УНИВЕРЗИТЕТ У НОВОМ САДУ
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Operators for Multi-Resolution Morse and Cell Complexes

Оператори за мулти-резолуционе комплексе Морза и ћелијске комплексе

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Rezime

Analiza topološke strukture skalarnih polja je u osnovi mnogih savremenih i tradicionalnih oblasti istraživanja, kao što su modelovanje geografskih terena i analiza oblika molekula, analiza volumetrijskih skupova podataka dobijenih merenjima uz pomoć senzora ili simulacijom (na primer, analiza mešanja fluida) u raznim domenima primene, uključujući medicinu, biologiju, meteorologiju, geologiju, da navedemo samo neke.

Teorija Morza (M. Morse) [84,86] se bavi izučavanjem veza izmedju kritičnih tačaka skalarnog polja f definisanog na mnogostrukosti M, i topološke strukture (oblika) mnogostrukosti M. U neprekidnom slučaju, teorija Morza predstavlja prirodan i intuitivan alat za analiziranje strukture skalarnog polja, kao i načina da se ta struktura kompaktno predstavi kroz dekompoziciju domena M skalarnog polja f u ćelijske komplekse koji se zovu (rastući i opadajući) kompleksi Morza.

Kompleksi Morza su sačinjeni od oblasti pridruženih kritičnim tačkama polja. Integralne linije koje konvergiraju ka kritičnoj tački p indeksa i formiraju i-dimenzionalnu ćeliju, koja se zove *opadajuća ćelija* tačke p. Dualno, integralne linije koje polaze iz p formiraju njenu $(n-i)$-dimenzionalnu *rastuću* ćeliju. Opadajuće i rastuće ćeliju definišu dekompoziciju mnogostrukosti M na opadajuće i rastuće komplekse Morza, koji se označavaju sa Γ_d i Γ_a, redom. Njihov presek definiše komplekse Morz-Smejla (S. Smale). Slika 2.3 ilustruje primer integralnih linija, rastućeg i opadajućeg kompleksa Morza, kao i kompleksa Morz-Smejla za dvodimenzionalno skalarno polje.

Kompleksi Morza nalaze široku primenu u analizi i modelovanju oblika (geografskih terena, 3D objekata, grafika skalarnih 3D polja), a našli su primenu i u razumevanju i analizi osobina skalarnih polja u vizualizaciji naučnih podataka. Kompleksi Morza se mogu reprezentovati pomoću bilo koje strukture podataka za reprezentaciju ćelijskih kompleksa. Mi smo u [24, 27] definisali grafovsku strukturu podataka za reprezentaciju topološke strukture kompleksa Morza i Morz-Smejla, koju smo nazvali *Morzov graf incidencije* (Morse Incidence Graph - MIG).

Prvi pristup reprezentaciji topološke strukture dvodimenzionalnih skalarnih polja je predožio Kejli (A. Cayley) 1859. godine [18], u okviru analize topografskih površi. Teren (grafik dvodimenzionalnog skalarnog polja) je opisan i analiziran pomoću konturnih i integralnih linija polja. Primećeno je da do

svakog sedla dolaze dve integralne linije koje ga spajaju sa minimumima, da iz svakog sedla polaze dve integralne linije koje ga spajaju sa maksimumima, kao i da su integralne linije koje spajaju dve sedla nestabilne i mogu se eliminisati malom perturbacijom polja. Dekompozicije domena polja na oblasti koje su pokrivene integralnim linijama koje imaju zajedničku početnu odnosno krajnju tačku su kasnije nazvane rastući odnosno opadajući kompleksi Morza. Iste pojmove je nezavisno otkrio Maksvel (J. C. Maxwell) 1870. godine [85], i proširio ih je uvodjenjem detaljnije klasifikacije sedlastih tačaka, i izučavanjem preseka opadajućih i rastućih kompleksa Morza. Takodje je ustanovio relaciju izmedju broja kritičnih tačaka različitih indeksa dvodimenzionalnog polja. Morz je 1925. godine počeo da izučava zavisnost izmedju kritičnih tačaka n-dimenzionalnog skalarnog polja i topologije (homologije) domena polja [87].

U primenama, neprekidna skalarna polja u 2D i 3D su obično diskretizovana, odnosno zadata su svojim vrednostima na diskretnom skupu tačaka koji čini temena regularne kvadratne odnosno kubne mreže, ili neregularne mreže trouglova odnosno tetraedara. Ovaj opis baziran na geometriji nije u stanju da na kompaktan način reprezentuje topološku strukturu polja definisanu njegovim kritičnim tačkama i načinom na koji su te tačke medjusobno povezane. Osim što je kompaktan, opis topološke strukture takodje podržava prisup zasnovan na znanju za analizu, vizualizaciju i razumevanje ponašanja skalarnog polja (u prostoru i vremenu) kao što se zahteva na primer u vizualnom istrazivanju baza podataka.

U literaturi su razvijeni mnogi algoritmi za segmentaciju domena M polja f, odnosno za izračunavanje aproksimacije dekompozicije domena M na komplekse Morza i Morz-Smejla, kako na regularnim 2D [6, 98, 99] i 3D [57, 106] mrežama, tako i na trougaonim [7, 14, 19, 43, 52, 77, 91, 102] i tetraedarskim [50, 51, 60, 64, 95] mrežama. Mi smo dali pregled algoritama za segmentaciju 2D i 3D skalarnih polja u [37] i [33], redom.

Zajednička odlika ovih algoritama u 2D je da se u izračunatoj aproksimaciji kompleksa Morz-Smejla može desiti da se dve ili više integralnih linija spoje i zajedno konvergiraju prema istom minimumu ili maksimumu. Ekvivalentno, može se desiti da u aproksimaciji kompleksa Morza tri ili više 2-ćelija ima zajedničku tačku koja nije teme (0-ćelija) kompleksa. Drugim rečima, u kompleksima Morza, 1-ćelije se mogu geometrijski delimično poklapati, a da se smatra da su kombinatorno disjunktne [13].

Iako kompleksi Morza reprezentuju topološku strukturu skalarnog polja na kompaktniji način nego polazan skup podataka, ipak postoji potreba da

se ti kompleksi pojednostave i da se time time smanji veličina reprezentacije topološke strukture polja. Razlozi za to su dvostruki. Prvo, danas dostupni skupovi naučnih podataka su uglavnom dati kao konačan skup tačaka iz dvodimenzionalnog ili trodimenzionalnog domena M na kojima je data vrednost skalarnog polja f. Takvi skupovi podataka se dobijaju na dva načina: merenjem ili simulacijom. U oba slučaja, dobijeni podaci su neprecizni zbog prisustva buke (grešaka mernih instrumenata odnosno grešaka numeričkih postupaka). Pojednostavljivanje kompleksa Morza se može posmatrati kao eliminacija takvih inherentih grešaka. Drugo, u poslednje vreme se primećuje sve veći rast kompleksnosti i količine dostupnih naučnih skupova podataka. To dovodi do porasta veličine kompleksa Morza i Morz-Smejla koji reprezentuju topološku strukturu tih podataka u meri koja prevazilazi računarske mogućnosti za njihovu efikasnu obradu. Zbog svega navedenog, istraživanje simplifikacije kompleksa Morza i Morz-Smejla dobija na značaju poslednjih godina.

U literaturi je predložen operator za simplifikaciju funkcija Morza, koji se naziva *kancelacija*. Ovaj operator je definisan u okviru teorije Morza [84]. Osobine ovog operatora su izučavane za 2D polja u [13, 14, 52, 91, 102, 112]. Kancelacija eliminiše dve kritične tačke p i q skalarnog polja f sukcesivnih indeksa i i $i+1$ koje su povezane jedinstvenom integralnom linijom. U dimenzijama većim of dva, kancelacija povećava broj integralnih linija (separatrisa) koje spajaju kritične tačke sukcesivnih indeksa, i time komplikuje strukturu polja.

Efekat kancelacije na komplekse Morz-Smejla za 3D polja je izučavan u [62]. Pokazano je da posle kancelacije 1-sedla i 2-sedla broj ćelija u kompleksima Morz-Smejla može da se poveća. Mi smo izučavali efekat kancelacije na opadajuće i rastuće komplekse Morza u [22], i pokazali smo da je efekat kancelacije lakše opisati i razumeti na kompleksima Morza, nego na kompleksima Morz-Smejla. U kompleksima Morza, svaka ćelija r dimenzije najviše i koja je bila na granici $(i+1)$-dimenzionalne ćelije p je posle kancelacije na granici svake ćelije t dimenzije najmanje $(i+1)$ koja je sadržavala i-dimenzionalnu ćeliju q na svojoj granici pre kancelacije. Ako je r i-ćelija u direktnoj granici $(i+1)$-ćelije p i ako je t $(i+1)$-ćelija u direktnoj ko-granici i-ćelije q, onda je višestrukost $mult'(r, \partial t)$ i-ćelije r u direktnoj granici $(i+1)$-ćelije t posle kancelacije jednaka $mult'(r, \partial t) = mult(r, \partial t) + mult(r, \partial p) \cdot mult(q, \partial t)$, gde je sa $mult$ označena višestrukost ćelije u odgovarajućoj direktnoj granici ∂ pre kancelacije. U kompleksima Morz-Smejla, svakom takvom paru ćelija r

(dimenzije l, $0 \leq l \leq i$) i t (dimenzije m, $i+1 \leq m \leq n$) odgovara nova ćelija dimenzije $m-l$. Slika 2.10 ilustruje primer kancelacije 1-sedla i 2-sedla na 3D kompleksima Morza i Morz-Smejla.

Geometrijski, $mult(q, \partial t)$ disjunktnih kopija $(i+1)$-ćelije p i i-ćelije q postaje deo $(i+1)$-ćelije t posle kancelacije. Dakle, posle kancelacije se može desiti da se ćelije delimično poklapaju (geometrijski), a da se smatra da su (kombinatorno) disjunktne. Zbog toga smo u [26] uveli pojam kombinatornih ćelijskih kompleksa, kod kojih ćelije mogu da zadovoljavaju ovaj uslov.

U dimenzijama većim od dva, kancelacija u opštem slučaju povećava relaciju incidencije medju ćelijama kompleksa Morza i ne može se smatrati pravim operatorom za simplifikaciju. Zbog toga se u 3D posle kancelacije 1-sedla i 2-sedla vrše dodatne kancelacije koje uključuju ekstremne tačke (kancelacije minimuma i 1-sedla ili maksimuma i 2-sedla) [62]. Predloženo je nekoliko strategija koje imaju za cilj da odlože primenu kancelacija koje uvode veći broj incidencija u kompleksima Morza od unapred zadatog praga [61].

Napred navedeno nam je bilo motiv da definišemo operatore za simplifikaciju kompleksa Morza koji neće imati nepoželjne osobine kancelacije. U [24, 27], definisali smo operatore za simplifikaciju kompleksa Morza, koje smo nazvali *removal* i *contraction*.

Operator *removal*, koji smo definisali u [24, 27] i detaljnije izučavali u [26], i njegovo uopštenje operator *remove*, koji smo definisali u [32], su elementarni operatori za simplifikaciju funkcija Morza, kao i odgovarajućih kompleksa Morza i Morz-Smejla. Definicija tih operatora je nezavisna od dimenzije odgovarajućih kompleksa Morza i Morz-Smejla skalarnog polja f. Kao kancelacija, ovi operatori takodje eliminišu dve kritične tačke p i q funkcije f sukcesivnih indeksa i i $i+1$, dve ćelije p i q u opadajućim (odnosno rastućim) kompleksima Morza sukcesivnih dimenzija i i $i+1$ (odnosno $n-i$ i $n-i-1$), kao i dva temena p i q u kompleksima Morz-Smejla, ali pri tom uvek smanjuju broj integralnih linija koje povezuju preostale kritične tačke simplifikovane funkcije, smanjuju relaciju incidencije medju preostalim ćelijama u simplifikovanim opadajućim i rastućim kompleksima Morza, i smanjuju broj ćelija u simplifikovanim kompleksima Morz-Smejla.

Operator *remove* je definisan kao specijalan slučaj kancelacije i ima dve instance: $remove_{i,i+1}$ i $remove_{i,i-1}$. Operator $remove_{i,i+1}$, koji eliminiše kritičnu tačku p indeksa $i+1$ i kritičnu tačku q indeksa i, ima dva tipa. Prvi tip, koji označavamo sa $remove_{i,i+1}(q, p, p')$ je definisan ako je kritična tačka q indeksa i povezana integralnom linijom sa tačno dve kritične tačke p i p' indeksa

$i+1$, i pri tome postoji jedinstvena integralna linija koja spaja tačke q i p. U opadajućem kompleksu Morza (i dualno u rastućem kompleksu), operator $remove_{i,i+1}(q,p,p')$ je definisan ako se i-ćelija q javlja na granici tačno dve $(i+1)$-ćelije p i p', i pri tom se ćelija q javlja tačno jednom na granici ćelije p, odnosno $mult(q,\partial p)=1$.

Posle primene operatora $remove_{i,i+1}(q,p,p')$, svaka integralna linija koja je povezivala neku kritičnu tačku r indeksa najviše i sa kritičnom tačkom p indeksa $i+1$ postaje integralna linija koja povezuje tačku r sa tačkom p'.

Da bismo definisali efekat operatora $remove_{i,i+1}$ na opadajuće kompleksa Morza, posmatramo sledeće skupove ćelija u Γ_d: skup $Z=\{z_h, h=1,..,h_{max}\}$ $(i-1)$-ćelija u direktnoj granici ćelije q; skup $S=\{s_k, k=1,..,k_{max}\}$ $(i+2)$-ćelija u direktnoj ko-granici ćelije p; skup $R=\{r_j, j=1,..,j_{max}\}$ i-ćelija u direktnoj granici ćelije p.

Operator $remove_{i,i+1}(q,p,p')$ na opadajućem kompleksu Γ_d:

- briše ćelije p i q,
- briše svaku instancu ćelije q iz direktne ko-granice ćelije p' i svih ćelija iz Z,
- briše svaku instancu ćelije p iz direktne granice ćelija iz S,
- zamenjuje svaku instancu ćelije p u direktnoj ko-granici ćelije r iz R sa $mult(q,\partial p')$ instanci ćelije p'.

Drugim rečima, $mult'(r,\partial p') = mult(r,\partial p') + mult(r,\partial p)\cdot mult(q,\partial p')$, gde smo sa $mult$ i $mult'$ redom označili višestrukost ćelije u direktnoj granici ∂ pre i posle primene operatora $remove$. Primer primene operatora $remove_{1,2}(q,p,p')$ u 3D je ilustrovan na Slici 4.2. Posle primene operatora na opadajući kompleks Morza Γ_d, 2-ćelije p i p' iz Γ_d su postale nova 2-ćelija p' u simplifikovanom kompleksu Γ'_d.

Geometrijski, u opadajućem kompleksu Morza, $mult(q,\partial p')$ disjunktnih kopija $(i+1)$-dimenzionalne ćelije p (kao i i-dimenzionalne ćelije q) postaje deo $(i+1)$-dimenzionalne ćelije p'. Sve ćelije koje su bile na granici $(i+1)$-dimenzionalne ćelije p su na granici $(i+1)$-dimenzionalne ćelije p' posle primene operatora $remove$. U rastućem kompleksu Morza, $mult(q,\partial p')$ disjunktnih kopija $(n-i)$-dimenzionalne ćelije q postaje deo svake $(n-i)$-dimenzionalne ćelije r koja je imala $(n-i-1)$-dimenzionalnu ćeliju p na svojoj granici. Ko-granica $(n-i-1)$-dimenzionalne ćelije p (skup svih ćelija koje imaju p na svojoj granici) postaje deo ko-granice $(n-i-1)$-dimenzionalne ćelije p'.

Drugi tip operatora $remove_{i,i+1}$, kojeg označavamo sa $remove_{i,i+1}(q, p, \emptyset)$ je definisan ako je kritična tačka q indeksa i povezana integralnom linijom sa tačno jednom kritičnom tačkom p indeksa $i+1$, i pri tome postoji jedinstvena integralna linija koja spaja tačke q i p. U opadajućem kompleksu Morza (i dualno u rastućem kompleksu), operator $remove_{i,i+1}(q, p, \emptyset)$ je definisan ako se i-ćelija q javlja na granici tačno jedne $(i+1)$-ćelije p, i pri tom se ćelija q javlja tačno jednom na granici ćelije p.

Posle primene operatora $remove_{i,i+1}(q, p, \emptyset)$, svaka integralna linija koja je povezivala neku kritičnu tačku r indeksa najviše i sa kritičnom tačkom p indeksa $i+1$ postaje integralna linija koja povezuje tačku r sa nekom tačkom indeksa većeg od $i+1$. U opadajućem kompleksu Morza, $(i+1)$-dimenzionalna ćelija p (kao i i-dimenzionalna ćelija q) se brišu iz kompleksa. U rastućem kompleksu Morza, brišu se $(n-i)$-dimenzionalna ćelija q i $(n-i-1)$-dimenzionalna ćelija p.

