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THEOREMS OF LARGE DEVIATIONS FOR THE SUMS OF A RANDOM NUMBER OF INDEPENDENT RANDOM VARIABLES

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ATSITIKTINIO SKAIČIAUS NEPRIKLAUSOMŲ DĖMENŲ SUMOS DIDŽIŲJŲ NUOKRYPIŲ TEOREMOS

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Abstract

In probability theory, the topic of large deviations, i. e., approximation problems of the probabilities of rare events, have a significant place. To understand why rare events are important at all one only has to think of the events in an insurance mathematics, nuclear physics and etc., to be convinced that those events can have an enormous impact.

This thesis is concerned with a normal approximation to a distribution of the sum $Z_N = \sum_{j=1}^{N} a_j X_j,\ Z_0 = 0,\ 0 < a_j < \infty$, of a random number of summands N of independent identically distributed weighted random variables $\{X, X_j, j = 1, 2, ...\}$ that takes into consideration large deviations in both the Cramér zone (the characteristic functions of the summands of Z_N are analytic in a vicinity of zero) and the power Linnik zone (the growth of the moments of the summands does not ensure the analyticity of the characteristic functions). Here a non-negative integer-valued random variable N is independent of $\{X, X_j, j = 1, 2, ...\}$. In addition, the asymptotic expansion that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process is obtained. To solve the problems, the classical method of characteristic functions, cumulant and combinatorial methods are used.

Although, in probability theory the asymptotic behavior of tail probabilities for the sums of a random number of summands of random variables is a quite new problem, it was initiated in the *XX*th century, but there is a very extensive literature on mentioned problem. However, as it is known for the author of the dissertation, there are a few scientific works on theorems of large deviations for the sums of a random number of summands of independent random variables in case where the cumulant method is used.

The thesis consists of an introduction, three chapters, general conclusions, references, and a list of the author's publications. The introduction reveals the importance of the scientific problem, describes the tasks of the thesis, research methodology, scientific novelty, the practical significance of results. In the first chapter an overview of the problems is presented. The second chapter is devoted for obtaining an upper bound for the cumulants, theorems of large deviations and exponential inequalities for the standardized version of the sum Z_N. The instances of large deviations (the law of N is known; $a_j \equiv 1$; discount version of large deviations) are also analyzed in this chapter. In the third chapter, the asymptotic expansion of large deviations in the Cramér zone for the density function of the standardized compound Poisson process is considered.

Santrauka

Itin svarbi tikimybių teorijos dalis yra skirta didžiųjų nuokrypių problematikai, tai yra retai pasitaikančių įvykių tikimybių aproksimacijos uždaviniams. Norint suprasti, kodėl reti įvykiai yra apskritai tokie svarbūs, užtenka prisiminti retai pasitaikančius įvykius draudos matematikoje, branduolinėje fizikoje ir pan., kurie gali turėti didžiulį poveikį.

Disertacija yra skirta atsitiktinio dėmenų skaičiaus N nepriklausomų vienodai pasiskirsčiusių atsitiktinių dydžių $\{X, X_j, j = 1, 2, ...\}$ su svoriniais koeficientais $0 < a_j < \infty$ sumos $Z_N = \sum_{j=1}^{N} a_j X_j,\ Z_0 = 0$, skirstinio normaliosios aproksimacijos didžiųjų nuokrypių Kramero (sumos Z_N dėmenų charakteristinės funkcijos yra analizinės nulinio taško aplinkoje) ir laipsninėse Liniko zonose (dėmenų momentų augimas neužtikrina charakteristinės funkcijos analiziškumo) problemos sprendimui. Čia neneigiamas, sveikareikšmis atsitiktinis dydis N yra nepriklausomas nuo $\{X, X_j, j = 1, 2, ...\}$. Taip pat, yra gautas standartizuoto sudėtinio Puasono proceso tankio funkcijos asimptotinis skleidinys didžiųjų nuokrypių Kramero zonoje. Uždavinių sprendimui yra naudojami klasikinis charakteristinių funkcijų, kumuliantų ir kombinatorinis metodai.

Nors atsitiktinio dėmenų skaičiaus atsitiktinių dydžių sumų didžiųjų nuokrypių tikimybių asimptotinio elgesio tyrimas yra pakankamai naujas tikimybių teorijos uždavinys, kuris pradėtas nagrinėti XX a., tačiau yra paskelbtas įspūdingas kiekis mokslinių darbų, kuriuose nagrinėjama minėta problema. Ir vis dėlto, kiek disertacijos autorei yra žinoma, mokslinių darbų, skirtų atsitiktinio dėmenų skaičiaus, nepriklausomų atsitiktinių dydžių sumų didžiųjų nuokrypių teoremų gavimui, taikant kumuliantų metodą, yra mažai.

Disertaciją sudaro įvadas, trys pagrindiniai skyriai, bendrosios išvados, literatūros sąrašas, autoriaus publikacijų disertacijos tema sąrašas. Įvade atskleidžiama nagrinėjamos mokslinės problemos svarba, aprašomi darbo uždaviniai, tyrimo metodai, mokslinis naujumas, praktinė rezultatų reikšmė. Pirmame skyriuje pateikiama nagrinėjamos temos apžvalga. Antrasis skyrius skirtas sumos Z_N standartizuoto varianto kumuliantų viršutinių įverčių ir didžiųjų nuokrypių teoremų bei eksponentinių nelygybių gavimui. Šiame skyriuje taip pat analizuojami atskiri didžiųjų nuokrypių atvejai (kai N skirstinys yra žinomas; kai $a_j \equiv 1$ ir didžiųjų nuokrypių diskontavimo versija). Trečiame skyriuje yra nagrinėjamas standartizuoto sudėtinio Puasono proceso tankio funkcijos asimptotinis skleidinys didžiųjų nuokrypių Kramero zonoje.

Notation

Symbols

$\mathbf{P}(A)$ – the probability of a random event A;
F_X – the distribution function of random variable X;
p_X – the density function of random variable X;
f_X – the characteristic function of random variable X;
$\mathbf{E}$ – the mean;
$\mathbf{D}$ – the variance;
μ – the mean of random variable X;
σ^2 – the variance of random variable X;
$\mathbb{N}$ – the set of natural numbers;
$\mathbb{N}_0$ – the set of non-negative integer numbers;
$\mathbb{R}$ – the set of real numbers;
$[x]$ – the integer part of x;
Φ – standard normal distribution function;
ϕ – standard normal density function;
$\xrightarrow{\mathbf{P}}$ – the convergence in probability;
$\xrightarrow{d}$ – the convergence in distribution;

$*$ – the convolution operation;
$\sim$ – asymptotically equivalent;
$\simeq$ – approximately equal;
$\ll$ – much less than;
$\bar{a}$ – $0 < \bar{a} = \inf\{a_j, j = 1, 2, ...\} < \infty$, where $0 < a_j < \infty$;
a – $0 < a = \sup\{a_j, j = 1, 2, ...\} < \infty$;
$(b \vee c),\ b,\ c \in \mathbb{R}$ – $(b \vee c) = \max\{b, c\}$.

Abbreviations

$i.\,i.\,d.$ – independent identically distributed;
$r.\,n.\,s.$ – random number of summands;
$a.\,s.$ – almost sure.

Contents

Introduction

Scientific problem

The theory of large deviations deals with the probabilities of rare events that are exponentially small as a function of some parameter. For example, in insurance mathematics, such problems arise in the approximation for small probabilities of large claims that occur rarely.

The theory was originally created for sums of independent identically distributed (i. i. d.) random variables and then extended to a class of random processes (see, e. g., Cramér 1938; Ibragimov and Linnik 1965; Petrov 1975; Nagaev, S. V. 1979; Saulis and Statulevičius 1991; Dembo and Zeitouni 2010; Borovkov and Mogulskii 2012; Yang *et al.* 2012). This thesis is concerned with theorems of large deviations in both the Cramér and the power Linnik zones (see, e. g., Saulis and Statulevičius 1991) for a distribution of the sums of a random number of summands (r. n. s.) of i. i. d. and weighted random variables. Only the case of normal approximation is considered in the thesis.

Topicality of the work

The asymptotic behavior of the probabilities for the sums of a r. n. s. of random variables is a quite recent problem in probability theory. The first results were developed in the twentieth century, in the 1940s. Presently, there are many strong results on the approximation of tail probabilities for the aforementioned sums. Nevertheless, the theory of large deviations is still under rapid development, because of a large number of diverse and extremely complicated problems arising in various areas of mathematics and applications that require it's investigation (see, e. g., Shorgin 1998; Pragarauskas 2007; Touchette 2009; Korolev *et al.* 2011; Borovkov and Mogulskii 2012; Foss *et al.* 2013).

Research object

The research object of this thesis is the sum of a r. n. s. of i. i. d. random variables with positive weights (weighted random sum). Throughout the thesis, it is assumed that the non-negative integer-valued index of the sum is independent of the considered random variables.

The aim and tasks of the dissertation

The aim of this dissertation is a normal approximation to a distribution of the standardized sum of a r. n. s. of i. i. d. weighted random variables that takes into consideration large deviations in both the Cramér and the power Linnik zones. The results are obtained for two cases: where the mean of considered random variables is zero, and where it is non-zero. In the thesis, the following problems are examined:

1. To evaluate the upper estimate for cumulants of the standardized weighted random sum in the case where the i. i. d. random variables satisfy S. N. Bernstein's condition (see, e. g., Saulis and Statulevičius 1991) and under some additional assumptions for the cumulants of a sum of a r. n. s. of positive weights (see, e. g. Kasparavičiūtė and Saulis 2013).
2. To obtain exact large deviation ratios and to analyze the asymptotic behavior (convergence to the unit) of that ratios for a distribution function of the standardized weighted random sum.
3. To derive exponential inequalities for the probability of large deviations for aforementioned sum.
4. To consider instances of large deviations where the law of the random number of summands is known (is a binomial random variable, and is homogeneous, or mixed Poisson process); where all weights are equal to a unit, and the discounted version of large deviations.

5. To obtain asymptotic expansion that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process.

Applied methods

Solutions to the problems of this thesis are obtained by first using general lemmas presented in (Rudzkis *et al.* 1978; Bentkus, R. and Rudzkis 1980; Saulis 1980), accordingly, on exact ratios of large deviations, exponential inequalities for large deviation probabilities, and asymptotic expansion of the density function for an arbitrary random variable with zero mean and unit variance.

Among the existing methods for large deviations (see, e. g., Saulis and Statulevičius 1991; Jensen 1995; Borovkov 1999; Fatalov 2011, 2010; Gao and Zhao 2011), we rely on the cumulant method that was proposed by S. V. Statulevičius (1966) and developed by R. Rudzkis, L. Saulis, and V. Statulevičius (1978), as it is a powerful method that permits the systematic investigation of large deviations for various statistics.

To obtain asymptotic expansion that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process along with the cumulant method, the classical method of characteristic functions is used. In addition, based on S. V. Statulevičius's known estimates for characteristic functions (see Statulevičius 1965) of an arbitrary random variable, the structure of the remainder term of aforementioned asymptotic expansion is obtained.

The combinatorial method is used to evaluate the upper estimates for the cumulants of the standardized weighted random sum.

Scientific novelty

There is a very extensive literature on approximation of tail probabilities for random sums, under different assumptions and with various applications (see, e. g., Aksomaitis 1965; Statulevičius 1967; Saulis 1978; Embrechts *et al.* 1985; Faÿ *et al.* 2006; Saulis and Deltuvienė 2007; Robert and Segers 2008; Yang *et al.* 2013; Foss *et al.* 2013). However, among scientific works there are no works – excepting publications (Kasparavičiūtė and Saulis 2010, 2011a, 2011b, 2013) by the author of this dissertation together with L. Saulis – for normal approximation that take into consideration large deviations in both the Cramér and the power Linnik zones for the sum of a r. n. s. of i. i. d. weighted random variables in case where the cumulant method is used.

To prove theorems of large deviations in both the Cramér and the power Linnik zones for the distribution function of the standardized weighted random sum, to obtain exponential inequalities for large deviation probabilities of aforementioned random sum, to derive asymptotic expansions that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process, when cumulant, characteristic functions and saddle-point methods are used, are rather complicated problems that were solved for the first time.

Besides, it should be emphasized that in the thesis, i. i. d. weighted random variables are considered, which constitute an intermediate variant between identical and non-identical distributed random variables. In addition, in order to obtain upper bounds for the cumulants of the sum of a r. n. s., combinatorial method is used.

Practical value of the results

The sums of a random number of independent random variables appear as models in many applied problems, for instance, in insurance, economic theory, finance mathematics, random walks, queuing theory, network theory, (see, e. g., Bening *et al.* 1997; Mikosh 2009; Korolev *et al.* 2011; Foss *et al.* 2013). In addition, the theory of large deviations is one of the most active research fields in probability theory, with many applications to areas such as statistical inference, queuing systems, communication networks, information theory, risk-sensitive control, partial differential equations, statistical mechanics, physics (see, e. g., Chernoff 1956; Varadhan 2003a,b; Feng and Kurtz 2006; Touchette 2009; Borovkov and Mogulskii 2012).

Questions related to extremal events play an increasingly important role in both financial and insurance applications (see, e. g., Pham 2010; Mikosh 2009; Korolev *et al.* 2011). In finance, large deviations arise in various contexts. They occur in risk management for computing the probability of large losses in a portfolio subject to market risk as well as the default probabilities for a portfolio under credit risk.

Statements presented for defence

1. A suitable bound for the cumulants of the standardized sum of a r. n. s. of i. i. d. weighted random variables.
2. Theorems on large deviations in both the Cramér and the power Linnik zones for a distribution function of aforementioned sum.
3. Exponential inequalities for the probability of large deviations for the standardized weighted random sum.
4. Asymptotic expansion that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process.

Approval of the work results

Four papers on the topic of the dissertation (see List of author's scientific publications on the topic of the dissertation: 117) have been published in refereed scientific journals: *Acta Applicandae Mathematicae* (Thomson ISI Web of Knowledge), *Nonlinear Analysis: Modelling and Control* (Thomson ISI Web of Science), and the *Lithuanian Mathematical Journal*. Intermediate research results were reported at 10 scientific conferences and approved in 4 seminars, of which 2 conferences and 1 seminar are international. The essential presentations are as follows:

1. Theorems on large deviations for the sum of a random number of summands, *International conference on Probability theory and it's applications*, Moscow, Russia, 2012.
2. Approximation of small probabilities of the sums of random number of summands, *International conference on Applied mathematics and approximation theory*, Ankara, Turkey, 2012.
3. Asymptotic analysis in the large deviation zones for the distribution and density functions of the random sums, *XXX International Seminar on Stability Problems for Stochastic Models*, Svetlagorsk, Russia, 2012.
4. On large deviations for compound mixed Poisson process *LMD 54th conference*, Vilnius, 2013.
5. Local limit theorems for the sums of a random number of summands *LMD 53rd conference*, Klaipėda, 2012.
6. The discounted version on large deviations for randomly indexed sum of random variables, *LMD 52nd conference*, Vilnius, 2011.

7. Theorems on large deviations for the sums of a random number of summands. *LMD 51st conference*, Šiauliai, 2010.

The scope of the scientific work

This thesis consists of an introduction, three chapters, general conclusions, references, and a list of the author's publications. The total scope of the dissertation is 118 pages, 352 mathematical expressions, 182 items of reference.

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I want to express my gratitude to everyone who supported me so that I could complete this work. Firstly, I am grateful to my scientific supervisor Prof Dr Habil Leonas Saulis for his patience and his helpful and insightful suggestions. He provided me with the opportunity to learn and always encouraged me to do my very best.

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1

Historical overview of large deviation theorems

The biggest and possibly most important part of probability theory consists of the limit theorems, where theorems of large deviations (see, e. g., Ibragimov and Linnik 1965; Saulis and Statulevičius 1991; Petrov 1995; Ramasubramanian 2008; Varadhan 2008) have a significant place.

The theory of large deviations for the sums of random variables $X_j, j = 1, 2, ...$ was developed by well known scientists in the works (Chintshine 1929; Esscher 1932; Smirnov 1933; Cramér 1938; Feller 1943, 1969; Petrov 1953, 1954, 1963, 1964, 1975; Linnik 1960, 1961a; 1961b, 1962; Ibragimov and Linnik 1965; Ibragimov 1967; Zolotarev 1962; Nagaev, S. V. 1963, 1973, 1979; Nagajev, S. V. and Fuk 1971; Nagaev, S. V. and Sakoyan 1976; Borovkov 1964; Borovkov and Mogulskij 1978, 1980; Statulevičius 1965, 1966, 1979; Nagaev, A. V. 1967, 1969; Saulis 1969, 1979, 1980; Prokhorov 1972; Osipov 1972; Misevičius and Saulis 1973; Bikelis and Žemaitis 1974, Rudzkis *et al.* 1978, 1979; Bentkus, R. and Rudzkis 1980; Mogulskii 1980; Bentkus, V. 1986; Rudzkis 1989) and others, by well known scientists. The most widely studied cases are the following:

- The *Cramér condition* is satisfied: there exist $h > 0$, such that

$$\mathbf{E}e^{h|X_j|} < \infty, \tag{1.1}$$

i. e., the characteristic functions

$$f_{X_j}(u) = \mathbf{E}e^{iuX_j} = \int_{-\infty}^{\infty} e^{iux} dF_{X_j}(x), \qquad u \in \mathbb{R} \tag{1.2}$$

of the summands X_j are analytic in a vicinity of the point $u = 0$ (see, e. g., Cramér 1938; Feller 1943; Petrov 1954; Statulevičius 1966; Petrov and Robinson 2007);

- *Linnik condition* is satisfied: there exist $h > 0$ and $0 < \gamma < 1$, such that
$$\mathbf{E}e^{h|X_j|^\gamma} < \infty, \tag{1.3}$$
i. e., all the moments of summands are finite but their growth does not ensure the analyticity of the characteristic functions (1.2) in a vicinity of the point $u = 0$ (see, e. g., Linnik 1961a,b; Zolotarev 1962; Nagaev, S. V. 1963, Saulis and Statulevičius 1991);
- The case of so-called moderate deviations, where the summands have only the finite number of moments (this case was first studied by Rubin and Sethuraman 1965, see also Gao and Zhao 2011);
- The case where the Cramér and Linnik conditions are not satisfied, but the behavior of the distribution tails of summands is regular enough (see, e. g., Nagaev, S. V. 1963; Heyde 1968; Nagaev, A. V. 1969; Tkachuk 1975; Mikosh and Wintenberger 2013).

It should be pointed out that the strong law of large numbers and the central limit theorem, the versatile classical limit theorems of probability theory, concern typical events. As large deviation estimates deal with probabilities of rare events, more subtle methods are needed. Moreover, context specific techniques play a major role, although there are quite a few general principles (see, e. g. Ramasubramanian 2008; Varadhan 2008).

Varadhan (2008) provides the following historical overview of work on large deviations. The origin of large deviation theory goes back to Scandinavian actuaries by Esscher (1932) who were interested in the analysis of risk in the insurance industry. A general large deviations for sums of independent random variables was established by Cramér (1938). The result for empirical distributions of independent identically distributed (i. i. d.) random variables is due to Sanov (1957). The generalization to Markov chains and processes can be found in several papers (see, e. g., Donsker and Varadhan 1975, 1976, 1983). The results concerning small random perturbations of deterministic systems go back, e. g., to Freidlin and Wentzell (1998)). There are lecture notes by Varadhan (1984), texts by Ellis (2006), Dembo and Zeitouni (2010) and, most recently, by Feng and Kurtz (2006). These cover a broad spectrum of topics in large deviation theory. Large deviations in the context of hydrodynamic scaling are discussed, e. g., in the exposition (Varadhan 1996), and large deviations for random walks are discussed in (Varadhan 2003a; Borovkov and Mogulskii 2012), as well as the references in (Dembo and Zeitouni 2010). For a general survey on large deviations and entropy appears in (Varadhan 2003b).

The review (Touchette 2009) presents many problems and results in statistical mechanics, and shows how these can be formulated and derived within the context of large deviation theory. The problems and results treated cover a wide range of physical systems, including equilibrium many-particle systems, noise-perturbed dynamics, and non-equilibrium systems, as well as multi-fractals, disordered systems, and chaotic systems.

In this chapter we provide a broad description of research on the topic of this dissertation based on an exhaustive analysis of the literature. We pay the greatest attention to the results used for solving the problems posed in this dissertation. For instance, we rely more on (Ibragimov and Linnik 1965; Statulevičius 1967; Petrov 1995; Saulis 1978, 1980, 1981; Saulis and Deltuvienė 2007; Saulis and Statulevičius 1991). In the first section, we give a central result of large deviation theory that is among the most frequently applied: *Cramér's theorem*. Most of Section 1.1 is devoted to general lemmas on large deviations and asymptotic expansions in the zones of large deviations in the case where cumulant method is used (see, e. g., Saulis and Statulevičius 1991). Section 1.2 presents a broad overview of the asymptotic behavior of sums of a random number of summands (r.n.s).

1.1. Theory of large deviations for the sums of non-random number of summands

Theory of large deviations concerns the rates at which probabilities of certain events decay as a natural parameter in the problem varies. A specific example will best illustrate this.

Assume that we have a family $\{X, X_j, j = 1, 2, ...\}$ of i. i. d. random variables that has a common distribution with mean and finite positive variance:

$$\mu = \mathbf{E}X, \qquad \sigma^2 = \mathbf{D}X < \infty, \qquad F_X(x) = \mathbf{P}(X < x), \quad x \in \mathbb{R}. \quad (1.4)$$

In addition,

$$\mathbf{E}X^k = \frac{1}{i^k}\frac{d^k}{du^k} f_X(u)\Big|_{u=0}, \qquad \Gamma_k(X) = \frac{1}{i^k}\frac{d^k}{du^k}\ln f_X(u)\Big|_{u=0} \tag{1.5}$$

denote the *k*th-order moments and cumulants, $k = 1, 2, ...$, where $f_X(u)$ is the characteristic function (1.2) of the random variable X. Here the existence of $\Gamma_k(X)$ up to the order k must be implied by the existence of all the k*th*-order absolute moments of X. Let us consider the sum

$$S_n = \sum_{j=1}^{n} X_j. \tag{1.6}$$

According to the central limit theorem, the standardized sum

$$\tilde{S}_n = \frac{S_n - \mathbf{E}S_n}{\sqrt{\mathbf{D}S_n}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

with mean $\mathbf{E}\tilde{S}_n = 0$ and variance $\mathbf{D}\tilde{S}_n = 1$ has a limiting standard normal distribution. In particular,

$$\lim_{n\to\infty} F_{\tilde{S}_n}(x) = \Phi(x)$$

uniformly in x, where $F_{\tilde{S}_n}(x) = \mathbf{P}(\tilde{S}_n < x) = \mathbf{P}(S_n - n\mu < \sqrt{n}\sigma x)$, and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy \tag{1.7}$$

is the standard normal distribution function. While the convergence is uniform in x, this does not say much if x is large. Thus, there is significant interest in the behavior of the ratio

$$\frac{1 - F_{\tilde{S}_n}(x)}{1 - \Phi(x)}$$

when $x \to \infty$. Whether this ratio tends to 1 even when $x \to \infty$, depends on how rapidly x is becoming large. In the case where $x \ll \sqrt{n}$, the ratio tends to 1 under suitable conditions, while if $x \simeq \sqrt{n}$, it does not. In the instance where $x \ll \sqrt{n}$, we have *moderate deviations*, i. e., the refinements of the central limit theorem (see, e. g., Gao and Zhao 2011 and the references therein). *Large deviations*, which occur when $x \simeq \sqrt{n}$, are different. It is best to think of them as estimating the probabilities $1 - F_{\tilde{S}_n}(x) = \mathbf{P}(S_n - n\mu \geq$

$\sigma\sqrt{n}x)$, $F_{\tilde{S}_n}(-x) = \mathbf{P}(S_n - n\mu < -\sigma\sqrt{n}x)$, as $x \to \infty$, which are called *probabilities of large deviations*.

The event $\{\tilde{S}_n > x\}$ is a typical rare event of interest in insurance (see, e. g., Ramasubramanian 2008). For example, X_j, $j = 1, 2, ...$, might denote the claim amount for policy holder j in a given year, in which case S_n denotes the total claim amount of n policy holders. Assuming a large portfolio for the insurance company (that is, assuming that n is very large), any estimate of $\mathbf{P}(\tilde{S}_n \geq x)$ gives information about the right tail of the total claim amount payable by the company in a year. For more illustrations from insurance and finance see, e. g., (Ramasubramanian 2008; Pham 2010). For a detailed account of insurance models, also see (Rolski *et al.* 2001).

The asymptotic behavior of large deviations is a fairly new problem in probability theory. The first limit theorems for the probabilities of large deviations of sums of independent random variables were obtained in (Chintshine 1929). A theorem on the probability of large deviations for Bernoulli trials was presented in (Smirnov 1933). The first fundamental theorem of large deviations for the sums of i. i. d. random variables was proved by Cramér (1938), who showed that the rate function is the convex conjugate of the logarithm of the moment generating function of the underlying common distribution.

The behavior of the ratios

$$U_n(x) = \frac{1 - F_{\tilde{S}_n}(x)}{1 - \Phi(x)}, \qquad V_n(x) = \frac{F_{\tilde{S}_n}(-x)}{\Phi(-x)}, \qquad x \in [0, \tau_n], \tag{1.8}$$

where τ_n is a non-decreasing function such that $\tau_n \to \infty$, $n \to \infty$, are of significant interest. It follows from the central limit theorem that

$$U_n \to 1, \qquad V_n \to 1 \tag{1.9}$$

uniformly by $x = [0, \tau_n]$, where $\tau_n = O(1)$, $n \to \infty$. If the ratios (1.9) do hold in the interval $x \in [0, \tau_n]$ when $\tau_n \to \infty$, we call the interval a *zone of normal convergence*. Cramér (1938) noticed that for large deviation problems, particular conditions are needed for the random variable's moments. The cases that have been the most studied are those where the summands of the sums satisfy either the Cramér condition (1.1), or the Linnik condition (1.3).

In the field of large deviations, the work (Cramér 1938) occupies a significant place. The work presents, Cramér's theorem, a central result of large deviation theory and one of the most frequently applied. Cramér's theorem has been extended and generalized in several directions, for example, to sums of dependent random variables or to general sequences of random variables. Feller

(1943), Petrov (1975, 1954), and Statulevičius (1966) studied the probability of large deviations for sums of independent non-identically distributed random variables subject to Cramér's condition. Feller (1943) extended Cramér's theorem to sequences of not necessarily identically distributed random variables under restrictive conditions. See (Feller 1969) for an account of Cramér's theorem in the context of the central limit problem. Cramér's theorem was also expanded by Petrov see (Petrov 1954, 1953) and also (Petrov 1975, 1963, 1964). Petrov obtained the optimal result for large deviation theorems in the Cramér zone when the summands of the sum (1.6) are i. i. d. random variables. A detailed proof of Cramér's theorem can be found in (Petrov 1995: 178).

Let $\mu = 0$, without loss of generality. Petrov's strengthened version of Cramér's theorem is as follows:

Theorem 1.1. *If i. i. d. random variables* $\{X, X_j, j = 1, 2, ...\}$ *satisfy Cramér's condition* (1.1)*, then for all* $x \geq 0$*,* $x = o(\sqrt{n})$ *as* $n \to \infty$*, the ratios*

$$U_n(x) = \exp\Big\{\frac{x^3}{\sqrt{n}}\lambda\Big(\frac{x}{\sqrt{n}}\Big)\Big\}\Big(1 + O\Big(\frac{x+1}{\sqrt{n}}\Big)\Big), \tag{1.10}$$

$$V_n(x) = \exp\Big\{-\frac{x^3}{\sqrt{n}}\lambda\Big(-\frac{x}{\sqrt{n}}\Big)\Big\}\Big(1 + O\Big(\frac{x+1}{\sqrt{n}}\Big)\Big) \tag{1.11}$$

hold, where $\lambda(u) = \sum_{k=0}^{\infty} \lambda_k u^k$ *is a power series whose coefficients* λ_k *depend on the cumulants* (1.5) *of the random variable* X.

The series $\lambda(u)$ converges for sufficiently small values of $|u|$. It is called *Cramér's series* and appears in many results related to large deviations. Results (1.10), (1.11) first were given in (Petrov 1954) together with a generalization to the case of non-identically distributed random variables.

Cramér's theorem is valid not only for real-valued random variables, but also for $\mathbb{R}^d$-valued random vectors, and even for some infinite-dimensional random vectors. For a more detailed account of generalizations of Cramér's theorem, see in (Ramasubramanian 2008).

The analysis of the asymptotic behavior of the ratios U_n and V_n that are defined by (1.8) is much more complicated, when the Cramér's condition (1.1) is not satisfied. Asymptotic convergence to a unit and the rate of convergence of large deviation ratios (1.8) when i. i. d. summands of the sum (1.6) satisfy the Linnik condition (1.3), were thoroughly investigated in (Linnik 1961a,b; Zolotarev 1962; Nagaev, S. V. 1963). Linnik developed a new method (see, e. g., Linnik 1961a,b) that yields general results when Cramér's condition fails, and it has been extended by Petrov (1963, 1964), Wolf (1970), and Osipov (1972). Nagajev, S. V. and Fuk (1971), Nagaev, S. V. (1973), and Nagaev, S. V. and Sakoyan (1976) generalized large deviation theorems in the power Linnik zones. These results are presented in the monograph by Ibragimov and Linnik (1965) and in the survey paper (Nagaev, S. V. 1979). In these works and others, large deviation theorems have been obtained by the rather complicated analytical saddle-point method (see, e. g., in Jensen 1995) and, as a rule, for sums of i. i. d. random variables. This is the simplest case that allows one to conceive the general view of large deviation probabilities.

Linnik's generalized version of Cramér's theorem (see Ibragimov and Linnik 1965: 307) is as follows :

Theorem 1.2. *If i. i. d. random variables* $\{X, X_j,\ j = 1, 2, ...\}$ *satisfy the Linnik condition* (1.3) *with* $h = 1$, $\gamma = 4\nu/(2\nu + 1)$, $0 < \nu < 1/2$, *and there exists a function* $\rho(n)$ *such that* $\lim_{n\to\infty} \rho(n) = +\infty$, *then in the interval* $0 \le x \le n^{\nu}/\rho(n)$, *the ratios*

$$U_n(x) = \exp\Big\{\frac{x^3}{\sqrt{n}}\lambda^{[q]}\Big(\frac{x}{\sqrt{n}}\Big)\Big\}\Big(1 + O\Big(\frac{x+1}{\sqrt{n}}\Big)\Big),$$
$$V_n(x) = \exp\Big\{-\frac{x^3}{\sqrt{n}}\lambda^{[q]}\Big(-\frac{x}{\sqrt{n}}\Big)\Big\}\Big(1 + O\Big(\frac{x+1}{\sqrt{n}}\Big)\Big)$$

are valid. Here $\lambda^{[q]}(u) = \sum_{k=0}^{q} \lambda_k u^k$, *where* q *is a non-negative integer such that*

$$\frac{q+1}{2(q+3)} \le \nu < \frac{q+2}{2(q+4)}.$$

The next major step in addressing problems of large deviation theorems was made when Statulevičius (1966) proposed the method of cumulants to consider large deviation probabilities for various statistics. The method of cumulants provided a way to obtain large deviation theorems for sums of independent and dependent random variables, polynomials forms, multiple stochastic integrals of random processes, and polynomial statistics in both the Cramér and the power Linnik zones. The monograph (Saulis and Statulevičius 1991) addresses these issues.

The cumulant method was developed by Rudzkis, Saulis, and Statulevičius (1978) where a general lemma of large deviations for an arbitrary random variable X with the regular behavior of it's cumulants (see condition (S_γ) below) was proved. Let us say that the random variable X with mean $\mu = 0$ and variance $\sigma^2 = \mathbf{E}X^2 = 1$ satisfies *S. V. Statulevičius' condition* (S_γ): there exist

$\gamma \geq 0$ and $\Delta > 0$ such that

$$|\Gamma_k(X)| \leq \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \qquad k = 3, 4, \ldots . \tag{S_γ}$$

When $\gamma = 0$, condition (S_γ) ensures the analyticity of the generating function $\varphi_X(z)$ in the domain $|z| < \Delta$. Thus in this case, theorems of large deviations in the Cramér zone are demonstrated. If $\gamma > 0$, then the generating function $\varphi_X(z)$ is not analytical, and theorems of large deviations are demonstrated for the power Linnik zones.

Condition (S_γ) can be easily verified for various multi-linear forms and is therefore very convenient for asymptotic analysis of large deviations for various statistics.

Let

$$\Delta_\gamma = c_\gamma \Delta^{\frac{1}{1+2\gamma}}, \qquad c_\gamma = \frac{1}{6}\Big(\frac{\sqrt{2}}{6}\Big)^{\frac{1}{1+2\gamma}}, \tag{1.12}$$

and let θ_i, with or without an index $i = 1, 2, \ldots$, denote a quantity (not always the same one) whose modulus is at most 1.

