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Monotone Modal Logic & Friends

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AIX MARSEILLE UNIVERSITE

Abstract

Ecole doctorale de mathématiques et informatique de Marseille (ED 184)

Doctor of Computer Science

Monotone Modal Logic & Friends

by Sabine FRITTELLA

The present thesis focuses on Monotone Modal Logic and closely related logics from the point of view of Correspondence Theory and Proof Theory.

The first part of the thesis establishes a formal connection between algorithmic correspondence theory and certain dual characterization results for finite lattices, similar to Nation's characterization of a hierarchy of pseudovarieties of finite lattices progressively generalizing finite distributive lattices. This formal connection is established through monotone modal logic. Specifically, we adapt the correspondence algorithm ALBA to the setting of monotone modal logic, and we use a certain duality-induced encoding of finite lattices as monotone neighbourhood frames to translate lattice terms into formulas in monotone modal logic.

The second part of the thesis extends the theory of display calculi to Baltag-Moss-Solecki's logic of Epistemic Actions and Knowledge (EAK), Monotone Modal Logic (MML), and Propositional Dynamic Logic (PDL). Our results include several cut-elimination metatheorems, which generalize the original metatheorem of Belnap in different and mutually independent dimensions. The two main generalizations of display calculi treated in the thesis are: the generalization from single type to multi-type languages, and from the full or relativized display property to no display property.

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To my mother.

Prologue

Introduction

Monotone modal logic. Monotone modal logic is a strengthening of classical modal logic¹ which requires the modal operators to satisfy the following monotonicity rule:

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}.$$

Monotone modal logics have been studied since the beginning of modal logic in its modern form (cf. [Lew18, LL32]) by formal philosophers such as Lewis in order to formalize the notion of strict implication. Of the logical systems $S1$ - $S5$ proposed by Langford and Lewis in [LL32], only $S4$ and $S5$ are normal modal logics. The interest in monotone modal logics as formal frameworks adequate to capture epistemic and deontic reasoning is witnessed by the work of Prior [Pri54, Pri58, Pri62], Lemmon [Lem57], von Wright [vW51, vW53, vW56, vW68, vW71].

To briefly expand on the type of objections against normality raised in the context of epistemic and deontic logic, we mention Lemmon's argument in [Lem57], where he proposes several axiomatic systems of epistemic and deontic modal logics. The rule of necessitation is not included in them since it causes the presence in the logic of theorems of the form $\Box\phi$. In the context of the interpretation of the \Box -operator as moral obligation or scientific but not logical necessity, Lemmon's systems are in line with the view that *nothing should be a scientific law or a moral obligation as a matter of logic*.

Besides epistemic and deontic logic, within the same area of philosophical logic, non-normal modal logics and monotone modal logic in particular have been linked also to the analysis of counterfactuals [Lew73], via the standard semantic environment of

¹Chellas' classical modal logic (cf. [Che80, Definition 8.1]), is defined as the smallest logic on a classical propositional base containing the axiom $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ and closed under the rule

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}.$$

neighbourhood frames (more on this topic below). After the introduction of Kripke semantics [Kri63], the axiomatic frameworks of normal modal logic became prevalent, to the point that normal modal logics became the default formal framework for modal logic.

However, in recent years, the interest on non-normal modal logics, and monotone modal logics in particular, has been renewed from diverse directions. To mention only some directions, in recent years, both deontic logic and epistemic logic have found new areas of application. For instance, deontic logic has been increasingly relevant to the field of system specification in computer science. In this context, the deontic interpretation of modal operators (not anymore ‘moral obligation’, but more in general ‘norm’) is useful e.g. to enforce a different management between soft and hard constraints in planning and scheduling problems. Using modal operators to express soft constraints as norms brings the advantage that norms can be violated without creating an inconsistency in the formal specification, in contrast to violations of hard constraints (cf. e.g. [CJ96]). In this context, the formalization of obligation in terms of normal modal operators is not considered adequate, since it gives rise to paradoxes or counterintuitive interpretations. Also, formal frameworks based on epistemic logic have been increasingly used in artificial intelligence to capture sophisticated aspects of informational dynamics. Also in these contexts, objections to formalizations in terms of normal modal operators have been raised.

Finally, in the last twenty years, logics have been introduced, aimed at capturing various phenomena such as concurrency in computer science (cf. Concurrent Propositional Dynamic Logic [Gol92a]), rational reasoning in multi-agent environments (cf. Coalition Logic [Pau02] and Game Logic [Par85]), or probabilistic reasoning (cf. Modal Logic for Probability [Her03]). In all these contexts, not only the assumption of normality, but also the distributivity of modalities over binary joins and meets is problematic. For instance, in Game Logic, an agent can have a strategy to reach a state where either p_1 or p_2 hold, without it being sufficient to establish which of the p_i will hold. Another example arises in settings in which the intended meaning of $\Box\phi$ is ϕ holds with a probability greater than some threshold $0 \leq r \leq 1$. Assume that the probability that p holds is $\frac{3}{4}$, that the probability that q holds is $\frac{3}{4}$, and that $r = \frac{2}{3}$. Then we have that $\Box p \wedge \Box q$ holds, but $\Box(p \wedge q)$ does not, given that the probability that $p \wedge q$ holds is $\frac{9}{16}$. For more details about monotone modal logic and neighbourhood semantics, the reader is referred to [Han03] and [Pac07].

A rich mathematical theory. Neighbourhood models, also known as Scott-Montague models (cf. [Sco70, Mon70]), generalize Kripke models, and have become the standard semantic environment for those modal logics such that the Kripke-valid principles $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$ (multiplicativity) and $\Box p \rightarrow \Box(p \vee q)$ (monotonicity) are not valid. In a neighbourhood frame, each state is associated with a collection of subsets of the universe (called neighbourhoods), and this is how the accessibility relation is specified. A neighbourhood frame is formally defined as a tuple $\mathbb{F} := (W, \tau)$ such that W is a non-empty set and $\tau : W \rightarrow \mathcal{P}\mathcal{P}W$, and a neighbourhood model is a tuple $\mathbb{M} := (\mathbb{F}, \nu)$ such that \mathbb{F} is a neighbourhood frame and $\nu : \text{AtProp} \rightarrow \mathcal{P}W$ is a valuation. A modal formula $\Box\phi$ is true at a given state of a neighbourhood model iff the truth set of ϕ is a neighbourhood of that state. Chellas' classical modal logic (cf. footnote 1) is sound and complete w.r.t. neighbourhood models.

In the environment of neighbourhood models and frames, it is possible to establish correspondence-type facts similar to those holding in Kripke frames. Here below, we report on the best known of them: for any $X \subseteq W$, let $\tau^{-1}[X] := \{w \in W \mid X \in \tau(w)\}$.

(N)	$\Box\top$	(n)	$\forall w[W \in \tau(w)]$
(P)	$\neg\Box\perp$	(p)	$\forall w[\emptyset \notin \tau(w)]$
(M)	$\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$	(m)	$\forall w\forall X\forall Y[(X \in \tau(w) \ \& \ X \subseteq Y) \Rightarrow Y \in \tau(w)]$
(C)	$\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$	(c)	$\forall w\forall X\forall Y[(X \in \tau(w) \ \& \ Y \in \tau(w)) \Rightarrow X \cap Y \in \tau(w)]$
(D)	$\Box\phi \rightarrow \Diamond\phi$	(d)	$\forall w\forall X[X \in \tau(w) \Rightarrow X^c \notin \tau(w)]$
(T)	$\Box\phi \rightarrow \phi$	(t)	$\forall w\forall X[X \in \tau(w) \Rightarrow w \in X]$
(B)	$\phi \rightarrow \Box\Diamond\phi$	(b)	$\forall w\forall X[w \in X \Rightarrow (\tau^{-1}[X^c])^c \in \tau(w)]$
(4)	$\Box\Box\phi \rightarrow \phi$	(iv)	$\forall w\forall X\forall Y[(X \in \tau(w) \ \& \ \forall x[x \in X \Rightarrow Y \in \tau(x)]) \Rightarrow Y \in \tau(w)]$
(4')	$\Box\phi \rightarrow \Box\Box\phi$	(iv')	$\forall w\forall X\forall Y[X \in \tau(w) \Rightarrow \tau^{-1}[X] \in \tau(w)]$
(5)	$\Diamond\phi \rightarrow \Box\Diamond\phi$	(v)	$\forall w\forall X[X \notin \tau(w) \Rightarrow (\tau^{-1}[X])^c \in \tau(w)]$

In particular, monotone modal logic, which can be equivalently characterized as classical modal logic extended with the axiom (M) above, is sound and strongly complete with respect to the class of *monotone neighbourhood frames*, i.e. the class of neighbourhood frames satisfying condition (m) above (see [Han03]).

Any Kripke frame $\mathbb{F} = (W, R)$ can be recast as a (special) monotone neighbourhood frame simply by defining the neighbourhood map $\sigma_{\mathbb{F}} : W \rightarrow \mathcal{P}\mathcal{P}W$ by the assignment $w \mapsto \{X \subseteq W \mid R[w] \subseteq X\}$. Conversely, a neighbourhood frame (W, σ) contains its core if $\bigcap \sigma(w) \in \sigma(w)$ for every $w \in W$. Clearly, for every Kripke frame \mathbb{F} , the monotone neighbourhood frame $\mathbb{N}_{\mathbb{F}} = (W, \sigma_{\mathbb{F}})$ contains its core. A monotone neighbourhood frame $\mathbb{N} = (W, \sigma)$ containing its core gives rise to the Kripke frame $\mathbb{F}_{\mathbb{N}} = (W, R_{\mathbb{N}})$ such that $R_{\mathbb{N}}[w] = \bigcap \sigma(w)$ for each $w \in W$. It is straightforward to see that $\mathcal{F} = \mathbb{F}_{\mathbb{N}_{\mathbb{F}}}$ and $\mathbb{N} = \mathbb{N}_{\mathbb{F}_{\mathbb{N}}}$.

for every Kripke frame \mathbb{F} and every monotone neighbourhood frame \mathbb{N} containing its core.

The link between monotone modal logics and normal logics has been studied a lot, both in the sense that normal modal logic has been used to simulate monotone modal logic [KW99], and in the sense that results and techniques from normal modal logic have been extended to the non-normal case: for instance, in [Doš89], duality results have been established between categories of neighbourhood frames and categories of modal algebras (understood as Boolean algebras with an additional unary operation), which extend results of Goldblatt [Gol76] and Thomason [Tho75] about categories of relational frames for normal modal logic. Building on the seminal work [Pau99], in [HK04] and [HKP07] the coalgebraic perspective on normal modal logic has been extended to non-normal modal logics. Indeed, neighbourhood structures have been equivalently recast as coalgebras for the functor $F := 2^{2^{(-)}}$ resulting from the composition of the contravariant powerset functor with itself. In this setting, the coalgebraic equivalence notions of F -bisimulation, behavioural equivalence and neighbourhood bisimulation have been studied, and the analogues of van Benthem's characterization theorem and Hennessy-Milner theorem have been proved.

Other recent directions explore the link between neighbourhood semantics and (formal) topology. For instance, in [dAFH09], Lewis' neighbourhood semantics for counterfactual logic [Lew73] has been extended to a topos-theoretic setting.

A logic embedded in many logics. Summing up the discussion so far, monotone modal logic lies at the intersection of many issues, pertaining to philosophy, mathematics, theoretical computer science, artificial intelligence and social science, as is witnessed by the fact that non-normal modal connectives occur in very different formal systems such as game logics [Par85, PP03], epistemic logics [vDvdHK03, Hei96], probabilistic and dynamic logics [Gol92b, Her03], deontic logics [CJ96]. Hence, we believe that it is important and useful to study monotone modal logic not in isolation, but in the context of all these different settings, and to develop methods which can be transferred from one logical system to another. This is the approach taken in the present thesis. Specifically, monotone modal logic is studied at the crossroads of two very general methodologies pertaining to two different areas of logic: algebraic *correspondence theory* (developed within Stone duality for nonclassical logics), and the theory of *display calculi*, developed within structural proof theory, and motivated both by algebraic and philosophical considerations. Algebraic and algorithmic correspondence theory has been adapted to monotone modal logic, and this adaptation has been used as a basis for the formalization of a correspondence result about finite lattices. On the other hand, the theory of

display calculi has been extended to a multi-type setting, which makes it possible to treat smoothly and uniformly a wide range of difficulties arising in the proof theory of dynamic logics such as Baltag-Moss-Solecki's logic of epistemic actions and knowledge. Within the multi-type setting, a generalization has been defined to deal with the specific feature of monotone modal logic, namely the fact that monotone modalities do not have adjoints per definition. In its turn, this generalization for monotone modal logic can be applied to solve certain hurdles in multi-type calculi for other dynamic logics such as game logic. This brief outline hopefully shows the usefulness of adopting a synthetic approach in the study of families of logics, and in particular, the key mediating role played by monotone modal logic in these different contexts.

In the remainder of the present introduction, we give an outline of algebraic and algorithmic correspondence, and of display calculi.

Correspondence theory. Sahlqvist theory has a long history in modal logic, going back to [Sah75] and [vB85]. The Sahlqvist theorem in [Sah75] gives a syntactic definition of a class of modal formulas, the *Sahlqvist class*, each member of which defines an elementary (i.e. first order definable) class of frames and is canonical. These are two very important properties: the canonicity of an axiomatization guarantees the strong Kripke completeness of its associated logic, and elementarity guarantees all the advantages, both computational and theoretical, of first-order logic over second-order logic. Both canonicity and elementarity turn out to be algorithmically undecidable, taken singularly and in combination [CC06], so a decidable approximation, like the Sahlqvist class, is very desirable.

Over the years, many extensions, variations and analogues of this result have appeared. For instance, algebraic/topological proofs in [SV89] and [Jón94], constructive canonicity in [GM97], variations of the correspondence language in [vB06], Sahlqvist-type results for hybrid logics in [tCMV05] and μ -calculus in [CFPS14] and [vBBH12], enlargements of the Sahlqvist class to e.g. the *inductive* formulas of [GV06].

From its onset, Sahlqvist theory has had a distinct algorithmic flavour to it, given that the Sahlqvist-van Benthem algorithm of [vB85] was used to effectively compute the first-order frame correspondents of Sahlqvist formulas. Recently, new algorithmic methods for correspondence and canonicity have greatly enlarged the original Sahlqvist class. In particular, the SQEMA algorithm (see [CGV06]) is guaranteed to succeed in computing the first order correspondent of each member of a class of modal formulas, the inductive formulas, significantly larger than the Sahlqvist class. Moreover, inductive formulas are also proven to be canonical.

In [CP12], algorithmic canonicity and correspondence results analogous to the ones appearing in [CGV06] have been obtained for the language of distributive modal logic, a modal logic framework the propositional base of which is the logic of distributive lattices. A critical feature of this direction is that the dualities and adjunctions between the relational and the algebraic semantics of these logics have made it possible to distill the order-theoretic and algebraic significance of the SQEMA reduction steps from the model theoretic setting, and hence to recast them into an algebraic setting which is more general than the boolean one, and which can be extended even more to the setting of general (i.e. not necessarily distributive) lattices [CP14]. Taking stock of these results gave rise to the so-called unified correspondence theory, a framework within which correspondence results can be formulated and proved abstracting away from specific logical signatures, and only in terms of the order-theoretic properties of the algebraic interpretations of logical connectives. A specific feature of the unified correspondence approach is the use of existing dualities to adapt correspondence results to possibly different semantic environments for a given logic. For more details about this connection and the ensuing generalizations, the reader is referred to [CPS] and [CGP14].

In Chapter 1 of the present thesis, we give an application of unified correspondence theory. Namely, we adapt the algorithm ALBA so that it fits an algebraic environment arising from finite lattices and which provides an interpretation for a monotone modal language. As a consequence of this adaptation, we are able to obtain certain characterization results about classes of finite lattices as instances of the modified ALBA reductions.

Display Calculi. Nuel Belnap introduced the first display calculus, which he calls *Display Logic* [Bel82], as a proof-theoretic framework designed to capture in a modular way many different logics in one uniform setting. The main technical advantage of display calculi is that they provide a setting to state and prove generic (i.e. canonical) cut elimination metatheorems. These are theorems which guarantee a given calculus to enjoy cut elimination, provided it satisfies certain conditions (reported here in Section 2.3), the verification of which is relatively straightforward. The proof of these metatheorems consists in defining *uniform* transformation steps in the cut elimination proof à la Gentzen. This uniformity is the reason why we can say that Belnap-style cut elimination (i.e. cut elimination obtained as the consequence of such a metatheorem) is to Gentzen-style cut elimination what canonicity is to completeness.

Display calculi have given adequate proof-theoretic accounts of logics which have notoriously been difficult to treat with other approaches. Among them, besides classical

and (bi-)intuitionistic logic [Bel82], are linear logic [Bel90], modal and tense (bi-)intuitionistic logic [GPT10, Gor96, Kra96, Wan94], substructural logics [Gor98], relevant logics [Res98], or the logic of relation algebras [Gor97]. In the specific case of modal logic, it has been extremely difficult to extend traditional Gentzen sequent calculi so as to obtain a uniform and general proof theory encompassing the numerous extensions of the minimal normal modal logic K . One important feature of display calculi is that there is a neat division of labour between the *introduction rules* for the logical connectives, and the *structural rules*, which encode special properties of given logical connectives, both taken singularly and in their interaction. Axiomatic extensions of a given logic are typically captured in a modular way by the adding structural rules corresponding to the additional axioms to the basic calculus. In [Kra96], Kracht accounted for a large class of such modal extensions uniformly and modularly in the context of display calculi, by characterizing a subclass of Sahlqvist modal formulas (the so-called *primitive formulas*), each of which can be encoded as a structural rule in display calculus.

As to the specific design features of display calculi, Belnap took inspiration from Gentzen's basic observations on structural rules. Indeed, in the standard Gentzen formulation, the comma symbol ‘,’ separating formulas in the precedent and in the succedent of sequents can be recognized as a metalinguistic connective, of which the structural rules define the behaviour. Belnap took this idea further by admitting not only the comma, but also several other connectives to keep formulas together in a structure, and called them *structural connectives*. Just like the comma in standard Gentzen sequents is interpreted contextually (that is, as conjunction when occurring on the left-hand side and as disjunction when occurring on the right-hand side), each structural connective typically corresponds to a pair of logical connectives, and is interpreted as one or the other of them contextually (more of this in Sections 2.2.2 and 2.6.1). Structural connectives maintain relations with one another, the most fundamental of which take the form of adjunctions and residuations. These relations make it possible for the calculus to enjoy the powerful property which gives it its name, namely, the *display property*.

Definition 0.1. A proof system enjoys the *display property* iff for every sequent $X \vdash Y$ and every substructure Z of either X or Y , the sequent $X \vdash Y$ can be equivalently transformed, using the rules of the system, into a sequent which is either of the form $Z \vdash W$ or of the form $W \vdash Z$, for some structure W . In the first case, Z is *displayed in precedent position*, and in the second case, Z is *displayed in succedent position*. The invertible rules enabling this equivalent rewriting are called *display postulates*.

To illustrate the fundamental role played by the display property in the transformation steps of the cut elimination metatheorem, consider the elimination step of the following cut application, in which the cut formula is principal on both premises of the cut.

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_1 \quad \vdots \pi_2 \quad \vdots \pi \\
\frac{X_1 \vdash A \quad X_2 \vdash B}{X_1, X_2 \vdash A \wedge B} \quad \frac{A, B \vdash Y}{A \wedge B \vdash Y} \\
\hline
X_1, X_2 \vdash Y
\end{array}
\rightsquigarrow
\begin{array}{c}
\vdots \pi \\
\frac{\frac{\frac{\vdots \pi_1 \quad X_1 \vdash A}{X_1 \vdash A} \quad \frac{A, B \vdash Y}{A \vdash Y < B}}{X_1 \vdash Y < B}}{\vdots \pi_2 \quad X_2 \vdash B} \\
\frac{X_2 \vdash B \quad \frac{X_1 \vdash Y < B}{X_1, B \vdash Y}}{B \vdash X_1 > Y} \\
\hline
\frac{X_2 \vdash X_1 > Y}{X_1, X_2 \vdash Y}
\end{array}
\end{array}$$

The dashed lines in the proof tree on the right-hand side correspond to applications of display postulates. Clearly, this transformation step has been made possible because the display postulates disassemble, as it were, compound structures so as to give us access to the immediate subformulas of the original cut formula, and then reassemble them so as to ‘put things back again’. Hence, it is possible to break down the original cut into two cut applications on the immediate subformulas, as required by the original Gentzen strategy.

At this point, it is natural to ask the following question. If the modal operators in monotone modal logic are supposed *not* to preserve joins and meets, then it is not reasonable to assume that they have adjoints. However, it clearly emerges from the example above that the display postulates essentially encode adjunctions/residuations. So how is it possible to account for monotone modal logics within the display calculi methodology?

Indeed, the price to pay for the beautiful display property is that the language needs to include additional connectives which are essential for the bookkeeping required by its enforcement. The presence of these additional connectives immediately brings about conservativity issues. In many cases, these issues can be resolved by appealing to semantic arguments (we will see an example of this happy situation in Chapter 2). In other cases, semantic arguments are not applicable, but conservativity can be proved nonetheless, thanks to excellent syntactic circumstances (this is the case of the calculus of Chapter 3). In the case of monotone modal logic, we have given up the display property, and opted for a calculus which is *display-type* but not really display. Indeed, in Chapter 5 we introduce a calculus which includes some display postulates but not enough of them to guarantee reduction steps like the one illustrated above. Nonetheless, and most importantly, we will show in Chapter 4 that it is still possible to prove a Belnap-style cut elimination metatheorem for display-type calculi. Finally, in the case of the calculus of Chapter 6, the conservativity is still open, and we will detail a proof strategy to achieve it.

Synopsis

In Chapter 1, we report on the results in [FPS14]. We adapt the algorithm ALBA to monotone modal logic, and obtain certain dual characterization results for finite lattices as instances of successful reductions of the modified ALBA algorithm.

In Chapter 2, we report on the results in [FGK⁺14c]. We provide an analysis of the existing proof systems for dynamic epistemic logic from the viewpoint of proof-theoretic semantics. We discuss the main features of display calculi, and focus our attention on Wansing's notion of proper display calculi. We generalize this notion to that of quasi-proper display calculi, and prove its corresponding Belnap-style cut elimination metatheorem. We introduce the display calculus D'.EAK for Baltag-Moss-Solecki's logic of Epistemic Actions and Knowledge. This calculus revises and improves a previous calculus D.EAK given in [GKP13], for which cut elimination Belnap-style could not be proven. A common feature of D.EAK and D'.EAK is the presence of the adjoints of the dynamic modal operators. Although these adjoints are not naturally interpretable in the standard Kripke semantics of updated models, they have a natural interpretation in the alternative but equivalent final coalgebra semantics for EAK. We prove that D'.EAK is sound w.r.t. the final coalgebra semantics, complete and conservative w.r.t. EAK, and is a quasi-proper display calculus, hence it enjoys cut elimination via the corresponding Belnap-style metatheorem.

In Chapter 3, we report on the results in [FGK⁺14a]. We generalize display calculi to a multi-type setting. We introduce the multi-type counterpart of quasi-proper display calculi, referred to as quasi-proper multi-type display calculi, and prove their corresponding Belnap-style cut elimination metatheorem. We introduce a multi-type display calculus for EAK, referred to as Dynamic Calculus for EAK. In this calculus, the parameters indexing the dynamic and epistemic modal operators in the original language of EAK are taken as terms, and the unary modal operators are encoded as binary operations taking arguments of different types. We prove that the Dynamic Calculus for EAK is sound w.r.t. the final coalgebra, complete and conservative w.r.t. EAK and is a quasi-proper multi-type display calculus, hence it enjoys cut elimination via the corresponding Belnap-style metatheorem.

In Chapter 4, we report on the results in [FGK⁺14b]. We discuss a generalization of the multi-type display setting in which the display property (either full or relativized) is dropped. This setting imports ideas from Sambin's Basic Logic [BFS00], and compensates the lack of display property by pivoting on the so-called *visibility property*. We introduce the notion of quasi-proper display-type calculi in the multi-type setting, and prove its associated Belnap-style cut elimination metatheorem.

In Chapter 5, we introduce a more compact version of the setting in the previous chapter. This version is single-type and ‘proper’ instead of quasi-proper (that is, the shape of the axioms is restricted as in Belnap’s original paper, rather than more general, as e.g. in the proof systems for EAK). We introduce a display-type calculus for monotone modal logic, and prove that it is sound and complete w.r.t. the basic monotone modal logic, and is a proper display-type calculus, hence it enjoys cut elimination via the corresponding Belnap-style metatheorem.

In Chapter 6, we report on the results in [FGKP14]. We introduce a multi-type display calculus for Propositional Dynamic Logic (PDL). This calculus is complete w.r.t. PDL, and is a proper multi-type display calculus, hence enjoys cut elimination via the corresponding Belnap-style metatheorem. We discuss the soundness of its rules w.r.t. the standard semantics, and its conservativity problem, which is still open.

Appendix A provides the proof of the main result in Chapter 1, that is Lemma 1.34.

Appendix B lists the Belnap-style cut elimination metatheorems discussed in Part II.

Appendices C, D, and E collect material pertaining to Chapter 2. Appendix C provides some derived rules of the calculus D’.EAK. Appendix D collects most transformation steps in the verification that D’.EAK is a quasi-proper display calculus (cf. Section 2.6.3). In Appendix E, we prove the completeness of D’.EAK w.r.t. the Hilbert presentation of EAK (cf. Sections 2.4.1 and 2.4.2) by deriving the axioms of (the intuitionistic version of) EAK in D’.EAK.

Appendices F and G refer to Chapter 3. In Appendix F, we complete the proof of the Belnap-style cut elimination (cf. Section 3.5.2) for the Dynamic Calculus for EAK given in Section 3.2. In Appendix G, we prove the completeness of the Dynamic Calculus for EAK w.r.t. the Hilbert presentation of D’.EAK (cf. Section 2.4) by deriving the axioms of (the intuitionistic version of) EAK in the Dynamic Calculus.

Appendix H provides the rules of the display-type sequent calculus for monotone modal logic introduced in Chapter 5.

Appendices I, J and K refer to Chapter 6. Appendix I provides the rules for the propositional base of the display calculus for PDL. In Appendix J, we complete the proof of the Belnap-style cut elimination (cf. Section 6.6) for the display calculus for PDL given in Section 6.3. In Appendix K, we prove that the display calculus for PDL given in Section 6.3 is complete w.r.t. the Hilbert presentation of PDL.

Table 1 gives a schematic view of the different kinds of calculi we study in Part II of the present thesis, and the logics to which they apply.

		Quasi-Proper	Proper
Display	Single-Type	EAK (cf. Chapter 2)	Primitive modal logics [Kra96]
	Multi-Type	EAK (cf. Chapter 3)	PDL (cf. Chapter 6)
Display-Type	Single-Type		MML (cf. Chapter 5)
	Multi-Type	Chapter 4	

TABLE 1: Overview of main notions in Part II of the present thesis

Contribution

Chapter 1 is based on [FPS14]. The initial idea of this work, that certain characterization results for finite lattices can be understood as correspondence phenomena, was formulated by Luigi Santocanale already in [San09]. He also conjectured that the correspondence techniques revolving around ALBA could be usefully adapted to this problem. The author of the present thesis, in collaboration with Alessandra Palmigiano, introduced the basic adaptation of the algorithm ALBA (cf. [CP12]) to monotone modal logic (cf. Section 1.3). The introduction of the further enhancements of ALBA for the specific finite lattice environment (cf. Section 1.4) and the application to the D^+ -chains (cf. Section 1.5 and Appendix A) are contributions of the present author.

Chapter 2 is based on [FGK⁺14c]. A precursor of this paper is [GKP13], in which Giuseppe Greco, Alessandra Palmigiano and Alexander Kurz introduce a calculus for EAK which is rather similar to the one appearing in [FGK⁺14c], but which fails to satisfy all the conditions of Belnap’s original metatheorem. In particular, the cut elimination for this calculus is proved Gentzen-style. The present author contributed to the introduction of the notion of quasi properly display calculus and to the proof of its associated cut-elimination metatheorem. A collective and crucial realization emerging from those first discussions was that we needed to abandon the strategy of trying to appeal to existing metatheorems and rather introduce a new design, which then resulted in the notion of quasi-proper display calculus. The present author contributed to all the developments which make the calculus introduced in [FGK⁺14c] a technical improvement over the one of [GKP13]. The preliminary Sections 2.1 and 2.4 elaborate material discussed in Vlasta Sikimić’s master thesis [Sik13]; the present version incorporates the feedback received from the reviewers of the paper. Section 2.5 on the coalgebra semantics was mainly developed by Alexander Kurz. The present author contributed, at an earlier or later stage, to each section of the paper, with a special stress on the new notion of quasi-proper display calculi (cf. Section 2.3), and the proof of its associated Belnap-style metatheorem.

Chapter 3 is based on [FGK⁺14a]. The idea of multi-type calculus already appeared in Vlasta Sikimić's thesis [Sik13], in which a preliminary version appears of the calculus introduced in [FGK⁺14a] and an attempt at a proof of cut elimination Belnap-style, by appealing to an existing metatheorem. However, as discussed above, the strategy of appealing to existing metatheorems turned out to be unfeasible, both for the single-type calculus, and a fortiori for the multi-type calculus. Again, while the present author's contributions cover each section of the paper in different stages of its development, the main contributions focus on the development of the notion of quasi-proper multi-type display calculi and its associated metatheorem (cf. Section 3.2), the modifications of the rules of the Dynamic Calculus for EAK (cf. Section 3.3), the soundness (cf. Section 3.4) and the conservativity (cf. Section 3.6).

Chapter 4 is based on [FGK⁺14b], and Chapter 5 reports on work not yet published. Introducing a display calculus for monotone modal logic was one of the main initial motivations for the present author, and an important technical step towards the development of proof-theoretic semantically adequate calculi for such logics as Parikh's game logic. The initial idea on how to overcome the technical difficulties specific to monotone modal logic was Giuseppe Greco's, and consisted in making use of the visibility notion as in [BFS00]. Giuseppe Greco and the present author studied the setting of display-type calculi together, developing in particular the metatheorem specific to display-type calculi (cf. Chapter 4). The present author then developed the calculus and the proof of the metatheorem most directly applicable to monotone modal logic (cf. Chapter 5).

Chapter 6 is based on [FGKP14]. As was the case with the previous many-author papers, this paper grew from the close collaboration and interaction of all the authors, so that each author contributed in a widespread way to the paper in its present form. As to some features of the design of the calculus (cf. Section 6.3), Alessandra Palmigiano had the idea to introduce actions and transitive actions as separate types, and to model the proof-theoretic behaviour of the connective $(\cdot)^+$ making use of the fact that its semantic interpretation, i.e. the transitive closure operation, is left adjoint to the embedding of transitive actions into actions ; Giuseppe Greco developed the theory around the fixed point axiom and the induction axiom, and the present author developed the other technicalities of the design of the calculus: the test, the choice, the sequential composition and the virtual adjoints. In collaboration with Alessandra Palmigiano, the present author proved the soundness of the rules involving no virtual adjoints (cf. Section 6.4), and, in collaboration with Giuseppe Greco, proved the completeness of the calculus (cf. Section 6.5).

Résumé en français

Introduction

Logique modale monotone. La logique modale monotone est la plus petite logique modale classique² dont les opérateurs modaux satisfont la règle d'inférence *RM*:

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}.$$

Les logiques modales monotones sont étudiées depuis l'apparition de la logique modale sous sa forme moderne (cf. [Lew18, LL32]) par des philosophes formels tels que Lewis dans le but de formaliser la notion d'implication stricte. Parmi les systèmes logiques *S* 1-*S* 5 proposés par Langford et Lewis dans [LL32], seulement *S* 4 et *S* 5 sont des logiques normales. Les travaux de Prior [Pri54, Pri58, Pri62], Lemmon [Lem57], et von Wright [vW51, vW53, vW56, vW68, vW71] témoignent de l'intérêt porté aux logiques modales monotones comme des systèmes formels adéquats pour représenter des raisonnements épistémiques et déontiques.

Par ailleurs, il existe plusieurs objections émises contre l'utilisation de logique modale normale dans le contexte de la logique épistémique et déontique. Mentionnons les arguments de Lemmon dans [Lem57] où il propose plusieurs systèmes d'axiomes pour les logiques épistémiques et déontiques. Ces systèmes ne contiennent pas la règle de nécessité car elle entraîne la présence dans la logique de théorèmes de la forme $\Box\phi$. Lorsque l'opérateur \Box est interprété comme une obligation morale ou scientifique mais pas comme une nécessité logique, les systèmes proposés par Lemmon sont en accord avec l'idée que *rien ne devrait être une loi scientifique ou une obligation morale d'un point vue logique*.

² La logique modale classique introduite par Chellas (cf. [Che80, Definition 8.1]) est définie comme la plus petite logique dont la base propositionnelle est classique, contenant l'axiome $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ et fermée par rapport à la règle

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}.$$

En plus des logiques épistémiques et déontiques, dans le même domaine de la logique philosophique, les logiques non-normales et en particuliers les logiques monotones ont aussi été liées à l'étude des contre-factuels [Lew73], via l'environnement de la sémantique standard des structures de voisinages. Après l'introduction de la sémantique de Kripke [Kri63], le système axiomatique de la logique modale normale a été très utilisé, à tel point que les logiques modales normales sont devenues le système formel par défaut pour la logique modale.

Cependant, ces dernières années, l'intérêt pour les logiques modales non-normales, et en particulier pour les logiques modales monotones s'est exprimé dans différentes directions. Pour n'en mentionner que quelques unes, récemment, la logique épistémique et la logique déontique ont toutes deux trouvé de nouveaux domaines d'applications. Par exemple, la logique déontique est devenue de plus en plus pertinente dans le domaine de la spécification de systèmes en informatique. Dans ce contexte, l'interprétation déontique des opérateurs modaux (non plus "l'obligation morale", mais de façon plus générale "la norme") est utile par exemple pour appliquer une gestion différente des contraintes faibles et dures dans les problèmes de planning et d'ordonnancement. Utiliser les opérateurs modaux pour exprimer les contraintes faibles en terme de normes a l'avantage que les normes peuvent être enfreintes sans créer une inconsistance dans la spécification formelle, contrairement à la violation de contraintes dures (cf. par exemple [CJ96]). Dans ce contexte, la formalisation de l'obligation à l'aide d'opérateurs modaux normaux n'est pas considérée adéquate, puisque cela engendre des paradoxes ou des interprétations contre-intuitives. De même, les systèmes formels basés sur la logique épistémique sont de plus en plus utilisés en intelligence artificielle pour modéliser des aspects sophistiqués de la dynamique de l'information. Dans ce contexte, des objections contre les formalisations en termes d'opérateurs modaux normaux sont aussi apparues.

Finalement, durant les vingt dernières années, des logiques ont été introduites dont le but est de modéliser différents phénomènes tels que la concurrence en informatique (cf. Concurrent Propositional Dynamic Logic [Gol92a]), les raisonnements rationnels dans des environnements contenant plusieurs agents (cf. *Coalition Logic* [Pau02] et *Game Logic* [Par85]), ou les raisonnements probabilistes (cf. *Modal Logic for Probability* [Her03]). Dans tous ces contextes, non seulement l'hypothèse de normalité, mais aussi la distributivité des opérateurs modaux sur les suprema et les infima binaires sont problématiques. Par exemple, en logique des jeux, un agent peut avoir une stratégie pour atteindre un état tel que p_1 ou p_2 est vraie sans que sa stratégie ne lui permette de déterminer laquelle des propositions p_i sera vraie. Un autre exemple provient du cadre dans lequel la signification de l'opérateur \square est " ϕ est vraie avec une probabilité supérieure à un certain seuil $0 \leq r \leq 1$ ". Supposons que la probabilité que p soit vraie est $\frac{3}{4}$, que la probabilité que q soit vraie soit $\frac{3}{4}$, et que $r = \frac{2}{3}$. Alors, nous avons que

$\Box p \wedge \Box q$ est vraie, mais $\Box(p \wedge q)$ ne l'est pas étant donné que la probabilité que $p \wedge q$ soit vraie est $\frac{9}{16}$. Pour plus de détails sur la logique modale monotone et la sémantique des voisinages, se référer à [Han03] et [Pac07].

Une théorie mathématique riche. Les structures de voisinages, aussi connues sous le nom de modèles de Scott et Montague (cf. [Sco70, Mon70]), généralisent les modèles de Kripke et sont devenues l'environnement sémantique standard des logiques pour lesquelles l'axiome de multiplicité $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$ et/ou l'axiome de monotonie $\Box p \rightarrow \Box(p \vee q)$ ne sont pas valides. Dans une structure de voisinages, chaque état est associé à une collection de sous-ensembles de l'univers (appelés voisinages) qui définissent la relation d'accessibilité. Une structure de voisinage est formellement définie comme un couple $\mathbb{F} := (W, \tau)$ tel que W est un ensemble non-vide et $\tau : W \rightarrow \mathcal{P}\mathcal{P}W$. Un modèle de voisinages est un couple $\mathbb{M} := (\mathbb{F}, \nu)$ tel que \mathbb{F} est une structure de voisinages et $\nu : \text{AtProp} \rightarrow \mathcal{P}W$ est une valuation. Une formule modale $\Box\varphi$ est vraie à un état donné du modèle de voisinages si et seulement si l'ensemble des valeurs de vérité de φ est un voisinage de cet état. La logique modale classique introduite par Chellas (cf. note de bas de page numéro 2) est correcte et complète par rapport aux modèles de voisinages.

Dans l'environnement des modèles et structures de voisinages, il est possible d'établir des résultats de correspondance similaires à ceux établis pour les structures de Kripke. Ci-dessous, nous rappelons les plus connus d'entre eux: pour tout $X \subseteq W$, soit $\tau^{-1}[X] := \{w \in W \mid X \in \tau(w)\}$,

(N)	$\Box \top$	(n)	$\forall w [W \in \tau(w)]$
(P)	$\neg \Box \perp$	(p)	$\forall w [\emptyset \notin \tau(w)]$
(M)	$\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$	(m)	$\forall w \forall X \forall Y [(X \in \tau(w) \ \& \ X \subseteq Y) \Rightarrow Y \in \tau(w)]$
(C)	$\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$	(c)	$\forall w \forall X \forall Y [(X \in \tau(w) \ \& \ Y \in \tau(w)) \Rightarrow X \cap Y \in \tau(w)]$
(D)	$\Box\phi \rightarrow \Diamond\phi$	(d)	$\forall w \forall X [X \in \tau(w) \Rightarrow X^c \notin \tau(w)]$
(T)	$\Box\phi \rightarrow \phi$	(t)	$\forall w \forall X [X \in \tau(w) \Rightarrow w \in X]$
(B)	$\phi \rightarrow \Box\Diamond\phi$	(b)	$\forall w \forall X [w \in X \Rightarrow (\tau^{-1}[X^c])^c \in \tau(w)]$
(4)	$\Box\Box\phi \rightarrow \phi$	(iv)	$\forall w \forall X \forall Y [(X \in \tau(w) \ \& \ \forall x [x \in X \Rightarrow Y \in \tau(x)]) \Rightarrow Y \in \tau(w)]$
(4')	$\Box\phi \rightarrow \Box\Box\phi$	(iv')	$\forall w \forall X \forall Y [X \in \tau(w) \Rightarrow \tau^{-1}[X] \in \tau(w)]$
(5)	$\Diamond\phi \rightarrow \Box\Diamond\phi$	(v)	$\forall w \forall X [X \notin \tau(w) \Rightarrow (\tau^{-1}[X])^c \in \tau(w)]$

En particulier, la logique modale monotone peut être définie de façon équivalente comme la logique modale classique enrichie avec l'axiome (M) ci-dessus. Elle est correcte et fortement complète par rapport à la classe des structures de voisinages monotones, i.e. la classe des structures de voisinages satisfaisant la condition (m) ci-dessus (voir [Han03]).

Toute structure de Kripke $\mathbb{F} = (W, R)$ peut être redéfinie comme une structure (spéciale) de voisinages monotone simplement en définissant la fonction de voisinages $\sigma_{\mathbb{F}} : W \rightarrow \mathcal{P}\mathcal{P}W$ avec l'application $w \mapsto \{X \subseteq W \mid R[w] \subseteq X\}$. Réciproquement, une structure de voisinages (W, σ) contient son noyau si $\bigcap \sigma(w) \in \sigma(w)$ pour chaque $w \in W$. Manifestement, pour chaque structure de Kripke \mathbb{F} , la structure de voisinages monotone $\mathbb{N}_{\mathbb{F}} = (W, \sigma_{\mathbb{F}})$ contient son noyau. Une structure de voisinages monotone $\mathbb{N} = (W, \sigma)$ contenant son noyau donne lieu à une structure de Kripke $\mathbb{F}_{\mathbb{N}} = (W, R_{\mathbb{N}})$ telle que $R_{\mathbb{N}}[w] = \bigcap \sigma(w)$ pour chaque $w \in W$. Il est facile de voir que $\mathbb{F} = \mathbb{F}_{\mathbb{N}_{\mathbb{F}}}$ et $\mathbb{N} = \mathbb{N}_{\mathbb{F}_{\mathbb{N}}}$ pour chaque structure de Kripke \mathbb{F} et pour chaque structure de voisinages monotone \mathbb{N} contenant son noyau.

Le lien entre les logiques modales monotones et les logiques normales a été beaucoup étudié. En effet, les logiques modales normales ont été utilisées pour simuler les logiques modales monotones [KW99], et les résultats et les techniques de la logique modale normale ont été étendus au cas des logiques non-normales. Par exemple, dans [Doš89], des résultats de dualité ont été établis entre des catégories de structures de voisinages et des catégories d'algèbres modales (i.e. algèbres de Boole avec un opérateur unaire supplémentaire) ; ces résultats étendent ceux de Goldblatt [Gol76] et Thomason [Tho75] sur les catégories de structures relationnelles pour les logiques modales normales. A partir des travaux de Pauly [Pau99], la perspective coalgébrique pour la logique modale normale a été étendue aux logiques modales non-normales. dans [HK04] et [HKP07]. En effet, les structures de voisinages peuvent être redéfinies de façon équivalente comme des coalgèbres pour le foncteur $F := 2^{2^{(-)}}$ résultant de la composition du foncteur contravariant "ensemble des parties" avec lui-même. Dans ce cadre de travail, les notions d'équivalences de F -bisimulation, équivalence comportementale et bisimulation de voisinages ont été étudiées, et les analogues du théorème de caractérisation de van Benthem et du théorème de Hennessy-Milner ont été prouvés.

D'autres directions de recherche récentes explorent le lien entre la sémantique des voisinages et la topologie (formelle). Par exemple, dans [dAFH09], la sémantique des voisinages de Lewis pour la logique contre-factuelle [Lew73] a été étendue au cadre de la théorie des topos.

Une logique entrelacée avec de nombreuses logiques. En résumé de la discussion menée jusque là, la logique modale monotone se situe à l'intersection de nombreuses problématiques appartenant à la philosophie, les mathématiques, l'informatique, l'intelligence artificielle et les sciences sociales, comme le montre le fait que les connecteurs modaux non-normaux apparaissent au sein de systèmes formels très différents tels que la logique des jeux [Par85, PP03], des logiques épistémiques [vDvdHK03, Hei96], des logiques

probabilistes et dynamiques [Gol92b, Her03], des logiques déontiques [CJ96]. D'où, nous pensons qu'il est important et utile d'étudier la logique modale monotone non pas de façon isolée, mais dans le contexte de ces différents environnements, et de développer des méthodes qui peuvent être transférées d'un système logique à un autre. Ceci est l'approche choisie dans cette thèse. Plus spécifiquement, la logique modale monotone est étudiée au croisement de deux méthodologies très générales appartenant à deux domaines différents de la logique: la *théorie de la correspondance* algébrique (développée dans le cadre de la dualité de Stone pour les logiques non-classiques), et la théorie des *display* calculs, développée dans le cadre de la théorie des preuves structurelle, et motivée par des considérations algébriques et philosophiques. D'une part, la théorie de la correspondance algébrique et algorithmique a été adaptée à la logique modale monotone, et cette adaptation a été utilisée comme une base pour la formalisation d'un résultat de correspondance pour les treillis finis. D'autre part, la théorie des *display* calculs a été étendue au cadre des types multiples, ce qui rend possible de traiter sans problèmes et uniformément un large champ de difficultés qui apparaissent dans la théorie des preuves des logiques dynamiques telles que la logique épistémique de Baltag-Moss-Solecki pour les actions épistémiques et la connaissance. Dans le cadre de la théorie des preuves avec les types multiples, une généralisation a été développée pour prendre en compte les particularités de la logique modale monotone, à savoir le fait que les modalités monotones, par définition, n'ont pas d'adjoints. A son tour, cette généralisation pour la logique modale monotone peut être utilisée pour franchir certains obstacles dans les calculs pour d'autres logiques dynamiques telles que la logique des jeux. Nous espérons que cette brève synthèse montre l'utilité d'adopter une approche synthétique dans l'étude des familles de logiques, et en particulier, le rôle clé joué par la logique modale monotone dans ces différents contextes.

Dans le reste de cette introduction, nous présentons les grandes lignes concernant la correspondance algébrique et algorithmique, et les *display* calculs.

Théorie de la correspondance. La théorie de Sahlqvist a une longue histoire dans la logique modale monotone, datant de [Sah75] et [vB85]. Le théorème de Sahlqvist dans [Sah75] donne une définition syntaxique d'une classe de formules modales, nommée la classe de Sahlqvist : chaque élément de cette classe définit une classe élémentaire (i.e. définissable dans la logique du premier ordre) de structures, et cette classe est canonique. Ce sont deux propriétés très importantes : la canonicité d'une axiomatisation garantit la complétude forte de Kripke de ses logiques associées, et l'élémentarité garantit tous les avantages, calculatoires et théoriques, de la logique du premier ordre par rapport à la logique du second ordre. A la fois la canonicité et l'élémentarité se trouvent être algorithmiquement non décidables qu'elles soient prises séparément ou ensemble

[CC06]. Ainsi, avoir une approximation décidable comme la classe de Sahlqvist est très recherché.

Au cours des années, plusieurs extensions et variantes de ce résultat sont apparues ; par exemple, des preuves algébriques/topologiques dans [SV89] et [Jón94], de la canonicité constructive dans [GM97], des variations du langage de correspondance dans [vB06], des résultats de type Sahlqvist pour les logiques hybrides dans [tCMV05] et pour le μ -calcul dans [CFPS14] et [vBBH12], des extensions de la classe de Sahlqvist, par exemple, aux formules *inductives* dans [GV06].

Depuis le commencement, la théorie de Sahlqvist a eu un aspect algorithmique marqué, car l'algorithme de Sahlqvist-van Benthem [vB85] fut utilisé pour calculer les structures du premier ordre correspondant aux formules de Sahlqvist. Récemment, de nouvelles méthodes algorithmiques pour la correspondance et la canonicité ont grandement élargi la classe de Sahlqvist d'origine. En particulier, l'algorithme SOEMA (voir [CGV06]) calcule à coup sûr le correspondant du premier ordre de chaque élément de la classe des formules inductives, qui est une classe de formules modales largement plus grande que la classe de Sahlqvist. De plus, il a été prouvé que les formules inductives sont aussi canoniques.

Dans [CP12], des résultats de canonicité et de correspondance analogues à ceux dans [CGV06] ont été obtenus pour le langage de la logique modale distributive, qui est une logique modale dont la base propositionnelle est la logique des treillis distributifs. Les dualités et adjonctions existantes entre les sémantiques relationnelles et algébriques de la logique modale normale et de la logique modale distributive ont rendu possible la retranscription des étapes de réduction de l'algorithme SOEMA (établi dans l'environnement de la théorie des modèles) en un algorithme similaire (établi dans un environnement algébrique). Ce résultat a lui-même été généralisé aux treillis, qui ne sont pas nécessairement distributifs [CP14].

La théorie unifiée de la correspondance a émergé de l'ensemble de ses résultats. La théorie unifiée de la correspondance est un cadre de travail dans lequel les résultats de correspondance peuvent être énoncés et prouvés de manière abstraite, sans se référer à une signature logique donnée. Les résultats sont alors exprimés seulement en fonction des propriétés des interprétations algébriques des connecteurs logiques. La correspondance unifiée permet d'utiliser des dualités existantes pour adapter des résultats de correspondance à différentes sémantiques d'une logique donnée. Pour plus de détails à ce sujet, se reporter à [CPS] et [CGP14]. Dans le Chapitre 1 de cette thèse, nous présentons une application de la théorie de la correspondance unifiée : nous utilisons un résultat de dualité entre les treillis finis et des structures similaires aux structures de voisinages et nous adaptons l'algorithme ALBA pour la logique modale monotone et plus

spécifiquement au cas des treillis finis. Cette adaptation de l'algorithme ALBA permet d'obtenir une caractérisation de certaines classes de treillis finis.

Display Calculs. Nuel Belnap introduisit le premier display calcul qu'il nomma *Display Logic* [Bel82], comme un environnement pour la théorie des preuves, conçu pour traiter d'une façon modulaire et uniforme de nombreuses logiques. Le principal avantage technique des *display* calculs est qu'ils fournissent un environnement pour énoncer et prouver de façon générique (i.e. canonique) des méta-théorèmes d'élimination de coupure. Ces méta-théorèmes garantissent qu'un calcul donné satisfait l'élimination de la coupure dans la mesure où il vérifie certaines conditions (c.f. Section 2.3) dont la vérification est aisée. La preuve de ces méta-théorèmes consiste à définir des étapes *uniformes* de transformation dans la preuve de l'élimination de la coupure de type Gentzen. Cette uniformité est la raison pour laquelle nous pouvons dire que l'élimination de la coupure de type Belnap (i.e. l'élimination de la coupure obtenue comme la conséquence d'un de ces méta-théorèmes) est à l'élimination de la coupure de type Gentzen ce que la canonicité est à la complétude.

Les *display* calculs ont permis d'avoir une meilleure compréhension, du point de vue de la théorie des preuves, de logiques qui sont notoirement difficiles à traiter avec d'autres approches. Parmi ces logiques on trouve, en plus des logiques classique et (bi-)intuitionniste [Bel82], la logique linéaire [Bel90], la logique modale et temporelle (bi-)intuitionniste [GPT10, Gor96, Kra96, Wan94], des logiques substructurelles [Gor98], des logiques pertinentes [Res98], ou la logique des algèbres relationnelles [Gor97]. Dans le cas de la logique modale, il a été extrêmement difficile d'étendre les calculs des séquents traditionnels à la Gentzen afin d'obtenir un théorie des preuves uniforme et générale englobant les nombreuses extensions de la logique modale normale minimale K . Une caractéristique importante des *display* calculs est qu'il y a une division du travail nette entre les *règles d'introduction* pour les connecteurs logiques, et les *règles structurelles* qui encodent des propriétés spécifiques à des connecteurs logiques donnés qu'ils soient considérés seuls ou dans leurs interactions. Les extensions axiomatiques d'une logique donnée sont typiquement traitées en ajoutant au calcul de base des règles structurelles correspondant aux axiomes ajoutés. Dans [Kra96], Kracht traite de façon uniforme et modulaire une large classe d'extensions de la logique modale dans le contexte des *display* calculs en caractérisant une sous-classe de formules modales de Sahlqvist (nommées *formules primitives*), dont chaque formule peut être encodée par une règle structurelle dans le *display* calcul.

En ce qui concerne les propriétés spécifiques aux *display* calculs, Belnap s'est inspiré des observations de Gentzen sur les règles structurelles. En effet, dans la formulation

standard de Gentzen, la virgule ‘,’ séparant les formules en position “précédente” et en position “succédente” peut être reconnue comme un connecteur méta-linguistique dont les règles structurelles définissent le comportement. Belnap généralisa cette idée en admettant non seulement la virgule mais aussi d’autres connecteurs pour garder les formules ensemble dans une structure et les appela *connecteurs structurels*. Tout comme la virgule, qui dans le calcul des séquents à la Gentzen est interprétée en fonction du contexte (i.e. comme une conjonction quand elle apparaît à gauche du séquent et comme une disjonction quand elle apparaît à droite du séquent), chaque connecteur structurel correspond généralement à une paire de connecteurs logiques et est interprété comme l’un ou l’autre en fonction du contexte (c.f. Sections 2.2.2 and 2.6.1). Les connecteurs structurels entretiennent des relations les uns avec les autres, dont la plus fondamentale prend la forme d’adjonctions ou de résiduations. Ces relations permettent au calcul de satisfaire la très importante *propriété de display* qui lui donne son nom.

Definition 0.2. Un système de preuve satisfait la *propriété de display* si et seulement si pour chaque séquent $X \vdash Y$ et chaque sous-structure Z de X ou Y , le séquent $X \vdash Y$ peut être transformé de façon équivalente, en utilisant les règles du système, en un séquent qui est soit de la forme $Z \vdash W$ soit de la forme $W \vdash Z$, et telle que W est une structure. Dans le premier cas, Z est isolé en *position précédente*, et dans le second, Z est isolé en *position succédente*. Les règles inversibles permettant cette réécriture équivalente sont appelées postulats de *display*.

Pour illustrer le rôle fondamental joué par la propriété de display dans les étapes de transformation du méta-théorème d’élimination de la coupure, considérons l’étape d’élimination de l’application de la coupure ci-dessous dans laquelle la formule de coupure est principale dans les deux prémisses de la coupure.

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \\
 X_1 \vdash A \\
 \hline
 X_1, X_2 \vdash A \wedge B
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi_2 \\
 X_2 \vdash B \\
 \hline
 X_1, X_2 \vdash B
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi \\
 A, B \vdash Y \\
 \hline
 A \wedge B \vdash Y
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{c}
 \vdots \pi \\
 \begin{array}{c}
 \vdots \pi_1 \\
 X_1 \vdash A \\
 \hline
 X_1 \vdash Y < B
 \end{array}
 \quad
 \begin{array}{c}
 A, B \vdash Y \\
 \hline
 A \vdash Y < B
 \end{array} \\
 \hline
 X_1 \vdash Y < B \\
 \hline
 \begin{array}{c}
 \vdots \pi_2 \\
 X_2 \vdash B \\
 \hline
 X_1, X_2 \vdash B
 \end{array}
 \quad
 \begin{array}{c}
 X_1 \vdash Y < B \\
 \hline
 X_1, B \vdash Y \\
 \hline
 B \vdash X_1 > Y
 \end{array} \\
 \hline
 X_2 \vdash B \quad B \vdash X_1 > Y \\
 \hline
 X_2 \vdash X_1 > Y \\
 \hline
 X_1, X_2 \vdash Y
 \end{array}
 \end{array}$$

Les lignes en pointillés dans l’arbre de preuve sur la droite correspondent à des applications de postulats de *display*. Cette étape de transformation est possible car les postulats de *display* désassemblent les structures composées afin de nous donner accès aux sous-formules qui constituent immédiatement la formule de coupure et les ré-assemblent afin de ‘remettre les choses ensemble’. Ainsi, il est possible de casser la coupure d’origine en deux applications de la coupure sur les sous-formules immédiates, comme réclamé par la stratégie originelle de Gentzen.

Il est alors naturel de se poser la question suivante : si les opérateurs modaux en logique modale monotone sont supposés ne pas préserver les suprema et les infima, alors il n'est pas raisonnable de supposer qu'ils ont des adjoints. Cependant, il apparaît clairement à partir de l'exemple ci-dessus que les postulats de *display* encodent essentiellement les adjonctions/résiduations. Ainsi, comment est-il possible de prendre en compte les logiques modales monotones dans la méthodologie des *display* calculs ?

En effet, le prix à payer pour l'élégante propriété de *display* est que le langage doit inclure des connecteurs additionnels qui sont essentiels pour la comptabilité requise par son application. La présence de ces connecteurs additionnels soulève immédiatement des problèmes de conservativité. Dans de nombreux cas, ces problèmes peuvent être résolus en utilisant des arguments sémantiques (nous verrons un exemple de cette heureuse situation dans le Chapitre 2). Dans d'autres cas, les arguments sémantiques ne sont pas utilisables, mais la conservativité peut néanmoins être prouvée grâce à d'excellentes propriétés syntaxiques (cela est le cas pour le calcul du Chapitre 3). Dans le cas de la logique modale monotone, nous avons abandonné la propriété de *display*, et opté pour un calcul "type *display*" mais qui ne satisfait pas la propriété de *display*. En effet, dans le Chapitre 5, nous introduisons un calcul qui contient certains postulats de *display*, mais pas suffisamment pour garantir les étapes de réduction comme celles illustrées précédemment. De plus, nous montrerons dans le Chapitre 4 qu'il est encore possible de prouver un méta-théorème d'élimination de la coupure pour les calculs de type *display*. Finalement, dans le cas du calcul du Chapitre 6, la conservativité est encore un problème ouvert, et nous présenterons une stratégie pour la prouver.

Synopsis

Dans le Chapitre 1, nous présentons les résultats de [FPS14]. Nous adaptons l'algorithme ALBA à la logique modale monotone, and obtenons des résultats de caractérisations duales pour les treillis finis en utilisant l'algorithme ALBA modifié.

Dans le Chapitre 2, nous présentons des résultats de [FGK⁺14c]. Nous analysons les systèmes de preuves existants pour la logique épistémique dynamique du point de vue de la sémantique de la théorie des preuves. Nous discutons les caractéristiques principales des *display* calculs, et nous concentrons notre attention sur la notion de *display* calcul propre introduite par Wansing. Nous généralisons cette notion à celle de *display* calcul quasi-propre, et nous prouvons le méta-théorème d'élimination de la coupure de type Belnap correspondant. Nous introduisons le *display* calcul D'.EAK pour la logique de Baltag-Moss-Solecki, *Epistemic Actions and Knowledge*. Ce calcul est une amélioration du calcul D.EAK présenté dans [GKP13], pour lequel l'élimination de la coupure

n'avait pas pu être prouvée en utilisant la méthode introduite par Belnap. La caractéristique commune aux calculs D.EAK et D'.EAK est la présence des adjoints des opérateurs dynamiques. Bien que ces adjoints ne soient pas directement interprétables dans la sémantique standard de Kripke pour les modèles actualisés, ils ont une interprétation naturelle dans la sémantique, alternative mais équivalente, de la coalgèbre finale pour EAK. Nous prouvons que D'.EAK est correct par rapport à la sémantique de la coalgèbre finale, et complet et conservatif par rapport à EAK, et que c'est un *display* calcul quasi-propre, qui il satisfait de fait l'élimination de la coupure via le méta-théorème de type Belnap correspondant.

Dans le Chapitre 3, nous présentons les résultats de [FGK⁺14a]. Nous généralisons les *display* calculs aux langages typés. Nous introduisons l'équivalent des *display* calculs quasi-propres pour les langages typés que nous nommons *display* calculs quasi-propres multi-types, et nous prouvons le méta-théorème d'élimination de la coupure de type Belnap correspondant. Nous introduisons un *display* calcul typé pour EAK, dénommé Calcul Dynamique pour EAK. Dans ce calcul, les paramètres indexant les opérateurs modaux dynamiques et épistémiques dans le langage originel de EAK sont vus comme des termes et les opérateurs modaux unaires sont codés par des opérations binaires prenant des arguments de types différents. Nous prouvons que le Calcul Dynamique pour EAK est correct par rapport à la sémantique pour la coalgèbre finale, et complet et conservatif par rapport à EAK et que EAK est un *display* calcul quasi-propre multi-type qui satisfait donc l'élimination de la coupure via le méta-théorème de type Belnap correspondant.

Dans le Chapitre 4, nous présentons les résultats de [FGK⁺14b]. Nous discutons une généralisation des *display* calculs multi-types dans laquelle nous abandonnons la propriété de *display*. Dans ce contexte, nous importons des idées de [BFS00], et compensons le manque de propriétés *display* en renforçant la propriété de visibilité. Nous introduisons la notion de calculs quasi-propres de type *display* dans le cadre des types multiples, et nous prouvons le méta-théorème d'élimination de la coupure de type Belnap qui lui est associé.

Dans le Chapitre 5, nous introduisons une version plus compacte de l'environnement introduit dans le chapitre précédent. Cette version ne comporte qu'un seul type et est 'propre' au lieu d'être 'quasi-propre' (i.e. la forme des axiomes est restreinte comme dans l'article original de Belnap, au lieu d'être plus générale comme, par exemple, dans les systèmes de preuves pour EAK). Nous introduisons un calcul de type *display* pour la logique modale monotone, nous prouvons qu'il est correct et complet par rapport à la logique monotone de base. Nous montrons de plus que c'est un calcul propre de type

display et donc qu'il a la propriété d'élimination de la coupure via le méta-théorème de type Belnap correspondant.

Dans le Chapitre 6, nous présentons les résultats de [FGKP14]. Nous introduisons un *display* calcul multi-types pour la Logique Propositionnelle Dynamique (PDL). Ce calcul est complet par rapport à PDL, et est un *display* calcul propre multi-type, il a donc la propriété d'élimination de la coupure via le méta-théorème de type Belnap correspondant. Nous discutons la correction de ses règles par rapport à la sémantique standard, et la question de sa conservativité qui est encore ouverte.

L'Appendice A fournit la preuve du résultat principal du Chapitre 1, qui est le Lemme 1.34.

L'Appendice B liste les méta-théorèmes d'élimination de la coupure de type Belnap discutés dans la Partie II.

Les Appendices C, D, et E collectent de la documentation concernant le Chapitre 2. L'Appendice C fournit certaines règles dérivées du calcul D'.EAK. L'Appendice D liste la plupart des étapes de transformation de la vérification du fait que D'.EAK est un *display* calcul quasi-propre (cf. Section 2.6.3). Dans l'Appendice E, nous prouvons la complétude de D'.EAK par rapport à la présentation de Hilbert de EAK (cf. Sections 2.4.1 et 2.4.2) en dérivant les axiomes de (la version intuitionniste de) EAK dans D'.EAK.

Les Appendices F et G se réfèrent au Chapitre 3. Dans l'Appendice F, nous complétons la preuve de l'élimination de la coupure (cf. Section 3.5.2) pour le Calcul Dynamique pour EAK donné dans la Section 3.2. Dans l'Appendice G, nous prouvons la complétude du Calcul Dynamique pour EAK par rapport à la présentation de Hilbert de D'.EAK (cf. Section 2.4) en dérivant les axiomes de (la version intuitionniste de) EAK dans le Calcul Dynamique.

L'Appendice H fournit les règles du calcul des séquents de type *display* pour la logique modale monotone introduite dans le Chapitre 5.

Les Appendices I, J et K se réfèrent au Chapitre 6. L'Appendice I fournit les règles de la base propositionnelle du *display* calcul pour PDL. Dans l'Appendice J, nous complétons la preuve de l'élimination de la coupure (cf. Section 6.6) pour le *display* calcul pour PDL donné dans la Section 6.3. Dans l'Appendice K, nous prouvons que le *display* calcul pour PDL donné dans la Section 6.3 est complet par rapport à la présentation de Hilbert de PDL.

Part I

Correspondence Theory

Chapter 1

Dual Characterizations for Finite Lattices via Correspondence Theory for Monotone Modal Logic

1.1 Introduction

Dual characterization results for finite lattices. The present chapter builds on a duality for finite lattices, established by Santocanale [San09]. The structures dually equivalent to finite lattices are referred to as *join-presentations*, and are certain triples (X, \leq, \mathcal{M}) such that (X, \leq) is a finite poset, and $\mathcal{M} : X \rightarrow \mathcal{P}\mathcal{P}X$. In [San09], it has been pointed out—and indicated as a worthwhile research direction—that the existence of this duality makes it possible to investigate systematic *dual characterization* results, between equations or inequalities in the algebraic language of lattices on one side, and first-order conditions in the language of join-presentations on the other. One significant instance of such systematic dual characterizations has been developed in the same paper, between a class of inequalities in the language of lattices and a corresponding class of first-order conditions. Both classes are parametric in the class of finite trees (cf. [San09, Proposition 8.5]). This result generalizes Nation’s [Nat90, Section 5] stating that a certain class of finite lattices¹ is a pseudovariety, and is similar to Semenova’s results [Sem05].

From modal logic to unified correspondence theory. Modal logic is an area in which systematic dual characterization results have been extensively developed, giving rise to a very rich theory—the so called *modal correspondence theory*—which has been investigated for almost forty years. Modal correspondence theory was originally developed in a

¹Namely, the finite lattices such that the length of their *D-chains* has a uniform upper bound.

purely model-theoretic way [vB85]. However, correspondence-related phenomena have been studied in an algebraic framework subsuming duality theory since the early 90s [JT51], and very recently, a unified correspondence framework has emerged [CGP14], which is based on duality, and uniformly extends correspondence theory to many non-classical logics. One of the main tools developed by this theory is an algorithm (actually various cognate versions of it, cf. [CP12]) or calculus for correspondence, called ALBA, which mechanizes dual characterization meta-arguments. In particular, as discussed in [CP12] and [CGP14], the core of ALBA is the encoding of a general meta-argument for correspondence known in the literature as the *minimal valuation argument* into a rule which relies on the *Ackermann lemma* [Ack35]. The algorithm ALBA takes in input formulas or inequalities in a given propositional language and, whenever it succeeds², it computes the *first-order correspondent* of the given formula or inequality, i.e. a first-order sentence which holds in a given structure exactly when the given propositional formula or inequality is valid in the dual algebra of that structure. The general theory also provides the syntactic characterization of a class of formulas/inequalities for each logic, the so-called inductive formulas/inequalities, on which the algorithm is guaranteed to uniformly succeed. For each language, inductive inequalities form the largest such class syntactically defined so far in the literature.

Aim of the chapter. Given the availability of this theory, it seems natural to try and understand dual characterization results such as [San09, Proposition 8.5] as instances of a more general unified correspondence mechanism. This is what the present chapter aims at doing, by establishing a novel dual characterization result similar to Nation's. Our result paves the way to the mechanization and systematization of dual characterizations such as the one in [San09].

Methodology: basic algorithmic correspondence for monotone modal logic. Our approach is based on an adaptation of the algorithm/calculus ALBA of [CP12] to the case of *monotone modal logic*. This adaptation is necessary, since some of the rules in the standard version of the algorithm would not be sound for the modal connectives of monotone modal logic, and is one of the contributions of the present chapter. The adapted ALBA is semantically justified in the general environment of two-sorted frames (cf. Section 1.3), which are general structures that can encode monotone neighbourhood frames as special cases. As their name suggests, two-sorted frames are relational structures based on two domains. Normal modal operators can be associated in the standard way with

²It is well known [CC06] that the problem of whether a formula admits a first-order correspondent is undecidable.

the binary relations on two-sorted frames. Monotone modal operators can be then interpreted on two-sorted frames as the composition of some of these normal modalities. This provides the basic semantic environment for the adapted ALBA.

Correspondence theory for monotone modal logic has already been studied in [Han03], where a class of monotone modal formulas which are guaranteed to have a first order correspondent has been identified. However, the class of inductive inequalities corresponding to the ALBA setting is strictly larger than the one in [Han03].

Enhancing ALBA. However, the translations of inequalities such as Nation's [Nat90] and as the ones treated in the present chapter fall outside the inductive class. Hence, another contribution of the present chapter is the addition of special rules which are sound on the specific semantic setting arising from finite lattices. Interestingly, an Ackermann-type rule features among these additional rules, the soundness of which cannot be straightforwardly explained in terms of the Ackermann lemma, but which however still intuitively encodes a minimal valuation argument.

Organization of the chapter. In Section 1.2, we collect preliminaries on the duality between finite lattices and join-presentations, the language and neighbourhood semantics of monotone modal logic, the duality-induced 'standard translation' of lattice terms into terms in the language of monotone modal logic, and the algorithm for correspondence ALBA. In Section 1.3, we adapt the algorithm ALBA specifically to monotone modal logic via the introduction of two-sorted frames. In Section 1.4, we enhance the adapted ALBA by introducing additional rules, and prove their soundness w.r.t. the semantic environment of so-called enriched two-sorted frames which can be naturally associated with finite lattices. In Section 1.5, upper bounds on the length of D^+ -chains (cf. Definition 1.32) are obtained as a reduction of the enhanced ALBA. Section 1.6 collects the conclusions and further directions. The proof of a technical lemma appears in Appendix A.

1.2 Preliminaries

The aim of the present section is to collect preliminaries belonging to diverse fields of logic, and to connect them so as to set the stage for the main result. In the next subsection, we report on a duality on objects for finite lattices, which has been introduced in [San09]. Given that the structures dual to finite lattices can be naturally associated with monotone neighbourhood frames, and given that monotone neighbourhood frames

are standard models for monotone modal logic, the duality presented in Section 1.2.1 serves as a basis for the definition of a standard translation between lattice terms and monotone modal logic formulas. In Section 1.2.2, we recall the basic definitions about monotone modal logic and neighbourhood frames, and we show how we can represent a finite lattice as a monotone neighbourhood frame. In Section 1.2.3, we define a standard translation between lattice terms and formulas of monotone modal logic, and show that this translation adequately transfers and reflects the validity of lattice inequalities on any finite lattice L , and the validity of their standard translations on the monotone neighbourhood frames associated with L . Finally, in Section 1.2.4, we give an informal presentation of the algorithm for correspondence ALBA, introduced in [CP12], for correspondence for normal modal logic.

1.2.1 Dual equivalence for finite lattices

In the present subsection, we report on the object-part of a dual equivalence between finite lattices and certain poset-based structures (cf. Definition 1.4). Our presentation is based on [San09]. These structures will turn out to be special neighbourhood frames, and hence the existence of this duality provides the bridge between the propositional logic of lattices and monotone modal logic.

In what follows, L will denote a finite lattice. Elements of L will be denoted a, b, \dots . Throughout this chapter, the letters i, j, k will be reserved for join-irreducible elements of L (the set of which is $J(L)$), and m, n for meet-irreducible elements of L (the set of which is $M(L)$), respectively. Recall that an element $j \neq \perp$ of L is *join-irreducible* iff $j = a \vee b$ implies that either $j = a$ or $j = b$ for all $a, b \in L$. Order-dually, an element $m \neq \top$ of L is *meet-irreducible* iff $m = a \wedge b$ implies that either $m = a$ or $m = b$ for all $a, b \in L$. A subset $C \subseteq L$ is a *join-cover* of $a \in L$ if $a \leq \bigvee C$.

For any poset (S, \leq) , its associated *refinement relation*, denoted \ll , is defined on the set $\mathcal{P}_f(S)$ of finite subsets of S by the following stipulation:

$$A \ll B \quad \text{iff} \quad \text{for every } a \in A \text{ there exists some } b \in B \text{ such that } a \leq b. \quad (1.1)$$

Equivalently,

$$A \ll B \quad \text{iff} \quad \downarrow A \subseteq \downarrow B,$$

where $\downarrow C := \{x \in S \mid x \leq c \text{ for some } c \text{ in } C\}$ for every $C \subseteq S$. Throughout the chapter, we say that a join-cover C of a is *minimal* if it is an \leq -antichain, and if, for any \leq -antichain $D \subseteq L$, $(a \leq \bigvee D \text{ and } D \ll C)$ imply $D = C$. A join-cover of a is *trivial* if it contains a . We

can easily show that any join-irreducible element $j \in J(L)$ has only one trivial minimal join-cover, that is, the singleton $\{j\}$.

Direct presentations and their closure operators. In the present paragraph, we define direct presentations, and introduce a closure operator over these presentations which is a key ingredient of the duality on objects between finite lattices and reflexive and transitive presentations (see paragraph below).

Definition 1.1. A *presentation* is a triple (X, \leq, \mathcal{M}) such that (X, \leq) is a poset, and $\mathcal{M} : X \rightarrow \mathcal{P}\mathcal{P}X$. A presentation is

- *monotone* if for all $x, y \in X$, any $C \subseteq X$, if $y \leq x$ and $C \in \mathcal{M}(x)$, then $D \ll C$ for some $D \in \mathcal{M}(y)$;
- *reflexive* if for each $x \in X$, there exists some $C \in \mathcal{M}(x)$ such that $C \ll \{x\}$;
- *transitive* if for every $x \in X$ and every $C \subseteq X$, if $C \in \mathcal{M}(x)$ then for every collection $\{D_c \mid c \in C\}$ such that $D_c \in \mathcal{M}(c)$ for every $c \in C$, there exists some $E \in \mathcal{M}(x)$ such that $E \ll \bigcup_{c \in C} D_c$;
- *direct* if it is monotone, reflexive, and transitive.

Recall that a *downset* of (X, \leq) is a subset $S \subseteq X$ such that for all $x, y \in X$, if $y \leq x$ and $x \in S$ then $y \in S$. Let $\mathcal{D}(X, \leq)$ denote the set of downsets of (X, \leq) .

For any presentation $\mathbb{A} = (\mathbb{X}, \mathcal{M})$ where $\mathbb{X} := (X, \leq)$ is a poset, the assignment $\overline{cl}_{\mathbb{A}} : \mathcal{D}\mathbb{X} \rightarrow \mathcal{D}\mathbb{X}$ is defined as follows: for any $S \in \mathcal{D}\mathbb{X}$,

$$\overline{cl}_{\mathbb{A}}(S) := \{x \in X \mid D \subseteq S \text{ for some } D \in \mathcal{M}(x)\}. \quad (1.2)$$

Lemma 1.2. *For any direct presentation $\mathbb{A} = (X, \leq, \mathcal{M})$, the map $\overline{cl}_{\mathbb{A}}$ is a closure operator.*

Proof. We first prove that the map $\overline{cl}_{\mathbb{A}}$ is well-defined. Fix $S \in \mathcal{D}\mathbb{X}$, $x \in \overline{cl}_{\mathbb{A}}(S)$ and $y \in X$. Assume that $y \leq x$. Since $x \in \overline{cl}_{\mathbb{A}}(S)$, there is some $C_x \in \mathcal{M}(x)$ such that $C_x \subseteq S$. In addition, since \mathbb{A} is monotone, $y \leq x$ implies that there exists some $C_y \in \mathcal{M}(y)$ such that $C_y \ll C_x$. By definition of \ll , we have that $C_y \subseteq S$ because S is a downset. Thus $y \in \overline{cl}_{\mathbb{A}}(S)$. This finishes the proof that $\overline{cl}_{\mathbb{A}}(S)$ is a downset. Hence the map $\overline{cl}_{\mathbb{A}}$ is well-defined.

To prove that $\overline{cl}_{\mathbb{A}}$ is a closure operator, we need to show that $\overline{cl}_{\mathbb{A}}$ is order-preserving, and that $S \subseteq \overline{cl}_{\mathbb{A}}(S)$ and $\overline{cl}_{\mathbb{A}}(\overline{cl}_{\mathbb{A}}(S)) \subseteq \overline{cl}_{\mathbb{A}}(S)$ for any $S \in \mathcal{D}\mathbb{X}$. It is immediate to see that $\overline{cl}_{\mathbb{A}}$ is order-preserving.

Since \mathbb{A} is reflexive, there is some $C \in \mathcal{M}(x)$ such that $C \subseteq \downarrow x$. Moreover, $x \in S$ implies that $\downarrow x \subseteq S$. Hence, by definition of $\overline{cl}_{\mathbb{A}}$, we have that $x \in \overline{cl}_{\mathbb{A}}(S)$ for any $x \in S$, that is $S \subseteq \overline{cl}_{\mathbb{A}}(S)$.

It remains to be shown that $\overline{cl}_{\mathbb{A}}(\overline{cl}_{\mathbb{A}}(S)) \subseteq \overline{cl}_{\mathbb{A}}(S)$ for any $S \in \mathcal{D}\mathbb{X}$. Let $x \in \overline{cl}_{\mathbb{A}}(\overline{cl}_{\mathbb{A}}(S))$. By definition of $\overline{cl}_{\mathbb{A}}$, there exists some $D \in \mathcal{M}(x)$ such that $D \subseteq \overline{cl}_{\mathbb{A}}(S)$. Then any $d \in D$ is an element of $\overline{cl}_{\mathbb{A}}(S)$. Thus, for each $d \in D$ there exists some $E_d \in \mathcal{M}(d)$ such that $E_d \subseteq S$. Since \mathbb{A} is transitive, there is some $C \in \mathcal{M}(x)$, such that $C \ll \bigcup_{d \in D} E_d$. Thus $C \subseteq S$, and, by definition of $\overline{cl}_{\mathbb{A}}(S)$, this proves that $x \in \overline{cl}_{\mathbb{A}}(S)$. This completes the proof that $\overline{cl}_{\mathbb{A}}$ is a closure operator. \square

Definition 1.3. For any direct presentation $\mathbb{A} = (X, \leq, \mathcal{M})$, a downset $S \subseteq X$ is *closed* if $S = \overline{cl}_{\mathbb{A}}(S)$. The closure of a downset $S \subseteq X$ is the set $\overline{cl}_{\mathbb{A}}(S)$. In the following, whenever it causes no confusion, we denote the closure of a downset S by \overline{S} .

Notice that for any direct presentation $\mathbb{A} = (X, \leq, \mathcal{M})$, we can extend the closure operator $\overline{cl}_{\mathbb{A}}$ to sets, as follows:

$$\begin{aligned} cl_{\mathbb{A}} : \mathcal{P}X &\longrightarrow \mathcal{P}X \\ S &\longmapsto \overline{cl}_{\mathbb{A}}(\downarrow_{\leq} S). \end{aligned} \tag{1.3}$$

Since, $\overline{cl}_{\mathbb{A}}$ and \downarrow_{\leq} are closure operators on downsets and on sets respectively, we can easily prove the $cl_{\mathbb{A}}$ is a closure operator too.

Join-presentation of a finite lattice.

Definition 1.4. The *join-presentation*³ of a lattice L is the presentation $(J(L), \leq, \mathcal{M})$ such that $(J(L), \leq)$ is the poset of the join-irreducible elements of L with the order induced by L , and \mathcal{M} is the map $J(L) \longrightarrow \mathcal{P}\mathcal{P}J(L)$ assigning any j to the collection of its minimal join-covers.

Lemma 1.5 (cf. Lemma 4.2 in [San09]). *For any finite lattice L , the join-presentation $(J(L), \leq, \mathcal{M})$ associated with L is a direct presentation.*

More generally, we can associate every element a of a lattice L with the set $\mathcal{M}(a)$ of its minimal join covers. The following lemma lists some properties of $\mathcal{M} : L \longrightarrow \mathcal{P}\mathcal{P}J(L)$.

Lemma 1.6 (cf. [San09], page 5). *Let (L, \leq) be a finite lattice. For all $a \in L$, $j \in J(L)$, $C \in \mathcal{M}(a)$ and $Y \subseteq L$,*

1. $C \subseteq J(L)$, and C is an \leq -antichain;
2. $\mathcal{M}(a)$ is a \ll -antichain;

³Join-presentations are also referred to as *OD-graphs* in the literature (cf. [Nat90, San09]).

3. if $a \leq \bigvee Y$, then there exists some $D \in \mathcal{M}(a)$ such that $D \ll Y$;
4. $\{j\} \in \mathcal{M}(j)$.

For every finite lattice L , let \mathfrak{Q}_L be the lattice of the closed downsets of the join presentation $(J(L), \leq, \mathcal{M})$ associated with L .

Proposition 1.7 (cf. [Nat90]). *Every finite lattice L is isomorphic to the lattice \mathfrak{Q}_L as above.*

The following lemmas will be useful in the remainder of the chapter.

Lemma 1.8 (Lemma 4.2 in [San09]). *For any finite lattice L and any $j \in J(L)$, the downset $\downarrow_{J(L)} j$ is a closed subset of the join-presentation $(J(L), \leq, \mathcal{M})$ associated with L .*

Lemma 1.9. *Let L be a finite lattice, and $(J(L), \leq, \mathcal{M})$ be its join-presentation. For any $j, k \in J(L)$ and any \leq -antichain $C \subseteq J(L)$, if $C \in \mathcal{M}(j)$ and $k \in C$,*

1. $j \notin \overline{\downarrow_{\leq}(C \setminus k)}$,
2. $k \notin \overline{\downarrow_{\leq}(C \setminus k)}$,
3. $j \notin \{k' \in J(L) \mid k' < k\}$,
4. *there is no $D \in \mathcal{M}(j)$ such that $D \subseteq \overline{\downarrow_{\leq j}(C \setminus k) \cup \{k' \in J(L) \mid k' < k\}}$.*

Proof. Fix $C \in \mathcal{M}(j)$ and $k \in C$. As to item 1. Since C is a minimal cover of j , the sets $C \setminus k$ and $\downarrow_{\leq j}(C \setminus k)$ are not covers of j . Hence $j \notin \overline{\downarrow_{\leq}(C \setminus k)}$.

We show item 2 by contradiction. Assume that $k \in \overline{\downarrow_{\leq}(C \setminus k)}$. By the definition of closure, this implies that there exists some $D \in \mathcal{M}(k)$ such that $D \subseteq \downarrow_{\leq}(C \setminus k)$. The following chain of inequalities holds

$$\begin{aligned}
 j &\leq \bigvee C && (C \in \mathcal{M}(j)) \\
 &= \bigvee ((C \setminus k) \cup \{k\}) \\
 &= (\bigvee (C \setminus k)) \vee k \\
 &\leq \bigvee (C \setminus k) \vee \bigvee D && (D \in \mathcal{M}(k)) \\
 &= \bigvee ((C \setminus k) \cup D),
 \end{aligned}$$

which shows that the set $(C \setminus k) \cup D$ is a cover of j . Hence, there exists a minimal cover $C' \in \mathcal{M}(j)$ that refines it, i.e. such that $C' \ll (C \setminus k) \cup D$. By the definition of \ll , this means that $C' \subseteq \downarrow_{\leq}((C \setminus k) \cup D)$. Since $D \subseteq \downarrow_{\leq}(C \setminus k)$, we have that $\downarrow_{\leq}((C \setminus k) \cup D) = \downarrow_{\leq}(C \setminus k)$, which proves that $C' \subseteq \downarrow_{\leq}(C \setminus k)$. This proves that $j \in \overline{\downarrow_{\leq}(C \setminus k)}$, which contradicts item 1.

Item 3 immediately follows from the definition of a minimal cover.

As to item 4, suppose for contradiction that there exists some $D \in \mathcal{M}(j)$ such that $D \subseteq \overline{\downarrow_{\leq_j}(C \setminus k) \cup \{k' \in J(L) \mid k' < k\}}$. Then, for any $d \in D$, there exists some $k_d \in \overline{\downarrow_{\leq_j}(C \setminus k) \cup \{k' \in J(L) \mid k' < k\}}$ such that $d \leq k_d$. If $k_d \in \{k' \in J(L) \mid k' < k\}$, then $k_d < k$. If $k_d \notin \{k' \in J(L) \mid k' < k\}$, then $k_d \in \overline{\downarrow_{\leq_j}(C \setminus k)}$ and there is some $E_d \in \mathcal{M}(d)$ such that $E_d \ll C \setminus k$. Since the join-presentation of L is a transitive presentation, the set

$$E := \bigcup \{E_d \mid d \in D \text{ and } k_d \notin \{k' \in J(L) \mid k' < k\}\} \cup \bigcup \{k_d \mid d \in D \text{ and } k_d < k\}$$

is a cover of j . Hence, there exists some $E' \in \mathcal{M}(j)$ such that $E' \ll E$. Since $E \ll C$ and the relation \ll is transitive, this implies that $E' \ll C$. Hence, to finish the proof, it is enough to show that $E' \neq C$, which would contradict the minimality of C . Since

$$k \notin \overline{\downarrow_{\leq_j}(C \setminus k) \cup \{k' \in J(L) \mid k' < k\}} \quad \text{and} \quad \downarrow_{\leq_j} E' \subseteq \downarrow_{\leq_j} E \subseteq \overline{\downarrow_{\leq_j}(C \setminus k) \cup \{k' \in J(L) \mid k' < k\}},$$

we have that $k \notin E'$. Since, by assumption, $k \in C$, this proves that $E' \neq C$ as required. \square

1.2.2 An environment for correspondence

The structures described in the previous subsection are very close to neighbourhood frames (we will expand on this at the end of the present subsection). Neighbourhood frames are well known to provide a state-based semantics for monotone modal logic (see [Han03]). Hence, as discussed in [CGP14], the duality between lattices and join presentations induces a correspondence-type relation between the propositional language and logic of lattices, and a fragment of the language of monotone modal logic.

In the present section we collect the basic ingredients of this correspondence: the languages, their interpretations, and a syntactic translation which may be regarded as a kind of standard translation between the language of lattices and the monotone modal language.

Definition 1.10. The language of lattice terms \mathcal{L}_{Latt} over the set of variables AtProp is as usual given by the following syntax

$$\varphi ::= \perp \mid \top \mid p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi,$$

with $p \in \text{AtProp}$.

Definition 1.11. The language of monotone modal logic \mathcal{L}_{MML} over the set of variables AtProp is recursively defined as follows:

$$\varphi ::= \perp \mid \top \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid (\exists\forall)\varphi \mid (\forall\exists)\varphi.$$

Definition 1.12. A *neighbourhood frame* is a tuple $\mathbb{F} = (X, \sigma)$ such that X is a set and $\sigma : X \rightarrow \mathcal{P}\mathcal{P}X$ is a map. For any $x \in X$, any element $N \in \sigma(x)$ is called a *neighbourhood* of x . A neighbourhood frame \mathbb{F} is *monotone* if for any $x \in X$, the collection $\sigma(x)$ is an upward closed subset of $(\mathcal{P}X, \subseteq)$. A *neighbourhood model* is a tuple $\mathbb{M} = (\mathbb{F}, \nu)$ such that $\mathbb{F} = (X, \sigma)$ is a neighbourhood frame and $\nu : \text{AtProp} \rightarrow \mathcal{P}X$ is a valuation.

Definition 1.13. For any neighbourhood model $\mathbb{M} = (\mathbb{F}, \nu)$ and any $w \in X$, the *satisfaction* of any formula $\varphi \in \mathcal{L}_{MML}$ in \mathbb{M} at w is defined recursively as follows:

$$\begin{aligned} \mathbb{M}, w \Vdash \perp & \quad \text{never} \\ \mathbb{M}, w \Vdash \top & \quad \text{always} \\ \mathbb{M}, w \Vdash p & \quad \text{iff } w \in \nu(p) \\ \mathbb{M}, w \Vdash \neg\varphi & \quad \text{iff } \mathbb{M}, w \not\Vdash \varphi \\ \mathbb{M}, w \Vdash \varphi \vee \psi & \quad \text{iff } \mathbb{M}, w \Vdash \varphi \text{ or } \mathbb{M}, w \Vdash \psi \\ \mathbb{M}, w \Vdash \varphi \wedge \psi & \quad \text{iff } \mathbb{M}, w \Vdash \varphi \text{ and } \mathbb{M}, w \Vdash \psi \\ \mathbb{M}, w \Vdash (\exists\forall)\varphi & \quad \text{iff } \text{there exists some } C \in \sigma(w) \text{ such that, for each } c \in C, \text{ we have } \mathbb{M}, c \Vdash \varphi \\ \mathbb{M}, w \Vdash (\forall\exists)\varphi & \quad \text{iff } \text{for each } C \in \sigma(w) \text{ there exists some } c \in C \text{ such that we have } \mathbb{M}, c \Vdash \varphi. \end{aligned}$$

The above definition of local satisfaction naturally extends to *global satisfaction* as follows: for any formula $\varphi \in \mathcal{L}_{MML}$,

$$\mathbb{M} \Vdash \varphi \quad \text{iff} \quad \mathbb{M}, w \Vdash \varphi \text{ for any } w \in X.$$

The notions of *local* and *global validity* are defined as follows: for any formula $\varphi \in \mathcal{L}_{MML}$, any neighbourhood frame $\mathbb{F} = (X, \sigma)$, and any $w \in X$,

$$\mathbb{F}, w \Vdash \varphi \quad \text{iff} \quad (\mathbb{F}, \nu), w \Vdash \varphi \text{ for any valuation } \nu : \text{AtProp} \rightarrow X.$$

$$\mathbb{F} \Vdash \varphi \quad \text{iff} \quad (\mathbb{F}, \nu) \Vdash \varphi \text{ for any valuation } \nu : \text{AtProp} \rightarrow X.$$

All the above definitions of satisfaction and validity can be naturally extended to \mathcal{L}_{MML} -inequalities as follows: for all formulas $\varphi, \psi \in \mathcal{L}_{MML}$, and any neighbourhood model $\mathbb{M} = (\mathbb{F}, \nu)$,

$$\mathbb{M} \Vdash \varphi \leq \psi \quad \text{iff} \quad \mathbb{M}, w \Vdash \varphi \text{ implies } \mathbb{M}, w \Vdash \psi \text{ for any } w \in X.$$

$$\mathbb{F} \Vdash \varphi \leq \psi \quad \text{iff} \quad \text{for any valuation } \nu \text{ and any } w \in X, \text{ if } (\mathbb{F}, \nu), w \Vdash \varphi \text{ then } (\mathbb{F}, \nu), w \Vdash \psi.$$

Remark 1.14. We notice that the definition above is usually adopted only for monotone neighbourhood frames and not for arbitrary neighbourhood frames. Under this definition, any neighbourhood frame behaves like a monotone one. Adopting this definition, rather than the usual one, is more advantageous for the present treatment, in that it will make it possible to equivalently describe any monotone neighbourhood frame only in terms of the minimal neighbourhoods of its states, as detailed in the following paragraph.