Druga instanca operatora *remove*, koju označavamo sa $remove_{i,i-1}$, eliminiše kritičnu tačku q indeksa i i kritičnu tačku p indeksa $i-1$. Operator $remove_{i,i-1}$ takodje ima dva tipa, u oznaci $remove_{i,i-1}(q, p, p')$ i $remove_{i,i-1}(q, p, \emptyset)$, koji su dualni operatorima $remove_{i,i+1}(q, p, p')$ i $remove_{i,i+1}(q, p, \emptyset)$, redom.

Osim operatora za simplifikaciju *removal* i *contraction*, u [26] smo definisali i inverzne operatore *insertion* i *expansion* za rafinaciju funkcije Morza, i odgovarajućih kompleksa Morza i Morz-Smejla. U [32] smo definisali uopštenje ovih operatora za rafinaciju. Novi operator, nazvan *insert*, je inverzan operatoru za simplifikaciju *remove*, od kojeg ima suprotan efekat.

Operator *insert* kreira dve nove kritične tačke skalarnog polja f, dve ćelije sukcesivnog indeksa u opadajućim i rastućim kompleksima Morza, i dva temena u kompleksima Morz-Smejla.

Operator $insert_{i,i+1}(q, p, p')$ je odredjen novim ćelijama (i-ćelijom q i $(i+1)$-ćelijom p), $(i+1)$-ćelijom p' (koja će biti u direktnoj ko-granici ćelije q), i-ćelijama iz skupa R (koje će biti u direktnoj granici ćelije p), $(i+2)$-ćelijama iz skupa S (koje će biti u direktnoj ko-granici ćelije p), $(i-1)$-ćelijama iz skupa Z (koje će biti u direktnoj granici ćelije q), kao i višestrukostima $mult'$ ćelija koje su u rafinisanom kompleksu Γ' u direktnoj granici ili ko-granici bar jedne od dve kreirane ćelije p i q.

Operator $insert_{i,i+1}(q, p, p')$ se može primeniti na kompleks Γ_d ako se ćelija p', kao i sve ćelije iz skupova R, S i Z, nalaze u Γ_d. Takodje, višestrukost *mult* svake ćelije r iz R u direktnoj granici ćelije p' mora da bude veća ili jednaka od proizvoda $mult'(r, \partial p) \cdot mult'(q, \partial p')$ višestrukosti ćelije r iz R u

direktnoj granici ćelije p i višestrukosti ćelije q u direktnoj granici ćelije p' posle rafinacije. Operator *insert* na kompleksu Γ_d se može prikazati kao modifikacija koja zamenjuje skup $\{p'\} \cup R \cup Z \cup S$ ćelija skupom $\{q, p, p'\} \cup R \cup Z \cup S$ i ažurira višestrukosti ćelija: višestrukost $mult'$ ćelije r iz R u direktnoj granici ćelije p' posle rafinacije je data sa $mult'(r, \partial p') = mult(r, \partial p') - mult'(r, \partial p) \cdot mult'(q, \partial p')$. Slika 4.11 ilustruje primer primene operatora $insert_{1,2}(q, p, p')$ na opadajući kompleks Morza u 3D.

Operator za rafinaciju $insert_{i,i+1}(q, p, \emptyset)$, inverzan operatoru za simplifikaciju $remove_{i,i+1}(q, p, \emptyset)$, može da se primeni na kompleks Γ_d ako su sve ćelije iz skupa $R \cup Z \cup S$ ćelija koje će biti u direktnoj granici ili ko-granici neke od dve kreirane ćelije u Γ_d. Operator kreira ćelije q i p i ažurira odgovarajuće višestrukosti.

Operatori za rafinaciju $insert_{i,i-1}(q, p, p')$ i $insert_{i,i-1}(q, p, \emptyset)$, inverzni operatorima za simplifikaciju $remove_{i,i-1}(q, p, p')$ i $remove_{i,i-1}(q, p, \emptyset)$ redom, su dualni.

Skup operatora za simplifikaciju i rafinaciju ima značajnu osobinu da čini bazu za skup operatora koji modifikuju komplekse Morza. To znači da ovi operatori čine minimalan skup operatora takav da se svaki operator koji modifikuje komplekse Morza na mnogostrukosti M na pogodan način može izraziti kao kombinacija operatora iz baze. U [26] smo pokazali kako se makro-operator, definisan na trodimenzionalnim kompleksima, koji se sastoji od kancelacije 1-sedla i 2-sedla, i niza kancelacija koje eliminišu incidencije uvedene takvom kancelacijom tako što eliminišu ekstremne tačke, može izraziti kao niz operatora *remove*, koji u svakom koraku smanjuje relaciju incidencije na kompleksima Morza.

U [25], pokazali smo kako se niz operatora *remove* na kompleksima Morza proizvoljne dimenzije moze kompaktno predstaviti u obliku hijerarhijske grafovske strukture podataka, koju smo nazvali *augmented cancellation forest*. Ispitali smo neke osobine te strukture i dali smo postupak za njenu konstrukciju.

U okviru geometrijskog modelovanja, dvodimenzionalni i trodimenzionalni oblici se najčešće predstavljaju pomoću simplicijalnih ili ćelijskih kompleksa. U literaturi je predložen veliki broj struktura podataka za takve komplekse, kao i operatora za njihovu modifikaciju. U [28], dali smo pregled i klasifikaciju struktura podataka za ćelijske komplekse, kao i operatora na njima.

U [29], definisali smo operatore za simplifikaciju i rafinaciju ćelijskih kompleksa, koji ne menjaju homologiju kompleksa. Ti operatori su uopštenje operatora *remove* i *insert*. Oni brišu ili dodaju par ćelija u kompleks. Takodje

smo definisali operatore za simplifikaciju i rafinaciju ćelijskih kompleksa, koji menjaju homologiju kompeksa tako što brišu jednu ćeliju iz kompleksa ili tako što dodaju jednu ćeliju u kompleks. Operatori koji menjaju homologiju trodimenzionalnih kompleksa su ilustrovani na slici 5.2. Pokazali smo da skup definisanih operatora čini bazu za skup operatora na ćelijskim kompleksima proizvoljne dimenzije, i pokazali smo kako se poznati operatori predloženi u literaturi mogu izraziti preko definisanih operatora.

Multi-rezoluciona reprezentacija topološke strukture skalarnih polja i oblika je važna u mnogim domenima primene. Ona je nezaobilazan element pri interaktivnoj analizi i istraživanju terena, statičih i dinamičkih volumetrijskih skupova podataka. Koristi se za analiziranje njihovih značajnih osobine pri različitim nivoima detalja i za smanjivanje veličine njihove reprezentacije.

Naš rad na multi-rezolucionoj reprezentaciji kompleksa Morza i ćelijskih kompleksa je sproveden u nekoliko faza. U [23] smo predložili multi-rezolucioni model za trodimenzionalne komplekse Morza baziran na operatorima *removal* i *contraction*. U [32], predložili smo multi-rezolucioni model za komplekse Morza proizvoljne dimenzije, baziran na operatorima *remove* i *insert*. On će umnogome povećati mogućnosti za analizu i razumevanje statičnih i dinamičnih oblika i volumetrijskih skupova podataka koji se mogu modelovati kao 3D i 4D skalarna polja. Taj model je suštinski isti kao multi-rezolucioni model za ćelijske komplekse baziran na operatorima koji ne menjaju homologiju kompleksa, koji smo predložili u [34]. Konačno, u [36] smo predložili multi-rezolucioni model za ćelijske komplekse koji je baziran kako na operatorima koji ne menjaju tako i na onima koji menjaju homologiju kompleksa. Ovaj poslednji model smo nazvali *Multi-rezolucioni ćelijski kompleks* (Multi-Resolution Cell Complex - $MRCC$).

$MRCC$ se konstruiše počevši od kompleksa Γ u punoj rezoluciji, tako što se iterativno primenjuju operatori za simplifikaciju, u redosledu koji je odredjen nekim kriterijumom koji zavisi od primene. Kompleks Γ_B koji se dobija kao rezultat primene niza operatora za simplifikaciju je bazni kompleks najniže rezolucije koji reprezentuje polazni kompleks Γ, i on čini prvu komponentu multi-rezolucionog modela.

Drugu komponentu čini skup $\mathcal{M}$ operatora za rafinaciju, inverznih operatorima za simplifikaciju pomoću kojih je dobijen bazni kompleks Γ_B.

Treću komponentu čini relacija direktne zavisnosti medju operatorima za rafinaciju iz skupa $\mathcal{M}$. Obeležićemo sa μ_0 rafinaciju koja kreira kompleks Γ_B, i smatraćemo da je ona element skupa $\mathcal{M}$.

Mi smo relaciju zavisnosti definisali na sledeći način: Neka su $\mu = insert(q, p, p')$ i μ^* dve rafinacije iz $\mathcal{M}$, i neka rafinacija μ koja čuva homologiju kreira ćelije p i q (neka rafinacija μ koja menja homologiju kreira ćeliju p). Relacija direktne zavisnosti medju modifikacijama iz skupa $\mathcal{M}$ je definisana na sledeći način: modifikacija μ *direktno zavisi* od modifikacije μ^* ako i samo ako

- μ^* kreira bar jednu ćeliju u direktnoj granici ili ko-granici ćelije p ili ćelije q (ako μ ne menja homologiju kompleksa);
- μ^* kreira bar jednu ćeliju u direktnoj granici ćelije p (ako μ menja homologiju kompleksa).

$MRCC$ se definiše kao uredjena trojka $(\Gamma_B, \mathcal{M}, \mathcal{R})$, gde je $\mathcal{R}$ relacija direktne zavisnosti. Relacija zavisnosti je tranzitivno zatvaranje relacije direktne zavisnosti. Ona je relacija parcijalnog poretka, jer se svaka ćelija kreira samo jednom modifikacijom u $\mathcal{M}$. Fleksibilnost multi-rezolucionog modela zavisi od načina na koji je definisana relacija zavisnosti.

U $MRCC$-u je organizovan veliki broj reprezentacija kompleksa Γ u različitim rezolucijama, koje se mogu dobiti tako što se na bazni kompleks Γ_B u najnižoj rezoluciji primeni niz $U = (\mu_0, \mu_1, .., \mu_k)$ operatora za rafinaciju. Iz $MRCC$-a, odnosno iz skupa $\mathcal{M}$, se moze izabrati podskup operatora za rafinaciju koji je saglasan sa relacijom poretka odredjenom relacijom direktne zavisnosti. Primenom tih operatora na bazni kompleks Γ_B u najnižoj rezoluciji se može dobiti široka lepeza kompleksa u kojima je rezolucija (definisana odgovarajućim kriterijumom) uniformna ili varijabilna.

Primer $MRCC$-a u 2D je ilustrovan na slici 5.6 u obliku usmerenog acikličnog grafa. Čvorovi grafa predstavljaju operatore za rafinaciju, dok grane predstavljaju relaciju direktne zavisnosti.

Uporedili smo dvodimenzionalnu instancu $MRCC$-a baziranu na operatorima koji ne menjaju homologiju kompleksa sa hijerarhijskim i multi-rezolucionim modelom za reprezentaciju kompleksa Morse-Smale-a koji su predloženi u literaturi u [14] i [15], redom. Ova dva modela su bazirana na kancelaciji [52,84], koja se u dvodimenzionalnom slučaju svodi na operator *remove*. Može se pokazati da je relacija zavisnosti koja definiše $MRCC$ manje restriktivna od one u [14]. Što se tiče relacije zavisnosti koja definiše $MRCC$ i relacije zavisnosti u [15], nijedna nije podskup druge.

U [31] smo predložili algoritam za simplifikaciju kompleksa Morza i MIG-a koji reprezentuje njegovu topološku strukturu, i izložili smo neke rezultate

eksperimenata na 2D i 3D skupovima podataka, uključujući vreme izvršenja algoritma i potreban memorijski prostor. Rezultate implementacije multi-rezolucionog modela za komplekse Morza smo izložili u [32], gde smo pokazali kako se iz modela mogu konstruisati kompleksi uniformne ili varijabilne rezolucije. Rezultate implementacije multi-rezolucionog modela za ćelijske komplekse baziranog na operatorima koji ne menjaju homologiju kompleksa smo predstavili u [34]. Rezultati primene multi-rezolucionog modela na izračunavanje homologije ćelijskih kompleksa su predstavljeni u [36], gde smo pokazali kako se pomoću tog modela mogu propagirati generatori homologije, izračunati na baznom kompleksu, na komplekse proizvoljne rezolucije. Implementacija multi-rezolucionog modela i algoritmi za upite sa uniformnom ili varijabilnom rezolucijom su ostvareni u saradnji sa Federicom Iuricichem. Primena multi-rezolucionog modela na izračunavanje homologije ćelijskih kompleksa je ostvarena u saradnji sa Uldericom Fugacciem. Definicija multi-rezolucionog modela je potpuno nezavisna od dimenzije polja, i obezbedjuje okvir za manipulaciju i ispitivanje topološke strukture oblika i skalarnih polja pri različitim rezolucijama.

Struktura disertacije

Disertacija je podeljena u šest poglavlja.

Prvo poglavlje je uvodno. U njemu je ukratko izložen kontekst u kojem je sprovedeno istraživanje prikazano u tezi, i data je motivacija za njega koja potiče iz različitih oblasti primene.

U drugom poglavlju su izloženi neki osnovni pojmovi korišćeni u tezi. Prvo je data definicija ćelijskih kompleksa, koji su jedan od osnovnih pojmova algebarske topologije. Zatim su dati osnovni pojmovi i teoreme teorije Morza: funkcije Morza, kritične tačke funkcije, dekompozicija domena M funkcije f na rastuće i opadajuće komplekse Morza. Opisan je operator za simplifikaciju kompleksa Morza, koji se naziva kancelacija.

Treće poglavlje sadrži kratak pregled stanja u nekim relevantnim oblastima istraživanja. Predstavljeni su različiti pristupi reprezentaciji topološke strukture skalarnih polja pomoću kompleksa Morza i Morz-Smejla i algoritmi za izračunavanje aproksimacije tih kompleksa počevši od diskretnog skupa tačaka na kojima je data vrednost polja. Predstavljeni su pristupi, predloženi u literaturi za dvodimenzionalna skalarna polja, za primenu kancelacije na konstruisanje hijerarhijskih i multi-rezolucionih modela kompleksa Morza i Morz-Smejla. Takodje je dat pregled postojećih operatora za modifikaciju dvodimenzionalnih i trodimenzionalnih ćelijskih kompleksa.

Četvrto i peto poglavlje sadrže originalne rezultate teze. U četvrtom poglavlju su definisani novi operatori za simplifikaciju i rafinaciju funkcija i kompleksa Morza proizvoljne dimenzije, koji su nazvani *remove* i *insert*, redom. Detaljno je opisan efekat tih operatora, kako na relaciju incidencije tako i na geometriju ćelija u kompleksima Morza. Operator za simplifikaciju *remove*, za razliku od kancelacije, smanjuje ne samo broj ćelija u kompleksima Morza, nego smanjuje i relaciju incidencije medju ćelijama. Pokazano je da definisani operatori čine bazu za skup svih operatora na kompleksima Morza na mnogostrukosti M. Operator *remove* je uporedjen sa kancelacijom, i pokazano je kako se makro-operator, koji se sastoji od kancelacije 1-sedla i 2-sedla i niza kancelacija koje eliminišu ekstremne tačke, može izraziti preko operatora *remove*. Predstavljena je hijerarhijska grafovska struktura podataka za reprezentovanje niza operatora *remove*, i dat je postupak za njeno izračunavanje.

U petom poglavlju su definisani operatori za simplifikaciju i rafinaciju

ćelijskih kompleksa proizvoljne dimenzije. Oni se mogu podeliti na dve klase: na operatore koji ne menjaju homologiju kompleksa, i na one koje je menjaju. Operatori koji ne menjaju homologiju kompleksa su definisani tako da je njihov efekat na ćelijske komplekse isti kao efekat operatora *remove* i *insert* na ćelijske komplekse. Pokazano je da definisani operatori čine bazu za skup svih operatora na ćelijskim kompleksima, i pokazano je kako se neki poznati i široko korišćeni operatori na trodimenzionalnim ćelijskim kompleksima mogu izraziti preko operatora iz baze. Definisan je multi-rezolucioni model za ćelijske komplekse proizvoljne dimenzije, predstavljen je postupak za njegovu konstrukciju i ispitane su neke njegove važne osobine. Pokazano je kako se iz multi-rezolucionog modela može dobiti veliki broj reprezentacija ćelijskih kompleksa uniformne ili varijabilne rezolucije. Multi-rezolucioni model baziran na operatorima koji ne menjaju homologiju kompleksa je suštinski isti kao multi-rezolucioni model za komplekse Morza baziran na operatorima *remove* i *insert*. Dvodimenzionalna instanca takvog modela je uporedjena sa postojećim hijerarhijskim i multi-rezolucionim modelima za dvodimenzionalna skalarna polja.

Šesto poglavlje je zaključno i sadrži neke smernice za dalja istraživanja.