Lemma 1.1. *(Rudzkis et al. 1978) If an arbitrary random variable X with $\mu = 0$ and $\mathbf{E}X^2 = 1$ satisfies condition (S_γ), then the ratios of large deviations*

$$\begin{aligned} \frac{1 - F_X(x)}{1 - \Phi(x)} &= \exp\{L_\gamma(x)\}\Big(1 + \theta_1 f(x)\frac{x+1}{\Delta_\gamma}\Big), \\ \frac{F_X(-x)}{\Phi(-x)} &= \exp\{L_\gamma(-x)\}\Big(1 + \theta_2 f(x)\frac{x+1}{\Delta_\gamma}\Big) \end{aligned} \tag{1.13}$$

are valid in the interval $0 \leq x < \Delta_\gamma$. Here

$$f(x) = \frac{60\big(1 + 10\Delta_\gamma^2 \exp\{-(1 - x/\Delta_\gamma)\sqrt{\Delta_\gamma}\}\big)}{1 - x/\Delta_\gamma},$$

$$L_\gamma(x) = \sum_{3 \leq k < r} \tilde{\lambda}_k x^k + \theta_3\Big(\frac{x}{\Delta_\gamma}\Big)^3, \qquad r = \begin{cases} 2 + \frac{1}{\gamma}, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases}$$

The coefficients $\tilde{\lambda}_k$ (expressed by cumulants of the random variable X) coincide with the coefficients of the Cramér-Petrov series (Petrov 1975) given by the formula

$$\tilde{\lambda}_k = -b_{k-1}/k, \tag{1.14}$$

where b_k are defined by the series of equations

$$\sum_{r=1}^{j}\frac{1}{r!}\Gamma_{r+1}(X)\sum_{\substack{j_1+\ldots+j_r=j\\ j_i\geq 1}}\prod_{i=1}^{r}b_{j_i}=\begin{cases}1, & j=1,\\ 0, & j=2,3,\ldots\,.\end{cases} \tag{1.15}$$

In particular,

$$\begin{aligned}\tilde{\lambda}_2 &= -\frac{1}{2},\\ \tilde{\lambda}_3 &= \frac{1}{6}\Gamma_3(X),\\ \tilde{\lambda}_{*,4} &= \frac{1}{24}(\Gamma_4(X)-3\Gamma_3^2(X)),\\ \tilde{\lambda}_{*,5} &= \frac{1}{120}(\Gamma_5(X)-10\Gamma_3(X)\Gamma_4(X)+15\Gamma_3^3(X)),\ldots\,.\end{aligned}$$

For the coefficients $\tilde{\lambda}_k$, the estimate

$$|\tilde{\lambda}_k|\leq\frac{2}{k}\Big(\frac{16}{\Delta}\Big)^{k-2}((k+1)!)^{\gamma} \tag{1.16}$$

holds, and therefore

$$L_\gamma(x)\leq\frac{x^3}{2(x+8\Delta_\gamma)},\qquad L_\gamma(-x)\geq-\frac{x^3}{3\Delta_\gamma}.$$

Frequently, instead of precise equalities of large deviations, less precise exponential inequalities have been used. These were proved in (Bentkus, R. and Rudzkis 1980).

Lemma 1.2. *(R. Bentkus, R. Rudzkis 1980) Assume that for an arbitrary random variable X with $\mu=0$ there exist quantities $\gamma\geq 0$, $H>0$, and $\bar{\Delta}>0$ such that*

$$|\Gamma_k(X)|\leq\Big(\frac{k!}{2}\Big)^{1+\gamma}\frac{H}{\bar{\Delta}^{k-2}},\qquad k=2,3,\ldots\,.$$

Then for all $x\geq 0$,

$$\mathbf{P}(\pm X\geq x)\leq\exp\Big\{-\frac{x^2}{2(H+(x/\bar{\Delta}^{1/(1+2\gamma)}))^{(1+2\gamma)/(1+\gamma)}}\Big\}. \tag{1.17}$$

The proofs of Lemmas 1.1, 1.2 can be found, e. g., in (Saulis and Statulevičius 1991: 20–40). The general Lemmas 1.1, 1.2 rendered an opportunity to consider exponential inequalities, large deviation theorems for various statistics. Based on the general lemma of large deviations, Rudzkis *et al.* (1978) obtained large deviation theorems for sums of independent non-identically distributed random variables with regular behavior of it's cumulants. Also see (Rudzkis *et al.* 1979), where large deviation theorems for such sums in terms of Lyapunov fractions are presented.

Consider independent non-identically distributed random variables X_1, X_2,..., X_n, $n \geq 1$, with $\mu_j = \mathbf{E}X_j = 0$ and $\sigma_j^2 = \mathbf{D}X_j = \mathbf{E}X_j^2 < \infty$, $j = 1, 2, ..., n$. Let

$$B_n^2 = \sum_{j=1}^{n} \sigma_j^2, \qquad \tilde{S}_n = \frac{S_n}{B_n}, \quad S_n = \sum_{j=1}^{n} X_j, \qquad F_{\tilde{S}_n}(x) = \mathbf{P}(\tilde{S}_n < x),$$

where $B_n > 0$. In addition, let $\max\{b, c\} := (b \vee c)$, $b, c \in \mathbb{R}$. Let us say that the random variables X_j with $\mu_j = 0$ and $\sigma_j^2 < \infty$ satisfy condition (B_γ): there exist $\gamma \geq 0$ and $K > 0$ such that

$$|\mathbf{E}X_j^k| \leq (k!)^{1+\gamma} K^{k-2} \sigma_j^2, \qquad k = 3, 4, ... \,. \tag{B_γ}$$

Theorem 1.3. *(Rudzkis et al. 1978) Let the random variables X_j, $j = 1, 2, ..., n$, satisfy condition (B_γ). Then*

$$|\varGamma_k(\tilde{S}_n)| \leq \frac{(k!)^{1+\gamma}}{\Delta_n^{k-2}}, \qquad k = 2, 3, ...,$$

where

$$\Delta_n = B_n / K_n, \qquad K_n = (K \vee \max_{1 \leq j \leq n} \sigma_j\}.$$

In addition, the ratios (1.13) *and bound* (1.17) *hold for* $X := \tilde{S}_n$ *with*

$$\Delta_\gamma = c_\gamma \Delta_n^{1/(1+2\gamma)}, \qquad H = 2^{1+\gamma}, \qquad \bar{\Delta} = \Delta_n,$$

where c_γ *is defined by* (1.12).

Corollary 1.1. *Let the random variables* X_j, $j = 1, 2, ..., n$, *satisfy condition* (B_γ). *Then*

$$\lim_{n \to \infty} \frac{1 - F_{\tilde{S}_n}(x)}{1 - \Phi(x)} = 1, \qquad \lim_{n \to \infty} \frac{F_{\tilde{S}_n}(-x)}{\Phi(-x)} = 1, \tag{1.18}$$

hold for $x \geq 0$, $x = o(\Delta_n^{\nu(\gamma)})$ *as* $\Delta_n \to \infty$, *where* $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$. *If all the moments of the random variables* X_j *up to order* $r = [1/\gamma] + 2$ *inclusively coincide with the corresponding moments of the normal distribution, then the ratios* (1.18) *are true for* $x \geq 0$, $x = o(\Delta_n^{1/(1+2\gamma)})$.

The last part of the assertion is meaningful only for $0 < \gamma < 1$. The proofs of Theorem 1.3 and Corollary 1.1 can be found, e. g., in (Saulis and Statulevičius 1991: 44–45).

The cumulant method is the proper method for the analysis of large deviation probabilities $\mathbf{P}(S_n \geq B_n x)$ in the case of dependent summands X_j, $j = 1, 2, \ldots$. For further details and references, see (Saulis and Statulevičius 1991). A completely new method for normal approximation taking large deviations into account was presented by Bentkus, V. (1986). This method is good for the investigation of martingales and is based, following an idea offered by Lindeberg in 1992, on the proof of the central limit theorem. Bentkus, V. (1986) considered large deviations in Banach spaces. Bentkus, V. (2004) obtained a new type of large deviation inequality. Račkauskas (1995) studied a normal approximation taking large deviations for martingales into account, by combining S. V. Statulevičius' cumulant method with V. Benkus' method.

Local limit theorems for densities taking into account large deviations in the scheme of summation of random variables under the Cramér condition were obtained by Rikhter (1957), Linnik (1961b, 1962), Nagaev, S. V. (1962), Zolotarev (1962), Petrov (1963, 1964), Nagaev, A. V. (1967). The Cramér condition is not imposed, e. g., in (Linnik 1961b, 1962; Zolotarev 1962; Nagaev, S. V. 1962; Petrov 1963, 1964; Nagaev, A. V. 1967; Wolf 1970). The book (Petrov 1975) examines asymptotic expansions in integral and local limit theorems with uniform and non-uniform estimates of the remainder term, and also lists extensive references on local limit theorems and asymptotic expansions. Asymptotic expansions for large deviations were first obtained by Kubilius (1964). Asymptotic expansions in the zones of large deviations have been studied, e. g., in (Cramér 1938; Petrov 1954, 1975; Linnik 1960, 1961a; Zolotarev 1962; Nagaev, S. V. 1963; Borovkov 1964; Statulevičius 1965; Ibragimov 1967; Nagaev, A. V. 1967, 1969; Bikelis 1967; Pipiras and Statulevičius 1968; Saulis 1969, 1973, 1991, 1996, 1999; Wolf 1970; Misevičius and Saulis 1973; Nakas and Saulis 1973; Bikelis and Žemaitis 1974, 1976; Osipov 1978; Jakševičius 1983–1985; Borovkov and Mogulskii 2000; Deltuvienė and Saulis 2003a, 2003b).

Without elaborating on all of the studies of asymptotic expansions, we present some results that have been used to solve the problems posed in this dissertation. Thus, let us consider the asymptotic expansion presented in (Saulis 1980) for distribution density of an arbitrary random variable which cumulants exhibit regular behavior.

Suppose that for an arbitrary random variable X with mean $\mu = 0$, variance $\sigma^2 = 1$, and distribution function $F_X(x) = \mathbf{P}(X < x)$ for all $x \in \mathbb{R}$, there exists a density function such that

$$\sup_x p_X(x) < \infty. \tag{D}$$

Let Θ denote the set of all points on the line, at which $p_X(x)$ either is continuous or has a discontinuity of the first kind, and in the latter case assume that

$$p(x_0) = (p(x_0 - 0) + p(x_0 + 0))/2.$$

Let $X(h)$, $h = h(x) > 0$, be an arbitrary random variable conjugate to X, accordingly, with the respective density and characteristic functions

$$p_{X(h)}(x) = \frac{\exp\{hx\} p_X(x)}{\varphi_X(h)}, \qquad f_{X(h)}(u) = \frac{\varphi_X(h + iu)}{\varphi_X(h)}, \tag{1.19}$$

where

$$\varphi_X(h) = \int_{-\infty}^{\infty} e^{hx} p_X(x) dx \tag{1.20}$$

is the generating function for the random variable X. Here $h \geq 0$ is the solution of the equation

$$x = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \mathit{\Gamma}_k(X) h^{k-1}.$$

Let

$$f_\gamma^*(u) = \begin{cases} \sum\limits_{k=0}^{s} \left(\frac{3}{2}\right)^k \frac{x^k}{k!} f_X^{(k)}(u), & \gamma > 0, \\ f_{X(h)}(u), & \gamma = 0. \end{cases} \tag{1.21}$$

Moreover, $f_X^{(0)}(u) = f_X(u)$, $s = 2[(1/2)(\Delta^2/18)^{1/(1+2\gamma)}] - 2$, where $\mathit{\Delta}$ is defined by condition (S_γ) with $\gamma \geq 0$. Set

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \tag{1.22}$$

$$c_1(\gamma) = 2\pi + 6^\gamma 25^3 \sqrt{2\pi}/\Delta, \tag{1.23}$$

$$\varepsilon(\gamma, \Delta) = \frac{1}{12}\Big(1 - \frac{x}{\Delta_\gamma}\Big)\Delta_\gamma, \qquad 0 \leq x < \Delta_\gamma, \tag{1.24}$$

where Δ_γ is defined by (1.12). Furthermore, set

$$\begin{aligned} q(r,\gamma) &= \Big(\frac{3\sqrt{2e}}{2}\Big)^r + 8(r+2)^2 6^{\gamma(r-1)} 4^{3(r+1)} \\ &\quad \cdot((r+1)!)^{\gamma(r-1)}\Gamma\Big(\frac{3r+1}{2}\Big), \qquad (1.25)\\ r^*(x,\Delta) &= \Big(1+9((m+2)!)^\gamma 16^{m-1} c_\gamma^{m+1-r}\frac{1}{m+1}\Big(\frac{x}{\Delta}\Big)^r\Big) \\ &\quad \cdot\Big(1+46\Delta_\gamma \exp\Big\{-\frac{1}{2}\Big(1-\frac{x}{\Delta_\gamma}\Big)\sqrt{\Delta_\gamma}\Big\}\Big)\Big(1-\frac{x}{\Delta_\gamma}\Big), \qquad (1.26)\\ r^*(x,\Delta) &\equiv 0 \text{ as } \gamma=0, \end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, $m = (1+\gamma)+r+1$, $\gamma > 0$, $r \geq 1$. If $\alpha = n \in \mathbb{N}$, then $\Gamma(n) = (n-1)!$.

Lemma 1.3. *(Saulis 1980) If the random variable X satisfies conditions (S_γ) and (D), then for each integer $r \geq 1$ in the interval $0 \leq x < \Delta_\gamma$, the relation*

$$\begin{aligned} \frac{p_X(x)}{\phi(x)} &= \exp\big\{L_m(x)\big\}\Bigg(1+\sum_{v=0}^{r-3} M_v(x) + \theta_1 q(r,\gamma)\Big(\frac{x+1}{\Delta}\Big)^{r-2} \\ &\quad +\theta_2 c_1(\gamma)\Delta_\gamma^{3/2}\exp\Big\{-\frac{1}{72}\Big(1-\frac{x}{\Delta_\gamma}\Big)\sqrt{\Delta_\gamma}\Big\} \\ &\quad +\theta_3\int_{|u|\geq\varepsilon(\gamma,\Delta)}|f_\gamma^*(u)|du\Bigg)(1+\theta_4 r^*(x,\Delta)) \end{aligned}$$

holds. Here $\phi(x)$, $f_\gamma^*(u)$, $c_1(\gamma)$, $\varepsilon(\gamma,\Delta)$, $q(r,\gamma)$, $r^*(x,\Delta)$ are defined, respectively, by (1.22)–(1.26). *And*

$$L_m(x) = \sum_{3\leq k<m}\tilde{\lambda}_k x^k, \qquad m = \begin{cases} 1+r+\frac{1}{\gamma}, & \gamma>0, \\ \infty, & \gamma=0. \end{cases}$$

The coefficients $\tilde{\lambda}_k$ (expressed by cumulants of the random variable X) coincide with the coefficients of the Cramér-Petrov series (Petrov 1975) given by the formulas (1.14), (1.15) *(see the estimate* (1.16) *of $\tilde{\lambda}_k$ too). For the polynomials $M_v(x)$, the formula*

$$M_v(x) = \sum_{k=0}^{v} K_k(x) Q_{v-k}(x)$$

with

$$K_v(x) = \sum \prod_{m=1}^{v} \frac{1}{k_m!}\big(- \tilde{\lambda}_{m+2} x^{m+2}\big)^{k_m}, \qquad K_0(x) \equiv 1,$$
$$Q_v(x) = \sum H_{v+2r}(x) \prod_{m=1}^{v} \frac{1}{k_m!}\left(\frac{\Gamma_{m+2}(X)}{(m+2)!}\right)^{k_m}, \quad Q_0(x) \equiv 1,$$

holds, where the summation is taken over all non-negative integer solutions of the equation $k_1 + 2k_2 + ... + vk_v = v$. *And* $H_r(x)$ *is Chebyshev-Hermite polynomial*

$$H_r(x) = (-1)^r e^{\frac{x^2}{2}} \frac{d^r}{dx^r} e^{-\frac{x^2}{2}}. \tag{1.27}$$

To prove Lemma 1.3, characteristic functions and the saddle-point method were used along with the cumulant method (see, e. g., in Saulis and Statulevičius 1991: 155).

Note that to determine the structure of the *reminder term*

$$R(h) = \int_{|u| \geq \varepsilon(\gamma,\Delta)} |f^*_\gamma(u)| du \tag{1.28}$$

for the asymptotic expansion in both $\gamma = 0$ and $\gamma > 0$, the known estimates of Statulevičius (1965) for the characteristic functions of an arbitrary random variable may be used (see Lemmas 1–3 in (Statulevičius 1965), or Lemmas 3.1–3.3 in Section 3.1). Based on Lemma 1.3 and under condition (B_γ) with $\gamma = 0$, asymptotic expansion in the Cramér zone of large deviations for the density function of the sum of independent non-identically distributed random variables X_j, $j = 1, 2, ...$, with $\mu_j = 0$, $\sigma_j^2 = \mathbf{E}X_j^2 < \infty$, $j = 1, 2, ..., n$, and density functions such that

$$\sup_x p_{X_j}(x) \leq A_j < \infty, \qquad A_j > 0, \tag{D'}$$

have been presented in (Saulis 1991) (also see Theorem 6.1 in Saulis and Statulevičius 1991: 180), were the structure of the remainder term in the case where $\gamma = 0$ was delivered.

Asymptotic expansions of large deviations in the Cramér and the power Linnik zones for the density function have been generalized in (Deltuvienė and Saulis 2001, 2003b) by considering asymptotic expansions in the zones of large deviations for the density function of sums of independent random variables in a triangular array scheme. These results improve on the results on sums of random variables with weights in Book (1973).

Asymptotic expansions for the distribution functions of sums of independent non-identically distributed random variables that take into consideration large deviations in both the Cramér and the power Linnik zones have been studied in (Saulis 1999), where the results was first obtained, primarily by applying the general lemma in (Saulis 1996) of asymptotic expansion in the zone of large deviations for distribution function of an arbitrary random variable with regular behavior of it's cumulants. Deltuvienė and Saulis (2003a) studied asymptotic expansions in a triangular array scheme for the distribution functions of sums of independent random variables that take into consideration large deviations in both the Cramér and the power Linnik zones.

Note that the probabilities of large deviations in the Cramér and the power Linnik zones can be investigated in terms of Lyapunov fractions (see Saulis 1999). Thus, the probabilities of large deviations in such zones mainly depend not on individual properties but rather on the average properties of summands, as emphasized in (Rudzkis *et al.* 1979; Saulis 1998).

1.2. Limit theorems for compound sums

The classical theory of limit theorems examines the non-random index $n \geq 1$ in the sum $S_n = X_1 + ... + X_n$ as a random variables $X_j, j = 1, 2, ...$, degenerated at the point n. Replacement of the number n by a non-negative integer-valued random variable N is natural. Throughout thesis we shall assume that the distribution of N depends on some parameter, for example, when N is a homogeneous Poisson process, the distribution of N depends on $t \geq 0$, and we denote N as N_t in this case. In addition, the mean, variance and the distribution of N would be denoted by

$$\alpha = \mathbf{E}N, \qquad \beta^2 = \mathbf{D}N, \qquad \mathbf{P}(N = s) = q_s, \qquad s \in \mathbb{N}_0. \tag{1.29}$$

Let us consider the *random (compound) sum*

$$S_N = \sum_{j=1}^{N} X_j \tag{1.30}$$

of independent random variables X_j. For definiteness, it is assumed that $S_0 = 0$. The most frequently considered case is when N is independent of i. i. d. summands X_j. In simple terms, the random sum (1.30) is a partial sum in which the deterministic index n of the partial sum S_n is replaced by N.

The asymptotic behaviors of compound sums have been investigated in the theory of probability and stochastic processes for some time now: (see, e. g., Kolmogorov and Prokhorov 1949; Renyi 1960; Blum *et al.* 1962; Aksomaitis 1965, 1967, 1973; Siradzhinov and Orazov 1966; Gnedenko 1967; Statulevičius 1967; Nagaev, S. V. 1968; Gnedenko and Fahim 1969; Feller 1971; Szasz 1972a,b; Paulauskas 1972; Berzhintskas *et al.* 1973; Bernotas 1976; Batirov *et al.* 1977; Saulis 1978, 1981; Embrechts *et al.* 1985; Sakalauskas 1985, 1988; Kruglov 1988; Kruglov and Korolev 1990b; Gnedenko and Korolev 1996; Korolev and Shevtsova 2012; Sunklodas 2012 and the references therein) all of which have appeared since the results in (Robbins 1948a).

Suppose that $N := N_n$, $n \in \mathbb{N}$, are independent of the random variables X_j. In general, the number of the terms N_n should satisfy some conditions. To illustrate this we can recall that Robbins (1948a) used the condition $\mathbf{E}N_n \to \infty$ as $n \to \infty$. To confer, the condition $N_n/n \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$ was applied in Feller's theorem for random sums (see, e. g., Feller 1971). Here $\xrightarrow{\mathbf{P}}$ means convergence in probability. In addition, Rychlik and Szynal (1972) required the condition $N_n \xrightarrow{\mathbf{P}} \infty$ as $n \to \infty$. These classical conditions for N_n stand in the following relationship:

$$N_n/n \xrightarrow{\mathbf{P}} 1 \text{ as } n \to \infty \Rightarrow N_n \xrightarrow{\mathbf{P}} \infty \text{ as } n \to \infty \Rightarrow \mathbf{E}N_n \to \infty \text{ as } n \to \infty.$$

The problem of finding conditions for which the limit relationships of the sequence $\{S_n, n \in \mathbb{N}\}$ are transferred to $\{S_{N_n}, n \in \mathbb{N}\}$ has been solved in more restricted cases, for example, in (Renyi 1960; Blum *et al.* 1962; Wittenberg 1964) where the independence of $X_j, j = 1, 2, ..., n$, and N_n is not presupposed, so additional restrictions on the convergence of the sequence $\{N_n, n \in \mathbb{N}\}$ are imposed. Renyi (1960) proved that if $N_n/n \xrightarrow{\mathbf{P}} Y$, where Y is a discrete positive variable, and $S_n/n^{1/2}$ converges in law to normal, then $S_{N_n}/N_n^{1/2}$ as well. (Blum *et al.* 1962) obtained this result under the weaker assumption that Y is an arbitrary positive random variable. (Wittenberg 1964) treated the case where S_n and S_{N_n} are close in the sense of Kolmogorov-Smirnov distance, the greatest vertical distance between distribution functions.

It should be noted that a qualitative leap in the theory of summation of a r. n. s N of i. i. d. random variables was made in Gnedenko's 1969 paper and those of his followers' (see, e. g., Gnedenko and Fahim 1969; Szasz and Freyer 1971; Szasz 1972a,b), which obtained sufficient and necessary conditions for the convergence of distributions of (1.30) in the scheme of series. It was the first fundamental result. Gnedenko and Fahim (1969) proved the transfer theorem for sums of a r. n. s. of real-valued random variables. Suppose that for every $n \in \mathbb{N}$, $\{X_{n,k}, k \in \mathbb{N}\}$ is a sequence of i. i. d. random variables and that $\{N_n, n \in \mathbb{N}\}$ is a sequence of positive integer-valued random variables such that N_n and $\{X_{n,k}, k \in \mathbb{N}\}$. Gnedenko and Fahim (1969) proved that if

$$\sum_{k=1}^{k_n} X_{n,k} \xrightarrow{d} \mu \text{ and } \frac{N_n}{k_n} \xrightarrow{d} \rho \text{ as } n \to \infty$$

for some probability distributions μ and ρ on $\mathbb{R}$ and $(0, \infty)$, respectively, then

$$\sum_{k=1}^{N_n} X_{n,k} \xrightarrow{d} \int_0^\infty \mu^t d\rho(t),$$

where $\xrightarrow{d}$ denotes convergence in distribution and μ^t denotes the t-fold convolution power of the necessarily infinitely divisible distribution μ, which is well-defined by the Lévy-Chintchine formula. A partial converse to the transfer theorem with necessary conditions under strong additional assumptions was obtained in (Szasz and Freyer 1971), while Szasz (1972b) not only eliminated these assumptions but also obtained necessary as well as necessary and sufficient conditions. The aim of the investigations in (Szasz 1972a) was to give both necessary and sufficient conditions for the convergence of the distribution of random sums with the random indices independent of the sequence of non-identically distributed summands in the double array scheme.

There is considerable ongoing interest in random limit theorems of this kind for numerous applications, and there currently exists a vast literature on transfer theorems generalizing the transfer theorem of Gnedenko. Without providing a complete list, we refer to the literature cited in (Szasz and Freyer 1971; Szasz 1972a,b; Kruglov and Korolev 1990b; Gnedenko and Korolev 1996; Korolev and Kruglov 1998; Peter 2012).

Questions related to limit distributions of normed, centered random sums (1.30) fall under non-random centering and random centering (centering of random sums S_{N_n} by real numbers a_n, $n \in \mathbb{N}$ is non-random centering, while random centering would involve a_{N_n}). Obviously, random sums may be normed both by constants which is non-random norming and by random variables which is random norming. Letting b_n stand for a real number, various kinds of random sums have been considered: $(S_{N_n} - a_{N_n})/b_{N_n}$ are sums with random centering and random norming, $(S_{N_n} - a_n)/b_{N_n}$ are sums with non-random centering and random norming, $(S_{N_n} - a_n)/b_n$ are sums with non-random centering and non-random norming. Possibly the first result for limit distributions of random sums (1.30) of i. i. d. random variables with non-random centering appeared in (Robbins 1948a). As discussed in (Finkelstein *et al.* 1994; Korolev and Kruglov 1998), the class of limiting distributions for random sums of non-centered random variables normed by constants when the summands are i. i. d. and have finite variance were considered in (Robbins 1948a; Robbins 1948b). Robbins (1948a) found sufficient conditions for the weak convergence of growing compound sums and showed that, depending on whether the sums are centered by constants or not, the limit distributions have the form of location or scale mixtures of the laws that are limits for non-random sums of the same summands.

Assume that $\{X, X_n, n \in \mathbb{N}\}$ is a sequence of i. i. d. random variables with a common distribution function $F_X = \mathbf{P}(X < x)$, mean $\mu = \mathbf{E}X$ (not necessarily equal to zero), and finite common variance $\sigma^2 = \mathbf{D}X > 0$. Let $\{N_n, n \in \mathbb{N}\}$ be a sequence of positive integer-valued random variables that are independent of $\{X, X_j, j = 1, 2, ...\}$ and such that $N_n \xrightarrow{\mathbf{P}} \infty$ as $n \to \infty$. Under these assumptions, Robbins proved that if

$$\frac{N_n - \mathbf{E}N_n}{\sqrt{\mathbf{D}N_n}} \xrightarrow{d} \text{(some) } U, \tag{1.31}$$

then

$$\frac{S_{N_n} - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} \text{(some) } Y$$

and $F_Y = \Phi * F_U$, where Φ is the standard normal distribution function (1.7) and $*$ denotes the convolution operation. Y is a mixture of normals, the mean being mixed by U.

A large number of generalizations and refinements of Robbins' results have been obtained (see, e. g., Siradzhinov and Orazov 1966; Nagaev, S. V. 1968; Paulauskas 1972; Rychlik and Walczyński 2001; Bernotas 1976; Kruglov 1988; Finkelstein *et al.* 1994). As discussed in (Finkelstein *et al.* 1994), Kruglov (1988) obtained a generalization of Robbins' theorem under the assumption that $\mathbf{E}N_n^2 < \infty$. It was proved that a necessary and sufficient condition for

$$\frac{S_{N_n} - \mu\mathbf{E}N_n}{\sqrt{\sigma^2\mathbf{E}N_n + \mu^2\mathbf{D}N_n}} \xrightarrow{d} \text{(some) } Y$$

is that (1.31) in which case $F_Y = \Phi * F_U$. Independently of this work Finkelstein and Tucker (1990) obtained a similar generalization of Robbin's result, and it's converse, but without moment assumptions on N_n. Kruglov and Korolev (1990a) obtained necessary and sufficient conditions for the convergence of non-randomly centered random sums of non-identically distributed random summands in the double array scheme. A fairly complete study of earlier results connected with random centering appears in the monograph (Kruglov and Korolev 1990b). Without providing a detailed list of studies connected with random or non-random centering and norming of (1.30), we refer, e. g., to (Kruglov 1988, 1998; Kruglov and Korolev 1990a,b; Korolev and Kruglov 1998; Finkelstein and Tucker 1990; Finkelstein *et al.* 1994; Gnedenko and Korolev 1996; Rychlik and Walczyński 2001 and the literature therein).

Siradzhinov and Orazov (1966) were the first to solve the problem of estimating the deviation of the distribution of a random sum S_N of a r. n. s. N of i. i. d. random variables. Their estimate was improved in (Nagaev, S. V. 1968). However, the results in these papers are more limited than in the paper (Englund 1983), where an estimate of the reminder term in the normal approximation for compound sums (1.30) of i. i. d. random variables is constructed for the case where the means of the summands are not equal to zero. Sakalauskas (1985) provided some estimates of the accuracy of the approximation of compound sums of i. i. d. random variables by the scale mixtures of stable laws which turn out to be limiting. The problem of estimating the deviation of the distribution of random sum (1.30) in the case of differently distributed random variables was solved, e. g., in (Paulauskas 1972; Berzhintskas *et al.* 1973; Batirov *et al.* 1977; Batirov and Manevich 1983; Rychlik 1985; Korolev 1988). According to (Korolev 1988), the uniform estimates obtained by transferring the known estimates for a nonrandom index to randomly indexed sums in the (Batirov *et al.* 1977; Batirov and Manevich 1983; Englund 1983; Rychlik 1985) have some disadvantages, though they have an optimal structure in the sense that there are examples (Englund 1983; Paulauskas 1972) showing that the distribution of random sums S_N do not converge to the normal law if the majorants of the reminder term specified by the assertion in (Batirov *et al.* 1977; Batirov and Manevich 1983; Englund 1983; Rychlik 1985) do not tend toward zero. Korolev (1988) considered integral and local uniform estimates of the accuracy of normal approximations for the distributions of (1.30) free of the disadvantages of the works (Batirov *et al.* 1977; Batirov and Manevich 1983; Englund 1983; Rychlik 1985) (for more details see Korolev 1988).

Again, without a detailed exposition on the convergence rates in various metrics in the central limit theorem for the sums (1.30) of a r. n. s. (in addition to the above papers) we cite, e. g., (Sakalauskas 1988; Kruglov and Korolev 1990b; Gnedenko and Korolev 1996; Bening *et al.* 1997; 2012; Chaidee and

Tuntapthai 2009; Shang 2011; Nefedova and Shevtsova 2013; Sunklodas 2012 and the references therein).

Since in most cases the accurate distribution for the sum S_N of a r. n. s. is not available, deriving asymptotic relationship for it's tail probability $\mathbf{P}(S_N \geq x)$ is important. Such asymptotic results often appear in actuarial situations (see, e. g., Bening and Korolev 2002; Embrechts *et al.* 1997; Foss *et al.* 2013; Korolev *et al.* 2011; Mikosh 2009; Pragarauskas 2007).

In recent years a very extensive literature has appeared on the approximation of tail probabilities of random sums (1.30) of i.i.d or non-identically distributed random variables under different assumptions and with various applications. The tail of the compound distribution depends on the tails of the non-negative integer-valued random number of terms N and of the summands X_j, $j = 1, 2, ...$, themselves. First, let us define light-tailed and heavy-tailed distributions (see Definitions 1.1–1.3 or, e. g., Mikosh 2009; Foss *et al.* 2013).

In probability theory, heavy-tailed distributions are probability distributions whose tails are not exponentially bounded. There is still some discrepancy in the use of the term "heavy-tailed". Some authors use the term to refer to those distributions for which not all the power moments are finite, and some others refer to those distributions that do not have a finite variance. Occasionally, "heavy-tailed" is used for any distribution that has heavier tails than the normal distribution. The usage of the term "heavy-tailed distribution" varies according to the area of interest, but it is frequently taken to correspond to an absence of (positive) exponential moments.

Definition 1.1. *The distribution of a random variable X with distribution function $F(x) = \mathbf{P}(X < x)$, $x \in \mathbb{R}$ is said to be heavy-tailed if* $\mathbf{E}\exp\{\gamma X\} = \infty$, *for all* $\gamma > 0$.

The counterpart is given by the following definition.

Definition 1.2. *The distribution of a random variable X with distribution function $F(x)$, $x \in \mathbb{R}$ is said to be light-tailed if* $\mathbf{E}\exp\{\gamma X\} < \infty$, *for some* $\gamma > 0$.

If distributions of X and N are light-tailed, then saddle-point approximation techniques can be used to analyze the tail of S_N (see, e. g., Embrechts *et al.* 1985; Jensen 1995).

We also provide the following definition which is the most general in use.

Definition 1.3. *A distribution function $F(x)$ is heavy-tailed if and only if*

$\limsup_{x\to\infty} \bar{F}(x)\exp\{\gamma x\} = \infty$, *for all* $\gamma > 0$. *Here* $\bar{F}(x) = 1 - F(x)$.

Definition 1.3 includes all distributions encompassed by the alternative definitions, as well as those distributions such as log-normal that possess all their power moments.

Heavy-tailed probability distributions are an important component in the modeling of many stochastic systems (see, e. g., Embrechts *et al.* 1997; Faÿ *et al.* 2006; Robert and Segers 2008). They are essential for describing risk processes in finance and also for insurance premia pricing. They are frequently used to accurately model inputs and outputs of computer and data networks and service facilities such as call centers. In addition, such distributions occur naturally in models of epidemiological spread. Examples of heavy(light)-tailed distributions and subclasses (the distribution function has consistent, dominated variation, is long-tailed, sub-exponential) of heavy-tailed distributions may be found, e. g., in (Chistyakov 1964; Cline 1994; Mikosh 2009; Gao *et al.* 2012; Yang *et al.* 2012; Foss *et al.* 2013).

We will omit here a detailed exposition of results on heavy-tailed distributions as we are interested in normal approximation to the distribution of compound sums that take into consideration large deviations both in the Cramér and the power Linnik zones (see, Section 1.1). Thus, only as an illustration, let us consider some results on tails of random sums (1.30) in case of the heavy-tailed distributions.

