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# MOMENT PROBLEM FOR THE PERIODIC ZETA-FUNCTION

Doctoral dissertation

Physical sciences, mathematics (01P)

Vilnius, 2014

The scientific work was carried out in 2010–2014 at Vilnius University

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Daktaro disertacija

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Disertacija rengta 2010–2014 metais Vilniaus universitete

**Mokslinis vadovas:**

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# Introduction

The periodic zeta-function in a half-plane of absolute convergence is defined by an ordinary Dirichlet series with coefficients $e^{2\pi i\lambda m}$, where $\lambda$ is a real parameter. In the thesis, problems related to the moments of the periodic zeta-function are considered.

## Actuality

In the theory of the value distribution, various approaches are applied. One of the most popular approaches is the investigation of moments of zeta-functions. This is motivated by a fact that, in various problems, the individual values of zeta-functions can be replaced by their mean values. Moreover, in some cases, the asymptotics of moments of zeta-functions implies probabilistic limit theorems. Therefore, the majority of analytic number theorists are concerned to the moment problem of zeta-functions. The first significant results in the field were obtained by the famous G. H. Hardy and J. E. Littlewood (an asymptotic formula for the mean square of the Riemann zeta-function) and A. E. Ingham (an asymptotic formula for the fourth power moment of the Riemann zeta-function). The moment problem for the Riemann zeta-function was developed by D. R. Heath-Brown, M. Jutila, A, Ivič, A. Selberg. Later, important results were obtained by F. V. Atkinson, K. Ramachandra, K. Matsumoto, T. Meurman, J. B. Conrey, A. Ghosh, J. Steuding. Recently, K. Soundararajan proposed new ideas for the moment problem, and improved a series of results [44].

Moments of zeta-functions also have a nice tradition in Lithuania. J. Kubilius, A. Maknys, A. Bulota, A. Matuliauskas considered the moments of zeta-functions of algebraic number fields. A. Laurinčikas and his students R. Garunkštis, D. Šiaučiūnas, S. Zamarys, J. Karaliūnaitė, R. Ivanauskaitė investigated the moments of the Riemann zeta-function, Dirichlet $L$-functions, zeta-functions of cusp forms, Dirichlet series with periodic coefficients and other zeta-functions, and applied the results obtained.

## Aims and problems

The aim of the thesis is to obtain asymptotic formulae for some analytic objects related to the periodic zeta-function. The problems are the following.

1. To prove the Atkinson-type formula with a new error term in the critical strip for the periodic zeta-function with rational parameter.

2. To prove a mean square formula for the error term in the Atkinson-type formula on the critical line for the periodic zeta-function.

3. To prove a mean square formula for the error term in the Atkinson-type formula in the critical strip for the periodic zeta-function.

4. To obtain an asymptotic formula for the fourth power moment of the periodic zeta-function.

## Methods

For the proof of the Atkinson-type formula, modifications of the methods of Atkinson, Matsumoto and Meurman are applied. For the mean square formulae, the method of Heath-Brown and Ivič is developed. For the fourth power moment, the method of approximate functional equations is used.

## Novelty

All results of the thesis are new. The Atkinson-type formula in the critical strip for the periodic zeta-function was considered by J. Karaliūnaitė, however, in the thesis, this formula is given with a corrected new error term with respect to the parameter.

# History of the problem

The moment problem in analytic number theory, first of all, is related to the investigations of value-distribution of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$. We remind that the function $\zeta(s)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The moment problem of $\zeta(s)$ consists of finding the asymptotics or estimates for the quantities

$$I_k(\sigma, T) \stackrel{\text{def}}{=} \int_0^T |\zeta(\sigma + it)|^{2k} dt, \ \sigma \geq \frac{1}{2}, \ k \geq 0,$$

as $T \to \infty$. In some problems, the information on $I_k(\sigma, T)$ successfully replaces the individual values of the function $\zeta(s)$. This is clearly illustrated by the relation of the moment problem to the Lindelöf hypothesis which asserts that, for every $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2} + it\right) = O_\varepsilon(|t|^\varepsilon), \ |t| \geq t_0.$$

It is not difficult to prove, see, for example [45], that the the Lindelöf hypothesis is equivalent, for every $\varepsilon > 0$, to the estimate

$$I_k\left(\frac{1}{2}, T\right) = O_\varepsilon(T^{1+\varepsilon}), \ k \in \mathbb{N}.$$

On the other hand, the investigation of moments of zeta-functions is a very complicated but interesting problem of analytic number theory. In the theory of the function $\zeta(s)$, there exists a conjecture that

$$I_k\left(\frac{1}{2}, T\right) \sim c(k) T (\log T)^{k^2} \tag{0.1}$$

as $T \to \infty$ with some constants $c(k)$, however, it is proved only for few values of $k$. G. H. Hardy and J. E. Littlewood [9] obtained that $c(1) = 1$. A. E. Ingham proved [14] that $c(2) = \frac{1}{2\pi^2}$. Moreover, (0.1) is true [28] for

$$k = \frac{a}{\sqrt{\log \log T}}$$

with $c(k) = 1$ for positive $a$ bounded by a constant. Also, various estimates for $I_k(\frac{1}{2}, T)$ are known. Very precise results in this direction were obtained by D. R. Heath-Brown. In [12], he proved that the bound

$$I_k\left(\frac{1}{2}, T\right) \geq c_k T (\log T)^{k^2} \tag{0.2}$$

with some $c_k > 0$ holds for all rational $k$. Under the Riemann hypothesis (RH) ($\zeta(s) \neq 0$, for $\sigma > \frac{1}{2}$), the latter estimate was earlier obtained by K. Ramachandra [43] for all $k > 0$. Moreover, in [12], it was proved the upper bound

$$I_k\left(\frac{1}{2}, T\right) \leq c_k T (\log T)^{k^2}$$

for $k = \frac{1}{m}$, $m \in \mathbb{N}$, and, under RH, for $0 < k < 2$. Using RH, J. B. Conrey and A. Ghosh [2] obtained the inequality

$$I_k\left(\frac{1}{2}, T\right) \geq (\hat{c}(k) + o(1)) T (\log T)^{k^2}$$

with explicitly given $\hat{c}(k)$ for all $k > 0$, and D. R. Heath-Brown in [13] gave, for $0 < k < 2$, the bound

$$I_k\left(\frac{1}{2}, T\right) \leq \left(\frac{2}{(k^2+1)(2-k)} \hat{c}(k) + o(1)\right) T (\log T)^{k^2}.$$

There exist conjectures that [3]

$$c(3) = \frac{42}{9!} \prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right)$$

and [4]

$$c(4) = \frac{24024}{14!} \prod_p \left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right),$$

and, in general, [24]

$$c(k) = \frac{1}{\Gamma(1+k^2)} \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{j=0}^{\infty} \frac{d_k^2(p^j)}{p^j} \prod_{j=0}^{k-1} \frac{j!}{(j+k)!},$$

where

$$d_k(p^j) = \frac{k(k+1)...(k+j-1)}{j!}.$$

Similar results for $I_k\left(\frac{1}{2}+\frac{1}{l_T},T\right)$ with $l_T\to\infty$ were obtained by A. Laurinčikas in [28] and [29]. In [29], it was conjectured that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+\frac{1}{l_T}+it\right)\right|^{2k} dt = b(k)\min(l_T,\log T)^{k^2}(1+o(1))$$

as $T\to\infty$ with some constants $b(k)$.

Many attention is devoted to the asympotics of $I_1\left(\frac{1}{2},T\right)$ . Let $\gamma_0$ denote the Euler constant, i.e.,

$$\gamma_0=\lim_{n\to\infty}\left(\sum_{m\le n}\frac{1}{m}-\log n\right)=0,577215...,$$

and

$$E(T)=I_1\left(\frac{1}{2},T\right)-T\log\frac{T}{2\pi}-(2\gamma_0-1)T.$$

The classical result [45] asserts that $E(T)=O(T^{\frac{1}{2}+\varepsilon})$ with every $\varepsilon>0$. The best known estimate

$$E(T)=O\left(T^{\frac{131}{416}}(\log T)^{\frac{32587}{8320}}\right)$$

was obtained in [46]. The conjectural bound is $O(T^{\frac{1}{4}+\varepsilon})$ with every $\varepsilon>0$.

The function $E(T)$ was studied by many number theorists. This can be explained not only by itself interest of $E(T)$ but also its close relation to the famous Dirichlet problem on the estimate of the quantity

$$\Delta(x)=\sum_{m\le x}d(m)-x(\log x+2\gamma_0-1),$$

where

$$d(m)=\sum_{d|m}1,\ \ m\in\mathbb{N},$$

is the divisor function. The results can be found in [16] and [17].

Various methods for the investigation of $E(T)$ are known. F. V. Atkinson in [1] proposed a new approach which allows to obtain an explicit formula for $E(T)$ with a small error term. Let $c_1<c_2$ be two positive constants, $c_1T<N<c_2T$, and

$$N_1=N_1(T,N)=\frac{T}{2\pi}+\frac{N}{2}-\sqrt{\frac{N^2}{4}+\frac{NT}{2\pi}}.$$

Define the functions

$$\operatorname{arsinh}(x) = \log(x + \sqrt{1+x^2}),$$

$$f(T,m) = 2T\operatorname{arsinh}\left(\sqrt{\frac{\pi m}{2T}}\right) + \sqrt{2\pi m T + \pi^2 m^2} - \frac{\pi}{4}.$$

Then the Atkinson theorem is of the following form [1].

**Theorem A.** *The formula*

$$\begin{aligned} E(T) \quad &= \frac{1}{\sqrt{2}} \sum_{m \le N} \frac{(-1)^m d(m)}{\sqrt{m}} \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi m}{2\pi}}\right)\right)^{-1} \left(\frac{T}{2\pi m} + \frac{1}{4}\right)^{-\frac{1}{4}} \cos(f(T,m)) \\ &-2 \sum_{m \le N_1} \frac{d(m)}{\sqrt{m}} \left(\log \frac{T}{2\pi m}\right)^{-1} \cos\left(T \log \frac{T}{2\pi m} - T + \frac{\pi}{4}\right) + O(\log^2 T) \end{aligned}$$

*holds.*

The proof of Theorem A with some small corrections is also given in the monograph [16]. We see that the Atkinson formula expresses the error term $E(T)$ by some rather simple elementary functions. This allows a more precise examination of $E(T)$.

The papers [19], [20], [38] and [40] are devoted to modified versions of Theorem A. A. Laurinčikas gave [26], [27] a version of the Atkinson formula near the critical line.

T. Meurman gave [37] a generalization of the Atkinson formula for Dirichlet $L$-functions. H. Ishikawa and K. Matsumoto proved [15] the Atkinson-type formula for the product of $\zeta(s)$ and a Dirichlet polynomial.

We note that the Atkinson formula is a useful tool in the theory of the Riemann zeta-function. This formula is used to obtain various estimates for the error term in the formula for $I_1(\sigma,T)$, and to continue the investigations of $I_k(\sigma,T)$. For example, D. R. Heath-Brown applied [11] Theorem A for the estimate of $I_6\left(\frac{1}{2},T\right)$. He proved that

$$I_6\left(\frac{1}{2},T\right) = O(T^2(\log T)^{17}).$$

More mean value results for the Riemann zeta-function can be found in the excellent survey papers [34] and [18].

In the thesis, the Atkinson type formula is discussed for the periodic zeta-function $\zeta_\lambda(s)$, $\lambda \in \mathbb{R}$. The function $\zeta_\lambda(s)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s}.$$

For $\lambda \in \mathbb{Z}$, the function $\zeta_\lambda(s)$ reduces to the Riemann zeta-function. On the other hand, $\zeta_\lambda(s)$ is closely related to the Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \ \sigma > 1,$$

where $\alpha$, $0 < \alpha \leq 1$, is a fixed parameter. Namely, we have that

$$\zeta_\lambda(s) = e^{2\pi i \lambda} L(\lambda, 1, s), \ \sigma > 1. \tag{0.3}$$

Since the function $L(\lambda, 1, s)$ with $\lambda \notin \mathbb{Z}$ is entire one [30], we also have that the function $\zeta_\lambda(s)$ with $\lambda \notin \mathbb{Z}$ has analytic continuation to the whole complex plane. In view of the periodicity of the coefficients $e^{2\pi i \lambda m}$, we may suppose without loss of generality that $0 < \lambda \leq 1$.

The function $\zeta_\lambda(s)$ is not so important in analytic number theory than $\zeta(s)$, however, it is a rather interesting analytic object depending on the parameter $\lambda$, and occurs in various problems. For example, $\zeta_\lambda(s)$ is used in the mean square formula of $L(\lambda, \alpha, s)$ with respect to the parameter $\alpha$ [30].

**Theorem B.** *Suppose that $\frac{1}{2} < \sigma < 1$ is fixed and $t > 1$. Then, for any $\lambda \in \mathbb{R}$,*

$$\int_0^1 |L(\lambda, \alpha, \sigma+it) - \alpha^{-\sigma-it}|^2 d\alpha = \frac{1}{2\sigma-1} + 2\Gamma(2\sigma-1) Re\left(\zeta_\lambda(2\sigma-1)\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\right)$$

$$-2Re\left(e^{-2\pi i \lambda}\zeta_\lambda(\sigma+it) - 1\right) + O(t^{-1}).$$

In [30], analogous formulae are also obtained for $\sigma = \frac{1}{2}$ and $\sigma = 1$.

In virtue of (0.3), the moments of the function $\zeta_\lambda(s)$ coincide with those of the function $L(\lambda, 1, s)$. Therefore, the theorems of Section 4.2 from [30] imply the following results. Denote by $\zeta(s, \alpha)$, $0 < \alpha \leq 1$, the Hurwitz zeta-function, i.e., $\zeta(s, \alpha) = L(\lambda, \alpha, s)$ with $\lambda \in \mathbb{Z}$.

**Theorem C.** *Suppose that $0 < \lambda < 1$ and $\frac{1}{2} < \sigma < 1$. Then, as $T \to \infty$,*

$$\int_0^T |\zeta_\lambda(\sigma+it)|^2 dt = \zeta(2\sigma)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma}\zeta(2-2\sigma, \lambda)T^{2-2\sigma}$$

$$+O(T^{1-\sigma}\log T)+O(T^{\frac{\sigma}{2}}).$$

Define the constant $c(\lambda)$ by

$$\sum_{m=0}^{n}\frac{1}{m+\lambda}=\log n+c(\lambda)+O\left(\frac{1}{n}\right),\ n\to\infty,$$

and $c(1)$ is Euler constant $\gamma_0$.

**Theorem D.** *Suppose that* $0<\lambda<1$. *Then, as* $T\to\infty$,

$$\int_0^T\left|\zeta_\lambda\left(\frac{1}{2}+it\right)\right|^2dt=T\log T+T(c(\lambda)+\gamma-1-\log 2\pi)+O(T^{\frac{1}{2}}\log T).$$

For the proof of Theorems C and D, an approximate functional equation for the Lerch zeta-function [7] is applied.

A more interesting and complicated mean square problem for the function $\zeta_\lambda(s)$ is the following. Suppose that $\lambda$ is a rational number, i.e., $\lambda=\frac{a}{q}$ with integers $a$ and $q$, $1\le a\le q$. Similarly to the case of Dirichlet $L$-functions, one can consider the mean square value

$$\sum_{a=1}^{q}\int_0^T|\zeta_{\frac{a}{q}}(\sigma+it)|^2dt. \tag{0.4}$$

The later problem with $\sigma=\frac{1}{2}$ was begun to study in [23], and analogue of Theorem A was obtained. Define

$$E(q,T)=\sum_{a=1}^{q}\int_0^T\left|\zeta_{\frac{a}{q}}\left(\frac{1}{2}+it\right)\right|^2dt-qT\left(\log\frac{qT}{2\pi}+2\gamma_0-1\right).$$

Let $c_1$ and $c_2$ be two positive constants, $c_1<c_2$, such that $c_1T<N<c_2T$, and

$$N_1=N_1(q,T,N)=q\left(\frac{T}{2\pi}+\frac{qN}{2}-\left(\left(\frac{qN}{2}\right)+\frac{qNT}{2\pi}\right)^{\frac{1}{2}}\right).$$

Moreover, let

$$f(T,m,q)=2T\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)+\sqrt{2\pi qmT+\pi^2q^2m^2}-\frac{\pi}{4}$$

and

$$g(T,m,q)=T\log\left(\frac{qT}{2\pi m}\right)-T+\frac{\pi}{4}.$$

So, we see that $f(T,m,q)=f(T,mq)$ and $g(T,m,q)=g(T,\frac{m}{q})$ in the notation of Theorem A. Define two sums

$$\sum\nolimits_1(q,T) \quad = \frac{1}{\sqrt{2q}}\sum_{m\le N}\frac{(-1)^{qm}d(m)}{\sqrt{m}}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{-\frac{1}{4}}$$

$$\times\cos f(T,m,q)$$

and

$$\sum\nolimits_2(q,T)=-\frac{2}{\sqrt{q}}\sum_{m\le N_1}\frac{d(m)}{\sqrt{m}}\left(\log\frac{qT}{2\pi m}\right)^{-1}\cos g(T,m,q).$$

Then the following analogue of Theorem A is true [23].

**Theorem E.** *Suppose that $q\le T$. Then*

$$E(q,T)=q\left(\sum\nolimits_1(q,T)+\sum\nolimits_2(q,T)\right)+O(\sqrt{q}\log^2 T)+O(qT^{-1}).$$

Note that, in the case $q=1$, Theorem E completely coincides with Theorem A. We also observe that the proof of Theorem E requires a bound $q\le T$.

K. Matsumoto obtained [33] an analogue of the Atkinson formula for fixed $\sigma$ in the range $\frac{1}{2}<\sigma<\frac{3}{4}$. Namely, he proved an explicit formula for the quantity

$$E_\sigma(T)=I_1(\sigma,T)-\zeta(2\sigma)T-(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}.$$

The role of the divisor function in this formula is played by the generalized divisor function

$$\sigma_a(m)=\sum_{d|m}d^a,\ a\in\mathbb{C}.$$

Let $N$ and $N_1$, and $f(T,m)$ and $g(T,m)$ be the same as in Theorem A. Define two sums

$$\sum\nolimits_{1,\sigma}(T)=2^{\sigma-1}\left(\frac{\pi}{T}\right)^{\sigma-\frac{1}{2}}\sum_{m\le N}\frac{(-1)^m\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}$$

$$\times\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi m}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi m}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos(f(T,m))$$

and

$$\sum\nolimits_{2,\sigma}(T)=-2\left(\frac{2\pi}{T}\right)^{\sigma-\frac{1}{2}}\sum_{m\le N_1}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\frac{T}{2\pi m}\right)^{-1}\cos(g(T,m)).$$

Then the Matsumoto theorem [33] is of the form.

**Theorem F.** *Suppose that* $\frac{1}{2} < \sigma < \frac{3}{4}$*. Then*

$$E_\sigma(T) = \sum\nolimits_{1,\sigma}(T) + \sum\nolimits_{2,\sigma}(T) + R(T),$$

*where* $R(T) = O(\log T)$ *with the O-constant depending only on* $\sigma$*.*

In [35], the formula of Theorem F was extended to the interval $\frac{1}{2} < \sigma < 1$. For this, a very complicated approach different from that of Atkinson was developed.

J. Karaliūnaitė , in [21] and in her thesis [22], also considered a version of Theorem E for fixed $\sigma$, $\frac{1}{2} < \sigma < 1$. In the proof of Atkinson type formula for the error term of the quantity (0.4), some new problems arise from the involving of the parameter $q$ which can grow together with $T$. Unfortunately, as it was detected by a careful reading, the occurrence of the parameter $q$ in the formula of [21] is not correct. Therefore, our first aim was to reprove an analogue of Theorem E for fixed $\sigma$, $\frac{1}{2} < \sigma < \frac{3}{4}$, and Chapter 1 of the thesis is devoted to the latter problem. Let

$$E_\sigma(q,T) = \sum_{a=1}^{q} \int_0^T |\zeta_{\frac{a}{q}}(\sigma+it)|^2 dt - q\zeta(2\sigma)T - \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}.$$

Preserving the notation of Theorem E, define two sums

$$\sum\nolimits_{1,\sigma}(q,T) = 2^{\sigma-1}q^{1-\sigma}\left(\frac{T}{\pi}\right)^{\frac{1}{2}-\sigma} \sum_{m\le N} \frac{(-1)^{qm}\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}$$
$$\times \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)\right)^{-1} \left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{-\frac{1}{4}} \cos(f(T,m,q))$$

and

$$\sum\nolimits_{2,\sigma}(q,T) = -2q^{1-\sigma}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \sum_{m\le N_1} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \left(\log\left(\frac{qT}{2\pi m}\right)\right)^{-1} \cos(g(T,m,q)).$$

Now we state the main theorem of Chapter 1 obtained in [6].

