leq E\}$. The subcomplex of $\mathrm{Del}(\mathrm{Prin}(G))$ consisting of cells on the lattice points $P(E)$ consists of all Delaunay cells on these lattice points. Recall $\mathrm{Del}(\mathrm{Prin}(G))$ is a tiling of the ambient space. Therefore this subcomplex forms a space which is homotopy equivalent to the polytope $Q(E)$ itself, and therefore is contractible. □

5.3 *Labeling* $\mathrm{Del}(\mathrm{Prin}(G))$ *and the minimal free resolution of* $\mathbf{I}_G$

Let $\mathbf{T} = K[\mathbf{x}, \mathbf{x}^{-1}]$ denote the Laurent polynomial ring in variables $\{x_v : v \in V(G)\}$. Clearly $\mathbf{T}$ is a module over $\mathbf{R}$. Consider the $\mathbf{R}$-submodule $\mathbf{U}_G \subset \mathbf{T}$ generated by Laurent monomials $\{\mathbf{x}^D : D \in \mathrm{Prin}(G)\}$. This Laurent monomial module $\mathbf{U}_G$ may be thought of as the "universal cover" of $\mathbf{I}_G$ and many question about $\mathbf{I}_G$ can be reduced to questions about $\mathbf{U}_G$. For example, the free resolutions of $\mathbf{U}_G$ and $\mathbf{I}_G$ are closely related. See [9] for an extensive study of this relation. Since the only effective divisor in $\mathrm{Prin}(G)$ is the all-0 divisor, the results of [9] apply to our situation.

Consider the cell decomposition $\mathrm{Del}(\mathrm{Prin}(G))$. By Theorem 5.2.6 the set of 0-dimensional cells in $\mathrm{Del}(\mathrm{Prin}(G))$ is precisely $\mathrm{Prin}(G)$. We will label each 0-cell

$D \in \mathrm{Prin}(G)$ by the Laurent monomials $\mathbf{x}^D$. As usual, we let the label of any other cell to be the least common multiple of the labels of its vertices. This labeled cell complex leads to a complex of free $\mathrm{Div}(G)$-graded $\mathbf{R}$-modules

$$\mathcal{F}_G := \mathcal{F}_{\mathrm{Del}(\mathrm{Prin}(G))} = \bigoplus_{\emptyset \neq F \in \mathrm{Del}(\mathrm{Prin}(G))} \mathbf{R}(-\mathbf{m}_F)$$

where $\mathbf{m}_F$ denotes the monomial label of the face F. Let $[F]$ denote the generator of $\mathbf{R}(-\mathbf{m}_F)$. The differential of $\mathcal{F}_G$ is the homogenized differential (boundary) operator of the cell complex $\mathrm{Del}(\mathrm{Prin}(G))$:

$$\partial([F]) = \sum_{\substack{\mathrm{codim(F,F')=1} \\ F' \subset F}} \varepsilon(F, F') \frac{\mathbf{m}_F}{\mathbf{m}_{F'}} \; [F'] \tag{13}$$

where $\varepsilon(F, F') \in \{-1, +1\}$ denotes the incidence function indicating the orientation of F' in the boundary of F.

Lemma 5.3.1.

(i) Let $\mathbf{a} \in \mathrm{Div}^0_{\mathbb{R}}(G)$. *Then* $\mathbf{a}(v) = \sum_{e_+ = v} \zeta_e(\mathbf{a})$.

(ii) Let $F = C_{\mathbf{a}}$ *be a cell in* $\mathrm{Del}(\mathrm{Prin}(G))$ *corresponding to a point* $\mathbf{a} \in \mathrm{Div}^0_{\mathbb{R}}(G)$ *(i.e.* $\mathbf{a} \in \mathrm{rel.int}(F)$*). Then* $\mathbf{m}_F = \mathbf{x}^E$ *where* $E \in \mathrm{Div}(G)$ *is defined by*

$$E(v) = \sum_{e_+ = v} \lceil \zeta_e(\mathbf{a}) \rceil \ . \tag{14}$$

(iii) For distinct faces $F' \subsetneq F$ *of* $\mathrm{Del}(\mathrm{Prin}(G))$ *we have* $\mathbf{m}_F \neq \mathbf{m}_{F'}$.

Proof. (i) By Lemma 5.2.3(i) we may write $\mathbf{a} = \Delta(f)$ for some $f \in C^1(G, \mathbb{R})$. By definition we have $\Delta(f) = \sum_v \sum_{e_+ = v} (f(e_+) - f(e_-))(v)$. Therefore, it follows from Lemma 5.2.3(ii) that $\mathbf{a}(v) = \sum_{e_+ = v} \zeta_e(\mathbf{a})$.

(ii) follows from (i) and the fact that open cells in $\mathrm{Del}(\mathrm{Prin}(G))$ correspond precisely to equivalence classes of points, where $\mathbf{a} \sim \mathbf{b}$ if and only if $\lceil \zeta_e(\mathbf{a}) \rceil = \lceil \zeta_e(\mathbf{b}) \rceil$ for all $e \in \mathbb{E}(G)$ (Remark 5.2.7(ii)).

(iii) Let $F = C_{\mathbf{a}}$ for $\mathbf{a} \in \operatorname{rel.int}(F)$ and $F' = C_{\mathbf{a}'}$ for $\mathbf{a}' \in \operatorname{rel.int}(F')$. Since $\mathbf{a}'$ is in F as well, it satisfies $\zeta_e(\mathbf{a}') \leq \lceil \zeta_e(\mathbf{a}) \rceil$ for all $e \in \mathbb{E}(G)$. Therefore we have $\lceil \zeta_e(\mathbf{a}') \rceil \leq \lceil \zeta_e(\mathbf{a}) \rceil$. But since $F' \neq F$ there must exist some e such that $\zeta_e(\mathbf{a}') \in \mathbb{Z}$ but $\zeta_e(\mathbf{a}) \notin \mathbb{Z}$ and therefore $\lceil \zeta_e(\mathbf{a}') \rceil < \lceil \zeta_e(\mathbf{a}) \rceil$. The result now follows from part (ii) because for this edge, by (14), the exponent of x_{e_+} in $\mathbf{m}_{F'}$ must be strictly less than the exponent of x_{e_+} in $\mathbf{m}_F$. □

Theorem 5.3.2. *The complex $(\mathcal{F}_G, \partial)$ is a minimal* $\operatorname{Div}(G)$*-graded free resolution of the module $\mathbf{U}_G$ over $\mathbf{R}$.*

Proof. We need to show two things:

(i) $(\mathcal{F}_G, \partial)$ is exact, i.e. $(\mathcal{F}_G, \partial)$ is a cellular free resolution of $\mathbf{U}_G$.

(ii) For distinct faces $F' \subsetneq F$ of $\operatorname{Del}(\operatorname{Prin}(G))$ with $\operatorname{codim}(F, F') = 1$ we have $\mathbf{m}_F \neq \mathbf{m}_{F'}$, i.e. no unit of $\mathbf{R}$ appears in differential maps and the resolution $(\mathcal{F}_G, \partial)$ is minimal.

By [9, Proposition 1.2], we know (i) is equivalent to

(i') For each $E \in \operatorname{Div}(G)$, the subcomplex of $\operatorname{Del}(\operatorname{Prin}(G))$ on the lattice points $\{D \in \operatorname{Prin}(G) : D \leq E\}$ is acyclic over the field K, i.e. its reduced homology $\widetilde{H}_i$ with K coefficients vanishes for all $i \geq 0$.

(i') follows from Lemma 5.2.8 and (ii) follows from Lemma 5.3.1(iii). □

From Theorem 5.3.2 and [9, Corollary 3.7] we immediately obtain the following theorem.

Theorem 5.3.3. *The quotient cell complex* $\operatorname{Del}(\operatorname{Prin}(G))/\operatorname{Prin}(G)$ *supports a* $\operatorname{Pic}(G)$*-graded minimal free resolution for I_G.*

Example 5.3.4. Consider the graph K_3 with a fixed orientation as in Figure 4.

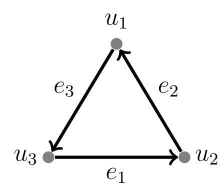

Figure 4: Graph K_3 and a fixed orientation $\mathcal{O}$

The lattice $\mathrm{Prin}(G)$ is two dimensional and is depicted in Figure 5. This lattice "lives in" $C_0(G, \mathbb{R}) = \mathrm{span}\{(u_1), (u_2), (u_3)\} \cong \mathbb{R}^3$. In the picture $c_1 = \Delta(\chi_{u_1}) = 2(u_1) - (u_2) - (u_3)$, $c_2 = \Delta(\chi_{u_2}) = -(u_1) + 2(u_2) - (u_3)$, and $c_3 = \Delta(\chi_{u_3}) = -(u_1) - (u_2) + 2(u_3)$.

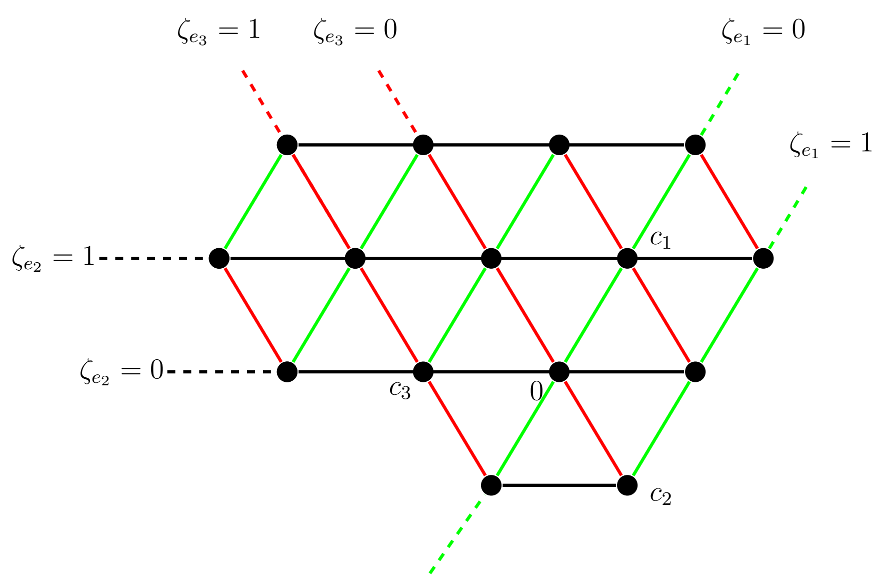

Figure 5: The lattice $(\mathrm{Prin}(G), \langle \cdot, \cdot \rangle_{\mathrm{en}})$ and the associated cellular decomposition of the ambient space $\mathrm{Div}^0_{\mathbb{R}}(G)$

The cell decomposition $\mathrm{Del}(\mathrm{Prin}(G))$ is the Delaunay decomposition of $\mathrm{Div}^0_{\mathbb{R}}(G)$ with respect to the principal lattice and the energy distance (Theorem 5.2.6(iv)) which coincides with the infinite hyperplane arrangement (11). The quotient cell complex $\mathrm{Del}(\mathrm{Prin}(G))/\,\mathrm{Prin}(G)$ of the torus has one 0-cell $\{v\}$ (orbit of the origin), three 1-cells $\{e, e', e''\}$ (orbits of green, red, and black edges), and two 2-cells $\{f, f'\}$

(orbits of upward and downward triangles).

In Figure 6 we have chosen a fundamental domain for the lattice, and have labeled all cells of this fundamental domain according to the recipe described in the beginning of §5.3 or, equivalently, in Lemma 5.3.1(ii). For simplicity we have used x_i instead of x_{u_i}. The labeled cell complex in Figure 6 is enough to completely describe a minimal free resolution for both $\mathbf{I}_G$ and $\mathbf{U}_G$. Concretely, the minimal resolution of $\mathbf{I}_G$ is as follows:

$$0 \to \mathbf{R}(-\mathbf{m}_f) \oplus \mathbf{R}(-\mathbf{m}_{f'}) \xrightarrow{\partial_2} \mathbf{R}(-\mathbf{m}_e) \oplus \mathbf{R}(-\mathbf{m}_{e'}) \oplus \mathbf{R}(-\mathbf{m}_{e''}) \xrightarrow{\partial_1} \mathbf{R}(-\mathbf{m}_v) \ .$$

As usual, assume $[F]$ denotes the generator of $\mathbf{R}(-\mathbf{m}_F)$. Let

$$\mathbf{m}_e = x_1^2 \ , \quad \mathbf{m}_{e'} = x_1x_2 \ , \quad \mathbf{m}_{e''} = x_2^2 \ ,$$

$$\mathbf{m}_f = x_1^2x_2 \ , \quad \mathbf{m}_{f'} = x_1x_2^2 \ .$$

The homogenized differential operator (see (13)) (∂_1, ∂_2) of the cell complex is described as follows:

$$\begin{aligned}
\partial_1([e]) &= \frac{x_1^2}{1}[v] - \frac{x_1^2}{\frac{x_1^2}{x_2x_3}}[v] = (x_1^2 - x_2x_3)[v] \ , \\
\partial_1([e']) &= \frac{x_1x_2}{\frac{x_1x_2}{x_3^2}}[v] - \frac{x_1x_2}{1}[v] = (x_3^2 - x_1x_2)[v] \ , \\
\partial_1([e'']) &= \frac{x_2^2}{\frac{x_2^2}{x_1x_3}}[v] - \frac{x_2^2}{1}[v] = (x_1x_3 - x_2^2)[v] \ , \\
\partial_2([f]) &= \frac{x_1^2x_2}{x_1^2}[e] - \frac{x_1^2x_2}{\frac{x_1^2x_2}{x_3}}[e''] + \frac{x_1^2x_2}{x_1x_2}[e'] = x_2[e] - x_3[e''] + x_1[e'] \ , \\
\partial_2([f']) &= \frac{x_1x_2^2}{\frac{x_1x_2^2}{x_3}}[e] - \frac{x_1x_2^2}{x_2^2}[e''] + \frac{x_1x_2^2}{x_1x_2}[e'] = x_3[e] - x_1[e''] + x_2[e'] \ .
\end{aligned}$$

Clearly $\mathbf{I}_G$ is the image of ∂_1 after identifying $[v]$ with $1 \in \mathbf{R}$ (see Theorem 3.2.2). Note that, since the labeling is compatible with the action of the lattice, any other fundamental domain would give rise to the exact same description of the differential maps.

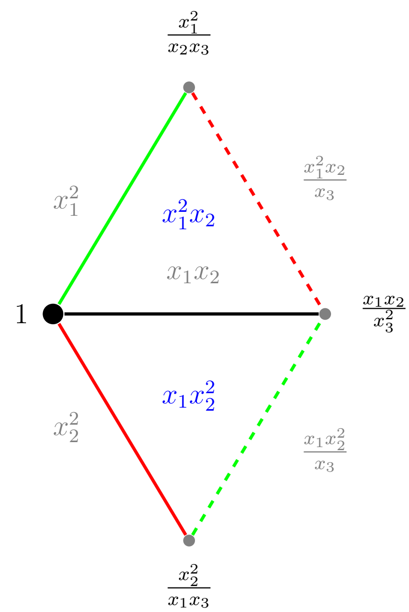

Figure 6: A choice of fundamental domain with labels

Remark 5.3.5. It follows from the computation

$$\begin{aligned}\langle \Delta(f), \Delta(g)\rangle_{\text{en}} &= \sum_{v\in V(G)} f(u)\Delta(g)(u) \\ &= \sum_{v\in V(G)} f(u)(\partial_{\mathcal{O}} d_{\mathcal{O}} g)(u) \\ &= \sum_{e\in \mathbb{E}(G)} (d_{\mathcal{O}} f)(e)(d_{\mathcal{O}} g)(e)\end{aligned}$$

that there is an isometry between the principal lattice $(\text{Prin}(G), \langle\cdot,\cdot\rangle_{\text{en}})$ and the *cut lattice* (lattice of integral cocyles) $(L(G), \langle\cdot,\cdot\rangle)$ defined in §6.2. It is natural to ask whether there are other ideals defined directly in terms of the cut lattice and, if so, whether there are nice relations between these ideals. These questions will be answered in this work (see §10, especially Remark 10.0.8).

Remark 5.3.6. It is possible to give a polyhedral cellular free resolution of the ideal $\mathbf{M}_G^q$ using the local picture at the origin of $\text{Del}(\text{Prin}(G))$ (or, alternatively, using the graphic hyperplane arrangement – see Remark 5.2.7(ii)) and study its Gröbner relation with $\mathbf{I}_G$, similar to what we will do for $\mathbf{O}_G^q$ in relation to $\mathbf{J}_G$ in §7. Instead, we will show (in §10) that one could alternatively relate $\mathbf{I}_G$ to $\mathbf{J}_G$ and $\mathbf{M}_G^q$ to $\mathbf{O}_G^q$ via a regular sequence. As a corollary, this gives an alternate way to describe polyhedral cellular free resolutions of all these ideals and to compare their Betti numbers.

Remark 5.3.7. The minimal free resolution of $\mathbf{M}_G^q$ is a Koszul complex when G is a tree because $\mathbf{M}_G^q$ is generated by the variables $\{x_v : v \neq q\}$ (see Theorem 3.2.2). When G is a complete graph, the minimal free resolution of $\mathbf{M}_G^q$ is given by a Scarf complex (see, e.g., [53, Corollary 6.9]).

CHAPTER VI

GRAPHS, ARRANGEMENTS, AND INTEGRAL CUTS

6.1 Graphic arrangements and connected partitions

Following [32], we define the *graphic hyperplane arrangement* as follows. An important feature that we want to emphasize in this section is that this arrangement naturally "lives in" the Euclidean space $C^0(G,\mathbb{R})$, i.e. the vector space of all real-valued functions on $V(G)$ endowed with the bilinear form

$$\langle f_1, f_2 \rangle = \sum_{v \in V(G)} f_1(v) f_2(v) \ .$$

Recall that $C^1(G,\mathbb{R})$ denotes the vector space of all real-valued functions on $\mathbb{E}(G)$, and $d\colon C^0(G,\mathbb{R}) \to C^1(G,\mathbb{R})$ denotes the usual coboundary map.

For each edge $e \in \mathbb{E}(G)$, let $\mathcal{H}_e \subset C^0(G,\mathbb{R})$ denote the hyperplane

$$\mathcal{H}_e = \{ f \in C^0(G,\mathbb{R}) : (df)(e) = 0 \} \ .$$

Note that $\mathcal{H}_{\bar{e}} = \mathcal{H}_e$. Consider the arrangement

$$\mathcal{H}'_G = \{ \mathcal{H}_e : e \in \mathbb{E}(G) \}$$

in $C^0(G,\mathbb{R})$. Since G is connected, we know $\bigcap_{e \in \mathbb{E}(G)} \mathcal{H}_e$ is the 1-dimensional space of constant functions on $V(G)$, which is the same as the kernel of d. We define the *graphic arrangement* corresponding to G, denoted by $\mathcal{H}_G$, to be the restriction of $\mathcal{H}'_G$ to the hyperplane

$$(\mathrm{Ker}(d))^{\perp} = \{ f \in C^0(G,\mathbb{R}) : \sum_{v \in V(G)} f(v) = 0 \} \ . \tag{15}$$

The intersection poset of $\mathcal{H}_G$ (i.e. the collection of nonempty intersections of hyperplanes $\mathcal{H}_e$ ordered by reverse inclusion) is naturally isomorphic to the poset of connected partitions of G (i.e. partitions of $V(G)$ whose blocks induce connected subgraphs). See, e.g., [32, p.112].