Assume that the summands of the compound sum S_N are i. i. d. random variables $\{X, X_j, j = 1, 2, ...\}$ with mean and finite positive variance

$$\mu = \mathbf{E}X, \qquad 0 < \sigma^2 = \mathbf{D}X < \infty.$$

Moreover, assume that the non-negative integer-valued random variable N is independent of the mentioned summands. As important result in the literature, in the case of i. i. d. random variables $\{X, X_j, j = 1, 2, ...\}$ with the distribution function $F(x) = \mathbf{P}(X < x)$ and finite positive mean $\mu = \mathbf{E}X$ is the following. Assume that the distribution function $F(x)$ is sub-exponential (this heavy-tailed distribution class was introduced in Chistyakov 1964), i. e., $\lim_{x\to\infty} \overline{F^{*2}}(x)/\bar{F}(x) = 2$, where $F^{*2}(x)$ denotes the 2-fold convolution of the distribution function $F(x)$ with itself. If, in addition, N is light-tailed, then (see, e. g., Embrechts *et al.* 1997)

$$\mathbf{P}(S_N > x) \sim \alpha\bar{F}(x), \qquad x \to \infty.$$

Here $\sim$ means asymptotically equivalent. In more detail, for two positive functions $u(x)$ and $v(x)$, we write $u(x) \sim v(x)$ if $\lim_{x\to\infty} u(x)/v(x) = 1$. Aleškevičienė *et al.* (2008) considered the case where $\bar{F}(x) = o(\bar{H}(x))$ and $H(x)$ has consistent variation, i. e., $\lim_{y\nearrow 1} \limsup_{x\to\infty} \bar{H}(xy)/\bar{H}(x) = 1$ (this regularity property was first introduced by Cline (1994) and is called *intermediate regular variation*). Aleškevičienė *et al.* (2008) proved that

$$\mathbf{P}(S_N > x) \sim \bar{H}(x\mu^{-1}), \qquad x \to \infty,$$

generalizing the corresponding result in (Robert and Segers 2008). Robert and Segers (2008) concentrated on the converse case when the tail of (1.30) is dominated by the tail of N. This case is relevant, for instance, for earthquake insurance, for featuring a potentially large number of bounded claims, or for cases of individual unobserved heterogeneity, as well as in queueing theory (see Resnick 1992; Faÿ *et al.* 2006; Robert and Segers 2008). Aleškevičienė *et al.* (2008) considered applications to the asymptotic behavior of the finite-time ruin probability in a compound renewal risk model. The compound renewal risk model was first introduced by Tang *et al.* (2001), and it has since been extensively investigated by many researchers (for example, see Gao *et al.* 2012; Yang *et al.* 2013; Lin and Shen 2013 and the references therein). Yang *et al.* (2012) considered the asymptotic behavior of the tail probability for the random sum S_N with negatively dependent increments on $\mathbb{R}$ in three cases: $\mathbf{P}(N > x) = o(\mathbf{P}(X > x))$ and the distribution function $F(x)$ is dominatedly varying (this heavy-tailed distribution class was introduced by Feller see, e. g., Feller (1971)) that is, $\limsup_{x\to\infty} \bar{F}(xy)/\bar{F}(x) < \infty$, for every fixed $y > 0$; $\mathbf{P}(X > x) = o(\mathbf{P}(N > x))$ and the distribution function of N is dominatedly varying; and the tails of X and N are comparable and dominatedly varying. The technical restrictions on the distribution functions of X and N have been minimized. (He *et al.* 2013) considered asymptotic lower bounds of precise large deviations with non-negative and dependent random variables and discussed these in a multi-risk model. The paper considered the case where it does not matter whether the distributions of the random variables are heavy-tailed or light-tailed. The work extended and improved the results presented in (Konstantinides and Loukissas 2011; Loukissas 2012).

As has already been mentioned, we are specifically interested in normal approximation to the distributions of random sums that takes into consideration large deviations in both the Cramér and the power Linnik zones. The papers (Aksomaitis 1965, 1967, 1973; Statulevičius 1967; Saulis 1978, 1981; Saulis and Deltuvienė 2007) address large deviations in the Cramér zone for distributions of random sums. Of existing methods on large deviations (see,

e. g., Saulis and Statulevičius 1991; Jensen 1995; Borovkov 1999; Fatalov 2011, 2010; Gao and Zhao 2011), we rely on the cumulant method. However, there are only a small number of papers (see, e. g., Statulevičius 1967; Saulis 1978, 1981; Saulis and Deltuvienė 2007; Kasparavičiūtė and Saulis 2010, 2011a, 2011b, 2013) on normal approximation taking into account large deviations for the distribution of the sums of a r. n. s. in the case where cumulant method is used.

Recall that we assumed that the distribution of non-negative integer-valued random variable N depends on some parameter. Statulevičius (1967) proved a theorem on large deviations in the Cramér zone for the distribution function $F_{S_N}(\mu\alpha + x\sigma\sqrt{\alpha})$ in case of i. i. d. summands and where $\alpha = \mathbf{E}N \to \infty$. The proof was obtained using the lemma (see Statulevičius 1966) on large deviations for the distribution $F_X(\mu+x\sigma)$ of an arbitrary random variable that has finite moments of any order. In addition, it was assumed that the following conditions are satisfied: there exist nonnegative numbers H_1, H_2, K_1, K_2 such that

$$|\mathbf{E}(X-\mu)^k| \leq k! H_1 K_1^{k-2}\sigma^2, \qquad k = 3, 4, ...,$$

and

$$|\mathit{\Gamma}_k(N)| \leq k! H_2 K_2^{k-1}\alpha^{1+(k-1)\epsilon}, \qquad k = 1, 2, ... \,. \tag{1.32}$$

However, in the proof of the theorem on large deviations for $F_{S_N}(\mu\alpha+x\sigma\sqrt{\alpha})$ was assumed that without loss of generality $\mu = 0$, but while the mean may be assumed to be zero without loss of generality when the number $n \in \mathbb{N}$ of summands is non-random, this is not so for sums of a random number N of terms. Aksomaitis (1965) considered large deviation theorems in case where $N := N_t$, $t \geq 0$, and $\mathbf{E}N_t = \alpha_t \to \infty$ as $t \to \infty$. Moreover, instead of condition (1.32), the stronger condition

$$|\mathit{\Gamma}_k(N_t)| \leq C^k \alpha_t^{1+(k-1)\epsilon}, \qquad k = 2, 3, ...$$

was used, where $C > 0$ and $\epsilon \geq 0$.

Later, Saulis (1978) presented large deviations for the distribution function of the maximum of a random number of sums of i. i. d. random variables in the case where $\mu = 0$. In addition, the following Theorem 1.4 on large deviations in the Cramér zone for $F_{S_N}(\sigma\sqrt{\alpha}x)$ has been established.

Theorem 1.4. *(Saulis 1978) If* $\mu = 0$ *and the conditions* (1.33), (K)*: there exist quantities* $K_2 > 0$, $\epsilon \geq 0$, *and* $A > 0$ *such that*

$$|\mathit{\Gamma}_k(N)| \leq k! K_2^{k-1}\alpha^{1+(k-1)\epsilon}, \qquad k = 1, 2, ..., \tag{1.33}$$

$$\int_{-\infty}^{\infty} e^{hx} dF_X(x) < \infty, \qquad |h| < A, \tag{K}$$

are satisfied, then in the interval $\max\{1, 2\sqrt{3.5\epsilon \ln \alpha}\} \leq x = O(\alpha^{(1-\epsilon)/2})$ *the relation*

$$\frac{1 - F_{S_N}(\sigma\sqrt{\alpha}x)}{1 - \Phi(x)} = \exp\{L(x)\}\Big(1 + O\Big(\frac{x}{\alpha^{(1-\epsilon)/2}}\Big) + O\Big[\Big(1 + \frac{1}{x}\max\Big\{\frac{\beta}{\sqrt{\alpha}}, \frac{\beta^2}{\alpha}\Big\}\Big)\exp\Big\{-\frac{x^2}{14}\Big\}\Big]\Big),$$

holds as $\alpha \to \infty$. *Here* $L(x) = \sum_{k=3}^{\infty} c_k x^k$ *is a Cramér power series with coefficients* c_k *that are expressed in terms of the cumulants of* S_N.

Under the same conditions as in (Statulevičius 1967) and using the cumulant method, Saulis (1981) considered large deviations in the Cramér zone for a sum of a random number of random vectors (for more detail see, e. g., Saulis and Statulevičius 1991: 203–206).

Let us denote the standardized version of (1.30) as

$$\tilde{S}_N = \frac{S_N - \mathbf{E}S_N}{\sqrt{\mathbf{D}S_N}}, \qquad \mathbf{D}S_N > 0, \tag{1.34}$$

with mean $\mathbf{E}\tilde{S}_N = 0$ and variance $\mathbf{D}\tilde{S}_N = 1$. Saulis and Deltuvienė (2007) considered the asymptotic behavior of large deviation theorems for the distribution function $F_{\tilde{S}_N}(x)$ (see Theorem 1.5) and exponential inequalities of the probability $\mathbf{P}(\pm\tilde{S}_N \geq x)$ (see Theorem 1.6) when $\mu \neq 0$. It was assumed that the random variable X satisfies the conditions: there exist quantities $K > 0$, $K_1 > 0$, $\epsilon \geq 0$ such that

$$|\mathbf{E}X^k| \leq k! K^{k-2} \mathbf{E}X^2, \qquad k = 3, 4, ..., \tag{$\bar{B}_0$}$$

and

$$|\varGamma_k(N)| \leq (1/2) k! K_1^{k-2} (\beta^2)^{1+(k-2)\epsilon}, \qquad k = 2, 3, \ldots . \tag{1.35}$$

Let

$$\varDelta_N = \sqrt{\sigma^2\alpha + \mu^2\beta^2}/L_N, \quad L_N = \big(3K_1|\mu|\beta^{2\epsilon} \vee \big(1 \vee \sigma/|\mu|\big)4M\big),$$

where $M = 2\max\{K, \sigma\}$. Assume that $\varDelta_N \to \infty$ if $\beta \to \infty$.

Theorem 1.5. *(Saulis and Deltuvienė 2007) Assume that the random variables X and N satisfy conditions $(\bar{B}_0)$ and (1.35). Then*

$$\frac{1 - F_{\tilde{S}_N}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{S}_N}(-x)}{\Phi(-x)} \to 1$$

hold for x, $x \geq 0$, $x = o(\Delta_N^{1/3})$ as $\beta \to \infty$ when $0 \leq \epsilon < 1/2$.

Theorem 1.6. *(Saulis and Deltuvienė 2007) Let X and N satisfy the conditions of Theorem 1.5. Then*

$$\mathbf{P}(\pm\tilde{S}_N \geq x) \leq \begin{cases} \exp\{-x^2/12\}, & 0 \leq x \leq 3\mathit{\Delta}_N, \\ \exp\{-x\mathit{\Delta}_N/4\}, & x \geq 3\mathit{\Delta}_N. \end{cases}$$

1.3. Conclusions of Chapter 1

Let us consider the *weighted random (compound) sum*

$$Z_N = \sum_{j=1}^{N} a_j X_j \tag{1.36}$$

of a r. n. s. of i. i. d. weighted random variables. Here $0 < a_j < \infty$. Again, we suppose that N is independent of $\{X, X_j, j = 1, 2, ...\}$ and, for definiteness, we assume that $Z_0 = 0$. In addition, it is assumed that the distribution of N depends on some parameter.

1. It follows from the Chapter 1 that is a very extensive literature on approximation of tail probabilities for random sums of a r. n. s., under different assumptions and with various applications. However, among scientific works there are no works – excepting the papers (Kasparavičiūtė and Saulis 2010, 2011a, 2011b, 2013) by the author of this thesis and L. Saulis – for normal approximation that take into consideration large deviations in both the Cramér and the power Linnik zones for the standardized weighted compound sum

$$\tilde{Z}_N = \frac{Z_N - \mathbf{E}Z_N}{\sqrt{\mathbf{D}Z_N}}, \qquad \mathbf{D}Z_N > 0, \tag{1.37}$$

in two cases: $\mu \neq 0$ and $\mu = 0$ (see Chapter 2).
2. The papers (Kasparavičiūtė and Saulis 2010, 2011a, 2011b, 2013) gen-

eralize the previously mentioned results of (Statulevičius 1967; Saulis 1978; Saulis and Deltuvienė 2007) in the Cramér zone. It should be emphasized that the papers (Kasparavičiūtė and Saulis 2011a, 2011b, 2013) consider the instance where the characteristic function of the separate summand of $\tilde{Z}_N$ is not analytic in a vicinity of zero, while the papers (Statulevičius 1967; Saulis 1978; Saulis and Deltuvienė 2007) considered only the instance where the characteristic function of (1.37) in the case where $a_j \equiv 1$ are analytic in a vicinity of zero.

2

Theorems of large deviations for random sums

Assume that $\{X, X_j,\ j = 1, 2, ...\}$ is a family of i. i. d. random variables. Throughout N denotes a non-negative integer-valued random variable, besides we shall assume that the distribution of N depends on some parameter. For example, when N is a homogeneous Poisson process (see Definition 2.2 in Subsection 2.4.1), the distribution of N depends on $t \geq 0$, such that $t \to \infty$, and we denote N as N_t in this case. We also note that the probability space on which the random variables N and $\{X, X_j, j = 1, 2, ...\}$ appear is the same. Obviously, the assumption of the existence of a probability space on which all the random variables with the specified properties appear does not restrict the generality.

Consider weighted random (compound) sum

$$Z_N = \sum_{j=1}^{N} a_j X_j,$$

of a r. n. s. of i. i. d. weighted random variables. Recall that $0 < a_j < \infty$. Throughout, we assume that N is independent of $\{X, X_j, j = 1, 2, ...\}$, and for definiteness, we suppose that $Z_0 = 0$. For instance, in ruin theory the weights are interpreted as discount factors and the sequence X_j as the net returns of an insurance company, for the purpose of analyzing the probability of ruin in either a finite or an infinite time. In general, random weighted sums appear in the analysis of random stochastic equations, and they have applications in many areas.

In this chapter, we are interested in the normal approximation for the distribution of a standardized weighted random sum

$$\tilde{Z}_N = \frac{Z_N - \mathbf{E}Z_N}{\sqrt{\mathbf{D}Z_N}}, \qquad \mathbf{D}Z_N > 0,$$

that takes into consideration large deviations in both the Cramér and the power Linnik zones in the case where cumulant method (see Section 1.1 or the monograph by Saulis and Statulevičius (1991)) is used. Let us recall that, to the best of our knowledge, the scientific literature on aforementioned problem has no studies except for the papers (Kasparavičiūtė and Saulis 2013, 2011a,b, 2010). We also refer the reader to the papers (Statulevičius 1967; Saulis 1978, 1981; Saulis and Deltuvienė 2007) that consider an instance of aforementioned problem, more detail, that address normal approximation taking into consideration large deviations in the Cramér zone for the distribution of the sums (1.34).

Such probability characteristics as the mean, variance, and moments of higher order play an important role in the analysis of the asymptotic properties of random sums (for more detail see, e. g., Kruglov and Korolev 1990b). Accordingly, Section 2.1 is devoted to the main probability characteristics of the examined compound random variables and weighted random sum (1.36). In addition, the conditions for moments and cumulants are also presented.

To achieve theorems of large deviations, the cumulant method (see Section 1.1) is used. Thus the combinatorial method is used in Section 2.2 to evaluate the suitable bounds for the *k*th-order cumulants, $k = 3, 4, ...$, of the standardized weighted compound sum (1.37). Section 2.3 lists theorems of: large deviations, in both the Cramér and the power Linnik zones, comparing the behavior of probabilities of large deviations of standardized weighted random sum (1.37) against the standard normal distribution (1.7); the exponential inequalities for the probability $\mathbf{P}(\pm\tilde{Z}_N \geq x)$. In addition, this section also considers the absolute error estimate for the normal approximation to the distribution of (1.37). Instances of large deviations are presented in Section 2.3, including the cases where the number of summands obeys the binomial law, is homogeneous, or is a mixed Poisson process, as well as, the cases where $a_j \equiv 1$, $j = 1, 2, ...$, and $a_j \equiv v^j$, $0 < v < 1$.

2.1. Main conditions and probability characteristics

Recall that (1.4) denotes the mean, finite positive variance, and distribution function of X as

$$\mu = \mathbf{E}X, \qquad 0 < \sigma^2 = \mathbf{D}X < \infty, \qquad F_X(x) = \mathbf{P}(X < x), \quad x \in \mathbb{R}.$$

In addition, the *k*th-order moments and cumulants of X are defined by

$$\mathbf{E}X^k = \frac{1}{i^k}\frac{d^k}{du^k} f_X(u)\Big|_{u=0}, \quad \Gamma_k(X) = \frac{1}{i^k}\frac{d^k}{du^k}\ln f_X(u)\Big|_{u=0}, \quad k = 1, 2, ...,$$

respectively, where $f_X(u) = \mathbf{E}exp\{iuX\}$, $u \in \mathbb{R}$ is the characteristic function of the random variable X. Here $\Gamma_1(X) = \mathbf{E}X$ and $\Gamma_2(X) = \mathbf{D}X$.

We say that the random variable X with $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$ if there exist constants $\gamma \geq 0$ and $K > 0$ such that

$$|\mathbf{E}(X-\mu)^k| \leq (k!)^{1+\gamma} K^{k-2}\sigma^2, \qquad k = 3, 4, \dots . \qquad (\bar{B}_\gamma)$$

Condition $(\bar{B}_\gamma)$ is a generalization of S. N. Bernstein's familiar condition

$$|\mathbf{E}X^k| \leq (1/2)k!K^{k-2}\sigma^2, \qquad k = 2, 3, \dots , \qquad (B_0)$$

where it is assumed that $\mu = 0$. Condition $(\bar{B}_\gamma)$ ensures the existence of all order moments of the random variable X. Taking into account the fact that $\Gamma_k(X) = \Gamma_k(X - \mu)$, $k = 3, 4, ...$ and using Lemma 3.1 in (Saulis and Statulevičius 1991: 42), we take up the position that

Proposition 2.1. *(Saulis and Statulevičius 1991) If the random variable X satisfies condition $(\bar{B}_\gamma)$, then*

$$|\Gamma_k(X)| \leq (k!)^{1+\gamma} M^{k-2}\sigma^2, \qquad M = 2\max\{\sigma, K\}, \ k = 2, 3, \dots . \quad (2.1)$$

To define the mean and the variance of Z_N, we first introduce the following compound random variables $T_{N,r}$:

$$T_{N,r} = \sum_{j=1}^{N} a_j^r, \qquad r \in \mathbb{N}, \qquad (2.2)$$

where $0 < a_j < \infty$. For definiteness, we assume $T_{0,r} = 0$ for any fixed r. Clearly, $T_{N,0} = N$, in case where $r = 0$. Here N is non-negative, integer-valued random variable with the mean, variance and the distribution

$$\alpha = \mathbf{E}N, \qquad \beta^2 = \mathbf{D}N, \qquad \mathbf{P}(N = s) = q_s, \quad s \in \mathbb{N}_0.$$

It is easy to verify that the probability characteristics of $T_{N,r}$ are expressed through the characteristics of the sum

$$T_{s,r} = \sum_{j=1}^{s} a_j^r, \qquad s \in \mathbb{N}, \tag{2.3}$$

where the number of summands s is non-random. For instance, the mean, variance, and second moment are as follows:

$$\mathbf{E}T_{N,r} = \sum_{s=0}^{\infty} q_s \mathbf{E}(T_{N,r}|N = s) = \sum_{s=1}^{\infty} q_s T_{s,r}, \tag{2.4}$$

$$\mathbf{D}T_{N,r} = \mathbf{E}T_{N,r}^2 - (\mathbf{E}T_{N,r})^2, \tag{2.5}$$

$$\mathbf{E}T_{N,r}^2 = \sum_{s=0}^{\infty} q_s \mathbf{E}(T_{N,r}^2|N = s) = \sum_{s=1}^{\infty} q_s T_{s,r}^2. \tag{2.6}$$

In addition, the characteristic function for $T_{N,r}$ is

$$f_{T_{N,r}}(u) = \sum_{s=0}^{\infty} q_s \mathbf{E}\big(e^{iuT_{N,r}}|N = s\big) = \sum_{s=0}^{\infty} q_s e^{iuT_{s,r}}, \quad u \in \mathbb{R}. \tag{2.7}$$

We shall also need the following proposition:

Lemma 2.1. *Suppose that the functions $y = y(x)$ and $z = z(x)$ have derivatives of order $k \geq 1$. Then*

$$\frac{d^k}{dx^k} z(y(x)) = k! \sum\nolimits_1^* \frac{d^m}{dy^m} z(y)\Big|_{y=y(x)} \prod_{j=1}^{k} \frac{1}{m_j!} \Big(\frac{1}{j!} \frac{d^j}{dx^j} y(x)\Big)^{m_j}, \tag{2.8}$$

where the summation $\sum_1^*$ is carried out over all non-negative integer solutions $(m_1, m_2, ..., m_k)$ of the equation

$$\begin{cases} m_1 + 2m_2 + ... + km_k = k, \\ m_1 + m_2 + ... + m_k = m, \end{cases} \tag{2.9}$$

where $0 \leq m_1, ..., m_k \leq k$, and $1 \leq m \leq k$.

This proposition can be established by induction and appears, for instance, in (Petrov 1995: 170) (see Lemma 5.6), where citations for other references can also be found.

We note that $f_{T_{N,r}}(u)|_{u=0} = 1$ and

$$\frac{d^m \ln y}{dy^m}\Big|_{y=1} = (-1)^{m-1}(m-1)!, \qquad m = 1, 2, \dots .$$

Hence, using (2.7) and (2.8) together with the definition (1.5) of the kth-order cumulants we derive the equality

$$\begin{aligned} \mathit{\Gamma}_k(T_{N,r}) &= \frac{1}{i^k}\frac{d^k}{du^k}\ln f_{T_{N,r}}(u)\Big|_{u=0} \\ &= k!\sum\nolimits_1^* \frac{d^m}{dy^m}\ln y\Big|_{y=f_{T_{N,r}}(u)} \prod_{j=1}^k \frac{1}{m_j!}\Big(\frac{1}{j!}\frac{d^j}{i^j dy^j} f_{T_{N,r}}(u)\Big)^{m_j}\Big|_{u=0} \\ &= k!\sum\nolimits_1^* \frac{(-1)^{m-1}(m-1)!}{m_1!\cdot\ldots\cdot m_k!}\prod_{j=1}^k \Big(\frac{1}{j!}\mathbf{E}T_{N,r}^j\Big)^{m_j}, \qquad (2.10) \end{aligned}$$

which expresses the cumulants $\mathit{\Gamma}_k(T_{N,r})$ of arbitrary order k through the kth-order moments

$$\mathbf{E}T_{N,r}^k = \frac{1}{i^k}\frac{d^k}{du^k} f_{T_{N,r}}(u)\Big|_{u=0} = \sum_{s=1}^{\infty} q_s T_{s,r}^k, \qquad k = 1, 2, \dots . \qquad (2.11)$$

Aforementioned equality can be obtained using equality (1.34) in (Saulis and Statulevičius 1991: 8).

Setting $0 < a = \sup\{a_j,\ j = 1, 2, ...\} < \infty$, we note

$$\mathbf{E}T_{N,r}^k \le a^{k(r-l)}\mathbf{E}T_{N,l}^k, \quad 0 \le l \le r,\ r = 1, 2, ...,\ k = 1, 2, \dots . \qquad (2.12)$$

We proceed to show that for (2.5), the following estimate is valid:

$$\mathbf{D}T_{N,r} \le a^{2(r-l)}\mathbf{D}T_{N,l}, \qquad 0 \le l \le r. \qquad (2.13)$$

Indeed, it can be obtained quite-easily. Observe that we can rewrite (2.4) as follows:

$$\mathbf{E}T_{N,r} = \sum_{s=j}^{\infty} q_s a_j^r = \sum_{j=1}^{\infty}\mathbf{P}(N \ge j) a_j^r, \qquad (2.14)$$

where

$$\mathbf{P}(N \geq j) = \sum_{s=j}^{\infty} q_s, \qquad 0 < \mathbf{P}(N \geq j) < 1. \tag{2.15}$$

Since

$$\sum_{s=2}^{\infty} \sum_{\substack{j,n=1 \\ j \neq n}}^{s} a_j^r a_n^r q_s = 2 \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} \mathbf{P}(N \geq n) a_n^r a_j^r,$$

we also can rewrite (2.4) as follows

$$\begin{aligned} \mathbf{E}T_{N,r}^2 &= \sum_{s=1}^{\infty} \Big(\sum_{j=1}^{s} a_j^r \Big)^2 q_s = \sum_{s=1}^{\infty} \sum_{j=1}^{s} a_j^{2r} q_s + \sum_{s=2}^{\infty} \sum_{\substack{j,n=1 \\ j \neq n}}^{s} a_j^r a_n^r q_s \\ &= \sum_{j=1}^{\infty} \mathbf{P}(N \geq j) a_j^{2r} + 2 \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} \mathbf{P}(N \geq n) a_n^r a_j^r. \end{aligned} \tag{2.16}$$

Hence, by

$$(\mathbf{E}T_{N,r})^2 = \sum_{j=1}^{\infty} (\mathbf{P}(N \geq j))^2 a_j^{2r} + 2 \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} \mathbf{P}(N \geq n) a_n^r \mathbf{P}(N \geq j) a_j^r,$$

and from (2.5), (2.14), and (2.16) we have

$$\begin{aligned} \mathbf{D}T_{N,r} = \mathbf{E}T_{N,r}^2 - (\mathbf{E}T_{N,r})^2 &= \sum_{j=1}^{\infty} (1 - \mathbf{P}(N \geq j)) \mathbf{P}(N \geq j) a_j^{2r} \\ &+ 2 \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} \mathbf{P}(N \geq n) a_n^r (1 - \mathbf{P}(N \geq j)) a_j^r. \end{aligned} \tag{2.17}$$

Subsequent, application of $a_j^r a_n^r \leq a^{2(r-l)} a_j^l a_n^l$ and $a_j^{2r} \leq a^{2(r-l)} a_j^{2l}$, $0 \leq l \leq r$, $r = 1, 2, ...$, yields (2.13). Furthermore, by virtue of (2.12), similar but rather complicated calculations can show that for (2.10), the estimate

$$|\Gamma_k(T_{N,r})| \leq a^{k(r-l)} |\Gamma_k(T_{N,l})|, \quad k = 1, 2, ..., \quad 0 \leq l \leq r. \tag{2.18}$$

holds for $r = 1, 2, \ldots$. Obviously, here

$$|\Gamma_k(T_{N,l})| \leq k! \sum\nolimits_1^* (m-1)! \prod_{j=1}^{k} \frac{1}{m_j!} \Big(\frac{1}{j!} \mathbf{E}T_{N,l}^j \Big)^{m_j}, \qquad k = 1, 2, ...,$$

where summation $\sum_1^*$ is carried out over all non-negative integer solutions $(m_1, m_2, ..., m_k)$, $0 \leq m_1, ..., m_k \leq k$, $1 \leq m \leq k$, of equation (2.9).

We will see later (see the proof of Lemma 2.2 in Section 2.2: 40) that from (2.33) and together with (2.20), it follows that if we need to estimate upper bounds for the *k*th-order cumulants of the standardized weighted random sum (1.37), we must impose conditions not only for the *k*th-order cumulants of the random variable X but, depending on the case: $\mu \neq 0$ or $\mu = 0$, of $T_{N,1}$ or $T_{N,2}$ as well. Consequently, we assume that the compound random variables $T_{N,1}$ and $T_{N,2}$ defined by (2.2) with $r = 1, 2$ satisfy the conditions (L) and (L_0), respectively: there exist constants $K_1 > 0$, $K_2 > 0$ and $\epsilon \geq 0$ such that

$$|\Gamma_k(T_{N,1})| \leq \frac{1}{2} k! K_1^{k-2} (\mathbf{D}T_{N,1})^{1+(k-2)\epsilon}, \qquad k = 2, 3, ..., \qquad (L)$$

$$|\Gamma_k(T_{N,2})| \leq k! K_2^{k-1} (\mathbf{E}T_{N,2})^{1+(k-1)\epsilon}, \qquad k = 1, 2, ... \,. \qquad (L_0)$$

Throughout the rest of thesis, we use the first condition (L) when $\mu \neq 0$, and the second condition (L_0) when $\mu = 0$.

In isolated instance ($a_j \equiv 1, j = 1, 2, ...$), these conditions are imposed for the *k*th-order cumulants of the random variable N (see conditions (1.32) and (1.33), and also instances where N is distributed according to the binomial law, is homogeneous, or is a mixed Poisson process).

It is easily seen that, by virtue of conditions (L) and (L_0) with $0 \leq \epsilon < 1/2$, the cumulants $\Gamma_k(T_{N,1}/\sqrt{\mathbf{D}T_{N,1}})$ and $\Gamma_k(T_{N,2}/\sqrt{\mathbf{E}T_{N,2}})$ decrease as $\mathbf{D}T_{N,1} \to \infty$ and $\mathbf{E}T_{N,2} \to \infty$, respectively.

Let us now define the main probability characteristics of the weighted random sum Z_N that is defined by (1.36). The sum Z_N is a partial sum where the deterministic index s of the partial sum $Z_s = \sum_{j=1}^{s} a_j X_j,\ s \in \mathbb{N}$, of i. i. d. weighted random variables has been replaced by the random variable N. Application of (2.4)–(2.6) leads to

$$\mathbf{E}Z_N = \sum_{s=0}^{\infty} q_s \mathbf{E}(Z_s | N = s) = \mu \mathbf{E}T_{N,1}, \qquad (2.19)$$

$$\mathbf{E}Z_N^2 = \mathbf{E}X^2 \mathbf{E}T_{N,2} + \mu^2(\mathbf{E}T_{N,1}^2 - \mathbf{E}T_{N,2}) = \sigma^2 \mathbf{E}T_{N,2} + \mu^2 \mathbf{E}T_{N,1}^2.$$

Consequently,

$$\mathbf{D}Z_N = \mathbf{E}Z_N^2 - (\mathbf{E}Z_N)^2 = \sigma^2 \mathbf{E}T_{N,2} + \mu^2 \mathbf{D}T_{N,1}. \qquad (2.20)$$

Although the random variables X_j under consideration are absolutely continuous, Z_N may be not absolutely continuous, due to the degenerate distribution function

$$F_{Z_0}(x) = F_0(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Therefore, certain difficulties may appear in the formulation of problems related to local limit theorems, as the distribution function for all $x \in \mathbb{R}$,

$$F_{Z_N}(x) = \sum_{s=0}^{\infty} q_s \mathbf{P}(Z_N < x | N = s) = \sum_{s=0}^{\infty} q_s \mathbf{P}(Z_s < x),$$

is not absolutely continuous. Since we are considering large deviation theorems (the case where $x > 0$), the distribution function for (1.36) will therefore be

$$F_{Z_N}(x) = q_0 + \sum_{s=1}^{\infty} q_s F_{Z_s}(x), \qquad x > 0. \tag{2.21}$$

Since N is independent of the i. i. d. random variables $\{X, X_j, j = 1, 2, ...\}$, given (1.5), we derive that the characteristic function

$$\begin{aligned} f_{Z_N}(u) = \mathbf{E}e^{iuZ_N} &= \sum_{s=0}^{\infty} q_s \mathbf{E}\big(e^{iuZ_N} \big| N = s\big) = \sum_{s=0}^{\infty} q_s f_{Z_s}(u) \\ &= \sum_{s=0}^{\infty} q_s e^{\ln f_{Z_s}(u)} = \sum_{s=0}^{\infty} q_s \exp\Big\{\sum_{k=1}^{\infty} \frac{\Gamma_k(Z_s)}{k!}(iu)^k\Big\} \qquad (2.22) \\ &= \sum_{s=0}^{\infty} q_s \exp\Big\{\sum_{k=1}^{\infty} \frac{T_{s,k}\Gamma_k(X)}{k!}(iu)^k\Big\}, \qquad u \in \mathbb{R}, \qquad (2.23) \end{aligned}$$

of Z_N exists if the *k*th-order cumulants (1.5) of X exist. Here $T_{s,k} = \sum_{j=1}^{s} a_j^k$.

2.2. The upper estimates for the *k*th order cumulants

Let us consider the standardized sum (1.37) of a r. n. s. of i. i. d. weighted random variables:

$$\tilde{Z}_N = \frac{Z_N - \mathbf{E}Z_N}{\sqrt{\mathbf{D}Z_N}}, \qquad Z_N = \sum_{j=1}^{N} a_j X_j,$$

with $\mathbf{E}\tilde{Z}_N = 0$ and $\mathbf{D}\tilde{Z}_N = 1$. Here $0 < a_j < \infty$, $j = 1, 2, ...$, and $\mathbf{E}Z_N$ and $\mathbf{D}Z_N > 0$ are defined by (2.19) and (2.20). Recall that non-negative integer valued random variable N is independent of $\{X, X_j, j = 1, 2, ...\}$. In addition, the distribution of N depends on some parameter.

To obtain large deviation theorems for $\tilde{Z}_N$, the cumulant method proposed by Statulevičius (1966) and generalized by R. Rudzkis, L. Saulis, S. V. Statulevičius (1978) (for detailes, see Section 1.1: 9, or monograph Saulis and Statulevičius 1991) is used. It is a powerful method that enables investigation of large deviations for random sums of both independent and dependent random variables.

Since we are interested not only in the convergence to the normal distribution but also in a more accurate asymptotic analysis of the distribution $F_{\tilde{Z}_N}(x) = \mathbf{P}(\tilde{Z}_N < x)$, we must first find a suitable bound for the *k*th-order cumulants of $\tilde{Z}_N$. Lemma 2.2 below presents the accurate upper estimate for $|\Gamma_k(\tilde{Z}_N)|$in two cases: $\mu \neq 0$ and $\mu = 0$.

Define the abbreviations $(b \vee c) = \max\{b, c\}$, $b,\ c \in \mathbb{R}$, $0 < \bar{a} = \inf\{a_j, j = 1, 2, ...\} < \infty$ and recall that $0 < a = \sup\{a_j, j = 1, 2, ...\} < \infty$.

Lemma 2.2. *Suppose the random variable X with variance $0 < \sigma^2 < \infty$ fulfills condition $(\bar{B}_\gamma)$ and that the random variables $T_{N,1}$ and $T_{N,2}$ defined by* (2.2) *satisfy conditions (L) and (L_0), respectively. Then*

$$|\Gamma_k(\tilde{Z}_N)| \leq \frac{(k!)^{1+\gamma}}{\Delta_*^{k-2}}, \qquad k = 3, 4, ..., \tag{2.24}$$

where

$$\Delta_* = \begin{cases} \Delta_N, & \text{if } \mu \neq 0, \\ \Delta_{N,0}, & \text{if } \mu = 0. \end{cases} \tag{2.25}$$

Here

$$\Delta_N = \frac{\sqrt{\mathbf{D}Z_N}}{L_N}, L_N = 2\Big(\frac{a}{\bar{a}}\Big)^2\Big(\frac{a}{\bar{a}}K_1|\mu|(\mathbf{D}T_{N,1})^{\epsilon}\vee\Big(1\vee\frac{\bar{a}\sigma}{2a|\mu|}\Big)aM\Big), \tag{2.26}$$

where $\mathbf{D}Z_N$ is defined by (2.20)*, and*

$$\Delta_{N,0} = \frac{\sqrt{\mathbf{D}Z_N}}{L_{N,0}}, \qquad L_{N,0} = 2\big(1 \vee K_2(\mathbf{E}T_{N,2})^{\epsilon}\big)\big((1/2) \vee a\big)M, \tag{2.27}$$

where $\mathbf{D}Z_N$ is defined by (2.20) *with $\mu = 0$. The constants K_1, K_2, ϵ, and M are defined by conditions (L), (L_0),* (2.1)*, and $\mathbf{D}T_{N,1}, \mathbf{E}T_{N,2}$ are defined by*

(2.5) *and* (2.4).