Finite monotone neighbourhood frames and finite neighbourhood frames. Our main focus of interest in the present chapter are finite lattices and their related structures, which are also finite. For any finite monotone neighbourhood frame $\mathbb{F} = (X, \sigma : X \rightarrow \mathcal{P}\mathcal{P}X)$, the collection $\sigma(x)$, which is an upset of $\mathcal{P}X$, is uniquely identified by the subcollection of its \subseteq -minimal elements. Hence, any \mathbb{F} as above can be equivalently represented as the neighbourhood frame $\mathbb{F}^* := (X, \sigma^*)$ where $\sigma^* : X \rightarrow \mathcal{P}\mathcal{P}X$ maps each state x to the \subseteq -minimal elements of the collection $\sigma(x)$. Conversely, any finite neighbourhood frame $\mathbb{F} = (X, \sigma)$ can be associated with a monotone neighbourhood frame $\mathbb{F}' := (X, \sigma')$ where $\sigma'(x) = \uparrow_{\subseteq} \sigma(x)$ for any $x \in X$, and moreover, $(\sigma^*)' = \sigma$ for any finite monotone neighbourhood frame. This correspondence extends to models as follows: for any finite monotone neighbourhood model $\mathbb{M} = (\mathbb{F}, \nu)$, let $\mathbb{M}^* := (\mathbb{F}^*, \nu)$ denote its associated finite neighbourhood model. Conversely, for any finite neighbourhood model $\mathbb{M} = (\mathbb{F}, \nu)$, let $\mathbb{M}' := (\mathbb{F}', \nu)$ denote its associated finite monotone neighbourhood model. Thanks to the slightly non-standard definition of the interpretation of \mathcal{L}_{MML} -formulas adopted in the present paper (cf. Definition 1.13 and Remark 1.14), this equivalent representation behaves well with respect to the interpretation of the monotone modal operators. Indeed, it is easy to show that for every $\varphi \in \mathcal{L}_{MML}$, every finite monotone neighbourhood model \mathbb{M} , and every finite neighbourhood model \mathbb{N} ,

$$\mathbb{M}, w \Vdash \varphi \quad \text{iff} \quad \mathbb{M}^*, w \Vdash \varphi \quad \text{and} \quad \mathbb{N}, w \Vdash \varphi \quad \text{iff} \quad \mathbb{N}', w \Vdash \varphi$$

The proof is done by induction on φ . We do not give it in full, and only report on the case of \mathbb{M} and the connectives $(\exists\forall)$ and $(\forall\exists)$.

$$\begin{aligned} \mathbb{M}, w \Vdash (\exists\forall)\varphi & \quad \text{iff} \quad \text{there exists some } C \in \sigma(w) \text{ such that } C \subseteq \nu(\varphi) \\ & \quad \text{iff} \quad \text{there exists some } C \in \min_{\subseteq} \sigma(w) \text{ such that } C \subseteq \nu(\varphi) \\ & \quad \text{iff} \quad \mathbb{M}^*, w \Vdash (\exists\forall)\varphi. \end{aligned}$$

$$\begin{aligned}
 \mathbb{M}, w \Vdash (\forall\exists)\varphi & \text{ iff } \text{for each } C \in \sigma(w), C \cap v(\varphi) \neq \emptyset \\
 & \text{ iff } \text{for each } C \in \min_{\subseteq}\sigma(w), C \cap v(\varphi) \neq \emptyset \\
 & \text{ iff } \mathbb{M}^*, w \Vdash (\exists\forall)\varphi.
 \end{aligned}$$

Join-presentations as monotone neighbourhood frames. Join-presentations (cf. Definition 1.4) of finite lattices bear a very close resemblance to neighbourhood frames. This resemblance can be spelled out more precisely, which is what we are going to do next.

For any finite lattice L , let $(J(L), \leq, \mathcal{M})$ be its join-presentation. The monotone neighbourhood frame associated with L is the tuple $\mathbb{F}_L := (J(L), \sigma_{\mathcal{M}} : J(L) \longrightarrow \mathcal{P}\mathcal{P}J(L))$ such that for each $j \in J(L)$,

$$\sigma_{\mathcal{M}}(j) := \{S \in \mathcal{P}\mathcal{P}J(L) \mid C \subseteq S \text{ for some } C \in \mathcal{M}(j)\}. \quad (1.4)$$

Clearly, $\sigma_{\mathcal{M}}(j)$ is upward-closed, hence the construction above is well defined. Moreover, since $\mathcal{M}(j)$ is a \ll -antichain (see Lemma 1.6.2), for all C and C' in $\mathcal{M}(j)$, if $C \subseteq C'$ then $C = C'$. This immediately implies that $\mathcal{M}(j)$ is the collection $\min_{\subseteq}\sigma_{\mathcal{M}}(j)$ of the \subseteq -minimal elements of $\sigma_{\mathcal{M}}(j)$.

Notice that the construction associating a neighbourhood frame with the join-presentation of a finite lattice L , involves a loss of information. Namely, the order \leq_J on the set $J(L)$ of the join-irreducible elements of L cannot be retrieved from the neighbourhood frame \mathbb{F}_L .

For every lattice L , we are only interested in valuations on \mathbb{F}_L which are the dual counterparts of assignments on L . Recall that L is isomorphic to the lattice \mathcal{Q}_L of closed sets of the join-presentation associated with L . Hence, we are only interested in valuations mapping atomic propositions to *closed* subsets, rather than to arbitrary subsets of \mathbb{F}_L . This motivates the following definition.

Definition 1.15. For any finite lattice L , let a *model* on \mathbb{F}_L be a tuple $\mathbb{M}_L = (\mathbb{F}_L, v^*)$ such that $v^* : \text{AtProp} \longrightarrow \mathcal{Q}_L$. We refer to such maps as *closed valuations*. Then, abusing terminology, the local and global validity of formulas and inequalities on \mathbb{F}_L will be understood relative to closed valuations, that is:

$$\begin{aligned}
 \mathbb{F}_L, j \Vdash \varphi & \text{ iff } (\mathbb{F}_L, v^*), j \Vdash \varphi \text{ for any closed valuation } v^*. \\
 \mathbb{F}_L \Vdash \varphi & \text{ iff } \mathbb{F}_L, j \Vdash \varphi \text{ for any } j \in J(L). \\
 \mathbb{F}_L \Vdash \varphi \leq \psi & \text{ iff } \text{for any closed valuation } v^* \text{ and any } j \in J(L), \\
 & \text{if } (\mathbb{F}_L, v^*), j \Vdash \varphi \text{ then } (\mathbb{F}_L, v^*), j \Vdash \psi.
 \end{aligned}$$

Let us spell out in detail the correspondence between assignments on L and closed valuations on \mathbb{F}_L . Clearly, given a set of variables AtProp , closed valuations of AtProp on \mathbb{F}_L can be identified with assignments of AtProp on \mathcal{Q}_L . The isomorphism $\mathcal{Q} : L \rightarrow \mathcal{Q}_L$ defined by the mapping $a \mapsto \{j \in J(L) \mid j \leq a\}$, with inverse defined by the mapping $S \mapsto \bigvee_L S$, induce bijections between assignments on L and assignments on \mathcal{Q}_L , defined by post-composition. That is, any assignment $v : \text{AtProp} \rightarrow L$ gives rise to the assignment $v^* : \text{AtProp} \rightarrow \mathcal{Q}_L$, such that for any $x \in \text{AtProp}$,

$$v^*(x) := \{j \in J(L) \mid j \leq v(x)\}. \quad (1.5)$$

The inverse correspondence maps any assignment/closed valuation $u : \text{AtProp} \rightarrow \mathcal{Q}_L$ to an assignment $u' : \text{AtProp} \rightarrow L$ such that for any $x \in \text{AtProp}$,

$$u'(x) := \bigvee_L u(x). \quad (1.6)$$

Thus, $v^{*'} = v$, and $u'^* = u$ for any assignment v on L and any assignment u on \mathcal{Q}_L . Hence, for all lattice terms s and t over AtProp , for any assignment v on L and any assignment u on \mathcal{Q}_L ,

$$L, v \models s \leq t \quad \text{iff} \quad \mathcal{Q}_L, v^* \models s \leq t, \quad (1.7)$$

$$L, u' \models s \leq t \quad \text{iff} \quad \mathcal{Q}_L, u \models s \leq t. \quad (1.8)$$

1.2.3 The standard translation

Thanks to the duality of the previous subsection, and to the correspondence environment introduced above, we are now in a position to define the ‘standard translation’ ST from the language of lattices to the language of monotone modal logic. The aim of this translation is to have, for any lattice term t , any finite lattice L , any $j \in J(L)$ and any $v : \text{AtProp} \rightarrow L$,

$$L, v \models j \leq t \quad \text{iff} \quad \mathbb{F}_L, v^*, j \Vdash ST(t), \quad (1.9)$$

where v^* is defined as in the discussion after Definition 1.15.

The definition of ST pivots on the duality between lattices and join-presentations. Namely, any given interpretation of a lattice term t on a finite lattice L translates to an interpretation of t into the lattice \mathcal{Q}_L of the closed sets of the join presentation $(J(L), \leq, \mathcal{M})$ associated with L , via the fact that L is isomorphic to \mathcal{Q}_L . Then, by dually characterizing the interpretation of t in \mathcal{Q}_L , we retrieve the interpretation of t into the join presentation $(J(L), \leq, \mathcal{M})$. In its turn, this interpretation boils down to the satisfaction clause, on \mathbb{F}_L , of certain formulas belonging to a *fragment* of monotone modal logic, which can be

recursively defined as follows:

$$\varphi ::= \perp \mid \top \mid p \mid \varphi \wedge \varphi \mid (\exists \forall)(\varphi \vee \varphi).$$

Let us define ST by the following recursion:

$$\begin{aligned} ST(p) &= p \\ ST(\top) &= \top \\ ST(\perp) &= \perp \\ ST(t \wedge s) &= ST(t) \wedge ST(s) \\ ST(t \vee s) &= (\exists \forall)(ST(t) \vee ST(s)). \end{aligned}$$

The definition above recasts [San09, Definition 7.1] into the language of monotone modal logic.

In what follows, we will find it useful to expand our propositional language with individual variables of a different sort than propositional variables. These new variables, denoted \mathbf{j}, \mathbf{k} , possibly with sub- and superscripts, are to be interpreted as join-irreducible elements of finite lattices. Let Nom (for nominals) be the collection of such variables, and let $\text{Var} := \text{AtProp} \cup \text{Nom}$. Finite lattice assignments from Var are maps $v : \text{Var} \rightarrow L$ such that $v(\mathbf{j}) \in J(L)$ for every $\mathbf{j} \in \text{Nom}$. Each such lattice assignment corresponds to a valuation from Var to \mathbb{F}_L as described in the discussion at the end of Section 1.2.2.

Proposition 1.16. *Let L be a finite lattice which is different from the one-element lattice. Then, for any lattice term t over AtProp , any $j \in J(L)$, and any assignment $v : \text{Var} \rightarrow L$ with $v(\mathbf{j}) = j$,*

$$L, v \models \mathbf{j} \leq t \quad \text{iff} \quad \mathbb{F}_L, v^*, j \Vdash ST(t), \quad (1.10)$$

Proof. By induction on t . If $t = \top, \perp$, then the statement is clearly true. If $t = p \in \text{AtProp}$, then $ST(p) = p$. Then, the following chain of logical equivalences holds:

$$\begin{aligned} L, v \models \mathbf{j} \leq p &\quad \text{iff} \quad v(\mathbf{j}) \leq_L v(p) \\ &\quad \text{iff} \quad \{k \in J(L) \mid k \leq v(\mathbf{j})\} \subseteq \{k \in J(L) \mid k \leq v(p)\} \quad (v(\mathbf{j}) \in \{k \in J(L) \mid k \leq v(\mathbf{j})\}) \\ &\quad \text{iff} \quad v^*(\mathbf{j}) \subseteq v^*(p) \quad (\text{definition of } v^*) \\ &\quad \text{iff} \quad v(\mathbf{j}) \in v^*(p) \quad (v^*(p) \text{ is a downset}) \\ &\quad \text{iff} \quad \mathbb{F}_L, v^*, j \Vdash p. \quad (v(\mathbf{j}) = j) \end{aligned}$$

The inductive step $t = t_1 \wedge t_2$ straightforwardly follows from the induction hypothesis.

As for the case $t = t_1 \vee t_2$, assume that the equivalence (1.10) holds for t_1 and t_2 , for every $c \in J(L)$ and for any $v : \mathbf{Var} \rightarrow L$. As discussed in the previous subsection (see equation (1.7)), we have

$$L, v \models \mathbf{j} \leq t_1 \vee t_2 \quad \text{iff} \quad \mathfrak{Q}_L, v^* \models \mathbf{j} \leq t_1 \vee t_2.$$

Let us recall that the meet \wedge^* and join \vee^* of \mathfrak{Q}_L are respectively defined as follows: for all $S, T \in \mathfrak{Q}_L$,

$$T \wedge^* S = T \cap S \quad \text{and} \quad T \vee^* S = \overline{T \cup S}$$

where $\overline{T \cup S}$ is defined in (1.2), that is: $\overline{T \cup S} = \{x \in X \mid \exists C \in \mathcal{M}(x) : C \subseteq T \cup S\}$. Hence, the following chain of logical equivalences holds:

$$\begin{aligned} & \mathfrak{Q}_L, v^* \models \mathbf{j} \leq t_1 \vee t_2 \\ \text{iff} & \quad v^*(\mathbf{j}) \subseteq v^*(t_1 \vee t_2) \\ \text{iff} & \quad j = v(\mathbf{j}) \in v^*(t_1 \vee t_2) \\ \text{iff} & \quad j \in \overline{v^*(t_1) \cup v^*(t_2)} \\ \text{iff} & \quad \text{there exists some } C \in \mathcal{M}(j) \text{ such that } c \in v^*(t_1) \text{ or } c \in v^*(t_2) \text{ for all } c \in C \\ \text{iff} & \quad \text{there exists some } C \in \mathcal{M}(j) \text{ such that } \downarrow c \subseteq v^*(t_1) \text{ or } \downarrow c \subseteq v^*(t_2) \text{ for all } c \in C, \end{aligned}$$

where $\downarrow c := \{k \in J(L) \mid k \leq c\}$. For any $c \in J(L)$, let u_c be the \mathbf{j} -variant of v^* such that $u_c(\mathbf{j}) = \downarrow c$. Hence, the previous clause can be equivalently rewritten as follows:

$$\text{there exists some } C \in \mathcal{M}(j) \text{ such that for all } c \in C, \quad \mathfrak{Q}_L, u_c \models \mathbf{j} \leq t_1 \quad \text{or} \quad \mathfrak{Q}_L, u_c \models \mathbf{j} \leq t_2.$$

By equation (1.8), the clause above can equivalently rewritten as follows:

$$\text{there exists some } C \in \mathcal{M}(j) \text{ such that for all } c \in C, \quad L, u'_c \models \mathbf{j} \leq t_1 \quad \text{or} \quad L, u'_c \models \mathbf{j} \leq t_2.$$

By the induction hypothesis, the clause above is equivalent to the following one:

$$\begin{aligned} & \text{there exists some } C \in \mathcal{M}(j) \text{ such that for all } c \in C, \\ & \mathbb{F}_L, (u'_c)^*, u'_c(\mathbf{j}) \Vdash ST(t_1) \quad \text{or} \quad \mathbb{F}_L, (u'_c)^*, u'_c(\mathbf{j}) \Vdash ST(t_2). \end{aligned}$$

Moreover, as discussed after Definition 1.15, we have that $u'_c(\mathbf{j}) = \bigvee_L u_c(\mathbf{j}) = \bigvee_L \downarrow c = c$, and $(u'_c)^* = u_c$. Hence, the clause above can be simplified as follows:

$$\text{there exists some } C \in \mathcal{M}(j) \text{ such that for all } c \in C, \quad \mathbb{F}_L, u_c, c \Vdash ST(t_1) \quad \text{or} \quad \mathbb{F}_L, u_c, c \Vdash ST(t_2),$$

and then as follows:

there exists some $C \in \mathcal{M}(j)$ such that for all $c \in C$, $c \in u_c(ST(t_1))$ or $c \in u_c(ST(t_2))$.

Since t_1 and t_2 are lattice terms over AtProp , no nominal variable occurs in them, and hence $u_c(ST(t_1)) = v^*(ST(t_1))$ and $u_c(ST(t_2)) = v^*(ST(t_2))$. Thus, we can equivalently rewrite the clause above as follows:

there exists some $C \in \mathcal{M}(j)$ such that for all $c \in C$, $c \in v^*(ST(t_1))$ or $c \in v^*(ST(t_2))$.

By (1.4), and since $\mathcal{M}(j) = \min_{\subseteq} \sigma_{\mathcal{M}(j)}$ (see discussion below (1.4)), the condition above is equivalent to

there exists some $S \in \sigma_{\mathcal{M}(j)}$ such that for all $c \in S$, $c \in v^*(ST(t_1))$ or $c \in v^*(ST(t_2))$.

By definition, this is equivalent to

$$\mathbb{F}_L, v^*, j \Vdash (\exists \forall)(ST(t_1) \vee ST(t_2)),$$

as required. □

The following corollary gives semantic justification to the standard translation, and provides the mathematical basis for our general approach of obtaining dual characterization results for finite lattices as instances of correspondence arguments in the language of monotone modal logic. Recall that, by Definition 1.15,

$$\begin{aligned} \mathbb{F}_L \Vdash \varphi \leq \psi \quad \text{iff} \quad & \text{for any closed valuation } v^* \text{ and any } j \in J(L), \\ & \text{if } (\mathbb{F}_L, v^*), j \Vdash \varphi \text{ then } (\mathbb{F}_L, v^*), j \Vdash \psi. \end{aligned}$$

Corollary 1.17. *Let L be a finite lattice. Then, for every lattice term t and s ,*

$$L \models t \leq s \quad \text{iff} \quad \mathbb{F}_L \Vdash ST(t) \leq ST(s).$$

Proof. Notice that finite lattices are join-generated by their join-irreducible elements. Hence, the condition $L \models t \leq s$ is equivalent to the following:

$$\text{for any assignment } v : \text{AtProp} \longrightarrow L, \text{ and for any } j \in J(L), \text{ if } j \leq v(t) \text{ then } j \leq v(s). \quad (1.11)$$

Clause (1.11) is equivalent to the following condition holding for any $j \in J(L)$, and for any valuation $v : \text{AtProp} \cup \text{Nom} \rightarrow L$ such that $v(\mathbf{j}) = j$:

$$\text{if } L, v \models \mathbf{j} \leq t, \text{ then } L, v \models \mathbf{j} \leq s. \quad (1.12)$$

By Proposition (1.16), clause (1.12) is equivalent to:

$$\text{if } \mathbb{F}_L, v^*, v(\mathbf{j}) \Vdash ST(t), \text{ then } \mathbb{F}_L, v^*, v(\mathbf{j}) \Vdash ST(s). \quad (1.13)$$

Next, we claim that clause (1.13) holding for any $j \in J(L)$ and for any valuation $v : \text{AtProp} \cup \text{Nom} \rightarrow L$ such that $v(\mathbf{j}) = j$ is equivalent to the following:

$$\begin{aligned} &\text{for any } j \in J(L), \text{ and for any closed valuation } u : \text{AtProp} \rightarrow \mathcal{Q}_L, \\ &\text{if } \mathbb{F}_L, u, j \Vdash ST(t), \text{ then } \mathbb{F}_L, u, j \Vdash ST(s). \end{aligned}$$

The latter condition is equivalent to $\mathbb{F}_L \Vdash ST(t) \leq ST(s)$, as desired.

To finish the proof, let us prove the claim. For the direction from top to bottom, fix a closed valuation $u : \text{AtProp} \rightarrow \mathcal{Q}_L$ such that $\mathbb{F}_L, u, j \Vdash ST(t)$ and let $v : \text{AtProp} \cup \text{Nom} \rightarrow L$ coincide with u' on AtProp (cf. (1.6)) and be such that $v(\mathbf{j}) = j$. By assumption, (1.13) holds for our choice of v . Since $(u')^* = u$, we have that v^* coincides with u on AtProp , hence $\mathbb{F}_L, v^*, v(\mathbf{j}) \Vdash ST(t)$. Then, by (1.13), $\mathbb{F}_L, v^*, v(\mathbf{j}) \Vdash ST(s)$. Since v^* coincides with u on AtProp , we have $\mathbb{F}_L, u, j \Vdash ST(s)$ as required. The direction from bottom to top is proved similarly. \square

1.2.4 An informal presentation of the algorithm ALBA

In the present subsection, we illustrate how ALBA works. Our presentation is based on [CP12, CGP14, CFPS14]. Rather than presenting the algorithm formally, in what follows we will run ALBA on one of the best known examples in correspondence theory, namely $\diamond\Box p \rightarrow \Box\diamond p$. It is well known that for every Kripke frame $\mathcal{F} = (W, R)$,

$$\mathcal{F} \Vdash \diamond\Box p \rightarrow \Box\diamond p \quad \text{iff} \quad \mathcal{F} \models \forall xyz (Rxy \wedge Rxz \rightarrow \exists u (Ryu \wedge Rzu)).$$

As is discussed at length in [CP12, CGP14], every piece of the argument used to prove this correspondence on Kripke frames can be translated by duality to their complex algebras (cf. [BdRV01, Definition 5.21]), which, as is well known, are complete atomic boolean algebras with operators. We will show how this is done in the case of the example above. First of all, the above validity condition on \mathcal{F} translates to its complex algebra \mathbb{A} as $\llbracket \diamond\Box p \rrbracket \subseteq \llbracket \Box\diamond p \rrbracket$ for every assignment of p into \mathbb{A} , so this validity clause

can be rephrased as follows:

$$\mathbb{A} \models \forall p[\diamond \Box p \leq \Box \diamond p]. \quad (1.14)$$

Since, in a complete atomic boolean algebra, every element is both the join of the completely join-prime elements (the set of which is denoted $J^\infty(\mathbb{A})$) below it and the meet of the completely meet-prime elements (the set of which is denoted $M^\infty(\mathbb{A})$) above it, the condition above can be equivalently rewritten as follows:

$$\mathbb{A} \models \forall p[\bigvee\{i \in J^\infty(\mathbb{A}) \mid i \leq \Box \diamond p\} \leq \bigwedge\{m \in M^\infty(\mathbb{A}) \mid \Box \diamond p \leq m\}].$$

By elementary properties of least upper bounds and greatest lower bounds in posets (cf. [DP02]), this condition is true if and only if every element in the join is less than or equal to every element in the meet. Thus, the condition above can be equivalently rewritten as:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \diamond \Box p \ \& \ \Box \diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

where the variables \mathbf{i} and \mathbf{m} range over $J^\infty(\mathbb{A})$ and $M^\infty(\mathbb{A})$ respectively. Since this presentation is geared towards the treatment in Section 1.5, we find it useful to slightly depart from the standard treatment in [CP12] and eliminate the conominal \mathbf{m} as follows. First, notice that the clause above is clearly equivalent to the following clause:

$$\mathbb{A} \models [\exists p \exists \mathbf{i} \exists \mathbf{m}(\mathbf{i} \leq \diamond \Box p \ \& \ \Box \diamond p \leq \mathbf{m} \ \& \ \mathbf{i} \not\leq \mathbf{m})] \Rightarrow \text{false}.$$

Second, notice that, in any complete atomic boolean algebra \mathbb{A} , for each $i \in J^\infty(\mathbb{A})$ and each $m \in M^\infty(\mathbb{A})$, one has $i \not\leq m$ iff $m = \kappa(i)$, where $\kappa(i) = \bigvee\{j \in J^\infty(\mathbb{A}) \mid j \neq i\} \in M^\infty(\mathbb{A})$. Hence, the clause above is equivalent to the following clause:

$$\mathbb{A} \models [\exists p \exists \mathbf{i}(\mathbf{i} \leq \diamond \Box p \ \& \ \Box \diamond p \leq \kappa(\mathbf{i}))] \Rightarrow \text{false}. \quad (1.15)$$

Since \mathbb{A} is in particular atomistic, the element of \mathbb{A} interpreting $\Box p$ is the join of the completely join-prime elements below it. Hence, if $i \in J^\infty(\mathbb{A})$ and $i \leq \diamond \Box p$, because \diamond is completely join-preserving on \mathbb{A} , we have that

$$i \leq \diamond(\bigvee\{j \in J^\infty(\mathbb{A}) \mid j \leq \Box p\}) = \bigvee\{\diamond j \mid j \in J^\infty(\mathbb{A}) \text{ and } j \leq \Box p\},$$

which implies that $i \leq \diamond j_0$ for some $j_0 \in J^\infty(\mathbb{A})$ such that $j_0 \leq \Box p$. Hence, we can equivalently rewrite the validity clause (1.15) as follows:

$$\mathbb{A} \models [\exists p \exists \mathbf{i}(\exists \mathbf{j}(\mathbf{i} \leq \diamond \mathbf{j} \ \& \ \mathbf{j} \leq \Box p) \ \& \ \Box \diamond p \leq \kappa(\mathbf{i}))] \Rightarrow \text{false},$$

and then as follows:

$$\mathbb{A} \models \forall p \forall i \forall j [(i \leq \diamond j \ \& \ j \leq \square p \ \& \ \square \diamond p \leq \kappa(i)) \Rightarrow \text{false}].$$

Now we observe that the operation \square preserves arbitrary meets in \mathbb{A} , which is in particular a complete lattice. By the general theory of adjunction in complete lattices, this is equivalent to \square being a right adjoint (cf. [DP02, proposition 7.34]). It is also well known that the left adjoint of \square is the operation \blacklozenge , which can be thought of as the backward looking diamond of tense logic. Hence the condition above can be equivalently rewritten as:

$$\mathbb{A} \models \forall p \forall i \forall j [(i \leq \diamond j \ \& \ \blacklozenge j \leq p \ \& \ \square \diamond p \leq \kappa(i)) \Rightarrow \text{false}],$$

and then as follows:

$$\mathbb{A} \models \forall i \forall j [(i \leq \diamond j \ \& \ \exists p (\blacklozenge j \leq p \ \& \ \square \diamond p \leq \kappa(i))) \Rightarrow \text{false}]. \quad (1.16)$$

At this point we are in a position to eliminate the variable p and equivalently rewrite the previous condition as follows:

$$\mathbb{A} \models \forall i \forall j [(i \leq \diamond j \ \& \ \square \diamond \blacklozenge j \leq \kappa(i)) \Rightarrow \text{false}]. \quad (1.17)$$

Let us justify this equivalence: for the direction from top to bottom, fix an interpretation v , and assume that $\mathbb{A}, v \models i \leq \diamond j$ and $\mathbb{A}, v \models \square \diamond \blacklozenge j \leq \kappa(i)$. Consider the p -variant v^* of v such that $v^*(p) = \blacklozenge j$. Then it can be easily verified that $\mathbb{A}, v^* \models i \leq \diamond j$ and $\mathbb{A}, v^* \models \blacklozenge j \leq p$ and $\mathbb{A}, v^* \models \square \diamond p \leq \kappa(i)$, which by assumption leads to an inconsistency.

Conversely, fix an interpretation v such that $\mathbb{A}, v \models i \leq \diamond j$ and $\mathbb{A}, v \models \exists p (\blacklozenge j \leq p \ \& \ \square \diamond p \leq \kappa(i))$. Then, by monotonicity, the antecedent of (1.17) holds under v , which leads again to an inconsistency. This is an instance of the following result, known as *Ackermann's lemma* ([Ack35], see also [CGV06]):

Lemma 1.18. *Let $\alpha, \beta(p), \gamma(p)$ be L -formulas, such that α is p -free, β is positive and γ is negative in p . For any assignment v on an L -algebra \mathbb{A} , the following are equivalent:*

1. $\mathbb{A}, v \models \beta(\alpha/p) \leq \gamma(\alpha/p)$;
2. there exists a p -variant v^* of v such that $\mathbb{A}, v^* \models \alpha \leq p$ and $\mathbb{A}, v^* \models \beta(p) \leq \gamma(p)$.

The proof is similar to that of [CP12, Lemma 4.2]. Whenever, in a reduction, we reach a shape in which the lemma above (or its order-dual) can be applied, we say that the condition is in *Ackermann shape*.

By the definition of $\kappa(\mathbf{i})$, the inequality $\diamond\Box\blacklozenge\mathbf{j} \leq \kappa(\mathbf{i})$ is equivalent to $\mathbf{i} \not\leq \diamond\Box\blacklozenge\mathbf{j}$. Hence, clause (1.17) can be equivalently rewritten as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{j} [(\mathbf{i} \leq \diamond\mathbf{j} \ \& \ \mathbf{i} \not\leq \diamond\Box\blacklozenge\mathbf{j}) \Rightarrow \text{false}], \quad (1.18)$$

and then as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{j} [\mathbf{i} \leq \diamond\mathbf{j} \Rightarrow \mathbf{i} \leq \diamond\Box\blacklozenge\mathbf{j}]. \quad (1.19)$$

By the atomicity of \mathbb{A} , the clause above is equivalent to:

$$\mathbb{A} \models \forall \mathbf{j} [\diamond\mathbf{j} \leq \diamond\Box\blacklozenge\mathbf{j}]. \quad (1.20)$$

By again applying the fact that \Box is a right adjoint we obtain

$$\mathbb{A} \models \forall \mathbf{j} [\blacklozenge\blacklozenge\mathbf{j} \leq \blacklozenge\mathbf{j}]. \quad (1.21)$$

Recalling that \mathbb{A} is the complex algebra of $\mathcal{F} = (W, R)$, this gives $\forall w (R[R^{-1}[w]] \subseteq R^{-1}[R[w]])$. Notice that $R[R^{-1}[w]]$ is the set of all states $x \in W$ which have a predecessor z in common with w , while $R^{-1}[R[w]]$ is the set of all states $x \in W$ which have a successor in common with w . This can be spelled out as

$$\forall x \forall w (\exists z (Rzx \wedge Rzw) \rightarrow \exists y (Rxy \wedge Rwy))$$

or, equivalently,

$$\forall z \forall x \forall w ((Rzx \wedge Rzw) \rightarrow \exists y (Rxy \wedge Rwy))$$

which is the familiar Church-Rosser condition.

1.3 Algorithmic correspondence for monotone modal logic

A key intermediate step of the present chapter is to adapt the algorithm or calculus for correspondence ALBA to monotone modal logic. The interest of this adaptation is independent from the applications to the theory of finite lattices. So, for the sake of modularity and generality, we work in a more abstract setting than the one associated with finite lattices, to which this adaptation will be applied. The general strategy underlying this adaptation is to exploit the well known fact that the ‘exists/for all’ and ‘for all/exists’ quantification patterns in the standard interpretation of the monotone modal operators make it possible to regard monotone modal operators as suitable concatenations of *normal* modalities. This same observation inspired Helle Hansen’s syntactic translation [Han03, Definition 5.7] on which her Sahlqvist correspondence theorem for

monotone modal logic is based. The present section is aimed at making all this precise. In the next subsection, we introduce two-sorted frames, their associated normal modal language, and first order correspondence language. We also spell out the relationship between two-sorted frames and monotone neighbourhood frames, which allows to interpret monotone modal logic on two-sorted frames. In Section 1.3.2, we introduce the basic adaptation of ALBA to the normal modal language of two-sorted frames.

1.3.1 Two-sorted frames

Definition 1.19. A *two-sorted frame* is a structure $\mathbb{X} = \langle X, Y, R_{XY}, R_{YX} \rangle$ such that X and Y are sets, $R_{XY} \subseteq X \times Y$ and $R_{YX} \subseteq Y \times X$.

The existence of the equivalent representation of any finite monotone neighbourhood frame \mathbb{F} in terms of the finite neighbourhood frame \mathbb{F}^* (cf. paragraph page 36) implies that we can equivalently encode any monotone neighbourhood frame \mathbb{F} as the following two-sorted structure (X, Y, R_{XY}, R_{YX}) , such that $Y = \mathcal{P}X$, and for every $x \in X$ and $y \in Y$,

$$xR_{XY}y \text{ iff } y \in \min_{\subseteq} \sigma(x) \quad \text{and} \quad yR_{YX}x \text{ iff } x \in y.$$

The definitions above imply that $R_{XY}[x] = \min_{\subseteq} \sigma(x)$ for any $x \in X$, and $R_{YX}[y] = y$ for any $y \in Y$. In the remainder of the chapter, for any relation $S \subseteq X \times Y$, we sometimes use the symbols xS and Sy to denote the sets $S[x]$ and $S^{-1}[y]$ respectively.

As is well known, each of the two relations R_{XY} and R_{YX} gives rise to a pair of semantic normal modal operators:

$$\begin{array}{ll} \langle R_{XY} \rangle : \mathcal{P}Y \longrightarrow \mathcal{P}X & [R_{XY}] : \mathcal{P}Y \longrightarrow \mathcal{P}X \\ T \longmapsto R_{XY}^{-1}[T] & T \longmapsto (R_{XY}^{-1}[T^c])^c \\ \\ \langle R_{YX} \rangle : \mathcal{P}X \longrightarrow \mathcal{P}Y & [R_{YX}] : \mathcal{P}X \longrightarrow \mathcal{P}Y \\ S \longmapsto R_{YX}^{-1}[S] & S \longmapsto (R_{YX}^{-1}[S^c])^c \end{array}$$

where

$$\begin{array}{ll} R_{XY}^{-1}[T] := \{x \in X \mid xR_{XY} \cap T \neq \emptyset\} & (R_{XY}^{-1}[T^c])^c := \{x \in X \mid xR_{XY} \subseteq T\} \\ R_{YX}^{-1}[S] := \{y \in Y \mid yR_{YX} \cap S \neq \emptyset\} & (R_{YX}^{-1}[S^c])^c := \{x \in X \mid xR_{YX} \subseteq S\}. \end{array}$$

Definition 1.20. The *complex algebra* of the two-sorted frame \mathbb{X} as above is the tuple

$$(\mathcal{P}X, \mathcal{P}Y, \langle R_{XY} \rangle, [R_{XY}], \langle R_{YX} \rangle, [R_{YX}]).$$

To make definitions and calculations more readable we introduce the following convention: we note \leq the order on $\mathcal{P}X$ and \leq the order on $\mathcal{P}Y$.

Two-sorted frames and their complex algebras will be used as (nonstandard) models for the modal language \mathcal{L}_{MML} over AtProp (cf. Definition 1.11), the definition of which we report here for the reader's convenience:

$$\phi ::= \perp \mid \top \mid p \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid (\exists\forall)\phi \mid (\forall\exists)\phi.$$

Definition 1.21. A *two-sorted model* is a tuple $M = (\mathbb{X}, \nu)$ such that \mathbb{X} is a two-sorted frame, and ν is a map $\text{AtProp} \rightarrow \mathcal{P}X$.

Given a valuation ν , its associated extension function is defined by induction as follows:

$$\begin{aligned} \llbracket \perp \rrbracket_{\nu, X} &= \emptyset \\ \llbracket \top \rrbracket_{\nu, X} &= X \\ \llbracket p \rrbracket_{\nu, X} &= \nu(p) \\ \llbracket \neg\phi \rrbracket_{\nu, X} &= \llbracket \phi \rrbracket_{\nu, X}^c \\ \llbracket \phi \vee \psi \rrbracket_{\nu, X} &= \llbracket \phi \rrbracket_{\nu, X} \cup \llbracket \psi \rrbracket_{\nu, X} \\ \llbracket \phi \wedge \psi \rrbracket_{\nu, X} &= \llbracket \phi \rrbracket_{\nu, X} \cap \llbracket \psi \rrbracket_{\nu, X} \\ \llbracket (\exists\forall)\phi \rrbracket_{\nu, X} &= \langle R_{XY} \rangle [R_{YX}] \llbracket \phi \rrbracket_{\nu, X} & (*) \\ \llbracket (\forall\exists)\phi \rrbracket_{\nu, X} &= [R_{XY}] \langle R_{YX} \rangle \llbracket \phi \rrbracket_{\nu, X} & (**) \end{aligned}$$

1.3.2 Basic ALBA on two-sorted frames

In order to adapt ALBA to the setting of two-sorted frames, we need to define the symbolic language which ALBA will manipulate. Analogously to what has been done in [CP12], let us introduce the language \mathcal{L}^+ as follows:

$$\begin{aligned} \varphi ::= \perp \mid \top \mid p \mid \mathbf{j} \mid \mathbf{m} \mid \underline{\mathbf{j}} \mid \underline{\mathbf{m}} \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \searrow \psi \mid \varphi \rightarrow \psi \mid \\ \langle R_{XY} \rangle \varphi \mid [R_{XY}] \varphi \mid \langle R_{YX} \rangle \varphi \mid [R_{YX}] \varphi \mid [R_{XY}^{-1}] \varphi \mid \langle R_{XY}^{-1} \rangle \varphi \mid [R_{YX}^{-1}] \varphi \mid \langle R_{YX}^{-1} \rangle \varphi, \end{aligned}$$

where $p \in \text{AtProp}$, $\mathbf{j} \in \text{Nom}_X$, $\underline{\mathbf{j}} \in \text{Nom}_Y$, $\mathbf{m} \in \text{CNom}_X$, $\underline{\mathbf{m}} \in \text{CNom}_Y$. The language above is shaped on the complex algebra of two-sorted frames. In particular, the variables in Nom_X and Nom_Y are to be interpreted as atoms of $\mathcal{P}X$ and $\mathcal{P}Y$ respectively, and the variables in CNom_X and CNom_Y are to be interpreted as coatoms of $\mathcal{P}X$ and $\mathcal{P}Y$. Moreover, the interpretation of the modal operators is the natural one suggested by the notation and indeed we are using the same symbols to denote both the operators and their interpretations. Finally, clauses (*) and (**) in Definition 1.21 justifies the definition of

the obvious translation from formulas of \mathcal{L}_{MML} to formulas in \mathcal{L}^+ . In what follows, we introduce the ALBA rules which are sound on general two-sorted structures.

Adjunction and residuation rules. It is well known that, in the setting of boolean algebras, the interpretation of the conjunction \wedge has a right residual, which is the interpretation of the implication, \rightarrow , and the interpretation of the disjunction \vee has a left residual, which is the interpretation of the subtraction \setminus . Thus, the following rules are sound and invertible in the two boolean algebras associated with any two-sorted structure:

$$\frac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \rightarrow \gamma} RS \wedge \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha \setminus \beta \leq \gamma} RS \vee$$

Moreover, it follows from very well known facts in modal logic that, for any two-sorted structure, $\langle R_{XY} \rangle$ (resp. $[R_{XY}]$) has a right (resp. left) adjoint, which is $[R_{YX}]$ (resp. $\langle R_{YX} \rangle$). Thus, the following rules are sound and invertible on any two-sorted structure:

$$\frac{\langle R_{XY} \rangle \alpha \leq \beta}{\alpha \leq [R_{XY}^{-1}] \beta} AJ \langle R_{XY} \rangle \qquad \frac{\alpha \leq [R_{XY}] \beta}{\langle R_{XY}^{-1} \rangle \alpha \leq \beta} AJ [R_{XY}]$$

$$\frac{\langle R_{YX} \rangle \alpha \leq \beta}{\alpha \leq [R_{YX}^{-1}] \beta} AJ \langle R_{YX} \rangle \qquad \frac{\alpha \leq [R_{YX}] \beta}{\langle R_{YX}^{-1} \rangle \alpha \leq \beta} AJ [R_{YX}]$$

Approximation rules. The soundness and invertibility of the rules below straightforwardly follows from the complete join- (resp. meet-)preservation properties of the modalities $\langle R_{XY} \rangle$, $[R_{XY}]$, $\langle R_{YX} \rangle$ and $[R_{YX}]$, and also from the fact that the boolean algebras $\mathcal{P}X$ and $\mathcal{P}Y$ are both completely join-generated by their completely join-irreducible elements and completely meet-generated by their completely meet-irreducible elements. For more details on this the reader is referred to [CP12].

$$\frac{\mathbf{i} \leq \langle R_{XY} \rangle \alpha}{\exists \mathbf{j} (\mathbf{i} \leq \langle R_{XY} \rangle \mathbf{j} \ \& \ \mathbf{j} \leq \alpha)} AP \langle R_{XY} \rangle \qquad \frac{[R_{XY}] \alpha \leq \mathbf{m}}{\exists \mathbf{n} (\alpha \leq \mathbf{n} \ \& \ [R_{XY}] \mathbf{n} \leq \mathbf{m})} AP [R_{XY}]$$

$$\frac{\mathbf{i} \leq \langle R_{YX} \rangle \alpha}{\exists \mathbf{j} (\mathbf{i} \leq \langle R_{YX} \rangle \mathbf{j} \ \& \ \mathbf{j} \leq \alpha)} AP \langle R_{YX} \rangle \qquad \frac{[R_{YX}] \alpha \leq \mathbf{m}}{\exists \mathbf{n} (\alpha \leq \mathbf{n} \ \& \ [R_{YX}] \mathbf{n} \leq \mathbf{m})} AP [R_{YX}]$$

Splitting rules. The following rules reflect the fact that the logical conjunction and disjunction are respectively interpreted with the greatest lower bound and least upper bound lattice operations, and hence are sound and invertible.

$$\frac{\varphi \leq \psi_1 \wedge \psi_2}{\varphi \leq \psi_1 \ \& \ \varphi \leq \psi_2} SP \wedge \qquad \frac{\psi_1 \vee \psi_2 \leq \varphi}{\psi_1 \leq \varphi \ \& \ \psi_2 \leq \varphi} SP \vee$$

Ackermann rules. The soundness and invertibility of the following rules (here below is the right-Ackermann rule) has been discussed in [CP12, Lemmas 4.2 and 4.3].

$$\frac{\exists p \left[\&_{i=1}^n \{ \alpha_i \leq p \} \ \& \ \&_{j=1}^m \{ \beta_j(p) \leq \gamma_j(p) \} \right]}{\&_{j=1}^m \{ \beta_j(\bigvee_{i=1}^n \alpha_i) \leq \gamma_j(\bigvee_{i=1}^n \alpha_i) \}} (RAR)$$

where p does not occur in $\alpha_1, \dots, \alpha_n$, the formulas $\beta_1(p), \dots, \beta_m(p)$ are positive in p , and $\gamma_1(p), \dots, \gamma_m(p)$ are negative in p . Here below is the left-Ackermann rule:

$$\frac{\exists p \left[\&_{i=1}^n \{ p \leq \alpha_i \} \ \& \ \&_{j=1}^m \{ \beta_j(p) \leq \gamma_j(p) \} \right]}{\&_{j=1}^m \{ \beta_j(\bigwedge_{i=1}^n \alpha_i) \leq \gamma_j(\bigwedge_{i=1}^n \alpha_i) \}} (LAR)$$

where p does not occur in $\alpha_1, \dots, \alpha_n$, the formulas $\beta_1(p), \dots, \beta_m(p)$ are negative in p , and $\gamma_1(p), \dots, \gamma_m(p)$ are positive in p .

Boolean tautologies. Clearly, the appropriate boolean and lattice tautologies justify the soundness and invertibility of the following rules. For the sake of conciseness, some of these rules will be given as formula-rewriting rules rather than as equivalences between inequalities.

$$\begin{array}{cccc} \frac{\varphi \vee \perp}{\varphi} \vee \perp & \frac{\varphi \vee (\psi_1 \wedge \psi_2)}{(\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2)} D \vee \wedge & \frac{\neg \neg \phi}{\phi} TNN & \frac{A \leq B}{(A \wedge B) = A} BA \wedge \\ \frac{\varphi \wedge \top}{\varphi} \wedge \top & \frac{\varphi \wedge (\psi_1 \vee \psi_2)}{(\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2)} D \wedge \vee & \frac{x \vee (y \searrow x)}{x \vee y} T \vee & \frac{B \leq A}{(A \vee B) = A} BA \vee \\ \frac{\varphi \vee \psi}{\psi \vee \varphi} C \vee & \frac{(\varphi \wedge \psi) \wedge \chi}{\psi \wedge (\varphi \wedge \chi)} A \wedge & \frac{x \wedge (x \rightarrow y)}{x \wedge y} T \wedge & \frac{\varphi \wedge \psi \leq \perp}{\varphi \leq \neg \psi} T \wedge \perp \\ \frac{\varphi \wedge \psi}{\psi \wedge \varphi} C \wedge & \frac{(\varphi \vee \psi) \vee \chi}{\psi \vee (\varphi \vee \chi)} A \vee & \frac{\xi \wedge (\varphi \searrow \psi) \leq \chi}{\xi \wedge \varphi \leq \psi \vee \chi} T \searrow & \frac{\neg(\varphi \vee \psi)}{\neg \varphi \wedge \neg \psi} DM \end{array}$$

Behaviour of atoms. In any complete atomic boolean algebra, $\kappa(j)$ coincides with $\neg j$ for each completely join-irreducible element j . Thus, the following rules are sound and invertible in the two boolean algebras associated with any two-sorted structure:

$$\begin{array}{ccc} \frac{\mathbf{j} \wedge s \leq \perp}{s \leq \kappa(\mathbf{j})} AtCoat1 & \frac{\mathbf{j} \wedge s \leq \kappa(\mathbf{j})}{s \leq \kappa(\mathbf{j})} AtCoat2 & \frac{\mathbf{j} \leq s \vee t \quad s \leq \kappa(\mathbf{j})}{\mathbf{j} \leq t \quad s \leq \kappa(\mathbf{j})} MT \\ \frac{\mathbf{j} \wedge s \leq \perp}{s \leq \underline{\kappa(\mathbf{j})}} AtCoat1 & \frac{\mathbf{j} \wedge s \leq \underline{\kappa(\mathbf{j})}}{s \leq \underline{\kappa(\mathbf{j})}} AtCoat2 & \frac{\mathbf{j} \leq s \vee t \quad s \leq \underline{\kappa(\mathbf{j})}}{\mathbf{j} \leq t \quad s \leq \underline{\kappa(\mathbf{j})}} MT \end{array}$$

Logical rules. Finally, we find it useful to stress that ALBA is able to perform elementary equivalent simplifications such as those represented in the rules below:

$$\frac{\phi \leq \psi}{\phi \leq \psi} \text{bis} \quad \frac{A = B \quad t(A) \leq s(A)}{A = B \quad t(B) \leq s(B)} \text{Sub} \quad \frac{\varphi \leq \psi \quad \psi \leq \chi}{\varphi \leq \psi \quad \psi \leq \chi \quad \varphi \leq \chi} \text{TR}$$

where $t(B)$ and $s(B)$ are obtained by replacing occurrences of A with B in t and s respectively.

Rules for normal modalities. The soundness and invertibility of the following rules TBD , TDB and TNM straightforwardly follows from well known validities for classical normal modal logic. The soundness and invertibility of TRR^{-1} immediately follows from the definition of the semantics of $\langle R \rangle$ and $\langle R^{-1} \rangle$.

$$\frac{[R]X}{\neg \langle R \rangle \neg X} \text{TBD} \quad \frac{\neg \langle R \rangle \neg X}{[R]X} \text{TDB} \quad \frac{\underline{\mathbf{j}} \leq \langle R \rangle \underline{\mathbf{j}}}{\underline{\mathbf{j}} \leq \langle R^{-1} \rangle \underline{\mathbf{j}}} \text{TRR}^{-1} \quad \frac{X \leq \langle R \rangle A \quad X \leq [R]B}{X \leq \langle R \rangle (A \wedge B) \quad X \leq [R]B} \text{TNM}.$$

1.4 Enhancing the algorithm for correspondence

We are working towards being able to account for Nation's characterisation in [Nat90] as an instance of algorithmic correspondence for the monotone modal logic language defined in Definition 1.11. As we saw in Section 1.2.3, the validity of a lattice inequality on any finite lattice L corresponds to the validity of the standard translation (cf. page 1.2.3) of the given inequality on the join-presentation \mathbb{F}_L associated with L restricted to closed valuations (cf. Definition 1.15). However, the version of ALBA for monotone neighbourhood frames defined in the previous section is not equipped to recognize closed valuations and properly treat them. Therefore, in the present section, we enhance the environment of two-sorted frames with an extra relation which encodes the order on the join-presentation \mathbb{F}_L . On this environment, additional ALBA rules can be shown to be sound, thanks to which closed valuations can be accounted for.

1.4.1 Enriched two-sorted frames

In the present subsection, we introduce the enriched two-sorted frames, and we show that the join-presentation of any finite lattice can be equivalently represented as an enriched two-sorted frame.

Definition 1.22. An *enriched two-sorted frame* is a structure $\mathbb{E} = \langle X, Y, R_{XY}, R_{YX}, R_{XX} \rangle$ such that $\langle X, Y, R_{XY}, R_{YX} \rangle$ is a two-sorted frame (cf. Definition 1.19), and $R_{XX} \subseteq X \times X$. An enriched two-sorted frame is

- *ordered* if R_{XX} is a partial order;
- *minimal* if
 - it is ordered,
 - $xR_{XY}y$ implies that the set $yR_{YX} = \{x' \in X \mid yR_{YX}x'\}$ is a R_{XX} -antichain for every $x \in X$ and $y \in Y$,
 - the collection $\{yR_{YX} \mid y \in xR_{XY}\}$ is a \ll -antichain for any $x \in X$,

where \ll is the refinement relation associated with the partial order (X, R_{XX}) (cf. (1.1)).

- *monotone* if for all $x, x' \in X$, and for each $y \in Y$, if $x'R_{XX}x$ and $xR_{XY}y$, then $y'R_{YX} \ll yR_{YX}$ for some $y' \in x'R_{XY}$;
- *reflexive* if for every $x \in X$ there exists some $y \in Y$ such that $xR_{XY}y$ and $yR_{YX} \ll \{x\}$;
- *transitive* if for every $x \in X$ and $y \in Y$, if $y \in xR_{XY}$ and $y_{x'} \in x'R_{XY}$ for some $x' \in yR_{YX}$ then there exists some $y' \in Y$ such that $xR_{XY}y'$ and $y'R_{YX} \ll \bigcup\{y_{x'}R_{YX} \mid x' \in yR_{YX}\}$;
- *direct* if it is ordered, minimal, monotone, reflexive and transitive;

Definition 1.23. Any join-presentation $\mathbb{A}_L := (J(L), \leq_J, \mathcal{M} : J(L) \rightarrow \mathcal{P}\mathcal{P}J(L))$ can be equivalently represented as an enriched two-sorted frame $\mathbb{E}_L := (X, Y, R_{XY}, R_{YX}, R_{XX})$ by setting

$$\begin{aligned} X &:= J(L), & Y &:= \{S \in \mathcal{P}J(L) \mid S \text{ is a } \leq_J \text{-antichain}\}, \\ R_{XY} &:= \{(x, y) \in X \times Y \mid y \in \mathcal{M}(x)\}, & R_{YX} &:= \exists, & \text{and} & R_{XX} = \leq_J. \end{aligned}$$

It can be easily verified that for every finite lattice L , the enriched two-sorted frame \mathbb{E}_L is direct.

Similarly to what has been discussed at the beginning of Section 1.3.1 (cf. page 46), each of the three relations R_{XY} , R_{YX} , and R_{XX} gives rise to a pair of semantic normal modal operators:

$$\begin{array}{ll} \langle R_{XY} \rangle : \mathcal{P}Y \longrightarrow \mathcal{P}X & [R_{XY}] : \mathcal{P}Y \longrightarrow \mathcal{P}X \\ T \longmapsto R_{XY}^{-1}[T] & T \longmapsto (R_{XY}^{-1}[T^c])^c \\ \\ \langle R_{YX} \rangle : \mathcal{P}X \longrightarrow \mathcal{P}Y & [R_{YX}] : \mathcal{P}X \longrightarrow \mathcal{P}Y \\ S \longmapsto R_{YX}^{-1}[S] & S \longmapsto (R_{YX}^{-1}[S^c])^c \\ \\ \langle R_{XX} \rangle : \mathcal{P}X \longrightarrow \mathcal{P}X & [R_{XX}] : \mathcal{P}X \longrightarrow \mathcal{P}X \\ S \longmapsto R_{XX}^{-1}[S] & S \longmapsto (R_{XX}^{-1}[S^c])^c \end{array}$$

where

$$\begin{aligned}
 R_{XY}^{-1}[T] &:= \{x \in X \mid xR_{XY} \cap T \neq \emptyset\} & (R_{XY}^{-1}[T^c])^c &:= \{x \in X \mid xR_{XY} \subseteq T\} \\
 R_{YX}^{-1}[S] &:= \{y \in Y \mid yR_{YX} \cap S \neq \emptyset\} & (R_{YX}^{-1}[S^c])^c &:= \{y \in Y \mid yR_{YX} \subseteq S\} \\
 R_{XX}^{-1}[S] &:= \{x \in X \mid xR_{XX} \cap S \neq \emptyset\} & (R_{XX}^{-1}[S^c])^c &:= \{x \in X \mid xR_{XX} \subseteq T\}.
 \end{aligned}$$

Definition 1.24. The *complex algebra* of the enriched two-sorted frame \mathbb{E} as above is the tuple

$$(\mathcal{P}X, \mathcal{P}Y, \langle R_{XY} \rangle, [R_{XY}], \langle R_{YX} \rangle, [R_{YX}], \langle R_{XX} \rangle, [R_{XX}]).$$

Definition 1.25. An *enriched two-sorted model* is a tuple $\mathbb{M} = (\mathbb{E}, \nu)$ such that \mathbb{X} is an enriched two-sorted frame, and ν is a map $\text{AtProp} \rightarrow \mathcal{P}X$.

The advantage of moving from the language of two-sorted frames to the language of enriched two-sorted frames is that the closure operator cl defined on direct presentations (see (1.3)) can be expressed in the modal language associated with enriched two-sorted frames. Indeed, unravelling the definitions involved, it is not difficult to see that for each subset S ,

$$cl(S) = \overline{\downarrow_{\leq_J} S} = (\exists \forall) \downarrow_{\leq_J} S = \langle \triangleleft \rangle [\exists] \langle \leq_J \rangle S. \quad (1.22)$$

Recall that any assignment ν on a given finite lattice L uniquely gives rise to the assignment ν^* on \mathbb{E}_L defined by $\nu^*(p) := \{j \in J(L) \mid j \leq_L \nu(p)\}$ for every $p \in \text{AtProp}$. Then it can be readily verified that the following identity is satisfied for every $p \in \text{AtProp}$:

$$\nu^*(p) = \langle \triangleleft \rangle [\exists] \langle \leq_J \rangle \nu^*(p).$$

The semantic identity above suggests the following definition:

Definition 1.26. A valuation ν on an enriched two-sorted model \mathbb{E} is *closed* if

$$\nu(p) = \langle R_{XY} \rangle [R_{YX}] \langle R_{XX} \rangle \nu(p)$$

for every $p \in \text{AtProp}$. An enriched two-sorted model is *closed* if its associated valuation is closed.

Thus, in the case a given enriched two-sorted model $\mathbb{M} = (\mathbb{E}_L, \nu)$ for some finite lattice L , the fact that the valuation ν arises from a lattice assignment on L can be expressed in the modal language of enriched two-sorted frames by means of the satisfaction of the identity $p = \langle R_{XY} \rangle [R_{YX}] \langle R_{XX} \rangle p$ for every $p \in \text{AtProp}$.

Definition 1.27. An enriched two-sorted model is *ordered* if its underlying enriched two-sorted frame is ordered and its associated valuation assigns every $p \in \text{AtProp}$ to a downset of (X, R_{XX}) .

1.4.2 Correspondence rules for enriched two-sorted frames

In the present subsection, we show the soundness of the following extra rules on enriched two-sorted frames.

$$\frac{\langle R_{XX} \rangle \mathbf{j} \wedge s \leq \kappa(\mathbf{j})}{s \leq \kappa(\mathbf{j})} \text{Atom}R_{XX}$$

Lemma 1.28. *The rule $\text{Atom}R_{XX}$ is sound and invertible on ordered enriched two-sorted models.*

Proof. Fix an ordered enriched two-sorted model $\mathbb{M} = (\mathbb{E}, \nu)$. Let $x \in X$ such that $\nu(\mathbf{j}) = \{x\}$, and assume that $\langle R_{XX} \rangle \mathbf{j} \wedge s \leq \kappa(\mathbf{j})$ is satisfied on \mathbb{M} . This means that $x \notin (\downarrow_{R_{XX}} x) \cap \nu(s)$, which implies that $x \notin \nu(s)$. This condition is equivalent to $s \leq \kappa(\mathbf{j})$ being satisfied.

Conversely, assume that $s \leq \kappa(\mathbf{j})$ is satisfied on \mathbb{M} as above. This is equivalent to $x \notin \nu(s)$. Since by assumption $\nu(s)$ is a downset of (X, R_{XX}) , we have that $x \notin \nu(s)$ iff $\downarrow_{R_{XX}} x \not\subseteq \nu(s)$. Hence $x \notin \downarrow_{R_{XX}} x \cap \nu(s)$, which is equivalent to $\langle R_{XX} \rangle \mathbf{j} \wedge s \leq \kappa(\mathbf{j})$ being satisfied on \mathbb{M} , as required. \square

$$\frac{\mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C} \quad \mathbf{k} \leq \langle \epsilon \rangle \mathbf{C}}{\mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C} \quad \mathbf{k} \leq \langle \epsilon \rangle \mathbf{C} \quad \langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (\langle \epsilon \rangle \mathbf{C} \setminus \mathbf{k}) \leq \kappa(\mathbf{k})} \text{MinCov2}$$

Lemma 1.29. *The rule MinCov2 is sound and invertible on every closed model $\mathbb{M} = (\mathbb{E}_L, \nu)$ such that $\mathbb{E}_L = (J(L), \mathcal{P}J(L), \triangleleft, \exists, \leq_J)$ is the enriched two-sorted frame associated with some finite lattice L (cf. Definition 1.23).*

Proof. The direction from bottom to top is immediate. Conversely, assume that the inequalities $\mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C}$ and $\mathbf{k} \leq \langle \epsilon \rangle \mathbf{C}$ are satisfied on \mathbb{M} . Let $j, k \in J(L)$ and $C \subseteq J(L)$ such that $\nu(\mathbf{j}) = \{j\}$, $\nu(\mathbf{k}) = \{k\}$ and $\nu(\mathbf{C}) = \{C\}$. Hence, $C \in \mathcal{M}(j)$ and $k \in C$. By Lemma 1.9.2, this implies that $k \notin \overline{\downarrow_{\leq}(C \setminus k)}$, which is equivalent to the satisfaction of the inequality $\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (\langle \epsilon \rangle \mathbf{C} \setminus \mathbf{k}) \leq \kappa(\mathbf{k})$ on \mathbb{M} . \square

Lemma 1.30. *Let s be a \mathcal{L}^+ -term. For every closed model $\mathbb{M} = (\mathbb{E}_L, \nu)$ such that $\mathbb{E}_L = (J(L), \mathcal{P}J(L), \triangleleft, \exists, \leq_J)$ is the enriched two-sorted frame associated with some finite lattice L (cf. Definition 1.23),*

$$\mathbb{M} \Vdash (S1) \quad \text{iff} \quad \mathbb{M} \Vdash (S2),$$

where

$$(S1) := \left(\begin{array}{l} \mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C} \\ \mathbf{k} \leq \langle \in \rangle \mathbf{C} \\ \langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k} \leq \kappa(\mathbf{k}) \\ \langle \leq_J \rangle \mathbf{k} \wedge s \leq \kappa(\mathbf{k}) \end{array} \right),$$

$$(S2) := \left(\begin{array}{l} \mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C} \\ \mathbf{k} \leq \langle \in \rangle \mathbf{C} \\ \langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k} \leq \kappa(\mathbf{k}) \\ \langle \leq_J \rangle \mathbf{k} \wedge s \leq \kappa(\mathbf{k}) \\ \mathbf{j} \wedge \langle \triangleleft \rangle [\exists] (\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{k} \wedge s)) \leq \perp \end{array} \right).$$

Proof. The right to left direction is immediate, since (S1) is a subset of (S2). Assume that (S1) is satisfied on \mathbb{M} . Let $j, k \in J(L)$ and $C \subseteq J(L)$ such that $v(\mathbf{j}) = \{j\}$, $v(\mathbf{k}) = \{k\}$ and $v(\mathbf{C}) = \{C\}$. The assumptions imply that

$$C \in \mathcal{M}(j), \quad k \in C, \quad k \notin \downarrow_{\leq_J} j \cap \downarrow_{\leq_J} k, \quad k \notin \downarrow_{\leq_J} k \cap v(\phi).$$

It is enough to show that

$$j \notin v(\langle \triangleleft \rangle [\exists] (\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{k} \wedge s))).$$

Unravelling the definitions of $\langle \triangleleft \rangle$ and $[\exists]$, the condition above is equivalent to the following:

$$\begin{array}{l} \text{there exists no } D \subseteq J(L) \text{ such that } D \in \mathcal{M}(j) \text{ and} \\ D \subseteq v'(\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{k} \wedge s)). \end{array} \quad (1.23)$$

The conditions $k \notin \downarrow_{\leq_J} j \cap \downarrow_{\leq_J} k$ and $k \notin \downarrow_{\leq_J} k \cap v(\phi)$ respectively imply that $k \notin \downarrow_{\leq_J} j$ and $k \notin v'(s)$. Hence, the following chain of inclusions holds:

$$\begin{array}{l} v'(\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{k} \wedge s)) \\ = v'(\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k})) \cup (v'(\langle \leq_J \rangle \mathbf{j}) \cap v'(\langle \leq_J \rangle \mathbf{k})) \cup (v'(\langle \leq_J \rangle \mathbf{k}) \cap v'(s)) \\ = v'(\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k})) \cup (\downarrow_{\leq_J} j \cap \downarrow_{\leq_J} k) \cup (\downarrow_{\leq_J} k \cap v'(s)) \quad (\text{by definition of } \langle \leq_J \rangle) \\ \subseteq v'(\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k})) \cup (\downarrow_{\leq_J} k \setminus \{k\}) \cup (\downarrow_{\leq_J} k \cap v'(s)) \quad (k \notin \downarrow_{\leq_J} j) \\ \subseteq v'(\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k})) \cup (\downarrow_{\leq_J} k \setminus \{k\}) \cup (\downarrow_{\leq_J} k \setminus \{k\}) \quad (k \notin v'(s)) \\ \subseteq v'(\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle (\langle \in \rangle \mathbf{C} \setminus \mathbf{k})) \cup (\downarrow_{\leq_J} k \setminus \{k\}) \\ \subseteq \overline{\downarrow_{\leq_J} (\mathbf{C} \setminus k)} \cup (\downarrow_{\leq_J} k \setminus \{k\}). \quad (\text{cf. (1.22)}) \end{array}$$

The fact that $C \in \mathcal{M}(j)$ and $k \in C$ implies, by Lemma 1.9.4, that there is no cover of j which is included in the set $\overline{\downarrow_{\leq j}(C \setminus k)} \cup (\downarrow_{\leq j} k \setminus \{k\})$. This implies that (1.23) holds as required. \square

1.4.3 Closed right Ackermann rule

In the present section, we are going to prove the soundness of the following special version of the Ackermann rule.

$$\frac{\exists p \left[\&_{i=1}^n \{\alpha_i \leq p\} \ \& \ \&_{j=1}^m \{\beta_j(p) \leq \gamma_j(p)\} \right]}{\&_{j=1}^m \{\beta_j(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle \bigvee_{i=1}^n \alpha_i) \leq \gamma_j(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle \bigvee_{i=1}^n \alpha_i)\}} \text{(RAcl)}$$

Lemma 1.31 (Right Ackermann Lemma for closed models). *The rule (RAcl) is sound and invertible on closed enriched two-sorted models $\mathbb{M} = (\mathbb{E}_L, \nu)$ such that \mathbb{E}_L is the enriched two-sorted frame associated with some finite lattice L .*

Proof. Fix a closed enriched two-sorted model $\mathbb{M} = (\mathbb{E}_L, \nu)$ such that \mathbb{E}_L is the enriched two-sorted frame associated with some finite lattice L . For the direction from bottom to top, assume that for every $1 \leq j \leq m$,

$$\nu(\beta_j(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle \bigvee_{i=1}^n \alpha_i)) \subseteq \nu(\gamma_j(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle \bigvee_{i=1}^n \alpha_i)).$$

Let ν' be the p -variant of ν such that $\nu'(p) = \langle \triangleleft \rangle [\exists] \langle \leq_X \rangle \nu(\bigvee_{i=1}^n \alpha_i)$. As discussed in Section 1.4.1, the composition $\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle$ is the operator that maps each set to the closure of its downset. Hence $\nu'(p)$ is a closed set and ν' is a closed valuation. Since α_i does not contain p , we have that $\nu'(\alpha_i) = \nu(\alpha_i)$, and hence

$$\nu'(\alpha_i) \leq \nu(\alpha_1) \vee \dots \vee \nu(\alpha_n) \leq \langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (\nu(\alpha_1) \vee \dots \vee \nu(\alpha_n)) = \nu'(p).$$

This shows that

$$\nu'(\alpha_i) \leq \nu'(p), \text{ for all } 1 \leq i \leq n.$$

Moreover, for all $1 \leq j \leq m$, we have

$$\nu'(\beta_j(p)) = \nu(\beta_j(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (\alpha_1 \vee \dots \vee \alpha_n) / p)) \leq \nu(\gamma_j(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (\alpha_1 \vee \dots \vee \alpha_n) / p)) = \nu'(\gamma_j(p)).$$

For the implication from top to bottom, we make use of the fact that the β_i are monotone (since positive) in p , while the γ_i are antitone (since negative) in p . Since the α_i do not contain p , and ν is a p -variant of ν' , we have $\nu(\alpha_i) = \nu'(\alpha_i) \leq \nu'(p)$, for all $1 \leq i \leq n$; hence, $\nu(\alpha_1) \vee \dots \vee \nu(\alpha_n) \leq \nu'(p)$. Since ν' is a closed valuation, $\nu'(p)$ is a closed set, and $\nu(\alpha_1) \vee \dots \vee \nu(\alpha_n) \leq \nu'(p)$ implies

that

$$\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (v(\alpha_1) \vee \dots \vee v(\alpha_n)) \leq v'(p).$$

Hence,

$$v(\beta_i(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (\alpha_1 \vee \dots \vee \alpha_n) / p)) \leq v'(\beta_i(p)) \leq v'(\gamma_i(p)) \leq v(\gamma_i(\langle \triangleleft \rangle [\exists] \langle \leq_X \rangle (\alpha_1 \vee \dots \vee \alpha_n) / p)).$$

□

1.5 Characterizing uniform upper bounds on the length of D^+ -chains in finite lattices

Definition 1.32. Let L be a finite lattice and let $\mathbb{E}_L = (J(L), \mathcal{P}J(L), \triangleleft, \exists, \leq_J)$ be its associated enriched two-sorted frame (cf. Definition 1.23). Consider the binary relation $D^+ \subseteq J(L) \times J(L)$ defined as follows: for any $j, k \in J(L)$,

$$jD^+k \quad \text{iff} \quad j \triangleleft C, k \in C \text{ and } k \not\leq j \text{ for some } C \in \mathcal{P}J(L).$$

A D^+ -chain of length l is a sequence (j_0, \dots, j_l) of elements of $J(L)$ such that $j_i D^+ j_{i+1}$ for each $0 \leq i \leq (l-1)$.

A notion similar to the one defined above has been used in [Nat90] and [Sem05] to define a hierarchy of varieties of lattices progressively generalising the variety of distributive lattices. More discussion about the similarities and differences between Nation and Semenova's notion of D -chain and the one above can be found in Section 1.6. In [San09], the result in [Nat90] and [Sem05] has been generalized, and the existence of a Sahlqvist-type correspondence mechanism underlying it has been observed. The main motivation of the present chapter is to provide a formal framework where this observation can be precisely spelled out, and in the present section, we are ready to obtain a result similar to Nation's by means of an ALBA reduction.

Fix enumerations of variables x_n, y_n for $n \in \mathbb{N}$. Consider the following family of lattice inequalities:

$$\{t_n \leq s_n \mid n \in \mathbb{N}\},$$

such that the lattice terms t_n and s_n are recursively defined as follows:

$$\begin{aligned} t_0 &:= x_0 & t_{n+1} &:= x_{n+1} \wedge (y_{n+1} \vee t_n) \\ s_0 &:= \perp & s_{n+1} &:= x_{n+1} \wedge (y_{n+1} \vee (x_{n+1} \wedge x_n) \vee s_n). \end{aligned}$$

The aim of this section is proving the following proposition:

Proposition 1.33. *For any finite lattice L and any $n \in \mathbb{N}$,*

$$L \models t_n \leq s_n \quad \text{iff} \quad \text{there is no } D^+ \text{-chain of length } n \text{ in } L.$$

Proof. For $n = 0$, we need to prove that, if L is a finite lattice, $L \models x_0 \leq \perp$ iff there is no D^+ -chain of length 0 in L . This is clear, since the only finite lattice L such that $L \models x_0 \leq \perp$ is the one-element lattice, which is the only finite lattice which has no join-irreducible element.

Let $n + 1 \geq 1$. By Corollary 1.17,

$$L \models t_{n+1} \leq s_{n+1} \quad \text{iff} \quad \mathbb{E}_L \Vdash ST(t_{n+1}) \leq ST(s_{n+1}),$$

where $\mathbb{E}_L = (J(L), \mathcal{P}J(L), \triangleleft, \ni, \leq_J)$ is the enriched two-sorted frame associated with L (cf. Definition 1.23), and the validity on the right-hand side of the equivalence is understood in terms of satisfaction for every closed valuation.

We will provide ALBA^l reductions for each $n + 1 \geq 1$ and each inequality $t_{n+1} \leq s_{n+1}$. Since all the ALBA^l rules are sound and invertible on \mathbb{E}_L , the reduction will output a condition in the first order correspondence language of \mathbb{E}_L , which is equivalent to the validity of the input inequality on L , and which will express the existence of no D -chains of length $n + 1$ in L .

First of all, using the standard translation introduced in Section 1.2.2, the lattice terms t_{n+1} and s_{n+1} translate into the following monotone modal logic formulas:

$$\begin{aligned} ST(t_0) &:= x_0 & ST(t_{n+1}) &:= x_{n+1} \wedge t'_{n+1} & \text{with } t'_{n+1} &:= (\exists \forall)(y_{n+1} \vee ST(t_n)) \\ ST(s_0) &:= \perp & ST(s_{n+1}) &:= x_{n+1} \wedge s'_{n+1} & \text{with } s'_{n+1} &:= (\exists \forall)(y_{n+1} \vee (x_{n+1} \wedge x_n) \vee ST(s_n)). \end{aligned}$$

Using the notation introduced in Section 1.2.2, the \mathcal{L}_{MML} -terms above can be translated into the modal language of enriched two-sorted frames (cf. Section 1.4.1) as indicated below. For the sake of simplicity, we use the symbols t_n and s_n also to indicate the translations of the original lattice terms.

$$\begin{aligned} t_0 &= x_0 & t_{n+1} &= x_{n+1} \wedge t'_{n+1} & \text{with } t'_{n+1} &= \langle \triangleleft \rangle [\ni](y_{n+1} \vee t_n) \\ s_0 &= \perp & s_{n+1} &= x_{n+1} \wedge s'_{n+1} & \text{with } s'_{n+1} &= \langle \triangleleft \rangle [\ni](y_{n+1} \vee (x_{n+1} \wedge x_n) \vee s_n) \end{aligned}$$

Let \bar{x} stand for the list of variables x_n, \dots, x_0 , and \bar{y} stand for the list of variables y_n, \dots, y_1 . ALBA^l transforms the input inequality $t_{n+1} \leq s_{n+1}$ into the following quasi-inequality (cf. Section 1.2.4):

$$\forall x_{n+1}, \forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq t_{n+1} \\ s_{n+1} \leq \kappa(\mathbf{j}_{n+1}) \end{array} \right) \Rightarrow \text{false} \right).$$

Since $t_{n+1} = x_{n+1} \wedge t'_{n+1}$ and $s_{n+1} = x_{n+1} \wedge s'_{n+1}$, we can rewrite the quasi-inequality above as:

$$\forall x_{n+1}, \forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq x_{n+1} \wedge t'_{n+1} \\ x_{n+1} \wedge s'_{n+1} \leq \kappa(\mathbf{j}_{n+1}) \end{array} \right) \Rightarrow \text{false} \right).$$

Applying the rule ($SP\wedge$) to the first inequality yields:

$$\forall x_{n+1}, \forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq x_{n+1} \\ \mathbf{j}_{n+1} \leq t'_{n+1} \\ x_{n+1} \wedge s'_{n+1} \leq \kappa(\mathbf{j}_{n+1}) \end{array} \right) \Rightarrow \text{false} \right).$$

Notice that $x_{n+1} \notin \text{Var}(t'_{n+1})$ and s'_{n+1} is monotone in x_{n+1} . Thus we can apply the Ackermann rule ($RACl$) to eliminate x_{n+1} via the substitution $x_{n+1} \leftarrow \langle \leq_J \rangle \mathbf{j}_{n+1}$.

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq t'_{n+1} \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge s'_{n+1}(\langle \leq_J \rangle \mathbf{j}_{n+1} / x_{n+1}) \leq \kappa(\mathbf{j}_{n+1}) \end{array} \right) \Rightarrow \text{false} \right).$$

Recall that \mathbb{E}_L is an ordered enriched two-sorted frame and closed valuations assign variables to downsets. Hence, by Lemma 1.28, the quasi-inequality above is equivalent to the quasi-inequality below by applying the rule ($AtomR_{XX}$).

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq t'_{n+1} \\ s'_{n+1}(\langle \leq_J \rangle \mathbf{j}_{n+1} / x_{n+1}) \leq \kappa(\mathbf{j}_{n+1}) \end{array} \right) \Rightarrow \text{false} \right).$$

By Lemma 1.34, the quasi-inequality above is equivalent to

$$\forall \mathbf{j}_{n+1}, \dots, \mathbf{j}_0, \forall \mathbf{C}_n, \mathbf{C}_{n-1}, \dots, \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \mathbf{j}_n \leq \perp \\ \dots \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \mathbf{j}_0 \leq \perp \end{array} \right) \Rightarrow \text{false} \right). \quad (1.24)$$

Notice that, for $0 \leq i \leq n$, the following inequalities:

$$\mathbf{j}_{i+1} \leq \langle \triangleleft \rangle \mathbf{C}_i, \quad \mathbf{j}_i \leq \langle \in \rangle \mathbf{C}_i, \quad \langle \leq_J \rangle \mathbf{j}_{i+1} \wedge \mathbf{j}_i \leq \perp$$

are respectively equivalent to the following atomic formulas in the first order correspondence language of enriched two-sorted frames (cf. Section 1.4.1):

$$j_{i+1} \triangleleft C_i, \quad j_i \in C_i, \quad j_i \not\leq j_{i+1}.$$

By Definition 1.32, the conditions above yield $j_{i+1}D^+j_i$ for each $0 \leq i \leq n$. Hence the quasi-inequality (1.24) is equivalent to the following quasi-inequality:

$$\forall j_{n+1}, \dots, j_0 \quad [(j_{n+1}D^+j_n \dots j_1D^+j_0) \Rightarrow \text{false}],$$

which expresses the condition that there is no D^+ -chain of length $n + 1$. \square

The proof of the proposition above relies on the following lemma, the proof of which can be found in Appendix A.

Lemma 1.34. *For every $n \geq 1$, ALBA^l succeeds on the quasi-inequality*

$$\forall x_{n-1}, \dots, x_0, \forall y_n, \dots, y_0, \forall \mathbf{j}_n, \left(\left(\begin{array}{l} \mathbf{j}_n \leq t'_n \\ s'_n(\langle \leq_J \rangle \mathbf{j}_n / x_n) \leq \kappa(\mathbf{j}_n) \end{array} \right) \Rightarrow \text{false} \right),$$

and produces

$$\forall \mathbf{j}_n, \dots, \mathbf{j}_0, \forall \mathbf{C}_{n-1}, \dots, \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_n \leq \langle \triangleleft \rangle \mathbf{C}_{n-1} \\ \mathbf{j}_{n-1} \leq \langle \in \rangle \mathbf{C}_{n-1} \\ \langle \leq_J \rangle \mathbf{j}_n \wedge \mathbf{j}_{n-1} \leq \perp \\ \dots \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \mathbf{j}_0 \leq \perp \end{array} \right) \Rightarrow \text{false} \right).$$

1.6 Conclusions and further directions

Conclusions. In the present chapter, the algorithmic correspondence theory revolving around ALBA (cf. [CP12, CGP14]) has been adapted and extended, so as to provide an adequate environment in which to formalize the observation (cf. [San09]) of the existence of a Sahlqvist-type mechanism underlying dual characterization results for finite lattices.

The treatment of lattice inequalities in the setting of ALBA is mediated by monotone modal logic, thanks to the existence of a duality-on-objects between finite lattices and join-presentations (cf. Definition 1.4), and the fact that join-presentations are closely related to (monotone) neighbourhood frames.

A key step towards the main result of the present chapter is the adaptation of ALBA to monotone modal logic, semantically justified by the introduction of two-sorted structures and their associated correspondence language. In this setting, the Sahlqvist correspondence theory of [Han03] can be embedded and generalized.

Comparison with Nation's results. As mentioned early on, our result is similar to Nation's dual characterization of uniform upper bounds on the length of D -chains in finite lattices:

Definition 1.35. Let L be a finite lattice. Let $D \subseteq J(L) \times J(L)$ be the binary relation defined as follows: for any $j, k \in J(L)$,

$$jDk \quad \text{iff} \quad j \triangleleft C, k \in C \text{ and } k \neq j \text{ for some } C \in \mathcal{P}J(L).$$

A D -chain of length l is a sequence (j_0, \dots, j_l) of elements of $J(L)$ such that $j_i D j_{i+1}$ for each $0 \leq i \leq (l - 1)$.

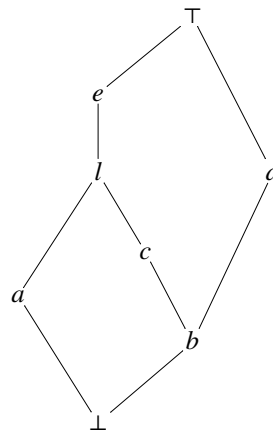
The dual characterization of Section 1.5 is different from Nation's [Nat90] and is not covered by the result in [San09], which generalizes Nation's. As far as we know, it is original.

Clearly, D^+ is included in D for any finite lattice L . Hence, the validity on a finite lattice L of Nation's inequalities for a given n is a sufficient condition for L having D^+ -chains of length at most n . However, in the remainder of the paragraph, we are going to show that this upper bound is not accurate. Indeed, the maximal length of D^+ -chains starting from a given join-irreducible element in a lattice can be strictly smaller than the one for D -chains starting from the same join-irreducible element. Consider the lattice L the Hasse diagram of which is given by the figure. In this example, it can be easily verified that

$$\begin{aligned} J(L) &:= \{a, b, c, d, e\}, \\ \mathcal{M}(c) &:= \{\{c\}, \{a, b\}\}, \\ \mathcal{M}(e) &:= \{\{e\}, \{a, d\}, \{c, d\}\}. \end{aligned}$$

The only D^+ -chain starting from e is eD^+d , whereas there are D -chains of length 2 starting from e , for instance $eDcDb$.

Further directions. The present chapter is a first step towards the fully-fledged automatization of dual characterization results for finite lattices. Significant extensions

FIGURE 1.1: The Hasse diagram of the lattice L

of Nation's dual characterization results appear e.g. in [Sem05] and [San09, Proposition 8.5]. Hence, natural directions worth pursuing are (a) extending the results of the present chapter so as to account for [San09, Proposition 8.5], and (b) analyzing the technical machinery introduced in the present chapter from an algorithmic perspective. The latter point involves e.g. establishing whether the present set of rules is minimal, or whether some rules can actually be derived.

Related to both these directions, but more on the front of methodology, are outstanding open questions about Lemma A.1. This lemma provides the soundness and invertibility of a rule by means of which variable elimination is effected via instantiation. So far, all rules of this type in ALBA have been proved sound and invertible thanks to one or another version of Ackermann's lemma. However, it is not clear whether Lemma A.1 can be accounted for in terms of Ackermann's lemma, and hence whether the rule justified in Lemma A.1 can be regarded as an Ackermann-type rule. Moreover, while Lemma A.1 is rooted and has an intuitive understanding in the semantics of minimal coverings, at the moment it is not clear whether and how more general versions of this rule can be formulated, which would be of a wider applicability. Giving answers to these questions would significantly enlarge the scope of algorithmic correspondence theory, and is also a worthwhile future direction.

Part II

Proof Theory

Chapter 2

A Proof-Theoretic Semantic Analysis of Dynamic Epistemic Logic

2.1 Introduction

In recent years, driven by applications in areas spanning from program semantics to game theory, the logical formalisms pertaining to the family of *dynamic logics* [HKT00, vDvdHK07] have been very intensely investigated, giving rise to a proliferation of variants.

Typically, the language of a given dynamic logic is an expansion of classical propositional logic with an array of modal-type *dynamic operators*, each of which takes an *action* as a parameter. The set of actions plays in some cases the role of a set of indexes or parameters; in other cases, actions form a quantale-type algebra. When interpreted in relational models, the formulas of a dynamic logic express properties of the model encoding the present state of affairs, as well as the pre- and post-conditions of a given action. Actions formalize transformations of one model into another one, the updated model, which encodes the state of affairs after the action has taken place.

Dynamic logics have been investigated mostly w.r.t. their semantics and complexity, while their proof-theoretic aspects have been comparatively not so prominent. However, the existing proposals of proof systems for dynamic logics witness a varied enough array of methodologies that a methodological evaluation is now timely.

The starting point of the present chapter is precisely the evaluation of the current proposals of proof-systems for the best-known dynamic epistemic logics from the viewpoint of *proof-theoretic semantics*.

Proof-theoretic semantics [SH13] is a theory of meaning which assigns formal proofs or derivations an autonomous semantic content. That is, formal proofs are treated as entities in terms of which meaning can be accounted for. Proof-theoretic semantics has been very influential in an area of research in structural proof theory which aims at defining the meaning of logical connectives in terms of an analysis of the behaviour of the logical connectives inside the derivations of a given proof system. Such an analysis is possible only in the context of proof systems which perform well w.r.t. certain criteria. Hence, one of the main themes in this area is to identify design criteria which both guarantee that the proof system enjoys certain desirable properties such as normalization or cut-elimination, and which make it possible to speak about the proof-theoretic meaning for given logical connectives.

An analysis of dynamic logics from a proof-theoretic semantic viewpoint is beneficial both for dynamic logics and for structural proof theory. Indeed, such an analysis provides dynamic logics with sound methodological and foundational principles, and with an entirely novel perspective on the topic of dynamics and change, which is *independent* from the dominating model-theoretic methods. Moreover, such an analysis provides structural proof theory with a novel array of case studies against which to test the generality of its proof-theoretic semantic principles, and with the opportunity to extend its modus operandi to still uncharted settings, such as the multi-type calculi introduced in Chapter 3.

Motivated by this proof theoretic semantic analysis, we propose a display calculus for the logic of Epistemic Knowledge and Actions and develop its theory.

Organization and results. In Section 2.2, we introduce the basic ideas of proof-theoretic semantics, as well as some of the principles in structural proof theory that were inspired by it, and we explain their consequences and spirit, in view of their applications in the following sections. In Section 2.3, we prove a generalization of Belnap’s cut elimination metatheorem. In Section 2.4, we review some of the most significant proposals of proof systems for dynamic epistemic logics, focusing mainly on the logic of Public Announcements (PAL) [Pla07] and the logic of Epistemic Knowledge and Actions (EAK) [BMS99], and we critically reflect on them in the light of the principles of proof-theoretic semantics stated in Section 2.2, in particular in Section 2.4.4, we focus on the display-type calculus D.EAK for PAL/EAK introduced in [GKP13]: we highlight its critical issues—the main of which being that a smooth (*Belnap-style*) proof of cut elimination is not readily available for it. In Section 2.5, we expand on the final coalgebra semantics for D.EAK, since it is relevant for the subsequent developments of the present and the next chapter. In Section 2.6, we propose a revised version of D.EAK, discuss why the revision is more

adequate for proof-theoretic semantics, and finally prove the cut elimination theorem for the revised version as a consequence of the metatheorem proven in Section 2.6.3. In Section 2.7, we collect some conclusions and indicate further directions. Most of the proofs and derivations are collected in Appendices C, D and E.

2.2 Preliminaries on proof-theoretic semantics and Display Calculi

In the present section, we review and discuss the proof-theoretic notions which will be used in the further development of the chapter. In the following subsection, we outline the conceptual foundations of proof-theoretic semantics; in Subsection 2.2.2, Belnap-style display calculi will be discussed; in Subsection 2.2.3 a refinement of Belnap's analysis, due to Wansing, will be reported on. Our presentation is certainly not exhaustive, and will limit itself to targeting the issues needed in the further development of the thesis. The reader is referred to [SH13, SH06] for a detailed presentation of proof-theoretic semantics, and to [Wan98, Wan00] for a discussion of proof-theoretic semantic principles in structural proof theory.

2.2.1 Basic ideas in proof-theoretic semantics

Proof-theoretic semantics is a line of research which covers both philosophical and technical aspects, and is concerned with methodological issues. Proof-theoretic semantics is based on the idea that a purely inferential theory of meaning is possible. That is, that the meaning of expressions (in a formal language or in natural language) can be captured purely in terms of the proofs and the inference rules which participate in the generation of the given expression, or in which the given expression participates. This *inferential* view is opposed to the mainstream *denotational* view on the theory of meaning, and is influential in e.g. linguistics, linking up to the idea, commonly attributed to Wittgenstein, that 'meaning is use'. In proof theory, this idea links up with Gentzen's famous observation about the introduction and elimination rules of his natural deduction calculi:

'The introductions represent, as it were, the definitions of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only in the sense afforded it by the introduction of that symbol'. ([Gen69] p. 80)

In the proof-theoretic semantic literature, this observation is brought to its consequences: rather than viewing proofs as entities the meaning of which is dependent on denotation, proof-theoretic semantics assigns proofs (in the sense of formal deductions) an *autonomous semantic role*; that is, proofs are entities in terms of which meaning can be accounted for.

Proof-theoretic semantics has inspired and unified much of the research in structural proof theory focusing on the purely inferential characterization of logical constants (i.e. logical connectives) in the setting of a given proof system.

2.2.2 Display calculi

Display calculi are among the approaches in structural proof theory aimed at the uniform development of an inferential theory of meaning of logical constants aligned with the ideas of proof-theoretic semantics. Display calculi have been successful in giving adequate proof-theoretic accounts of logics—such as modal logics and substructural logics—which have notoriously been difficult to treat with other approaches. In particular, the contributions in this line of research which are most relevant to our analysis are Belnap’s [Bel82], Wansing’s [Wan98], Goré’s [Gor98], and Restall’s [Res00].

Display Logic. Nuel Belnap introduced the first display calculus, which he calls *Display Logic* [Bel82], as a sequent system augmenting and refining Gentzen’s basic observations on structural rules. Belnap’s refinement is based on the introduction of a special syntax for the constituents of each sequent. Indeed, his calculus treats sequents $X \vdash Y$ where X and Y are so-called *structures*, i.e. syntactic objects inductively defined from formulas using an array of special connectives. Belnap’s basic idea is that, in the standard Gentzen formulation, the comma symbol ‘,’ separating formulas in the precedent and in the succedent of sequents can be recognized as a metalinguistic connective, of which the structural rules define the behaviour.

Belnap took this idea further by admitting not only the comma, but also several other connectives to keep formulas together in a structure, and called them *structural connectives*. Just like the comma in standard Gentzen sequents is interpreted contextually (that is, as conjunction when occurring on the left-hand side and as disjunction when occurring on the right-hand side), each structural connective typically corresponds to a pair of logical connectives, and is interpreted as one or the other of them contextually (more of this in Sections 2.5 and 2.6.1). Structural connectives maintain relations with one another, the most fundamental of which take the form of adjunctions and residuations. These relations make it possible for the calculus to enjoy the powerful property

which gives it its name, namely, the *display property*. Before introducing it formally, let us agree on some auxiliary definitions and nomenclature: *structures* are defined much in the same way as formulas, taking formulas as atomic components and closing under the given structural connectives; therefore, each structure can be uniquely associated with a generation tree. Every node of such a generation tree defines a *substructure*. A *sequent* $X \vdash Y$ is a pair of structures X, Y . The display property was introduced by Belnap, see Theorem 3.2 of [Bel82] (where $X \vdash Y$ is called a consecution and X the antecedent and Y the consequent):

Definition 2.1. A proof system enjoys the *display property* iff for every sequent $X \vdash Y$ and every substructure Z of either X or Y , the sequent $X \vdash Y$ can be equivalently transformed, using the rules of the system, into a sequent which is either of the form $Z \vdash W$ or of the form $W \vdash Z$, for some structure W . In the first case, Z is *displayed in precedent position*, and in the second case, Z is *displayed in succedent position*. The rules enabling this equivalent rewriting are called *display postulates*.

Thanks to the fact that display postulates are based on adjunction and residuation, in display calculi exactly one of the two alternatives mentioned in the definition above occurs. In other words, in a system enjoying the display property, any substructure of any sequent $X \vdash Y$ is always displayed either only in precedent position or only in succedent position. This is why we can talk about occurrences of substructures in *precedent* or in *succedent* position, even if they are nested deep within a given sequent, as illustrated in the following example:

$$\frac{\frac{\frac{Y \vdash X > Z}{X; Y \vdash Z}}{Y; X \vdash Z}}{X \vdash Y > Z}$$

In the derivation above, the structure X is on the right side of the turnstile, but it is displayable on the left, and therefore is in precedent position. As discussed in Section 4 (on page 7), the display property is a crucial technical ingredient for display calculi cut elimination metatheorem, but it is also at the basis of Belnap's methodology for characterizing operational connectives: according to Belnap, any logical connective should be introduced *in isolation*, i.e., when it is introduced, the context on the side it has been introduced must be empty. The display property guarantees that this condition is not too restrictive.

Canonical cut elimination. In [Bel82], a metatheorem is proven, which gives sufficient conditions in order for a sequent calculus to enjoy cut elimination.¹ This metatheorem

¹Note that, as Belnap observed on pag. 389 in [Bel82]: 'The eight conditions are supposed to be a reminiscent of those of Curry' in [Cur63].

captures the essentials of the Gentzen-style cut elimination procedure, and is the main technical motivation for the design of Display Logic. Belnap’s metatheorem gives a set of eight conditions on sequent calculi, which are relatively easy to check, since most of them are verified by inspection on the shape of the rules. Together, these conditions guarantee that the cut is eliminable in the given sequent calculus, and that the calculus enjoys the subformula property. When Belnap’s metatheorem can be applied, it provides a much smoother and more modular route to cut elimination than the Gentzen-style proofs. Moreover, as we will see later, a Belnap style cut elimination theorem is robust with respect to adding structural rules and with respect to adding new logical connectives, whereas a Gentzen-style cut elimination proof for the modified system cannot be deduced from the old one, but must be proved from scratch.

In a slogan, we could say that Belnap-style cut elimination is to ordinary cut elimination what canonicity is to completeness: indeed, canonicity provides a *uniform strategy* to achieve completeness. In the same way, the conditions required by Belnap’s metatheorem ensure that *one and the same* given set of transformation steps is enough to achieve Gentzen-style cut elimination for any system satisfying them.²

In what follows, we review and discuss eight conditions which are stronger in certain respects than those in [Bel82],³ and which define the notion of *proper display calculus* in [Wan98].⁴

C₁: Preservation of formulas. This condition requires each formula occurring in a premise of a given inference to be the subformula of some formula in the conclusion of that inference. That is, structures may disappear, but not formulas. This condition is not included in the list of sufficient conditions of the cut elimination metatheorem, but, in the presence of cut elimination, it guarantees the subformula property of a system. Condition C₁ can be verified by inspection on the shape of the rules.