It is well-known that there is a one-to-one correspondence between acyclic orientations of G and the regions of $\mathcal{H}_G$ (see, e.g., [32, Lemma 7.1 and Lemma 7.2]). Given any function $f \in C^0(G, \mathbb{R})$ one can label each vertex v with the real number $f(v)$. In this way we obtain an acyclic partial orientation of G by directing v to u if $f(u) < f(v)$. Recall this means we have an acyclic orientation on the graph G/f obtained by contracting all unoriented edges (i.e. all edges $\{u, v\}$ with $f(u) = f(v)$).

We are mainly interested in acyclic orientations of G with a *unique source* at $q \in V(G)$. For this purpose, we fix a real number $c > 0$ and define

$$\mathcal{H}^{q,c} = \{f \in C^0(G, \mathbb{R}) : f(q) = -c\} \ .$$

The restriction of the arrangement $\mathcal{H}_G$ to $\mathcal{H}^{q,c}$ will be denoted by $\mathcal{H}_G^{q,c}$. We denote the *bounded complex* (i.e. the polyhedral complex consisting of bounded cells) of $\mathcal{H}_G^{q,c}$ by $\mathcal{B}_G^{q,c}$.

Remark 6.1.1.

(i) By (15), the restriction of $\mathcal{H}_G$ to $\mathcal{H}^{q,c}$ coincides with the restriction of $\mathcal{H}_G$ to

$$(\mathcal{H}^{q,c})' = \{f \in C^0(G, \mathbb{R}) : \sum_{v \neq q} f(v) = c\} \ .$$

(ii) We will see in §7.5 (e.g. Lemma 7.5.2(ii)) that it is most natural (although not necessary) to choose $0 < c < 1$.

The following lemma relates regions of $\mathcal{B}_G^{q,c}$ to acyclic orientations with unique source at q (see also [32, Theorem 7.3]).

Lemma 6.1.2. *Each $f \in \mathcal{B}_G^{q,c}$ gives an acyclic partial orientation of G with a unique source at q. In particular $f(v) \geq f(q)$ for any edge $\{v, q\} \in E(G)$.*

Proof. Since we are considering the orientation on G/f we may assume $f(u) \neq f(v)$ for any $\{u, v\} \in E(G)$. Since any acyclic orientation of G has at least one source vertex[1], it suffices to show that no vertex $v \neq q$ can be a source in the orientation corresponding to f.

Let w be a vertex such that $f(w)$ is maximum (i.e. $f(w) \geq f(v)$ for all $v \in V(G)$). To obtain a contradiction, assume $s \neq q$ is a source and therefore $f(v) > f(s)$ for all $\{v, s\} \in V(G)$.

Recall that χ_v denotes the characteristic function of $v \in V(G)$. It follows that

$$f_t = f + t(\chi_w - \chi_s) \in C^0(G, \mathbb{R})$$

also belongs to the same cell as f for *any* $t \geq 0$. This is because:

- $f_t(q) = f(q) = -c$: note that $s \neq q$ by assumption. Moreover, since $f(q) = -c$ and $\sum_{v \neq q} f(v) = c > 0$, there must be at least one vertex v with $f(v) > 0 > f(q)$. Therefore $f(q)$ cannot be maximum among $f(v)$'s, which means $w \neq q$.

- $\sum_{v \in V(G)} f_t(v) = \sum_{v \in V(G)} f(v) + t - t = 0$.

- If $\{u, v\} \in E(G)$, we have $f_t(u) > f_t(v)$ if and only if $f(u) > f(v)$. Note that f_t and f differ only in places w and s. So this claim follows from the fact that $f_t(w) = f(w) + t \geq f(w)$ and $f(s) \geq f(s) - t = f_t(s)$.

However, not all f_t for $t \geq 0$ can be contained in the bounded complex because they constitute a ray in $C^0(G, \mathbb{R})$ emanating from f. □

Remark 6.1.3. It follows (see also [32, Corollary 7.3]) that the number of i-dimensional cells in $\mathcal{B}_G^{q,c}$ is equal to the number of acyclic partial orientations of G with $(i + 2)$ (connected) components having a unique source at q. For an example, see Example 7.5.5.

[1] It is an elementary fact that *any* acyclic orientation of G has at least one source and one sink.

6.2 *Lattice of integral cuts and graphic infinite arrangements*

Fix an arbitrary orientation $\mathcal{O} \subset \mathbb{E}(G)$. Consider the restricted coboundary map $d_{\mathcal{O}} : C^0(G,\mathbb{Z}) \to C^1_{\mathcal{O}}(G,\mathbb{Z})$ and the usual bilinear form on $C^1_{\mathcal{O}}(G,\mathbb{Z})$ defined by

$$\langle g_1, g_2 \rangle = \sum_{e \in \mathcal{O}} g_1(e) g_2(e) \ . \tag{16}$$

The *lattice of integral cuts* (with respect to the orientation $\mathcal{O}$) is by definition the group of integral coboundaries $\mathrm{Image}(d_{\mathcal{O}})$ inside $C^1_{\mathcal{O}}(G,\mathbb{Z})$ with its bilinear form induced from (16). It is denoted by $L(G,\mathcal{O})$. When the orientation is clear we simply denote it by $L(G)$.

Remark 6.2.1. Consider the (unrestricted) coboundary map

$$d : C^0(G,\mathbb{Z}) \to C^1(G,\mathbb{Z}) \cong C^1_{\mathcal{O}}(G,\mathbb{Z}) \oplus C^1_{\bar{\mathcal{O}}}(G,\mathbb{Z}) \ .$$

Its image $\Lambda = \mathrm{Image}(d)$ is isomorphic to the lattice $\{(a,-a) : a \in L(G,\mathcal{O})\}$. The choice of the orientation $\mathcal{O}$ gives a splitting of $C^1(G,\mathbb{Z})$ and of Λ.

We may identify $C^0(G,\mathbb{Z})$ with $\mathbb{Z}^{V(G)}$ and $C^1_{\mathcal{O}}(G,\mathbb{Z})$ with $\mathbb{Z}^{\mathcal{O}}$. If we also fix a labeling on the vertices and edges of the graph, then $d_{\mathcal{O}}$ is represented by the matrix B^T, where B is the $n \times m$ vertex-edge incidence matrix of G. In this case, the lattice of integral cuts $L(G)$ is $\mathrm{Image}(B^T) \hookrightarrow \mathbb{Z}^m$. It is a well-known classical fact (due to Poincaré) that the matrix B is totally unimodular in the sense of Example 4.2.3 (see, e.g., [10, Proposition 5.3]). Therefore Theorem 4.3.1(iii) and Remark 4.3.2 apply to this situation. The Delaunay cell decomposition corresponding to the lattice $L(G)$ will be denoted by $\mathrm{Del}(L(G))$.

Here we list some properties of $L(G)$ from [59]. Elements of $L(G)$ are integral 1-coboundaries. A 1-coboundary is called *elementary* if it has minimal nonempty support in $L(G)$. An elementary element $f \in L(G)$ is called *primitive* if $f(v) \in \{-1, 0, +1\}$ for all $v \in V(G)$. It follows from the total unimodularity that every elementary element of $L(G)$ is an integral multiple of a primitive element of $L(G)$ (see, e.g., [59, §1 and §5]). Primitive elements of $L(G)$ correspond precisely to *bonds* (i.e. minimal edge-cuts, or, equivalently, edge-cuts connecting two connected subgraphs) (see, e.g., [59, §1.3]). If $f, g \in L(G)$, we say that g *conforms* to f if $f(e)g(e) > 0$ for all $e \in \mathcal{O}$ with $g(e) \neq 0$. For any $0 \neq f \in L(G)$, there exists a primitive element conforming to f ([59, 1.23]). Moreover, f can be represented as a sum of primitive elements, each conforming to f ([59, 1.24]).

CHAPTER VII

GRAPHIC ORIENTED MATROID IDEAL AND LAWRENCE IDEAL

We next study some natural ideals associated to the cell complexes introduced in §6. See [8] and [49] for a more general study of such constructions.

7.1 Graphic oriented matroid ideal

An *oriented hyperplane arrangement* is a real hyperplane arrangement along with a choice of a "positive side" for each hyperplane. Equivalently, one may fix a set of linear forms vanishing on hyperplanes to fix the "orientation". For any oriented hyperplane arrangement one can define (see [49]) the associated *oriented matroid ideal*: let $\{h_j\}$ be m nonzero linear forms defining the hyperplane arrangement $\mathcal{A}$ with hyperplanes $\mathcal{H}_j = \{\mathbf{p} \in V : h_j(\mathbf{p}) = c_j\}$ in a real affine space V. The oriented matroid ideal associated to $\mathcal{A}$ is the ideal in $2m$ variables of the form:

$$\mathbf{O}_{\mathcal{A}} = \langle \mathbf{m}(\mathbf{p}) : \mathbf{p} \in V \rangle \subset K[\mathbf{w}, \mathbf{z}]$$

where for each $\mathbf{p} \in V$

$$\mathbf{m}(\mathbf{p}) = \prod_{h_i(\mathbf{p}) > c_i} w_i \prod_{h_i(\mathbf{p}) < c_i} z_i \ .$$

Note that any two points in the relative interior of a cell will give rise to the same monomial.

Consider the hyperplane arrangement $\mathcal{H}_G^{q,c}$ (defined in §6.1) which is contained in a codimension 2 affine subspace of $C^0(G, \mathbb{R})$. Fixing an orientation $\mathcal{O}$ of the graph G will fix the linear forms $(df)(e) = f(e_+) - f(e_-)$ for $e \in \mathcal{O}$ and gives an orientation to the hyperplane arrangement $\mathcal{H}_G^{q,c}$. The oriented matroid ideal associated to this oriented hyperplane arrangement $\mathcal{H}_G^{q,c}$ will be denoted by $\mathbf{O}_G^q$ (instead of $\mathbf{O}_{\mathcal{H}_G^{q,c}}$) and will be called the *graphic oriented matroid ideal* associated to G and q. It follows from the discussion in §6.1 that this ideal is independent of the choice of the real number $c > 0$. In this situation, we may consider the variables $\mathbf{w}$ as $\{y_e : e \in \mathcal{O}\}$ and the variables $\mathbf{z}$ as $\{y_{\bar{e}} : e \in \mathcal{O}\}$ and then $\mathbf{O}_G^q \subset \mathbf{S}$.

7.2 Graphic Lawrence ideal

For any embedded integral lattice $L \hookrightarrow \mathbb{Z}^m$ one can define (see [58, Chapter 7]) a binomial ideal $\mathbf{J}_L$ in $2m$ variables, called the *Lawrence ideal* of L, by the following formula:

$$\mathbf{J}_L = \langle \mathbf{w}^{a^+}\mathbf{z}^{a^-} - \mathbf{w}^{a^-}\mathbf{z}^{a^+} : a^+, a^- \in \mathbb{N}^m,\ a = a^+ - a^- \in L \rangle \subset K[\mathbf{w}, \mathbf{z}] \ .$$

When the lattice L is unimodular, the Lawrence ideal $\mathbf{J}_L$ is called unimodular ([8]).

For simplicity, the unimodular Lawrence ideal associated to the unimodular lattice of integral cuts $L(G)$ will be denoted by $\mathbf{J}_G$ (instead of $\mathbf{J}_{L(G)}$) and will be called the *graphic Lawrence ideal* of G. Again, we may consider the variables $\mathbf{w}$ as $\{y_e : e \in \mathcal{O}\}$ and the variables $\mathbf{z}$ as $\{y_{\bar{e}} : e \in \mathcal{O}\}$ and then $\mathbf{J}_G \subset \mathbf{S}$.

7.3 Labeling $\mathcal{B}_G^{q,c}$ and the minimal free resolution of $\mathbf{O}_G^q$

The bounded polyhedral cell complex $\mathcal{B}_G^{q,c}$ (defined in §6.1) supports a minimal free resolution for the ideal $\mathbf{O}_G^q$. To see this, we need to label the vertices of $\mathcal{B}_G^{q,c}$ appropriately: each vertex $f \in \mathcal{B}_G^{q,c}$ is labeled by the monomial

$$\mathbf{m}(f) = \prod_{\substack{e \in \mathbb{E}(G) \\ (df)(e) > 0}} y_e \ . \tag{17}$$

Remark 7.3.1. Fixing an orientation $\mathcal{O}$ will result in the factorization of $\mathbf{m}(f)$ as

$$\mathbf{m}(f) = \prod_{\substack{e\in\mathcal{O}\\ f(e_+)-f(e_-)>0}} y_e \prod_{\substack{e\in\mathcal{O}\\ f(e_-)-f(e_+)>0}} y_{\bar{e}} \ .$$

In this way, we obtain a labeling of all cells by the least common multiple construction. It is easily seen that the label of any cell will be $\mathbf{m}(f)$ (as in (17)) for any point f in the relative interior of that cell.

The following result is an application of [49, Theorem 1.3(b)] for the hyperplane arrangement $\mathcal{H}_G^q$.

Theorem 7.3.2. *The labeled polyhedral cell complex $\mathcal{B}_G^{q,c}$ gives a $C^1(G,\mathbb{Z})$-graded minimal free resolution for $\mathbf{O}_G^q$. In particular, $\mathbf{O}_G^q$ is minimally generated by the monomials $\mathbf{m}(f)$, as f ranges over the vertices of $\mathcal{B}_G^{q,c}$.*

The fact that there is no unit in the corresponding differential maps is immediate from the description of the labelings. All subcomplexes $(\mathcal{B}_G^{q,c})_{\leq \mathbf{m}}$ are in fact contractible, by a result of Björner and Ziegler ([14, Theorem 4.5.7]). See [49] for more details, and Example 7.5.5 and Figure 11 for an example.

7.4 Labeling $\mathrm{Del}(L(G))$ and the minimal free resolution of $\mathbf{J}_G$

Fix an arbitrary orientation $\mathcal{O} \subset \mathbb{E}(G)$ of G and consider the lattice of integral cuts $L(G)$ as in §6.2. As we have already discussed, it comes equipped with a canonical polyhedral cell decomposition of the ambient real vector space $L(G)_{\mathbb{R}} = L(G) \otimes \mathbb{R} = \mathrm{Image}(d_{\mathcal{O}} \colon C^0(G,\mathbb{R}) \to C^1_{\mathcal{O}}(G,\mathbb{R}))$. This polyhedral cell decomposition, denoted by $\mathrm{Del}(L(G))$, can be thought of as an infinite hyperplane arrangement (Theorem 4.3.1(iii)), or more naturally, as the Delaunay decomposition of the ambient space with respect to the lattice $L(G)$ and the metric induced by its natural pairing (16) (See Remark 4.3.2(ii)). We make this a labelled cell complex by assigning the label

$$\mathbf{b}(a) = \prod_{e \in \mathbb{E}(G)} y_e^{a(e)} \tag{18}$$

to each vertex $a \in L(G) \hookrightarrow C^1(G, \mathbb{R})$.

Remark 7.4.1. Fixing an orientation $\mathcal{O}$ will result in the factorization of this Laurent monomial as

$$\mathbf{b}(a) = \prod_{e \in \mathcal{O}} y_e^{a(e)} \prod_{e \in \bar{\mathcal{O}}} y_e^{-a(e)} = \prod_{e \in \mathcal{O}} y_e^{a(e)} / \prod_{e \in \mathcal{O}} y_{\bar{e}}^{a(e)}$$

for $a \in L(G)$.

As usual, we extend the labeling to all faces by the least common multiple rule. The associated complex of free $C^1(G, \mathbb{Z})$-graded $\mathbf{S}$-modules (see §5.1) is not $\mathbf{S}$-finite. By [8, Theorem 3.1] this complex is a minimal cellular free resolution of the (Laurent) monomial module generated by the labels of the lattice points in $L(G)$. This Laurent monomial module can be thought of as the "universal cover" of $\mathbf{J}_G$; the Delaunay cell complex is invariant under the translation by $L(G)$ (Theorem 4.3.1 and Remark 4.3.2), and the labeling is also compatible with this action. So we obtain a well-defined finite cell complex on the quotient torus $L(G)_{\mathbb{R}}/L(G)$, which we denote by $\mathrm{Del}(L(G))/L(G)$. The following theorem is an application of [8, Theorem 3.5] (or [9, Theorem 3.2]) to our setting.

Theorem 7.4.2. *The quotient cell complex* $\mathrm{Del}(L(G))/L(G)$ *supports a* $(C^1(G, \mathbb{Z})/\Lambda)$*-graded minimal free resolution for* $\mathbf{J}_G$.

Here Λ is the image of the (unrestricted) coboundary map $d : C^0(G, \mathbb{Z}) \to C^1(G, \mathbb{Z})$ (see Remark 6.2.1).

7.5 Gröbner relation between $\mathbf{J}_G$ and $\mathbf{O}_G^q$

Recall that the hyperplane arrangement $\mathcal{H}_G^{q,c}$ is naturally sitting inside $C^0(G, \mathbb{R})$, and the Delaunay decomposition $\mathrm{Del}(L(G))$ is an infinite hyperplane arrangement naturally sitting inside $C^1_{\mathcal{O}}(G,\mathbb{R})$. The obvious map between these ambient spaces is the (restricted) coboundary map $d_{\mathcal{O}}\colon C^0(G,\mathbb{R}) \to C^1_{\mathcal{O}}(G,\mathbb{R})$. As we will see, this map relates the corresponding hyperplane arrangements and cell complexes, and this relation translates into precise algebraic relations between $\mathbf{J}_G$ and $\mathbf{O}^q_G$.

First note that $\mathrm{Ker}(d) = \mathrm{Ker}(d_{\mathcal{O}})$ is the 1-dimensional space of constant functions on $V(G)$, and we have

$$L(G)_{\mathbb{R}} = \mathrm{Image}(d_{\mathcal{O}}) \cong C^0(G,\mathbb{R})/\operatorname{Ker}(d) \cong C^0(G,\mathbb{R}) \cap (\mathrm{Ker}(d))^{\perp} \ .$$

Let $e \in \mathbb{E}(G)$. Under the induced isomorphism $d_{\mathcal{O}}\colon C^0(G,\mathbb{R}) \cap (\mathrm{Ker}(d))^{\perp} \xrightarrow{\sim} L(G)_{\mathbb{R}}$, the hyperplane

$$\mathcal{H}_e|_{(\mathrm{Ker}(d))^{\perp}} = \{f \in C^0(G,\mathbb{R}) : (df)(e) = 0\} \cap (\mathrm{Ker}(d))^{\perp}$$

is mapped to the hyperplane

$$\mathcal{G}_e = \{a \in L(G)_{\mathbb{R}} : \varphi_e(a) = 0\} \ ,$$

where φ_e is the restriction of the functional $e = e^{**} \in C_1(G,\mathbb{Z})$ to $L(G)_{\mathbb{R}}$. By Example 4.2.3, Proposition 4.3.1(iii), and Remark 4.3.2(ii), the hyperplanes $\mathcal{G}_e$ are precisely the hyperplanes passing through the origin in $\mathrm{Del}(L(G))$.