Proof of Lemma 2.2. First, observe that $f_{Z_N}(u)|_{u=0} = 1$, where $f_{Z_N}(u)$ is defined by (2.23), and recall that

$$\frac{d^m \ln y}{dy^m}\Big|_{y=1} = (-1)^{m-1}(m-1)!, \qquad m = 1, 2, \ldots .$$

According to Lemma 2.1 together with definition (1.5) of the kth-order moments and cumulants, we can assert that for all $k = 1, 2, \ldots$,

$$\begin{aligned} \Gamma_k(Z_N) &= \frac{d^k}{i^k du^k} \ln f_{Z_N}(u)\Big|_{u=0} \\ &= k! \sum\nolimits_1^* \frac{d^m}{dy^m} \ln y\Big|_{y=f_{Z_N}(u)} \prod_{j=1}^{k} \frac{1}{m_j!} \Big(\frac{1}{j!} \frac{d^j}{i^j dy^j} f_{Z_N}(u)\Big)^{m_j}\Big|_{u=0} \\ &= \sum\nolimits_1^* \frac{(-1)^{m-1}(m-1)!}{m_1! \cdot \ldots \cdot m_k!} \prod_{j=1}^{k} \Big(\frac{1}{j!} \mathbf{E} Z_N^j\Big)^{m_j}. \end{aligned} \tag{2.28}$$

is valid. Here $\sum_1^*$ is a summation over all non-negative integer solutions $(m_1, m_2, \ldots, m_k)$ of equation (2.9). Further, using (2.8), (2.23) together with (2.11) for all $j = 1, 2, \ldots$ gives us

$$\begin{aligned} \mathbf{E} Z_N^j &= \sum_{s=1}^{\infty} q_s j! \sum\nolimits_2^* \frac{1}{\eta_1! \cdot \ldots \cdot \eta_j!} \prod_{n=1}^{j} \Big(\frac{1}{n!} T_{s,n} \Gamma_n(X)\Big)^{\eta_n} \\ &= j! \sum\nolimits_2^* \frac{\mathbf{E}\big(T_{N,1}^{\eta_1} \cdot \ldots \cdot T_{N,j}^{\eta_j}\big)}{\eta_1! \cdot \ldots \cdot \eta_j!} \prod_{n=1}^{j} \Big(\frac{1}{n!} \Gamma_n(X)\Big)^{\eta_n}, \end{aligned} \tag{2.29}$$

where

$$\mathbf{E}\big(T_{N,1}^{\eta_1} \cdot \ldots \cdot T_{N,j}^{\eta_j}\big) = \sum_{s=0}^{\infty} q_s \big(T_{s,1}^{\eta_1} \cdot \ldots \cdot T_{s,j}^{\eta_j}\big), \quad T_{s,j} = \sum_{r=1}^{s} a_r^j, \quad j = 1, 2, \ldots .$$

Here $\sum_2^*$ is taken over all non-negative integer solutions $(\eta_1, \ldots, \eta_j)$ of the equation

$$\begin{cases} \eta_1 + 2\eta_2 + \ldots + j\eta_j = j, \\ \eta_1 + \eta_2 + \ldots + \eta_j = \eta. \end{cases}$$

Here $0 \le \eta_1, \ldots, \eta_j \le j$, and $1 \le \eta \le j$. Consequently, substituting (2.29) into

(2.28) produces

$$\Gamma_k(Z_N) = k! \sum\nolimits_1^* \frac{(-1)^{m-1}(m-1)!}{m_1! \cdot ... \cdot m_k!} \prod_{j=1}^{k} \cdot \Big(\sum\nolimits_2^* \frac{\mathbf{E}\big(T_{N,1}^{\eta_1} \cdot ... \cdot T_{N,j}^{\eta_j}\big)}{\eta_1! \cdot \eta_2! \cdot ... \cdot \eta_j!} \prod_{n=1}^{j} \Big(\frac{1}{n!}\Gamma_n(X)\Big)^{\eta_n}\Big)^{m_j}. \tag{2.30}$$

In particular,

$$\begin{aligned}
\Gamma_1(Z_N) &= \Gamma_1(T_{N,1})\Gamma_1(X) = \mathbf{E}Z_N,\\
\Gamma_2(Z_N) &= \Gamma_2(T_{N,1})(\Gamma_1(X))^2 + \mathbf{E}T_{N,2}\Gamma_2(X) = \mathbf{D}Z_N,\\
\Gamma_3(Z_N) &= \Gamma_3(T_{N,1})(\Gamma_1(X))^3 + 3cov(T_{N,1},T_{N,2})\Gamma_1(X)\Gamma_2(X)\\
&\quad + \mathbf{E}T_{N,3}\Gamma_3(X),\\
\Gamma_4(Z_N) &= \Gamma_4(T_{N,1})(\Gamma_1(X))^4 + 4cov(T_{N,1},T_{N,3})\Gamma_1(X)\Gamma_3(X)\\
&\quad + 6(\Gamma_1(X))^2\Gamma_2(X)\big(cov(T_{N,1}^2,T_{N,2}) - 2\mathbf{E}T_{N,1}cov(T_{N,1},T_{N,2})\big)\\
&\quad + 3\mathbf{D}T_{N,2}(\Gamma_2(X))^2 + \mathbf{E}T_{N,4}\Gamma_4(X),\ \ldots .
\end{aligned}$$

Note that

$$T_{N,j}^{\eta_j} \le a^{(j-2)\eta_j} T_{N,2}^{\eta_j} \quad \textit{as } j \ge 2,$$

where $0 < a = \sup\{a_j, j = 1,2,...\} < \infty$. In addition,

$$T_{N,1}^{\eta_1} \le \bar{a}^{-\eta_1} T_{N,2}^{\eta_1},$$

where $0 < \bar{a} = \inf\{a_j, j = 1,2,...\} < \infty$. Thus

$$|\Gamma_k(Z_N)| \le k! \sum\nolimits_1^* \frac{(-1)^{m-1}(m-1)!}{m_1! \cdot ... \cdot m_k!} \prod_{j=1}^{k} \cdot \Big(\sum\nolimits_2^* \frac{(a/\bar{a})^{\eta_1}\mathbf{E}T_{N,2}^{\eta}}{\eta_1! \cdot ... \cdot \eta_j!} \prod_{n=1}^{j} \Big(\frac{1}{n!}a^{n-2}|\Gamma_n(X)|\Big)^{\eta_n}\Big)^{m_j}, \tag{2.31}$$

for $k = 1,2,...$. Here $\mathbf{E}T_{N,2}^{\eta}$ is defined by (2.11) with $r = 2$ and $k = \eta$. In particular,

$$\begin{aligned}
|\Gamma_1(Z_N)| &\le (1/\bar{a})\Gamma_1(T_{N,2})|\Gamma_1(X)|,\\
|\Gamma_2(Z_N)| &\le \mathbf{E}T_{N,2}\Gamma_2(X) + (1/\bar{a})^2\mathbf{D}T_{N,2}(\Gamma_1(X))^2,
\end{aligned}$$

$$
\begin{aligned}
|\Gamma_3(Z_N)| &\le (1/\bar{a})^3 \Gamma_3(T_{N,2}) |\Gamma_1(X)|^3 + 3(a/\bar{a}) \mathbf{D} T_{N,2} |\Gamma_1(X)| \Gamma_2(X) \\
&\quad + a \mathbf{E} T_{N,2} |\Gamma_3(X)|, \\
|\Gamma_4(Z_N)| &\le (1/\bar{a})^4 \Gamma_4(T_{N,2}) (\Gamma_1(X))^4 + 4(a/\bar{a}) \mathbf{D} T_{N,2} |\Gamma_1(X)| |\Gamma_3(X)| \\
&\quad + 6(1/\bar{a})^2 \Gamma_3(T_{N,2}) (\Gamma_1(X))^2 \Gamma_2(X) \\
&\quad + 3 \mathbf{D} T_{N,2} (\Gamma_2(X))^2 + a^2 \mathbf{E} T_{N,2} \Gamma_4(X), \ \dots .
\end{aligned}
$$

Let us note that according to Lemma 2.1

$$
\begin{aligned}
& k! \sum\nolimits_1^* \frac{(-1)^{m-1}(m-1)!}{m_1! \cdot \ldots \cdot m_k!} \prod_{j=1}^{k} \Big(\sum\nolimits_2^* \frac{\mathbf{E} T_{N,2}^{\eta}}{\eta_1! \cdot \ldots \cdot \eta_j!} \prod_{n=1}^{j} \Big(\frac{1}{n!} \Gamma_n(X) \Big)^{\eta_n} \Big)^{m_j} \\
&= k! \sum\nolimits_1^* \frac{d^m}{dy^m} \ln y \Big|_{y = f_{T_{N,2}} \left(\frac{1}{i} \ln f_X(u) \right)} \\
&\quad \cdot \prod_{j=1}^{k} \frac{1}{m_j!} \Big(\frac{1}{j!} \frac{d^j}{i^j dy^j} f_{T_{N,2}} (\ln f_X(u)/i) \Big)^{m_j} \Big|_{u=0} \\
&= k! \sum\nolimits_1^* \frac{d^m}{i^m dy^m} \ln f_{T_{N,2}}(y) \Big|_{y=0} \prod_{j=1}^{k} \frac{1}{m_j!} \Big(\frac{1}{j!} \frac{d^j}{i^j dy^j} \ln f_X(u) \Big|_{u=0} \Big)^{m_j} \\
&= k! \sum\nolimits_1^* \frac{\Gamma_m(T_{N,2})}{m_1! \cdot \ldots \cdot m_k!} \prod_{j=1}^{k} \frac{1}{m_j!} \Big(\frac{1}{j!} \Gamma_j(X) \Big)^{m_j}, \qquad (2.32)
\end{aligned}
$$

where

$$
\begin{aligned}
f_{T_{N,2}} \Big(\frac{1}{i} \ln f_X(u) \Big) &= \sum_{s=0}^{\infty} q_s \exp\{ T_{s,2} \ln f_X(u) \} \\
&= \sum_{s=0}^{\infty} q_s \exp \Big\{ T_{s,2} \sum_{k=1}^{\infty} \frac{\Gamma_k(X)}{k!} (iu)^k \Big\}.
\end{aligned}
$$

Here $\Gamma_m(T_{N,2})$ is defined by (2.10). Hence (2.31) we rewrite as follows:

$$
|\Gamma_k(Z_N)| \le k! \sum\nolimits_1^* \frac{|\Gamma_m(T_{N,2})|}{m_1! \cdot \ldots \cdot m_k!} \Big(\frac{a}{\bar{a}} \Big)^{m_1} \prod_{j=1}^{k} \Big(\frac{1}{j!} a^{j-2} |\Gamma_j(X)| \Big)^{m_j}, \quad (2.33)
$$

for $k = 1, 2, \dots$.

Now let us consider the case where $\mu \neq 0$. Separating the summand of the sum

$\sum_1^*$ in case where $m_1 = ... = m_{k-1} = 0$, $m_k = 1$, from (2.33) we derive

$$|\varGamma_k(Z_N)| \le k!(R_1 + R_2), \qquad k = 2, 3, ..., \tag{2.34}$$

where

$$R_1 = \frac{1}{k!} a^{k-2} |\varGamma_1(T_{N,2})| |\varGamma_k(X)|,$$
$$R_2 = \sum\nolimits_3^* \frac{|\varGamma_{\tilde m}(T_{N,2})|}{m_1! \cdot ... \cdot m_{k-1}!} \Big(\frac{|\mu|}{\bar a}\Big)^{m_1} \prod_{j=2}^{k-1} \Big(\frac{1}{j!} a^{j-2} |\varGamma_j(X)|\Big)^{m_j}.$$

Here $\sum_3^*$ is taken over all the non-negative integer solutions $(m_1, m_2, ..., m_{k-1})$ of the equation

$$\begin{cases} m_1 + 2m_2 + ... + (k-1)m_{k-1} = k, \\ \qquad m_1 + m_2 + ... + m_{k-1} = \tilde m, \end{cases} \tag{2.35}$$

where $0 \le m_1, ..., m_{k-1} \le k$, $2 \le \tilde m \le k$.
Now we turn to estimating R_1 and R_2. Condition (2.1) yields

$$R_1 \le (k!)^\gamma (aM)^{k-2} \sigma^2 \mathbf{E} T_{N,2}, \qquad k = 3, 4, ..., \tag{2.36}$$

Application of (2.1), (L), and (2.18) with $r = 2$, $l = 1$ implies

$$R_2 \le \frac{1}{2} \mathbf{D} T_{N,1} \sum\nolimits_3^* \frac{\tilde m!}{m_1! \cdot ... \cdot m_{k-1}!} (K_1 (\mathbf{D} T_{N,1})^\epsilon)^{\tilde m - 2}$$
$$\cdot\, a^{\tilde m} \Big(\frac{|\mu|}{\bar a}\Big)^{m_1} \prod_{j=2}^{k-1} ((j!)^\gamma (aM)^{j-2} \sigma^2)^{m_j}, \qquad k = 3, 4, ... \,. \tag{2.37}$$

We also need the equality

$$g_k = \sum\nolimits_1^* \frac{(m_1 + ... + m_k)!}{m_1! \cdot ... \cdot m_k!} = 2^{k-1}, \qquad k = 1, 2, ... \,. \tag{2.38}$$

It is assumed, by convention, that $g_0 = 1$. This equality can be obtained on the basis of the generating function that generates the sequence of numbers $< g_l >$

$$G(\omega) = \sum_{l \ge 0} g_l \omega^l = \sum_{l \ge 0} (\omega + \omega^2 + ... + \omega^k + ...)^l = \frac{1 - \omega}{1 - 2\omega}, \qquad |\omega| < \frac{1}{2}.$$

Therefore,

$$G(\omega) = \frac{1}{1-2\omega} - \frac{\omega}{1-2\omega} = \sum_{l\geq 0} 2^l\omega^l - \sum_{l\geq 0} 2^l\omega^{l+1}$$
$$= 1 + \sum_{l\geq 1}(2^l - 2^{l-1})\omega^l = 1 + \sum_{l\geq 1} 2^{l-1}\omega^l.$$

Consequently,

$$\sum\nolimits_3^* \frac{(m_1 + ... + m_{k-1})!}{m_1!\cdot ...\cdot m_{k-1}!} = 2^{k-1} - 1, \qquad k = 2, 3, ... \,. \tag{2.39}$$

Because of the inequality $b!\cdot c! \leq (b+c)!$, where b and c are non-negative integers, we have that for any solution $(m_1, m_2, ..., m_{k-1})$ of the equation (2.35), the inequality

$$\prod_{j=1}^{k-1} (j!)^{m_j} \leq (k-1)!, \qquad k = 2, 3, ..., \tag{2.40}$$

is valid. Next, recalling that $\mu \neq 0$ let us to evaluate

$$a^{\tilde{m}}\Big(\frac{|\mu|}{\bar{a}}\Big)^{m_1}\prod_{j=2}^{k-1}((aM)^{j-2}\sigma^2)^{m_j}$$
$$\leq \Big(\frac{a|\mu|}{\bar{a}}\Big)^{\tilde{m}}\Big(\frac{\bar{a}M\sigma}{2|\mu|}\Big)^{m_2+m_3+...+m_{k-1}}(aM)^{m_3+2m_4+...+(k-3)m_{k-1}}$$
$$\leq \Big(\frac{a|\mu|}{\bar{a}}\Big)^{\tilde{m}}\Big(\Big(1\vee\frac{\sigma\bar{a}}{2|\mu|a}\Big)aM\Big)^{k-\tilde{m}}, \qquad k = 2, 3, ..., \tag{2.41}$$

as $\sigma \leq M/2$, and $0 \leq m_2 + m_3 + ... + m_{k-1} \leq k - \tilde{m}$, where $0 \leq m_1, ..., m_{k-1} \leq k$, $2 \leq \tilde{m} \leq k$. Here M and $\tilde{m}$ are defined, respectively, by (2.1) and (2.35).
Using (2.37) and (2.39)–(2.41) for $k = 3, 4, ...$, we derive

$$R_2 \leq (k!)^\gamma\mu^2\mathbf{D}T_{N,1}$$
$$\cdot\Big(2\Big(\frac{a}{\bar{a}}\Big)^2\Big(\frac{a}{\bar{a}}K_2|\mu|(\mathbf{D}T_{N,1})^\epsilon \vee \Big(1\vee\frac{\sigma\bar{a}}{2|\mu|a}\Big)aM\Big)\Big)^{k-2}. \tag{2.42}$$

Consequently, from (2.34), (2.36), and (2.42) follows that

$$|\varGamma_k(Z_N)| \leq (k!)^{1+\gamma}L_N^{k-2}\mathbf{D}Z_N, \qquad k = 3, 4, ..., \tag{2.43}$$

where $\mathbf{D}Z_N$ and L_N are defined by (2.20) and (2.26), respectively.

Now let us consider the case where $\mu = 0$, and $0^0 = 1$. Referring to (2.33), we obtain

$$|\varGamma_k(Z_N)| \leq k! \sum\nolimits_4^* |\varGamma_{\bar{m}}(T_{N,2})| \prod_{j=2}^{k} \frac{1}{m_j!} \Big(\frac{1}{j!} a^{j-2} |\varGamma_j(X)| \Big)^{m_j}, \qquad (2.44)$$

for $k = 2, 3, ...$, where $\sum_4^*$ is a summation over all the non-negative integer solutions $(m_2, m_3, ..., m_k)$ of the equation

$$\left\{ \begin{array}{r} 2m_2 + ... + km_k = k, \\ m_2 + ... + m_k = \bar{m}. \end{array} \right. \qquad (2.45)$$

Here $1 \leq \bar{m} \leq k$. Further, using (2.1), (L_0), and (2.44), we obtain

$$\begin{aligned} |\varGamma_k(Z_N)| \quad \leq \quad & k! \mathbf{E}T_{N,2} \sum\nolimits_4^* \frac{\bar{m}!}{m_2! \cdot ... \cdot m_k!} \\ & \cdot (K_2 (\mathbf{E}T_{N,2})^{\epsilon})^{\bar{m}-1} \prod_{j=2}^{k} ((j!)^{\gamma} (aM)^{j-2} \sigma^2)^{m_j}. \quad (2.46) \end{aligned}$$

Noting that $b! \cdot c! \leq (b+c)!$, (2.38) gives, for any solution $(m_2, ..., m_k)$ of equation (2.45)

$$\prod_{j=2}^{k} (s!)^{m_j} \leq k!, \qquad \sum\nolimits_4^* \frac{\bar{m}!}{m_2! \cdot ... \cdot m_k!} \leq 2^{k-2}, \quad k = 2, 3, ... \, . \qquad (2.47)$$

Since

$$\prod_{j=2}^{k} ((aM)^{j-2} \sigma^2)^{m_j} = \sigma^{2\bar{m}} (aM)^{k-2\bar{m}}, \qquad k = 2, 3, ..., \qquad (2.48)$$

substituting (2.47) and (2.48) into (2.46) and recalling that $\sigma \leq M/2$, we conclude that

$$|\varGamma_k(Z_N)| \leq (k!)^{1+\gamma} \mathbf{D}Z_N L_{N,0}^{k-2}, \qquad k = 3, 4, ..., \qquad (2.49)$$

where $L_{N,0}$ is defined by (2.27) and $\mathbf{D}Z_N$ is defined by (2.20) with $\mu = 0$.

To complete the proof of Lemma 2.2, it is sufficient to use (2.43), (2.49) and then, by noting that

$$\mathit{\Gamma}_k(\tilde{Z}_N) = \frac{\mathit{\Gamma}_k(Z_N - \mathbf{E}Z_N)}{(\mathbf{D}Z_N)^{k/2}} = \frac{\mathit{\Gamma}_k(Z_N)}{(\mathbf{D}Z_N)^{k/2}}, \qquad k = 2, 3, ...,$$

we arrive at (2.24). □

Note that, by Leonov (1964), for the convergence to standard normal distribution under the conditions $\mathit{\Gamma}_1(\tilde{Z}_N) = 0$, $\mathit{\Gamma}_2(\tilde{Z}_N) = 1$, it is sufficient that

$$\mathit{\Gamma}_k(\tilde{Z}_N) \to 0 \qquad \text{as} \quad \mathit{\Delta}_* \to \infty,$$

for every $k = 3, 4, ...$. Here Δ_* is defined by (2.25).

It follows from estimate (2.24) that $\tilde{Z}_N$ satisfies S. V. Statulevičius' condition (S_γ) with $\mathit{\Delta} := \mathit{\Delta}_*$.

2.3. Theorems for large deviations

Since the accurate upper bounds (2.24) for the *k*th-order cumulants of the standardized sum $\tilde{Z}_N$ have been derived, to prove theorems of large deviations and exponential inequalities we have to use general lemmas presented in (Rudzkis *et al.* 1978; Bentkus, R. and Rudzkis 1980), respectively, about large deviations and exponential inequalities for an arbitrary random variable with zero mean and unit variance.

Recall that $\mathit{\Delta}_*$ is defined by (2.25). If $\gamma = 0$, then based on (2.24) the generating function $\varphi_{\tilde{Z}_N}(z)$ is analytical in the domain $|z| < \Delta_*$ which in turn guarantees that the large deviation theorems are applicable to the distribution function $F_{\tilde{Z}_N}(x)$ or, if it exists, to the density function $F_{\tilde{Z}_N}(x)$ for $0 \le x < \Delta_*$ (i. e., in Cramér zone). If $\gamma > 0$, then the generating function $\varphi_{\tilde{Z}_N}(z)$ is no longer analytical and we can only prove large deviation theorems for $0 \le x < \Delta_*$ (i. e., in the power Linnik zone).

Let

$$\mathit{\Delta}_{*,\gamma} = c_\gamma {\mathit{\Delta}_*}^{1/(1+2\gamma)}, \qquad c_\gamma = \frac{1}{6}\Big(\frac{\sqrt{2}}{6}\Big)^{1/(1+2\gamma)}. \tag{2.50}$$

We will use θ (with or without an index) to denote a value, not always the same, that does not exceed 1 in modulus.

Theorems 2.1, 2.2, 2.3 and Corollaries 2.1, 2.2 present the exact large deviations, in both the Cramér and power Linnik zones equivalent for the tails (left and right tails) of $\tilde{Z}_N$; asymptotic convergence of large deviation ratios to a unit; non-asymptotic exponential inequalities for the probability of large deviations of $\tilde{Z}_N$; normal approximation with an explicit non-asymptotic estimate of the "distance" between the distribution of $\tilde{Z}_N$ and the standard Gaussian distribution $\Phi(x)$ which is defined by (1.7).

Theorem 2.1. *If the random variable X with variance $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$ and the random variables $T_{N,1}$, $T_{N,2}$ satisfy conditions (L) and (L_0), respectively, then in the interval*

$$0 \le x < \Delta_{*,\gamma},$$

the ratios of large deviations

$$\begin{aligned} \frac{1 - F_{\tilde{Z}_N}(x)}{1 - \Phi(x)} &= \exp\{L_{*,\gamma}(x)\}\Big(1 + \theta_1 f(x)\frac{x+1}{\Delta_{*,\gamma}}\Big), \\ \frac{F_{\tilde{Z}_N}(-x)}{\Phi(-x)} &= \exp\{L_{*,\gamma}(-x)\}\Big(1 + \theta_2 f(x)\frac{x+1}{\Delta_{*,\gamma}}\Big) \end{aligned} \tag{2.51}$$

are valid, where

$$f(x) = \frac{60\big(1 + 10\Delta_{*,\gamma}^2 \exp\{-(1 - x/\Delta_{*,\gamma})\sqrt{\Delta_{*,\gamma}}\}\big)}{1 - x/\Delta_{*,\gamma}},$$

$$L_{*,\gamma}(x) = \sum_{3 \le k < r} \lambda_{*,k} x^k + \theta_3\Big(\frac{x}{\Delta_{*,\gamma}}\Big)^3, \quad r = \begin{cases} 2 + \frac{1}{\gamma}, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases} \tag{2.52}$$

The coefficients $\tilde{\lambda}_{*,k}$ (expressed by cumulants of $\tilde{Z}_N$ defined by (1.37)) coincide with the coefficients of the Cramér-Petrov series (Petrov 1975) given by the formula

$$\tilde{\lambda}_{*,k} = -b_{*,k-1}/k, \tag{2.53}$$

where the $b_{*,k}$ are determined successively from the equations

$$\sum_{r=1}^{j} \frac{1}{r!} \Gamma_{r+1}(\tilde{Z}_N) \sum_{j_1 + \ldots + j_r = j,\ j_i \ge 1} \prod_{i=1}^{r} b_{*,j_i} = \begin{cases} 1, & j = 1, \\ 0, & j = 2, 3, \ldots . \end{cases}$$

In particular,

$$\begin{aligned} \tilde{\lambda}_{*,2} &= -1/2, \\ \tilde{\lambda}_{*,3} &= \Gamma_3(\tilde{Z}_N)/6, \end{aligned}$$

$$\begin{aligned}\tilde{\lambda}_{*,4} &= (\varGamma_4(\tilde{Z}_N) - 3\varGamma_3^2(\tilde{Z}_N))/24,\\ \tilde{\lambda}_{*,5} &= (\varGamma_5(\tilde{Z}_N) - 10\varGamma_3(\tilde{Z}_N)\varGamma_4(\tilde{Z}_N) + 15\varGamma_3^3(\tilde{Z}_N))/120.\end{aligned}$$

For $\tilde{\lambda}_{*,k}$, the estimate

$$|\tilde{\lambda}_{*,k}| \le \frac{2}{k}\Big(\frac{16}{\varDelta_*}\Big)^{k-2}((k+1)!)^{\gamma}, \qquad k = 2, 3, ...$$

is valid. Therefore,

$$L_{*,\gamma}(x) \le \frac{x^3}{2(x + 8\varDelta_{*,\gamma})}, \qquad L_{*,\gamma}(-x) \ge -\frac{x^3}{3\varDelta_{*,\gamma}}.$$

***Proof of Theorem* 2.1**. Theorem 2.1 is proved using Lemma 2.2 and follows directly from the general Lemma 1.1 on large deviations. Clearly, $\tilde{Z}_N$ satisfies S. V. Statulevičius' condition (S_γ) with the parameter $\varDelta := \varDelta_*$. Accordingly, Lemma 1.1 yields the assertion of Theorem 2.1. □

Theorem 2.2. *Under the conditions of Theorem* 2.1*, the ratios*

$$\frac{1 - F_{\tilde{Z}_N}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{Z}_N}(-x)}{\Phi(-x)} \to 1 \tag{2.54}$$

hold for $x \ge 0$,

$$x = \begin{cases} o((\mathbf{D}T_{N,1})^{((1/2)-\epsilon)\nu(\gamma)}) \text{ when } \mu \ne 0,\\ o((\mathbf{E}T_{N,2})^{((1/2)-\epsilon)\nu(\gamma)}) \text{ when } \mu = 0,\end{cases} \tag{2.55}$$

if $\mathbf{D}T_{N,1} \to \infty$ or $\mathbf{E}T_{N,2} \to \infty$ (depending on the case: $\mu \ne 0$ or $\mu = 0$) when $0 \le \epsilon < 1/2$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$.

***Proof of Theorem* 2.2**. The first part of Theorem 2.2 follows immediately if we use the definition of $L_{*,\gamma}(x)$, $\gamma \ge 0$ by relation (2.52). Next, we shall prove that $L_{*,\gamma}(x) \to 0$ and $x/\Delta_{*,\gamma} \to 0$ as $\Delta_* \to \infty$, where Δ_* and $\Delta_{*,\gamma}$ are defined by (2.25) and (2.50), respectively. We begin by considering the case where $\mu \ne 0$. Let us recall the definition (2.26) in Δ_N. It follows that

$$\begin{aligned}\Delta_N &= \frac{\sqrt{\sigma^2\mathbf{E}T_{N,2} + \mu^2\mathbf{D}T_{N,1}}}{2(a/\bar{a})^2\Big((a/\bar{a})K_1|\mu|(\mathbf{D}T_{N,1})^{\epsilon} \vee \big(1 \vee \sigma\bar{a}/(2a|\mu|)\big)aM\Big)}\\ &\ge C_1(\mathbf{D}T_{N,1})^{(1/2)-\epsilon}, \qquad C_1 = \bar{a}^3/(2a^3K_1) > 0,\end{aligned}$$

if $(1 \vee \sigma\bar{a}/(2a|\mu|))aM \leq (a/\bar{a})K_1|\mu|(\mathbf{D}T_{N,1})^{\epsilon}$, and

$$\Delta_N = \frac{\sqrt{\sigma^2 \mathbf{E}T_{N,2} + \mu^2 \mathbf{D}T_{N,1}}}{2(a/\bar{a})^2\Big((a/\bar{a})K_1|\mu|(\mathbf{D}T_{N,1})^{\epsilon} \vee \big(1 \vee \sigma\bar{a}/(2|\mu|)\big)aM\Big)}$$
$$a \geq \bar{C}_1\sqrt{\mathbf{D}T_{N,1}}, \qquad \bar{C}_1 = \bar{a}^2|\mu|/\big(2\big(1 \vee \sigma\bar{a}/(2a|\mu|)\big)a^3M\big) > 0,$$

if $(1\vee\sigma\bar{a}/(2a|\mu|))aM \geq (a/\bar{a})K_1|\mu|(\mathbf{D}T_{N,1})^{\epsilon}$. Thus $\Delta_N \to \infty$ as $\mathbf{D}T_{N,1} \to \infty$ when $0 \leq \epsilon < 1/2$. Here M, $\mathbf{E}T_{N,2}$, $\mathbf{D}T_{N,1}$ and K_1 are defined, respectively, by (2.1), (2.4), (2.5) and (L), and $0 < a = \sup\{a_j, j = 1, 2, ...\} < \infty$, $0 < \bar{a} = \inf\{a_j, j = 1, 2, ...\} < \infty$.

Taking into account estimate (2.24) and equality (2.53), we obtain that for all $x = o((\mathbf{D}T_{N,1})^{((1/2)-\epsilon)v})$, $1 - (1 \vee \gamma) \leq 0$ with $0 \leq \epsilon < 1/2$,

$$\lambda_{*,3}x^3 = \frac{1}{6}\Gamma_3(\tilde{Z}_N)x^3 = o((\mathbf{D}T_{N,1})^{((1/2)-p)(3v-1)})$$
$$= o((\mathbf{D}T_{N,1})^{2v((1/2)-p)(1-(1\vee\gamma))}) = o(1)$$

as $\mathbf{D}T_{N,1} \to \infty$. Here $v = v(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geq 0$. On the other hand, recalling the definition of $\Delta_{*,\gamma}$ for $\gamma - (1 \vee \gamma) \leq 0$, we have

$$\frac{x}{\Delta_{*,\gamma}} = o((\mathbf{D}T_{N,1})^{((1/2)-\epsilon)(v-1/(1+2\gamma))})$$
$$= o((\mathbf{D}T_{N,1})^{2v((1/2)-\epsilon)(\gamma-(1\vee\gamma))/(1+2\gamma)}) = o(1)$$

as $\mathbf{D}T_{N,1} \to \infty$. Now we turn to the case where $\mu = 0$. By virtue of (2.27), we obtain

$$\Delta_{N,0} = \frac{\sigma\sqrt{\mathbf{E}T_{N,2}}}{2(1 \vee K_2(\mathbf{E}T_{N,2})^{\epsilon})((1/2) \vee a)M}$$
$$\geq C_2(\mathbf{E}T_{N,2})^{(1/2)-\epsilon}, \qquad C_2 = \sigma/\big(2K_2M((1/2) \vee a)\big) > 0,$$

if $K_2(\mathbf{E}T_{N,2})^{\epsilon})((1/2) \vee a)M \geq 1$, and

$$\Delta_{N,0} = \frac{\sigma\sqrt{\mathbf{E}T_{N,2}}}{2(1 \vee K_2(\mathbf{E}T_{N,2})^{\epsilon})((1/2) \vee a)M}$$
$$\geq \bar{C}_2\sqrt{\mathbf{E}T_{N,2}}, \qquad \bar{C}_2 = \sigma/\big(2M((1/2) \vee a)\big) > 0,$$

if $K_2(\mathbf{E}T_{N,2})^{\epsilon})((1/2) \vee a)M \leq 1$. Hence $\Delta_{N,0} \to \infty$ as $\mathbf{E}T_{N,2} \to \infty$ when $0 \leq \epsilon < 1/2$. Here K_2 is defined by (L_0). Therefore, it is easily checked that in the same way as in the case where $\mu \neq 0$, for all $x = o((\mathbf{E}T_{N,2})^{((1/2)-\epsilon)v})$ with $0 \leq p < 1/2$,

$$\lambda_{*,3}x^3 = o((\mathbf{E}T_{N,2})^{2v((1/2)-\epsilon)(1-(1\vee\gamma))}) = o(1) \text{ as } \mathbf{E}T_{N,2} \to \infty,$$

in the case where $1-(1\vee\gamma) \leq 0$, and

$$\frac{x}{\Delta_{*,\gamma}} = o((\mathbf{E}T_{N,2})^{2v((1/2)-\epsilon)(\gamma-(1\vee\gamma))/(1+2\gamma)}) = o(1) \text{ as } \mathbf{E}T_{N,2} \to \infty,$$

in the case where $\gamma-(1\vee\gamma) \leq 0$.
Finally, depending on the case: $\mu \neq 0$, or $\mu = 0$, for all $x \geq 0$ defined by (2.55), we derive that $L_{*,\gamma}(x) \to 0$ as $\mathbf{D}T_{N,1} \to \infty$ or $\mathbf{E}T_{N,2} \to \infty$, when $0 \leq \epsilon < 1/2$. □

Remark 2.1. It follows from Theorem 2.2 that in the case where $\gamma = 0$, the ratios (2.54) hold for $x \geq 0$ such that

$$x = \begin{cases} o((\mathbf{D}T_{N,1})^{(1/2-\epsilon)/3}) \text{ when } \mu \neq 0, \\ o((\mathbf{E}T_{N,2})^{(1/2-\epsilon)/3}) \text{ when } \mu = 0, \end{cases}$$

if $\mathbf{D}T_{N,1} \to \infty$ or $\mathbf{E}T_{N,2} \to \infty$ when $0 \leq \epsilon < 1/2$.