**Theorem 1.1.** *Suppose that* $\frac{1}{2} < \sigma < \frac{3}{4}$*. Then for* $q \le T$*,*

$$E_\sigma(q,T) = \sum\nolimits_{1,\sigma}(q,T) + \sum\nolimits_{2,\sigma}(q,T) + R(q,T),$$

*where* $R(q,T) = O(q^{\frac{7}{4}-\sigma}\log T)$ *with the O-constant depending only on* $\sigma$*.*

Chapter 2 of the thesis is devoted to the mean square of $E(q,T)$. A similar problem for the Riemann zeta-function was introduced by D. R. Heath-Brown [10]. In [10], see also [16], he obtained, as $T\to\infty$, the formula

$$\int_2^T E^2(t)dt = \frac{2T^{\frac{3}{2}}}{3\sqrt{2\pi}}\sum_{m=1}^{\infty}\frac{d^2(m)}{m^{\frac{3}{2}}} + O(T^{\frac{5}{4}}\log^4 T). \tag{0.5}$$

Using Theorem E, in Chapter 2 of the thesis, we prove the following generalization of formula (0.5).

**Theorem 2.1.** *For $T\to\infty$ and $q\le\frac{1}{8}T$,*

$$\int_2^T E^2(q,t)dt = \frac{2\sqrt{q}T^{\frac{3}{2}}}{3\sqrt{2\pi}}\sum_{m=1}^{\infty}\frac{d^2(m)}{m^{\frac{3}{2}}} + O(T^{\frac{5}{4}}q^{\frac{3}{4}}\log^4 T).$$

*For $q=o(\frac{T}{\log^{16}T})$, the above formula is asymptotic.*

If $q=1$, then $E(q,t)=E(t)$. Thus, Theorem 2.1 contains formula (0.5).

Chapter 3 of the thesis deals with the mean square of $E_\sigma(q,T)$. There a generalization of the formula for the mean square of $E_\sigma(t)$ is presented. In [33], K. Matsumoto proved that, for $\frac{1}{2}<\sigma<\frac{3}{4}$,

$$\begin{aligned}\int_2^T E_\sigma^2(t)dt &= \frac{2}{5-4\sigma}(2\pi)^{2\sigma-\frac{3}{2}}\frac{\zeta^2(\frac{3}{2})}{\zeta(3)}\zeta\left(\frac{5}{2}-\sigma\right)\zeta\left(\frac{1}{2}+2\sigma\right)T^{\frac{5}{2}-2\sigma}\\ &\quad +O\left(T^{\frac{7}{4}-\sigma}\log T\right).\end{aligned}$$

In [36], the error term in the above formula has been replaced by $O(T)$. Moreover, in [33], it was obtained that

$$\int_2^T E_{\frac{3}{4}}^2(t)dt = \frac{\zeta^2(\frac{3}{2})\zeta(2)}{\zeta(3)}T\log T + O(T(\log T)^{\frac{1}{2}}),$$

and, for $\frac{3}{4}<\sigma<1$,

$$\int_2^T E_\sigma^2(t)dt = O(T).$$

**Theorem 3.1.** *Let $\sigma$, $\frac{1}{2}<\sigma<\frac{3}{4}$, be fixed. Then, for $T\to\infty$ and $q\le T^{1-\frac{4\sigma}{3}-\varepsilon}$ with every $\varepsilon$>0,*

$$\int_2^T E_\sigma^2(q,t)dt = 2(5-4\sigma)^{-1}(2\pi)^{2\sigma-\frac{3}{2}}q^{\frac{3}{2}-2\sigma}T^{\frac{5}{2}-2\sigma}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac{5}{2}-2\sigma}}$$

$$+O(q^{\frac{11}{4}-2\sigma}T^{\frac{7}{4}-\sigma}\log T).$$

**Remark 3.2.** *If $q \le T^{\frac{3}{5}-\frac{4\sigma}{5}-\varepsilon}$ with arbitrary $\varepsilon > 0$, then the equality of Theorem 3.1 is asymptotic.*

For $q = 1$, we have the Matsumoto result.

In the last chapter of the thesis, Chapter 4, asymptotic formulae for the fourth power moment of the periodic zeta-function in the critical strip are obtained. These formulae depend on the arithmetics of the parameter $\lambda$. The case of irrational and rational $\lambda$ are discussed.

The monograph [45], Chapter VII, contains the following limit theorem.

**Theorem G.** *Suppose that $\frac{1}{2} < \sigma < 1$. Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T|\zeta(\sigma+it)|^4dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}.$$

Theorem G was generalized in [31] for the function $\zeta_\lambda(s)$.

**Theorem H.** *Suppose that the parameter $\lambda$ is irrational, $0 < \lambda < 1$. Then, for $\frac{1}{2} < \sigma < 1$,*

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T|\zeta_\lambda(\sigma+it)|^4dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2}\frac{\sin^2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}.$$

In Section 4.1, the rate of convergence in Theorem H is estimated.

**Theorem 4.1.** *Suppose that the parameter $\lambda$ is irrational, $0 < \lambda < 1$, $\frac{1}{2} < \sigma < 1$ and $T \to \infty$. Then, for every $\varepsilon > 0$,*

$$\int_1^T|\zeta_\lambda(\sigma+it)|^4dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2}\frac{\sin^2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}\right) + O(T^{\frac{3}{2}-\sigma+\varepsilon}).$$

The case of rational parameter $\lambda$ is more complicated, and the analogues of Theorems H and 4.1 are valid in a more narrow region than in the case of irrational $\lambda$. In [32], the following analogue of Theorem H was given.

**Theorem I.** *Suppose that the parameter $\lambda$ is rational, $0 < \lambda < 1$. Then, for $\frac{3}{4} < \sigma < 1$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}.$$

Section 4.2 of the thesis contains the improvement of Theorem I with the rate of convergence.

**Theorem 4.2.** *Suppose that the number $\lambda$ is rational, $0 < \lambda < 1$, $\frac{3}{4} < \sigma < 1$ and $T \to \infty$. Then, for every $\varepsilon > 0$,*

$$\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}\right) + O(T^{\frac{7}{4}-\sigma+\varepsilon}).$$

For the proof of Theorems H, I, 4.1 and 4.2 the approximate functional equation for the function $\zeta_\lambda(s)$ is applied.

# Approbation

The results of the thesis were presented at the MMA (Mathematical Modelling and Analysis) conferences (MMA 2012, June 6 - 9, 2012, Tallinn, Estonia), (MMA 2013, May 27 - 30, Tartu, Estonia), at the 27th Journées Arithmétiques (June 27-July 1, 2011, Vilnius University, Faculty of Mathematics and Informatics, Lithuania), at the International Number Theory Conference (September 8-13, 2013, Šiauliai University, Lithuania), at the Conferences of Lithuanian Mathematical Society (2008, 2011, 2012, 2013), as well as at the seminars of Number Theory in Vilnius University.

# Principal publications

The main results of the thesis are published in the following papers:


1. S. Černigova, One estimate related to the periodic zeta-function, Liet. Matem. Rink. LMD darbai, **51**(2010), 25-30.

2. S. Černigova, On the periodic zeta-function with rational parameter, Liet. Matem. Rink. LMD darbai, **52**(2011), 1-6.

3. S. Černigova, The moments of the periodic zeta-function, Proceedings XII International Conference Algebra and Number Theory: Modern Problems and Application, dedicated to 80-th aniversary of Professor V. N.Latyshev, Tula, (2014), 250-253.

4. S. Černigova, A. Laurinčikas. On the mean square of the periodic zeta-function, in: Anal. Probab. Methods Number Thneory, Kubilius Memorial Volume, A. Laurinčikas et al (Eds), TEV, Vilnius, (2012), 91-99.

5. S. Černigova, A. Laurinčikas, The Atkinson type formula for the periodic zeta-function, Chebysh sb., **XIV**(2)(46)(2013), 180-199.

6. S. Černigova, A. Laurinčikas, On the mean square of the periodic zeta-function. II, Nonlinear Analysis: Modelling and Control (to appear).

## Conferences abstracts


1. S. Černigova, On moments of the periodic zeta-function, Abstracts of 27th Journées Arithmétiques, 2011, p. 14.

2. S. Černigova, A mean square formula for the periodic zeta-function, Abstracts of MMA2012, June 6-9, 2012, Tallinn, Tallinn University of Technology, p. 29.

3. S. Černigova, The mean square of the periodic zeta-function, Abstracts of MMA2013 and AMOE2013, May 27-30, 2013, Tartu, Institute of Mathematics of the University of Tartu, p. 23.

# Acknowledgment

I would like to thank my supervisor Professor Antanas Laurinčikas for various support during my doctoral studies. Also I would like to thank the members of the Department of Probability Theory and Number Theory of Vilnius University for support and useful discussion. I express my gratitude to Professor Kohji Matsumoto for sending me a lot of reprints and for indications some references.

# Chapter 1

# Atkinson type formula for the periodic zeta-function

Let $0 < \lambda \le 1$ be a fixed parameter. We remind that the periodic zeta-function $\zeta_\lambda(s)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

and by analytic continuation elsewhere. For $\lambda = 1$, the function $\zeta_\lambda(s)$ becomes the Riemann zeta-function $\zeta(s)$, thus, it has the unique simple pole at the point $s = 1$ with residue 1. For $0 < \lambda < 1$, the function $\zeta_\lambda(s)$ is entire one.

This chapter is devoted to the Atkinson type formula for the error term in the averaged mean square formula for the periodic zeta-function with rational parameter $\lambda$ in the critical strip.

## 1.1. Statement of the Atkinson-type formula

Let $a$ and $q$ be integers, $1 \le a \le q$. For $\frac{1}{2} < \sigma < 1$, define

$$E_\sigma(q, T) \quad = \sum_{a=1}^{q} \int_0^T |\zeta_{\frac{a}{q}}(\sigma + it)|^2 dt - q\zeta(2\sigma)T$$

$$-\frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma},$$

where, as usual, $\Gamma(s)$ is the Euler gamma-function.

Let $c_1T < N < c_2T$ with some positive constants $c_1 < c_2$. Define

$$N_1 = N_1(q,N,T) = q\left(\frac{T}{2\pi}+\frac{qN}{2}-\left(\left(\frac{qN}{2}\right)^2+\frac{qNT}{2\pi}\right)^{\frac{1}{2}}\right),$$

denote by $\sigma_\alpha(m)$, $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$, the generalized divisor function, i.e.,

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha.$$

Define the functions

$$\operatorname{arsinh}(x) = \log(x+\sqrt{1+x^2}),$$

$$f(T,m,q) = 2T\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)+\sqrt{2\pi qmT+\pi^2q^2m^2}-\frac{\pi}{4}$$

and

$$g(T,m,q) = T\log\left(\frac{qT}{2\pi m}\right)-T+\frac{\pi}{4},$$

and let

$$\begin{aligned}\sum\nolimits_{1,\sigma}(q,T) &= 2^{\sigma-1}q^{1-\sigma}\left(\frac{T}{\pi}\right)^{\frac{1}{2}-\sigma}\sum_{m\le N}\frac{(-1)^{qm}\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)\right)^{-1}\\ &\times\left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos\left(f(T,m,q)\right)\end{aligned}$$

and

$$\begin{aligned}\sum\nolimits_{2,\sigma}(q,T) &= -2q^{1-\sigma}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma}\sum_{m\le N_1}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\left(\frac{qT}{2\pi m}\right)\right)^{-1}\\ &\times\cos\left(g(T,m,q)\right).\end{aligned}$$

**Theorem 1.1.** *Suppose that $\frac{1}{2} < \sigma < \frac{3}{4}$. Then, for $q \le T$,*

$$E_\sigma(q,T) = \sum\nolimits_{1,\sigma}(q,T)+\sum\nolimits_{2,\sigma}(q,T)+R(q,T),$$

*where $R(q,T) = O(q^{\frac{7}{4}-\sigma}\log T)$ with the $O$-constant depending only on $\sigma$.*

If $q = 1$, then we have the Atkinson formula for the Riemann zeta-function obtained in [33].

## 1.2. Lemmas

**Lemma 1.2.** Let $\alpha \neq 1$, $\beta$, $\gamma$ and $T \in \mathbb{R}_+$, $k \in \mathbb{R}$, $|k| \geq 1$, $0 < a < \frac{1}{2}$, $a < \frac{T}{8\pi|k|}$ and $b \geq T$. Then, for every $\varepsilon > 0$,

$$\int_a^b \frac{\exp\{iT\log\frac{1+y}{y} + 2\pi kiy\}dy}{y^\alpha(1+y)^\beta(\log\frac{1+y}{y})^\gamma} = \delta(k)(2k\sqrt{\pi})^{-1}T^{\frac{1}{2}}V^{-\gamma}U^{-\frac{1}{2}}\left(U-\frac{1}{2}\right)^{-\alpha}\left(U+\frac{1}{2}\right)^{-\beta}$$

$$\times\exp\left\{iTV + 2\pi ikU - \pi ik + \frac{\pi i}{4}\right\} + O(a^{1-\alpha}T^{-1}) + O(b^{\gamma-\alpha-\beta}|k|^{-1}) + R(T,k)$$

uniformly for $|\alpha - 1| > \varepsilon$, where

$$U = \left(\frac{T}{2\pi k} + \frac{1}{4}\right)^{\frac{1}{2}},$$

$$V = 2\text{arsinh}\left(\sqrt{\frac{\pi k}{2T}}\right),$$

$$R(T,k) = \begin{cases} T^{\frac{\gamma-\alpha-\beta}{2}-\frac{1}{4}}|k|^{-\frac{\gamma-\alpha-\beta}{2}-\frac{5}{4}} & \text{if} \quad |k| \ll T, \\ T^{-\frac{1}{2}-\alpha}|k|^{\alpha-1} & \text{if} \quad |k| \gg T, \end{cases}$$

and

$$\delta(k) = \begin{cases} 1 & \text{if} \quad k > 0, \\ 0 & \text{if} \quad k < 0. \end{cases}$$

The lemma is Lemma 2 of [1], see also Lemma 15.1 of [16]. In the above form, the lemma is stated in [33].

For $a$, $b$, $\alpha \in \mathbb{R}_+$, and $m$, $q \in \mathbb{N}$, define

$$I\left(a, b; \pm, \frac{m}{q}, \alpha\right) = \int_a^b x^{-\alpha}\left(\text{arsinh}\left(x\sqrt{\frac{\pi q}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{\frac{1}{4}}\left(\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{\frac{1}{2}} + \frac{1}{2}\right)^{-1}$$

$$\times\exp\left\{i\left(\pm 4\pi x\sqrt{\frac{m}{q}} - 2T\text{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right) - \left(2\pi x^2 T + \pi^2 x^4\right)\right)\right\}dx.$$

**Lemma 1.3.** Let $c_1\sqrt{qT} < a < c_2\sqrt{qT}$ with fixed $0 < c_1 < c_2$. Then

$$I\left(a, b; \pm, \frac{m}{q}, \alpha\right) = 4\pi\delta T^{-1}\left(\frac{m}{q}\right)^{\frac{\alpha-1}{2}}\left(\log\left(\frac{Tq}{2\pi m}\right)\right)^{-1}\left(\frac{T}{2\pi} - \frac{m}{q}\right)^{\frac{3}{2}-\alpha}$$

$$\times \exp\left\{i\left(T - T\log\left(\frac{Tq}{2\pi m}\right) - \frac{2\pi m}{q} + \frac{\pi}{4}\right)\right\}$$

$$+O\left(\delta\left(\frac{m}{q}\right)^{\frac{\alpha-1}{2}}\left(\frac{T}{2\pi} - \frac{m}{q}\right)^{1-\alpha} T^{-\frac{3}{2}}\right)$$

$$+O\left(T^{-\frac{\alpha}{2}}\min\left(1, \left|a - \left(a^2 - \frac{2T}{\pi}\right)^{\frac{1}{2}} \pm 2\sqrt{\frac{m}{q}}\right|^{-1}\right)\right)$$

$$+O\left(b^{-\alpha}\left(\frac{n}{q}\right)^{\frac{1}{2}} + O\left(\frac{T}{b}\right)^{-1}\right)$$

$$+O\left(e^{-CT-C\sqrt{\frac{mT}{q}}}\right)$$

with a large constant $C > 0$, where

$$\delta = \begin{cases} 1 & \text{if} \quad m \le \frac{Tq}{2\pi},\ ma^2 \le \left(\frac{Tq^2}{2\pi} - mq\right)^2 \le mb^2 \text{ and the double sign takes } +, \\ 0 & \text{otherwise.} \end{cases}$$

The lemma is a slight modification of Lemma 3 from [1], see also Lemma 15.2 of [16]. The statement of the lemma follows that of Lemma 4 of [33].

The next lemmas are related to the function $\sigma_{1-2\sigma}(m)$. Let

$$D_\sigma(x) = \sum_{m \le x}{}' \sigma_{1-2\sigma}(m),$$

where the sign "$'$" means that the last term in the sum is to be halved if $x \in \mathbb{N}$. Define $\Delta_{1-2\sigma}(x)$ by

$$\Delta_{1-2\sigma}(x) = D_\sigma(x) - \zeta(2\sigma)x - \frac{\zeta(2-2\sigma)x^{2-2\sigma}}{2-2\sigma} + \frac{\zeta(2\sigma-1)}{2}.$$

**Lemma 1.4.** For every $\varepsilon > 0$,

$$\Delta_{1-2\sigma}(x) = O(x^{\frac{1}{4\sigma+1}+\varepsilon}).$$

The lemma is Lemma 2 from [33].

We recall that a series is boundedly convergent if it converges almost everywhere and has bounded partial sums.

**Lemma 1.5.** We have

$$\Delta_{1-2\sigma}(x) = \frac{x^{\frac{3}{4}-\sigma}}{\sqrt{2}\pi} \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}$$

$$\times \left(\cos\left(4\pi\sqrt{mx} - \frac{\pi}{4}\right) - (32\pi\sqrt{mx})^{-1}(16(1-\sigma)^2 - 1)\sin\left(4\pi\sqrt{mx} - \frac{\pi}{4}\right)\right)$$

$$+O(x^{-\frac{1}{4}-\sigma}),$$

the series being boundedly convergent in any fixed finite interval of $x$.

The lemma is Lemma 1 of [33], and is a result of [41] and [8].