Recall from §6.1 that the hyperplane arrangement $\mathcal{H}^{q,c}_G$ has another hyperplane defined by

$$(\mathcal{H}^{q,c})'|_{(\mathrm{Ker}(d))^{\perp}} = \{f \in C^0(G,\mathbb{R}) : \sum_{v\neq q} f(v) = c\} \cap (\mathrm{Ker}(d))^{\perp} \ . \tag{19}$$

The real vector space $L(G)_{\mathbb{R}}$ is spanned by $\{d_{\mathcal{O}}(\chi_v) : v \neq q\}$. Under the induced isomorphism $d_{\mathcal{O}}\colon C^0(G,\mathbb{R}) \cap (\mathrm{Ker}(d_{\mathcal{O}}))^{\perp} \xrightarrow{\sim} L(G)_{\mathbb{R}}$, the hyperplane (19) is mapped to the affine hyperplane

$$\mathcal{G}^{q,c} = \{a \in C^1(G,\mathbb{R}) : a = \sum_{v\neq q} f(v) d_{\mathcal{O}}(\chi_v) \text{ with } \sum_{v\neq q} f(v) = c\} \ .$$

This is a hyperplane passing through all points $\{c \cdot d_{\mathcal{O}}(\chi_v) : v \neq q\}$.

We denote the restriction of the arrangement $\{\mathcal{G}_e\}_{e\in\mathbb{E}(G)}$ to the affine hyperplane $\mathcal{G}^{q,c}$ by $\mathcal{G}_G^{q,c}$. It follows that $\mathcal{G}_G^{q,c}$, upto a linear transformation, coincides with the arrangement $\mathcal{H}_G^{q,c}$, and therefore its bounded complex, which we denote by $\mathcal{A}_G^{q,c}$, may be identified with $\mathcal{B}_G^{q,c}$.

Next we show that these geometric considerations nicely relate the labeling of $\mathcal{B}_G^{q,c}$ by monomials (described in §7.3) with the natural labeling of $\mathcal{A}_G^{q,c}$ induced by $\mathrm{Del}(L(G))$ (described in §7.4). For this purpose, we will see that it is most natural to assume $0 < c < 1$. With this assumption, if the hyperplane $\mathcal{G}^{q,c}$ intersects a Delaunay cell C, then C must contain the origin. By the least common multiple labeling rule, this means that all such cells C have monomial labels in $\mathbf{S}$.

To concretely describe these induced monomial labels, it suffices to find the labels of the vertices in $\mathcal{G}_G^{q,c}$ induced from the labels of the rays in the central hyperplane arrangement $\{\mathcal{G}_e : e \in \mathbb{E}(G)\}$. These rays correspond to bonds $d_{\mathcal{O}}(\chi_B)$ for $B \subset V(G)$ (see §6.2). Such a ray intersects $\mathcal{G}^{q,c}$ if and only if for some real number $t > 0$ we have

$$td_{\mathcal{O}}(\chi_B) = \sum_{v\neq q} f(v) d_{\mathcal{O}}(\chi_v) \ ,$$

or equivalently

$$d_{\mathcal{O}}(t\chi_B - \sum_{v\neq q} f(v)\chi_v) = 0 \ .$$

Since the kernel of $d_{\mathcal{O}}$ consists of constant functions we must have

$$t\sum_{v\in B}\chi_v - \sum_{v\neq q} f(v)\chi_v = k\sum_{v}\chi_v \tag{20}$$

for some constant $k \in \mathbb{R}$.

We claim that $q \notin B$. Indeed, if $q \in B$, then evaluating (20) at q we obtain $k = t$ and therefore

$$t\sum_{v\in B^c}\chi_v = -\sum_{v\neq q} f(v)\chi_v \ .$$

This implies that $f(v) = -t < 0$ for $v \in B^c$ and $f(v) = 0$ for $v \in B \backslash \{q\}$. But this is impossible because $\sum_{v \neq q} f(v) = c$ by assumption.

Since $q \notin B$, by evaluating (20) at q we obtain $k = 0$ and therefore

$$t \sum_{v \in B} \chi_v = \sum_{v \neq q} f(v) \chi_v \ ,$$

which implies that $f(v) = t$ for $v \in B$ and $f(v) = 0$ for $v \in B^c \backslash \{q\}$. Since $\sum_{v \neq q} f(v) = c$, we must have $t = \frac{c}{|B|}$. Conversely, for any nonempty subset $B \subset V(G) \backslash \{q\}$, the ray corresponding to the simple cut $d_{\mathcal{O}}(\chi_B)$ intersects $\mathcal{G}^{q,c}$ at the point $\frac{c}{|B|} d_{\mathcal{O}}(\chi_B)$. If we fix $0 < c < 1$, then we always have $0 < \frac{c}{|B|} < 1$ which means that the point of intersection belongs to a cell in $\mathrm{Del}(L(G))$ containing the origin. We summarize these observations in the following proposition.

Proposition 7.5.1. *Let $\emptyset \neq B \subset V(G)$. The ray corresponding to the bond $d_{\mathcal{O}}(\chi_B)$ intersects $\mathcal{G}^{q,c}$ if and only if $q \notin B$. If $0 < c < 1$, then the point of intersection belongs to a cell in* $\mathrm{Del}(L(G))$ *containing the origin.*

The vertices of $\mathcal{A}_G^{q,c}$ are the points of intersections with these rays. For each vertex of $\mathcal{A}_G^{q,c}$ we may assign the label corresponding to the 1-dimensional cell of $\mathrm{Del}(L(G))$ containing that vertex. If we assume $0 < c < 1$, this is a (non-Laurent) monomial label that coincides with the labeling rule for $\mathcal{B}_G^{q,c}$ described in §7.3. From this point of view, it is straightforward to describe these labels combinatorially.

Lemma 7.5.2. *For any $A \subsetneq V(G)$ with $q \in A$ the following holds.*

(i) The label of the point $d_{\mathcal{O}}(\chi_{A^c})$ in the labeled complex $\mathrm{Del}(L(G))$ *is*

$$\mathbf{b}(d_{\mathcal{O}}(\chi_{A^c})) = \frac{\prod_{e \in \mathbb{E}(A^c, A)} y_e}{\prod_{e \in \mathbb{E}(A, A^c)} y_e} \ .$$

(ii) For $0 < c < 1$, the induced label on the vertex $\mathcal{A}_G^{q,c}$ corresponding to the bond $d_{\mathcal{O}}(\chi_{A^c})$ is

$$\prod_{e \in \mathbb{E}(A^c, A)} y_e \ .$$

Proof. (i) By (18) we have

$$\begin{aligned}\mathbf{b}(d_{\mathcal{O}}(\chi_{A^c})) &= \prod_{e\in\mathbb{E}(G)} y_e^{d(\chi_{A^c})(e)} \\ &= \prod_{e\in\mathbb{E}(G)} y_e^{\chi_{A^c}(e_+)-\chi_{A^c}(e_-)} \\ &= \frac{\prod_{e\in\mathbb{E}(A^c,A)} y_e}{\prod_{e\in\mathbb{E}(A,A^c)} y_e}\,.\end{aligned}$$

(ii) The label of the origin is $\mathbf{b}(\mathbf{0}) = 1$. Therefore, by the least common multiple construction, the label of the one-dimensional cell $\{\mathbf{0}, d_{\mathcal{O}}(\chi_{A^c})\}$ in $\mathrm{Del}(L(G))$ is $\prod_{e\in\mathbb{E}(A^c,A)} y_e$. The result now follows from Proposition 7.5.1. □

Since the labeled complex $\mathcal{A}_G^{q,c}$ (for $0 < c < 1$) coincides with the labeled complex $\mathcal{B}_G^{q,c}$, we might as well think of the ideal $\mathbf{O}_G^q$ as constructed from $\mathcal{A}_G^{q,c}$. The advantage of this point of view is a precise Gröbner relation between $\mathbf{O}_G^q$ and $\mathbf{J}_G$ coming from the described relation of $\mathcal{A}_G^{q,c}$ and $\mathrm{Del}(L(G))$.

Lemma 7.5.3. *Intersection of cells in* $\mathrm{Del}(L(G))$ *with the hyperplane* $\mathcal{G}^{q,c}$ *induces a bijection between* $(i+1)$*-dimensional cells of* $\mathrm{Del}(L(G))/L(G)$ *and* i*-dimensional cells of* $\mathcal{A}_G^{q,c}$ *for all* $0 \le i \le n-2$.

Proof. It suffices to only consider cells in $\mathrm{Del}(L(G))$ containing the origin; all other cells in $\mathrm{Del}(L(G))$ can be obtained by translating such cells by $L(G)$. The primitive (or indecomposable) elements of $L(G)$ correspond to bonds (see §6.2). Therefore the vertex set of any cell in $\mathrm{Del}(L(G))$ containing the origin is of the form $\{\mathbf{0}\} \cup P$ for some $P \subset \{d_{\mathcal{O}}(\chi_B) : \emptyset \neq B \subset V(G)\}$. Since $d_{\mathcal{O}}(\chi_{B^c}) = -d_{\mathcal{O}}(\chi_B)$, it suffices to restrict our attention to the case where $P \subset \{d_{\mathcal{O}}(\chi_B) : \emptyset \neq B \subset V(G), q \notin B\}$. By Proposition 7.5.1, these are precisely those cells that have nonempty intersection with $\mathcal{G}^{q,c}$. □

Proposition 7.5.4.

(i) A generating set for the ideal $\mathbf{J}_G$ *is*

$$\left\{ \prod_{e \in \mathbb{E}(A^c, A)} y_e - \prod_{e \in \mathbb{E}(A, A^c)} y_e : A \subsetneq V(G), q \in A \right\}.$$

If we consider only those subsets A *of* $V(G)$ *such that both* $G[A]$ *and* $G[A^c]$ *are connected, then we have a minimal generating set for* $\mathbf{J}_G$.

(ii) The minimal generating set in part (i) is also a Gröbner basis with respect to any *term order (i.e. is a universal Gröbner basis).*

(iii) A minimal generating set for the ideal $\mathbf{O}_G^q$ *is*

$$\left\{ \prod_{e \in \mathbb{E}(A^c, A)} y_e : A \subsetneq V(G), q \in A, G[A] \text{ and } G[A^c] \text{ are connected} \right\}.$$

(iv) $\mathbf{O}_G^q$ *is the initial ideal of* $\mathbf{J}_G$ *with respect to any term order* $\prec_q$ *with the property that*

$$\prod_{e \in \mathbb{E}(A, A^c)} y_e \prec_q \prod_{e \in \mathbb{E}(A^c, A)} y_e$$

for every $A \subsetneq V(G)$ *with* $q \in A$ *such that both* $G[A]$ *and* $G[A^c]$ *are connected.*

Proof. (i) It follows from the discussion in §5.1, Theorem 7.4.2 and [9, proof of Theorem 3.2] that a minimal generating set for $\mathbf{J}_G$ is given by binomials

$$\frac{\mathbf{m}_F}{\mathbf{m}_{F'}} - \frac{\mathbf{m}_F}{\mathbf{m}_{\mathbf{0}}},$$

where F is in a fundamental set of representatives of 1-cells in $\mathrm{Del}(L(G))$ connecting $\mathbf{0}$ to $F = d_{\mathcal{O}}(\chi_{A^c})$ for $A \subsetneq V(G)$ and $q \in A$.

By Lemma 7.5.2(i), we have

$$\mathbf{m}_{F'} = \mathbf{b}(d_{\mathcal{O}}(\chi_{A^c})) = \frac{\prod_{e \in \mathbb{E}(A^c, A)} y_e}{\prod_{e \in \mathbb{E}(A, A^c)} y_e}, \quad \mathbf{m}_{\mathbf{0}} = 1,$$

$$\mathbf{m}_F = \mathrm{lcm}(\mathbf{m}_{F'}, \mathbf{m}_{\mathbf{0}}) = \prod_{e \in \mathbb{E}(A^c, A)} y_e$$

and therefore

$$\frac{\mathbf{m}_F}{\mathbf{m}_{F'}} - \frac{\mathbf{m}_F}{\mathbf{m}_\mathbf{0}} = \prod_{e \in \mathbb{E}(A,A^c)} y_e - \prod_{e \in \mathbb{E}(A^c,A)} y_e \ .$$

The rest of part (i) is immediate.

(ii) follows from the general fact that in any Lawrence ideal, a minimal binomial generating set is a Gröbner basis with respect to any term order ([58, Theorem 7.1]). In our concrete situation, one can also easily verify (as in the proof of Theorem 3.2.2 given in [48, Theorem 5.1]) that the S-polynomial of the two binomials corresponding to the cuts (A, A^c) and (B, B^c) can be reduced to zero by the binomials corresponding to the cuts $(A \backslash B, (A \backslash B)^c)$ and $(B \backslash A, (B \backslash A)^c)$.

(iii) It follows from the discussion in §5.1, Theorem 7.3.2, and the fact that the labeled cell complex $\mathcal{A}_G^{q,c}$ coincides with the labeled complex $\mathcal{B}_G^{q,c}$, that a minimal generating set for $\mathbf{O}_G^q$ is given by the monomials $\mathbf{m}_F$ as F varies over the vertices of the bounded cell complex $\mathcal{A}_G^{q,c}$. By Proposition 7.5.1 and Lemma 7.5.2(ii), these labels are precisely of the form

$$\prod_{e \in \mathbb{E}(A^c,A)} y_e$$

for $A \subsetneq V(G)$ with $q \in A$ such that the edges between (A, A^c) form a bond.

(iv) follows from (ii) and (iii). □

Example 7.5.5. Consider the graph G depicted in Figure 7 with the fixed orientation $\mathcal{O}$. Let q be the distinguished (red) vertex at the bottom. Acyclic partial orientations of G with unique source at q are depicted in Figures 8–10.

Figure 7: Graph G and a fixed orientation $\mathcal{O}$

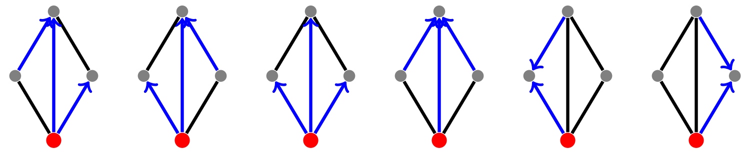

Figure 8: Acyclic partial orientations with 2 components

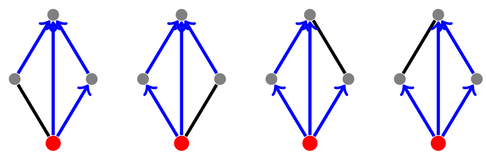

Figure 9: Acyclic partial orientations with 3 components

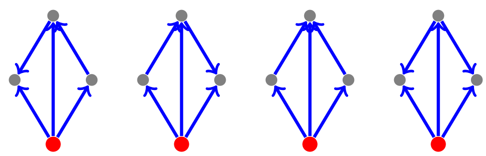

Figure 10: Acyclic partial orientations with 4 components

Consider the arrangement $\mathcal{H}'_G = \{\mathcal{H}_{e_1}, \ldots, \mathcal{H}_{e_5}\}$. The graphic arrangement $\mathcal{H}_G^{q,c}$ (for some $c > 0$) is two-dimensional and is depicted in Figure 11. Its bounded complex $\mathcal{B}_G^{q,c}$ is the bounded part of this figure. Recall that the graphic arrangement "lives in" $C^0(G, \mathbb{R})$, which may be identified with $\mathbb{R}^4$ after fixing a labeling of the vertices. For each hyperplane labeled $\mathcal{H}_e$, the small arrow next to it denotes the side where $(df)(e) > 0$. The hyperplane $\mathcal{H}_{\bar{e}}$ coincides with $\mathcal{H}_e$, but its arrow will be reversed. We have also labeled the 0-cells according to (17).

The polynomial ring $\mathbf{S}$ has 10 variables:

$$\{y_e, y_{\bar{e}} : e \in \mathcal{O}\} = \{y_{e_1}, y_{e_2}, y_{e_3}, y_{e_4}, y_{e_5}; y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_3}, y_{\bar{e}_4}, y_{\bar{e}_5}\} \ .$$

By Theorem 7.3.2, the associated oriented matroid ideal $\mathbf{O}_G^q$ is minimally generated by the labels of the 0-cells:

$$\mathbf{O}_G^q = \langle y_{\bar{e}_1} y_{e_4} y_{e_5}, y_{e_2} y_{e_3} y_{e_5}, y_{\bar{e}_3} y_{e_4}, y_{\bar{e}_1} y_{e_3} y_{e_5}, y_{e_1} y_{e_2}, y_{e_2} y_{e_4} y_{e_5} \rangle \ . \tag{21}$$

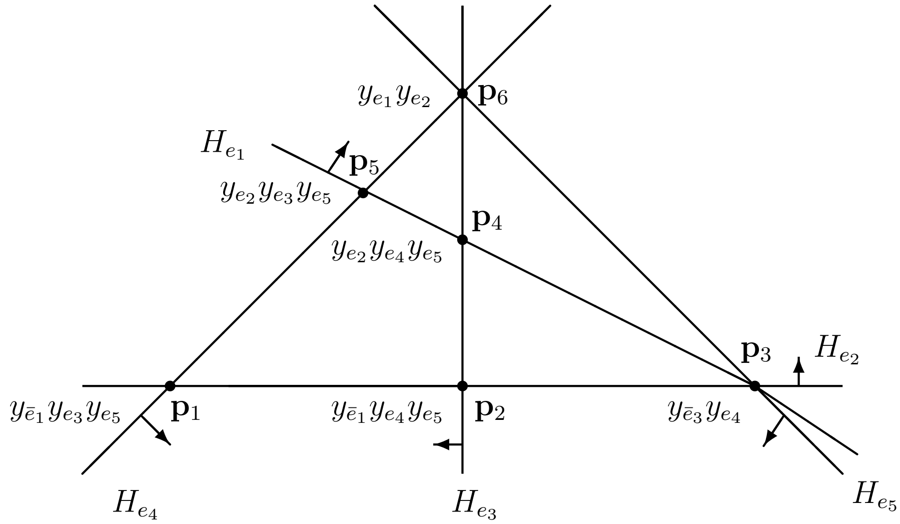

Figure 11: $\mathcal{H}_G^q$, $\mathcal{B}_G^q$, and the monomial labels on the vertices

Note that the indices appearing in the minimal generating set correspond precisely to the oriented edges leaving the connected partition containing q (i.e. the blue edges in Figure 8). This is what we expect by Proposition 7.5.4(iii).