Theorem 2.3. *Let X with variance $0 < \sigma^2 < \infty$, $T_{N,1}$ and $T_{N,2}$ satisfy conditions $(\bar{B}_\gamma)$, (L) and (L_0), respectively. Then for all $x \geq 0$,*

$$\mathbf{P}(\pm\tilde{Z}_N \geq x) \leq \exp\Big\{-\frac{x^2}{2(2^{1+\gamma}+(x/\Delta_*^{1/(1+2\gamma)}))^{(1+2\gamma)/(1+\gamma)}}\Big\}. \quad (2.56)$$

***Proof of Theorem* 2.3.** The proof of Theorem 2.3 is obtained by virtue of general Lemma 1.2, where the inequality (1.17) holds with $H = 2^{1+\gamma}$, $\Delta := \Delta_*$. □

Corollary 2.1. *Under the conditions of Theorem 2.3, the exponential inequalities*

$$\mathbf{P}(\pm\tilde{Z}_N \geq x) \leq \begin{cases} \exp\big\{-x^2/8\big\}, & 0 \leq x \leq (2^{(1+\gamma)^2}\Delta_*)^{1/(1+2\gamma)}, \\ \exp\big\{-(x\Delta_*)^{1/(1+\gamma)}/4\big\}, & x \geq (2^{(1+\gamma)^2}\Delta_*)^{1/(1+2\gamma)}, \end{cases}$$

are valid.

***Proof of Corollary* 2.1.** If $x \leq (2^{(1+\gamma)^2}\Delta_*)^{1/(1+2\gamma)}$, then

$H \geq (x/\Delta_*^{1/(1+2\gamma)})^{(1+2\gamma)/(1+\gamma)}$, and therefore the right-hand side of the inequality in Corollary 2.1 does not exceed $\exp\{-x^2/(4 \cdot 2^{1+\gamma})\}$. The second case is obtained analogously. □

Corollary 2.2. *If X with variance $0 < \sigma^2 < \infty$, $T_{N,1}$ and $T_{N,2}$ satisfy conditions $(\bar{B}_\gamma)$, (L) and (L_0), respectively, then*

$$\sup_x |F_{\tilde{Z}_N}(x) - \Phi(x)| \leq \frac{4.4}{\Delta_{*,\gamma}}. \tag{2.57}$$

***Proof of Corollary* 2.2.** Corollary 2.2 follows by Lemma 2.2 and Corollary 3 in (Saulis 1996: 291). As $\tilde{Z}_N$ defined by (1.37) satisfies S. V. Statulevičius' condition (S_γ) with the parameter $\Delta := \Delta_*$. Accordingly, Corollary 3 in (Saulis 1996) yields Corollary 2.2. □

Remark 2.2. Note that it is possible to obtain large deviation theorems for the sum Z_N using only one of the conditions (L) or (L_0), under some additional assumptions. For example, if $\mathbf{D}T_{N,1} \geq \mathbf{E}T_{N,2}$, then it is enough to use (L_0). And in this case, the ratios (2.54) are valid in both cases: $\mu \neq 0$, or $\mu = 0$, for $x \geq 0$, $x = o((\mathbf{E}T_{N,2})^{((1/2)-\epsilon)\nu(\gamma)})$, if $\mathbf{E}T_{N,2} \to \infty$ when $0 \leq \epsilon < 1/2$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$.

Remark 2.3. Assume N is non-random: $N = n \in \mathbb{N}$. Then

$$T_{N,r} = T_{n,r} = \sum_{j=1}^{n} a_j^r, \qquad r \in \mathbb{N},$$

where $T_{N,r}$ is defined by (2.2). Thus in accordance with (2.4) and (2.10), we have

$$\mathbf{E}T_{N,r} = T_{n,r}, \qquad \Gamma_k(T_{n,r}) = 0, \qquad k = 2, 3, \dots . \tag{2.58}$$

Consequently, taking (2.19) and (2.20) into account, we get

$$\mathbf{E}Z_n = \mu T_{n,1}, \qquad \mathbf{D}Z_n = \sigma^2 T_{n,2}.$$

Equality (2.30) and condition (2.1) yield

$$|\Gamma_k(\tilde{Z}_n)| \leq \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \qquad \Delta = \frac{\sqrt{\mathbf{D}Z_n}}{aM}, \qquad k = 3, 4, \dots, \tag{2.59}$$

where $0 < a = \sup\{a_j, j = 1, 2, \dots\} < \infty$, and M is defined by (2.1).

The upper estimate (2.59) coincides with the estimate (15) presented in

(Saulis 1979: 280) for i. i. d. random variables. In this instance, estimate (15) holds with the parameters $\Delta_n := \Delta$, $\bar{B}_n^2 := \mathbf{D}Z_n$, and $\gamma_n := a$.

Note that $\Delta = C\sqrt{T_{n,2}}$, where $C = \sigma/(aM) > 0$. Therefore, in consideration of the proof of Theorem 2.2, the ratios (2.54) are valid for $x \geq 0$ such that $x = o(T_{n,2}^{\nu(\gamma)/2})$, if $T_{n,2} \to \infty$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geq 0$.

2.4. Instances of large deviations

The non-negative integer-valued random number N of summands in weighted compound sum Z_N which is defined by (1.36) can obey various probability laws. Subsection 2.4.1 includes remarks about N obeys the binomial law, is a Poisson process, or is a mixed Poisson process. Accordingly, definitions and additional notations for the aforementioned processes are listed first. For these, we mostly followed (Mikosh 2009), however we also refer to (Embrechts *et al.* 1997; Faÿ *et al.* 2006; Gnedenko and Korolev 1996; Grändell 1997; Pragarauskas 2007; Rolski *et al.* 2001) and (Bening and Korolev 2002).

Large deviations in the case where $a_j \equiv 1$, $j = 1, 2, \ldots$, are considered in Subsection 2.4.2 where they are compared with the known results for large deviations for compound sums S_N, that are defined by as defined by (1.30), of i. i. d. random variables. Subsections 2.4.3–2.4.5 consider large deviation theorems for the binomial random sums and the most prominent processes, the compound Poisson and compound mixed Poisson processes. The last subsection is concerned with the discounted version of large deviations, including remarks about the cases where N obeys the binomial law, is a compound Poisson process, or is a mixed Poisson process.

2.4.1. Definitions and remarks

In stochastic theory a random number of summands N are often assumed to follow the Poisson law. As noted in (Mikosh 2009), the Poisson process has very desirable theoretical properties that have been collected for several decades and a long tradition in applied probability and stochastic process theory. The Poisson process is currently used in actuarial science to count numbers of claims, although it is perhaps not the most realistic process when it comes to fitting real-life claim arrival times. Other models for N are modifications of the Poisson process that yield greater flexibility in one way or an other.

Let us define the Poisson process. We say that a non-negative integer-valued random variable Y obeys a Poisson distribution with the parameter

$\lambda > 0$ $\big(Y \sim \mathcal{P}(\lambda)\big)$ if

$$\mathbf{P}(Y = s) = e^{-\lambda}\frac{\lambda^s}{s!}, \qquad s \in \mathbb{N}_0. \tag{2.60}$$

We suppose that $Y = 0$ almost surely (a. s.) has a $\mathcal{P}(0)$ distribution.

Definition 2.1. *A stochastic process $N := N_t$, $t \geq 0$, is said to be a Poisson process if the following conditions hold:*

1^0 *The process starts at zero: $N_0 = 0$ a. s.*

2^0 *The process has independent increments: for any t_j, $j = 0, 1, ..., n$, $n \geq 1$, such that $0 = t_0 < t_1 < ... < t_n$, the increments $N_{t_1} - N_{t_0}, ..., N_{t_n} - N_{t_{n-1}}$, are mutually independent.*

3^0 *There exists a non-decreasing right-continuous function Λ: $[0,\infty) \to [0,\infty)$ with $\Lambda(0) = 0$ such that the increments $N_{t'} - N_t$ for $0 \leq t < t^{'} < \infty$ have a Poisson distribution $\mathcal{P}(\Lambda(t^{'}) - \Lambda(t))$. We call Λ the mean value function of N.*

4^0 *With probability 1, the sample paths $N_t(\omega)$, $t \geq 0$, of the process N are right-continuous for $t \geq 0$ and have limits from the left for $t > 0$. We say that N has cádlág sample paths.*

A Poisson random variable Y is determined only by it's mean value (=variance): $\lambda = \mathbf{E}Y = \mathbf{D}Y$, which is a rare property. Hence, in order to determine the distribution of the Poisson process N_t, it suffices to know it's mean value function. If the mean value function is absolutely continuous, i. e., for any $t < t^{'}$ and for some non-negative measurable function λ_t,

$$\Lambda(t^{'}) - \Lambda(t) = \int_t^{t^{'}} \lambda_y dy, \qquad t < t^{'},$$

then we say that the Poisson process N_t has *the intensity or rate function* λ_t. In this instance Λ is a continuous function.

For the amazing properties of the Poisson process, we refer the reader to, e. g., (Bening and Korolev 2002; Mikosh 2009; Pragarauskas 2007).

Definition 2.2. *A process with a following linear mean value function $\Lambda(t) = \lambda t$, $t \geq 0$, for some $\lambda > 0$, is said to be a homogeneous Poisson process, and otherwise it is an inhomogeneous Poisson process. The quantity $\lambda > 0$ is the intensity or rate of the homogeneous Poisson process.*

A homogeneous Poisson process plays a major role in insurance mathematics (for a detailed discussion see, e. g., Mikosh 2009).

Definition 2.3. *A homogeneous Poisson process with $\lambda = 1$ is called a standard homogeneous Poisson process.*

The mean value function Λ can be interpreted as operational time or the inner clock of the Poisson process. If the process N_t is homogeneous, time evolves linearly: $\Lambda(t^{'}) - \Lambda(t) = \Lambda(t^{'} + h) - \Lambda(t + h)$ for any $h > 0$ and $0 = t < t^{'} < \infty$. If N_t has a non-constant intensity function λ_t, time "slows down" or "speeds up" according to the magnitude of λ_t. In an insurance context, a non-constant λ_t may refer to seasonal effects or trends.

A homogeneous Poisson process N_t

1^0 has cádlág sample paths;

2^0 starts at zero;

3^0 has independent and stationary increments, i. e., $\mathbf{P}(N_{t^{'}} - N_t = s) = \mathbf{P}(N_{t^{'}+h} - N_{t+h} = s) \sim \mathcal{P}(\lambda(t^{'} - t))$, for any $0 \leq t < t^{'} < \infty$ and $h > 0$ – the Poisson increment parameter only depends on the length of the interval, not on it's location;

4^0 is $\mathcal{P}(\lambda t)$ distributed for every $t > 0$.

A process on $[0, \infty)$ with properties 1^0–3^0 is called *a Lévy process* (refer to Mikosh 2009: 335). The homogeneous Poisson process is one of the prime examples of a Lévy process with applications in various areas such as queuing theory, finance, insurance and stochastic networks, to name a few.

Proposition 2.2. *Let N be a Poisson process with the mean value function Λ, and let $N^{'}$ be a standard homogeneous Poisson process. Then the following statements hold:*

(1) The process $N^{'}_{\Lambda(t)}$, $t \geq 0$, is Poisson with mean value function Λ.

(2) If Λ is continuous and increasing with $\lim_{t\to\infty} \Lambda(t) = \infty$, then $N_{\Lambda^{-1}}$, $t \geq 0$, is a standard homogeneous Poisson process.

This result immediately follows from the definition of a Poisson process and may by verified, for example, by following the proofs of Lemmas 3.1.5, 3.1.6 in (Pragarauskas 2007: 42), or the notations in (Mikosh 2009: 14–15).

Definition 2.4. *Let $N^{'}$ be a standard homogeneous Poisson process and let $\Lambda(t)$, $t \geq 0$, be a non-negative, non-decreasing process that is independent* *of the process $N^{'}$. Moreover, assume that $\Lambda(0) = 0$ a. s. and $\Lambda(t) < \infty$ a. s. for any $t \geq 0$. Then the random process $N_t := N^{'}_{\Lambda(t)}$, $t \geq 0$, is called a Cox process, or a doubly stochastic Poisson process.*

In other words, a Cox process is a Poisson process where the mean value function Λ is random. In an insurance context, Λ is usually defined by the formula

$$\Lambda(t) = \int_0^t \lambda_y dy, \qquad t \geq 0,$$

where λ_t is a right-continuous, non-negative, integrable random process. If $\lambda_t \geq c > 0$, $t \geq 0$, then Λ is continuous and increasing with $\lim_{t\to\infty} \Lambda(t) = \infty$ a. s. The Cox process is more appropriately used as a claim arrival process as unpredictable random environmental factors like catastrophic events should be based on a specific stochastic process. The doubly stochastic Poisson process provides flexibility by not only letting the intensity depend on time but also allowing it to be a random process.

Set $t > 0$ and suppose that $\mathbf{E}\Lambda^2(t) < \infty$, $\mathbf{D}\Lambda(t) > 0$. Taking Definition 2.4 and (2.60) into account (for more detail see, e. g., Pragarauskas 2007: 51, or Korolev *et al.* 2011: 362–363), we have

$$\mathbf{E}N^{'}_{\Lambda(t)} = \int_0^\infty \mathbf{E}N^{'}_y dF_{\Lambda(t)}(y) = \int_0^\infty y dF_{\Lambda(t)}(y) = \mathbf{E}\Lambda(t), \quad (2.61)$$

$$\mathbf{E}N^{'2}_{\Lambda(t)} = \int_0^\infty (y + y^2) dF_{\Lambda(t)}(y) = \mathbf{E}\Lambda(t) + \mathbf{E}\Lambda^2(t), \quad (2.62)$$

and thus

$$\mathbf{D}N^{'}_{\Lambda(t)} = \mathbf{E}\Lambda(t)+\mathbf{D}\Lambda(t) = \mathbf{E}N^{'}_{\Lambda(t)}\big(1+\mathbf{D}\Lambda(t)/\mathbf{E}\Lambda(t)\big) > \mathbf{E}N^{'}_{\Lambda(t)}. \quad (2.63)$$

The property that $\mathbf{D}N^{'}_{\Lambda(t)} > \mathbf{E}N^{'}_{\Lambda(t)}$ for any $t > 0$ with $\Lambda(t) > 0$ is called *over-dispersion*. This is one of the major differences between the Cox process and the Poisson process where $\mathbf{E}N_t = \mathbf{D}N_t$.

Now let us consider a special Cox process – a mixed Poisson process. Such processes have proved useful, for example, in medical statistics, where every sample path represents the medical history of a particular patient who has his/her own mean value function.

Definition 2.5. *Let $N^{'}$ be a standard homogeneous Poisson process and let Λ be the mean value function of a Poisson process on $(0, \infty]$. Let $\theta > 0$ a. s. be a (non-degenerate) random variable independent of $N^{'}$. Then the process $N_t :=$*

$N^{'}_{\theta\Lambda(t)}$, $t \geq 0$, *is said to be a mixed Poisson process with mixing variable* θ.

For instance, in an insurance context, a mixed Poisson process is introduced as a claim number process if one does not believe in one particular Poisson process as generating process for claim arrivals. θ can represent different factors of influence on an insurance portfolio.

Extensive detailed treatments of mixed Poisson processes and their properties appear in (Bening and Korolev 2002; Grändell 1997; Korolev *et al.* 2011). For the properties that the mixed Poisson process inherits from the Poisson process and for the properties that it loses, we refer to (Mikosh 2009: 68-69). For a gentle introduction to point processes and generalized Poisson processes we refer to (Embrechts *et al.* 1997). For a rigorous treatment at a moderate level see, e. g. (Resnick 1992). Various interpretations of considered processes are available, e. g., in (Bening and Korolev 2002; Embrechts *et al.* 1997; Faÿ *et al.* 2006; Grändell 1997; Korolev *et al.* 2011; Pragarauskas 2007; Resnick 1992; Rolski *et al.* 2001).

Remark 2.4. Recall the probability characteristics $\mathbf{E}T_{N,1}$, $\mathbf{E}T_{N,2}$ and $\mathbf{D}T_{N,1}$ that are defined by (2.14) and (2.17) with $r = 1, 2$:

$$\mathbf{E}T_{N,1} = \sum_{j=1}^{\infty}\mathbf{P}(N \geq j)a_j, \qquad \mathbf{E}T_{N,2} = \sum_{j=1}^{\infty}\mathbf{P}(N \geq j)a_j^2,$$

$$\begin{aligned}\mathbf{D}T_{N,1} &= \sum_{j=1}^{\infty}(1-\mathbf{P}(N \geq j))\mathbf{P}(N \geq j)a_j^2 \\ &\quad + 2\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty}(1-\mathbf{P}(N \geq j))\mathbf{P}(N \geq n)a_j a_n,\end{aligned}$$

where the compound variables $T_{N,1}$, $T_{N,2}$ are defined by (2.2) with $r = 1, 2$, and $\mathbf{P}(N \geq j) = \sum_{s=j}^{\infty} q_s$, $0 < \mathbf{P}(N \geq j) < 1$, $j = 1, 2, ...$, $q_s = \mathbf{P}(N = s)$, $s \in \mathbb{N}_0$.

Suppose that the mean (2.19) and the variance (2.20) of the weighted random sum $Z_N = \sum_{j=1}^{N} a_j X_j$, $Z_0 = 0$, $0 < a_j < \infty$, where a non-negative integer-valued random index N is independent of the i. i. d., weighted random variables $\{X, X_j, j = 1, 2, ...\}$, as well as, conditions (L) and (L_0) for the cumulants of $T_{N,1}$ and $T_{N,2}$, respectively, hold with aforementioned probability characteristics $\mathbf{E}T_{N,1}$, $\mathbf{E}T_{N,2}$ and $\mathbf{D}T_{N,1}$. Then it immediately follows from

Lemma 2.2 and Theorem 2.2 that in the cases where N obeys the binomial law, is a compound Poisson process, or is a mixed Poisson process, the upper bound of $|\mathit{\Gamma}_k(\tilde{Z}_N)|$, $k = 3, 4, ...$ defined by (2.24) and, consequently, the convergence of large deviation ratios to a unit (2.54) are valid with $\mathbf{E}T_{N,2}$ and $\mathbf{D}T_{N,1}$, where $\mathbf{P}(N \geq j)$, $j = 1, 2, ...$, defined by (2.15), depends on the law in question.

Example 2.1. Assume that $N := N_t$, $t \geq 0$, is the most popular Poisson process – the homogeneous Poisson process with a linear mean value function $\Lambda(t) = \lambda t, t \geq 0$, for some $\lambda > 0$ (see Definition 2.2) and with the distribution

$$q_s = \mathbf{P}(N_t = s) = e^{-\lambda t}(\lambda t)^s/s!, \qquad s \in \mathbb{N}_0, \tag{2.64}$$

due to (2.60). If $N_t, t \geq 0$ is a homogeneous Poisson process, then

$$\mathbf{P}(N_t \geq j) = \sum_{s=j}^{\infty} q_s = e^{-\lambda t} \sum_{s=j}^{\infty} \frac{(\lambda t)^s}{s!} = 1 - Q(j, \lambda t). \tag{2.65}$$

Here $Q(m, x) = \Gamma(m, x)/\Gamma(m)$ is the *regularized Gamma function*, and $\Gamma(m, x) = \int_x^\infty e^{-y} y^{m-1} dy$ is *the upper incomplete Gamma function*. If m is a positive integer, then $\Gamma(m) = (m-1)!$. Note that when $m > 0$ is an integer, $Q(m, \lambda)$ is the cumulative distribution function for Poisson random variables. Accordingly, if Y is a Poisson random variable with intensity $\lambda > 0$, then

$$P(Y < m) = \sum_{i<m} e^{-\lambda} \frac{\lambda^i}{i!} = \frac{\Gamma(m, \lambda)}{(m-1)!} = Q(m, \lambda).$$

Example 2.2. Let us consider a special Cox process – the mixed Poisson process $N_t := N'_{\Lambda(t)}$, $t > 0$, where the mean value function $\Lambda(t)$ is a general random process with non-decreasing sample paths, independent of the standard Poisson process N' (see Definitions 2.2–2.5). By a mixed Poisson distribution with the mixing distribution $F_{\Lambda(t)}(x) = \mathbf{P}(\Lambda(t) < x)$, we mean (see, e. g., Korolev *et al.* 2011, 2012)

$$q_s = \mathbf{P}(N_t = s) = \frac{1}{s!} \int_0^\infty e^{-x} x^s dF_{\Lambda(t)}(x), \qquad s \in \mathbb{N}_0, \tag{2.66}$$

following the Definition 2.5 of the mixed Poisson process.

The most well-known and most widely used mixed Poisson distribution is *the negative binomial distribution* that is generated by the mixing Gamma distribution. To elaborate, assume that $\Lambda(t)$ is distributed according to the Gamma law with the positive parameters $(n_t,\ b_t)$ and density function

$$p_{\Lambda(t)}(x) = \frac{b_t^{n_t}}{\Gamma(n_t)} x^{n_t-1} e^{-b_t x}, \qquad x > 0, \tag{2.67}$$

where $\Gamma(n_t) = \int_0^\infty x^{n_t-1} e^{-x} dx$ is Gamma function. Obviously, by virtue of (2.66) and (2.67),

$$q_s = \frac{\Gamma(n_t+s)}{s!\Gamma(n_t)} p^{n_t} (1-p)^s, \qquad p = \frac{b_t}{1+b_t}, \quad s \in \mathbb{N}_0. \tag{2.68}$$

Hence, N_t is distributed according to the negative binomial law with the probability (2.68) and parameters $0 < p < 1$, $n_t > 0$. This process is called a *negative binomial or Pólya process* and is often used in insurance and other dynamic population models. If n_t is a positive integer, then negative binomial distribution is called *a Pascal distribution* and, in case where $n_t \equiv 1$, it is called *a geometric distribution*.

The negative binomial process was first used in the form of a mixed Poisson distribution by Greenwood and Yule (1920) to model the frequencies of accidents. Other examples of mixed Poisson process distributions can be found in, e. g., (Bening and Korolev 2002; Grändell 1997; Korolev *et al.* 2011).

According to (2.68),

$$\mathbf{P}(N_t \geq j) = \sum_{s=0}^{\infty} \frac{\Gamma(n_t+s+j)}{(j+s)!\Gamma(n_t)} p^{n_t} (1-p)^{s+j} = \frac{\Gamma(n_t+j)}{j!\Gamma(n_t)} \frac{b_t^{n_t}}{(1+b_t)^{n_t+j}}$$
$$\cdot\ {}_2F_1\big(1, n_t+j; j+1; 1/(1+b_t)\big), \tag{2.69}$$

where

$${}_2F_1(b;c;d;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{b^{\bar{k}} c^{\bar{k}}}{d^{\bar{k}}}$$

is the hypergeometric function known as the *Gaussian function*. Here $x^{\bar{k}}$ is used for *the rising factorial*

$$x^{\bar{k}} = x(x+1)(x+2)(x+k-1) = \Gamma(x+k)/\Gamma(x).$$

If $N := N_p$ is distributed according to the geometric law with $q_s = p(1-p)^s$ and $0 < p < 1$, $s \in N_0$, namely, $n_t \equiv 1$, then $\mathbf{P}(N_p \geq j) = 1/(1+b_t)^j$ as a consequence of (2.69).

Example 2.3. Suppose that $N := N_n$, $n \in \mathbb{N}_0$, is distributed according to *the binomial law* with

$$q_s = C_n^s \bar{p}^s (1-\bar{p})^{n-s}, \quad C_n^s = \frac{n!}{s!(n-s)!}, \quad 0 < \bar{p} < 1, \ s \in \mathbb{N}_0. \tag{2.70}$$

In this instance,

$$\mathbf{P}(N_n \geq j) = C_n^j \, {}_2F_1\big(1, j-n; j+1; \bar{p}/(1-\bar{p})\big)\bar{p}^j (1-\bar{p})^{n-j}. \tag{2.71}$$

Remark 2.5. Based on Remark 2.4, together with (2.65), (2.69), and (2.71), the convergence of the large deviation ratios to a unit (2.54) depends on the parameters (n_t , b_t, n and p) of the process or probability law considered in Examples 2.1–2.3.

To clarify the assertion in Remark 2.5, let us consider some instances of large deviations for the distribution of weighted random sum Z_N when $a_j \equiv 1$ or $a_j \equiv v^j$, where $0 < v < 1$ and $j = 1, 2, \ldots$.

2.4.2. Large deviations in summations without weights

Let us suppose that the non-negative bounded weights a_j, $j = 1, 2, 3, \ldots$, are equal to one. That is, let us consider the sum

$$S_N = \sum_{j=1}^{N} X_j, \qquad S_0 = 0,$$

of a r. n. s. of i. i. d. random variables $\{X, X_j, j = 1, 2, \ldots\}$ with the mean, variance, and distribution function

$$\mu = \mathbf{E}X, \qquad 0 < \sigma^2 = \mathbf{D}X < \infty, \qquad F_X(x) = \mathbf{P}(X < x), \quad x \in \mathbb{R}.$$

Recall that the non-negative integer-valued random variable N with

$$\alpha = \mathbf{E}N, \qquad \beta^2 = \mathbf{D}N, \qquad \mathbf{P}(N = s) = q_s, \qquad s \in \mathbb{N}_0,$$

is independent of $\{X, X_j, j = 1, 2, \ldots\}$. If $a_j \equiv 1$, then we can rewrite (2.4)–(2.6) (see Section 2.1: 35) as follows:

$$\mathbf{E}T_{N,r} = \alpha, \qquad \mathbf{D}T_{N,r} = \mathbf{E}N^2 - \alpha^2 = \beta^2,$$

where $\mathbf{E}T_{N,r}^2 = \mathbf{E}N^2$, given $T_{N,r} = N$ and $T_{s,r} = s$, $r \in \mathbb{N}_0$. Here $T_{N,r}$ and $T_{s,r}$ are defined by (2.2) and (2.3), respectively. Hence, it follows from (2.19) –(2.21) that

$$\mathbf{E}S_N = \mu\alpha, \qquad \mathbf{D}S_N = \sigma^2\alpha + \mu^2\beta^2, \tag{2.72}$$

and

$$F_{S_N}(x) = q_0 + \sum_{s=1}^{\infty} q_s F_X^{*s}(x), \qquad x > 0,$$

where $F_X^{*s}(x)$ is the s-fold convolution of the distribution function $F_X(x)$ of the random variable X with itself. For more details about the probability characteristics of S_N see, e. g., (Korolev *et al.* 2011).

A suitable bound for the *k*th-order cumulants, $k = 3, 4, ...$, of

$$\tilde{S}_N = \frac{S_N - \mathbf{E}S_N}{\sqrt{\mathbf{D}S_N}}, \qquad \mathbf{D}S_N > 0,$$

follows from Lemma 2.2 on the upper estimate for the *k*th-order cumulants, $k = 3, 4, ...$, of $\tilde{Z}_N$ in the case where $a_j \equiv 1$. Here $\tilde{Z}_N$ is defined by (1.37).

Corollary 2.3. *Assume that the random variable* X *with variance* $0 < \sigma^2 < \infty$ *fulfills condition* $(\bar{B}_\gamma)$. *Also assume that the non-negative integer-valued random variable* N *satisfies conditions* (1.35) *and* (1.33). *Then*

$$|\Gamma_k(\tilde{S}_N)| \le \frac{(k!)^{1+\gamma}}{\Delta_*^{k-2}}, \qquad k = 3, 4, ..., \tag{2.73}$$

where Δ_* *is defined by* (2.25) *with*

$$\Delta_N = \sqrt{\mathbf{D}S_N}/L_N, \quad L_N = 2\big(K_1|\mu|\beta^{2\epsilon} \vee \big(1 \vee \sigma/(2|\mu|)\big)M\big) \tag{2.74}$$

when $\mu \neq 0$, *and*

$$\Delta_{N,0} = \sqrt{\mathbf{D}S_N}/L_{N,0}, \quad L_{N,0} = 2(1 \vee K_2\alpha^\epsilon)M \tag{2.75}$$

when $\mu = 0$. *Here* $\mathbf{D}S_N$ *and* $M > 0$ *are defined by* (2.72) *and* (2.1), *respectively, and in the case where* $\mu = 0$ *holds with* $\sigma^2 = \mathbf{E}X^2$.

Thanks to the accurate upper bounds (2.73) for the *k*th-order cumulants of the standardized sum (1.34), we may conclude that the assertions of the following Corollaries 2.4, 2.5 follow directly from Theorems 2.1–2.3 and Corollary 2.2 (see Section 2.3).

Corollary 2.4. *If the random variables X and N satisfy the conditions* $(\bar{B}_\gamma)$ *and* (1.35), (1.33)*, respectively, then the large deviation equalities* (2.51)*, exponential inequality* (2.56)*, and absolute error estimate* (2.57) *are valid with* $\tilde{Z}_N := \tilde{S}_N$ *and* Δ_**, respectively, defined by* (1.34) *and* (2.25) *with* (2.74) *and* (2.75).

Corollary 2.5. *Under the conditions of Corollary* 2.4*, the ratios*

$$\frac{1 - F_{\tilde{S}_N}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{S}_N}(-x)}{\Phi(-x)} \to 1 \tag{2.76}$$

hold for $x \geq 0$*,*

$$x = \begin{cases} o((\beta^{(1-2\epsilon)\nu(\gamma)}) & \text{as } \mu \neq 0, \\ o((\alpha^{((1/2)-\epsilon)\nu(\gamma)}) & \text{as } \mu = 0, \end{cases}$$

if $\beta \to \infty$ *or* $\alpha \to \infty$ *(depending on the case:* $\mu \neq 0$ *or* $\mu = 0$*) when* $0 \leq \epsilon < 1/2$*. Here* $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$*,* $\gamma \geq 0$.

Large deviation theorems in the Cramér zone for the sum S_N under conditions (1.35) and (1.33) and when the cumulant method is used have been investigated by (Statulevičius 1967; Saulis and Deltuvienė 2007). Corollaries 2.4, 2.5 in the case where $\mu \neq 0$ and $\gamma = 0$ coincide with the results obtained in the paper (Saulis and Deltuvienė 2007). For more details, see in Section 1.2: 21, where an overview of large deviations for distributions of random sums is given.

Remark 2.6. Assume that the number of summands in the sum S_N is nonrandom, i. e., $N = n \in \mathbb{N}$. Then from (1.29) and (2.58) in the case where $a_j \equiv 1$, we have $\alpha = n$ and $\Gamma_m(N) = 0$, $m = 2, 3, \ldots$. Thus,

$$\mathbf{E} S_n = n\mu, \qquad \mathbf{D} S_n = n\sigma^2$$

as a consequence of (2.72). Then, taking (2.32) with $T_{N,2} = N$ and (2.1) into account, we arrive at

$$|\Gamma_k(\tilde{S}_n)| \leq \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \qquad \Delta = \frac{\sqrt{\mathbf{D} S_n}}{M}, \qquad k = 3, 4, \ldots .$$

It is clear that $\Delta = C\sqrt{n}$, $C = \sigma/M > 0$. Obviously, it follows from Corollary 2.5 that the convergence of ratios to a unite (2.76) is valid for $x \geq 0$, $x = o(n^{\nu(\gamma)/2})$, if $n \to \infty$.

2.4.3. Large deviation theorems for compound Poisson processes

In the continuous dynamic models of an insurance stock,

$$R_t = R_0 + P_t - S_{N_t}, \qquad t \geq 0,$$

can express the surplus R_t at time t (see, e. g., Pragarauskas 2007). Here R_0 is the initial reserve and P_t is the total premium received up to time t. That is the company sells insurance policies and receives a premium according to P_t. The sum

$$S_{N_t} = \sum_{j=1}^{N_t} X_j, \qquad S_0 = 0, \tag{2.77}$$

is the total claim amount process in the time interval $[0, t]$. In this example, $X_j,\ j = 1, 2, ...$, denotes the *j*th claim, and $N := N_t$ is the number of claims by time t.

S_{N_t} shares various properties with the partial sum process. For example, asymptotic properties such as the central limit theorem and the strong law of large numbers are analogous for the two processes (for more detail, see Mikosh 2009).

Assume that $N := N_t$, $t \geq 0$, is the most popular Poisson process the homogeneous Poisson process with the linear mean value function $\Lambda(t) = \lambda t$, $t \geq 0$, for some $\lambda > 0$, and the distribution (2.64) (see Example 2.1). In addition,

$$\alpha_t = \mathbf{E}N_t = \lambda t, \qquad \beta_t^2 = \mathbf{D}N_t = \lambda t, \tag{2.78}$$

by virtue of properties of the Poisson process. If $N := N_t$ is a homogeneous Poisson process, then S_{N_t} is *a compound Poisson process* with the mean and variance

$$\mathbf{E}S_{N_t} = \mu\lambda t, \qquad \mathbf{D}S_{N_t} = \lambda t(\sigma^2 + \mu^2), \tag{2.79}$$

as a consequence of (2.72) and (2.78).

Central limit problems for Poisson random sums have been addressed, for example, in Bening *et al.* (1997); Korolev and Shevtsova (2012); Nefedova and Shevtsova (2011); Sunklodas (2009), also see the books (Embrechts *et al.* 1997; Gnedenko and Korolev 1996; Korolev *et al.* 2011; Mikosh 2009) and references therein. Exponential inequalities, probabilities of large deviations for compound Poisson sums under different assumptions are available, e. g., in (Bening and Korolev 2002; Bonin 2003; Embrechts *et al.* 1985; Michel 1993;

Mita 1997; Shorgin 1998). The tail behavior of random sums under heavy-tailed distributions with applications in mathematical finance and insurance, such as ruin probability in the Cramér–Lundberg model (see, e. g., Mikosh 2009: 12) of risk theory, applications in queueing, random walk, generalized renewal theory, where compound Poisson sums appear can be found, for example, in (Embrechts *et al.* 1997; Faÿ *et al.* 2006; Resnick 1992; Robert and Segers 2008; Rolski *et al.* 2001; Tang *et al.* 2001).