## 1.3. Formula for $E_\sigma(q, T)$

Let $u$ and $v$ be complex variables, $Reu > 1$ and $Rev > 1$. Then we have

$$\begin{aligned} \sum_{a=1}^{q} \zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v) &= \sum_{a=1}^{q}\sum_{m=1}^{\infty} \frac{e^{2\pi i \frac{a}{q} m}}{m^u} \sum_{n=1}^{\infty} \frac{e^{-2\pi i \frac{a}{q} n}}{n^v} \\ &= q\zeta(u+v) + \sum_{a=1}^{q} \sum_{\substack{m=1 \\ m\neq n}}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i \frac{a}{q}(m-n)}}{m^u n^v}. \end{aligned} \tag{1.1}$$

Since

$$\sum_{a=1}^{q} e^{2\pi i \frac{a}{q}(m-n)} = \begin{cases} q & \text{if} \quad m \equiv n \pmod q, \\ 0 & \text{if} \quad m \not\equiv n \pmod q, \end{cases}$$

we have from (1.1) that

$$\sum_{a=1}^{q} \zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v) = q(\zeta(u+v) + f_q(u,v) + f_q(v,u)), \tag{1.2}$$

where

$$f_q(u,v) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{1}{m_1^u (m_1 + qm_2)^v}.$$

Using the Poisson summation formula [16] and properties of the gamma-function $\Gamma(s)$, we find that, for $Re(u+v)>2$ and $Reu<0$,

$$f_q(u,v)=\frac{\zeta(u+v-1)\Gamma(u+v-1)\Gamma(1-u)}{q^{u+v-1}\Gamma(v)}+g_q(u,v), \tag{1.3}$$

where

$$g_q(u,v)=\frac{2}{q^{u+v-1}}\sum_{m=1}^{\infty}\sigma_{1-u-v}(m)\int_0^{\infty}\frac{\cos(2\pi mqy)dy}{y^u(1+y)^v}.$$

We need the analytic continuation for $g_q(u,v)$ to a certain region lying in $0<Reu<1$, $0<Rev<1$. Suppose that we have such an analytic continuation. Then, in view of (1.2) and (1.3), we find that

$$\sum_{a=1}^{q}\zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v)=q\left(\zeta(u+v)+\frac{\zeta(u+v-1)\Gamma(u+v-1)\Gamma(1-u)}{q^{u+v-1}\Gamma(v)}\right.$$
$$\left.+\frac{\zeta(u+v-1)\Gamma(u+v-1)\Gamma(1-v)}{q^{u+v-1}\Gamma(u)}\right)+q\left(g_q(u,v)+g_q(v,u)\right).$$

In the latter equality, we take $u=\sigma+it$ and $v=2\sigma-u=\sigma-it$. Then, using the estimate [35]

$$\int_0^T\left(\frac{\Gamma(1-\sigma-it)}{\Gamma(\sigma-it)}+\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\right)dt=\frac{\sin(\pi\sigma)}{1-\sigma}T^{2-2\sigma}+O(T^{-2\sigma}),$$

we obtain that

$$\sum_{a=1}^{q}\int_0^T|\zeta_{\frac{a}{q}}(\sigma+it)|^2dt=q\zeta(2\sigma)T+\frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}$$
$$-iq\int_{\sigma-iT}^{\sigma+iT}g_q(u,2\sigma-u)du+O(qT^{-2\sigma}). \tag{1.4}$$

Now we consider the function $g_q(u,2\sigma-u)$. Define

$$h(u,x)=2\int_0^{\infty}\frac{\cos(2\pi xy)dy}{y^u(1+y)^{2\sigma-u}}.$$

Then, by the definition of $g_q(u,v)$,

$$g_q(u,2\sigma-u)=\frac{1}{q^{2\sigma-1}}\sum_{m=1}^{\infty}\sigma_{1-2\sigma}(m)h(u,mq). \tag{1.5}$$

Suppose that $N \in \mathbb{N}$, and let $X = N + \frac{1}{2}$. Then, by the definition of $D_{1-2\sigma}(x)$ and $\Delta_{1-2\sigma}(x)$, we have that

$$\sum_{m>N} \sigma_{1-2\sigma}(m) h(u, mq) = \int_X^\infty h(u, qx) dD_{1-2\sigma}(x)$$

$$= \int_X^\infty (\zeta(2\sigma) + \zeta(2 - 2\sigma) x^{1-2\sigma}) h(u, qx) dx$$

$$+ \int_X^\infty h(u, qx) d\Delta_{1-2\sigma}(x)$$

$$= -\Delta(X) h(u, qX) - \int_X^\infty \Delta_{1-2\sigma}(x) \frac{\partial h(u, qx)}{\partial x} dx$$

$$+ \int_X^\infty (\zeta(2\sigma) + \zeta(2 - 2\sigma) x^{1-2\sigma}) h(u, qx) dx.$$

This and (1.5) show that

$$g_q(u, 2\sigma - u) = \frac{1}{q^{2\sigma-1}} \sum_{m \le N} \sigma_{1-2\sigma}(m) h(u, mq) - \frac{1}{q^{2\sigma-1}} \Delta_{1-2\sigma}(X) h(u, qX)$$

$$- \frac{1}{q^{2\sigma-1}} \int_X^\infty \Delta_{1-2\sigma}(x) \frac{\partial h(u, qx)}{\partial x} dx$$

$$+ \frac{1}{q^{2\sigma-1}} \int_X^\infty (\zeta(2\sigma) + \zeta(2 - 2\sigma) x^{1-2\sigma}) h(u, qx) dx$$

$$\stackrel{\text{def}}{=} g_{q,1}(u) - g_{q,2}(u) - g_{q,3}(u) + g_{q,4}(u). \qquad (1.6)$$

By the definition, the function $h(u, x)$ is analytic in the $Re u < 1$. Therefore, the functions $g_{q,1}(u)$ and $g_{q,2}(u)$ also are analytic in the latter region.

Using Lemma 1.4 and estimate [1]

$$\frac{\partial h(u, x)}{\partial x} = O(x^{Re u - 2}),$$

we obtain that

$$\frac{1}{q^{2\sigma-1}} \int_X^\infty \Delta_{1-2\sigma}(x) \frac{\partial h(u, qx)}{\partial x} dx \ll q^{Re u - 2\sigma} \int_X^\infty x^{Re u + \frac{1}{4\sigma+1} - 2 + \varepsilon} dx,$$

and the integral is convergent for $Reu < 1 - \frac{1}{4\sigma+1}$. Since $1 - \frac{1}{4\sigma+1} > \sigma$ for $\sigma < \frac{3}{4}$, we have that the function $g_{q,3}(u)$ is analytic in the region including the line $Reu = \sigma$.

It is easily seen that

$$\begin{aligned} g_{q,4}(u) &= \frac{1}{q^{2\sigma-1}} \int_X^\infty (\zeta(2\sigma) + \zeta(2-2\sigma)x^{1-2\sigma}) \\ &\times \left( \int_0^{i\infty} \frac{e^{2\pi iqxy} dy}{y^u(1+y)^{2\sigma-u}} + \int_0^{-i\infty} \frac{e^{-2\pi iqxy} dy}{y^u(1+y)^{2\sigma-u}} \right) dx. \end{aligned}$$

Suppose that $Reu < 0$. Then

$$\frac{1}{q^{2\sigma-1}} \int_X^\infty \left( (\zeta(2\sigma) + \zeta(2-2\sigma)x^{1-2\sigma}) \int_0^{i\infty} \frac{e^{2\pi iqxy} dy}{y^u(1+y)^{2\sigma-u}} \right) dx$$

$$= \frac{1}{2\pi iq^{2\sigma}} (\zeta(2\sigma) + \zeta(2-2\sigma)x^{1-2\sigma}) \int_0^{i\infty} \frac{e^{2\pi iqxy} dy}{y^{u+1}(1+y)^{2\sigma-u}} \Big|_X^\infty$$

$$- \frac{1}{2\pi q^{2\sigma}} \int_X^\infty \left( (\zeta(2-2\sigma)(1-2\sigma)x^{-2\sigma}) \int_0^\infty \frac{e^{2\pi iqxy} dy}{y^{u+1}(1+y)^{2\sigma-u}} \right) dx$$

$$= - \frac{1}{2\pi iq^{2\sigma}} \left( \zeta(2\sigma) + \zeta(2-2\sigma)X^{1-2\sigma} \right) \int_0^{i\infty} \frac{e^{2\pi iqXy} dy}{y^{u+1}(1+y)^{2\sigma-u}}$$

$$- \frac{\zeta(2-2\sigma)(1-2\sigma)}{2\pi iq^{2\sigma}} \int_X^\infty dx \int_0^{i\infty} \frac{e^{2\pi iqy} dy}{y^{u+1}(x+y)^{2\sigma-u}}$$

$$= - \frac{1}{2\pi iq^{2\sigma}} \left( \zeta(2\sigma) + \zeta(2-2\sigma)X^{1-2\sigma} \right) \int_0^\infty \frac{e^{2\pi iqXy} dy}{y^{u+1}(1+y)^{2\sigma-u}}$$

$$- \frac{\zeta(2-2\sigma)(1-2\sigma)X^{1-2\sigma}}{2\pi iq^{2\sigma}(2\sigma-u-1)} \int_0^\infty \frac{e^{2\pi iqXy} dy}{y^{u+1}(1+y)^{2\sigma-u-1}}.$$

Similarly, we find that

$$\frac{1}{q^{2\sigma-1}} \int_X^\infty \left( (\zeta(2\sigma) + \zeta(2-2\sigma)x^{1-2\sigma}) \int_0^{-i\infty} \frac{e^{-2\pi iqxy} dy}{y^{u+1}(1+y)^{2\sigma-u}} \right) dx$$

$$= \frac{1}{2\pi iq^{2\sigma-2}} (\zeta(2\sigma) + \zeta(2-2\sigma)X^{1-2\sigma}) \int_0^\infty \frac{e^{-2\pi iqXy} dy}{y^{u+1}(1+y)^{2\sigma-u}}$$

$$+ \frac{\zeta(2-2\sigma)(1-2\sigma)X^{1-2\sigma}}{2\pi iq^{2\sigma}(2\sigma-u-1)} \int_0^\infty \frac{e^{-2\pi iqXy} dy}{y^{u+1}(1+y)^{2\sigma-u-1}}.$$

The later two equalities yield

$$g_{q,4}(u) = - \frac{1}{\pi q^{2\sigma}} (\zeta(2\sigma) + \zeta(2-2\sigma)X^{1-2\sigma}) \int_0^\infty \frac{\sin(2\pi qXy) dy}{y^{u+1}(1+y)^{2\sigma-u}}$$

$$-\frac{\zeta(2-2\sigma)(1-2\sigma)X^{1-2\sigma}}{\pi q^{2\sigma-2}(2\sigma-u-1)}\int_0^\infty \frac{\sin(2\pi qXy)dy}{y^{u+1}(1+y)^{2\sigma-u-1}}. \tag{1.7}$$

The above integrals are convergent absolutely for $Reu < 1$. Thus, we have analytic continuation for $g_{q,4}(u)$ to the suitable region. Consequently, (1.4) is true for $\frac{1}{2} < \sigma < \frac{3}{4}$.

From (1.4) we find that, for $\frac{1}{2} < \sigma < \frac{3}{4}$,

$$E_\sigma(q,T) = -iq\int_{\sigma-iT}^{\sigma+iT} g_q(u, 2\sigma-u) + O(qT^{-2\sigma}).$$

Therefore, in view of (1.6),

$$E_\sigma(q,T) = -iq^{2-2\sigma}(G_{q,1} - G_{q,2} - G_{q,3} + G_{q,4}) + O(qT^{-2\sigma}), \tag{1.8}$$

where

$$\begin{aligned} G_{q,1} \quad &= 2\sum_{m\le N}\sigma_{1-2\sigma}(m)\int_0^\infty \left(\frac{\cos(2\pi qmy)}{(1+y)^{2\sigma}}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^u du\right)dy \\ &= 4i\sum_{m\le N}\sigma_{1-2\sigma}(m)\int_0^\infty \frac{\cos(2\pi qmy)\sin(T\log\frac{1+y}{y})dy}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}}, \end{aligned}$$

$$G_{q,2} = 4i\Delta_{1-2\sigma}(X)\int_0^\infty \frac{\cos(2\pi qXy)\sin(T\log\frac{1+y}{y})dy}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}},$$

$$\begin{aligned} G_{q,3} \quad &= 4i\int_X^\infty \Delta_{1-2\sigma}(x)\frac{\partial}{\partial x}\left(\int_0^\infty \frac{\cos(2\pi qxy)\sin(T\log\frac{1+y}{y})dy}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}}\right)dx \\ &= 4i\int_X^\infty \Delta_{1-2\sigma}(x)\frac{\partial}{\partial x}\left(\int_0^\infty \frac{\cos(2\pi qy)\sin(T\log\frac{x+y}{y})dy}{y^\sigma(x+y)^\sigma x^{1-2\sigma}\log\frac{x+y}{y}}\right)dx \\ &= 4i\int_X^\infty \Delta_{1-2\sigma}(x)\int_0^\infty \frac{\cos(2\pi qy)}{y^\sigma}\left(\frac{(2\sigma-1)x^{2\sigma-2}\sin(T\log\frac{x+y}{y})}{(x+y)^\sigma\log\frac{x+y}{y}}\right. \\ &\left.+\frac{x^{2\sigma-1}T\cos(T\log\frac{x+y}{y})}{(x+y)^{\sigma+1}} - \frac{\sigma x^{2\sigma-1}\sin(T\log\frac{x+y}{y})}{(x+y)^{\sigma+1}\log\frac{x+y}{y}} - \frac{x^{2\sigma-1}\sin(T\log\frac{x+y}{y})}{(x+y)^{\sigma+1}\log^2\frac{x+y}{y}}\right)dxdy \\ &= 4i\int_X^\infty \frac{\Delta_{1-2\sigma}(x)}{x}\left(\int_0^\infty \frac{\cos(2\pi qxy)}{y^\sigma(1+y)^{\sigma+1}\log\frac{1+y}{y}}\left(T\cos\left(T\log\frac{1+y}{y}\right)\right.\right. \end{aligned}$$

$$+\sin\left(T\log\frac{1+y}{y}\right)\left((2\sigma-1)(1+y)-\sigma-\frac{1}{\log\frac{1+y}{y}}\right)\Bigg)\,dy\Bigg)\,dx,$$

$$G_{q,4} = -\frac{2i}{\pi q}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})\int_0^\infty \frac{\sin(2\pi qXy)\sin(T\log\frac{1+y}{y})}{y^{\sigma+1}(1+y)^\sigma\log\frac{1+y}{y}}$$
$$+\frac{(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}}{\pi q}\int_0^\infty\left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}}\int_{\sigma-iT}^{\sigma+iT}\frac{(\frac{1+y}{y})^u du}{u-2\sigma+1}\right)dy.$$

## 1.4. Proof of Theorem 1.1.

By (1.8), it suffices to evaluate $G_{q,1}-G_{q,4}$. For evaluation of $G_{q,1}$, we apply Lemma 1.2 with $\alpha=\beta=\sigma$, $\gamma=1$, $k=qm$ and $k=-qm$. Then taking $T\ll N\ll T$ gives

$$G_{q,1} = 2^{\sigma-1}q^{\sigma-1}\left(\frac{\pi}{T}\right)^{\sigma-\frac{1}{2}} i\sum_{m\le N}\sigma_{1-2\sigma}(m)m^{\sigma-1}V^{-1}U^{-\frac{1}{2}}\sin(TV+2\pi qmU-\pi qm+\frac{\pi}{4})$$
$$+O(\max(T^{\frac{1}{4}-\sigma}q^{-\frac{7}{4}+\sigma},T^{-\frac{1}{2}}))$$
$$=2^{\sigma-1}q^{\sigma-1}\left(\frac{\pi}{T}\right)^{\sigma-\frac{1}{2}} i\sum_{m\le N}(-1)^{qm}\sigma_{1-2\sigma}(m)m^{\sigma-1}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi mq}{2\pi}}\right)\right)^{-1}$$
$$\times\left(\frac{T}{2\pi mq}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos\left(2T\operatorname{arsinh}\left(\sqrt{\frac{\pi mq}{2\pi}}\right)+2\pi qm\left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{\frac{1}{2}}-\frac{\pi}{4}\right)$$
$$+O\left(\max\left(T^{\frac{1}{4}-\sigma}q^{-\frac{7}{4}+\sigma},T^{-\frac{1}{2}}\right)\right). \qquad (1.9)$$

For $G_{q,2}$, it is sufficient to obtain an estimate. Lemma 1.2 implies that

$$G_{q,2} = O(\Delta_{1-2\sigma}(X)q^{\sigma-1}T^{\frac{1}{2}-\sigma}X^{\sigma-1}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qX}{2T}}\right)\right)^{-1}$$
$$\times\left(\frac{T}{2\pi Xq}+\frac{1}{4}\right)^{-\frac{1}{4}}+O(\Delta_{1-2\sigma}(X)T^{-\frac{3}{2}}q^{\sigma-1}).$$

Therefore, in view of Lemma 1.4,

$$G_{q,2}=O\left(T^{\frac{1-4\sigma}{2(4\sigma+1)}+\varepsilon}q^{\sigma-1}(\log q)^{-1}\right)+O\left(T^{\frac{1}{1-4\sigma}-\frac{3}{2}+\varepsilon}q^{\sigma-1}\right)=O\left(T^{\frac{1-4\sigma}{2(4\sigma+1)}+\varepsilon}q^{\sigma-1}\right). \qquad (1.10)$$

Now we will deal with $G_{q,4}$. First we observe that, in virtue of the residue theorem, for $0 < y \le 1$,

$$\int_{\sigma-iT}^{\sigma+iT} \frac{\left(\frac{1+y}{y}\right)^u du}{u-2\sigma+1} = 2\pi i Res_{u=2\sigma-1}(...) - \left(\int_{\sigma+iT}^{-\infty+iT} + \int_{-\infty-iT}^{\sigma-iT}\right)\left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}$$
$$= 2\pi i \left(\frac{1+y}{y}\right)^{2\sigma-1} + O(T^{-1}y^{-\sigma}).$$

Moreover, for $y \ge 1$,

$$\int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1} = O\left(\int_{\sigma-iT}^{\sigma+iT} \left|\frac{du}{u-2\sigma+1}\right|\right) = O(\log T).$$

Thus,

$$\int_0^\infty \left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right) dy = \left(\int_0^1 + \int_1^\infty\right)(...)dy$$
$$= 2\pi i \int_0^1 \frac{\sin(2\pi qXy)}{y^{2\sigma}} dy + O\left(T^{-1}\int_0^1 \frac{|\sin(2\pi qXy)|dy}{y^{\sigma+1}}\right)$$
$$+ \int_1^\infty \left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right) dy.$$

We have that

$$2\pi i \int_0^1 \frac{\sin(2\pi qXy)dy}{y^{2\sigma}} = 2\pi i \int_0^\infty \frac{\sin(2\pi qXy)}{y^{2\sigma}} dy + O(T^{-1}q^{-1})$$
$$= 2\pi i (2\pi qX)^{2\sigma-1} \int_0^\infty \frac{\sin y dy}{y^{2\sigma}} + O(T^{-1}q^{-1})$$
$$= (2\pi)^{2\sigma}(qX)^{2\sigma-1} i \frac{\pi}{2\Gamma(2\sigma)\sin(\pi\sigma)} + O(T^{-1}q^{-1}),$$

$$T^{-1}\int_0^1 \frac{\sin(2\pi qXy)dy}{y^{\sigma+1}} = O\left(T^{-1}qX \int_0^{(qX)^{-1}} \frac{dy}{y^\sigma}\right) + O\left(T^{-1}\int_{(qX)^{-1}}^\infty \frac{dy}{y^{\sigma+1}}\right) = O(q^\sigma T^{\sigma-1}),$$

and, in view of the estimate

$$\int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1} = O(\log T),$$

$$\int_1^\infty \left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^u\frac{du}{u-2\sigma+1}\right)dy$$

$$= \left(-\frac{\cos(2\pi qXy)}{2\pi qXy(1+y)^{2\sigma-1}}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^u\frac{du}{u-2\sigma+1}\right)\Big|_1^\infty$$

$$-\int_1^\infty \left(\frac{\cos(2\pi qXy)}{2\pi qXy^2(1+y)^{2\sigma-1}}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^u\frac{du}{u-2\sigma+1}\right)dy$$

$$+(1-2\sigma)\int_1^\infty \left(\frac{\cos(2\pi qXy)}{2\pi qXy(1+y)^{2\sigma}}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^u\frac{du}{u-2\sigma+1}\right)dy$$

$$-\int_1^\infty \left(\frac{\cos(2\pi qXy)}{2\pi qXy(1+y)^{2\sigma-1}}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^{u-1}\frac{du}{y^2(u-2\sigma+1)}\right)dy$$

$$= O(q^{-1}T^{-1}\log T).$$

All these estimates show that the second term in the formula for $G_{q,4}$ is

$$i\pi(2\pi)^{2\sigma-1}(1-2\sigma)q^{2\sigma-2}\frac{1}{\Gamma(2\sigma)\sin(\pi\sigma)} + O(q^{\sigma-1}T^{1-\sigma}). \tag{1.11}$$

For the evaluation of the first term of $G_{q,4}$, we apply the second mean value theorem and Lemma 1.2. We write the integral as

$$\int_0^\infty (...)dy = \left(\int_0^{(2qX)^{-1}} + \int_{(2qX)^{-1}}^\infty\right)(...)dy.$$