The lattice of integral cuts $L(G)$ is 3-dimensional. Instead of drawing it, we may directly write a minimal generating set for $\mathbf{J}_G$ using Proposition 7.5.4(i):

$$\mathbf{J}_G = \langle y_{\bar{e}_1}y_{e_4}y_{e_5} - y_{e_1}y_{\bar{e}_4}y_{\bar{e}_5}, y_{e_2}y_{e_3}y_{e_5} - y_{\bar{e}_2}y_{\bar{e}_3}y_{\bar{e}_5}, y_{\bar{e}_3}y_{e_4} - y_{e_3}y_{\bar{e}_4}, y_{\bar{e}_1}y_{e_3}y_{e_5} - y_{e_1}y_{\bar{e}_3}y_{\bar{e}_5},$$

$$y_{e_1}y_{e_2} - y_{\bar{e}_1}y_{\bar{e}_2}, y_{e_2}y_{e_4}y_{e_5} - y_{\bar{e}_2}y_{\bar{e}_4}y_{\bar{e}_5}\rangle \ .$$

The first term in each binomial is the dominant term for the term order $\prec_q$. The bounded complex $\mathcal{B}_G^q$ has six 0-cells $\{\mathbf{p}_1, \ldots, \mathbf{p}_6\}$, nine 1-cells $\{E_1, \ldots, E_9\}$, and four 2-cells $\{F_1, \ldots, F_4\}$. These numbers correspond to the acyclic orientations of Figure 8, Figure 9, and Figure 10, as well as the Betti numbers of $\mathbf{O}_G^q$ and $\mathbf{J}_G$. Moreover, $\mathcal{B}_G^q$ supports a minimal free resolution for $\mathbf{O}_G^q$. To explicitly describe this minimal resolution, let

$$E_1 = \{\mathbf{p}_1, \mathbf{p}_2\}, \quad E_2 = \{\mathbf{p}_2, \mathbf{p}_3\}, \quad E_3 = \{\mathbf{p}_1, \mathbf{p}_5\}, \quad E_4 = \{\mathbf{p}_2, \mathbf{p}_4\}, \quad E_5 = \{\mathbf{p}_3, \mathbf{p}_4\}$$

$$E_6 = \{\mathbf{p}_4, \mathbf{p}_5\}, \quad E_7 = \{\mathbf{p}_5, \mathbf{p}_6\}, \quad E_8 = \{\mathbf{p}_4, \mathbf{p}_6\}, \quad E_9 = \{\mathbf{p}_3, \mathbf{p}_6\} \ ,$$

$$F_1 = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4, \mathbf{p}_5\}, \quad F_2 = \{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}, \quad F_3 = \{\mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6\}, \quad F_4 = \{\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_6\} \ .$$

We extend the labeling on the vertices to the whole $\mathcal{B}_G^q$ by the least common multiple construction. For example,

$$\mathbf{m}_{E_2} = y_{\bar{e}_1} y_{\bar{e}_3} y_{e_4} y_{e_5}, \ \mathbf{m}_{E_4} = y_{\bar{e}_1} y_{e_2} y_{e_4} y_{e_5}, \ \mathbf{m}_{E_5} = y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5}, \ \mathbf{m}_{E_6} = y_{e_2} y_{e_3} y_{e_4} y_{e_5},$$

$$\mathbf{m}_{F_2} = y_{\bar{e}_1} y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5} \ .$$

Then the minimal resolution of $\mathbf{O}_G^q$ is as follows.

$$0 \to \bigoplus_{i=1}^{4} \mathbf{S}(-\mathbf{m}_{F_i}) \xrightarrow{\partial_2} \bigoplus_{i=1}^{9} \mathbf{S}(-\mathbf{m}_{E_i}) \xrightarrow{\partial_1} \bigoplus_{i=1}^{6} \mathbf{S}(-\mathbf{m}_{\mathbf{p}_i}) \xrightarrow{\partial_0} \mathbf{S} \twoheadrightarrow \mathbf{S}/\mathbf{O}_G^q \ .$$

As usual, assume $[F]$ denotes the generator of $\mathbf{S}(-\mathbf{m}_F)$. The homogenized differential operator of the cell complex $(\partial_0, \partial_1, \partial_2)$ is as described in (13). For example

$$\partial_0([\mathbf{p}_i]) = \mathbf{m}_{\mathbf{p}_i} = \mathbf{m}(\mathbf{p}_i) \ ,$$

$$\partial_1([E_6]) = \frac{y_{e_2} y_{e_3} y_{e_4} y_{e_5}}{y_{e_2} y_{e_4} y_{e_5}} [\mathbf{p}_4] - \frac{y_{e_2} y_{e_3} y_{e_4} y_{e_5}}{y_{e_2} y_{e_3} y_{e_5}} [\mathbf{p}_5] = y_{e_3}[\mathbf{p}_4] - y_{e_4}[\mathbf{p}_4] \ ,$$

$$\begin{aligned} \partial_2([F_2]) &= \frac{y_{\bar{e}_1} y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5}}{y_{\bar{e}_1} y_{\bar{e}_3} y_{e_4} y_{e_5}} [E_2] - \frac{y_{\bar{e}_1} y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5}}{y_{\bar{e}_1} y_{e_2} y_{e_4} y_{e_5}} [E_4] + \frac{y_{\bar{e}_1} y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5}}{y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5}} [E_5] \\ &= y_{e_2}[E_2] - y_{\bar{e}_3}[E_4] + y_{\bar{e}_1}[E_5] \ . \end{aligned}$$

Although $\mathbf{J}_G$ has the same Betti table as $\mathbf{O}_G^q$, it is not possible to read the minimal free resolution for $\mathbf{J}_G$ directly from $\mathcal{B}_G^q$; one really needs to consider the cell decomposition of the torus $L(G)_{\mathbb{R}}/L(G)$.

Example 7.5.6. Consider the graph K_3 with a fixed orientation as in Figure 12.

The lattice of integral cuts $L(G)$ is two-dimensional and is depicted in Figure 13. This picture should be compared with Figure 5 (see Remark 5.3.5). This lattice

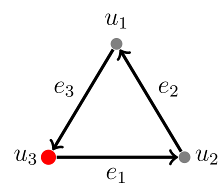

Figure 12: Graph K_3 and a fixed orientation $\mathcal{O}$

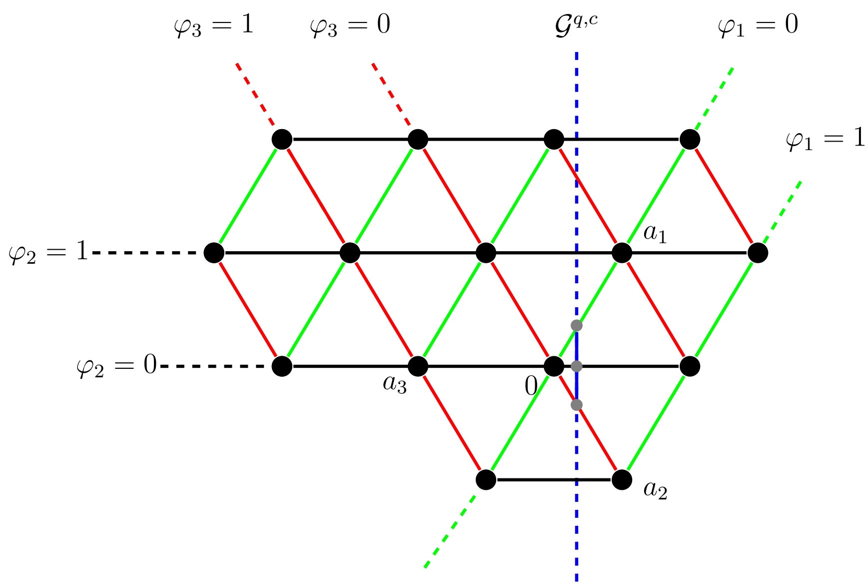

Figure 13: Cut lattice $L(G)$

"lives in" $C^1_{\mathcal{O}}(G, \mathbb{R}) = \text{span}\{e_1^*, e_2^*, e_3^*\} \cong \mathbb{R}^3$. In the picture $a_1 = d_{\mathcal{O}}(\chi_{u_1}) = e_2^* - e_3^*$, $a_2 = d_{\mathcal{O}}(\chi_{u_2}) = e_1^* - e_2^*$, and $a_3 = d_{\mathcal{O}}(\chi_{u_3}) = e_3^* - e_1^*$.

The cell decomposition $\text{Del}(L(G))$ is the Delaunay decomposition of $L(G)_{\mathbb{R}}$ with respect to the cut lattice and the usual Euclidean metric (cf. Remark 4.3.2(ii)), which coincides with an infinite hyperplane arrangement (Theorem 4.3.1(ii) and §6.2). The hyperplanes at the origin are defined by $\varphi_i = e_i|_{L(G)_{\mathbb{R}}} = 0$. The quotient cell decomposition $\text{Del}(L(G))/L(G)$ of the torus $L(G)_{\mathbb{R}}/L(G)$ has one 0-cell $\{\mathbf{p}\}$ (the orbit of the origin), three 1-cells $\{E, E', E''\}$ (the orbits of the green, red, and black edges), and two 2-cells $\{F, F'\}$ (the orbits of the upward and downward triangles). Assume that $q = u_3$ is the distinguished vertex. The hyperplane $\mathcal{G}^{q,c}$ is the hyperplane passing through points ca_1 and ca_2. In the figure c is roughly $\frac{1}{3}$. The bounded complex of the intersection of this hyperplane with the arrangement at the origin is denoted by a solid blue segment. This is $\mathcal{A}_G^{q,c}$, which is combinatorially equivalent to $\mathcal{B}_G^{q,c}$ (via the coboundary map).

In Figure 14, we have chosen a fundamental domain for the lattice, and have labeled all cells of this fundamental domain according to the recipe described in §7.4. This labeling induces a labeling on $\mathcal{A}_G^{q,c}$ (compatible with the labeling of $\mathcal{B}_G^{q,c}$) which is also given in the figure. The labelled cell complexes in Figure 14 are enough to completely describe minimal free resolutions for $\mathbf{J}_G$ and for $\mathbf{O}_G$. Concretely, the minimal resolution of $\mathbf{J}_G$ is as follows:

$$0 \to \mathbf{S}(-\mathbf{m}_F) \oplus \mathbf{S}(-\mathbf{m}_{F'}) \xrightarrow{\partial_2} \mathbf{S}(-\mathbf{m}_E) \oplus \mathbf{S}(-\mathbf{m}_{E'}) \oplus \mathbf{S}(-\mathbf{m}_{E''}) \xrightarrow{\partial_1} \mathbf{S}(-\mathbf{m}_{\mathbf{p}}) \ .$$

As usual, assume $[F]$ denotes the generator of $\mathbf{S}(-\mathbf{m}_F)$. The labels of cells in $\mathrm{Del}(L(G))/L(G)$ are:

$$\mathbf{m}_E = y_{e_2}y_{\bar{e}_3} \ , \quad \mathbf{m}_{E'} = y_{e_1}y_{\bar{e}_3} \ , \quad \mathbf{m}_{E''} = y_{e_1}y_{\bar{e}_2} \ ,$$

$$\mathbf{m}_F = y_{e_1}y_{e_2}y_{\bar{e}_3} \ , \quad \mathbf{m}_{F'} = y_{e_1}y_{\bar{e}_2}y_{\bar{e}_3} \ .$$

The homogenized differential operator (see (13)) of the cell complex (∂_1, ∂_2) is described as follows:

$$\partial_1([E]) = \frac{y_{e_2}y_{\bar{e}_3}}{1}[\mathbf{p}] - \frac{y_{e_2}y_{\bar{e}_3}}{\frac{y_{e_2}y_{\bar{e}_3}}{y_{\bar{e}_2}y_{e_3}}}[\mathbf{p}] = (y_{e_2}y_{\bar{e}_3} - y_{\bar{e}_2}y_{e_3})[\mathbf{p}] \ ,$$

$$\partial_1([E']) = \frac{y_{e_1}y_{\bar{e}_3}}{\frac{y_{e_1}y_{\bar{e}_3}}{y_{\bar{e}_1}y_{e_3}}}[\mathbf{p}] - \frac{y_{e_1}y_{\bar{e}_3}}{1}[\mathbf{p}] = (y_{\bar{e}_1}y_{e_3} - y_{e_1}y_{\bar{e}_3})[\mathbf{p}] \ ,$$

$$\partial_1([E'']) = \frac{y_{e_1}y_{\bar{e}_2}}{\frac{y_{e_1}y_{\bar{e}_2}}{y_{\bar{e}_1}y_{e_2}}}[\mathbf{p}] - \frac{y_{e_1}y_{\bar{e}_2}}{1}[\mathbf{p}] = (y_{\bar{e}_1}y_{e_2} - y_{e_1}y_{\bar{e}_2})[\mathbf{p}] \ ,$$

$$\partial_2([F]) = \frac{y_{e_1}y_{e_2}y_{\bar{e}_3}}{y_{e_2}y_{\bar{e}_3}}[E] - \frac{y_{e_1}y_{e_2}y_{\bar{e}_3}}{\frac{y_{e_1}y_{e_2}y_{\bar{e}_3}}{y_{e_3}}}[E''] + \frac{y_{e_1}y_{e_2}y_{\bar{e}_3}}{y_{e_1}y_{\bar{e}_3}}[E'] = y_{e_1}[E] - y_{e_3}[E''] + y_{e_2}[E'] \ ,$$

$$\partial_2([F']) = \frac{y_{e_1}y_{\bar{e}_2}y_{\bar{e}_3}}{\frac{y_{e_1}y_{\bar{e}_2}y_{\bar{e}_3}}{y_{\bar{e}_1}}}[E] - \frac{y_{e_1}y_{\bar{e}_2}y_{\bar{e}_3}}{y_{e_1}y_{\bar{e}_2}}[E''] + \frac{y_{e_1}y_{\bar{e}_2}y_{\bar{e}_3}}{y_{e_1}y_{\bar{e}_3}}[E'] = y_{\bar{e}_1}[E] - y_{\bar{e}_3}[E''] + y_{\bar{e}_2}[E'] \ .$$

Clearly $\mathbf{J}_G$ is the image of ∂_1 after identifying $[\mathbf{p}]$ with 1 (see Proposition 7.5.4). Since the labeling is compatible with the action of the lattice, any translation of this fundamental domain would give rise to the exact same description of the differential maps.

The minimal resolution of $\mathbf{O}_G^q$ can be read from the bounded complex $\mathcal{A}_G^{q,c}$. If we identify the name of each cell in $\mathcal{A}_G^{q,c}$ with the name of the associated cell in $\mathrm{Del}(L(G))$, we have

$$0 \to \mathbf{S}(-\mathbf{m}_F) \oplus \mathbf{S}(-\mathbf{m}_{F'}) \xrightarrow{\tilde{\partial}_1} \mathbf{S}(-\mathbf{m}_E) \oplus \mathbf{S}(-\mathbf{m}_{E'}) \oplus \mathbf{S}(-\mathbf{m}_{E''}) \xrightarrow{\tilde{\partial}_0} \mathbf{S} \ ,$$

where

$$\tilde{\partial}_0([E]) = \mathbf{m}_E = y_{e_2}y_{\bar{e}_3} \ ,$$

$$\tilde{\partial}_0([E']) = \mathbf{m}_{E'} = y_{e_1}y_{\bar{e}_3} \ ,$$

$$\tilde{\partial}_0([E'']) = \mathbf{m}_{E''} = y_{e_1}y_{\bar{e}_2} \ ,$$

$$\tilde{\partial}_1([F]) = \frac{y_{e_1}y_{e_2}y_{\bar{e}_3}}{y_{e_2}y_{\bar{e}_3}}[E] - \frac{y_{e_1}y_{e_2}y_{\bar{e}_3}}{y_{e_1}y_{\bar{e}_3}}[E'] = y_{e_1}[E] - y_{e_2}[E'] \ ,$$

$$\tilde{\partial}_1([F']) = \frac{y_{e_1}y_{\bar{e}_2}y_{\bar{e}_3}}{y_{e_1}y_{\bar{e}_3}}[E'] - \frac{y_{e_1}y_{\bar{e}_2}y_{\bar{e}_3}}{y_{e_1}y_{\bar{e}_2}}[E''] = y_{\bar{e}_2}[E'] - y_{\bar{e}_3}[E''] \ .$$

The ideal $\mathbf{O}_G^q$ is the image of $\tilde{\partial}_0$ (see Proposition 7.5.4). This example is, of course, closely related to Example 5.3.4. The general relationship between these two constructions is explained in Remark 10.0.8.

7.6 Potential theory and Gröbner weight functionals for $\mathbf{J}_G$

Let $C_0(G,\mathbb{R})$ denote the real vector space spanned by $V(G)$, and let $C_1(G,\mathbb{R})$ denote the real vector space spanned by $\mathbb{E}(G)$. The usual boundary operator $\partial\colon C_1(G,\mathbb{R}) \to C_0(G,\mathbb{R})$ is defined by

$$(\partial(\sigma))(v) = \sum_{e_+=v} \sigma(e) - \sum_{e_-=v} \sigma(e) \ .$$

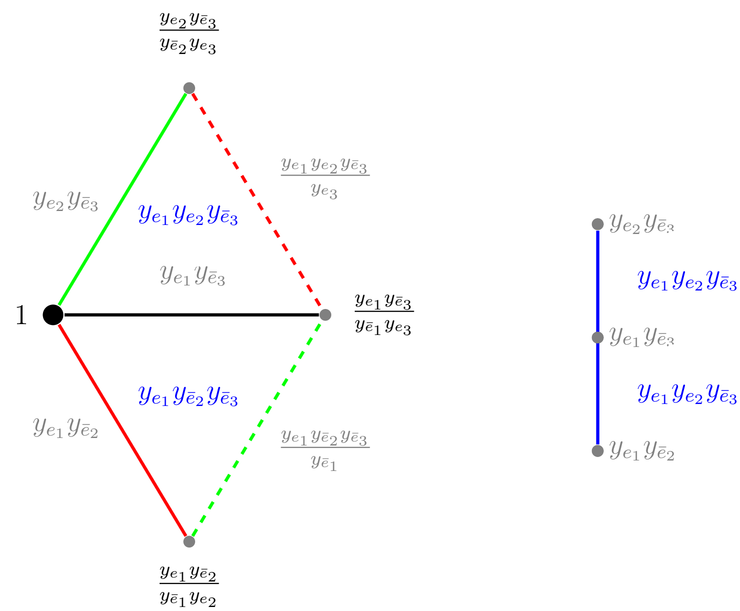

Figure 14: A choice of fundamental domain with labels (left) , $\mathcal{A}_G^{q,c}$ with its induced labels (right)

An element $\sigma \in C_1(G, \mathbb{R})$ gives a map $\sigma \colon C^1(G, \mathbb{Z}) \to \mathbb{R}$ by sending f to $f(\sigma)$. So it may be thought of as a weight functional for the ideal $\mathbf{J}_G$. Our next goal is to study the weight functionals $\sigma \in C_1(G, \mathbb{R})$ that represent the term order $\prec_q$ in Proposition 7.5.4(iv). For our application, a very important class of examples arises from weight functionals representing $<_q$ for $\mathbf{I}_G$ as studied in §3.3 (see Lemma 3.3.1, Definition 3.3.3, or (6)).

Proposition 7.6.1. *Let $\vartheta \in C^0(G, \mathbb{R})$ be any weight functional representing $<_q$ for $\mathbf{I}_G$ (i.e. $\mathbf{M}_G^q = \mathrm{in}_\vartheta(\mathbf{I}_G)$). Then the 1-chain $\sigma \in C_1(G, \mathbb{R})$ defined by*

$$\sigma(e) = \vartheta(e_+) \quad \textit{for all } e \in \mathbb{E}(G)$$

represents a term order $\prec_q$ for $\mathbf{J}_G$ with $\mathbf{O}_G^q = \mathrm{in}_\sigma(\mathbf{J}_G)$.