C₂: Shape-alikeness of parameters. This condition is based on the relation of *congruence* between *parameters* (i.e., non-active parts) in inferences; the congruence relation is an equivalence relation which is meant to identify the different occurrences of the same formula or substructure along the branches of a derivation [Bel82, Section 4], [Res00, Definition 6.5]. Condition C₂ requires that congruent parameters be occurrences of the same structure. This can be understood as a condition on the *design* of the rules of the system if the congruence relation is understood as part of the specification of each

²The relationship between canonicity and Belnap-style cut elimination is in fact more than a mere analogy, see [Kra96, Theorem 20].

³See also [Bel90, Res00] and the ‘second formulation’ of condition C6/7 in Section 4.4 of [Wan98].

⁴See the ‘first formulation’ of conditions C6, C7 in Section 4.1 of [Wan98].


given rule; that is, each rule of the system comes with an explicit specification of which elements are congruent to which (and then the congruence relation is defined as the reflexive and transitive closure of the resulting relation). In this respect, C_2 is nothing but a sanity check, requiring that the congruence is defined in such a way that indeed identifies the occurrences which are intuitively “the same”.

C_3 : Non-proliferation of parameters. Like the previous one, also this condition is actually about the definition of the congruence relation on parameters. Condition C_3 requires that, for every inference (i.e. rule application), each of its parameters is congruent to at most one parameter in the conclusion of that inference. Hence, the condition stipulates that for a rule such as the following,

$$\frac{X \vdash Y}{X, X \vdash Y}$$

the structure X from the premise is congruent to *only one* occurrence of X in the conclusion sequent. Indeed, the introduced occurrence of X should be considered congruent only to itself. Moreover, given that the congruence is an equivalence relation, condition C_3 implies that, within a given sequent, any substructure is congruent only to itself.

Remark 2.2. Conditions C_2 and C_3 make it possible to follow the history of a formula along the branches of any given derivation. In particular, C_3 implies that the the history of any formula within a given derivation has the shape of a tree, which we refer to as the *history-tree* of that formula in the given derivation. Notice, however, that the history-tree of a formula might have a different shape than the portion of the underlying derivation corresponding to it; for instance, the following application of the Contraction rule gives rise to a bifurcation of the history-tree of A which is absent in the underlying branch of the derivation tree, given that Contraction is a unary rule.

$$\frac{\begin{array}{c} \vdots \\ A, A \vdash X \end{array}}{A \vdash X}$$


C_4 : Position-alikeness of parameters. This condition bans any rule in which a (sub)structure in precedent (resp. succedent) position in a premise is congruent to a (sub)structure in succedent (resp. precedent) position in the conclusion.

C_5 : Display of principal constituents. This condition requires that any principal occurrence be always either the entire antecedent or the entire consequent part of the sequent

in which it occurs. In the following section, a generalization of this condition will be discussed, in view of its application to the main focus of interest of the present chapter.

The following conditions C_6 and C_7 are not reported below as they are stated in the original paper [Bel82], but as they appear in [Wan98, Section 4.1]. More about this difference is discussed in Section 2.7.2.

C_6 : Closure under substitution for succedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in succedent position. Condition C_6 ensures, for instance, that if the following inference is an application of the rule R :

$$\frac{(X \vdash Y)([A]_i^{suc} \mid i \in I)}{(X' \vdash Y')[A]^{suc}} R$$

and $([A]_i^{suc} \mid i \in I)$ represents all and only the occurrences of A in the premiss which are congruent to the occurrence of A in the conclusion⁵, then also the following inference is an application of the same rule R :

$$\frac{(X \vdash Y)([Z/A]_i^{suc} \mid i \in I)}{(X' \vdash Y')[Z/A]^{suc}} R$$

where the structure Z is substituted for A .

This condition caters for the step in the cut elimination procedure in which the cut needs to be “pushed up” over rules in which the cut-formula in succedent position is parametric. Indeed, condition C_6 guarantees that, in the picture below, a well-formed subtree $\pi_1[Y/A]$ can be obtained from π_1 by replacing any occurrence of A corresponding to a node in the history tree of the cut-formula A by Y , and hence the following transformation step is guaranteed go through uniformly and “canonically”:

$$\frac{\begin{array}{c} \vdots \pi'_1 \\ X' \vdash A \end{array} \quad \frac{\begin{array}{c} \vdots \pi_1 \\ X \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ A \vdash Y \end{array}}{X \vdash Y}}{\frac{\begin{array}{c} \vdots \pi'_1 \\ X' \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ A \vdash Y \end{array}}{X' \vdash Y}} \rightsquigarrow \frac{\begin{array}{c} \vdots \pi'_1 \\ X' \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ A \vdash Y \end{array}}{X' \vdash Y} \quad \begin{array}{c} \vdots \pi_1 \\ X \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ A \vdash Y \end{array}}{X \vdash Y} \quad \begin{array}{c} \vdots \pi_1[Y/A] \\ X \vdash Y \end{array}$$

if each rule in π_1 verifies condition C_6 .

⁵Clearly, if $I = \emptyset$, then the occurrence of A in the conclusion is congruent to itself.

C₇: Closure under substitution for precedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in precedent position. Condition C₇ can be understood analogously to C₆, relative to formulas in precedent position. Therefore, for instance, if the following inference is an application of the rule R :

$$\frac{(X \vdash Y)([A]_i^{pre} \mid i \in I)}{(X' \vdash Y')[A]^{pre}} R$$

then also the following inference is an instance of R :

$$\frac{(X \vdash Y)([Z/A]_i^{pre} \mid i \in I)}{(X' \vdash Y')[Z/A]^{pre}} R$$

Similarly to what has been discussed for condition C₆, condition C₇ caters for the step in the cut elimination procedure in which the cut needs to be “pushed up” over rules in which the cut-formula in precedent position is parametric.

C₈: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C₈ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of cut involving proper subformulas of the original cut-formulas.

Rules introducing logical connectives. In display calculi, these rules, sometimes referred to as *operational rules* as opposed to the structural rules, typically occur in two flavors: operational rules which translate one structural connective in the premises in the corresponding connective in the conclusion, and operational rules in which both the operational connective and its structural counterpart are introduced in the conclusion. An example of this pattern is provided below for the case of the modal operator ‘diamond’:

$$\frac{\circ A \vdash X}{\diamond A \vdash X} \diamond_L \quad \frac{X \vdash A}{\circ X \vdash \diamond A} \diamond_R$$

This introduction pattern obeys very strict criteria, which will be expanded on in the next subsection. From this example, it is clear that the introduction rules capture the

rock bottom behavior of the logical connective in question; additional properties (for instance, normality, in the case in point), which might vary depending on the logical system, are to be captured at the level of additional (purely structural) rules. This enforces a clear-cut division of labour between operational rules, which only encode the basic proof-theoretic meaning of logical connectives, and structural rules, which account for all extra relations and properties, and which can be modularly added or removed, thus accounting for the space of logics.

Summing up, the two main benefits of display calculi are a “canonical” proof of cut elimination, and an explicit and modular account of logical connectives.

2.2.3 Wansing’s criteria

In [Wan98, subsection 1.3], referring to the well known idea that ‘a proof-theoretic semantics exemplifies the Wittgensteinian slogan that meaning is use’, Wansing stresses that, for this slogan to serve as a conceptual basis for a general inferential theory of meaning, ‘use’ should be understood as ‘*correct use*’. The consequences of the idea of meaning as correct use then precipitate into the following principles for the introduction rules for operational connectives, which he discusses in the same subsection and which are reported below. These principles are hence to be understood as the general requirements a (sequent-style) proof system needs to satisfy in order to encode the correct use, and hence for being suitable for proof-theoretic semantics.

Separation. This principle requires that the introduction rules for a given connective f should not exhibit any other connective rather than f . Hence the meaning of a given operational connective cannot be dependent from any other operational connectives. For instance, the following rule does not satisfy *separation*:

$$\frac{\Box\Gamma \vdash A, \Diamond\Delta}{\Box\Gamma \vdash \Box A, \Diamond\Delta}$$

This criterion does not ban the possibility of defining composite connectives; however, it ensures that the dependence relation between connectives creates no vicious circles. In fact, as it is formulated, this criterion is much stronger, since it requires every connective to be independent of any other.

Isolation. This is a stronger requirement than separation, and stipulates that, in addition, the precedent (resp. succedent) of the conclusion sequent in a left (resp. right) introduction rule must not exhibit any structure operation. In [Bel82], Belnap explains

this requirement by remarking that an introduction rule with nonempty context on the principal side would fail to account for the meaning of the logical connective involved in a context-independent way.

Segregation. This is an even stronger requirement than isolation, and stipulates that, in addition, also the auxiliary formulas in the premise(s) must occur within an empty context. This property appears under the name of *visibility* in [BFS00].⁶

Weak symmetry. This requirement stipulates that each introduction rule for a given connective f should either belong to a set of rules $(f \vdash)$ which introduce f on the left-hand side of the turnstile \vdash in the conclusion sequent, or to a set of rules $(\vdash f)$ which introduce f on the right-hand side of the turnstile \vdash in the conclusion sequent. Understanding the either-or as exclusive disjunction, this criterion prevents an operational connective to be introduced on both sides by the application of one and the same rule. Thus, weak symmetry stipulates that the sets $(f \vdash)$ and $(\vdash f)$ be disjoint. However, weak symmetry does not exclude that either $(f \vdash)$ or $(\vdash f)$ be empty.

Symmetry. This condition strengthens weak symmetry by requiring both $(f \vdash)$ and $(\vdash f)$ to be nonempty for each connective f . Rather than a requirement on individual rules, this principle is a requirement on the set of the introduction rules for any given connective. Notice that symmetry does not exclude the possibility of having, for instance, two rules that introduce a given connective on the left and one that introduces it on the right side of the turnstile.

Weak explicitness. An introduction rule for f is *weakly explicit* if f occurs only in the conclusion of a rule and not in its premisses.

Explicitness. An introduction rule for f is *explicit* if it is weakly explicit and in addition to this, f appears *only once* in the conclusion of the rule.

The following principles are of a more global nature, which involves the proof system as a whole:

⁶In Chapter 4, following ideas from [BFS00], the visibility property is identified as an essential ingredient to generalise Belnap's metatheorem beyond display calculi.

Unique characterization. This principle requires each logical connective to be uniquely characterized by its behaviour in the system, in the following sense. Let Λ be a logical system with a syntactic presentation S in which f occurs. Let S^* be the result of rewriting f everywhere in S as f^* , and let $\Lambda\Lambda^*$ be the system presented by the union SS^* of S and S^* in the combined language with both f and f^* . Let A_f denote a formula (in this language) that contains a certain occurrence of f , and let A_{f^*} denote the result of replacing this occurrence of f in A_f by f^* . The connectives f and f^* are *uniquely characterized* in $\Lambda\Lambda^*$ (cfr. [Wan98, Subsubsection 1.4]) if for every formula A_f in the language of $\Lambda\Lambda^*$, A_f is provable in SS^* iff A_{f^*} is provable in SS^* .

Došen’s principle. Hilbert style presentations are modular in the following sense: if Λ_1 and Λ_2 are finitely axiomatizable logics over the same language and Λ_1 is stronger than Λ_2 , then an axiomatization of Λ_2 can be obtained from one of Λ_1 by adding finitely many axioms to it. This makes it possible to modularly generate all finite axiomatic extensions of a given logic. Although it is arguably more difficult to achieve an analogous degree of modularity in the sequent calculi presentation, a principle aimed to achieve it has been advocated by Wansing under the name of *Došen’s principle* (cfr. [Wan98, Subsubsection 1.5]): “The rules for the logical operations are never changed; all changes are made in the structural rules”. Thus, suitable finite axiomatic extensions of a given logic L can be captured by adding structural rules to the proof system associated with L . Display calculi are particularly suitable to implement Došen’s principle. As remarked early on, besides featuring structural rules which encode properties of single structural connectives (which is the case e.g. of the rule exchange), display calculi typically feature rules which concern the interaction between different structural connectives (the adjunction between two structural connectives is an example of the latter type of rule, see for instance the rules applied in the example on page 69).

Cut-eliminability. Finally, Wansing considers the eliminability of the *cut* rule as an important requirement for the proof-theoretic semantics of logical connectives.

2.3 Belnap-style metatheorem for quasi-proper display calculi

In the present section, we discuss a slight extension of Wansing’s notion of proper display calculus (cf. Section 2.2.2), and prove its associated Belnap-style cut elimination metatheorem. The cut elimination for the calculus D’.EAK introduced in Section 2.6.3 (see also Appendix D) will be derived as an instance of the metatheorem below.

2.3.1 Quasi-proper display calculi

Definition 2.3. A sequent calculus is a *quasi proper display calculus* if it verifies conditions C_1 , C_2 , C_3 , C_4 , C_6 , C_7 , C_8 of Section 2.2.2, and moreover it satisfies the following conditions C'_5 , C''_5 and C''_8 :

C'_5 : Quasi-display of principal constituents. If a formula A is principal in the conclusion sequent s of a derivation π , then A is in display, unless π consists only of its conclusion sequent s (i.e. s is an axiom).

C''_5 : Display-invariance of axioms. If a display rule can be applied to an axiom s , the result of that rule application is again an axiom.

C''_8 : Closure of axioms under cut. If $X \vdash A$ and $A \vdash Y$ are axioms, then $X \vdash Y$ is again an axiom.

Notice that condition C_5 in Section 2.2.2 is stronger than both C'_5 and C''_5 , and that the strength of condition C'_5 is intermediate between that of C_5 and of the following one, appearing in [Res00, Definition 6.8]:

C'''_5 : Single principal constituents. This condition requires that, in the conclusion of any rule, there be at most one *non-parametric* formula—which is the formula introduced by the application of the rule in question—unless the rule is an axiom.

The above condition C'''_5 is introduced in [Res00] within a setting accounting for sequent calculi which do not necessarily enjoy the full display property. The calculi considered in [Res00] are such that the introduction rules do not need to enjoy the requirement of isolation (cf. Chapter 6), and the (multiple) cut rule applies at any depth. The calculus introduced in Section 2.6.1 enjoys the full display property, therefore the following cut rule, in which both cut formulas occur in isolation:

$$\frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \textit{Cut}$$

will be taken as primitive in it without loss of generality, as is standardly done in display calculi. However, the calculus in Section 2.6.1 fails to enjoy the property of *isolation*, which typically plays a role in the cut elimination metatheorem for display calculi, and

indeed appears in [Wan98] as condition C_5 . In the next subsection, we show that, even when the cut rule is the one above, requiring the combination of C'_5 and C''_5 suffices.⁷

2.3.2 Belnap-style metatheorem

The aim of the present subsection is to prove the following theorem:

Theorem 2.4. *Any calculus satisfying conditions $C_2, C_3, C_4, C'_5, C''_5, C_6, C_7, C_8$, and C''_8 enjoys cut elimination. If C_1 is also satisfied, then the calculus enjoys the subformula property.*

Proof. This is a generalization of the proof in [Wan02, Section 3.3, Appendix A]. For the sake of conciseness, we will expand only on the parts of the proof which depart from that treatment.

Our original derivation is

$$\frac{\begin{array}{c} \vdots \pi_1 \\ X \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ A \vdash Y \end{array}}{X \vdash Y}$$

Principal stage: both cut formulas are principal. There are three subcases.

If the end sequent $X \vdash Y$ is identical to the conclusion of π_1 (resp. π_2), then we can eliminate the cut simply replacing the derivation above with π_1 (resp. π_2).

If the premises $X \vdash A$ and $A \vdash Y$ are axioms, then, by C''_8 , the conclusion $X \vdash Y$ is an axiom, therefore the cut can be eliminated by simply replacing the original derivation with $X \vdash Y$.

If one of the two premises of the cut in the original derivation is not an axiom, then, by C_8 , there is a proof of $X \vdash Y$ which uses the same premise(s) of the original derivation and which involves only cuts on proper subformulas of A .

Parametric stage: at least one cut formula is parametric. There are two subcases: either one cut formula is principal or they are both parametric.

Consider the subcase in which one cut formula is principal. W.l.o.g. we assume that the cut-formula A is principal in the the left-premise $X \vdash A$ of the cut in the original proof (the other case is symmetric). As discussed in Remark 2.2, conditions C_2 and C_3 make it possible to consider the history-tree of the right-hand-side cut formula A in π_2 . The situation can be pictured as follows:

⁷In Chapter 3, we give a metatheorem which is based on a different tradeoff: on the one hand, we will not require the full display property, but on the other hand we will require a condition close to segregation.

$$\begin{array}{c}
\begin{array}{ccc}
\vdots \pi_{2,i} & \vdots \pi_{2,j} & \vdots \pi_{2,k} \\
\underline{A}_i \vdash Y_i & (X_j \vdash Y_j)[\underline{A}_j]^{pre} & (X_k \vdash Y_k)[\overline{A}_k]^{pre} \\
\ddots & \vdots & \ddots \\
\vdots \pi_1 & \ddots \vdots \pi_2 & \\
\hline
X \vdash A & & A \vdash Y \\
X \vdash Y & &
\end{array}
\end{array}$$

where, for $i, j, k \in \{1, \dots, n\}$, the nodes

$$\underline{A}_i \vdash Y_i, \quad (X_j \vdash Y_j)[\underline{A}_j]^{pre}, \quad \text{and} \quad (X_k \vdash Y_k)[\overline{A}_k]^{pre}$$

represent the three ways in which the leaves A_i , A_j and A_k in the history-tree of A in π_2 can be introduced, and which will be discussed below. The notation \underline{A} and (resp. \overline{A}) indicates that the given occurrence is principal (resp. parametric). Notice that condition C_4 guarantees that all occurrences in the history of A are in precedent position in the underlying derivation tree.

Let A_l be introduced as a parameter (as represented in the picture above in the conclusion of $\pi_{2,k}$ for $A_l = A_k$). Assume that $(X_k \vdash Y_k)[\overline{A}_k]$ is the conclusion of an application *inf* of the rule *Ru* (for instance, in the calculus of Section 2.6.1, this situation arises if A_k has been introduced with an application of Weakening). Since A_k is a leaf in the history-tree of A , we have that A_k is congruent only to itself in $X_k \vdash Y_k$. Hence, C_7 implies that it is possible to substitute X for A_k by means of an application of the same rule *Ru*. That is, $(X_k \vdash Y_k)[\overline{A}_k]$ can be replaced by $(X_k \vdash Y_k)[X/\overline{A}_k]$.

Let A_l be introduced as a principal formula. The corresponding subcase in [Wan02] splits into two subsubcases: either A_l is introduced in display or it is not.

If A_l is in display (as represented in the picture above in the conclusion of $\pi_{2,i}$ for $A_l = A_i$), then we form a subderivation using π_1 and $\pi_{2,i}$ and applying cut as the last rule.

If A_l is not in display (as represented in the picture above in the conclusion of $\pi_{2,j}$ for $A_l = A_j$), then condition C'_5 implies that $(X_j \vdash Y_j)[\underline{A}_j]^{pre}$ is an axiom (so, in particular, there is at least another occurrence of A in succedent position), and C''_5 implies that some axiom $\underline{A}_j \vdash Y'_j$ exists, which is display-equivalent to the first axiom, and in which A_j occurs in display. Let π' be the derivation which transforms $\underline{A}_j \vdash Y'_j$ into $(X_j \vdash Y_j)[\underline{A}_j]^{pre}$. We form a subderivation using π_1 and $\underline{A}_j \vdash Y'_j$ and joining them with a cut application, then attaching $\pi'[X/A_j]^{pre}$ below the new cut.

The transformations just discussed explain how to transform the leaves of the history tree of A . Finally, condition C_7 implies that substituting X for each occurrence of A in the history tree of

the cut formula A in π_2 (or in a display-equivalent proof π') gives rise to an admissible derivation $\pi_2[X/A]^{pre}$ (use C_6 for the symmetric case).

Summing up, this procedure generates the following proof tree:

$$\begin{array}{c}
 \vdots \pi_1 \\
 X \vdash A \quad \frac{\underline{A}_j \vdash Y'[\underline{A}]^{suc}}{X \vdash Y'[\underline{A}]^{suc}} \\
 \hline
 \vdots \pi_1 \quad \vdots \pi_{2,i} \\
 \frac{X \vdash A \quad \underline{A}_i \vdash Y_i}{X \vdash Y_i} \quad \vdots \pi' [X/A]^{pre} \quad \vdots \pi_{2,k} \\
 \frac{\quad \quad \quad (X_j \vdash Y_j)[X/A_j]^{pre}[\underline{A}]^{suc} \quad (X_k \vdash Y_k)[\overline{X}/\overline{A}_k]^{pre}}{\quad \quad \quad \vdots} \\
 \vdots \\
 \vdots \vdots \vdots \pi_2[X/A]^{pre} \\
 \hline
 X \vdash Y
 \end{array}$$

If, in the original derivation, the history-tree of the cut formula A (in the right-hand-side premise of the given cut application) contains at most one leaf A_l which is principal, then the height of the new cuts is lower than the height of the original cut.

If, in the original derivation, the history-tree of the cut formula A (in the right-hand-side premise of the given cut application) contains more than one leaf A_l which is principal, then we cannot conclude that the height of the new cuts is always lower than the height of the original cut (for instance, in the calculus introduced in Section 2.6.1, this situation may arise when two ancestors of a cut formula are introduced as principal, and then are identified via an application of the rule Contraction). In this case, we observe that in each newly introduced application of the cut rule, both cut formulas are principal. Hence, we can apply the procedure described in the Principal stage and transform the original derivation in a derivation in which the cut formulas of the newly introduced cuts have strictly lower complexity than the original cut formula.

Finally, as to the subcase in which both cut formulas are parametric, consider a proof with at least one cut. The procedure is analogous to the previous case. Namely, following the history of one of the cut formulas up to the leaves, and applying the transformation steps described above, we arrive at a situation in which, whenever new applications of cuts are generated, in each such application at least one of the cut formulas is principal. To each such cut, we can apply (the symmetric version of) the Parametric stage described so far. \square

2.4 Dynamic Epistemic Logics and their proof systems

In the present section, we first review the two best known logical systems in the family of dynamic epistemic logics, namely public announcement logic (PAL) [Pla07], and the logic of epistemic actions and knowledge (EAK) [BMS99], focusing mainly on the latter one. Our presentation in Subsection 2.4.1 is different but equivalent to the original version from [BMS99] (without common knowledge), and rather follows the presentation given in [MPS13] and in [GKP13]. In Subsection 2.4.2, we present the intuitionistic version of EAK. In Subsections 2.4.3 and 2.4.4 we discuss their existing proof-theoretic formalizations, particularly in relation to the viewpoint of proof-theoretic semantics, and mention the system D.EAK as a promising approximation of a setting for proof-theoretic semantics. Finally, in Subsection 2.5, we discuss the final coalgebra semantics, since this is a semantic environment in which all connectives of the language of D.EAK (and of its improved version D'.EAK) can be naturally interpreted.

2.4.1 The logic of epistemic actions and knowledge

The logic of epistemic actions and knowledge (further on EAK) is a logical framework which combines a multi-modal classical logic with a dynamic-type propositional logic. Static modalities in EAK are parametrized with agents, and their intended interpretation is epistemic, that is, $\langle a \rangle A$ intuitively stands for ‘agent a thinks that A might be the case’. Dynamic modalities in EAK are parametrized with epistemic *action-structures* (defined below) and their intended interpretation is analogous to that of dynamic modalities in e.g. Propositional Dynamic Logic. That is, $\langle \alpha \rangle A$ intuitively stands for ‘the action α is executable, and after its execution A is the case’. Informally, action structures loosely resemble Kripke models, and encode information about epistemic actions such as e.g. public announcements, private announcements to a group of agents, with or without (actual or suspected) wiretapping, etc. Action structures consist of a finite nonempty domain of action-states, a designated state, binary relations on the domain for each agent, and a precondition map. Each state in the domain of an action structure α represents the possible appearance of the epistemic action encoded by α . The designated state represents the action actually taking place. Each binary relation of an action structure represents the type, or degree, of uncertainty entertained by the agent associated with the given binary relation about the action taking place; for instance, the agents’ knowledge, ignorance, suspicions. Finally, the precondition function maps each state in the domain to a formula, which is intended to describe the state of affairs under which it is possible to execute the (appearing) action encoded by the given state. This formula

encodes the *preconditions* of the action-state. The reader is referred to [BMS99] for further intuition and concrete examples.

Let AtProp be a countable set of atomic propositions, and Ag be a nonempty set (of agents). The set \mathcal{L} of formulas A of the logic of epistemic actions and knowledge (EAK), and the set $\text{Act}(\mathcal{L})$ of the *action structures* α over \mathcal{L} are defined simultaneously as follows:

$$A := p \in \text{AtProp} \mid \neg A \mid A \vee A \mid \langle a \rangle A \mid \langle \alpha \rangle A \quad (\alpha \in \text{Act}(\mathcal{L}), a \in \text{Ag}),$$

where an *action structure* over \mathcal{L} is a tuple $\alpha = (K, k, (\alpha_a)_{a \in \text{Ag}}, \text{Pre}_\alpha)$, such that K is a finite nonempty set, $k \in K$, $\alpha_a \subseteq K \times K$ and $\text{Pre}_\alpha : K \rightarrow \mathcal{L}$.

The symbol $\text{Pre}(\alpha)$ stands for $\text{Pre}_\alpha(k)$. For each action structure α and every $i \in K$, let $\alpha_i := (K, i, (\alpha_a)_{a \in \text{Ag}}, \text{Pre}_\alpha)$. Intuitively, the family of action structures $\{\alpha_i \mid k\alpha_a i\}$ encodes the uncertainty of agent a about the action $\alpha = \alpha_k$ that is actually taking place. Perhaps the best known epistemic actions are *public announcements*, formalized as action structures α such that $K = \{k\}$, and $\alpha_a = \{(k, k)\}$ for all $a \in \text{Ag}$. The logic of public announcements (PAL) [Pla07] can then be subsumed as the fragment of EAK restricted to action structures of the form described above. The connectives \top , \perp , \wedge , \rightarrow and \leftrightarrow are defined as usual.

Standard models for EAK are relational structures $M = (W, (R_a)_{a \in \text{Ag}}, V)$ such that W is a nonempty set, $R_a \subseteq W \times W$ for each $a \in \text{Ag}$, and $V : \text{AtProp} \rightarrow \mathcal{P}(W)$. The interpretation of the static fragment of the language is standard. For every Kripke frame $\mathcal{F} = (W, (R_a)_{a \in \text{Ag}})$ and each action structure α , let the Kripke frame $\coprod_\alpha \mathcal{F} := (\coprod_K W, ((R \times \alpha)_a)_{a \in \text{Ag}})$ be defined as follows: $\coprod_K W$ is the $|K|$ -fold coproduct of W (which is set-isomorphic to $W \times K$), and $(R \times \alpha)_a$ is a binary relation on $\coprod_K W$ defined as

$$(w, i)(R \times \alpha)_a(u, j) \quad \text{iff} \quad wR_a u \quad \text{and} \quad i\alpha_a j.$$

For every model M and each action structure α , let

$$\coprod_\alpha M := (\coprod_\alpha \mathcal{F}, \coprod_K V)$$

be such that $\coprod_\alpha \mathcal{F}$ is defined as above, and $(\coprod_K V)(p) := \coprod_K V(p)$ for every $p \in \text{AtProp}$. Finally, let the *update* of M with the action structure α be the submodel

$$M^\alpha := (W^\alpha, (R_a^\alpha)_{a \in \text{Ag}}, V^\alpha)$$

of $\coprod_{\alpha} M$ the domain of which is the subset

$$W^{\alpha} := \{(w, j) \in \coprod_K W \mid M, w \Vdash \text{Pre}_{\alpha}(j)\}.$$

Given this preliminary definition, formulas of the form $\langle \alpha \rangle A$ are interpreted as follows:

$$M, w \Vdash \langle \alpha \rangle A \quad \text{iff} \quad M, w \Vdash \text{Pre}_{\alpha}(k) \text{ and } M^{\alpha}, (w, k) \Vdash A.$$

The model M^{α} is intended to encode the (factual and epistemic) state of affairs after the execution of the action α . Summing up, the construction of M^{α} is done in two stages: in the first stage, as many copies of the original model M are taken as there are ‘epistemic potential appearances’ of the given action (encoded by the action states in the domain of α); in the second stage, states in the copies are removed if their associated original state does not satisfy the preconditions of their paired action-state.

A complete axiomatization of EAK consists of copies of the axioms and rules of the minimal normal modal logic K for each modal operator, either epistemic or dynamic, plus the following (interaction) axioms:

$$\langle \alpha \rangle p \leftrightarrow (\text{Pre}(\alpha) \wedge p); \tag{2.1}$$

$$\langle \alpha \rangle \neg A \leftrightarrow (\text{Pre}(\alpha) \wedge \neg \langle \alpha \rangle A); \tag{2.2}$$

$$\langle \alpha \rangle (A \vee B) \leftrightarrow (\langle \alpha \rangle A \vee \langle \alpha \rangle B); \tag{2.3}$$

$$\langle \alpha \rangle \langle \mathbf{a} \rangle A \leftrightarrow (\text{Pre}(\alpha) \wedge \bigvee \{\langle \mathbf{a} \rangle \langle \alpha_i \rangle A \mid k\alpha_{\mathbf{a}}i\}). \tag{2.4}$$

The interaction axioms above can be understood as attempts at defining the meaning of any given dynamic modality $\langle \alpha \rangle$ in terms of its interaction with the other connectives. In particular, while axioms (2.2) and (2.3) occur also in other dynamic logics such as PDL, axioms (2.1) and (2.4) capture the specific behaviour of epistemic actions. Specifically, axiom (2.1) encodes the fact that epistemic actions do not change the factual state of affairs, and axiom (2.4) plausibly rephrases the fact that ‘after the execution of α , agent \mathbf{a} thinks that A might be the case’ in terms of ‘there being some epistemic appearance of α to \mathbf{a} such that \mathbf{a} thinks that, after its execution, A is the case’. An interesting aspect of these axioms is that they work as rewriting rules which can be iteratively used to transform any EAK-formula into an equivalent one free of dynamic modalities. Hence, the completeness of EAK follows from the completeness of its static fragment, and EAK is not more expressive than its static fragment. However, and interestingly, there is an exponential gap in succinctness between equivalent formulas in the two languages [Lut06].

Action structures are one among many possible ways to represent actions. Following [GKP13], we prefer to keep a black-box perspective on actions, and to identify agents with the indistinguishability relation they induce on actions; so, in the remainder of the article, the role of the action-structures α_i for *kai* will be played by actions β such that $\alpha\beta$, allowing us to reformulate (2.4) as

$$\langle\alpha\rangle\langle a\rangle A \leftrightarrow (Pre(\alpha) \wedge \bigvee \{\langle a\rangle\langle\beta\rangle A \mid \alpha\beta\}).$$

2.4.2 The intuitionistic version of EAK

In [MPS13, KP13], an analysis of PAL and EAK has been given from the point of view of algebraic semantics, resulting in the definition of the intuitionistic counterparts of PAL and EAK. In the present subsection, we briefly review the definition of the latter one, as it reveals a more subtle interaction between the various modalities, thus preparing the ground for the even richer picture that will arise from the proof-theoretic analysis.

Let $AtProp$ be a countable set of atomic propositions, and let Ag be a nonempty set (of agents). The set $\mathcal{L}(m\text{-IK})$ of the formulas A of the multi-modal version $m\text{-IK}$ of Fischer Servi's intuitionistic modal logic IK are inductively defined as follows:

$$A := p \in AtProp \mid \perp \mid A \vee A \mid A \wedge A \mid A \rightarrow A \mid \langle a\rangle A \mid [a]A \quad (a \in Ag)$$

The logic $m\text{-IK}$ is the smallest set of formulas in the language $\mathcal{L}(m\text{-IK})$ (where $\neg A$ abbreviates as usual $A \rightarrow \perp$) containing the axioms in Table 2.1 (page 85) and closed under modus ponens and necessitation rules.

To define the language of the intuitionistic counterpart of EAK, let $AtProp$ be a countable set of atomic propositions, and let Ag be a nonempty set. The set $\mathcal{L}(IEAK)$ of the formulas A of the intuitionistic logic of epistemic actions and knowledge (IEAK), and the set $Act(\mathcal{L})$ of the *action structures* α over \mathcal{L} are defined simultaneously as follows:

$$A := p \in AtProp \mid \perp \mid A \rightarrow A \mid A \vee A \mid A \wedge A \mid \langle a\rangle A \mid [a]A \mid \langle \alpha\rangle A \mid [\alpha]A,$$

where $a \in Ag$, and an *action structure* α over $\mathcal{L}(IEAK)$ is defined in just the same way as action structures in Section 2.4.1. Then, the logic IEAK is defined in a Hilbert-style presentation which includes the axioms and rules of $m\text{-IK}$ plus the Fischer Servi axioms FS1 and FS2 for each dynamic modal operator, plus the axioms and rules in Table 2.2 (page 86).

Axioms

$$\begin{aligned}
& A \rightarrow (B \rightarrow A) \\
& (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\
& A \rightarrow (B \rightarrow A \wedge B) \\
& A \wedge B \rightarrow A \\
& A \wedge B \rightarrow B \\
& A \rightarrow A \vee B \\
& B \rightarrow A \vee B \\
& (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C)) \\
& \perp \rightarrow A \\
& [a](A \rightarrow B) \rightarrow ([a]A \rightarrow [a]B) \\
& \langle a \rangle(A \vee B) \rightarrow \langle a \rangle A \vee \langle a \rangle B \\
& \neg \langle a \rangle \perp \\
\text{FS1} \quad & \langle a \rangle(A \rightarrow B) \rightarrow ([a]A \rightarrow \langle a \rangle B) \\
\text{FS2} \quad & \langle \langle a \rangle A \rightarrow [a]B \rangle \rightarrow [a](A \rightarrow B)
\end{aligned}$$

Inference Rules

$$\begin{aligned}
\text{MP} \quad & \text{if } \vdash A \rightarrow B \text{ and } \vdash A, \text{ then } \vdash B \\
\text{Nec} \quad & \text{if } \vdash A, \text{ then } \vdash [a]A
\end{aligned}$$

TABLE 2.1: Axioms and inference rules of the logic m- IK **2.4.3 Proof theoretic formalisms for PAL and DEL**

In the present subsection, we discuss the most relevant existing proof-theoretic accounts [BvdHdL10, NM10, NM11, BCS07, DST13, Auc10, Auc11, AMS12] for the logic of public announcements [Pla07] and for the logic of epistemic actions and knowledge [BMS99].

Labelled tableaux for PAL. In [BvdHdL10], a labelled tableaux system is proposed for public announcement logic. This system is sound and complete with respect to the semantics of PAL. Moreover, the computational complexity of this tableaux system is shown to be optimal for satisfiability checking in the language of PAL. The system manipulates triples, called labelled formulas, of the form $\langle \mu, n, \phi \rangle$ such that μ is a (possibly empty) list of PAL-formulas, n is a natural number, and ϕ is a PAL-formula. Intuitively, the tuple $\langle \mu, n \rangle$ stands for an epistemic state of the model updated with a sequence of announcements encoded by μ . To give a closer impression of this tableaux system, consider the following rule:

$$R\widehat{K} \frac{\langle (\alpha_1, \dots, \alpha_k), n, \neg K_a A \rangle}{\langle \epsilon, n', \neg[\alpha_1] \dots [\alpha_k] A \rangle : \langle a, n, n' \rangle} \quad n' \text{ fresh}$$

Interaction Axioms

$$\begin{aligned}\langle \alpha \rangle p &\leftrightarrow Pre(\alpha) \wedge p \\ [\alpha] p &\leftrightarrow Pre(\alpha) \rightarrow p\end{aligned}$$

$$\begin{aligned}\langle \alpha \rangle \perp &\leftrightarrow \perp \\ \langle \alpha \rangle \top &\leftrightarrow Pre(\alpha) \\ [\alpha] \top &\leftrightarrow \top \\ [\alpha] \perp &\leftrightarrow \neg Pre(\alpha)\end{aligned}$$

$$\begin{aligned}[\alpha](A \wedge B) &\leftrightarrow [\alpha]A \wedge [\alpha]B \\ \langle \alpha \rangle(A \wedge B) &\leftrightarrow \langle \alpha \rangle A \wedge \langle \alpha \rangle B\end{aligned}$$

$$\begin{aligned}\langle \alpha \rangle(A \vee B) &\leftrightarrow \langle \alpha \rangle A \vee \langle \alpha \rangle B \\ [\alpha](A \vee B) &\leftrightarrow Pre(\alpha) \rightarrow (\langle \alpha \rangle A \vee \langle \alpha \rangle B)\end{aligned}$$

$$\begin{aligned}\langle \alpha \rangle(A \rightarrow B) &\leftrightarrow Pre(\alpha) \wedge (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \\ [\alpha](A \rightarrow B) &\leftrightarrow \langle \alpha \rangle A \rightarrow \langle \alpha \rangle B\end{aligned}$$

$$\begin{aligned}\langle \alpha \rangle \langle a \rangle A &\leftrightarrow Pre(\alpha) \wedge \bigvee \{ \langle a \rangle \langle \beta \rangle A \mid \alpha a \beta \} \\ [\alpha] \langle a \rangle A &\leftrightarrow Pre(\alpha) \rightarrow \bigvee \{ \langle a \rangle \langle \beta \rangle A \mid \alpha a \beta \}\end{aligned}$$

$$\begin{aligned}[\alpha] [a] A &\leftrightarrow Pre(\alpha) \rightarrow \bigwedge \{ [a] \langle \beta \rangle A \mid \alpha a \beta \} \\ \langle \alpha \rangle [a] A &\leftrightarrow Pre(\alpha) \wedge \bigwedge \{ [a] \langle \beta \rangle A \mid \alpha a \beta \}\end{aligned}$$

Inference Rules

$$\text{Nec} \quad \text{if } \vdash A, \text{ then } \vdash [\alpha]A$$

TABLE 2.2: Axioms and inference rules of the logic IEAK

This rule can be read as follows: if a state n does not satisfy $K_a A$ after the sequence of announcements $\alpha_1, \dots, \alpha_k$, then at least one of its R_a -successor states n' in the original model, represented by the tuple $\langle \epsilon, n' \rangle$ in the rule, must survive the updates and not satisfy A . Hence, $\langle \epsilon, n' \rangle$ must satisfy the formula $\langle \alpha_1 \rangle \dots \langle \alpha_k \rangle \neg A$, which is classically equivalent to $\neg [\alpha_1] \dots [\alpha_k] A$.

Clearly, rules such as this one incorporate the relational semantics of PAL. This is not satisfactory from the point of view of proof-theoretic semantics, since it prevents these rules from providing an independent contribution to the meaning of the logical connectives. A second issue, of a more technical nature, is that the statement of this rule is grounded on the classical interdefinability between the box-type and diamond-type modalities. This implies that if we dispense with the classical propositional base, we

would need to reformulate this rule. Hence the calculus is non-modular in the sense discussed in Section 2.2.3.

Labelled sequent calculi for PAL. In [NM10] and [NM11], cut-free labelled sequent calculi for PAL are introduced with truthful and non-truthful announcements, respectively. Also in this case, the statement of the rules of these calculi incorporates the relational semantics. For instance, this is illustrated here below for the case of truthful announcements.

$$\frac{w :^{\mu, \alpha} A, w :^{\mu} [\alpha]A, w :^{\mu} \alpha, \Gamma \vdash \Delta}{w :^{\mu} [\alpha]A, w :^{\mu} \alpha, \Gamma \vdash \Delta} L[] :^{\mu} \quad \frac{w :^{\mu} \alpha, \Gamma \vdash \Delta, w :^{\mu, \alpha} A}{\Gamma \vdash \Delta, w :^{\mu} [\alpha]A} R[] :^{\mu}$$

In the rules above, symbols such as $w :^{\mu} A$ can be rearranged and then understood as the labelled formulas $\langle \mu, w, A \rangle$ in the tableaux system presented before. The only difference is that w is an individual variable which stands for a given state of a relational structure, and not for a natural number; however, this difference is completely nonessential. Under this interpretation, it is clear that e.g. the rule $L[] :^{\mu}$ encodes the relational satisfaction clause of $[\alpha]A$, when α is a truthful announcement. The following rules are also part of the calculi.

$$\frac{v : A, w : K_a A, w R_a v, \Gamma \vdash \Delta}{w : K_a A, w R_a v, \Gamma \vdash \Delta} LK_a \quad \frac{w R_a v, \Gamma \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : K_a A} RK_a$$

Besides the individual variables w and v , the rules above feature the binary relation symbol R_a encoding the epistemic uncertainty of the agent a . Since the relational semantics is imported in the definitions of the rules, the same issues pointed out in the case of the tableaux system appear also here. On the other hand, importing the relational semantics allows for some remarkable extra power. Indeed, the interaction axiom (2.4) can be derived from the four rules above, which deal with static and dynamic modalities in complete independence of one another.

Merging different logics. In [BCS07] and [DST13], sequent calculi have been defined for dynamic logics arising in an algebraic way, motivated by program semantics, with a methodology introduced by [AV93]. Essentially, this approach is based on the idea of merging a linear-type logic of actions (more precisely, [Moo95]) with a classical or intuitionistic logic of propositions. Following the treatment of [AV93], this logic arises semantically as the logic of certain quantale-modules, namely of maps $\star : M \times Q \rightarrow M$, preserving complete joins in each coordinate, where Q is a quantale and M is a complete join-semilattice. Each $q \in Q$ induces a completely join-preserving operation $(-\star q) : M \rightarrow$

M , which, by general order-theoretic facts, has a unique right adjoint $[q] : M \rightarrow M$. That is, for every $m, m' \in M$,

$$m \star q \leq m' \text{ iff } m \leq [q]m'. \quad (2.5)$$

Intuitively, the elements of Q are actions (or rather, inverses of actions), and M is an algebra interpreting propositions, which in the best known cases arises as the complex algebra of some relational structure, and therefore will be e.g. a complete and atomic Boolean algebra with operators. Thus the framework of [BCS07] and [DST13] is vastly more general than dynamic epistemic logic as it is usually understood. A remarkable feature of this setting is that the dynamic operations which are intended as the interpretation of the primitive dynamic connectives arise in this setting as adjoints of “more primitive” operations; thus, and much more importantly, every dynamic modality comes with its adjoint. Moreover, every epistemic modality (parametrized as usual with an agent) comes in two copies: one as an operation on Q and one as an operation on M , and these two copies are stipulated to interact in a suitable way. More formally, the semantic structures are defined as tuples $(M, Q, \{f_A\}_{A \in Ag})$, where M and Q are as above, and for every agent A , f_A is a pair of completely join preserving maps $(f_A^M : M \rightarrow M, f_A^Q : Q \rightarrow Q)$ such that the following three conditions hold:

$$f_A^Q(q \cdot q') \leq f_A^Q(q) \cdot f_A^Q(q') \quad (2.6)$$

$$f_A^M(m \star q) \leq f_A^M(m) \star f_A^Q(q) \quad (2.7)$$

$$1 \leq f_A^Q(1). \quad (2.8)$$

Intuitively, for every agent A , the operation f_A^M is the diamond-type modal operator encoding the epistemic uncertainty of A , and f_A^Q is the diamond-type modal operator encoding the epistemic uncertainty of A about the action that is actually taking place. Given this understanding, condition (2.7) hardcodes the following well-known DEL-axiom in the semantic structures above:

$$\bigwedge \{ [A][q']m \mid qAq' \} \vdash [q][A]m. \quad (2.9)$$

where the notation qAq' means that the action q' is indistinguishable from q for the agent A . In (2.7), the element $f_A^Q(q)$ encodes the join of all such actions. Because \star is bilinear, we get:

$$f_A^M(m) \star f_A^Q(q) = f_A^M(m) \star \bigvee_Q \{ q' \mid qAq' \} = \bigvee_M \{ f_A^M(m) \star q' \mid qAq' \}.$$

Hence, (2.7) can be equivalently rewritten in the form of a rule as follows:

$$\frac{\bigvee \{f_A^M(m) \star q' \mid qAq'\} \vdash m'}{f_A^M(m \star q) \vdash m'}$$

Applying adjunction to the premise and to the conclusion gets us to:

$$\frac{m \vdash \wedge \{[A][q']m' \mid qAq'\}}{m \vdash [q][A]m'}$$

Finally, rewriting the rule above back as an inequality gets us to (2.9). The first pioneering proposal is the sequent calculus developed in [BCS07]. This calculus manipulates two kinds of sequents: Q-sequents, of the form $\Gamma \vdash_Q q$, where q is an action and Γ is a sequence of actions and agents, and M-sequents, of the form $\Gamma \vdash_M m$, where m is a proposition and Γ is a sequence of propositions, actions and agents. These different entailment relations need to be brought together by means of rules of hybrid type, such as the left one below.

$$\frac{m' \vdash_M m \quad \Gamma_Q \vdash_Q q}{[q]m', \Gamma_Q \vdash_M m} \text{DyL} \quad \frac{\Gamma, q \vdash_M m}{\Gamma \vdash_M [q]m} \text{DyR}$$

As to the soundness of the rule *DyL*, let us identify the logical symbols with their interpretation, assume that the inequalities $m \leq m'$ and $\Gamma_Q \leq q$ are satisfied on given M and Q respectively,⁸ and prove that $[q]m', \Gamma_Q \leq m$ in M . Indeed,

$$[q]m' \star \Gamma_Q \leq [q]m' \star q \leq m' \leq m.$$

The first inequality follows from $\Gamma_Q \leq q$ and \star being order-preserving in its second coordinate; the second inequality is obtained by applying the right-to-left direction of (2.5) to the inequality $[q]m' \leq [q]m'$; the last inequality holds by assumption. The soundness of *DyR* follows likewise from the left-to-right direction of (2.5).

This calculus is shown to be both sound and complete w.r.t. this algebraic semantics. The setting illustrated above is powerful enough that sufficiently many epistemic actions can be encoded in it to support the formalisation of various variants of the Muddy Children Puzzle in which children might be cheating. However, cut elimination for this system has not been proven.

In [DST13], a similar framework is presented which exploits the same basic ideas, and results in a system with more explicit proof-theoretic performances and which is shown

⁸where Γ_Q now stands for a suitable product in Q of the interpretations of its individual components.

to be cut-free. However, like its previous version, this system focuses on a logic semantically arising from an algebraic setting which is vastly more general than the usual relational setting. The issue about how it precisely restricts to the usual setting, and hence how the usual DEL-type logics can be captured within this more general calculus, is left largely implicit. The semantic setting of [BCS07], where propositions are interpreted as elements of a right module M on a quantale Q , specialises in [DST13] to a setting in which $M = (\mathbb{A}, \{\Box_A, \blacklozenge_A : A \in Ag\})$, where \mathbb{A} is a Heyting algebra and, for every agent A , the modalities \Box_A and \blacklozenge_A are adjoint to each other. Notice that \blacklozenge_A , which in the classical case is defined as $\neg\Box_A\neg$, cannot be expressed any more in this way, and needs to be added as a primitive connective, which has not been done in [DST13].

As mentioned before, the design of this calculus gives a more explicit account than its previous version to certain technical aspects which come from the semantic setting; for instance, the semantic setting motivating both papers features two domains of interpretation (one for the actions and one for the propositions), which are intended to give rise to two consequence relations which are to be treated on a par and then made to interact. In [BCS07], the calculus manipulates sequents which are made of heterogeneous components. For instance, in action-sequents $\Gamma \vdash_Q q$, the precedent Γ is a sequence in which both actions and agents may occur. Since Γ is to be semantically interpreted as an element of Q , they need to resort to a rather clumsy technical solution which consists in interpreting, e.g. the sequence (q, A, q') as the element $f_A^Q(q) \cdot q'$. In [DST13], the calculus is given in a *deep-inference* format; namely, rules of this calculus make it possible to manipulate formulas inside a given context. This more explicit bookkeeping makes it possible to prove the cut elimination, following the original Gentzen strategy. However, the presence of two different consequence relations and the need to account for their interaction calls for the development of an extensive theory-of-contexts, in which no less than five different types of contexts need to be introduced. This also causes a proliferation of rules, since the possibility of performing some inferences depends on the type of context under which they are to be performed.

Calculi for updates. In [Auc10], a formal framework accounting for dynamic revisions or updates is introduced, in which the revisions/updates are formalized using the turnstile symbol. This framework has aspects similar to Hoare logic: indeed, it manipulates sequent-type structures of the form $\phi, \phi' \Vdash \phi''$, such that ϕ and ϕ'' are formulas of proposition-type, and ϕ' is a formula of event-type. This formalism has also common aspects to [BCS07] and [DST13]: indeed, both proposition-type and event-type (i.e. action-type) formulas allow epistemic modalities for each agent, respectively accounting for the agent's epistemic uncertainty about the world and about the actions actually taking place.

In [AMS12] and [Auc11], three formal calculi are introduced, manipulating the syntactic structures above. Given that the turnstile encodes the update rather than a consequence relation or entailment, the syntactic structures above are not sequents in a proper sense. Rather than sequent calculi, these calculi should be rather regarded as being of natural deduction-type. As such, the design of these calculi presents many issues from a proof-theoretic semantic viewpoint; to mention only one, multiple connectives are introduced at the same time, for instance in the following rule:

$$\frac{\phi, \phi' \vdash \phi''}{\langle Bj \rangle(\phi \wedge Pre(p')), \langle Bj \rangle(\phi' \wedge p') \vdash \langle Bj \rangle \phi''} R_5.$$

These calculi are shown to be sound and complete w.r.t. three semantic consequence relations, respectively.

2.4.4 First attempt at a display calculus for EAK

In [GKP13], a display-style sequent calculus D.EAK has been introduced, which is sound with respect to the final coalgebra semantics (cf. Section 2.5), and complete w.r.t. EAK, of which it is a conservative extension. Moreover, Gentzen-style cut elimination holds for D.EAK. Finally, this system is defined independently of the relational semantics of EAK, and therefore is suitable for a fine-grained proof-theoretic semantic analysis.

Here below, we are not going to report on it in detail, but we limit ourselves to mention the structural rules which capture the specific features of EAK:

Structural Rules with Side Conditions

$$\begin{array}{c} \text{reduce}_L \frac{Pre(\alpha); \{\alpha\}A \vdash X}{\{\alpha\}A \vdash X} \quad \text{reduce}_R \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash \{\alpha\}A} \\ \\ \text{swap-in}_L \frac{Pre(\alpha); \{\alpha\}\{\mathbf{a}\}X \vdash Y}{Pre(\alpha); \{\mathbf{a}\}\{\beta\}_{\alpha\mathbf{a}\beta} X \vdash Y} \quad \text{swap-in}_R \frac{Y \vdash Pre(\alpha) > \{\alpha\}\{\mathbf{a}\}X}{Y \vdash Pre(\alpha) > \{\mathbf{a}\}\{\beta\}_{\alpha\mathbf{a}\beta} X} \\ \\ \text{swap-out}_L \frac{\left(Pre(\alpha); \{\mathbf{a}\}\{\beta\} X \vdash Y \mid \alpha\mathbf{a}\beta \right)}{Pre(\alpha); \{\alpha\}\{\mathbf{a}\}X \vdash ;(Y \mid \alpha\mathbf{a}\beta)} \quad \text{swap-out}_R \frac{\left(Y \vdash Pre(\alpha) > \{\mathbf{a}\}\{\beta\} X \mid \alpha\mathbf{a}\beta \right)}{;(Y \mid \alpha\mathbf{a}\beta) \vdash Pre(\alpha) > \{\alpha\}\{\mathbf{a}\}X} \end{array}$$

The *swap-out* rules do not have a fixed arity; they have as many premises as there are actions β such that $\alpha\mathbf{a}\beta$. In the conclusion, the symbol $;(Y \mid \alpha\mathbf{a}\beta)$ refers to a string $(\dots(Y; Y); \dots; Y)$ with n occurrences of Y , where $n = |\{\beta \mid \alpha\mathbf{a}\beta\}|$.

Operational Rules with Side Conditions

$$\text{reverse}_L \frac{Pre(\alpha); \{\alpha\}A \vdash X}{Pre(\alpha); [\alpha]A \vdash X} \quad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash Pre(\alpha) > \langle \alpha \rangle A} \text{reverse}_R$$

The main issues of D.EAK from the point of view of Wansing's criteria are linked with the presence of the formula $Pre(\alpha)$: namely, the *swap-in* and *swap-out* rules violate the principle that all parametric variables should occur unrestricted. Indeed, the occurrences of the formula $Pre(\alpha)$ in these rules is easily seen to be parametric, since $Pre(\alpha)$ occurs both in the premises and in the conclusion. Since $Pre(\alpha)$ is (the metalinguistic abbreviation of) a formula, it is a structure of a very restricted shape. As to the *swap-out* rules, it is not difficult to see, e.g. semantically (cf. [KP13, Definition 4.2.]), that the occurrences of $Pre(\alpha)$ can be removed both in the premises and in the conclusion without affecting either the soundness of the rule or the proof power of the system; this entirely remedies the problem. Likewise, as to *swap-in*, it is not difficult to see that the occurrences of $Pre(\alpha)$ can be removed in the premises, but not in the conclusion. However, even modified in this way, the *swap-in* rules would not be satisfactory. Indeed, the new form of *swap-in* would introduce $Pre(\alpha)$ in the conclusion. Since $Pre(\alpha)$ is a metalinguistic abbreviation of a formula which as such has no other specific restrictions, the occurrence of $Pre(\alpha)$ in the conclusion of *swap-in* must also be regarded as parametric. However, we still would not be able to substitute arbitrary structures for it, which is the source of the problem. This problem would be solved if $Pre(\alpha)$ could be expressed, as a structure, purely in terms of the parameter α and structural constants (but no structural variables). If this was the case, *swap-in* would encode the relations between all these logical constants, and all the occurring structural variables would be unrestricted.

Secondly, the rules *reduce* violate condition C_1 : indeed, in each of them, a formula in the premisses, namely $Pre(\alpha)$, is not a subformula of any formula occurring in the conclusion. Together with the cut elimination, condition C_1 guarantees the subformula property (cf. [Bel82, Theorem 4.3]), but is not itself essential for the cut elimination, and indeed, cut elimination has been proven for D.EAK (albeit not *à la* Belnap). The specific way in which *reduce* violates C_1 is also not a very serious one. Indeed, if the formula $Pre(\alpha)$ could be expressed in a structural way, this violation would disappear.

This solution cannot be implemented in D.EAK because the language of D.EAK does not have enough expressivity to talk about $Pre(\alpha)$ in any other way than as an arbitrary formula, which needs to be introduced via weakening or via identity (if atomic). Being able to account for $Pre(\alpha)$ in a satisfactory way from a proof-theoretic semantic perspective would require being able to state rules which, for any α , would introduce $Pre(\alpha)$

specifically, thus capturing its proof-theoretic meaning. Thus, by having structural and operational rules for $Pre(\alpha)$, we would solve many problems in one stroke: on the one hand, we would gain the *practical* advantage of achieving the satisfaction of C_1 , thus guaranteeing the subformula property; on the other hand, and more importantly, from a *methodological* perspective, we would be able to have a setting in which the occurrences of $Pre(\alpha)$ are not to be regarded as side formulas, but rather, they would occur as *structures*, on a par with all the other structures they would be interacting with.

Finally, the only operational rules violating Wansing's *separation* principle (cf. Section 2.2.3) are the *reverse* rules:

$${}_{rev_L} \frac{Pre(\alpha); \{\alpha\}A \vdash X}{Pre(\alpha); [\alpha]A \vdash X} \qquad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash Pre(\alpha) > \langle \alpha \rangle A} {}_{rev_R}$$

Here again, the problem comes from the fact that the language is not expressive enough to capture the principles encoded in the rules above at a purely structural level. In this operational formulation, these rules are to participate, in our view improperly, in the proof-theoretic meaning of the connectives $[\alpha]$ and $\langle \alpha \rangle$. Thus, it would be desirable that the rules above could be either derived, so that they disappear altogether, or alternatively, be reformulated as structural rules.

2.5 Final coalgebra semantics of dynamic logics

In order to provide a justification for the soundness of the display postulates involving the dynamic connectives, in [GKP13] the final coalgebra was used as a semantic environment for the calculus D.EAK. Specifically, the final coalgebra was there used to show that D.EAK is sound, and conservatively extends EAK. In the present section, we briefly review the needed preliminaries on the final coalgebra, and then the interpretation of EAK-formulas in the final coalgebra, which we will use in Section 2.6.2 to show that D'.EAK is sound, and conservatively extends EAK.⁹

2.5.1 The final coalgebra

The general notion of a coalgebra, as an arrow

$$W \rightarrow FW$$

⁹This semantics specifically applies to the classical base. Analogous ideas can be developed for weaker propositional bases, but in the present chapter we do not pursue them further.

is given w.r.t. a functor $F : C \rightarrow C$ on an arbitrary category C , and much of the theory of coalgebras is devoted to establishing results on coalgebras parametric in that functor F . For example, important notions such as bisimilarity and Hennessy-Milner logics can be given for arbitrary functors on the category of sets (and many other concrete categories). But even if one is interested, as in our case here, only in one particular functor, the notion of a final coalgebra is of value, as we are going to see.

Aczel [Acz88] observed that coalgebras

$$W \rightarrow \mathcal{P}W$$

for the powerset functor \mathcal{P} (which maps a set W to the set $\mathcal{P}W$ of subsets of W) are exactly Kripke frames. Indeed, a map $W \rightarrow \mathcal{P}W$ equivalently encodes a binary relation R on W . More importantly, the category theoretic notion of a coalgebra morphism coincides with the notion of bounded (or p-) morphism in modal logic, and the coalgebraic notion of bisimulation coincides with the notion in modal logic. This observation generalises easily to Kripke models over a set AtProp of atomic propositions and with multiple relations indexed by a set of agents Ag , which are exactly coalgebras

$$W \rightarrow (\mathcal{P}W)^{\text{Ag}} \times \in^{\text{AtProp}}.$$

As shown by [AM89], one can construct a ‘universal model’ \mathbb{Z} by taking the disjoint union of all coalgebras \mathbb{M} and quotienting by bisimilarity. This coalgebra \mathbb{Z} is *final*, that is, for any coalgebra \mathbb{M} there is exactly one morphism $\mathbb{M} \rightarrow \mathbb{Z}$. The property of finality characterises \mathbb{Z} up to isomorphism.

\mathbb{Z} may be a proper class. In [AM89], any functor F on sets is extended to classes and it is shown that the extended functor always has a final coalgebra, constructed as the bisimilarity collapse of the disjoint union of all coalgebras. In [Bar93], the same construction is recast in terms of an inaccessible cardinal, staying inside the set-theoretic universe without using classes. In [AMV05] these results are generalized from sets to other similar categories such as posets, and in [AMV04], it is shown that any functor F on classes is the extension of a functor F on sets.

\mathbb{Z} classifies bisimilarity. The importance of the theorems above is not merely the existence of the final coalgebra. Since all of these theorems involve two functors, one on ‘large’ sets extending another one on ‘small’ sets, and since one is interested in the notion of bisimilarity associated with the small functor, the existence of a final coalgebra for the large functor is not in itself the result one is interested in. But it is a fact, expressed for

example as the small subcoalgebra lemma in [AM89], that in all of the constructions above, the final coalgebra for the large functor classifies the notion of bisimilarity associated with the small functor. In other words, passing from small to large does not extend—up to bisimilarity—the range of available models.

Frame conditions on \mathbb{Z} . Often, one is interested in Kripke models satisfying additional frame conditions such as reflexivity, transitivity, equivalence, etc. A sufficient condition for the existence of a final coalgebra under such additional conditions is that these conditions can be formulated by modal axioms or rules, see [Kur02, Kur01] for details.

2.5.2 Final coalgebra semantics of modal logic

Summing up the discussion in the previous subsection, there is a one-to-one correspondence between subsets of the final coalgebra and unary predicates invariant under bisimilarity. Therefore, whenever we know that A is a formula invariant under bisimilarity, we may declare the subset $\llbracket A \rrbracket_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid \mathbb{Z}, z \Vdash A\}$ of the final coalgebra as the (final) semantics of A and recover $\llbracket A \rrbracket_M \subseteq W$ as

$$\llbracket A \rrbracket_M = f^{-1}(\llbracket A \rrbracket_{\mathbb{Z}}), \quad (2.10)$$

where f is the unique homomorphism

$$f : M \rightarrow \mathbb{Z}$$

provided by the property of \mathbb{Z} being final. Let us note that this approach is quite general: it only needs a notion of bisimilarity tied to the morphisms of some category (see [KR05] for a general definition) and a notion of modal formula whose semantics is invariant under this notion of bisimilarity.

Final coalgebra semantics of dynamic modalities. Dynamic logics add to Kripke semantics a facility for updating the Kripke model interpreting a formula. Typically, despite seemingly increasing the expressiveness of modal logic, such dynamic logics also enjoy bisimulation invariance and can therefore be interpreted in the final coalgebra.

Whereas the Kripke semantics of an action α is a relation between pointed models, the final coalgebra semantics of an action α is simply a relation on the carrier Z of the final coalgebra \mathbb{Z} . The precise relationship between Kripke semantics and final coalgebra

semantics of actions is as follows. Let us write

$$z \alpha_Z z'$$

to express that the two points z, z' of the final coalgebra are related by α , formalising that in z the action α can happen and has z' as a successor. Then $z \alpha_Z z'$ iff there are pointed models (\mathbb{M}, w) and (\mathbb{M}', w') related by the action α such that the unique morphisms $\mathbb{M} \rightarrow \mathbb{Z}$ and $\mathbb{M}' \rightarrow \mathbb{Z}$ map w to z and w' to z' .

Specific desiderata for epistemic actions. The specific feature of epistemic actions versus arbitrary actions is that epistemic actions do not change the factual states of affairs. Semantically, this motivates the additional requirement that if $\alpha_Z \subseteq Z \times Z$ is the interpretation of an epistemic action α and $z, z' \in Z$ are such that $z \alpha_Z z'$, then

$$\{p \in \text{AtProp} \mid z \Vdash p\} = \{p \in \text{AtProp} \mid z' \Vdash p\}.$$

Adjoints of dynamic modalities. To semantically justify the full display property of display calculi for dynamic logics, adjoints need to be available not only for the standard modalities, but also for the dynamic ones. Now, it is well known that modalities induced by a relation come in adjoint pairs. Let us recall

Proposition 2.5. *Every relation $R \subseteq X \times Y$ gives rise to the modal operators*

$$\langle R \rangle, [R] : PY \rightarrow PX \text{ and } \langle R^\circ \rangle, [R^\circ] : PX \rightarrow PY$$

defined as follows: for every $V \subseteq X$ and every $U \subseteq Y$,

$$\begin{aligned} \langle R \rangle U &= \{x \in X \mid \exists y . xRy \ \& \ y \in U\} & [R]U &= \{x \in X \mid \forall y . xRy \Rightarrow y \in U\} \\ \langle R^\circ \rangle V &= \{y \in Y \mid \exists x . xRy \ \& \ x \in V\} & [R^\circ]V &= \{y \in Y \mid \forall x . xRy \Rightarrow x \in V\}. \end{aligned}$$

These operators come in adjoint pairs:

$$\langle R \rangle U \subseteq V \text{ iff } U \subseteq [R^\circ]V \tag{2.11}$$

$$\langle R^\circ \rangle V \subseteq U \text{ iff } V \subseteq [R]U. \tag{2.12}$$

In order to apply this proposition to dynamic modalities, we need to consider the relation corresponding to an action α . Kripke semantics suggests to consider α as a relation on all pointed Kripke models (\mathbb{M}, w) , but this would introduce a two-tiered semantics: with the semantics of an ordinary modality given by a relation on the carrier of a model \mathbb{M}

Structural connective	if in precedent position	if in succedent position
I	\top	\perp
$A ; B$	$A \wedge B$	$A \vee B$
$A > B$	$A \triangleright B$	$A \rightarrow B$
$\{a\}A$	$\langle a \rangle A$	$[a]A$
$\widehat{a}A$	$\widehat{a}A$	$\overline{a}A$
$\{\alpha\}A$	$\langle \alpha \rangle A$	$[\alpha]A$
$\widehat{\alpha}A$	$\widehat{\alpha}A$	$\overline{\alpha}A$

TABLE 2.3: Translation of structural connectives into logical connectives

and the semantics of a dynamic modality given by a relation on the set of all pointed models (\mathbb{M}, w) . In the final coalgebra semantics all relations are relations on the final coalgebra \mathbb{Z} and we can directly apply the above proposition to both static and dynamic modalities (with the X and Y of the proposition being the carrier of the final coalgebra).

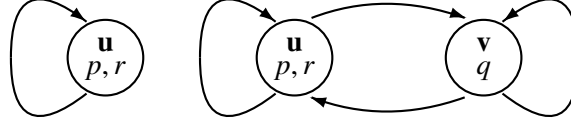
Soundness of the display postulates. Let us expand on how to interpret display-type structures and sequents in the final coalgebra. Structures will be translated into formulas, and formulas will be interpreted as subsets of the final coalgebra. In order to translate structures as formulas, structural connectives need to be translated as logical connectives; to this effect, structural connectives are associated with pairs of logical connectives and any given occurrence of a structural connective is translated as one or the other, according to which side of the sequent the given occurrence can be displayed on as main connective, as reported in Table 2.3. These logical connectives in turn are interpreted in the final coalgebra in the standard way. For example,

$$\begin{aligned} \llbracket \langle \alpha \rangle A \rrbracket_{\mathbb{Z}} &= \langle \alpha_{\mathbb{Z}} \rangle \llbracket A \rrbracket_{\mathbb{Z}} & \llbracket [\alpha] A \rrbracket_{\mathbb{Z}} &= [\alpha_{\mathbb{Z}}] \llbracket A \rrbracket_{\mathbb{Z}} \\ \llbracket \widehat{\alpha} A \rrbracket_{\mathbb{Z}} &= \langle \alpha_{\mathbb{Z}}^{\circ} \rangle \llbracket A \rrbracket_{\mathbb{Z}} & \llbracket \overline{\alpha} A \rrbracket_{\mathbb{Z}} &= [\alpha_{\mathbb{Z}}^{\circ}] \llbracket A \rrbracket_{\mathbb{Z}} \end{aligned}$$

where the notation on the right-hand sides refers to the one defined in Proposition 2.5.

Sequents $A \vdash B$ will be interpreted as inclusions $\llbracket A \rrbracket_{\mathbb{Z}} \subseteq \llbracket B \rrbracket_{\mathbb{Z}}$; rules $(A_i \vdash B_i \mid i \in I) / C \vdash D$ will be interpreted as implications of the form “if $\llbracket A_i \rrbracket_{\mathbb{Z}} \subseteq \llbracket B_i \rrbracket_{\mathbb{Z}}$ for every $i \in I$, then $\llbracket C \rrbracket_{\mathbb{Z}} \subseteq \llbracket D \rrbracket_{\mathbb{Z}}$ ”. As a direct consequence of the adjunctions (2.11) and (2.12), the following display postulates are sound under the interpretation above.

$$\frac{\{\alpha\}X \vdash Y}{X \vdash \widehat{\alpha}Y} \begin{matrix} \{\alpha\} \\ \widehat{\alpha} \end{matrix} \qquad \frac{X \vdash \{\alpha\}Y}{\widehat{\alpha}X \vdash Y} \begin{matrix} \widehat{\alpha} \\ \{\alpha\} \end{matrix}$$

FIGURE 2.1: The models \mathbb{M}^α and \mathbb{M} .

Remark. On the other hand, standard Kripke models are not in general closed under (the interpretations of) α and α° . As a direct consequence of this fact, we can show that e.g. the display postulate $\left(\frac{\{\alpha\}}{\bar{\alpha}}\right)$ is not sound if we interpret it in a Kripke model \mathbb{M} for any interpretation of formulas of the form $\bar{\alpha} B$ in \mathbb{M} .

Indeed, consider the model \mathbb{M} represented on the right-hand side of the Figure 2.1 and let the action α be so that updating (\mathbb{M}, u) gives the model \mathbb{M}^α depicted on the left-hand side of the figure. In other words, α is the public announcement (cf. [BMS99]) of the atomic proposition r . Further, let $A := [a]p$ and $B := q$, where a is the agent whose equivalence relation is depicted by the arrows of the figure. Let $i : \mathbb{M}^\alpha \hookrightarrow \mathbb{M}$ be the submodel injection map. Clearly, $\llbracket [a]p \rrbracket_{\mathbb{M}} = \emptyset$, which implies that the inclusion $\llbracket [A]_{\mathbb{M}} \subseteq \llbracket \bar{\alpha} B \rrbracket_{\mathbb{M}}$ trivially holds for any interpretation of $\bar{\alpha} B$ in \mathbb{M} ; however, $i[\llbracket [a]p \rrbracket_{\mathbb{M}^\alpha}] = \{u\}$, hence $\llbracket \langle \alpha \rangle [a]p \rrbracket_{\mathbb{M}} = V(r) \cap \{u\} = \{u\} \not\subseteq \{v\} = \llbracket [q]_{\mathbb{M}}$, which falsifies the inclusion $\llbracket \langle \alpha \rangle A \rrbracket_{\mathbb{M}} \subseteq \llbracket [B]_{\mathbb{M}}$. This proves our claim.

Related work. Final coalgebra semantics for dynamic logics was employed by Gerbrandy and Groeneveld [GG97], Gerbrandy [Ger99], Baltag [Bal03], and Cîrstea and Sadrzadeh [CS07]. Adjoints of dynamic modalities with Kripke semantics were considered in Baltag, Coecke, Sadrzadeh [BCS07]. To guarantee the soundness of the rules involving the adjoints, they have to close the Kripke models under actions, which amounts, from our point of view, to generating a subcoalgebra of the final coalgebra closed under actions. The arguments reported here in favour of the final coalgebra semantics for treating dynamic modalities with their adjoints are taken from [GKP13].

2.6 Proof-Theoretic Semantics for EAK

In the present section, we introduce the calculus $D'.\text{EAK}$ for the logic EAK, which is a revised and improved version of the calculus $D.\text{EAK}$ discussed in Section 2.4.4. We argue that $D'.\text{EAK}$ satisfies the requirements discussed in Section 2.2.3. On the basis of this, we propose $D'.\text{EAK}$ as an adequate calculus from the viewpoint of proof-theoretic semantics. We also verify that $D'.\text{EAK}$ is a quasi proper display calculus (cf. Definition 2.3), and hence its cut elimination theorem follows from Theorem 2.4.

2.6.1 The calculus D'.EAK

As is typical of display calculi, D'.EAK manipulates sequents of type $X \vdash Y$, where X and Y are structures, i.e. syntactic objects inductively built from formulas using *structural connectives*, or *proxies*. Every proxy is typically associated with two logical (operational) connectives, and is interpreted *contextually* as one or the other of the two, depending on whether it occurs in precedent or in succedent position (cf. Definition 2.1). The design of D'.EAK follows Došen's principle (cf. Section 2.2.3); consequently, D'.EAK is modular along many dimensions. For instance, the space of the versions of EAK on nonclassical bases, down to e.g. the Lambek calculus, can be captured by suitably removing structural rules. Moreover, also w.r.t. static modal logic, the space of properly displayable normal modal logics (cf. [Kra96]) can be reconstructed by adding or removing structural rules in a suitable way. Finally, different types of interaction between the dynamic and the epistemic modalities can be captured by changing the relative structural rules.

In order to highlight this modularity, we will present the system piecewise. First we give rules for the propositional base, divided into structural rules and operational rules; then we do the same for the static modal operators; finally, we introduce the rules for the dynamic modalities.

In the table below, we give an overview of the logical connectives of the propositional base and their proxies.

Structural symbols	<		>		;		I	
Operational symbols	−<	←	−>	→	∧	∨	⊤	⊥

Table 2.4 (on page 100) contains the structural rules for the propositional base.

The top-to-bottom direction of each I-rule is a special case of the corresponding *weakening* rule. However, we state them all the same for the sake of modularity, since they might still be part of a calculus for a substructural logic without weakening. The *weakening* rules are not given in the usual shape; the present version has the advantage that the new structure is introduced in isolation; nevertheless, the standard version is derivable from the display postulates, as shown below:

$$\frac{\frac{X \vdash Z}{Y \vdash Z < X}}{Y ; X \vdash Z}$$

Having both versions of *weakening* as primitive rules is useful for reducing the size of derivations. In the following table, we include the display postulates linking the structural connective ; with > and <:

Display Postulates

$$\begin{array}{c}
 (, <) \frac{X; Y \vdash Z}{X \vdash Z < Y} \quad \frac{Z \vdash X; Y}{Z < Y \vdash X} (<,) \\
 (, >) \frac{X; Y \vdash Z}{Y \vdash X > Z} \quad \frac{Z \vdash X; Y}{X > Z \vdash Y} (>,)
 \end{array}$$

In the current presentation, more connectives with their associated rules are accounted for than in [GKP13]. The additional rules can be proved to be derivable from the remaining ones in the presence of the rules *exchange* E_L and E_R . Likewise, as is well known, by dispensing with *contraction*, *weakening* and *associativity*, an even wider array of connectives would ensue (for instance, dispensing with *weakening* and *contraction* would separate the additive and the multiplicative versions of each connective, etc.). We are not going to expand on these well known ideas any further, but only point out that, in the context of the whole system that we are going to introduce below, this would give a modular account of different versions of EAK with different substructural

Structural Rules

$$\begin{array}{c}
 Id \frac{}{p \vdash p} \quad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y} Cut \\
 I_L^1 \frac{X \vdash Y}{I \vdash Y < X} \quad \frac{X \vdash Y}{X < Y \vdash I} I_R^1 \\
 I_L^2 \frac{X \vdash Y}{I \vdash X > Y} \quad \frac{X \vdash Y}{Y > X \vdash I} I_R^2 \\
 IW_L \frac{I \vdash X}{Y \vdash X} \quad \frac{X \vdash I}{X \vdash Y} IW_R \\
 W_L^1 \frac{X \vdash Z}{Y \vdash Z < X} \quad \frac{X \vdash Z}{X < Z \vdash Y} W_R^1 \\
 W_L^2 \frac{X \vdash Z}{Y \vdash X > Z} \quad \frac{X \vdash Z}{Z > X \vdash Y} W_R^2 \\
 C_L \frac{X; X \vdash Y}{X \vdash Y} \quad \frac{Y \vdash X; X}{Y \vdash X} C_R \\
 E_L \frac{Y; X \vdash Z}{X; Y \vdash Z} \quad \frac{Z \vdash X; Y}{Z \vdash Y; X} E_R \\
 A_L \frac{X; (Y; Z) \vdash W}{(X; Y); Z \vdash W} \quad \frac{W \vdash (Z; Y); X}{W \vdash Z; (Y; X)} A_R
 \end{array}$$

TABLE 2.4: Structural rules for the propositional base of D'.EAK

logics as propositional base. The calculus introduced here is amenable to this line of investigation. A natural question in this respect would be to relate these ensuing proof formalisms with the semantic settings of [BCS07].

In line with this modular perspective on the propositional base for EAK, the classical base is obtained by adding the so-called *Grishin* rules (following e.g. [Gor00]), encoding validities which are classical but not intuitionistic:

Grishin rules

$$Gri_L \frac{X > (Y; Z) \vdash W}{(X > Y); Z \vdash W} \quad \frac{W \vdash X > (Y; Z)}{W \vdash (X > Y); Z} Gri_R$$

This modular treatment can be regarded as an application of Došen's principle: calculi for versions of EAK with stronger and stronger propositional bases are obtained by progressively adding structural rules, but keeping the same operational rules. As a consequence, cut elimination for the different versions will follow immediately from the cut elimination metatheorem without having to verify condition C_8 again.

The following table shows the operational rules for the propositional base:

Operational Rules

$$\begin{array}{c} \perp_L \frac{}{\perp \vdash I} \quad \frac{X \vdash I}{X \vdash \perp} \perp_R \\ \\ \top_L \frac{I \vdash X}{\top \vdash X} \quad \frac{}{I \vdash \top} \top_R \\ \\ \wedge_L \frac{A; B \vdash Z}{A \wedge B \vdash Z} \quad \frac{X \vdash A \quad Y \vdash B}{X; Y \vdash A \wedge B} \wedge_R \\ \\ \vee_L \frac{A \vdash X \quad B \vdash Y}{A \vee B \vdash X; Y} \quad \frac{Z \vdash A; B}{Z \vdash A \vee B} \vee_R \\ \\ \leftarrow_L \frac{B \vdash Y \quad X \vdash A}{B \leftarrow A \vdash Y < X} \quad \frac{Z \vdash B < A}{Z \vdash B \leftarrow A} \leftarrow_R \\ \\ <_L \frac{B < A \vdash Z}{B < A \vdash Z} \quad \frac{Y \vdash B \quad A \vdash X}{Y < X \vdash B < A} <_R \\ \\ \rightarrow_L \frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X > Y} \quad \frac{Z \vdash A > B}{Z \vdash A \rightarrow B} \rightarrow_R \\ \\ >_L \frac{A > B \vdash Z}{A > B \vdash Z} \quad \frac{A \vdash X \quad Y \vdash B}{X > Y \vdash A > B} >_R \end{array}$$

As is well known, in the presence of *exchange*, the connectives \leftarrow and \prec are identified with \rightarrow and \succ , respectively. Notice that the rules \perp_R and \top_L are derivable in the presence of *weakening* and the I-rules. An example of such a derivation is given below:

$$\frac{\frac{\frac{X \vdash I}{I \succ X \vdash \perp}}{X \vdash I; \perp}}{X \prec \perp \vdash I}}{X \vdash \perp}$$

The rules for the normal epistemic modalities can be added to the system above or to any of its variants discussed early on. To this end, the language is now expanded with two contextual proxies and four operational connectives for every agent a , as follows:

Structural symbols	{a}		\overline{a}	
Operational symbols	$\langle a \rangle$	[a]	\widehat{a}	\overline{a}

The proxies {a} and \overline{a} are translated into diamond-type modalities when occurring in precedent position and into box-type modalities when occurring in succedent position. The structural rules, the display postulates, and the operational rules for the static modalities are respectively given in the following three tables:

Structural Rules

$$\begin{array}{c}
\begin{array}{c}
nec_L^{ep} \frac{I \vdash X}{\{a\}I \vdash X} \quad \frac{X \vdash I}{X \vdash \{a\}I} nec_R^{ep} \\
epnec_L \frac{I \vdash X}{\overline{a}I \vdash X} \quad \frac{X \vdash I}{X \vdash \overline{a}I} epnec_R \\
FS_L^{ep} \frac{\{a\}Y \succ \{a\}Z \vdash X}{\{a\}(Y \succ Z) \vdash X} \quad \frac{Y \vdash \{a\}X \succ \{a\}Z}{Y \vdash \{a\}(X \succ Z)} FS_R^{ep} \\
mon_L^{ep} \frac{\{a\}X; \{a\}Y \vdash Z}{\{a\}(X; Y) \vdash Z} \quad \frac{Z \vdash \{a\}Y; \{a\}X}{Z \vdash \{a\}(Y; X)} mon_R^{ep} \\
epFS_L \frac{\overline{a}Y \succ \overline{a}X \vdash Z}{\overline{a}(Y \succ X) \vdash Z} \quad \frac{Y \vdash \overline{a}X \succ \overline{a}Z}{Y \vdash \overline{a}(X \succ Z)} epFS_R \\
epmon_L \frac{\overline{a}X; \overline{a}Y \vdash Z}{\overline{a}(X; Y) \vdash Z} \quad \frac{Z \vdash \overline{a}Y; \overline{a}X}{Z \vdash \overline{a}(Y; X)} epmon_R
\end{array}
\end{array}$$

Notice that the *mon*-rules (the soundness of which is due to the monotonicity of $\langle \alpha \rangle$ and $[\alpha]$) are derivable from the *FS*-rules in the presence of non restricted *weakening* and *contraction*.

The *FS*-rules above encode the following Fischer Servi-type axioms:

$$\begin{aligned} \langle a \rangle A \rightarrow [a]B \vdash [a](A \rightarrow B) & \quad \widehat{a}A \rightarrow \overline{a}B \vdash \overline{a}(A \rightarrow B) \\ \langle a \rangle(A \succ B) \vdash [a]A \succ \langle a \rangle B & \quad \widehat{a}(A \succ B) \vdash \overline{a}A \succ \widehat{a}B. \end{aligned}$$

These axioms encode the link between $\langle a \rangle$ and $[a]$ (and \widehat{a} and \overline{a}), namely, that they are interpreted semantically using the same relation in a Kripke frame. This link can be alternatively expressed by *conjugation axioms*, given below both in the diamond- and in the box-version:

$$\langle a \rangle A \wedge B \vdash \langle a \rangle(A \wedge \widehat{a}B) \quad \widehat{a}A \wedge B \vdash \widehat{a}(A \wedge \langle a \rangle B), \quad (2.13)$$

$$[a](\overline{a}A \vee B) \vdash (A \vee [a]B) \quad \overline{a}([a]A \vee B) \vdash (A \vee \overline{a}B), \quad (2.14)$$

which in turn can be encoded in the following *conjugation* rules:

$$\text{conj} \frac{\{a\}(X; \widehat{a}Y) \vdash Z}{\{a\}X; Y \vdash Z} \quad \frac{X \vdash \{a\}(Y; \overline{a}Z)}{X \vdash \{a\}Y; Z} \text{conj}$$

$$\text{conj} \frac{\overline{a}(X; \{a\}Y) \vdash Z}{\overline{a}X; Y \vdash Z} \quad \frac{X \vdash \overline{a}(Y; \{a\}Z)}{X \vdash \overline{a}Y; Z} \text{conj}$$

The *conj*-rules and the *FS*-rules can be shown to be interderivable thanks to the following display postulates.

Display Postulates

$$(\{a\}, \widehat{a}) \frac{\{a\}X \vdash Y}{X \vdash \widehat{a}Y} \quad \frac{X \vdash \{a\}Y}{\widehat{a}X \vdash Y} (\overline{a}, \{a\})$$

The display postulates above directly come from the fact in the final coalgebra semantic for EAK the dynamic connectives $[\alpha]$ and $\langle \alpha \rangle$ are parts of *adjoint pairs*. Specifically, we have the following adjunction relations $\langle \alpha \rangle \dashv \overline{\alpha}$ and $\widehat{\alpha} \dashv [\alpha]$: for all formulas A, B ,

$$\langle \alpha \rangle A \vdash B \text{ iff } A \vdash \overline{\alpha}B \quad \widehat{\alpha}A \vdash B \text{ iff } A \vdash [\alpha]B \quad (2.15)$$

The reader is referred to Section 2.5 for a detailed discussion.

Operational Rules

$$\begin{array}{c}
\langle a \rangle_L \frac{\{a\}A \vdash X}{\langle a \rangle A \vdash X} \quad \frac{X \vdash A}{\{a\}X \vdash \langle a \rangle A} \langle a \rangle_R \\
[a]_L \frac{A \vdash X}{[a]A \vdash [a]X} \quad \frac{X \vdash \{a\}A}{X \vdash [a]A} [a]_R \\
\widehat{a}_L \frac{\widehat{a}A \vdash X}{\widehat{a}A \vdash X} \quad \frac{X \vdash A}{\widehat{a}X \vdash \widehat{a}A} \widehat{a}_R \\
\overline{a}_L \frac{A \vdash X}{\overline{a}A \vdash \overline{a}X} \quad \frac{X \vdash \overline{a}A}{X \vdash \overline{a}A} \overline{a}_R
\end{array}$$

The rules presented so far are essentially adaptations of display calculi of Goré's [Gor00]. Let us turn to the dynamic part of the calculus D'.EAK: the language is now expanded by adding, for each action α :

- two contextual proxies, together with their four corresponding operational unary connectives;
- one constant symbol and its corresponding structural proxy:

Structural symbols	$\{\alpha\}$	$\widehat{\alpha}$	Φ_α
Operational symbols	$\langle \alpha \rangle$	$[\alpha]$	1_α

As in the previous version D.EAK, the proxies $\{\alpha\}$ and $\widehat{\alpha}$ are translated into diamond-type modalities when occurring in precedent position, and into box-type modalities when occurring in succedent position. An important difference between D.EAK and D'.EAK is the introduction of the structural and operational constants Φ_α and 1_α ; indeed, the additional expressivity they provide is used to capture the proof-theoretic behaviour of the metalinguistic abbreviation $Pre(\alpha)$ at the object-level. As was the case of $Pre(\alpha)$ in D.EAK, the rules below will be such that the proxy Φ_α can occur only in precedent position. Hence, the Φ_α can never be interpreted as anything else than 1_α . However, a natural way to extend D'.EAK would be to introduce an operational constant 0_α , intuitively standing for the postconditions of α for each action α , and dualize the relevant rules so as to capture the behaviour of postconditions. In the present chapter, this expansion is not pursued any further.

The two tables below introduce the structural rules for the dynamic modalities which are analogous to those for the static modalities given early on.

Structural Rules

$$\begin{array}{c}
nec_L^{dyn} \frac{I \vdash X}{\{\alpha\} I \vdash X} \quad \frac{X \vdash I}{X \vdash \{\alpha\} I} nec_R^{dyn} \\
\\
dynnec_L \frac{I \vdash X}{\underline{\alpha} I \vdash X} \quad \frac{X \vdash I}{X \vdash \underline{\alpha} I} dynnec_R \\
\\
FS_L^{dyn} \frac{\{\alpha\} Y > \{\alpha\} Z \vdash X}{\{\alpha\} (Y > Z) \vdash X} \quad \frac{Y \vdash \{\alpha\} X > \{\alpha\} Z}{Y \vdash \{\alpha\} (X > Z)} FS_R^{dyn} \\
\\
mon_L^{dyn} \frac{\{\alpha\} X; \{\alpha\} Y \vdash Z}{\{\alpha\} (X; Y) \vdash Z} \quad \frac{Z \vdash \{\alpha\} Y; \{\alpha\} X}{Z \vdash \{\alpha\} (Y; X)} mon_R^{dyn} \\
\\
dynFS_L \frac{\underline{\alpha} Y > \underline{\alpha} X \vdash Z}{\underline{\alpha} (Y > X) \vdash Z} \quad \frac{Y \vdash \underline{\alpha} X > \underline{\alpha} Z}{Y \vdash \underline{\alpha} (X > Z)} dynFS_R \\
\\
dynmon_L \frac{\underline{\alpha} X; \underline{\alpha} Y \vdash Z}{\underline{\alpha} (X; Y) \vdash Z} \quad \frac{Z \vdash \underline{\alpha} Y; \underline{\alpha} X}{Z \vdash \underline{\alpha} (Y; X)} dynmon_R
\end{array}$$

Analogous considerations as those made for the epistemic *FS*- and *mon*-rules apply to the dynamic *FS*- and *mon*-rules above, also in relation to analogous *conjugation* rules.

Display Postulates

$$(\{\alpha\}, \underline{\alpha}) \frac{\{\alpha\} X \vdash Y}{X \vdash \underline{\alpha} Y} \quad \frac{Y \vdash \{\alpha\} X}{\underline{\alpha} Y \vdash X} (\underline{\alpha}, \{\alpha\})$$

Next, we introduce the structural rules which are to capture the specific behaviour of epistemic actions

Atom

$$\frac{}{\Gamma p \vdash \Delta p} atom$$

where Γ and Δ are arbitrary finite sequences of the form $(\alpha_1) \dots (\alpha_n)$, such that each (α_j) is of the form $\{\alpha_j\}$ or of the form $\underline{\alpha}_j$, for $1 \leq j \leq n$. Intuitively, the *atom* rules capture the requirement that epistemic actions do not change the factual state of affairs (in the Hilbert-style presentation of EAK, this is encoded in the axiom (2.1) in Section 2.4.1).

Structural Rules for Epistemic Actions

$$\begin{array}{c}
\frac{X \vdash Y}{\{\alpha\}X \vdash \{\alpha\}Y} \text{ balance} \\
\\
\text{comp}_L^\alpha \frac{\{\alpha\} \widetilde{\alpha} X \vdash Y}{\Phi_\alpha; X \vdash Y} \quad \frac{X \vdash \{\alpha\} \widetilde{\alpha} Y}{X \vdash \Phi_\alpha > Y} \text{comp}_R^\alpha \\
\\
\text{reduce}'_L \frac{\Phi_\alpha; \{\alpha\}X \vdash Y}{\{\alpha\}X \vdash Y} \quad \frac{Y \vdash \Phi_\alpha > \{\alpha\}X}{Y \vdash \{\alpha\}X} \text{reduce}'_R \\
\\
\text{swap-in}'_L \frac{\{\alpha\}\{\mathbf{a}\}X \vdash Y}{\Phi_\alpha; \{\mathbf{a}\}\{\beta\}_{\alpha\mathbf{a}\beta} X \vdash Y} \quad \frac{Y \vdash \{\alpha\}\{\mathbf{a}\}X}{Y \vdash \Phi_\alpha > \{\mathbf{a}\}\{\beta\}_{\alpha\mathbf{a}\beta} X} \text{swap-in}'_R \\
\\
\text{swap-out}'_L \frac{(\{\mathbf{a}\}\{\beta\} X \vdash Y \mid \alpha\mathbf{a}\beta)}{\{\alpha\}\{\mathbf{a}\}X \vdash ;(Y \mid \alpha\mathbf{a}\beta)} \quad \frac{(Y \vdash \{\mathbf{a}\}\{\beta\} X \mid \alpha\mathbf{a}\beta)}{;(Y \mid \alpha\mathbf{a}\beta) \vdash \{\alpha\}\{\mathbf{a}\}X} \text{swap-out}'_R
\end{array}$$

The *swap-in'* rules are unary and should be read as follows: if the premise holds, then the conclusion holds relative to any action β such that $\alpha\mathbf{a}\beta$. The *swap-out'* rules do not have a fixed arity; they have as many premises¹⁰ as there are actions β such that $\alpha\mathbf{a}\beta$. In the conclusion, the symbol $;(Y \mid \alpha\mathbf{a}\beta)$ refers to a string $(\dots(Y; Y); \dots; Y)$ with n occurrences of Y , where $n = \|\beta \mid \alpha\mathbf{a}\beta\|$. The *swap-in* and *swap-out* rules encode the interaction between dynamic and epistemic modalities as it is captured by the interaction axioms in the Hilbert style presentation of EAK (cf. (2.4) in Section 2.4.1 and similarly in Section 2.4.2). The *reduce* rules encode well-known EAK validities such as $\langle\alpha\rangle A \rightarrow (Pre(\alpha) \wedge \langle\alpha\rangle A)$.