Then

$$\int_0^{(2qX)^{-1}} (...)dy \le 2\pi qX \int_0^\beta \frac{\sin(T\log\frac{1+y}{y})y^{1-\sigma}(1+y)^{1-\sigma}}{y(1+y)\log\frac{1+y}{y}}dy$$

$$= \frac{2\pi qX\beta^{1-\sigma}(1+\beta)^{1-\sigma}}{\log\frac{1+\beta}{\beta}}\int_\alpha^\beta \frac{\sin(T\log\frac{1+y}{y})dy}{y(1+y)}$$

$$= \frac{2\pi qX\beta^{1-\sigma}(1+\beta)^{1-\sigma}}{\log\frac{1+\beta}{\beta}}\left(T^{-1}\cos\left(T\log\frac{1+y}{y}\right)\right)\Big|_\alpha^\beta = O(q^\sigma T^{\sigma-1}),$$

where $0 \le \alpha \le \beta \le (2qX)^{-1}$. Moreover, an application of Lemma 1.2 gives the estimate

$$\int_{(2qX)^{-1}}^\infty (...)dy = O(q^\sigma T^{\sigma-1}).$$

From these estimates and (1.11), we obtain that

$$G_{q,4} = i\pi(2\pi)^{2\sigma-1}(1-2\sigma)q^{2\sigma-2}\frac{1}{\Gamma(2\sigma)\sin(\pi\sigma)} + O(q^{\sigma-1}T^{\sigma-1}). \tag{1.12}$$

The most complicated is the integral $G_{q,3}$. We apply Lemma 1.2 again and find that, for $x \gg T$,

$$\int_0^\infty \frac{\cos(2\pi qxy)}{y^\sigma(1+y)^{\sigma+1}\log\frac{1+y}{y}}\left(T\cos\left(T\log\frac{1+y}{y}\right)\right) + \sin\left(T\log\frac{1+y}{y}\right)$$
$$\times\left((2\sigma-1)(1+y) - \sigma - \left(\log\frac{1+y}{y}\right)^{-1}\right)dy = i2^{2\sigma-1}\pi^{\sigma-\frac{1}{2}}q^{\sigma-1}x^{\sigma-1}T^{\frac{3}{2}-\sigma}$$
$$\times\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{-\frac{1}{4}}\left(\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)^{-1}$$
$$\times\cos\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right) + 2\pi qx\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}} - \pi qx + \frac{\pi}{4}\right) + O(q^{\sigma-1}T^{\frac{1}{2}-\sigma}x^{\sigma-1}).$$

Hence,

$$G_{q,3} = i2^{\sigma-1}\pi^{\sigma-\frac{1}{2}}q^{\sigma-1}T^{\frac{3}{2}-\sigma}\int_X^\infty \frac{\Delta_{1-2\sigma}(x)}{x^{2-\sigma}}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right)\right)^{-1}$$
$$\times\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{-\frac{1}{4}}\left(\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)^{-1}$$
$$\times\cos\left(2T\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right) + 2\pi qx\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{-\frac{1}{2}} - \pi qx + \frac{\pi}{4}\right)dx$$
$$+O\left(q^{\sigma-1}T^{\frac{1}{2}-\sigma}\int_X^\infty \frac{\Delta_{1-2\sigma}(x)}{x}dx\right). \tag{1.13}$$

In remains to evaluate and estimate the latter integrals.

Using Lemma 1.4 and the restriction $\frac{1}{2} < \sigma < \frac{3}{4}$, we obtain that

$$q^{\sigma-1}T^{\frac{1}{2}-\sigma}\int_X^\infty \frac{\Delta_{1-2\sigma}(x)dx}{x^{2-\sigma}} = O(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}). \tag{1.14}$$

For the evaluation of the first integral in (1.13), we apply Lemma 1.5 and the argument proposed in [33] to avoid the problem arising from the bounded convergence of the series in Lemma 1.5. Thus, by (1.13) and (1.14),

$$G_{q,3} \quad = iq^{\sigma-\frac{3}{4}}\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\sigma}\lim_{b\to\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\int_{\sqrt{qX}}^b x^{-\frac{3}{2}}\left(\cos\left(4\pi x\sqrt{\frac{m}{q}}-\frac{\pi}{4}\right) - \left(32\pi x\sqrt{\frac{m}{q}}\right)^{-1}\right.$$

$$\times \left(16\,(1-\sigma)^2-1\right)\sin\left(4\pi x-\sqrt{\frac{m}{q}}-\frac{\pi}{4}\right)\Bigg)$$

$$\times\left(\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{-\frac{1}{4}}\left(\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^2+\frac{1}{2}\right)^{-1}$$

$$\times\cos\left(2T\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)+(2\pi x^2T+\pi^2x^4)^{\frac{1}{2}}-\pi x^2+\frac{\pi}{4}\right)dx$$

$$+O\left(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}\right).$$

In the notation of Lemma 1.3., this can be rewritten in the form

$$\begin{aligned}G_{q,3} \quad &= iq^{\sigma-\frac{3}{4}}\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\sigma}\lim_{b\to\infty}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\\ &\times\Bigg(ReI\left(\sqrt{qX},b;-,\frac{m}{q},\frac{3}{2}\right)+ImI\left(\sqrt{qX},b;+,\frac{m}{q},\frac{3}{2}\right)+\left(32\pi x\sqrt{\frac{m}{q}}\right)^{-1}\\ &\times\left(16(1-\sigma)^2-1\right)\left(Im\left(\sqrt{qX},b;+,\frac{m}{q},\frac{5}{2}\right)+ReI\left(\sqrt{qX},b;+,\frac{m}{q},\frac{5}{2}\right)\right)\Bigg)\\ &+O\left(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}\right).\end{aligned}\tag{1.15}$$

Define

$$Z=q\left(\frac{T}{2\pi}+\frac{qX}{2}\right)-\left(\left(\frac{qX}{2}\right)^2+\frac{qXT}{2\pi}\right)^{\frac{1}{2}}.$$

Then an application of Lemma 1.3. with $\alpha=\frac{3}{2}$ and $\alpha=\frac{5}{2}$, and $a=\sqrt{qX}$ for (1.15) yields

$$G_{q,3}=iq^{\sigma-\frac{3}{4}}\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\sigma}\lim_{b\to\infty}\Bigg(4\pi q^{-\frac{1}{4}}T^{-1}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}$$

$$\times\left(\log\left(\frac{Tq}{2\pi m}\right)^{-1}\right)\cos\left(T\log\left(\frac{Tq}{2\pi m}\right)-T+\frac{\pi}{4}\right)$$

$$+O\left(q^{-\frac{1}{4}}T^{-1}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\left(\frac{Tq}{2\pi m}\right)\right)^{-1}\left(\frac{T}{2\pi}-\frac{m}{q}\right)^{-1}\right)$$

$$+O\left(q^{-\frac{1}{4}}T^{-\frac{3}{2}}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\frac{T}{2\pi}-\frac{m}{q}\right)^{-\frac{1}{2}}\right)$$

$$+O\left(b^{-\frac{3}{2}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\left(\left(\frac{m}{q}\right)^{\frac{1}{2}}+O\left(\frac{T}{b}\right)\right)^{-1}\right)$$

$$+O\left(e^{-CT}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}e^{-C\sqrt{\frac{mT}{q}}}\right)$$

$$+O\left(T^{-\frac{3}{4}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min\left(1,\left|(q,X)^{\frac{1}{2}}-\left(qX+\frac{2T}{\pi}\right)^{\frac{1}{2}}+2\sqrt{\frac{m}{q}}\right|^{-1}\right)\right)$$

$$+O\left(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}\right). \tag{1.16}$$

Since $\frac{1}{2}<\sigma<\frac{3}{4}$, we have that

$$b^{-\frac{3}{2}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}}}\left(\left(\frac{m}{q}\right)^{\frac{1}{2}}+\left(\frac{T}{b}\right)\right)^{-1}\to 0 \tag{1.17}$$

as $b\to\infty$.

From the definition of $Z$, it follows that $Z\ll T$. Thus, $\frac{Tq}{2\pi}-Z\gg Tq$. Therefore,

$$T^{-1}q^{-\frac{1}{4}}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\left(\frac{Tq}{2\pi m}\right)\right)^{-1}\left(\frac{T}{2\pi}-\frac{m}{q}\right)^{-1}$$

$$\ll T^{-2}q^{-\frac{1}{4}}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\ll T^{\sigma-2}q^{-\frac{1}{4}} \tag{1.18}$$

in view of the estimate

$$\sum_{m\le x}\sigma_{1-2\sigma}(x)\ll x,\ \ x>0. \tag{1.19}$$

Similary, we find that

$$T^{-\frac{3}{2}}q^{-\frac{1}{4}}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\frac{T}{2\pi}-\frac{m}{q}\right)^{-\frac{1}{2}}\ll T^{\sigma-2}q^{-\frac{1}{4}}. \tag{1.20}$$

Since $q\ll T$,

$$e^{CT}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}e^{-C\sqrt{\frac{Tm}{q}}}=O(e^{-c_1}T) \tag{1.21}$$

with some $c_1>0$. We have that

$$\left(\frac{1}{2}\sqrt{q^2X+\frac{2Tq}{\pi}}-\frac{1}{2}\sqrt{q^2X}\right)^2=\frac{q^2X}{2}+\frac{Tq}{2\pi}-q\sqrt{\frac{q^2T^2}{4}+\frac{qXT}{2\pi}}=Z.$$

Thus,

$$\begin{aligned}
&T^{-\frac{3}{4}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min\left(1,\left|(q,X)^{\frac{1}{2}}-\left(qX+\frac{2T}{\pi}\right)^{\frac{1}{2}}+2\sqrt{\frac{m}{q}}\right|^{-1}\right)\\
&\ll q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min(1,|\sqrt{m}-\sqrt{Z}|^{-1})\\
&=q^{\frac{1}{2}}T^{-\frac{3}{4}}\left(\sum_{m\le\frac{Z}{2}}+\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}+\sum_{Z-\sqrt{Z}<m\le Z+\sqrt{Z}}+\sum_{Z+\sqrt{Z}<m\le 2Z}+\sum_{m>2Z}\right)\\
&\times\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min(1,|\sqrt{m}-\sqrt{Z}|^{-1}). \qquad (1.22)
\end{aligned}$$

Clearly, in view of (1.19) and $Z\ll T$,

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m\le\frac{Z}{2}}(...)\ll q^{\frac{1}{2}}T^{-\frac{5}{4}}\sum_{m\le\frac{Z}{2}}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}=q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}, \qquad (1.23)$$

$$\begin{aligned}
&q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}(...)\ll q^{\frac{1}{2}}T^{-\frac{5}{4}}\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}(\sqrt{Z}-\sqrt{m})^{-1}\\
&\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}\sigma_{1-2\sigma}(m)(Z-m)^{-1}\\
&\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\sum_{\sqrt{Z}\le m\le\frac{Z}{2}}\sigma_{1-2\sigma}(Z-m)m^{-1}\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\log T, \qquad (1.24)
\end{aligned}$$

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{Z-\sqrt{Z}<m\le Z+\sqrt{Z}}(...)\ll q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{Z-\sqrt{Z}<m\le Z+\sqrt{Z}}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}} \qquad (1.25)$$

by using Lemma 1.4,

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{Z+\sqrt{Z}<m\le 2Z}(...)\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\log T, \qquad (1.26)$$

and

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m>2Z}(...)\ll q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m>2Z}\frac{\sigma_{1-2\sigma}}{m^{\frac{7}{4}-\sigma}}\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}. \qquad (1.27)$$

Finally, combining (1.16) - (1.18) and (1.20) - (1.27), we obtain that

$$\begin{aligned} G_{q,3} \quad &= 2iq^{\sigma-1}\left(\frac{2\pi}{T}\right)^{\sigma-\frac{1}{2}} \sum_{m\le Z} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \left(\log\left(\frac{Tq}{2\pi m}\right)\right)^{-1} \\ &\times \cos\left(T\log\left(\frac{Tq}{2\pi m}\right) - T + \frac{\pi}{4}\right) + O(q^{\sigma-\frac{1}{4}}\log T). \end{aligned}$$

Thus, from this, (1.8) - (1.10) and (1.12), Theorem 1.1 follows because $Z$ can be replaced by $N_1$ with a negligible error.

# Chapter 2

# Mean square of the function $E(q,T)$

In this chapter, we obtain a formula for

$$\int_2^T E^2(q,t)dt,$$

where

$$E(q,T)=\sum_{a=1}^{q}\int_0^T|\zeta_{\frac{a}{q}}(\frac{1}{2}+it)|^2dt-qT(\log\frac{qT}{2\pi}+2\gamma_0-1),$$

and $a$, $q$ are as in Chapter 1, $\gamma_0$ is the Euler constant. More precisely, we prove the following formula.

**Theorem 2.1.** *For $T\to\infty$ and $q\le\frac{1}{8}T$,*

$$\int_2^T E^2(q,t)dt=\frac{2\sqrt{q}T^{\frac{3}{2}}}{3\sqrt{2\pi}}\sum_{m=1}^{\infty}\frac{d^2(m)}{m^{\frac{3}{2}}}+O(T^{\frac{5}{4}}q^{\frac{3}{4}}\log^4 T).$$

*For $q=o(\frac{T}{\log^{16}T})$, the above formula is asymptotic.*

For the proof of Theorem 2.1., we apply the Atkinson-type formula for $E(q,T)$ obtained in [23] with slightly changed error term.

**Remark 2.2.** *If $q=1$, then*

$$E(q,T)=\int_0^T|\zeta(\frac{1}{2}+it)|^2dt-T\log\frac{T}{2\pi}-(2\gamma_0-1)T,$$

*and Theorem 2.1. contains the result of D. R. Heath-Brown [10].*

## 2.1. Some lemmas

In this section, we state some estimates which will be used in the proof of Theorem 2.1. The first of them is connected to exponential integrals.

**Lemma 2.3.** Let $g_j(t)$, $j = 1, ..., k$, and $f(t)$ be real-valued continuous monotonic functions on $[a, b]$, and let $f(t)$ have a continuous monotonic derivative on $[a, b]$. If $|g_j(t)| \leq M_j$, $j = 1, ..., k$, and $|f'(t)| \geq M_0^{-1}$ on $[a, b]$, then

$$\left| \int_a^b \prod_{j=1}^{k} g_j(t) e^{2\pi i f(t)} dt \right| \leq 2^{k+3} \prod_{j=0}^{k} M_j.$$

Proof of the lemma is given in [16], Lemma 15.3.

The next lemmas are concerned with the divisor function $d(m)$. Let $x \geq 2$.

**Lemma 2.4.** For every $\varepsilon > 0$,

$$\sum_{m \leq x} \sum_{n \leq x} \frac{d(m)d(n)}{(mn)^{\frac{3}{4}}} |\sqrt{m} - \sqrt{n}|^{-1} \ll x^{\varepsilon}.$$

Proof of the lemma can be found in [16], 362 p., 467 p.

**Lemma 2.5.** We have

$$\sum_{m \leq x} \sum_{n \leq x} \frac{d(m)d(n)}{\sqrt{mn}} \left| \log \frac{m}{n} \right|^{-1} \ll x \log^4 x.$$

The estimate of the lemma is given in [16], 469p.

**Lemma 2.6.** The estimate

$$\sum_{m \leq x} \frac{d^2(m)}{m} \ll \log^4 x$$

holds.

Proof of the lemma is given, for example, in [42].

## 2.2. Proof of Theorem 2.1.

We apply the method of D. R. Heath-Brown [10] taking into account the dependence of $E(q,t)$ on the parameter $q$ which can increase together with $T$. We recall that $q \le \frac{1}{8}T$. Denote by $R(q,T)$ the error term in the formula of Theorem E, thus

$$R(q,T) = O(\sqrt{q}\log^2 T). \tag{2.1}$$

Then we have by Theorem E that

$$\begin{aligned}\int_T^{2T} E^2(q,t)dt \quad &= q^2 \int_T^{2T} \sum\nolimits_1^2(q,t)dt + 2q\int_T^{2T} \sum\nolimits_1(q,t)(q\sum\nolimits_2(q,t) + R(q,t))dt \\ &+ \int_T^{2T} (q\sum\nolimits_2(q,t) + R(q,T))^2 dt. \end{aligned} \tag{2.2}$$

We take $N = T$ in Theorema E. Then

$$\sum\nolimits_1(q,t) = \frac{1}{\sqrt{2}} \sum_{m\le T} \frac{(-1)^{qm} d(m)}{\sqrt{qm}} g_1(t,qm) g_2(t,qm)\cos(f(t,qm)),$$

where, for brevity, we write

$$g_1(t,qm) = \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2t}}\right)\right)^{-1},$$

$$g_2(t,qm) = \left(\frac{t}{2\pi qm} + \frac{1}{4}\right)^{-\frac{1}{4}}.$$

Therefore,

$$\begin{aligned}\sum\nolimits_1^2(q,t) \quad &= \frac{1}{2q}\sum_{m\le T}\sum_{n\le T} \frac{(-1)^{qm+qn} d(m)d(n)}{\sqrt{mn}} \\ &\times g_1(t,qm)g_1(t,qn)g_2(t,qm)g_2(t,qn)\cos(f(t,qm))\cos(f(t,qn)) \\ &= \frac{1}{4q}\sum_{m\le T}\sum_{n\le T} \frac{(-1)^{qm+qn} d(m)d(n)}{\sqrt{mn}} \\ &\times g_1(t,qm)g_1(t,qn)g_2(t,qm)g_2(t,qn)\end{aligned}$$

$$\times(\cos(f(t,qm)+f(t,qn))+\cos(f(t,qm)-f(t,qn))). \tag{2.3}$$

Denote by $S_1(q,t)$ a part of $\sum_1^2(q,t)$ in(2.3) with $m=n$. Then

$$\int_T^{2T} S_1(q,t)dt = \frac{1}{4q}Re\sum_{m\le T}\frac{d^2(m)}{m}\int_T^{2T} g_1^2(t,qm)g_2^2(t,qm)e^{2if(t,qm)}dt$$
$$+\frac{1}{4q}\sum_{m\le T}\frac{d^2(m)}{m}\int_T^{2T} g_1^2(t,qm)g_2^2(t,qm)dt. \tag{2.4}$$

We observe that, for $t\in[T,2T]$,

$$g_1(t,qm)\ll\begin{cases}\sqrt{\frac{T}{mq}} & \text{if} \quad m\le\frac{T}{4q},\\ 1 & \text{if} \quad m>\frac{T}{4q},\end{cases} \tag{2.5}$$

and

$$g_2(t,qm)\ll\begin{cases}\left(\frac{mq}{T}\right)^{\frac14} & \text{if} \quad m\le\frac{T}{4q},\\ 1 & \text{if} \quad m>\frac{T}{4q}.\end{cases} \tag{2.6}$$

Moreover,

$$f'(t,qm)=2\text{arsinh}\left(\sqrt{\frac{\pi mq}{2t}}\right). \tag{2.7}$$

Thus, for $t\in[T,2T]$,

$$|f'(t,qm)|\gg\begin{cases}\sqrt{\frac{mq}{T}} & \text{if} \quad m\le\frac{T}{4q},\\ 1 & \text{if} \quad m>\frac{T}{4q}.\end{cases} \tag{2.8}$$

The estimates (2.5), (2.6) and (2.8) together with Lemma 2.3 show that

$$\sum_{m\le T}\frac{d^2(m)}{m}\int_T^{2T} g_1^2(t,qm)g_2^2(t,qm)e^{2if(t,qm)}dt$$
$$\ll\frac{T}{q}\sum_{m\le\frac{T}{4q}}\frac{d^2(m)}{m^2}+\sum_{\frac{T}{4q}<m\le T}\frac{d^2(m)}{m}\ll\frac{T}{q}\log^4 T \tag{2.9}$$

in virtue of Lemma 2.6.