Proof. By Proposition 7.5.4, the term order $\prec_q$ is characterized by requiring

$$\prod_{e \in \mathbb{E}(A, A^c)} y_e \prec_q \prod_{e \in \mathbb{E}(A^c, A)} y_e$$

for every $A \subsetneq V(G)$, where $q \in A$ with $G[A]$ and $G[A^c]$ connected. Since (see Lemma 7.5.2)

$$\frac{\prod_{e\in\mathbb{E}(A^c,A)} y_e}{\prod_{e\in\mathbb{E}(A,A^c)} y_e} = \prod_{e\in\mathbb{E}(G)} y_e^{d(\chi_{A^c})(e)} ,$$

we have $\mathbf{O}_G^q = \mathrm{in}_\sigma(\mathbf{J}_G)$ if and only if

$$\sigma(d(\chi_{A^c})) = \sum_{e\in\mathbb{E}(G)} \sigma(e)\cdot(d(\chi_{A^c}))(e) > 0 \tag{22}$$

for all bonds $d(\chi_{A^c})(e)$ associated to $A \subsetneq V(G)$ with $q \in A$. Since ∂ is the adjoint to d, (22) is equivalent to

$$\sum_{v\in V(G)} (\partial(\sigma))(v)\cdot\chi_{A^c}(v) > 0\ . \tag{23}$$

Since $\sigma(e) = \vartheta(e_+)$, we have

$$\begin{aligned}(\partial(\sigma))(v) &= \sum_{e_+=v}\sigma(e) - \sum_{e_-=v}\sigma(e)\\ &= \sum_{e_+=v}\vartheta(e_+) - \sum_{e_-=v}\vartheta(e_+)\\ &= \deg(v)\vartheta(v) - \sum_{\{u,v\}\in E(G)}\vartheta(u)\\ &= \Delta(\vartheta)(v)\ .\end{aligned}$$

Therefore (see (4))

$$\sum_{v\in V(G)} (\partial(\sigma))(v)\cdot\chi_{A^c}(v) = \sum_{v\in V(G)} \Delta(\vartheta)(v)\cdot\chi_{A^c}(v) > 0$$

and (23) holds. □

Definition 7.6.2. Let $\vartheta_q \in C^0(G,\mathbb{Z})$ denote the non-negative, integral functional defined in Definition 3.3.3. We denote by λ_q the associated non-negative, integral weight functional in $C_1(G,\mathbb{R})$ defined by

$$\lambda_q(e) = \vartheta_q(e_+) \quad \text{for all } e \in \mathbb{E}(G)$$

as in Proposition 7.6.1 .

7.7 Gröbner cone of $\mathbf{O}_G^q$

Next we will describe the Gröbner cone associated to $\mathbf{O}_G^q$. As in §3.4, this cone is intimately related to potential theory and Green's functions.

The description of this cone is most elegant when G does not have a cut vertex. Cut vertices introduce linear subspaces in the Gröbner cone and are slightly tedious (but similar) to deal with. Throughout this section, we will therefore assume that G is 2-vertex-connected. This condition is equivalent to assuming that the lattice $L(G)$ is indecomposable ([2, Proposition 4]).

Proposition 7.7.1. *Assume G is 2-vertex-connected. Then $\sigma \in C_1(G, \mathbb{R})$ represents a term order $\prec_q$ for $\mathbf{J}_G$ with $\mathbf{O}_G^q = \mathrm{in}_\sigma(\mathbf{J}_G)$ if and only if for all $p \in V(G)\backslash\{q\}$ we have*

$$\beta_p := (\partial(\sigma))(p) > 0 \ .$$

Proof. We have already seen that $\sigma \in C_1(G, \mathbb{R})$ represents a term order $\prec_q$ for $\mathbf{J}_G$ with $\mathbf{O}_G^q = \mathrm{in}_\sigma(\mathbf{J}_G)$ if and only if (23) holds for all bonds $d(\chi_{A^c})(e)$ associated to $A \subsetneq V(G)$ with $q \in A$. Since we have assumed there is no cut vertex, the star of every vertex gives a bond, so it is necessary (setting $A^c = \{p\}$ for $p \neq q$ in (23)) to have $\beta_p = (\partial(\sigma))(p) > 0$. This condition is also sufficient because then for any bond $d(\chi_{A^c})(e)$ associated to $A \subsetneq V(G)$ with $q \in A$, we get

$$\sum_{v \in V(G)} (\partial(\sigma))(v) \cdot \chi_{A^c}(v) = \sum_{v \in V(G)} \beta_v \cdot \sum_{p \in A^c} \chi_p(v) = \sum_{p \in A^c} \beta_p > 0$$

and (23) holds. □

Therefore $\sigma \in C_1(G, \mathbb{R})$ is a solution to $\partial(\sigma) = \beta$ for $\beta = \sum_{p \in V(G)} \beta_v(v)$ in $\mathrm{Div}^0(G)$ with $\beta_p > 0$ for $p \neq q$.

After identifying $C_1(G, \mathbb{R})$ with $C^1(G, \mathbb{R})$ (by sending e to e^*) we have the orthogonal ("Hodge") decomposition

$$C_1(G, \mathbb{R}) \cong \mathrm{Ker}(\partial) \oplus \mathrm{Image}(d) \ .$$

Let $\sigma = \sigma' + \sigma''$ for $\sigma' \in \mathrm{Ker}(\partial)$ and $\sigma'' = d(\psi) \in \mathrm{Image}(d)$ for $\psi \in C^0(G, \mathbb{R})$. Then $\partial(\sigma) = \beta$ if and only if $\partial d(\psi) = \partial(\sigma'') = \beta$. By Remark 3.0.1 $\partial d = 2\Delta$, so

$$\Delta\psi = \frac{1}{2}\beta \ .$$

It follows from the definition of the Green's function $j_q(p, v)$, together with the fact that the Laplacian operator has a one dimensional zero-eigenspace generated by $\mathbf{1}$, that:

$$\psi = \frac{1}{2} \sum_{p \in V(G)} \beta_p j_q(p, \cdot) + k \cdot \mathbf{1}$$

for some constant $k \in \mathbb{R}$. Therefore

$$\sigma(e) = \sigma'(e) + \sigma''(e) = \sigma'(e) + (d(\psi))(e) = \sigma'(e) + \frac{1}{2} \sum_{p \in V(G)} \beta_p (j_q(p, e_+) - j_q(p, e_-)) \ .$$

We summarize these observations in the following theorem.

Theorem 7.7.2. *Assume G is 2-vertex connected. The 1-chain $\sigma \in C_1(G, \mathbb{R})$ represents $\prec_q$ for $\mathbf{J}_G$ if and only if there exist $\sigma' \in \mathrm{Ker}(\partial)$ and real numbers $\beta'_p > 0$ (for $p \in V(G)$) such that*

$$\sigma(e) = \sigma'(e) + \sum_{p \in V(G)} \beta'_p (j_q(p, e_+) - j_q(p, e_-))$$

for all $e \in \mathbb{E}(G)$.

In other words σ (up to an element of the "extended cycle space" $\mathrm{Ker}(\partial)$) is in the interior of the cone generated by the vectors $(j_q(p, e_+) - j_q(p, e_-))_{e \in \mathbb{E}(G)}$ for various $p \in V(G)$. It is easy, using [6, Construction 3.1], to show that these vectors are independent.

CHAPTER VIII

REGULAR SEQUENCES, MINIMAL FREE RESOLUTIONS, AND FLAT FAMILIES

8.1 "Nice" gradings and Nakayama's lemma for polynomial rings

Let S be a polynomial ring over K in r variables $\{z_1, \ldots, z_r\}$. Let $\mathfrak{m}$ denote the ideal consisting of all polynomials with zero constant term. Let M be a finitely generated $\mathbb{Z}$-graded module over S. Nakayama's lemma for $\mathbb{Z}$-graded polynomial rings is the statement that $\mathfrak{m}M = M$ implies $M = 0$.

The proof of this lemma is significantly simpler than the proof of the analogues statement for local rings; taking i to be the least integer such that $M_i \neq 0$, we see that the graded piece M_i cannot appear in $\mathfrak{m}M$, so $\mathfrak{m}M \neq M$ unless $M = 0$.

The above version of Nakayama's lemma is a statement about $\mathbb{Z}$-graded polynomial rings and modules. It naturally extends to other gradings, provided that the grading is "nice". Let A be an abelian group, and assume the polynomial ring S is endowed with an A-valued degree map (semigroup homomorphism) $\deg_A \colon \mathbb{N}^r \to \mathsf{A}$. Let $S_{\mathbf{a}}$ denote the K-vector space consisting of all homogeneous polynomials having degree $\mathbf{a} \in \mathsf{A}$. Then S has the direct sum decomposition

$$S = \bigoplus_{\mathbf{a} \in \mathsf{A}} S_{\mathbf{a}}$$

satisfying $S_{\mathbf{a}} \cdot S_{\mathbf{b}} \subseteq S_{\mathbf{a}+\mathbf{b}}$.

Definition 8.1.1. We call an A-grading of S "nice" if there exists a group homomorphism $u' \colon \mathsf{A} \to \mathbb{Z}$ such that the semigroup homomorphism $u := u' \circ \deg_A \colon \mathbb{N}^r \to \mathbb{Z}$ has the following properties

(i) $u(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{N}^r$,

(ii) $u(\mathbf{v}) = 0$ if and only if $\mathbf{v} = (0, 0, \ldots, 0)$.

For a "nice" A-grading of S, we automatically have $S_{\mathbf{0}} = K$. This is because $S_{\mathbf{0}}$ is spanned by the set of all monomials $\mathbf{z}^{\mathbf{v}}$ satisfying $\deg_A(\mathbf{v}) = \mathbf{0}$. Since $u'(\mathbf{0}) = 0$, it follows that $u(\mathbf{v}) = 0$ and (ii) implies that $\mathbf{v} = (0, 0, \ldots, 0)$. It follows that, when we have a "nice" grading, $\bigoplus_{\mathbf{a} \in \mathsf{A} \setminus \{\mathbf{0}\}} S_{\mathbf{a}}$ coincides with the maximal ideal $\mathfrak{m}$ consisting of all polynomials with zero constant term.

It is clear that the usual (coarse) $\mathbb{Z}$-grading is "nice" in the above sense. The following example generalizes the (fine) $\mathbb{Z}^r$-grading.

Example 8.1.2. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$ be an integral positive (i.e. $\omega_i \in \mathbb{Z}_{>0}$) weight vector. Let $\mathbf{e}_i$ denote the standard vector having 1 in position i and 0 elsewhere. Consider the grading $\deg_\omega \colon \mathbb{N}^r \to \bigoplus_{i=1}^r \mathbb{Z}\omega_i\mathbf{e}_i$ defined by sending $\mathbf{e}_i$ to $\omega_i\mathbf{e}_i$. This is a "nice" grading. Indeed, let $u' \colon \bigoplus_{i=1}^r \mathbb{Z}\omega_i\mathbf{e}_i \to \mathbb{Z}$ be the group homomorphism defined by sending $\omega_i\mathbf{e}_i$ to ω_i. Then the induced map $u \colon \mathbb{N}^r \to \mathbb{Z}$ is defined by sending $\mathbf{e}_i$ to ω_i, and (i) and (ii) immediately follow from the positivity of the ω_i's. These are the "positive multigradings" in the sense of [46, Definition 8.7].

Example 8.1.3. Consider the polynomial ring $\mathbf{R} = K[\mathbf{x}]$ in variables $\{x_v : v \in V(G)\}$. Each monomial is of the form $\mathbf{x}^D$ for some effective divisor $D \in \mathrm{Div}_+(G)$. Consider the $\mathrm{Pic}(G)$-grading defined by the semigroup homomorphism $\mathrm{Div}_+(G) \to \mathrm{Pic}(G)$ sending D to its equivalence class $[D]$. This is a "nice" grading via the map $u' \colon \mathrm{Pic}(G) \to \mathbb{Z}$ sending $[D]$ to $\deg([D]) = \sum_v D(v)$. This is a well-defined homomorphism because all principal divisors have degree 0. The induced map $u \colon \mathrm{Div}_+(G) \to \mathbb{Z}$ sends the effective divisor D to $\deg(D) = \sum_v D(v)$. It is immediate that (i) and (ii) hold. See [48, Section 2.2] for more details. Note that if G is not a tree then $\mathrm{Pic}(G)$ contains torsion elements. This example shows that our definition is robust enough to handle gradings with groups which are not necessarily torsion-free.

Let S be graded by A. An S-module M is called A-*graded* if it is endowed with a decomposition $M = \bigoplus_{\mathbf{a}\in\mathsf{A}} M_{\mathbf{a}}$ as a direct sum of graded components such that $S_{\mathbf{a}} M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathsf{A}$.

Lemma 8.1.4 (Nakayama's lemma for "nicely" graded polynomial rings)**.** *Assume S is a polynomial ring endowed with a "nice" A-grading. Let M be a finitely generated A-graded S-module. Then $\mathfrak{m}M = M$ implies $M = 0$.*

Proof. Suppose $M \neq 0$. Let $u' \colon \mathsf{A} \to \mathbb{Z}$ be as in Definition 8.1.1. Write $M = \bigoplus_{\mathbf{a}\in\mathsf{A}} M_{\mathbf{a}}$. For any graded piece $M_{\mathbf{a}}$, let $\ell(M_{\mathbf{a}})$ denote the integer $u'(\mathbf{a})$. Let $\ell(M) = \min_{\mathbf{a}\in\mathsf{A}} \ell(M_{\mathbf{a}})$. Since M is assumed to be finitely generated, $\ell(M) > -\infty$. Since $u'(r) \geq 1$ for all $r \in \mathfrak{m} = \bigoplus_{\mathbf{a}\neq\mathbf{0}} M_{\mathbf{a}}$, we have $\ell(\mathfrak{m}M) > \ell(M)$ and therefore $\mathfrak{m}M \neq M$.

□

8.2 Regular sequences and homogeneous systems of parameters

Recall that for a commutative ring S and an S-module M, an element $s \in S$ is called a *nonzerodivisor* on M if $sm = 0$ implies $m = 0$ for $m \in M$. An *M-regular sequence* is a sequence $s_1, \ldots, s_d \in S$ such that

(i) $M/(s_1, \ldots, s_d)M \neq 0$,

(ii) s_i is a nonzerodivisor on $M/(s_1, \ldots, s_{i-1})M$ for $i = 1, \ldots, d$.

Remark 8.2.1. In our application S will always be a "nicely" graded polynomial ring, $M \neq 0$ will be a finitely generated graded S-module, and the s_i's will be polynomials with zero constant term. In this situation (i) is automatically satisfied. This follows from Lemma 8.1.4: if $s_i \in \mathfrak{m}$ and $M/(s_1, \ldots, s_d)M = 0$, then we must have $M = \mathfrak{m}M$. But Nakayama's lemma would then imply that $M = 0$.

Lemma 8.2.2. *Assume that $s_1, \ldots, s_d \in S$ is an M-regular sequence and $\varepsilon_1, \ldots, \varepsilon_d$ are units in S. Then $\varepsilon_1 s_1, \ldots, \varepsilon_d s_d$ is also an M-regular sequence.*

Proof. Clearly $\epsilon_1 s_1$ is a nonzerodivisor on M. We need to show that $\epsilon_i s_i$ is a nonzerodivisor on $M/(\varepsilon_1 s_1, \ldots, \varepsilon_{i-1}s_{i-1})M$ for all $i > 1$. First note that $(\varepsilon_1 s_1, \ldots, \varepsilon_{i-1}s_{i-1})M = (s_1, \ldots, s_{i-1})M$. Assume that $(\varepsilon_i s_i)m \in (s_1, \ldots, s_{i-1})M$ for some $m \in M$, or $(\varepsilon_i s_i)m = \sum_{j=1}^{i-1} s_j m_j$ for some $m_j \in M$. Then $s_i m \in (\varepsilon_1^{-1}s_1, \ldots, \varepsilon_{i-1}^{-1}s_{i-1})M = (s_1, \ldots, s_{i-1})M$, which is a contradiction because s_i is a nonzerodivisor on $M/(s_1, \ldots, s_{i-1})M$. □

Lemma 8.2.3. *Let S be a ring, M be an S-module, and N be a flat S-module. If $s_1, \ldots, s_d \in S$ is an M-regular sequence then $s_1, \ldots, s_d$ is also an $(M \otimes_S N)$-regular sequence, provided that $(s_1, \ldots, s_d)(M \otimes_S N) \neq (M \otimes_S N)$.*

For a proof see, e.g., [17, Proposition 1.1.2].

It is not necessarily true that every permutation of the s_i's is again a regular sequence. For example, $xy, xz, y-1$ is a regular sequence for $K[x, y, z]$ (as a module over itself), but $xy, y-1, xz$ is not a regular sequence. However, in situations where Nakayama's lemma apply, permutation of a regular sequence is allowed. The following theorem, for local rings, is proved in [17, Proposition 1.1.6].

Theorem 8.2.4. *Let S be a polynomial ring endowed with a "nice" A-grading. Let M be a finitely generated A-graded S-module. Assume $s_1, \ldots, s_d$ is an M-regular sequence consisting of elements in $\mathfrak{m}$. Then any permutation of $s_1, \ldots, s_d$ is also an M-regular sequence.*

Proof. It suffices to show that if s_1, s_2 is an M-regular sequence then s_2, s_1 is also an M-regular sequence (see [17, proof of Proposition 1.1.6]).

- s_2 is a nonzerodivisor on M: let N denote the kernel of the map $M \to M$ sending m to $s_2 m$. For each $z \in N$ we have $s_2 z = 0$ and therefore $s_2 z + s_1 M = s_1 M$. Since s_2 is a nonzerodivisor on $M/s_1 M$ by assumption, we must have $z \in s_1 M$ or $z = s_1 z'$ for some $z' \in M$. But then $s_1(s_2 z') = s_2(s_1 z') = 0$ and since s_1 is a nonzerodivisor on M we must have $s_2 z' = 0$ and $z' \in N$. So we have shown that $N \subseteq s_1N$ and therefore $N = s_1N$. Since $s_1 \in \mathfrak{m}$ by assumption, we obtain $N = s_1N \subseteq \mathfrak{m}N \subseteq N$ or $\mathfrak{m}N = N$, and by Lemma 8.1.4 we get $N = 0$, which is what we want.

- s_1 is a nonzerodivisor on M/s_2M: if $s_1(z + s_2M) = s_2M$ for some $z \in M$, then $s_1z \in s_2M$ or $s_1z = s_2z'$ for some $z' \in M$. But then $s_2z' \in s_1M$, or equivalently $s_2(z' + s_1M) = s_1M$. Since s_2 is a nonzerodivisor on M/s_1M, this means that $z' \in s_1M$, so $z' = s_1m$ for some $m \in M$. But $s_1z = s_2s_1m$ implies $z = s_2m$ because s_1 is a nonzerodivisor on M. Therefore $z \in s_2M$, which is what we want.