Finally, the operational rules for $\langle\alpha\rangle$, $[\alpha]$, and 1_α are given in the table below:

Operational Rules

$$\begin{array}{c}
\langle\alpha\rangle_L \frac{\{\alpha\}A \vdash X}{\langle\alpha\rangle A \vdash X} \quad \frac{X \vdash A}{\{\alpha\}X \vdash \langle\alpha\rangle A} \langle\alpha\rangle_R \\
\\
[\alpha]_L \frac{A \vdash X}{[\alpha]A \vdash \{\alpha\}X} \quad \frac{X \vdash \{\alpha\}A}{X \vdash [\alpha]A} [\alpha]_R \\
\\
1_{\alpha L} \frac{\Phi_\alpha \vdash X}{1_\alpha \vdash X} \quad \frac{}{\Phi_\alpha \vdash 1_\alpha} 1_{\alpha R}
\end{array}$$

¹⁰The *swap-out* rule could indeed be infinitary if action structures were allowed to be infinite, which in the present setting, as in [BMS99], is not the case.

2.6.2 Properties of D'.EAK

Soundness. The calculus D'.EAK can be readily shown to be sound with respect to the final coalgebra semantics. The general procedure has been outlined in Section 2.5. The soundness of most of the rules of D'.EAK can be shown entirely analogously to the soundness of the corresponding rules in D.EAK, which is outlined in [GKP13].

As for rules not involving $\widehat{\alpha}$, we will rely on the following observation, which is based on the invariance of EAK-formulas under bisimulation (cf. Section 2.4.1):

Lemma 2.6. *The following are equivalent for all EAK-formulas A and B :*

- (1) $\llbracket A \rrbracket_Z \subseteq \llbracket B \rrbracket_Z$;
- (2) $\llbracket A \rrbracket_M \subseteq \llbracket B \rrbracket_M$ for every model M .

Proof. The direction from (2) to (1) is clear; conversely, fix a model M , and let $f : M \rightarrow Z$ be the unique arrow; then (1) immediately implies that $\llbracket A \rrbracket_M = f^{-1}(\llbracket A \rrbracket_Z) \subseteq f^{-1}(\llbracket B \rrbracket_Z) = \llbracket B \rrbracket_M$. \square

In the light of the lemma above, and using the translations provided in Table 2.3, the soundness of unary rules $A \vdash B/C \vdash D$ not involving $\widehat{\alpha}$, such as *balance*, $\langle \alpha \rangle_R$ and $[\alpha]_L$, can be straightforwardly checked as implications of the form “if $\llbracket A \rrbracket_M \subseteq \llbracket B \rrbracket_M$ on every model M , then $\llbracket C \rrbracket_M \subseteq \llbracket D \rrbracket_M$ on every model M ”. As an example, let us check the soundness of *balance*: Let A, B be EAK-formulas such that $\llbracket A \rrbracket_M \subseteq \llbracket B \rrbracket_M$ on every model M . Let us fix a model M , and show that $\llbracket \langle \alpha \rangle A \rrbracket_M \subseteq \llbracket [\alpha] B \rrbracket_M$. As discussed in [KP13, Section 4.2], the following identities hold in any standard model:

$$\llbracket \langle \alpha \rangle A \rrbracket_M = \llbracket \text{Pre}(\alpha) \rrbracket_M \cap \iota_k^{-1}[i[\llbracket A \rrbracket_{M^\alpha}]], \quad (2.16)$$

$$\llbracket [\alpha] A \rrbracket_M = \llbracket \text{Pre}(\alpha) \rrbracket_M \Rightarrow \iota_k^{-1}[i[\llbracket A \rrbracket_{M^\alpha}]], \quad (2.17)$$

where the map $i : M^\alpha \rightarrow \coprod_\alpha M$ is the submodel embedding, and $\iota_k : M \rightarrow \coprod_\alpha M$ is the embedding of M into its k -colored copy. Letting $g(-) := \iota_k^{-1}[i[-]]$, we need to show that

$$\llbracket \text{Pre}(\alpha) \rrbracket_M \cap g(\llbracket A \rrbracket_{M^\alpha}) \subseteq \llbracket \text{Pre}(\alpha) \rrbracket_M \Rightarrow g(\llbracket B \rrbracket_{M^\alpha}).$$

This is a direct consequence of the Heyting-valid implication “if $b \leq c$ then $a \wedge b \leq a \rightarrow c$ ”, the monotonicity of g , and the assumption that $\llbracket A \rrbracket_M \subseteq \llbracket B \rrbracket_M$ holds on every model, hence on M^α .

Actually, for all rules $(A_i \vdash B_i \mid i \in I)/C \vdash D$ not involving $\widehat{\alpha}$ except *balance*, $\langle \alpha \rangle_R$ and $[\alpha]_L$, stronger soundness statements can be proven of the form “for every model M , if $\llbracket A_i \rrbracket_M \subseteq \llbracket B_i \rrbracket_M$ for every $i \in I$, then $\llbracket C \rrbracket_M \subseteq \llbracket D \rrbracket_M$ ” (this amounts to the soundness

w.r.t. the standard semantics). This is the case for all display postulates not involving $\widehat{\alpha}$, the soundness of which boils down to the well known adjunction conditions holding in every model M . As to the remaining rules not involving $\widehat{\alpha}$, thanks to the following general principle of *indirect (in)equality*, the stronger soundness condition above boils down to the verification of inclusions which interpret validities of IEAK [KP13], and hence, a fortiori, of EAK. Same arguments hold for the Grishin rules, except that their soundness boils down to classical but not intuitionistic validities.

Lemma 2.7. (Principle of indirect inequality) *Tfae for any preorder P and all $a, b \in P$:*

- (1) $a \leq b$;
- (2) $x \leq a$ implies $x \leq b$ for every $x \in P$;
- (3) $b \leq y$ implies $a \leq y$ for every $y \in P$.

As an example, let us verify $s\text{-out}_L$: fix a model M , fix EAK-formulas A and B , and assume that for every action β , if $\alpha a \beta$ then $\llbracket \langle a \rangle \langle \beta \rangle A \rrbracket_M \subseteq \llbracket B \rrbracket_M$, i.e., that $\bigcup \{ \llbracket \langle a \rangle \langle \beta \rangle A \rrbracket_M \mid \alpha a \beta \} \subseteq \llbracket B \rrbracket_M$; we need to show that $\llbracket \langle \alpha \rangle \langle a \rangle A \rrbracket_M \subseteq \llbracket B \rrbracket_M$. By the principle of indirect inequality, it is enough to show that $\llbracket \langle \alpha \rangle \langle a \rangle A \rrbracket_M \subseteq \bigcup \{ \llbracket \langle a \rangle \langle \beta \rangle A \rrbracket_M \mid \alpha a \beta \}$. Indeed, since axiom (2.4) is valid on any model, we have:

$$\llbracket \langle \alpha \rangle \langle a \rangle A \rrbracket_M \subseteq \llbracket \text{Pre}(\alpha) \rrbracket_M \cap \bigcup \{ \llbracket \langle a \rangle \langle \beta \rangle A \rrbracket_M \mid \alpha a \beta \} \subseteq \bigcup \{ \llbracket \langle a \rangle \langle \beta \rangle A \rrbracket_M \mid \alpha a \beta \}.$$

The soundness of the operational rules of 1_α is immediate; the soundness of atom can be proven directly on the final coalgebra by induction on the length of Γ and Δ using the fact, mentioned on page 96, that epistemic actions do not change the valuations of atomic formulae. For instance, as to the base case of this induction, let us argue for the soundness of $\widehat{\alpha} p \vdash p$ and $p \vdash \widehat{\alpha} p$: indeed, let $\alpha_Z \subseteq Z \times Z$ be the interpretation of the epistemic action α on the final coalgebra, then the left-hand side of the atomsequent above is interpreted as the set $\alpha_Z[\llbracket p \rrbracket_Z]$. Because of the assumption on α_Z mentioned above it immediately follows that $\alpha_Z[\llbracket p \rrbracket_Z] \subseteq \llbracket p \rrbracket_Z$, and $\alpha_Z[\llbracket p \rrbracket_Z^c] \subseteq \llbracket p \rrbracket_Z^c$. The former inclusion gives the soundness of $\widehat{\alpha} p \vdash p$, while the latter is equivalent to $\llbracket p \rrbracket_Z \subseteq (\alpha_Z[\llbracket p \rrbracket_Z^c])^c$, which gives the soundness of $p \vdash \widehat{\alpha} p$.

The soundness of the comp rules is given in Appendix C.2.

Finally, the soundness of the rules which do involve $\widehat{\alpha}$ remains to be shown. The soundness of the display postulates immediately follows from Proposition 2.5. As an example, let us verify the soundness of ${}^{dyn}FS_L$: translating the structures into formulas, it boils down to verifying that, for all EAK-formulas A, B and C , if $\llbracket \widehat{\alpha} A \rrbracket_Z \succ \llbracket \widehat{\alpha} B \rrbracket_Z \subseteq \llbracket C \rrbracket_Z$, then $\llbracket \widehat{\alpha} (A \succ B) \rrbracket_Z \subseteq \llbracket C \rrbracket_Z$. By applying the appropriate adjunction rules, the implication above is equivalent to the following implication: if $\llbracket B \rrbracket_Z \subseteq \llbracket [\alpha](\widehat{\alpha} A \vee C) \rrbracket_Z$

then $\llbracket B \rrbracket_Z \subseteq \llbracket A \vee [\alpha]C \rrbracket_Z$. By applying the principle of indirect inequality, we are reduced to showing the inclusion

$$\llbracket [\alpha](\overline{\alpha}A \vee C) \rrbracket_Z \subseteq \llbracket A \vee [\alpha]C \rrbracket_Z,$$

which is the soundness of the box-version of a conjugation condition (see the shape of (2.14) for epistemic modalities), and is true in Z since $\overline{\alpha}$ is interpreted as $[\alpha^\circ]$.

Completeness and conservativity. The completeness of $D'.EAK$ w.r.t. the Hilbert presentation of EAK (cf. Sections 2.4.1 and 2.4.2) is achieved by showing that the axioms of (the intuitionistic version of) EAK are derivable in $D'.EAK$. These derivations are collected in Appendix E.

Again, as was the case for $D.EAK$, the fact that $D'.EAK$ is a conservative extension of EAK can be argued as follows: let A, B be EAK -formulas such that $A \vdash_{D'.EAK} B$, and let \mathbb{Z} be the final coalgebra. By the soundness of $D'.EAK$ w.r.t. the \mathbb{Z} , this implies that $\llbracket A \rrbracket_Z \subseteq \llbracket B \rrbracket_Z$, which, by the bisimulation invariance of EAK (cf. [GKP13, Lemma 1]), implies that $\llbracket A \rrbracket_M \subseteq \llbracket B \rrbracket_M$ for every Kripke model M , which, by the completeness of EAK w.r.t. the standard Kripke semantics, implies that $A \vdash_{EAK} B$.

Adequacy of $D'.EAK$ w.r.t. Wansing's criteria. It is easy to see that the calculus $D'.EAK$ enjoys the display property (cf. Definition 2.1). Like its previous version, $D'.EAK$ is defined independently of the relational semantics of EAK , and therefore is suitable for a fine-grained proof-theoretic semantic analysis. It can be readily verified by inspection that all operational rules satisfy Wansing's criteria of *separation*, *symmetry* and *explicitness* (cf. Section 2.2.3).

Moreover, a clear-cut division of labour has been achieved between the operational rules, which are to encode the proof-theoretic meaning of the new connectives, and the structural rules, which are to express the relations entertained between the different connectives by way of their proxies.

Another important proof-theoretic feature of $D'.EAK$ is modularity. As discussed in Section 2.6.1, by suitably removing structural rules for the propositional base of $D'.EAK$, the substructural versions of EAK can be modularly defined. Moreover, by adding structural rules corresponding to properly displayable modal logics (cf. [Kra96]), different assumptions can be captured on the behaviour of the epistemic modalities.¹¹

¹¹Note that *Balance*, *comp*, *reduce*, *swap-in* and *swap-out* are the only specific structural rules for epistemic actions; the *monotonicity* and *Fischer-Servi* rules respectively encode the conditions that box and diamond are monotone and interpreted by means of the same relation; the *necessitation* can be considered as a special case of atom and IW can be eliminated if, e.g., $\perp \vdash \langle \alpha \rangle \perp$ and $[\alpha] \top \vdash \top$ are introduced as zeroary rules.

Notwithstanding the fact that the old *reverse* rules, offending *segregation*, are derived rules in D'.EAK, still the system D'.EAK does not satisfy *segregation*. However, the only rule in D'.EAK offending *segregation* is *atom* because one of the two principal formulas in each *atom* axioms might not occur in display. Even if the most rigid proof-theoretic semantic principle is not met, D'.EAK is a quasi-proper display calculus, and hence it enjoys Belnap-style cut elimination, as will be shown in the next subsection.

2.6.3 Belnap-style cut elimination for D'.EAK

In the present subsection, we prove that D'.EAK is a quasi proper display calculus (cf. Section 2.3.1), that is, the rules of D'.EAK satisfy conditions $C_1, C_2, C_3, C_4, C'_5, C''_5, C_6, C_7, C_8$. By Theorem 2.4, this is enough to establish that the calculus enjoys the cut elimination and the subformula property.

The rules *reverse* are now derivable, and all the rules with the side condition $Pre(\alpha)$ have been reformulated so as to either remove $Pre(\alpha)$ altogether, or to replace it with its structural counterpart. This has been achieved by expanding the language so that the meta-linguistic abbreviation $Pre(\alpha)$ can be replaced by an operational constant and its corresponding structural connective. Hence, it can be readily verified that all rules are closed under simultaneous substitution of arbitrary structures for congruent parameters, which satisfies conditions C_6 and C_7 . It is easy to see that the operational rules for 1_α and the *comp* rules satisfy the criteria C_1 – C_7 . The *atom* axioms can be readily seen to verify condition C''_5 as given in Section 2.3.1.

Finally, as to condition C_8 , let us show the cases involving the new connective 1_α . All the other cases are reported in Appendix D.

$$\frac{\frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{\Phi_\alpha \vdash X}{1_\alpha \vdash X}}{\Phi_\alpha \vdash X}}{\Phi_\alpha \vdash X} \rightsquigarrow \frac{\vdots \pi}{\Phi_\alpha \vdash X}$$

2.7 Conclusions and further directions

2.7.1 Conclusions

In the present chapter, we provide an analysis, conducted adopting the viewpoint of proof-theoretic semantics, of the state-of-the-art deductive systems for dynamic epistemic logic, focusing mainly on Baltag-Moss-Solecki's logic of epistemic actions and

knowledge (EAK). We start with an overview of the general research agenda in proof-theoretic semantics, and then we focus on display calculi, as a proof-theoretic paradigm which has been successful in accounting for difficult logics, such as modal logics and substructural logics. We discuss the requirements which a proof system should satisfy to provide adequate proof-theoretic semantics to logical constants, and, as an original contribution, we introduce the notion of quasi proper display calculus, and prove its corresponding Belnap-style cut elimination metatheorem. We then evaluate the main existing proof systems for PAL/EAK according to the previously discussed requirements. As the second original contribution, we propose a revised version of one such system, namely of the system D.EAK (cf. Section 2.4.4), and we argue that our revised system D'.EAK adequately meets the proof-theoretic semantic requirements for all the logical constants involved. We also show that D'.EAK is sound w.r.t. the final coalgebra semantics, complete w.r.t. EAK, of which it is a conservative extension. These three facts together guarantee that D'.EAK exactly captures EAK. Finally, we verify that D'.EAK is a quasi proper display calculus. Hence, the generalized metatheorem applies, and D'.EAK is thus shown to enjoy Belnap-style cut elimination (which was not argued for in the case of the original system D.EAK) and the subformula property. The main ingredient of this revision is an expansion of the language of the original system, aimed at achieving an *independent* proof-theoretic account of the preconditions $Pre(\alpha)$. This account is independent both in the sense that it is given purely in terms of the resources of D'.EAK, and in the sense that the metalinguistic abbreviation $Pre(\alpha)$ is treated as a first-class citizen of the revised system. Indeed, $Pre(\alpha)$ is endowed with both an operational and a structural representation, both of which well-behaving.

2.7.2 Further directions

Uniform proof-theoretic account for dynamic logics. As we mentioned early on, the results collected in the present chapter form the basis of a larger research program aimed at providing dynamic logics with a uniform proof-theoretic account. As we will see in the following chapters, this methodology has been extended to monotone modal logic and PDL. Other interesting case studies are Concurrent Propositional Dynamic Logic [Gol92b], Game Logic [Par85], Coalition Logic [Pau01, Pau02], Concurrent Dynamic Epistemic Logic [vDvdHK03], and variants of Dynamic epistemic logics with non-normal epistemic operators.

Multi-type display-style calculi. The metatheorem proven in the present chapter applies to a class of display calculi (the quasi-proper display calculi) which generalize Wansing's notion of proper display calculi by relaxing the property of isolation. However, in both

quasi proper and proper display calculi, rules are required to be closed under simultaneous substitution of *arbitrary* structures for congruent formulas. This requirement occurs in a weaker form in both the original [Bel82, Theorem 4.4] and in some of its subsequent versions [Bel90, Res00, Wan98]. Indeed, these metatheorems apply to display calculi admitting rules for which the closure under substitution may be *not arbitrary*, but restricted to structures satisfying certain conditions. This weaker requirement primarily concerns rules; however, it is encoded in the notion of *regular formula* and asks every formula to be regular. The condition given in terms of regular formulas is key to accounting for important logics such as linear logic. On the other hand, it ingeniously relies on very special features of the signature of linear logic, and hence it is of difficult application outside that setting. We conjecture that logics such as linear logic can be alternatively accounted for by display-type calculi all the rules of which are closed under simultaneous substitution of arbitrary structures for parametric operational terms (formulas). We conjecture that this is possible thanks to the introduction of a suitable *multi-type* environment, in which every derivable sequent/consecution is required to be *type-uniform* (i.e., both the antecedent and the consequent of any sequent/consecution must belong to the same type). The requirement formulated in terms of regular formulas would then be encoded in the multi-type setting in terms of the condition that, in each given rule, parametric constituents (of a given and unambiguously determined type) can be uniformly replaced by structures which are *arbitrary within that same type*, so as to obtain instances of the same rule. An example of such a multi-type environment is introduced in Chapter 3. The adaptation of the multi-type setting to the case of linear logic is work in progress.

Chapter 3

Multi-Type Display Calculus for Dynamic Epistemic Logic

3.1 Introduction

In the previous chapter, we have analyzed Baltag-Moss-Soilecki's logic of Epistemic Actions and Knowledge (EAK) from a proof-theoretic semantic perspective, and in particular we have been confronted with the fact that, as is the case of other dynamic logics, the hurdles preventing its standard proof-theoretic development are due precisely to the very features which make it interesting and suitable for applications, such as e.g. its not being closed under uniform substitution, or the existence of certain interactions between logical connectives which cannot be expressed within the language itself.

Indeed, EAK prominently features non schematic axioms such as

$$[\alpha]p \leftrightarrow (Pre(\alpha) \rightarrow p),$$

where the variable p ranges over atomic propositions, and $Pre(\alpha)$ is a meta-linguistic abbreviation for an arbitrary formula, and axioms such as

$$[\alpha][a]A \leftrightarrow (Pre(\alpha) \rightarrow \bigwedge \{[a][\beta]A \mid \alpha a \beta\}),$$

in which the extra-linguistic label $\alpha a \beta$ expresses the fact that actions α and β are indistinguishable for agent a .

Difficulties posed by features such as these caused the existing proposals of calculi in the literature to be often ad hoc, not easily generalizable e.g. to other logics, and more in general lacking a smooth proof-theoretic behaviour. In particular, the difficulty in

smoothly transferring results from one logic to another is a problem in itself, since logics such as EAK typically come in large families. Hence, proof-theoretic approaches which uniformly apply to each logic in a given family are in high demand.

Contribution. The present chapter focuses on the core technical aspects of a proof-theoretic methodology and set-up closely linked to Belnap’s display calculi [Bel82]. Specifically, the main contribution here is the introduction of a methodology for the design of display calculi based on *multi-type languages*. In the case study provided by EAK, we start by observing that having to resort to the label $\alpha a \beta$ is symptomatic of the fact that the language of EAK lacks the necessary expressivity to autonomously capture the piece of information encoded in the label.

In order to provide the desired additional expressivity, we introduce a language in which not only formulas are generated from formulas and actions (as it happens in the symbol $\langle \alpha \rangle A$) and formulas are generated from formulas and agents (as it happens in the symbol $\langle a \rangle A$), but also *actions* are generated from the interaction between *agents* and *actions*, which is precisely what the label $\alpha a \beta$ is about.

In the multi-type language for EAK introduced in the present chapter, each generation step mentioned above is explicitly accounted for via special connectives taking arguments of *different types*. In principle, more than one alternative is possible in this respect; our choice for the present setting consists of the following types: Ag for agents, Fnc for functional actions, Act for actions, and Fm for formulas. Hence, the present setting introduces a separation between functional, i.e. deterministic actions, of type Fnc, and possibly non-deterministic actions, of type Act (see discussion at the end of Section 3.3).

The proposed calculus provides an interesting and in our opinion very promising *methodological* platform towards the uniform development of a general proof-theoretic account of all dynamic logics, and also, from a purely structurally proof-theoretic viewpoint, for clarifying and sharpening the formulation of criteria leading to the statement and proof of meta-theoretic results such as Belnap-style cut elimination (see Section 3.7).

Organization and results. For preliminaries on display calculi, EAK, the intuitionistic version of EAK and the (single-type) display calculus D’.EAK, the reader is referred to Sections 2.2.2, 2.4.1, 2.4.2, and 2.6.1 respectively. In Section 3.2, we sketch the general features of the environment of multi-type display calculi, extend Wansing’s definition of quasi-proper display calculi to the multi-type setting, and prove the corresponding

extension of Belnap’s cut elimination metatheorem. In Section 3.3, we propose a novel display calculus for EAK, which we refer to as *Dynamic Calculus*, and which concretely exemplifies the notion of multi-type display calculus. In Sections 3.4, 3.5 and 3.6, we prove that the Dynamic Calculus adequately captures EAK, and enjoys Belnap-style cut elimination. In Section 3.7, we collect some conclusions and indicate further directions. The routine proofs and derivations are collected in the Appendices F and G.

3.2 Multi-type calculi, and cut elimination metatheorem

The present section is aimed at introducing the environment of multi-type display calculi. Our treatment will be very general, and in particular, no signature will be specified. However, the calculus introduced in Section 3.3 is a concrete instantiation of this abstract description.

3.2.1 Multi-type calculi

Our starting point is a propositional language, the terms of which form n pairwise disjoint types $T_1 \dots T_n$, each of which with its own signature. We will use a, b, c and x, y, z to respectively denote operational and structural terms of unspecified (possibly different) type. Further, we assume that operational connectives and structural connectives are given *both* within each type *and* also between different types, so that the display property holds.

In the applications we have in mind, the need will arise to support types that are semantically ordered by inclusion. For example, in Section 3.3 we will introduce, beside the type Fm of formulas, two types Fnc and Act of functional and general actions, respectively. The need for enforcing the distinction between functional and general actions in the specific situation of Section 3.3 arises because of the presence of the rule *balance* (see page 134 for more details on this topic). The semantic point of view suggests to treat Fnc as a proper subset of Act , but our syntactic stipulations, although will be sound w.r.t. this state of affairs, will be tuned for the more general situation in which the sets Fnc and Act are disjoint. This is convenient as each term can be assigned a unique type *unambiguously*. This is a crucial requirement for the Belnap-style cut elimination theorem of the next section, and will be explicitly stated in condition C'_2 below.

Definition 3.1. A sequent $x \vdash y$ is *type-uniform* if x and y are of the same type T . In this case, we will say that $x \vdash y$ is of type T .

A fundamental and very natural desideratum for rules in a multi-type display calculus is that they preserve type-uniformity, that is, each rule should be such that if all the premises are type uniform, then the conclusion is type uniform. As we will see, all rules in the multi-type calculus introduced in Section 3.3 preserve type uniformity.

Finally, in a display calculus, the cut rule is typically of the following form:

$$\frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \textit{Cut}$$

where X, Y are structures and A is a formula. This translates straightforwardly to the multi-type environment, by the stipulation that cut rules of the form

$$\frac{x \vdash a \quad a \vdash y}{x \vdash y} \textit{Cut}$$

are allowed in the given multi-type system for each type. These cut rules will be asked to satisfy the following additional requirement:

Definition 3.2. A rule is *strongly type-uniform* if its premises and conclusion are of the same type.

3.2.2 Relativized display property

As discussed in Section 4 (on page 7), the full display property is a key ingredient in the proof of the cut elimination metatheorem. Indeed, it enables a system enjoying it to meet Belnap's condition C_8 of the cut elimination metatheorem. However, it turns out that an analogously good behaviour can be guaranteed of any sequent calculus enjoying the following weaker property:

Definition 3.3. A proof system enjoys the *relativized display property* iff for every derivable sequent $X \vdash Y$ and every substructure Z of either X or Y , the sequent $X \vdash Y$ can be transformed, using the rules of the system, into a logically equivalent sequent which is either of the form $Z \vdash W$ or of the form $W \vdash Z$, for some structure W .

The calculus defined in Section 3.3 does not enjoy the full display property, but does enjoy the relativized display property above (more about this in Sections 3.3 and 3.6), which enables it to verify the condition C'_8 (see Section 3.2.3). More details about it are collected in Appendix F. Finally, notice that the definition of substructures in precedent or succedent position within each sequent can be given in a way which does not rely on the full display property. It is enough to rely on the polarity of the coordinates of each structural connective: if these polarities are assigned, then for any sequent $X \vdash Y$,

if Z is a substructure of X , then Z is in precedent (resp. succedent) position if, in the generation tree of X , the path from Z to the root goes through an even (resp. odd) number of coordinates with negative polarity. If Z is a substructure of Y , then Z is in succedent (resp. precedent) position if, in the generation tree of Y , the path from Z to the root goes through an even (resp. odd) number of coordinates with negative polarity.

3.2.3 Quasi-proper multi-type display calculi

In Chapter 2, to show that Belnap-style cut elimination holds for the display calculus $D'.EAK$, the definition of quasi-proper display calculi is given (generalizing Wansing's definition of properly displayable calculi [Wan98, Section 4.2]), and its corresponding Belnap style metatheorem is discussed. We are working towards the proof that the multi-type display calculus introduced in Section 3.3 enjoys cut elimination Belnap-style. The aim of the present subsection is then to extend the notion of quasi-proper display calculi to the multi-type environment. Let a *quasi-proper multi-type display calculus* be any display calculus in a multi-type language satisfying the following list of conditions¹:

C_1 : preservation of operational terms. Each operational term occurring in a premise of an inference rule *inf* is a subterm of some operational term in the conclusion of *inf*.

C_2 : Shape-alikeness of parameters. Congruent parameters² are occurrences of the same structure.

C'_2 : Type-alikeness of parameters. Congruent parameters have exactly the same type. This condition bans the possibility that a parameter changes type along its history.

C'_3 : Restricted non-proliferation of parameters. Each parameter in an inference rule *inf* is congruent to at most one constituent in the conclusion of *inf*. This restriction does not need to apply to parameters of any type T such that the only applications of cut with cut terms of type T are of the following shapes:

¹See Chapter 2 for a discussion on C'_5 and C''_5 .

²The congruence relation is an equivalence relation which is meant to identify the different occurrences of the same formula or substructure along the branches of a derivation [Bel82, Section 4], [Res00, Definition 6.5]. Condition C_2 can be understood as a condition on the *design* of the rules of the system if the congruence relation is understood as part of the specification of each given rule; that is, each rule of the system should come with an explicit specification of which elements are congruent to which (and then the congruence relation is defined as the reflexive and transitive closure of the resulting relation). In this respect, C_2 is nothing but a sanity check, requiring that the congruence is defined in such a way that indeed identifies the occurrences which are intuitively “the same”.

$$\frac{\begin{array}{c} \vdots \\ X \vdash a \end{array} \quad a \vdash a}{X \vdash a} \quad \frac{a \vdash a \quad \begin{array}{c} \vdots \\ a \vdash Y \end{array}}{a \vdash Y}$$

C₄: Position-alikeness of parameters. Congruent parameters are either all antecedent or all succedent parts of their respective sequents.

C'₅: Quasi-display of principal constituents. If an operational term a is principal in the conclusion sequent s of a derivation π , then a is in display, unless π consists only of its conclusion sequent s (i.e. s is an axiom).

C''₅: Display-invariance of axioms. If a display rule can be applied to an axiom s , the result of that rule application is again an axiom.

C'₆: Closure under substitution for succedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in succedent position, *within each type*.

C'₇: Closure under substitution for precedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in precedent position, *within each type*.

Condition C₆ (and likewise C'₇) ensures, for instance, that if the following inference is an application of the rule R :

$$\frac{(x \vdash y)([a]_i^{suc} \mid i \in I)}{(x' \vdash y')[a]^{suc}} R$$

and $([a]_i^{suc} \mid i \in I)$ represents all and only the occurrences of the operational term a in the premiss which are congruent to the occurrence of a in the conclusion³, then also the following inference is an application of the same rule R :

$$\frac{(x \vdash y)([z/a]_i^{suc} \mid i \in I)}{(x' \vdash y')[z/a]^{suc}} R$$

where the structure z is substituted for a , and z and a have the same type.

³Clearly, if $I = \emptyset$, then the occurrence of a in the conclusion is congruent to itself.

C'₈: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, and hence each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C'₈ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of the cut rule, involving proper subterms of the original operational cut-term. In addition to this, specific to the multi-type setting is the requirement that the new application(s) of the cut rule be also *strongly type-uniform* (cf. condition C₁₀ below).

C''₈: Closure of axioms under cut. If $x \vdash a$ and $a \vdash y$ are axioms, then $x \vdash y$ is again an axiom.

C₉: Type-uniformity of derivable sequents. Each derivable sequent is type-uniform.

C₁₀: Strong type-uniformity of cut rules. All cut rules are strongly type-uniform (cf. Definition 3.2).

3.2.4 Belnap-style metatheorem for multi-types

In the present subsection, we state and prove the Belnap-style metatheorem which we will appeal to when establishing the cut elimination Belnap-style for the calculus we will introduce in the next section.

Theorem 3.4. *Any multi-type display calculus satisfying $C_2, C'_2, C'_3, C_4, C'_5, C''_5, C'_6, C'_7, C'_8, C''_8, C_9$ and C_{10} is cut-admissible. If also C_1 is satisfied, then the calculus enjoys the subformula property.*

Proof. This is a generalization of the proof in [Wan02, Section 3.3, Appendix A]. For the sake of conciseness, we will expand only on the parts of the proof which depart from that treatment. As usual, the proof is done by induction on the ordered pair of parameters given by the complexity of the cut term and the height of the cut. Our original derivation is

$$\frac{\begin{array}{c} \vdots \pi_1 \\ x \vdash a \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ a \vdash y \end{array}}{x \vdash y}$$

Principal stage: both cut formulas are principal. There are three subcases.

If the end sequent $x \vdash y$ is identical to the conclusion of π_1 (resp. π_2), then we can eliminate the cut simply replacing the derivation above with π_1 (resp. π_2).

If the premises $x \vdash a$ and $a \vdash y$ are axioms, then, by C''_8 , the conclusion $x \vdash y$ is an axiom, therefore the cut can be eliminated by simply replacing the original derivation with $x \vdash y$.

If one of the two premises of the cut in the original derivation is not an axiom, then, by C'_8 , there is a proof of $x \vdash y$ which uses the same premise(s) of the original derivation and which involves only strongly uniform cuts on proper subterms of a .

Parametric stage: at least one cut term is parametric. There are two subcases: either one cut term is principal or they are both parametric.

Consider the subcase in which one cut term is principal. W.l.o.g. we assume that the cut-term a is principal in the left-premise $x \vdash a$ of the cut in the original proof (the other case is symmetric). We can assume w.l.o.g. that the conclusion of the cut is different from either of its premises. Then, conditions C_2 and C'_3 make it possible to trace the history-tree of the occurrences of the cut-term a in π_2 (cf. Remark 2.2 on page 71), and by conditions C'_2 and C_4 , any ancestor of a is of the same type and in the same position (that is, is in precedent position). The situation can be pictured as follows:

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_{2,i} \\
 \underline{a_i \vdash y_i}
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \pi_{2,j} \\
 (x_j \vdash y_j)[\underline{a_j}]^{pre}
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \pi_{2,k} \\
 (x_k \vdash y_k)[\bar{a}_k]^{pre}
 \end{array}
 \\
 \vdots \qquad \qquad \vdots \qquad \qquad \vdots
 \\
 \vdots \qquad \qquad \vdots \qquad \qquad \vdots
 \\
 \qquad \qquad \qquad \vdots \pi_2
 \\
 \begin{array}{c}
 \vdots \pi_1 \\
 \underline{x \vdash a}
 \end{array}
 \qquad
 \begin{array}{c}
 a \vdash y
 \end{array}
 \\
 \hline
 x \vdash y
 \end{array}$$

where, for $i, j, k \in \{1, \dots, n\}$, the nodes

$$\underline{a}_i \vdash y_i, \quad (x_j \vdash y_j)[\underline{a}_j]^{pre}, \quad \text{and} \quad (x_k \vdash y_k)[\bar{a}_k]^{pre}$$

represent the three ways in which the leaves a_i , a_j and a_k in the history-tree of a in π_2 can be introduced, and which will be discussed below. The notation \underline{a} (resp. \bar{a}) indicates that the given occurrence is principal (resp. parametric). Notice that condition C_4 guarantees that all occurrences in the history of a are in precedent position in the underlying derivation tree, and condition C'_2 guarantees that the type of a never changes along its history. Let a_l be introduced as a parameter (as represented in the picture above in the conclusion of $\pi_{2,k}$ for $a_l = a_k$). Assume

that $(x_k \vdash y_k)[\bar{a}_k]^{pre}$ is the conclusion of an application *inf* of the rule *Ru* (for instance, in the calculus of Section 3.3, this situation arises if a_k is of type *Fm* and has been introduced with an application of Weakening, or if a_k is of type *Fnc* and has been introduced with an application of *Atom*, or *Balance*). Since a_k is a leaf in the history-tree of a , we have that a_k is congruent only to itself in $x_k \vdash y_k$. Notice that the assumption that every derivable sequent is type-uniform (C₉), and the type-alikeness of parameters (C’₂) imply that the sequent a_1, a_k and x have the same type. Hence, C’₇ implies that it is possible to substitute x for a_k by means of an application of the same rule *Ru*. That is, $(x_k \vdash y_k)[\bar{a}_k]$ can be replaced by $(x_k \vdash y_k)[\bar{x}/\bar{a}_k]$.

Let a_l be introduced as a principal formula. The corresponding subcase in [Wan02] splits into two subsubcases: either a_l is introduced in display or it is not.

If a_l is in display (as represented in the picture above in the conclusion of $\pi_{2,i}$ for $a_l = a_i$), then we form a subderivation using π_1 and $\pi_{2,i}$ and applying cut as the last rule. The assumptions that the original cut is strongly type-uniform (C₁₀), that every derivable sequent is type-uniform (C₉), and the type-alikeness of parameters (C’₂) imply that the sequent $\underline{a}_i \vdash y_i$ is of the same type as the sequents $x \vdash a$ and $a \vdash y$. Hence, the new cut is strongly type-uniform.

If a_l is not in display (as represented in the picture above in the conclusion of $\pi_{2,j}$ for $a_l = a_j$), then condition C’₅ implies that $(x_j \vdash y_j)[\underline{a}_j]^{pre}$ is an axiom, and C’₅ implies that some axiom $\underline{a}_j \vdash y'_j$ exists, which is display-equivalent to the first axiom, and in which a_j occurs in display. Let π' be the derivation which transforms $\underline{a}_j \vdash y'_j$ into $(x_j \vdash y_j)[\underline{a}_j]^{pre}$. We form a subderivation using π_1 and $\underline{a}_j \vdash y'_j$ and joining them with a cut application, then attaching $\pi'[x/a_j]^{pre}$ below the new cut.

The transformations just discussed explain how to transform the leaves of the history tree of a . Finally, since, as discussed above, x has the same type of a , condition C’₇ implies that substituting x for each occurrence of a in the history tree of the cut term a in π_2 (and in each occurring π' as above) gives rise to an admissible derivation $\pi_2[x/a]^{pre}$ (use C’₆ for the symmetric case).

Summing up, this procedure generates the following proof tree:

$$\begin{array}{c}
 \vdots \pi_1 \\
 x \vdash a \quad \underline{a}_j \vdash y' \\
 \hline
 x \vdash y' [a]^{suc} \\
 \vdots \pi' [x/a]^{pre} \quad \vdots \pi_{2,k} \\
 (x_j \vdash y_j)[x/a_j]^{pre} \quad (x_k \vdash y_k)[\bar{x}/\bar{a}_k]^{pre} \\
 \vdots \quad \vdots \\
 \vdots \vdots \vdots \pi_2 [x/a]^{pre} \\
 x \vdash y
 \end{array}$$

We observe that in each newly introduced application of the cut rule, both cut terms are principal. Hence, we can apply the procedure described in the Principal stage and transform the original derivation in a derivation in which the cut terms of the newly introduced cuts have strictly lower complexity than the original cut terms. When the newly introduced applications of cut are of lower height than the original one, we do not need to resort to the Principal stage.⁴

Finally, as to the subcase in which both cut terms are parametric, consider a proof with at least one cut. The procedure is analogous to the previous case. Namely, following the history of one of the cut terms up to the leaves, and applying the transformation steps described above, we arrive at a situation in which, whenever new applications of cuts are generated, in each such application at least one of the cut formulas is principal. To each such cut, we can apply (the symmetric version of) the Parametric stage described so far.

□

3.3 The Dynamic Calculus for EAK

As mentioned in the introduction, the key idea is to introduce a language in which not only formulas are generated from formulas and actions (as it happens in the symbol $\langle \alpha \rangle A$) and formulas are generated from formulas and agents (as it happens in the symbol $\langle a \rangle A$), but also *actions* are generated from the interactions between *agents* and *actions*.

An algebraically motivated introduction. In the present section, we define a *multi-type language* into which the language of (I)EAK translates, and in which each generation step mentioned above is explicitly accounted for via special binary connectives taking arguments of *different types*. More than one alternative is possible in this respect; our choice for the present setting consists of the following types: Ag for agents, Fnc for functional actions, Act for actions, and Fm for formulas. We also stipulate that Ag, Act, Fm and Fnc are pairwise disjoint. The new connectives, and their types, are:

$$\Delta_0, \blacktriangle_0 : \text{Fnc} \times \text{Fm} \rightarrow \text{Fm} \quad (3.1)$$

$$\Delta_1, \blacktriangle_1 : \text{Act} \times \text{Fm} \rightarrow \text{Fm} \quad (3.2)$$

$$\Delta_2, \blacktriangle_2 : \text{Ag} \times \text{Fm} \rightarrow \text{Fm} \quad (3.3)$$

$$\Delta_3, \blacktriangle_3 : \text{Ag} \times \text{Fnc} \rightarrow \text{Act} \quad (3.4)$$

⁴ This is for instance the case if, in the original derivation, the history-tree of the cut term a (in the right-hand-side premise of the given cut application) contains at most one leaf a_i which is principal. However, the procedure described above in the Parametric stage does not always produce cuts of lower height. For instance, in the calculus introduced in Section 3.3, this situation may arise when two ancestors of a cut term of type Fm are introduced as principal along the same branch, and then are identified via an application of the rule Contraction.

We stipulate that the interpretations of the connectives are maps preserving existing joins in each coordinate (see below) with algebras as domains and codomains suitable to interpret (functional) actions, formulas, and agents respectively. For instance, suitable choices for domains of interpretation for formulas can be complete atomic Boolean algebras or perfect Heyting algebras (cf. [KP13]); in the setting of e.g. epistemic action logic (cf. [vDvdHK07]), following [BCS07], the domain of interpretation for actions can be a quantale or a relation algebra (of which the functional actions can be a sub-monoid). In the setting of EAK, in which no algebraic structure is required of actions and agents, a suitable domain of interpretation can be a complete join-semilattice, which is completely join-generated by a given subset (interpreting the functional actions), and the domain of interpretation of agents can be a set.⁵

In Section 3.4, the final coalgebra Z (more details on the final coalgebra in Section 2.5) is taken as semantic environment for the Dynamic Calculus. In this setting, the boolean algebra $\mathcal{P}Z$ is taken as the domain of interpretation for Fm-type terms, Fnc-type terms are interpreted as graphs of partial functions on Z , subject to certain restrictions, and the domain of interpretation of Act-type terms is the complete \cup -semilattice generated by the domain of interpretation of Fnc.

In all the domains of interpretation which are complete lattices (i.e. the algebras interpreting terms of type Fm and Act), the fact that the interpretation of each connective Δ and \blacktriangle is completely join-preserving in its second coordinate implies that it has a right adjoint in its second coordinate. These right adjoints provide natural interpretation for the following additional connectives:

$$\rightarrow_0, \rightarrow_0 : \text{Fnc} \times \text{Fm} \rightarrow \text{Fm} \quad (3.5)$$

$$\rightarrow_1, \rightarrow_1 : \text{Act} \times \text{Fm} \rightarrow \text{Fm} \quad (3.6)$$

$$\rightarrow_2, \rightarrow_2 : \text{Ag} \times \text{Fm} \rightarrow \text{Fm}. \quad (3.7)$$

The assumptions above imply that Δ_1 and \blacktriangle_1 have right adjoints also in their first coordinate. Hence, each of the following connectives can be naturally interpreted, in the setting above, as the right adjoint of Δ_1 and \blacktriangle_1 respectively:

$$\blacktriangleleft_1, \triangleleft_1 : \text{Fm} \times \text{Fm} \rightarrow \text{Act}. \quad (3.8)$$

Intuitively, for all formulas A, B , the term $B \blacktriangleleft_1 A$ denotes the weakest epistemic action γ such that, if A was true before γ was performed, then B is true after any successful execution of γ . This is also related to Vaughn Pratt's notion of weakest preserver (cf.

⁵Notice also that for other dynamic logics the domain of interpretation of agents might be endowed with some algebraic structure; for instance, in the case of game logic (cf. [PP03]), the set of agents consists of two elements, on which a negation-type operation can be assumed.

[Pra91, Section 4.2]) However, we cannot assume that more adjoints exist, which would provide semantic interpretation for the following symbols:

$$\begin{aligned}
\leftarrow_0, \leftarrow_0 & : \text{Fm} \times \text{Fm} \rightarrow \text{Fnc} \\
\leftarrow_2, \leftarrow_2 & : \text{Fm} \times \text{Fm} \rightarrow \text{Ag} \\
\leftarrow_3, \leftarrow_3 & : \text{Act} \times \text{Fnc} \rightarrow \text{Ag} \\
\rightarrow_3, \rightarrow_3 & : \text{Ag} \times \text{Act} \rightarrow \text{Fnc}.
\end{aligned}$$

Virtual adjoints. We adopt the following notational convention about the three different shapes of arrows introduced so far. Arrows with a straight tail (\rightarrow and \rightarrow) stand for connectives which have a semantic counterpart and which are included in the language of the Dynamic Calculus (see the grammar of operational terms on page 127); arrows with no tail (e.g. \leftarrow and \leftarrow) do have a semantic interpretation but are *not* included in the language, and arrows with a squiggly tail (\rightsquigarrow , \leftarrow , \rightarrow and \leftarrow) stand for syntactic objects, called *virtual adjoints*, which do not have a semantic interpretation, but will play an important role, namely guaranteeing the Dynamic Calculus to enjoy the relativized display property (cf. Definition 3.3). In what follows, virtual adjoints will be introduced *only* as structural connectives. That is, they will not correspond to any operational connective, and they will not appear actively in any rule schema other than the display postulates (cf. Definition 2.1). As will be shown in Section 3.6, these limitations keep the calculus sound even if virtual adjoints do not have an independent semantic interpretation.

The $\Delta \dashv \rightarrow$ and $\blacktriangle \dashv \rightarrow$ adjunction relations stipulated above translate into the following clauses for every agent a , every functional action α , every action γ , and every formula A :

$$\alpha \Delta_0 A \leq B \text{ iff } A \leq \alpha \rightarrow_0 B \quad \alpha \blacktriangle_0 A \leq B \text{ iff } A \leq \alpha \rightarrow_0 B \quad (3.9)$$

$$\gamma \Delta_1 A \leq B \text{ iff } A \leq \gamma \rightarrow_1 B \quad \gamma \blacktriangle_1 A \leq B \text{ iff } A \leq \gamma \rightarrow_1 B \quad (3.10)$$

$$a \Delta_2 A \leq B \text{ iff } A \leq a \rightarrow_2 B \quad a \blacktriangle_2 A \leq B \text{ iff } A \leq a \rightarrow_2 B. \quad (3.11)$$

The adjunction relations $\Delta_1 \dashv \leftarrow_1$ and $\blacktriangle_1 \dashv \leftarrow_1$ translate into the following clauses for every action γ and every formula A :

$$\gamma \Delta_1 A \leq B \text{ iff } \gamma \leq B \leftarrow_1 A \quad \gamma \blacktriangle_1 A \leq B \text{ iff } \gamma \leq B \leftarrow_1 A. \quad (3.12)$$

As we will see, the display postulates corresponding to triangle- and arrow-shaped connectives are modelled over the conditions (3.9)-(3.12) above. Also the display postulates involving virtual adjoints are shaped in the same way, which explains their name.

Translating D'.EAK into the multi-type setting. The intended link between the language of D'.EAK (cf. Section 2.6.1) and the language of the Dynamic Calculus is illustrated in the following table:

$\langle \alpha \rangle A$	becomes	$\alpha \Delta_0 A$		$\widehat{\alpha} A$	becomes	$\alpha \blacktriangle_0 A$
$\langle a \rangle A$	becomes	$a \Delta_2 A$		$\widehat{a} A$	becomes	$a \blacktriangle_2 A$
$[\alpha] A$	becomes	$\alpha \rightarrow_0 A$		$\overline{\alpha} A$	becomes	$\alpha \rightarrow_0 A$
$[a] A$	becomes	$a \rightarrow_2 A$		$\overline{a} A$	becomes	$a \rightarrow_2 A$
1_α	becomes	$\alpha \Delta_0 \top$				

The table above can be extended to the definition of a formal translation between the operational language of D'.EAK and that of the Dynamic Calculus, simply by preserving the non modal propositional fragment. We omit the details of this straightforward inductive definition. In Section 3.4, this translation will be elaborated on, and the interpretation of the language of the Dynamic Calculus in the final coalgebra will be defined so that the translation above preserves the validity of sequents. In the light of this translation, the adjunction conditions in clauses (3.9) correspond to the adjunction conditions (2.15) in D'.EAK, which, in their turn, motivate the display postulates reported on in Section 2.5:

$$\langle \alpha \rangle \dashv \overline{\alpha} \quad \widehat{\alpha} \dashv [\alpha].$$

The connectives Δ_3 and \blacktriangle_3 have no counterpart in the language of D'.EAK, but the introduction of \blacktriangle_3 is exactly what brings the additional expressiveness we need in order to eliminate the label. Indeed, we stipulate that for every a and α as above,

$$a \blacktriangle_3 \alpha = \bigvee \{ \beta \mid \alpha a \beta \}. \quad (3.13)$$

A way to understand this stipulation is in the light of the discussion in Section 2.4.3 after clause (2.8). There, in the context of a discussion about the proof system in [BCS07], the link between the semantic condition $f_A^M(m \star q) \leq f_A^M(m) \star f_A^Q(q)$ (cf. [BCS07, Definitions 2.2(2) and 2.3]) and the axiom (2.4)—which in [BCS07] was left implicit—is made more explicit, by understanding the action $f_A^Q(q)$ as the join, taken in Q , of all the actions q' which are indistinguishable from q for the agent A . In the present setting, the stipulation (3.13) says that $a \blacktriangle_3 \alpha$ encodes exactly the same information encoded in $f_A^Q(q)$, namely, the non-deterministic choice between all the actions that are indistinguishable from α for the agent a .

Additional conditions. As was the case in the setting of D'EAK, in order to express in this new language that e.g. $\langle \alpha \rangle$ and $[\alpha]$ are “interpreted over the same relation”, Sahlqvist correspondence theory (cf. e.g. [CP12, CPS, CGP14] for a state-of-the-art-treatment) provides us with two alternatives: one of them is that we impose the following Fischer Servi-type conditions to hold for every $\mathbf{a} \in \mathbf{Ag}$, $\alpha \in \mathbf{Fnc}$, $\gamma \in \mathbf{Act}$ and $A, B \in \mathbf{Fm}$:

$$\begin{array}{ll}
(\alpha \Delta_0 A) \rightarrow (\alpha \rightarrow_0 B) \leq \alpha \rightarrow_0 (A \rightarrow B) & (\alpha \blacktriangle_0 A) \rightarrow (\alpha \rightarrow_0 B) \leq \alpha \rightarrow_0 (A \rightarrow B) \\
(\gamma \Delta_1 A) \rightarrow (\gamma \rightarrow_1 B) \leq \gamma \rightarrow_1 (A \rightarrow B) & (\gamma \blacktriangle_1 A) \rightarrow (\gamma \rightarrow_1 B) \leq \gamma \rightarrow_1 (A \rightarrow B) \\
(\mathbf{a} \Delta_2 A) \rightarrow (\mathbf{a} \rightarrow_2 B) \leq \mathbf{a} \rightarrow_2 (A \rightarrow B) & (\mathbf{a} \blacktriangle_2 A) \rightarrow (\mathbf{a} \rightarrow_2 B) \leq \mathbf{a} \rightarrow_2 (A \rightarrow B).
\end{array}$$

$$\begin{array}{ll}
\alpha \Delta_0 (A \succ B) \leq (\alpha \rightarrow_0 A) \succ (\alpha \Delta_0 B) & \alpha \blacktriangle_0 (A \succ B) \leq (\alpha \rightarrow_0 A) \succ (\alpha \blacktriangle_0 B) \\
\gamma \Delta_1 (A \succ B) \leq (\gamma \rightarrow_1 A) \succ (\gamma \Delta_1 B) & \gamma \blacktriangle_1 (A \succ B) \leq (\gamma \rightarrow_1 A) \succ (\gamma \blacktriangle_1 B) \\
\mathbf{a} \Delta_2 (A \succ B) \leq (\mathbf{a} \rightarrow_2 A) \succ (\mathbf{a} \Delta_2 B) & \mathbf{a} \blacktriangle_2 (A \succ B) \leq (\mathbf{a} \rightarrow_2 A) \succ (\mathbf{a} \blacktriangle_2 B).
\end{array}$$

To see that the conditions above correspond to the usual Fischer Servi axioms in standard modal languages, one can observe that the conditions in the first and third line above are images, under the translation discussed above, of the Fischer Servi axioms reported on in Section 2.4.2). The second alternative is to impose that, for every $0 \leq i \leq 2$, the connectives Δ_i and \blacktriangle_i yield *conjugated* diamonds (cf. discussion in Section 2.6.2); that is, the following inequalities hold for all $\mathbf{a} \in \mathbf{Ag}$, $\alpha, \beta \in \mathbf{Fnc}$, and $A, B \in \mathbf{Fm}$:

$$\begin{array}{ll}
(\alpha \Delta_0 A) \wedge B \leq \alpha \Delta_0 (A \wedge \alpha \blacktriangle_0 B) & (\alpha \blacktriangle_0 A) \wedge B \leq \alpha \blacktriangle_0 (A \wedge \alpha \Delta_0 B) \\
(\gamma \Delta_1 A) \wedge B \leq \gamma \Delta_1 (A \wedge \gamma \blacktriangle_1 B) & (\gamma \blacktriangle_1 A) \wedge B \leq \gamma \blacktriangle_1 (A \wedge \gamma \Delta_1 B) \\
(\mathbf{a} \Delta_2 A) \wedge B \leq \mathbf{a} \Delta_2 (A \wedge \mathbf{a} \blacktriangle_2 B) & (\mathbf{a} \blacktriangle_2 A) \wedge B \leq \mathbf{a} \blacktriangle_2 (A \wedge \mathbf{a} \Delta_2 B).
\end{array}$$

$$\begin{array}{ll}
\alpha \rightarrow_0 (A \vee \alpha \rightarrow_0 B) \leq (\alpha \rightarrow_0 A) \vee B & \alpha \rightarrow_0 (A \vee \alpha \rightarrow_0 B) \leq (\alpha \rightarrow_0 A) \vee B \\
\gamma \rightarrow_1 (A \vee \gamma \rightarrow_1 B) \leq (\gamma \rightarrow_1 A) \vee B & \gamma \rightarrow_1 (A \vee \gamma \rightarrow_1 B) \leq (\gamma \rightarrow_1 A) \vee B \\
\mathbf{a} \rightarrow_2 (A \vee \mathbf{a} \rightarrow_2 B) \leq (\mathbf{a} \rightarrow_2 A) \vee B & \mathbf{a} \rightarrow_2 (A \vee \mathbf{a} \rightarrow_2 B) \leq (\mathbf{a} \rightarrow_2 A) \vee B.
\end{array}$$

The conditions in the first and third line above are images, under the translation discussed above, of the conjugation conditions reported on in Section 2.6.2.

The operational language, formally. Let us introduce the operational terms of the multi-type language by the following simultaneous induction, based on sets \mathbf{AtProp} of atomic

propositions, Fnc of functional actions, and Ag of agents:

$$\begin{aligned}
\text{Fm} \ni A &::= p \mid \perp \mid \top \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \succ A \mid \\
&\quad \alpha \Delta_0 A \mid \alpha \rightarrow_0 A \mid \gamma \Delta_1 A \mid \gamma \rightarrow_1 A \mid \mathbf{a} \Delta_2 A \mid \mathbf{a} \rightarrow_2 A \mid \\
&\quad \alpha \blacktriangle_0 A \mid \alpha \rightarrow_0 A \mid \gamma \blacktriangle_1 A \mid \gamma \rightarrow_1 A \mid \mathbf{a} \blacktriangle_2 A \mid \mathbf{a} \rightarrow_2 A \\
\text{Fnc} \ni \alpha &::= \alpha \\
\text{Act} \ni \gamma &::= \mathbf{a} \blacktriangle_3 \alpha \mid \mathbf{a} \Delta_3 \alpha \\
\text{Ag} \ni \mathbf{a} &::= \mathbf{a}
\end{aligned}$$

The fundamental difference between the language above and the language of D'.EAK is that, in D'.EAK, agents and actions are *parametric indexes* in the construction of formulas, which are the only first-class citizens. In the present setting, however, each type lives on a par with any other. Because of the relative simplicity of the EAK setting, two of the four types are attributed no algebraic structure at the operational level. However, it is not difficult to enrich the algebraic structure of those types with sensible and intuitive operations: for instance, the skip and crash actions are functional, and parallel and sequential composition and iteration on functional actions preserve functionality, hence can be added to the array of constructors for Fnc. As a consequence of the fact that each type is a first-class citizen, as we will see shortly, four types of structures will be defined, and the turnstile symbol in the sequents of this calculus will be interpreted in the appropriate domain.

On the meta-linguistic labels $\alpha\mathbf{a}\beta$. Let us illustrate how the label $\alpha\mathbf{a}\beta$ can be subsumed when translating D'.EAK-formulas in the multi-type language. Consider for example (the intuitionistic counterparts of) the following axiom (cf. (2.4)):

$$(\text{Pre}(\alpha) \rightarrow \bigwedge \{[\mathbf{a}][\beta]A \mid \alpha\mathbf{a}\beta\}) \rightarrow [\alpha][\mathbf{a}]A$$

By applying the translation above we get:

$$(\alpha \Delta_0 \top \rightarrow \bigwedge \{\mathbf{a} \rightarrow_2 (\beta \rightarrow_0 A) \mid \alpha\mathbf{a}\beta\}) \rightarrow \alpha \rightarrow_0 (\mathbf{a} \rightarrow_2 A).$$

Since (the semantic interpretation of) \rightarrow_2 is completely meet-preserving in the second coordinate, the clause above is semantically equivalent to the following one:

$$(\alpha \Delta_0 \top \rightarrow [\mathbf{a} \rightarrow_2 \bigwedge \{\beta \rightarrow_0 A \mid \alpha\mathbf{a}\beta\}]) \rightarrow \alpha \rightarrow_0 (\mathbf{a} \rightarrow_2 A).$$

The next step is the only place of the chapter in which we will need to assume that (the domains of interpretation of Fcn and Act are such that) $\text{Fcn} \subseteq \text{Act}$. Under this

assumption, \rightarrow_0 can be taken as the restriction of \rightarrow_1 . By general order-theoretic facts (see e.g. [DP02]), the latter is completely join-reversing in its first coordinate. Hence, we can equivalently rewrite the clause above as follows:

$$(\alpha \Delta_0 \top \rightarrow [a \rightarrow_2 (\bigvee \{\beta \mid \alpha a \beta\} \rightarrow_1 A)]) \rightarrow \alpha \rightarrow_0 (a \rightarrow_2 A).$$

Now we apply the stipulation (3.13) and get the following :

$$(\alpha \Delta_0 \top \rightarrow [a \rightarrow_2 ((a \blacktriangle_3 \alpha) \rightarrow_1 A)]) \rightarrow \alpha \rightarrow_0 (a \rightarrow_2 A). \quad (3.14)$$

An analogous argument justifies that the following axiom:

$$\langle \alpha \rangle \langle a \rangle A \rightarrow (Pre(\alpha) \wedge \bigvee \{\langle a \rangle \langle \beta \rangle A \mid \alpha a \beta\})$$

corresponds to:

$$\alpha \Delta_0 (a \Delta_2 A) \rightarrow (\alpha \Delta_0 \top \wedge a \Delta_2 [(a \blacktriangle_3 \alpha) \Delta_1 A]). \quad (3.15)$$

Without appealing to $Fcn \subseteq Act$, we could take the correspondences above as primitive stipulations.

Structural language, formally. As discussed in the preliminaries, display calculi manipulate two closely related languages: the operational and the structural. Let us introduce the structural language of the Dynamic Calculus, which as usual matches the operational language, although in the present case not in the same way as in D'.EAK. We have *formula-type structures*, *functional action-type structures*, *action-type structures*, *agent-type structures*, defined by simultaneous recursion as follows:

$$\begin{aligned} FM \ni X ::= & A \mid I \mid X ; X \mid X > X \mid \\ & F \triangle_0 X \mid F \triangleright_0 X \mid \Gamma \triangle_1 X \mid \Gamma \triangleright_1 X \mid A \triangle_2 X \mid A \triangleright_2 X \mid \\ & F \blacktriangle_0 X \mid F \blacktriangleright_0 X \mid \Gamma \blacktriangle_1 X \mid \Gamma \blacktriangleright_1 X \mid A \blacktriangle_2 X \mid A \blacktriangleright_2 X \end{aligned}$$

$$FNC \ni F ::= \alpha \mid X \triangleleft_0 X \mid X \blacktriangleleft_0 X \mid A \triangleright_3 \Gamma \mid A \blacktriangleright_3 \Gamma$$

$$ACT \ni \Gamma ::= A \blacktriangle_3 F \mid A \triangle_3 F \mid X \triangleleft_1 X \mid X \blacktriangleleft_1 X$$

$$AG \ni A ::= a \mid X \triangleleft_2 X \mid X \blacktriangleleft_2 X \mid \Gamma \triangleleft_3 F \mid \Gamma \blacktriangleleft_3 F.$$

The propositional base. As is typical of display calculi, each operational connective corresponds to one structural connective. In particular, the propositional base connectives behave exactly as in D'.EAK, but for the sake of self-containment, we are going to report on these rules below:

Structural symbols	<		>		;		I
Operational symbols	<-	←	>-	→	∧	∨	⊤ ⊥

Structural Rules

$$\begin{array}{c}
 Id \frac{}{p \vdash p} \quad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \text{Cut} \\
 \\
 I_L^1 \frac{X \vdash Y}{I \vdash Y < X} \quad \frac{X \vdash Y}{X < Y \vdash I} I_R^1 \\
 \\
 I_L^2 \frac{X \vdash Y}{I \vdash X > Y} \quad \frac{X \vdash Y}{Y > X \vdash I} I_R^2 \\
 \\
 IW_L \frac{I \vdash X}{Y \vdash X} \quad \frac{X \vdash I}{X \vdash Y} IW_R \\
 \\
 W_L^1 \frac{X \vdash Z}{Y \vdash Z < X} \quad \frac{X \vdash Z}{X < Z \vdash Y} W_R^1 \\
 \\
 W_L^2 \frac{X \vdash Z}{Y \vdash X > Z} \quad \frac{X \vdash Z}{Z > X \vdash Y} W_R^2 \\
 \\
 C_L \frac{X; X \vdash Y}{X \vdash Y} \quad \frac{Y \vdash X; X}{Y \vdash X} C_R \\
 \\
 E_L \frac{Y; X \vdash Z}{X; Y \vdash Z} \quad \frac{Z \vdash X; Y}{Z \vdash Y; X} E_R \\
 \\
 A_L \frac{X; (Y; Z) \vdash W}{(X; Y); Z \vdash W} \quad \frac{W \vdash (Z; Y); X}{W \vdash Z; (Y; X)} A_R
 \end{array}$$

Display Postulates

$$\begin{array}{c}
 (, <) \frac{X; Y \vdash Z}{X \vdash Z < Y} \quad \frac{Z \vdash X; Y}{Z < Y \vdash X} (<, ;) \\
 \\
 (, >) \frac{X; Y \vdash Z}{Y \vdash X > Z} \quad \frac{Z \vdash X; Y}{X > Z \vdash Y} (>, ;)
 \end{array}$$

The classical base is obtained by adding the so-called *Grishin* rules (following e.g. [Gor00]), which encode classical, but not intuitionistic validities:

$$Gri_L \frac{X > (Y ; Z) \vdash W}{(X > Y) ; Z \vdash W} \quad \frac{W \vdash X > (Y ; Z)}{W \vdash (X > Y) ; Z} Gri_R$$

Operational Rules

$$\begin{array}{c} \perp_L \frac{}{\perp \vdash I} \quad \frac{X \vdash I}{X \vdash \perp} \perp_R \\ \\ \top_L \frac{I \vdash X}{\top \vdash X} \quad \frac{}{I \vdash \top} \top_R \\ \\ \wedge_L \frac{A ; B \vdash Z}{A \wedge B \vdash Z} \quad \frac{X \vdash A \quad Y \vdash B}{X ; Y \vdash A \wedge B} \wedge_R \\ \\ \vee_L \frac{A \vdash X \quad B \vdash Y}{A \vee B \vdash X ; Y} \quad \frac{Z \vdash A ; B}{Z \vdash A \vee B} \vee_R \\ \\ \leftarrow_L \frac{B \vdash Y \quad X \vdash A}{B \leftarrow A \vdash Y < X} \quad \frac{Z \vdash B < A}{Z \vdash B \leftarrow A} \leftarrow_R \\ \\ \prec_L \frac{B < A \vdash Z}{B \prec A \vdash Z} \quad \frac{Y \vdash B \quad A \vdash X}{Y < X \vdash B \prec A} \prec_R \\ \\ \rightarrow_L \frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X > Y} \quad \frac{Z \vdash A > B}{Z \vdash A \rightarrow B} \rightarrow_R \\ \\ \succ_L \frac{A > B \vdash Z}{A \succ B \vdash Z} \quad \frac{A \vdash X \quad Y \vdash B}{X > Y \vdash A \succ B} \succ_R \end{array}$$

Rules for heterogeneous connectives. Unlike what was the case in the setting of D'.EAK, in the present setting, each heterogeneous structural connective is associated with at most one operational connective, as illustrated in the following table: for $0 \leq i \leq 3$ and $j \in \{0, 2\}$,

Structural symbols	\triangle_i	\blacktriangle_i	\triangleleft_j	\blacktriangleright_j
Operational symbols	Δ_i	\blacktriangle_i	\rightarrow_j	\rightarrow_j

That is, structural connectives are to be interpreted as usual in a context-sensitive way, but the present language lacks the operational connectives which would correspond to them on one of the two sides. This is of course because in the present setting we do not

need them. However, in a setting in which they would turn out to be needed, it would not be difficult to introduce the missing operational connectives. We can now introduce the operational rules for heterogeneous connectives. Let x, y stand for structures of an undefined type, and let a, b denote operational terms of the appropriate type. Then, for $0 \leq i \leq 3$,

$$\begin{array}{c} \Delta_{iL} \frac{a \triangle_i b \vdash z}{a \Delta_i b \vdash z} \quad \frac{x \vdash a \quad y \vdash b}{x \triangle_i y \vdash a \Delta_i b} \Delta_{iR} \\ \blacktriangle_{iL} \frac{a \blacktriangle_i b \vdash z}{a \blacktriangle_i b \vdash z} \quad \frac{x \vdash a \quad y \vdash b}{x \blacktriangle_i y \vdash a \blacktriangle_i b} \blacktriangle_{iR} \end{array}$$

and for $0 \leq i \leq 2$,

$$\begin{array}{c} \rightarrow_{iL} \frac{x \vdash a \quad B \vdash Y}{a \rightarrow_i B \vdash x \triangleright_i Y} \quad \frac{Z \vdash a \triangleright_i B}{Z \vdash a \rightarrow_i B} \rightarrow_{iR} \\ \rightarrow_{iL} \frac{x \vdash a \quad B \vdash Y}{a \rightarrow_i B \vdash x \blacktriangleright_i Y} \quad \frac{Z \vdash a \blacktriangleright_i B}{Z \vdash a \rightarrow_i B} \rightarrow_{iR} \end{array}$$

where B, Y, Z are formula-type operational and structural terms. Clearly, the rules in the two tables above for $i = 0, 2$ yield the operational rules for the dynamic and epistemic modal operators under the translation given early on. Notice that each sequent is always interpreted in one domain. However, since heterogeneous connectives take arguments of different types (which justifies their name), premises of binary rules are of course interpreted in different domains.

Axioms will be given in three types⁶, as follows:

$$a \vdash a \quad \alpha \vdash \alpha \quad p \vdash p \quad \perp \vdash \perp \quad I \vdash \top$$

where the first and second axioms from the left are of type Ag and Fnc respectively, and the remaining ones are of type Fm. A generalization of $p \vdash p$ will be added below to the system (see *atom* axiom on page 133).

Further, we allow the following *strongly type-uniform* (cf. Definition 3.2) cut rules on operational terms:

$$\frac{A \vdash a \quad a \vdash B}{A \vdash B} \quad \frac{F \vdash \alpha \quad \alpha \vdash G}{F \vdash G} \quad \frac{\Gamma \vdash \gamma \quad \gamma \vdash \Delta}{\Gamma \vdash \Delta} \quad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}$$

⁶Indeed, there is no axiom schema for atomic terms of type Act, because the language does not admit them.

Next, we give the display postulates for heterogeneous connectives. In what follows, let x, y, z stand for structures of an undefined type. Then, for $0 \leq i \leq 2$,

$$(\Delta_i, \rightarrow_i) \frac{x \triangle_i y \vdash z}{y \vdash x \blacktriangleright_i z} \quad \frac{x \blacktriangle_i y \vdash z}{y \vdash x \blacktriangleright_i z} (\blacktriangle_i, \rightarrow_i)$$

For $i = 1$, we also have

$$(\Delta_1, \blacktriangleleft_1) \frac{x \triangle_1 y \vdash z}{x \vdash z \blacktriangleleft_1 y} \quad \frac{x \blacktriangle_1 y \vdash z}{x \vdash z \blacktriangleleft_1 y} (\blacktriangle_1, \blacktriangleleft_1)$$

The display postulates above involve structural connectives each of which has a semantic interpretation. In the following display postulates, the squiggly arrows are not semantically justified: they are the *virtual adjoints*, informally introduced at the beginning of the present Section 3.3, which will be discussed in detail in Section 3.6. For each $i = 0, 2, 3$, we have:

$$(\Delta_i, \blacktriangleleft_i) \frac{x \triangle_i y \vdash z}{x \vdash z \blacktriangleleft_i y} \quad \frac{x \blacktriangle_i y \vdash z}{x \vdash z \blacktriangleleft_i y} (\blacktriangle_i, \blacktriangleleft_i)$$

and for $i = 3$,

$$(\Delta_3, \rightsquigarrow_3) \frac{x \triangle_3 y \vdash z}{y \vdash x \rightsquigarrow_3 z} \quad \frac{x \blacktriangle_3 y \vdash z}{y \vdash x \rightsquigarrow_3 z} (\blacktriangle_3, \rightsquigarrow_3)$$

Notice that sequents occurring in each display postulate involving heterogeneous connectives are *not* of the same type. However, it is easy to see that the display postulates preserve the type-uniformity (cf. Definition 3.1); that is, if the premise of any instance of a display postulate is a type-uniform sequent, then so is its conclusion. Next, the *necessitation*, *conjugation*, *Fischer Servi*, and *monotonicity* rules: for $0 \leq i \leq 2$,

$$\begin{array}{c}
\begin{array}{c}
(nec_i \Delta) \frac{I \vdash W}{x \triangle_i I \vdash W} \\
(nec_i \blacktriangle) \frac{I \vdash W}{x \blacktriangle_i I \vdash W} \\
(conj_i \Delta) \frac{x \triangle_i ((x \blacktriangle_i Y); Z) \vdash W}{Y; (x \triangle_i Z) \vdash W} \\
(conj_i \blacktriangle) \frac{x \blacktriangle_i ((x \triangle_i Y); Z) \vdash W}{Y; (x \blacktriangle_i Z) \vdash W} \\
(FS_i \Delta) \frac{(x \triangleright_i Y) > (x \triangle_i Z) \vdash W}{x \triangle_i (Y > Z) \vdash W} \\
(FS_i \blacktriangle) \frac{(x \blacktriangleright_i Y) > (x \blacktriangle_i Z) \vdash W}{x \blacktriangle_i (Y > Z) \vdash W} \\
(mon_i \Delta) \frac{(x \triangle_i Y); (x \triangle_i Z) \vdash W}{x \triangle_i (Y; Z) \vdash W} \\
(mon_i \blacktriangle) \frac{(x \blacktriangle_i Y); (x \blacktriangle_i Z) \vdash W}{x \blacktriangle_i (Y; Z) \vdash W}
\end{array}
\qquad
\begin{array}{c}
\frac{W \vdash I}{W \vdash x \triangleright_i I} (nec_i \rightarrow) \\
\frac{W \vdash I}{W \vdash x \blacktriangleright_i I} (nec_i \rightarrow) \\
\frac{W \vdash x \triangleright_i ((x \blacktriangleright_i Y); Z)}{W \vdash Y; (x \triangleright_i Z)} (conj_i \rightarrow) \\
\frac{W \vdash x \blacktriangleright_i ((x \triangleleft_i Y); Z)}{W \vdash Y; (x \blacktriangleright_i Z)} (conj_i \rightarrow) \\
\frac{W \vdash (x \triangleleft_i Y) > (x \triangleright_i Z)}{W \vdash x \triangleright_i (Y > Z)} (FS_i \rightarrow) \\
\frac{W \vdash (x \blacktriangleleft_i Y) > (x \blacktriangleright_i Z)}{W \vdash x \blacktriangleright_i (Y > Z)} (FS_i \rightarrow) \\
\frac{W \vdash (x \triangleright_i Y); (x \triangleright_i Z)}{W \vdash x \triangleright_i (Y; Z)} (mon_i \rightarrow) \\
\frac{W \vdash (x \blacktriangleright_i Y); (x \blacktriangleright_i Z)}{W \vdash x \blacktriangleright_i (Y; Z)} (mon_i \rightarrow)
\end{array}
\end{array}$$

Next, we introduce the rules translating the interaction axioms between dynamic and epistemic modalities. In what follows we omit the subscripts, since the reading is unambiguous.

$$\begin{array}{c}
\text{swap-out}_L \frac{(A \blacktriangle F) \blacktriangle (A \blacktriangle X) \vdash Y}{A \blacktriangle (F \blacktriangle X) \vdash Y} \qquad \text{swap-out}_R \frac{X \vdash (A \blacktriangle F) \blacktriangleright (A \blacktriangleright Y)}{X \vdash A \blacktriangleright (F \blacktriangleright Y)} \\
\text{swap-in}_L \frac{A \blacktriangle (F \blacktriangle X) \vdash Y}{(A \blacktriangle F) \blacktriangle (A \blacktriangle ((F \triangle I); X)) \vdash Y} \qquad \text{swap-in}_R \frac{X \vdash A \blacktriangleright (F \blacktriangleright Y)}{X \vdash (A \blacktriangle F) \blacktriangleright (A \blacktriangleright ((F \triangle I) > Y))}
\end{array}$$

The structure $(A \blacktriangle F)$ in the *swap*-rules above has absorbed the labels $\alpha\beta$ in the corresponding *swap*-rules of D'.EAK. Moreover, new *swap-out* rules are unary, whereas the corresponding ones in D'.EAK are of a non-fixed arity.

The following *atom* axiom translates the *atom* axiom of D'.EAK:

$$F_1 \circ (F_2 \circ \dots (F_n \circ p) \dots) \vdash G_1 \triangleright (G_2 \triangleright \dots (G_m \triangleright p) \dots)$$

where $F_1, \dots, F_n, G_1, \dots, G_m \in \text{FNC}$, $\circ \in \{\triangle_0, \blacktriangle_0\}$, $\triangleright \in \{\triangleright_0, \blacktriangleright_0\}$ and $n, m \in \mathbb{N}$. In what follows, we sometimes indicate the *atom* axiom with the shorter symbol $\Phi p \vdash \Psi p$.

Notice the following difference between the present *atom* axiom and the one of D'.EAK (cf. page 105): the structural variables F s and G s (which are typically instantiated as operational variables α and β of type Fnc) translate what in the *atom* axiom of D'.EAK were indexes for logical connectives, whereas in the Dynamic Calculus, the operational variables contained in any instantiation of the F s and G s are first-class citizens, on the same ground as the operational variable p of type Fm. Hence we need to stipulate whether the introduction of each of these variables is parametric or not. As is customary in the literature on display calculi (cf. [Bel82, Definition 4.1]), we stipulate that the only principal variables in *atom* are the p s, and all the other variable occurrences are parametric.

Finally, the following *balance* rule:

$$\frac{X \vdash Y}{F \triangle_0 X \vdash F \triangleright_0 Y}$$

is sound only for $F \in \text{FNC}$, and cannot be extended to an arbitrary actions.⁷ In this rule, every variable occurrence is parametric, and each occurrence of F is only congruent to itself.

Justifying the two types of actions. As discussed in the introduction, one of the initial aims of the present chapter was introducing a formal framework expressive enough so as to capture at the object-level the information encoded in the meta-linguistic label $\alpha a \beta$. From the order-theoretic analysis at the beginning of the present section, it emerged that the additional expressivity encoded in the connective \blacktriangle_3 and its interpretation (3.13) requires a semantic environment which cannot be restricted to functional actions. The introduction of the general type Act serves this purpose. However, the fact that the rule *balance* is only sound for functional actions is the reason why both types Fnc and Act are needed in order for the Dynamic Calculus to satisfy conditions C'₆ and C'₇ of Section 3.2.3. Indeed, the distinct type Fnc allows for the rule *balance* to be formulated so that all parametric variables occur unrestricted within each type.

⁷To see this, notice that this rule instantiates to

$$\frac{p \vdash p}{F \triangle_0 p \vdash F \triangleright_0 p}$$

If the rule *balance* is to be sound, the validity of the premise implies the validity of the conclusion, which is the translation of the sequent $\langle F \rangle p \vdash [F]p$, which is equivalent to the axiom $\langle F \rangle p \rightarrow [F]p$. It is a well known fact from Sahlqvist theory that the latter axiom corresponds to the condition that the binary relation associated with $\langle F \rangle$ and $[F]$ is the graph of a partial function.

3.4 Soundness

In the present section, we discuss the soundness of the rules of the Dynamic Calculus and prove that those which do not involve virtual adjoints (cf. Section 3.3) are sound with respect to the final coalgebra semantics. In Section 2.5, basic facts about the final coalgebra have been collected and is explained in detail how the rules of display calculi are to be interpreted in the final coalgebra. Here we will briefly recall some basics, and refer the reader to Section 2.5 for a complete discussion.

Structures will be translated into operational terms of the appropriate type, and operational terms will be interpreted according to their type. Specifically, each atomic proposition p is assigned to a subset $\llbracket p \rrbracket$ of the final coalgebra Z , each agent a a binary relation $a_Z = \llbracket a \rrbracket$ on Z representing as usual a 's uncertainty about the world, and each functional actions α is assigned a functional (i.e. deterministic) relation $\alpha_Z = \llbracket \alpha \rrbracket \subseteq Z \times Z$ subject to the restriction defining the specific feature of epistemic actions, namely, that for all $z, z' \in Z$, if $z \alpha_Z z'$, then $z \in \llbracket p \rrbracket$ iff $z' \in \llbracket p \rrbracket$ for every atomic proposition p .

Further, each agent a is associated with an auxiliary binary relation a_{Fnc} on the domain of interpretation of Fnc , which is the collection of graphs of partial functions having subsets of Z as domain and range. For each agent a , the relation a_{Fnc} represents a 's uncertainty about which action takes place).

In order to translate structures as operational terms, structural connectives need to be translated as logical connectives. To this effect, non-modal structural connectives are associated with pairs of logical connectives, and any given occurrence of a structural connective is translated as one or the other, according to its (antecedent or succedent) position. The following table illustrates how to translate each propositional structural connective of type FM, in the upper row, into one or the other of the logical connectives corresponding to it on the lower row: the one on the left-hand (resp. right-hand) side, if the structural connective occurs in precedent (resp. succedent) position.

Structural symbols	<	>	;	I		
Operational symbols	⋄	⋅	∧	∨	⊤	⊥

Recall that, in the Boolean setting treated here, the connectives $\⋄$ and $\⋅$ are interpreted as $A \⋄ B := A \wedge \neg B$ and $A \⋅ B := \neg A \wedge B$.

The soundness of structural and operational rules which only involve active components of type FM has been discussed in Section 2.6.1 and is here therefore omitted.

As to the heterogeneous connectives, their translation into the corresponding operational connectives is indicated in the table below, to be understood similarly to the one above, where the index i ranges over $\{0, 1, 2, 3\}$ for the triangles and over $\{0, 1, 2\}$ for the arrows.

Structural symbols	\triangle_i	\blacktriangle_i	\triangleright_i	\blacktriangleright_i
Operational symbols	Δ_i	\blacktriangle_i	\rightarrow_i	\blackrightarrow_i

The interpretation of the heterogeneous connectives involving formulas and agents corresponds to that of the well-known forward and backward modalities discussed in Sections 2.5.2 and 2.6.1 (below on the right-hand side we recall the notation of D'.EAK):

$$\begin{aligned}
\llbracket \mathbf{a} \Delta_2 A \rrbracket &= \{z \in Z \mid \exists z'. z \mathbf{a}_Z z' \ \& \ z' \in \llbracket A \rrbracket\} && \langle \mathbf{a} \rangle A \\
\llbracket \mathbf{a} \blacktriangle_2 A \rrbracket &= \{z \in Z \mid \exists z'. z' \mathbf{a}_Z z \ \& \ z' \in \llbracket A \rrbracket\} && \widehat{\mathbf{a}} A \\
\llbracket \mathbf{a} \rightarrow_2 A \rrbracket &= \{z \in Z \mid \forall z'. z \mathbf{a}_Z z' \Rightarrow z' \in \llbracket A \rrbracket\} && [\mathbf{a}] A \\
\llbracket \mathbf{a} \blackrightarrow_2 A \rrbracket &= \{z \in Z \mid \forall z'. z' \mathbf{a}_Z z \Rightarrow z' \in \llbracket A \rrbracket\} && \overline{\mathbf{a}} A
\end{aligned}$$

The connectives Δ_0 , \rightarrow_0 , \blacktriangle_0 , \blackrightarrow_0 , involving formulas and functional actions, are interpreted in the same way, replacing the relation \mathbf{a}_Z with the deterministic relations α_Z . From the definitions above, it immediately follows for any $\alpha \in \text{Fnc}$, we have $\llbracket \alpha \Delta_0 \top \rrbracket = \text{dom}(\alpha_Z)$, where the set

$$\text{dom}(\alpha_Z) := \{z \in Z \mid \exists z' (z' \in Z \ \& \ z \alpha_Z z')\}$$

is the *domain* of α_Z .

It can also be readily verified that, after having fixed the relations interpreting all α and \mathbf{a} , the translation of Section 3.3 preserves the semantic interpretation, that is, $\llbracket A \rrbracket = \llbracket A' \rrbracket$ for any D'.EAK formula A , where A' denotes the translation of A in the language of the Dynamic Calculus.

The auxiliary relations $\mathbf{a}_{\text{Fnc}} = \llbracket \mathbf{a} \rrbracket^{\text{Fnc}}$ are used to define the interpretations of Δ_3 - and \blacktriangle_3 -operational terms. Following 3.13, we let

$$\begin{aligned}
\llbracket \mathbf{a} \blacktriangle_3 \alpha \rrbracket &= \bigcup \{G \mid \alpha_Z \mathbf{a}_{\text{Fnc}} G\}, \\
\llbracket \mathbf{a} \Delta_3 \alpha \rrbracket &= \bigcup \{G \mid G \mathbf{a}_{\text{Fnc}} \alpha_Z\}.
\end{aligned}$$

The connectives Δ_1 , \rightarrow_1 , \blacktriangle_1 , \blackrightarrow_1 , involving Act-type operational terms γ , are interpreted in the same way as the 0- and 2- indexed connectives, replacing the relation \mathbf{a}_Z with the interpretation of the appropriate operational term γ of type Act.

The soundness of all operational rules for heterogeneous connectives immediately follows from the fact that their semantic counterparts as defined above are monotone or antitone in each coordinate.

The soundness of the rule *balance* immediately follows from the fact that the functional actions are interpreted as deterministic relations (for more details cf. Section 2.6.2).

The soundness of the cut-rules follows from the transitivity of the inclusion relation in the domain of interpretation of each type.

The soundness of the *Atom* axioms is argued similarly to that of the *Atom* axioms of the system D'EAK, crucially using the fact that epistemic actions do not change the factual states of affairs (cf. Section 2.6.2).

The display rules $(\Delta_i, \rightarrow_i)$ and $(\blacktriangle_i, \rightarrow_i)$ for $0 \leq i \leq 2$, and $(\Delta_1, \blacktriangleleft_1)$ and $(\blacktriangle_1, \blacktriangleleft_1)$ are sound as the semantics of the triangle and arrow connectives form adjoint pairs.

On the other hand, in the display rules $(\Delta_3, \rightsquigarrow_3)$, $(\blacktriangle_3, \rightsquigarrow_3)$, $(\Delta_i, \blacktriangleleft_i)$ and $(\blacktriangle_i, \blacktriangleleft_i)$ for $i = 0, 2, 3$, the arrow-connectives are what we call *virtual adjoints* (cf. Section 3.3), that is, they do not have a semantic interpretation. In the next section, we will account for the fact that their presence in the calculus is safe.

Soundness of necessitation, conjugation, Fischer Servi, and monotonicity rules is straightforward and proved as in Section 2.6.2. In the remainder of the section, we discuss the soundness of the new rules swap-in and swap-out recalled below.

Fact 3.5. The following defining clause for the interpretation of \blacktriangle_1 -operational terms

$$\llbracket \gamma \blacktriangle_1 A \rrbracket = \{z \in Z \mid \exists z'. z' \gamma_Z z \ \& \ z' \in \llbracket A \rrbracket\}$$

immediately implies that the semantic interpretation of \blacktriangle_1 is completely \cup -preserving in its first coordinate.

Proof. If $\gamma_Z = \bigcup_{i \in I} \beta_i$, then clearly $z' \gamma_Z z$ iff $z' \beta_i z$ for some $i \in I$. □

As to the soundness of *swap-out_L*, assume that the structures \mathbf{A} , F , X and Y have been given the following interpretations, according to their type, as discussed above: $\mathbf{a}_Z \subseteq Z \times Z$, \mathbf{a}_{Fnc} is a binary relation on graphs of partial functions on Z , F_Z is a functional relation on Z , and $X_Z, Y_Z \subseteq Z$. Let

$$\mathbf{a}_Z \blacktriangle_3 F_Z := \bigcup \{\beta \mid F_Z \mathbf{a}_{\text{Fnc}} \beta\}.$$

Assume that the premise of *swap-out_L* is satisfied. That is:

$$\widehat{\mathbf{a}_Z \blacktriangle_3 F_Z} \widehat{\mathbf{a}_Z} X_Z \subseteq Y_Z,$$

where the symbols $\widehat{\mathbf{a}_Z \blacktriangle_3 F_Z}$ and $\widehat{\mathbf{a}_Z}$ denote the semantic diamond operations associated with the converses of the relations $\mathbf{a}_Z \blacktriangle_3 F_Z$ and \mathbf{a}_Z respectively. Then, the following chain of equivalences holds:

$$\begin{aligned} \widehat{\mathbf{a}_Z \blacktriangle_3 F_Z} \widehat{\mathbf{a}_Z} X_Z \subseteq Y_Z & \text{ iff } \bigcup \{ \widehat{G} \widehat{\mathbf{a}_Z} X_Z \mid F_Z \mathbf{a}_{\text{Fnc}} G \} \subseteq Y_Z & \text{(fact 3.5)} \\ & \text{ iff } \widehat{G} \widehat{\mathbf{a}_Z} X_Z \subseteq Y_Z \text{ for every } G \text{ s.t. } F_Z \mathbf{a}_{\text{Fnc}} G \\ & \text{ iff } X_Z \subseteq [\mathbf{a}_Z][G]Y_Z \text{ for every } G \text{ s.t. } F_Z \mathbf{a}_{\text{Fnc}} G \\ & \text{ iff } X_Z \subseteq \bigcap \{ [\mathbf{a}_Z][G]Y_Z \mid F_Z \mathbf{a}_{\text{Fnc}} G \} \\ \text{hence } X_Z & \subseteq (\text{dom}(F_Z))^c \cup \bigcap \{ [\mathbf{a}_Z][G]Y_Z \mid F_Z \mathbf{a}_{\text{Fnc}} G \}. \end{aligned}$$

Consider the new variables p, q, \mathbf{a}, α , and β_i for each G_i such that $F_Z \mathbf{a}_{\text{Fnc}} G_i$. Let us stipulate that $\llbracket p \rrbracket := X_Z$, $\llbracket q \rrbracket := Y_Z$, $\llbracket \mathbf{a} \rrbracket := \mathbf{a}_Z$, $\llbracket \alpha \rrbracket := F_Z$, and $\llbracket \beta_i \rrbracket := G_i$. Hence $\llbracket \text{Pre}(\alpha) \rrbracket = \llbracket \alpha \Delta_0 \top \rrbracket = \text{dom}(F_Z)$. Therefore, the computation above can continue as follows:

$$\begin{aligned} X_Z & \subseteq (\text{dom}(F_Z))^c \cup \bigcap \{ [\mathbf{a}_Z][G]Y_Z \mid F_Z \mathbf{a}_{\text{Fnc}} G \}. \\ \text{iff } \llbracket p \rrbracket & \subseteq \llbracket \text{Pre}(\alpha) \rightarrow \bigwedge \{ [\mathbf{a}][\beta]q \mid \alpha \mathbf{a} \beta \} \rrbracket \\ \text{iff } \llbracket p \rrbracket & \subseteq \llbracket [\alpha][\mathbf{a}]q \rrbracket \\ \text{iff } X_Z & \subseteq [F_Z][\mathbf{a}_Z]Y_Z \\ \text{iff } \widehat{\mathbf{a}_Z} \widehat{F_Z} X_Z & \subseteq Y_Z \end{aligned}$$

which completes the proof of the soundness of swap-out_L . The proof of the soundness of the remaining swap-rules is similar.

3.5 Completeness and cut elimination

In 3.5.1, we discuss the completeness of the Dynamic Calculus w.r.t. the final coalgebra semantics. We show that the translation (cf. Section 3.3) of each of the EAK axioms is derivable in the Dynamic Calculus. Our proof is indirect, and relies on the fact that EAK is complete w.r.t. the final coalgebra semantics, and that the translation preserves the semantic interpretation on the final coalgebra (as discussed in Section 3.4). In 3.5.2, we show that the Dynamic Calculus is a quasi-proper display calculus (cf. Section 3.2.3). By Theorem 3.4, this is enough to establish that the calculus enjoys cut elimination and the subformula property.

3.5.1 Derivable rules and completeness

In what follows, a and α are atomic variables (and also the generic operational terms) of type Ag and Fnc respectively, and A, B are generic operational terms of type Fm. Since the reading is unambiguous, in the remainder of the present chapter the indexes of the heterogeneous connectives are dropped.