Abstract

Morse and Morse-Smale complexes have been recognized as a suitable model for representing topological information extracted from discrete scalar fields. They induce a subdivision of the domain M of a scalar field f into regions associated with the critical points of f, and compactly represent the topology of M. We define a simplification *remove* operator on Morse complexes, which works in arbitrary dimensions, and we define its inverse refinement *insert* operator. We describe how simplification and refinement operators affect Morse complexes on M, and we show that these operators form a complete set of atomic operators to create and update Morse complexes on M. Thus, any operator that modifies Morse complexes on M can be expressed as a suitable sequence of the atomic simplification and refinement operators we have defined. We compare the 3D instance of our operator with the existing simplification operator proposed in the literature. We define a graph-based data structure for representing a sequence of simplifications on Morse complexes. The simplification and refinement operators provide a suitable basis for the construction of a multi-resolution representation of Morse complexes which contains a large number of topological representations of the scalar field over its domain M, at both uniform and variable resolutions.

We define homology-preserving and homology-modifying simplification and refinement operators on cell complexes in arbitrary dimensions, and we show that they form a basis for the update operators on cell complexes. We show how various widely used operators on cell complexes can be expressed in terms of our operators.

We propose a dimension-independent multi-resolution model for cell complexes, that we call the *Multi-Resolution Cell Complex (MRCC)*, and we investigate its important properties. The *MRCC* based on homology-preserving operators is essentially the same as the multi-resolution model for Morse complexes based on *remove* and *insert* operators. We compare its two-dimensional instance with the existing hierarchical models for two-dimensional scalar fields.

Contents

1

Introduction

The focus of this thesis are the update operators on Morse and cell complexes.

Morse complexes have been introduced in computer graphics for the analysis of 2D scalar fields, and specifically for terrain modeling and analysis, where the domain is a region in the plane, and the scalar field is the elevation function. Recently, these complexes have been considered as a tool to analyze also 3D functions. They are used in scientific visualization, where data are obtained through measurements of scalar field values over a volumetric domain, or through simulation, such as the analysis of mixing fluids. With an appropriate selection of the scalar function, Morse complexes are also used for segmenting molecular models to detect cavities and protrusions, which influence interactions between proteins. Morse complexes of the distance function have been used in shape matching and retrieval.

Structural problems in Morse complexes, like over-segmentation in the presence of noise, or efficiency issues arising due to the very large size of the available input data sets, can be faced and solved by defining simplification operators on those complexes. Morse complexes can be simplified by applying an operator called *cancellation*. In general, a cancellation complicates the structure of Morse complexes by increasing the incidence relation between its cells, which renders it unfeasible in dimensions higher than two.

Cell (and simplicial) complexes are the most common way to discretize geometric shapes, such as static and dynamic 3D objects, or surfaces and hypersurfaces describing the behavior of scalar or vector fields. Representations for these complexes are at the heart of modeling and simulation tools in a variety of application domains, such as computer graphics, computer aided design, fi-

nite element analysis, animation, scientific visualization, and geographic data processing. The literature on operators for building and updating cell complexes is vast but quite disorganized; many sets of basis operators on 3D cell complexes have been proposed.

We have defined a set of simplification *remove* and refinement *insert* operators on Morse and Morse-Smale complexes, and we have shown that these operators form a basis for the set of operators on Morse complexes on a manifold M. We have compared the *remove* simplification operator with the existing simplification operator on Morse and Morse-Smale complexes, called *cancellation*. We have shown that the *remove* operator, unlike cancellation, always reduces the incidence relation on the Morse complexes, and decreases the number of cells in the Morse-Smale complexes. We have defined a graph-based data structure for encoding a sequence of simplification *remove* operators.

We have defined a set of basis modeling operators on cell complexes, which are not necessarily Morse complexes of a Morse-Smale function. These operators can be classified as homology-preserving and homology-modifying ones. Homology-preserving operators extend the *remove* and *insert* operators, which are defined on Morse complexes.

Based on these operators, we have defined a multi-resolution model for cell complexes, and we have investigated its important properties. The multi-resolution model based on homology-preserving operators is basically the same as the multi-resolution model for Morse complexes based on *remove* and *insert*. We have compared its 2D instance with existing hierarchical and multi-resolution models for 2D scalar fields.

In Section 2, we recall some background notions on Morse theory. We give also the definition of complete and admissible Morse complexes, which we have introduced in [26].

In Section 3, we give a brief overview of the related work on topological representation of scalar fields through Morse and Morse-Smale complexes, on their computation, simplification and multi-resolution representation, and on update operators on cell complexes. We have given a survey on data structures and operators on cell complexes in [28].

In Sections 4 and 5, we review the original work on topological update operators on Morse and cell complexes, and their use in multi-resolution modeling. In Section 4, we review simplification *remove* and refinement *insert* operators on Morse complexes that we have defined in [32]. They generalize the simplification and refinement operators that we have introduced and studied

in [24, 26, 27]. We show that these operators form a basis for the set of update operators on Morse complexes on a manifold M. We compare in detail the 3D instance of the *remove* operator with the existing simplification operator on Morse complexes called *cancellation*. We review a hierarchical graph-based data structure that we have introduced in [25], which encodes a sequence of *remove* operators on Morse complexes.

In Section 5, we review the homology-preserving and homology-modifying update operators on cell complexes that we have defined in [29]. We show that these operators form a basis for the set of update operators on cell complexes, and we compare them with the existing update operators on cell complexes. Based on these operators, we have defined a multi-resolution model for cell complexes in [32], and we have implemented the multi-resolution model on a graph-based representation of cell complexes based on homology-preserving operators in [34] in collaboration with Federico Iuricich. In [36], we have used this multi-resolution model to compute homology of a cell complex at various uniform or variable resolution in collaboration with Ulderico Fugacci.

In Section 6, we give some possible research directions to extend the work presented here.

2

Background Notions

We briefly review some basic notions on cell complexes (for more details on algebraic topology, see [1, 65, 81, 88]) and on Morse theory in the case of n-manifolds (for more details on Morse theory, see [84, 86, 87]).

2.1 Cell Complexes

Intuitively, a cell complex is a collection of basic elements, called *cells*, which cover a domain in the Euclidean space $\mathbb{R}^m$.

An (open) *k-dimensional cell* (k-cell) p (also denoted as $p^{(k)}$) in the Euclidean space $\mathbb{R}^m$, $1 \leq k \leq m$, is a subset of $\mathbb{R}^m$ homeomorphic to an open k-dimensional ball $B^k = \{x \in \mathbb{R}^k : ||x|| < 1\}$ ($||x||$ denotes the magnitude or the norm of the vector x). A 0-cell is a point in $\mathbb{R}^m$. A closed k-cell $\overline{p}^k$ is a homeomorphic image of a closed k-dimensional ball $\overline{B}^k = \{x \in \mathbb{R}^k : ||x|| \leq 1\}$ The integer k is called the *dimension* of the k-cell.

Informally, a cell complex in $\mathbb{R}^m$ is a finite set Γ of cells in $\mathbb{R}^m$ such that

1. the cells in Γ are pairwise disjoint,

2. for each cell $p \in \Gamma$, the boundary of p is a disjoint union of cells in Γ.

The maximum d of dimensions of cells in Γ is the *dimension* of Γ, and Γ is called a *cell d-complex*.

More formally, a (combinatorial) n-dimensional *cell complex* (a cell n-complex) [76] on a set X is defined as the equivalence class of cell structures (X, Φ). A cell structure (X, Φ) on the set X (or a cellular decomposition of

the set X) is a pair composed of X and a collection Φ of (continuous) maps φ of closed Euclidean cells into X such that:

1. each map $\varphi \in \Phi$ with domain $\overline{p}^k$ is injective on the interior $p^{(k)}$ of $\overline{p}^{(k)}$,
2. the images of the maps $\varphi \in \Phi$ are disjoint and their union is X,
3. for each map $\varphi \in \Phi$ with domain $\overline{p}^{(k)}$, φ maps the boundary of $\overline{p}^{(k)}$ into the union of images of maps $\psi' \in \Phi$ with domain $\overline{p}^{(j)}$, $j \leq k-1$.

The *relative boundary* of a k-cell $\overline{p}^{(k)}$ in a cell complex Γ is a subset of $\overline{p}^{(k)}$ corresponding to the $(k-1)$-sphere $S^{k-1} = \{x \in E^k : ||x|| = 1\}$ under φ.

We make the following assumptions: (i) The image of the boundary of $\overline{p}^{(k)}$ under $\varphi \in \Phi$ meets finitely many images of maps $\psi' \in \Phi$ with domain $\overline{p}^{(j)}$, $j \leq k-1$. (ii) If the image of map $\varphi \in \Phi$ with domain $\overline{p}^{(k)}$ contains a point in the image of map $\psi' \in \Phi$ with domain $\overline{p}^{(j)}$, $j \leq k-1$, then it contains all points in the image of map $\psi' \in \Phi$. (iii) On the image of map $\psi' \in \Phi$, the number of connected components of ψ^{-1} is constant. This number is called the *multiplicity* (denoted as *mult*) of the image of map φ' in the boundary of the image of map φ.

Two cell structures (X, Φ) and (X, Φ') are *equivalent* if there is a one-to-one correspondence between Φ and Φ' such that the corresponding maps differ only by a reparametrization of their domain.

A subset Λ of Γ is called a *subcomplex* of Γ if and only if Λ is a cell complex. The k-skeleton of Γ is the subcomplex of Γ that consists of all cells of Γ of dimension less than or equal to k. A cell complex Γ is *normal* if each cell $\overline{p}^{(k)}$ in Γ carries a subcomplex.

The set of all cells in the relative boundary of an (open) k-cell $p^{(k)}$ in Γ is called the *(combinatorial) boundary* of $p^{(k)}$. The boundary of a 0-cell is empty. The *(combinatorial) co-boundary* of $p^{(k)}$ consists of those cells in Γ that have $p^{(k)}$ in their combinatorial boundary. If p is a k-cell, then the *immediate (combinatorial) boundary* $\partial_k\ p$ (also denoted as ∂p) of p is a multi-set that consists of all the $(k-1)$-cells p' in the boundary of p, $1 \leq k \leq n$, where each cell is taken with the multiplicity, denoted as $mult(p', \partial p)$, equal to the number of times the cell p' appears in the immediate boundary of p. Dually, the *immediate (combinatorial) co-boundary* of p is a multi-set that consists of all the $(k+1)$-cells in the co-boundary of p, $0 \leq k \leq n-1$, where each cell is taken with the multiplicity equal to the number of times it appears in the immediate co-boundary of p. The co-boundary of an n-cell in a cell n-complex

is empty. If the $(k-1)$-cell p' is in the immediate boundary of the k-cell p with multiplicity $mult(p', \partial p)$, then the k-cell p is in the immediate co-boundary of the $(k-1)$-cell p' with the same multiplicity. An h-cell p' in the boundary of a k-cell p, $0 \leq h \leq k$, is called an *h-face* of p, and p is called a *coface* of p', denoted as $p' \leq p$. The cells γ and γ' are said to be *incident*.

The (open) *star* $St(p)$ of a cell p in Γ is the set of all cells in Γ that have p on their boundary. In other words, $St(p)$ is equal to the co-boundary of p. The closed star $\overline{St}(p)$ is the closure of the star $St(p)$ of p ($St(p)$ plus all the faces of $St(p)$). The *link* $Lk(\gamma)$ is the closed star of p minus the star of the closure of p ($Lk(p) = \overline{St}(p) \backslash St(\overline{p})$). The link of p is a subcomplex of Γ, while the star of p is in general not a subcomplex.

The *domain* (or *carrier*) $\Delta\Gamma$ of a cell complex Γ is the subset of $\mathbb{R}^m$ spanned by the cells of Γ (the cells in Γ cover $\Delta\Gamma$). We will consider cell complexes Γ such that $\Delta\Gamma$ is homeomorphic to a smooth compact manifold M without boundary.

The Euler-Poincaré formula expresses the necessary validity condition of a cell complex with manifold or non-manifold carrier [2]. The Euler-Poincaré formula for a cell d-complex Γ (with or without boundary, of homogenous or non-homogenous dimension) with n_i i-cells states that

$$\sum_{i=0}^{d}(-1)^i n_i = n_0 - n_1 + .. + (-1)^d n_d = \sum_{i=0}^{d}(-1)^i \beta_i = \beta_0 - \beta_1 + .. + (-1)^d \beta_d.$$

Here, β_i is the ith Betti number of Γ, and it measures the number of independent non-bounding i-cycles in Γ, i.e., the number of independent i-holes. The alternating sum $n_0 - n_1 + .. + (-1)^d n_d$ is denoted as $\chi(\Gamma)$, and is called the *Euler-Poincaré characteristic* of Γ.

If Γ is a (manifold or non-manifold) cell complex embedded in $\mathbb{R}^3$, then $\beta_3 = 0$ and the number of 0-cells (vertices), 1-cells (edges), 2-cells (faces) and 3-cells (volumes) is usually denoted as v, e, f and c, respectively. For a cell 3-complex in $\mathbb{R}^3$,

$$v - e + f - c = \beta_0 - \beta_1 + \beta_2. \tag{2.1}$$

For a cell 2-complex in $\mathbb{R}^3$, also $c = 0$ and

$$v - e + f = \beta_0 - \beta_1 + \beta_2. \tag{2.2}$$

Intuitively, in $\mathbb{R}^3$, β_0 is the number of connected components, β_1 is the number of holes (non-bounding 1-cycles), and β_2 is the number of cavities (non-bounding 2-cycles) in Γ.

For a manifold cell 2-complex Γ whose carrier is the boundary of a solid object in $\mathbb{R}^3$, the Euler-Poincaré formula states that

$$v - e + f = 2(s - g). \tag{2.3}$$

Here, s is the total number of shells (connected components of Γ) and g is the genus of the boundary surface.

If Γ is a cell 2-complex homeomorphic to a 2-sphere, then

$$v - e + f = 1 - 0 + 1 = 2. \tag{2.4}$$

If Γ is a 1-dimensional cell complex (a graph), called a wire frame, then

$$v - e = \beta_0 - \beta_1. \tag{2.5}$$

Two distinct k-cells p_1 and p_2 are said to be *adjacent* if and only if (i) $k = 0$ or there exists a $(k-1)$-cell of Γ, which is a face of both p_1 and p_2, and (ii) $k = n$ or there exists a $(k+1)$-cell of Γ, which is a coface of both p_1 and p_2. For example, two 0-cells p_1 and p_2 are adjacent if there is a 1-cell connecting them. Two n-cells p_1 and p_2 are adjacent if they share an $(n-1)$-face.

A d-complex Γ^* is called a *space dual* of a d-complex Γ if there is a one-to-one mapping from Γ onto Γ^* such that (i) p is a k-cell in Γ if and only if its image is a $(d-k)$-cell in Γ^*, and (ii) cells p_1 and p_2 are adjacent in Γ if and only if their images are adjacent in Γ^*.

2.2 Morse Theory

Morse theory is a powerful tool to capture the topological structure of shapes and scalar fields by establishing the relationships between the topology of a manifold M, and the critical points of a scalar (real-valued) function defined on M.

We review basic notions on Morse theory for C^2-differentiable functions. More details can be found in [84, 86, 100]. Recall that an n-manifold is a Hausdorff space with countable base, in which each point has a neighborhood homeomorphic to $\mathbb{R}^n$.

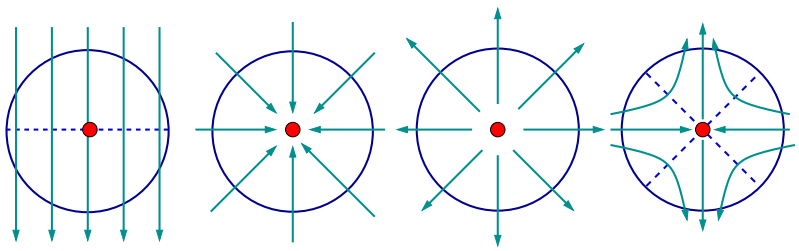


Figure 2.1: [26] Neighborhoods of non-degenerate critical points in the 2D case. Arrowed lines represent integral lines, dotted lines represent iso-lines. A regular point (a), a local maximum (b), a local minimum (c), and a simple saddle (d).

2.2.1 Morse Functions

Let f be a C^2-differentiable real-valued function (a scalar field) defined over a closed differentiable manifold M. A point $p \in M$ is a *critical point* of f if and only if the gradient $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$ (in some local coordinate system around p) of f vanishes at p. Function f is said to be a *Morse function* if all its critical points are non-degenerate (the Hessian matrix $Hess_p f = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=1}^{n}$ of the second derivatives of f at p is non-singular). The number of negative eigenvalues of $Hess_p f$ is called the *index* of a critical point p. The corresponding eigenvectors show the directions in which f is decreasing. If the index of p is i, $0 \leq i \leq n$, p is called an *i-saddle*. A 0-saddle, or an n-saddle, is also called a *minimum*, or a *maximum*, respectively. Non-critical points are called *regular points*. Figures 2.1 and 2.2 illustrate a neighborhood of critical points in two and three dimensions, respectively.

For a Morse function f, there is a neighborhood U of each critical point $p = (p_1, p_2, ..., p_n)$ of f, in which f can be expressed in the *standard form* as

$$f(x_1, x_2, \ldots, x_n) = f(p_1, p_2, ..., p_n) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2.$$

This representation of the scalar field f is valid for all $(x_1, x_2, \ldots, x_n) \in U$. The number i of the minus signs in the above equality is equal to the number of the negative eigenvalues of $Hess_p f$ (the index of the critical point p). This representation implies that the critical points of a Morse function f are isolated, and if M is compact then f has finitely many critical points.