Remark 8.2.1 completes the proof. $\square$

Consider polynomial rings with $\mathbb{Z}$-gradings. In this $\mathbb{Z}$-graded setting, an *h.s.o.p.* (homogeneous system of parameters) for M is defined as a set $\{\theta_1, \ldots, \theta_{\dim(M)}\} \subset S$ of homogeneous elements of positive degree such that $\dim(M/(\theta_1, \ldots, \theta_{\dim(M)})M) = 0$. Here $\dim(\cdot)$ denotes the Krull dimension. Equivalently, $\{\theta_1, \ldots, \theta_d\} \subset S$ is an h.s.o.p if and only if $d = \dim(M)$ and M is a finitely generated $K[\theta_1, \ldots, \theta_d]$-module. Clearly the property of being an h.s.o.p. does not change under permutation.

By definition, $\operatorname{depth}(M)$ is the length of the longest homogeneous M-regular sequence. In general $\operatorname{depth}(M) \leq \dim(M)$. If $\operatorname{depth}(M) = \dim(M)$, then M is called *Cohen-Macaulay*.

Theorem 8.2.5. *Assume M has an h.s.o.p. Then M is Cohen-Macaulay if an only if every h.s.o.p. is an M-regular sequence.*

For a proof see, e.g., [57, p.35].

An *l.s.o.p.* (linear system of parameters) for M is an h.s.o.p., all of whose elements have degree one.

8.3 *Linear systems of parameters and squarefree monomial ideals*

Consider the polynomial ring $K[\mathbf{z}]$ in variables $\mathbf{z} = \{z_1, \dots, z_r\}$. Monomial ideals are the $\mathbb{N}^r$-graded ideals of $K[\mathbf{z}]$. An ideal is squarefree if it is generated by squarefree monomials. Given an abstract simplicial complex Σ, the squarefree monomial ideal in $K[\mathbf{z}]$ defined as

$$I_\Sigma = \langle \mathbf{z}^\tau : \tau \notin \Sigma \rangle$$

is called the *Stanley-Reisner ideal* of Σ. The *Stanley-Reisner ring* (or *face ring*) $K[\Sigma]$ is, by definition, $K[\mathbf{z}]/I_\Sigma$. In fact, this gives a bijective correspondence between squarefree monomial ideals inside $K[\mathbf{z}]$ and abstract simplicial complexes on the vertices $\{z_1, \dots, z_r\}$ (see, e.g., [46, Chapter 1]). The simplicial complex Σ is called Cohen-Macaulay if $K[\Sigma]$ is Cohen-Macaulay. A (pure) "shellable" simplicial complex is Cohen-Macaulay (see, e.g., [57, Chapter III] or [46, Chapter 13]). In general, $\dim(K[\Sigma])$ is equal to the maximal cardinality of the faces of Σ (see, e.g., [57, p.53]).

Given a degree one element $\theta = \sum_i \alpha_i z_i$ and a face $\tau \in \Sigma$, by *restriction of* θ *to* τ we mean

$$\theta|_\tau = \sum_{z_i \in \tau} \alpha_i z_i \ .$$

For squarefree monomial ideals, there is a nice characterization of l.s.o.p. which was first given in [37].

Lemma 8.3.1. *Let $K[\Sigma]$ be a Stanley-Reisner ring of Krull dimension d, and let $\{\theta_1, \dots, \theta_d\} \subset K[\Sigma]$ be a set of elements of degree one. Then the following are equivalent:*

(i) $\{\theta_1, \dots, \theta_d\}$ is an l.s.o.p. for $K[\Sigma]$,

(ii) for every facet τ of Σ the restrictions $\theta_1|_\tau, \dots, \theta_d|_\tau$ span a vector space of dimension equal to $|\tau|$ (the cardinality of τ).

For a proof see, e.g., [57, pp.81-82].

8.4 Regular sequences and free resolutions

Let S be a polynomial ring with its usual $\mathbb{Z}$-grading, and M be a graded S-module. Assume that

$$\mathcal{F}\colon 0 \to \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$$

is a graded free resolution. We may form the free complex of $S/(s)$-modules

$$\mathcal{F} \otimes_S S/(s)\colon 0 \to \cdots \to F_i \otimes_S S/(s) \xrightarrow{\varphi_i \otimes \mathrm{id}} F_{i-1} \otimes_S S/(s) \to \cdots \to F_0 \otimes_S S/(s) \ .$$

The following theorem is a slight generalization of [28, Lemma 3.15] (see also [17, Proposition 1.1.5]).

Theorem 8.4.1. *Assume s is a nonzerodivisor on S and on M. Then*

(i) $\mathcal{F} \otimes_S S/(s)$ is a free resolution of $M/(s)M$.

(ii) If $s \in \mathfrak{m}$ (i.e. if s has zero constant term) and if $\mathcal{F}$ is a minimal *free resolution of M, then $\mathcal{F} \otimes_S S/(s)$ is a minimal free resolution of $M/(s)M$.*

(iii) If s is homogeneous of positive degree, then the $\mathbb{Z}$-graded Betti numbers of M over S coincide with the $\mathbb{Z}$-graded Betti numbers of $M/(s)M$ over the graded ring $S/(s)$.

(iv) If $s \in \mathfrak{m}$ and if $\mathcal{F}$ is a minimal cellular free resolution *of M, then $\mathcal{F} \otimes_S S/(s)$ is a* minimal cellular free resolution *of $M/(s)M$.*

Proof. (i) We compute the homology of the complex $\mathcal{F} \otimes_S S/(s)$. By definition, this homology is computed by the Tor functor:

$$\mathrm{Tor}_i^S(M, S/(s)) = \mathrm{Ker}(\varphi_i \otimes \mathrm{id}) / \mathrm{Image}(\varphi_{i+1} \otimes \mathrm{id}) \ .$$

To compute $\mathrm{Tor}_i^S(M, S/(s))$ consider the exact sequence (for the nonzerodivisor s on S)

$$0 \to S \xrightarrow{s} S \xrightarrow{\epsilon} S/(s) \to 0$$

which can be seen as a free resolution of $S/(s)$. Tensoring this resolution with M on the left we obtain the complex

$$0 \to M \xrightarrow{\mathrm{id}\otimes s} M \xrightarrow{\mathrm{id}\otimes\epsilon} M \otimes_S S/(s) \to 0 \ .$$

Again by definition, $\mathrm{Tor}_0^S(M, S/(s)) = M \otimes_S S/(s) = M/(s)M$, $\mathrm{Tor}_1^S(M, S/(s)) = \mathrm{Ker}(\mathrm{id} \otimes s) = 0$ (because s is a nonzerodivisor on M), and $\mathrm{Tor}_i^S(M, S/(s)) = 0$ for $i > 1$.

(ii) $\mathcal{F}$ is a minimal free resolution of M if and only if there are no S-units in the matrices corresponding to φ_i ($i \geq 1$). The matrix corresponding to $\varphi_i \otimes \mathrm{id}$ in $\mathcal{F} \otimes_S S/(s)$ is the same as the matrix corresponding to φ_i, except that its entries are considered as elements in $S/(s)$. If an entry u is a unit in $S/(s)$ then there exists $u' \in S$ such that $(u + (s))(u' + (s)) = 1 + (s)$, or equivalently $uu' - 1 \in (s)$. But this is not possible because u and u' are homogeneous of positive degree and $s \in \mathfrak{m}$.

(iii) It follows from part (ii) that a minimal free resolution of M turns into a minimal free resolution of $M/(s)M$. When s is homogeneous, the degrees of the graded parts remain the same.

(iv) Assume the minimal free resolution $\mathcal{F}$ is supported on a labeled cell complex $\mathcal{D}$. Then $\mathcal{F} \otimes_S S/(s)$ is supported on the same cell complex whose labels are now considered as elements of $S/(s)$.

We remark that one can use Theorem 8.4.1 repeatedly and obtain a similar result for regular sequences.

8.5 Regular sequences and flat families

The purpose of this section is to give a generalization (and a complete proof) of [27, Proposition 15.15] in Proposition 8.5.2

Let S be a polynomial ring in r variables $\{z_1, \ldots, z_r\}$, and let $S[t]$ be the polynomial ring with one extra indeterminate t over S. Let $\omega \in \mathrm{Hom}(\mathbb{Z}^r, \mathbb{Z})$ be an integral weight functional. For any $g = \sum_i u_i m_i \in S$, where u_i's are nonzero constants in K and m_i's are some monomials in S, we define $\deg_\omega(g) := \max \omega(m_i)$. The "lift" of g to $S[t]$ with respect to ω is

$$\tilde{g} = t^b g(t^{-\omega(z_1)} z_1, \ldots, t^{-\omega(z_r)} z_r)$$

in which $b = \deg_\omega(g)$. For any ideal $I \subset S$ we define the ideal

$$\tilde{I} = \langle \tilde{g} : g \in I \rangle \subset S[t] \ .$$

It follows from the definition that

$$(S[t]/\tilde{I})/t(S[t]/\tilde{I}) \cong S/\operatorname{in}_\omega(I) \ . \tag{24}$$

In other words, $\tilde{g}$ modulo t is precisely $\operatorname{in}_\omega(g)$.

For a proof of the following result, see [27, Theorem 15.17].

Theorem 8.5.1. *For any ideal $I \subset S$,*

(i) The $K[t]$-algebra $S[t]/\tilde{I}$ is a free (and thus flat) $K[t]$-module.

(ii) The map

$$\varphi \colon (S[t]/\tilde{I}) \otimes_{K[t]} K[t, t^{-1}] \to (S/I)[t, t^{-1}]$$

induced by

$$z_i \mapsto t^{\omega(z_i)} z_i$$

gives an isomorphism of $K[t]$-algebras.

Note that for the map φ we have $\varphi(\tilde{g}) = t^b g$ (where $b = \deg_\omega(g)$) and $\varphi(\tilde{I}) = I$.

Proposition 8.5.2. *Let I be a graded ideal and ω be a positive integral weight functional. Assume that $f_1, \ldots, f_d \in S$ are such that*

$$\operatorname{in}_\omega(f_1), \ldots, \operatorname{in}_\omega(f_d)$$

is an $(S/\operatorname{in}_\omega(I))$*-regular sequence. Then*

$$f_1, \ldots, f_d$$

is an (S/I)*-regular sequence.*

Proof. Let $M = S[t]/\tilde{I}$. By Theorem 8.5.1(i), M is a free $K[t]$-module. Therefore t is a nonzerodivisor on M. By (24), $M/tM \cong S/\operatorname{in}_\omega(I)$ and $\tilde{g}$ modulo t equals $\operatorname{in}_\omega(g)$ for all $g \in S$. Therefore, the hypothesis is precisely the statement that $t, \tilde{f}_1, \ldots, \tilde{f}_d$ is an M-regular sequence. But $t, \tilde{f}_1, \ldots, \tilde{f}_d$ are all homogeneous elements with respect to the "nice" grading of M defined by $\deg(z_i) = \omega(z_i)$ and $\deg(t) = 1$ (see Example 8.1.2). Therefore, by Theorem 8.2.4, the permutation $\tilde{f}_1, \ldots, \tilde{f}_d, t$ is also an M-regular sequence. The module $K[t, t^{-1}]$ is the localization of $K[t]$ with respect to t, therefore it is a flat $K[t]$-module. By Lemma 8.2.3 (and Remark 8.2.1), $\tilde{f}_1, \ldots, \tilde{f}_d$ is also a regular sequence on $M' := M \otimes_{K[t]} K[t, t^{-1}]$.

By Theorem 8.5.1 (ii), the map

$$\varphi\colon M' \to (S/I)[t, t^{-1}] \quad \textit{with} \quad z_i \mapsto t^{\omega(z_i)} z_i$$

is an isomorphism. Therefore $\varphi(\tilde{f}_1) = t^{b_1} f_1, \ldots, \varphi(\tilde{f}_d) = t^{b_d} f_d$ is a regular sequence on $(S/I)[t, t^{-1}]$ (here $b_i = \deg_\omega(f_i)$). Since t^{b_i}'s are units in $(S/I)[t, t^{-1}]$, by Lemma 8.2.2, $f_1, \ldots, f_d$ is also a regular sequence on $(S/I)[t, t^{-1}]$. Therefore $f_1, \ldots, f_d$ is a regular sequence on (S/I). □

Remark 8.5.3. When $f_1, f_2, \ldots, f_d$ are $\mathbb{Z}$-homogeneous, the positivity of ω in Proposition 8.5.2 is unnecessary. This is because ω and $\omega + c\mathbf{1}$ (for any $c > 0$) behave the same on these $\mathbb{Z}$-homogeneous forms.

CHAPTER IX

REGULAR SEQUENCES FOR $\mathbf{O}_G^q$ AND $\mathbf{J}_G$

9.1 Linear system of parameters for $\mathbf{O}_G^q$

The ideal $\mathbf{O}_G^q \subset \mathbf{S}$ is a squarefree monomial ideal. Let Σ_G^q denote its associated simplicial complex on $2m$ vertices $\{y_e : e \in \mathbb{E}(G)\}$.

For each spanning tree T of G, let $\mathcal{O}_T$ denote the orientation of T with a unique source at q (i.e. the orientation obtained by orienting all paths away from q). For an example, see Figure 17.

Proposition 9.1.1.

(i) The number of facets of Σ_G^q is the same as the number of spanning trees of G. For each spanning tree T, the corresponding facet τ_T is:

$$\tau_T = \{y_e : e \in \mathbb{E}(G) \backslash \mathcal{O}_T\} \ .$$

(ii) For each spanning tree T of G, let $P_T = \langle y_e : e \in \mathcal{O}_T \rangle$. The minimal prime decomposition of $\mathcal{O}_G^q$ is

$$\mathbf{O}_G^q = \bigcap_T P_T \ ,$$

the intersection being over all spanning trees of G.

(iii) For each facet τ of Σ_G^q we have $|\tau| = 2m - n + 1$. Therefore

$$\dim(K[\Sigma_G^q]) = 2m - n + 1 \ .$$

(iv) Σ_G^q is Cohen-Macaulay.

Proof. (i) By Proposition 7.5.4, we know that $\mathbf{O}_G^q$ is generated by monomials of the form $\prod_{e\in\mathbb{E}(A^c,A)} y_e$, where $q \in A \subsetneq V(G)$ and $\mathbb{E}(A^c, A) \subset \mathbb{E}(G)$ denotes the set of oriented edges from A to its complement A^c.

First we show that for each spanning tree T, the monomial $m_T := \prod_{e\in\mathbb{E}(G)\backslash\mathcal{O}_T} y_e$ does not belong to $\mathbf{O}_G^q$. Clearly $m_T \in \mathbf{O}_G^q$ if and only if m_T is divisible by one of the given generators $\prod_{e\in\mathbb{E}(A^c,A)} y_e$. But

$$\prod_{e\in\mathbb{E}(A^c,A)} y_e \mid \prod_{e\in\mathbb{E}(G)\backslash\mathcal{O}_T} y_e \qquad \Longleftrightarrow \qquad \mathbb{E}(A^c, A) \subseteq (\mathbb{E}(G)\backslash\mathcal{O}_T)\ .$$

However, it follows from the definition of $\mathcal{O}_T$ that it must contain some element of $\mathbb{E}(A^c, A)$ for any A. This shows that $\tau_T = \{y_e : e \in \mathbb{E}(G)\backslash\mathcal{O}_T\}$ is a face in the simplicial complex Σ_G^q.

Next we show that τ_T must be a facet; for $f \in \mathcal{O}_T$ removing f from the tree gives a partition of $V(T) = V(G)$ into two connected subsets B and B^c with $f_- \in B$ and $f_+ \in B^c$. Then the monomial $m_T \cdot y_f$ is divisible by $\prod_{e\in\mathbb{E}(B^c,B)} y_e$.

It remains to show that for any monomial $m = \prod_{e\in F} y_e$ that does not belong to $\mathbf{O}_G^q$ we have $F \subseteq (\mathbb{E}(G)\backslash\mathcal{O}_T)$ for some spanning tree T. To show this, we repeatedly use the fact that m is not divisible by generators of the form $\prod_{e\in\mathbb{E}(A^c,A)} y_e$ for various A, and construct a spanning tree T. This procedure is explained in Algorithm 1. Note that if $\prod_{e\in F} y_e$ is not divisible by $\prod_{e\in\mathbb{E}(A^c,A)} y_e$ then there exists an $e \in \mathbb{E}(A^c, A)$ such that $e \notin F$. The orientation $\mathcal{O}_T$ is also induced by Algorithm 1.

(ii) follows from (i) and [46, Theorem 1.7].

(iii) follows from (i) and the fact that $\dim(K[\Sigma_G^q])$ is equal to the maximal cardinality of the faces of Σ_G^q.

(iv) The Krull dimension of $K[\Sigma_G^q] = \mathbf{S}/\mathbf{O}_G^q$ is $2m - n + 1$ by part (iii). By the Auslander–Buchsbaum formula (for graded rings and modules, see [33, page 437]),

$$\operatorname{depth}(\mathbf{S}/\mathbf{O}_G^q) = \operatorname{depth}(\mathbf{S}) - \operatorname{pd}_{\mathbf{S}}(\mathbf{S}/\mathbf{O}_G^q) = 2m - n + 1$$

Input:
A monomial $m = \prod_{e \in F} y_e$ not belonging to $\mathbf{O}_G^q$.
Output:
A spanning tree T such that $F \subseteq (\mathbb{E}(G) \backslash \mathcal{O}_T)$.

Initialization:
$A = \{q\}$,
$T = \emptyset$.

while $A \neq V(G)$ **do**
  Find an oriented edge e such that $e \in E(A, A^c)$ and $e \notin F$,
  $T = T \cup \{e\}$,
  $A = A \cup \{e_+\}$,
end
Output T.

Algorithm 1: Finding a facet containing a given monomial not belonging to $\mathbf{O}_G^q$

because $\mathrm{pd}_{\mathbf{S}}(\mathbf{S}/\mathbf{O}_G^q) = n-1$ by Theorem 7.3.2. Therefore $\dim(\mathbf{S}/\mathbf{O}_G^q) = \mathrm{depth}(\mathbf{S}/\mathbf{O}_G^q)$ and $K[\Sigma_G^q]$ is Cohen-Macaulay. □

Remark 9.1.2.

(i) Proposition 9.1.1(iii) can be strengthened; the simplicial complex Σ_G^q is in fact shellable. Since $\mathbf{J}_G$ is the lattice ideal associated to the free abelian group $\Lambda = \mathrm{Image}(d)$, it is a toric ideal (in the sense of [58, Chapter 4]). Σ_G^q is precisely the *initial complex* of $\mathbf{J}_G$ with respect to $\prec_q$ (in the sense of [58, Chapter 8]). Let $\sigma \in C_1(G, \mathbb{R})$ be any weight functional representing the term order $\prec_q$ for $\mathbf{J}_G$ (e.g. ϑ_q of Definition 7.6.2 – see also §7.7). By [58, Theorem 8.3] σ provides us with a regular triangulation of Σ_G^q. This is accomplished by "lifting" each point y_e into the next dimension by the height $\sigma(e)$, and then projecting back the lower face of the resulting positive cone. This is a unimodular triangulation because the ideal $\mathbf{O}_G^q$ is squarefree ([58, Corollary 8.9]). The associated Gröbner fan studied in §7.6 coincides with the associated secondary fan of this triangulation.

It is well-known that given any regular triangulation, one can obtain shelling orders using the *line shelling* technique (see, e.g., [23, Theorem 9.5.10]).