Under the stipulations above, the translations of the rules *reduce'* from D'.EAK (cf. Section 2.6.1, page 106) can be derived in the Dynamic Calculus as follows.

$$\text{Dis}_{0\Delta} \frac{\frac{\frac{\alpha \triangle I ; \alpha \triangle A \vdash X}{\alpha \triangle (I ; A) \vdash X}}{I ; A \vdash \alpha \blacktriangleright X}}{I \vdash \alpha \blacktriangleright X < A}}{A \vdash \alpha \blacktriangleright X}}{\alpha \triangle A \vdash X}$$

Also the translations of the *comp* rules are derivable in the Dynamic Calculus as follows.

$$\text{conj}_{0\Delta} \frac{\frac{\frac{\frac{\alpha \triangle (\alpha \blacktriangle X) \vdash Y}{\alpha \blacktriangle X \vdash \alpha \blacktriangleright Y}}{I \vdash \alpha \blacktriangle X > \alpha \blacktriangleright Y}}{\alpha \blacktriangle X ; I \vdash \alpha \blacktriangleright Y}}{\alpha \triangle (\alpha \blacktriangle X ; I) \vdash Y}}{X ; \alpha \triangle I \vdash Y}}{\alpha \triangle I ; X \vdash Y}$$

Let us derive the axiom (3.14):

$$\text{swap-out}_L \frac{\frac{\frac{\frac{\frac{\frac{a \vdash a \quad \alpha \vdash \alpha}{a \blacktriangle \alpha \vdash a \blacktriangle \alpha} \quad A \vdash A}{a \vdash a \quad (a \blacktriangle \alpha) \rightarrow A \vdash (a \blacktriangle \alpha) \triangleright A}}{a \rightarrow ((a \blacktriangle \alpha) \rightarrow A) \vdash a \triangleright ((a \blacktriangle \alpha) \triangleright A)}}{a \blacktriangle (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) \vdash ((a \blacktriangle \alpha) \triangleright A)}}{((a \blacktriangle \alpha) \blacktriangle (a \blacktriangle (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)))) \vdash A}}{a \blacktriangle (\alpha \blacktriangle (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))) \vdash A}}{\frac{\frac{\frac{\alpha \blacktriangle (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) \vdash a \triangleright A}{\alpha \blacktriangle (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) \vdash a \rightarrow A}}{a \rightarrow ((a \blacktriangle \alpha) \rightarrow A) \vdash \alpha \triangleright (a \rightarrow A)}}{a \rightarrow ((a \blacktriangle \alpha) \rightarrow A) \vdash \alpha \rightarrow (a \rightarrow A)}}$$

Let us derive the axiom (3.15):

$$\begin{array}{c}
\frac{\frac{\frac{\frac{a \vdash a \quad \alpha \vdash \alpha}{a \blacktriangle \alpha \vdash a \blacktriangle \alpha}}{A \vdash A \quad a \blacktriangle \alpha \vdash a \blacktriangle \alpha}}{a \vdash a \quad (a \blacktriangle \alpha) \triangle A \vdash (a \blacktriangle \alpha) \triangle A}}{a \triangle ((a \blacktriangle \alpha) \triangle A) \vdash a \triangle ((a \blacktriangle \alpha) \triangle A)}}{(a \blacktriangle \alpha) \triangle A \vdash (a \blacktriangleright (a \triangle ((a \blacktriangle \alpha) \triangle A)))}} \\
\frac{A \vdash (a \blacktriangle \alpha) \blacktriangleright (a \blacktriangleright (a \triangle ((a \blacktriangle \alpha) \triangle A)))}{A \vdash a \blacktriangleright (\alpha \blacktriangleright (a \triangle ((a \blacktriangle \alpha) \triangle A)))} \text{ swap-out}_R \\
\frac{a \triangle A \vdash \alpha \blacktriangleright (a \triangle ((a \blacktriangle \alpha) \triangle A))}{a \triangle A \vdash \alpha \blacktriangleright (a \triangle ((a \blacktriangle \alpha) \triangle A))}}{\frac{\alpha \triangle (a \triangle A) \vdash a \triangle ((a \blacktriangle \alpha) \triangle A)}{\alpha \triangle (a \triangle A) \vdash a \triangle ((a \blacktriangle \alpha) \triangle A)}}
\end{array}$$

A slight difference between the setting of [DST13] and the present setting is that in that paper only the dynamic boxes are allowed in the object language, even if their propositional base is taken as non classical. In the present setting however, both the dynamic boxes and diamonds are taken as primitive connectives. When moving to a propositional base which is weaker than the Boolean one, also the diamond/box interaction axioms such as the following one become primitive:

$$[\alpha] \langle a \rangle A \leftrightarrow 1_\alpha \rightarrow \bigvee \{ \langle a \rangle \langle \beta \rangle A \mid \alpha a \beta \}.$$

The axiom above translates as:

$$\alpha \rightarrow (a \triangle A) \leftrightarrow \alpha \triangle \top \rightarrow a \triangle ((a \blacktriangle \alpha) \triangle A).$$

The derivations (of the translations) of the remaining axioms have been relegated to the appendix.

3.5.2 Belnap-style cut elimination, and subformula property

In the present subsection, we prove that the Dynamic Calculus for EAK is a quasi-proper display calculus (cf. Section 3.2.3). By Theorem 3.4, this is enough to establish that the calculus enjoys the cut elimination and the subformula property. Conditions C_1 , C_2 , C_4 , C'_5 , C'_6 , C'_7 and C_{10} are straightforwardly verified by inspecting the rules and are left to the reader.

Condition C''_5 can be straightforwardly argued by observing that the only axioms to which a display postulate can be applied are of the *atom* form: $\Phi p \vdash \Psi p$. In this case, the only applicable display postulates are those rewriting \triangleleft - or \blacktriangleleft -structures into \blacktriangleright - and \triangleright -structures and vice versa, which indeed preserve the *atom* shape. Condition C''_8 is straightforwardly verified by inspection on the axioms. Condition C'_2 can be straightforwardly verified by inspection on the rules, for instance by observing that the domains and codomains of adjoints are rigidly determined.

The following proposition shows that condition C_9 is met:

Proposition 3.6. *Any derivable sequent in the Dynamic Calculus for EAK is type-uniform.*

Proof. We prove the proposition by induction on the height of the derivation. The base case is verified by inspection; indeed, the following axioms are type-uniform by definition of their constituents:

$$a \vdash a \quad \alpha \vdash \alpha \quad \Phi p \vdash \Psi p \quad \perp \vdash \top \quad \top \vdash \perp$$

As to the inductive step, one can verify by inspection that all the rules of the Dynamic Calculus preserve type-uniformity, and that the Cut rules are strongly type-uniform. \square

As to condition C'_3 , all parameters in all but the *swap-in* rules satisfy the condition of non-proliferation. In each *swap-in* rule, the parameters of type Ag and Fnc in the premise are congruent to *two* parameters in the conclusion. However, it is not difficult to see that in each derivation, each application of any cut rule

$$\frac{\begin{array}{c} \vdots \\ x \vdash a \end{array} \quad \begin{array}{c} \vdots \\ a \vdash y \end{array}}{x \vdash y}$$

of type Ag or Fnc must be such that the structure x reduces to the atomic term a . Indeed, because the sequent $x \vdash a$ is derivable, by Proposition 3.6 it must be type uniform, that is, the structure x needs to be of type AG if a is, or of type FNC if a is. If x was not atomic, then its main structural connective would be a squiggly arrow $\triangleleft\sim$ or $\blacktriangleleft\sim$. Because these connectives do not have any operational counterpart, such a structure cannot have been introduced by an application of an operational rule. Hence, the only remaining possibility is that it has been introduced via a display postulate. But also this case is impossible, since in display postulates introduce these connectives only in the succedent, and x is in precedent position. This finishes the verification of condition C'3.

Finally, the verification steps for C'8 are collected in Appendix F.

3.6 Conservativity

In the definition of the language of the Dynamic Calculus, we have adopted a rather inclusive policy. That is, the operational language includes almost all the logical symbols which could be assigned a natural interpretation purely on the basis of reasonable assumptions on the order-theoretic properties of the domains of interpretation of the various types of terms, the only exception being the connectives \blacktriangleleft_1 and \triangleleft_1 , which are excluded from the language although they are semantically justified. A very useful and powerful consequence of the fact that the Dynamic Calculus enjoys cut elimination Belnap-style is that this cut elimination is then inherited by the subcalculi corresponding to each fragment of the operational language of the Dynamic Calculus which verify as they stand the assumptions of Theorem 3.4. However, the question is still open about whether these subcalculi interact with each other in unwanted ways when their proof power is concerned: for any two such fragments $\mathcal{L}_1 \subseteq \mathcal{L}_2$, does the subcalculus corresponding to \mathcal{L}_2 conservatively extend the one corresponding to \mathcal{L}_1 ? Typically, the absence of unwanted interactions among subcalculi is deduced from having cut elimination, and soundness and completeness w.r.t. a given semantics. This way, in Chapter 2 it is also shown that the system D'.EAK conservatively extends EAK.

However, this strategy is not immediately applicable to the setting of the Dynamic Calculus, due to the structural symbols referred to as *virtual adjoints*, which are easily recognizable, since they are shaped like arrows with a squiggly tail: $\blacktriangleleft\sim$, $\sim\triangleright$ etc. Virtual adjoints have no semantic justification, and hence, the rules in which they specifically occur (that is, the display postulates relative to them) cannot be justified on semantic grounds. The reason for including virtual adjoints in the language of the Dynamic Calculus is for it to enjoy the relativized display property, discussed in Section 3.2.2, which is key to guarantee the crucial condition C'8, requiring the existence of a way to solve

the principal stage of the cut elimination theorem (cf. Section 3.2.4, see also Appendix F).

When discussing virtual adjoints in Section 3.3, we claimed that, since they are only introduced in a derivation by way of display postulates and do not specifically intervene in any other structural rule, their presence in the calculus does not add unwanted proof power compared to D'.EAK (and hence to EAK). This is the sense in which the introduction of the virtual adjoints can be regarded as syntactically sound. The aim of the present section is to prove this claim.

A general and very powerful method for proving the conservativity of display calculi has been introduced in [CDGT13a, CDGT13b] for the full intuitionistic linear logic. This method involves no less than two translations, one from the given display calculus into an intermediate shallow inference nested sequent calculus, and another one from the intermediate calculus into a deep inference nested sequent calculus. This method is very intricate, requiring the verification of hundreds of cases which account for every possible interaction between the shallow and the deep calculus. The intricacy of this proof was such that the correctness of the results in [CDGT13a, CDGT13b] has been established by formalizing them in the proof assistant Isabelle/HOL, as reported in [DCGT14].

However, in the present section, a much smoother proof of conservativity is given for the Dynamic Calculus for EAK, which does not rely on any nested sequent calculus. Rather, the proof below relies on very specific and uncommon features of the design of the Dynamic Calculus for EAK. In a sense, the very fact that such a smooth proof is possible witnesses how uncommonly well behaved EAK is.

Definition 3.7. A sequent $x \vdash y$ is *severe* if in the generation trees of either x or y there are occurrences of structural connectives to which no display postulates can be applied. Such occurrences will be referred to as *severe*.

Clearly, the definition above makes sense only in the context of calculi which, as is the case of the Dynamic Calculus, do not enjoy the full display property (cf. Definition 2.1). It can be easily verified that, in the specific case of the Dynamic Calculus, there are only two types of severe occurrences: triangle-type connectives rooting a structure in succedent position, and arrow-type connectives rooting a structure in precedent position. Examples of severe sequents then are $(a \triangleright \alpha) \triangle A \vdash B$ and $A \vdash a \triangle B$.

Lemma 3.8. *Any rule in the Dynamic Calculus preserves the severity of sequents. That is, if a rule is applied to a severe sequent, the conclusion of that rule application is also severe.*

Proof. By inspection on the rules. □

Fact 3.9. Let $x \vdash y$ be a sequent of type AG or FNC, in the full language of Dynamic Calculus, which is derivable by means of a derivation π in which no application of Weakening, Necessitation, Balance or Atom introduce occurrences of virtual adjoints. Then $x = a$ for some operational term a of the appropriate type.

Proof. If x is not an atomic structure, then the grammar of AG and FNC prescribes that x has a virtual adjoint as a main connective. However, the assumptions imply that such structures can be introduced only by way of applications of display postulates, which introduce them in succedent position. Hence, given the assumptions, there is no way in which such a connective can be introduced in precedent position. \square

Lemma 3.10. *Let $X \vdash Y$ be a sequent of type FM which is derivable in the Dynamic Calculus by means of a derivation π in which no application of Weakening, Necessitation, Balance or Atom introduce occurrences of virtual adjoints. Then a derivation π' of $X \vdash Y$ exists every node of which (hence the conclusion in particular) is free of virtual adjoints.*

Proof. Let s be some node/sequent in π where the given virtual adjoint has been introduced. Since virtual adjoints in the Dynamic Calculus are all virtual “right adjoints”, and since, by assumption, they are introduced only by way of applications of display postulates, the given virtual adjoint is the main connective in the succedent of the sequent s . Moreover, virtual adjoints are main connectives of structures of type AG and FNC. By type uniformity, this implies that the sequent s is either of type AG or FNC, and therefore s cannot be the conclusion of π . Some rule R must exist which takes s as a premise. It can be easily verified by inspection that R cannot coincide with any structural rule in the Dynamic Calculus which is neither a cut of the appropriate type nor a display postulate, since all structural rules different from Cut and display postulates have premises of type FM. We can also assume w.l.o.g. that R is not an application of Cut. Indeed, by Fact 3.9, s is of the form $a \vdash x$, with a being an operational term, and x being a non-atomic structure by assumption. Hence, if R was a Cut-application, the inference must be of the form

$$\frac{\begin{array}{c} \vdots \\ y \vdash a \end{array} \quad \begin{array}{c} \vdots \\ a \vdash x \end{array}}{y \vdash x}$$

Because Cut rules in the Dynamic Calculus are strongly type-regular, also $y \vdash a$ would be a (derivable) sequent of type AG or FNC, hence Fact 3.9 applies to $y \vdash a$. That is, y must be atomic, and because $y \vdash a$ is derivable, y must coincide with a . Hence, the conclusion of that Cut application is again s . This shows that if R was Cut, w.l.o.g. we would be able remove that application from the proof tree. The remaining options are that R coincides with an introduction rule of some heterogeneous logical connective. Recall that, by Fact 3.9, s is of the form $a \vdash x$, where a is an operational term. Then, it can be verified by inspection that no heterogeneous rule

is applicable if x is not an atomic structure, which is not the case of the sequent s , as discussed above. Finally, since the left-hand side of s is atomic, no other display postulates are applicable to s but the converse direction of the same display postulate which had introduced the virtual adjoint and which makes it disappear. Therefore, the refinement π' of π consists in removing these double and redundant applications of display postulates. \square

Lemma 3.11. *If inf is an application of Balance, Atom, Necessitation or Weakening in which some occurrence of a virtual adjoint is introduced, then the conclusion of inf is a severe sequent.*

Proof. As to Balance, Atom and Necessitation _{i} with $i = 0, 2$, notice that each of these rules introduces a structure x of type FNC or AG in precedent position. If a virtual adjoint is introduced as a substructure of x , then x is non-atomic, and it can be immediately verified by inspecting the syntax of FNC and AG that the main connective of x is an arrow-type connective, which would then be in precedent position. Hence, the resulting sequent is severe.

As to Weakening and Necessitation₁, notice that these rules introduce structures x of type FM and ACT respectively. Recall that virtual adjoints root structures of type FNC or AG. Hence, if some virtual adjoint occurs in the generation tree of x , it cannot occur at the root of x . Hence, the virtual adjoint must occur in the scope of some other structural connective. Notice that the heterogeneous connectives are the only ones which can take as argument a structure rooted in a virtual adjoint. We claim that either the virtual adjoint occurs in precedent position (which would be enough to conclude that the conclusion of inf is severe), or under the scope of some structural connective to which no display postulate can be applied. Assume that the virtual adjoint occurs in succedent position. If its immediate ancestor in the generation tree of X is a triangle-type connective, then these connectives are in succedent position too, and hence no display postulate can be applied to them, which makes the conclusion of inf severe, as required. Similarly, if the immediate ancestor of the virtual adjoint is an arrow-type connective, then it can be easily checked by inspection that these connectives take structures of type FNC or AG exclusively in their antitone coordinate (that is, on the flat side of the arrow). Hence, the arrow-type connective is in precedent position, and hence no display postulate can be applied to it, as required. \square

Corollary 3.12. *Let $A' \vdash B'$ be a sequent of type Fm in the language of the Dynamic Calculus, such that A' and B' are, respectively, images of some D' .EAK-formulas A and B under the translation of Section 3.3. If $A' \vdash B'$ is derivable in the Dynamic Calculus, then $A \vdash B$ is derivable in D' .EAK.*

Proof. Let π be a derivation of $A' \vdash B'$ in the Dynamic Calculus. By assumption, $A' \vdash B'$ is not severe. Hence, no rule application in π can introduce severe sequents, since these, by Lemma 3.8, would then propagate till the conclusion. Hence in particular, by Lemma 3.11,

in π there cannot be any applications of Balance, Atom, Necessitation or Weakening in which some occurrence of a virtual adjoint is introduced. Therefore, by Lemma 3.10, a derivation π' of $A' \vdash B'$ exists in which no virtual adjoints occur. By the results collected in Section 3.4, the derivation π' is sound w.r.t. the final coalgebra semantics. Hence $A' \vdash B'$ is satisfied on the final coalgebra semantics. Since, as discussed in Section 3.4, $\llbracket A \rrbracket = \llbracket A' \rrbracket$ and $\llbracket B \rrbracket = \llbracket B' \rrbracket$, this implies that $A \vdash B$ is satisfied. Since $D'.\text{EAK}$ is complete w.r.t. the final coalgebra semantics, a $D'.\text{EAK}$ -derivation of $A \vdash B$ exists. \square

3.7 Conclusions and further directions

The main contribution of the present chapter is the definition of a display calculus which smoothly encompasses the most proof-theoretically impervious features of Baltag Moss and Solecki's logic of epistemic actions and knowledge. Besides being well performing (it adequately captures EAK and enjoys Belnap-style cut elimination), this calculus provides an interesting and in our opinion very promising *methodological* platform towards the uniform development of a general proof-theoretic account of all dynamic logics, and also, from a purely structurally proof-theoretic viewpoint, for clarifying and sharpening the formulation of criteria leading to the statement and proof of meta-theoretic results such as Belnap-style cut elimination, or conservativity issues.

Seminal approaches. The starting point of this methodology is to introduce enough syntactic devices, both at the operational and at the structural level, so that the parameters indexing logical connectives can be accounted for in the system as *terms* in the language of choice. This gives rise to the definition of multi-type languages, endowed with connectives which manage the interaction of the different types. This approach appears seminally in both [BCS07] and [DST13]; however, in neither paper it is fully explored: in [BCS07] there is no theory of contexts governing the interaction of different types, and in [DST13], this interaction is clarified, but only at the metalinguistic level.

Refinements of Belnap's conditions, and type-uniformity. In Section 2.7, we formulated the conjecture that the multi-type setting will prove useful for a smoother treatment of Wansing's and Belnap's *regularity* requirements (cf. conditions C_6/C_7 in [Wan98], [Bel90, Section 2]) for the Belnap-style cut elimination, via the notion of type-uniformity (Definition 3.1). In [Bel82], Belnap motivates his condition C_7 ⁸ saying that “rules need not be wholly closed under substitution of structures for congruent formulas which are

⁸Recall that Belnap's condition C_7 corresponds to Wansing's cons-regularity for formulas occurring in precedent position. An analogous explanation holds of course for the ant-regularity condition of formulas in succedent position.

antecedent parts, but they must be closed enough.” Then he explains that *closed enough* refers to the closure under substitution of formulas A for structures X such that a derivation is available in the system for the sequent $X \vdash A$, in which the occurrence of A in the conclusion is principal. The crucial observation is that, even if a system is not defined *a priori* as multi-type, it can be regarded as a multi-type setting: indeed, the type of A can be defined as consisting of all the structures X such that the shape of derivation alluded to above exists. Then, condition C_6/C_7 can be equivalently reformulated as the requirement that rules should be closed under uniform substitution within each type. Notice that, under the stipulations above, different types must be separated by at least one structural rule. For instance, in the Dynamic Calculus for EAK, the rule *balance* separates *Fnc* from *Act*. In conclusion, our conjecture is that Wansing’s and Belnap’s conditions C_6/C_7 boil down to a type-uniformity requirement in a context in which types are not given explicitly. The observations above indicate that type-uniformity is a desirable design requirement for general dynamic calculi, and in particular for the development of an adequate proof theory for dynamic logics, particularly in view of a uniform path to Belnap-style cut elimination.

Non-proliferation. Our analysis towards Belnap-style cut elimination led us to refine and weaken various aspects of the cut elimination metatheorem. For instance, the requirement of non-proliferation of parameters for quasi-proper multi-type display calculi applies only to types the grammar of which is rich enough that allows non-trivial cut applications, that is, applications of cut the conclusion of which is different from both premises. The case study of EAK allows such a simple grammar on functional actions and agents that these two types are not subject to the restriction of non-proliferation. This in turn makes it possible to include the *swap-in* rules in the calculus, in which every occurring parameter of a type which can proliferate does indeed proliferate. Introducing some nontrivial grammar on functional actions (e.g. sequential composition) would make the restriction of non-proliferation applicable to this type, and hence would make *swap-in* not suitable anymore.

Expanding the signature. Notwithstanding the concerns about *swap-in*, the multi-type language provides the opportunity to consider various natural expansions of the language of actions. Early on, we argued that the connective \blacktriangleleft_1 which takes formulas in both coordinate as arguments and delivers an action, has the following natural interpretation: for all formulas A, B , the term $B \blacktriangleleft_1 A$ denotes the weakest epistemic action γ such that, if A was true before γ was performed, then B is true after any successful execution of γ . This connective seems particularly suited to explore epistemic capabilities and planning.

Chapter 4

Display-Type Calculi via Visibility

4.1 Introduction

In Chapter 2, we have introduced the final coalgebra as a semantic setting for EAK alternative to the standard one, with the purpose of giving an independent justification of the existence of the adjoints of the dynamic modal operators associated with epistemic actions. This alternative semantics was crucial to argue that the calculus $D'.EAK$ is sound w.r.t. the original language of EAK.

In Chapter 3, we justified the soundness of the multi-type calculus w.r.t. EAK thanks to very special circumstances which can be easily lost when moving to other logics. Hence, nice and smooth as it is, the conservativity result obtained there does not give rise to a uniform and modular methodology.

Our main motivation in the present line of research is precisely to extend the benefits of display calculi in a uniform and modular way to the wide family of dynamic logics. In the present chapter, we address a crucial intermediate step towards this goal, namely the issue of how to account for logics the semantics of which, as is the case of monotone modal logic, does not support the existence of enough adjoints so as to safely justify the design of any calculus with the display property, even relativized as in Definition 3.2.2. In the present chapter, we explore the possibility of dropping the display property, either full or relativized, but still work in a setting for which the all-important Belnap-style cut elimination metatheorem can be formulated and proved.

Of course, dropping one property calls for some readjustments, if we are to make the metatheorem work. As to the first readjustment, the present set-up relies on a partial strengthening of Belnap's condition C_5 (Display of principal constituents) which is inspired by Sambin's work on Basic logic [BFS00]. The rules of the Basic Logic calculus

verify the so-called *visibility* or *segregation* property, which we report in the formulation of [BFS00]:

We thus say that a rule satisfies *visibility* if it operates on a formula (or two formulae) only if it is (they are) the only formula(e), either in the antecedent or in the succedent of a sequent. Formally, visibility is the property that all active formulae (secondary or principal formulae, in Gentzen's terminology) are isolated, or visible, all passive contexts (not on the same side of any active formula) are free.

To illustrate this technical point, notice that Belnap's condition C₅ does not exclude the possibility of designing introduction rules of the following shape:

$$\frac{(X_1 \vdash Y_1)[A]^{p/s} \quad (X_2 \vdash Y_2)[B]^{p/s}}{X \vdash A * B} \quad \frac{(U \vdash V)[A*B]^{[A]^{s/p}, [B]^{s/p}}}{A * B \vdash Z}$$

The condition on the design of these rules which we wish to emphasize here is that the position (either precedent or succedent) of each active occurrence A and B in one rule needs to be the opposite of its position in the other rule. If this requirement is satisfied, then the display property takes care of almost everything else:¹ for instance, if both A and B occur in succedent position in the rule above on the left and in precedent position in the rule above on the right, the reduction step in the cut elimination metatheorem goes as follows:

$$\frac{\frac{\pi_1}{(X_1 \vdash Y_1)[A]} \quad \frac{\pi_2}{(X_2 \vdash Y_2)[B]}}{X \vdash A * B} \quad \frac{\pi}{(U \vdash V)[A*B]} \quad \rightsquigarrow \quad \frac{\frac{\pi_2}{(X_2 \vdash Y_2)[B]} \quad \frac{\frac{\frac{\pi_1}{(X_1 \vdash Y_1)[A]}{X' \vdash A} \quad \frac{\pi}{(U \vdash V)[A*B]}{A \vdash U'[B]}}{X' \vdash U'[B]} \quad \frac{B \vdash V'[X']}{B \vdash V'[X']}}{Y' \vdash V'[X']}}{X \vdash Z}$$

Namely, the (relativized) display property makes it possible to access (the active occurrences of) the immediate subformulas of the original cut formula, and break down the original cut into cut applications of strictly lower rank.

Enforcing visibility in our running example amounts to replacing the rules above by weaker rules such as the following ones:

$$\frac{X_1 \vdash A \quad X_2 \vdash B}{X_1 * X_2 \vdash A * B} \quad \frac{A * B \vdash Z}{A * B \vdash Z}$$

¹The double dashed line in the proof tree after the reduction step hides a series of transformations which cannot be accounted for only in terms of display rules. The analysis leading to the explicit underpinning of these transformation steps has a long and nontrivial history, and is insightful and interesting in its own right, but going deeper into it is not directly useful to the point we are trying to illustrate in this example, which is the interplay between display property and visibility.

This more restricted shape of rules, in the context of the transformation steps pictured above, frees ourselves from the need of having to rely on display rules to access cut subformulas e.g. under the subtrees π_1 and π_2 , but does not help us with π .

The second readjustment is then giving up the display-type cut rule in which both cut formulas are in display, and adopting more general cut rules, namely *surgical cut rules*, of the following form:

$$\frac{Z \vdash A \quad (X \vdash Y)[A]^{pre}}{(X \vdash Y)[Z/A]^{pre}} \quad \frac{(X \vdash Y)[A]^{suc} \quad A \vdash Z}{(X \vdash Y)[Z/A]^{suc}}$$

These readjustments make it possible to perform the following reduction step, which does not rely on any display postulate:

$$\frac{\frac{\frac{\pi_1}{X_1 \vdash A} \quad \frac{\pi_2}{X_2 \vdash B}}{X_1 * X_2 \vdash A * B} \quad \frac{\pi}{A * B \vdash Y}}{X_1 * X_2 \vdash Y} \rightsquigarrow \frac{\frac{\pi_2}{X_2 \vdash B} \quad \frac{\frac{\pi_1}{X_1 \vdash A} \quad \frac{\pi}{A * B \vdash Y}}{X_1 * B \vdash Y}}{X_1 * X_2 \vdash Y}$$

In fact, in what follows, we will adopt a hybrid approach which mixes and balances Sambin's visibility and Belnap's display property. Namely, in those cases in which visibility is allowed not to hold, we will require enough display postulates for the cut elimination step affected by the change to go through.

Organization and results. For preliminaries on multi-type calculi the reader is referred to Section 3.2.1. In Section 4.2, we list a set of conditions generalising the (proper) display calculi of [Wan02]. In Section 4.3, we discuss how these conditions guarantee the cut elimination metatheorem for the multi-type calculi enjoying them.

4.2 Quasi-proper multi-type display-type calculi

A multi-type calculus is a *quasi-proper* display-type calculus if it satisfies the following list of conditions:

C₁: Preservation of operational terms. Each operational term occurring in a premise of an inference rule *inf* is a subterm of some operational term in the conclusion of *inf*.

C₂: Shape-alikeness of parameters. Congruent parameters (i.e. non-active terms in the application of a rule) are occurrences of the same structure.

C₂': **Type-alikeness of parameters.** Congruent parameters have exactly the same type. This condition bans the possibility that a parameter changes type along its history.

C₃: **Non-proliferation of parameters.** Each parameter in an inference rule *inf* is congruent to at most one constituent in the conclusion of *inf*.

C₄: **Position-alikeness of parameters.** Congruent parameters are either all precedent or all succedent parts of their respective sequents. In the case of calculi enjoying the display property, precedent and succedent parts are defined in the usual way (see [Bel82]). Otherwise, these notions can still be defined by induction on the shape of the structures, by relying on the polarity of each coordinate of the structural connectives.

C₅': **Quasi-display of principal constituents.** If an operational term *a* is principal in the conclusion sequent *s* of a derivation π , then *a* is in display, unless π consists only of its conclusion sequent *s* (i.e. *s* is an axiom).

C₅'': **Display-invariance of axioms.** If *a* is principal in an axiom *s*, then *a* can be isolated by applying Display Postulates and the new sequent is still an axiom.

C₆': **Closure under substitution for succedent parts within each type.** Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in succedent position, *within each type*.

C₇': **Closure under substitution for precedent parts within each type.** Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in precedent position, *within each type*.

C₈': **Eliminability of matching principal constituents.** This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C₈' requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of the cut rule, involving proper subterms of the original

operational cut-term. In addition to this, specific to the multi-type setting is the requirement that the new application(s) of the cut rule be also *type-uniform* (cf. condition C'_{10} below).

C'''_8 : Closure of axioms under surgical cut. If $(x \vdash y)([a]^{pre}, [a]^{suc})$, $a \vdash z[a]^{suc}$ and $v[a]^{pre} \vdash a$ are axioms, then $(x \vdash y)([a]^{pre}, [z/a]^{suc})$ and $(x \vdash y)([v/a]^{pre}, [a]^{suc})$ are again axioms.

C_9 : Type-uniformity of derivable sequents. Each derivable sequent is type-uniform.

C'_{10} : Preservation of type-uniformity of cut rules. All cut rules preserve type-uniformity (cf. Definition 3.1).

4.3 Cut elimination metatheorem

Theorem 4.1. *Any multi-type sequent calculus satisfying $C_2, C'_2, C_3, C_4, C'_5, C''_5, C'_6, C'_7, C'_8, C'''_8, C_9$ and C'_{10} is cut-admissible. If also C_1 is satisfied, then the calculus enjoys the subformula property.*

Proof. We follow the proof in [Wan02, Section 3.3, Appendix A]. For the sake of conciseness, we will expand only on the parts of the proof which depart from the treatment in [Wan02]. In particular, we consider elimination of surgical cuts (cf. page 151). As to the principal move, the only difference concerns the case of a surgical cut application both premises of which are axioms. Condition C'''_8 guarantees that this cut application can be eliminated. The remaining principal moves go exactly as in [Wan02], thanks to C'_8 . As to the parametric moves, we are in the following situation:

$$\frac{\begin{array}{c} \vdots \pi_{2.1} \\ (x_1 \vdash y_1)[a_{u_1}]^{pre} \end{array} \quad \cdots \quad \begin{array}{c} \vdots \pi_{2.n} \\ (x_n \vdash y_n)[a_{u_n}]^{pre} \end{array}}{\begin{array}{c} \vdots \pi_1 \\ z \vdash a \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ (x \vdash y)[a]^{pre} \end{array}} \frac{}{(x \vdash y)[z]^{pre}}$$

where $(x \vdash y)[z]^{pre}[w]^{suc}$ means that z and w respectively occur in precedent and succedent position in $x \vdash y$, and the cut term a is parametric in the conclusion of π_2 (the other case is symmetric).

Conditions C_2 , C'_2 , C_3 and C_4 make it possible to follow the history of that occurrence of a , since these conditions enforce that the history takes the shape of a tree, of which we consider each leaf. Let a_{u_i} (abbreviated to a_u from now on) be one such uppermost-occurrence in the history-tree of the parametric cut term a occurring in π_2 , and let $\pi_{2,i}$ be the subderivation ending in the sequent $x_i \vdash y_i$, in which a_u is introduced.

Wansing's case (1) splits into two subcases: (1a) a_u is introduced in display; (1b) a_u is not introduced in display. Condition C'_5 guarantees that (1b) can only be the case when a_u has been introduced via an axiom.

If (1a), then we can perform the following transformation:

$$\frac{\frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\vdots \pi_2}{(x \vdash y)[a]^{pre}}}{(x \vdash y)[z]^{pre}} \quad \frac{\frac{\vdots \pi_{2,i}}{a_u \vdash y_i} \quad \frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\vdots \pi_{2,i}}{a_u \vdash y_i}}{z \vdash y_i}}{\vdots \pi_2 [z/a]^{pre}}}{(x \vdash y)[z]^{pre}} \rightsquigarrow$$

where $\pi_2[z/a]^{pre}$ is the derivation obtained by substituting z for every occurrence in the history of a . Notice that the assumption that a is parametric in the conclusion of π_2 and that a_u is principal in *inf* imply that π_2 has more than one node, and hence the transformation above results in a cut application of strictly lower height. Moreover, the assumptions that the original cut preserves type-uniformity (C_{10}), that every derivable sequent is type-uniform (C_9), and the type-alikeness of parameters (C'_2) imply that the sequent $a_u \vdash y_i$ is of the same type as the sequents $z \vdash a$. Hence, in particular, the new cut preserves type-uniformity. Finally, condition C'_7 implies that the substitution of z for a in π_2 gives rise to an admissible derivation $\pi_2[z/a]^{pre}$ in the calculus (use C'_6 for the symmetric case). If (1b), i.e. if a_u is the principal formula of an axiom, the situation is illustrated below in the derivation on the left-hand side:

$$\frac{\frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\vdots \pi_2}{(x \vdash y)[a]^{pre}}}{(x \vdash y)[z]^{pre}} \quad \frac{\frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\vdots \pi_2}{(x \vdash y)[a]^{pre}}}{(x \vdash y)[z]^{pre}} \quad \frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\vdots \pi_2}{(x \vdash y)[a]^{pre}}}{z \vdash y'[a]^{suc}}}{z \vdash y'[a]^{suc}}}{(x_i \vdash y_i)[a_u]^{pre}[a]^{suc}} \quad \frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\vdots \pi_2}{(x \vdash y)[a]^{pre}}}{z \vdash y'[a]^{suc}}}{z \vdash y'[a]^{suc}}}{(x_i \vdash y_i)[z/a_u]^{pre}[a]^{suc}} \quad \frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\frac{\vdots \pi_1}{z \vdash a} \quad \frac{\vdots \pi_2}{(x \vdash y)[a]^{pre}}}{z \vdash y'[a]^{suc}}}{z \vdash y'[a]^{suc}}}{(x \vdash y)[z]^{pre}} \rightsquigarrow$$

where $(x_i \vdash y_i)[a_u]^{pre}[a]^{suc}$ is an axiom. Then, condition C''_5 implies that some sequent $a_u \vdash y'[a]^{suc}$ exists, which is display-equivalent to the first axiom, and in which a_u occurs in display. This new sequent can be either identical to $(x_i \vdash y_i)[a_u]^{pre}[a]^{suc}$, in which case we proceed as in case (1a), or it can be different, in which case, condition C''_5 guarantees that it is an axiom as well.

Further, if π is the derivation consisting of applications of display postulates which transform the latter axiom into the former, then let $\pi' = \pi[z/a]^{pre}$. As discussed when treating (1a), the assumptions imply that π_2 has more than one node, so the transformation described above results in a cut application of strictly lower height. Moreover, the assumptions that the original cut preserves type-uniformity, that every derivable sequent is type-uniform, and the type-alikeness of parameters imply that the sequent $a_u \vdash y'[a]^{suc}$ is of the same type as the sequent $z \vdash a$. Hence, the new cut is strongly type-uniform. Finally, condition C'_7 implies that substituting z for a in π_2 and in π gives rise to admissible derivations $\pi_2[z/a]$ and π' in the calculus (use C'_6 for the symmetric case).

As to Wansing's case (2), assume that a_u has been introduced as a parameter in the conclusion of $\pi_{2,i}$ by an application *inf* of the rule *Ru*. For instance, in the calculus of Section 3.3, this situation can arise if a is of type formula Fm and it is introduced by *weakening*, or if a is of type functional actions Fnc and it is introduced by the rule *balance* or *atom*. Since a_u is a leaf in the history-tree of a , this implies that a_u is congruent only to itself in $\pi_{2,i}$. Hence, conditions C'_7 , the assumption that the original cut is quasi strongly type-uniform, and the type-alikeness of parameters (C'_2) imply that the sequent $(x_i \vdash y_i)[a_u]^{pre}$ can be replaced in the conclusion of $\pi_{2,i}$ by the sequent $(x_i \vdash y_i)[z/a_u]^{pre}$ by means of an application of the same rule *Ru*. Let $\pi'_{2,i}$ be the resulting derivation.

Therefore, the transformation below yields a derivation where π_1 is not used at all and the cut disappears.

$$\frac{\begin{array}{c} \vdots \pi_{2,i} \\ (x_i \vdash y_i)[a_u]^{pre} \\ \vdots \pi_1 \quad \vdots \pi_2 \\ z \vdash a \quad (x \vdash y)[a]^{pre} \\ \hline (x \vdash y)[z]^{pre} \end{array}}{\quad} \rightsquigarrow \frac{\begin{array}{c} \vdots \pi'_{2,i} \\ (x_i \vdash y_i)[z/a_u]^{pre} \\ \vdots \pi_2[z/a]^{pre} \\ (x \vdash y)[z]^{pre} \end{array}}{\quad}$$

From this point on, the proof proceeds like in [Wan02]. It is useful to emphasise that the need to combine principal and parametric moves arises in multi-type settings such as the Dynamic Calculus for EAK introduced in Section 3.3 not only because of contraction or additive rules, but also due to the presence of structural rules such as

$$\frac{(x \blacktriangle Y); (x \blacktriangle Z) \vdash W}{x \blacktriangle (Y; Z) \vdash W} \quad \frac{W \vdash (x \triangle Y) > (x \triangleright Z)}{W \vdash x \triangleright (Y > Z)}$$

□

Chapter 5

Display-Type Calculus for Monotone Modal Logic

5.1 Introduction

In the present chapter, we adapt the display-type methodology discussed in the previous chapter to the case study of monotone modal logic. This setting is simpler for two reasons: firstly, we can dispense with the introduction of types; secondly, we do not need to introduce axioms of a more complex type (such as the *atom* axioms discussed in Chapters 2 and 3). Since this is the most common and familiar situation, we take the opportunity of a closer look at this case study to formulate shorter and better fitting versions of the cut elimination theorem as well.

Organization and results. In Section 5.2, we collect preliminaries on monotone modal logic, display calculi and the visibility property as discussed in [BFS00]. In Section 5.3, we introduce proper display-type calculi in terms of the properties ensuring the cut elimination metatheorem, and we outline the proof of their metatheorem. In Section 5.4, we introduce a concrete instance of display-type calculus, which is designed to account for monotone modal logic; we show that this calculus is sound and complete w.r.t. the standard monotone neighbourhood semantics, and that its cut elimination result follows from the metatheorem of Section 5.3. More details are collected in the Appendix H.

5.2 Preliminaries

In the present section, we collect the preliminaries on Monotone Modal Logic and on the sequent calculi needed for our treatment.

5.2.1 Syntax and semantics of monotone modal logic

In the present subsection, we recall the syntax and semantic of monotone modal logic (MML).

Let AtProp be a countable set of atomic propositions. The set \mathcal{L} of the formulas A of Monotone Modal Logic (MML) is defined as follows:

$$A ::= p \in \text{AtProp} \mid \neg A \mid A \vee A \mid (\exists\forall)A.$$

Monotone modal logics (cf. [Che80, Han03]) are classical modal logics such that the modal operator $(\exists\forall)$ is required to satisfy the following condition: for all formulas A and B ,

$$A \rightarrow B \quad \text{implies} \quad (\exists\forall)A \rightarrow (\exists\forall)B. \quad (\mathbf{RM})$$

It is well known (cf. [Che80], Theorem 8.11) that monotone modal logics can be equivalently defined as being closed under the following axiom:

$$(\exists\forall)(A \wedge B) \rightarrow (\exists\forall)A \wedge (\exists\forall)B$$

and satisfying the following condition: for all formulas A and B ,

$$A \leftrightarrow B \quad \text{implies} \quad (\exists\forall)A \leftrightarrow (\exists\forall)B. \quad (\mathbf{RE})$$

As the propositional base of MML is classical, we can define the modality $(\forall\exists)$ as the dual of $(\exists\forall)$, that is: $(\forall\exists) := \neg(\exists\forall)\neg$. If we consider weaker propositional bases than classical propositional logic, $(\forall\exists)$ must be taken as primitive.

It is well known (cf. [Che80, Section 9.2.]) that MML cannot be adequately captured by Kripke semantics, and that the basic monotone modal logic M is sound and complete w.r.t. monotone neighbourhood frames (see [Han03]):

Definition 5.1. A *neighbourhood frame* is a tuple $\mathbb{F} = (W, \sigma)$ such that W is a set, and $\sigma : W \rightarrow \mathcal{P}\mathcal{P}W$ is a map. For any $w \in W$, an element $N \in \sigma(w)$ is called a *neighbourhood* of w . A neighbourhood frame \mathbb{F} is *monotone* if for any $w \in W$, the collection $\sigma(w)$ is an upward

closed subset of $(\mathcal{P}W, \subseteq)$, that is, if $X \in \sigma(w)$ and $X \subseteq Y \subseteq W$, then $Y \in \sigma(w)$. A *(monotone) neighbourhood model* is a tuple $\mathbb{M} = (\mathbb{F}, V)$, such that $\mathbb{F} = (W, \sigma)$ is a (monotone) neighbourhood frame, and $V : \text{AtProp} \rightarrow \mathcal{P}W$.

Given the stipulations above, the set-theoretic interpretation of the non-modal connectives is the usual one. Modal connectives are interpreted as follows:

$$\begin{aligned} M, w \Vdash (\exists V)A & \quad \text{iff} \quad \exists N \in \mathcal{P}W (N \in \sigma(w) \text{ and } \forall n \in N, M, n \Vdash A) \\ M, w \Vdash (\forall \exists)A & \quad \text{iff} \quad \forall N \in \mathcal{P}W (N \in \sigma(w) \text{ implies } \exists n \in N, M, n \Vdash A) \end{aligned}$$

5.2.2 Visibility and cut elimination

Visibility property. In [BFS00], a uniform methodology was introduced to generate (restricted) standard sequent calculi. This methodology is aimed at providing a proof-theoretic setting in which certain weakenings of classical logic (which include intuitionistic, linear and quantum logic) could be accounted for in a modular way. Key aspects of this methodology concern the definition of the introduction rules for each connective (which rely on so-called principle of *reflection*), but also—and more importantly for the developments in the paper—the achievement of a uniform cut elimination theorem, to which the so-called *visibility* property is key. Visibility is a requirement on the design of the rules introducing logical connectives, which restricts active contexts to be empty.¹ That is, visibility requires both the auxiliary formulas and the principal formula in any introduction rule to constitute the whole antecedent or the whole consequent of the sequent in which they occur. This property puts very heavy constraints on the shape of the introduction rules. However, in [BFS00], it was observed that the visibility property ensures a smooth cut elimination strategy for calculi which are otherwise subject to very heavy restrictions (e.g. calculi which do not contain structural rules such as weakening, contraction or exchange).

Canonical cut elimination without display. In fact, the remark in [BFS00] about the importance of visibility for the cut elimination strategy applies *mutatis mutandis* also for a *display-type calculus*, that is, a calculus for sequents of the form $X \vdash Y$ with X and Y being structures in the sense of display calculi, but in which there might not be enough display postulates to guarantee the display property either full (cf. Definition 2.1) or relativized (cf. Definition 3.3). In Section 5.4, we introduce a display-type calculus for monotone modal logic, enjoying the visibility property but not the display property

¹Recall that *active* formulas are either the principal formula of the conclusion sequent, or the auxiliary formula(s) in the premise(s) of an introduction rule. The context on the same (resp. opposite) side of active formulas is referred to as the *active* (resp. *passive*) context.

either full or relativized. The cut elimination for this calculus will be shown to follow from a ‘canonical’ cut elimination metatheorem.

5.3 Proper display-type calculi, and their metatheorem

In the present section, we discuss the cut elimination metatheorem in the setting of proper display-type calculi. This setting is simpler than the one of Section 4.2 for two reasons: firstly, it is single-typed; secondly, axioms are restricted to the original shape admitted in Belnap’s paper [Bel82]. In what follows, we define proper display-type calculi, and in 5.3.2, we prove their corresponding cut elimination metatheorem. The most notable difference between the conditions defining properly displayable calculi in [Wan02] and the ones below concern conditions C_5 (see below).

5.3.1 Proper display-type calculi

A display-type calculus is *proper* if it satisfies the following list of conditions:

C_1 : Preservation of formulas. Each formula occurring in a premise of an inference rule *inf* is a subterm of some operational term in the conclusion of *inf*.

C_2 : Shape-alikeness of parameters. Congruent parameters (i.e. non-active structures in the application of a rule) are occurrences of the same structure.

C_3 : Non-proliferation of parameters. Each parameter in an inference rule *inf* is congruent to at most one constituent in the conclusion of *inf*.

C_4 : Position-alikeness of parameters. Congruent parameters are either all antecedent or all succedent parts of their respective sequents. In the case of calculi enjoying the display property, antecedent and succedent parts are defined in the usual way (see [Bel82]). Otherwise, these notions can still be defined by induction on the shape of the structures, by relying on the polarity of each coordinate of the structural connectives.

C_5^* : Display of active constituents. If a formula A is active in the application of any rule, then A is in display.

C₆: Closure under substitution for succedent parts. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas occurring in succedent position.

C₇: Closure under substitution for precedent parts. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas occurring in precedent position.

The intended meaning of conditions C₆ (and likewise C₇) can be explained by the following diagram:

$$\frac{(X \vdash Y)[A]^{suc}}{(X' \vdash Y')[A]^{suc}} R \rightsquigarrow \frac{(X \vdash Y)[Z]^{suc}}{(X' \vdash Y')[Z]^{suc}} R$$

Any rule R should be such that, for any parametric occurrence of a formula A which is in succedent position, if A is substituted for an arbitrary structure Z both in the premise(s) and in the corresponding place in the conclusion, the resulting inference should always be justified as an application of the rule R .

C₈: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C₈ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of the cut rule, involving proper subterms of the original operational cut-term.

Remark 5.2. The setting above is on the one hand more restrictive than the one in [Wan02] and on the other more permissive. Specifically, while we are dealing with a more general shape of cut, we require condition C₅^{*} to be more restrictive than the corresponding condition C₅ in [Wan02]. Indeed, C₅^{*} requires that *all* the active formulas in each rule to be in display, and not just the principal ones.

5.3.2 Cut elimination metatheorem for proper display-type calculi

Theorem 5.3. *In any display-type calculus satisfying C₂, C₃, C₄, C₅^{*}, C₆, C₇, C₈, the surgical cut rules can be eliminated. Display-type calculi which in addition satisfy C₁ enjoy the subformula property.*

Proof. We follow the proof scheme of [Wan02, Section 3.3, Appendix A].

Principal stage: both cut formulas are principal. Without loss of generality, we can assume that the cut in the original proof is a Left Cut (the proof for a Right Cut is symmetric), that is we are going to consider the following situation

$$\frac{\begin{array}{c} \vdots \pi_1 \\ Z \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ (X \vdash Y)[A]^{pre} \end{array}}{(X \vdash Y)[Z/A]^{pre}}$$

There are two subcases.

If the end sequent $(X \vdash Y)[Z/A]^{pre}$ is identical to the conclusion of π_1 or π_2 , then we can eliminate the cut simply replacing the derivation above with π_1 (resp. π_2). A special case of this situation arises when both the premises $Z \vdash A$ and $(X \vdash Y)[A]^{pre}$ are axioms.

If the end sequent $(X \vdash Y)[Z/A]^{pre}$ is different from the conclusion of both π_1 and π_2 , then by C_8 , there is a proof of $(X \vdash Y)[Z/A]^{pre}$ which uses the same premise(s) of the original derivation and which involves only cuts on proper subformulas of A . Notice that the part of condition C_5^* which requires that not only the principal formula, but also the active premises occur in display makes sure that the new cut applications are surgical.

Parametric stage: at least one cut formula is parametric. There are two subcases: either one cut formula is principal or they are both parametric.

Consider the subcase in which one cut formula is principal. W.l.o.g. we assume that the cut-formula A is principal and in display in the the left-premise $X \vdash A$ of the cut in the original proof (the other case is symmetric). The situation is as follows

$$\frac{\begin{array}{c} \vdots \pi_1 \\ Z \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_{2.1} \\ \underline{A}_1 \vdash Y_1 \end{array} \quad \cdots \quad \begin{array}{c} \vdots \pi_{2.n} \\ (X_n \vdash Y_n)[\bar{A}_n]^{pre} \end{array}}{\begin{array}{c} \vdots \pi_2 \\ (X \vdash Y)[A]^{pre} \end{array}} \quad \frac{\quad}{(X \vdash Y)[Z]^{pre}}$$

where $(X_i \vdash Y_i)[A_i]$ with $i \in \{1, \dots, n\}$ denote the sequents in which the uppermost ancestor A_i of A in π_2 was introduced. The notation \underline{A}_i means that this occurrence is principal, and \bar{A}_j means that this occurrence is parametric. Conditions C_2 and C_3 make it possible to follow the history

of the right-hand-side cut formula A which takes the form of a tree (see discussion in Section 2.2.2, Remark 2.2).

Assume that A_i is introduced in $(X_i \vdash Y_i)[\bar{A}_i]$ as a parameter by an application *inf* of the rule Ru (for instance, in the calculus of Section 5.4, this situation can arise if A_i was introduced by Weakening). Since A_i is a leaf in the history-tree of A , we have that A_i is congruent only to itself in $X_i \vdash Y_i$. Hence, C_7 implies that it is possible to substitute Z for A_i by means of an application of the same rule Ru , i.e. $(X_i \vdash Y_i)[\bar{A}_i]$ can be replaced by $(X_i \vdash Y_i)[Z/\bar{A}_i]$ (note that if all the uppermost ancestors A_i are parametric, then the transformation gives a new derivation in which the given application of cut disappears and no new cut has been introduced).

Assume that A_i is introduced in $(X_i \vdash Y_i)[\underline{A}_i]^{pre}$ as a principal formula. By C_5^* , A_i is in display. Then we form a subderivation using π_1 and $\pi_{2,i}$ and applying a surgical cut as the last rule.

The transformations just discussed explain how to transform the leaves of the history tree of A . Finally, condition C_7 implies that substituting Z for each occurrence of A in the history tree of the cut formula A in π_2 gives rise to an admissible derivation $\pi_2[Z/A]^{pre}$ (use C_6 for the symmetric case).

Summing up, this procedure generates the following proof tree:

$$\begin{array}{c}
 \begin{array}{c} \vdots \pi_1 \\ Z \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_{2,i} \\ \underline{A}_1 \vdash Y_1 \end{array} \\
 \text{Cut} \frac{\quad}{Z \vdash Y_1} \quad \cdots \quad \begin{array}{c} \vdots \pi_{2,n} \\ (X_n \vdash Y_n)[\bar{Z}]^{pre} \end{array} \\
 \vdots \vdots \vdots \pi_2[Z/A]^{pre} \\
 (X \vdash Y)[Z]^{pre}
 \end{array}$$

We observe that in each newly introduced application of the cut rule, both cut terms are principal. Hence, we can apply the procedure described in the Principal stage and transform the original derivation in a derivation in which the cut terms of the newly introduced cuts have strictly lower complexity than the original cut terms. When the newly introduced applications of cut are of lower height than the original one, we do not need to resort to the Principal stage.²

Finally, as to the subcase in which both cut formulas are parametric, consider a proof with at least one cut. The procedure is analogous to the previous case. Namely, following the history of one of the cut formulas up to the leaves, and applying the transformation steps described above, we arrive at a situation in which, whenever new applications of cuts are generated, in each such

² This is for instance the case if, in the original derivation, the history-tree of the cut formula A (in the right-hand-side premise of the given cut application) contains at most one leaf A_l which is principal. However, the procedure described above in the Parametric stage does not always produce cuts of lower height. For instance, in the calculus introduced in Section 5.4, this situation may arise when two ancestors of a cut formula are introduced as principal along the same branch, and then are identified via an application of the rule Contraction.

application at least one of the cut formulas is principal. To each such cut, we can apply (the symmetric version of) the Parametric stage described so far. \square

5.4 A calculus for monotone modal logic

5.4.1 The basic calculus for MML

In the present subsection, we introduce the display-type calculus for MML. Since the propositional base is classical and very well known in the display calculi literature, we collect the structural and operational rules for the propositional fragment in the Appendix H. Here below, we focus on the rules concerning the modalities. As the modalities interact with negation, we recall rules for negation below.

Since we are working towards a display-type calculus, we need to introduce the structural language, which as usual matches the operational language.

$$A ::= \perp \mid \top \mid p \in \text{AtProp} \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid (\exists \forall)A \mid (\forall \exists)A$$

$$X ::= A \mid \mathbf{I} \mid *X \mid X ; X \mid \circ X \mid X > X \mid X < X$$

As is the case of standard display calculi, each operational connective corresponds to one structural connective. The following table illustrates the correspondence between structural symbols (on the first line) and operational symbols (on the second line).

\mathbf{I}	$*$	$;$	$<$	$>$	\circ
\top \perp	\neg \neg	\wedge \vee	$(-<)$ (\leftarrow)	$(>-)$ \rightarrow	$(\exists\forall)$ $(\forall\exists)$

In what follows, we introduce the rules for the calculus. These rules are standard and very well-known in the display calculi literature, and the only conspicuous aspect of this calculus is in fact the *absence* of the display postulates for the modal operators, which is the reason why the calculus does not enjoy the display property either full or relativized.

Cut rules. Since we are not including the display postulates for the modal operators, we need a more general type of cut rule than the one typically occurring in display calculi. Below we use following *surgical* cut rules:

$$Cut_L \frac{Z \vdash A \quad (X \vdash Y)[A]^{pre}}{(X \vdash Y)[Z]^{pre}} \quad \frac{(X \vdash Y)[A]^{suc} \quad A \vdash Z}{(X \vdash Y)[Z]^{suc}} Cut_R$$

Structural rules for negation and for the interaction negation-modalities.

$$* \frac{\frac{X \vdash Y}{* Y \vdash * X}}{X \vdash ** Y} \quad \text{swap}_L \frac{\frac{\circ * X \vdash Y}{* \circ X \vdash Y}}{Y \vdash \circ * X} \quad \text{swap}_R$$

Operational rules for negation and modalities.

$$\neg_L \frac{*A \vdash X}{\neg A \vdash X} \quad \frac{X \vdash *A}{X \vdash \neg A} \neg_R$$

$$(\forall\exists)_L \frac{\frac{\circ A \vdash X}{(\forall\exists)A \vdash X}}{(\forall\exists)_L} \quad \frac{X \vdash A}{\circ X \vdash (\forall\exists)A} (\forall\exists)_R$$

$$(\exists\forall)_L \frac{\frac{A \vdash X}{(\exists\forall)A \vdash \circ X}}{(\exists\forall)_L} \quad \frac{X \vdash \circ A}{X \vdash (\exists\forall)A} (\exists\forall)_R$$

5.4.2 Soundness

In the present section, we outline the soundness of the calculus introduced in the previous section w.r.t. the monotone neighbourhood semantics. Structures will be translated into formulas, and formulas will be interpreted in monotone neighbourhood models in the usual way. In order to translate structures as formulas, structural connectives need to be translated as logical connectives; to this effect, any given occurrence of a structural connective is translated as one or the other of its associated logical connectives, according to table 5.1. This translation takes place according to whether the substructure having the given connective as its root is in precedent or succedent position. In display calculi enjoying the full display property, the notion of precedent and succedent position can be defined in terms of whether each substructure is displayed as precedent or succedent part, as described in Section 2.2.2. Since the present calculus does not enjoy the display property, we need to define the same notion in a different way.

Indeed, for a sequent $X \vdash Y$ the notion of precedent and succedent position for substructures can be defined inductively by means of the polarity of the structural connectives by labelling the nodes of the generation trees of X and Y as follows:

- we start by labelling the root of X as precedent and the root of Y as succedent;
- nodes inherit the same (resp. opposite) label of their direct ancestors if the coordinate in the scope of which they occur is positive (resp. negative).

Structural connective	if in precedent position	if in succedent position
I	\top	\perp
$A ; B$	$A \wedge B$	$A \vee B$
$*A$	$\neg A$	$\neg A$
$A > B$	$A \succ B$	$A \rightarrow B$
$A < B$	$A \prec B$	$A \leftarrow B$
$\circ A$	$(\forall \exists)A$	$(\exists \forall)A$

TABLE 5.1: Translation of structural connectives into logical connectives

- \circ and $;$ are positive in each coordinate. $*$ is negative, $>$ is negative in its first coordinate and positive in its second, and $<$ is positive in its first coordinate and negative in its second.

For example, in the sequent $*(A > (B > E)) \vdash C ; *D$, the formulas A, B , and D are in precedent position, and C and E are in succedent position.

In each monotone neighbourhood model \mathbb{M} , sequents $A \vdash B$ will be interpreted as inclusions $\llbracket A \rrbracket_{\mathbb{M}} \subseteq \llbracket B \rrbracket_{\mathbb{M}}$; rules $(A_i \vdash B_i \mid i \in I) / C \vdash D$ will be interpreted as implications of the form “if $\llbracket A_i \rrbracket_{\mathbb{M}} \subseteq \llbracket B_i \rrbracket_{\mathbb{M}}$ for every $i \in I$, then $\llbracket C \rrbracket_{\mathbb{M}} \subseteq \llbracket D \rrbracket_{\mathbb{M}}$ ”. As an example, let us verify the soundness of the rule $(\forall \exists)_L$. Fix a monotone neighbourhood model \mathbb{M} , assume that $A \vdash B$ is satisfied on \mathbb{M} , that is $\llbracket A \rrbracket_{\mathbb{M}} \subseteq \llbracket B \rrbracket_{\mathbb{M}}$, and let us show that $(\forall \exists)A \vdash (\forall \exists)B$ is satisfied on \mathbb{M} , that is, $\llbracket (\forall \exists)A \rrbracket_{\mathbb{M}} \subseteq \llbracket (\forall \exists)B \rrbracket_{\mathbb{M}}$. Fix $x \in \llbracket (\forall \exists)A \rrbracket_{\mathbb{M}}$. Hence, for any $N \in \sigma(x)$, there is some $n \in N$ such that $n \in \llbracket A \rrbracket_{\mathbb{M}}$. As $\llbracket A \rrbracket_{\mathbb{M}} \subseteq \llbracket B \rrbracket_{\mathbb{M}}$, we have that $n \in \llbracket B \rrbracket_{\mathbb{M}}$. Hence, for any $N \in \sigma(x)$, there is some $n \in N$ such that $n \in \llbracket B \rrbracket_{\mathbb{M}}$, which shows that $x \in \llbracket (\forall \exists)B \rrbracket_{\mathbb{M}}$, as required. The soundness of the rule $(\exists \forall)_L$ is proved similarly.

Let us verify the soundness of the rule $swap_L$. Fix a monotone neighbourhood model \mathbb{M} , assume that $\llbracket (\forall \exists)\neg A \rrbracket_{\mathbb{M}} \subseteq \llbracket B \rrbracket_{\mathbb{M}}$ and let us show that $\llbracket \neg(\exists \forall)A \rrbracket_{\mathbb{M}} \subseteq \llbracket B \rrbracket_{\mathbb{M}}$. It is enough to show that for any state x in the model, $x \in \llbracket \neg(\exists \forall)A \rrbracket_{\mathbb{M}}$ iff $x \in \llbracket (\forall \exists)\neg A \rrbracket_{\mathbb{M}}$. Indeed, $x \in \llbracket \neg(\exists \forall)A \rrbracket_{\mathbb{M}}$ iff $x \notin \llbracket (\exists \forall)A \rrbracket_{\mathbb{M}}$ iff there is no $N \in \sigma(x)$ such that $N \subseteq \llbracket A \rrbracket_{\mathbb{M}}$. This is equivalent to saying that for any $N \in \sigma(x)$ there exists some $n \in N$ such that $n \notin \llbracket A \rrbracket_{\mathbb{M}}$, that is, $n \in \llbracket \neg A \rrbracket_{\mathbb{M}}$. This is equivalent to $x \in \llbracket (\forall \exists)\neg A \rrbracket_{\mathbb{M}}$. Hence the rule $swap_L$ is sound, as required. The proof that $swap_R$ is sound is similar.

Finally, let us verify the soundness of the rule Cut_L . Fix a monotone neighbourhood model \mathbb{M} , assume that $\llbracket \tau_1(Z) \rrbracket_{\mathbb{M}} \subseteq \llbracket A \rrbracket_{\mathbb{M}}$ and $\llbracket \tau_1(X) \rrbracket_{\mathbb{M}} \subseteq \llbracket \tau_2(Y) \rrbracket_{\mathbb{M}}$, where $\tau_1(Z), \tau_1(X)$ and $\tau_2(Y)$ are the formulas respectively translating the structures Z, X and Y as outlined above, and the formula A is in precedent position in $X \vdash Y$. Hence, two cases may occur: if A is a substructure of X , then X is positive in this specific occurrence of A . Hence, it can be readily shown that the term function associated with $\tau_1(X)$ is monotone in

this specific occurrence of A . Since we assume that $\llbracket \tau_1(Z) \rrbracket_{\mathbb{M}} \subseteq \llbracket A \rrbracket_{\mathbb{M}}$, this implies that $\llbracket \tau_1(X)[Z/A] \rrbracket_{\mathbb{M}} \subseteq \llbracket \tau_1(X)[A] \rrbracket_{\mathbb{M}} \subseteq \llbracket \tau_2(Y) \rrbracket_{\mathbb{M}}$, as required. Similarly, if A occurs in Y , it can be readily shown that the term function associated with $\tau_2(Y)$ is antitone in this specific occurrence of A , and hence $\llbracket \tau_1(X) \rrbracket_{\mathbb{M}} \subseteq \llbracket \tau_2(Y)[A] \rrbracket_{\mathbb{M}} \subseteq \llbracket \tau_2(Y)[Z/A] \rrbracket_{\mathbb{M}}$, as required.

5.4.3 Completeness

To give an indirect proof of the completeness of the calculus introduced in Section 5.4 it is enough to show that all the axioms of the Hilbert style presentation of the monotonic modal logic (cf. Section 5.2.1) are theorems and that all the rules are derived or admissible rules. In fact, according to the first presentation of the minimal monotone modal logic, there are no axioms, and the only rule is the condition (RM). This rule is immediately seen to be derivable by means of the following derivation.

$$\frac{\frac{\frac{\frac{A \vdash A \quad B \vdash B}{A \rightarrow B \vdash A > B}}{\vdash A \rightarrow B}}{\vdash A > B}}{A \vdash B}}{(\exists \forall)A \vdash \circ B}}{(\exists \forall)A \vdash (\exists \forall)B}}{\frac{\vdash (\exists \forall)A > (\exists \forall)B}{\vdash (\exists \forall)A \rightarrow (\exists \forall)B}}$$

In what follows, we also include the derivations of the box and diamond version of the monotonicity axioms and the inter derivability of modalities.

Monotonicity of the modalities

$$\frac{\frac{\frac{A \vdash A}{A; B \vdash A}}{A \wedge B \vdash B}}{(\exists \forall)(A \wedge B) \vdash \circ B} \quad \frac{\frac{\frac{B \vdash B}{A; B \vdash B}}{A \wedge B \vdash B}}{(\exists \forall)(A \wedge B) \vdash \circ B} \quad \frac{\frac{\frac{A \vdash A}{A; B \vdash A}}{A \wedge B \vdash A}}{\circ A \wedge B \vdash (\forall \exists)A} \quad \frac{\frac{\frac{B \vdash B}{A; B \vdash B}}{A \wedge B \vdash B}}{\circ A \wedge B \vdash (\forall \exists)B}}{(\forall \exists)(A \wedge B) \vdash (\forall \exists)A} \quad \frac{(\forall \exists)(A \wedge B) \vdash (\forall \exists)A}{(\forall \exists)(A \wedge B) \vdash (\forall \exists)A} \quad \frac{(\forall \exists)(A \wedge B) \vdash (\forall \exists)B}{(\forall \exists)(A \wedge B) \vdash (\forall \exists)B}}{(\exists \forall)(A \wedge B); (\exists \forall)(A \wedge B) \vdash (\exists \forall)A \wedge (\exists \forall)B} \quad \frac{(\exists \forall)(A \wedge B) \vdash (\exists \forall)A}{(\exists \forall)(A \wedge B) \vdash (\exists \forall)A} \quad \frac{(\exists \forall)(A \wedge B) \vdash (\exists \forall)B}{(\exists \forall)(A \wedge B) \vdash (\exists \forall)B}}{(\exists \forall)(A \wedge B) \vdash (\exists \forall)A \wedge (\exists \forall)B}$$

$$\frac{\frac{\frac{A \vdash A}{A \vdash A; B}}{A \vdash A \vee B}}{(\exists \forall)A \vdash \circ A \vee B} \quad \frac{\frac{\frac{B \vdash B}{B \vdash A; B}}{B \vdash A \vee B}}{(\exists \forall)B \vdash \circ A \vee B} \quad \frac{\frac{\frac{A \vdash A}{A \vdash A; B}}{A \vdash A \vee B}}{\circ A \vdash (\forall \exists)(A \vee B)} \quad \frac{\frac{\frac{B \vdash B}{B \vdash A; B}}{B \vdash A \vee B}}{\circ B \vdash (\forall \exists)(A \vee B)} \quad \frac{(\exists \forall)A \vdash (\exists \forall)(A \vee B)}{(\exists \forall)A \vdash (\exists \forall)(A \vee B)} \quad \frac{(\exists \forall)B \vdash (\exists \forall)(A \vee B)}{(\exists \forall)B \vdash (\exists \forall)(A \vee B)} \quad \frac{(\forall \exists)A \vdash (\forall \exists)(A \vee B)}{(\forall \exists)A \vdash (\forall \exists)(A \vee B)} \quad \frac{(\forall \exists)B \vdash (\forall \exists)(A \vee B)}{(\forall \exists)B \vdash (\forall \exists)(A \vee B)} \quad \frac{(\forall \exists)A \vee (\forall \exists)B \vdash (\forall \exists)(A \vee B); (\exists \forall)A \vee (\exists \forall)B}{(\forall \exists)A \vee (\forall \exists)B \vdash (\forall \exists)(A \vee B)} \quad \frac{(\forall \exists)A \vee (\forall \exists)B \vdash (\forall \exists)(A \vee B)}{(\forall \exists)A \vee (\forall \exists)B \vdash (\forall \exists)(A \vee B)} \quad \frac{(\exists \forall)A \vee (\exists \forall)B \vdash (\exists \forall)(A \vee B)}{(\exists \forall)A \vee (\exists \forall)B \vdash (\exists \forall)(A \vee B)} \quad \frac{(\exists \forall)A \vee (\exists \forall)B \vdash (\exists \forall)(A \vee B)}{(\exists \forall)A \vee (\exists \forall)B \vdash (\exists \forall)(A \vee B)}$$

Inter-derivability of the modalities

$$\begin{array}{c}
\frac{A \vdash A}{*A \vdash *A} \\
\frac{*A \vdash *A}{\neg A \vdash *A} \\
\frac{\neg A \vdash *A}{A \vdash * \neg A} \\
\frac{A \vdash * \neg A}{(\exists \forall)A \vdash \circ * \neg A} \\
\frac{(\exists \forall)A \vdash \circ * \neg A}{(\exists \forall)A \vdash * \circ \neg A} \\
\frac{(\exists \forall)A \vdash * \circ \neg A}{\circ \neg A \vdash * (\exists \forall)A} \\
\frac{\circ \neg A \vdash * (\exists \forall)A}{(\forall \exists) \neg A \vdash * (\exists \forall)A} \\
\frac{(\forall \exists) \neg A \vdash * (\exists \forall)A}{(\exists \forall)A \vdash * (\forall \exists) \neg A} \\
\frac{(\exists \forall)A \vdash * (\forall \exists) \neg A}{(\exists \forall)A \vdash \neg (\forall \exists) \neg A}
\end{array}
\qquad
\begin{array}{c}
\frac{A \vdash A}{*A \vdash *A} \\
\frac{*A \vdash *A}{*A \vdash \neg A} \\
\frac{*A \vdash \neg A}{\circ * A \vdash (\forall \exists) \neg A} \\
\frac{\circ * A \vdash (\forall \exists) \neg A}{* \circ A \vdash (\forall \exists) \neg A} \\
\frac{* \circ A \vdash (\forall \exists) \neg A}{* (\forall \exists) \neg A \vdash \circ A} \\
\frac{* (\forall \exists) \neg A \vdash \circ A}{* (\forall \exists) \neg A \vdash (\exists \forall)A} \\
\frac{* (\forall \exists) \neg A \vdash (\exists \forall)A}{\neg (\forall \exists) \neg A \vdash (\exists \forall)A}
\end{array}$$

5.4.4 Cut elimination

Principal stage: the two cut formulas are principal. We consider a surgical cut, but as we have visibility, if the cut formula A is principal in π_1 and π_2 , then the cut has to be in isolation. Hence, in the principal stage, we have to show how to reduce the complexity of the cut, when the cut is in isolation and the cut formulas are both principal.

$$\begin{array}{c}
\frac{\frac{\vdots \pi_1}{Z \vdash \circ A} \quad \frac{\vdots \pi_2}{A \vdash Y}}{Z \vdash (\exists \forall)A} \quad \frac{A \vdash Y}{(\exists \forall)A \vdash \circ Y} \text{Cut} \rightsquigarrow \frac{\vdots \pi_1}{Z \vdash \circ A} \quad \frac{\vdots \pi_2}{A \vdash Y} \text{Cut}_R \\
\frac{\frac{\vdots \pi_1}{X \vdash A} \quad \frac{\vdots \pi_2}{Y \vdash B} \quad \frac{\vdots \pi_3}{A, B \vdash Z}}{X, Y \vdash A \wedge B} \quad \frac{A, B \vdash Z}{A \wedge B \vdash Z} \rightsquigarrow \text{Cut}_L \frac{\vdots \pi_2}{Y \vdash B} \quad \text{Cut}_L \frac{\frac{\vdots \pi_1}{X \vdash A} \quad \frac{\vdots \pi_3}{A, B \vdash Z}}{X, B \vdash Z} \\
\frac{\frac{\vdots \pi_1}{Z \vdash *A} \quad \frac{\vdots \pi_2}{*A \vdash Y}}{Z \vdash \neg A} \quad \frac{*A \vdash Y}{\neg A \vdash Y} \rightsquigarrow \text{Cut}_L \frac{\vdots \pi_2}{*A \vdash Y} \quad \frac{\vdots \pi_1}{Z \vdash *A}}{\frac{*Y \vdash A}{Z \vdash **Y}} \\
\frac{\frac{\vdots \pi_1}{Z \vdash A > B} \quad \frac{\vdots \pi_2}{X \vdash A} \quad \frac{\vdots \pi_3}{B \vdash Y}}{Z \vdash A \rightarrow B} \quad \frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X > Y} \rightsquigarrow \text{Cut}_L \frac{\frac{\vdots \pi_2}{X \vdash A} \quad \frac{\vdots \pi_1}{Z \vdash A > B}}{Z \vdash X > B} \quad \frac{\vdots \pi_3}{B \vdash Y} \text{Cut}_R \\
\end{array}$$

Chapter 6

Multi-Type Display Calculus for Propositional Dynamic Logic

6.1 Introduction

In the present chapter, we extend the multi-type display methodology to the full language of Propositional Dynamic Logic (PDL).

An important technical solution, which was adopted also in Chapter 3 but for different reasons, is that actions are assigned *two* distinct types: Act for general actions, and TAct for transitive actions (we take positive iteration rather than iteration as primitive).

The introduction of different types lays the ground for overcoming some of the difficulties which typically make the proof-theoretic treatment of dynamic logics not straightforward, and have hindered the smooth transfer of results from one dynamic logic to another. In the specific case of PDL, the main hurdle lies in the encoding of the induction axiom into a structural rule. Due to the inductive ‘loop’ (by which we mean a given parametric formula occurring both in antecedent and in consequent position), the induction axiom cannot straightforwardly be captured at the structural level, given that structures mean different things depending on their (antecedent or consequent) position in a sequent. However, as it is well known (cf. [HKT00]), the induction axiom reflects, in the context of *formulas*, the fact that the *positive iteration* + (resp. *iteration*, also known as *Kleene star* *) semantically is the (reflexive and) transitive closure operator *on actions*. This fact can (and should) be captured by finitary (structural) rules *purely in the context of actions*, that is, without formulas playing a mediating role. As in the case of EAK (cf. Chapter 3), the multi-type environment can be used to provide the

additional expressivity needed to capture the informational contents of various dynamic axioms in the context most naturally suited to support them.

As discussed in the previous chapters, the multi-type methodology aims at generating calculi enjoying the following features, as witnessed e.g. by the Dynamic Calculus of Chapter 3: (1) a neat division of labour, required by general proof-theoretic semantic principles, between operational and structural rules; (2) a neat division of labour between structural rules describing the properties *pertinent to each type*, and structural rules describing the *interaction* between different types; (3) all rules enjoying closure under uniform substitution of parametric operational terms for arbitrary structures within each type.

Feature (3) is crucial to the definition of the *multi-type* version of *proper display calculi* (cf. Wansing's definition [Wan98, Section 4.1]. See Section 6.2.2 for more on this). The cut elimination result for any such calculus follows straightforwardly from the corresponding Belnap-style metatheorem 3.4.

The multi-type display calculus for PDL treated in the present chapter has a design similar to the one in Chapter 3, from which features analogous to (1) and (3) above will follow. This calculus is shown to be a proper display calculus (cf. Definition 6.1), and hence its cut elimination and subformula property smoothly follow from the Belnap-style metatheorem 3.4.

However, the present system does not enjoy feature (2) above. Indeed, the present setting is not yet expressive enough to capture transitive closure at the *structural* level. Hence, the induction axiom is captured by means of infinitary rules, and making use of the mediating role of formulas. We conjecture that being able to express transitive closure at the structural level is key to dispensing with the infinitary rules, which is our next goal for future developments in this line research.

Organization and results. In Section 6.2, we collect the relevant preliminaries on PDL, and we recall the generalization of the notion of proper display calculi to the multi-type setting, and its corresponding extension of Belnap's cut elimination metatheorem. In Section 6.3, we introduce the multi-type display calculus for PDL. In Section 6.4, we discuss the soundness of some of the rules w.r.t. the standard semantics. In Section 6.5, we prove that this calculus is complete w.r.t. PDL. In Section 6.6, we prove that it enjoys the Belnap-style cut elimination. In Section 6.7, we discuss the different techniques available to prove that the calculus is conservative. In Section 6.8, we collect some conclusions and indicate further directions. Most of the proofs and derivations are collected in Appendices J and K.

6.2 Basic facts and definitions

In the present section, we collect some basic facts: in subsection 6.2.1, we briefly review the Hilbert-style axiomatization of Propositional Dynamic Logic (PDL); in subsection 6.2.2 we introduce the definition of proper multi-type display calculi, and state their cut elimination metatheorem. This definition and theorem are more compact versions of the ones of Chapter 3, hence they will not be expanded on, but the Belnap-style cut elimination for the Dynamic Calculus for PDL (cf. Section 6.6) will be shown on the basis of this more compact definition.

6.2.1 Propositional Dynamic Logic

In our review of PDL, we will loosely follow [HKT00]. However, our presentation differs from that in [HKT00] in some respects, which will be discussed below.

Let AtProp and AtAct be countable and disjoint sets of atomic propositions and atomic actions, respectively. The set \mathcal{L} of the formulas A of Propositional Dynamic Logic (PDL), and the set $\text{Act}(\mathcal{L})$ of the actions α over \mathcal{L} are defined simultaneously as follows:

$$A ::= p \in \text{AtProp} \mid \neg A \mid A \vee A \mid \langle \alpha \rangle A \quad (\alpha \in \text{Act}(\mathcal{L})),$$

$$\alpha ::= a \in \text{AtAct} \mid \alpha ; \alpha \mid \alpha \cup \alpha \mid A? \mid \alpha^+ \quad (A \in \mathcal{L}).$$

Let the symbols \wedge and \perp be defined as usual, that is $A \wedge B := \neg(\neg A \vee \neg B)$ and $\perp := A \wedge \neg A$ for some formula $A \in \mathcal{L}$.

Models for this language are tuples $M = (W, \{R_a \mid a \in \text{AtAct}\}, V)$, such that $R_a \subseteq W \times W$ for each $a \in \text{AtAct}$ and $V(p) \subseteq W$ for each $p \in \text{AtProp}$. For each non atomic $\alpha \in \text{Act}(\mathcal{L})$, the relation $R_\alpha \subseteq W \times W$ is defined recursively as follows:

$$\begin{aligned} R_{\alpha ; \beta} &::= R_\alpha \circ R_\beta &= \{(u, v) \mid \exists w \in W((u, w) \in R_\alpha \text{ and } (w, v) \in R_\beta)\} \\ R_{\alpha \cup \beta} &::= R_\alpha \cup R_\beta &= \{(u, v) \mid (u, v) \in R_\alpha \text{ or } (u, v) \in R_\beta\} \\ R_{\alpha^+} &::= (R_\alpha)^+ &= \bigcup_{n \geq 1} R_\alpha^n \\ R_{A?} &::= \{(w, w) \mid w \in V(A)\}. \end{aligned}$$

where R_α^n is defined as follows: $R_\alpha^1 := R$ and $R_\alpha^{n+1} := R_\alpha^n \circ R_\alpha$ for every $n \geq 1$. Given the stipulations above, the definitions of satisfiability and validity of propositions are the usual ones in modal logic; in particular the evaluation of formulas of the form $\langle \alpha \rangle A$ (resp. $[\alpha]A$) makes use of the corresponding relation R_α :

$$\begin{aligned}
M, w \Vdash \langle \alpha \rangle A & \text{ iff } \exists v \in W(wR_\alpha v \text{ and } M, w \Vdash A) \\
M, w \Vdash [\alpha]A & \text{ iff } \forall v \in W(\text{if } wR_\alpha v \text{ then } M, w \Vdash A)
\end{aligned}$$

The following list of axioms and rules constitutes a sound and complete Hilbert-style deductive system for PDL with box as primitive modality and diamond defined as usual $\langle \alpha \rangle A := \neg[\alpha]\neg A$. We list the *Fix-Point Axiom* (also called *Mix Axiom*) and the *Induction Axiom* (also called *Seegerberg Axiom*) in the version with $*$ (cf. [Seg82, FL79, HKT00]) and with $+$ (cf. [Har13a, Har13b]). For proofs of the completeness of PDL, the reader is referred to [KP81]. For the sake of the developments of the following sections, we take the axiomatisation of PDL with positive iteration $+$, rather than with Kleene star $*$.

Box-axioms

K	$\vdash [\alpha](A \rightarrow B) \rightarrow ([\alpha]A \rightarrow [\alpha]B)$
Choice	$\vdash [\alpha \cup \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$
Composition	$\vdash [\alpha; \beta]A \leftrightarrow [\alpha][\beta]A$
Test	$\vdash [A?]B \leftrightarrow (A \rightarrow B)$
Distributivity	$\vdash [\alpha](A \wedge B) \leftrightarrow [\alpha]A \wedge [\alpha]B$
<hr/>	
Fix Point $*$	$\vdash [\alpha^*]A \leftrightarrow A \wedge [\alpha][\alpha^*]A$
Induction $*$	$\vdash A \wedge [\alpha^*](A \rightarrow [\alpha]A) \rightarrow [\alpha^*]A$
<hr/>	
Fix Point $+$	$\vdash [\alpha^+]A \leftrightarrow [\alpha]A \wedge [\alpha][\alpha^+]A$
Induction $+$	$\vdash ([\alpha]A \wedge [\alpha^+](A \rightarrow [\alpha]A)) \rightarrow [\alpha^+]A$

Inference Rules

Modus Ponens	if $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$
$[\alpha]$ -Intro	if $\vdash A$, then $\vdash [\alpha]A$

As was done in Chapter 3, motivated by the fact that the calculus of Section 6.3 is modular and can be easily rearranged to take propositional bases which are strictly weaker than the Boolean one, both box and diamond operators will be taken as primitive (see also [Gol92b] for an axiomatisation of PDL with independent modalities).

Diamond-axioms

Choice	$\vdash \langle \alpha \cup \beta \rangle A \leftrightarrow \langle \alpha \rangle A \vee \langle \beta \rangle A$
Composition	$\vdash \langle \alpha ; \beta \rangle A \leftrightarrow \langle \alpha \rangle \langle \beta \rangle A$
Test	$\vdash \langle A? \rangle B \leftrightarrow A \wedge B$
Distributivity	$\vdash \langle \alpha \rangle (A \vee B) \leftrightarrow \langle \alpha \rangle A \vee \langle \alpha \rangle B$
Fix point *	$\vdash \langle \alpha^* \rangle A \leftrightarrow A \vee \langle \alpha \rangle \langle \alpha^* \rangle A$
Induction *	$\vdash \langle \alpha^* \rangle A \leftrightarrow A \vee \langle \alpha^* \rangle (\neg A \wedge \langle \alpha \rangle A)$
Fix point +	$\vdash \langle \alpha^+ \rangle A \leftrightarrow \langle \alpha \rangle A \vee \langle \alpha \rangle \langle \alpha^+ \rangle A$
Induction +	$\vdash \langle \alpha^+ \rangle A \rightarrow \langle \alpha \rangle A \vee \langle \alpha^+ \rangle (\neg A \wedge \langle \alpha \rangle A)$

Inference Rules

Modus Ponens	if $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$
$\langle \alpha \rangle$ -Intro	if $\vdash A \rightarrow \perp$, then $\vdash \langle \alpha \rangle A \rightarrow \perp$

The language of PDL will be also extended with the modalities $\overline{\langle \alpha \rangle}$ and $\widehat{\langle \alpha \rangle}$ which are adjoint to $\langle \alpha \rangle$ and $[\alpha]$ respectively for each action α . These modalities correspond to the *converse operator* $(\cdot)^{-1}$ with the following semantics: for any action α ,

$$R_{\alpha^{-1}} := R_{\alpha}^{-1},$$

where $R^{-1} := \{(u, v) \mid (v, u) \in R\}$ for every relation R . As mentioned in [HKT00], the converse operator is useful to talk about ‘running a program backward’ or reversing actions, although this is not always possible in practice. The modal operators $\overline{\langle \alpha \rangle}$ and $\widehat{\langle \alpha \rangle}$ are semantically interpreted in the standard way using the relation R_{α}^{-1} . The relevant additional rules are reported below (see also [Har13a, Har13b] for an analogous extension).

Inference Rules

$(\langle \alpha \rangle \dashv \overline{\langle \alpha \rangle})$ -Adj.	$\vdash \langle \alpha \rangle A \rightarrow B$ iff $\vdash A \rightarrow \overline{\langle \alpha \rangle} B$
$(\widehat{\langle \alpha \rangle} \dashv [\alpha])$ -Adj.	$\vdash A \rightarrow [\alpha] B$ iff $\vdash \widehat{\langle \alpha \rangle} A \rightarrow B$

6.2.2 Proper multi-type display calculi, and their cut elimination metatheorem

Definition 6.1. Let a *proper display multi-type calculus* be any display-type calculus in a multi-type language satisfying the following list of conditions:

C₁: preservation of operational terms. Each operational term occurring in a premise of an inference rule *inf* is a subterm of some operational term in the conclusion of *inf*.

C₂: Shape-alikeness of parameters. Congruent parameters are occurrences of the same structure.

C'₂: Type-alikeness of parameters. Congruent parameters have exactly the same type. This condition bans the possibility that a parameter changes type along its history.

C₃: Non-proliferation of parameters. Each parameter in an inference rule *inf* is congruent to at most one constituent in the conclusion of *inf*.

C₄: Position-alikeness of parameters. Congruent parameters are either all antecedent or all succedent parts of their respective sequents.

C₅: Display of principal constituents. If an operational term *a* is principal in the conclusion sequent *s* of a derivation π , then *a* is in display.

C'₆: Closure under substitution for succedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in succedent position, *within each type*.

C'₇: Closure under substitution for precedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in precedent position, *within each type*.

C'₈: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C'₈ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of the cut rule, involving proper subterms of the original operational cut-term. In addition to this, specific to the multi-type setting is the requirement that the new application(s) of the cut rule be also *strongly type-uniform* (cf. condition C₁₀ below).

C₉: Type-uniformity of derivable sequents. Each derivable sequent is type-uniform.

C₁₀: Strong type-uniformity of cut rules. All cut rules are strongly type-uniform (cf. Definition 3.2).

Theorem 6.2. *Any multi-type display calculus satisfying $C_2, C'_2, C_3, C_4, C_5, C'_6, C'_7, C'_8, C_9$ and C_{10} is cut-admissible. If also C_1 is satisfied, then the calculus enjoys the subformula property.*

The proof of the theorem above is similar to the proof of Theorem 3.4.

6.3 Language and rules

As mentioned in the introduction, the key idea is to introduce a language in which actions are not accounted for as *parameters* indexing the dynamic connectives, but as *logical terms* in their own right. In the present section, we define a *multi-type language* into which the language of PDL translates, and in which the different types interact via special unary or binary connectives. The present setting consists of the following types: Act for actions, TAct for transitive actions, and Fm for formulas. We stipulate that Act, TAct and Fm are pairwise disjoint.

An algebraically motivated introduction. Similarly to the binary connectives introduced in Chapter 3, the following binary connectives (referred to as *heterogeneous connectives*) facilitate the interaction between the two types of actions and the formulas:

$$\Delta_0, \blacktriangle_0 : \text{TAct} \times \text{Fm} \rightarrow \text{Fm} \quad (6.1)$$

$$\Delta_1, \blacktriangle_1 : \text{Act} \times \text{Fm} \rightarrow \text{Fm}. \quad (6.2)$$

We think of the connectives above as being semantically interpreted as maps preserving existing joins in each coordinate (see below), between algebras suitable to interpret general actions, transitive actions, and formulas respectively. For instance, suitable domains of interpretation for formulas can be complete atomic Boolean algebras or perfect Heyting algebras; suitable domains of interpretation for actions (in different versions of PDL) can be quantal frames [Res06, Chapter III.2], or Kleene algebras with tests (KATs) [HKT00, page 421], appropriate subalgebras of which can serve as domains of interpretation for transitive actions (possibly w.r.t. to certain restrictions of the signature).

Connected to the standard relational semantic setting for PDL outlined in Section 6.2.1, for any relational model M based on the set W , the complex algebra based on $\mathcal{P}W$

is taken as the domain of interpretation for Fm-type terms, and Act-type (resp. TAct-type) terms are interpreted as (transitive) relations on W . This way, in particular, for any model M , the domain of interpretation of TAct is the relation algebra based on the complete lattice $\mathcal{T}(W \times W)$ of the transitive relations on W (indeed, $\mathcal{T}(W \times W)$ is a sub- \cap -semilattice of $\mathcal{P}(W \times W)$).

A natural requirement of the algebraic environment outlined above, which is verified by the algebras arising from the standard semantic setting of Kripke models, is that the interpretations of these heterogeneous connectives are *actions*, i.e., that (the domains of interpretation of) both Act and TAct induce module structures (cf. [Res06, Chapter II.2]) on (the domain of interpretation of) Fm. That is, the following conditions hold for all $\alpha, \beta \in \text{Act}$, $\gamma, \delta \in \text{TAct}$, and $A \in \text{Fm}$,

$$\gamma \Delta_0(\delta \Delta_0 A) = (\gamma ; \delta) \Delta_0 A \quad \alpha \Delta_1(\beta \Delta_1 A) = (\alpha ; \beta) \Delta_1 A \quad (6.3)$$

$$\gamma \blacktriangle_0(\delta \blacktriangle_0 A) = (\delta ; \gamma) \blacktriangle_0 A \quad \alpha \blacktriangle_1(\beta \blacktriangle_1 A) = (\beta ; \alpha) \blacktriangle_1 A. \quad (6.4)$$

In the semantic contexts mentioned above, the fact that the interpretations of the connectives Δ_i and \blacktriangle_i for $i = 0, 1$ are completely join-preserving in both coordinates implies that each of them has right adjoint in each coordinate. In particular, the following additional connectives have a natural interpretation as the right adjoints of Δ_i and \blacktriangle_i for $i = 0, 1$ in their second coordinate:

$$\rightarrow_0, \dashv_0 : \text{TAct} \times \text{Fm} \rightarrow \text{Fm} \quad (6.5)$$

$$\rightarrow_1, \dashv_1 : \text{Act} \times \text{Fm} \rightarrow \text{Fm}. \quad (6.6)$$

Also, the following connectives can be naturally interpreted in the setting above, as right adjoints of Δ_1 and \blacktriangle_1 in their first coordinate:

$$\blacktriangleleft_1, \triangleleft_1 : \text{Fm} \times \text{Fm} \rightarrow \text{Act}. \quad (6.7)$$

Intuitively, for all formulas A, B , the term $B \blacktriangleleft_1 A$ denotes the weakest action α such that, if A was true before α was performed, then B is true after any successful execution of α . The connectives above, when restricted to the diagonal subset of $\text{Fm} \times \text{Fm}$, are the ones Pratt described as the *weakest preservers* in [Pra91].

Virtual adjoints, part 1. However, Δ_0 and \blacktriangle_0 cannot be assumed to have right adjoints in their first coordinate (the reason for this will be discussed in part 2 below). Hence,

$\langle \alpha \rangle A$	becomes	$\alpha \Delta_1 A$	$\widehat{\langle \alpha \rangle} A$	becomes	$\alpha \blacktriangle_1 A$
$[\alpha] A$	becomes	$\alpha \rightarrow_1 A$	$\overline{[\alpha]} A$	becomes	$\alpha \rightarrow_1 A$
$\langle \alpha^+ \rangle A$	becomes	$\alpha^+ \Delta_0 A$	$\widehat{\langle \alpha^+ \rangle} A$	becomes	$\alpha^+ \blacktriangle_0 A$
$[\alpha^+] A$	becomes	$\alpha^+ \rightarrow_0 A$	$\overline{[\alpha^+]} A$	becomes	$\alpha^+ \rightarrow_0 A$

TABLE 6.1: Translating box- and diamond-formulas of PDL into multi-type terms.

the following connectives cannot be assigned a natural interpretation:

$$\blackleftarrow_0, \blackleftarrow_0 : \text{Fm} \times \text{Fm} \rightarrow \text{TAct}. \quad (6.8)$$

We adopt the following notational convention about the three different shapes of arrows introduced so far. Arrows with straight tails (\rightarrow and \blackrightarrow) stand for connectives which have a semantic counterpart and which are included in the language of the Dynamic Calculus (see the grammar of operational terms on page 182); arrows with no tail (e.g. \blackleftarrow and \blackleftarrow) do have a semantic interpretation but are *not* included in the language at the operational level, and arrows with squiggly tails (\blackleftarrow and \blackleftarrow) stand for syntactic objects, called *virtual adjoints*, which do not have a semantic interpretation, but will play an important role, namely guaranteeing the dynamic calculus to enjoy the relativized display property (cf. Definition 3.3).

In what follows, virtual adjoints will be introduced *only* as structural connectives. That is, they will not correspond to any operational connective, and they will not appear actively in any rule schema other than the display postulates (cf. Definition 2.1).