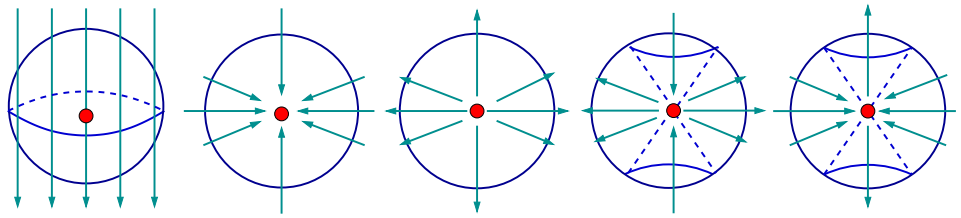


Figure 2.2: [26] Neighborhoods of non-degenerate critical points in the 3D case. Arrowed lines represent integral lines, dotted lines indicate cone-like iso-surfaces. A regular point (a), a local maximum (b), a local minimum (c), a 1-saddle (d) and a 2-saddle (e).

A vector field X is a *gradient-like* vector field [84] for a Morse function f on a manifold M if

1. $X \cdot \nabla f > 0$ away from the critical points of f, and
2. in some neighborhood of each i-saddle p of f, $X = -2x_1\frac{\partial}{\partial x_1} - .. - 2x_i\frac{\partial}{\partial x_i} + 2x_{i+1}\frac{\partial}{\partial x_{i+1}} + .. + 2x_n\frac{\partial}{\partial x_n}$ in some local coordinate system around p.

If we denote as m_i the number of i-saddles of f, and as $\chi(M)$ the *Euler-Poincaré characteristic* of M, then the strong and the weak Morse inequalities are satisfied, that is, for $i = 1, .., n$,

$$m_i - m_{i-1} + ... + (-1)^i m_0 \geq \beta_i - \beta_{i-1} + ... + (-1)^i \beta_0,$$

$$m_i \geq \beta_i.$$

The last strong Morse inequality (for $i = n$) is an equality. This implies that the Euler-Poincaré characteristic $\chi(M)$ of M, defined as the alternating sum of the Betti numbers of M ($\chi(M) = \sum_{i=0}^{n}(-1)^i\beta_i$), is equal to the alternating sum of the number of i-saddles of f ($\chi(M) = \sum_{i=0}^{n}(-1)^i m_i$), i.e.,

$$\chi(M) = \beta_0 - \beta_1 + ... + (-1)^n\beta_n = m_0 - m_1 + ... + (-1)^n m_n.$$

An *integral line* of a function f is a maximal path that is everywhere tangent to the gradient ∇f of f. Thus, an integral line follows the direction in which

the function has the maximum increasing growth. More formally, an *integral line* through a point $p \in M$ is a curve $c_p(t)$ in M such that

$$\frac{dc_p}{dt}(t) = \nabla f, \quad c_p(0) = p.$$

If p is a critical point of f, then an integral line through p is p itself. Two integral lines $c_p(t)$ and $c_q(t)$ are either disjoint or the same. Each integral line $c(t)$ starts at a critical point $r = \lim_{t \to -\infty} c(t)$ of f, called its *origin*, and ends at another critical point $s = \lim_{t \to \infty} c(t)$ of f, called its *destination*. We say that the integral line $c(t)$ connects points r and s (and that it is incident to r and s), although it does not contain these points. An integral line that connects an i-saddle and an $(i+1)$-saddle, $0 \leq i \leq n-1$, is called a *separatrix* or a *separatrix line*.

Each 1-saddle is connected to exactly two (not necessarily distinct) minima, and each $(n-1)$-saddle is connected to exactly two (not necessarily distinct) maxima. Each i-saddle, $2 \leq i \leq n$, is connected to an arbitrary number of m-saddles, $0 \leq m < i \leq n$, and each i -saddle, $0 \leq i \leq n-2$, is connected to an arbitrary number of l-saddles, $0 \leq i < l \leq n$. In 2D, a saddle is connected exactly to two minima and two maxima. In 3D, a 1-saddle is connected to two minima and to an arbitrary number of 2-saddles and maxima, and a 2-saddle is connected to two maxima and to an arbitrary number of 1-saddles and minima.

2.2.2 Morse and Morse-Smale Complexes

Integral lines that converge to a critical point p of index i (together with p) cover an (open) i-cell called the *stable (descending) cell* $W^s(p)$ of p. Dually, integral lines that originate at p cover an $(n-i)$-cell called the *unstable (ascending) cell* $W^u(p)$ of p. More formally,

$$W^s(p) = \{p\} \cup \{q \in M : \lim_{t \to +\infty} c_q(t) = p\},$$

$$W^u(p) = \{p\} \cup \{q \in M : \lim_{t \to -\infty} c_q(t) = p\}.$$

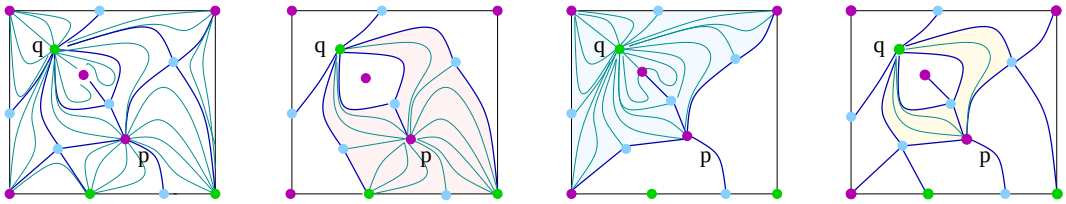
The descending cells (or manifolds) are pairwise disjoint, they cover M, and the boundary of every cell is a union of lower-dimensional cells. The descending cells decompose M into a cell complex, called the *descending Morse*

complex of f on M, denoted as Γ_d. Dually, the ascending cells form the *ascending Morse complex* Γ_a of f on M. A maximum corresponds to an n-cell in the descending complex and to a 0-cell in the ascending complex. Dually, a minimum corresponds to a 0-cell in the descending complex and to an n-cell in the ascending complex.

Two cells p and q are incident in the Morse complexes if and only if the corresponding critical points p and q are connected through an integral line of f. If a cell p is on the boundary of a cell q in the descending complex Γ_d, then for the corresponding critical points p and q, $f(p) < f(q)$.

A Morse function f is called a *Morse-Smale function* if and only if each non-empty intersection of a descending and an ascending cell is transversal. This means that each connected component of the intersection (if it exists) of the descending i-cell of an i-saddle p, and the ascending $(n-j)$-cell of a j-saddle q, $i \geq j$, is an $(i-j)$-cell. The connected components of the intersection of descending and ascending cells of a Morse-Smale function f decompose M into a *Morse-Smale complex*.

Each 0-cell in the Morse-Smale complex corresponds to a critical point of f. Each i-cell in the Morse-Smale complex, $1 \leq i \leq n$, is covered by the integral lines connecting a k-saddle p to a $(k+i)$-saddle q, $0 \leq k \leq n-i$. Figure 2.3 illustrates the correspondence between Morse and Morse-Smale complexes in the 2D case.



(a) (b) (c) (d)

Figure 2.3: [26] (a) Integral lines and critical points of a scalar field f defined on a 2D domain. Green, blue, and magenta points denote maxima, saddles, and minima, respectively. (b) Ascending Morse complex with integral lines originating at the minimum p. (c) Descending Morse complex with integral lines converging to the maximum q. (d) Morse-Smale complex with integral lines originating at p and converging to q.

In 2D, each 2-cell in a Morse-Smale complex is related to a maximum p and a minimum q, as it is obtained as the intersection of the descending 2-cell

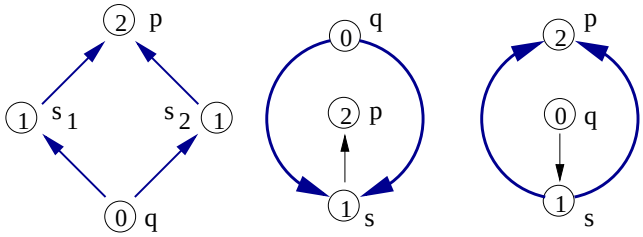


Figure 2.4: The three possible types of 2-cells in the Morse-Smale complex of a Morse-Smale function f in 2D. The numbers indicate the index of the corresponding critical point.

of p and the ascending 2-cell of q. Such 2-cell is quadrangular, with vertices of index 0,1,2,1 (q, s_1, p, s_2), in this order along the boundary. These 2-cells are obtained as the union of two triangles, namely q, s_1, p and q, s_2, p, where s_1 and s_2 are the saddles connected through an integral line to both p and q. Saddles s_1 and s_2 are not necessarily distinct, thus it is possible that $s_1 = s_2 = s$. In [89], it has been shown that, for a Morse-Smale function f, there are three different types of 2-cells in the Morse-Smale complex of f, which are illustrated in Figure 2.4.

In 3D, each 3-cell of the Morse-Smale complex is related to a maximum p and a minimum q: it is obtained as the intersection of the descending 3-cell of p and the ascending 3-cell of q. Such 3-cell is the union of tetrahedra of the form q, s_1, s_2, p, where s_1 and s_2 are 1-saddles and 2-saddles connected through an integral line to both p and q. These 1- and 2-saddles may belong to more than two such different tetrahedra.

If f is a Morse-Smale function, then there is no integral line connecting two different critical points of f of the same index. The ascending complex Γ_a and the descending complex Γ_d of f are space duals. Each i-saddle p of f, $0 \leq i \leq n$, corresponds to

1. the vertex p in the Morse-Smale complex of f,

2. the descending i-cell of p in Γ_d, and

3. the ascending $(n-i)$-cell of p in Γ_a.

If p is an i-cell and q is an $(i+1)$-cell in the descending Morse complex, then the multiplicity $mult(p, \partial q)$ of the i-cell p in the boundary of the $(i+1)$-cell

q is equal to the number of integral lines connecting the i-saddle p and the $(i+1)$-saddle q.

2.2.3 Complete and Admissible Morse Complexes

We review *complete* and *admissible* Morse complexes, which we have defined in [26].

The majority of topological models and data structures for representing and handling subdivided objects, like quasi-Morse complexes [51,52], incidence graphs [49], combinatorial maps [17], n-G maps [73], cell-tuples [16], actually represent admissible cell complexes. Here, we introduce formally the definition of such complexes.

We define a Morse function f to be a *complete* Morse function if

1. each i-saddle is connected to at least one $(i-1)$-saddle, $1 \leq i \leq n$, and
2. each i-saddle is connected to at least one $(i+1)$-saddle, $0 \leq i \leq n-1$.

Recall that there is a one-to-one correspondence between (1) the i-saddles of f, (2) the i-cells in the descending complex of f and (3) the $(n-i)$-cells in the ascending complex of f, $0 \leq i \leq n$. Two cells in the descending Morse complex (and dually in the ascending Morse complex) are incident if and only if the corresponding critical points are connected through an integral line of f. Thus, both in the ascending and the descending Morse complexes of a complete Morse function f:

1. each i-cell is bounded by at least one $(i-1)$-cell, $1 \leq i \leq n$, and
2. each i-cell bounds at least one $(i+1)$-cell, $0 \leq i \leq n-1$.

We call the Morse complexes of a complete Morse function f *complete* Morse complexes.

A Morse function f can be complete without being Morse-Smale, and it can be Morse-Smale but not complete. We illustrate this in the two examples in Figure 2.5. Recall that a Morse function f is a Morse-Smale function if each non-empty intersection of a descending and an ascending cell is transversal.

Figure 2.5 (a) illustrates a descending Morse complex of a function defined on a torus. It has one minimum p, one maximum q, and two saddles r and t. There are two integral lines starting, and two integral lines ending at each

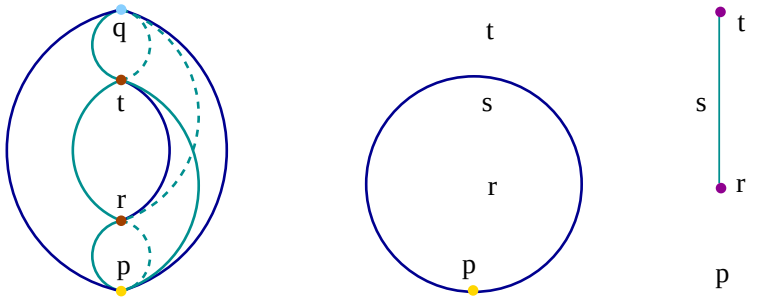


Figure 2.5: [26] (a) Critical points and integral lines of a Morse function f defined on a torus. Function f is complete but it is not Morse-Smale, as there is an integral line of f connecting two critical points, r and t, of the same index. Subdivision of a 3-sphere ($\mathbb{R}^3$ compactified by a point at infinity) into (b) descending cells (two 3-cells r and t, one bubble-like 2-cell s in blue, and one 0-cell p in yellow) and (c) ascending cells (two 0-cells r and t in magenta, one 1-cell s in cyan, and one 3-cell p), induced by a Morse-Smale function f, which is not complete, since, for example, the descending 2-cell s has no 1-cell on its boundary.

saddle. Two integral lines starting at t connect the saddle t to the maximum q, and two integral lines ending at r connect the minimum p to the saddle r. One of the integral lines ending at t connects the minimum p to the saddle t, and one of the integral lines starting at r connects the saddle r to the maximum q. There is one integral line starting at r and ending at t, connecting the saddle r to the saddle t. This function is clearly complete, but it is not Morse-Smale, since the descending 1-cell of t and the ascending 1-cell of r partially overlap, and thus do not intersect transversally (there is an integral line of f connecting two different saddles, r and t, of the same index).

Figure 2.5 (b) and (c) illustrates the descending and ascending Morse complex, respectively, of a function f defined on a 3-sphere ($\mathbb{R}^3$ compactified by a point at infinity). It has two maxima r and t, one 2-saddle s, no 1-saddles, and one minimum p. In the descending complex, the two 3-cells r and t are bounded by the only 2-cell s. The 2-cell s is bounded by the only 0-cell p and is not bounded by any 1-cell. The 0-cell p does not bound any 1-cell. In the dual ascending complex, illustrated in Figure 2.5 (c), the only 3-cell p is bounded by the only 1-cell s. It is not bounded by any 2-cell. The 1-cell s is bounded

by two 0-cells r and t. The intersection of the ascending and the descending cells is transversal, and the function is Morse-Smale. It is not complete, since for example in the descending complex there are no 1-cells on the boundary of the 2-cell s.

We say that a complete Morse-Smale function is *admissible*. We call the Morse complexes defined by an admissible function *admissible Morse complexes*.

The topological structure of admissible Morse functions and the corresponding ascending and descending Morse complexes can be represented in the *Incidence Graph (IG)* [49], which is an incidence-based explicit data structure for n-dimensional cell complexes. The topological information captured is the set of incidence relations among cells that differ by one dimension. Formally, the *IG* encodes all the cells of any given n-dimensional cell complex Γ, and for each i-cell p, its immediate boundary, and immediate co-boundary relations, namely (i) for each i-cell p, where $0 < i \leq n$, all the $(i-1)$-cells in the immediate boundary of p, and (ii) for each i-cell p, where $0 \leq i < n$, all the $(i+1)$-cells in the immediate co-boundary of p. A similar notion is that of the *Hasse diagram* describing the complex [72], the difference being that in the Hasse diagram each arc connecting an i-cell and an $(i-1)$-cell in its boundary is oriented from the i-cell towards the $(i-1)$-cell.

In [23, 24], we have extended the notion of the incidence graph to represent not only cell complexes (admissible descending and ascending Morse complexes), but also the critical points of the corresponding admissible scalar field f and the integral (separatrix) lines connecting them. The *Morse Incidence Graph (MIG)* is a multigraph $G = (N, A)$ where:

1. the set of nodes N is partitioned into $n+1$ subsets N_0, N_1,...,N_n, such that there is a one-to-one correspondence between the nodes in N_i (which we call the *i-nodes*) and the i-saddles of f (and thus the i-cells of Γ_d and the $(n-i)$-cells of Γ_a);

2. there are k arcs joining an i-node p with an $(i+1)$-node q if and only if there are k integral lines connecting the i-saddle p and the $(i+1)$-saddle q. The integer k is equal to the number of times the i-cell p appears in the immediate boundary of the $(i+1)$-cell q in Γ_d, i.e., to the multiplicity $mult(p, \partial q)$ of the i-cell p in the immediate boundary of the $(i+1)$-cell q in Γ_d.

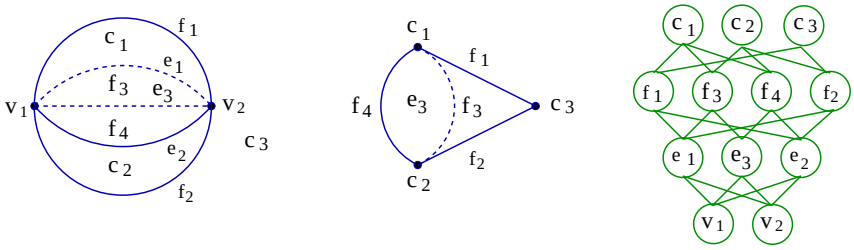


Figure 2.6: [24] A descending 3D Morse complex, the dual ascending complex, and the corresponding MIG.