(ii) A minimal free resolution of the Alexander dual of $\mathbf{O}_G^q$ can be obtained by the constructions given in [7] or in [25].

Example 9.1.3. Consider the graph in Example 7.5.5. For the spanning tree in Figure 15 we have

$$\tau_T = \{y_e : e \in \mathbb{E}(G)\} \backslash \{y_{e_1}, y_{e_3}, y_{e_4}\} = \{y_{e_2}, y_{e_5}, y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_3}, y_{\bar{e}_4}, y_{\bar{e}_5}\} \ ,$$

(which is the same as τ_8 in Example 9.1.5). Moreover, $P_T = \langle y_{e_1}, y_{e_3}, y_{e_4} \rangle$.

Figure 15: Spanning tree T and its orientation $\mathcal{O}_T$

We are now ready to give a particularly nice l.s.o.p. for $\mathbf{O}_G^q$. Note that since $K[\Sigma_G^q] = \mathbf{S}/\mathbf{O}_G^q$ is Cohen-Macaulay, every h.s.o.p. (in particular every l.s.o.p.) is regular (Theorem 8.2.5).

First we introduce some notation. For each $v \in V(G)$ we choose a distinguished incoming edge to v and denote it by e_v. In other words, we fix a distinguished subset $\{e_v : v \in V(G) \ \} \subset \mathbb{E}(G)$ of cardinality n in such a way that $(e_v)_+ = v$.

For each v define the set of linear forms

$$\mathcal{L}_v = \{y_e - y_{e_v} : \ e \in \mathbb{E}(G) \ , e \neq e_v \ , e_+ = (e_v)_+ = v\}$$

and let

$$\mathcal{L} = \bigcup_{v \in V(G)} \mathcal{L}_v \ . \qquad (25)$$

We also let

$$\mathcal{L}^{(q)} = \mathcal{L} \cup \{y_{e_q}\} \ .$$

Clearly, $|\mathcal{L}_v| = \deg(v) - 1$ for $v \in V(G)$, $|\mathcal{L}| = 2m - n$, and $|\mathcal{L}^{(q)}| = 2m - n + 1$.

Proposition 9.1.4. *The set $\mathcal{L}^{(q)}$ forms an l.s.o.p. (and thus a regular sequence) for $K[\Sigma_G^q] = \mathbf{S}/\mathbf{O}_G^q$.*

Proof. We will use the criterion in Lemma 8.3.1. Note that by Proposition 9.1.1(iii), $\dim K[\Sigma_G^q] = |\mathcal{L}^{(q)}|$. For each facet τ and each vertex $v \neq q$, by Proposition 9.1.1(i), all but one variable y_e with $e^+ = v$ appear in τ. Again by Proposition 9.1.1(i), all variables y_e with $e^+ = q$ appear in τ. It follows that the dimension of the vector space spanned by the restrictions of forms in $\mathcal{L}^{(q)}$ to the facet τ is equal to $\sum_v (\deg(v) - 1) + 1 = 2m - n + 1$ which is equal to $|\tau|$ by Proposition 9.1.1(iii), and the conditions in Lemma 8.3.1 are satisfied. $\square$

Example 9.1.5. For the graph in Example 7.5.5, $\mathbf{O}_G^q$ is the Stanley-Reisner ideal of the simplicial complex Σ_G^q given by facets

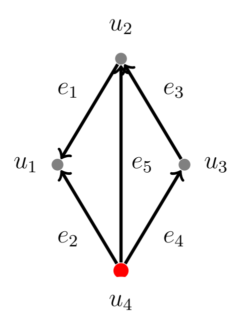

Figure 16: Graph G and a fixed orientation

$$\tau_1 = \{y_{e_1}, y_{e_3}, y_{e_4}, y_{e_5}, y_{\bar{e}_2}, y_{\bar{e}_4}, y_{\bar{e}_5}\},\ \tau_2 = \{y_{e_1}, y_{e_3}, y_{e_4}, y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_4}, y_{\bar{e}_5}\},$$

$$\tau_3 = \{y_{e_2}, y_{e_3}, y_{e_4}, y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_4}, y_{\bar{e}_5}\},\ \tau_4 = \{y_{e_1}, y_{e_3}, y_{e_5}, y_{\bar{e}_2}, y_{\bar{e}_3}, y_{\bar{e}_4}, y_{\bar{e}_5}\},$$

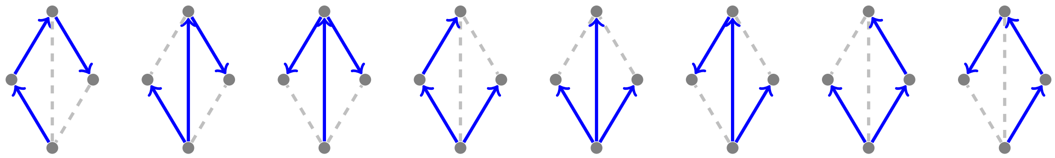

Figure 17: Spanning trees T and orientations $\mathcal{O}_T$ corresponding to $\tau_1, \tau_2, \ldots, \tau_8$

$$\tau_5 = \{y_{e_1}, y_{e_3}, y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_3}, y_{\bar{e}_4}, y_{\bar{e}_5}\},\ \tau_6 = \{y_{e_2}, y_{e_3}, y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_3}, y_{\bar{e}_4}, y_{\bar{e}_5}\},$$

$$\tau_7 = \{y_{e_1}, y_{e_5}, y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_3}, y_{\bar{e}_4}, y_{\bar{e}_5}\},\ \tau_8 = \{y_{e_2}, y_{e_5}, y_{\bar{e}_1}, y_{\bar{e}_2}, y_{\bar{e}_3}, y_{\bar{e}_4}, y_{\bar{e}_5}\}.$$

See Proposition 9.1.1(i), Example 9.1.3, and Figure 17.

If we choose $\{e_1, e_3, e_4, \bar{e}_4\}$ as our distinguished set of incoming edges to vertices, we have

$$\mathcal{L}_{u_1} = \{y_{e_2} - y_{e_1}\}\ , \quad \mathcal{L}_{u_2} = \{y_{\bar{e}_1} - y_{e_3}, y_{e_5} - y_{e_3}\}\ ,$$

$$\mathcal{L}_{u_3} = \{y_{\bar{e}_3} - y_{e_4}\}\ , \quad \mathcal{L}_{u_4} = \{y_{\bar{e}_2} - y_{\bar{e}_4}, y_{\bar{e}_5} - y_{\bar{e}_4}\}\ .$$

Therefore

$$\mathcal{L} = \bigcup_{v \in V(G)} \mathcal{L}_v = \{y_{e_2} - y_{e_1}, y_{\bar{e}_1} - y_{e_3}, y_{e_5} - y_{e_3}, y_{\bar{e}_3} - y_{e_4}, y_{\bar{e}_2} - y_{\bar{e}_4}, y_{\bar{e}_5} - y_{\bar{e}_4}\}$$

and

$$\mathcal{L}^{(q)} = \mathcal{L} \cup \{y_{\bar{e}_4}\}\ .$$

Note that $|\mathcal{L}^{(q)}| = 7 = 2 \times 5 - 4 + 1$. The restrictions of linear forms of $\mathcal{L}^{(q)}$ to τ_1 are

$$\mathcal{L}^{(q)}|_{\tau_1} = \{-y_{e_1}, -y_{e_3}, y_{e_5} - y_{e_3}, -y_{e_4}, y_{\bar{e}_2}, y_{\bar{e}_5}, y_{\bar{e}_4}\}\ ,$$

which span a vector space of dimension $|\tau_1| = 7 = 2 \times 5 - 4 + 1$. Similarly, the restrictions of the linear forms of $\mathcal{L}^{(q)}$ to the other τ_i's span a vector space of dimension $|\tau_i|$.

9.2 *Linear system of parameters for* $\mathbf{J}_G$

Next we use Proposition 8.5.2 to give a regular sequence for $\mathbf{S}/\mathbf{J}_G$.

Proposition 9.2.1. *The set* $\mathcal{L}$ *forms a regular sequence for* $\mathbf{S}/\mathbf{J}_G$*.*

Proof. Let $\lambda_q \in C_1(G, \mathbb{R})$ be the integral, non-negative weight functional defined in Definition 7.6.2. Any element of $\mathcal{L}$ is of the form $g = y_e - y_{e_v}$ with $e_+ = (e_v)_+ = v$ for some $v \in V(G)$. Since $\lambda_q(e) = \lambda_q(e_v)$ depends only on v by the construction in Proposition 7.6.1, we obtain $\mathrm{in}_{\lambda_q}(g) = g$ and $\tilde{g} = g$. Therefore $\{\mathrm{in}_{\lambda_q}(g) : g \in \mathcal{L}\} = \mathcal{L}$ which is a regular sequence on $\mathbf{S}/\,\mathrm{in}_{\lambda_q}(\mathbf{J}_G) = \mathbf{S}/\mathbf{O}_G^q$ by Proposition 9.1.4. So we may apply Proposition 8.5.2 to conclude that $\mathcal{L}$ is a $(\mathbf{S}/\mathbf{J}_G)$-regular sequence. $\square$

Remark 9.2.2. It follows from Theorem 8.2.5 and Proposition 11.1.1 that $\mathcal{L}$ also forms a (partial) l.s.o.p. for $\mathbf{S}/\mathbf{J}_G$.

CHAPTER X

$\mathbf{I}_G$ FROM $\mathbf{J}_G$ AND $\mathbf{M}_G^q$ FROM $\mathbf{O}_G^q$

A common and powerful technique in the theory of divisors on graphs and chip-firing games is to relate divisors to orientations. Given an orientation, one can form a divisor from the associated indegrees or outdegrees (see, e.g., [15, Theorem 2.3], [5, Theorem 3.3], [36], [48], and [1]). Algebraically, there is a good justification for the strength of this method related to the regular sequences studied in §9.

Recall that $\mathbf{R} = K[\mathbf{x}]$ denotes the polynomial ring in n variables $\{x_v : v \in V(G)\}$ and $\mathbf{S} = K[\mathbf{y}]$ denotes the polynomial ring in $2m$ variables $\{y_e : e \in \mathbb{E}(G)\}$. There is a canonical surjective K-algebra homomorphism

$$\phi \colon \mathbf{S} \to \mathbf{R}$$

defined by sending y_e to x_{e_+} for all $e \in \mathbb{E}(G)$. The kernel of this map is precisely the ideal generated by $\mathcal{L}$ (defined in (25)), which we denote by $\mathfrak{a} = \langle \mathcal{L} \rangle$. The induced isomorphism

$$\bar{\phi} \colon \mathbf{S}/\mathfrak{a} \xrightarrow{\sim} \mathbf{R}$$

is the "algebraic indegree map", and it relates the ideals $\mathbf{I}_G$ and $\mathbf{M}_G^q$ to the ideals $\mathbf{J}_G$ and $\mathbf{O}_G^q$.

Proposition 10.0.3.

(i) $\bar{\phi}(\mathbf{J}_G + \mathfrak{a}) = \mathbf{I}_G$. *In other words* $\bar{\phi}$ *induces an isomorphism* $(\mathbf{S}/\mathbf{J}_G) \otimes_{\mathbf{S}} (\mathbf{S}/\mathfrak{a}) \cong \mathbf{R}/\mathbf{I}_G$.

(ii) $\bar{\phi}(\mathbf{O}_G^q + \mathfrak{a}) = \mathbf{M}_G^q$. *In other words* $\bar{\phi}$ *induces an isomorphism* $(\mathbf{S}/\mathbf{O}_G^q) \otimes_{\mathbf{S}} (\mathbf{S}/\mathfrak{a}) \cong \mathbf{R}/\mathbf{M}_G^q$.

Proof. The map $\bar{\phi}$ sends $\prod_{e\in\mathbb{E}(A^c,A)} y_e + \mathfrak{a}$ to $\mathbf{x}^{D(A^c,A)}$. So the proposition immediately follows from examining the generating sets described in Theorem 3.2.2 and in Proposition 7.5.4. □

Remark 10.0.4. The variables y_e with $e_+ = q$ do not appear in the support of any element of $\mathbf{O}_G^q$ (see Theorem 7.5.4(iii)). Likewise, the variable x_q does not appear in the support of any element of $\mathbf{M}_G^q$ (see Theorem 3.2.2). Therefore we also have an isomorphism $\bar{\phi}(\mathbf{O}_G^q + \langle \mathcal{L}^{(q)} \rangle) = \bar{\phi}(\mathbf{O}_G^q + \mathfrak{a} + \langle y_{e_q} \rangle) \cong \mathbf{M}_G^q + \langle x_q \rangle$. In other words $(\mathbf{S}/\mathbf{O}_G^q) \otimes_{\mathbf{S}} (\mathbf{S}/\langle \mathcal{L}^{(q)} \rangle) \cong \tilde{\mathbf{R}}/\mathbf{M}_G^q$, where $\tilde{\mathbf{R}} = K[\{x_v\}_{v \neq q}]$.

Theorem 10.0.5.

(i) The polyhedral cell complex $\mathcal{B}_G^{q,c}$ (equivalently, $\mathcal{A}_G^{q,c}$) supports a Div(G)*-graded (and $\mathbb{Z}$-graded) minimal free resolution for $\mathbf{M}_G^q$.*

(ii) The quotient labeled cell complex Del$(L(G))/L(G)$ *supports a* Pic(G)*-graded (and $\mathbb{Z}$-graded) minimal free resolution for $\mathbf{I}_G$.*

(iii) The $\mathbb{Z}$-graded Betti diagrams of $\mathbf{J}_G$, $\mathbf{I}_G$, $\mathbf{O}_G^q$, and $\mathbf{M}_G^q$ coincide.

Proof. (i) By Theorem 7.3.2, we know that $\mathcal{B}_G^{q,c}$ gives a $C^1(G,\mathbb{Z})$-graded minimal free resolution for $\mathbf{S}/\mathbf{O}_G^q$. The same statement is true if we replace $\mathcal{B}_G^{q,c}$ with $\mathcal{A}_G^{q,c}$ by the discussion in §7.5. By Theorem 8.4.1(iv) and Proposition 9.1.4, if we replace all the labels $\mathbf{m}_F$ with $\mathbf{m}_F + \mathfrak{a}$, we obtain a minimal cellular free resolution for $(\mathbf{S}/\mathbf{O}_G^q)/ \otimes_{\mathbf{S}} (\mathbf{S}/\mathfrak{a}) \cong \mathbf{R}/\mathbf{M}_G^q$ (see Proposition 10.0.3(ii)). Alternatively we could replace all labels $\mathbf{m}_F$ with $\mathbf{m}_F + \langle \mathcal{L}^{(q)} \rangle$ to obtain a minimal cellular free resolution for $(\mathbf{S}/\mathbf{O}_G^q) \otimes_{\mathbf{S}} (\mathbf{S}/\langle \mathcal{L}^{(q)} \rangle) \cong \tilde{\mathbf{R}}/\mathbf{M}_G^q$. The new labels are easily seen to be Div(G) and $\mathbb{Z}$-homogeneous, and the resulting minimal free resolution is Div(G) and $\mathbb{Z}$-graded.

(ii) follows similarly from Theorem 7.4.2, Theorem 8.4.1(iv), Proposition 9.2.1, and Proposition 10.0.3(i).

(iii) The fact that the (ungraded) Betti numbers of $\mathbf{J}_G$ and $\mathbf{O}_G^q$ coincide follows from Lemma 7.5.3. By the labeling compatibility described in Lemma 7.5.2 the $\mathbb{Z}$-graded Betti numbers of $\mathbf{J}_G$ and $\mathbf{O}_G^q$ coincide as well. Since all elements of $\mathcal{L}$ are homogeneous (linear) forms, the relabeling of cells described above (in passing from $\mathbf{J}_G$ to $\mathbf{I}_G$ and from $\mathbf{O}_G^q$ to $\mathbf{M}_G^q$) does not change the $\mathbb{Z}$-degrees (see also Theorem 8.4.1(iii)). Therefore the $\mathbb{Z}$-graded Betti diagrams of all four ideals coincide. □

Remark 10.0.6. Recall from Remark 6.1.3 that the number of i-dimensional cells in $\mathcal{B}_G^{q,c}$ is equal to the number of acyclic partial orientations of G with $(i+2)$ (connected) components having a unique source at q. So one immediately obtains a combinatorial description of the (ungraded) Betti numbers in terms of acyclic partial orientations. This interpretation for the Betti numbers of I_G was conjectured in [52] and proved in [48] and [41].

Example 10.0.7. We return to Examples 7.5.5. We described the sequence $\mathcal{L}^{(q)}$ in Example 9.1.5. For simplicity we let $x_i = x_{u_i}$. By sending $\{y_{e_2}, y_{e_1}\}$ to x_1, $\{y_{\bar{e}_1}, y_{e_5}, y_{e_3}\}$ to x_2, and $\{y_{\bar{e}_3}, y_{e_4}\}$ to x_3, $\mathbf{O}_G^q$ in (21) is sent to the ideal

$$\langle x_2^2x_3, x_1x_2^2, x_3^2, x_2^3, x_1^2, x_1x_2x_3\rangle$$

which is precisely $\mathbf{M}_G^q = \mathrm{in}_{<_q}(\mathbf{I}_G)$ by Theorem 3.2.2(ii). The minimal cellular free resolution of $\mathbf{M}_G^q$ is obtained from the minimal cellular free resolution of $\mathbf{O}_G^q$ (described in Examples 7.5.5) by "relabeling" (i.e. by replacing each y_e with x_{e_+}). We first relabel the complex in Figure 11 to obtain Figure 18. The resulting labeled complex gives a minimal free resolution for $\mathbf{M}_G^q$ which is precisely the minimal free resolution of $\mathbf{O}_G^q$ "relabeled". Concretely, we first extend the labels $\mathbf{m}'(\mathbf{p}_i)$ on the vertices to the whole of $\mathcal{B}_G^q$ by the least common multiple construction. For example,

$$\mathbf{m}_{E_2} = y_{\bar{e}_1}y_{\bar{e}_3}y_{e_4}y_{e_5} \mapsto \mathbf{m}'_{E_2} = x_2^2x_3^2\ ,$$

$$\mathbf{m}_{E_4} = y_{\bar{e}_1}y_{e_2}y_{e_4}y_{e_5} \mapsto \mathbf{m}'_{E_4} = x_1x_2^2x_3\ ,$$

$$\mathbf{m}_{E_5} = y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5} \mapsto \mathbf{m}'_{E_5} = x_1 x_2 x_3^2 \ ,$$

$$\mathbf{m}_{E_6} = y_{e_2} y_{e_3} y_{e_4} y_{e_5} \mapsto \mathbf{m}'_{E_6} = x_1 x_2^2 x_3 \ ,$$

$$\mathbf{m}_{F_2} = y_{\bar{e}_1} y_{e_2} y_{\bar{e}_3} y_{e_4} y_{e_5} \mapsto \mathbf{m}'_{F_2} = x_1 x_2^2 x_3^2 \ .$$

The minimal resolution of $\mathbf{M}_G^q$ is as follows.

$$0 \to \bigoplus_{i=1}^{4} \mathbf{R}(-\mathbf{m}'_{F_i}) \xrightarrow{\partial'_2} \bigoplus_{i=1}^{9} \mathbf{R}(-\mathbf{m}'_{E_i}) \xrightarrow{\partial'_1} \bigoplus_{i=1}^{6} \mathbf{R}(-\mathbf{m}'_{\mathbf{p}_i}) \xrightarrow{\partial'_0} \mathbf{R} \twoheadrightarrow \mathbf{R}/\mathbf{M}_G^q \ .$$

Assume $[[F]]$ denotes the generator of $\mathbf{R}(-\mathbf{m}'_F)$. The homogenized differential operator of the cell complex $(\partial'_0, \partial'_1, \partial'_2)$ is as described in (13). For example:

$$\partial'_0([[\mathbf{p}_i]]) = \mathbf{m}'_{\mathbf{p}_i} = \mathbf{m}'(\mathbf{p}_i) \ ,$$

$$\partial'_1([[E_6]]) = x_2[[\mathbf{p}_4]] - x_3[[\mathbf{p}_4]] \ ,$$

$$\partial'_2([[F_2]]) = x_1[[E_2]] - x_3[[E_4]] + x_2[[E_5]] \ .$$

Although $\mathbf{J}_G$ and $\mathbf{I}_G$ have the same Betti table as $\mathbf{O}_G^q$ and $\mathbf{M}_G^q$, it is not possible to read the minimal free resolutions for $\mathbf{J}_G$ or $\mathbf{I}_G$ directly from $\mathcal{B}_G^q$; one really needs to consider the cell decomposition of $L(G)_\mathbb{R}/L(G)$ or of $\mathrm{Div}^0_\mathbb{R}(G)/\operatorname{Prin}(G)$.