The $\Delta \dashv \blackrightarrow$, \blackleftarrow and $\blacktriangle \dashv \blackrightarrow$, \blackleftarrow adjunction relations stipulated above translate into the following clauses for every action α , every transitive action δ , and all formulas A and B :

$$\delta \Delta_0 A \leq B \text{ iff } A \leq \delta \blackrightarrow_0 B \quad \delta \blacktriangle_0 A \leq B \text{ iff } A \leq \delta \rightarrow_0 B \quad (6.9)$$

$$\alpha \Delta_1 A \leq B \text{ iff } A \leq \alpha \blackrightarrow_1 B \quad \alpha \blacktriangle_1 A \leq B \text{ iff } A \leq \alpha \rightarrow_1 B \quad (6.10)$$

$$\alpha \Delta_1 A \leq B \text{ iff } \alpha \leq B \blackleftarrow_1 A \quad \alpha \blacktriangle_1 A \leq B \text{ iff } \alpha \leq B \blackleftarrow_1 A. \quad (6.11)$$

Translating PDL into the multitype language, part 1. The intended link between the dynamic connectives of PDL and the multi-type language informally outlined above is illustrated in Table 6.1. This table will be extended to account for the disambiguation of the action-only connectives (see further on). This yields the definition of a formal translation between the language of PDL (possibly extended with adjoints) and that of the Dynamic Calculus, simply by preserving the non modal propositional fragment. We omit the details of this straightforward inductive definition. In Section 6.4, this translation will be elaborated on, and the interpretation of the language of the Dynamic

Calculus will be defined so that the translation above preserves the validity of sequents. In the light of this translation, the adjunction conditions in clauses (6.9) and (6.10) correspond to the following adjunction conditions:

$$\langle \alpha^+ \rangle \dashv \overline{\alpha^+} \quad \widehat{\alpha^+} \dashv [\alpha^+] \quad \langle \alpha \rangle \dashv \overline{\alpha} \quad \widehat{\alpha} \dashv [\alpha].$$

Transitive closure as left adjoint. The other key idea of the design of this calculus is to shape the proof-theoretic behaviour of the iteration connective in PDL on the order-theoretic behaviour of the transitive closure. Namely, it is well known (cf. [DP02, 7.28]) that the map associating each binary relation on a given set W with its transitive closure can be characterized order-theoretically as the left adjoint of the inclusion map $\iota : \mathcal{T}(W \times W) \hookrightarrow \mathcal{P}(W \times W)$. Indeed, for every $R \in \mathcal{P}(W \times W)$ and every $T \in \mathcal{T}(W \times W)$,

$$R^+ \subseteq T \text{ iff } R \subseteq \iota(T).$$

This motivates the introduction of *two* different types of actions: they are needed in order to properly express this adjunction. Thus, we consider the following pair of adjoint maps:

$$(\cdot)^+ : \text{Act} \rightarrow \text{TAct} \quad (6.12)$$

$$(\cdot)^- : \text{TAct} \rightarrow \text{Act}. \quad (6.13)$$

The $(\cdot)^+ \dashv (\cdot)^-$ adjunction relation stipulated above translates into the following clause for every action α , and every transitive action δ :

$$\alpha^+ \leq \delta \text{ iff } \alpha \leq \delta^-. \quad (6.14)$$

Type-disambiguation of action-parameters. We aim at designing a multi-type calculus which verifies condition C'_2 about type-alikeness of parameters. The stipulation that TAct and Act are disjoint is motivated by this goal, but this alone is not enough. We also need to introduce several copies of sequential composition and non deterministic choice, as follows:

$$\cup_{1, ;_1} : \text{Act} \times \text{Act} \rightarrow \text{Act} \quad (6.15)$$

$$\cup_{2, ;_2} : \text{TAct} \times \text{Act} \rightarrow \text{Act} \quad (6.16)$$

$$\cup_{3, ;_3} : \text{Act} \times \text{TAct} \rightarrow \text{Act} \quad (6.17)$$

$$\cup_{4, ;_4} : \text{TAct} \times \text{TAct} \rightarrow \text{Act}. \quad (6.18)$$

Adjoints for action-connectives. When actions are interpreted e.g. in $\mathcal{P}(W \times W)$ for some set W , the natural interpretation of $;$ ₁ (resp. of \cup ₁) is completely join-preserving (resp. meet-preserving) in each coordinate. This implies that their right (resp. left) adjoints exist in each coordinate, hence the following connectives have a natural interpretation:

$$\supset\text{-}_1, \text{-}\subset\text{-}_1 : \text{Act} \times \text{Act} \rightarrow \text{Act} \quad (6.19)$$

$$\rightarrow\text{-}_1, \leftarrow\text{-}_1 : \text{Act} \times \text{Act} \rightarrow \text{Act}. \quad (6.20)$$

Here below, the conditions relative to these adjunctions: for all $\alpha, \beta, \gamma \in \text{Act}$,

$$\alpha \leq \beta \cup\text{-}_1 \gamma \text{ iff } \beta \supset\text{-}_1 \alpha \leq \gamma \text{ iff } \alpha \text{-}\subset\text{-}_1 \gamma \leq \beta \quad (6.21)$$

$$\alpha ;\text{-}_1 \beta \leq \gamma \text{ iff } \beta \leq \alpha \rightarrow\text{-}_1 \gamma \text{ iff } \alpha \leq \gamma \leftarrow\text{-}_1 \beta. \quad (6.22)$$

Also, residuated operations exist for the j -indexed variants, $j \in \{2, 3\}$, of $;$ in their Act -coordinate, and for all j -indexed variants of \cup in both coordinates. These operations provide a natural interpretation for the following connectives:

$$\supset\text{-}_2, \rightarrow\text{-}_2 : \text{TAct} \times \text{Act} \rightarrow \text{Act} \quad (6.23)$$

$$\text{-}\subset\text{-}_3, \leftarrow\text{-}_3 : \text{Act} \times \text{TAct} \rightarrow \text{Act} \quad (6.24)$$

$$\text{-}\subset\text{-}_2, \supset\text{-}_3 : \text{Act} \times \text{Act} \rightarrow \text{TAct} \quad (6.25)$$

$$\supset\text{-}_4, : \text{TAct} \times \text{Act} \rightarrow \text{TAct} \quad (6.26)$$

$$\text{-}\subset\text{-}_4 : \text{Act} \times \text{TAct} \rightarrow \text{TAct}. \quad (6.27)$$

The adjunction clauses relative to the connectives above are analogous to those displayed in (6.21)–(6.22) relative to the connectives of their same shape.

Virtual adjoints, part 2. Since the j -indexed (resp. 0-indexed) variants of $;$ (resp. of Δ and \blacktriangle) are to be regarded as restrictions of their 1-indexed counterpart, they cannot be assumed to be completely join-preserving in their TAct -coordinates. This point is somewhat delicate, so it is worth being expanded on. In the standard semantic setting, the domains of interpretation of Act - and Tact -terms are the algebras $\mathcal{P}(W \times W)$ and $\mathcal{T}(W \times W)$, the domains of which are respectively given by all the binary relations and all the transitive relations on a given set W . As mentioned early on, $\mathcal{T}(W \times W)$ is a sub \cap -semilattice of $\mathcal{P}(W \times W)$, and hence it is itself a complete lattice. However, for every $\mathcal{X} \subseteq \mathcal{T}(W \times W)$, we have that $\bigvee \mathcal{X}$ in $\mathcal{T}(W \times W)$ coincides with the transitive closure $(\bigcup \mathcal{X})^+$ of $\bigcup \mathcal{X}$. This means in particular that, while meets in $\mathcal{T}(W \times W)$ coincide with

$$\begin{array}{lcl}
\alpha ; \beta & \rightsquigarrow & \alpha ;_1 \beta \\
\alpha^+ ; \beta & \rightsquigarrow & \alpha^+ ;_2 \beta \\
\alpha ; \beta^+ & \rightsquigarrow & \alpha ;_3 \beta^+ \\
\alpha^+ ; \beta^+ & \rightsquigarrow & \alpha^+ ;_4 \beta^+
\end{array}
\left\| \begin{array}{lcl}
\alpha \cup \beta & \rightsquigarrow & \alpha \cup_1 \beta \\
\alpha^+ \cup \beta & \rightsquigarrow & \alpha^+ \cup_2 \beta \\
\alpha \cup \beta^+ & \rightsquigarrow & \alpha \cup_3 \beta^+ \\
\alpha^+ \cup \beta^+ & \rightsquigarrow & \alpha^+ \cup_4 \beta^+
\end{array} \right.$$

TABLE 6.2: Type-disambiguation of action-parameters in PDL.

meets in $\mathcal{P}(W \times W)$, joins in $\mathcal{T}(W \times W)$ are in general different from joins in $\mathcal{P}(W \times W)$, or equivalently, $\mathcal{T}(W \times W)$ is not a sub \cup -semilattice of $\mathcal{P}(W \times W)$. This implies that if the j -indexed (resp. 0-indexed) variants of $;$ (resp. of Δ and \blacktriangle) are to be regarded as restrictions of their 1-indexed counterpart, they will preserve the joins of $\mathcal{P}(W \times W)$ but not necessarily those of $\mathcal{T}(W \times W)$. This explains why the j -indexed variants of $;$ for $j \neq 1$ (resp. Δ_0 and \blacktriangle_0) cannot be assumed to be completely join-preserving in their TAct-coordinates. This implies that they do not have right adjoints in their TAct-coordinates.¹ Hence, the following connectives, which are also referred to as *virtual adjoints*, are not semantically justified:

$$\leftarrow_2, \rightarrow_3 : \text{Act} \times \text{Act} \rightarrow \text{TAct} \quad (6.28)$$

$$\rightarrow_4, : \text{TAct} \times \text{Act} \rightarrow \text{TAct} \quad (6.29)$$

$$\leftarrow_4 : \text{Act} \times \text{TAct} \rightarrow \text{TAct}. \quad (6.30)$$

$$\triangleleft_0, \blacktriangleleft_0 : \text{Fm} \times \text{Fm} \rightarrow \text{TAct}. \quad (6.31)$$

Again, as discussed in Section 3.3, virtual adjoints are important to guarantee the dynamic calculus for PDL to enjoy the relativized display property, which in turn guarantees the calculus to verify the condition C'_8 , crucial to the Belnap-style cut elimination metatheorem. However, to ensure that the virtual adjoints do not add unwanted proof power to the calculus, they will be added to the language only at the structural level, and will occur only in the display rules.

Translating PDL into the multitype language, part 2. To have a complete account of how PDL formulas are to be translated into formulas of the multi-type language, Table 6.2 integrates Table 6.1.

¹Precisely because meets in $\mathcal{T}(W \times W)$ coincide with meets in $\mathcal{P}(W \times W)$, (the interpretations of) all the j -indexed variants of \cup for $2 \leq j \leq 4$ are completely meet-preserving in each coordinate, and hence they do have adjoints in each coordinate. A case *sui generis* is the one of the connective \triangleleft_0 , which denotes the right adjoint of the test operator $?_0$ regarded as a map into transitive actions. Notice that, whenever \mathcal{X} is a collection of subsets of the diagonal relation $1_W = \{(z, z) \mid z \in W\}$, the join of \mathcal{X} in $\mathcal{T}(W \times W)$ does coincide with $\bigcup \mathcal{X}$. Hence, (the interpretation of) $?_0$ is completely join preserving, which implies that \triangleleft_0 is semantically justified.

The different copies of connectives introduced above are needed for the calculus to satisfy condition C'_2 about the type-alikeness of parameters. However, in concrete derivations, as soon as the type of the atomic constituents is clearly identifiable, the subscripts can be dropped. The disambiguation will also involve the action-type constants, which will be introduced only as structural connectives, but not as operational ones. Specifically, the structural constants Φ_0 and Φ_1 (when occurring in antecedent position) both correspond to the action skip, regarded as a transitive action or as a general action, respectively. Likewise, the structural constants \mathbb{T}_0 and \mathbb{T}_1 , when occurring in antecedent (resp. succedent position) both correspond to the action top (resp. crash) regarded as a transitive action, and as a general action, respectively.

Axiomatizing PDL in the multitype language. Given the translation based on Tables 6.1 and 6.2, the original axioms of PDL can be translated as indicated below. In what follows, the variables a, b denote terms of type Act or TAct, the variables A, B denote terms of type Fm and α is a term of type Act. For every $1 \leq j \leq 4$ and $i = 0, 1$,

Box-axioms

Choice	$(a \cup_j b) \rightarrow A$	$\dashv\vdash$	$(a \rightarrow A) \wedge (b \rightarrow A)$
Composition	$(a ; j b) \rightarrow A$	$\dashv\vdash$	$a \rightarrow (b \rightarrow A)$
Test	$A?_i \rightarrow B$	$\dashv\vdash$	$A \rightarrow B$
Distributivity	$a \rightarrow (A \wedge B)$	$\dashv\vdash$	$(a \rightarrow A) \wedge (a \rightarrow B)$
Fix point +	$\alpha^+ \rightarrow A$	$\dashv\vdash$	$(\alpha \rightarrow A) \wedge (\alpha \rightarrow (\alpha^+ \rightarrow A))$
Induction +	$\alpha^+ \rightarrow A$	\dashv	$(\alpha \rightarrow A) \wedge (\alpha^+ \rightarrow (A \rightarrow (\alpha \rightarrow A)))$

Diamond-axioms

Choice	$(a \cup_j b) \Delta A$	$\dashv\vdash$	$(a \Delta A) \vee (b \Delta A)$
Composition	$(a ; j b) \Delta A$	$\dashv\vdash$	$a \Delta (b \Delta A)$
Test	$A?_i \Delta B$	$\dashv\vdash$	$A \wedge B$
Distributivity	$a \Delta (A \vee B)$	$\dashv\vdash$	$(a \Delta A) \vee (a \Delta B)$
Fix point +	$\alpha^+ \Delta A$	$\dashv\vdash$	$(\alpha \Delta A) \vee \alpha \Delta \alpha^+ \Delta A$
Induction +	$\alpha^+ \Delta A$	\vdash	$(\alpha \Delta A) \vee (\alpha^+ \Delta (\neg A \wedge (\alpha \Delta A)))$

Note that the subscripts of the arrow- and triangle-shaped connectives are completely determined by the type of their arguments in the first coordinate, and hence they have been omitted.

Additional conditions. As done and discussed in the setting of Chapter 3, in order to express in the multi-type language that e.g. $\langle \alpha \rangle$ and $[\alpha]$ are “interpreted over the same

relation”, Sahlqvist correspondence theory (cf. e.g. [CP12, CPS, CGP14] for a state-of-the-art-treatment) provides us with two alternatives: one of them is that we impose the following Fischer Servi-type conditions [FS84] to hold for all a of type Act or TAct and $A, B \in \text{Fm}$: for $i = 0, 1$,

$$\begin{aligned} (a \Delta_i A) \rightarrow (a \rightarrow_i B) &\leq \delta \rightarrow_i (A \rightarrow B) & (a \blacktriangleright_i A) \rightarrow (a \rightarrow_i B) &\leq a \rightarrow_i (A \rightarrow B) \\ a \Delta_i (A \succ B) &\leq (a \rightarrow_i A) \succ (a \Delta_0 B) & a \blacktriangleright_i (A \succ B) &\leq (a \rightarrow_i A) \succ (a \blacktriangleright_i B). \end{aligned}$$

To see that the conditions above correspond to the usual Fischer Servi axioms in standard modal languages, one can observe that the conditions in the first and third line above are images, under the translation discussed above, of the Fischer Servi axioms reported on e.g. in Section 2.6.1. The second alternative is to impose that, for every $0 \leq i \leq 2$, the connectives Δ_i and \blacktriangleright_i yield *conjugated* diamonds (cf. discussion in Section 2.6.2); that is, the following inequalities hold for all a of type Act or TAct and $A, B \in \text{Fm}$:

$$\begin{aligned} (a \Delta_i A) \wedge B &\leq a \Delta_i (A \wedge (a \blacktriangleright_i B)) & (a \blacktriangleright_i A) \wedge B &\leq a \blacktriangleright_i (A \wedge (a \Delta_0 B)) \\ a \rightarrow_i (A \vee (a \rightarrow_i B)) &\leq (a \rightarrow_i A) \vee B & a \rightarrow_i (A \vee (a \rightarrow_i B)) &\leq (a \rightarrow_i A) \vee B. \end{aligned}$$

The operational language, formally. Let us introduce the operational terms of the multi-type language by the following simultaneous induction, based on sets AtProp of atomic propositions, and AtAct of atomic actions:

$$\begin{aligned} \text{Fm} \ni A &::= p \in \text{AtProp} \mid \perp \mid \top \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \succ A \mid \\ &\quad \delta \Delta_0 A \mid \delta \rightarrow_0 A \mid \alpha \Delta_1 A \mid \alpha \rightarrow_1 A \mid \\ &\quad \delta \blacktriangleright_0 A \mid \delta \rightarrow_0 A \mid \alpha \blacktriangleright_1 A \mid \alpha \rightarrow_1 A \\ \text{Act} \ni \alpha &::= \pi \in \text{AtAct} \mid \delta^- \mid A?_1 \mid \\ &\quad \alpha ;_1 \alpha \mid \delta ;_2 \alpha \mid \alpha ;_3 \delta \mid \delta ;_4 \delta \mid \\ &\quad \alpha \cup_1 \alpha \mid \delta \cup_2 \alpha \mid \alpha \cup_3 \delta \mid \delta \cup_4 \delta \\ \text{TAct} \ni \delta &::= \alpha^+ \mid A?_0 \end{aligned}$$

Structural language, formally. As discussed in the preliminaries, display calculi manipulate two closely related languages: the operational and the structural. Let us introduce the structural language of the Dynamic Calculus, which as usual matches the operational language. We have *formula-type* structures, *transitive action-type structures*, *action-type*

structures, defined by simultaneous recursion as follows:

$$\begin{aligned} \text{FM} \ni X ::= & A \mid I \mid X, X \mid X > X \mid X < X \mid \Pi \dot{\mathbf{I}}_1 \mid \Delta \dot{\mathbf{I}}_0 \mid \\ & \Delta \Delta_0 X \mid \Delta \triangleright_0 X \mid \Pi \Delta_1 X \mid \Pi \triangleright_1 X \mid \\ & \Delta \blacktriangle_0 X \mid \Delta \blacktriangleright_0 X \mid \Pi \blacktriangle_1 X \mid \Pi \blacktriangleright_1 X \end{aligned}$$

$$\begin{aligned} \text{ACT} \ni \Pi ::= & \alpha \mid \mathbb{T}_1 \mid \Phi_1 \mid \Delta^\ominus \mid X?_1 \mid \\ & \Pi ;_1 \Pi \mid \Pi >_1 \Pi \mid \Pi <_1 \Pi \mid \Delta ;_2 \Pi \mid \Delta >_2 \Pi \mid \Pi ;_3 \Delta \mid \Pi <_3 \Delta \mid \Delta ;_4 \Delta \mid \\ & \Pi \emptyset_1 \Pi \mid \Pi \supset_1 \Pi \mid \Pi \subset_1 \Pi \mid \Delta \emptyset_2 \Pi \mid \Delta \supset_2 \Pi \mid \Pi \emptyset_3 \Delta \mid \Pi \subset_3 \Delta \mid \Delta \emptyset_4 \Delta \mid \\ & X \triangleleft_1 X \mid X \blacktriangleleft_1 X \end{aligned}$$

$$\begin{aligned} \text{TACT} \ni \Delta ::= & \mathbb{T}_0 \mid \Phi_0 \mid \Pi^\oplus \mid X?_0 \mid \\ & \Pi \leftarrow_2 \Pi \mid \Pi \rightsquigarrow_3 \Pi \mid \Delta \rightsquigarrow_4 \Pi \mid \Pi \leftarrow_4 \Delta \mid \\ & \Pi \subset_2 \Pi \mid \Pi \supset_3 \Pi \mid \Delta \supset_4 \Pi \mid \Pi \subset_4 \Delta \mid \\ & X \triangleleft\sim_0 X \mid X \blacktriangleleft\sim_0 X \end{aligned}$$

The propositional base. As is typical of display calculi, each operational connective corresponds to one structural connective. In particular, the propositional base connectives behave exactly as in Chapters 2 and 3, and their corresponding rules are reported in Appendix I.²

Structural symbols	I	,	<	>				
Operational symbols	\top	\perp	\wedge	\vee	(\leftarrow)	(\leftarrow)	(\rightarrow)	\rightarrow

Action connectives, part 1. As to the 0-ary and binary action-type connectives the table below provides the connection between structural and operational connectives for $1 \leq j \leq 4$, $h = 1, 2$, and $k = 1, 3$. The indexes of the structural connectives are omitted.³

²The operational connectives \top , \perp , \wedge , \vee , \rightarrow belong to the language of the most common axiomatizations of propositional classical logic. The operational connectives in brackets \leftarrow , \leftarrow , \rightarrow are mentioned in the table for the sake of exhaustiveness. In particular, \leftarrow and \rightarrow (resp. \leftarrow and \rightarrow) are interderivable in the presence of the rule *exchange*, and the same is true of the dual connectives \leftarrow and \rightarrow . The latter two connectives are known as *subtraction* or *disimplication*. The formula $A \rightarrow B$ (resp. $A \leftarrow B$) is classically equivalent to $\neg A \wedge B$ (resp. $A \wedge \neg B$).

³The operational connectives in brackets are given for the sake of completeness, but they do not belong to the language of the most common axiomatizations of PDL considered here. See [Har13a, Har13b, Pra91] for some extensions of the language and their interpretations.

Structural symbols	\mathbb{I}	\emptyset	\supset	\subset	
Operational symbols	(τ)	(\perp)	\cup	$(\supset -_j)$	$(\subset -_j)$

Structural symbols	Φ	$;$	$>$	$<$
Operational symbols	(1)	$;$	(\rightarrow_h)	(\leftarrow_k)

Heterogeneous connectives. Similarly to Chapter 3, the heterogeneous structural connectives correspond one-to-one with the operational ones, as illustrated in the following table: for $i = 0, 1$,

Structural symbols	\triangle_i	\blacktriangleright_i	\blacktriangleleft_i
Operational symbols	Δ_i	\rightarrow_i	(\triangleleft_i)

Structural symbols	\blacktriangle_i	\triangleright_i	\triangleleft_i
Operational symbols	\blacktriangleright_i	\rightarrow_i	(\triangleleft_i)

That is, the structural connectives are to be interpreted in a context-sensitive way, but the present language lacks the operational connectives which would correspond to them on one or both of the two sides. This is of course because in the present setting we do not need them. However, in a setting in which they would turn out to be needed, it would not be difficult to introduce the missing operational connectives.⁴ The operational rules for the heterogeneous connectives are essentially the same as the analogous rules given in Section 3.3: in what follows, let x, y and a respectively stand for structural and operational terms of a type which can be either TAct or Act, and let Y, Z and B respectively stand for structural and operational terms of type Fm; then, for $i = 0, 1$,

Actions-Propositions Operational Rules

$$\begin{array}{l} \Delta_{iL} \frac{a \triangle_i B \vdash Z}{a \Delta_i B \vdash Z} \quad \frac{x \vdash a \quad Y \vdash B}{x \triangle_i Y \vdash a \Delta_i B} \Delta_{iR} \\ \blacktriangle_{iL} \frac{a \blacktriangle_i B \vdash Z}{a \blacktriangleright_i B \vdash Z} \quad \frac{x \vdash a \quad Y \vdash B}{x \blacktriangle_i Y \vdash a \blacktriangleright_i B} \blacktriangle_{iR} \\ \rightarrow_{iL} \frac{x \vdash a \quad B \vdash Y}{a \rightarrow_i B \vdash x \triangleright_i Y} \quad \frac{Z \vdash a \triangleright_i B}{Z \vdash a \rightarrow_i B} \rightarrow_{iR} \\ \rightarrow_{iL} \frac{x \vdash a \quad B \vdash Y}{a \rightarrow_i B \vdash x \blacktriangleright_i Y} \quad \frac{Z \vdash a \blacktriangleright_i B}{Z \vdash a \rightarrow_i B} \rightarrow_{iR} \end{array}$$

⁴See [Har13a, Har13b, Pra91] for some extensions of the language and their interpretations.

Clearly, the rules above yield the operational rules for the dynamic modal operators under the translation given early on. Notice that each sequent is always interpreted in one domain; however, since the connectives take arguments of different types (and in this sense we are justified in referring to them as *heterogeneous connectives*), premises of binary rules are of course interpreted in different domains.

Identity and cut rules. Axioms will be given in each type; here below, $\pi \in \text{AtAct}$, and $p \in \text{AtProp}$:

Identity Rules

$$\pi \text{ Id } \frac{}{\pi \vdash \pi} \quad p \text{ Id } \frac{}{p \vdash p}$$

where the first axiom is of type Act, and the second one is of type Fm.

Further, we allow the following *strongly type-uniform* cut rules on the operational terms:

Cut Rules

$$\frac{\Gamma \vdash \delta \quad \delta \vdash \Delta}{\Gamma \vdash \Delta} \delta \text{ Cut} \quad \frac{\Pi \vdash \alpha \quad \alpha \vdash \Sigma}{\Pi \vdash \Sigma} \alpha \text{ Cut} \quad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y} A \text{ Cut}$$

Display postulates for heterogeneous connectives. Recall that x is a structural variable of type TAct or Act, and Y and Z are structural variables of type Fm; for $i = 0, 1$,

Actions-Propositions Display Postulates

$$\begin{array}{cc} \Delta_i \triangleright_i \frac{x \triangleleft_i Y \vdash Z}{Y \vdash x \blacktriangleright_i Z} & \frac{x \blacktriangleleft_i Y \vdash Z}{Y \vdash x \blacktriangleright_i Z} \blacktriangleleft_i \triangleright_i \\ \Delta_1 \blacktriangleleft_1 \frac{\pi \triangleleft_1 Y \vdash Z}{\pi \vdash Z \blacktriangleleft_1 Y} & \frac{\pi \blacktriangleleft_1 Y \vdash Z}{\pi \vdash Z \blacktriangleleft_1 Y} \blacktriangleleft_1 \blacktriangleleft_1 \\ \Delta_0 \blacktriangleleft_0 \frac{\delta \triangleleft_0 Y \vdash Z}{\delta \vdash Z \blacktriangleleft_0 Y} & \frac{\delta \blacktriangleleft_0 Y \vdash Z}{\delta \vdash Z \blacktriangleleft_0 Y} \blacktriangleleft_0 \blacktriangleleft_0 \end{array}$$

Notice that sequents occurring in each display postulate above are *not* of the same type. However, it is easy to see that the display postulates preserve the type-uniformity (cf. Definition 3.1); that is, if the premise of any instance of a display postulate is a type-uniform sequent, then so is its conclusion.

Necessitation, Conjugation, Fischer Servi, and Monotonicity rules. For $i = 0, 1$,

Necessitation Rules

$$nec_i \Delta \frac{I \vdash W}{x \triangle_i I \vdash W} \quad \frac{I \vdash W}{x \blacktriangle_i I \vdash W} nec_i \blacktriangle$$

The following rules are derivable from the ones above using the display postulates:

$$nec_i \triangleright \frac{W \vdash I}{W \vdash x \triangleright_i I} \quad \frac{W \vdash I}{W \vdash x \blacktriangleright_i I} nec_i \blacktriangleright$$

Conjugation Rules

$$(conj_i \Delta) \frac{x \triangle_i ((x \blacktriangle_i Y); Z) \vdash W}{Y; (x \triangle_i Z) \vdash W} \quad \frac{W \vdash x \triangleright_i ((x \blacktriangleright_i Y); Z)}{W \vdash Y; (x \triangleright_i Z)} (conj_i \rightarrow)$$

$$(conj_i \blacktriangle) \frac{x \blacktriangle_i ((x \triangle_i Y); Z) \vdash W}{Y; (x \blacktriangle_i Z) \vdash W} \quad \frac{W \vdash x \blacktriangleright_i ((x \triangleright_i Y); Z)}{W \vdash Y; (x \blacktriangleright_i Z)} (conj_i \rightarrow)$$

The rules above are interderivable with the following Fischer-Servi rules using the appropriate display postulates:

$$FS_i \Delta \frac{(x \triangleright_i Y) > (x \triangle_i Z) \vdash W}{x \triangle_i (Y > Z) \vdash W} \quad \frac{W \vdash (x \triangle_i Y) > (x \triangleright_i Z)}{W \vdash x \triangleright_i (Y > Z)} FS_i \triangleright$$

$$FS_i \blacktriangle \frac{(x \blacktriangleright_i Y) > (x \blacktriangle_i Z) \vdash W}{x \blacktriangle_i (Y > Z) \vdash W} \quad \frac{W \vdash (x \blacktriangle_i Y) > (x \blacktriangleright_i Z)}{W \vdash x \blacktriangleright_i (Y > Z)} FS_i \blacktriangleright$$

The following rules encode the fact that both arrow- and triangle-shaped heterogeneous connectives are order preserving in their second coordinate.

Monotonicity Rules

$$mon_i \Delta \frac{(x \triangle_i Y), (x \triangle_i Z) \vdash W}{x \triangle_i (Y, Z) \vdash W} \quad \frac{W \vdash (x \triangleright_i Y), (x \triangleright_i Z)}{W \vdash x \triangleright_i (Y, Z)} mon_i \triangleright$$

$$mon_i \blacktriangle \frac{(x \blacktriangle_i Y), (x \blacktriangle_i Z) \vdash W}{x \blacktriangle_i (Y, Z) \vdash W} \quad \frac{W \vdash (x \blacktriangleright_i Y), (x \blacktriangleright_i Z)}{W \vdash x \blacktriangleright_i (Y, Z)} mon_i \blacktriangleright$$

Action rules. The following rules encode conditions (6.3) and (6.4). For $1 \leq j \leq 4$, the subscripts for $\triangle, \blacktriangle, \triangleright, \blacktriangleright$ are omitted since they are uniquely determined by j , and x, y are structural variables of the suitable action- or transitive action-type.

Actions Rules

$$\text{act}_{j\triangle} \frac{x \triangle (y \triangle Z) \vdash W}{(x ;_j y) \triangle Z \vdash W} \quad \text{act}_{j\blacktriangle} \frac{x \blacktriangle (y \blacktriangle Z) \vdash W}{(y ;_j x) \blacktriangle Z \vdash W}$$

The following rules are derivable from the ones above using the display postulates:

$$\frac{W \vdash x \triangleright (y \triangleright Z)}{W \vdash (x ;_j y) \triangleright Z} \text{act}_{j\triangleright} \quad \frac{W \vdash x \blacktriangleright (y \blacktriangleright Z)}{W \vdash (y ;_j x) \blacktriangleright Z} \text{act}_{j\blacktriangleright}$$

Rules for test and iteration. Also the *unary* heterogeneous structural connectives correspond one-to-one with the operational ones, as illustrated in the following table (the indices are omitted):

Structural symbols	?	$(\cdot)^\oplus$	$(\cdot)^\ominus$
Operational symbols	?	$(\cdot)^+$	$(\cdot)^-$

The operational rules for these connectives are given in the table below, where $i = 0, 1$, and x is a structural variable of suitable action-type or transitive action-type, uniquely determined so as to satisfy type-regularity.

Test and Iteration Operational Rules

$$?_L^i \frac{A?_i \vdash x}{A?_i \vdash x} \quad \frac{X \vdash A}{X?_i \vdash A?_i} ?_R^i \quad +_L \frac{\alpha^\oplus \vdash \Delta}{\alpha^+ \vdash \Delta} \quad \frac{\Psi \vdash \alpha}{\Psi^\oplus \vdash \alpha^+} +_R \quad -_L \frac{\delta \vdash \Delta}{\delta^- \vdash \Delta^\ominus} \quad \frac{\Psi \vdash \delta^\ominus}{\Psi \vdash \delta^-} -_R$$

Test and Iteration Display Postulates

$$?_{\mathfrak{L}}^i \frac{X?_i \vdash x}{X \vdash x \mathfrak{L}_i} \quad \oplus \ominus \frac{\Pi^\oplus \vdash \Delta}{\Pi \vdash \Delta^\ominus}$$

Test Structural Rules

$$?_{\Delta i} \frac{X, Y \vdash Z}{X?_i \triangle_i Y \vdash Z} \quad \frac{X, Y \vdash Z}{Y?_i \blacktriangle_i X \vdash Z} ?_{\blacktriangle i}$$

The following rules are display equivalent to the ones above.

$$?_{\triangleright i} \frac{Y \vdash X > Z}{Y \vdash X?_i \triangleright_i Z} \quad \frac{Y \vdash X > Z}{Y \vdash X?_i \blacktriangleright_i Z} ?_{\blacktriangleright i}$$

Absorption and promotion/demotion rules.

Absorption Rules

$$\begin{array}{l}
 \text{abs } 1 \frac{\Pi \vdash \Delta^\ominus \quad \Sigma \vdash \Delta^\ominus}{\Pi ;_1 \Sigma \vdash \Delta^\ominus} \quad \frac{\Delta \vdash \Gamma \quad \Xi \vdash \Gamma}{\Delta ;_4 \Xi \vdash \Gamma} \text{abs } 4 \\
 \text{abs } 2 \frac{\Gamma \vdash \Delta \quad \Sigma \vdash \Delta^\ominus}{\Gamma ;_2 \Sigma \vdash \Delta^\ominus} \quad \frac{\Sigma \vdash \Delta^\ominus \quad \Gamma \vdash \Delta}{\Sigma ;_3 \Gamma \vdash \Delta} \text{abs } 3
 \end{array}$$

Promotion/Demotion Rules

$$\begin{array}{l}
 \text{pro/dem } 2;1 \frac{\Gamma ;_2 \Sigma \vdash \Pi}{\Gamma^\ominus ;_1 \Sigma \vdash \Pi} \quad \frac{\Pi \vdash \Gamma \quad \emptyset_2 \Sigma}{\Pi \vdash \Gamma^\ominus \quad \emptyset_1 \Sigma} \text{pro/dem } 2\emptyset 1 \\
 \text{pro/dem } 3;1 \frac{\Sigma ;_3 \Gamma \vdash \Pi}{\Sigma ;_1 \Gamma^\ominus \vdash \Pi} \quad \frac{\Pi \vdash \Sigma \quad \emptyset_3 \Gamma}{\Pi \vdash \Sigma \quad \emptyset_1 \Gamma^\ominus} \text{pro/dem } 3\emptyset 1 \\
 \text{pro/dem } 4;2 \frac{\Delta ;_4 \Gamma \vdash \Pi}{\Delta ;_2 \Gamma^\ominus \vdash \Pi} \quad \frac{\Pi \vdash \Delta \quad \emptyset_4 \Gamma}{\Pi \vdash \Delta \quad \emptyset_2 \Gamma^\ominus} \text{pro/dem } 4\emptyset 2 \\
 \text{pro/dem } 4;3 \frac{\Delta ;_4 \Gamma \vdash \Pi}{\Delta^\ominus ;_3 \Gamma \vdash \Pi} \quad \frac{\Pi \vdash \Delta \quad \emptyset_4 \Gamma}{\Pi \vdash \Delta^\ominus \quad \emptyset_3 \Gamma} \text{pro/dem } 4\emptyset 3 \\
 \text{pro/dem } ? \frac{X?_0 \vdash \Delta}{X?_1 \vdash \Delta^\ominus}
 \end{array}$$

$$\text{dem } \triangle \frac{\Pi^\oplus \triangle_0 X \vdash Y}{\Pi \triangle_1 X \vdash Y} \quad \text{dem } \blacktriangle \frac{\Pi^\oplus \blacktriangle_0 X \vdash Y}{\Pi \blacktriangle_1 X \vdash Y}$$

Using the rules above and the Display Postulates, the following rules are derivable:

$$\begin{array}{l}
 \text{dem } \triangleright \frac{X \vdash \Pi^\oplus \triangleright_0 Y}{X \vdash \Pi \triangleright_1 Y} \quad \frac{X \vdash \Pi^\oplus \blacktriangleright_0 Y}{X \vdash \Pi \blacktriangleright_1 Y} \text{dem } \blacktriangleright \\
 \text{dem } \triangleleft \frac{\Pi^\oplus \vdash X \triangleleft_0 Y}{\Pi \vdash X \triangleleft_1 Y} \quad \frac{\Pi^\oplus \vdash X \blacktriangleleft_0 Y}{\Pi \vdash X \blacktriangleleft_1 Y} \text{dem } \blacktriangleleft
 \end{array}$$

Fixed point structural rules. The following rules correspond to the fixed point axioms.

Fixed Point Structural Rules

$$\text{FP } \triangle \frac{\Pi \triangle_1 X \vdash Y \quad (\Pi ;_3 \Pi^\oplus) \triangle_1 X \vdash Y}{\Pi^\oplus \triangle_0 X \vdash Y} \quad \frac{\Pi \blacktriangle_1 X \vdash Y \quad (\Pi ;_3 \Pi^\oplus) \blacktriangle_1 X \vdash Y}{\Pi^\oplus \blacktriangle_0 X \vdash Y} \text{FP } \blacktriangle$$

Using the rules above and the Display Postulates, the following rules are derivable:

$$\begin{array}{c}
 FP \triangleright \frac{X \vdash \Pi \triangleright_1 Y \quad X \vdash (\Pi ;_3 \Pi^\oplus) \triangleright_1 Y}{X \vdash \Pi^\oplus \triangleright_0 Y} \quad \frac{X \vdash \Pi \blacktriangleright_1 Y \quad X \vdash (\Pi ;_3 \Pi^\oplus) \blacktriangleright_1 Y}{X \vdash \Pi^\oplus \blacktriangleright_0 Y} FP \blacktriangleright \\
 \\
 FP \triangleleft \frac{\Pi \vdash Y \triangleleft_1 X \quad (\Pi ;_3 \Pi^\oplus) \vdash Y \triangleleft_1 X}{\Pi^\oplus \vdash Y \triangleleft_{\sim 0} X} \quad \frac{\Pi \vdash Y \blacktriangleleft_1 X \quad (\Pi ;_3 \Pi^\oplus) \vdash Y \blacktriangleleft_1 X}{\Pi^\oplus \vdash Y \blacktriangleleft_{\sim 0} X} FP \blacktriangleleft
 \end{array}$$

The infinitary *iteration* rules are given below:

Omega-Iteration Structural Rules

$$\omega \triangle \frac{\left(\Pi^{(n)} \triangleleft_1 X \vdash Y \mid n \geq 1 \right)}{\Pi^\oplus \triangleleft_0 X \vdash Y} \quad \frac{\left(\Pi^{(n)} \blacktriangleleft_1 X \vdash Y \mid n \geq 1 \right)}{\Pi^\oplus \blacktriangleleft_0 X \vdash Y} \omega \blacktriangle$$

Using the rules above and the Display Postulates, the following rules are derivable:

$$\begin{array}{c}
 \omega \triangleright \frac{\left(X \vdash \Pi^{(n)} \triangleright_1 Y \mid n \geq 1 \right)}{X \vdash \Pi^\oplus \triangleright_0 Y} \quad \frac{\left(X \vdash \Pi^{(n)} \blacktriangleright_1 Y \mid n \geq 1 \right)}{X \vdash \Pi^\oplus \blacktriangleright_0 Y} \omega \blacktriangleright \\
 \\
 \omega \triangleleft \frac{\left(\Pi^{(n)} \vdash Y \triangleleft_1 X \mid n \geq 1 \right)}{\Pi^\oplus \vdash Y \triangleleft_{\sim 0} X} \quad \frac{\left(\Pi^{(n)} \vdash Y \blacktriangleleft_1 X \mid n \geq 1 \right)}{\Pi^\oplus \vdash Y \blacktriangleleft_{\sim 0} X} \omega \blacktriangleleft
 \end{array}$$

Rules for action constants. For the following rules, $j = 1, 2$ and $k = 1, 3$. Moreover, x, y, z are structural variables of the suitable action- or transitive action-type. The index on \mathbb{T} is omitted because it is uniquely determined by j and k .

\mathbb{T} -Rules

$$\begin{array}{c}
 \frac{x \vdash y}{x \vdash \mathbb{T} \emptyset_j y} \mathbb{T}_{1R}^j \quad \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \mathbb{T} \emptyset_3 \Gamma} \mathbb{T}_{1R}^3 \quad \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \mathbb{T} \emptyset_4 \Gamma} \mathbb{T}_{1R}^4 \\
 \\
 \frac{x \vdash y}{x \vdash y \emptyset_k \mathbb{T}} \mathbb{T}_{2R}^k \quad \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \Gamma \emptyset_2 \mathbb{T}} \mathbb{T}_{2R}^2 \quad \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \Gamma \emptyset_4 \mathbb{T}} \mathbb{T}_{2R}^4
 \end{array}$$

For the following rules, $j = 1, 2$ and $k = 1, 3$. Moreover, x, y, z are structural variables of the suitable action- or transitive action-type. The index on \mathbb{T} is omitted because it is uniquely determined by j and k .

Φ -Rules

$$\begin{array}{ccc} \Phi_{1L}^j \frac{x \vdash y}{\Phi ;_j x \vdash y} & \Phi_{1L}^3 \frac{\Delta \vdash \Gamma}{\Phi ;_3 \Delta \vdash \Gamma^\ominus} & \Phi_{1L}^4 \frac{\Delta \vdash \Gamma}{\Phi ;_4 \Delta \vdash \Gamma^\ominus} \\ \Phi_{2L}^k \frac{x \vdash y}{x ;_k \Phi \vdash y} & \Phi_{2L}^2 \frac{\Delta \vdash \Gamma}{\Delta ;_2 \Phi \vdash \Gamma^\ominus} & \Phi_{2L}^4 \frac{\Delta \vdash \Gamma}{\Delta ;_4 \Phi \vdash \Gamma^\ominus} \end{array}$$

Structural rules for binary action connectives. For the following rules (cf. [Har13a, Har13b] for Weakening w.r.t. sequential composition), $j = 1, 2$ and $k = 1, 3$. Moreover, x, y, z are structural variables of the suitable action- or transitive action-type.

Weakening Rules for Actions

$$\begin{array}{ccc} \frac{x \vdash y}{x \vdash z \wp_j y} W_{1R}^h & \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \Pi \wp_3 \Gamma} W_{1R}^3 & \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \Gamma' \wp_4 \Gamma} W_{1R}^4 \\ \frac{x \vdash y}{x \vdash y \wp_k z} W_{2R}^l & \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \Gamma \wp_2 \Pi} W_{2R}^2 & \frac{\Delta \vdash \Gamma}{\Delta^\ominus \vdash \Gamma \wp_4 \Gamma'} W_{2R}^4 \end{array}$$

For the following rules, $k = 1, 4$ and x, y, z are structural variables of the suitable action- or transitive action-type required by type-regularity.

Contraction Rule for Actions

$$\frac{y \vdash x \wp_k x}{y \vdash x} C_R^k$$

Additional *contraction* rules can be derived using the *promotion/demotion* rules. For the following rules, $k = 1, 4$. Moreover, x, y, z are structural variables of the suitable action- or transitive action-type required by type-regularity.

Exchange Rules for Actions

$$\frac{\Sigma \vdash \Delta \wp_2 \Pi}{\Sigma \vdash \Pi \wp_3 \Delta} E_R^{2\wp_3} \quad \frac{z \vdash x \wp_k y}{z \vdash y \wp_k x} E_R^{k\wp_k}$$

For the following rules, the indices are omitted, under the convention that they span over all the combinations allowed by the grammar, by type-regularity and by type-alikeness of parameters. The variables x, y, z are of the suitable action- or transitive action-type.

Associativity Rules for Actions

$$A_L \frac{x ;_j (y ;_k z) \vdash w}{(x ;_j y) ;_k z \vdash w} \quad A_R \frac{w \vdash (z \wp_k y) \wp_l x}{w \vdash z \wp_l (y \wp_k x)}$$

Dynamics and non-deterministic choice. In the following *choice* rules, the index on \emptyset is uniquely determined and is omitted. These rules encode the fact that \blacktriangleleft_1 and \triangleleft_1 are monotone in their first coordinate and antitone in their second coordinate.

Structural Rules for Non-Deterministic Choice

$$\begin{array}{c} \text{choice } \blacktriangleleft_1 \frac{\Psi \vdash (Y \blacktriangleleft_1 X) \emptyset (Z \blacktriangleleft_1 X)}{\Psi \vdash (Y, Z) \blacktriangleleft_1 X} \quad \frac{\Psi \vdash (Y \triangleleft_1 X) \emptyset (Z \triangleleft_1 X)}{\Psi \vdash (Y, Z) \triangleleft_1 X} \text{choice } \triangleleft_1 \\ \text{choice } \blacktriangleleft^1 \frac{\Psi \vdash (X \blacktriangleleft_1 Y) \emptyset (X \blacktriangleleft_1 Z)}{\Psi \vdash X \blacktriangleleft_1 (Y, Z)} \quad \frac{\Psi \vdash (X \triangleleft_1 Y) \emptyset (X \triangleleft_1 Z)}{\Psi \vdash X \triangleleft_1 (Y, Z)} \text{choice } \triangleleft^1 \end{array}$$

More rules on non-deterministic choice and sequential composition. For the following rules, $1 \leq k \leq 4$. Moreover, x, y, z are structural variables of the suitable action- or transitive action-type.

Display Postulates for Non-Deterministic Choice and Sequential Composition

$$\frac{z \vdash x \emptyset_k y}{x \supset_k z \vdash y} \quad \frac{z \vdash x \emptyset_k y}{z \subset_k y \vdash x} \quad \frac{x ;_j y \vdash z}{y \vdash x \succ_j z} \quad \frac{x ;_j y \vdash z}{x \vdash z \prec_j y}$$

Finally, the rules for the operational connectives \cup_j and $;$ are given below. For the following rules, $1 \leq j \leq 4$, and the variables x, y and f, g respectively denote structural and operational terms of suitable type uniquely determined by j and by term-uniformity.

Operational Rules for Non-Deterministic Choice and Sequential Composition

$$\cup_L^j \frac{f \vdash x \quad g \vdash y}{f \cup_j g \vdash x \emptyset_j y} \quad \cup_R^j \frac{x \vdash f \emptyset_j g}{x \vdash f \cup_j g} \quad ;_L^j \frac{f ;_j g \vdash x}{f ;_j g \vdash x} \quad ;_R^j \frac{x \vdash f \quad y \vdash g}{x ;_j y \vdash f ;_j g}$$

6.4 Soundness

In the present section, we discuss the soundness of the rules of the dynamic calculus and prove that those which do not involve virtual adjoints (cf. Section 6.3) are sound with respect to the standard relational semantics. As we will see, the interpretation of the multi-type language which we are about to define preserves the translation from the standard PDL language to the multi-type one, which was outlined in Tables 6.1 and 6.2.

A *model* for the multi-type language for PDL is a tuple $N = (W, \nu)$ such that W is a nonempty set, and ν is a variable assignment from $\text{AtProp} \cup \text{AtAct}$ mapping each $p \in$

AtProp to a subset $\llbracket p \rrbracket_V \subseteq W$, and each $\pi \in \text{AtAct}$ to a binary relation $R_\pi \subseteq W \times W$. Clearly, these models bijectively correspond to standard Kripke models for PDL: indeed, for every standard Kripke model $M = (W, \mathcal{R}, V)$ such that $\mathcal{R} = \{R_\pi \mid \pi \in \text{AtAct}\}$, let $N_M := (W, v_M)$, where $v_M(p) = V(p)$ for every $p \in \text{AtProp}$, and $v_M(\pi) = R_\pi$ for every $\pi \in \text{AtAct}$. Conversely, for every $N = (W, v)$ as above, let $M_N := (W, \mathcal{R}_N, V_N)$ such that $\mathcal{R}_N := \{v(\pi) \mid \pi \in \text{AtAct}\}$, and $V_N(p) = v(p)$ for every $p \in \text{AtProp}$. It is immediate to verify that $N_{M_N} = N$ and $M_{N_M} = M$ for every M and N as above. Clearly each model N as above gives rise to algebras $\mathcal{P}(W)$, $\mathcal{P}(W \times W)$ and $\mathcal{T}(W \times W)$, which provide suitable domains of interpretations of terms of type Fm, Act and TAct, respectively.

Structures will be translated into operational terms of the appropriate type, and operational terms will be interpreted according to their type. In order to translate structures as operational terms, structural connectives need to be translated as logical connectives. To this effect, non-modal, propositional structural connectives are associated with pairs of logical connectives, and any given occurrence of a structural connective is translated as one or the other, according to its (antecedent or succedent) position. The following table illustrates how to translate each propositional structural connective of type FM, in the upper row, into one or the other of the logical connectives corresponding to it on the lower row: the one on the left-hand (resp. right-hand) side, if the structural connective occurs in precedent (resp. succedent) position.

Structural symbols	$<$		$>$		$;$		I	
Operational symbols	\prec	\leftarrow	\succ	\rightarrow	\wedge	\vee	\top	\perp

Recall that, in the Boolean setting treated here, the connectives \prec and \succ are interpreted as $A \prec B := A \wedge \neg B$ and $A \succ B := \neg A \wedge B$. The soundness of structural and operational rules which only involve active components of type FM has been discussed in Chapter 2 and is here therefore omitted.

The following table illustrates, with the reading indicated above, how to translate each action-type structural connective. Notice that some of the operational connectives in the table below are not included in the operational language of the dynamic calculus for PDL. However, as discussed in Section 6.3, the operational symbols below are the ones endowed with a semantic justification (so although the indexes are omitted, it is understood that the table below refers to no virtual adjoints). So even if they are not included in the language, they are used the present section to facilitate the semantic interpretation of structures occurring in sequents. Notice also that the structural connectives below have a semantic interpretation only when occurring in precedent (resp. succedent) position. Hence, not every structure is going to be semantically interpretable. However, as we will see, this is enough for checking the soundness of the rules.

Structural symbols	\mathbb{I}	\emptyset	\supset	\subset
Operational symbols	\perp	\cup	$\supset-$	$-\subset$

Structural symbols	Φ	$;$	$>$	$<$
Operational symbols	1	$;$	\rightarrow	\leftarrow

Structural symbols	$?$	$\dot{?}$	$(\cdot)^\oplus$	$(\cdot)^\ominus$
Operational symbols	$?$	$\dot{?}$	$(\cdot)^+$	$(\cdot)^-$

The interpretation of the connectives above corresponds to the standard one discussed in Sections 6.2 and 6.3. Below, a and b are operational terms of type Act or TAct, α , δ and A are operational terms of type Act, TAct and Fm respectively, and the indexes are omitted.

$$\begin{aligned} \llbracket a \cup b \rrbracket_v &= \{(z, z') \in W \times W \mid (z, z') \in \llbracket a \rrbracket_v \text{ or } (z, z') \in \llbracket b \rrbracket_v\} \\ \llbracket a ; b \rrbracket_v &= \{(z, z') \in W \times W \mid \exists w. (z, w) \in \llbracket a \rrbracket_v \ \& \ (w, z') \in \llbracket b \rrbracket_v\} \\ \llbracket \perp \rrbracket_v &= \{(z, z') \in W \times W \mid (z, z') \neq (z, z')\} = \emptyset \\ \llbracket 1 \rrbracket_v &= \{(z, z) \in W \times W \mid z \in W\} \\ \llbracket A? \rrbracket_v &= \{(z, z) \in W \times W \mid z \in \llbracket A \rrbracket_v\} \\ \llbracket a \dot{?} \rrbracket_v &= \{z \in W \mid (z, z) \in \llbracket a \rrbracket_v\} \\ \llbracket \alpha^+ \rrbracket_v &= \bigcup_{n \geq 1} \llbracket \alpha \rrbracket_v^n \\ \llbracket \delta^- \rrbracket_v &= \llbracket \delta \rrbracket_v \\ \llbracket a \rightarrow b \rrbracket_v &= \{(z, z') \in W \times W \mid \forall w. ((w, z) \in \llbracket a \rrbracket_v \Rightarrow (w, z') \in \llbracket b \rrbracket_v)\} \\ \llbracket a \leftarrow b \rrbracket_v &= \{(z, z') \in W \times W \mid \forall w. ((z', w) \in \llbracket b \rrbracket_v \Rightarrow (z, w) \in \llbracket a \rrbracket_v)\} \\ \llbracket a \supset- b \rrbracket_v &= \{(z, z') \in W \times W \mid (z, z') \in \llbracket b \rrbracket_v \ \& \ (z, z') \notin \llbracket a \rrbracket_v\} = \llbracket b -\subset a \rrbracket_v \end{aligned}$$

Given this standard interpretation, the verification of the soundness of the pure-action rules is straightforward, and is omitted.

As to the heterogeneous connectives, their translation into the corresponding operational connectives is indicated in the table below, to be understood similarly to the one above, where the index i ranges over $\{0, 1\}$.

Structural symbols	\triangle_i	\blacktriangle_i	\triangleright_i	\blacktriangleright_i	\triangleleft_1	\blacktriangleleft_1
Operational symbols	Δ_i	\blacktriangle_i	\rightarrow_i	\rightarrow_i	\triangleleft_1	\blacktriangleleft_1

The interpretation of the heterogeneous connectives involving formulas and actions corresponds to that of the well known forward and backward modalities discussed in Section 6.3 (below on the right-hand side we recall the notation in the standard language of PDL with adjoint modalities):

$$\begin{aligned}
\llbracket \alpha \Delta_1 A \rrbracket &= \{z \in W \mid \exists z'. z R_\alpha z' \ \& \ z' \in \llbracket A \rrbracket\} && \langle \alpha \rangle A \\
\llbracket \alpha \blacktriangle_1 A \rrbracket &= \{z \in W \mid \exists z'. z' R_\alpha z \ \& \ z' \in \llbracket A \rrbracket\} && \widehat{\alpha} A \\
\llbracket \alpha \rightarrow_1 A \rrbracket &= \{z \in W \mid \forall z'. z R_\alpha z' \Rightarrow z' \in \llbracket A \rrbracket\} && [\alpha] A \\
\llbracket \alpha \rightarrow_1 A \rrbracket &= \{z \in W \mid \forall z'. z' R_\alpha z \Rightarrow z' \in \llbracket A \rrbracket\} && \overline{\alpha} A
\end{aligned}$$

The connectives $\Delta_0, \rightarrow_0, \blacktriangle_0, \rightarrow_0$, involving formulas and transitive actions, are interpreted in the same way, replacing the relation R_α with the appropriate transitive relations R_δ . Finally, the following syntactic adjoints can be given an interpretation as follows:

$$\begin{aligned}
\llbracket B \triangleleft_1 A \rrbracket_v &= \{(z, z') \in W \times W \mid z \in \llbracket A \rrbracket_v \Rightarrow z' \in \llbracket B \rrbracket_v\} \\
\llbracket B \blacktriangleleft_1 A \rrbracket_v &= \{(z, z') \in W \times W \mid z' \in \llbracket A \rrbracket_v \Rightarrow z \in \llbracket B \rrbracket_v\}
\end{aligned}$$

It can also be readily verified that the translation of Section 6.3 preserves the semantic interpretation, that is, $\llbracket A \rrbracket_M = \llbracket A' \rrbracket_{N_M}$ for every Kripke model M and any PDL-formula A , where A' denotes the translation of A in the language of the dynamic calculus.

The soundness of all operational rules for heterogeneous connectives immediately follows from the fact that their semantic counterparts as defined above are monotone or antitone in each coordinate.

The soundness of the cut-rules follows from the transitivity of the inclusion relation in the domain of interpretation of each type.

The display rules $(\Delta_i, \rightarrow_i)$ and $(\blacktriangle_i, \rightarrow_i)$ for $0 \leq i \leq 1$, and $(\Delta_1, \blacktriangleleft_1)$ and $(\blacktriangle_1, \triangleleft_1)$ are sound as the semantics of the triangle and arrow connectives form adjoint pairs.

On the other hand, in the display rules $(\Delta_0, \blacktriangleleft_0)$ and $(\blacktriangle_0, \triangleleft_0)$, the arrow-connectives are what we call *virtual adjoints* (cf. Section 6.3), that is, they do not have a semantic interpretation. In the next section, we will discuss a proof method to show that their presence in the calculus is safe.⁵ Soundness of necessitation, conjugation, Fischer Servi, and monotonicity rules is straightforward and proved as in Section 2.6.2. In the remainder of the section, we discuss the soundness of the fixed point and omega-rules. As to the soundness of FP_Δ , fix a model $N = (W, \nu)$, assume that the structures X, Y and Π

⁵At the moment, this is still a conjecture.

have been assigned interpretations, denoted (abusing notation) $\llbracket X \rrbracket_v, \llbracket Y \rrbracket_v \subseteq W$ and $R = \llbracket \Pi \rrbracket_v \subseteq W \times W$, and that the premises of $FP\Delta$ are satisfied, that is:

$$R^{-1}[\llbracket X \rrbracket_v] \subseteq \llbracket Y \rrbracket_v \quad \text{and} \quad (R \circ R^+)^{-1}[\llbracket X \rrbracket_v] \subseteq \llbracket Y \rrbracket_v.$$

We need to show that $(R^+)^{-1}[\llbracket X \rrbracket_v] \subseteq \llbracket Y \rrbracket_v$. By definition, $R^+ = \bigcup_{n \geq 1} R^n$, where $R^1 = R$, and $R^{n+1} = R \circ R^n$. Hence, $(R^+)^{-1}[\llbracket X \rrbracket_v] = \bigcup_{n \geq 1} (R^n)^{-1}[\llbracket X \rrbracket_v]$. Therefore it is enough to show that, for every $n \geq 1$,

$$(R^n)^{-1}[\llbracket X \rrbracket_v] \subseteq \llbracket Y \rrbracket_v.$$

This is shown by induction on n . Both the base and the induction cases follow by the assumptions. The soundness of the remaining FP -rules is shown similarly, and is omitted.

As to the soundness of the rule $\omega\Delta$, fix N , let $\llbracket X \rrbracket_v, \llbracket Y \rrbracket_v$ and $R = \llbracket \Pi \rrbracket_v$ as above. The assumption that the premises of $\omega\Delta$ are all satisfied boil down to the inclusion $(R^n)^{-1}[\llbracket X \rrbracket_v] \subseteq \llbracket Y \rrbracket_v$ holding for every $n \geq 1$. Hence,

$$(R^+)^{-1}[\llbracket X \rrbracket_v] = \bigcup_{n \geq 1} (R^n)^{-1}[\llbracket X \rrbracket_v] \subseteq \llbracket Y \rrbracket_v,$$

as required. The soundness of the remaining ω -rules is shown similarly, and is omitted.

6.5 Completeness

In the present section, we discuss the completeness of the Dynamic Calculus for PDL w.r.t. the semantics of Section 6.4. We show that the translation (cf. Section 6.3) of each of the PDL axioms is derivable in the Dynamic Calculus. Unlike what we did in Chapter 3, here we need to consider all possible version of the axioms arising from the disambiguation procedure. Our completeness proof is indirect, and relies on the fact that PDL is complete w.r.t. the standard Kripke semantics, and that the translation preserves the semantic interpretation on the standard models (as discussed in Section 6.4).

In the present section, we restrict our attention to deriving the box-versions of the fix point and induction axioms for PDL. The derivations of the remaining box-axioms for PDL are collected in Appendix K. The diamond-axioms can be also derived without appealing to the classical box/diamond interdefinability. These derivations follow a similar pattern to the ones given below and in Appendix K; the details are omitted.

Box-Fix point $+ (\alpha \rightarrow A) \wedge (\alpha \rightarrow (\alpha^+ \rightarrow A)) \dashv\vdash \alpha^+ \rightarrow A$

$$\begin{array}{c}
 \frac{\frac{\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+} \quad A \vdash A}{\alpha \rightarrow A \vdash \alpha \triangleright A}}{\alpha \blacktriangle \alpha \rightarrow A \vdash A} \quad \frac{\frac{\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+} \quad A \vdash A}{\alpha \rightarrow A \vdash \alpha^\oplus \triangleright A}}{\alpha \rightarrow (\alpha^+ \rightarrow A) \vdash \alpha \triangleright (\alpha^\oplus \triangleright A)} \\
 \frac{\frac{\alpha \blacktriangle \alpha \rightarrow (\alpha^+ \rightarrow A) \vdash \alpha^\oplus \triangleright A}{\alpha^\oplus \blacktriangle (\alpha \blacktriangle \alpha \rightarrow (\alpha^+ \rightarrow A)) \vdash A}}{(\alpha; \alpha^\oplus) \blacktriangle \alpha \rightarrow (\alpha^+ \rightarrow A) \vdash A} \\
 FP \blacktriangle \frac{\alpha \blacktriangle \alpha \rightarrow A \vdash A}{\alpha^\oplus \blacktriangle (\alpha \rightarrow A, \alpha \rightarrow (\alpha^+ \rightarrow A)) \vdash A} \\
 \frac{\alpha^\oplus \vdash A \triangleleft (\alpha \rightarrow A, \alpha \rightarrow (\alpha^+ \rightarrow A))}{\alpha^+ \vdash A \triangleleft (\alpha \rightarrow A, \alpha \rightarrow (\alpha^+ \rightarrow A))} \\
 \frac{\alpha^+ \blacktriangle (\alpha \rightarrow A, \alpha \rightarrow (\alpha^+ \rightarrow A)) \vdash A}{\alpha \rightarrow A, \alpha \rightarrow (\alpha^+ \rightarrow A) \vdash \alpha^+ \triangleright A} \\
 \frac{\alpha \rightarrow A, \alpha \rightarrow (\alpha^+ \rightarrow A) \vdash \alpha^+ \rightarrow A}{\alpha \rightarrow A \wedge \alpha \rightarrow (\alpha^+ \rightarrow A) \vdash \alpha^+ \rightarrow A} \\
 \\
 \frac{\frac{\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+} \quad \frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+}}{\alpha^\oplus; \alpha^+ \vdash \alpha^{+\ominus}}}{(\alpha^\oplus; \alpha^+)^\oplus \vdash \alpha^+} \quad A \vdash A \\
 \frac{\alpha^+ \rightarrow A \vdash (\alpha^\oplus; \alpha^+)^\oplus \triangleright_0 A}{\alpha^+ \rightarrow A \vdash (\alpha^\oplus; \alpha^+) \triangleright_1 A} \\
 \frac{\alpha^+ \rightarrow A \vdash \alpha^\oplus \triangleright (\alpha^+ \triangleright_0 A)}{\alpha^+ \rightarrow A \vdash \alpha^\oplus \triangleright (\alpha^+ \rightarrow A)} \\
 \frac{\alpha^+ \rightarrow A \vdash \alpha \triangleright_1 (\alpha^+ \rightarrow A)}{\alpha^+ \rightarrow A \vdash \alpha \rightarrow (\alpha^+ \rightarrow A)} \\
 CL \frac{\alpha^+ \rightarrow A, \alpha^+ \rightarrow A \vdash (\alpha \rightarrow A) \wedge (\alpha \rightarrow (\alpha^+ \rightarrow A))}{\alpha^+ \rightarrow A \vdash (\alpha \rightarrow A) \wedge (\alpha \rightarrow (\alpha^+ \rightarrow A))}
 \end{array}$$

Box-Induction + $(\alpha \multimap A) \wedge (\alpha^+ \multimap (A \rightarrow (\alpha \multimap A))) \vdash \alpha^+ \multimap A$

The following (incomplete) derivation takes us to the point in which the infinitary rule $\omega \blacktriangleleft$ is applied:

$$\frac{\left(\begin{array}{c|c} \pi_n & \\ \vdots & \\ \alpha^{(n)} \blacktriangle (\alpha \multimap A, \alpha^+ \multimap (A \rightarrow (\alpha \multimap A))) \vdash A & n \geq 1 \end{array} \right)}{\omega \blacktriangle} \frac{\frac{\alpha^\oplus \blacktriangle (\alpha \multimap A, \alpha^+ \multimap (A \rightarrow (\alpha \multimap A))) \vdash A}{\alpha^\oplus \vdash A \blacktriangleleft (\alpha \multimap A, \alpha^+ \multimap (A \rightarrow (\alpha \multimap A)))}}{\alpha^+ \vdash A \blacktriangleleft (\alpha \multimap A, \alpha^+ \multimap (A \rightarrow (\alpha \multimap A)))}}{\frac{\alpha^+ \blacktriangle (\alpha \multimap A, \alpha^+ \multimap (A \rightarrow (\alpha \multimap A))) \vdash A}{\alpha \multimap A, \alpha^+ \multimap (A \rightarrow (\alpha \multimap A)) \vdash \alpha^+ \blacktriangleright A}}{\frac{(\alpha \multimap A) \wedge (\alpha^+ \multimap (A \rightarrow (\alpha \multimap A))) \vdash \alpha^+ \blacktriangleright A}{(\alpha \multimap A) \wedge (\alpha^+ \multimap (A \rightarrow (\alpha \multimap A))) \vdash \alpha^+ \multimap A}}{\frac{[\alpha]A \wedge [\alpha^+](A \rightarrow [\alpha]A) \vdash [\alpha^+]A}{\text{notation}}}}$$

To complete the proof we are reduced to showing that each premise of the application of the $\omega \blacktriangleleft$ rule is derivable, that is:

Proposition 6.3. *The following sequent is derivable for any $n \geq 1$:*

$$\alpha^{(n)} \blacktriangle (\alpha \multimap A, \alpha^+ \multimap (A \rightarrow (\alpha \multimap A))) \vdash A.$$

In what follows, the abbreviations below will be useful:

- let $\alpha^{(\odot n)}(-)$ abbreviate $\underbrace{\alpha \odot (\alpha \odot \dots (\alpha \odot (-)) \dots)}_n$, for $\odot \in \{\triangleleft, \blacktriangle, \blacktriangleright, \blacktriangleright\}$;
- let $\alpha^{(\cdot n)}(-)$ abbreviate $\underbrace{\alpha \cdot (\alpha \cdot \dots (\alpha \cdot (-)) \dots)}_n$, for $\cdot \in \{\Delta, \blacktriangle, \multimap, \rightarrow\}$;
- let $\alpha^{(n)}$ and α^n abbreviate $\underbrace{\alpha ; (\alpha ; \dots (\alpha ; \alpha) \dots)}_n$ and $\underbrace{\alpha ; (\alpha ; \dots (\alpha ; \alpha) \dots)}_n$, respectively.

Lemma 6.4. *Let $B = A \rightarrow (\alpha \multimap A)$. The following sequent is derivable for each $n \geq 1$:*

$$\alpha^{(\multimap n)}(A), \alpha^{(\multimap n)}(B) \vdash \alpha^{(\blacktriangleright n+1)}(A).$$

Proof. The statement is proved by the following schematic derivation.

$$\begin{array}{c}
\frac{\frac{\alpha \vdash \alpha \quad A \vdash A}{\alpha \rightarrow A \vdash \alpha \triangleright A}}{n-1 \text{ appl's of } \rightarrow_L \quad \frac{\alpha \vdash \alpha \quad \alpha \rightarrow A \vdash \alpha \triangleright A}{\alpha^{(\rightarrow^n)}(A) \vdash \alpha^{(\triangleright^n)}A}}{n \text{ appl's of } \triangleright \blacktriangle \quad \frac{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) \vdash A \quad \frac{\alpha \vdash \alpha \quad A \vdash A}{\alpha \rightarrow A \vdash \alpha \triangleright A}}{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) \vdash \alpha^{(\triangleright^n)}A}}{\text{notation} \quad \frac{A \rightarrow (\alpha \rightarrow A) \vdash \alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) > (\alpha \triangleright A)}{B \vdash \alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) > (\alpha \triangleright A)}}} \\
\frac{n \text{ appl's of } \rightarrow_L \quad \frac{\alpha \vdash \alpha \quad B \vdash \alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) > (\alpha \triangleright A)}{\alpha^{(\rightarrow^n)}(B) \vdash \alpha^{(\triangleright^n)}(\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) > (\alpha \triangleright A)}}}{n \text{ appl's of } \triangleright \blacktriangle \quad \frac{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(B)) \vdash \alpha^{(\triangleright^n)}(\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) > (\alpha \triangleright A)}{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(B)) \vdash \alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)) > (\alpha \triangleright A)}}} \\
\frac{(\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A))), (\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(B)) \vdash \alpha \triangleright A)}{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)), \alpha^{(\rightarrow^n)}(B) \vdash \alpha \triangleright A} \quad n \text{ appl's of } \blacktriangle \text{ dis} \\
\frac{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A)), \alpha^{(\rightarrow^n)}(B) \vdash \alpha \triangleright A}{\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B) \vdash \alpha^{(\triangleright^{n+1})}(A)} \quad n \text{ appl's of } \blacktriangle \triangleright
\end{array}$$

□

Corollary 6.5. *Let $B = A \rightarrow (\alpha \rightarrow A)$. The following sequent is derivable for each $n \geq 1$:*

$$\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B) \vdash \alpha^{(\rightarrow^{n+1})}(A).$$

Proof. The schematic derivation in the proof of Lemma 6.4 shows in particular that a derivation for the following sequent exists:

$$\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B)) \vdash \alpha \triangleright A.$$

Then, the desired derivation can be obtained by prolonging that derivation with n alternations of \rightarrow_R and $\blacktriangle \triangleright$, as follows:

$$\begin{array}{c}
\frac{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B)) \vdash \alpha \triangleright A}{\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B)) \vdash \alpha \rightarrow A} \\
\frac{\alpha^{(\blacktriangle^{n-1})}(\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B)) \vdash \alpha \triangleright (\alpha \rightarrow A)}{\alpha^{(\blacktriangle^{n-1})}(\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B)) \vdash \alpha \rightarrow (\alpha \rightarrow A)} \\
\frac{\alpha^{(\blacktriangle^{n-1})}(\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B)) \vdash \alpha \rightarrow (\alpha \rightarrow A)}{\alpha^{(\rightarrow^n)}(A), \alpha^{(\rightarrow^n)}(B) \vdash \alpha^{(\rightarrow^{n+1})}(A)}
\end{array}$$

□

Lemma 6.6. *Let $B = A \rightarrow (\alpha \rightarrow A)$. The following sequent is derivable for each $n \geq 1$:*

$$\alpha^{(\rightarrow^1)}(A), \alpha^{(\rightarrow^1)}(B), \dots, \alpha^{(\rightarrow^{n-1})}(B), \alpha^{(\rightarrow^n)}(B) \vdash \alpha^{(\triangleright^{n+1})}(A).$$

Proof. Fix $n \geq 1$, let X_{n+1} abbreviate $\alpha^{(\triangleright^{n+1})}(A)$, and for each $1 \leq i < n$, let C_i and D_i abbreviate $\alpha^{(\rightarrow^i)}(A)$ and $\alpha^{(\rightarrow^i)}(B)$, respectively. By Corollary 6.5, a derivation π_i of $C_i, D_i \vdash C_{i+1}$

is available for each $1 \leq i < n$, and by Lemma 6.4, a derivation π_n of $C_n, D_n \vdash X_{n+1}$ is also available. Then the following derivation, which essentially consists in $n - 1$ applications of Cut, proves the statement:

$$\begin{array}{c}
\begin{array}{c} \pi_1 \\ \vdots \\ C_1, D_1 \vdash C_2 \end{array} \quad \begin{array}{c} \pi_2 \\ \vdots \\ C_2, D_2 \vdash C_3 \\ C_2 \vdash C_3 < D_2 \end{array} \quad \begin{array}{c} \pi_3 \\ \vdots \\ C_3, D_3 \vdash C_4 \\ C_3 \vdash C_4 < D_3 \end{array} \quad \begin{array}{c} \pi_i \\ \vdots \\ C_n, D_n \vdash X_{n+1} \\ C_n \vdash X_{n+1} < D_n \end{array} \quad \begin{array}{c} \pi_n \\ \vdots \\ C_n, D_n \vdash X_{n+1} \\ C_n \vdash X_{n+1} < D_n \end{array} \\
\hline
\begin{array}{c} C_1, D_1 \vdash C_2 \\ C_1, D_1 \vdash C_3 < D_2 \\ C_1, D_1, D_2 \vdash C_3 \\ C_1, D_1, D_2 \vdash C_4 < D_3 \\ C_1, D_1, D_2, D_3 \vdash C_4 \\ \vdots \\ C_1, D_1, \dots, D_{n-1} \vdash X_{n+1} < D_n \\ C_1, D_1, \dots, D_{n-1}, D_n \vdash X_{n+1} \end{array} \\
\hline
\alpha^{(\rightarrow^1)}(A), \alpha^{(\rightarrow^1)}(B), \dots, \alpha^{(\rightarrow^{n-1})}(B), \alpha^{(\rightarrow^n)}(B) \vdash \alpha^{(\triangleright^{n+1})}(A)
\end{array}$$

□

Lemma 6.7. *The following sequent is derivable for each $n \geq 1$ and every formula C :*

$$\alpha^+ \rightarrow C \vdash \alpha^{(\rightarrow^n)}(C).$$

Proof. For $n = 1$, the following derivation proves the statement:

$$\begin{array}{c}
\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+} \quad C \vdash C \\
\hline
\alpha^+ \rightarrow C \vdash \alpha^\oplus \triangleright C \\
\hline
\alpha^+ \rightarrow C \vdash \alpha \triangleright C \\
\hline
\alpha^+ \rightarrow C \vdash \alpha \rightarrow C
\end{array}$$

For $n \geq 2$, the following schematic derivation proves the statement:

$$\begin{array}{c}
\frac{\frac{\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+}}{\alpha \vdash \alpha^{+\ominus}}}{\alpha \vdash \alpha^{+\ominus}} \quad \frac{\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+}}{\alpha \vdash \alpha^{+\ominus}} \quad \cdots \quad \frac{\frac{\frac{\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+}}{\alpha \vdash \alpha^{+\ominus}}}{\alpha \vdash \alpha^{+\ominus}}}{\alpha \vdash \alpha^{+\ominus}} \quad \frac{\frac{\alpha \vdash \alpha}{\alpha^\oplus \vdash \alpha^+}}{\alpha \vdash \alpha^{+\ominus}}}{\alpha \vdash \alpha^{+\ominus}} \text{ abs} \\
\frac{\alpha; \alpha^{(n-1)} \vdash \alpha^{+\ominus}}{(\alpha; \alpha^{(n-1)})^\oplus \vdash \alpha^+} \quad n-2 \text{ appl's of } abs \\
\hline
\frac{\alpha^+ \rightarrow C \vdash (\alpha; \alpha^{(n-1)})^\oplus \triangleright C}{\alpha^+ \rightarrow C \vdash (\alpha; \alpha^{(n-1)}) \triangleright C} \text{ act } \triangleright \\
\frac{\alpha^+ \rightarrow C \vdash \alpha \triangleright (\alpha^{(n-1)}) \triangleright C}{\alpha \blacktriangle \alpha^+ \rightarrow C \vdash \alpha^{(n-1)} \triangleright C} \triangleright \blacktriangle \\
\frac{\alpha \blacktriangle \alpha^+ \rightarrow C \vdash \alpha^{(n-1)} \triangleright C}{\alpha \blacktriangle \alpha^+ \rightarrow C \vdash (\alpha; \alpha^{(n-2)}) \triangleright C} \text{ notation} \\
\frac{\alpha \blacktriangle (\alpha^{(n-2)}(\alpha^+ \rightarrow C)) \vdash \alpha \triangleright C}{\alpha \blacktriangle (\alpha^{(n-2)}(\alpha^+ \rightarrow C)) \vdash \alpha \rightarrow C} (*) \\
\frac{\alpha \blacktriangle (\alpha^{(n-2)}(\alpha^+ \rightarrow C)) \vdash \alpha \rightarrow C}{\alpha^{(n-2)}(\alpha^+ \rightarrow C) \vdash \alpha \triangleright \alpha \rightarrow C} \blacktriangle \triangleright \\
\frac{\alpha^{(n-2)}(\alpha^+ \rightarrow C) \vdash \alpha \triangleright \alpha \rightarrow C}{\alpha^+ \rightarrow C \vdash \alpha^{(n-1)}(C)} (**).
\end{array}$$

(*) $n - 2$ alternating applications of the structural rule $act \triangleright$ and of the display postulate for the connectives \triangleright and \blacktriangle .

(**) $n - 1$ alternating applications of the operational rule \rightarrow_R and of the display postulate for the connectives \blacktriangle and \triangleright . □

Lemma 6.8. *Let $B = A \rightarrow (\alpha \rightarrow A)$. The following sequent is derivable for each $n \geq 1$:*

$$\alpha^{(\rightarrow^1)}(A), \alpha^+ \rightarrow B \vdash \alpha^{(\triangleright^n)}(A).$$

Proof. Fix $n \geq 1$, let X_{n+1} abbreviate $\alpha^{(\triangleright^{n+1})}(A)$, let D^+ abbreviate $\alpha^+ \rightarrow B$, and for each $1 \leq i \leq n$, let C_i and D_i abbreviate $\alpha^{(\rightarrow^i)}(A)$ and $\alpha^{(\rightarrow^i)}(B)$, respectively. By Lemma 6.6, a derivation π_n of the sequent

$$C_1, D_1, \dots, D_n \vdash X_{n+1}$$

is available for each $n \geq 1$ and for $B = A \rightarrow (\alpha \rightarrow A)$.

By Lemma 6.7, for each $1 \leq i \leq n$, a derivation π'_i of the sequent $\alpha^+ \rightarrow C \vdash \alpha^{(\rightarrow^i)}(C)$ is available for any C , so in particular for $C = B$ we get a derivation of

$$D^+ \vdash D_i.$$

Applying Cut $n - 1$ times, the following derivation proves the statement:

$$\begin{array}{c}
\pi'_n \quad \pi'_i \quad \pi'_1 \quad \pi_n \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
D^+ \vdash D_n \quad \dots \quad \frac{C_1, D_1, \dots, D_n \vdash X_{n+1}}{D_1 \vdash Y_1} \text{ } D_1 \text{ in display} \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
\frac{D^+ \vdash Y_1}{D_2 \vdash Y_2} \text{ } D_2 \text{ in display} \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
\frac{D^+ \vdash Y_n}{D_n \vdash Y_n} \text{ } n-2 \text{ sequences of display and Cut} \\
\frac{\dots}{\text{Cut}_{n-1}} \\
\frac{D^+ \vdash Y_n}{C_1, \underbrace{D^+, \dots, D^+}_{n} \vdash X_{n+1}} \text{ } \text{Display} \\
\frac{\dots}{C_1, D^+ \vdash X_{n+1}} \text{ } \text{Contraction}
\end{array}$$

□

Now we can finish the proof of Proposition 6.3 as follows:

Proof. By Lemma 6.8, a derivation of the sequent $\alpha^{(\rightarrow^1)}(A), \alpha^+ \rightarrow B \vdash \alpha^{(\triangleright^n)}(A)$ exists; then the desired derivation is obtained by prolonging that derivation as shown below.

$$\begin{array}{c}
\alpha^{(\rightarrow^1)}(A), \alpha^+ \rightarrow B \vdash \alpha^{(\triangleright^n)}(A) \\
\hline
n \text{ appl's of } \triangleright \blacktriangle \\
\alpha^{(\blacktriangle^n)}(\alpha^{(\rightarrow^1)}(A), \alpha^+ \rightarrow B) \vdash A \\
\hline
\alpha^{(n)} \blacktriangle (\alpha^{(\rightarrow^1)}(A), \alpha^+ \rightarrow B) \vdash A \\
n \text{ appl's of } act \blacktriangle
\end{array}$$

□

6.6 Cut elimination

In the present section, we prove that the multi-type display calculus for PDL is a proper display calculus (cf. Definition 6.1). By Theorem 6.2, this is enough to establish that the calculus enjoys the cut elimination and the subformula property. Conditions $C_1, C_2, C_3, C_4, C_5, C'_5, C'_6, C'_7$ and C_{10} are straightforwardly verified by inspecting the rules and are left to the reader. Condition C'_2 can be straightforwardly verified by inspection on the rules, for instance by observing that the domains and codomains of adjoints are rigidly determined.

The following proposition shows that condition C_9 is met:

Proposition 6.9. *Any derivable sequent in the calculus for PDL is type-uniform.*

Proof. We prove the proposition by induction on the height of the derivation. The base case is verified by inspection; indeed, the following axioms are type-uniform by definition of their constituents:

$$d \vdash d \quad a \vdash a \quad p \vdash p \quad \perp \vdash I \quad I \vdash \top$$

As to the inductive step, one can verify by inspection that all the rules of the calculus preserve type-uniformity, and that the Cut rules are strongly type-uniform. \square

Finally, the verification steps for C'_8 are collected in Appendix J.

6.7 The open issue of conservativity

In the present section, we expand on the difficulties encountered in the proof of conservativity for the Dynamic Calculus for PDL.

Semantic argument. The main avenue to prove the conservativity of a display calculus w.r.t. the original logic that the calculus is meant to capture is semantic. Namely, if the original logic is complete w.r.t. a given semantics, then it is enough to prove that every rule is sound w.r.t. that semantics. This is not possible in the case of the Dynamic Calculus for PDL, since some display rules are not interpretable in the semantics due to the presence of virtual adjoints (cf. Section 6.3). This situation is analogous to that of the Dynamic Calculus for EAK.

Syntactic elimination of virtual adjoints. In the setting of the Dynamic Calculus for EAK, our proof was syntactical, and its pivot step was showing that any valid proof-tree the root of which is operational and of type F_m can be rewritten into a valid proof-tree involving no virtual adjoints (cf. Section 3.6). This process of removing virtual adjoints could take place essentially because the action-grammar of EAK was very poor. In the case of PDL, because the presence of the iteration in the grammar of actions, this fact is not true. However, the fact that virtual adjoints occur in essential ways in derivation trees of operational sequents $A \vdash B$ of type F_m does not imply per se that the calculus is not sound w.r.t. the original language.

Display-Type Calculi. Another option would be modifying the Dynamic Calculus for PDL so as to make it a display-type calculus rather than a display calculus. The modifications would require removing all the rules involving virtual adjoints and replacing the cut rules with suitable surgical cuts. However, in order to obtain a complete calculus,

certain rules which are derived in the original Dynamic Calculus would also need to be added. This is the case of the following rule:

$$\frac{\alpha^\oplus \blacktriangle X \vdash Y}{\alpha^+ \blacktriangle X \vdash Y}$$

Unfortunately, the rule above violates visibility. Hence, cut elimination cannot be proved for the resulting calculus via Theorem 4.1.

Conservativity via translation. In [DCGT14] and [CDGT13b], the conservativity issue for a display calculus for Full Intuitionistic Linear Logic (FILL) was resolved with a technique which we intend to adapt to PDL. This adaptation is still work in progress. In what follows, we report on the proof strategy adopted in [DCGT14], and discuss its possible adaptations. The main steps in the proof strategy are:

1. define a sound and complete display calculus for an extension of the logic with additional adjunctions. The extension considered in [DCGT14] is Bi-Intuitionistic Linear Logic (BiILL).
2. translate the display calculus to a shallow inference nested sequent calculus.
3. translate the shallow inference nested sequent calculus to a deep inference nested sequent calculus.
4. prove that the deep inference nested sequent calculus is sound with respect to the original logic. In the case of [CFPS14], the authors prove that the deep inference nested sequent calculus is sound with respect to FILL.

The adaptation of this technique to the setting of PDL is not straightforward. For instance, the first translation transforms logical connectives into meta-linguistic data structures such as \vdash without losing information. The naive adaptation of this step to the setting of PDL would make us lose information. This direction is still work in progress.

6.8 Conclusions

The calculus introduced in the present chapter is an attempt at extending the methodology of display calculi to a fully-fledged PDL-type setting. Previous attempts in this direction (e.g. [Wan98]) exclude both the Kleene star and the positive iteration. Accounting for these operations is proof-theoretically challenging, and indeed, the existing

proposals in the literature, also outside the display calculus methodology, typically witness a trade-off between achieving *syntactical* full cut elimination at the price of having infinitary rules in the system (e.g. [Pog10]), or dispensing with infinitary rules at the price of achieving cut elimination modulo analytic cut(s) (e.g. unpublished manuscript [Har]). The present proposal aims at paving the way for escaping this trade-off. Indeed, our starting point is the basic understanding that the induction axioms/induction rules (which are the main hurdle to a smooth proof-theoretic treatment of PDL) ingeniously encode *by means of formulas* a piece of information which by rights pertains to *actions*; namely, they encode the relation between an action and its (reflexive and) transitive closure. This encoding is done either by resorting to infinitary axioms/rules, or by introducing some forms of ‘loops’ (i.e. formulas appearing both in the antecedent and in the consequent of an implication). Each of these two ways gives rise to issues which hinder a smooth proof-theoretic treatment of PDL. Taken together, these two alternatives are at the basis of the trade-off we wish to escape. Our idea for a solution (which needs to be perfected) involves introducing enough expressivity in the language so that formulas are not to be relied upon anymore to encode a piece of information which strictly speaking pertains purely to *actions*, and neither pertains to formulas, nor to the interaction between formulas and actions.

In particular, we aim at describing the proof-theoretic behaviour of the positive iteration operation $+$ in terms of the order-theoretic behaviour of the transitive closure. Namely, we make use of the well known fact that the map associating each binary relation on a given set W to its transitive closure can be characterized order-theoretically as the left adjoint of the inclusion map sending the transitive relations on W into $\mathcal{P}(W \times W)$. The introduction of *two* different types of actions is then motivated by the need to properly express this adjunction. Thus, we expand the language, both at the structural and at the operational level, with the following pair of adjoint maps:

$$(\cdot)^+ : \text{Act} \rightarrow \text{TAct} \quad (\cdot)^- : \text{TAct} \rightarrow \text{Act}.$$

The adjunction relation $(\cdot)^+ \dashv (\cdot)^-$ is not enough to capture the informational content of the transitive closure. The missing pieces are: (1) the map $(\cdot)^-$ being an order-embedding; (2) the fact that the TAct-type elements are transitive, i.e. $\delta; \delta \vdash \delta$ for each $\delta \in \text{TAct}$. Neither piece of information is captured at the operational level. Indeed, we can only prove

$$\frac{\frac{\delta \vdash \delta}{\delta^- \vdash \delta^\ominus}}{\delta^{-\ominus} \vdash \delta}$$

Hence, we had to resort to the omega-induction rules (which, besides being infinitary, take the form of interaction rules between formula-type and action-type terms) to encode the transitive closure and derive the induction axioms. We conjecture that being able to express transitive closure at the structural level is key to dispensing with the infinitary rules, which is our next goal for future developments in this line research.

Finally, it is perhaps worth stressing that considerations such as the ones just made above can be made in a meaningful way only in the context of a multi-type environment in which actions and formulas enjoy equal standing as first-class citizens. Thus, the multi-type approach can also function as a ‘conceptual tool’, by means of which technical difficulties such as the ones mentioned above can be explained in terms of problems of *expressivity*. In their turn, properties and considerations involving different degrees of expressivity can then be sharpened and made precise.

Conclusions

Contributions

The original contributions of the present thesis belong to two directions, one pertaining to unified correspondence theory, the other to the theory of display calculi. As to the first direction, a characterization result about finite lattices has been established as a novel application of unified correspondence theory. Towards this result, the algorithm ALBA has been adapted to a semantic setting for monotone modal logic, and a duality-based translation has been defined between lattice inequalities and inequalities in the language of monotone modal logic. As to the second direction, the proof-theoretic methodology of display calculi has been extended to Baltag-Moss-Solecki's logic of Epistemic Actions and Knowledge (EAK), Monotone Modal Logic (MML) and Propositional Dynamic Logic (PDL). Due precisely to the distinguishing features which make these logics useful in various fields of science, the proof-theoretic treatment of each of these logics has proved to be very challenging. The results collected in Part II of the present thesis are embedded in a wider program aimed at developing good proof calculi for dynamic logics, where by a 'good proof calculus' we mean one enjoying a series of properties which go beyond the most basic ones, which are soundness and completeness w.r.t. the original logic, and cut elimination. Indeed, 'good proof calculi' should be *modular*, that is, amenable to the development of methods which transfer easily from one logic to another. In the thesis, for instance, much attention has been dedicated to the Belnap-style cut elimination, i.e. the statement and proof of metatheorems guaranteeing any system to enjoy cut elimination provided it satisfies certain conditions which are relatively easy to check. But also, following a basic idea from proof-theoretic semantics, 'good proof calculi' for a given logic should develop an *independent* perspective on that logics. That is, one which is independent of existing semantic frameworks, and does not make use of meta-linguistic resources such as abbreviations and labels. In the thesis, we generalize display calculi to a multi-type setting. Besides being in line with the general principles of proof-theoretic semantics, the multi-type environment is a very effective conceptual tool, which makes it possible to calibrate the expressivity of a language in several ways. For instance, in

the thesis, we were able to absorb the labels in the formulation of the interaction axioms in EAK precisely by calibrating the expressivity of its corresponding multi-type environment. But also, and perhaps more subtly, the multi-type environment is of help as a diagnostic tool. For instance, the dynamic calculus for PDL brings two main outstanding issues: the conservativity and the omega rules. In a sense, it could be said that these two issues are both symptomatic of the fact that the omega rule uses the type formula to encode a property (the transitive closure) which has nothing to do with formulas. We believe that disentangling these two types is key to improving on the design of this calculus and on its overall performances.

Further research.

Correspondence theory.

The results in Chapter 1 pave the way to the systematic formalization of dual characterization results for finite lattices. Significant extensions of Nation's dual characterization results appear e.g. in [Sem05] and [San09, Proposition 8.5]. Hence, natural directions worth pursuing concern on the one hand the generalization of the results of the present thesis so as to account for [San09, Proposition 8.5], and (b) analyzing the technical machinery introduced in the present paper from an algorithmic perspective. The latter point involves e.g. establishing whether the present set of rules is minimal, or whether some rules can actually be derived.

Related to both these directions, but more on the front of methodology, are outstanding open questions about Lemma A.1. This lemma provides the soundness and invertibility of a rule by means of which variable elimination is effected via instantiation. So far, all rules of this type in ALBA have been proved sound and invertible thanks to one or another version of Ackermann's lemma. However, it is not clear whether Lemma A.1 can be accounted for in terms of Ackermann's lemma, and hence whether the rule justified in Lemma A.1 can be regarded as an Ackermann-type rule. Moreover, while Lemma A.1 is rooted and has an intuitive understanding in the semantics of minimal coverings, at the moment it is not clear whether and how more general versions of this rule can be formulated, which would be of a wider applicability. Giving answers to these questions would significantly enlarge the scope of algorithmic correspondence theory, and is also a worthwhile future direction.

Proof theory.

Non-proper display calculi. In [Wan98], a difference was drawn between proper display calculi and general display calculi. Roughly speaking, all rules of proper display calculi are required to be schematic (i.e. closed under uniform substitution) whereas this requirement is relaxed in the case of general display calculi. The best known example of a non-proper display calculus appears in [Bel90], motivated by the treatment of the exponential connectives in linear logic. This treatment is ingenious, but it does not transfer easily to other logics. We conjecture that a *proper* display calculus can be designed for linear logic in the multi-type setting. Our basic idea consists in introducing two types of terms: the linear ones, for which the contraction rule is banned, and the Boolean ones, for which the contraction rule is allowed. Then, the exponential connectives can be regarded as toggles in between these types. This direction is work in progress. For expanded discussions see Sections 2.7 and 3.7.