Each node is labeled with the critical point (or equivalently, the descending and the dual ascending cell) it represents.

Alternatively, the Morse incidence graph (MIG) can be defined as a labeled simple graph $G = (N, A, \psi)$, where N is the set of nodes, A is the set of arcs, and $\psi : A \rightarrow \mathbb{N}$ is the labeling function, such that:

1. the set of nodes N is partitioned into $n+1$ subsets N_0, N_1,...,N_n, such that there is a one-to-one correspondence between the nodes in N_i (the *i-nodes*) and the i-saddles of f (and thus the i-cells of Γ_d, and the $(n-i)$-cells of Γ_a);

2. there is an arc joining an i-node p with an $(i+1)$-node q if and only if there is an integral line of f connecting the i-saddle p and the $(i+1)$-saddle q (if the i-cell p appears in the immediate boundary of the $(i+1)$-cell q in Γ_d).

3. the label $\psi(p, q)$ of the arc joining an i-node p with an $(i+1)$-node q is equal to the number of integral lines connecting the i-saddle p and the $(i+1)$-saddle q. It is equal to the number of times the i-cell p appears in the immediate boundary of the $(i+1)$-cell q, i.e., to the multiplicity $mult(p, \partial q)$ of the i-cell p in the immediate boundary of the $(i+1)$-cell q in Γ_d.

If f is a Morse-Smale function, then the MIG provides also a combinatorial representation of the 1-skeleton of its Morse-Smale complex. Figure 2.6 illustrates the descending Morse complex and the corresponding MIG in 3D.

2.2.4 Cancellation Operator

The *cancellation* operator simplifies a Morse function f defined on a manifold M by eliminating its critical points in pairs [84]. It transforms the canceled critical points into regular points by transforming the function f into a new Morse function g. This is achieved by modifying the gradient field of f around the integral line connecting the two canceled critical points. Two critical points p and q can be canceled if and only if the following two conditions are both satisfied:

1. p and q are of index i and $i+1$, respectively, $0 \leq i \leq n-1$, and
2. there is a unique integral line connecting p and q.

If p and q can be cancelled, then in a neighborhood of the unique integral line connecting p and q in some coordinates in which $p = (0, 0, .., 0)$ and $q = (0, ..0, 1, 0, ..0)$ (1 is at the $(i+1)$st place) the gradient-like vector field of f can be written as $X = -2x_1\frac{\partial}{\partial x_1} - .. - 2x_i\frac{\partial}{\partial x_i} + 2v(x_{i+1})\frac{\partial}{\partial x_{i+1}} + .. + 2x_n\frac{\partial}{\partial x_n}$, where $v(x_{i+1})$ is a C^∞-differentiable function defined on $(-\delta, 1+\delta)$ $(\delta > 0)$, $v(t) = t$ in a neighborhood of 0, $v(t) = 1 - t$ in a neighborhood of 1 and $v(t) > 0$ for $0 < t < 1$.

The cancellation of the i-saddle p and the $(i+1)$-saddle q perturbs the gradient-like vector field X to the vector field $Y = -2x_1\frac{\partial}{\partial x_1} - .. - 2x_i\frac{\partial}{\partial x_i} + 2v_{x_1^2+..+x_n^2}(x_{i+1})\frac{\partial}{\partial x_{i+1}} + .. + 2x_n\frac{\partial}{\partial x_n}$ where $v_{x_1^2+..+x_n^2}(x_{i+1})$ is obtained from a family $\{v_s(x_{i+1})\}_s$ such that: (i) each $v_s(x_{i+1})$ is a C^∞-differentiable function on $(-\delta, 1+\delta)$, (ii) for a sufficiently small $\nu > 0$, the functions $v_s(x_{i+1})$ are defined for $-\nu < s < 2\nu$, (iii) for $s \geq \nu$, $v_s(x_{i+1}) = v(x_{i+1})$, (iv) for $s \leq 0$, $v_s(x_{i+1}) = v_0(x_{i+1})$ and $v_0(x_{i+1}) < 0$ for $x_{i+1} \in (-\delta, 1+\delta)$, (v) for $x_{i+1} < -\frac{\delta}{2}$ or $x_{i+1} > 1 + \frac{\delta}{2}$, $v_s(x_{i+1}) = v(x_{i+1})$ for any s.

The vector field Y is a gradient-like vector field of a Morse function g, which coincides with f everywhere on M except in a small neighborhood of the integral line connecting p and q. Thus, the critical points of g are the same in position and in index as the critical points of f with the exception of p and q. Neither the function g obtained from f after the cancellation of p and q, nor its gradient, are uniquely determined by the cancellation. Nevertheless, the cancellation of p and q affects the set of integral lines starting or ending at either p or q in an unambiguous way.

In Figure 2.7, we illustrate a neighborhood of the unique integral line connecting critical points p and q before and after the cancellation of p and q.

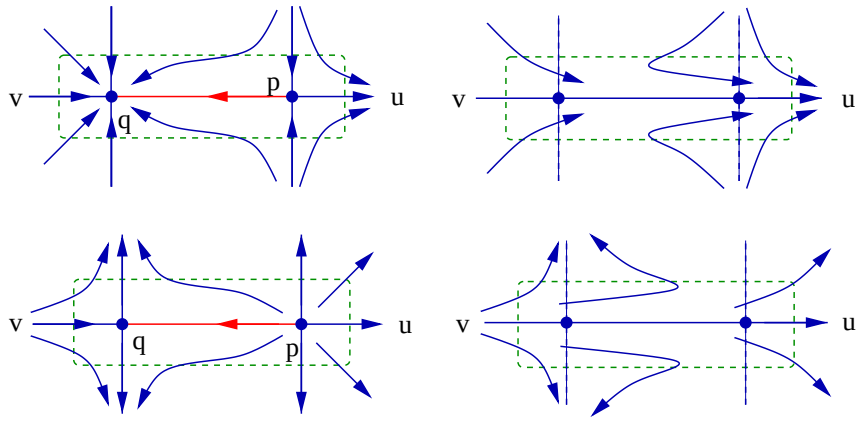


Figure 2.7: [26] A neighborhood (dotted green) of the integral line (red) connecting critical points p and q of index i and $i+1$, respectively, before and after the cancellation of p and q. Integral lines that converged to q or p before the cancellation converge to u after the cancellation (top). Integral lines that originated at p or q before the cancellation originate at v after the cancellation (bottom).

The set of integral lines that converged to q or to p before the cancellation is transformed into a set of integral lines converging to critical points u of index $j > i$ that were the destination of integral lines starting at p before the cancellation (see Figure 2.7 (top)). Similarly, the set of integral lines that originated at q or at p before the cancellation is transformed into a set of integral lines originating at critical points v of index $k < i+1$ that were the origin of integral lines ending at q before the cancellation (see Figure 2.7 (bottom)).

The conditions on the feasibility of the cancellation on the Morse complexes of f are obtained from the conditions on the feasibility of the cancellation on the scalar field f. Recall that two cells in the Morse complexes of f are incident if and only if the corresponding critical points are connected through an integral line of f.

In the descending (and, symmetrically, ascending) Morse complex, two cells p and q can be simultaneously eliminated if the following conditions are both satisfied:

1. p is an i-cell and q is an $i+1$-cell, and

2. the cell p appears exactly once in the immediate boundary of the cell q

(the multiplicity $mult(p, \partial q)$ of the i-cell p in the immediate boundary of the $(i+1)$-cell q is equal to 1).

The first condition translates the fact that p and q are critical points of index i and $i+1$, respectively. The second condition specifies that critical points p and q are connected through exactly one integral line.

The Morse and Morse-Smale complexes of f are determined by the integral lines of f. Thus, the effect of the cancellation of p and q on these complexes can be deduced from the effect of the cancellation on the set of integral lines starting or ending at p or q. We describe the effect of the cancellation on the descending complexes, ascending complexes, and on Morse-Smale complexes.

Let us denote as

1. u_j, $j = 1...k_j$, the cells in the co-boundary of the i-cell p in Γ_d, $u_j \neq q$, and

2. v_h, $h = 1, ...k_h$, the cells in the boundary of the $(i+1)$-cell q in Γ_d, $v_h \neq p$.

The cancellation of p and q changes the connectivity (the incidence relation) of the descending complex Γ_d, as each cell v_h, $h = 1, ...k_h$, becomes part of the boundary of each cell u_j, $j = 1...k_j$.

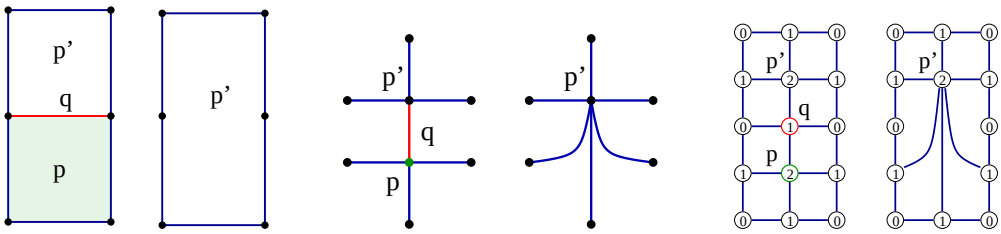
If we denote as

1. r_i, $i = 1...k_i$, the $(i+1)$-cells in the immediate co-boundary of the i-cell p in Γ_d, $r_i \neq q$, and

2. t_l, $l = 1, ...k_l$, the i-cells in the immediate boundary of the $(i+1)$-cell q in Γ_d, $t_l \neq p$,

then the effect of the cancellation of p and q on the immediate boundary relation is as follows: each i-cell t_l will belong to the immediate boundary of each $(i+1)$-cell r_i with the multiplicity increased by the product of the multiplicity of the i-cell p in the immediate boundary of the $(i+1)$-cell r_i and the multiplicity of the i-cell t_l in the immediate boundary of the $(i+1)$-cell q. In other words, $mult'(t_l, \partial r_i) = mult(t_l, \partial r_i) + mult(p, \partial r_i) \cdot mult(t_l, \partial q)$, where $mult$ and $mult'$ denote the multiplicities in the descending Morse complex Γ and in the simplified descending Morse complex Γ', respectively.

The cancellation of p and q changes also the geometry of some cells in the descending complex Γ_d. If



(a) (b) (c)

Figure 2.8: [26] Portion of a 2D descending Morse complex (a), ascending Morse complex (b), and Morse-Smale complex (c), before and after the cancellation of the maximum p and the saddle q. Numbers indicate the index of the corresponding critical point.

1. s_i, $i = 1...k_i$, are the $(i+1)$-cells in the immediate co-boundary of the i-cell p in Γ_d, $s_i \neq q$, and

2. z_l, $l = 1, ...k_l$, are the i-cells in the immediate boundary of the $(i+1)$-cell q in Γ_d, $z_l \neq p$,

then in the descending complex Γ_d, each $(i+1)$-cell s_i is merged with $mult(p, \partial s_i)$ disjoint copies of the $(i+1)$-cell q (and the i-cell p).

Dually, in the ascending complex Γ_a, the effect of the cancellation of the i-cell p and the $(i+1)$-cell q is to merge $mult(z_l, \partial q)$ disjoint copies of the $(n-i)$-cell p and the $(n-i-1)$-cell q into each $(n-i)$-cell z_l. These copies are infinitesimally close, and are considered to be combinatorially disjoint.

If q is a maximum (an n-cell in Γ_d) and p is an $(n-1)$-saddle, (an $(n-1)$-cell in Γ_d), then $k_i = 1$ and $mult(p, \partial s_1) = 1$. This means that the unique other n-cell s_1 in the co-boundary of the n-cell q in Γ_d is different from q. Dually, if p is a minimum (a 0-cell in the descending complex Γ_d) and q is a 1-saddle (a 1-cell in Γ_d), then $k_l = 1$ and $mult(z_1, \partial q) = 1$. This means that the unique other 0-cell z_1 in the boundary of q in Γ_d is different from p.

In the Morse-Smale complex, there is a new k-cell for each two critical points u and v of f that become connected through an integral line after the cancellation and that differ in index by k (for each two cells u and v that become incident to each other in the Morse complexes after the cancellation and that differ in dimension by k).

In 2D, there are two instances of the cancellation. One cancels a saddle and a maximum, the other cancels a saddle and a minimum.

The *maximum-saddle* cancellation is defined if the maximum p and the saddle q are connected through a unique integral line. This implies that the saddle q is connected through a unique integral line to exactly two different maxima p and p'.

In the descending complex Γ_d, p and p' correspond to 2-cells and q corresponds to the 1-cell shared by p and p'. The maximum-saddle cancellation, which eliminates the maximum p and the saddle q, is feasible on Γ_d if in a cyclic order of edges in the boundary of q, edge p appears only once ($mult(p, \partial q)$=1). The effect of the cancellation consists of deleting the 1-cell corresponding to q and thus merging the 2-cells p and p' (see Figure 2.8 (a)). Each 1-cell (and each 0-cell) on the boundary of p becomes a part of the boundary of p'.

In the ascending complex Γ_a, p and p' correspond to 0-cells and q corresponds to a 1-cell bounded by p and p'. The cancellation of the maximum p and the saddle q is feasible on Γ_a if the 1-cell q is not a loop with endpoint p. The effect of the maximum-saddle cancellation on Γ_a consists of collapsing the 1-cell corresponding to q and thus merging p into p' (see Figure 2.8 (b)). Each 1-cell (and each 2-cell) in the co-boundary of p becomes incident to p'.

In the Morse-Smale complex, p, p' and q are 0-cells, and q is connected to p and to p' through a 1-cell. After the cancellation, p and q are deleted, together with all the 1-cells incident in q (see Figure 2.8 (c)). All the 1-cells and 2-cells incident in p and not incident in q become incident in p'.

The effect of the *minimum-saddle* cancellation is completely dual. It corresponds to the contraction of a 1-cell in the descending complex and to the removal of a 1-cell in the ascending complex. The effect on the Morse-Smale complex is exactly the same as described for the maximum-saddle cancellation.

In higher dimensions, there are two types of the cancellation: one cancels an extremum and a saddle (a maximum and an $(n-1)$-saddle, or a minimum and a 1-saddle), the other cancels two saddle points. The first type of the cancellation can be thought of as the merging of cells.

The cancellation of a maximum p and an $(n-1)$-saddle q is feasible if q is connected through a unique integral line to exactly two different maxima p and p'. In the descending Morse complex, it removes the $(n-1)$-cell q, thus merging the n-cell p into the n-cell p'.

The cancellation of a minimum p and a 1-saddle q is feasible if q is connected through a unique integral line to exactly two different minima p and p'. In the descending complex Γ_d, it contracts the 1-cell q with the effect of collapsing the 0-cell p on the 0-cell p'. Dually, in the ascending Morse complex, the

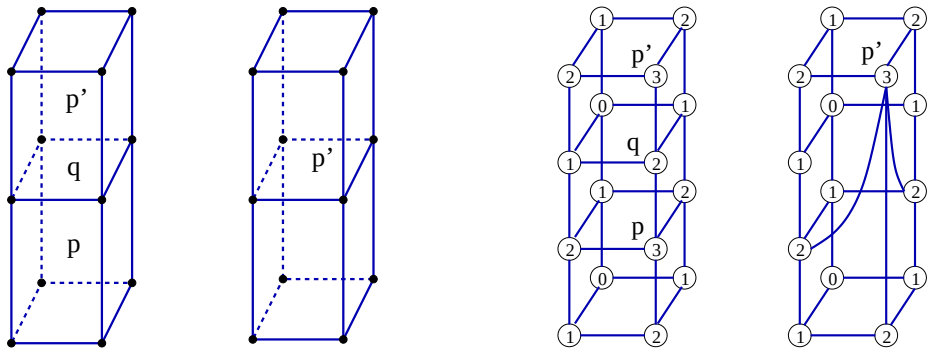


Figure 2.9: [26] A portion of the 3D descending Morse complex (a), and the Morse-Smale complex (b) before and after the cancellation of the maximum p and the 2-saddle q. The numbers indicate the index of the corresponding critical point.

cancellation of a minimum p and a 1-saddle q removes the $(n-1)$-cell q and merges the n-cell p with the n-cell p'; the cancellation of a maximum p and an $(n-1)$-saddle q contracts the 1-cell q and collapses the 0-cell p on the 0-cell p'.

On the Morse-Smale complex, the effect of the cancellation of a maximum p and an $(n-1)$-saddle q consists of deleting all the cells incident in q and not incident in either p or p', and merging p and q into p' by contracting the 1-cells connecting p to q and q to p'. The effect of the cancellation of a minimum p and a 1-saddle q on the Morse-Smale complex is completely similar.

Figure 2.9 shows the effect of the cancellation of a maximum p and a 2-saddle q on a 3D descending Morse complex and on the corresponding Morse-Smale complex. In the descending Morse complex Γ_d, the 2-cell q is removed, and the 3-cell p is merged with the unique 3-cell p' incident in q and different from p. In the Morse-Smale complex, all cells incident in q and not incident in either p or p' are deleted, and the 1-cells connecting p and q, and q and p' are contracted. Thus, all the remaining cells that were incident in p become incident in p'.