The second integral in the right-hand side of (2.4) will give the main term of the formula of Theorem 2.1. Therefore, we have to use for its evaluation more precise expressions for $g_1(t,qm)$ and $g_2(t,qm)$. It is easily seen that, for $t\in[T,2T]$,

$$g_1^2(t,qm)=\begin{cases}\frac{2t}{\pi mq}+O(1) & \text{if} \quad m\le\frac{T}{4q},\\ O(1) & \text{if} \quad m>\frac{T}{4q},\end{cases}$$

and

$$g_2^2(t,qm) = \begin{cases} \left(\frac{2\pi qm}{t}\right)^{\frac{1}{2}} + O\left(\frac{qm}{t}\right)^{\frac{3}{2}} & \text{if} \quad m \le \frac{T}{4q}, \\ O(1) & \text{if} \quad m > \frac{T}{4q}. \end{cases}$$

Therefore, for $t \in [T, 2T]$,

$$g_1^2(t,qm) g_2^2(t,qm) = \begin{cases} 2^{\frac{3}{2}} \sqrt{\frac{t}{\pi mq}} + O\left(\sqrt{\frac{qm}{t}}\right) & \text{if} \quad m \le \frac{T}{4q}, \\ O(1) & \text{if} \quad m > \frac{T}{4q}. \end{cases}$$

Hence, for every $\varepsilon > 0$,

$$\begin{aligned} &\frac{1}{4q} \sum_{m \le T} \frac{d^2(m)}{m} \int_T^{2T} g_1^2(t,qm) g_2^2(t,qm) dt = \frac{1}{\sqrt{2\pi} q^{\frac{3}{2}}} \sum_{m \le \frac{T}{4q}} \frac{d^2(m)}{m^{\frac{3}{2}}} \int_T^{2T} t^{\frac{1}{2}} dt \\ &+ O\left( \sqrt{\frac{1}{Tq}} \sum_{m \le \frac{T}{4q}} \frac{d^2(m)}{\sqrt{m}} \int_T^{2T} dt \right) + O\left( \frac{1}{q} \sum_{\frac{T}{4q} < m \le T} \frac{d^2(m)}{m} \int_T^{2T} dt \right) = \\ &\frac{2}{3\sqrt{2\pi} q^{\frac{3}{2}}} \sum_{m \le \frac{T}{4q}} \frac{d^2(m)}{m^{\frac{3}{2}}} ((2T)^{\frac{3}{2}} - T^{\frac{3}{2}}) + O\left( \sqrt{\frac{T}{q}} \sum_{m \le \frac{T}{4q}} \frac{d^2(m)}{\sqrt{m}} \right) \\ &+ O\left( \frac{T}{q} \sum_{m \le T} \frac{d^2(m)}{m} \right) = \frac{2}{3\sqrt{2\pi} q^{\frac{3}{2}}} \sum_{m=1}^{\infty} \frac{d^2(m)}{m^{\frac{3}{2}}} ((2T)^{\frac{3}{2}} - T^{\frac{3}{2}}) \\ &+ O\left( \left( \frac{T}{q} \right)^{1+\varepsilon} \right) + O\left( \frac{T}{q} \log^4 T \right) \qquad (2.10) \end{aligned}$$

in virtue of Lemma 2.6.

Now let $S_2(q,t)$ be a part of $\sum_1^2(q,t)$ in (2.3) with $m \ne n$. In this case, we need a lower estimate for

$$f'(t,qm) \pm f'(t,qn).$$

Using (2.7), we find that, for $t \in [T, 2T]$,

$$|f'(t,qm) \pm f'(t,qn)| \gg \begin{cases} \sqrt{\frac{q}{T}} |\sqrt{m} \pm \sqrt{n}| & \text{if} \quad m \le \frac{T}{4q}, n \le \frac{T}{4q}, \\ |\log \frac{m}{n}| & \text{if} \quad m > \frac{T}{4q}, n > \frac{T}{4q}, \\ \min\left( \sqrt{\frac{q}{T}} |\sqrt{m} \pm \sqrt{n}|, |\log \frac{m}{n}|, 1 \right) & \text{if} \quad m \le \frac{T}{4q}, n > \frac{T}{4q} \\ & \text{or} \quad n \le \frac{T}{4q}, m > \frac{T}{4q}. \end{cases}$$

Therefore, taking into account estimates (2.5) and (2.6), and applying Lemmas 2.3 - 2.5, we obtain

$$\begin{aligned}\int_T^{2T} S_2(q,t)dt \quad &\ll \frac{T}{q^2}\sum_{m\le\frac{T}{4q}}\sum_{n\le\frac{T}{4q}}\frac{d(m)d(n)}{(mn)^{\frac{3}{4}}}|\sqrt{m}-\sqrt{n}|^{-1}\\ &+\frac{1}{q}\sum_{\frac{T}{4q}<m\le T}\sum_{\frac{T}{4q}<n\le T}\frac{d(m)d(n)}{\sqrt{mn}|\log\frac{m}{n}|}\\ &+\frac{1}{q^{\frac{3}{4}}}\sum_{m\le T}\sum_{n\le T}\frac{d(m)d(n)}{(mn)^{\frac{3}{4}}}|\sqrt{m}-\sqrt{n}|^{-1}\\ &+\frac{T^{\frac{1}{4}}}{q^{\frac{5}{4}}}\sum_{m\le T}\sum_{n\le T}\frac{d(m)d(n)}{\sqrt{mn}|\log\frac{m}{n}|}\\ &+\frac{T^{\frac{1}{4}}}{q^{\frac{5}{4}}}\sum_{m\le T}\sum_{n\le T}\frac{d(m)d(n)}{\sqrt{mn}}\\ &\ll \frac{T^{1+\varepsilon}}{q^{2+\varepsilon}}+\frac{T}{q}\log^4 T+\frac{T^{\varepsilon}}{q^{\frac{3}{4}}}+\frac{T^{\frac{5}{4}}}{q^{\frac{5}{4}}}\log^4 T\\ &\ll \left(\frac{T}{q}\right)^{\frac{5}{4}}\log^4 T.\end{aligned}$$

This, together with (2.4), (2.9) and (2.10) show that

$$\int_T^{2T} {\sum}_1^2(q,t)dt = \frac{2}{3\sqrt{2\pi}q^{\frac{3}{2}}}\sum_{m=1}^{\infty}\frac{d^2(m)}{m^{\frac{3}{2}}}((2T)^{\frac{3}{2}}-T^{\frac{3}{2}})+O\left(\frac{T}{q}\right)^{\frac{5}{4}}\log^4 T. \tag{2.11}$$

It remains to estimate the integral

$$\int_T^{2T} {\sum}_2^2(q,t)dt.$$

We have that

$$\begin{aligned}{\sum}_2^2(q,t) \quad &= \frac{4}{q}\sum_{m\le N_1}\sum_{n\le N_1}\frac{d(m)d(n)}{\sqrt{mn}}\left(\log\frac{qt}{2\pi m}\right)^{-1}\left(\log\frac{qt}{2\pi n}\right)^{-1}\cos(g(t,qm))\cos(g(t,qn))\\ &= \frac{2}{q}\sum_{m\le N_1}\sum_{n\le N_1}\frac{d(m)d(n)}{\sqrt{mn}}\left(\log\frac{qt}{2\pi m}\right)^{-1}\left(\log\frac{qt}{2\pi n}\right)^{-1}\\ &\times(\cos(g(t,qm)+g(t,qn))+\cos(g(t,qm)-g(t,qn))),\end{aligned} \tag{2.12}$$

where

$$g(t, qm) = t\log\frac{qt}{2\pi m} - t + \frac{2\pi m}{q} + \frac{\pi}{4},$$

and

$$N_1 = N_1(q,t,T) = q\left(\frac{t}{2\pi} + \frac{qT}{2} - \left(\left(\frac{qT}{2}\right)^2 + \frac{qTt}{2\pi}\right)^{\frac{1}{2}}\right).$$

We have that

$$N_1 \le q\left(\frac{t}{2\pi} + \frac{qT}{2} - \frac{qT}{2}\left(1 + \frac{t}{\pi qT} - \frac{t^2}{2(\pi qT)^2}\right)\right) = \frac{t^2}{4\pi^2 T}.$$

Therefore, for $m \le N_1$ and $t \in [T, 2T]$, $\frac{qt}{2\pi m} \ge \pi q$, hence

$$\left(\log\frac{qt}{2\pi m}\right)^{-1} \gg 1. \tag{2.13}$$

Moreover, for $m \le N_1$, $n \in N_1$, $m \ne n$, and $t \in [T, 2T]$,

$$(g(t,qm) + g(t,qn))' = \left(t\log\frac{(qt)^2}{4\pi^4 mn} - 2t + \frac{2\pi}{q}(m+n) + \frac{\pi}{2}\right)' = \log\frac{(qt)^2}{4\pi^2 mn}$$

$$= \log\frac{qt}{2\pi m} + \log\frac{qt}{2\pi n} \gg \left|\log\frac{qt}{2\pi m} - \log\frac{qt}{2\pi n}\right| \gg \left|\log\frac{m}{n}\right| \tag{2.14}$$

and

$$(g(t,qm) - g(t,qn))' \quad = \quad \left(t\log\frac{m}{n} + \frac{2\pi(m-n)}{q}\right)' \gg \left|\log\frac{m}{n}\right|. \tag{2.15}$$

Denote by $Z_1(q,t)$ a part of $\sum_2^2(q,t)$ in(2.12) with $m \ne n$, and let $T_1 \ge T$ be such that $N_1(q,t,T) \ge \max(m,n)$ for $t \ge T_1$. Then an application of Lemmas 2.3 and 2.5, and (2.12) - (2.15) yields

$$\begin{aligned}\int_T^{2T} Z_1(q,t)dt \quad &= \frac{2}{q}\int_T^{2T} \sum\sum_{\substack{m\le N_1(q,t,T), n\le N_1(q,t,T)\\ m\ne n}} \frac{d(m)d(n)}{\sqrt{mn}}\left(\log\frac{qt}{2\pi m}\right)^{-1}\left(\log\frac{qt}{2\pi n}\right)^{-1} \\ &\times(\cos(g(t,qm)+g(t,qn)) + \cos(g(t,qm)-g(t,qn))dt \\ &= \frac{2}{q}\sum\sum_{\substack{m\le N_1(q,2T,T), n\le N_1(q,2T,T)\\ m\ne n}} \frac{d(m)d(n)}{\sqrt{mn}}\int_{T_1}^{T}\left(\log\frac{qt}{2\pi m}\right)^{-1}\left(\log\frac{qt}{2\pi n}\right)^{-1}\end{aligned}$$

$$\times(\cos(g(t,qm)+g(t,qn))+\cos(g(t,qm)-g(t,qn))dt$$

$$\ll \frac{1}{q}\sum_{\substack{m\le T,n\le T\\ m\ne n}}\sum \frac{d(m)d(n)}{\sqrt{mn}}\left|\log\frac{m}{n}\right| \ll \frac{T}{q}\log^4 T. \tag{2.16}$$

Let $Z_2(q,t)$ denote a part of $\sum_2^2(q,t)$ in (2.12) with $m=n$. Then, by Lemma 2.6, we find that

$$\int_T^{2T} Z_2(q,t)dt \ll \frac{1}{q}\sum_{m\le T}\frac{d^2(m)}{m}\int_T^{2T} dt \ll \frac{T}{q}\log^4 T.$$

This, (2.12) and (2.16) give the bound

$$\int_T^{2T} q^2\sum\nolimits_2^2(q,t)dt \ll Tq\log^4 T. \tag{2.17}$$

Obviously, by (2.1),

$$\int_T^{2T} R^2(q,t)dt \ll Tq\log^4 T.$$

Therefore, in view of (2.17),

$$\int_T^{2T}(q\sum\nolimits_2(q,t)+R(q,t))^2 dt \ll \int_T^{2T} q^2\sum\nolimits_2^2(q,t)dt + \int_T^{2T} R^2(q,t)dt \ll Tq\log^4 T. \tag{2.18}$$

Applying the Cauchy-Schwarz inequality, we deduce from(2.11) and (2.18) the estimate

$$\int_T^{2T} q\sum\nolimits_1(q,t)\left(q\sum\nolimits_2(q,t)+R(q,t)\right)dt \ll q\left(\int_T^{2T}\sum\nolimits_1^2(q,t)dt\right)^{\frac{1}{2}}$$

$$\times\left(\int_T^{2T}\left(q\sum\nolimits_2(q,t)+R(q,t)\right)^2 dt\right)^{\frac{1}{2}} \ll (T^{\frac{3}{4}}q^{\frac{1}{4}}+T^{\frac{5}{8}}q^{\frac{3}{8}}\log^2 T)$$

$$\times T^{\frac{1}{2}}q^{\frac{1}{2}}\log^2 T \ll T^{\frac{5}{4}}q^{\frac{3}{4}}\log^2 T.$$

Combining this with (2.2), (2.11) and (2.18) gives the formula

$$\int_T^{2T} E^2(q,t)dt = \frac{2\sqrt{q}}{3\sqrt{2\pi}}\sum_{m=1}^{\infty}\frac{d^2(m)}{m^{\frac{3}{2}}}((2T)^{\frac{3}{2}}-T^{\frac{3}{2}})+O(T^{\frac{5}{4}}q^{\frac{3}{4}}\log^4 T).$$

Now we take in the latter formula $T2^{-j}$, $j=1,2,...,[\log_2 T]+1$ in place of $T$, and sum the results over $j$. This shows that

$$\int_2^T E^2(q,t) = \frac{2\sqrt{q}}{3\sqrt{2\pi}}T^{\frac{3}{2}}\sum_{m=1}^{\infty}\frac{d^2(m)}{m^{\frac{3}{2}}}+O(T^{\frac{5}{4}}q^{\frac{3}{4}}\log^4 T). \tag{2.19}$$

If $q \leq \frac{T\psi_T}{\log^{16} T}$, where $\psi_T = o(1)$ as $T \to \infty$, then, clearly,

$$T^{\frac{5}{4}} q^{\frac{3}{4}} \log^4 T = T^{\frac{3}{2}} q^{\frac{1}{2}} T^{-\frac{1}{4}} q^{\frac{1}{4}} \log^4 T \leq T^{\frac{3}{2}} q^{\frac{1}{2}} \psi_T^{\frac{1}{4}} = o(T^{\frac{3}{2}} q^{\frac{1}{2}})$$

as $T \to \infty$.

*Proof of Remark 2.2.* Since the series

$$\sum_{m=1}^{\infty} \frac{d^2(m)}{m^{\frac{3}{2}}}$$

is convergent, the first term in the right-hand side of (2.19) admits the estimate $\gg (T^{\frac{3}{2}} q^{\frac{1}{2}})$. Thus, we have an asymptotic formula.

# Chapter 3

# Mean square of the function $E_\sigma(q,T)$

In this chapter, we extend Theorem 2.1 to the strip $\{s\in\mathbb{C}:\frac{1}{2}<\sigma<\frac{3}{4}\}$. We remind that

$$\begin{aligned}E_\sigma(q,T) &= \sum_{a=1}^{q}\int_0^T |\zeta_{\frac{a}{q}}(\sigma+it)|^2dt - q\zeta(2\sigma)T\\ &-\frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}.\end{aligned}$$

The main result of the chapter is the following statement.

**Theorem 3.1** *Let $\sigma$, $\frac{1}{2}<\sigma<\frac{3}{4}$, be fixed. Then, for $T\to\infty$ and $q\le T^{1-\frac{4\sigma}{3}-\varepsilon}$ with every $\varepsilon>0$,*

$$\begin{aligned}\int_2^T E_\sigma^2(q,t)dt &= 2(5-4\sigma)^{-1}(2\pi)^{2\sigma-\frac{3}{2}}q^{\frac{3}{2}-2\sigma}T^{\frac{5}{2}-2\sigma}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac{5}{2}-2\sigma}}\\ &+O(q^{\frac{11}{4}-2\sigma}T^{\frac{7}{4}-\sigma}\log T).\end{aligned}$$

**Remark 3.2.** *If $q\le T^{\frac{3}{5}-\frac{4\sigma}{5}-\varepsilon}$ with arbitrary $\varepsilon>0$, then the equality of Theorem 3.1 is asymptotic.*

## 3.1. Some estimates

In this section, we present some known estimates that will be applied in the proof of Theorem 3.1.

**Lemma 3.3.** The estimates

$$\sum_{m\le x}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}\ll x^{2\sigma-1}$$

and

$$\sum_{m\le x}\frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}}\ll x^{2\sigma-1}$$

are true.

*Proof of Lemma 3.3.* Since $1-2\sigma<0$, the series over primes

$$\sum_{p}\frac{|\sigma^2_{1-2\sigma}(p)-1|\log p}{p}$$

is convergent. Therefore, from general mean values theorems for multiplicative functions, see, for example, [25], we obtain that

$$\sum_{m\le x}\sigma^2_{1-2\sigma}(m)\ll x.$$

This and summing by parts give the estimate of the lemma.

The second estimate of the lemma is obtained similarly.

**Lemma 3.4.** For every $\varepsilon>0$,

$$\sum_{\substack{m\le x\\ m\ne n}}\sum_{n\le x}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{\frac{5}{4}-\sigma}}|\sqrt{m}-\sqrt{n}|^{-1}\ll x^{2\sigma-1+\varepsilon}.$$

*Proof of Lemma 3.4.* We have that

$$\sum_{\substack{m\le x\\ m\ne n}}\sum_{n\le x}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{\frac{5}{4}-\sigma}}|\sqrt{m}-\sqrt{n}|^{-1}$$

$$\ll\sum_{n<m\le x}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{\frac{5}{4}-\sigma}}(\sqrt{m}-\sqrt{n})^{-1}$$

$$\ll\sum_{n\le\frac{m}{2}}+\sum_{n>\frac{m}{2}}.$$

We observe that, for $n \le \frac{m}{2}$, we have that $m - n \ge \frac{m}{2}$, thus, $(m-n)^{-1} \ll m^{-1}$ for $n \le \frac{m}{2}$. Therefore, using Lemma 3.3 and partial summation, we find

$$\begin{aligned}
\sum_{n \le \frac{m}{2}} &\ll \sum_{m \le x} \frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}} \sum_{n \le \frac{m}{2}} \frac{\sigma_{1-2\sigma}(n)}{n^{\frac{5}{4}-\sigma}} \left(\sqrt{m} + \sqrt{n}\right) (m-n)^{-1} \\
&\ll \sum_{m \le x} \frac{\sigma_{1-2\sigma}(m)}{m^{\frac{3}{4}-\sigma}} \sum_{n \le \frac{m}{2}} \frac{\sigma_{1-2\sigma}(n)}{n^{\frac{5}{4}-\sigma}} (m-n)^{-1} \\
&\ll \sum_{m \le x} \frac{\sigma_{1-2\sigma}(m)}{m^{\frac{7}{4}-\sigma}} \sum_{n \le \frac{m}{2}} \frac{\sigma_{1-2\sigma}(n)}{n^{\frac{5}{4}-\sigma}} \\
&\ll \sum_{m \le x} \frac{\sigma_{1-2\sigma}(m)}{m^{\frac{7}{4}-\sigma}} m^{\sigma - \frac{1}{4}} = \sum_{m \le x} \frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \ll x^{2\sigma-1},
\end{aligned}$$

and, for every $\varepsilon > 0$,

$$\sum_{n > \frac{m}{2}} \ll \textstyle\sum_{m \le x} \frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \sum_{\frac{m}{2} < n < m} \frac{\sigma_{1-2\sigma}(n)}{m-n} \ll x^{2\sigma - 1 + \varepsilon}.$$

**Lemma 3.5.** We have

$$\sum_{\substack{m \le x \\ m \ne n}} \sum_{n \le x} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \left| \log \frac{m}{n} \right|^{-1} \ll x^{2\sigma} \log x.$$

*Proof of Lemma 3.5.* In view of Lemma 3.3 and the estimate [16]

$$\sum_{m \le x} \left| \log \frac{m}{n} \right|^{-1} \ll x + n \log x,$$

we find that the considered sum is estimated as

$$\begin{aligned}
&\ll \sum_{\substack{m \le x \\ m \ne n}} \sum_{n \le x} \left( \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} + \frac{\sigma_{1-2\sigma}^2(n)}{n^{2-2\sigma}} \right) \left| \log \frac{m}{n} \right|^{-1} \\
&\ll \sum_{\substack{m \le x \\ m \ne n}} \sum_{n \le x} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \left| \log \frac{m}{n} \right|^{-1} \\
&\ll \sum_{m \le x} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \sum_{n \le x} \left| \log \frac{m}{n} \right|^{-1} \\
&\ll x^{2\sigma} + \log x \sum_{m \le x} \frac{\sigma_{1-2\sigma}^2(m)}{m^{1-2\sigma}} \ll x^{2\sigma} \log x.
\end{aligned}$$