Recall that the papers (Kasparavičiūtė and Saulis 2013, 2011a), have examined large deviation theorems for the distribution of the standardized weighted compound sum (1.37) in both the Cramér and the power Linnik zones in case where the cumulant method is used. These papers also discuss the instances of large deviations for the distribution of compound Poisson process.

According to (2.64),

$$f_{S_{N_t}}(u) = \mathbf{E}e^{iuS_{N_t}} = \sum_{s=0}^{\infty} q_s f_X^s(u) = e^{-\lambda t(1-f_X(u))}, \quad u \in \mathbb{R}, \tag{2.80}$$

where $f_X(u)$ is the characteristic function (1.2) of the random variable X. Hence, it follows from (2.79) and (2.80) that

$$\begin{aligned} f_{\tilde{S}_{N_t}}(u) &= \exp\left\{\lambda t\left(f_X\left(\frac{u}{\sqrt{\mathbf{D}S_{N_t}}}\right) - 1 - i\mu\frac{u}{\sqrt{\mathbf{D}S_{N_t}}}\right)\right\} \\ &= f^{\lambda t}_{S_{N_1}-\mu}\left(\frac{u}{\sqrt{\mathbf{D}S_{N_t}}}\right), \end{aligned} \tag{2.81}$$

where

$$\tilde{S}_{N_t} = \frac{S_{N_t} - \mathbf{E}S_{N_t}}{\sqrt{\mathbf{D}S_{N_t}}}, \qquad \mathbf{D}S_{N_t} > 0, \tag{2.82}$$

is the standardized compound Poisson process. Moreover,

$$f_{S_{N_1}-\mu}\left(\frac{u}{\sqrt{\mathbf{D}S_{N_t}}}\right) = \exp\left\{f_X\left(\frac{u}{\sqrt{\mathbf{D}S_{N_t}}}\right) - 1 - i\mu\frac{u}{\sqrt{\mathbf{D}S_{N_t}}}\right\}$$

is the characteristic function of the random variable $S_{N_1} - \mu = \sum_{j=1}^{N_1} X_j - \mu$, with N_1 a Poisson random variable with the parameter 1. $\mathbf{E}(S_{N_1} - \mu) = 0$ and $\mathbf{D}(S_{N_1} - \mu) = \mathbf{E}X^2$. As the papers (Bening *et al.* 1997; Korolev and Zhukov 2000) have discussed, the representation (2.81) shows that the asymptotic behavior of (2.81) as $\lambda t \to \infty$ is similar to that of the characteristic function of

the random variable
$$\frac{1}{\sqrt{\lambda t(\sigma^2+\mu^2)}}\sum_{j=1}^{n} X_j$$
as $n \asymp \lambda t \to \infty$ (we write $u(x) \asymp v(x)$ for real functions $u(x)$ and $v(x)$ if $u(x) = O(v(x))$ and $v(x) = O(u(x))$), where the X_j are independent random variables The asymptotic properties of Poisson random sums are to a great extent similar to the corresponding properties of sums of the same random variables with a non-random number of summands (see, for example, Gnedenko and Kolmogorov 1954). To confirm these words, it suffices to recall the method of accompanying infinitely divisible distributions (see, e. g., Gnedenko and Korolev 1996). However, this analogy is not absolutely continuous as, for example, the distribution function

$$F_{S_{N_t}}(x) = e^{-\lambda t}F_0(x) + \sum_{s=1}^{\infty} q_s F_X^{*s}(x), \qquad x \in \mathbb{R}, \tag{2.83}$$

of the compound Poisson process is not absolutely continuous for all $x \in \mathbb{R}$ because of the presence of an atom at zero. Here $F_X^{*s}(x)$ is the s-fold convolution of the distribution function $F_X(x)$ of the random variable X with itself, and $F_0(x)$ is the distribution function with a single unit jump at zero.

Proposition 2.3. *If the random variable X with $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$ and N_t, $t \geq 0$, is the homogeneous Poisson process with the probability* (2.64)*, then*

$$|\Gamma_k(\tilde{S}_{N_t})| \leq \frac{(k!)^{1+\gamma}}{\Delta_t^{k-2}}, \ \Delta_t = \frac{\sqrt{\lambda t(\sigma^2+\mu^2)}}{K}, \ K > 0, \ k = 3, 4, \dots . \tag{2.84}$$

Proof of Proposition 2.3. (2.80) and (1.5) gives us

$$\Gamma_k(S_{N_t}) = \frac{d^k}{i^k du^k} \ln f_{S_{N_t}}(u)\Big|_{u=0} = \lambda t \mathbf{E}X^k, \qquad k = 1, 2, \dots . \tag{2.85}$$

Based on $(\bar{B}_\gamma)$, for the kth-order moments $\mathbf{E}X^k$ of the random variable X with $0 < \sigma^2 < \infty$ we use the following condition

$$|\mathbf{E}X^k| \leq (k!)^{1+\gamma} K^{k-2} \mathbf{E}X^2, \qquad k = 3, 4, \dots .$$

Therefore, $\Gamma_k(\tilde{S}_{N_t}) = \Gamma_k(S_{N_t})/(\mathbf{D}S_{N_t})^{k/2}$, $k = 2, 3, \dots$, yield (2.84). Here $\mathbf{D}S_{N_t}$ is defined by (2.79). □

Remark 2.7. Clearly,

$$\varGamma_k(N_t) = \frac{d^k}{i^k du^k} \ln f_{N_t}(u)\Big|_{u=0} = \lambda t, \qquad k = 1, 2, ...,$$

as according to (2.64), the characteristic function of the homogeneous Poisson process is $f_{N_t}(u) = \exp\{-\lambda t(1 - \exp\{iu\})\}$. Therefore, we can assert that conditions (1.33) and (1.35) hold with $\epsilon = 0$ and $K_1 = K_2 = 1$. So the use of Proposition 2.3 immediately gives that the upper bound of $\varGamma_k(S_{N_t})$ satisfies inequality (2.73) with $N := N_t$ being the homogeneous Poisson process and with $\varDelta_* := \varDelta_{*,t}$ which would be defined by (2.25), and in this instant holds with

$$\varDelta_{N_t} = \sqrt{\lambda t(\sigma^2 + \mu^2)}/L_1, \quad L_1 = 2\big(|\mu| \vee \big(1 \vee \sigma/(2|\mu|)\big)M\big) \textit{ as } \mu \neq 0,$$
$$\varDelta_{N_t,0} = \sigma\sqrt{\lambda t}/L_2, \qquad L_2 = 2M \textit{ as } \mu = 0.$$

Here $M = 2(K \vee \sigma)$.

We note that the upper estimate (2.84) for the *k*th-order cumulants of the standardized Poisson process (2.82) is more accurate then the one mentioned in Remark 2.7.

Corollary 2.6. *If the random variable X with variance $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$ and N_t, $t \geq 0$, is a homogeneous Poisson process, then the large deviation equalities* (2.51), *exponential inequality* (2.56), *and absolute error estimate* (2.57) *are valid with $\tilde{S}_N := \tilde{S}_{N_t}$ and $\varDelta_* := \Delta_t$, where $\tilde{S}_{N_t}$ and Δ_t are defined, respectively, by* (2.82) *and* (2.84).

Corollary 2.7. *Under the conditions of Corollary* 2.6, *the ratios*

$$\frac{1 - F_{\tilde{S}_{N_t}}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{S}_{N_t}}(-x)}{\Phi(-x)} \to 1$$

hold in both cases: when $\mu \neq 0$ and when $\mu = 0$, for $x \geq 0$, $x = o(t^{\nu(\gamma)/2})$, if $t \to \infty$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geq 0$.

The assertions of Corollary 2.6 follow directly from Proposition 2.3 and Corollary 2.4. The proof of Corollary 2.7 comes from the proof of Theorem 2.2 in the instance where $a_j \equiv 1$, $j = 1, 2, ...$ and $N := N_t$, $t \geq 0$, is a homogeneous Poisson process. Following the proof of Theorem 2.2 with $\tilde{S}_N := \tilde{S}_{N_t}$ and $\varDelta_* := \varDelta_t$, it is obvious that is enough to show that $\Delta_t \to \infty$

as $t \to \infty$, where Δ_t is defined by (2.84). In fact, we have $\varDelta_t \geq Ct^{1/2}$, where $C = \sqrt{\lambda(\mu^2 + \sigma^2)}/K > 0$. Thus, $\Delta_t \to \infty$ as $t \to \infty$.

Remark 2.8. Thanks to Proposition 2.3, application of Corollary 2.2 immediately leads us to conclude that in cases where $a_j \equiv 1$, $j = 1, 2, ...$, and $N := N_t$ is a homogeneous Poisson process, the upper estimate of the normal approximation to the distribution of $\tilde{S}_{N_t}$ in the Cramér zone ($\gamma = 0$) is

$$\sup_x |F_{\tilde{S}_{N_t}}(x) - \Phi(x)| \leq \frac{4.4K}{\sqrt{(\mu^2 + \sigma^2)\lambda t}}.$$

Note that the absolute constant 4.4 in this upper bound may be sharpened. Indeed, Korolev and Shevtsova (2012) presented sharpened upper bounds for the absolute constant in the Berry-Esseen inequality for Poisson and mixed Poisson random sums. (Korolev and Shevtsova 2012: 97) proved the following theorem for the case where $N := N_\lambda$ is a Poisson random variable with rate $\lambda > 0$:

Theorem 2.4. *(Korolev and Shevtsova 2012) Under $\beta^3 = \mathbf{E}|X|^3 < \infty$ for any $\lambda > 0$, the following inequality holds:*

$$\sup_x |F_{\tilde{S}_{N_\lambda}}(x) - \Phi(x)| \leq \frac{0.3041\beta^3}{(\mu^2 + \sigma^2)^{3/2}\sqrt{\lambda}}.$$

See this paper also for a references on a rather interesting history of the so-called Berry-Essen inequality and for the problem of establishing the best value of the absolute constant in it.

According to Theorem 2.4 and condition (B_0),

$$\sup_x |F_{\tilde{S}_{N_t}}(x) - \Phi(x)| \leq \frac{0.9123K}{\sqrt{(\mu^2 + \sigma^2)\lambda t}}, \quad K, \sigma, \lambda > 0, \ t \geq 0.$$

2.4.4. Large deviation theorems for compound mixed Poisson processes

Suppose that $N_t := N'_{\Lambda(t)}$, $t > 0$, is a mixed Poisson process with the distribution defined by (2.66) (see Example 2.2). Here the mean value function $\Lambda(t)$ is a general random process with non-decreasing sample paths, independent of the standard Poisson process N' (see Definition 2.3). Let as recall that the most well-known and most widely used mixed Poisson distribution is the negative binomial distribution with the following probability and parameters $0 < p < 1, n_t > 0$:

$$q_s = \frac{\Gamma(n_t + s)}{s!\Gamma(n_t)} p^{n_t}(1-p)^s, \qquad p = \frac{b_t}{1+b_t}, \quad s \in \mathbb{N}_0.$$

In this case, $\Lambda(t)$ is distributed according to the Gamma law with positive parameters $(n_t,\ b_t)$ and the density function defined by (2.67). Thus,

$$\alpha_t = \mathbf{E}\Lambda(t) = n_t/b_t, \qquad \hat{\beta}_t^2 = \mathbf{D}\Lambda(t) = n_t/b_t^2. \tag{2.86}$$

As previously mentioned, N_t is called a negative binomial or Pólya process and is often used in insurance and other dynamic population models. If n_t is a positive integer, then the negative binomial distribution is called Pascal distribution, and in the case where $n_t = 1$ it is called a geometric distribution. According to (2.86) and (2.61)–(2.63),

$$\mathbf{E}N_t = \alpha_t, \qquad \beta_t^2 = \mathbf{D}N_t = \alpha_t + \hat{\beta}_t^2 = \alpha_t(1 + \alpha_t/n_t) > \alpha_t. \tag{2.87}$$

A complete list of references for work on the bounds of the tails of compound Cox or mixed Poisson processes, and compound negative binomial distributions would be overly long, and to the best of our knowledge there are no papers on normal approximation taking into account large deviations for the aforementioned compound distributions when the cumulant method is used. Thus, without elaborating, we refer, e. g., to (Bening and Korolev 2002; Cai and Garrido 2000; Embrechts *et al.* 1985; Frolov 2009; Gordon and Xiaodong 1997; Grändell 1997; Kong and Shen 2009) and the references therein.

Let us consider the compound mixed Poisson process S_{N_t} denoted by (2.77) with N_t being a mixed Poisson process for each $t > 0$, independent of the i. i. d. random variables $\{X,\, X_j,\, j = 1, 2, ...\}$ and with the mean, variance, and distribution function that are denoted by (1.4). Based on (2.72) and (2.87),

$$\mathbf{E}S_{N_t} = \mu\alpha_t, \qquad \mathbf{D}S_{N_t} = \hat{\beta}_t^2\mu^2 + \alpha_t(\sigma^2 + \mu^2), \tag{2.88}$$

where α_t and $\hat{\beta}_t^2$ are defined by (2.86). Our aim is to consider large deviations in both the Cramér and the power Linnik zones for the distribution of $\tilde{S}_{N_t}$ that is denoted by (2.82), with N_t being a mixed Poisson process.

Recall that for this purpose, the upper estimates for the *k*th-order cumulants of N_t are first required.

Proposition 2.4. *Assume that* $\Lambda(t) > 0,\ t > 0,$ *is distributed according to* *the Gamma law* (2.67) *with the parameters* $n_t > 0$*,* $0 < b_t \leq 1$*. Then for the kth-order cumulants of the mixed Poisson process* N_t*, the upper estimate holds:*

$$\varGamma_k(N_t) \leq (k-1)!\frac{n_t}{2}\Big(\frac{2}{b_t}\Big)^k, \qquad k = 1, 2, \dots . \tag{2.89}$$

Proof of Proposition 2.4. Pursuant to (2.67), the characteristic function for $\Lambda(t)$ is

$$f_{\Lambda(t)}(u) = \mathbf{E}e^{iu\Lambda(t)} = (1 - iu/b_t)^{-n_t}, \qquad u \in \mathbb{R}. \tag{2.90}$$

From this, the definition (1.5) of the *k*th-order cumulants leads to

$$\varGamma_k(\Lambda(t)) = \frac{1}{i^k}\frac{d^k}{du^k}\ln f_{\Lambda(t)}(u)\Big|_{u=0} = (k-1)!n_t/b_t^k, \quad k = 1, 2, \dots . \tag{2.91}$$

Further, in view of (2.68) together with (2.90),

$$f_{N_t}(u) = \mathbf{E}e^{iuN_t} = \Big(\frac{p}{(1-(1-p)e^{iu})}\Big)^{n_t} = f_{\Lambda(t)}\Big(\frac{1}{i}\ln f_{N_1}(u)\Big), \tag{2.92}$$

where $0 < p < 1$ is defined by (2.68), and N_1 is distributed according to the Poisson law with the unit parameter. It is clear that $f_{N_1}(u) = \exp\{\exp\{iu\} - 1\}$, by (2.60). Consequently, $\varGamma_k(N_1) = 1$, $k = 1, 2, \dots$. Thus based on Lemma 2.1, using (1.5) together with (2.92), we have that the upper estimate for the *k*th-order cumulants of the mixed Poisson process N_t is

$$\begin{aligned}\varGamma_k(N_t) &= k!\sum\nolimits_1^{*}\frac{\varGamma_m(\Lambda(t))}{m_1!\cdot\ldots\cdot m_k!}\prod_{j=1}^{k}\Big(\frac{1}{j!}\Big)^{m_j} \\ &= \sum_{l=0}^{k-1} c_{k-l}^{(k)}\varGamma_{k-l}(\Lambda(t)), \qquad k = 1, 2, \dots . \end{aligned}\tag{2.93}$$

where $\sum_1^*$ is the summation over all the non-negative integer solutions $0 \leq m_1, \dots, m_k \leq k$ of the equation (2.9), $m_1 + \dots + m_k = m$, and $1 \leq m \leq k$. The integers $c_j^{(k)} \geq 1$, $j = 1, 2, \dots, k$, are Stirling numbers of the second kind that may be determined, e. g., from

$$c_{k-l}^{(k)} = k!\sum\nolimits_1^{**}\frac{1}{m_1!\cdot\ldots\cdot m_k!}\prod_{j=1}^{k}\Big(\frac{1}{j!}\Big)^{m_j}, \qquad l = 0, 1, \dots, k-1, \tag{2.94}$$

where $\sum_1^{**}$ is the same summation as $\sum_1^*$ but choosing $m = k - l$, $0 \leq l \leq$

$k-1$. For instance,

$$\begin{aligned} c_1^{(k)} &= c_k^{(k)} = 1, \qquad k = 1,2,..., \\ c_{k-1}^{(k)} &= k(k-1)/2, \qquad c_2^{(k)} = 2^{k-1}-1, \qquad k = 2,3,\dots . \end{aligned}$$

Substituting (2.91) into (2.93) and by noting that $(1/b_t)^{k-l} \le (1/b_t)^k$ as $b_t \le 1$, $0 \le l \le k-1$, $k = 1,2,...$, leads to (2.89), since

$$\sum_{l=0}^{k-1} c_{k-l}^{(k)}(k-l-1)! \le (k-1)!2^{k-1}, \qquad k = 1,2,\dots . \tag{2.95}$$

□

Proposition 2.5. *If the random variable X with variance $0 < \sigma^2 < \infty$ fulfills condition $(\bar{B}_\gamma)$ and the mixed Poisson process N_t, $t > 0$ with the probability* (2.68) *satisfies condition* (2.89)*, then*

$$|\varGamma_k(\tilde{S}_{N_t})| \le \frac{(k!)^{1+\gamma}}{\varDelta_{*,t}^{k-2}}, \quad \varDelta_{*,t} = \frac{b_t\sqrt{\mathbf{D}S_{N_t}}}{L_j}, \; j = 1,2, \; k = 3,4,... \tag{2.96}$$

when $0 < b_t \le 1$. *Here*

$$L_1 = 2\Big(2|\mu| \vee \Big(1 \vee \frac{\sigma}{2|\mu|}\Big)M\Big) \text{ when } \mu \ne 0, \; L_2 = 2M \text{ when } \mu = 0, \tag{2.97}$$

where $\mathbf{D}S_{N_t}$ and $M > 0$ are defined, respectively, by (2.88) *and* (2.1)*. In case where $\mu = 0$, $\mathbf{D}S_{N_t}$ and M stand with $\sigma^2 = \mathbf{E}X^2$.*

Proof of Proposition 2.5. First let us consider the case where $\mu \ne 0$. Based on (2.32) in the instance when $T_{N,2} := N_t$ together with (2.1) and (2.89), we have

$$\begin{aligned} |\varGamma_k(S_{N_t})| \;\le\; & (k!)^{1+\gamma}M^{k-2}\alpha_t\sigma^2 + k!n_t \sum\nolimits_2^* \frac{(\tilde{m}-1)!}{m_1!\cdot\ldots\cdot m_{k-1}!}\frac{2^{\tilde{m}-1}}{b_t^{\tilde{m}}} \\ \cdot \; & |\mu|^{m_1}\prod_{j=2}^{k-1}((j!)^\gamma M^{j-2}\sigma^2)^{m_j}, \qquad k = 2,3,..., \end{aligned} \tag{2.98}$$

where $\sum_2^*$ is the summation over all the non-negative integer solutions $0 \le m_1,...,m_{k-1} \le k$ of the equation (2.35) with $m_1 + ... + m_{k-1} = \tilde{m}$, and $2 \le \tilde{m} \le k$. Clearly, $n_t/b_t^{\tilde{m}} = \hat{\beta}_t^2/b_t^{\tilde{m}-2} < \beta_t^2/b_t^{\tilde{m}-2}$ as $\beta_t^2 > \hat{\beta}_t^2$. Here $\hat{\beta}_t^{\,2}$ and β_t^2

are defined, respectively, by (2.86) and (2.87). Consequently, recalling (2.39)–(2.41) with $a = \sup\{a_j, j = 1, 2, ...\} \equiv 1$ and $\bar{a} = \inf\{a_j, j = 1, 2, ...\} \equiv 1$, we arrive at the estimate of (2.98):

$$|\Gamma_k(S_{N_t})| \leq (k!)^{1+\gamma} \mathbf{D}S_{N_t}(L_1/b_t)^{k-2}, \qquad k = 2, 3, ..., \tag{2.99}$$

since $0 < b_t \leq 1$. Here $\mathbf{D}S_{N_t}$ and L_1 are defined by (2.88) and (2.97), respectively.

Now let us consider the case where $\mu = 0$, supposing that $0^0 = 1$. Taking into consideration (2.32) (in the instance when $T_{N,2} := N_t$) along with (2.1) and (2.89) gives us

$$\begin{aligned} |\Gamma_k(S_{N_t})| \quad &\leq \quad k!\alpha_t \sum\nolimits_3^* \frac{(\bar{m}-1)!}{m_2! \cdot ... \cdot m_k!} \Big(\frac{2}{b_t}\Big)^{\bar{m}-1} \\ &\cdot \quad \prod_{j=2}^{k} ((j!)^\gamma \sigma^2 M^{j-2})^{m_j}, \qquad k = 2, 3, ..., \end{aligned} \tag{2.100}$$

where α_t is denoted by (2.87), and $\sum_3^*$ is the summation over all the non-negative integer solutions $0 \leq m_2, ..., m_k \leq k$ of the equation (2.45), where $m_2 + ... + m_k = \bar{m}$ and $1 \leq \bar{m} \leq k$. The inequality (2.100) together with (2.47) and (2.48) (in the instance where $a \equiv 1$) ensures that

$$|\Gamma_k(S_{N_t})| \leq (k!)^{1+\gamma} \mathbf{D}S_{N_t}(L_2/b_t)^{k-2}, \qquad k = 2, 3, ... \tag{2.101}$$

when $0 < b_t \leq 1$ and $\sigma \leq M/2$. Here $\mathbf{D}S_{N_t}$ and $M > 0$ stand with $\mu = 0$, and L_2 is defined by (2.97). Consequently, the inequality $\Gamma_k(\tilde{S}_{N_t}) = \Gamma_k(S_{N_t})/(\mathbf{D}S_{N_t})^{k/2}$, $k = 2, 3, ...$, together with (2.99) and (2.101) leads to (2.96). □

Since the kth-order upper estimates of $|\Gamma_k(\tilde{S}_{N_t})|$ have been derived, applying Corollary 2.4 immediately confirms the assertion of Corollary 2.8.

Corollary 2.8. *If the random variable X with variance $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$ and mixed Poisson process N_t, $t > 0$, with the probability* (2.68) *fulfills condition* (2.89)*, then the large deviation equalities* (2.51)*, exponential inequality* (2.56)*, and absolute error estimate* (2.57) *are valid with $\tilde{S}_N := \tilde{S}_{N_t}$, $\Delta_* := \Delta_{*,t}$, where $\tilde{S}_{N_t}$ is defined by* (2.82) *with N_t being a mixed Poisson process, and $\Delta_{*,t}$ is defined by* (2.96).

Assume that $n_t b_t \to \infty$ as $t \to \infty$.

Corollary 2.9. *Under the conditions of Corollary* 2.8*, the ratios*

$$\frac{1 - F_{\tilde{S}_{N_t}}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{S}_{N_t}}(-x)}{\Phi(-x)} \to 1 \tag{2.102}$$

hold in both cases: $\mu \neq 0$ and $\mu = 0$, for $x \geq 0$, $x = o((n_t b_t)^{\nu(\gamma)/2})$ as $n_t b_t \to \infty$ as $t \to \infty$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geq 0$.

The assertion of Corollary 2.9 is derived from the proof of Theorem 2.2, in the instance where $a_j \equiv 1$, $j = 1, 2, ...$ and $N := N_t$, $t > 0$, is a mixed Poisson process with the probability (2.68).
Really, examining the proof of Theorem 2.2 with $Z_N := S_{N_t}$ and $\Delta_* := \Delta_{*,t}$, and letting $\epsilon \equiv 1$, it is obvious that is enough to show that $\Delta_{*,t} \to \infty$ as $n_t b_t \to \infty$, where $\Delta_{*,t}$ is defined by (2.96).
Recalling the definitions of $\mathbf{D}S_{N_t}$ and $\Delta_{*,t}$, by (2.88) and (2.96), respectively, and noting that $\beta_t^2 \geq 2\alpha_t$ as $\hat{\beta}_t^2 \geq \alpha_t$ and $0 < b_t \leq 1$, we have $\varDelta_{N_{*,t}} = \sqrt{\mathbf{D}S_{N_t}}/L_j \geq C_j(n_t b_t)^{1/2}$, $j = 1, 2$, where $C_1 = \sqrt{2\mu^2 + \sigma^2}/L_1 > 0$, $C_2 = \sigma/L_2 > 0$. Thus, supposing that $n_t b_t \to \infty$ as $t \to \infty$, we have that in this instance $\Delta_{*,t} \to \infty$.

Note that it follows, from Corollary 2.9 that whether ratios (2.102) hold depends on the parameters $n_t > 0$ and $b_t > 0$ of Gamma distribution (2.67).

Now we will consider some instances. In order to follow Corollary 2.3 directly, we should derive conditions for the *k*th-order cumulants of the mixed Poisson process, that are similar to (1.33) and (1.35). The relations (2.103) and (2.105) described in the following remark can be used for this purpose.

Remark 2.9. Assume that the mean value function $\Lambda(t)$, $t > 0$, is distributed according to the Gamma law (2.67) with the parameters $n_t > 0$ and $0 < b_t \leq 1$ related by

$$n_t := \Big(\frac{n}{b_t^{2(1-\bar{\epsilon})}}\Big)^{1/\bar{\epsilon}}, \qquad n \in \mathbb{N}, \quad 0 < \bar{\epsilon} \leq 1, \tag{2.103}$$

then

$$\varGamma_k(N_t) \leq 2(k-1)!\Big(\frac{2}{\sqrt{n}}\Big)^{k-2}(\beta_t^2)^{1+\bar{\epsilon}(k-2)/2}, \qquad k = 2, 3, \dots . \tag{2.104}$$

If the parameters $n_t > 0$, $0 < b_t \leq 1$ are related by

$$n_t := \Big(\frac{n}{b_t^{1-\epsilon}}\Big)^{1/\epsilon}, \qquad n \in \mathbb{N},\ 0 < \epsilon \leq 1, \tag{2.105}$$

then

$$\Gamma_k(N_t) \leq (k-1)!\Big(\frac{2}{n}\Big)^{k-1}\alpha_t^{1+\epsilon(k-1)}, \qquad k=1,2,\dots . \tag{2.106}$$

Indeed, by virtue of $\hat{\beta}_t^2$ as defined by (2.86), we can rewrite condition (2.89) in the following way:

$$\Gamma_k(N_t) \leq 2(k-1)!(2/\sqrt{n_t})^{k-2}(\hat{\beta}_t^2)^{1+(k-2)/2},\ k=1,2,\dots . \tag{2.107}$$

Since relation (2.103) holds,

$$\hat{\beta}_t/\sqrt{n_t} = 1/b_t = \hat{\beta}_t^{\bar{\epsilon}}/\sqrt{n}, \qquad n\in\mathbb{N}, \quad 0<\bar{\epsilon}\leq 1, \tag{2.108}$$

$$\alpha_t = (n/b_t^{2-\bar{\epsilon}})^{1/\bar{\epsilon}},\ \hat{\beta}_t^2 = (n/b_t^2)^{1/\bar{\epsilon}},\ \beta_t^2 = \hat{\beta}_t^2\big(1+\sqrt{n}/\hat{\beta}_t^{\bar{\epsilon}}\big). \tag{2.109}$$

Consequently, (2.107) and (2.108), noting that $\hat{\beta}_t^2 < \beta_t^2$, lead to (2.104).

The second part of Remark 2.9 follows in a similar way. Let us rewrite condition (2.89) as follows:

$$\Gamma_k(N_t) \leq (k-1)!(2/n_t)^{k-1}\alpha_t^{1+(k-1)}, \qquad k=1,2,\dots \tag{2.110}$$

using α_t which is defined by (2.86). Since relation (2.105) holds, we have

$$\alpha_t/n_t = 1/b_t = \alpha_t^{\epsilon}/n, \qquad n\in\mathbb{N}, \quad 0<\epsilon\leq 1, \tag{2.111}$$

$$\alpha_t = (n/b_t)^{1/\epsilon}, \quad \hat{\beta}_t^2 = (n/b_t^{1+\epsilon})^{1/\epsilon}, \quad \beta_t^2 = \alpha_t\big(1+\alpha_t^{\epsilon}/n\big). \tag{2.112}$$

To complete the proof of (2.106), it suffices to use (2.110) and (2.111).

Remark 2.10. It follows from (2.104) and (2.106) that N_t, $t>0$ satisfies condition (1.35) with $K_1 := 2/\sqrt{n}$ and $\epsilon := \bar{\epsilon}/2$ such that $0<\bar{\epsilon}\leq 1$, and it also satisfies condition (1.33) with $K_2 := 2/n$ and $0<\epsilon\leq 1$, $n\in\mathbb{N}$. Thus, clearly, applying Corollary 2.3 directly shows that the majorating upper estimate for the *k*th-order cumulants of a standardized compound mixed Poisson process $\tilde{S}_{N_t}$ satisfies inequality (2.73) with $N := N_t$, where Δ_* is defined by (2.25) with

$$\Delta_{N_t} = \frac{\sqrt{\mathbf{D}S_{N_t}}}{L_{N_t}},\ \ L_{N_t} = 2\Big(2|\mu|\frac{\beta_t^{\bar{\epsilon}}}{\sqrt{n}} \vee \Big(1\vee\frac{\sigma}{2|\mu|}\Big)M\Big) \ \textit{as}\ \mu\neq 0, \tag{2.113}$$

$$\Delta_{N_t,0} = \frac{\sqrt{\mathbf{D}S_{N_t}}}{L_{N_t,0}},\ \bar{L}_{N_t,0} = \frac{2\alpha_t^{\epsilon}}{nM}\ \textit{as}\ \mu=0. \tag{2.114}$$

Here $\mathbf{D}S_{N_t}$, $\beta_t^{\bar{\epsilon}}$, and α_t^{ϵ} are defined by (2.88), (2.109), and (2.112), respectively, and $0 < \bar{\epsilon} \leq 1$, $0 < \epsilon \leq 1$, and $M > 0$ are defined by conditions (2.103), (2.105), and (2.1), respectively. In the case where $\mu = 0$, $\mathbf{D}S_{N_t}$ and M stand with $\sigma^2 = \mathbf{E}X^2$.

Obviously, to obtain accurate upper estimates for $|\Gamma_k(\tilde{S}_{N_t})|$, $k = 3, 4, ...$, in both cases: when $\mu \neq 0$ and when $\mu = 0$, it is enough to use one of the conditions (2.104) or (2.106).

Remark 2.11. In accordance with the previous remarks, it is clear, based on the proof of Corollary 2.9 and assuming that condition (2.104) or (2.106) holds and $n \in \mathbb{N}$ is fixed, the convergence of large deviation ratios to a unit (2.102) would be valid in both: when $\mu \neq 0$ and when $\mu = 0$, for $x \geq 0$: $x = o((1/b_t)^{\nu(\gamma)(1-2\epsilon)/(2\epsilon)})$ with $0 < \epsilon < 1/2$ in the case where relation (2.105) holds, and for $x = o((1/b_t)^{\nu(\gamma)(2-3\epsilon)/(2\epsilon)})$ with $0 < \bar{\epsilon} < 2/3$ in the case where relation (2.103) holds, as $b_t \to 0$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geq 0$.

Example 2.4. Assume that $b_t = 1/t \leq 1$. Then, by (2.105), (2.111), and (2.112),

$$n_t = (nt^{1-\epsilon})^{1/\epsilon}, \qquad \alpha_t = (nt)^{1/\epsilon}, \qquad \hat{\beta}_t^2 = (nt^{1+\epsilon})^{1/\epsilon}.$$

And according to (2.103), (2.108), and (2.109),

$$n_t = (nt^{2(1-\bar{\epsilon})})^{1/\bar{\epsilon}}, \qquad \alpha_t = (nt^{2-\bar{\epsilon}})^{1/\bar{\epsilon}}, \qquad \hat{\beta}_t^2 = (nt^2)^{1/\bar{\epsilon}}.$$

Obviously, based on Remark 2.11, the convergence of ratios to a unit (2.102) will hold as $t \to \infty$.

Remark 2.12. If $b_t = 1/t \leq 1$, $\epsilon = \bar{\epsilon} = 1$, then according to both relations: (2.105) and (2.103), gives the same result

$$n_t = n, \qquad \alpha_t = nt, \qquad \hat{\beta}_t^2 = nt^2, \qquad \beta_t^2 = nt^2(1 + t^{-1}). \tag{2.115}$$

Obviously, in this instance, due to Remark 2.11, the convergence of ratios to a unit (2.102) are not valid in the case where $t \to \infty$. However, if we would suppose that t is fixed, and $n_t = n \to \infty$, then according to Propositions 2.4, 2.5 and the proof of Corollary 2.9, we can immediately derive that (2.102) hold for $x \geq 0$ such that $x = o(n^{\nu(\gamma)/2})$ as $n \to \infty$.

Let us note that the aforementioned assertion also can be verified by using directly the Corollary 2.3. For that it is enough to observe that conditions (1.35), (1.33) hold with $K_1 = K_2 = 2t$ and $\epsilon = 0$ in instance where $n_t = n$, as (2.89) can be rewritten in the following ways:

$$\mathit{\Gamma}_k(N_n) \leq (k-1)!2^{k-1}nt^k = (k-1)!(2t)^{k-1}\alpha_n, \qquad k = 1, 2, ...,$$

on the other hand,

$$\mathit{\Gamma}_k(N_n) \leq (k-1)!2^{k-1}nt^k = 2(k-1)!(2t)^{k-2}\hat{\beta}_n^2, \qquad k = 1, 2, ...$$

when $t \geq 1$. Here $N_t := N_n$ is mixed Poisson process with the parameters $b_t = 1/t$, $n_t = n$, where t is fixed. $\alpha_t := \alpha_n$ and $\hat{\beta}_t^2 := \hat{\beta}_n^2$ are defined by (2.115).