Cancellations that do not involve an extremum are more complex. In 2D, all cancellations involve an extremum, while already in 3D there are cancellations involving a 1-saddle and a 2-saddle. These latter cannot be interpreted as the merging of cells in the descending or ascending Morse complexes.

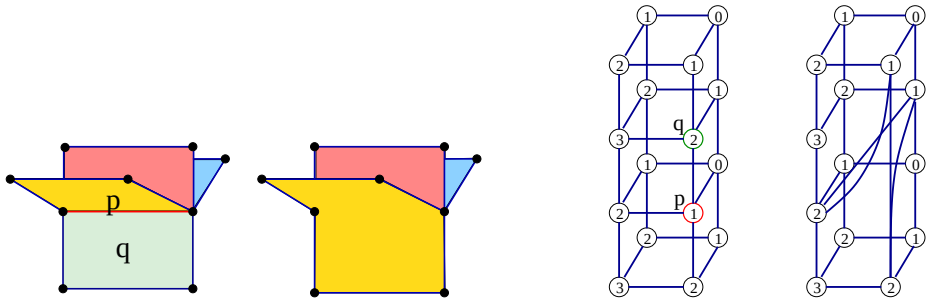


Figure 2.10: [26] The cancellation of the 1-saddle p and the 2-saddle q in 3D on (a) a portion of the descending Morse complex, and (b) a portion of the Morse-Smale complex. The numbers indicate the index of the corresponding critical point.

The cancellation of a 1-saddle p and a 2-saddle q on a 3D descending Morse complex is illustrated in Figure 2.10 (a). The 1-cell p and the 2-cell q are deleted. Each 1-cell that was in the immediate boundary of the 2-cell q (with the exception of the 1-cell p) becomes part of the immediate boundary of each 2-cell (with the exception of the 2-cell q) in the immediate co-boundary of the 1-cell p. Geometrically, each 2-cell incident in the 1-cell p (in the immediate co-boundary of p) is extended to include a disjoint copy of the 2-cell q (and of the 1-cell p), keeping an infinitesimal separation between these copies. Each 1-cell and each 0-cell that was in the boundary of the 2-cell q (with the exception of the 1-cell p) becomes part of the boundary of each 2-cell and each 3-cell incident in p (with the exception of the 2-cell q).

The effect of a cancellation of a 1-saddle p and a 2-saddle q on a Morse-Smale complex is illustrated in Figure 2.10 (b). The 0-cells corresponding to p and q are deleted from the complex, as well as all the 1-cells incident in either of them. The problem here is that there is an arbitrary number of 2-saddles connected through an integral line to p and an arbitrary number of 1-saddles connected to q. Each 2-saddle (and each 3-saddle) that was adjacent to p, becomes adjacent to each 1-saddle (and each 0-saddle) that was adjacent to q. The number of the 0-cells in the Morse-Smale complex (critical points of f) decreases by two, but the number of the 1-cells (and of the higher-dimensional cells) in the Morse-Smale complex may increase. Let s_i, $i = 1, ..., k_i$ be the 2-saddles adjacent to p and let z_l, $l = 1, ..., k_l$ be the 1-saddles adjacent to

q. Then, the operator deletes each 1-cell connecting p to s_i, and each 1-cell connecting q to z_l, and thus it deletes $k_i + k_l$ 1-cells, but it inserts a 1-cell for each pair (s_i, z_l), and thus it inserts a total of $k_i \cdot k_l$ 1-cells. Similarly, the number of 2-cells and 3-cells in the Morse-Smale complex may increase after the 1-saddle-2-saddle cancellation. There is a new 2-cell for each pair (s_i, r_h), where r_h is a 0-saddle (minimum) adjacent to q and not adjacent to p, and for each pair (t_j, z_l), where t_j is a 3-saddle (maximum) adjacent to p and not adjacent to q. Finally, there is a new 3-cell for each pair (t_j, r_h).

Each such new cell in the Morse-Smale complex corresponds to a new pair of incident cells in the descending (and in the ascending) Morse complex.

Let us consider another example of a sequence consisting of two cancellations, illustrated in Figure 2.11. A portion of the Morse complex before the cancellations is illustrated in Figure 2.11 (left). There are five 2-cells, which separate four 3-cells. The 2-cell f_1 is bounded by the 1-cells (v_1, v_2), (v_2, v_6), (v_6, v_5), (v_5, v_1), the 2-cell f_2 is bounded by the 1-cells (v_3, v_4), (v_4, v_6), (v_6, v_5), (v_5, v_3), the 2-cell f_3 is bounded by the 1-cells (v_7, v_8), $(v_8, v_{10}$, (v_{10}, v_9), (v_9, v_7), the 2-cell f_4 is bounded by the 1-cells (v_7, v_8), (v_8, v_{12}), (v_{12}, v_{11}), (v_{11}, v_8), while the 2-cell f_5 is bounded by the 1-cells (v_5, v_6), (v_6, v_8), (v_8, v_7), (v_7, v_5). The 2-cell f_1 separates the 3-cells V_1 and V_2, the 2-cell f_2 separates the 3-cells V_2 and V_3, the 2-cell f_3 separates the 3-cells V_1 and V_4, the 2-cell f_4 separates the 3-cells V_3 and V_4, while the 2-cell f_5 separates the 3-cells V_1 and V_3.

After the cancellation of the 1-cell (v_5, v_6) and the 2-cell f_5, the 1-cells in the immediate boundary of the 2-cell f_1 are (v_1, v_2), (v_2, v_6), (v_6, v_8), (v_8, v_7), (v_7, v_5), (v_5, v_1) (in this order), the 1-cells in the immediate boundary of f_2 are (v_3, v_4), (v_4, v_6), (v_6, v_8), (v_8, v_7), (v_7, v_5), (v_5, v_3) (in this order), the 2-cell f_1 separates the 3-cells V_1 and V_2 (as was the case before the cancellation), and the 2-cell f_2 separates the 3-cells V_2 and V_3 (as was the case before the cancellation). The 3-cells V_1 and V_3 are no longer adjacent (they are not separated by a common 2-cell). This is illustrated in Figure 2.11 (b).

After the cancellation of the 1-cell (v_7, v_8) and the 2-cell f_1, the 1-cells in the immediate boundary of the 2-cell f_2 are (v_3, v_4), (v_4, v_6), (v_6, v_8), (v_8, v_6), (v_6, v_2), (v_2, v_1), (v_1, v_5), (v_5, v_7), (v_7, v_5), (v_5, v_4) (in this order). Thus, the 1-cells (v_6, v_8) and (v_5, v_7) have the multiplicity equal to 2 in the immediate boundary of the 2-cell f_1 ($mult((v_6, v_8), \partial f_1) = mult((v_5, v_7), \partial f_1) = 2$). The 1-cells in the immediate boundary of the 2-cell f_3 are (v_{10}, v_9), (v_9, v_7), (v_7, v_5), (v_5, v_1), (v_1, v_2), (v_2, v_6), (v_6, v_8), (v_8, v_{10}) (in this order), and the 1-cells in the immediate boundary of the 2-cell f_4 are (v_{11}, v_{12}), (v_{12}, v_8), (v_8, v_6), (v_6, v_2),

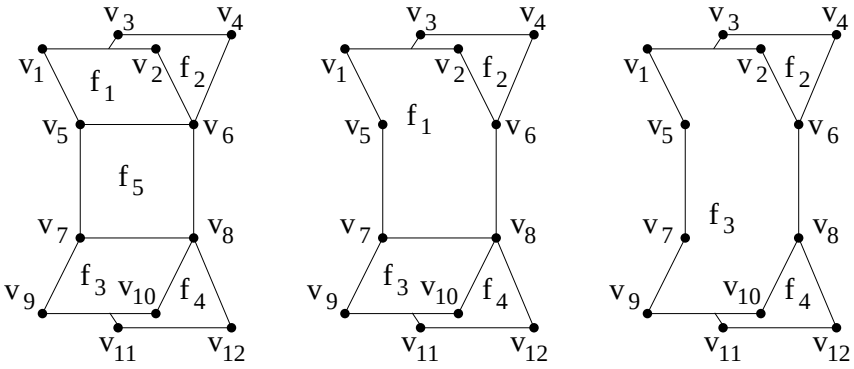


Figure 2.11: A portion of the descending Morse complex (a). After the cancellation of the 1-cell (v_5, v_6) and the 2-cell f_5 (b). After the cancellation of the 1-cell (v_7, v_8) and the 2-cell f_1 (c).

(v_2, v_1), (v_1, v_5), (v_5, v_7), (v_7, v_{11}) (in this order). The 2-cell f_3 separates the 3-cells V_4 and V_1, the 2-cell f_2 separates the 3-cells V_2 and V_3, and the 2-cell f_4 separates the 3-cells V_3 and V_4, as was the case before the cancellation. The 3-cells V_1 and V_2 are no longer adjacent.

Note that in 2D, the cancellation of a 1-cell and a 2-cell is also called edge removal, and the cancellation of a 0-cell and a 1-cell is also called edge contraction. These are the well known Euler operators, which are widely used both in boundary-based solid modeling [79], and in the framework of combinatorial maps and n-G maps. In solid modeling, they are called KEV (Kill Edge and Vertex) and KEF (Kill Edge and Face), while the inverse operators are called MEV (Make Edge and Vertex) and MEF (Make Edge and Face). In the framework of combinatorial maps the cancellation is called the (dual) edge contraction [17]. In 3D combinatorial maps, cancellation of a 1-cell and a 2-cell is defined only in a special situation, when a face is bounded by two different edges, and when no volume and no vertex are removed by the cancellation, thus avoiding the folding of faces [66]. All these operators maintain the Euler formula $\chi(\Gamma) = \sum_{i=0}^{n}(-1)^i c_i$, where c_i is the number of the i-cells in Γ. They change the number of the cells in the complex, but they do not change the Betti numbers or the Euler characteristic of the complex. We give a more detailed review of Euler operators in Section 3.4.1.

3

State of the Art

We review the related work on different research areas, which are relevant to the work presented here, namely: (*i*) topological representations of scalar fields in 2D and 3D, (*ii*) algorithms that assume a discretization of the domain of the scalar field as a manifold simplicial or cell complex and extract critical points and an approximation of Morse and Morse-Smale complexes of the field, (*iii*) simplification algorithms on Morse and Morse-Smale complexes, (*iv*) hierarchical and multi-resolution representations of the topology of 2D scalar fields encoded through the Morse and Morse-Smale complexes, and (*v*) update operators on cell complexes.

3.1 Topological Representations for 2D and 3D Scalar Fields

The first technique for representing the topological structure of a 2D scalar field has been proposed in [18], in the framework of analyzing topographic surfaces. Contour lines and integral lines have been used in [18] to describe and analyze terrains (2D scalar fields defined over a subset of a plane). It has been noted that in 2D each saddle is the origin and destination of exactly two integral lines, and that a slight perturbation of the scalar field f can eliminate integral lines connecting two saddles. The same notions have been discovered independently in [85], and have been extended to include the notion of the Morse-Smale complex in 2D, obtained as the intersection of the ascending and descending Morse complexes. The distinction between the different types of

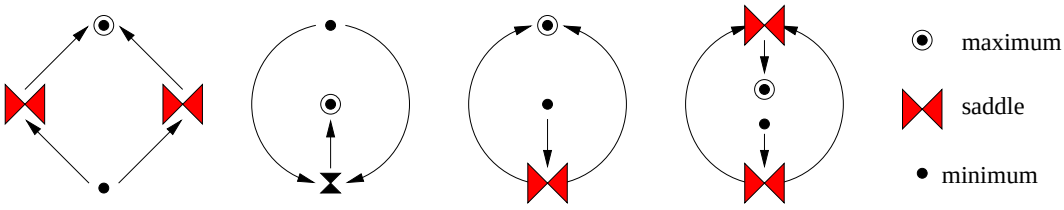


Figure 3.1: The slope districts of the CPCG.

saddles has been made (those that merge two components of the lower level set into one, and those that split one such component in two) and a relationship between the number of the critical points of f (the number of extrema and the number of saddles) has been established for the 2D case. In [87], the study of Morse theory proper has been initiated as a study of the relationships between the critical points of an nD scalar field f defined over a manifold M, and the topology (homotopy type) of M.

In [93, 111, 112], a definition of the *weighted surface network* is given, as a weighted, directed, tripartite planar graph $G = (N_0, N_1, N_2; E)$. For a given Morse-Smale scalar field f, the nodes in N_0, N_1 and N_2 correspond to minima, saddles and maxima of f, the arcs correspond to integral lines connecting them and the regions correspond to the 2-cells in the Morse-Smale complex of f. The weight of an arc is equal to the absolute difference in function values between its two end nodes. The notion of the *critical net* [102] is similar.

The *Critical Point Configuration Graph* (*CPCG*) [89] is a generalization of the critical net for a Morse function f defined on the closure of a simply-connected open set in the plane. It is not supposed that f is a Morse-Smale function, so some integral lines may connect two saddles. Minimal cycles of edges of the CPCG partition the domain of f into regions, called *slope districts*. There are four different types of slope districts, as illustrated in Figure 3.1. The first three are quadrangles (possibly glued along the edges) with nodes of index 1,0,1,2 respectively (saddle, minimum, saddle, maximum). These quadrangles correspond to the possible types of 2-cells in the Morse-Smale complex. The first type occurs most frequently in the real data, the second and third type correspond to an isolated mountain and an isolated crater, respectively. The last type of a slope district occurs only if f is not a Morse-Smale function. It is unstable, in the sense that a small perturbation of the scalar field f would replace the integral lines connecting two saddles with integral lines connecting those saddles to extrema.

The notion of a *quasi-Morse-Smale complex* in 2D and 3D has been introduced in [52] and [51], respectively, as a discrete counterpart of a Morse-Smale complex, with the aim of capturing the combinatorial structure of the Morse-Smale complex of a Morse-Smale function f defined over a manifold M, without making reference to the function f. In 2D, a quasi-Morse-Smale complex is defined as a complex (a graph embedded in M), in which the set of vertices can be partitioned into three sets U, V and W (corresponding to minima, saddles and maxima, respectively, of a Morse-Smale function f) and the set of edges can be partitioned into two sets A and B, such that: (i) there is no edge connecting two vertices from $U \cup W$, or two vertices from V, (ii) edges in A have endpoints in $U \cup V$ and edges in B have endpoints in $V \cup W$, (iii) each vertex $p \in V$ belongs to four edges, which alternate between A and B in a cyclic order around p, (iv) all regions of Γ are quadrangles, with vertices from U, V, W, V, in this order along the boundary.

In 3D, a complex Γ is a quasi-Morse-Smale complex if it is a subdivision of M, in which the set of vertices can be partitioned into four sets U, V, X and Y (corresponding to minima, 1-saddles, 2-saddles and maxima of a Morse-Smale function f), the set of edges can be partitioned into three sets R, S, and T and the set of regions can be partitioned into two sets P and Q, such that: (i) the edges from R, S, and T connect vertices from U and V, V and X, and X and Y, respectively, and the regions from P and Q are quadrangles with the nodes from U, V, X, V, and V, X, Y, X, in that order, respectively, around the boundary, (ii) there are no vertices within the arcs, quadrangles and 3-cells, (iii) each edge in S is in the boundary of four quadrangles, which in a cyclic order alternate between P and Q.

3.2 Algorithms for Computing Morse and Morse-Smale Complexes

The problem of extracting critical points and integral lines connecting them, as well as segmenting the domain of the scalar field f into regions associated with the critical points, has a long history. The first attempts to the decomposition of the domain of a scalar field f have been for 2D scalar fields representing images or terrains. Recently, some algorithms or 3D scalar fields have been proposed. It is assumed that the values of f are given at a discrete set of vertices forming either a *regular square grid (RSG)* [6, 92, 99, 104, 105] or an

irregular triangular network (TIN). Thus, the domain of a discrete scalar field is usually the set of vertices of a cubical or a simplicial complex. We will briefly review only the algorithms working on simplicial complexes. For a survey of the work in this area, see [11].

Almost all the existing methods extract the critical points of a scalar field f as a first step, based on the values of the function f at a vertex p and at the points in some neighborhood of p. In [8, 52, 90], the vertices of the triangulation are classified as regular, minimum, maximum or saddle according to their lower star $St^-(p)$. The *lower star* of a vertex p is the subset of the star of p which contains simplices that have p as their highest vertex. Thus, $St^-(p) = \{\sigma \in St(p) : f(q) \geq f(p), q \leq \sigma\}$. If a vertex p is a minimum (maximum), then its lower star is empty (coincides with the star). Otherwise, let k be the number of connected components in $St^-(p) - \{p\}$. If $k = 1$, then p is a regular vertex, and if $k > 1$, then p is a saddle point of multiplicity $k-1$ (a $(k-1)$-fold saddle). A procedure to decompose a k-fold saddle into k simple saddles has been proposed in [102]. It is similar to the procedure called unfolding of multiple saddles in [52].

Algorithms for decomposing the domain M of a 2D scalar field f into an approximation of a Morse or of a Morse-Smale complex can be classified as *boundary-based* [7, 14, 52, 91, 102], or *region-based* [19, 77]. Boundary-based algorithms compute the critical points and trace the integral lines of f (or their approximations), which start at saddle points and converge to minima and maxima of f. Region-based methods (including the watershed-based ones [96]) grow the 2D cells, defined by the minima and maxima of f, starting from those critical points.