## 3.2. Proof of Theorem 3.1.

For brevity, we write

$$\sum\nolimits_1 (q,T) = \sum\nolimits_{1,\sigma} (q,T)$$

and

$$\sum\nolimits_2 (q,T) = \sum\nolimits_{2,\sigma} (q,T).$$

Then by Theorem 1.1, we have that

$$\begin{aligned} \int_T^{2T} E_\sigma^2(q,t)dt \quad &= \int_T^{2T} \sum\nolimits_1^2 (q,T)dt + 2\int_T^{2T} \sum\nolimits_1 (q,T) \left(\sum\nolimits_2 (q,T) + R(q,t)\right) dt \\ &+ \int_T^{2T} \left(\sum\nolimits_2 (q,T) + R(q,t)\right)^2 dt. \end{aligned} \tag{3.1}$$

For brevity, we write

$$g_1(t,qm) = \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2t}}\right)\right)^{-1},$$

$$g_2(t,qm) = \left(\frac{t}{2\pi qm} + \frac{1}{4}\right)^{-\frac{1}{4}}$$

and

$$f(t,qm) = 2t\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2t}}\right) + \sqrt{2\pi qmt + t^2q^2m^2} - \frac{\pi}{4}.$$

Then, taking $N = T$ in Theorem 1.1, we have that

$$\begin{aligned} \sum\nolimits_1 (q,t) \quad &= 2^{\sigma-1}q^{1-\sigma}\left(\frac{t}{\pi}\right)^{\frac{1}{2}-\sigma} \sum_{m\le T} \frac{(-1)^{qm}\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \\ &\times g_1(t,qm)g_2(t,qm)\cos(f(t,qm)). \end{aligned}$$

Hence,

$$\sum\nolimits_1^2 (q,t) \quad = 2^{2\sigma-2}q^{2-2\sigma}\left(\frac{t}{\pi}\right)^{1-2\sigma} \sum_{m\le T}\sum_{n\le T} \frac{(-1)^{qm+qn}\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}}$$

$$\times g_1(t,qm)g_1(t,qn)g_2(t,qm)g_2(t,qn)\cos(f(t,qm))\cos(f(t,qn))$$

$$= 2^{2\sigma-3}q^{2-2\sigma}\left(\frac{t}{\pi}\right)^{1-2\sigma}\sum_{m\le T}\sum_{n\le T}\frac{(-1)^{qm+qn}\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}}$$

$$\times g_1(t,qm)g_1(t,qn)g_2(t,qm)g_2(t,qn)$$

$$\times(\cos(f(t,qm)+f(t,qn))+\cos(f(t,qm)-f(t,qn))). \tag{3.2}$$

Let $S_1(q,t)$ be the part of $\sum_1^2(q,t)$ in (3.2) with $m=n$. Then

$$\int_T^{2T} S_1(q,t)dt \quad = 2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}Re\sum_{m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}$$

$$\times\int_T^{2T} t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)e^{2if(t,qm)}dt$$

$$+2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}\sum_{m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}\int_T^{2T} t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)dt. \tag{3.3}$$

It is not difficult to see that, for $t\in[T,2T]$,

$$g_1(t,qm)\ll\begin{cases}\sqrt{\frac{T}{mq}} & \text{if} \quad m\le\frac{T}{4q},\\ 1 & \text{if} \quad m>\frac{T}{4q},\end{cases} \tag{3.4}$$

and

$$g_2(t,qm)\ll\begin{cases}\left(\frac{mq}{T}\right)^{\frac14} & \text{if} \quad m\le\frac{T}{4q},\\ 1 & \text{if} \quad m>\frac{T}{4q}.\end{cases} \tag{3.5}$$

Moreover, for $t\in[T,2T]$,

$$|f'(t,qm)|\gg\begin{cases}\sqrt{\frac{mq}{T}} & \text{if} \quad m\le\frac{T}{4q},\\ 1 & \text{if} \quad m>\frac{T}{4q}.\end{cases} \tag{3.6}$$

In view of estimates (3.4) - (3.6), using Lemmas 2.3 and 3.3, we find that

$$2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}Re\sum_{m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}$$

$$\times\int_T^{2T} t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)e^{2if(t,qm)}dt$$

$$\ll q^{1-2\sigma}T^{2-2\sigma}\sum_{m\le\frac{T}{4q}}\frac{\sigma_{1-2\sigma}^2(m)}{m^{3-2\sigma}}$$

$$+q^{2-2\sigma}T^{1-2\sigma}\sum_{\frac{T}{4q}<m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}\le q^{1-2\sigma}T^{2-2\sigma}+q^{2-2\sigma}. \tag{3.7}$$

For the second term in the right-hand side of (3.3), we use more precise estimates for the functions $g_1(t,qm)$ and $g_2(t,qm)$, namely, for $t\in[T,2T]$,

$$g_1^2(t,qm)=\begin{cases}\frac{2t}{\pi mq}+O(1) & \text{if} \quad m\le\frac{T}{4q},\\ O(1) & \text{if} \quad m>\frac{T}{4q},\end{cases}$$

and

$$g_2^2(t,qm)=\begin{cases}\left(\frac{2\pi qm}{t}\right)^{\frac12}+O\left(\left(\frac{qm}{t}\right)^{\frac32}\right) & \text{if} \quad m\le\frac{T}{4q},\\ O(1) & \text{if} \quad m>\frac{T}{4q}.\end{cases}$$

Hence, for $t\in[T,2T]$,

$$t^{1-2\sigma}g_1(t,qm)g_2(t,qm)=\begin{cases}2^{\frac23}t^{\frac32-2\sigma}\left(\pi mq\right)^{-\frac12}+O\left(t^{\frac12-2\sigma}\left(qm\right)^{\frac12}\right) & \text{if} \quad m\le\frac{T}{4q},\\ O(t^{1-2\sigma}) & \text{if} \quad m>\frac{T}{4q}.\end{cases}$$

Therefore,

$$2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}\sum_{m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}\int_T^{2T}t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)dt$$

$$=2^{2\sigma-\frac32}\pi^{2\sigma-\frac32}q^{\frac32-2\sigma}\sum_{m\le\frac{T}{4q}}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac52-2\sigma}}\int_T^{2T}t^{\frac32-2\sigma}dt$$

$$+O\left(q^{\frac52-2\sigma}\sum_{m\le\frac{T}{4q}}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac32-2\sigma}}\int_T^{2T}t^{\frac12-2\sigma}dt\right)$$

$$+O\left(q^{2-2\sigma}\sum_{\frac{T}{4q}<m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}\int_T^{2T}t^{1-2\sigma}dt\right)$$

$$=\frac{2^{2\sigma-\frac32}\pi^{2\sigma-\frac32}q^{\frac32-2\sigma}}{\frac52-2\sigma}+\sum_{m\le\frac{T}{4q}}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac52-2\sigma}}((2T)^{\frac52-2\sigma}-T^{\frac52-2\sigma})$$

$$+O\left(q^{\frac52-2\sigma}T^{\frac32-2\sigma}\sum_{m\le\frac{T}{4q}}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac32-2\sigma}}\right)+O\left(q^{2-2\sigma}T^{2-2\sigma}\sum_{\frac{T}{4q}<m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}\right)$$

$$= \frac{2^{2\sigma-\frac{3}{2}}\pi^{2\sigma-\frac{3}{2}}q^{\frac{3}{2}-2\sigma}}{\frac{5}{2}-2\sigma} + \sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac{5}{2}-2\sigma}}((2T)^{\frac{5}{2}-2\sigma} - T^{\frac{5}{2}-2\sigma}) + O(q^{2-2\sigma}T) \tag{3.8}$$

by application of Lemma 3.3.

Denote by $S_2(q,t)$ the part of $\sum_1^2(q,t)$ in (3.2) with $m \neq n$. In this case, we use the estimate, for $t \in [T, 2T]$,

$$f'(t,qm) \pm f'(t,qn) \gg \begin{cases} \sqrt{\frac{q}{T}}|\sqrt{m} \pm \sqrt{n}| & \text{if} \quad m \leq \frac{T}{4q}, n \leq \frac{T}{4q}, \\ \left|\log\frac{m}{n}\right| & \text{if} \quad m > \frac{T}{4q}, n > \frac{T}{4q}, \\ 1 & \text{if} \quad m \leq \frac{T}{4q}, n > \frac{T}{4q} \text{ or } m > \frac{T}{4q}, n \leq \frac{T}{4q}. \end{cases}$$

This, Lemmas 2.3, 3.3 - 3.5 and estimates (3.4), (3.5) give the estimate

$$\begin{aligned}\int_T^{2T} S_2(q,t)dt \quad & \ll q^{1-2\sigma}T^{2-2\sigma}\sum_{\substack{m\leq\frac{T}{4q} \\ m\neq n}}\sum_{n\leq\frac{T}{4q}}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(m)}{(mn)^{\frac{5}{4}-\sigma}}|\sqrt{m}-\sqrt{n}|^{-1} \\ & + q^{2-2\sigma}T^{1-2\sigma}\sum_{\substack{\frac{T}{4q}<m\leq T \\ m\neq n}}\sum_{\frac{T}{4q}<m\leq T}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}\left|\log\frac{m}{n}\right|} \\ & + q^{\frac{7}{4}-2\sigma}T^{\frac{5}{4}-2\sigma}\sum_{\substack{m\leq T \\ m\neq n}}\sum_{n\leq T}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\ & \ll q^{2-4\sigma}T^{1+\varepsilon} + q^{2-2\sigma}T\log T + q^{\frac{7}{4}-2\sigma}T^{\frac{1}{4}} \ll q^{2-2\sigma}T^{1+\varepsilon}.\end{aligned}$$

This, (3.3), (3.7) and (3.8) show that

$$\begin{aligned}\int_T^{2T}\sum\nolimits_1^2(q,t)dt = 2^{2\sigma-\frac{3}{2}}\left(\frac{5}{2}-2\sigma\right)^{-1}\pi^{2\sigma-\frac{3}{2}}q^{\frac{3}{2}-2\sigma} \\ \times\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac{5}{2}-2\sigma}}((2T)^{\frac{5}{2}-2\sigma} - T^{\frac{5}{2}-2\sigma}) + O(q^{2-2\sigma}T^{1+\varepsilon}).\end{aligned} \tag{3.9}$$

Now we will estimate

$$\int_T^{2T}\sum\nolimits_2^2(q,t)dt.$$

By the definition of $\sum_2(q,t)$, we have that

$$\sum\nolimits_2^2(q,t) = 4q^{2-2\sigma}\left(\frac{t}{2\pi}\right)^{1-2\sigma}\sum_{m\leq N_1}\sum_{n\leq N_1}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}}$$

$$\times \left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1} \left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1} \cos(g(t,qm))\cos(g(t,qn))$$

$$= 2q^{2-2\sigma}\left(\frac{t}{2\pi}\right)^{1-2\sigma} \sum_{m\le N_1}\sum_{n\le N_1} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}}$$

$$\times \left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1} \left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1}$$

$$\times(\cos(g(t,qm)+g(t,qn))+\cos(g(t,qm)-g(t,qn))), \tag{3.10}$$

where

$$g(t,qm) = t\log\left(\frac{qt}{2\pi m}\right) - t + \frac{\pi}{4},$$

and

$$N_1 = N_1(q,t,T) = q\left(\frac{t}{2\pi} + \frac{qT}{2} - \left(\left(\frac{qT}{2}\right)^2 + \frac{qTt}{2\pi}\right)^{\frac{1}{2}}\right).$$

It is not difficult to see, that

$$N_1 \le \frac{t^2}{4\pi^2 T}.$$

Therefore, for $m \le N_1$ and $t \in [T, 2T]$,

$$\frac{qt}{2\pi m} \ge \pi q > 3.$$

This implies the estimate

$$\left(\log\frac{qt}{2\pi m}\right)^{-1} \ll 1. \tag{3.11}$$

Moreover, by the definition of $g(t,qm)$, we find that

$$(g(t,qm) \pm g(t,qn))' \gg \left|\log\frac{m}{n}\right|. \tag{3.12}$$

Denote by $Z_1(q,t)$ the part of $\sum_1^2(q,t)$ in (3.10) with $m \ne n$, and let $T_1 \ge T$ be such that $N_1(q,t,T) \ge \max(m,n)$ for $t \ge T_1$. Then we deduce from Lemmas 2.3 and 3.5, and from estimates (3.10) - (3.12) that

$$\int_T^{2T} Z_1(q,t)dt \quad = 2q^{2-2\sigma}(2\pi)^{2\sigma-1} \sum_{m\le N_1(q,t,T)}\sum_{n\le N_1(q,t,T)} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}}$$

$$\times t^{1-2\sigma}\left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1}\left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1}$$

$$\times(\cos(g(t,qm)+g(t,qn))+\cos(g(t,qm)-g(t,qn)))dt$$

$$= 2q^{2-2\sigma}(2\pi)^{2\sigma-1}\sum_{m\le N_1(q,2T,T)}\sum_{n\le N_1(q,2T,T)}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}}$$

$$\times\int_{T_1}^{2T} t^{1-2\sigma}\left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1}\left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1}$$

$$\times(\cos(g(t,qm)+g(t,qn))+\cos(g(t,qm)-g(t,qn)))dt$$

$$\ll q^{2-2\sigma}T^{1-2\sigma}\sum_{m\le T}\sum_{n\le T}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}}\left|\log\frac{m}{n}\right|^{-1}$$

$$\ll q^{2-2\sigma}T\log T. \tag{3.13}$$

Now let $Z_2(q,t)$ be the part of $\sum_2^2(q,t)$ in (3.10) with $m=n$. Then, by Lemma 3.3., we find that

$$\int_T^{2T} Z_2(q,t)dt \ll q^{2-2\sigma}T^{1-2\sigma}\sum_{m\le T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \ll q^{2-2\sigma}T.$$

This, (3.10) and (3.3) show that

$$\int_T^{2T} \sum\nolimits_2^2(q,t)dt \ll q^{2-2\sigma}T\log T. \tag{3.14}$$

Clearly, in virtue of the estimate

$$R(q,t)=O(q^{\frac{7}{4}-\sigma}\log T),$$

$$\int_T^{2T} R^2(q,t)dt \ll Tq^{\frac{7}{2}-2\sigma}\log^2 T.$$

This, together with (3.14), gives

$$\int_T^{2T}\left(\sum\nolimits_2(q,t)+R(q,t)\right)^2 dt \quad \ll \int_T^{2T}\sum\nolimits_2^2(q,t)dt+\int_T^{2T} R^2(q,t)dt$$

$$\ll q^{\frac{7}{2}-2\sigma} T \log^2 T. \tag{3.15}$$

Moreover, the Cauchy-Schwarz inequality and (3.9), (3.15) imply the estimate

$$\int_T^{2T} \sum\nolimits_1(q,t)\left(\sum\nolimits_2(q,t)+R(q,t)\right)dt$$

$$\ll \left(\int_T^{2T} \sum\nolimits_1^2(q,t)dt\right)^{\frac{1}{2}} \left(\int_T^{2T} \left(\sum\nolimits_2(q,t)+R(q,t)\right)^2 dt\right)^{\frac{1}{2}}$$

$$\ll q^{\frac{11}{4}-2\sigma} T^{\frac{7}{4}-\sigma} \log T.$$

This estimate together with (3.1), (3.9) and (3.15) gives

$$\begin{aligned}\int_T^{2T} E_\sigma^2(q,t)dt &= 2^{2\sigma-\frac{3}{2}}\left(\frac{5}{2}-2\sigma\right)^{-1}\pi^{2\sigma-\frac{3}{2}} q^{\frac{3}{2}-2\sigma} \\ &\times \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac{5}{2}-2\sigma}}\left((2T)^{\frac{5}{2}-2\sigma}-T^{\frac{5}{2}-2\sigma}\right)+O(q^{\frac{7}{2}-2\sigma}T\log^2 T) \\ &+O(q^{\frac{11}{4}-2\sigma}T^{\frac{7}{4}-\sigma}\log T).\end{aligned} \tag{3.16}$$

Since $q \ll T^{1-\frac{4\sigma}{3}-\varepsilon}$, we have that the first error term in (3.16) is smaller than the second.

Now, taking $\frac{T}{2}, \frac{T}{2^2}$, ... in place of $T$ and summing, we obtain the theorem.

*Proof of Remark 3.2.* Clearly, $\frac{3}{5}-\frac{4\sigma}{5} < 1-\frac{4\sigma}{3}$ for $\sigma < \frac{3}{4}$. Moreover, for $\sigma < \frac{3}{4}$,

$$\sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}^2(m)}{m^{\frac{5}{2}-2\sigma}} < \infty$$

and we have that, for $q \leq \frac{3}{5}-\frac{4\sigma}{5}-\varepsilon$,

$$q^{\frac{11}{4}-2\sigma}T^{\frac{7}{4}-\sigma} = o(q^{\frac{3}{2}-2\sigma}T^{\frac{5}{2}-2\sigma})$$

as $T \to \infty$.

# Chapter 4

# Asymptotic formulae for the fourth power moment of the periodic zeta-function in the critical strip

In this chapter, we obtain asymptotic formulae with estimated error term for the fourth power moment of the periodic zeta-function $\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt$ in the critical strip $\frac{1}{2} < \sigma < 1$. We investigate two cases when the parameter $\lambda$, $0 < \lambda < 1$, is irrational and rational.

We will prove the following two theorems.

**Theorem 4.1.** *Suppose that the parameter $\lambda$ is irrational, $0 < \lambda < 1$, $\frac{1}{2} < \sigma < 1$ and $T \to \infty$. Then, for every $\varepsilon > 0$,*

$$\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}\right) + O(T^{\frac{3}{2}-\sigma+\varepsilon}).$$

**Theorem 4.2.** *Suppose that the number $\lambda$ is rational, $0 < \lambda < 1$, $\frac{3}{4} < \sigma < 1$ and $T \to \infty$. Then, for every $\varepsilon > 0$,*

$$\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}\right) + O(T^{\frac{7}{4}-\sigma+\varepsilon}).$$

## 4.1. Approximate functional equation

The proof of Theorems 4.1 and 4.2 is based on the approximate functional equation for the function $\zeta_\lambda(s)$.

Denote by $[u]$ the integer part of $u$, and define

$$p(t) = \left[\sqrt{\frac{t}{2\pi}} - 1\right], q(t) = \left[\sqrt{\frac{t}{2\pi}}\right], g(\lambda, t) = 2\sqrt{\frac{t}{2\pi}} - p(t) - q(t) - \lambda - 1,$$

$$f(\lambda, t) = -\frac{t}{2\pi}\log\frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(1-\lambda^2) + p(t) - q(t)$$

$$+2\sqrt{\frac{t}{2\pi}}(q(t) - p(t) + \lambda - 1) - \frac{1}{2}(q(t) + p(t)) - \lambda(1 + q(t) - p(t))$$

and

$$\psi(t) = \frac{\cos\pi(\frac{t^2}{2} - t - \frac{1}{8})}{\cos\pi t}.$$

**Lemma 4.3.** Suppose that $0 < \lambda < 1$, $0 \le \sigma \le 1$ and $t \ge t_0 > 0$. Then

$$\begin{aligned}\zeta_\lambda(s) \quad &= \sum_{1\le m\le p(t)} \frac{e^{2\pi i\lambda m}}{m^s} + \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{it+\frac{\pi i}{4}} \sum_{0\le m\le q(t)} \frac{1}{(m+\lambda)^{1-s}} \\ &+ \left(\frac{t}{2\pi}\right)^{-\frac{\sigma}{2}} e^{\pi i f(\lambda,t)+2\pi i\lambda}\psi(g(\lambda,t)) + O(t^{\frac{\sigma}{2}-1}).\end{aligned}$$

*Proof of Lemma 4.3.* The assertion of the lemma follows from an approximate functional equation for the Lerch zeta-function, see [7], [30] and equality (0.3).

From Lemma 4.3 we have that, for $0 \le \sigma \le 1$,

$$\zeta_\lambda(s) = S_1(s) + S_2(s) + O(t^{-\frac{1}{4}}), \tag{4.1}$$

where

$$S_1(s) = \sum_{1\le m\le p(t)} \frac{e^{2\pi i\lambda m}}{m^s}$$

and

$$S_2(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{it+\frac{\pi i}{4}} \sum_{0\le m\le q(t)} \frac{1}{(m+\lambda)^{1-s}}.$$

## 4.2. Asymptotics for the fourth power moment of the sum $S_1(s)$

In this section, we investigate the asymptotics for the fourth power moment of the sum $S_1(s)$.