2.4.5. Large deviation theorems for random binomial sums

Let us consider the random sum (1.30) assuming that $N := N_n$, $n \in \mathbb{N}_0$, is distributed according to the binomial law, with the probability

$$q_s = C_n^s \bar{p}^s (1-\bar{p})^{n-s}, \qquad C_n^s = \frac{n!}{s!(n-s)!}, \quad 0 < \bar{p} < 1, \ s \in \mathbb{N}_0.$$

In addition,

$$\alpha = n\bar{p}, \qquad \beta^2 = n\bar{p}(1-\bar{p}). \tag{2.116}$$

Let us recall that N_n is independent of the i. i. d. random variables $\{X, X_j, j = 1, 2, ...\}$. Thus, according to (2.72) and (2.116), we have

$$\mathbf{E}S_{N_n} = \mu n\bar{p}, \qquad \mathbf{D}S_{N_n} = n\bar{p}(\sigma^2 + \mu^2(1-\bar{p})). \tag{2.117}$$

Proposition 2.6. *Assume that N_n is distributed according to the binomial law with the probability* (2.70) *and parameters* $0 < \bar{p} < 1$, $n \in \mathbb{N}$. *Then*

$$|\mathit{\Gamma}_k(N_n)| \leq (k-1)!2^{k-1}n\bar{p}, \qquad k = 1, 2, ... \,. \tag{2.118}$$

Proof of Proposition 2.6. According to (2.70), the characteristic function of N_n is

$$f_{N_n}(u) = \mathbf{E}e^{iuN_n} = (1-(1-\bar{p})e^{iu})^n, \qquad u \in \mathbb{R}.$$

Thus based on Lemma 2.1 and using (1.5), we get

$$|\mathit{\Gamma}_k(N_n)| \leq \mathit{\Gamma}_k(N_t) = \sum_{l=0}^{k-1} c_{k-l}^{(k)} \mathit{\Gamma}_{k-l}(\Lambda((n)) \leq (k-1)!2^{k-1}n\bar{p}, \ k = 1, 2, ...,$$

due to (2.91), (2.93), and (2.95), where is assumed that $\Lambda(n)$ is distributed according to the Gamma law (2.67) with the parameters $n_t := n$, $b_t := 1/\bar{p} > 1$, and N_t, $t > 0$, is mixed Poisson process with the probability (2.68). Here the $c^{(k)}_{k-l}$ are defined by (2.94). □

Remark 2.13. It follows from (2.118) that N_n satisfies conditions (1.33), (1.35) with $\epsilon = 0$, $K_1 = 8/(1-\bar{p})$ and $K_2 = 2$.

Proposition 2.7. *If the random variable X with variance $0 < \sigma^2 < \infty$ fulfills condition $(\bar{B}_\gamma)$, and the binomial random variable N_n, $n \in \mathbb{N}$, satisfies condition* (2.118)*, then*

$$|\mathit{\Gamma}_k(\tilde{S}_{N_n})| \leq \frac{(k!)^{1+\gamma}}{\mathit{\Delta}^{k-2}_{*,n}}, \quad \mathit{\Delta}_{*,n} = \frac{\sqrt{\mathbf{D}S_{N_n}}}{L_j}, \; j = 1, 2, \; k = 3, 4, ..., \quad (2.119)$$

where

$$L_1 = \frac{2}{1-\bar{p}}\Big(2|\mu| \vee \Big(1 \vee \frac{\sigma}{2|\mu|}\Big)M\Big) \text{ as } \mu \neq 0, \; L_2 = 2M \text{ as } \mu = 0, \quad (2.120)$$

where $0 < \bar{p} < 1$. Here $\mathbf{D}S_{N_n}$ and $M > 0$ are defined, respectively, by (2.117) *and* (2.1)*. In the case where $\mu = 0$, $\mathbf{D}S_{N_n}$ and $M > 0$ hold with $\sigma^2 = \mathbf{E}X^2$.*

Proof of Proposition 2.7. Following the proof of Proposition 2.5 with $n_t := n$, $b_t := 1/\bar{p} > 1$ and noting that $1/(1-\bar{p}) \leq 1/(1-\bar{p})^{k-2}$, $k = 2, 3, ...$, we can see that (2.119) holds. □

Corollary 2.10. *If the random variable X with variance $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$, and the binomial random variable N_n, $n \in \mathbb{N}_0$ fulfills condition* (2.118)*, then the large deviation equalities* (2.51)*, exponential inequality* (2.56)*, and absolute error estimate* (2.57) *are valid with $\tilde{Z}_N := \tilde{S}_{N_n}$ and $\Delta_* := \Delta_{*,n}$, where $\tilde{S}_{N_n}$ is defined by* (1.34) *with $N := N_n$ being a binomial random variable, and $\Delta_{*,n}$ is defined by* (2.119).

Corollary 2.11. *Under the conditions of Corollary* 2.10*,*

$$\frac{1 - F_{\tilde{S}_{N_n}}(x)}{1-\Phi(x)} \to 1, \qquad \frac{F_{\tilde{S}_{N_n}}(-x)}{\Phi(-x)} \to 1 \quad (2.121)$$

hold in both cases: when $\mu \neq 0$ and when $\mu = 0$, for $x \geq 0$, $x = o(n^{\nu(\gamma)/2})$, if $n \to \infty$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geq 0$.

The statement of Corollary 2.11 follows from the proof of Theorem 2.2, considering the instance where $a_j \equiv 1, j = 1, 2, ...$ and $N := N_n$ is a binomial random variable. Since $\Delta_{*,n} = \sqrt{\mathbf{D}S_{N_n}}/L_j \geq \sqrt{n}C_j$, $j = 1, 2$, where $C_1 = \sqrt{\bar{p}(\sigma^2 + \mu^2(1-\bar{p}))}/L_1 > 0$, $C_2 = \sqrt{\bar{p}}\sigma/L_2 > 0$, then $\Delta_{*,n} \to \infty$ as $n \to \infty$. Thus it yields the statement of Corollary 2.11, due to the proof of the Theorem 2.2.

2.4.6. Discounted version of large deviations

Let us consider the discounted version of large deviations, i. e., let us assume that $a_j \equiv v^j$, $0 < v < 1$, $j = 1, 2, ...$. In this case we consider the random sum

$$Z_N = \sum_{j=1}^{N} v^j X_j, \qquad Z_0 = 0. \tag{2.122}$$

Recall that $\{X, X_j, j = 1, 2, ...\}$ is a family of i. i. d. random variables with the mean, positive variance and distribution function

$$\mu = \mathbf{E}X, \qquad \sigma^2 = \mathbf{D}X < \infty, \qquad F_X(x) = \mathbf{P}(X < x), \quad x \in \mathbb{R}.$$

In addition, the non-negative integer-valued random variable $N$with the mean, variance and distribution

$$\alpha = \mathbf{E}X, \qquad \beta^2 = \mathbf{D}X, \qquad \mathbf{P}(N = s) = q_s, \qquad s \in \mathbb{N}_0,$$

is independent of $\{X, X_j, j = 1, 2, ...\}$.

In the case where $a_j \equiv v^j$, $0 < v < 1$,

$$T_{N,r} = \sum_{j=1}^{N} v^{jr} = \frac{v^r(1-v^{rN})}{1-v^r}, \; T_{s,r} = \sum_{j=1}^{s} v^{jr} = \frac{v^r(1-v^{rs})}{1-v^r}, \tag{2.123}$$

where $r, s \in \mathbb{N}$. Is is assumed that $T_{0,r} = 0$. Clearly, $T_{N,0} = N$. Thus,

$$\mathbf{E}T_{N,r} = \frac{v^r}{1-v^r}(1-\mathbf{E}v^{rN}), \; \mathbf{E}T_{N,r}^s = \frac{v^{rs}}{(1-v^r)^s}\mathbf{E}(1-v^{rN})^s, \tag{2.124}$$

$$\mathbf{D}T_{N,r} = \frac{v^{2r}}{(1-v^r)^2}\mathbf{D}v^{rN}, \tag{2.125}$$

where

$$\mathbf{E}v^{rN} = \sum_{s=0}^{\infty} q_s v^{rs}, \qquad \mathbf{E}(1-v^{rN})^s = \sum_{s=0}^{\infty} q_s(1-v^{rs})^s, \quad (2.126)$$

$$\mathbf{D}v^{rN} = \mathbf{E}v^{2Nr} - (\mathbf{E}v^{Nr})^2. \quad (2.127)$$

Consequently, conditions (L) and (L_0), the mean (2.19) and the variance (2.20) of Z_N hold with $\mathbf{E}T_{N,1}$, $\mathbf{E}T_{N,2}$ and $\mathbf{D}T_{N,1}$ defined by (2.124) – (2.127) with $r = 1, 2$.

For the following, recall the abbreviation $(b \vee c) = \max\{b, c\},\ b,\ c \in \mathbb{R}$. And note that the statement of the thereunder Corollary 2.12 immediately follows from the proof of Lemma 2.2 in case where $a_j \equiv v^j$, $0 < v < 1$, $j = 1, 2, ...$, and the quantity $a/\bar{a}$ is replaced by $1 + v$.

Corollary 2.12. *Assume that the random variable X with variance $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$ and that the compound random variables $T_{N,1}$ and $T_{N,2}$ defined by (2.123) satisfy conditions, respectively, (L) and (L_0) with $\mathbf{D}T_{N,1}$, $\mathbf{E}T_{N,2}$ defined by (2.124)–(2.127). Then the upper estimates for the kth-order cumulants of $\tilde{Z}_N$ defined by (1.37) with $a_j \equiv v^j$, $0 < v < 1$, $j = 1, 2, ...$, satisfy inequality (2.24), where Δ_* is defined by (2.25) with*

$$\varDelta_N = \frac{\sqrt{\mathbf{D}Z_N}}{L_N},$$

$$L_N = 2(1+v)^2\Big((1+v)K_1|\mu|(\mathbf{D}T_{N,1})^{\epsilon} \vee \Big(1 \vee \frac{\sigma}{2|\mu|}\Big)vM\Big), \quad (2.128)$$

$$\varDelta_{N,0} = \frac{\sqrt{\mathbf{D}Z_N}}{L_{N,0}}, \quad L_{N,0} = 2(1 \vee K_2(\mathbf{E}T_{N,2})^{\epsilon})(1/2 \vee v)M. \quad (2.129)$$

Here $M = 2(K \vee \sigma)$, $K,\ \sigma > 0$, and $\mathbf{D}Z_N$, K_1, K_2, $\epsilon \geq 0$ are defined by (2.20), (L), (L_0). $\mathbf{D}Z_N$, $M > 0$ have $\sigma^2 = \mathbf{E}X^2$ in the case where $\mu = 0$.

Consequently, it immediately follows that Theorems 2.1–2.3 and Corollaries 2.1, 2.2 (see, Section 2.3: 48) hold with the same Δ_*, where $\varDelta_N$ and $\varDelta_{N,0}$ are defined, respectively, by (2.128) and (2.129).

Corollary 2.13. *Under the conditions of Theorem 2.1, the ratios*

$$\frac{1 - F_{\tilde{Z}_N}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{Z}_N}(-x)}{\Phi(-x)} \to 1 \quad (2.130)$$

hold for $x \geq 0$ *defined by* (2.55) *with* (2.124)–(2.127)*, if* $v \to 1$*, and* $\beta \to \infty$ *or* $\alpha \to \infty$ *(depending on the case, either* $\mu \neq 0$ *or* $\mu = 0$*) when* $0 \leq \epsilon < 1/2$. *Here* $0 < v < 1$, $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geq 0$.

We remark that the use of L'Hospital's rule yields

$$\begin{aligned}
\lim_{v\to 1} \mathbf{D}T_{N,1} &= \lim_{v\to 1} \frac{v}{v-1}(\mathbf{D}v^N + \mathbf{E}Nv^{2N} - \mathbf{E}v^N\mathbf{E}Nv^N) \\
&= \lim_{v\to 1}\big(\mathbf{D}v^N + 3(\mathbf{E}Nv^{2N} - \mathbf{E}v^N\mathbf{E}Nv^N) + 2\mathbf{E}N^2v^{2N} \\
&\quad -\mathbf{E}v^N\mathbf{E}N^2v^N - (\mathbf{E}Nv^N)^2\big) = \beta^2, \\
\lim_{v\to 1} \mathbf{E}T_{N,2} &= \lim_{v\to 1}\big(\mathbf{E}Nv^{2N} + \mathbf{E}v^{2N} - 1\big) = \alpha.
\end{aligned}$$

With these, the statement of Corollary 2.13 follows immediately from the proof of Theorem 2.2 if we use the definitions of $\mathbf{E}T_{N,2}$ and $\mathbf{D}T_{N,1}$ given by (2.124) – (2.127) and assume that $v \to 1$ and either $\beta \to \infty$ or $\alpha \to \infty$ (depending on the case, either $\mu \neq 0$ or $\mu = 0$).

Remark 2.14. Assume $N = \infty$. Thus, according to (2.123), $T_{N,r} = v^r/(1 - v^r)$, $r \in \mathbb{N}$. And hence, it follows that

$$\mathbf{E}T_{N,r} = \frac{v^r}{1-v^r}, \qquad \Gamma_m(T_{N,r}) = 0, \qquad m = 2, 3, ..., \tag{2.131}$$

due to (2.10). Consequently,

$$\mathbf{E}Z_N = \frac{\mu v}{1-v}, \qquad \mathbf{D}Z_N = \frac{\sigma^2 v^2}{1-v^2}. \tag{2.132}$$

Next, based on (2.132) together with (2.33) and (2.1), we have

$$|\Gamma_k(\tilde{Z}_N)| \leq \frac{v^{k-2}|\Gamma_1(T_{N,2})||\Gamma_k(X)|}{(\sigma^2\mathbf{E}T_{N,2})^{k/2}} \leq \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \; \Delta = \frac{\sigma}{M\sqrt{1-v^2}}, \tag{2.133}$$

for $k = 3, 4, \ldots$.

On the other hand, since the characteristic function of the compound sum (2.122) is $f_{Z_N}(u) = \prod_{j=1}^{\infty} f_X(v^j u)$, the definition (1.5) of the kth-order cumulants gives us

$$\Gamma_k(Z_N) = \sum_{j=1}^{\infty} \frac{1}{i^k}\frac{d^k}{du^k} \ln f_X(v^j u)\Big|_{u=0} = \frac{v^k}{1-v^k}\Gamma_k(X), \quad k = 1, 2, \ldots .$$

Next, application of condition (2.1), leads to

$$|\Gamma_k(\tilde{Z}_N)| \leq (k!)^{1+\gamma} \frac{(1+v)(M\sqrt{1-v^2})^{k-2}}{(1+v+v^2+...+v^{k-1})\sigma^{k-2}} \leq \frac{(k!)^{1+\gamma}H}{\bar{\Delta}^{k-2}}, \quad (2.134)$$

for $k = 3, 4, ...$, where

$$\bar{\Delta} = \frac{\sigma}{M\sqrt{1-v^2}}, \qquad H = \frac{1+v}{1+v+v^2}.$$

Comparing expression (2.133) with expression (2.134), we see that the first holds with $H < 1$. Assume that $\bar{\Delta} \leq \Delta_v = \sigma/(M\sqrt{1-v})$ and $H \geq (1 + v + v^2)^{-1}$, then the upper estimate (2.134) will coincide with the estimate (10) presented in (Saulis and Deltuvienė 2006: 221).

Clearly, $\Delta \geq C/\sqrt{1-v^2}$, where $C = \sigma/M > 0$. Here Δ is defined by (2.133). Hence, the proof of Theorem 2.2 leads us to assert that the large deviation ratios to a unit ratios (2.54) hold for $x \geq 0$ such that $x = o((1 - v^2)^{\nu(\gamma)/2})$ as $v \to 1$.

Let us consider some example where N obeys concrete probability laws.

Remark 2.15. If N is a binomial random variable or is a homogeneous or mixed Poisson process, then equalities (2.19) and (2.20), conditions (L) and (L_0), and consequently Corollaries 2.12, 2.13 are valid with $\mathbf{E}T_{N,1}$, $\mathbf{E}T_{N,2}$, and $\mathbf{D}T_{N,1}$ defined by (2.124)–(2.127) with $r = 1, 2$, where the values of $\mathbf{E}v^{rN}$, $\mathbf{D}v^{rN}$ coincide for the distribution under consideration.

Example 2.5. Suppose that $N := N_t$, $t \geq 0$, is a homogeneous Poisson process with the linear mean value function $\Lambda(t) = \lambda t$, $t \geq 0$, for some $\lambda > 0$, and the distribution

$$q_s = \mathbf{P}(N_t = s) = e^{-\lambda t}(\lambda t)^s/s!, \qquad s \in \mathbb{N}_0, \qquad 0 < q_s < 1.$$

Then,

$$\mathbf{E}v^{rN_t} = e^{-\lambda t(1-v^r)}, \qquad \mathbf{D}v^{rN_t} = e^{-\lambda t(1-v^{2r})}\big(1 - e^{-\lambda t(1-v^r)^2}\big), \ r \in \mathbb{N}_0.$$

Hence, based on Corollary 2.13 and (2.78), we see that in both cases: $\mu = 0$ and $\mu \neq 0$, (2.130) hold if $v \to 1$ and $t \to \infty$.

Example 2.6. Let us consider a mixed Poisson process $N_t := N'_{\Lambda(t)}$, $t > 0$, where $\Lambda(t)$ is distributed according to the Gamma law with the parameters

$(n_t,\ b_t)$, $n_t,\ b_t > 0$, $t > 0$, and the density function (2.67). Recall that in this instance,

$$q_s = \frac{\Gamma(n_t+s)}{s!\Gamma(n_t)} p^{n_t}(1-p)^s, \qquad p = \frac{b_t}{1+b_t}, \qquad s \in \mathbb{N}_0.$$

N_t is distributed according to the negative binomial law with parameters $0 < p < 1$ and $n_t > 0$, defined by (2.68). Hence,

$$\begin{aligned}
\mathbf{E}v^{rN} &= \frac{p^{n_t}}{(1-v^r(1-p))^{n_t}}, \\
\mathbf{D}v^{rN} &= \frac{p^{n_t}}{(1-v^{2r}(1-p))^{n_t}} - \frac{p^{2n_t}}{(1-v^r(1-p))^{2n_t}}.
\end{aligned}$$

Thus, recalling (2.87) and that $p = b_t/(1+b_t)$, it follows immediately from Corollary 2.13 that in both cases: $\mu \neq 0$ and $\mu = 0$, (2.130) hold as $v \to 1$ and $n_t/b_t \to \infty$. For example, we can suppose that $n_t := n \in \mathbb{N}$ is fixed and $p \to 0$ or, for example, $b_t = 1/t \to 0$, $t > 0$, as $t \to \infty$. Obviously, supposing that $n_t := n$, and b_t is fixed, it is possible to show that the ratios (2.130) hold as $n \to \infty$.

Example 2.7. Now let as assume that $N := N_n$, $n \in \mathbb{N}_0$, is distributed according to the binomial law, with

$$q_s = C_n^s \bar{p}^s(1-\bar{p})^{n-s}, \qquad C_n^s = \frac{n!}{s!(n-s)!}, \quad 0 < \bar{p} < 1,\ s \in \mathbb{N}_0. \quad (2.135)$$

In this instance,

$$\mathbf{E}v^{rN} = (1-\bar{p}(1-v^r))^n,\ \mathbf{D}v^{rN} = (1-\bar{p}(1-v^{2r}))^n - (1-\bar{p}(1-v^r))^{2n}.$$

Thus, due to Corollary 2.13 and (2.116), it follows that in both cases: $\mu \neq 0$ and $\mu = 0$, (2.130) hold as $v \to 1$ and $n \to \infty$.

2.5. Conclusions of Chapter 2

1. By using combinatorial method, the suitable bound (2.24) for the cumulants of the standardized weighted random sum $\tilde{Z}_N$ which is defined by (1.37) is obtained. It was assumed that the i. i. d. random summands satisfy the generalized S. N. Bernstein's condition $(\bar{B}_\gamma)$, $\gamma > 0$,

and random variables $T_{N,1}$ and $T_{N,2}$ defined by (2.2) with $r = 1, 2$ satisfy the conditions (L) and (L_0). The suitable bound and general Lemmas 1.1, 1.2 lead to proof theorems of large deviations in both the Cramér and the power Linnik zones and exponential inequalities.

2. It follows from Remarks 2.3, 2.6, 2.14 that Theorems 2.1–2.3 and Corollaries 2.4, 2.5, 2.12, 2.13 of large deviations for the distribution function of the sum $\tilde{Z}_N$ (the cases where $a_j \equiv 1$, $j = 1, 2...$, and v^j, $0 < v < 1$ also include) can be regarded as refinements of the theorems of large deviations and exponential inequalities for the sums of non-random numbers of summands.
3. The results of the Chapter 2 lead us to large deviation theorems for the standardized compound and mixed Poisson processes (see Subsection 2.4.3, 2.4.4) that are largely used in insurance and finance mathematics.

3

Local limit theorem for compound Poisson process

In this chapter we assume that i. i. d. random variables $\{X,\, X_j,\, j = 1, 2, ...\}$ with the mean $\mathbf{E}X = \mu$, finite, positive variance $0 < \mathbf{D}X = \sigma^2 < \infty$ and the distribution function $F_X(x) = \mathbf{P}(X < x)$ for all $x \in \mathbb{R}$, satisfy the condition $(\bar{B}_\gamma)$ with $\gamma = 0$: there exist constant $K > 0$ such that

$$|\mathbf{E}(X-\mu)^k| \le k! K^{k-2}\sigma^2, \qquad k = 3, 4, \dots . \qquad (\bar{B}_0)$$

In addition, by virtue of Proposition 2.1, we take up the position that

$$|\varGamma_k(X)| \le k! M^{k-2}\sigma^2, \qquad M = 2\max\{\sigma, K\}, \quad k = 3, 4, \dots . \qquad (3.1)$$

Throughout, along with the condition $(\bar{B}_0)$ we assume that for X there exist the density function

$$p_X(x) = \frac{d}{dx}F_X(x)$$

such that

$$\sup_x p_X(x) \le A < \infty, \qquad A > 0. \qquad (D')$$

Let us recall the compound Poisson process

$$S_{N_t} = \sum_{j=1}^{N_t} X_j, \qquad S_0 = 0,$$

with the mean and variance

$$\mathbf{E}S_{N_t} = \mu\lambda t, \qquad \mathbf{D}S_{N_t} = \lambda t(\sigma^2 + \mu^2) > 0.$$

Here N_t, $t \geq 0$, is a homogeneous Poisson process with a linear mean value function $\Lambda(t) = \lambda t$, $t \geq 0$, for some $\lambda > 0$ (see Definition 2.2 in Subsection 2.4.1). In addition, with the mean $\alpha_t = \mathbf{E}N_t$, variance $\beta_t^2 = \mathbf{D}N_t$ and the distribution $\mathbf{P}(N_t = s) = q_s$:

$$\alpha_t = \beta_t^2 = \lambda t, \qquad q_s = e^{-\lambda t}(\lambda t)^s/s!, \qquad s \in \mathbb{N}_0, \quad 0 < q_s < 1.$$

It is assumed that N_t is independent of i. i. d. random variables $\{X, X_j, j = 1, 2, ...\}$.

Local limit theorems for Poisson random sums are available, e. g., in (Korolev and Zhukov 2000), where the results for non-random sums presented in (Korolev and Zhukov 1998) are extended. For treatments of asymptotic expansions for Poisson random sums we refer the reader, for example, in (Babu *et al.* 2003; Bening and Korolev 2002; Gnedenko and Korolev 1996; Korolev *et al.* 2011).

The aim of this chapter is to extend asymptotic expansions that take into consideration large deviations in the Cramér zone ($\gamma = 0$) that are established in (Saulis 1991, Deltuvienė and Saulis 2001, 2003b) in another direction, that is to consider asymptotic expansion that take into consideration large deviations in the Cramér zone for the distribution density function of standardized compound Poisson process

$$\tilde{S}_{N_t} = \frac{S_{N_t} - \mathbf{E}S_{N_t}}{\sqrt{\mathbf{D}S_{N_t}}}.$$

The structure of the reminder term (3.19) of asymptotic expansion (3.21) also is determined (see Section 3.2).

Let us recall that certain difficulties may appear in the formulation of problems related to local limit theorems for compound Poisson process (2.77) as distribution function (2.83) of S_{N_t} is not continuous for all $x \in \mathbb{R}$, because of the presence of an atom at zero. Obviously, we consider the case where

$F_0(x) = 1$, $x > 0$, thus if (2.83) is differentiable, then

$$p_{S_{N_t}}(x) = \frac{d}{dx} F_{S_{N_t}}(x) = \sum_{s=0}^{\infty} q_s p_{X_1 + \ldots + X_s}(x), \qquad x > 0, \tag{3.2}$$

where $p_0(x) = 0$. In addition, we may conclude that the fulfillment of the condition (D') implies

$$\sup_x p_{S_{N_t}}(x) \leq A \sum_{s=0}^{\infty} q_s = A < \infty, \qquad A > 0.$$

Solution to the problem of this chapter is achieved by first using general Lemma 1.3 (see Section 1.1) presented in (Saulis 1980: 165) about asymptotic expansion for the density function of an arbitrary random variable with zero mean and unit variance and joining methods of the cumulant and characteristic functions (see, e. g., Saulis and Statulevičius 1991). To follow general Lemma 1.3, we have to estimate the *k*th-order cumulants of the standardized compound Poisson process (2.82) in the case where $\gamma = 0$. For that Proposition 2.3 (see, Subsection 2.4.3) in instance where $\gamma = 0$ should be used. Particularly, if the random variable X with variance $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_0)$, and N_t, $t \geq 0$, is a homogeneous Poisson process, then

$$|\Gamma_k(\tilde{S}_{N_t})| \leq k!/\Delta_t^{k-2}, \ \Delta_t = \sqrt{\lambda t(\sigma^2 + \mu^2)}/K, \ k = 3, 4, ..., \tag{3.3}$$

where $K > 0$ is defined by $(\bar{B}_0)$. $\sigma^2 = \mathbf{E}X^2$ in the case where $\mu = 0$.

Following, e. g., (Deltuvienė and Saulis 2001, 2003b; Saulis and Statulevičius 1991) in order to estimate the reminder term (3.19) of asymptotic expansion (3.21) along with aforementioned methods S. V. Statulevičius' known estimates for characteristic functions should be used (see Statulevičius 1965 or Lemmas 3.1–3.3 in Section 3.1).

3.1. Auxiliary lemmas

Let $X' = X - Y$ be an arbitrary, symmetrized random variable, where Y is independent of X and with the same distribution. Clearly, the distribution and characteristic functions of X' are as follows

$$F_{X'}(x) = \int_{-\infty}^{\infty} F_X(x+y) dF_X(x), \qquad f_{X'}(u) = |f_X(u)|^2.$$

Corresponding density will be denoted by $p_{X'}(x)$. Moreover, in this section, we assume that $\mathbf{E}X = 0$.

In the paper (Statulevičius 1965) the following lemmas were proved.

Lemma 3.1. *Let X be any random variable with density $p_X(x)$. Then for any collection $\mathfrak{M} = \{\Delta_i, A_i\}$ of non overlapping intervals Δ_i and positive constants $A_i < \infty$ for any $-\infty < u < \infty$ the estimate*

$$|f_X(u)| \le \exp\Big\{-\frac{u^2}{3}\sum_{i=1}^{\infty}\frac{\mathcal{Q}_i^3}{(|\Delta_i||u|+2\pi)^2A_i^2}\Big\}$$

holds, where

$$\mathcal{Q}_i = \int_{\Delta_i}\min\{A_i, p_{X'}(x)\}dx.$$

Corollary 3.1. *If $p_X(x) \le A < \infty$ and $\sigma^2 = \mathbf{E}X^2 < \infty$, then*

$$|f_X(u)| \le \exp\Big\{-\frac{u^2}{96}\frac{1}{(2\sigma|u|+\pi)^2A^2}\Big\}$$

for all $-\infty < u < \infty$, where $A > 0$.

Lemma 3.2. *Let a non-negative function $g(u)$, defined on the interval $[b, \infty)$, satisfies the Lipschitz condition $|g(u+s) - g(u)| \le K|s|$. Moreover, let*

$$V := \int_b^{\infty} g(u)du < \infty.$$

Then for any $\varepsilon > 0$ and any partition $b = u_0 < u_1 < ...$ of the interval $[b, \infty)$ with $\max\limits_{0\le k<\infty}(u_{k+1} - u_k) \le \varepsilon$ we have the inequality

$$\sum_{k=0}^{\infty}(\max_{u_k\le u\le u_{k+1}} g^2(u))\Delta u_k \le V(2K\varepsilon + 4\sup_{a\le u<\infty} g(u)),$$

where $\Delta u_k = u_{k+1} - u_k$.

For a while, let us assume that X_j, $j = 1, 2, ...,$ are independent, non-identically distributed random variables, and put

$$S_n = \sum_{j=1}^{n} X_j, \qquad B_n^2 = \sum_{j=1}^{n}\sigma_j^2.$$

Let

$$l_n(H_n) = \frac{1}{B_n^2}\sum_{j=1}^{n}\int_{|x|\le H_n} x^2 p_{X_j'}(x), \qquad H_n > 0.$$

and

$$J_n(u) = \sum_{j=1}^{n}\int_{-\infty}^{\infty} \langle xy\rangle^2 p_{X_j'}(x)dx,$$

where $\langle b\rangle$ denotes the distance of number b to the nearest integer.

Lemma 3.3. *For any $n \ge 1$ and $H_n > 0$, there exist a partition*

$$... < u_{-1}^{(n)} < u_0^{(n)} = 0 < u_1^{(n)} < u_2^{(n)} < ...$$

of the interval $(-\infty, \infty)$ satisfying the condition

$$\frac{1}{6H_n} \le \Delta u_k^{(n)} \le \frac{1}{4H_n}, \qquad \Delta u_k^{(n)} = u_{k+1}^{(n)} - u_k^{(n)}.$$

such that

$$J_n(u) \ge \frac{1}{2} l_n(H_n)(u - u_{k0}^{(n)})^2 B_n^2,$$

provided $u \in [u_k^{(n)}, u_{k+1}^{(n)}]$, where, for given n, u_{k0} is $u_k^{(n)}$ or $u_{k+1}^{(n)}$ depending on k.

The proofs of Lemmas 3.1, 3.2, 3.3 also can be found in (Saulis and Statulevičius 1991: 172–174).

3.2. Asymptotic expansion in large deviation Cramér zone for density function of the compound Poisson process

Let us note that according to the proof of general Lemma 1.3, we must first to denote the conjugate process of the compound Poisson process. Assume that the *conjugate compound Poisson process* can be denoted by (see Bonin 2003)

$$S_{N_t(h)}(h) = \sum_{j=1}^{N_t(h)} X_j(h), \tag{3.4}$$

where $N_t(h)$ and $X_j(h)$, $t \geq 0$, $h > 0$, are independent, besides the probability of $N_t(h)$ is

$$q_s(h) = \mathbf{P}(N_t(h) = s) = \exp\{-\lambda t \varphi_X(h)\}(\lambda t \varphi_X(h))^s/s!, \tag{3.5}$$

where

$$\varphi_X(h) = \int_{\infty}^{\infty} e^{hx} p_X(x)dx$$

is the generating function of the random variable X. The quantity h we will define later. The identification of $N_t(h)$ and $X(h)$ can be performed with the help of Laplace transform of $S_{N_t(h)}(h)$. Indeed, recall that an arbitrary conjugate random variable $X(h)$ of an arbitrary random variable X is defined by the density function (1.19):

$$p_{X(h)}(x) = \varphi_X^{-1}(h)e^{hx}p_X(x).$$

Thus the conjugate process of the compound Poisson process can be defined by using the density function (1.19) with $X(h) := S_{N_t(h)}(h)$ and $X := S_{N_t}$ (see, e. g., Saulis 1978; Bonin 2003; Korolev *et al.* 2011):

$$p_{S_{N_t(h)}(h)}(x) = \varphi_{S_{N_t}}^{-1}(h)e^{hx}p_{S_{N_t}}(x). \tag{3.6}$$

By virtue of (1.20) with (3.2) and (2.64), we can state that the generating function of S_{N_t} is

$$\begin{aligned} \varphi_{S_{N_t}}(h) &= \int_{-\infty}^{\infty} e^{hx} p_{S_{N_t}}(x)dx = \sum_{s=0}^{\infty} q_s \varphi_{X_1+\ldots+X_s}(h) = \sum_{s=0}^{\infty} q_s \varphi_X^s(h) \\ &= e^{-\lambda t} \sum_{s=0}^{\infty} (\lambda t \varphi_X(h))^s/s! = \exp\{-\lambda t(1-\varphi_X(h))\}. \end{aligned} \tag{3.7}$$

So by (1.2), (3.6) and (3.7)

$$\begin{aligned} f_{S_{N_t(h)}(h)}(u) = \varphi_{S_{N_t}}^{-1}(h) \int_{-\infty}^{\infty} e^{(h+iu)x} p_{S_{N_t}}(x)dx = \varphi_{S_{N_t}}^{-1}(h)\varphi_{S_{N_t}}(h+iu) \\ = \exp\{-\lambda t \varphi_X(h)(1 - f_{X(h)}(u))\}. \end{aligned} \tag{3.8}$$

Clearly,

$$f_{S_{N_t(h)}(h)}(u) = \exp\{-\lambda t \varphi_X(h)(1 - f_{X(h)}(u))\}$$

$$= e^{-\lambda t \varphi_X(h)} \sum_{s=0}^{\infty} (\lambda t \varphi_X(h) f_{X(h)}(u))^s / s! = \sum_{s=0}^{\infty} f^s_{X(h)}(u) q_s(h),$$

where $q_s(h)$ is defined by (3.5). Thus we obtained the characteristic function of the process $S_{N_t(h)}(h)$ which is defined by (3.4). Note that $q_s(0) = q_s$ as $\varphi_X(0) = 1$, where q_s is defined by (2.64).