Conservativity. As mentioned at various points early on, the issue of conservativity commonly occurs in display calculi, since enforcing the display property might require introducing connectives which do not belong to the language of the original logic which one wishes to capture. In the thesis, the conservativity issue was solved for all case studies but the Dynamic Calculus for PDL. In Section 6.7, we discussed how we plan to resolve the outstanding issue of the conservativity of the Dynamic Calculus for PDL, which is work in progress. But more in general, an important research direction in display calculi which we also intend to pursue is to state and prove metatheorems for conservativity, analogous to those for cut elimination.

Omega-rules. The second outstanding issue about PDL is whether it is possible to design a multi-type display calculus in which the induction axiom is captured by means of finite rules. As discussed in Section 6.8, our conjecture is that the present set-up is not expressive enough so as to be able to capture the information about transitive closure purely within the action and transitive action type. This direction is also work in progress.

More dynamic logics. The proof-theoretic results in the present thesis form the core of a methodology which is ready to be extended to logics such as Concurrent Propositional Dynamic Logic [Gol92b], Game Logic [Par85], Coalition Logic [Pau01, Pau02], Concurrent Dynamic Epistemic Logic [vDvdHK03], and variants of Dynamic epistemic logics with non-normal epistemic operators. At the present stage of development, we expect

that this extension will be a nontrivial task, but these case studies will be an opportunity to sharpen the general framework.

Appendices

Appendix A

Proof of Lemma 1.34

Lemma 1.34 For every $n \geq 1$, ALBA^l succeeds on the quasi-inequality

$$\forall x_{n-1}, \dots, x_0, \forall y_n, \dots, y_0, \forall \mathbf{j}_n, \left(\left(\begin{array}{l} \mathbf{j}_n \leq t'_n \\ s'_n(\langle \leq_X \rangle \mathbf{j}_n / x_n) \leq \kappa(\mathbf{j}_n) \end{array} \right) \Rightarrow \text{false} \right), \quad (\text{A.1})$$

and produces

$$\forall \mathbf{j}_n, \dots, \mathbf{j}_0, \forall \mathbf{C}_{n-1}, \dots, \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_n \leq \langle \triangleleft \rangle \mathbf{C}_{n-1} \\ \mathbf{j}_{n-1} \leq \langle \in \rangle \mathbf{C}_{n-1} \\ \langle \leq_J \rangle \mathbf{j}_n \wedge \mathbf{j}_{n-1} \leq \perp \\ \dots \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \mathbf{j}_0 \leq \perp \end{array} \right) \Rightarrow \text{false} \right).$$

Proof. By induction on n . If $n = 1$, then the quasi-inequality (A.1) has the following shape (cf. the definitions of t_n and s_n on page 57):

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle [\exists](y_1 \vee x_0) \\ \langle \triangleleft \rangle [\exists](y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \vee \perp \leq \kappa(\mathbf{j}_1) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rules (*AtCoat1*) and (*T* \wedge \perp) to the second inequality, we get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle [\exists](y_1 \vee x_0) \\ \mathbf{j}_1 \leq \neg \langle \triangleleft \rangle [\exists](y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} \right).$$

Now we can apply (*TDB*) and (*TBD*) to the second inequality, and get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle [\exists](y_1 \vee x_0) \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rule (*TNM*) to the first and second inequalities, we get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \langle [\exists] \rangle (y_1 \vee x_0) \wedge \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} \right).$$

We can apply the rule (*AP* \diamond) to the first inequality, and get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{C}_0 \leq [\exists](y_1 \vee x_0) \wedge \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rule (*SP* \wedge) to the second inequality, we get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{C}_0 \leq [\exists](y_1 \vee x_0) \\ \mathbf{C}_0 \leq \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} \right).$$

We can now apply the rule (*AJ* \square) to the second inequality and (*AP* \diamond) to the third inequality, and get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \langle \in \rangle \mathbf{C}_0 \leq y_1 \vee x_0 \\ \mathbf{C}_0 \leq \langle \exists \rangle \mathbf{j}_0 \\ \mathbf{j}_0 \leq \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rules (*DM*) and (*SP* \wedge) to the fourth inequality, we get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \langle \in \rangle \mathbf{C}_0 \leq y_1 \vee x_0 \\ \mathbf{C}_0 \leq \langle \exists \rangle \mathbf{j}_0 \\ \mathbf{j}_0 \leq \neg y_1 \\ \mathbf{j}_0 \leq \neg(\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0) \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \langle \exists \rangle \neg(y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rule (TRR^{-1}) to the third inequality, the rules ($T \wedge \perp$) and ($AtCoat1$) to the fourth and fifth inequalities, and the rules (TDB) and (TBD) to the last inequality, we get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \langle \in \rangle \mathbf{C}_0 \leq y_1 \vee x_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ y_1 \leq \kappa(\mathbf{j}_0) \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge x_0 \leq \kappa(\mathbf{j}_0) \\ \mathbf{j}_1 \leq \neg \langle \triangleleft \rangle [\exists](y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \end{array} \right) \Rightarrow \text{false} .$$

By applying the rule (TR) to the second and third inequalities and the rule ($T \wedge \perp$) to the last inequality, we get:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \langle \in \rangle \mathbf{C}_0 \leq y_1 \vee x_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq y_1 \vee x_0 \\ y_1 \leq \kappa(\mathbf{j}_0) \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge x_0 \leq \kappa(\mathbf{j}_0) \\ \mathbf{j}_1 \wedge \langle \triangleleft \rangle [\exists](y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \leq \perp \end{array} \right) \Rightarrow \text{false} .$$

By applying the rule (MT) to the fourth and fifth inequalities, the rule ($AtCoat1$) to the last inequality, and by exchanging the position of the second and third inequalities, the quasi-inequality above can be equivalently rewritten as follows:

$$\forall x_0, \forall y_1, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \in \rangle \mathbf{C}_0 \leq y_1 \vee x_0 \\ \mathbf{j}_0 \leq x_0 \\ y_1 \leq \kappa(\mathbf{j}_0) \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge x_0 \leq \kappa(\mathbf{j}_0) \\ \mathbf{j}_1 \wedge \langle \triangleleft \rangle [\exists](y_1 \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge x_0)) \leq \perp \end{array} \right) \Rightarrow \text{false} .$$

By lemma A.1 with the following instantiations¹

$$t := \top, \quad s := \perp, \quad \mathbf{j} := \mathbf{j}_1, \quad \mathbf{k} := \mathbf{j}_0, \quad \mathbf{C} := \mathbf{C}_0, \quad x := x_0,$$

¹Notice that $t := \top$ and $s := \perp$ reduce the inequalities $\mathbf{k} \leq t$, $\langle \in \rangle \mathbf{C} \leq y \vee t$ and $x \wedge s \leq \kappa(\mathbf{k})$ in the statement of Lemma A.1 to tautologies, and the inequality $\mathbf{j} \wedge \langle \triangleleft \rangle [\exists](y \vee (\langle \leq_J \rangle \mathbf{j} \wedge x) \vee (x \wedge s)) \leq \perp$ to $\mathbf{j} \wedge \langle \triangleleft \rangle [\exists](y \vee (\langle \leq_J \rangle \mathbf{j} \wedge x)) \leq \perp$.

the quasi-inequality above is equivalent to the following:

$$\forall x_0, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ cl(\langle \in \rangle \mathbf{C}_0 \setminus \mathbf{j}_0) \leq \kappa(\mathbf{j}_0) \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \langle \leq_J \rangle \mathbf{j}_0 \leq \kappa(\mathbf{j}_0) \\ \mathbf{j}_1 \wedge \langle \triangleleft \rangle [\exists](cl(\langle \in \rangle \mathbf{C}_0 \setminus \mathbf{j}_0) \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge \langle \leq_J \rangle \mathbf{j}_0)) \leq \perp \end{array} \right) \Rightarrow \text{false} \right),$$

where cl abbreviates the composition $\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle$. By applying the rule *MinCov2* bottom to top, the quasi-inequality above can be equivalently rewritten as follows

$$\forall x_0, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \langle \leq_J \rangle \mathbf{j}_0 \leq \kappa(\mathbf{j}_0) \\ \mathbf{j}_1 \wedge \langle \triangleleft \rangle [\exists](cl(\langle \in \rangle \mathbf{C}_0 \setminus \mathbf{j}_0) \vee (\langle \leq_J \rangle \mathbf{j}_1 \wedge \langle \leq_J \rangle \mathbf{j}_0)) \leq \perp \end{array} \right) \Rightarrow \text{false} \right),$$

By applying Lemma 1.30, we get:

$$\forall x_0, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \langle \leq_J \rangle \mathbf{j}_0 \leq \kappa(\mathbf{j}_0) \end{array} \right) \Rightarrow \text{false} \right),$$

By applying (*AtomR_{XX}*) to the third inequality, we get:

$$\forall x_0, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \leq \kappa(\mathbf{j}_0) \end{array} \right) \Rightarrow \text{false} \right),$$

By (*AtCoat1*) to the third inequality, we get:

$$\forall x_0, \forall \mathbf{j}_1, \mathbf{j}_0, \forall \mathbf{C}_0 \left(\left(\begin{array}{l} \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \mathbf{j}_0 \leq \perp \end{array} \right) \Rightarrow \text{false} \right),$$

which finishes the proof of the base case.

Induction step. Fix $n \geq 1$, and assume that the lemma holds for n . Recall that \bar{x} stands for the list of variables x_n, \dots, x_0 , and \bar{y} stands for the list of variables y_n, \dots, y_1 . Let us prove the lemma for the quasi-inequality

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1} \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq t'_{n+1} \\ s'_{n+1}(\langle \leq_J \rangle \mathbf{j}_{n+1} / x_{n+1}) \leq \kappa(\mathbf{j}_{n+1}) \end{array} \right) \Rightarrow \text{false} \right).$$

By the definitions on page 57, the quasi-inequality above can be rewritten into:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1} \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle [\exists](y_{n+1} \vee t_n) \\ \langle \triangleleft \rangle [\exists](y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \leq \kappa(\mathbf{j}_{n+1}) \end{array} \right) \Rightarrow \text{false} \right),$$

which, by applying the rules (*AtCoat1*) and (*T* \wedge \perp) to the second inequality, is equivalent to:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1} \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle [\exists](y_{n+1} \vee t_n) \\ \mathbf{j}_{n+1} \leq \neg \langle \triangleleft \rangle [\exists](y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rule (*TBD*) and (*TDB*) to the second inequality, we get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1} \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle [\exists](y_{n+1} \vee t_n) \\ \mathbf{j}_{n+1} \leq [\triangleleft] \langle \exists \rangle \neg (y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rule (*TNM*) to the first and second inequalities, we get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1} \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle ([\exists](y_{n+1} \vee t_n) \wedge \langle \exists \rangle \neg (y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n)) \\ \mathbf{j}_{n+1} \leq [\triangleleft] \langle \exists \rangle \neg (y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rule (*AP* \diamond) to the first inequality, we get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \forall \mathbf{C}_n \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{C}_n \leq [\exists](y_{n+1} \vee t_n) \wedge \langle \exists \rangle \neg (y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \\ \mathbf{j}_{n+1} \leq [\triangleleft] \langle \exists \rangle \neg (y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying the rule (*SP* \wedge) to the second inequality, we get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \forall \mathbf{C}_n \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{C}_n \leq [\exists](y_{n+1} \vee t_n) \\ \mathbf{C}_n \leq \langle \exists \rangle \neg (y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \\ \mathbf{j}_{n+1} \leq [\triangleleft] \langle \exists \rangle \neg (y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} \right).$$

We can now apply the rule ($AJ\Box$) to the second inequality and ($AP\Diamond$) to the third inequality, and get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \mathbf{j}_n, \forall \mathbf{C}_n \left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \langle \in \rangle \mathbf{C}_n \leq y_{n+1} \vee t_n \\ \mathbf{C}_n \leq \langle \exists \rangle \mathbf{j}_n \\ \mathbf{j}_n \leq \neg(y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \\ \mathbf{j}_{n+1} \leq [\triangleleft] \langle \exists \rangle \neg(y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} .$$

By applying the rules (DM) and ($SP\wedge$) to the fourth inequality, we get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \mathbf{j}_n, \forall \mathbf{C}_n \left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \langle \in \rangle \mathbf{C}_n \leq y_{n+1} \vee t_n \\ \mathbf{C}_n \leq \langle \exists \rangle \mathbf{j}_n \\ \mathbf{j}_n \leq \neg y_{n+1} \\ \mathbf{j}_n \leq \neg(\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \\ \mathbf{j}_n \leq \neg s_n \\ \mathbf{j}_{n+1} \leq [\triangleleft] \langle \exists \rangle \neg(y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} .$$

By applying the rule (TRR^{-1}) to the third inequality, the rules ($T \wedge \perp$) and ($AtCoat1$) to the fourth, fifth and sixth inequalities, and the rules (TDB) and (TBD) to the last inequality, we get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \mathbf{j}_n, \forall \mathbf{C}_n \left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \langle \in \rangle \mathbf{C}_n \leq y_{n+1} \vee t_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ y_{n+1} \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n \leq \kappa(\mathbf{j}_n) \\ s_n \leq \kappa(\mathbf{j}_n) \\ \mathbf{j}_{n+1} \leq \neg \langle \triangleleft \rangle [\exists](y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \end{array} \right) \Rightarrow \text{false} .$$

By applying the rule (TR) to the second and third inequalities and the rule ($T \wedge \perp$) to the last inequality, and by exchanging the position of the second and third inequalities, we get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \mathbf{j}_n, \forall \mathbf{C}_n \left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \langle \in \rangle \mathbf{C}_n \leq y_{n+1} \vee t_n \\ \mathbf{j}_n \leq y_{n+1} \vee t_n \\ y_{n+1} \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n \leq \kappa(\mathbf{j}_n) \\ s_n \leq \kappa(\mathbf{j}_n) \\ \mathbf{j}_{n+1} \wedge \langle \triangleleft \rangle [\exists](y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee s_n) \leq \perp \end{array} \right) \Rightarrow \text{false} .$$

By applying the rule (*MT*) to the fourth and fifth inequalities, and since by definition $t_n = x_n \wedge t'_n$ and $s_n = x_n \wedge s'_n$, the quasi-inequality above is equivalent to the quasi-inequality below:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \mathbf{j}_n, \forall \mathbf{C}_n$$

$$\left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \langle \in \rangle \mathbf{C}_n \leq y_{n+1} \vee (x_n \wedge t'_n) \\ \mathbf{j}_n \leq x_n \wedge t'_n \\ y_{n+1} \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n \leq \kappa(\mathbf{j}_n) \\ x_n \wedge s'_n \leq \kappa(\mathbf{j}_n) \\ \mathbf{j}_{n+1} \wedge \langle \triangleleft \rangle [\exists](y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee (x_n \wedge s'_n)) \leq \perp \end{array} \right) \Rightarrow \text{false} \right).$$

We can apply now the rules (*D* \vee \wedge) and (*SP* \wedge) to the third inequality, and the rule (*SP* \wedge) on the fourth inequality, and get:

$$\forall \bar{x}, \forall y_{n+1}, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \mathbf{j}_n, \forall \mathbf{C}_n$$

$$\left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \langle \in \rangle \mathbf{C}_n \leq y_{n+1} \vee x_n \\ \langle \in \rangle \mathbf{C}_n \leq y_{n+1} \vee t'_n \\ \mathbf{j}_n \leq x_n \\ \mathbf{j}_n \leq t'_n \\ y_{n+1} \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n \leq \kappa(\mathbf{j}_n) \\ x_n \wedge s'_n \leq \kappa(\mathbf{j}_n) \\ \mathbf{j}_{n+1} \wedge \langle \triangleleft \rangle [\exists](y_{n+1} \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge x_n) \vee (x_n \wedge s'_n)) \leq \perp \end{array} \right) \Rightarrow \text{false} \right).$$

By lemma A.1 with the following instantiations

$$t := t'_n, \quad s := s'_n, \quad \mathbf{j} := \mathbf{j}_{n+1}, \quad \mathbf{k} := \mathbf{j}_n, \quad \mathbf{C} := \mathbf{C}_n, \quad x := x_n,$$

the quasi-inequality above is equivalent to the following quasi-inequality:

$$\forall x_{n-1}, \dots, x_0, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \forall \mathbf{C}_n \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq t'_n \\ cl(\langle \in \rangle \mathbf{C}_n \setminus \mathbf{j}_n) \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \langle \leq_J \rangle \mathbf{j}_n \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_n \wedge s'_n \leq \kappa(\mathbf{j}_n) \\ \mathbf{j}_{n+1} \wedge \langle \triangleleft \rangle [\exists](cl(\langle \in \rangle \mathbf{C}_n \setminus \mathbf{j}_n) \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \langle \leq_J \rangle \mathbf{j}_n) \vee (\langle \leq_J \rangle \mathbf{j}_n \wedge s'_n)) \leq \perp \end{array} \right) \Rightarrow \text{false} \right).$$

where cl abbreviates the composition $\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle$. By applying the rule *MinCov2* bottom to top, the quasi-inequality above can be equivalently rewritten as follows

$$\forall x_{n-1}, \dots, x_0, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \forall \mathbf{C}_n \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq t'_n \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \langle \leq_J \rangle \mathbf{j}_n \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_n \wedge s'_n \leq \kappa(\mathbf{j}_n) \\ \mathbf{j}_{n+1} \wedge \langle \triangleleft \rangle [\exists](cl(\langle \in \rangle \mathbf{C}_n \setminus \mathbf{j}_n) \vee (\langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \langle \leq_J \rangle \mathbf{j}_n) \vee (\langle \leq_J \rangle \mathbf{j}_n \wedge s'_n)) \leq \perp \end{array} \right) \Rightarrow \text{false} \right).$$

By applying Lemma 1.30, we get:

$$\forall x_{n-1}, \dots, x_0, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \forall \mathbf{C}_n \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq t'_n \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \langle \leq_J \rangle \mathbf{j}_n \leq \kappa(\mathbf{j}_n) \\ \langle \leq_J \rangle \mathbf{j}_n \wedge s'_n \leq \kappa(\mathbf{j}_n) \end{array} \right) \Rightarrow \text{false} \right).$$

By applying (*AtomR_{XX}*) and (*AtCoat1*) to the fourth inequality, and (*AtomR_{XX}*) to the last inequality, we get:

$$\forall x_{n-1}, \dots, x_0, \forall \bar{y}, \forall \mathbf{j}_{n+1}, \forall \mathbf{C}_n \left(\left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \mathbf{j}_n \leq \perp \\ \mathbf{j}_n \leq t'_n \\ s'_n \leq \kappa(\mathbf{j}_n) \end{array} \right) \Rightarrow \text{false} \right).$$

Notice that the system above consists of a set of pure inequalities and a set of inequalities of the exact shape to which the induction hypothesis applies. Since a run of ALBA does not depend

on the presence of side pure inequalities, the induction hypothesis implies that ALBA^l succeeds on the system above, and outputs the pure quasi-inequality below, as required:

$$\forall \mathbf{j}_{n+1}, \mathbf{j}_n, \dots, \mathbf{j}_0, \forall \mathbf{C}_n, \mathbf{C}_{n-1}, \dots, \mathbf{C}_0 \left(\begin{array}{l} \mathbf{j}_{n+1} \leq \langle \triangleleft \rangle \mathbf{C}_n \\ \mathbf{j}_n \leq \langle \in \rangle \mathbf{C}_n \\ \langle \leq_J \rangle \mathbf{j}_{n+1} \wedge \mathbf{j}_n \leq \perp \\ \mathbf{j}_n \leq \langle \triangleleft \rangle \mathbf{C}_{n-1} \\ \mathbf{j}_{n-1} \leq \langle \in \rangle \mathbf{C}_{n-1} \\ \langle \leq_J \rangle \mathbf{j}_n \wedge \mathbf{j}_{n-1} \leq \perp \\ \dots \\ \mathbf{j}_1 \leq \langle \triangleleft \rangle \mathbf{C}_0 \\ \mathbf{j}_0 \leq \langle \in \rangle \mathbf{C}_0 \\ \langle \leq_J \rangle \mathbf{j}_1 \wedge \mathbf{j}_0 \leq \perp \end{array} \right) \Rightarrow \text{false} .$$

□

The lemma below proves the soundness of an Ackermann-type rule for the elimination of non-elementary variables which however cannot be explained in terms of Ackermann principles.

Lemma A.1. *Let t and s be monotone \mathcal{L}^+ -terms such that $x, y \notin \text{Var}(t)$. For every closed model $\mathbb{M} = (\mathbb{E}_L, \nu)$ such that $\mathbb{E}_L = (J(L), \mathcal{P}J(L), \triangleleft, \exists, \leq_J)$ is the enriched two-sorted frame associated with some finite lattice L (cf. Definition 1.23),*

$$\mathbb{M} \Vdash (S1) \quad \text{iff} \quad \mathbb{M} \Vdash (S2),$$

where

$$(S1) := \exists x \exists y \exists \mathbf{j} \exists \mathbf{k} \exists \mathbf{C} \left(\begin{array}{l} \mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C} \\ \mathbf{k} \leq \langle \in \rangle \mathbf{C} \\ \langle \in \rangle \mathbf{C} \leq y \vee x \\ \langle \in \rangle \mathbf{C} \leq y \vee t \\ \mathbf{k} \leq x \\ \mathbf{k} \leq t \\ y \leq \kappa(\mathbf{k}) \\ \langle \leq_J \rangle \mathbf{j} \wedge x \leq \kappa(\mathbf{k}) \\ x \wedge s \leq \kappa(\mathbf{k}) \\ \mathbf{j} \wedge \langle \triangleleft \rangle [\exists](y \vee (\langle \leq_J \rangle \mathbf{j} \wedge x) \vee (x \wedge s)) \leq \perp \end{array} \right),$$

$$(S2) := \exists j \exists k \exists C \left(\begin{array}{l} \mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C} \\ \mathbf{k} \leq \langle \in \rangle \mathbf{C} \\ \mathbf{k} \leq t \\ cl(\langle \in \rangle \mathbf{C} \setminus \mathbf{k}) \leq \kappa(\mathbf{k}) \\ \langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k} \leq \kappa(\mathbf{k}) \\ \langle \leq_J \rangle \mathbf{k} \wedge s \leq \kappa(\mathbf{k}) \\ \mathbf{j} \wedge \langle \triangleleft \rangle [\exists](cl(\langle \in \rangle \mathbf{C} \setminus \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{j} \wedge \langle \leq_J \rangle \mathbf{k}) \vee (\langle \leq_J \rangle \mathbf{k} \wedge s)) \leq \perp \end{array} \right),$$

and $cl(\phi)$ denotes $\langle \triangleleft \rangle [\exists] \langle \leq_J \rangle \phi$.

Proof. Assume that the conjunction of the inequalities in (S1) holds under v . Let v' be the (x, y) -variant of v such that $v'(x) = \langle \leq_J \rangle v(\mathbf{k})$ and $v'(y) = cl(v(\langle \in \rangle \mathbf{C} \setminus \mathbf{k}))$. Since the assignment v is closed, for any $z \in AtProp \setminus \{x, y\}$, the set $v'(z) = v(z)$ is closed. By definition, $v'(y)$ is closed, and $v'(x)$ is closed because for any finite lattice and any $k \in J(L)$, the downset $\downarrow_{\leq_J} k$ is a closed set (cf. Lemma 1.8). Thus v' is a closed assignment. In addition, $v'(x) \subseteq v(x)$. Indeed, the assumption that $\mathbf{k} \leq x$ holds under v and $v(x)$ being closed, hence a downset, imply that $\downarrow_{\leq_J} v(\mathbf{k}) \subseteq v(x)$, hence we have:

$$v'(x) = \langle \leq_J \rangle v(\mathbf{k}) = \downarrow_{\leq_J} v(\mathbf{k}) \subseteq v(x).$$

The first, second and third inequalities in (S2) hold under v' since they do not contain the variables x and y and coincide with the first, second and sixth inequalities in (S1), which by assumption hold under v . The satisfaction of the fifth and sixth inequalities in (S2) under v' is implied by monotonicity, since the eighth and ninth inequalities in (S1) are satisfied under v , and since $v'(x) \subseteq v(x)$. It remains to show that the fourth and seventh inequalities in (S2) hold under v' . Let $j, k \in J(L)$ and $C \subseteq J(L)$ such that $v(\mathbf{j}) = \{j\}$, $v(\mathbf{k}) = \{k\}$ and $v(\mathbf{C}) = \{C\}$. The assumption that $\mathbf{j} \leq \langle \triangleleft \rangle \mathbf{C}$ and $\mathbf{k} \leq \langle \in \rangle \mathbf{C}$ hold under v imply that $C \in \mathcal{M}(j)$ and $k \in C$. Hence, $\langle \in \rangle \mathbf{C} = C$. By Lemma 1.9.2, $k \notin \overline{\downarrow_{\leq_J}(C \setminus k)}$. Hence,

$$v'(cl(\langle \in \rangle \mathbf{C} \setminus \mathbf{k})) = \overline{\downarrow_{\leq_J}(C \setminus k)} \subseteq J(L) \setminus k = v'(\kappa(\mathbf{k})).$$

Thus the fourth inequality in (S2) holds under v' . As to the last inequality, it follows directly from the satisfaction of the previous inequalities under v' and Lemma 1.30.

Let us prove the converse implication. Assume that the conjunction of the inequalities in (S2) holds under v . Let v' be the (x, y) -variant of v such that $v'(x) := \langle \leq_J \rangle v(\mathbf{k})$ and $v'(y) := cl(\langle \in \rangle v(\mathbf{C}) \setminus v(\mathbf{k}))$. The first, second and sixth inequalities in (S1) hold under v' since they do not contain the variables x and y and coincide with the first, second and third inequalities in (S2), which by assumption hold under v . Since $v'(x) = v'(\langle \leq_J \rangle \mathbf{k}) = \downarrow_{\leq_J} v(\mathbf{k})$, the fifth inequality is

satisfied under v' . The satisfaction under v' of the eighth, ninth and tenth inequalities in (S1) immediately follows from the satisfaction of the fifth, sixth and seventh inequalities in (S2) respectively and the definition of v' .

It remains to be shown that the third, fourth and seventh inequalities in (S1) hold under v' . Let $j, k \in J(L)$ and $C \subseteq J(L)$ such that $v(\mathbf{j}) = \{j\}$, $v(\mathbf{k}) = \{k\}$ and $v(\mathbf{C}) = \{C\}$. The satisfaction of the first and second inequalities in (S1) under v' imply that $C \in \mathcal{M}(j)$ and $k \in C$, which imply by Lemma 1.9.2, that $k \notin v'(y)$. This implies that the seventh inequality in (S1) is satisfied under v' . By definition of v' and of the closure,

$$v'(\langle \epsilon \rangle \mathbf{C} \setminus \mathbf{k}) \subseteq cl(v'(\langle \epsilon \rangle \mathbf{C} \setminus \mathbf{k})) = v'(y).$$

In addition, by the satisfaction of the fifth and sixth inequalities in (S1) under v' , we have that $k \in v'(x)$ and $k \in v'(t)$. Hence

$$\langle \epsilon \rangle v'(\mathbf{C}) = (\langle \epsilon \rangle v'(\mathbf{C}) \setminus k) \cup \{k\} \subseteq v'(y) \cup v'(x)$$

and

$$\langle \epsilon \rangle v'(\mathbf{C}) = (\langle \epsilon \rangle v'(\mathbf{C}) \setminus k) \cup \{k\} \subseteq v'(y) \cup v'(t).$$

This finishes the proof that the third and fourth inequalities (S1) hold under the closed assignment v' . \square

Appendix B

The Cut Elimination Metatheorems

In the present appendix, we summarize the different cut elimination metatheorems mentioned and/or proved in the thesis. We first recall the definitions of the different conditions given by these theorems, then we give a table summarizing which theorem requires which conditions.

The different conditions are listed below:

C₁: Preservation of formulas. Each formula occurring in a premise of a given inference is the subformula of some formula in the conclusion of that inference.

C₁^{MT}: preservation of operational terms. Each operational term occurring in a premise of an inference rule *inf* is a subterm of some operational term in the conclusion of *inf*.

C₂: Shape-likeness of parameters. Congruent parameters are occurrences of the same structure.

C'₂: Type-likeness of parameters. Congruent parameters have exactly the same type.

C₃: Non-proliferation of parameters. Each parameter in an inference rule *inf* is congruent to at most one constituent in the conclusion of *inf*.

C₃: Restricted non-proliferation of parameters. Each parameter in an inference rule *inf* is congruent to at most one constituent in the conclusion of *inf*. This restriction does not need to apply to parameters of any type T such that the only applications of cut with cut terms of type T are of the following shapes:

$$\frac{\begin{array}{c} \vdots \\ X \vdash a \end{array} \quad a \vdash a}{X \vdash a} \quad \frac{a \vdash a \quad \begin{array}{c} \vdots \\ a \vdash Y \end{array}}{a \vdash Y}$$

C₄: Position-alikeness of parameters. Congruent parameters are either all antecedent or all succedent parts of their respective sequents.

C₅: Display of principal constituents. Any principal occurrence is always either the entire antecedent or the entire consequent part of the sequent in which it occurs.

C₅' : Quasi-display of principal constituents. If a formula *A* is principal in the conclusion sequent *s* of a derivation π , then *A* is in display, unless π consists only of its conclusion sequent *s* (i.e. *s* is an axiom).

C₅'': Display-invariance of axioms. If a display rule can be applied to an axiom *s*, the result of that rule application is again an axiom.

C₅*: Display of active constituents. If a formula *A* is active in the application of any rule, then *A* is in display.

C₆: Closure under substitution for succedent parts. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas occurring in succedent position.

C₆' : Closure under substitution for succedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in succedent position, *within each type*.

C₇: Closure under substitution for precedent parts. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas occurring in precedent position.

C'₇: Closure under substitution for precedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in precedent position, *within each type*.

C₈: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C₈ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of cut involving proper subformulas of the original cut-formulas.

C'₈: Eliminability of matching principal constituents. This condition goes exactly as the condition C₈, but in addition, specific to the multi-type setting is the requirement that the new application(s) of the cut rule be also *strongly type-uniform* as required by condition C₁₀ (or *type-uniform* as required by condition C'₁₀).

C''₈: Closure of axioms under cut. If $X \vdash A$ and $A \vdash Y$ are axioms, then $X \vdash Y$ is again an axiom.

C'''₈: Closure of axioms under surgical cut. If $(x \vdash y)([a]^{pre}, [a]^{suc})$, $a \vdash z[a]^{suc}$ and $v[a]^{pre} \vdash a$ are axioms, then $(x \vdash y)([a]^{pre}, [z/a]^{suc})$ and $(x \vdash y)([v/a]^{pre}, [a]^{suc})$ are again axioms.

C₉: Type-uniformity of derivable sequents. Each derivable sequent is type-uniform.

C₁₀: Strong type-uniformity of cut rules. All cut rules are strongly type-uniform (cf. Definition 3.2).

C'₁₀: Preservation of type-uniformity of cut rules. All cut rules preserve type-uniformity (cf. Definition 3.1).

In Table B.1 (page 229), we summarize the different cut elimination metatheorems. Each column corresponds to one theorem, as indicated in the following table.

P	Proper Display Calculus	see [Wan98]
Q	Quasi-Proper Display Calculus	Theorem 2.4
QM	Quasi-Proper Multi-Type Display Calculus	Theorem 3.4
QMDT	Quasi-Proper Multi-Type Display-Type Calculus	Theorem 4.1
PDT	Proper Display-Type Calculus	Theorem 5.3
PM	Proper Multi-Type Display Calculus	Theorem 6.2

Notice that in Table B.1, the check marks refer to the conditions as they are stated in the corresponding theorem and not to the conditions which might be implied by the assumptions of the theorem. Hence, for instance, condition C_3 implies condition C'_3 , however, there is only one check mark in the row corresponding to C'_3 .

	P	Q EAK	QM EAK	QMDT	PDT MML	PM PDL
C_1 : Preservation of formulas.	✓	✓			✓	
C_1^{MT} : Preservation of operational terms.			✓	✓		✓
C_2 : Shape-alikeness of parameters.	✓	✓	✓	✓	✓	✓
C'_2 : Type-alikeness of parameters.			✓	✓		✓
C_3 : Non-proliferation of parameters.	✓	✓		✓	✓	✓
C'_3 : Restricted non-proliferation of parameters.			✓			
C_4 : Position-alikeness of parameters.	✓	✓	✓	✓	✓	✓
C_5 : Display of principal constituents.	✓					✓
C'_5 : Quasi-display of principal constituents.		✓	✓	✓		
C''_5 : Display-invariance of axioms.		✓	✓	✓		
C^*_5 : Display of active constituents.					✓	
C_6 : Closure under substitution for succedent parts.	✓	✓			✓	
C'_6 : Closure under substitution for succedent parts within each type.			✓	✓		✓
C_7 : Closure under substitution for precedent parts.	✓	✓			✓	
C'_7 : Closure under substitution for precedent parts within each type.			✓	✓		✓
C_8 : Eliminability of matching principal constituents.	✓	✓			✓	
C'_8 : Eliminability of matching principal constituents.			✓	✓		✓
C''_8 : Closure of axioms under cut.		✓	✓			
C'''_8 : Closure of axioms under surgical cut.				✓		
C_9 : Type-uniformity of derivable sequents.			✓	✓		✓
C_{10} : Strong type-uniformity of cut rules.			✓			✓
C'_{10} : Preservation of type-uniformity of cut rules.				✓		

TABLE B.1: The cut elimination metatheorems

Appendix C

Special Rules in D'.EAK

C.1 Derived rules in D'.EAK

In the presence of the display postulates, the *conj*-rules are interderivable with the Fisher Servi rules. Indeed, let us show that the following rules

$$\text{conj} \frac{\{\alpha\}(X; \overline{\alpha} Y) \vdash Z}{\{\alpha\}X; Y \vdash Z} \quad \frac{Y \vdash \overline{\alpha} X > \overline{\alpha} Z}{Y \vdash \overline{\alpha}(X > Z)} \text{FS}$$

are interderivable:¹

$$\frac{\frac{\frac{\{\alpha\}(X; \overline{\alpha} Y) \vdash Z}{X; \overline{\alpha} Y \vdash \overline{\alpha} Z}}{\overline{\alpha} Y; X \vdash \overline{\alpha} Z}}{X \vdash \overline{\alpha} Y > \overline{\alpha} Z} \text{FS}}{X \vdash \overline{\alpha}(Y > Z)} \text{FS}}{\frac{\{\alpha\}X \vdash Y > Z}{Y; \{\alpha\}X \vdash Z}}{\{\alpha\}X; Y \vdash Z}} \quad \text{conj} \frac{\frac{\frac{Y \vdash \overline{\alpha} X > \overline{\alpha} Z}{\overline{\alpha} X; Y \vdash \overline{\alpha} Z}}{Y; \overline{\alpha} X \vdash \overline{\alpha} Z}}{\{\alpha\}(Y; \overline{\alpha} X) \vdash Z}}{\{\alpha\}Y; X \vdash Z}}{X; \{\alpha\}Y \vdash Z}}{\frac{\{\alpha\}Y \vdash X > Z}{Y \vdash \overline{\alpha}(X > Z)}}$$

Analogous derivations show that the pairs of rules in each row of the table below are interderivable:

¹Note that we are using exchange, but this rule is not required if we add the corresponding *Fisher-Servi* rule for the right-residuum of ‘;’ and the obvious *conjugation* rule with ‘X; {α}Y’ in a reversed order.

$$\begin{array}{c}
\text{conj} \frac{\widehat{\alpha}(X; \{\alpha\}Y) \vdash Z}{\widehat{\alpha}X; Y \vdash Z} \qquad \text{FS} \frac{Y \vdash \{\alpha\}X > \{\alpha\}Z}{Y \vdash \{\alpha\}(X > Z)} \\
\\
\text{conj} \frac{X \vdash \{\alpha\}(Y; \widehat{\alpha}Z)}{X \vdash \{\alpha\}Y; Z} \qquad \text{FS} \frac{\widehat{\alpha}Y > \widehat{\alpha}X \vdash Z}{\widehat{\alpha}(Y > X) \vdash Z} \\
\\
\text{conj} \frac{X \vdash \widehat{\alpha}(Y; \{\alpha\}Z)}{X \vdash \widehat{\alpha}Y; Z} \qquad \text{FS} \frac{\{\alpha\}Y > \{\alpha\}Z \vdash X}{\{\alpha\}(Y > Z) \vdash X}
\end{array}$$

Let us show that the rules “with side conditions” in D.EAK (cf. Section 2.4.4) can be derived from their corresponding rules in D'.EAK and the remaining part of the calculus.

An important benefit of the revised system is that the operational rules *reverse* (or more precisely their rewritings in the new notation), which were primitive in the old system, are now derivable using the new rules for Φ_α and 1_α and the new *reduce*. This supports our intuition that the rules *reverse* do not participate in the proof-theoretic meaning of the connectives $\langle \alpha \rangle$ and $[\alpha]$.

$$\begin{array}{c}
\frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{1_\alpha; \{\alpha\}A \vdash X}{1_\alpha \vdash X < \{\alpha\}A}}{\Phi_\alpha \vdash X < \{\alpha\}A} \\
\frac{\Phi_\alpha; \{\alpha\}A \vdash X}{\{\alpha\}A \vdash X} \\
\frac{A \vdash \widehat{\alpha}X}{[\alpha]A \vdash \{\alpha\}\widehat{\alpha}X} \\
\frac{[\alpha]A \vdash \{\alpha\}\widehat{\alpha}X}{[\alpha]A \vdash \Phi_\alpha > X} \text{ comp} \\
\frac{\Phi_\alpha; [\alpha]A \vdash X}{[\alpha]A; \Phi_\alpha \vdash X} \\
\frac{\Phi_\alpha \vdash [\alpha]A > X}{1_\alpha \vdash [\alpha]A > X} \\
\frac{[\alpha]A; 1_\alpha \vdash X}{1_\alpha; [\alpha]A \vdash X} \\
\\
\frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{X \vdash 1_\alpha > \{\alpha\}A}{1_\alpha; X \vdash \{\alpha\}A}}{\Phi_\alpha \vdash \{\alpha\}A < X} \\
\frac{\Phi_\alpha; X \vdash \{\alpha\}A}{X \vdash \Phi_\alpha > \{\alpha\}A} \text{ reduce}' \\
\frac{X \vdash \{\alpha\}A}{\widehat{\alpha}X \vdash A} \\
\frac{\widehat{\alpha}X \vdash A}{\{\alpha\}\widehat{\alpha}X \vdash \langle \alpha \rangle A} \\
\frac{\{\alpha\}\widehat{\alpha}X \vdash \langle \alpha \rangle A}{\Phi_\alpha; X \vdash \langle \alpha \rangle A} \text{ comp} \\
\frac{\Phi_\alpha; X \vdash \langle \alpha \rangle A}{X; \Phi_\alpha \vdash \langle \alpha \rangle A} \\
\frac{X; \Phi_\alpha \vdash \langle \alpha \rangle A}{\Phi_\alpha \vdash X > \langle \alpha \rangle A} \\
\frac{\Phi_\alpha \vdash X > \langle \alpha \rangle A}{1_\alpha \vdash X > \langle \alpha \rangle A} \\
\frac{1_\alpha \vdash X > \langle \alpha \rangle A}{X; 1_\alpha \vdash \langle \alpha \rangle A} \\
\frac{X; 1_\alpha \vdash \langle \alpha \rangle A}{1_\alpha; X \vdash \langle \alpha \rangle A} \\
\frac{1_\alpha; X \vdash \langle \alpha \rangle A}{X \vdash 1_\alpha > \langle \alpha \rangle A}
\end{array}$$

The old rules *reduce* are derivable as follows.

$$\begin{array}{c}
\text{reduce}' \frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{1_\alpha; \{\alpha\}A \vdash X}{1_\alpha \vdash X < \{\alpha\}A}}{\Phi_\alpha; \{\alpha\}A \vdash X} \\
\frac{\Phi_\alpha; \{\alpha\}A \vdash X}{\{\alpha\}A \vdash X} \\
\\
\frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{X \vdash 1_\alpha > \{\alpha\}A}{1_\alpha; X \vdash \{\alpha\}A}}{\Phi_\alpha \vdash X > \{\alpha\}A} \\
\frac{\Phi_\alpha; X \vdash \{\alpha\}A}{X; \Phi_\alpha \vdash \{\alpha\}A} \\
\frac{X; \Phi_\alpha \vdash \{\alpha\}A}{\Phi_\alpha; X \vdash \{\alpha\}A} \\
\frac{\Phi_\alpha; X \vdash \{\alpha\}A}{X \vdash \Phi_\alpha > \{\alpha\}A} \text{ reduce}' \\
\frac{X \vdash \Phi_\alpha > \{\alpha\}A}{X \vdash \{\alpha\}A}
\end{array}$$

The old *swap-in* rules are derivable in the revised calculus from the new *swap-in* rules as follows.

$$\begin{array}{c}
\frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{1_\alpha; \{\alpha\}\{a\}X \vdash Y}{1_\alpha \vdash Y < \{\alpha\}\{a\}X}}{\Phi_\alpha \vdash Y < \{\alpha\}\{a\}X} \\
\text{reduce}' \frac{\Phi_\alpha; \{\alpha\}\{a\}X \vdash Y}{\{\alpha\}\{a\}X \vdash Y} \\
\text{swap-in}' \frac{\Phi_\alpha; \{\alpha\}\{a\}X \vdash Y}{\Phi_\alpha; \{a\}\{\beta\}_{\alpha\beta} X \vdash Y} \\
\frac{\Phi_\alpha \vdash Y < \{a\}\{\beta\}_{\alpha\beta} X}{1_\alpha \vdash Y < \{a\}\{\beta\}_{\alpha\beta} X} \\
\frac{1_\alpha \vdash Y < \{a\}\{\beta\}_{\alpha\beta} X}{1_\alpha; \{a\}\{\beta\}_{\alpha\beta} X \vdash Y}
\end{array}
\qquad
\begin{array}{c}
\frac{Y \vdash 1_\alpha > \{\alpha\}\{a\}X}{1_\alpha; Y \vdash \{\alpha\}\{a\}X} \\
\frac{\Phi_\alpha \vdash 1_\alpha \quad 1_\alpha \vdash \{\alpha\}\{a\}X < Y}{\Phi_\alpha \vdash \{\alpha\}\{a\}X < Y} \\
\frac{\Phi_\alpha; Y \vdash \{\alpha\}\{a\}X}{Y \vdash \Phi_\alpha > \{\alpha\}\{a\}X} \text{reduce}' \\
\frac{Y \vdash \{\alpha\}\{a\}X}{Y \vdash \Phi_\alpha > \{a\}\{\beta\}_{\alpha\beta} X} \text{swap-in}'
\end{array}$$

The old *swap-out* rules (translated into D'.EAK) are derivable using the new *swap-out* rules:

$$\begin{array}{c}
\frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{1_\alpha; \{a\}\{\beta_1\} X \vdash Y \mid \alpha a \beta_1}{1_\alpha \vdash Y < \{a\}\{\beta_1\} X \mid \alpha a \beta_1} \quad \dots \quad \frac{1_\alpha; \{a\}\{\beta_n\} X \vdash Y \mid \alpha a \beta_n}{1_\alpha \vdash Y < \{a\}\{\beta_n\} X \mid \alpha a \beta_n}}{\Phi_\alpha \vdash Y < \{a\}\{\beta_1\} X \mid \alpha a \beta_1} \quad \dots \quad \frac{\Phi_\alpha \vdash 1_\alpha \quad 1_\alpha \vdash Y < \{a\}\{\beta_n\} X \mid \alpha a \beta_n}{\Phi_\alpha \vdash \{a\}\{\beta_n\} X \mid \alpha a \beta_n} \\
\text{reduce}' \frac{\Phi_\alpha; \{a\}\{\beta_1\} X \vdash Y \mid \alpha a \beta_1}{\{a\}\{\beta_1\} X \vdash Y \mid \alpha a \beta_1} \quad \dots \quad \frac{\Phi_\alpha; \{a\}\{\beta_n\} X \vdash Y \mid \alpha a \beta_n}{\{a\}\{\beta_n\} X \vdash Y \mid \alpha a \beta_n} \text{reduce}' \\
\frac{\{a\}\{a\}X \vdash ;(Y \mid \alpha a \beta)}{1_\alpha \vdash \{\alpha\}\{a\}X > ;(Y \mid \alpha a \beta)} \\
\frac{\{a\}\{a\}X; 1_\alpha \vdash ;(Y \mid \alpha a \beta)}{1_\alpha; \{a\}\{a\}X \vdash ;(Y \mid \alpha a \beta)}
\end{array}$$

$$\begin{array}{c}
\frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{Y \vdash 1_\alpha > \{a\}\{\beta_1\} X \mid \alpha a \beta_1}{1_\alpha; Y \vdash \{a\}\{\beta_1\} X \mid \alpha a \beta_1} \quad \dots \quad \frac{Y \vdash 1_\alpha > \{a\}\{\beta_n\} X \mid \alpha a \beta_n}{1_\alpha; Y \vdash \{a\}\{\beta_n\} X \mid \alpha a \beta_n}}{\Phi_\alpha \vdash 1_\alpha \quad 1_\alpha \vdash \{a\}\{\beta_1\} X \mid \alpha a \beta_1 < Y} \quad \dots \quad \frac{\Phi_\alpha \vdash 1_\alpha \quad 1_\alpha \vdash \{a\}\{\beta_n\} X \mid \alpha a \beta_n < Y}{\Phi_\alpha \vdash \{a\}\{\beta_n\} X \mid \alpha a \beta_n < Y} \\
\frac{\Phi_\alpha; Y \vdash \{a\}\{\beta_1\} X \mid \alpha a \beta_1}{Y \vdash \Phi_\alpha > \{a\}\{\beta_1\} X \mid \alpha a \beta_1} \quad \dots \quad \frac{\Phi_\alpha; Y \vdash \{a\}\{\beta_n\} X \mid \alpha a \beta_n}{Y \vdash \Phi_\alpha > \{a\}\{\beta_n\} X \mid \alpha a \beta_n} \text{reduce}' \\
\text{reduce}' \frac{Y \vdash \{a\}\{\beta_1\} X \mid \alpha a \beta_1}{; (Y \mid \alpha a \beta) \vdash \{a\}\{a\}X} \\
\frac{; (Y \mid \alpha a \beta) \vdash \{a\}\{a\}X}{1_\alpha \vdash ; (Y \mid \alpha a \beta) > \{a\}\{a\}X} \\
\frac{; (Y \mid \alpha a \beta); 1_\alpha \vdash \{a\}\{a\}X}{; (Y \mid \alpha a \beta); 1_\alpha \vdash \{\alpha\}\{a\}X} \\
\frac{; (Y \mid \alpha a \beta); 1_\alpha \vdash \{\alpha\}\{a\}X}{; (Y \mid \alpha a \beta) \vdash 1_\alpha > \{\alpha\}\{a\}X}
\end{array}$$

C.2 Soundness of *comp* rules in the final coalgebra

We address the reader to Section 2.5 and [GKP13] for details on the final coalgebra semantics for dynamic epistemic logic. To prove the soundness of the *comp* rules in the final coalgebra it suffices to check that for every formula A ,

$$[\alpha][\alpha^{-1}][A]_{\mathbb{Z}} \subseteq [\text{Pre}(\alpha) \rightarrow A]_{\mathbb{Z}} \text{ and } [\text{Pre}(\alpha); A]_{\mathbb{Z}} \subseteq \langle \alpha \rangle \langle \alpha^{-1} \rangle [A]_{\mathbb{Z}}.$$

We will make use of the following general fact:

Fact C.1. Let R be a binary relation on a set X and let R^{-1} be its converse. Then,

$$[\text{Dom}(R) \times \text{Dom}(R)] \cap \Delta_X \subseteq R; R^{-1},$$

where $\text{Dom}(R) = \{x \in X \mid xRy \text{ for some } y \in X\}$, and $\Delta_X = \{(x, x) \mid x \in X\}$.

Proof. Straightforward. □

Fact C.2. The following *comp* rules:

$$\frac{Y \vdash \{\alpha\} \widetilde{\alpha} X}{Y \vdash \text{Pre}(\alpha) > X} \quad \frac{\{\alpha\} \widetilde{\alpha} X \vdash Y}{\text{Pre}(\alpha); X \vdash Y}$$

are sound in the final coalgebra.

Proof.

$$\begin{aligned} \langle \alpha \rangle \langle \alpha^{-1} \rangle [A]_{\mathbb{Z}} &= \alpha^{-1}[\alpha[[A]_{\mathbb{Z}}]] \\ &= (\alpha; \alpha^{-1})[[A]_{\mathbb{Z}}] \\ &\supseteq S[[A]_{\mathbb{Z}}] \quad \text{Fact C.1} \\ &= \text{Dom}(\alpha) \cap [A]_{\mathbb{Z}} \\ &= [\text{Pre}(\alpha); A]_{\mathbb{Z}}, \end{aligned}$$

$$\begin{aligned} [\alpha][\alpha^{-1}][A]_{\mathbb{Z}} &= (\alpha^{-1}[(\alpha^{-1}[[A]_{\mathbb{Z}}])^c])^c \\ &= (\alpha^{-1}[\alpha[[A]_{\mathbb{Z}}^c]])^c \\ &= ((\alpha; \alpha^{-1})[[A]_{\mathbb{Z}}^c])^c \\ &\subseteq (S[[A]_{\mathbb{Z}}^c])^c \quad \text{Fact C.1} \\ &= (\text{Dom}(\alpha) \cap [A]_{\mathbb{Z}}^c)^c \\ &= \text{Dom}(\alpha)^c \cup [A]_{\mathbb{Z}} \\ &= [\text{Pre}(\alpha) \rightarrow A]_{\mathbb{Z}}, \end{aligned}$$

where $S = [\text{Dom}(R) \times \text{Dom}(R)] \cap \Delta_X$. □

Appendix D

Cut Elimination for D'.EAK

In the present section, we report on the remaining cases for the verification of condition C_8 for D'.EAK; these cases are needed already for the cut elimination á la Gentzen for D.EAK, but do not appear in [GKP13].

First we consider the *atom* rule (see page 105).

$$\frac{\Gamma p \vdash p \quad p \vdash \Delta p}{\Gamma p \vdash \Delta p} \rightsquigarrow \Gamma p \vdash \Delta p$$

Now we treat the introductions of the connectives of the propositional base (we also treat here the cases relative to the two additional arrows \leftarrow and \succ added to our presentation of D.EAK):

$$\frac{\frac{\frac{\vdots \pi}{I \vdash \top} \quad \frac{I \vdash X}{\top \vdash X}}{I \vdash X}}{\vdots \pi} \rightsquigarrow I \vdash X$$

$$\frac{\frac{\frac{\vdots \pi}{X \vdash \perp} \quad \perp \vdash I}{X \vdash I}}{\vdots \pi} \rightsquigarrow X \vdash I$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_1 \quad \vdots \pi_2 \quad \vdots \pi_3 \\
\frac{X \vdash A \quad Y \vdash B}{X; Y \vdash A \wedge B} \quad \frac{A; B \vdash Z}{A \wedge B \vdash Z} \\
\hline
X; Y \vdash Z
\end{array}
\rightsquigarrow
\begin{array}{c}
\vdots \pi_3 \\
\frac{\frac{\frac{\frac{\vdots \pi_1}{X \vdash A} \quad \frac{A; B \vdash Z}{B; A \vdash Z}}{A \vdash B > Z}}{X \vdash B > Z}}{B; X \vdash Z} \\
\frac{Y \vdash B}{X; B \vdash Z} \\
\frac{Y \vdash X > Z}{B \vdash X > Z} \\
\hline
X; Y \vdash Z
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_3 \quad \vdots \pi_1 \quad \vdots \pi_2 \\
\frac{Z \vdash B; A}{Z \vdash B \vee A} \quad \frac{B \vdash Y \quad A \vdash X}{B \vee A \vdash Y; X} \\
\hline
Z \vdash Y; X
\end{array}
\rightsquigarrow
\begin{array}{c}
\vdots \pi_3 \\
\frac{Z \vdash B; A}{Z \vdash A; B} \quad \vdots \pi_1 \\
\frac{A > Z \vdash B}{A > Z \vdash Y} \quad \frac{B \vdash Y}{Z \vdash A; Y} \\
\frac{Z \vdash A; Y}{Z \vdash Y; A} \quad \vdots \pi_2 \\
\frac{Y > Z \vdash A}{Y > Z \vdash X} \quad \frac{A \vdash X}{Z \vdash Y; X} \\
\hline
Z \vdash Y; X
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_1 \quad \vdots \pi_2 \quad \vdots \pi_3 \\
\frac{Y \vdash A > B}{Y \vdash A \rightarrow B} \quad \frac{X \vdash A \quad B \vdash Z}{A \rightarrow B \vdash X > Z} \\
\hline
Y \vdash X > Z
\end{array}
\rightsquigarrow
\begin{array}{c}
\vdots \pi_1 \\
\frac{Y \vdash A > B}{A; Y \vdash B} \quad \vdots \pi_3 \\
\frac{B \vdash Z}{A; Y \vdash Z} \\
\vdots \pi_2 \\
\frac{X \vdash A}{Y; A \vdash Z} \quad \frac{A \vdash Y > Z}{A \vdash Y > Z} \\
\frac{X \vdash Y > Z}{Y; X \vdash Z} \\
\frac{X; Y \vdash Z}{Y \vdash X > Z} \\
\hline
Y \vdash X > Z
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_1 \quad \vdots \pi_2 \quad \vdots \pi_3 \\
\frac{Y \vdash B < A}{Y \vdash B \leftarrow A} \quad \frac{B \vdash Z \quad X \vdash A}{B \leftarrow A \vdash Z < X} \\
\hline
Y \vdash Z < X
\end{array}
\rightsquigarrow
\begin{array}{c}
\vdots \pi_1 \\
\frac{Y \vdash B < A}{Y; A \vdash B} \quad \vdots \pi_3 \\
\frac{B \vdash Z}{Y; A \vdash Z} \\
\vdots \pi_2 \\
\frac{X \vdash A}{A; Y \vdash Z} \quad \frac{A \vdash Z < Y}{A \vdash Z < Y} \\
\frac{X \vdash Z < Y}{X; Y \vdash Z} \\
\frac{Y; X \vdash Z}{Y \vdash Z < X} \\
\hline
Y \vdash Z < X
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_2 \quad \vdots \pi_3 \quad \vdots \pi_1 \\
\frac{A \vdash Y \quad Z \vdash B}{Y > Z \vdash A > B} \quad \frac{A > B \vdash X}{A > B \vdash X} \\
\hline
Y > Z \vdash X
\end{array} \\
\rightsquigarrow \\
\begin{array}{c}
\vdots \pi_3 \quad \vdots \pi_1 \\
\frac{Z \vdash B \quad \frac{A > B \vdash X}{B \vdash A; X}}{Z \vdash A; X} \\
\frac{Z \vdash A; X}{Z \vdash X; A} \\
\frac{X > Z \vdash A \quad A \vdash Y}{X > Z \vdash Y} \\
\frac{Z \vdash X; Y}{Z \vdash Y; X} \\
\hline
Y > Z \vdash X
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_2 \quad \vdots \pi_3 \quad \vdots \pi_1 \\
\frac{Y \vdash B \quad A \vdash Z}{Y < Z \vdash B < A} \quad \frac{B < A \vdash X}{B < A \vdash X} \\
\hline
Y < Z \vdash X
\end{array} \\
\rightsquigarrow \\
\begin{array}{c}
\vdots \pi_3 \quad \vdots \pi_1 \\
\frac{Y \vdash B \quad \frac{B < A \vdash X}{B \vdash X; A}}{Y \vdash X; A} \\
\frac{Y \vdash X; A}{Y \vdash A; X} \\
\frac{Y < X \vdash A \quad A \vdash Z}{Y < X \vdash Z} \\
\frac{Y \vdash Z; X}{Y \vdash X; Z} \\
\hline
Y < Z \vdash Y
\end{array}
\end{array}$$

Now we turn to the part of D'.EAK with static modalities. We omit the proofs for $\widehat{\langle a \rangle}$ and $\widehat{[a]}$, because they are analogous to the transformations of $\langle a \rangle$ and $[a]$.

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_1 \quad \vdots \pi_2 \\
\frac{X \vdash A \quad \{a\}A \vdash Y}{\{a\}X \vdash \langle a \rangle A} \quad \frac{\{a\}A \vdash Y}{\langle a \rangle A \vdash Y} \\
\hline
\{a\}X \vdash Y
\end{array} \\
\rightsquigarrow \\
\begin{array}{c}
\vdots \pi_2 \\
\frac{X \vdash A \quad \frac{\{a\}A \vdash Y}{A \vdash \widehat{[a]}Y}}{X \vdash \widehat{[a]}Y} \\
\hline
\{a\}X \vdash Y
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_1 \quad \vdots \pi_2 \\
\frac{X \vdash \{a\}A \quad A \vdash Y}{X \vdash [a]A} \quad \frac{A \vdash Y}{[a]A \vdash \{a\}Y} \\
\hline
X \vdash \{a\}Y
\end{array} \\
\rightsquigarrow \\
\begin{array}{c}
\vdots \pi_1 \\
\frac{X \vdash \{a\}A \quad \vdots \pi_2}{\widehat{[a]}X \vdash A} \quad \frac{A \vdash Y}{A \vdash Y} \\
\hline
\widehat{[a]}X \vdash Y \\
\hline
X \vdash \{a\}Y
\end{array}
\end{array}$$

The transformations of the dynamic modalities are analogous to the ones of static modalities and, again, we only show them for $\langle \alpha \rangle$ and $[\alpha]$.

$$\begin{array}{c}
\begin{array}{c}
\vdots \pi_1 \quad \vdots \pi_2 \\
\frac{X \vdash A \quad \{\alpha\}A \vdash Y}{\{\alpha\}X \vdash \langle \alpha \rangle A} \quad \frac{\{\alpha\}A \vdash Y}{\langle \alpha \rangle A \vdash Y} \\
\hline
\{\alpha\}X \vdash Y
\end{array} \\
\rightsquigarrow \\
\begin{array}{c}
\vdots \pi_2 \\
\frac{X \vdash A \quad \frac{\{\alpha\}A \vdash Y}{A \vdash \widehat{[\alpha]}Y}}{X \vdash \widehat{[\alpha]}Y} \\
\hline
\{\alpha\}X \vdash Y
\end{array}
\end{array}$$

$$\frac{\frac{\frac{\vdots \pi_1}{X \vdash \{\alpha\}A}}{X \vdash [\alpha]A} \quad \frac{\frac{\vdots \pi_2}{A \vdash Y}}{[\alpha]A \vdash \{\alpha\}Y}}{X \vdash \{\alpha\}Y} \rightsquigarrow \frac{\frac{\frac{\vdots \pi_1}{X \vdash \{\alpha\}A}}{\underline{\widehat{\alpha}} X \vdash A} \quad \frac{\vdots \pi_2}{A \vdash Y}}{\underline{\widehat{\alpha}} X \vdash Y}}{X \vdash \{\alpha\}Y}$$

Appendix E

Completeness of D'.EAK

To prove, indirectly, the completeness of D'.EAK it is enough to show that all the axioms and rules of IEAK are theorems and, respectively, derived or admissible rules of D'.EAK. Below we show the derivations of the dynamic axioms and we leave the remaining axioms and rules to the reader.

- $\langle \alpha \rangle p \dashv\vdash 1_\alpha \wedge p$

$$\begin{array}{c}
 \frac{\Phi_\alpha \vdash 1_\alpha \quad \{\alpha\} p \vdash p}{\Phi_\alpha ; \{\alpha\} p \vdash 1_\alpha \wedge p} \\
 \text{reduce}' \frac{\frac{\frac{\frac{\frac{\Phi_\alpha \vdash 1_\alpha \quad \{\alpha\} p \vdash p}{\Phi_\alpha ; \{\alpha\} p \vdash 1_\alpha \wedge p}}{\{\alpha\} p \vdash 1_\alpha \wedge p}}{\langle \alpha \rangle p \vdash 1_\alpha \wedge p}}{\langle \alpha \rangle p \vdash 1_\alpha \wedge p}}{\langle \alpha \rangle p \vdash 1_\alpha \wedge p} \\
 \frac{\frac{\frac{\frac{\frac{\frac{\langle \alpha \rangle p \vdash p}{\langle \alpha \rangle p \vdash p}}{\{\alpha\} \langle \alpha \rangle p \vdash \langle \alpha \rangle p}}{\Phi_\alpha ; p \vdash \langle \alpha \rangle p}}{\Phi_\alpha \vdash \langle \alpha \rangle p < p}}{1_\alpha \vdash \langle \alpha \rangle p < p}}{1_\alpha ; p \vdash \langle \alpha \rangle p}}{1_\alpha \wedge p \vdash \langle \alpha \rangle p} \\
 \text{comp} \frac{\frac{\frac{\frac{\frac{\langle \alpha \rangle p \vdash p}{\langle \alpha \rangle p \vdash p}}{\{\alpha\} \langle \alpha \rangle p \vdash \langle \alpha \rangle p}}{\Phi_\alpha ; p \vdash \langle \alpha \rangle p}}{\Phi_\alpha \vdash \langle \alpha \rangle p < p}}{1_\alpha \vdash \langle \alpha \rangle p < p}}{1_\alpha ; p \vdash \langle \alpha \rangle p}}{1_\alpha \wedge p \vdash \langle \alpha \rangle p}
 \end{array}$$

- $[\alpha] p \dashv\vdash 1_\alpha \rightarrow p$

$$\begin{array}{c}
 \frac{p \vdash \langle \alpha \rangle p}{[\alpha] p \vdash \{\alpha\} \langle \alpha \rangle p} \\
 \text{comp} \frac{\frac{\frac{\frac{\frac{p \vdash \langle \alpha \rangle p}{[\alpha] p \vdash \{\alpha\} \langle \alpha \rangle p}}{[\alpha] p \vdash \Phi_\alpha > p}}{\Phi_\alpha ; [\alpha] p \vdash p}}{\Phi_\alpha \vdash p < [\alpha] p}}{1_\alpha \vdash p < [\alpha] p}}{1_\alpha ; [\alpha] p \vdash p}}{[\alpha] p \vdash 1_\alpha > p} \\
 \text{reduce}' \frac{\frac{\frac{\frac{\frac{1_\alpha \vdash p < [\alpha] p}{1_\alpha \vdash p < [\alpha] p}}{1_\alpha \rightarrow p \vdash \Phi_\alpha > \{\alpha\} p}}{\Phi_\alpha \vdash 1_\alpha \quad p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash [\alpha] p} \\
 \frac{\frac{\frac{\frac{\frac{1_\alpha \vdash p < [\alpha] p}{1_\alpha \vdash p < [\alpha] p}}{1_\alpha \rightarrow p \vdash \Phi_\alpha > \{\alpha\} p}}{\Phi_\alpha \vdash 1_\alpha \quad p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash [\alpha] p} \\
 \text{reduce}' \frac{\frac{\frac{\frac{\frac{1_\alpha \vdash p < [\alpha] p}{1_\alpha \vdash p < [\alpha] p}}{1_\alpha \rightarrow p \vdash \Phi_\alpha > \{\alpha\} p}}{\Phi_\alpha \vdash 1_\alpha \quad p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash [\alpha] p} \\
 \frac{\frac{\frac{\frac{\frac{1_\alpha \vdash p < [\alpha] p}{1_\alpha \vdash p < [\alpha] p}}{1_\alpha \rightarrow p \vdash \Phi_\alpha > \{\alpha\} p}}{\Phi_\alpha \vdash 1_\alpha \quad p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash \{\alpha\} p}}{1_\alpha \rightarrow p \vdash [\alpha] p}
 \end{array}$$

- $\langle \alpha \rangle \top \dashv\vdash 1_\alpha$

$$\begin{array}{c}
 \frac{\Phi_\alpha \vdash 1_\alpha}{\text{I} \vdash 1_\alpha < \Phi_\alpha} \\
 \frac{\text{I} \vdash 1_\alpha < \Phi_\alpha}{\{\alpha\} \text{I} \vdash 1_\alpha < \Phi_\alpha} \text{ nec} \\
 \frac{\{\alpha\} \text{I} \vdash 1_\alpha < \Phi_\alpha}{\text{I} \vdash \widehat{\alpha} (1_\alpha < \Phi_\alpha)} \\
 \frac{\text{I} \vdash \widehat{\alpha} (1_\alpha < \Phi_\alpha)}{\top \vdash \widehat{\alpha} (1_\alpha < \Phi_\alpha)} \\
 \frac{\top \vdash \widehat{\alpha} (1_\alpha < \Phi_\alpha)}{\{\alpha\} \top \vdash 1_\alpha < \Phi_\alpha} \\
 \frac{\{\alpha\} \top \vdash 1_\alpha < \Phi_\alpha}{\Phi_\alpha ; \{\alpha\} \top \vdash 1_\alpha} \\
 \frac{\Phi_\alpha ; \{\alpha\} \top \vdash 1_\alpha}{\{\alpha\} \top \vdash 1_\alpha} \text{ reduce}' \\
 \frac{\{\alpha\} \top \vdash 1_\alpha}{\langle \alpha \rangle \top \vdash 1_\alpha}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\text{I} \vdash \top}{\widehat{\alpha} \text{I} \vdash \top} \text{ nec} \\
 \frac{\widehat{\alpha} \text{I} \vdash \top}{\{\alpha\} \widehat{\alpha} \text{I} \vdash \langle \alpha \rangle \top} \\
 \frac{\{\alpha\} \widehat{\alpha} \text{I} \vdash \langle \alpha \rangle \top}{\Phi_\alpha ; \text{I} \vdash \langle \alpha \rangle \top} \text{ comp} \\
 \frac{\Phi_\alpha ; \text{I} \vdash \langle \alpha \rangle \top}{\Phi_\alpha \vdash \langle \alpha \rangle \top} \\
 \frac{\Phi_\alpha \vdash \langle \alpha \rangle \top}{1_\alpha \vdash \langle \alpha \rangle \top}
 \end{array}$$

- $[\alpha] \perp \dashv\vdash \neg 1_\alpha$

$$\begin{array}{c}
 \frac{\perp \vdash \text{I}}{\perp \vdash \widehat{\alpha} \text{I}} \text{ nec} \\
 \frac{\perp \vdash \widehat{\alpha} \text{I}}{[\alpha] \perp \vdash \{\alpha\} \widehat{\alpha} \text{I}} \\
 \frac{[\alpha] \perp \vdash \{\alpha\} \widehat{\alpha} \text{I}}{[\alpha] \perp \vdash \Phi_\alpha > \text{I}} \text{ comp} \\
 \frac{[\alpha] \perp \vdash \Phi_\alpha > \text{I}}{\Phi_\alpha ; [\alpha] \perp \vdash \text{I}} \\
 \frac{\Phi_\alpha ; [\alpha] \perp \vdash \text{I}}{\Phi_\alpha ; [\alpha] \perp \vdash \perp} \\
 \frac{\Phi_\alpha ; [\alpha] \perp \vdash \perp}{\Phi_\alpha \vdash \perp < [\alpha] \perp} \\
 \frac{\Phi_\alpha \vdash \perp < [\alpha] \perp}{1_\alpha \vdash \perp < [\alpha] \perp} \\
 \frac{1_\alpha \vdash \perp < [\alpha] \perp}{1_\alpha ; [\alpha] \perp \vdash \perp} \\
 \frac{1_\alpha ; [\alpha] \perp \vdash \perp}{[\alpha] \perp \vdash 1_\alpha > \perp} \\
 \frac{[\alpha] \perp \vdash 1_\alpha > \perp}{[\alpha] \perp \vdash 1_\alpha \rightarrow \perp} \\
 \frac{[\alpha] \perp \vdash 1_\alpha \rightarrow \perp}{[\alpha] \perp \vdash \neg 1_\alpha}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\perp \vdash \text{I}}{\perp \vdash \{\alpha\} \text{I}} \text{ nec} \\
 \frac{\perp \vdash \{\alpha\} \text{I}}{\widehat{\alpha} \perp \vdash \text{I}} \\
 \frac{\widehat{\alpha} \perp \vdash \text{I}}{\widehat{\alpha} \perp \vdash \perp} \\
 \frac{\widehat{\alpha} \perp \vdash \perp}{\perp \vdash \{\alpha\} \perp} \\
 \frac{\perp \vdash \{\alpha\} \perp}{1_\alpha \rightarrow \perp \vdash \Phi_\alpha > \{\alpha\} \perp} \\
 \frac{1_\alpha \rightarrow \perp \vdash \Phi_\alpha > \{\alpha\} \perp}{\neg 1_\alpha \vdash \Phi_\alpha > \{\alpha\} \perp} \\
 \frac{\neg 1_\alpha \vdash \Phi_\alpha > \{\alpha\} \perp}{\neg 1_\alpha \vdash \{\alpha\} \perp} \text{ reduce}' \\
 \frac{\neg 1_\alpha \vdash \{\alpha\} \perp}{\neg 1_\alpha \vdash [\alpha] \perp}
 \end{array}$$

- $\langle \alpha \rangle \perp \dashv\vdash \perp$

$$\begin{array}{c}
 \frac{\perp \vdash \text{I}}{\perp \vdash \widehat{\alpha} \text{I}} \text{ nec} \\
 \frac{\perp \vdash \widehat{\alpha} \text{I}}{\{\alpha\} \perp \vdash \text{I}} \\
 \frac{\{\alpha\} \perp \vdash \text{I}}{\{\alpha\} \perp \vdash \perp} \\
 \frac{\{\alpha\} \perp \vdash \perp}{\langle \alpha \rangle \perp \vdash \perp}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\perp \vdash \text{I}}{\perp \vdash \langle \alpha \rangle \perp}
 \end{array}$$

- $[\alpha] \top \dashv\vdash \top$

$$\begin{array}{c}
 \frac{\text{I} \vdash \top}{[\alpha] \top \vdash \top}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\text{I} \vdash \top}{\widehat{\alpha} \text{I} \vdash \top} \text{ nec} \\
 \frac{\widehat{\alpha} \text{I} \vdash \top}{\text{I} \vdash \{\alpha\} \top} \\
 \frac{\text{I} \vdash \{\alpha\} \top}{\top \vdash \{\alpha\} \top} \\
 \frac{\top \vdash \{\alpha\} \top}{\top \vdash [\alpha] \top}
 \end{array}$$

- $[\alpha](A \wedge B) \dashv\vdash [\alpha]A \wedge [\alpha]B$

$$\begin{array}{c}
\frac{\frac{A \vdash A}{A; B \vdash A}}{A \wedge B \vdash A} \quad \frac{\frac{B \vdash B}{A; B \vdash B}}{A \wedge B \vdash B} \\
\frac{[\alpha](A \wedge B) \vdash \{\alpha\}A}{[\alpha](A \wedge B) \vdash [\alpha]A} \quad \frac{[\alpha](A \wedge B) \vdash \{\alpha\}B}{[\alpha](A \wedge B) \vdash [\alpha]B} \\
\frac{[\alpha](A \wedge B); [\alpha](A \wedge B) \vdash [\alpha]A \wedge [\alpha]B}{[\alpha](A \wedge B) \vdash [\alpha]A \wedge [\alpha]B}
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{A \vdash A}{[\alpha]A \vdash \{\alpha\}A}}{\widetilde{\alpha}[\alpha]A \vdash A} \quad \frac{\frac{B \vdash B}{[\alpha]B \vdash \{\alpha\}B}}{\widetilde{\alpha}[\alpha]B \vdash B} \\
\text{mon} \frac{\widetilde{\alpha}[\alpha]A; \widetilde{\alpha}[\alpha]B \vdash A \wedge B}{\widetilde{\alpha}([\alpha]A; [\alpha]B) \vdash A \wedge B} \\
\frac{[\alpha]A; [\alpha]B \vdash \{\alpha\}(A \wedge B)}{[\alpha]A; [\alpha]B \vdash [\alpha](A \wedge B)} \\
\frac{[\alpha]A \wedge [\alpha]B \vdash [\alpha](A \wedge B)}{[\alpha]A \wedge [\alpha]B \vdash [\alpha](A \wedge B)}
\end{array}$$

- $\langle \alpha \rangle(A \wedge B) \dashv\vdash \langle \alpha \rangle A \wedge \langle \alpha \rangle B$

$$\begin{array}{c}
\frac{\frac{A \vdash A}{A; B \vdash A}}{A \wedge B \vdash A} \quad \frac{\frac{B \vdash B}{A; B \vdash B}}{A \wedge B \vdash B} \\
\frac{\{\alpha\}A \wedge B \vdash \langle \alpha \rangle A}{\langle \alpha \rangle(A \wedge B) \vdash \langle \alpha \rangle A} \quad \frac{\{\alpha\}A \wedge B \vdash \langle \alpha \rangle B}{\langle \alpha \rangle(A \wedge B) \vdash \langle \alpha \rangle B} \\
\frac{\langle \alpha \rangle(A \wedge B); \langle \alpha \rangle(A \wedge B) \vdash \langle \alpha \rangle A \wedge \langle \alpha \rangle B}{\langle \alpha \rangle(A \wedge B) \vdash \langle \alpha \rangle A \wedge \langle \alpha \rangle B}
\end{array}
\qquad
\begin{array}{c}
\text{balance} \frac{\frac{A \vdash A}{\{\alpha\}A \vdash \{\alpha\}A}}{\widetilde{\alpha}\{\alpha\}A \vdash A} \quad \text{balance} \frac{\frac{B \vdash B}{\{\alpha\}B \vdash \{\alpha\}B}}{\widetilde{\alpha}\{\alpha\}B \vdash B} \\
\text{mon} \frac{\widetilde{\alpha}\{\alpha\}A; \widetilde{\alpha}\{\alpha\}B \vdash A \wedge B}{\widetilde{\alpha}(\{\alpha\}A; \{\alpha\}B) \vdash A \wedge B} \\
\text{comp} \frac{\{\alpha\} \widetilde{\alpha}(\{\alpha\}A; \{\alpha\}B) \vdash \langle \alpha \rangle(A \wedge B)}{\Phi_\alpha; \{\alpha\}A; \{\alpha\}B \vdash \langle \alpha \rangle(A \wedge B)} \\
\text{reduce}' \frac{\Phi_\alpha; \{\alpha\}A \vdash \langle \alpha \rangle(A \wedge B) < \{\alpha\}B}{\langle \alpha \rangle A \vdash \langle \alpha \rangle(A \wedge B) < \{\alpha\}B} \\
\frac{\langle \alpha \rangle A \vdash \langle \alpha \rangle(A \wedge B) < \{\alpha\}B}{\langle \alpha \rangle A; \{\alpha\}B \vdash \langle \alpha \rangle(A \wedge B)} \\
\frac{\{\alpha\}B \vdash \langle \alpha \rangle A > \langle \alpha \rangle(A \wedge B)}{\langle \alpha \rangle B \vdash \langle \alpha \rangle A > \langle \alpha \rangle(A \wedge B)} \\
\frac{\langle \alpha \rangle A; \langle \alpha \rangle B \vdash \langle \alpha \rangle(A \wedge B)}{\langle \alpha \rangle A \wedge \langle \alpha \rangle B \vdash \langle \alpha \rangle(A \wedge B)}
\end{array}$$

- $\langle \alpha \rangle(A \vee B) \dashv\vdash \langle \alpha \rangle A \vee \langle \alpha \rangle B$

$$\begin{array}{c}
\frac{\frac{A \vdash A}{\{\alpha\}A \vdash \langle \alpha \rangle A}}{A \vdash \widetilde{\alpha}\langle \alpha \rangle A} \quad \frac{\frac{B \vdash B}{\{\alpha\}B \vdash \langle \alpha \rangle B}}{B \vdash \widetilde{\alpha}\langle \alpha \rangle B} \\
\frac{A \vee B \vdash \widetilde{\alpha}\langle \alpha \rangle A; \widetilde{\alpha}\langle \alpha \rangle B}{A \vee B \vdash \widetilde{\alpha}(\langle \alpha \rangle A; \langle \alpha \rangle B)} \\
\frac{\{\alpha\}A \vee B \vdash \langle \alpha \rangle A; \langle \alpha \rangle B}{\langle \alpha \rangle(A \vee B) \vdash \langle \alpha \rangle A; \langle \alpha \rangle B} \\
\frac{\langle \alpha \rangle(A \vee B) \vdash \langle \alpha \rangle A; \langle \alpha \rangle B}{\langle \alpha \rangle(A \vee B) \vdash \langle \alpha \rangle A \vee \langle \alpha \rangle B}
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{A \vdash A}{A \vdash A; B}}{A \vdash A \vee B} \quad \frac{\frac{B \vdash B}{B \vdash A; B}}{B \vdash A \vee B} \\
\frac{\{\alpha\}A \vdash \langle \alpha \rangle(A \vee B)}{\langle \alpha \rangle A \vdash \langle \alpha \rangle(A \vee B)} \quad \frac{\{\alpha\}B \vdash \langle \alpha \rangle(A \vee B)}{\langle \alpha \rangle B \vdash \langle \alpha \rangle(A \vee B)} \\
\frac{\langle \alpha \rangle A \vee \langle \alpha \rangle B \vdash \langle \alpha \rangle(A \vee B); \langle \alpha \rangle(A \vee B)}{\langle \alpha \rangle A \vee \langle \alpha \rangle B \vdash \langle \alpha \rangle(A \vee B)}
\end{array}$$

- $[\alpha](A \vee B) \dashv\vdash 1_\alpha \rightarrow (\langle\alpha\rangle A \vee \langle\alpha\rangle B)$

$$\begin{array}{c}
\frac{A \vdash A}{\{\alpha\}A \vdash \langle\alpha\rangle A} \quad \frac{B \vdash B}{\{\alpha\}B \vdash \langle\alpha\rangle B} \\
\frac{A \vdash \widetilde{\alpha}\langle\alpha\rangle A}{A \vee B \vdash \widetilde{\alpha}\langle\alpha\rangle A; \widetilde{\alpha}\langle\alpha\rangle B} \quad \frac{B \vdash \widetilde{\alpha}\langle\alpha\rangle B}{A \vee B \vdash \widetilde{\alpha}\langle\alpha\rangle A; \widetilde{\alpha}\langle\alpha\rangle B} \\
\frac{A \vee B \vdash \widetilde{\alpha}\langle\alpha\rangle A; \widetilde{\alpha}\langle\alpha\rangle B}{[\alpha](A \vee B) \vdash \{\alpha\} \widetilde{\alpha}(\langle\alpha\rangle A \vee \langle\alpha\rangle B)} \\
\frac{[\alpha](A \vee B) \vdash \{\alpha\} \widetilde{\alpha}(\langle\alpha\rangle A \vee \langle\alpha\rangle B)}{[\alpha](A \vee B) \vdash \Phi_\alpha > (\langle\alpha\rangle A \vee \langle\alpha\rangle B)} \text{comp}_R^\alpha \\
\frac{\Phi_\alpha; [\alpha](A \vee B) \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B}{\Phi_\alpha \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B < [\alpha](A \vee B)} \\
\frac{1_\alpha \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B < [\alpha](A \vee B)}{1_\alpha; [\alpha](A \vee B) \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B} \\
\frac{[\alpha](A \vee B) \vdash 1_\alpha > (\langle\alpha\rangle A \vee \langle\alpha\rangle B)}{[\alpha](A \vee B) \vdash 1_\alpha \rightarrow (\langle\alpha\rangle A \vee \langle\alpha\rangle B)}
\end{array}$$

$$\begin{array}{c}
\frac{A \vdash A}{\{\alpha\}A \vdash \{\alpha\}A} \quad \frac{B \vdash B}{\{\alpha\}B \vdash \{\alpha\}B} \\
\frac{\langle\alpha\rangle A \vdash \{\alpha\}A}{\langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \{\alpha\}A; \{\alpha\}B} \quad \frac{\langle\alpha\rangle B \vdash \{\alpha\}B}{\langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \{\alpha\}A; \{\alpha\}B} \\
\frac{\langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \{\alpha\}A; \{\alpha\}B}{\widetilde{\alpha}(\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash A; B} \\
\frac{\widetilde{\alpha}(\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash A; B}{\widetilde{\alpha}(\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash A \vee B} \\
\frac{\Phi_\alpha \vdash 1_\alpha \quad \langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \{\alpha\}(A \vee B)}{1_\alpha \rightarrow (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash \Phi_\alpha > \{\alpha\}(A \vee B)} \\
\frac{1_\alpha \rightarrow (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash \{\alpha\}(A \vee B)}{1_\alpha \rightarrow (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash [\alpha](A \vee B)} \text{reduce}'_R
\end{array}$$

- $\langle \alpha \rangle (A \rightarrow B) \dashv\vdash 1_\alpha \wedge (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B)$

$$\begin{array}{c}
\frac{\frac{A \vdash A}{\{\alpha\}A \vdash \{\alpha\}A} \quad \frac{B \vdash B}{\{\alpha\}B \vdash \langle \alpha \rangle B}}{\frac{\langle \alpha \rangle A \vdash \{\alpha\}A}{\widetilde{\alpha} \langle \alpha \rangle A \vdash A} \quad \frac{B \vdash \langle \alpha \rangle B}{B \vdash \widetilde{\alpha} \langle \alpha \rangle B}} \\
\frac{A \rightarrow B \vdash \widetilde{\alpha} \langle \alpha \rangle A > \widetilde{\alpha} \langle \alpha \rangle B}{A \rightarrow B \vdash \widetilde{\alpha} (\langle \alpha \rangle A > \langle \alpha \rangle B)} \\
\frac{\{\alpha\}(A \rightarrow B) \vdash \langle \alpha \rangle A > \langle \alpha \rangle B}{\Phi_\alpha \vdash 1_\alpha \quad \{\alpha\}(A \rightarrow B) \vdash \langle \alpha \rangle A \rightarrow \langle \alpha \rangle B} \\
\text{reduce}' \frac{\Phi_\alpha ; \{\alpha\}(A \rightarrow B) \vdash 1_\alpha \wedge (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B)}{\frac{\{\alpha\}(A \rightarrow B) \vdash 1_\alpha \wedge (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B)}{\langle \alpha \rangle (A \rightarrow B) \vdash 1_\alpha \wedge (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B)}} \\
\frac{\frac{A \vdash A}{\{\alpha\}A \vdash \langle \alpha \rangle A} \quad \frac{B \vdash B}{\{\alpha\}B \vdash \langle \alpha \rangle B} \text{balance}}{\frac{\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B \vdash \{\alpha\}A > \{\alpha\}B}{\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B \vdash \{\alpha\}(A > B)}} \\
\frac{\widetilde{\alpha} (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash A > B}{\widetilde{\alpha} (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash A \rightarrow B} \\
\text{comp}_L^\alpha \frac{\{\alpha\} \widetilde{\alpha} (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash \langle \alpha \rangle (A \rightarrow B)}{\Phi_\alpha ; (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash \langle \alpha \rangle (A \rightarrow B)} \\
\frac{(\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) ; \Phi_\alpha \vdash \langle \alpha \rangle (A \rightarrow B)}{\Phi_\alpha \vdash (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) > \langle \alpha \rangle (A \rightarrow B)} \\
\frac{1_\alpha \vdash (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) > \langle \alpha \rangle (A \rightarrow B)}{(\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) ; 1_\alpha \vdash \langle \alpha \rangle (A \rightarrow B)} \\
\frac{1_\alpha ; (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash \langle \alpha \rangle (A \rightarrow B)}{1_\alpha \wedge (\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash \langle \alpha \rangle (A \rightarrow B)}
\end{array}$$

- $[\alpha](A \rightarrow B) \dashv\vdash \langle \alpha \rangle A \rightarrow \langle \alpha \rangle B$

$$\begin{array}{c}
\frac{\frac{A \vdash A}{\{\alpha\}A \vdash \{\alpha\}A}}{\widetilde{\alpha}\{\alpha\}A \vdash A} \quad \frac{\frac{B \vdash B}{\{\alpha\}B \vdash \langle \alpha \rangle B}}{B \vdash \widetilde{\alpha}\langle \alpha \rangle B} \\
\frac{A \rightarrow B \vdash \widetilde{\alpha}\{\alpha\}A > \widetilde{\alpha}\langle \alpha \rangle B}{A \rightarrow B \vdash \widetilde{\alpha}(\{\alpha\}A > \langle \alpha \rangle B)} \\
\frac{[\alpha](A \rightarrow B) \vdash \{\alpha\}\widetilde{\alpha}(\{\alpha\}A > \langle \alpha \rangle B)}{[\alpha](A \rightarrow B) \vdash \Phi_\alpha > (\{\alpha\}A > \langle \alpha \rangle B)} \text{comp}_R^\alpha \\
\frac{\Phi_\alpha; [\alpha](A \rightarrow B) \vdash \{\alpha\}A > \langle \alpha \rangle B}{\{\alpha\}A; (\Phi_\alpha; [\alpha](A \rightarrow B)) \vdash \langle \alpha \rangle B} \\
\frac{(\{\alpha\}A; \Phi_\alpha); [\alpha](A \rightarrow B) \vdash \langle \alpha \rangle B}{[\alpha](A \rightarrow B); (\{\alpha\}A; \Phi_\alpha) \vdash \langle \alpha \rangle B} \\
\frac{\frac{\{\alpha\}A; \Phi_\alpha \vdash [\alpha](A \rightarrow B) > \langle \alpha \rangle B}{\Phi_\alpha; \{\alpha\}A \vdash [\alpha](A \rightarrow B) > \langle \alpha \rangle B}}{\{\alpha\}A \vdash [\alpha](A \rightarrow B) > \langle \alpha \rangle B} \text{reduce}' \\
\frac{\langle \alpha \rangle A \vdash [\alpha](A \rightarrow B) > \langle \alpha \rangle B}{[\alpha](A \rightarrow B); \langle \alpha \rangle A \vdash \langle \alpha \rangle B} \\
\frac{\langle \alpha \rangle A; [\alpha](A \rightarrow B) \vdash \langle \alpha \rangle B}{[\alpha](A \rightarrow B) \vdash \langle \alpha \rangle A > \langle \alpha \rangle B} \\
\frac{[\alpha](A \rightarrow B) \vdash \langle \alpha \rangle A > \langle \alpha \rangle B}{[\alpha](A \rightarrow B) \vdash \langle \alpha \rangle A \rightarrow \langle \alpha \rangle B}
\end{array}$$

$$\begin{array}{c}
\frac{A \vdash A}{\{\alpha\}A \vdash \langle \alpha \rangle A} \quad \frac{\frac{B \vdash B}{\{\alpha\}B \vdash \{\alpha\}B}}{\langle \alpha \rangle B \vdash \{\alpha\}B} \\
\frac{\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B \vdash \{\alpha\}A > \{\alpha\}B}{\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B \vdash \{\alpha\}(A > B)} \\
\frac{\widetilde{\alpha}(\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash A > B}{\widetilde{\alpha}(\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B) \vdash A \rightarrow B} \\
\frac{\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B \vdash \{\alpha\}(A \rightarrow B)}{\langle \alpha \rangle A \rightarrow \langle \alpha \rangle B \vdash [\alpha](A \rightarrow B)}
\end{array}$$

For ease of notation, in the following derivations we assume the actions β , such that $\alpha\beta$ form the set $\{\beta_i \mid 1 \leq i \leq n\}$.