**Lemma 4.4.** Suppose that $0 < \lambda < 1$, $\frac{1}{2} < \sigma < 1$ and $T \to \infty$. Then, for every $\varepsilon > 0$

$$\int_1^T |S_1(\sigma+it)|^4 dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}\right) + O(T^{\frac{3}{2}-\sigma+\varepsilon}) \quad (4.2)$$

*Proof of Lemma 4.4.* For the proof of this lemma, we use the method similar to that for the Riemann zeta-function [45]. We find that

$$\begin{aligned} |S_1(\sigma+it)|^4 &= \sum_{m_1} \frac{e^{2\pi i\lambda m_1}}{m_1^{\sigma+it}} \sum_{m_2} \frac{e^{2\pi i\lambda n_1}}{n_1^{\sigma+it}} \sum_{m_2} \frac{e^{2\pi i\lambda m_2}}{m_2^{\sigma-it}} \sum_{n_2} \frac{e^{2\pi i\lambda n_2}}{n_2^{\sigma-it}} \\ &= \sum_{m_1,n_1,m_2,n_2} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1m_2n_2)^\sigma} \left(\frac{m_2n_2}{m_1n_1}\right)^{it}, \end{aligned}$$

where in each sum we sum over $[1, m(t)]$. Let $T_1 = 2\pi max((m_1+1)^2, (n_1+1)^2, (m_2+1)^2, (n_2+1)^2)$. Then we have

$$\begin{aligned} \int_1^T |S_1(\sigma+it)|^4 dt &= \int_1^T \sum_{m_1,n_1,m_2,n_2} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1m_2n_2)^\sigma} \left(\frac{m_2n_2}{m_1n_1}\right)^{it} dt \\ &= \sum_{1\le m_1,n_1,m_2,n_2\le m(T)} \frac{e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1m_2n_2)^\sigma} \int_{T_1}^T \left(\frac{m_2n_2}{m_1n_1}\right)^{it} dt \\ &= \sum_{m_1n_1=m_2n_2}^{*} \frac{(T-T_1)e^{2\pi i\lambda(m_1+n_1-m_2-n_2)}}{(m_1n_1m_2n_2)^\sigma} \\ &\quad + O\left(\sum_{m_1n_1\ne m_2n_2}^{*} \frac{|\log\frac{m_2n_2}{m_1n_1}|^{-1}}{(m_1n_1m_2n_2)^\sigma}\right), \end{aligned} \quad (4.3)$$

where the star $*$ means, that the sum is taken over $m_1,\ n_1,\ m_2,\ n_2 \in [1, m(T)]$. Let $d(k) = \sum_{d/k} 1$, $k \in N$, be the divisor function, and $N(k)$ be the number of solutions of the equation $m_1 n_1 = m_2 n_2 = k$. Then, we have that $N(k) = d^2(k)$, if $k \leq u$, $m_1,\ n_1,\ m_2,\ n_2 \leq u$, and $N(k) \leq d^2(k)$, if $k \geq u$, $m_1,\ n_1,\ m_2,\ n_2 \leq u$. It is well known [45] that, for $\sigma > \frac{1}{2}$,

$$\sum_{k=1}^{\infty} \frac{d^2(k)}{k^{2\sigma}} = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}.$$

Also, it is well know that $d(k) = O_\varepsilon(k^\varepsilon)$ with every $\varepsilon > 0$. Therefore,

$$\begin{aligned}
&\sideset{}{^*}\sum_{m_1 n_1 = m_2 n_2} \frac{T e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} \\
&= T \sum_{m_1 n_1 = m_2 n_2 \leq m(T)} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} \\
&+ T \sideset{}{^*}\sum_{m(T) < m_1 n_1 = m_2 n_2 \leq m^2(T)} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} \\
&= T \sum_{m_1 n_1 = m_2 n_2 \leq m(T)} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} + O\left( T \sum_{k \geq m(T)} \frac{d^2(k)}{k^{2\sigma}} \right) \\
&= T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 = m_2 + n_2}} \frac{1}{(m_1 n_1 m_2 n_2)^\sigma} + T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} + O(T^{(3/2) - \sigma + \varepsilon}) \\
&= T \sum_{m_1 n_1 = m_2 n_2} \frac{1}{(m_1 n_1 m_2 n_2)^\sigma} - T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{1 - \cos 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1 m_2 n_2)^\sigma} \\
&+ iT \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{\sin 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1 m_2 n_2)^\sigma} + O(T^{(3/2) - \sigma + \varepsilon}) \\
&= T \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{1 - \cos 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1 m_2 n_2)^\sigma} \\
&+ iT \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{\sin 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1 m_2 n_2)^\sigma} + O(T^{(3/2) - \sigma + \varepsilon})
\end{aligned} \tag{4.4}$$

By definition of $T_1$ and symmetry

$$\begin{aligned}
\sideset{}{^*}\sum_{m_1 n_1 = m_2 n_2} \frac{T_1}{(m_1 n_1 m_2 n_2)^\sigma} &\leq 2\pi \sideset{}{^*}\sum_{m_1 n_1 = m_2 n_2} \frac{(m_1 + 1)^2 + (n_1 + 1)^2 + (m_2 + 1)^2 + (n_2 + 1)^2}{(m_1 n_1 m_2 n_2)^\sigma} \\
&= O\left( \sideset{}{^*}\sum_{m_1 n_1 = m_2 n_2} \frac{m_1^2}{(m_1 n_1 m_2 n_2)^\sigma} \right) = O\left( \sideset{}{^*}\sum_{m_1, n_1} \frac{m_1^2 d(m_1 n_1)}{(m_1 n_1)^{2\sigma}} \right)
\end{aligned}$$

$$= O\left(T^{\varepsilon} \sum_{m_1 \le m(T)} \frac{1}{m_1^{2\sigma-2}} \sum_{n_1 \le m(T)} \frac{1}{n_1^{2\sigma}}\right)$$

$$= O(T^{\varepsilon}(T^{(1/2)(3-2\sigma)})) = O(T^{(3/2)-\sigma+\varepsilon}). \tag{4.5}$$

Using the estimate [45]

$$\sum_{0<m} \sum_{<n \le T} \frac{1}{m^{\sigma} n^{\sigma} \log(n/m)} = O(T^{2-2\sigma} \log T), \ \frac{1}{2} \le \sigma < 1,$$

we find that

$$\sum_{m_1 n_1 \ne m_2 n_2}^{*} \frac{|\log(m_2 n_2 / m_1 n_1)|^{-1}}{(m_1 n_1 m_2 n_2)^{\sigma}} = O\left(\sum_{0<m<} \sum_{n \le m^2(T)} \frac{d(m)d(n)}{(mn)^{\sigma} \log(n/m)}\right)$$

$$= O\left(T^{\varepsilon} \sum_{0<m<} \sum_{n \le m^2(T)} \frac{1}{(mn)^{\sigma} \log(n/m)}\right) = O(T^{2-2\sigma+\varepsilon}). \tag{4.6}$$

From estimates (4.1) - (4.6) we have that

$$\int_1^T |S_1(\sigma+it)|^4 dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1 n_1)^{2\sigma}}\right) + O(T^{\frac{3}{2}-\sigma+\varepsilon}).$$

## 4.3. Estimates for the fourth power moment of the sum $S_2(s)$

We apply the following lemma.

**Lemma 4.5.** Suppose that $u_1, ..., u_r$ are complex numbers, $\lambda_1, ..., \lambda_r$ are distinct real numbers, and $\delta_m = min_{n \ne m}|\lambda_n - \lambda_m|$. Then

$$\sum_{m,n=1}^{r} u_m \overline{u}_n (\lambda_n - \lambda_m)^{-1} \ll \sum_{m=1}^{r} |u_m|^2 \delta_m^{-1}.$$

The lemma is called a modification of the Hilbert inequality, and was obtained by H. L. Mongomery and R. C. Vaughan in [39], Theorem 2.

**Lemma 4.6.** Let $\lambda$ be irrational, $0 < \lambda < 1$, $\frac{1}{2} < \sigma < 1$ and $T \to \infty$. Then for every $\varepsilon > 0$

$$\int_1^T |S_2(\sigma + it)|^4 dt = O_\lambda(T^{2-2\sigma+\varepsilon}).$$

*Proof of Lemma 4.6.* Denote

$$Z(T) = \int_1^T \left| \sum_{0 \le m \le q(t)} \frac{1}{(m+\lambda)^{1-\sigma-it}} \right|^4 dt.$$

Then

$$Z(T) = O\left( \int_1^T \left| \frac{1}{\lambda^{1-\sigma-it}} \right|^4 dt + Z_1(T) \right) = O_\lambda(T) + O(Z_1(T)), \tag{4.7}$$

where

$$Z_1(T) = \int_1^T \left| \sum_{1 \le m \le q(t)} \frac{1}{(m+\lambda)^{1-\sigma-it}} \right|^4 dt.$$

As in the case of $S_1(s)$ we have that

$$\begin{aligned} Z_1(T) \quad &= \int_1^T \sum_{m_1,n_1,m_2,n_2} \frac{1}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}} \\ &\times \left( \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right)^{it} dt, \end{aligned}$$

where every $m_1$, $n_1$, $m_2$, $n_2$ runs over the interval $[1, q(t)]$. Therefore,

$$\begin{aligned} Z_1(T) \quad &= \sum_{1 \le m_1,n_1,m_2,n_2 \le q(T)} \frac{1}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}} \\ &\times \int_{T_2}^T \left( \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right)^{it} dt, \end{aligned}$$

where $T_2 = 2\pi \max(m_1^2, n_1^2, m_2^2, n_2^2)$. Since $\lambda$ is irrational, we have that $(m_1+\lambda)(n_1+\lambda) = (m_2+\lambda)(n_2+\lambda)$ if and only if $m_1 n_1 = m_2 n_2$ and $m_1 + n_1 = m_2 + n_2$. Therefore,

$$\begin{aligned} Z_1(T) \quad &= O\left( \sideset{}{^*}\sum_{m_1 n_1 = m_2 n_2} \frac{T - T_2}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} \right. \\ &+ \left. \sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda) \ne (m_2+\lambda)(n_2+\lambda)} \frac{\left| \log\left( \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right) \right|^{-1}}{(m_1 n_1 m_2 n_2)^{1-\sigma}} \right). \end{aligned} \tag{4.8}$$

The star $*$ means that the summing runs over $m_1, n_1, m_2, n_2 \in [1, q(T)]$. It is easily seen that

$$\sum_{m_1 n_1 = m_2 n_2}^{*} \frac{T}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} = O\left( T \sum_{k \le q^2(T)} \frac{d^2(k)}{k^{2-2\sigma}} \right) = O\left(T^{2\sigma+\varepsilon}\right). \tag{4.9}$$

and

$$\sum_{m_1 n_1 = m_2 n_2}^{*} \frac{T_2}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} = O\left( \sum_{m_1, n_1 \le q(T)} \frac{m_1^2 d(m_1 n_1)}{(m_1 n_1)^{2-2\sigma}} \right)$$
$$= O\left( T^{\varepsilon} \left( \sum_{m_1 \le q(T)} m_1^{2\sigma} \sum_{n_1 \le q(T)} \frac{1}{n_1^{2-2\sigma}} \right) \right) = O(T^{2\sigma+\varepsilon}). \tag{4.10}$$

Moreover, when $(m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)$, then

$$\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right| \ge \min\left( \frac{c(\lambda)}{(m_1+\lambda)(n_1+\lambda)}, \frac{c(\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right),$$

where $c(\lambda)$ is a positive constant. From this ant Lemma 4.5 we have that

$$\sum_{\substack{m_1 n_1 \neq m_2 n_2 \\ (m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)}}^{*} \frac{\left| \log\left( \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right) \right|^{-1}}{(m_1 n_1 m_2 n_2)^{1-\sigma}}$$
$$= O_\lambda\left( T^{\varepsilon} \sum_{1 \le m \le q^2(T)} \frac{m}{m^{2-2\sigma}} \right) = O_\lambda(T^{2\sigma+\varepsilon}), \tag{4.11}$$

and

$$\sum_{\substack{m_1 n_1 = m_2 n_2 \\ (m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)}}^{*} \frac{\left| \log\left( \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right) \right|^{-1}}{(m_1 n_1 m_2 n_2)^{1-\sigma}}$$
$$= O_\lambda\left( \sum_{k \le q^2(T)} \frac{k d^2(k)}{k^{2-2\sigma}} \right) = O(T^{2\sigma+\varepsilon}). \tag{4.12}$$

Now estimates (4.8) - (4.12) imply

$$Z_1(t) = O_\lambda(T^{2\sigma+\varepsilon}).$$

Therefore, in view of (4.7),

$$Z(t) = O_\lambda(T^{2\sigma+\varepsilon}). \tag{4.13}$$

The definition of $S_2(s)$ and (4.13) show that

$$\int_1^T |S_2(\sigma+it)|^4 dt = O\left(\int_1^T t^{2-4\sigma} dZ(t)\right)$$

$$= O\left(t^{2-4\sigma}Z(t)|_1^T + (4\sigma-2)\int_1^T Z(t)t^{1-4\sigma}dt\right)$$

$$= O_\lambda\left(T^{2-2\sigma+\varepsilon} + \int_1^T t^{1-2\sigma+\varepsilon}dt\right) = O_\lambda(T^{2-2\sigma+\varepsilon}).$$

and the lemma is proved.

**Lemma 4.7** Suppose that $\lambda$ is rational, $0<\lambda<1$, $\frac{3}{4}<\sigma<1$ and $T\to\infty$. Then, for every $\varepsilon>0$

$$\int_1^T |S_2(\sigma+it)|^4 dt = O(T^{\frac{5}{2}-2\sigma+\varepsilon}).$$

*Proof of Lemma 4.7.* We preserve the notation used in the proof of Lemma 4.6. We start with the formula for $Z_1(T)$

$$Z_1(T) = \sum_{1\le m_1,n_1,m_2,n_2\le q(T)} \frac{1}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}} \times \int_{T_2}^T \left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right)^{it} dt. \tag{4.14}$$

We separate two cases

$$(m_1+\lambda)(n_1+\lambda) = (m_2+\lambda)(n_2+\lambda)$$

and

$$(m_1+\lambda)(n_1+\lambda) \ne (m_2+\lambda)(n_2+\lambda).$$

Obviously, in the first case we have

$$\int_{T_2}^T \left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right)^{it} dt = \int_{T_2}^T dt = T - T_2,$$

while, in the second case, the estimate

$$\int_{T_2}^T \left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right)^{it} dt = \int_{T_2}^T \exp\left\{it\log\left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right)\right\} dt$$

$$= \left( \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right)^{-1} \exp\left\{ it \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right\} \Big|_1^T$$

$$\ll \left| \log\left( \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right) \right|^{-1}$$

follows. From the last two estimates and (4.14) we obtain that

$$Z_1(T) \ll \sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)} \frac{T-T_2}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}}$$

$$+ \sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda)\neq(m_2+\lambda)(n_2+\lambda)} \frac{\left|\log\left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right)\right|^{-1}}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}}, \tag{4.15}$$

where the star "*" means that we sum over $m_1, n_1, m_2, n_2$ from the interval $[1, q(T)]$. I is not difficult to see that

$$T \sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)} \frac{1}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}}$$

$$\ll T \sum_{m\le q^2} \sum_{m-\sqrt{T}\le n\le m+\sqrt{T}} \frac{d(m)d(n)}{m^{2-2\sigma}}$$

$$\ll T^{\frac{3}{2}+\varepsilon} \sum_{m\le q^2} \frac{d(m)}{m^{2-2\sigma}} \ll T^{\frac{1}{2}+2\sigma+\varepsilon}. \tag{4.16}$$

and

$$\sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)} \frac{T_2}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}}$$

$$\ll \sum_{m_1,n_1} \sum_{<q(T)} \sum_{m_1n_1-\sqrt{T}\le n\le m_1n_1+\sqrt{T}} \frac{m_1^2 d(m_1n_1)d(n)}{(m_1n_1)^{2-2\sigma}}$$

$$\ll T^{\frac{1}{2}+\varepsilon} \left( \sum_{m_1\le q(T)} m_1^{2\sigma} \sum_{n_1\le q(T)} \frac{1}{n_1^{2-2\sigma}} \right) \ll T^{\frac{1}{2}+2\sigma+\varepsilon}. \tag{4.17}$$

For estimation of the second sum in (4.15), we apply Lemma 4.5. For a certain positive constant $c(\lambda)$, we have that

$$\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right| \ge c(\lambda) \min\left( \frac{1}{(m_1+\lambda)(n_1+\lambda)}, \frac{1}{(m_2+\lambda)(n_2+\lambda)} \right)$$

what is an analogue of the well known inequality

$$\left| \log \frac{n}{m} \right| > \min\left( \frac{1}{n}, \frac{1}{m} \right), \; m, n \in \mathbb{N},$$

with $m \neq n$. Therefore, the application of Lemma 4.5 yields, for every $\varepsilon > 0$,

$$\sum_{(m_1+\lambda)(n_1+\lambda)\neq(m_2+\lambda)(n_2+\lambda)}^{*} \frac{\left|\log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right|^{-1}}{(m_1n_1m_2n_2)^{1-\sigma}}$$

$$\ll_\lambda T^\varepsilon \sum_{1\leq m\leq q^2(T)} \frac{m}{m^{2-2\sigma}} \ll_\lambda T^{2\sigma+\varepsilon}.$$

From this and (4.15) - (4.17), we find that

$$Z_1(T) \ll_\lambda T^{\frac{1}{2}+2\sigma+\varepsilon}.$$

This estimate together with (4.7) shows that

$$Z(T) \ll_\lambda T^{\frac{1}{2}+2\sigma+\varepsilon} \tag{4.18}$$

with every $\varepsilon > 0$. Now, from the definitions of the functions $S_2(s)$ and $Z(T)$, and (4.18), we deduce that

$$\begin{aligned}
\int_1^T |S_2(\sigma+it)|^4 &\ll \int_1^T t^{2-4\sigma} \left| \sum_{0\leq m\leq q(t)} \frac{1}{(m+\lambda)^{1-\sigma-it}} \right|^4 dt \\
&= \int_1^T t^{2-4\sigma} dJ(t) = t^{2-4\sigma} J(t)|_1^T - (2-4\sigma)\int_1^T J(T) t^{1-4\sigma} dt \\
&\ll t^{2-4\sigma} t^{\frac{1}{2}+2\sigma+\varepsilon}|_1^T + \int_1^T t^{\frac{1}{2}+2\sigma+\varepsilon} t^{1-4\sigma} dt \\
&\ll t^{\frac{5}{2}+2\sigma+\varepsilon}|_1^T + \int_1^T t^{\frac{3}{2}-2\sigma+\varepsilon} dt \ll T^{\frac{5}{2}-2\sigma+\varepsilon} + \left.\frac{t^{\frac{5}{2}-2\sigma+\varepsilon}}{\frac{5}{2}-2\sigma+\varepsilon}\right|_1^T \\
&\ll T^{\frac{5}{2}-2\sigma+\varepsilon}.
\end{aligned}$$