The use of the definition (1.5) of the moments of X together with (1.19) and (1.20) produces the rth-order moments of $X(h)$

$$\begin{aligned} \mathbf{E}X^r(h) &= \varphi_X^{-1}(h) \frac{d^r}{i^r du^r} \varphi_X(h+iu)\Big|_{u=0} = \varphi_X^{-1}(h) \frac{d^r}{dh^r} \varphi_X(h) \\ &= \varphi_X^{-1}(h) \sum_{k=r}^{\infty} \frac{\mathbf{E}X^k h^{k-r}}{(k-r)!}, \qquad r = 1, 2, \ldots . \end{aligned} \tag{3.9}$$

Additionally, based on Lemma 2.1 in Section 2.1,

$$\begin{aligned} \mathit{\Gamma}_r(X(h)) &= r! \sum\nolimits_1^* \frac{d^m \ln y}{dy^m}\Big|_{y=\varphi_X(h+iu)} \prod_{j=1}^{r} \frac{1}{m_j!} \Big(\frac{1}{j!} \frac{d^j \varphi_X(h+iu)}{i^j du^j} \Big)^{m_j} \Big|_{u=0} \\ &= \frac{d^r}{dh^r} \ln \varphi_X(h) = \sum_{k=r}^{\infty} \frac{\mathit{\Gamma}_k(X)}{(k-r)!} h^{k-r}, \qquad r = 1, 2, \ldots, \end{aligned} \tag{3.10}$$

where $\sum_1^*$ is the summation over all the non-negative integer solutions $0 \le m_1, \ldots, m_r \le r$ of the equation (2.9), $m_1 + \ldots + m_r = m$, and $1 \le m \le r$. In particularly,

$$\mu(h) = \mathbf{E}X(h) = \mathit{\Gamma}_1(X(h)) = \sum_{k=1}^{\infty} \frac{\mathit{\Gamma}_k(X) h^{k-1}}{(k-1)!}, \tag{3.11}$$

$$\mathbf{E}X^2(h) = \varphi_X^{-1}(h) \sum_{k=2}^{\infty} \frac{\mathbf{E}X^k h^{k-2}}{(k-2)!}, \tag{3.12}$$

$$\sigma^2(h) = \mathbf{D}X(h) = \sum_{k=2}^{\infty} \frac{\mathit{\Gamma}_k(X) h^{k-2}}{(k-2)!}. \tag{3.13}$$

According to the definition (1.5) of the cumulants together with (3.8) and (3.9), we get

$$\mathit{\Gamma}_r(S_{N_t}(h)) = \lambda t \varphi_X(h) \mathbf{E}X^r(h) = \sum_{k=r}^{\infty} \frac{\mathit{\Gamma}_k(S_{N_t}) h^{k-r}}{(k-r)!}, \ r = 1, 2, \ldots, \tag{3.14}$$

where $\Gamma_k(S_{N_t})$ is defined by (2.85).

For the following, set

$$\tilde{S}_{N_t(h)}(h) = \frac{S_{N_t(h)}(h) - \mathbf{E}S_{N_t(h)}(h)}{\sqrt{\mathbf{D}S_{N_t(h)}(h)}}, \qquad \mathbf{D}S_{N_t(h)}(h) > 0, \tag{3.15}$$

where by (3.14),

$$\mathbf{E}S_{N_t(h)}(h) = \lambda t \varphi_X(h)\mathbf{E}X(h), \quad \mathbf{D}S_{N_t(h)}(h) = \lambda t \varphi_X(h)\mathbf{E}X^2(h). \tag{3.16}$$

Basing on (Ibragimov and Linnik 1965: 213–216) (or see for saddle-point method in Jensen 1995), to derive equation which gives the solution of $h = h(x) > 0$ we need to do the following calculations. By (3.6),

$$F_{S_{N_t}}(x) = \varphi_{S_{N_t}}(h) \int_{-\infty}^{x} e^{-hy} dF_{S_{N_t(h)}(h)}(y).$$

Thus,

$$F_{\tilde{S}_{N_t}}(x) = \varphi_{S_{N_t}}(h) \int_{-\infty}^{0} e^{-h(\sqrt{\mathbf{D}S_{N_t(h)}(h)}y + \mathbf{E}S_{N_t(h)}(h))} dF_{\tilde{S}_{N_t(h)}}(y),$$

as

$$F_{\tilde{S}_{N_t}}(y) = F_{S_{N_t}}(\sqrt{\mathbf{D}S_{N_t}}y + \mathbf{E}S_{N_t}),$$
$$F_{\tilde{S}_{N_t(h)}(h)}(y) = F_{S_{N_t(h)}(h)}\Big(\sqrt{\mathbf{D}S_{N_t(h)}(h)}y + \mathbf{E}S_{N_t(h)}(h)\Big),$$

when

$$x = \frac{\mathbf{E}S_{N_t(h)}(h)}{\sqrt{\mathbf{D}S_{N_t}}} - \frac{\mathbf{E}S_{N_t}}{\sqrt{\mathbf{D}S_{N_t}}}. \tag{3.17}$$

Hence, according to Ibragimov and Linnik (1965), the quantity $h = h(x) > 0$ should be defined as the solution of the equation (3.17).

Note that

$$f_{\tilde{S}_{N_t(h)}(h)}(u) = \exp\bigg\{ - i \frac{\mathbf{E}S_{N_t(h)}(h)u}{\sqrt{\mathbf{D}S_{N_t(h)}(h)}} \bigg\}$$
$$\cdot \exp\bigg\{ - \lambda t \varphi_X(h)\Big(1 - f_{X(h)}\Big(\frac{u}{\sqrt{\mathbf{D}S_{N_t(h)}(h)}}\Big)\Big)\bigg\}. \tag{3.18}$$

Let us denote

$$R_t(h) = \int_{|u|\ge U_t} |f_{\tilde{S}_{N_t(h)}(h)}(u)|du, \tag{3.19}$$

$$U_t = \frac{1}{12}\Big(1 - \frac{Kx}{\sqrt{\lambda t(\sigma^2+\mu^2)}}\Big)\frac{\sqrt{\lambda t(\sigma^2+\mu^2)}}{K}, \tag{3.20}$$

for

$$0 \le x < \frac{\sqrt{\lambda t(\sigma^2+\mu^2)}}{24K}, \qquad K > 0.$$

In addition,

$$q(m) = \Big(\frac{3\sqrt{2e}}{2}\Big)^m + 8(m+2)^2 4^{3(r+1)}\Gamma\Big(\frac{3m+1}{2}\Big), \qquad m \ge 1.$$

We will use θ_i, $i = 1, 2, ...$ (with or without an index) to denote a quantity, not always one and the same, that does not exceed 1 in modulus.

Theorem 3.1. *If X with $0 < \sigma^2 < \infty$ satisfies conditions $(\bar{B}_0)$, (D'), and N_t, $t \ge 0$, is a homogeneous Poisson process with the probability* (2.64)*, then for every $m \ge 3$, in the interval $0 \le x < \sqrt{\lambda t(\sigma^2+\mu^2)}/(24K)$, $K > 0$, the asymptotic expansion*

$$\frac{p_{\tilde{S}_{N_t}}(x)}{\phi(x)} = \exp\{L_t(x)\}\Bigg(1 + \sum_{k=0}^{m-3} M_{t,k}(x) + \theta_1 q(m)\Big(\frac{K(x+1)}{\sqrt{\lambda t(\sigma^2+\mu^2)}}\Big)^{m-2} + \theta_2 R_t(h)\Bigg) \tag{3.21}$$

is valid, where for the reminder term $R_t(h)$ which is defined by (3.19) *the estimate*

$$R_t(h) \le \frac{1}{c_1(h)U_t}\exp\{-c_1(h)U_t^2\} + c_2(h)\exp\{-\lambda t c_3(h)\} \tag{3.22}$$

as $\lambda t > 2$ holds. Here $h = h(x) > 0$ is the solution of the equation (3.17)*, and U_t is defined by* (3.20)*. In addition,*

$$c_1(h) = \sigma^2(h)/(\pi^2 \mathbf{E}X^2(h)), \tag{3.23}$$

$$c_2(h) = 12\pi\sqrt{2\pi}e^{2\varphi_X(h)}\frac{\sqrt{\mathbf{E}X^2(h)}}{\sigma(h)}A(\sqrt{2}\pi\sigma(h) + 4H(h)), \tag{3.24}$$

$$c_3(h) = \frac{\varphi_X(h)(1-e^{-c})^3}{16(\tau(h)+H(h))^2A^2(h)}, \qquad c > 0, \tag{3.25}$$

where $\varphi_X(h)$ *and* $A > 0$ *are defined, respectively, by* (1.20) *and* (D'). *In addition,* $H(h)$, $\tau(h)$, $A(h)$ *are defined by* (3.39), (3.47), (3.48), *respectively. For constants* $c_1(h)$, $c_2(h)$, $c_3(h)$ *estimates*

$$c_1(h) \geq c_1 = \sigma^2/(\mathbf{E}X^2\pi^2 g_3(\delta)), \tag{3.26}$$

$$c_2(h) \leq c_2 = 12\pi\sqrt{2\pi}\exp\left\{2\exp\{\delta^2 g_2(\delta_1)/8\}\right\}\sqrt{g_3(\delta)\mathbf{E}X^2}/\sigma \cdot MA(\pi\sqrt{2(1+g_1(\delta))}/2 + 8g^{1/2}(\delta)), \tag{3.27}$$

$$c_3(h) \geq c_3 = \frac{(1-e^{-c})^3}{16(\frac{c}{\delta_1-\delta} + \frac{\delta_1^2}{4(\delta_1-\delta)} + 2g^{1/2}(\delta))^2} \cdot \frac{1}{\exp\left\{\frac{\delta^2}{4}g_1(\delta_1) + \frac{\delta_1^2\delta}{2(\delta_1-\delta)} + \frac{2c\delta}{\delta_1-\delta}\right\}(MA)^2} \tag{3.28}$$

hold, where $M = 2\{\sigma \vee K\}$, *and*

$$g(\delta) = \frac{4g_2(\delta)+3(1-g_1^2(\delta))}{4(1-g_1(\delta))}, \qquad g_1(\delta) = \frac{2\delta(\delta^2-3\delta+3)}{(1-\delta)^3}, \tag{3.29}$$

$$g_2(\delta) = \frac{24}{(1-\delta)^5}, \qquad g_3(\delta) = \frac{1+g_1(\delta)}{1-g_1(\delta)}, \tag{3.30}$$

$$g_1(\delta_1) = \frac{1-3\delta_1}{1-\delta_1}, \qquad g_2(\delta_1) = \frac{1+\delta_1}{1-\delta_1}. \tag{3.31}$$

here $0 < \delta < 1 - \sqrt[3]{18}/3$, $0 < \delta_1 < 1/3$, $\delta_1 > \delta$.
Further, $L_t(x) = \sum_{l=3}^{\infty}\tilde{\lambda}_{t,l}x^l$, *where the coefficients* $\tilde{\lambda}_{t,l}$ *(expressed by cumulants of* $\tilde{S}_{N_t}$ *coincide with the coefficients of the Cramer series). They are determined by the relation* $\lambda_{t,l} = -b_{t,l-1}/l$, *and* $b_{t,l}$ *are identified successively from the equations*

$$\sum_{r=1}^{j}\frac{1}{r!}\Gamma_{r+1}(\tilde{S}_{N_t})\sum_{\substack{j_1+\ldots+j_r=j\\ j_i\geq 1}}\prod_{i=1}^{r}b_{t,j_i} = \begin{cases} 1, & j=1,\\ 0, & j=2,3,\ldots. \end{cases}$$

$$\begin{aligned} M_{t,k}(x) &= \sum_{l=0}^{k} K_{t,l}(x) Q_{t,k-l}(x), \\ K_{t,k}(x) &= \sum\nolimits_1^* \prod_{r=1}^{k} \frac{1}{m_r!} (-\tilde{\lambda}_{t,r+2} x^{r+2})^{m_r}, \qquad K_{t,0}(x) \equiv 1, \\ Q_{t,k}(x) &= \sum\nolimits_1^* H_{k+2m}(x) \prod_{r=1}^{k} \frac{1}{m_r!} \Big(\frac{\Gamma_{r+2}(\tilde{S}_{N_t})}{(r+2)!} \Big)^{m_r}, \quad Q_{t,0}(x) \equiv 1. \end{aligned}$$

where the summation $\sum_1^*$ *is taken over all non-negative, integer solutions* $(m_1, m_2, ..., m_k)$ *of the equation* (2.9), $m_1 + m_2 + ... + m_k = m$, $0 \leq m_1, ..., m_k \leq k$, $1 \leq m \leq k$. *Here* $H_r(x)$ *is the Chebyshev-Hermite polynomials* (1.27).

Remark 3.1. Note that (3.16) together with (2.79), (3.11) and $(\bar{B}_0)$ leads to the estimate of (3.17)

$$x = \frac{\lambda t}{\sqrt{\mathbf{D} S_{N_t}}} \sum_{k=2}^{\infty} \frac{\mathbf{E} X^k}{(k-1)!} h^{k-1} = \sqrt{\mathbf{D} S_{N_t}} h \Big(1 + \theta \frac{Kh(3-2Kh)}{(1-Kh)^2} \Big),$$

if $h < 1/K$. If $h \leq \delta/K$, then

$$x = \delta \Big(1 + \theta \frac{\delta(3-2\delta)}{(1-\delta)^2} \Big) \frac{\sqrt{\lambda t(\sigma^2+\mu^2)}}{K}.$$

If $\delta = 1/28$, then $x \leq \sqrt{\lambda t(\sigma^2+\mu^2)}/(24K)$. Additionally, if $\delta_1 = 1/25$, then

$$c_1 = \sigma^2/(1.6\pi^2 \mathbf{E} X^2), \; c_2 = 334 e^4 \sqrt{2\pi} \sqrt{\mathbf{E} X^2} MA/\sigma, \; c_3 = c/(MA)^2.$$

where $0 < c < 2 \cdot 10^{-8}$,

***Proof of Theorem* 3.1.** Let us recall, if for the random variable X the condition $(\bar{B}_0)$ is fulfilled and N_t, $t \geq 0$, is the homogeneous Poisson process with the probability (2.64), then for the kth-order cumulants $|\Gamma_k(\tilde{S}_{N_t})|$, $k = 3, 4, ...$, of the standardized compound Poisson process $\tilde{S}_{N_t}$ which is defined by (2.82) the upper estimate (3.3) holds. Thus, observe that $\tilde{S}_{N_t}$ satisfies S. V. Statulevičius' condition (S_γ) in the case where $\gamma = 0$ with the parameter $\Delta := \Delta_t$, where Δ_t is defined by (3.3). Accordingly, general Lemma 1.3 yields (3.21).

To finish the proof of Theorem 3.1 the estimate (3.22) of the reminder term

(3.19) should be verified.

Obviously,

$$R_t(h) = \int_{|u|\geq U_t} |f_{\tilde{S}_{N_t(h)}(h)}(u)|du = I_1 + I_2, \tag{3.32}$$

where

$$I_1 = \int_{U_t\leq|u|\leq\tilde{U}_t(h)} |f_{\tilde{S}_{N_t(h)}(h)}(u)|du, \quad I_2 = \int_{\tilde{U}_t(h)\leq|u|<\infty} |f_{\tilde{S}_{N_t(h)}(h)}(u)|du,$$

here $\tilde{U}_t(h)$ defined by (3.39). Suppose, that $X^{'}(h) = X(h) - Y(h)$ is a symmetrized, conjugate random variable, where the conjugate random variable $Y(h)$ is independent of $X(h)$ and with the same distribution. It is clear that the distribution and characteristic functions of $X^{'}(h)$ are as follows

$$F_{X'(h)}(x) = \int_{-\infty}^{\infty} F_{X(h)}(x+y)dF_{X(h)}(x), \qquad f_{X'(h)}(u) = |f_{X(h)}(u)|^2.$$

Corresponding density will be denoted by $p_{X'(h)}(x)$. Obviously, $\mathbf{D}X^{'}(h) = 2\sigma^2(h)$. Denote,

$$l_h(H(h)) = \frac{1}{\sigma^2(h)} \int_{|x|<H(h)} x^2 p_{X'(h)}(x)dx, \qquad H(h) > 0. \tag{3.33}$$

Since

$$1 - |f_{X(h)}(2\pi u)| \geq \frac{1}{2}(1 - |f_{X(h)}(2\pi u)|^2) := I_h(u), \tag{3.34}$$

by (3.8), we get

$$\begin{aligned} |f_{S_{N_t(h)}(h)}(2\pi u)| &\leq \exp\{-\lambda t \varphi_X(h)(1 - |f_{X(h)}(2\pi u)|)\} \\ &\leq \exp\{-\lambda t \varphi_X(h) I_h(u)\}, \end{aligned} \tag{3.35}$$

where

$$I_h(u) = \int_{-\infty}^{\infty} \sin^2(\pi u x) p_{X'(h)}(x)dx \geq 4u^2\sigma^2(h) l_h(1/(2|u|)). \tag{3.36}$$

Here $l_h(1/(2|u|))$ is defined by (3.33). Further,

$$l_h(H(h)) = \frac{1}{\sigma^2(h)} \int_{-\infty}^{\infty} x^2 p_{X'(h)}(x)dx - \frac{2}{\sigma^2(h)} \int_{H(h)}^{\infty} x^2 p_{X'(h)}(x)dx$$

$$\geq 2\Big(1 - \frac{2\mathbf{E}|X(h)-\mu(h)|^4}{\sigma^2(h)H^2(h)}\Big) \geq 1, \tag{3.37}$$

if $H(h) = 2\big(\mathbf{E}(X(h)-\mu(h))^4\big)^{1/2}/\sigma(h)$. The use of (3.18) and (3.35)–(3.37) gives

$$|f_{S_{N_t(h)}(h)}(2\pi u)| \leq \exp\{-\lambda t\varphi_X(h)4u^2\sigma^2(h)\} \quad as \quad |u| \leq 1/(2H(h)),$$

and

$$|f_{\tilde{S}_{N_t(h)}(h)}(u)| \leq \exp\Big\{-u^2\frac{\sigma^2(h)}{\pi^2\mathbf{E}X^2(h)}\Big\} \quad as \quad |u| \leq \tilde{U}_t(h), \tag{3.38}$$

where

$$\tilde{U}_t(h) = \frac{\pi\sqrt{\mathbf{D}S_{N_t(h)}(h)}}{H(h)}, \quad H(h) = \frac{2(\mathbf{E}(X(h)-\mu(h))^4)^{1/2}}{\sigma(h)}. \tag{3.39}$$

Here $\mu(h), \mathbf{E}X^2(h)$, and $\sigma^2(h)$ are defined by (3.11)–(3.13). And $\mathbf{D}S_{N_t(h)}(h)$ is defined by (3.16). Consequently,

$$\begin{aligned} I_1 &\leq \frac{2}{U_t}\int_{U_t}^{\tilde{U}_t(h)} |u|\exp\Big\{-u^2\frac{\sigma^2(h)}{\pi^2\mathbf{E}X^2(h)}\Big\}du \\ &\leq \frac{1}{c_1(h)U_t}\exp\{-c_1(h)U_t^2\}, \end{aligned} \tag{3.40}$$

according to (3.38). Here U_t and $c_1(h)$ are defined by (3.20) and (3.23).

If we put $n = 1$ and conjugate random variable $X(h)$ instead of X in Lemma 3.3, then we derive that for any $H(h) > 0$ there exist a partition

$$... < u_{-1} < u_0 = 0 < u_1 < u_2 < ...$$

of the interval $(-\infty, \infty)$ satisfying the condition

$$(6H(h))^{-1} \leq \Delta u_k \leq (4H(h))^{-1}, \qquad \Delta u_k = u_{k+1} - u_k. \tag{3.41}$$

such that

$$I_h(u) \geq \exp\{-2\sigma^2(h)l_h(H(h))(u-u_{k0})^2\}, \tag{3.42}$$

provided $u \in [u_k,\, u_{k+1}]$, where u_{k0} is u_k or u_{k+1} depending on k. Here $l_h(H(h))$ defined by (3.33). On the other hand, employing Lemma 3.1 gives:

if $X(h)$ has a density function such that $p_{X(h)}(x) \le A(h) < \infty$, then for any collection $\mathfrak{M}(h) = \{\Delta(h), A(h)\}$, of the interval $\Delta(h)$ and positive constant $A(h)$ the estimate

$$I_h(u) \ge \frac{Q^3(h)}{3(|\Delta(h)| + 2H(h))^2 A^2(h)}. \tag{3.43}$$

holds for all $|u| \ge 1/(2H(h))$, $H(h) > 0$. Here

$$Q(h) = \int_{\Delta(h)} \min\{A(h), p_{X'(h)}(x)\}dx.$$

Now let us estimate I_2:

$$\begin{aligned} I_2 &= 2\pi\sqrt{\mathbf{D}S_{N_t(h)}(h)} \int_{(2H(h))^{-1} \le |u| < \infty} |f_{S_{N_t(h)}(h)}(2\pi u)|du \\ &\le 2\pi\sqrt{\mathbf{D}S_{N_t(h)}(h)} \int_{(2H(h))^{-1} \le |u| < \infty} \exp\{-(\lambda t - 2)\varphi_X(h)I_h(u)\} \\ &\cdot \exp\{-2\varphi_X(h)(1 - |f_{X(h)}(2\pi u))|\}du, \end{aligned}$$

by (3.34) and (3.35) as $\lambda t > 2$, where $I_h(u)$ defined by (3.36). Hence observing that $\exp\{2\varphi_X(h)I_h(u)\} \le \exp\{2\varphi_X(h)\}$ due to (3.36), we arrive at

$$\begin{aligned} I_2 \quad &\le \quad 2\pi e^{2\varphi_X(h)}\sqrt{\mathbf{D}S_{N_t(h)}(h)} \int_{|u| \ge (2H(h))^{-1}} \exp\big\{ - \lambda t\varphi_X(h)I_h(u)\big\} \\ &\cdot \exp\{-\varphi_X(h)(1 - |f_{X(h)}(2\pi u)|^2)\}du. \end{aligned} \tag{3.44}$$

The next step is to estimate (3.44) for $(3/4)\lambda t\varphi_X(h)I_h(u)$, $(1/4)\lambda t\varphi_X(h)I_h(u)$ using, respectively, (3.43) and (3.42). According to (3.43) and (3.42),

$$\begin{aligned} I_2 &\le 2\pi\sqrt{2\pi}e^{2\varphi_X(h)}\sqrt{\mathbf{D}S_{N_t(h)}(h)} \exp\Big\{ - \frac{\lambda t\varphi_X(h)Q^3(h)}{4(|\Delta(h)| + 2H(h))^2 A^2(h)}\Big\} \\ &\cdot \sum_k \int_{u_k}^{u_{k+1}} \exp\{-\lambda t\varphi_X(h)\sigma^2(h)l_h(H(h))(u - u_{k0})^2/2\} \\ &\cdot \exp\{-\varphi_X(h)(1 - |f_{X(h)}(2\pi u)|^2)\} \\ &\le 2\pi\sqrt{2\pi}e^{2\varphi_X(h)}\frac{\sqrt{\mathbf{E}X^2(h)}}{\sigma(h)} \exp\Big\{ - \frac{\lambda t\varphi_X(h)Q^3(h)}{4(|\Delta(h)| + 2H(h))^2 A^2(h)}\Big\} \end{aligned}$$

$$\cdot \sum_k \sup_{u_k < u < u_{k+1}} \exp\{-\varphi_X(h)(1 - |f_{X(h)}(2\pi u)|^2)\}. \tag{3.45}$$

Further, let us find $\tau(h)$ such that

$$Q(h) = \int\limits_{|y| \le \tau(h)} p_{X'(h)}(y)dy \ge 1 - e^{-c}, \qquad c > 0. \tag{3.46}$$

It was proved in Theorem 6.1 in (Saulis and Statulevičius 1991: 185) that

$$\int\limits_{|y| \ge \tau(h)} p_{X'(h)}(y)dy \le \exp\{-(\tilde{A} - h)\tau(h)\}\varphi_{X'}(\tilde{A})\varphi_{X'}^{-1}(h),$$

if $\tilde{A} > h \ge 0$. Hence

$$\exp\{-(\tilde{A} - h)\tau(h)\}\varphi_{X'}(\tilde{A})\varphi_{X'}^{-1}(h) \le \exp\{-c\}, \qquad c > 0.$$

It is enough that

$$\tau(h) = \frac{c + \ln(\varphi_{X'}(\tilde{A})/\varphi_{X'}(h))}{\tilde{A} - h} > 0, \qquad \tilde{A} \ge h > 0, \qquad c > 0, \tag{3.47}$$

where $\varphi_{X'}(\tilde{A}), \varphi_{X'}(h)$ are defined by (1.20). Next, if $\Delta(h) =]-\tau(h), \tau(h)[$, then recalling (D'), we derive

$$p_{X'(h)}(y) = \varphi_{X'}^{-1}(h)\exp\{hy\}p_{X'}(y) \le A(h) < \infty,$$

where

$$A(h) = \varphi_{X'}^{-1}(h)\exp\{h\tau(h)\}A < \infty, \qquad c > 0. \tag{3.48}$$

It remains to evaluate $\sum_k \sup\limits_{u_k < u < u_{k+1}} \exp\{-\varphi_X(h)(1 - |f_{X(h)}(2\pi u)|^2)\}$. Remark that Lemma 3.2 holds with

$$g(u) = e^{-\varphi_X(h)(1 - |f_{X(h)}(2\pi u)|^2)} = e^{-\varphi_X(h)} \sum_{k=0}^{\infty} \frac{|f_{X(h)}(2\pi u)|^{2k}\varphi_X^k(h)}{k!}.$$

Here (see Saulis and Statulevičius 1991: 186),

$$|f_{X'(h)}(2\pi(u+s)) - f_{X'(h)}(2\pi u)| \le 2\pi s\Big(\int\limits_{-\infty}^{\infty} y^2 p_{X'(h)}(y)dy\Big)^{1/2}$$
$$= 2\sqrt{2}\pi\sigma(h)s.$$

Hence, $|g(u+s) - g(u)| \le 2\sqrt{2}\pi\sigma(h)s$. Accordingly, Lemma 3.2 holds with

$$K := \tilde{K}(h) = 2\sqrt{2}\pi\sigma(h), \quad V := V(h) = \int_{-\infty}^{\infty} g(u)du \le A, \qquad (3.49)$$

as

$$\int_{-\infty}^{\infty} |f_{X(h)}(2\pi u)|^2 du \le p_{X'(h)}(0) \le A.$$

Therefore, taking (3.2) into consideration, together with (3.41) and (3.49), we can write

$$\sum_k \sup_{u_k<u<u_{k+1}} \exp\{-\varphi_X(h)(1-|f_{X(h)}(2\pi u)|^2)\}$$
$$\le 6H(h)A\Big(\frac{4\pi\sqrt{2}\sigma(h)}{4H(h)}+4\Big) = 6A(\sqrt{2}\pi\sigma(h)+4H(h)). \qquad (3.50)$$

Substituting (3.46)–(3.48) and (3.50) into (3.45) we derive

$$I_2 \le c_2(h)\exp\{-\lambda t c_3(h)\}. \qquad (3.51)$$

where $c_2(h)$ and $c_3(h)$ are defined by (3.24), (3.25). Finally, (3.32) (3.40) (3.51) leads to (3.22).

Let us derive estimates (3.26)–(3.28). The use of (3.1), (3.13) and (3.10) gives

$$\sigma^2(h) = \sigma^2\Big(1+\theta\sum_{k=3}^{\infty}k(k-1)\delta^{k-2}\Big) = \sigma^2(1+\theta g_1(\delta)), \qquad (3.52)$$

$$\Gamma_4(X(h)) \le (\sigma M)^2\sum_{k=4}^{\infty}k(k-1)(k-2)(k-3)\delta^{k-4} = g_2(\delta)(\sigma M)^2, \quad (3.53)$$

if $0 \le h \le \delta/M$, $0<\delta<1$. Here $g_1(\delta)$ and $g_2(\delta)$ are defined by (3.29). Note that

$$\frac{\mathbf{E}(X(h)-\mu(h))^4}{\sigma^2(h)} = \frac{\Gamma_4(X(h))}{\sigma^2(h)} + 3\sigma^2(h).$$

Consequently in view of (3.52), (3.53) together with $\sigma \le M/2$, we evaluate

$$\frac{|\mathbf{E}(X(h)-\mu(h))^4|}{\sigma^2(h)} \le M^2 g(\delta), \tag{3.54}$$

where $g(\delta)$ defined by (3.30). Hence

$$H(h) \le 2Mg^{1/2}(\delta) \tag{3.55}$$

by (3.39) and (3.54). Here $g(\delta) > 0$, if $0 < \delta < 1 - \sqrt[3]{18}/3$.
Employing (3.9), $(\bar{B}_0)$, together with $K \le M/2 < M$, we imply

$$\begin{aligned} \mathbf{E}X^2(h) &= \varphi_X^{-1}(h)\Big(\mathbf{E}X^2 + \theta\mathbf{E}X^2\sum_{k=3}^{\infty}\frac{k!}{(k-2)!}(Kh)^{k-2}\Big) \\ &= \varphi_X^{-1}(h)\mathbf{E}X^2(1+\theta g_1(\delta)), \end{aligned} \tag{3.56}$$

if $0 \le h \le \delta/M$. Further, we will need the estimate

$$\begin{aligned} \varphi_{X-\mu}(z) &= \exp\Big\{\sum_{k=2}^{\infty}\frac{1}{k!}\varGamma_k(X)z^k\Big\} \\ &= \exp\Big\{\frac{1}{2}\sigma^2 z^2\Big(1 + 2\theta\sum_{k=3}^{\infty}(M|z|)^{k-2}\Big)\Big\} \\ &= \exp\Big\{\frac{1}{2}\sigma^2 z^2\Big(1 + \theta\frac{2\delta_1}{1-\delta_1}\Big)\Big\}, \end{aligned}$$

if $|z| \le \tilde{A} = \frac{\delta_1}{M}$, $0 < \delta_1 < 1$. Thus

$$\exp\Big\{\frac{1}{2}\sigma^2 z^2 g_1(\delta_1)\Big\} \le |\varphi_X(z)| \le \exp\Big\{\frac{1}{2}\sigma^2 z^2 g_2(\delta_1)\Big\}, \tag{3.57}$$

where $g_1(\delta_1)$ and $g_2(\delta_1)$ defined by (3.31). From (3.47) follows that $\delta \le \delta_1$. Besides $g_1(\delta_1) > 0$, if $0 < \delta_1 < 1/3$. The apply of (3.52), (3.56) and (3.57), gives

$$\frac{\mathbf{E}X^2(h)}{\sigma^2(h)} \le \frac{\mathbf{E}X^2 g_3(\delta)}{\varphi_X(h)\sigma^2} \le \frac{\mathbf{E}X^2}{\sigma^2} g_3(\delta), \tag{3.58}$$

for $h \ge 0$. Here $g_3(\delta)$ defined by (3.30). Note that $g_3(\delta) > 0$, if $0 < \delta < 1 - \sqrt[3]{18}/3$.
The next step is to estimate $\tau(h)$ and $A(h)$ defined, respectively, by (3.47) and (3.48). Recalling (3.57) and observing that $\tilde{A} = \delta_1/M$, $\sigma \le M/2$, $h \le$

$\delta/M < \tilde{A}$ let us to assert,

$$\tau(h) \leq \frac{c + \frac{1}{2}\sigma^2 \tilde{A}^2 g_2(\delta_1) + \frac{1}{2}\sigma^2 h^2 g_2(\delta_1)}{\tilde{A} - h} \leq \frac{Mc}{\delta_1 - \delta} + \frac{\delta_1^2 M}{4(\delta_1 - \delta)}, \quad (3.59)$$

and

$$A(h) \leq \exp\Big\{\frac{\delta_1^2 \delta}{4(\delta_1 - \delta)} + \frac{c\delta}{\delta_1 - \delta}\Big\} A, \quad (3.60)$$

where $c > 0$ and $A > 0$ are defined by (3.46) and (D'). Finally, employing (3.55)–(3.60), gives estimates (3.26)–(3.28). □

3.3. Conclusions of Chapter 3

1. The suitable bound (3.3) and general Lemma 1.3 lead to proof asymptotic expansion that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process (see Theorem 3.1). Additionally, S. V. Statulevičius' known estimates for characteristic functions lead to estimate the reminder term (3.19) of aforementioned asymptotic expansion (3.21).
2. The result on asymptotic expansion that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process extends asymptotic expansions for the density function of the sums of non-random number of summands considered in the works by Saulis (1991), Deltuvienė and Saulis (2001, 2003b).

General conclusions

1. Having explored instances of large deviations for a distribution of the standardized sum of a random number of summands of i. i. d. weighted random variables, it was noted that in the thesis obtained theorems of large deviations in both the Cramér and the power Linnik zones and exponential inequalities can be regarded as extension of the theorems of large deviations and exponential inequalities for the sums of non-random number of summands.
2. In the thesis, obtained theorems of large deviations in the Cramér zone and exponential inequalities for weighted random sum can be regarded as generalization of the works (Statulevičius 1967; Saulis 1978; Saulis and Deltuvienė 2007). It should be emphasized, as distinct from the aforementioned works, in the thesis, the instance where characteristic function of the separate summand of the sum of a r. n. s. is not analytic in a vicinity of zero is also considered.
3. The results of the thesis lead us to large deviation theorems for the standardized compound and mixed Poisson processes that are largely used in insurance and finance mathematics.
4. The result on asymptotic expansion that take into consideration large deviations in the Cramér zone for the density function of the standardized compound Poisson process extends asymptotic expansions for the density function of the sums of non-random number of summands.

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List of the author's scientific publications on the topic of the dissertation

In the reviewed scientific journals

Kasparavičiūtė, A.; Saulis, L. 2013. Large deviations for weighted random sums, *Nonlinear Analysis: Modelling and Control* 18(2): 129–142. ISSN 1392-5113. (Thomson ISI Web of Science).

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THEOREMS OF LARGE DEVIATIONS
FOR THE SUMS OF A RANDOM NUMBER
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ATSITIKTINIO SKAIČIAUS
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