Remark 10.0.8. There is an isometry between the principal lattice $(\operatorname{Prin}(G), \langle \cdot, \cdot \rangle_{\mathrm{en}})$ and the cut lattice $(L(G), \langle \cdot, \cdot \rangle)$ (Remark 5.3.5). So the Delaunay decompositions $\operatorname{Del}(\operatorname{Prin}(G))$ and $\operatorname{Del}(L(G))$ are combinatorially equivalent (compare Figure 5 with Figure 13) and the relabeling of cells in $\operatorname{Del}(L(G))$ described above correspond to the labels that were given to cells of $\operatorname{Del}(\operatorname{Prin}(G))$ in §5.3. Therefore the resolution of $\mathbf{I}_G$ described in Theorem 5.3.3 coincides with the resolution of $\mathbf{I}_G$ obtained from the resolution of $\mathbf{J}_G$ in Theorem 7.4.2 by "relabeling" as in Theorem 10.0.5. For example, the resolution of $\mathbf{I}_G$ described in Example 5.3.4 can alternatively be obtained from the resolution of $\mathbf{J}_G$ described in Example 7.5.6.

It is straightforward to give an alternate *proof* for Theorem 5.3.2 and Theorem 5.3.3 using these observations.

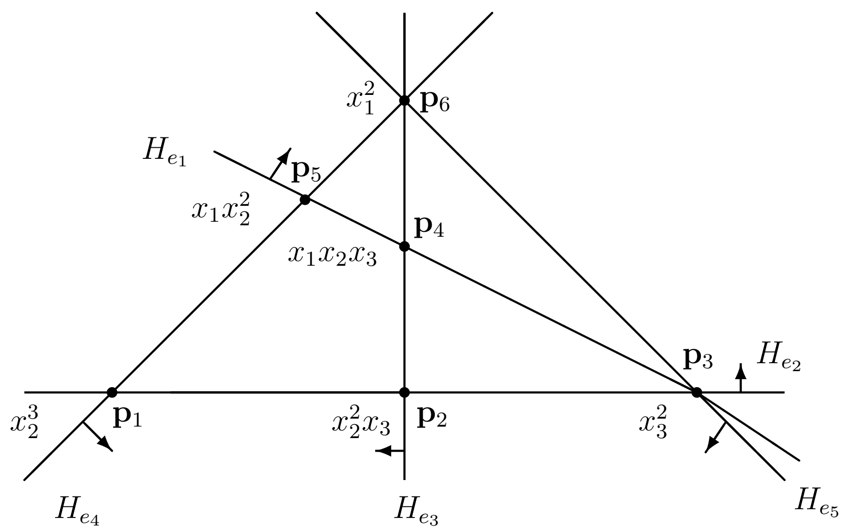

Figure 18: The relabeled bounded complex $\mathcal{B}_G^{q,c}$ giving a minimal free resolution of $\mathbf{M}_G^q$

CHAPTER XI

SOME CONSEQUENCES OF OUR MAIN RESULTS

11.1 Cohen-Macaulayness

For a polynomial ring S, a term order $<$ and an ideal $I \subset S$, it is known that S/I is Cohen-Macaulay if and only if $S/\operatorname{in}_<(I)$ is Cohen-Macaulay (see, e.g., [34, Corollary 3.3.5]).

Proposition 11.1.1. *The modules* $\mathbf{S}/\mathbf{O}_G^q$*,* $\mathbf{R}/\mathbf{M}_G^q$*,* $\mathbf{S}/\mathbf{J}_G$*, and* $\mathbf{R}/\mathbf{I}_G$ *are all Cohen-Macaulay.*

Proof. By Proposition 9.1.1(iv) we have that $\mathbf{S}/\mathbf{O}_G^q$ is Cohen-Macaulay.

For $\mathbf{R}/\mathbf{M}_G^q$, first observe that by Theorem 3.2.2 the variable x_q does not appear in the support of any of the given monomial generators of $\mathbf{M}_G^q$. This implies that $\operatorname{depth}(\mathbf{R}/\mathbf{M}_G^q) \geq 1$. On the other hand, $\dim(\mathbf{R}/\mathbf{M}_G^q) = 1$. One way to see this is the following: by Proposition 9.1.1(iii) we know that $\dim(\mathbf{S}/\mathbf{O}_G^q) = 2m - n + 1$. Since $\mathcal{L}^{(q)} = \mathfrak{a} \cup \{y_{e_q}\}$ is an l.s.o.p. for $\mathbf{S}/\mathbf{O}_G^q$, we deduce by Proposition 10.0.3(ii) that $\dim(\mathbf{R}/\mathbf{M}_G^q) = \dim((\mathbf{S}/\mathbf{O}_G^q) \otimes_{\mathbf{S}} (\mathbf{S}/\mathfrak{a})) = \dim((\mathbf{S}/\mathbf{O}_G^q)/\mathfrak{a}(\mathbf{S}/\mathbf{O}_G^q)) = 1$. Therefore $\mathbf{M}_G^q$ is also Cohen-Macaulay.

Since $\operatorname{in}_{\prec_q}(\mathbf{J}_G) = \mathbf{O}_G^q$ and $\operatorname{in}_{<_q}(\mathbf{I}_G) = \mathbf{M}_G^q$, we immediately conclude that $\mathbf{S}/\mathbf{J}_G$ and $\mathbf{R}/\mathbf{I}_G$ are also Cohen-Macaulay. □

11.2 Multiplicities

For a finitely generated (graded) module M of dimension $d > 0$ over a polynomial ring, the *multiplicity* of M is defined to be the leading coefficient of the Hilbert polynomial of M (i.e. the polynomial defining $i \mapsto \dim(M_i)$ for $i >> 0$). We will denote this quantity by $e(M)$. Since the Hilbert polynomial is completely determined by the Betti table (see, e.g., [46, Theorem 8.20 and Proposition 8.23]), the multiplicity is also determined by the Betti table. The following result easily follows.

Theorem 11.2.1.

$$e(\mathbf{S}/\mathbf{O}_G^q) = e(\mathbf{S}/\mathbf{J}_G) = e(\mathbf{R}/\mathbf{M}_G^q) = e(\mathbf{R}/\mathbf{I}_G) = \kappa(G) \ ,$$

where $\kappa(G)$ denotes the number of spanning trees of G.

Proof. All these ideals have the same Betti table and hence the same multiplicity. It suffices to compute the multiplicity of $\mathbf{S}/\mathbf{O}_G^q = K[\Sigma_G^q]$. By Proposition 9.1.1(ii), we have

$$\mathbf{O}_G^q = \bigcap_T P_T \ ,$$

the intersection being over all spanning trees of G. By Proposition 9.1.1(iii), we have $\dim(\mathbf{S}/\mathbf{O}_G^q) = 2m-n+1$. Also, for each spanning tree T we have $P_T = \langle y_e : e \in \mathcal{O}_T \rangle$ and therefore

$$\dim(\mathbf{S}/P_T) = 2m - n + 1 \quad \text{and} \quad e(\mathbf{S}/P_T) = 1 \ .$$

In this situation (see, e.g., [33, Lemma 5.3.11]) we have

$$e(\mathbf{S}/\mathbf{O}_G^q) = \sum_T e(\mathbf{S}/P_T) \ ,$$

the sum being over all spanning trees of G. □

For $\mathbf{R}/\mathbf{I}_G$, the multiplicity was recently computed in [50] using a different method.

11.3 Alexander dual of M_G^q and cocellular free resolution

In [42], Riemann-Roch theory for graphs is linked to Alexander duality for the ideal $\mathbf{M}_G^q$. Recall that $\mathbf{M}_G^q \subset \tilde{\mathbf{R}} = K[\{x_v\}_{v \neq q}]$ (see Remark 10.0.4). Here we quickly study the Alexander dual of $\mathbf{M}_G^q$ and use Theorem 7.3.2 to obtain its minimal cocellular free resolution.

We define the divisor

$$\mathbf{a} = \sum_{v \in V(G)} (\deg(v))(v) \ .$$

It follows from Theorem 3.2.2 and Theorem 10.0.5(i) that:

(i) $\mathbf{M}_G^q$ is generated in degree preceding $\mathbf{a}$.

(ii) $\mathbf{M}_G^q + \langle \{x_v^{\mathbf{a}(v)+1}\}_{v \neq q} \rangle = \mathbf{M}_G^q$; this this is because for each $v \neq q$ in $V(G)$, the star of the vertex v forms a cut and therefore $x_v^{\deg(v)} \in \mathbf{M}_G^q$.

(iii) All face labels in the labeled cell complex $B_G^{q;c}$ resolving $\mathbf{M}_G^q$ (as in Theorem 10.0.5(i)) divide $\mathbf{x}^{\mathbf{a}+\mathbf{1}}$. In fact a stronger statement is true; all vertex labels divide $\mathbf{x}^{\mathbf{a}}$.

Consider the cellular complex $\mathcal{B}_G^{q;c}$ with labels $\mathbf{m}'_F$ for cells F as in the proof of Theorem 10.0.5(i). Relabel each cell F with $\mathbf{x}^{\mathbf{a}+\mathbf{1}}/\mathbf{m}'_F$. For simplicity, let us call $\mathcal{B}_G^{q;c}$ with its new labels $\mathcal{D}$. Let $\mathcal{D}_{\leq \mathbf{a}}$ denote the subcomplex consisting of cells with labels dividing $\mathbf{a}$. Let $(\mathbf{M}_G^q)^{[\mathbf{a}]}$ denote the Alexander dual of $\mathbf{M}_G^q$ with respect to $\mathbf{a}$ ([46, Definition 5.20]). In this setting, [46, Theorem 5.37] gives the following result:

Proposition 11.3.1. *The polyhedral complex* $(\mathcal{D}_G)_{\leq \mathbf{a}}$ *supports a minimal (cocellular) resolution for the ideal* $(\mathbf{M}_G^q)^{[\mathbf{a}]}$.

This observation has been made (independently) in [26]. See [46, Section 5.3] for more details. Here we give an example to illustrate this result.

Example 11.3.2. The complex $(\mathcal{D}_G)_{\leq \mathbf{a}}$ associated to Example 10.0.7 and Figure 18 is depicted in Figure 19 in blue. The ideal $(\mathbf{M}_G^q)^{[\mathbf{a}]}$ is minimally generated as

$$(\mathbf{M}_G^q)^{[\mathbf{a}]} = \langle x_1x_2^2x_3^2, x_1x_2^3x_3, x_1^2x_2^2x_3, x_1^2x_2x_3^2 \rangle \ .$$

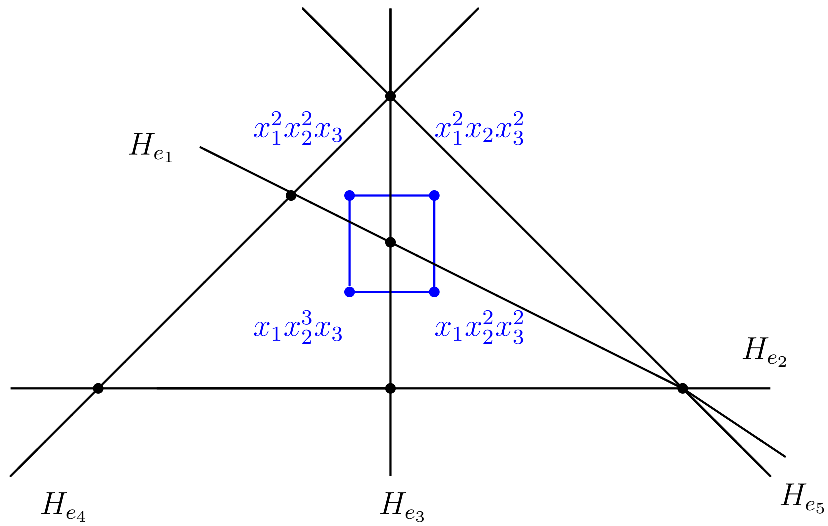

Figure 19: The bounded complex supporting a minimal free resolution of $(\mathbf{M}_G^q)^{[\mathbf{a}]}$

11.4 Graphic matroid ideal and h-vectors

Let $\tilde{\mathbf{S}} = K[\mathbf{z}]$ denote the polynomial ring in m variables $\{z_e : e \in E(G)\}$. There is a surjective K-algebra homomorphism

$$\pi \colon \mathbf{S} \to \tilde{\mathbf{S}}$$

defined by sending both y_e and $y_{\bar{e}}$ to z_e. The kernel of this map is the ideal generated by

$$\mathcal{K} = \{y_e - y_{\bar{e}} : e \in \mathcal{O}\}$$

for some fixed orientation $\mathcal{O}$. We will denote this kernel by $\mathfrak{b} = \langle \mathcal{K} \rangle$. We get an induced isomorphism

$$\tilde{\pi} \colon \mathbf{S}/\mathfrak{b} \to \tilde{\mathbf{S}}\ .$$

We define the (unoriented) *graphic matroid ideal* $\mathbf{Mat}_G \subset \tilde{S}$ to be the image of $\mathbf{O}_G^q + \mathfrak{b}$ under this isomorphism. Concretely, $\mathbf{Mat}_G$ is obtained from $\mathbf{O}_G^q$ by identifying the variables y_e and $y_{\bar{e}}$ and replacing them with z_e.

Lemma 11.4.1.

(i) $\mathcal{K}$ *forms a regular sequence for* $\mathbf{S}/\mathbf{O}_G^q$.

(ii) $\mathcal{B}_G^{q,c}$ *(equivalently* $\mathcal{A}_G^{q,c}$*) supports a minimal free resolution for* $\mathbf{Mat}_G$.

(iii) $\mathbf{Mat}_G$ *is independent of the choice of* q.

(iv) The $\mathbb{Z}$*-graded Betti diagram of* $\mathbf{Mat}_G$ *coincides with the* $\mathbb{Z}$*-graded Betti diagrams of* $\mathbf{J}_G$, $\mathbf{I}_G$, $\mathbf{O}_G^q$, *and* $\mathbf{M}_G^q$.

Proof. (i) follows from [49, Corollary 2.7]. Alternatively, by the explicit description of the facets in Proposition 9.1.1(i), the restriction of each linear form in $\mathcal{K}$ spans a vector space of dimension 1 and therefore the result follows from Lemma 8.3.1.

(ii) follows from (i) and Theorem 8.4.1(iv).

There are several ways to see (iii). For example, it follows from (ii) and the discussion in §7.5 (e.g. Proposition 7.5.4) that $\mathbf{Mat}_G$ is minimally generated by monomials

$$\{ \prod_{e \in E(A^c, A)} z_e : A \subsetneq V(G), G[A] \text{ and } G[A^c] \text{ are connected}\} \tag{26}$$

where $E(A^c, A)$ denotes the set of (unoriented) edges connecting $G[A]$ and $G[A^c]$. This description is independent of the choice of the base vertex q.

(iv) follows from Theorem 10.0.5 and Theorem 8.4.1(iii). □

It is a fact, essentially due to Hilbert, that the Hilbert series of a module is completely determined by its graded Betti table and its dimension. The numerator of the Hilbert series is called the h-polynomial. Its coefficients are obtained from the Betti numbers as an alternating sum and they form the h-vector (see, e.g, [46, Theorem 8.20 and Theorem 8.23]). So we immediately obtain, from Lemma 11.4.1(iv), the following result.

Lemma 11.4.2. *The* h*-vectors of* $\mathbf{S}/\mathbf{J}_G$, $\mathbf{R}/\mathbf{I}_G$, $\mathbf{S}/\mathbf{O}_G^q$, $\mathbf{R}/\mathbf{M}_G^q$, *and* $\tilde{\mathbf{S}}/\mathbf{Mat}_G$ *coincide.*

The ideal $\mathbf{Mat}_G$ has been extensively studied in the literature (see, e.g., [56], [57, Section III.3], [49, Section 3]). A more well known presentation of this ideal is by its prime decomposition; for each spanning tree T of G, let $I_T = \langle z_e : e \in T \rangle$. The minimal prime decomposition of $\mathbf{Mat}_G$ is

$$\mathbf{Mat}_G = \bigcap_T I_T \ , \tag{27}$$

the intersection being over all spanning trees of G. This can be proved the same way as Proposition 9.1.1(ii) (or can be deduced from it).

From (27) it is evident that $\mathbf{Mat}_G$ is the Stanley-Reisner ideal of the simplicial complex Σ of independent sets of the cographic matroid (i.e. the matroid whose bases are the complements of spanning trees of G). Therefore the h-polynomial of $\tilde{\mathbf{S}}/\mathbf{Mat}_G$ is precisely $T(1,y)$, where $T(x,y)$ is the Tutte polynomial of the graph ([13, page 236]). By Lemma 11.4.2, we obtain the following result:

Corollary 11.4.3. *$T(1,y)$ is the h-polynomial for $\mathbf{S}/\mathbf{J}_G$, $\mathbf{R}/\mathbf{I}_G$, $\mathbf{S}/\mathbf{O}_G^q$, $\mathbf{R}/\mathbf{M}_G^q$, and $\tilde{\mathbf{S}}/\mathbf{Mat}_G$.*

Postnikov and Shapiro in [53] prove this result for $\tilde{\mathbf{R}}/\mathbf{M}_G^q$ (equivalently, for $\mathbf{R}/\mathbf{M}_G^q$) by a combinatorial argument. Merino's work in [44] proves this result for $\mathbf{R}/\mathbf{I}_G$ using deletion-contraction methods. A bijective proof of Merino's result was later presented in [21] (see also [6]). We believe that Corollary 11.4.3 gives a unified and more conceptual proof of these results. Moreover, Merino's theorem (stating that $T(1,y)$ is the generating function for the number of q-reduced divisors in various degrees) is a straightforward consequence of Corollary 11.4.3 and Theorem 3.2.2.

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