## 4.4. Proof of Theorem 4.1

From (4.1), we find that

$$|\zeta_\lambda(s)|^4 = |S_1(s) + S_2(s) + O(t^{-\frac{1}{4}})|^4$$

$$= (S_1(s) + S_2(s) + O(t^{-\frac{1}{4}}))^2(\overline{S_1(s)} + \overline{S_2(s)} + O(t^{-\frac{1}{4}}))^2$$

$$= (S_1^2(s) + S_2^2(s) + 2S_1(s)S_2(s) + O(|S_1(s)|t^{-\frac{1}{4}}) + O(|S_2(s)|t^{-\frac{1}{4}})$$

$$+O(t^{-\frac{1}{2}})) \times (\overline{S_1^2(s)} + \overline{S_2^2(s)} + 2\overline{S_1(s)S_2(s)} + O(|S_1(s)|t^{-\frac{1}{4}})$$

$$+O(|S_2(s)|t^{-\frac{1}{4}}) + O(t^{-\frac{1}{2}}))$$

$$= |S_1(s)|^4 + S_1^2(s)\overline{S_2^2(s)} + 2S_1^2(s)\overline{S_1(s)S_2(s)} + \overline{S_1^2(s)}S_2^2(s) + |S_2(s)|^4$$

$$+2\overline{S_1(s)}S_2^2(s)\overline{S_2(s)} + 2S_1(s)\overline{S_1^2(s)}S_2(s) + 2S_1(s)\overline{S_2^2(s)}S_2(s)$$

$$+4|S_1(s)|^2|S_2(s)|^2 + O(|S_1(s)|^3t^{-\frac{1}{4}}) + O(|S_1(s)|^2|S_2(s)|t^{-\frac{1}{4}})$$

$$+O(|S_1(s)|^2t^{-\frac{1}{2}}) + O(|S_1(s)||S_2(s)|^2t^{-\frac{1}{4}}) + O(|S_2(s)|^3t^{-\frac{1}{4}})$$

$$+O(|S_2(s)|^2t^{-\frac{1}{2}}) + O(|S_1(s)||S_2(s)|t^{-\frac{1}{2}}) + O|S_1(s)|t^{-\frac{3}{4}})$$

$$+O(|S_2(s)|t^{-\frac{3}{4}}) + O(t^{-1}). \tag{4.19}$$

Using the Cauchy-Schwarz inequality, Lemmas 4.4 and 4.6, we obtain the estimates

$$\int_1^T (S_1^2(\sigma+it)\overline{S_2^2(\sigma+it)} + \overline{S_1^2(\sigma+it)}S_2^2(\sigma+it) + 4|S_1^2(\sigma+it)|^2|S_2^2(\sigma+it)|^2)dt$$
$$= O\left(\int_1^T |S_1(\sigma+it)|^4dt \int_1^T |S_2(\sigma+it)|^4dt\right)^{\frac{1}{2}} = O_\lambda(T^{\frac{3}{2}-\sigma+\varepsilon}),$$

$$\int_1^T (2S_1^2(\sigma+it)\overline{S_1(\sigma+it)+S_2(\sigma+it)} + S_1(\sigma+it)\overline{S_1^2(\sigma+it)}S_2(\sigma+it)$$

$$+2\overline{S_1(\sigma+it)}S_2^2(\sigma+it)\overline{S_2(\sigma+it)} + 2S_1(\sigma+it)\overline{S_2^2(\sigma+it)}S_2(\sigma+it))dt$$
$$= O\left(\int_1^T |S_1(\sigma+it)|^2|S_2(\sigma+it)|^2dt\right)^{\frac{1}{2}}$$

$$\times\left(\left(\int_1^T |S_1(\sigma+it)|^4 dt\right)^{\frac{1}{2}} + \left(\int_1^T |S_2(\sigma+it)|^4 dt\right)^{\frac{1}{2}}\right)$$

$$= O_\lambda\left(\int_1^T |S_1(\sigma+it)|^4 dt \int_1^T |S_2(\sigma+it)|^4 dt\right)^{\frac{1}{4}} \left(T^{\frac{1}{2}} + T^{1-\sigma+\varepsilon}\right)$$

$$= O_\lambda\left(T^{\frac{3}{4}-\frac{\sigma}{2}+\varepsilon}\left(T^{\frac{1}{2}} + T^{1-\sigma+\varepsilon}\right)\right) = O_\lambda\left(T^{\frac{5}{4}-\frac{\sigma}{2}+\varepsilon}\right),$$

$$\int_1^T |S_1(\sigma+it)|^3 t^{-\frac{1}{4}} dt = O\left(\int_1^T |S_1(\sigma+it)|^4 dt \int_1^T |S_1(\sigma+it)|^2 t^{-\frac{1}{2}} dt\right)^{\frac{1}{2}}$$

$$= O_\lambda\left(T^{\frac{3}{4}+\varepsilon}\right),$$

$$\int_1^T |S_2(\sigma+it)|^3 t^{-\frac{1}{4}} dt = O\left(\int_1^T |S_2(\sigma+it)|^4 dt \int_1^T |S_2(\sigma+it)|^2 t^{-\frac{1}{2}} dt\right)^{\frac{1}{2}}$$

$$= O_\lambda\left(T^{\frac{3}{2}-\frac{3}{2}\sigma+\varepsilon}\right),$$

$$\int_1^T |S_1(\sigma+it)||S_2(\sigma+it)|^2 t^{-\frac{1}{4}} dt = O\left(\int_1^T |S_1(\sigma+it)|^2 t^{-\frac{1}{2}} dt \int_1^T |S_2(\sigma+it)|^4 dt\right)^{\frac{1}{2}}$$

$$= O_\lambda\left(T^{\frac{5}{4}-\sigma+\varepsilon}\right),$$

$$\int_1^T |S_1(\sigma+it)|^2 |S_2(\sigma+it)| t^{-\frac{1}{4}} dt = O\left(\int_1^T |S_1(\sigma+it)|^4 dt \int_1^T |S_2(\sigma+it)|^2 t^{-\frac{1}{2}} dt\right)^{\frac{1}{2}}$$

$$= O_\lambda\left(T^{1-\frac{\sigma}{2}+\varepsilon}\right),$$

$$\int_1^T |S_1(\sigma+it)||S_2(\sigma+it)| t^{-\frac{1}{2}} dt = O\left(\int_1^T |S_1(\sigma+it)|^2 dt \int_1^T |S_2(\sigma+it)|^2 t^{-\frac{1}{2}} dt\right)^{\frac{1}{2}}$$

$$= O_\lambda\left(T^{\frac{3}{4}-\frac{\sigma}{2}+\varepsilon}\right).$$

The above estimates together with (4.19) and Lemmas 4.4 and 4.6 show that

$$\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1 n_1 m_2 n_2)^\sigma}\right) + O_\lambda(T^{\frac{3}{2}-\sigma+\varepsilon}).$$

## 4.5. Proof of Theorem 4.2

The proof of Theorem 4.2 is similar to that of Theorem 4.1 with one difference that, in place of Lemma 4.6, we apply Lemma 4.7.

Applying of Lemmas 4.4 and 4.7 together with the Cauchy-Schwarz inequality gives the estimates

$$\int_1^T (S_1^2(\sigma+it)\overline{S_2^2(\sigma+it)} + \overline{S_1^2(\sigma+it)}S_2^2(\sigma+it) + 4|S_1^2(\sigma+it)|^2|S_2^2(\sigma+it)|^2)dt$$

$$\ll \left(\int_1^T |S_1(\sigma+it)|^4 dt \int_1^T |S_2(\sigma+it)|^4 dt\right)^{\frac{1}{2}} \ll_\lambda T^{\frac{7}{4}-\sigma+\varepsilon},$$

$$\int_1^T (2S_1^2(\sigma+it)\overline{S_1(\sigma+it)S_2(\sigma+it)} + 2S_1(\sigma+it)\overline{S_1^2(\sigma+it)}S_2(\sigma+it)$$

$$+2\overline{S_1(\sigma+it)}S_2^2(\sigma+it)\overline{S_2(\sigma+it)} + 2S_1(\sigma+it)\overline{S_2^2(\sigma+it)}S_2(\sigma+it))dt$$

$$\ll \left(\int_1^T |S_1(\sigma+it)|^2|S_2(\sigma+it)|^2 dt\right)^{\frac{1}{2}}$$

$$\times\left(\left(\int_1^T |S_1(\sigma+it)|^4 dt\right)^{\frac{1}{2}} + \left(\int_1^T |S_2(\sigma+it)|^4 dt\right)^{\frac{1}{2}}\right)$$

$$\ll_\lambda \left(\int_1^T |S_1(\sigma+it)|^4 dt \int_1^T |S_2(\sigma+it)|^4 dt\right)^{\frac{1}{4}}$$

$$\times(T^{\frac{1}{2}} + T^{1-\sigma+\varepsilon}) \ll_\lambda (T^{\frac{7}{8}-\frac{\sigma}{2}+\varepsilon}(T^{\frac{1}{2}} + T^{1-\sigma+\varepsilon})) \ll_\lambda T^{\frac{11}{8}-\frac{\sigma}{2}+\varepsilon},$$

$$\int_1^T |S_1(\sigma+it)|^3 t^{-\frac{1}{4}} dt \ll \left(\int_1^T |S_1(\sigma+it)|^4 dt \int_1^T |S_1(\sigma+it)|^2 t^{-\frac{1}{2}} dt\right)^{\frac{1}{2}}$$

$$\ll T^{\frac{3}{4}+\varepsilon},$$

$$\int_1^T |S_2(\sigma+it)|^3 t^{-\frac{1}{4}} dt \ll \left(\int_1^T |S_2(\sigma+it)|^4 dt \int_1^T |S_2(\sigma+it)|^2 t^{-\frac{1}{2}} dt\right)^{\frac{1}{2}}$$

$$\ll_\lambda T^{\frac{15}{8}-\frac{3}{2}\sigma+\varepsilon},$$

$$\int_1^T |S_1(\sigma+it)||S_2(\sigma+it)|^2 t^{-\frac{1}{4}}dt \ll \left(\int_1^T |S_1(\sigma+it)|^2 t^{-\frac{1}{2}}dt \int_1^T |S_2(\sigma+it)|^4 dt\right)^{\frac{1}{2}}$$

$$\ll_\lambda T^{\frac{3}{2}-\sigma+\varepsilon},$$

$$\int_1^T |S_1(\sigma+it)|^2|S_2(\sigma+it)| t^{-\frac{1}{4}}dt \ll \left(\int_1^T |S_1(\sigma+it)|^4 dt \int_1^T |S_2(\sigma+it)|^2 t^{-\frac{1}{2}}dt\right)^{\frac{1}{2}}$$

$$\ll_\lambda T^{\frac{5}{4}-\frac{\sigma}{2}+\varepsilon},$$

$$\int_1^T |S_1(\sigma+it)S_2(\sigma+it)| t^{-\frac{1}{2}}dt \ll \left(\int_1^T |S_1(\sigma+it)|^2 t^{-\frac{1}{2}}dt \int_1^T |S_2(\sigma+it)|^2 t^{-\frac{1}{2}}dt\right)^{\frac{1}{2}}$$

$$\ll_\lambda T^{\frac{7}{8}-\frac{\sigma}{2}+\varepsilon}.$$

All these estimates, and Lemmas 4.4 and 4.7 lead to

$$\int_1^T |\zeta_\lambda(\sigma+it)|^4 dt = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1n_1=m_2n_2} \frac{\sin^2 \pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1m_2n_2)^\sigma}\right) + O(T^{\frac{7}{4}-\sigma+\varepsilon}).$$

# Conclusions

In the thesis, it was obtained that the periodic zeta-function $\zeta_\lambda(s)$ has the following properties:

1. For the error term $E_\sigma(q,T)$ of the formula for
$$\sum_{a=1}^{q}\int_0^T |\zeta_{\frac{a}{q}}(\sigma+it)|^2dt$$
the Atkinson-type formula is true with $\frac{1}{2}<\sigma<\frac{3}{4}$.

2. Asymptotic formulae for the mean squares of $E(q,T)$ ($E_{\frac{1}{2}}(q,T)=E(q,T)$) and $E_\sigma(q,T)$ with $\frac{1}{2}<\sigma<\frac{3}{4}$ are valid.

3. Asymptotic formulae for the fourth power moment of $\zeta_\lambda(s)$ are valid in the strip $\frac{1}{2}<\sigma<1$.

# Bibliography


[1] F. V. Atkinson, The mean-value of the Riemann zeta, Acta Math., **81**(1949), 353-376.

[2] J. B. Conrey, A. Ghosh, On mean values of the zeta-function, Mathematica, **31**(1984), 159-161.

[3] J. B. Conrey, A. Ghosh, A conjecture for the sixth power moment of the Riemann zeta-function, Int. Math. Res. Nit., **15**(1998), 775-780.

[4] J. B. Conrey, S. M. Gonek, High moments of the Riemann zeta-function, Duke Math. J., **107**(3)(2001), 577-604.

[5] S. Černigova, A. Laurinčikas, On the mean square of the periodic zeta-function, in: Anal. Probab. Methods Number Theory, A. Laurinčikas el al. (Eds), TEV Vilnius, (2012), 91-99.

[6] S. Černigova, A. Laurinčikas, The Atkinson type formula for the periodic zeta-function, Čebyševskij sbornik, XIV, **2**(46)(2013), 180-199.

[7] R. Garunkštis, A. Laurinčikas, J. Steuding, On the mean square of Lerchzeta-function, Liet. Matem. rink., **80**(2003), 47-60.

[8] J. L. Hafner, On the representation of the summatory functions of a class of arithmetical functions, Analytic Number Theory. Lecture Notes in Mathematics, **899**(1981), 148-165.

[9] G. H. Hardy, J. E. Littlewood, Contributions to the theory of the Riemann zeta-function and the distribution of primes, Acta Math. **41**(1918), 119-196.

[10] D. R. Heath-Brown, The mean value theorem for the Riemann zeta-function, Mathematika, **25**(1978), 177-184.

[11] D. R. Heath-Brown, The twelfth power moment of the Riemann zeta-function, Quart. J. Math. Oxford, **29**(1978), 443-462.

[12] D. R. Heath-Brown, Fractional moments of the Riemann zeta-function, J. London Math. Soc., **24**(2)(1981), 65-78.

[13] D. R. Heath-Brown, Fractional moments of the Riemann zeta-function. II, Quart. J. Math. Oxford, **44**(2)(1993),185-197.

[14] A. E. Ingham, Mean-value theorems in the theory of Riemann zeta-function, Proc. London Math. Soc., **27**(2)(1928), 273-300.

[15] H. Ishikawa, K. Matsumoto, An explicit formula of Atkinson type for the product of the Riemann zeta-function and a Dirichlet polynomial, Central European Journal of Mathematics, **9**(1)(2011), 102-126.

[16] A. Ivič, The Riemann zeta-function: The Theory of the Riemann zeta-function with applications, New York: Wiley, (1985).

[17] A. Ivič, Mean Values of the Riemann Zeta-Function, Springer-Verlag, Berlin, (1991).

[18] A. Ivič, The Mean Values of the Riemann Zeta-Function on the Critical Line, in: Analytic Number Theory, Approximation Theory, and Special Functions, (2014), 3-68.

[19] M. Jutila, Transformation formulae for Dirichlet polynomials, J. Number Theory, **18**(1984),135-156.

[20] M. Jutila, Atkinson's formula revisited, in: Voronoi's Impact on Modern Science. Book 1. Proc. Inst. Math. National Acad. Sc. Ukraine. V. 14. Kiev.,(1998), 137-154.

[21] J. Karaliūnaitė, The Atkinson formula for the periodic zeta-function in the critical strip, in: Voronoi's Impact on Modern Science, Book 4, vol 1. Proceedings of the 4th International Conference on Analytic Number Theory and Spatial Tessellations, Institute of Mathematics NAS of Ukraine, Kyiv., (2008), 48-58.

[22] J. Karaliūnaitė, Value distribution theorems for the periodic zeta function, Doctoral dissertation, Vilnius University, Vilnius, (2010).

[23] J. Karaliūnaitė, A. Laurinčikas, The Atkinson formula for the periodic zeta-function, Lith. Math. J., **47**(4)(2007), 504-516.

[24] Keating, N.C. Snaith, Random matrix theory and $L$-functions at $s = \frac{1}{2}$, Communications in Mathematical Physics, **214**(1)(2000), 91-110.

[25] J. Kubilius, The method of generating Dirichlet series in the theory of distribution of arithmetic functions I, Litovsk. Mat. Sb., Russian, **11**(1971),125-134.

[26] A. Laurinčikas, The Atkinson formula near the critical line, Analytic and Probabilistic Methods in Number Theory, New Trends in Probability and Statistics, 2. Vilnius: TEV, Utrecht: VSP (1992), 335-354.

[27] A. Laurinčikas, The Atkinson formula near the critical line. II, Liet. Mat. Rinkinys., **33**(3)(1993), 302-313.

[28] A. Laurinčikas, On the moments of the Riemann zeta-function near the critical line, Lithuanian Mathematical Journal, **35**(3)(1995), 262-283.

[29] A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer Academic Publishers, Dordrecht-Boston-London, (1996).

[30] A. Laurinčikas, R. Garunkštis, The Lerch Zeta-Function, Kluwer Academic Publisher, Dordrecht, (2002).

[31] A. Laurinčikas, D.Šiaučiūnas, On the fourth power moment of the function $\zeta_\lambda(s)$, Integral Transforms Spec. Funct., **18**(9)(2007), 629-638.

[32] A. Laurinčikas, D. Šiaučiūnas, On the fourth power moment of the function $\zeta_\lambda(s)$. II, Integral transforms and special functions,**22**(10)(2011), 759-765.

[33] K. Matsumoto, The mean square of the Riemann zeta-function in the critical strip, Japan J. Math., **15**(1)(1989), 1-13.

[34] K. Matsumoto, Recent developments in the mean square theory of the Riemann zeta and other zeta-functions, in: Number Theory, Trends Math., Birkhäuser, Boston,(2000), 241-286.

[35] K. Matsumoto, T. Meurman, The mean square of the Riemann zeta-function in the critical strip III, Acta Math., **64**(4)(1993), 357-382.

[36] K. Matsumoto, T. Meurman, The mean square of the Riemann zeta-function in the critical strip II, Acta Arithmetica **68**(4)(1994), 369-382.

[37] T. Meurman, A generalization of Atkinson's formula to $L$-functions, Acta Arith., **47**(1986), 351-370.

[38] T. Meurman, On the mean square of the Riemann zeta-function, Quart. J. Math. Oxford (2), **38**(1987), 337-343.

[39] H. L. Montgomery, R. C. Vaughan, Hilbert's inequality, J. London Math. Soc., **8**(2)(1974), 73-82.

[40] Y. Motohashi, A note on the mean value of the zeta and $L$-functions, IV, Proc. Japan Acad., **62A.**(1986), 311-313.

[41] A. Oppenheim, Some identities in the theory of numbers, Proc. London Math. Soc., **26**(2)(1927), 295-350.

[42] K. Prachar, Primzahlverteilung, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1957.

[43] K. Ramachandra, Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series. I, Hardy-Ramanujan J., **1**(1978), 1-15.

[44] K. Soundararajan, Moments of the Riemann zeta function, Ann. Math. (2), **170**(2)(2009), 981-993.

[45] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, second edition revised by D. R. Heath-Brown, Clarendon Press, Oxford (1986).

[46] N. Walt, A note on the mean square of $\left|\zeta(\frac{1}{2}+it)\right|$, J. London Math. Soc. (2), **82**(2010), 279-294.

# Notation

| | |
|---|---|
| $\mathbb{N}$ | set of all positive integers |
| $\mathbb{R}$ | set of all real numbers |
| $\mathbb{C}$ | set of all complex numbers |
| $i = \sqrt{-1}$ | imaginary unity |
| $s = \sigma + it$, $\sigma, t \in \mathbb{R}$ | complex variable |
| $Res$ | real part of $s$ |
| $Ims$ | imaginary part of $s$ |
| $\gamma_0$ | Euler constant |
| $d(m)$ | divisor function, $d(m) = \sum\limits_{d\mid m} 1$ |
| $\sigma_\alpha$ | generalized divisor function, $\sigma_\alpha(m) = \sum\limits_{d\mid m} d^\alpha$ |
| $f(x) \ll_\theta O(g(x))$, $x \in I$ | means that $\lvert f(x)\rvert \leq C_\theta g(x)$, $x \in I$ |
| $f(x) \ll_\theta g(x)$, $x \in I$ | means that $f(x) = O_\theta(g(x))$, $x \in I$ |
| $\zeta(s)$ | Riemann zeta-function defined, for $\sigma > 1$, by $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$, and by analytic continuation elsewhere |
| $L(\lambda, \alpha, s)$ | Lerch zeta-function with parameters $\lambda$ and $\alpha$ defined, for $\sigma > 1$, by $L(\lambda, \alpha, s) = \sum\limits_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}$, and by analytic continuation elsewhere |

$\zeta_\lambda(s)$ periodic zeta-function with parameter $\lambda$ defined,

for $\sigma > 1$, by

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

and by analytic continuation elsewhere

$\Gamma(s)$ Euler gamma-function defined,

for $\sigma > 1$ by

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-\mathrm{u}} \mathrm{u}^{\mathrm{s}-1} \mathrm{du},$$

and by analytic continuation elsewhere