- $\langle\alpha\rangle\langle\mathbf{a}\rangle A \vdash 1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta\rangle A \mid \alpha\beta\}$

$$\begin{array}{c}
 \frac{A \vdash A}{\{\beta_1\}A \vdash \langle\beta_1\rangle A} \quad \cdots \quad \frac{A \vdash A}{\{\beta_n\}A \vdash \langle\beta_n\rangle A} \\
 \text{swap-out}' \frac{\frac{\frac{A \vdash A}{\{\beta_1\}A \vdash \langle\beta_1\rangle A} \quad \cdots \quad \frac{A \vdash A}{\{\beta_n\}A \vdash \langle\beta_n\rangle A}}{\{\mathbf{a}\}\{\beta_1\}A \vdash \langle\mathbf{a}\rangle\langle\beta_1\rangle A} \quad \cdots \quad \frac{\frac{A \vdash A}{\{\beta_n\}A \vdash \langle\beta_n\rangle A}}{\{\mathbf{a}\}\{\beta_n\}A \vdash \langle\mathbf{a}\rangle\langle\beta_n\rangle A}}{\{\alpha\}\{\mathbf{a}\}A \vdash \langle\mathbf{a}\rangle\langle\beta_i\rangle A} \\
 \frac{\Phi_\alpha \vdash 1_\alpha \quad \frac{\{\alpha\}\{\mathbf{a}\}A \vdash \langle\mathbf{a}\rangle\langle\beta_i\rangle A}{\{\alpha\}\{\mathbf{a}\}A \vdash \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\}}}{\Phi_\alpha ; \{\alpha\}\{\mathbf{a}\}A \vdash 1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\}} \\
 \text{reduce}' \frac{\Phi_\alpha ; \{\alpha\}\{\mathbf{a}\}A \vdash 1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\}}{\{\alpha\}\{\mathbf{a}\}A \vdash 1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\}} \\
 \frac{\{\mathbf{a}\}A \vdash \widetilde{\alpha}(1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\})}{\langle\mathbf{a}\rangle A \vdash \widetilde{\alpha}(1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\})} \\
 \frac{\{\alpha\}\langle\mathbf{a}\rangle A \vdash 1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\}}{\langle\alpha\rangle\langle\mathbf{a}\rangle A \vdash 1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\}}
 \end{array}$$

- $1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta\rangle A \mid \alpha\beta\} \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A$

$$\begin{array}{c}
 \frac{A \vdash A}{\{\mathbf{a}\}A \vdash \langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{A \vdash A}{\{\mathbf{a}\}A \vdash \langle\mathbf{a}\rangle A} \\
 \text{swap-in}' \frac{\frac{\frac{A \vdash A}{\{\mathbf{a}\}A \vdash \langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{A \vdash A}{\{\mathbf{a}\}A \vdash \langle\mathbf{a}\rangle A}}{\{\alpha\}\{\mathbf{a}\}A \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{\frac{A \vdash A}{\{\mathbf{a}\}A \vdash \langle\mathbf{a}\rangle A}}{\{\alpha\}\{\mathbf{a}\}A \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A}}{\Phi_\alpha ; \{\mathbf{a}\}\{\beta_1\}A \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A} \quad \text{swap-in}' \\
 \frac{\{\mathbf{a}\}\{\beta_1\}A ; \Phi_\alpha \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\Phi_\alpha \vdash \{\mathbf{a}\}\{\beta_1\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{\{\mathbf{a}\}\{\beta_n\}A ; \Phi_\alpha \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\Phi_\alpha \vdash \{\mathbf{a}\}\{\beta_n\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \\
 \frac{\Phi_\alpha \vdash \{\mathbf{a}\}\{\beta_1\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A}{1_\alpha \vdash \{\mathbf{a}\}\{\beta_1\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{\Phi_\alpha \vdash \{\mathbf{a}\}\{\beta_n\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A}{1_\alpha \vdash \{\mathbf{a}\}\{\beta_n\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \\
 \frac{1_\alpha \vdash \{\mathbf{a}\}\{\beta_1\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\{\mathbf{a}\}\{\beta_1\}A ; 1_\alpha \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{1_\alpha \vdash \{\mathbf{a}\}\{\beta_n\}A > \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\{\mathbf{a}\}\{\beta_n\}A ; 1_\alpha \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A} \\
 \frac{1_\alpha ; \{\mathbf{a}\}\{\beta_1\}A \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\{\mathbf{a}\}\{\beta_1\}A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{1_\alpha ; \{\mathbf{a}\}\{\beta_n\}A \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\{\mathbf{a}\}\{\beta_n\}A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \\
 \frac{\{\beta_1\}A \vdash \widetilde{\alpha}(1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A)}{\langle\beta_1\rangle A \vdash \widetilde{\alpha}(1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A)} \quad \cdots \quad \frac{\{\beta_n\}A \vdash \widetilde{\alpha}(1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A)}{\langle\beta_n\rangle A \vdash \widetilde{\alpha}(1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A)} \\
 \frac{\langle\beta_1\rangle A \vdash \widetilde{\alpha}(1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A)}{\{\mathbf{a}\}\langle\beta_1\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \quad \cdots \quad \frac{\langle\beta_n\rangle A \vdash \widetilde{\alpha}(1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A)}{\{\mathbf{a}\}\langle\beta_n\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \\
 \frac{\langle\mathbf{a}\rangle\langle\beta_1\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\} \vdash \langle\mathbf{a}\rangle\langle\alpha\rangle A} \quad \cdots \quad \frac{\langle\mathbf{a}\rangle\langle\beta_n\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A}{\bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\} \vdash 1_\alpha > \langle\alpha\rangle\langle\mathbf{a}\rangle A} \\
 \frac{\bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\} \vdash \langle\mathbf{a}\rangle\langle\alpha\rangle A}{1_\alpha ; \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\} \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A} \\
 1_\alpha \wedge \bigvee \{\langle\mathbf{a}\rangle\langle\beta_i\rangle A\} \vdash \langle\alpha\rangle\langle\mathbf{a}\rangle A
 \end{array}$$

- $[\alpha]\langle a \rangle A \vdash \text{Pre}(\alpha) \rightarrow \bigvee \{ \langle a \rangle \langle \beta \rangle A \mid \alpha a \beta \}$

$$\begin{array}{c}
 \frac{A \vdash A}{\{\beta_1\}A \vdash \langle \beta_1 \rangle A} \quad \dots \quad \frac{A \vdash A}{\{\beta_n\}A \vdash \langle \beta_n \rangle A} \\
 \frac{\dots}{\{\mathbf{a}\}\{\beta_1\}A \vdash \langle \mathbf{a} \rangle \langle \beta_1 \rangle A} \quad \dots \quad \frac{\dots}{\{\mathbf{a}\}\{\beta_n\}A \vdash \langle \mathbf{a} \rangle \langle \beta_n \rangle A} \\
 \text{swap-out}' \quad \frac{\dots}{\{\alpha\}\{\mathbf{a}\}A \vdash \langle \mathbf{a} \rangle \langle \beta_i \rangle A} \\
 \frac{\dots}{\{\alpha\}\{\mathbf{a}\}A \vdash \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \\
 \frac{\dots}{\{\mathbf{a}\}A \vdash \widehat{\alpha} \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \\
 \frac{\dots}{\langle \mathbf{a} \rangle A \vdash \widehat{\alpha} \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \\
 \frac{\dots}{[\alpha]\langle \mathbf{a} \rangle A \vdash \langle \alpha \rangle \widehat{\alpha} \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \\
 \frac{\dots}{[\alpha]\langle \mathbf{a} \rangle A \vdash \Phi_\alpha > \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \text{comp}_k^\alpha \\
 \frac{\dots}{\Phi_\alpha ; [\alpha]\langle \mathbf{a} \rangle A \vdash \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \\
 \frac{\dots}{\Phi_\alpha \vdash \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A) < [\alpha]\langle \mathbf{a} \rangle A} \\
 \frac{\dots}{1_\alpha \vdash \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A) < [\alpha]\langle \mathbf{a} \rangle A} \\
 \frac{\dots}{1_\alpha ; [\alpha]\langle \mathbf{a} \rangle A \vdash \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \\
 \frac{\dots}{[\alpha]\langle \mathbf{a} \rangle A \vdash 1_\alpha > \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)} \\
 \frac{\dots}{[\alpha]\langle \mathbf{a} \rangle A \vdash 1_\alpha \rightarrow \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A)}
 \end{array}$$

- $\text{Pre}(\alpha) \rightarrow \bigvee \{ \langle a \rangle \langle \beta_i \rangle A \mid \alpha a \beta \} \vdash [\alpha]\langle a \rangle A$

$$\begin{array}{c}
 \frac{A \vdash A}{\{\mathbf{a}\}A \vdash \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{A \vdash A}{\{\mathbf{a}\}A \vdash \langle \mathbf{a} \rangle A} \\
 \frac{\dots}{\langle \alpha \rangle \{\mathbf{a}\}A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\langle \alpha \rangle \{\mathbf{a}\}A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \text{swap-in}' \\
 \frac{\dots}{\Phi_\alpha ; \{\mathbf{a}\}\{\beta_1\}A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\Phi_\alpha ; \{\mathbf{a}\}\{\beta_n\}A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \text{swap-in}' \\
 \frac{\dots}{\{\mathbf{a}\}\{\beta_1\}A \vdash \Phi_\alpha > \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\{\mathbf{a}\}\{\beta_n\}A \vdash \Phi_\alpha > \langle \alpha \rangle \langle \mathbf{a} \rangle A} \text{reduce}' \\
 \frac{\dots}{\{\mathbf{a}\}\{\beta_1\}A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\{\mathbf{a}\}\{\beta_n\}A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \text{reduce}' \\
 \frac{\dots}{\{\beta_1\}A \vdash \widehat{\alpha} \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\{\beta_n\}A \vdash \widehat{\alpha} \langle \alpha \rangle \langle \mathbf{a} \rangle A} \\
 \frac{\dots}{\langle \beta_1 \rangle A \vdash \widehat{\alpha} \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\langle \beta_n \rangle A \vdash \widehat{\alpha} \langle \alpha \rangle \langle \mathbf{a} \rangle A} \\
 \frac{\dots}{\{\mathbf{a}\}\langle \beta_1 \rangle A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\{\mathbf{a}\}\langle \beta_n \rangle A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \\
 \frac{\dots}{\langle \mathbf{a} \rangle \langle \beta_1 \rangle A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \quad \dots \quad \frac{\dots}{\langle \mathbf{a} \rangle \langle \beta_n \rangle A \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \\
 \frac{\dots}{\bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A) \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \\
 \frac{\dots}{\bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A) \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \\
 \frac{\Phi_\alpha \vdash 1_\alpha \quad \dots}{1_\alpha \rightarrow \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A) \vdash \Phi_\alpha > \langle \alpha \rangle \langle \mathbf{a} \rangle A} \\
 \frac{\dots}{1_\alpha \rightarrow \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A) \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A} \text{reduce}' \\
 \frac{\dots}{1_\alpha \rightarrow \bigvee (\langle \mathbf{a} \rangle \langle \beta_i \rangle A) \vdash [\alpha]\langle \mathbf{a} \rangle A}
 \end{array}$$

- $[\alpha][a]A \vdash \text{Pre}(\alpha) \rightarrow \bigwedge \{[a][\beta_i]A \mid \alpha a \beta\}$

$$\begin{array}{c}
\begin{array}{ccc}
\frac{A \vdash A}{[a]A \vdash \{a\}A} & \cdots & \frac{A \vdash A}{[a]A \vdash \{a\}A} \\
\frac{[\alpha][a]A \vdash \{\alpha\}\{a\}A}{[\alpha][a]A \vdash \Phi_\alpha > \{a\}\{\beta_1\}A} & \cdots & \frac{[\alpha][a]A \vdash \{\alpha\}\{a\}A}{[\alpha][a]A \vdash \Phi_\alpha > \{a\}\{\beta_n\}A} \\
\frac{\Phi_\alpha; [\alpha][a]A \vdash \{a\}\{\beta_1\}A}{\widehat{\alpha}(\Phi_\alpha; [\alpha][a]A) \vdash \{\beta_1\}A} & \cdots & \frac{\Phi_\alpha; [\alpha][a]A \vdash \{a\}\{\beta_n\}A}{\widehat{\alpha}(\Phi_\alpha; [\alpha][a]A) \vdash \{\beta_n\}A} \\
\frac{\widehat{\alpha}(\Phi_\alpha; [\alpha][a]A) \vdash \{\beta_1\}A}{\Phi_\alpha; [\alpha][a]A \vdash \{a\}\{\beta_1\}A} & \cdots & \frac{\widehat{\alpha}(\Phi_\alpha; [\alpha][a]A) \vdash \{\beta_n\}A}{\Phi_\alpha; [\alpha][a]A \vdash \{a\}\{\beta_n\}A} \\
\frac{\Phi_\alpha; [\alpha][a]A \vdash \{a\}\{\beta_1\}A}{\Phi_\alpha; [\alpha][a]A \vdash [a][\beta_1]A} & \cdots & \frac{\Phi_\alpha; [\alpha][a]A \vdash \{a\}\{\beta_n\}A}{\Phi_\alpha; [\alpha][a]A \vdash [a][\beta_n]A}
\end{array} \\
\text{swap-in'} & & \text{swap-in'} \\
\hline
& \vdash (\Phi_\alpha; [\alpha][a]A) \vdash \bigwedge ([a][\beta_i]A) \\
& \frac{\Phi_\alpha; [\alpha][a]A \vdash \bigwedge ([a][\beta_i]A)}{[\alpha][a]A; \Phi_\alpha \vdash \bigwedge ([a][\beta_i]A)} \\
& \frac{\Phi_\alpha \vdash [\alpha][a]A > \bigwedge ([a][\beta_i]A)}{1_\alpha \vdash [\alpha][a]A > \bigwedge ([a][\beta_i]A)} \\
& \frac{1_\alpha \vdash [\alpha][a]A > \bigwedge ([a][\beta_i]A)}{[\alpha][a]A; 1_\alpha \vdash \bigwedge ([a][\beta_i]A)} \\
& \frac{[\alpha][a]A; 1_\alpha \vdash \bigwedge ([a][\beta_i]A)}{1_\alpha; [\alpha][a]A \vdash \bigwedge ([a][\beta_i]A)} \\
& \frac{1_\alpha; [\alpha][a]A \vdash \bigwedge ([a][\beta_i]A)}{[\alpha][a]A \vdash 1_\alpha > \bigwedge ([a][\beta_i]A)} \\
& \frac{[\alpha][a]A \vdash 1_\alpha > \bigwedge ([a][\beta_i]A)}{[\alpha][a]A \vdash 1_\alpha \rightarrow \bigwedge ([a][\beta_i]A)}
\end{array}$$

- $\text{Pre}(\alpha) \rightarrow \bigwedge \{[a][\beta_i]A \mid \alpha a \beta\} \vdash [\alpha][a]A$

$$\begin{array}{c}
\begin{array}{ccc}
\frac{A \vdash A}{[\beta_1]A \vdash \{\beta_1\}A} & \cdots & \frac{A \vdash A}{[\beta_n]A \vdash \{\beta_n\}A} \\
\frac{[\alpha][\beta_1]A \vdash \{a\}\{\beta_1\}A}{\vdash ([a][\beta_1]A) \vdash \{\alpha\}\{a\}A} & \cdots & \frac{[\alpha][\beta_n]A \vdash \{a\}\{\beta_n\}A}{\vdash ([a][\beta_n]A) \vdash \{\alpha\}\{a\}A} \\
& & \text{swap-out'}
\end{array} \\
\frac{\Phi_\alpha \vdash 1_\alpha \quad \vdash ([a][\beta_i]A) \vdash \{\alpha\}\{a\}A}{\bigwedge ([a][\beta_i]A) \vdash \{\alpha\}\{a\}A} \\
\frac{1_\alpha \rightarrow \bigwedge ([a][\beta_i]A) \vdash \Phi_\alpha > \{\alpha\}\{a\}A}{1_\alpha \rightarrow \bigwedge ([a][\beta_i]A) \vdash \{\alpha\}\{a\}A} \text{reduce'} \\
\frac{\widehat{\alpha}(1_\alpha \rightarrow \bigwedge ([a][\beta_i]A)) \vdash \{a\}A}{\widehat{\alpha}(1_\alpha \rightarrow \bigwedge ([a][\beta_i]A)) \vdash [a]A} \\
\frac{1_\alpha \rightarrow \bigwedge ([a][\beta_i]A) \vdash \{\alpha\}[a]A}{1_\alpha \rightarrow \bigwedge ([a][\beta_i]A) \vdash [\alpha][a]A}
\end{array}$$

- $\langle \alpha \rangle [a]A \vdash \text{Pre}(\alpha) \wedge \bigwedge \{ [a][\beta_i]A \mid \alpha a \beta_i \}$

$$\begin{array}{c}
 \begin{array}{ccc}
 \frac{A \vdash A}{[a]A \vdash \{a\}A} & \cdots & \frac{A \vdash A}{[a]A \vdash \{a\}A} \\
 \text{balance} \frac{}{\{\alpha\}[a]A \vdash \{\alpha\}\{a\}A} & \cdots & \frac{}{\{\alpha\}[a]A \vdash \{\alpha\}\{a\}A} \\
 \text{swap-in}' \frac{}{\{\alpha\}[a]A \vdash \Phi_\alpha > \{a\}\{\beta_1\}A} & \cdots & \frac{}{\{\alpha\}[a]A \vdash \Phi_\alpha > \{a\}\{\beta_n\}A} \\
 \text{reduce}' \frac{}{\Phi_\alpha; \{\alpha\}[a]A \vdash \{a\}\{\beta_1\}A} & \cdots & \frac{}{\Phi_\alpha; \{\alpha\}[a]A \vdash \{a\}\{\beta_n\}A} \\
 \frac{}{\{\alpha\}[a]A \vdash \{a\}\{\beta_1\}A} & \cdots & \frac{}{\{\alpha\}[a]A \vdash \{a\}\{\beta_n\}A} \\
 \frac{}{\{\alpha\}[a]A \vdash \{a\}\{\beta_1\}A} & \cdots & \frac{}{\{\alpha\}[a]A \vdash \{a\}\{\beta_n\}A} \\
 \frac{}{\overline{\alpha} \{ \alpha \} [a] A \vdash \{ a \} \{ \beta_1 \} A} & \cdots & \frac{}{\overline{\alpha} \{ \alpha \} [a] A \vdash \{ a \} \{ \beta_n \} A} \\
 \frac{}{\overline{\alpha} \{ \alpha \} [a] A \vdash \{ \beta_1 \} A} & \cdots & \frac{}{\overline{\alpha} \{ \alpha \} [a] A \vdash \{ \beta_n \} A} \\
 \frac{}{\{\alpha\}[a]A \vdash \{a\}[\beta_1]A} & \cdots & \frac{}{\{\alpha\}[a]A \vdash \{a\}[\beta_n]A} \\
 \frac{}{\{\alpha\}[a]A \vdash [a][\beta_1]A} & \cdots & \frac{}{\{\alpha\}[a]A \vdash [a][\beta_n]A} \\
 \hline
 & ;(\{\alpha\}[a]A) \vdash \bigwedge ([a][\beta_i]A) & \\
 \hline
 \Phi_\alpha \vdash 1_\alpha & \{\alpha\}[a]A \vdash \bigwedge ([a][\beta_i]A) & \\
 \hline
 \text{reduce}' \frac{}{\Phi_\alpha; \{\alpha\}[a]A \vdash 1_\alpha \wedge \bigwedge ([a][\beta_i]A)} & & \\
 \frac{}{\{\alpha\}[a]A \vdash 1_\alpha \wedge \bigwedge ([a][\beta_i]A)} & & \\
 \frac{}{\langle \alpha \rangle [a]A \vdash 1_\alpha \wedge \bigwedge ([a][\beta_i]A)} & &
 \end{array}
 \end{array}$$

- $\text{Pre}(\alpha) \wedge \bigwedge \{ [a][\beta_i]A \mid \alpha a \beta_i \} \vdash \langle \alpha \rangle [a]A$

$$\begin{array}{c}
 \begin{array}{ccc}
 \frac{A \vdash A}{[\beta_1]A \vdash \{\beta_1\}A} & \cdots & \frac{A \vdash A}{[\beta_n]A \vdash \{\beta_n\}A} \\
 \frac{}{[a][\beta_1]A \vdash \{a\}\{\beta_1\}A} & \cdots & \frac{}{[a][\beta_n]A \vdash \{a\}\{\beta_n\}A} \\
 \hline
 & ;([a][\beta_i]A) \vdash \{\alpha\}\{a\}A & \\
 \hline
 & \bigwedge ([a][\beta_i]A) \vdash \{\alpha\}\{a\}A & \\
 \hline
 & \overline{\alpha} \bigwedge ([a][\beta_i]A) \vdash \{a\}A & \\
 \hline
 & \overline{\alpha} \bigwedge ([a][\beta_i]A) \vdash [a]A & \\
 \hline
 & \{\alpha\} \overline{\alpha} \bigwedge ([a][\beta_i]A) \vdash \langle \alpha \rangle [a]A & \\
 \hline
 & \Phi_\alpha; \bigwedge ([a][\beta_i]A) \vdash \langle \alpha \rangle [a]A & \\
 \hline
 & \bigwedge ([a][\beta_i]A); \Phi_\alpha \vdash \langle \alpha \rangle [a]A & \\
 \hline
 & \Phi_\alpha \vdash \bigwedge ([a][\beta_i]A) > \langle \alpha \rangle [a]A & \\
 \hline
 & 1_\alpha \vdash \bigwedge ([a][\beta_i]A) > \langle \alpha \rangle [a]A & \\
 \hline
 & \bigwedge ([a][\beta_i]A); 1_\alpha \vdash \langle \alpha \rangle [a]A & \\
 \hline
 & 1_\alpha; \bigwedge ([a][\beta_i]A) \vdash \langle \alpha \rangle [a]A & \\
 \hline
 & 1_\alpha \wedge \bigwedge ([a][\beta_i]A) \vdash \langle \alpha \rangle [a]A &
 \end{array}
 \end{array}$$

Appendix F

Cut Elimination for the Dynamic Calculus for EAK

Let us recall that C'_8 only concerns applications of the cut rules in which both occurrences of the given cut-term are *non parametric*. Notice that non parametric occurrences of atomic terms of type Fm involve an axiom on at least one premise, thus we are reduced to the following cases (the case of the constant \perp is symmetric to the case of \top and is omitted):

$$\frac{\Phi p \vdash p \quad p \vdash \Psi p}{\Phi p \vdash \Psi p} \rightsquigarrow \Phi p \vdash \Psi p \quad \frac{\begin{array}{c} \vdots \pi \\ I \vdash \top \\ \hline I \vdash X \end{array} \quad \frac{I \vdash X}{\top \vdash X}}{I \vdash X} \rightsquigarrow \begin{array}{c} \vdots \pi \\ I \vdash X \end{array}$$

Notice that non parametric occurrences of any given (atomic) operational term a of type Fnc or Ag are confined to axioms $a \vdash a$. Hence:

$$\frac{a \vdash a \quad a \vdash a}{a \vdash a} \rightsquigarrow a \vdash a$$

In each case above, the cut in the original derivation is strongly uniform by assumption, and is eliminated by the transformation. As to cuts on non atomic terms, let us restrict our attention to those cut-terms the main connective of which is $\Delta_i, \blacktriangle_i, \rightarrow_i, \dashv\rightarrow_i$ for $0 \leq i \leq 3$ (the remaining operational connectives are straightforward and left to the reader). In the derivations below, for every $0 \leq i \leq 3$, the symbols $\triangleright, \blacktriangleright, \triangleleft$ and \blacktriangleleft holds for the following structural symbols:

	$i = 0$	$i = 1$	$i = 2$	$i = 3$
\triangleright	\triangleright_0	\triangleright_1	\triangleright_2	$\sim\triangleright_3$
\blacktriangleright	\blacktriangleright_0	\blacktriangleright_1	\blacktriangleright_2	$\sim\blacktriangleright_3$
\triangleleft	$\triangleleft\sim_0$	\triangleleft_1	$\triangleleft\sim_2$	$\triangleleft\sim_3$
\blacktriangleleft	$\blacktriangleleft\sim_0$	\blacktriangleleft_1	$\blacktriangleleft\sim_2$	$\blacktriangleleft\sim_3$

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_0 \quad \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{x \vdash a \quad y \vdash b}{x \triangleleft_i y \vdash a \triangleleft_i b} \quad \frac{a \triangleleft_i b \vdash z}{a \triangleleft_i b \vdash z} \\
 \hline
 x \triangleleft_i y \vdash z
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_2 \\
 \frac{\vdots \pi_0 \quad \frac{a \triangleleft_i b \vdash z}{a \vdash z \blacktriangleleft_i b}}{x \vdash a \quad a \vdash z \blacktriangleleft_i b} \\
 \hline
 x \vdash z \blacktriangleleft_i b \\
 \hline
 \frac{\vdots \pi_1 \quad \frac{x \triangleleft_i b \vdash z}{b \vdash x \blacktriangleright_i z}}{y \vdash b \quad b \vdash x \blacktriangleright_i z} \\
 \hline
 y \vdash x \blacktriangleright_i z \\
 \hline
 x \triangleleft_i y \vdash z
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_0 \quad \vdots \pi_2 \\
 \frac{y \vdash a \triangleright_i b}{y \vdash a \rightarrow_i b} \quad \frac{x \vdash a \quad b \vdash z}{a \rightarrow_i b \vdash x \triangleright_i z} \\
 \hline
 y \vdash x \triangleright_i z
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_1 \\
 \frac{y \vdash a \triangleright_i b}{y \vdash a \triangleright_i b} \\
 \hline
 \frac{\vdots \pi_0 \quad \frac{a \blacktriangleleft_i y \vdash b}{a \vdash b \triangleleft_i y}}{x \vdash a \quad a \vdash b \triangleleft_i y} \\
 \hline
 x \vdash b \triangleleft_i y \\
 \hline
 \frac{x \blacktriangleleft_i y \vdash b}{x \blacktriangleleft_i y \vdash b} \quad \frac{\vdots \pi_2}{b \vdash z} \\
 \hline
 x \blacktriangleleft_i y \vdash z \\
 \hline
 y \vdash x \triangleright_i z
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_0 \quad \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{x \vdash a \quad y \vdash b}{x \blacktriangleleft_i y \vdash a \blacktriangleleft_i b} \quad \frac{a \blacktriangleleft_i b \vdash z}{a \blacktriangleleft_i b \vdash z} \\
 \hline
 x \blacktriangleleft_i y \vdash z
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_2 \\
 \frac{\vdots \pi_0 \quad \frac{a \blacktriangleleft_i b \vdash z}{a \vdash z \triangleleft_i b}}{x \vdash a \quad a \vdash z \triangleleft_i b} \\
 \hline
 x \vdash z \triangleleft_i b \\
 \hline
 \frac{\vdots \pi_1 \quad \frac{x \blacktriangleleft_i b \vdash z}{b \vdash x \triangleright_i z}}{y \vdash b \quad b \vdash x \triangleright_i z} \\
 \hline
 y \vdash x \triangleright_i z \\
 \hline
 x \blacktriangleleft_i y \vdash z
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \vdots \pi_1 \\
 \frac{y \vdash a \blacktriangleright_i b}{y \vdash a \rightarrow_i b} \\
 \hline
 \vdots \pi_0 \quad \vdots \pi_2 \\
 \frac{x \vdash a \quad b \vdash z}{a \rightarrow_i b \vdash x \blacktriangleright_i z} \\
 \hline
 y \vdash x \blacktriangleright_i z
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \vdots \pi_1 \\
 \frac{y \vdash a \blacktriangleright_i b}{a \triangleleft_i y \vdash b} \\
 \hline
 \vdots \pi_0 \\
 \frac{x \vdash a \quad a \vdash b \blacktriangleleft_i y}{x \vdash b \blacktriangleleft_i y} \\
 \hline
 \frac{x \triangleleft_i y \vdash b \quad b \vdash z}{x \triangleleft_i y \vdash z} \\
 \hline
 y \vdash x \blacktriangleright_i z
 \end{array}$$

In each case above, the cut in the original derivation is strongly uniform by assumption, and after the transformation, cuts of lower complexity are introduced which can be easily verified to be strongly uniform for each $0 \leq i \leq 3$.

Appendix G

Completeness for the Dynamic Calculus for EAK

To prove the completeness of the Dynamic Calculus it is enough to show that all the axioms and rules of IEAK are theorems and, respectively, derived or admissible rules of Dynamic Calculus. Below we show the derivations of the dynamic axioms.

- $\alpha \Delta p \dashv\vdash (\alpha \Delta \top) \wedge p$

$$\begin{array}{c}
 \frac{\frac{\frac{\alpha \vdash \alpha}{\alpha \Delta \top} \quad \text{I} \vdash \top}{\alpha \Delta \top \vdash \alpha \Delta \top} \quad \alpha \Delta p \vdash p}{(\alpha \Delta \top); (\alpha \Delta p) \vdash (\alpha \Delta \top) \wedge p} \\
 \frac{\alpha \Delta (\text{I}; p) \vdash (\alpha \Delta \top) \wedge p}{\text{I}; p \vdash \blacktriangleright (\alpha \Delta \top) \wedge p} \\
 \frac{\text{I}; p \vdash \blacktriangleright (\alpha \Delta \top) \wedge p}{p \vdash \blacktriangleright (\alpha \Delta \top) \wedge p} \\
 \frac{\alpha \Delta p \vdash (\alpha \Delta \top) \wedge p}{\alpha \Delta p \vdash (\alpha \Delta \top) \wedge p}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\alpha \blacktriangleright p \vdash p}{\text{I} \vdash (\alpha \blacktriangleright p) > p} \\
 \frac{\text{I} \vdash (\alpha \blacktriangleright p) > p}{\top \vdash (\alpha \blacktriangleright p) > p} \\
 \frac{\alpha \vdash \alpha \quad (\alpha \blacktriangleright p); \top \vdash p}{\alpha \Delta ((\alpha \blacktriangleright p); \top) \vdash \alpha \Delta p} \\
 \frac{\alpha \Delta ((\alpha \blacktriangleright p); \top) \vdash \alpha \Delta p}{(\alpha \Delta \top); p \vdash \alpha \Delta p} \\
 \frac{(\alpha \Delta \top); p \vdash \alpha \Delta p}{\alpha \Delta \top \vdash \alpha \Delta p < p} \\
 \frac{\alpha \Delta \top \vdash \alpha \Delta p < p}{\alpha \Delta \top; p \vdash \alpha \Delta p} \\
 \frac{\alpha \Delta \top; p \vdash \alpha \Delta p}{(\alpha \Delta \top) \wedge p \vdash \alpha \Delta p}
 \end{array}$$

- $\alpha \rightarrow p \Vdash (\alpha \Delta \top) \rightarrow p$

$$\begin{array}{c}
\frac{\frac{\frac{\alpha \vdash \alpha \quad p \vdash \alpha \blacktriangleright p}{\alpha \rightarrow p \vdash \alpha \triangleright (\alpha \blacktriangleright p)}}{\alpha \blacktriangle (\alpha \rightarrow p) \vdash \alpha \blacktriangleright p}}{\frac{I \vdash (\alpha \blacktriangle (\alpha \rightarrow p)) > \alpha \blacktriangleright p}{\top \vdash (\alpha \blacktriangle (\alpha \rightarrow p)) > \alpha \blacktriangleright p}}}{\frac{(\alpha \blacktriangle (\alpha \rightarrow p)); \top \vdash \alpha \blacktriangleright p}{\alpha \triangle ((\alpha \blacktriangle (\alpha \rightarrow p)); \top) \vdash p}}{\frac{(\alpha \rightarrow p); (\alpha \triangle \top) \vdash p}{\alpha \Delta \top \vdash (\alpha \rightarrow p) > p}}}{\frac{(\alpha \rightarrow p); (\alpha \Delta \top) \vdash p}{(\alpha \Delta \top); (\alpha \rightarrow p) \vdash p}}{\frac{\alpha \rightarrow p \vdash (\alpha \Delta \top) > p}{\alpha \rightarrow p \vdash (\alpha \Delta \top) \rightarrow p}}
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{\frac{\alpha \vdash \alpha \quad I \vdash \top}{\alpha \triangle I \vdash \alpha \Delta \top} \quad p \vdash \alpha \triangleright p}{(\alpha \Delta \top) \rightarrow p \vdash (\alpha \triangle I) > (\alpha \triangleright p)}}{(\alpha \Delta \top) \rightarrow p \vdash \alpha \triangleright (I > p)} \\
\frac{\alpha \blacktriangle ((\alpha \Delta \top) \rightarrow p) \vdash I > p}{\alpha \blacktriangle ((\alpha \Delta \top) \rightarrow p) \vdash p} \\
\frac{(\alpha \Delta \top) \rightarrow p \vdash \alpha \triangleright p}{(\alpha \Delta \top) \rightarrow p \vdash \alpha \rightarrow p}
\end{array}$$

- $\langle \alpha \rangle \top \Vdash 1_\alpha \rightsquigarrow \alpha \Delta \top \Vdash \alpha \Delta \top$

$$\frac{\frac{\alpha \vdash \alpha \quad \top \vdash \top}{\alpha \triangle \top \vdash \alpha \Delta \top}}{\alpha \Delta \top \vdash \alpha \Delta \top}$$

- $\alpha \rightarrow \perp \Vdash \alpha \Delta \top \rightarrow \perp$

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\perp \vdash I}{\alpha \vdash \alpha \quad \perp \vdash \alpha \blacktriangleright I}}{\alpha \rightarrow \perp \vdash \alpha \triangleright (\alpha \blacktriangleright I)}}{\alpha \blacktriangle (\alpha \rightarrow \perp) \vdash \alpha \blacktriangleright I}}{\frac{I \vdash (\alpha \blacktriangle (\alpha \rightarrow \perp)) > (\alpha \blacktriangleright I)}{\top \vdash (\alpha \blacktriangle (\alpha \rightarrow \perp)) > (\alpha \blacktriangleright I)}}}{\frac{\alpha \blacktriangle (\alpha \rightarrow \perp); \top \vdash \alpha \blacktriangleright I}{\alpha \triangle (\alpha \blacktriangle (\alpha \rightarrow \perp); \top) \vdash I}}{\frac{\alpha \rightarrow \perp; (\alpha \triangle \top) \vdash I}{\alpha \rightarrow \perp; (\alpha \triangle \top) \vdash \perp}}}{\frac{\frac{\frac{\alpha \triangle \top \vdash \alpha \rightarrow \perp > \perp}{\alpha \Delta \top \vdash \alpha \rightarrow \perp > \perp}}{\alpha \rightarrow \perp; \alpha \Delta \top \vdash \perp}}{\alpha \Delta \top; \alpha \rightarrow \perp \vdash \perp}}{\frac{\alpha \rightarrow \perp \vdash \alpha \Delta \top > \perp}{\alpha \rightarrow \perp \vdash \alpha \Delta \top \rightarrow \perp}}
\end{array}
\qquad
\begin{array}{c}
\frac{\frac{\frac{\perp \vdash I}{\perp \vdash \alpha \triangleright I}}{\alpha \blacktriangle \perp \vdash I}}{\frac{\alpha \vdash \alpha \quad I \vdash \top}{\alpha \triangle I \vdash \alpha \Delta \top} \quad \frac{\perp \vdash \alpha \triangleright \perp}{\perp \vdash \alpha \triangleright \perp}}}{\frac{\alpha \Delta \top \rightarrow \perp \vdash \alpha \triangle I > \alpha \triangleright \perp}{\alpha \Delta \top \rightarrow \perp \vdash \alpha \triangleright (I > \perp)}}}{\frac{\alpha \blacktriangle (\alpha \Delta \top \rightarrow \perp) \vdash I > \perp}{I; \alpha \blacktriangle (\alpha \Delta \top \rightarrow \perp) \vdash \perp}}}{\frac{I \vdash \perp < \alpha \blacktriangle (\alpha \Delta \top \rightarrow \perp)}{\alpha \blacktriangle (\alpha \Delta \top \rightarrow \perp) \vdash \perp}}{\frac{\alpha \Delta \top \rightarrow \perp \vdash \alpha \triangleright \perp}{\alpha \Delta \top \rightarrow \perp \vdash \alpha \rightarrow \perp}}
\end{array}$$

- $\alpha \Delta \perp \dashv\vdash \perp$

$$\frac{\frac{\frac{\perp \vdash I}{\perp \vdash \alpha \blacktriangleright I}}{\alpha \Delta \perp \vdash I}}{\alpha \Delta \perp \vdash \perp}}{\alpha \Delta \perp \vdash \perp} \quad \frac{\perp \vdash I}{\perp \vdash \alpha \Delta \perp}$$

- $\alpha \rightarrow \top \dashv\vdash \top$

$$\frac{\frac{I \vdash \top}{\alpha \rightarrow \top \vdash \top}}{\alpha \rightarrow \top \vdash \top} \quad \frac{\frac{\frac{I \vdash \top}{\alpha \blacktriangleright I \vdash \top}}{I \vdash \alpha \blacktriangleright \top}}{\top \vdash \alpha \blacktriangleright \top}}{\alpha \blacktriangleright \top \vdash \top}}{\top \vdash \alpha \blacktriangleright \top}}{\top \vdash \alpha \rightarrow \top}$$

- $\alpha \rightarrow (A \wedge B) \dashv\vdash \alpha \rightarrow A \wedge \alpha \rightarrow B$

$$\frac{\frac{\frac{\frac{A \vdash A}{A; B \vdash A}}{\alpha \vdash \alpha} \quad \frac{B \vdash B}{A; B \vdash B}}{\alpha \vdash \alpha} \quad \frac{A \wedge B \vdash A}{\alpha \rightarrow (A \wedge B) \vdash \alpha \blacktriangleright A}}{\alpha \rightarrow (A \wedge B) \vdash \alpha \rightarrow A} \quad \frac{\frac{\frac{B \vdash B}{A; B \vdash B}}{\alpha \vdash \alpha} \quad \frac{A \wedge B \vdash B}{\alpha \rightarrow (A \wedge B) \vdash \alpha \blacktriangleright B}}{\alpha \rightarrow (A \wedge B) \vdash \alpha \rightarrow B}}{\alpha \rightarrow (A \wedge B); \alpha \rightarrow (A \wedge B) \vdash \alpha \rightarrow A \wedge \alpha \rightarrow B}}{\alpha \rightarrow (A \wedge B) \vdash \alpha \rightarrow A \wedge \alpha \rightarrow B}$$

$$\frac{\frac{\frac{\alpha \vdash \alpha \quad A \vdash A}{\alpha \rightarrow A \vdash \alpha \blacktriangleright A}}{\alpha \blacktriangleright (\alpha \rightarrow A) \vdash A} \quad \frac{\frac{\alpha \vdash \alpha \quad B \vdash B}{\alpha \rightarrow B \vdash \alpha \blacktriangleright B}}{\alpha \blacktriangleright (\alpha \rightarrow B) \vdash B}}{\alpha \blacktriangleright (\alpha \rightarrow A); \alpha \blacktriangleright (\alpha \rightarrow B) \vdash A \wedge B}}{\alpha \blacktriangleright (\alpha \rightarrow A); \alpha \rightarrow B \vdash \alpha \blacktriangleright (A \wedge B)}}{\alpha \rightarrow A; \alpha \rightarrow B \vdash \alpha \blacktriangleright (A \wedge B)}}{\alpha \rightarrow A \wedge \alpha \rightarrow B \vdash \alpha \rightarrow (A \wedge B)}$$

- $\alpha \Delta(A \wedge B) \dashv\vdash \alpha \Delta A \wedge \alpha \Delta B$

$$\begin{array}{c}
\frac{\frac{\frac{A \vdash A}{A; B \vdash A}}{A \wedge B \vdash A} \quad \frac{\frac{B \vdash B}{A; B \vdash B}}{A \wedge B \vdash B}}{\frac{\frac{\alpha \vdash \alpha}{\alpha \Delta A \wedge B \vdash \alpha \Delta A} \quad \frac{\alpha \vdash \alpha}{\alpha \Delta A \wedge B \vdash \alpha \Delta B}}{\alpha \Delta(A \wedge B) \vdash \alpha \Delta A \wedge \alpha \Delta B}} \\
\frac{\alpha \vdash \alpha}{\alpha \Delta(A \wedge B); \alpha \Delta(A \wedge B) \vdash \alpha \Delta A \wedge \alpha \Delta B} \\
\frac{\alpha \vdash \alpha}{\alpha \Delta(A \wedge B) \vdash \alpha \Delta A \wedge \alpha \Delta B}
\end{array}$$

$$\begin{array}{c}
\text{balance} \frac{\frac{A \vdash A}{\alpha \Delta A \vdash \alpha \triangleright A}}{\alpha \blacktriangle(\alpha \Delta A) \vdash A} \quad \frac{\frac{B \vdash B}{\alpha \Delta B \vdash \alpha \triangleright B}}{\alpha \blacktriangle(\alpha \Delta B) \vdash B} \text{balance} \\
\frac{\frac{\frac{\alpha \blacktriangle(\alpha \Delta A); \alpha \blacktriangle(\alpha \Delta B) \vdash A \wedge B}{\alpha \blacktriangle(\alpha \Delta A; \alpha \Delta B) \vdash A \wedge B}}{\alpha \Delta(\alpha \blacktriangle(\alpha \Delta A; \alpha \Delta B)) \vdash \alpha \Delta(A \wedge B)}}{\alpha \blacktriangle(\alpha \Delta A; \alpha \Delta B) \vdash \alpha \blacktriangleright(\alpha \Delta(A \wedge B))} \\
\frac{\alpha \blacktriangle(\alpha \Delta A; \alpha \Delta B); I \vdash \alpha \blacktriangleright(\alpha \Delta(A \wedge B))}{\alpha \Delta(\alpha \blacktriangle(\alpha \Delta A; \alpha \Delta B)); I \vdash \alpha \Delta(A \wedge B)} \text{conj} \\
\frac{\alpha \Delta A; (\alpha \Delta B; \alpha \Delta I) \vdash \alpha \Delta(A \wedge B)}{\alpha \Delta B; \alpha \Delta I \vdash \alpha \Delta A > \alpha \Delta(A \wedge B)} \\
\frac{\alpha \Delta(B; I) \vdash \alpha \Delta A > \alpha \Delta(A \wedge B)}{B; I \vdash \alpha \blacktriangleright(\alpha \Delta A > \alpha \Delta(A \wedge B))} \\
\frac{I \vdash B > (\alpha \blacktriangleright(\alpha \Delta A > \alpha \Delta(A \wedge B)))}{B \vdash \alpha \blacktriangleright(\alpha \Delta A > \alpha \Delta(A \wedge B))} \\
\frac{\alpha \Delta B \vdash \alpha \Delta A > \alpha \Delta(A \wedge B)}{\alpha \Delta B \vdash \alpha \Delta A > \alpha \Delta(A \wedge B)} \\
\frac{\alpha \Delta A; \alpha \Delta B \vdash \alpha \Delta(A \wedge B)}{\alpha \Delta A \vdash \alpha \Delta(A \wedge B) < \alpha \Delta B} \\
\frac{\alpha \Delta A \vdash \alpha \Delta(A \wedge B) < \alpha \Delta B}{\alpha \Delta A; \alpha \Delta B \vdash \alpha \Delta(A \wedge B)} \\
\frac{\alpha \Delta A; \alpha \Delta B \vdash \alpha \Delta(A \wedge B)}{\alpha \Delta A \wedge \alpha \Delta B \vdash \alpha \Delta(A \wedge B)}
\end{array}$$

- $\alpha \Delta (A \vee B) \dashv\vdash \alpha \Delta A \vee \alpha \Delta B$

$$\begin{array}{c}
\frac{\frac{\alpha \vdash \alpha \quad A \vdash A}{\alpha \Delta A \vdash \alpha \Delta A}}{A \vdash \alpha \blacktriangleright (\alpha \Delta A)} \quad \frac{\frac{\alpha \vdash \alpha \quad B \vdash B}{\alpha \Delta B \vdash \alpha \Delta B}}{B \vdash \alpha \blacktriangleright (\alpha \Delta B)} \\
\frac{A \vee B \vdash \alpha \blacktriangleright (\alpha \Delta A); \alpha \blacktriangleright (\alpha \Delta B)}{A \vee B \vdash \alpha \blacktriangleright (\alpha \Delta A; \alpha \Delta B)} \\
\frac{\alpha \Delta A \vee B \vdash \alpha \Delta A; \alpha \Delta B}{\alpha \Delta (A \vee B) \vdash \alpha \Delta A; \alpha \Delta B} \\
\frac{\alpha \Delta (A \vee B) \vdash \alpha \Delta A; \alpha \Delta B}{\alpha \Delta (A \vee B) \vdash \alpha \Delta A \vee \alpha \Delta B}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{A \vdash A}{A > A \vdash B}}{A \vdash A; B} \quad \frac{\frac{B \vdash B}{B < B \vdash A}}{B \vdash A; B} \\
\frac{\alpha \vdash \alpha \quad \frac{A \vdash A \vee B}{A \vdash A \vee B}}{\alpha \Delta A \vdash \alpha \Delta (A \vee B)} \quad \frac{\alpha \vdash \alpha \quad \frac{B \vdash A \vee B}{B \vdash A \vee B}}{\alpha \Delta B \vdash \alpha \Delta (A \vee B)} \\
\frac{\alpha \Delta A \vee \alpha \Delta B \vdash \alpha \Delta (A \vee B); \alpha \Delta (A \vee B)}{\alpha \Delta A \vee \alpha \Delta B \vdash \alpha \Delta (A \vee B)}
\end{array}$$

- $\alpha \rightarrow (A \vee B) \dashv\vdash (\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B)$

$$\begin{array}{c}
\frac{\frac{\frac{\alpha \vdash \alpha}{\alpha \Delta A \vdash \alpha \Delta A} \quad \frac{A \vdash A}{\alpha \Delta B \vdash \alpha \Delta B}}{A \vdash \alpha \blacktriangleright (\alpha \Delta A) \quad B \vdash \alpha \blacktriangleright (\alpha \Delta B)}}{A \vee B \vdash \alpha \blacktriangleright (\alpha \Delta A); \alpha \blacktriangleright (\alpha \Delta B)}}{A \vee B \vdash \alpha \blacktriangleright (\alpha \Delta A; \alpha \Delta B)} \\
\frac{\frac{\alpha \Delta (A \vee B) \vdash \alpha \Delta A; \alpha \Delta B}{\alpha \Delta (A \vee B) \vdash \alpha \Delta A \vee \alpha \Delta B}}{\alpha \vdash \alpha \quad A \vee B \vdash \alpha \blacktriangleright (\alpha \Delta A \vee \alpha \Delta B)} \\
\frac{\alpha \rightarrow (A \vee B) \vdash \alpha \blacktriangleright (\alpha \blacktriangleright (\alpha \Delta A \vee \alpha \Delta B))}{\alpha \blacktriangle (\alpha \rightarrow (A \vee B)) \vdash \alpha \blacktriangleright (\alpha \Delta A \vee \alpha \Delta B)} \\
\frac{I \vdash \alpha \blacktriangle (\alpha \rightarrow (A \vee B)) > \alpha \blacktriangleright (\alpha \Delta A \vee \alpha \Delta B)}{(\alpha \blacktriangle (\alpha \rightarrow (A \vee B))); I \vdash \alpha \blacktriangleright (\alpha \Delta A \vee \alpha \Delta B)} \\
\frac{\alpha \Delta ((\alpha \blacktriangle (\alpha \rightarrow (A \vee B))); I) \vdash \alpha \Delta A \vee \alpha \Delta B}{(\alpha \rightarrow (A \vee B)); \alpha \Delta I \vdash \alpha \Delta A \vee \alpha \Delta B} \text{conj}_0\Delta \\
\frac{\alpha \Delta I \vdash \alpha \rightarrow (A \vee B) > \alpha \Delta A \vee \alpha \Delta B}{I \vdash \alpha \blacktriangleright (\alpha \rightarrow (A \vee B) > \alpha \Delta A \vee \alpha \Delta B)} \\
\frac{I \vdash \alpha \blacktriangleright (\alpha \rightarrow (A \vee B) > \alpha \Delta A \vee \alpha \Delta B)}{\top \vdash \alpha \blacktriangleright (\alpha \rightarrow (A \vee B) > \alpha \Delta A \vee \alpha \Delta B)} \\
\frac{\alpha \Delta \top \vdash \alpha \rightarrow (A \vee B) > \alpha \Delta A \vee \alpha \Delta B}{\alpha \Delta \top \vdash \alpha \rightarrow (A \vee B) > \alpha \Delta A \vee \alpha \Delta B} \\
\frac{(\alpha \rightarrow (A \vee B)); (\alpha \Delta \top) \vdash \alpha \Delta A \vee \alpha \Delta B}{(\alpha \Delta \top); (\alpha \rightarrow (A \vee B)) \vdash \alpha \Delta A \vee \alpha \Delta B} \\
\frac{(\alpha \rightarrow (A \vee B)) \vdash (\alpha \Delta \top) > \alpha \Delta A \vee \alpha \Delta B}{(\alpha \rightarrow (A \vee B)) \vdash (\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B)}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\frac{A \vdash A}{\alpha \Delta A \vdash \alpha \blacktriangleright A} \quad \frac{B \vdash B}{\alpha \Delta B \vdash \alpha \blacktriangleright B}}{\alpha \Delta A \vee \alpha \Delta B \vdash \alpha \blacktriangleright A; \alpha \blacktriangleright B}}{\alpha \Delta A \vee \alpha \Delta B \vdash \alpha \blacktriangleright (A; B)} \\
\frac{\alpha \blacktriangle (\alpha \Delta A \vee \alpha \Delta B) \vdash A; B}{\alpha \vdash \alpha \quad I \vdash \top \quad \alpha \blacktriangle (\alpha \Delta A \vee \alpha \Delta B) \vdash A \vee B} \\
\frac{\alpha \Delta I \vdash \alpha \Delta \top \quad \alpha \Delta A \vee \alpha \Delta B \vdash \alpha \blacktriangleright (A \vee B)}{(\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B) \vdash \alpha \Delta I > \alpha \blacktriangleright (A \vee B)} \\
\frac{(\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B) \vdash \alpha \blacktriangleright (I > (A \vee B))}{\alpha \blacktriangle ((\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B)) \vdash I > (A \vee B)} \\
\frac{I; \alpha \blacktriangle ((\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B)) \vdash A \vee B}{I \vdash (A \vee B) < \alpha \blacktriangle ((\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B))} \\
\frac{\alpha \blacktriangle ((\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B)) \vdash A \vee B}{(\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B) \vdash \alpha \blacktriangleright (A \vee B)} \\
\frac{(\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B) \vdash \alpha \blacktriangleright (A \vee B)}{(\alpha \Delta \top) \rightarrow (\alpha \Delta A \vee \alpha \Delta B) \vdash \alpha \rightarrow (A \vee B)}
\end{array}$$

- $\alpha \Delta (A \rightarrow B) \dashv\vdash (\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B)$

$$\begin{array}{c}
\frac{A \vdash A}{\alpha \Delta A \vdash \alpha \triangleright A} \quad \frac{\alpha \vdash \alpha \quad B \vdash B}{\alpha \Delta B \vdash \alpha \Delta B} \\
\frac{\alpha \Delta A \vdash \alpha \triangleright A}{\alpha \blacktriangle (\alpha \Delta A) \vdash A} \quad \frac{B \vdash \alpha \blacktriangleright (\alpha \Delta B)}{B \vdash \alpha \blacktriangleright (\alpha \Delta B)} \\
\frac{A \rightarrow B \vdash (\alpha \blacktriangle (\alpha \Delta A)) > (\alpha \blacktriangleright (\alpha \Delta B))}{A \rightarrow B \vdash \alpha \blacktriangleright (\alpha \Delta A > \alpha \Delta B)} \\
\frac{\alpha \vdash \alpha \quad I \vdash \top}{\alpha \Delta I \vdash \alpha \Delta \top} \quad \frac{\alpha \Delta (A \rightarrow B) \vdash \alpha \Delta A > \alpha \Delta B}{\alpha \Delta (A \rightarrow B) \vdash \alpha \Delta A \rightarrow \alpha \Delta B} \\
\frac{(\alpha \Delta I); \alpha \Delta (A \rightarrow B) \vdash (\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B)}{\alpha \Delta (I; A \rightarrow B) \vdash (\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B)} \\
\frac{I; A \rightarrow B \vdash \alpha \blacktriangleright ((\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B))}{I \vdash (\alpha \blacktriangleright ((\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B))) < (A \rightarrow B)} \\
\frac{A \rightarrow B \vdash \alpha \blacktriangleright ((\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B))}{\alpha \Delta (A \rightarrow B) \vdash (\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B)} \\
\frac{\alpha \Delta (A \rightarrow B) \vdash (\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B)}{\alpha \Delta (A \rightarrow B) \vdash (\alpha \Delta \top) \wedge (\alpha \Delta A \rightarrow \alpha \Delta B)}
\end{array}$$

$$\begin{array}{c}
\frac{B \vdash B}{\alpha \Delta B \vdash \alpha \triangleright B} \\
\frac{\alpha \vdash \alpha \quad A \vdash A}{\alpha \Delta A \vdash \alpha \Delta A} \quad \frac{\alpha \Delta B \vdash \alpha \triangleright B}{\alpha \Delta B \vdash \alpha \triangleright B} \\
\frac{\alpha \Delta A \rightarrow \alpha \Delta B \vdash \alpha \Delta A > \alpha \triangleright B}{\alpha \Delta A \rightarrow \alpha \Delta B \vdash \alpha \triangleright (A > B)} \\
\frac{\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B) \vdash A > B}{\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B) \vdash A \rightarrow B} \\
\frac{\alpha \Delta (\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B)) \vdash \alpha \Delta (A \rightarrow B)}{\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B) \vdash \alpha \blacktriangleright (\alpha \Delta (A \rightarrow B))} \\
\frac{I \vdash \alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B) > \alpha \blacktriangleright (\alpha \Delta (A \rightarrow B))}{\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B); I \vdash \alpha \blacktriangleright (\alpha \Delta (A \rightarrow B))} \\
\frac{\alpha \Delta (\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B)); I \vdash \alpha \Delta (A \rightarrow B)}{\alpha \Delta A \rightarrow \alpha \Delta B; (\alpha \Delta I) \vdash \alpha \Delta (A \rightarrow B)} \\
\frac{\alpha \Delta I \vdash \alpha \Delta A \rightarrow \alpha \Delta B > \alpha \Delta (A \rightarrow B)}{I \vdash \alpha \blacktriangleright (\alpha \Delta A \rightarrow \alpha \Delta B > \alpha \Delta (A \rightarrow B))} \\
\frac{\top \vdash \alpha \blacktriangleright (\alpha \Delta A \rightarrow \alpha \Delta B > \alpha \Delta (A \rightarrow B))}{\alpha \Delta \top \vdash \alpha \Delta A \rightarrow \alpha \Delta B > \alpha \Delta (A \rightarrow B)} \\
\frac{\alpha \Delta \top \vdash \alpha \Delta A \rightarrow \alpha \Delta B > \alpha \Delta (A \rightarrow B)}{\alpha \Delta A \rightarrow \alpha \Delta B; \alpha \Delta \top \vdash \alpha \Delta (A \rightarrow B)} \\
\frac{\alpha \Delta \top; \alpha \Delta A \rightarrow \alpha \Delta B \vdash \alpha \Delta (A \rightarrow B)}{\alpha \Delta \top \wedge (\alpha \Delta A \rightarrow \alpha \Delta B) \vdash \alpha \Delta (A \rightarrow B)}
\end{array}$$

- $\alpha \rightarrow (A \rightarrow B) \dashv\vdash \alpha \Delta A \rightarrow \alpha \Delta B$

$$\begin{array}{c}
\frac{\frac{\frac{A \vdash A}{\alpha \Delta A \vdash \alpha \triangleright A}}{\alpha \blacktriangle (\alpha \Delta A) \vdash A} \quad \frac{\frac{\alpha \vdash \alpha \quad B \vdash B}{\alpha \Delta B \vdash \alpha \Delta B}}{B \vdash \alpha \blacktriangleright (\alpha \Delta B)}}{A \rightarrow B \vdash \alpha \blacktriangle (\alpha \Delta A) > \alpha \blacktriangleright (\alpha \Delta B)} \\
\frac{\alpha \vdash \alpha \quad A \rightarrow B \vdash \alpha \blacktriangleright (\alpha \Delta A > \alpha \Delta B)}{\alpha \rightarrow (A \rightarrow B) \vdash \alpha \triangleright (\alpha \blacktriangleright (\alpha \Delta A > \alpha \Delta B))} \\
\frac{\alpha \blacktriangle (\alpha \rightarrow (A \rightarrow B)) \vdash \alpha \blacktriangleright (\alpha \Delta A > \alpha \Delta B)}{I \vdash \alpha \blacktriangle (\alpha \rightarrow (A \rightarrow B)) > \alpha \blacktriangleright (\alpha \Delta A > \alpha \Delta B)} \\
\frac{\alpha \blacktriangle (\alpha \rightarrow (A \rightarrow B)); I \vdash \alpha \blacktriangleright (\alpha \Delta A > \alpha \Delta B)}{\alpha \Delta (\alpha \blacktriangle (\alpha \rightarrow (A \rightarrow B)); I) \vdash \alpha \Delta A > \alpha \Delta B} \text{conj} \\
\frac{\alpha \rightarrow (A \rightarrow B); (\alpha \Delta I) \vdash \alpha \Delta A > \alpha \Delta B}{(\alpha \Delta I); \alpha \rightarrow (A \rightarrow B) \vdash \alpha \Delta A > \alpha \Delta B} \\
\frac{\alpha \Delta A; ((\alpha \Delta I); \alpha \rightarrow (A \rightarrow B)) \vdash \alpha \Delta B}{(\alpha \Delta A; (\alpha \Delta I)); \alpha \rightarrow (A \rightarrow B) \vdash \alpha \Delta B} \\
\frac{\alpha \Delta A; (\alpha \Delta I) \vdash \alpha \Delta B < \alpha \rightarrow (A \rightarrow B)}{\alpha \Delta (A; I) \vdash \alpha \Delta B < \alpha \rightarrow (A \rightarrow B)} \\
\frac{A; I \vdash \alpha \blacktriangleright (\alpha \Delta B < \alpha \rightarrow (A \rightarrow B))}{I \vdash A > (\alpha \blacktriangleright (\alpha \Delta B < \alpha \rightarrow (A \rightarrow B)))} \\
\frac{A \vdash \alpha \blacktriangleright (\alpha \Delta B < \alpha \rightarrow (A \rightarrow B))}{\alpha \Delta A \vdash \alpha \Delta B < \alpha \rightarrow (A \rightarrow B)} \\
\frac{\alpha \Delta A \vdash \alpha \Delta B < \alpha \rightarrow (A \rightarrow B)}{\alpha \Delta A \vdash \alpha \Delta B < \alpha \rightarrow (A \rightarrow B)} \\
\frac{\alpha \Delta A; \alpha \rightarrow (A \rightarrow B) \vdash \alpha \Delta B}{\alpha \rightarrow (A \rightarrow B) \vdash \alpha \Delta A > \alpha \Delta B} \\
\alpha \rightarrow (A \rightarrow B) \vdash \alpha \Delta A \rightarrow \alpha \Delta B
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\frac{\alpha \vdash \alpha \quad A \vdash A}{\alpha \Delta A \vdash \alpha \Delta A} \quad \frac{\frac{\alpha \vdash \alpha \quad B \vdash B}{\alpha \Delta B \vdash \alpha \Delta B}}{\alpha \Delta B \vdash \alpha \triangleright B}}{\alpha \Delta A \rightarrow \alpha \Delta B \vdash \alpha \Delta A > \alpha \triangleright B}}{\alpha \Delta A \rightarrow \alpha \Delta B \vdash \alpha \triangleright (A > B)} \\
\frac{\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B) \vdash A > B}{\alpha \blacktriangle (\alpha \Delta A \rightarrow \alpha \Delta B) \vdash A \rightarrow B} \\
\frac{\alpha \Delta A \rightarrow \alpha \Delta B \vdash \alpha \triangleright A \rightarrow B}{\alpha \Delta A \rightarrow \alpha \Delta B \vdash \alpha \rightarrow (A \rightarrow B)}
\end{array}$$

- $1_\alpha \wedge \bigvee \{ \langle \mathbf{a} \rangle \langle \beta \rangle A \mid \alpha \mathbf{a} \beta \} \vdash \langle \alpha \rangle \langle \mathbf{a} \rangle A \rightsquigarrow (\alpha \Delta \top) \wedge (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)) \vdash \alpha \Delta (\mathbf{a} \Delta A)$

$$\begin{array}{c}
\frac{\frac{\frac{\mathbf{a} \vdash \mathbf{a} \quad A \vdash A}{\alpha \vdash \alpha} \quad \frac{\mathbf{a} \Delta A \vdash \mathbf{a} \Delta A}{\alpha \Delta (\mathbf{a} \Delta A) \vdash \alpha \Delta (\mathbf{a} \Delta A)}}{\alpha \Delta (\mathbf{a} \Delta A) \vdash \alpha \Delta (\mathbf{a} \Delta A)}}{\mathbf{a} \Delta A \vdash \alpha \blacktriangleright (\alpha \Delta (\mathbf{a} \Delta A))} \\
\frac{\frac{\frac{\mathbf{a} \Delta A \vdash \alpha \blacktriangleright (\alpha \Delta (\mathbf{a} \Delta A))}{A \vdash \mathbf{a} \blacktriangleright (\alpha \blacktriangleright (\alpha \Delta (\mathbf{a} \Delta A)))} \quad \frac{\mathbf{a} \Delta A \vdash \alpha \blacktriangleright (\alpha \Delta (\mathbf{a} \Delta A))}{A \vdash (\mathbf{a} \blacktriangle \alpha) \blacktriangleright (\mathbf{a} \blacktriangleright ((\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A))))}}{\mathbf{a} \Delta A \vdash \alpha \blacktriangleright ((\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A)))} \text{ swap-in}_R \\
\frac{\frac{\frac{\mathbf{a} \blacktriangle \alpha \vdash (\mathbf{a} \blacktriangleright ((\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A)))) \blacktriangleleft A}{\mathbf{a} \blacktriangle \alpha \vdash (\mathbf{a} \blacktriangleright ((\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A)))) \blacktriangleleft A} \quad \frac{\mathbf{a} \blacktriangle \alpha \vdash (\mathbf{a} \blacktriangleright ((\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A)))) \blacktriangleleft A}{(\mathbf{a} \blacktriangle \alpha) \Delta A \vdash \mathbf{a} \blacktriangleright ((\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A)))}}{\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A) \vdash (\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A))} \\
\frac{\frac{\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A) \vdash (\alpha \Delta \mathbf{I}) > (\alpha \Delta (\mathbf{a} \Delta A))}{(\alpha \Delta \mathbf{I}); (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)) \vdash \alpha \Delta (\mathbf{a} \Delta A)}}{\frac{\frac{\frac{\alpha \Delta \mathbf{I} \vdash (\alpha \Delta (\mathbf{a} \Delta A)) < (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A))}{\mathbf{I} \vdash \alpha \blacktriangleright ((\alpha \Delta (\mathbf{a} \Delta A)) < (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)))} \quad \frac{\mathbf{I} \vdash \alpha \blacktriangleright ((\alpha \Delta (\mathbf{a} \Delta A)) < (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)))}{\top \vdash \alpha \blacktriangleright ((\alpha \Delta (\mathbf{a} \Delta A)) < (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)))}}{\alpha \Delta \top \vdash (\alpha \Delta (\mathbf{a} \Delta A)) < (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A))} \\
\frac{\frac{\alpha \Delta \top \vdash (\alpha \Delta (\mathbf{a} \Delta A)) < (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A))}{(\alpha \Delta \top); (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)) \vdash \alpha \Delta (\mathbf{a} \Delta A)}}{\frac{(\alpha \Delta \top); (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)) \vdash \alpha \Delta (\mathbf{a} \Delta A)}{(\alpha \Delta \top) \wedge (\mathbf{a} \Delta ((\mathbf{a} \blacktriangle \alpha) \Delta A)) \vdash \alpha \Delta (\mathbf{a} \Delta A)}}
\end{array}$$

- $[\alpha][a]A \vdash \text{Pre}(\alpha) \rightarrow \wedge\{[a][\beta]A \mid \alpha\beta\} \rightsquigarrow \alpha \rightarrow (a \rightarrow A) \vdash (\alpha \Delta \top) \rightarrow (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))$

$$\begin{array}{c}
\frac{\frac{\frac{a \vdash a \quad A \vdash A}{\alpha \vdash \alpha \quad a \rightarrow A \vdash a \blacktriangleright A}}{\alpha \rightarrow (a \rightarrow A) \vdash \alpha \blacktriangleright (a \blacktriangleright A)}}{\alpha \blacktriangle (\alpha \rightarrow (a \rightarrow A)) \vdash a \blacktriangleright A} \\
\frac{\frac{\frac{\frac{a \blacktriangle (\alpha \blacktriangle (\alpha \rightarrow (a \rightarrow A))) \vdash A}{(a \blacktriangle \alpha) \blacktriangle (a \blacktriangle ((\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A)))) \vdash A}}{\frac{a \blacktriangle \alpha \vdash A \blacktriangleleft (a \blacktriangle ((\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A))))}{a \blacktriangle \alpha \vdash A \blacktriangleleft (a \blacktriangle ((\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A))))}}{(a \blacktriangle \alpha) \blacktriangle (a \blacktriangle ((\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A)))) \vdash A}}}{\frac{a \blacktriangle ((\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A))) \vdash (a \blacktriangle \alpha) \blacktriangleright A}{a \blacktriangle ((\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A))) \vdash (a \blacktriangle \alpha) \rightarrow A}}{\frac{(\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A)) \vdash a \blacktriangleright ((a \blacktriangle \alpha) \rightarrow A)}{(\alpha \triangle I); (\alpha \rightarrow (a \rightarrow A)) \vdash a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)}}{\frac{\alpha \triangle I \vdash (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) < (\alpha \rightarrow (a \rightarrow A))}{I \vdash \alpha \blacktriangleright ((a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) < (\alpha \rightarrow (a \rightarrow A)))}}{\frac{\top \vdash \alpha \blacktriangleright ((a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) < (\alpha \rightarrow (a \rightarrow A)))}{\alpha \triangle \top \vdash (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) < (\alpha \rightarrow (a \rightarrow A))}}{\frac{\alpha \Delta \top \vdash (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) < (\alpha \rightarrow (a \rightarrow A))}{(\alpha \Delta \top); (\alpha \rightarrow (a \rightarrow A)) \vdash a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)}}{\frac{(\alpha \rightarrow (a \rightarrow A)) \vdash (\alpha \Delta \top) > (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))}{(\alpha \rightarrow (a \rightarrow A)) \vdash (\alpha \Delta \top) \rightarrow (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))}}
\end{array}$$

- $\langle \alpha \rangle [a]A \vdash \text{Pre}(\alpha) \wedge \wedge\{[a][\beta]A \mid \alpha\beta\} \rightsquigarrow \alpha \Delta (a \rightarrow A) \vdash (\alpha \Delta \top) \wedge (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))$

$$\begin{array}{c}
\frac{\frac{\frac{a \vdash a \quad A \vdash A}{a \rightarrow A \vdash a \blacktriangleright A}}{\alpha \triangle (a \rightarrow A) \vdash \alpha \blacktriangleright (a \blacktriangleright A)}}{\alpha \blacktriangle (\alpha \triangle (a \rightarrow A)) \vdash a \blacktriangleright A} \\
\frac{\frac{\frac{\frac{a \blacktriangle (\alpha \blacktriangle (\alpha \triangle (a \rightarrow A))) \vdash A}{(a \blacktriangle \alpha) \blacktriangle (a \blacktriangle ((\alpha \triangle I); (\alpha \triangle (a \rightarrow A)))) \vdash A}}{\frac{a \blacktriangle \alpha \vdash A \blacktriangleleft (a \blacktriangle ((\alpha \triangle I); (\alpha \triangle (a \rightarrow A))))}{a \blacktriangle \alpha \vdash A \blacktriangleleft (a \blacktriangle ((\alpha \triangle I); (\alpha \triangle (a \rightarrow A))))}}{(a \blacktriangle \alpha) \blacktriangle (a \blacktriangle ((\alpha \triangle I); (\alpha \triangle (a \rightarrow A)))) \vdash A}}}{\frac{a \blacktriangle ((\alpha \triangle I); (\alpha \triangle (a \rightarrow A))) \vdash (a \blacktriangle \alpha) \blacktriangleright A}{a \blacktriangle ((\alpha \triangle I); (\alpha \triangle (a \rightarrow A))) \vdash (a \blacktriangle \alpha) \rightarrow A}}{\frac{(\alpha \triangle I); (\alpha \triangle (a \rightarrow A)) \vdash a \blacktriangleright ((a \blacktriangle \alpha) \rightarrow A)}{(\alpha \triangle I); (\alpha \triangle (a \rightarrow A)) \vdash a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)}}{\frac{\alpha \triangle I \vdash (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) < (\alpha \rightarrow (a \rightarrow A))}{I \vdash \top}}{\frac{\alpha \triangle \top \vdash (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A)) < (\alpha \rightarrow (a \rightarrow A))}{\alpha \Delta (a \rightarrow A) \vdash (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))}}{\frac{(\alpha \rightarrow (a \rightarrow A)) \vdash (\alpha \Delta \top) \wedge (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))}{\alpha \triangle (a \rightarrow A) \vdash (\alpha \Delta \top) \wedge (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))}}{\frac{\alpha \Delta (a \rightarrow A) \vdash (\alpha \Delta \top) \wedge (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))}{\alpha \Delta (a \rightarrow A) \vdash (\alpha \Delta \top) \wedge (a \rightarrow ((a \blacktriangle \alpha) \rightarrow A))}}
\end{array}$$

- $Pre(\alpha) \wedge \wedge\{[a][\beta]A \mid \alpha a \beta\} \vdash \langle \alpha \rangle [a]A \rightsquigarrow (\alpha \Delta \top) \wedge a \rightarrow ((a \blacktriangle \alpha) \rightarrow A) \vdash \alpha \Delta (a \rightarrow A)$

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\frac{a \vdash a \quad \alpha \vdash \alpha}{a \blacktriangle \alpha \vdash a \blacktriangle \alpha} \quad A \vdash A}{a \vdash a \quad (a \blacktriangle \alpha) \rightarrow A \vdash (a \blacktriangle \alpha) \triangleright A}}{a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A) \vdash a \triangleright ((a \blacktriangle \alpha) \triangleright A)}}{a \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)) \vdash (a \blacktriangle \alpha) \triangleright A}}{\text{swap-out}_L \frac{(a \blacktriangle \alpha) \blacktriangle (a \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A))) \vdash A}{a \blacktriangle (\alpha \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A))) \vdash A}}{\frac{\alpha \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)) \vdash a \triangleright A}{\alpha \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)) \vdash a \rightarrow A}}{I \vdash \alpha \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)) > a \rightarrow A}} \\
\frac{\alpha \vdash \alpha \quad \alpha \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)); I \vdash a \rightarrow A}{\text{conj}_{\Delta} \frac{\alpha \Delta (\alpha \blacktriangle (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)); I) \vdash \alpha \Delta (a \rightarrow A)}{a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A); (\alpha \Delta I) \vdash \alpha \Delta (a \rightarrow A)}} \\
\frac{\alpha \Delta I \vdash a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A) > \alpha \Delta (a \rightarrow A)}{I \vdash \alpha \blacktriangleright (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)) > \alpha \Delta (a \rightarrow A)} \\
\frac{I \vdash \alpha \blacktriangleright (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)) > \alpha \Delta (a \rightarrow A)}{\top \vdash \alpha \blacktriangleright (a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A)) > \alpha \Delta (a \rightarrow A)} \\
\frac{\alpha \Delta \top \vdash a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A) > \alpha \Delta (a \rightarrow A)}{\alpha \Delta \top \vdash a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A) > \alpha \Delta (a \rightarrow A)} \\
\frac{a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A); \alpha \Delta \top \vdash \alpha \Delta (a \rightarrow A)}{(\alpha \Delta \top); a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A) \vdash \alpha \Delta (a \rightarrow A)} \\
(\alpha \Delta \top) \wedge a \rightarrow ((\alpha \blacktriangle \alpha) \rightarrow A) \vdash \alpha \Delta (a \rightarrow A)
\end{array}$$

Appendix H

The Proper Display-Type Calculus for Monotone Modal Logic

Structural Rules

$$Id \frac{}{p \vdash p}$$

$$I_L^1 \frac{X \vdash Y}{I; X \vdash Y} \quad \frac{Y \vdash X}{Y \vdash I; X} I_R^1$$

$$I_L^2 \frac{X \vdash Y}{X; I \vdash Y} \quad \frac{Y \vdash X}{Y \vdash X; I} I_R^2$$

$$IW_L \frac{I \vdash X}{Y \vdash X} \quad \frac{X \vdash I}{X \vdash Y} IW_R$$

$$W_L^1 \frac{Y \vdash Z}{X; Y \vdash Z} \quad \frac{Z \vdash X}{Z \vdash Y; X} W_R^1$$

$$W_L^2 \frac{X \vdash Z}{X; Y \vdash Z} \quad \frac{Z \vdash Y}{Z \vdash Y; X} W_R^2$$

$$C_L \frac{X; X \vdash Y}{X \vdash Y} \quad \frac{Y \vdash X; X}{Y \vdash X} C_R$$

$$E_L \frac{Y; X \vdash Z}{X; Y \vdash Z} \quad \frac{Z \vdash X; Y}{Z \vdash Y; X} E_R$$

$$A_L \frac{X; (Y; Z) \vdash W}{(X; Y); Z \vdash W} \quad \frac{W \vdash (Z; Y); X}{W \vdash Z; (Y; X)} A_R$$

Display Postulates

$$\begin{array}{c}
 (*,;) \frac{X \vdash Z; *Y}{X; Y \vdash Z} \quad \frac{Z; *Y \vdash X}{Z \vdash X; Y} (*,;) \\
 (;,*) \frac{X \vdash Z; *Y}{Y \vdash *X; Z} \quad \frac{Z; *Y \vdash X}{*X; Z \vdash Y} (;,*) \\
 (<,;) \frac{X \vdash Z < Y}{X; Y \vdash Z} \quad \frac{Z < Y \vdash X}{Z \vdash X; Y} (<,;) \\
 (;,>) \frac{X \vdash Z < Y}{Y \vdash X > Z} \quad \frac{Z < Y \vdash X}{X > Z \vdash Y} (;,>)
 \end{array}$$

Note that the following rules are derivable:

$$\frac{*X \vdash Y}{*Y \vdash X} \quad \frac{Y \vdash *X}{X \vdash *Y} \quad \frac{X \vdash Y}{**X \vdash Y}$$

The following table shows the operational rules for the propositional base:

Operational Rules

$$\begin{array}{c}
 \perp_L \frac{}{\perp \vdash I} \quad \frac{X \vdash I}{X \vdash \perp} \perp_R \\
 \top_L \frac{I \vdash X}{\top \vdash X} \quad \frac{}{I \vdash \top} \top_R \\
 \wedge_L \frac{A; B \vdash Z}{A \wedge B \vdash Z} \quad \frac{X \vdash A \quad Y \vdash B}{X; Y \vdash A \wedge B} \wedge_R \\
 \vee_L \frac{A \vdash X \quad B \vdash Y}{A \vee B \vdash X; Y} \quad \frac{Z \vdash A; B}{Z \vdash A \vee B} \vee_R \\
 \rightarrow_L \frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X > Y} \quad \frac{Z \vdash A > B}{Z \vdash A \rightarrow B} \rightarrow_R \\
 \neg_L \frac{X \vdash A}{\neg A \vdash *X} \quad \frac{A \vdash X}{*X \vdash \neg A} \neg_R \\
 (\forall\exists)_L \frac{\circ A \vdash X}{(\forall\exists)A \vdash X} \quad \frac{X \vdash A}{\circ X \vdash (\forall\exists)A} (\forall\exists)_R \\
 (\exists\forall)_L \frac{A \vdash X}{(\exists\forall)A \vdash \circ X} \quad \frac{X \vdash \circ A}{X \vdash (\exists\forall)A} (\exists\forall)_R
 \end{array}$$

Appendix I

The Calculus for the Propositional Base of PDL

Propositions Structural Rules

$$\begin{array}{ccc}
 I_L^1 \frac{X \vdash Y}{I \vdash Y < X} & \frac{X \vdash Y}{X < Y \vdash I} I_R^1 & I_L^2 \frac{X \vdash Y}{I \vdash X > Y} & \frac{X \vdash Y}{Y > X \vdash I} I_R^2 \\
 W_L^1 \frac{X \vdash Z}{Y \vdash Z < X} & \frac{X \vdash Z}{X < Z \vdash Y} W_R^1 & W_L^2 \frac{X \vdash Z}{Y \vdash X > Z} & \frac{X \vdash Z}{Z > X \vdash Y} W_R^2 \\
 C_L \frac{X, X \vdash Y}{X \vdash Y} & \frac{Y \vdash X, X}{Y \vdash X} C_R & A_L \frac{X, (Y, Z) \vdash W}{(X, Y), Z \vdash W} & \frac{W \vdash (Z, Y), X}{W \vdash Z, (Y, X)} A_R \\
 E_L \frac{Y, X \vdash Z}{X, Y \vdash Z} & \frac{Z \vdash X, Y}{Z \vdash Y, X} E_R & Gri_L \frac{X > (Y; Z) \vdash W}{(X > Y); Z \vdash W} & \frac{W \vdash X > (Y; Z)}{W \vdash (X > Y); Z} Gri_R
 \end{array}$$

The last rules in the table above, Gri_L and Gri_R , are known as Grishin's rules: here they are useful to force the classical behaviour of our propositional base (if we remove Gri , we will obtain a weaker logic cfr. []).

Propositions Display Postulates

$$\begin{array}{cc}
 \frac{X, Y \vdash Z}{Y \vdash X > Z} & \frac{Z \vdash X, Y}{X > Z \vdash Y} \\
 \frac{X, Y \vdash Z}{X \vdash Z < Y} & \frac{Z \vdash X, Y}{Z < Y \vdash X}
 \end{array}$$

Below we list the rules for the operational connective (note that the latest three connectives with the name of the rule in brackets are those which do not belong to the language of the axioms that we have implicitly chosen).

Propositions Operational Rules

$$\begin{array}{c}
\perp_L \frac{}{\perp \vdash \mathbf{I}} \quad \frac{X \vdash \mathbf{I}}{X \vdash \perp} \perp_R \qquad \top_L \frac{\mathbf{I} \vdash X}{\top \vdash X} \quad \frac{}{\mathbf{I} \vdash \top} \top_R \\
\wedge_L \frac{A, B \vdash X}{A \wedge B \vdash X} \quad \frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \wedge_R \quad \vee_L \frac{A \vdash X \quad B \vdash Y}{A \vee B \vdash X, Y} \quad \frac{X \vdash A, B}{X \vdash A \vee B} \vee_R \\
\rightarrow_L \frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X > Y} \quad \frac{X \vdash A > B}{X \vdash A \rightarrow B} \rightarrow_R \\
(\leftarrow_L) \frac{B \vdash Y \quad X \vdash A}{B \leftarrow A \vdash Y < X} \quad \frac{Z \vdash B < A}{Z \vdash B \leftarrow A} (\leftarrow_R) \\
(>_L) \frac{A > B \vdash Z}{A > B \vdash Z} \quad \frac{A \vdash X \quad Y \vdash B}{X > Y \vdash A > B} (>_R) \\
(<_L) \frac{B < A \vdash X}{B < A \vdash X} \quad \frac{Y \vdash B \quad A \vdash X}{Y < X \vdash B < A} (<_R)
\end{array}$$

Appendix J

Cut Elimination for PDL, Principal Stage

In the present subsection, we report on the verification of condition C₈' of the definition of quasi-proper multi-type display calculi (cf. Section 6.2.2).

Let us recall that C₈' only concerns applications of the cut rules in which both occurrences of the given cut-term are *non parametric*. Notice that non parametric occurrences of atomic terms of type Fm involve an axiom on at least one premise, thus we are reduced to the following cases (the case of the constant \perp is symmetric to the case of \top and is omitted):

$$\frac{p \vdash p \quad p \vdash p}{p \vdash p} \rightsquigarrow p \vdash p \qquad \frac{\begin{array}{c} \vdots \pi \\ \text{I} \vdash \top \quad \frac{\text{I} \vdash X}{\top \vdash X} \end{array}}{\text{I} \vdash X} \rightsquigarrow \begin{array}{c} \vdots \pi \\ \text{I} \vdash X \end{array}$$

Notice that non parametric occurrences of any given (atomic) operational term a of type Act and d of type Tact are confined to axioms $a \vdash a$ and $d \vdash d$, so the proofs are analogous to the previous case of operational term p of type Fm and they are omitted. In each of these cases, the cut in the original derivation is strongly-uniform by assumption, and is eliminated by the transformation. As to cuts on non atomic terms, let us now restrict our attention to those cut-terms the main connective of which is $\Delta_i, \blacktriangle_i, \rightarrow_i, \rightarrow_i$ for $0 \leq i \leq 1$. Here below we show the proofs only for the white heterogeneous connectives: the proofs for the black heterogeneous connectives are exactly the same modulo a uniform substitution of each white connective by the same black connective (both at the operational and structural level).

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_0 \quad \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{x \vdash a \quad y \vdash b}{x \Delta_i y \vdash a \Delta_i b} \quad \frac{a \Delta_i b \vdash z}{a \Delta_i b \vdash z} \\
 \hline
 x \Delta_i y \vdash z
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{y \vdash b \quad \frac{x \Delta_i b \vdash z}{b \vdash x \blacktriangleright_i z}}{y \vdash x \blacktriangleright_i z} \\
 \hline
 x \Delta_i y \vdash z
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_0 \quad \vdots \pi_2 \\
 \frac{y \vdash a \blacktriangleright_i b}{y \vdash a \rightarrow_i b} \quad \frac{x \vdash a \quad b \vdash z}{a \rightarrow_i b \vdash x \blacktriangleright_i z} \\
 \hline
 y \vdash x \blacktriangleright_i z
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{x \blacktriangleright_i y \vdash b \quad b \vdash z}{x \blacktriangleright_i y \vdash z} \\
 \hline
 y \vdash x \blacktriangleright_i z
 \end{array}
 \end{array}$$

In each of these cases, the cut in the original derivation is strongly-uniform by assumption, and after the transformation, cuts of lower complexity are introduced which can be easily verified to be strongly-uniform for each $0 \leq i \leq 1$.

Finally, let us consider the unary modalities test $?_i$ for $0 \leq i \leq 1$, positive iteration $+$ and its left adjoint $-$.

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{X \vdash A}{X ?_i \vdash A ?_i} \quad \frac{A ?_i \vdash Y}{A ?_i \vdash Y} \\
 \hline
 X ?_i \vdash Y
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{X \vdash A \quad \frac{A ?_i \vdash Y}{A \vdash Y \dot{\iota}_i}}{X \vdash Y \dot{\iota}_i} \\
 \hline
 X ?_i \vdash Y
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{\Pi \vdash \alpha}{\Pi^\oplus \vdash \alpha^+} \quad \frac{\alpha^\oplus \vdash \Delta}{\alpha^+ \vdash \Delta} \\
 \hline
 \Pi^\oplus \vdash \Delta
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{\Pi \vdash \alpha \quad \frac{\alpha^\oplus \vdash \Delta}{\alpha \vdash \Delta^\ominus}}{\Pi \vdash \Delta^\ominus} \\
 \hline
 \Pi^\oplus \vdash \Delta
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{\Pi \vdash \delta^\ominus}{\Pi \vdash \delta^-} \quad \frac{\delta \vdash \Delta}{\delta^- \vdash \Delta^\ominus} \\
 \hline
 \Pi \vdash \Delta^\ominus
 \end{array} \\
 \rightsquigarrow \\
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{\Pi \vdash \delta^\ominus}{\Pi^\oplus \vdash \delta} \quad \frac{\delta \vdash \Delta}{\delta \vdash \Delta} \\
 \hline
 \Pi^\oplus \vdash \Delta
 \end{array}
 \end{array}$$

In each case above, the cut in the original derivation is strongly-uniform by assumption, and after the transformation, cuts of lower complexity are introduced which can be easily verified to be strongly-uniform for each $0 \leq i \leq 1$ in the first proof and also for the remaining two proofs.

The remaining operational connectives are straightforward and left to the reader.

Appendix K

Completeness for PDL

Box-Choice $(\alpha \cup \beta) \rightarrow A \vdash (\alpha \rightarrow A) \wedge (\beta \rightarrow A)$

$$\begin{array}{c}
 \frac{\frac{\frac{\alpha \vdash \alpha}{\alpha \vdash \alpha} \text{w}_\emptyset}{\alpha \vdash \alpha \cup \beta} \quad A \vdash A}{(\alpha \cup \beta) \rightarrow A \vdash \alpha \triangleright A} \quad \frac{\frac{\frac{\beta \vdash \beta}{\beta \vdash \beta} \text{w}_\emptyset}{\beta \vdash \alpha \cup \beta} \quad A \vdash A}{(\alpha \cup \beta) \rightarrow A \vdash \beta \triangleright A} \\
 \frac{(\alpha \cup \beta) \rightarrow A \vdash \alpha \rightarrow A}{(\alpha \cup \beta) \rightarrow A \vdash \alpha \rightarrow A} \quad \frac{(\alpha \cup \beta) \rightarrow A \vdash \beta \rightarrow A}{(\alpha \cup \beta) \rightarrow A \vdash \beta \rightarrow A} \\
 \text{c}_L \frac{(\alpha \cup \beta) \rightarrow A, (\alpha \cup \beta) \rightarrow A \vdash (\alpha \rightarrow A) \wedge (\beta \rightarrow A)}{(\alpha \cup \beta) \rightarrow A \vdash (\alpha \rightarrow A) \wedge (\beta \rightarrow A)} \\
 \\
 \frac{\frac{\frac{\alpha \vdash \alpha}{\alpha \rightarrow A \vdash \alpha \triangleright A} \quad A \vdash A}{\alpha \blacktriangle \alpha \rightarrow A \vdash A} \quad \frac{\frac{\frac{\beta \vdash \beta}{\beta \rightarrow A \vdash \beta \triangleright A} \quad A \vdash A}{\beta \blacktriangle \beta \rightarrow A \vdash A}}{\alpha \vdash A \triangleleft \alpha \rightarrow A \quad \beta \vdash A \triangleleft \beta \rightarrow A} \\
 \frac{\alpha \cup \beta \vdash (A \triangleleft \alpha \rightarrow A) \wp (A \triangleleft \beta \rightarrow A)}{\alpha \cup \beta \vdash A \triangleleft (\alpha \rightarrow A, \beta \rightarrow A)} \text{choice } \triangleleft \\
 \frac{\alpha \cup \beta \blacktriangle (\alpha \rightarrow A, \beta \rightarrow A) \vdash A}{\alpha \rightarrow A, \beta \rightarrow A \vdash \alpha \cup \beta \triangleright A} \\
 \frac{(\alpha \rightarrow A) \wedge (\beta \rightarrow A) \vdash \alpha \cup \beta \triangleright A}{(\alpha \rightarrow A) \wedge (\beta \rightarrow A) \vdash (\alpha \cup \beta) \rightarrow A}
 \end{array}$$

Box-Composition $(\alpha; \beta) \rightarrow A \dashv\vdash \alpha \rightarrow (\beta \rightarrow A)$

$$\begin{array}{c}
 \frac{\frac{\alpha \vdash \alpha \quad \beta \vdash \beta}{\alpha; \beta \vdash \alpha; \beta} \quad A \vdash A}{(\alpha; \beta) \rightarrow A \vdash (\alpha; \beta) \triangleright A} \\
 \frac{(\alpha; \beta) \rightarrow A \vdash (\alpha; \beta) \triangleright A}{(\alpha; \beta) \rightarrow A \vdash \alpha \triangleright (\beta \triangleright A)} \text{act}_j \triangleright \\
 \frac{\alpha \blacktriangle (\alpha; \beta) \rightarrow A \vdash \beta \triangleright A}{\alpha \blacktriangle (\alpha; \beta) \rightarrow A \vdash \beta \rightarrow A} \\
 \frac{\alpha \blacktriangle (\alpha; \beta) \rightarrow A \vdash \beta \rightarrow A}{(\alpha; \beta) \rightarrow A \vdash \alpha \triangleright \beta \rightarrow A} \\
 \frac{(\alpha; \beta) \rightarrow A \vdash \alpha \triangleright \beta \rightarrow A}{(\alpha; \beta) \rightarrow A \vdash \alpha \rightarrow (\beta \rightarrow A)} \\
 \\
 \frac{\frac{\frac{\beta \vdash \beta \quad A \vdash A}{\beta \rightarrow A \vdash \beta \triangleright A} \quad \alpha \vdash \alpha}{\alpha \rightarrow (\beta \rightarrow A) \vdash \alpha \triangleright (\beta \triangleright A)} \text{act}_j \triangleright}{\alpha \rightarrow (\beta \rightarrow A) \vdash (\alpha; \beta) \triangleright A} \\
 \frac{\alpha \rightarrow (\beta \rightarrow A) \vdash (\alpha; \beta) \triangleright A}{(\alpha; \beta) \blacktriangle \alpha \rightarrow (\beta \rightarrow A) \vdash A} \\
 \frac{(\alpha; \beta) \blacktriangle \alpha \rightarrow (\beta \rightarrow A) \vdash A}{\alpha \Delta \beta \vdash A \triangleleft \alpha \rightarrow (\beta \rightarrow A)} \\
 \frac{\alpha \Delta \beta \vdash A \triangleleft \alpha \rightarrow (\beta \rightarrow A)}{\alpha \Delta \beta \blacktriangle \alpha \rightarrow (\beta \rightarrow A) \vdash A} \\
 \frac{\alpha \rightarrow (\beta \rightarrow A) \vdash \alpha \Delta \beta \triangleright A}{\alpha \rightarrow (\beta \rightarrow A) \vdash (\alpha \Delta \beta) \rightarrow A}
 \end{array}$$

Box-Test $A? \rightarrow B \dashv\vdash A \rightarrow B$

$$\begin{array}{c}
 \frac{A \vdash A \quad B \vdash B}{A \rightarrow B \vdash A > B} \text{?} \triangleright \\
 \frac{A \rightarrow B \vdash A? \triangleright B}{A? \blacktriangle A \rightarrow B \vdash B} \\
 \frac{A? \vdash B \triangleleft A \rightarrow B}{A? \vdash B \triangleleft A \rightarrow B} \\
 \frac{A? \blacktriangle A \rightarrow B \vdash B}{A \rightarrow B \vdash A? \triangleright B} \\
 \frac{A \rightarrow B \vdash A? \triangleright B}{A \rightarrow B \vdash A? \rightarrow B} \\
 \\
 \frac{A \vdash A}{A? \vdash A?} \quad \frac{B \vdash B}{B \vdash B} \\
 \frac{A? \rightarrow B \vdash A? \triangleright B}{A? \rightarrow B \vdash A > B} \text{?} \triangleright \\
 \frac{A? \rightarrow B \vdash A > B}{A? \rightarrow B \vdash A \rightarrow B}
 \end{array}$$

Box-Distributivity $\alpha \rightarrow (A \wedge B) \dashv\vdash \alpha \rightarrow A \wedge \alpha \rightarrow B$

$$\begin{array}{c}
 \text{w.} \frac{A \vdash A}{A, B \vdash A} \quad \text{w.} \frac{B \vdash B}{A, B \vdash B} \\
 \frac{\alpha \vdash \alpha \quad A \wedge B \vdash A}{\alpha \rightarrow (A \wedge B) \vdash \alpha \triangleright A} \quad \frac{\alpha \vdash \alpha \quad A \wedge B \vdash B}{\alpha \rightarrow (A \wedge B) \vdash \alpha \triangleright B} \\
 \frac{\alpha \rightarrow (A \wedge B) \vdash \alpha \triangleright A}{\alpha \rightarrow (A \wedge B) \vdash \alpha \rightarrow A} \quad \frac{\alpha \rightarrow (A \wedge B) \vdash \alpha \triangleright B}{\alpha \rightarrow (A \wedge B) \vdash \alpha \rightarrow B} \\
 \text{C}_L \frac{\alpha \rightarrow (A \wedge B), \alpha \rightarrow (A \wedge B) \vdash (\alpha \rightarrow A) \wedge (\alpha \rightarrow B)}{\alpha \rightarrow (A \wedge B) \vdash (\alpha \rightarrow A) \wedge (\alpha \rightarrow B)}
 \end{array}$$

$$\begin{array}{c}
\frac{\frac{\alpha \vdash \alpha \quad A \vdash A}{\alpha \rightarrow A \vdash \alpha \triangleright A} \quad \frac{\alpha \vdash \alpha \quad B \vdash B}{\alpha \rightarrow B \vdash \alpha \triangleright B}}{\alpha \blacktriangle \alpha \rightarrow A \vdash A \quad \alpha \blacktriangle \alpha \rightarrow B \vdash B} \\
\text{mon} \blacktriangle \frac{(\alpha \blacktriangle \alpha \rightarrow A), (\alpha \blacktriangle \alpha \rightarrow B) \vdash A \wedge B}{\alpha \blacktriangle (\alpha \rightarrow A, \alpha \rightarrow B) \vdash A \wedge B} \\
\frac{\alpha \rightarrow A, \alpha \rightarrow B \vdash \alpha \triangleright A \wedge B}{\alpha \rightarrow A \wedge \alpha \rightarrow B \vdash \alpha \triangleright A \wedge B} \\
\frac{\alpha \rightarrow A \wedge \alpha \rightarrow B \vdash \alpha \triangleright A \wedge B}{\alpha \rightarrow A \wedge \alpha \rightarrow B \vdash \alpha \rightarrow (A \wedge B)}
\end{array}$$

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