A common feature of boundary-based algorithms is that in the computed approximation of the Morse-Smale complex it may happen that two or more ascending integral lines merge, and converge to the same maximum. Similarly, two or more descending integral lines may merge, and converge to the same minimum. Equivalently, in the approximation of the Morse complexes computed by region-based algorithms, it may happen that three or more 2-cells meet at a point, which is not a 0-cell in the complex. Thus, in an approximation of the Morse complexes (computed either by boundary-based or by region-based algorithms) 1-cells may (geometrically) partially overlap, but they are considered to be (combinatorially) disjoint [13].

Among the methods for classification of critical points in 3D based on gridded data [57, 103, 106, 107], only the method in [103] classifies saddles in

more detail (multiple 1- or 2-saddles), while for unstructured 3D data, the method in [51] uses the Betti numbers of the lower link of p to characterize the configurations of the neighborhood of each vertex and, in this way, it recognizes multiple saddles composed of an arbitrary number of 1- and 2-saddles. For each vertex p of Σ, the lower link $Lk^-(p)$ of p is considered. It consists of the vertices q in the link $Lk(p)$ of p such that $f(q) < f(p)$, and of the simplexes of $Lk(p)$ defined by these vertices. The vertex p is classified as a minimum if its lower link is empty. It is classified as a maximum if its lower link is the same as its link $Lk(p)$. Otherwise, p is classified based on the reduced Betti numbers $\tilde{\beta}_0$ an $\tilde{\beta}_1$ of the lower link $Lk^-(p)$ as a critical point composed of $\tilde{\beta}_0$ 1-saddles and $\tilde{\beta}_1$ 2-saddles. The reduced Betti numbers $\tilde{\beta}_k$ are equal to the un-reduced Betti numbers β_k for $k \geq 1$. For empty complexes, $\tilde{\beta}_0 = 1$, and for non-empty complexes $\tilde{\beta}_0 = \beta_0 - 1$. Intuitively, the Betti numbers β_0 and β_1 count the number of connected components and the number of holes in the complex, respectively.

Algorithms for computing Morse and Morse-Smale complexes for 3D scalar fields can also be classified as boundary-based [50, 51] and region-based [30, 43, 64]. Discrete methods rooted in discrete Morse theory proposed by Forman [55, 56] are computationally more efficient [38, 60, 70, 95].

We have given a survey on algorithms for extracting Morse and Morse-Smale complexes in 2D and 3D in [37] and [33], respectively.

3.3 Simplification and Hierarchical Representation of Morse and Morse-Smale Complexes

One of the major issues that arise when computing a representation of the scalar field as the Morse or the Morse-Smale complex is over-segmentation. It is caused by the inherent presence of noise in the data sets. *Simplification algorithms* have been developed in the literature in order to eliminate less significant features from these complexes. Simplification is achieved by applying an operator, called *cancellation*, which has been described in Section 2.2.4. The cancellation is defined for smooth Morse functions [84]. It transforms a Morse function f into a Morse function g with fewer critical points. Thus, it transforms a Morse-Smale complex into another, with fewer vertices, and it

transforms a Morse complex into another, with fewer cells. It enables also the creation of a hierarchical representation.

The cancellation in 2D consists of collapsing a maximum-saddle pair into a maximum, or a minimum-saddle pair into a minimum. The cancellation operator on 2D Morse-Smale complexes has been investigated in [14,52,62,102, 112]. In [44], the cancellation operator in 2D has been extended to functions that may have multiple saddles and macro-saddles (saddles that are connected to each other).

In 3D, the cancellation consists of collapsing a maximum-2-saddle pair into a maximum, or a minimum-1-saddle pair into a minimum, plus the cancellation operator of a pair consisting of a 1-saddle and a 2-saddle. The effect of the cancellation of critical points on 3D Morse-Smale complexes has been investigated in [62]. In [22], we have investigated the effect of the cancellation on 3D Morse complexes and we have shown that the effect of the cancellation can be more naturally described on Morse complexes than on Morse-Smale ones. We have also shown how the cancellation affects the incidence graph representing Morse and Morse-Smale complexes.

There have been only two approaches in the literature to modify not only the Morse and Morse-Smale complexes by the cancellation, but to simplify also the scalar field f after the cancellation, i.e., to construct a function g that corresponds to the simplified complexes and thus to couple the topological simplification with the smoothing of f. Both approaches are for two-dimensional scalar fields and they modify the function f numerically. The approach in [14] uses Laplacian smoothing and produces a C^0 simplified field, while the approach in [109] produces a C^1 result. Another approach, presented in [54] for the 2D case, modifies the scalar field f combinatorially, by changing the order in which the vertices appear in their sorted list according to function values.

In [67], simplification of cubical complexes in arbitrary dimensions has been considered, with the aim to compute the homology of the complex by reducing it to a smaller complex having the same homology. Simplification operators used are called collapses and reductions. A *collapse* is our $remove_{i,i+1}$ operator of the second type (see Section 4), and a *reduction* is a cancellation (see Section 2.2.4), both applied on a cubical complex.

In [97], simplification operators have been defined in the framework of geometric modeling, without studying a (scalar) function defined on the object. The object is described through a decomposition into cells, which are not assumed to be topological cells (homeomorphic to a ball). Two simplification

operators have been defined: a topological simplification operator *join* is our $remove_{i,i+1}$ operator of the first type; a (non-topological) simplification operator *incorporate* produces cells that are not homeomorphic to a ball.

Topological simplification operators have been defined for 2D vector fields in [20]. For gradient vector fields (without periodic orbits) these operators reduce to the cancellation (which, in 2D, is equal to the *remove* operator).

Due to the large size and complexity of available scientific data sets, a hierarchical and a multi-resolution representation is crucial for their interactive exploration. Historically, a *hierarchical* representation of the 2D scalar field topology has been first investigated in the framework of Geographic Information Systems (GIS), where 2D scalar fields are usually called terrains, and the critical points and the separatrix lines connecting them are represented in the form of a critical net. Current hierarchical terrain models are just based on a progressive simplification process based on the cancellation operator applied to a simplicial mesh describing a terrain at full resolution [13, 52, 101, 112].

The pair of critical points to be removed must be selected in such a way that the resulting decomposition is still a Morse-Smale complex. This means that along each integral line connecting a minimum with a saddle, or a saddle with a maximum, the function f is increasing. In other words, the weights of the arcs in the corresponding generalized surface network are positive. When a saddle s together with an adjacent minimum (maximum) p is removed, the remaining saddles s_i that were connected to p will be connected to the other minimum (maximum) p' adjacent to s. This connection is possible if the elevation (function value) at s_i is higher (lower) than the elevation (function value) at p'. One way to ensure this is to express the weights of the edges (p', s_i) in terms of the weights of the edges (p', s), (s, p), and (p, s_i), as $w(p', s_i) = w(p', s) - w(s, p) + w(p, s_i)$, and take care that the negative summand $(-w(s, p))$ is not greater (in absolute value) than the sum of other two summands $(w(p', s) + w(p, s_i))$.

In [112], a minimum (maximum) p is chosen for cancellation together with its lowest (highest) adjacent saddle s. In [101], a pair of critical points (p, s) is chosen such that the difference in elevation between p and s is minimal among all (unsigned) differences in elevation between a saddle and an adjacent minimum, or a saddle and an adjacent maximum. In the approach described in [52], and later in [13] (where geometric considerations are taken into account), a saddle s is chosen together with its adjacent maximum at lower elevation, or its adjacent minimum at higher elevation. The order in

which the pairs of points are cancelled is determined based on the notion of persistence (see [53] for details). Different measures of importance of a critical point, defined by assigning a weight not only to the edges, but also to the vertices of the critical net, have been introduced in [112], such as the maximum, or the minimum or the sum of the differences in elevation between a maximum (minimum) and all of its adjacent saddles. It has been argued that the criteria using only elevation difference, and not taking into account the length of the corresponding integral lines or some other area- or slope-based measures, fail to provide enough information about the importance of the critical point [94]. In any case, the measure of importance of a point is application dependent, and cannot be expressed in absolute terms. In [41], a simplification algorithm guided by the morphology of the critical net has been proposed. Finally, in [98] a hierarchical critical net has been proposed based on an entirely different approach, which identifies hierarchies of ridges and valleys inside the critical net at maximum resolution.

A more concise and powerful representation can be obtained by combining the Morse-Smale decomposition (and the critical net) with a *multi-resolution* approach. It is based on the simplification operators incrementally applied to the critical net at maximum resolution and on the definition of the dependency relation between these operators, which can be organized into a multi-resolution representation as the one proposed in [46].

There have been several approaches in the literature to a multi-resolution representation of the topology of a scalar field in 2D [14, 15, 44]. In [14], a region of influence is associated with each cancellation, and two cancellations are defined to be independent if their regions of influence have an empty intersection. The large size of the region of influence induces many dependencies between the operators, and thus a small number of representations that can be obtained from the multi-resolution model. In [15], the region of influence of a cancellation has been significantly reduced, enabling a creation of a more flexible multi-resolution model for 2D scalar fields. The method in [44] creates a multi-resolution representation of a function with multiple saddles and saddles that can be connected to each other. Hierarchical watershed approaches have been developed to cope with the increase in size of both 2D and 3D images [10].

There have been two attempts in the literature to couple the multi-resolution topological model provided by Morse-Smale complexes with the multi-resolution model of the geometry of the underlying simplicial mesh. The approach in [14] first creates a hierarchy of Morse-Smale complexes by applying cancellation

operators to the full-resolution complex, and then, by Laplacian smoothing, it constructs the smoothed function corresponding to the simplified topology. The approach in [42] creates the hierarchy by applying half-edge contraction operator, which simplifies the geometry of the mesh. When necessary, the topological representation corresponding to the simplified coarser mesh is also simplified. The data structure encoding the geometrical hierarchy of the mesh, and the data structure encoding the topological hierarchy of the critical net are interlinked. The hierarchical critical net is used as a topological index to query the hierarchical representation of the geometry of the simplicial mesh.

3.4 Update Operators on Cell Complexes

We review briefly the topological operators designed for building and updating data structures representing cell complexes. We classify these operators as: *Homology-preserving* operators, which change the combinatorial description of the model by introducing or removing cells in the complex, but do not change its homology (expressed through the Euler-Poincaré characteristics and the Betti numbers); *Homology-modifying* operators, which introduce or remove topological entities in a way that changes the topology of the complex, but they do not influence the validity of the Euler-Poincaré formula. Initialization operators, which create an initial model starting from the empty set, may be considered as homology-modifying.

A review of data structures and update operators for cell complexes has been presented in [28]. A review of data structures for simplicial complexes can be found in [45].

3.4.1 Euler Operators on Cell Complexes

We adopt the naming convention widely used for Euler operators. The letters M and K stand for *Make* and *Kill* (create and delete) a topological entity. *Kill* operators are inverse to the corresponding *Make* ones.

In a wide variety of the basis Euler operators proposed in the literature, operators MEV (*Make Edge and Vertex*) and MEF (*Make Edge and Face*) are a part of virtually all of the bases. They have been proposed in [9] for modeling polyhedral models (surfaces) homeomorphic to a 2-sphere and satisfying formula 2.4. MEV operator and the dual MEF operator are illustrated in Figures 3.2 and 3.3, respectively.

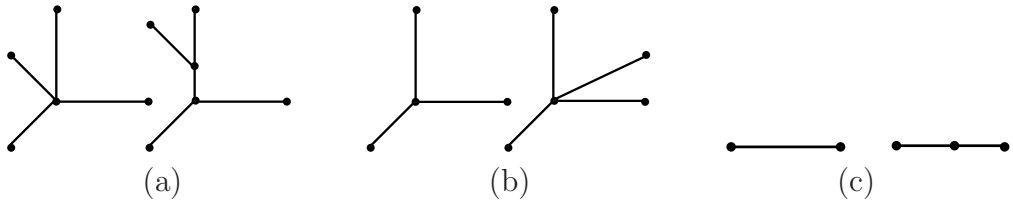


Figure 3.2: [28] Some instances of the MEV operator.

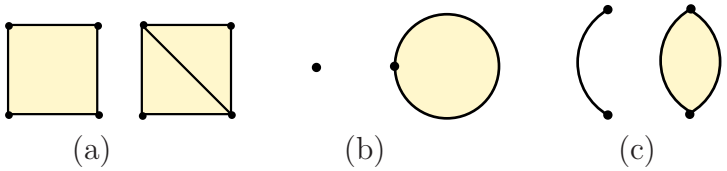


Figure 3.3: [28] Some instances of the MEF operator.

Euler Operators on Manifold Complexes

For a complex Γ representing an orientable manifold surface S bounding a solid object in 3D and satisfying Euler-Poincaré formula 2.3, several sets of basis operators have been proposed [3, 12, 48, 78, 79]. The connected components of S are called shells. The initialization operator is called MVFS (*Make Vertex, Face and Shell*). It creates a topological representation of a 2-sphere with minimum number of cells: one vertex and one face, which together define one shell.

In [48], two homology modifying operators are defined, called *glue* and $MEKF$ (*Make Edge, Kill Face*).

The *glue* operator merges two faces and deletes them both, together with all the edges and vertices on the boundary of one of the deleted faces. It corresponds to the connected sum operator on manifold surfaces. Two faces may be glued if they have the same number of vertices, and they have no edges in common. There are two instances of the glue operator, illustrated in Figure 3.4. If the two glued faces belong to the same shell, a handle (genus) is created, and the operator is called $KFMH$ (*Kill Face, Make Hole*). If the two glued faces belong to two different shells, one shell is deleted, and the operator is called KFS (*Kill Face and Shell*).

$MEKF$ operator in [12,48,78] operator joins two vertices belonging to two different faces through a new edge, merging the two faces into one face. The new edge belongs twice to the new joined face. The operator has three in-

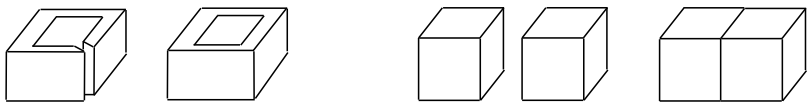


Figure 3.4: [28] The two instances of the glue operator: $KFMH$ (*Kill Face, Make Hole*) (a); KFS (*Kill Face and Shell*) (b).

stances, since it ca decrease the number β_0 of connected components, decrease the number β_2 of cavities, or increase the number β_1 of holes, as illustrated in Figure 3.5.

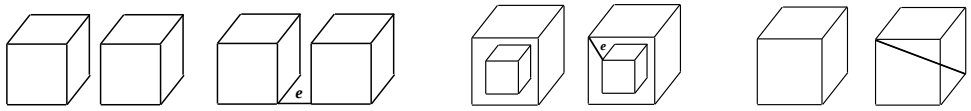


Figure 3.5: [28] The three instances of the $MEKF$ (*Make Edge, Kill Face*) operator.

In [12], the topology modifying operator is called $MRKF$ (*Make Ring, Kill Face*). It creates a ring and deletes a face from the model, by gluing a (simply connected) face to another face, thus deleting one face and making an (inner) ring in another face. It has two instances:

- $KFMRH$ (*Kill Face, Make Ring and Hole*) operator glues two faces belonging to the same shell, thus making a hole (genus) in the surface.
- $MRKFS$ (*Make Ring, Kill Face and Shell*) operator glues together two faces belonging to two different shells, thus merging two shells into one.

$MRKF$ operator is similar to the glue operator in [48], but it imposes looser conditions on the glued faces, and it deletes only one of the faces. If an edge is introduced connecting a vertex from one face to a vertex of the other face, then no inner ring is produced, and the remaining cell is a topological cell (homeomorphic to a disc).

Euler Operators on Non-Manifold Complexes

The earliest approaches to solid modeling considered a *wireframe model* (a graph). The homology modifying operator, called ME (*Make Edge*) [110],

creates a new edge, which connects two existing vertices. It either merges two components or creates a hole.

The first approach to modeling and updating the boundary of a non-manifold solid object, introduced in [108], has been extended in [71], where the set of basis operators has been proposed.

Homology-modifying operators, illustrated in Figure 3.6, are: $MVCc$ (*Make Vertex and Complex Cavity*), which makes a new vertex in the interior of a component, and a new cavity composed of a single vertex; $MECh$ (*Make Edge and Complex Hole*), which makes a new edge connecting two existing vertices on the same connected component in the complex, and creates a hole; $MFKCh$ (*Make Face, Kill Complex Hole*), which makes a face, which fills in and kills an existing complex hole (cycle of edges); and $MFCc$ (*Make Face and Complex Cavity*), which makes a face filling in a loop of edges that is not a cycle, and closes off a cavity.

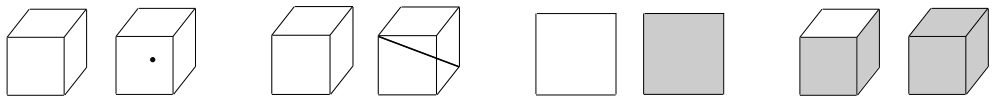


Figure 3.6: [28] Homology modifying operators in [71]: $MVCc$ (*Make Vertex and Cavity*) (a); $MECh$ (*Make Edge and Hole*) (b); $MFKCh$ (*Make Face, Kill Hole*) (c); $MFCc$ (*Make Face and Cavity*) (d).

For 3-complexes in $\mathbb{R}^3$ [82, 83], an additional homology-modifying operator is defined, called $MVlKCc$ (*Make Volume, Kill Complex Cavity*), which makes a volume that fills in a cavity.

In [58], the operators in [83] have been extended to complexes called *stratifications*, in which cells, called *strata*, are defined by analytic equalities and inequalities. Homology preserving operators, called *cell subdividers*, subdivide an n-cell by inserting into it an $(n-1)$-cell.

Homology modifying operators, called *global hole shapers*, either attach or detach a cell, thus creating or deleting a hole. There are two instances of this operator: the attached topological n-cell creates an n-hole or the detached topological n-cell creates an $(n-1)$-hole.

3.4.2 Splice Operator

The *splice* operator, defined in 2D in [59] and in 3D in [47], unifies the various Euler operators in a single operator.

In 2D, it takes as argument two edges. Depending on the cycles the two edges belong to, the splice operator can be either homology-preserving or homology-modifying. In the case when it is homology-preserving, splice can be expressed through homology-preserving Euler operators as a MEF followed by KEV, or as a MEV followed by KEF. Thus, it either increases the number of faces by one and decreases the number of vertices by one, or it decreases the number of faces by one and increases the number of vertices by one.

When it is homology-modifying, it either merges two connected components of the surface into one, or it creates a handle on the surface. In the first case, it can be expressed through homology-modifying Euler operator KFSME in [79], which connects the two connected components through an edge, followed by KEV, which contracts the edge made by KFSME. In the second case, it can be expressed through homology-modifying operator KFMEH, which connects two vertices on the same face through an edge, followed by KEV, which contracts the edge made by KFSME.

The *splice* operator in 3D does not always produce a valid complex. Because of this drawback, another homology-modifying operator, called *meld* has been introduced in [47]. It glues two faces with equal number of edges on their boundaries, similarly to the glue operator in [48], but it deletes only one of the glued faces, not both of them.

3.4.3 Handle Operators

The *handle operators* on a manifold cell 2-complex Γ triangulating a surface S have been introduced in [74]. They are based on the handlebody theory for surfaces, stating that any surface S can be obtained from a 2-ball by iteratively attaching handles (0-, 1- and 2-handles).

The initialization operator in [74] corresponds to attaching a 0-handle. It creates a new surface with one face, three edges and three vertices.

There are three operators that correspond to attaching a 1-handle. They identify two boundary edges of Γ (incident in exactly one face) with no vertices in common. If the two identified edges belong to two different components of Γ, then the number of connected components β_0 and of boundary curves (connected components of boundary edges) in Γ is decreased by one. If the two

identified edges belong to the same component and the same boundary curve of Γ, then the number β_1 of holes (independent 1-cycles) and the number of boundary curves in Γ is increased by 1. If the two identified edges belong to the same component and two different boundary curves of Γ, then the number β_1 of holes (independent 1-cycles) is increased by 1, and number of boundary curves in Γ is decreased by 1.

The operator that corresponds to the attachment of a 2-handle identifies two edges on the boundary of Γ with two vertices in common. It decreases the number β_1 of holes and the number of boundary curves in Γ by 1.

The analogous work in 3D in [75] uses the fact that each compact orientable 3-manifold S can be obtained by iteratively attaching handles (0-, 1-, 2- and 3-handles) to a 3-ball.

The operator that creates a new 3-ball (initialization operator) corresponds to the attachment of a 0-handle. Other operators identify two boundary faces (incident in exactly one 3-cell) of a cell 3-complex Γ triangulating a solid S.

The attachment of a 1-handle can be applied in three situations: if the two identified boundary faces are on different connected components of Γ, then the two components are merged into one; if the two identified faces belong to the same boundary surface component of Γ (connected component of boundary faces) and have no edges in common, then a hole is created; if the two identified faces belong to the different boundary surfaces of the same connected component of Γ, the operator can be realized only if Γ is embedded in a space of dimension greater than 3.

The attachment of a 2-handle corresponds to identifying two faces on the same boundary surface component of Γ that have some edges in common. The operator can create cavities and/or close holes in Γ. The attachment of a 3-handle is applicable if the two identified faces belong to the same boundary surface component and have all edges in common. This operator fills in the cavity formed by the two identified faces.

4

New Update Operators on Morse Complexes

In [24, 27], we have defined two dual dimension-independent simplification operators that we have called *removal* and *contraction*. These operators have been defined as a special case of the cancellation, and they have been obtained by imposing additional constraints on the feasibility of the cancellation. The constraints guarantee that the two operators behave like a cancellation involving an extremum: they do not increase the number of separatrix lines of the Morse function, they do not increase the incidence relation on the Morse complexes or the number of cells in the Morse-Smale ones. A removal behaves like a cancellation involving a maximum, and a contraction behaves like a cancellation involving a minimum.

In [26], we have defined the refinement *insertion* and *expansion* operators, inverse to the simplification *removal* and *contraction* operators, respectively. We have described the effect of these operators on the Morse function and on the corresponding Morse complexes. We have proved that the simplification *removal* and *contraction* operators, together with the inverse refinement *insertion* and *expansion* operators, form a minimal basis through which any operator on Morse complexes on a manifold M can be expressed.

In [32], we have defined a new simplification operator as a generalization of the *removal* and *contraction* simplification operators, by relaxing the constraints on the feasibility of the operators, and we have defined the inverse refinement operator. We have called the new simplification operator *remove*. It retains the properties of *removal* and *contraction* operators while having a

wider applicability domain. The operator *remove* has two instances, namely the $remove_{i,i+1}$ and the dual $remove_{i,i-1}$, which are a generalization of the *removal* and *contraction* operators, respectively. The new refinement operator has been called *insert*. It is a generalization of the *insertion* and *expansion* operators. It also has two instances, the $insert_{i,i+1}$ and the dual $insert_{i,i-1}$, inverse to the $remove_{i,i+1}$ and the dual $remove_{i,i-1}$, respectively.

Here, we recall the definition of the simplification *remove* operator, and of its inverse refinement *insert* operator, and we show the effect of these operators on Morse functions and on the corresponding Morse complexes in Sections 4.1 and 4.2, respectively. In Section 4.3, we prove that these operators form a basis for the set of operators on Morse complexes on a manifold M. In Section 4.4, we describe a dimension-independent data structure for representing a sequence of simplification operators. We have introduced this data structure in [25], and we have called it the *augmented cancellation forest*. In Section 4.5, we give a detailed comparison of the *remove* operator and cancellation in the 3D case.

4.1 Simplification Operator *remove*

We introduce the notion of the *combinatorial* Morse complex, as a cell complex in which the cells are allowed to partially overlap (geometrically), while they are considered to be disjoint (combinatorially). The motivation for introducing combinatorial Morse complexes is twofold:

- the majority of algorithms that compute an approximation of the Morse complexes produce combinatorial complexes (see Section 3.2),

- the cancellation of an i-cell p and an $(i+1)$-cell q on the descending Morse complex (and dually on the ascending complex) in the continuous case merges each $(i+1)$-cell in the co-boundary of p with a disjoint copy of the $(i+1)$-cell q (and the i-cell p), keeping an infinitesimal separation between these copies, thus producing a combinatorial complex (see Section 2.2.4).

We have defined the removal and contraction simplification operators, and their generalization the *remove* operator, on the combinatorial Morse complexes in the discrete case as the operators that simulate the effect on the Morse complexes of the same operators on a Morse function.

4.1.1 Operator $remove_{i,i+1}$

The $remove_{i,i+1}$ operator merges two $(i+1)$-saddles by deleting the i-saddle connected to both of them. It is feasible if the deleted i-saddle has a correct neighborhood structure through integral lines, as we explain below. The operator has two types, denoted as $remove_{i,i+1}(q,p,p')$ and $remove_{i,i+1}(q,p,\emptyset)$. We will give a definition of the $remove_{i,i+1}$ operator on the scalar field f and on the corresponding combinatorial Morse complexes, and we will give examples in 2D and 3D.

Definition 1 *Let p and q be an $(i+1)$-saddle and an i-saddle respectively, $1 \leq i \leq n-1$, such that*

- *the i-saddle q is connected through an integral line (a separatrix) to the $(i+1)$-saddle p and exactly one $(i+1)$-saddle p' different from p, and*
- *there is exactly one integral line connecting the i-saddle q to the $(i+1)$-saddle p.*

Then, the cancellation of p and q is called the $remove_{i,i+1}$ of the i-saddle q and the $(i+1)$-saddle p, denoted as $remove_{i,i+1}(q,p,p')$.

Definition 2 *Let p and q be an $(i+1)$-saddle and an i-saddle respectively, $1 \leq i \leq n-1$, such that*

- *the i-saddle q is connected through an integral line (separatrix) to exactly one $(i+1)$-saddle p, and*
- *there is exactly one integral line connecting the i-saddle q to the $(i+1)$-saddle p.*

Then, the cancellation of p and q is called the $remove_{i,i+1}$ of the i-saddle q and the $(i+1)$-saddle p, denoted as $remove_{i,i+1}(q,p,\emptyset)$.

The effect of the $remove_{i,i+1}(q,p,p')$ operator on the set of integral lines starting or ending at p or q, can be deduced from the effect of the cancellation of p and q on those lines. The $remove_{i,i+1}(q,p,p')$ transforms the set of integral lines converging to p or to q into a set of integral lines converging to p' (the unique $(i+1)$-saddle different from the $(i+1)$-saddle p, connected to the i-saddle q). Each critical point that was the origin of an integral line converging to p or to q becomes the origin of an integral line converging to p'.

The effect of the cancellation of the critical points p and q on the geometry of cells in the descending Morse complex Γ_d is as follows: the i-cell q is removed and (a disjoint copy of) the $(i+1)$-cell p is merged into the $(i+1)$-cell p' for each time the i-cell q appears in the immediate boundary of the $(i+1)$-cell p'. More formally, the only $(i+1)$-cell p' in the co-boundary of the lower-dimensional deleted cell (the i-cell q) is merged with k disjoint copies of the higher-dimensional deleted cell (the $(i+1)$-cell p), where k is the multiplicity of the lower-dimensional cancelled cell (the i-cell q) in the immediate boundary of the $(i+1)$-cell p' ($k = mult(q, \partial p')$).

The combinatorial effect of the operator is that the immediate boundary of the $(i+1)$-cell p is merged in the immediate boundary of the $(i+1)$-cell p' with the appropriate adjustment of the multiplicities. We denote as *mult* the multiplicities before the *remove*, and as $mult'$ the multiplicities after the *remove*. Each i-cell r that was in the immediate boundary of the $(i+1)$-cell p (with the exception of the i-cell q) with the multiplicity $mult(r, \partial p)$ belongs to the immediate boundary of the $(i+1)$-cell p' after the $remove_{i,i+1}(q, p, p')$ with the multiplicity increased by the product of the multiplicity of the i-cell r in the immediate boundary of the $(i+1)$-cell p and the multiplicity of the i-cell q in the immediate boundary of the $(i+1)$-cell p' ($mult'(r, \partial p') = mult(r, \partial p') + mult(r, \partial p) \cdot mult(q, \partial p')$).

We introduce the following notation for the cells in the combinatorial complex Γ_d (before the $remove_{i,i+1}(q, p, p')$ of the i-cell q and the $(i+1)$-cell p):

1. r_j, $j = 1, .., j_{max}$, are the i-cells (different from the i-cell q) in the immediate boundary of the $(i+1)$-cell p in Γ_d, $R = \{r_j, j = 1, .., j_{max}\}$,

2. z_h, $h = 1, .., h_{max}$, are the $(i-1)$-cells in the immediate boundary of the i-cell q in Γ_d, $Z = \{z_h, h = 1, .., h_{max}\}$, and

3. s_k, $k = 1, .., k_{max}$, are the $(i+2)$-cells in the immediate co-boundary of the $(i+1)$-cell p in Γ_d, $S = \{s_k, k = 1, .., k_{max}\}$.

We denote as $C_{q,p}$ the set of cells in the immediate boundary and the immediate co-boundary of the deleted cells q and p, $C_{q,p} = \{p'\} \cup R \cup Z \cup S$.

We define the $remove_{i,i+1}$ operator on the combinatorial (descending) Morse complex Γ_d so that it simulates the effect on the descending Morse complex of the $remove_{i,i+1}$ operator on the Morse function.

Definition 3 *The $remove_{i,i+1}(q, p, p')$ operator (of the first type), $1 \leq i \leq n-1$, on the descending combinatorial complex Γ_d is feasible if the i-cell q*

appears in the boundary of exactly two $(i+1)$-cells p and p', and it appears exactly once in the immediate boundary of the $(i+1)$-cell p ($mult(q, \partial p) = 1$).

The operator replaces the set $\{q, p\} \cup \{p'\} \cup R \cup Z \cup S = \{q, p\} \cup C_{q,p}$ of cells in Γ_d with the set $\{p'\} \cup R \cup Z \cup S = C_{q,p}$, producing the simplified complex Γ'_d. The immediate boundary and co-boundary relation for the cells in the simplified complex Γ'_d is the same as the immediate boundary and co-boundary relation for the corresponding cells in Γ_d, except that:

1. *each instance of the $(i+1)$-cell p is removed from the immediate boundary of each $(i+2)$-cell s_k in S, and from the immediate co-boundary of each i-cell r_j in R,*

2. *each instance of the i-cell q is removed from the immediate co-boundary of each $(i-1)$-cell z_h in Z, and from the immediate boundary of the new $(i+1)$-cell p', and*

3. *each instance of each i-cell r_j in R is added k times to the immediate boundary of the new $(i+1)$-cell p', where k is the multiplicity of the i-cell q in the immediate boundary of the $(i+1)$-cell p' ($mult'(r_j, \partial p') = mult(r_j, \partial p') + mult(r_j, \partial p) \cdot mult(q, \partial p')$).*

The new $(i+1)$-cell p' in Γ'_d is the union of the $(i+1)$-cell p' (before the $remove_{i,i+1}$ (q, p, p')), a disjoint copy of the $(i+1)$-cell p for each time the i-cell q appears in the immediate boundary of the $(i+1)$-cell p', and the i-cell q in Γ_d. Other cells in Γ'_d are the same as the corresponding cells in Γ_d.

The operator $remove_{i,i+1}(q, p, p')$ is defined on the dual combinatorial ascending Morse complex Γ_a in a dual manner. We introduce the notation:

1. r_j, $j = 1, .., j_{max}$, are the $(n-i)$-cells (different from the $(n-i)$-cell q) in the immediate co-boundary of the $(n-i-1)$-cell p in Γ_a, $R = \{r_j, j = 1, .., j_{max}\}$,

2. z_h, $h = 1, .., h_{max}$, are the $(n-i+1)$-cells in the immediate co-boundary of the $(n-i)$-cell q in Γ_a, $Z = \{z_h, h = 1, .., h_{max}\}$, and

3. s_k, $k = 1, .., k_{max}$, are the $(n-i-2)$-cells in the immediate boundary of the $(n-i-1)$-cell p in Γ_a, $S = \{s_k, k = 1, .., k_{max}\}$.

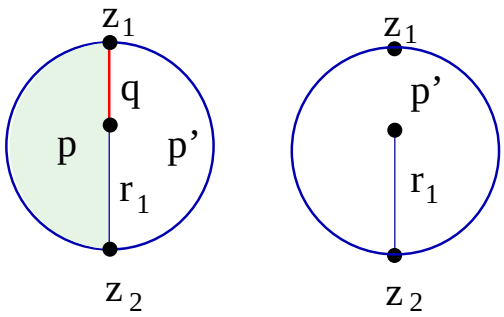


Figure 4.1: After the $remove_{1,2}(q, p, p')$, 1-cell r_1 appears two times in the immediate boundary of the 2-cell p'.

After the $remove_{i,i+1}(q, p, p')$, the $(n-i)$-cell q is contracted and the $(n-i-1)$-cell p is merged into the $(n-i-1)$-cell p'. A disjoint copy of the higher-dimensional deleted cell (the $(n-i)$-cell q) is merged into each $(n-i)$-cell in the immediate co-boundary of the lower-dimensional deleted cell (the $(n-i-1)$-cell p) for each time the higher-dimensional deleted cell q has the $(n-i-1)$-cell p' in its immediate boundary. Each $(n-i)$-cell r_j in R that had the $(n-i-1)$-cell p in its immediate boundary (with the exception of the $(n-i)$-cell q) has a copy of the $(n-i-1)$-cell p' in its immediate boundary with the multiplicity increased by the product of the multiplicity of the $(n-i-1)$-cell p in the immediate boundary of the $(n-i)$-cell r_j and the multiplicity of the $(n-i-1)$-cell p' in the immediate boundary of the $(n-i)$-cell q ($mult'(p', \partial r_j) = mult(p', \partial r_j) + mult(p, \partial r_j) \cdot mult(p', \partial q)$).

In 2D, there is just one $remove_{i,i+1}$ operator, the $remove_{1,2}(q, p, p')$, where q is a saddle, and p and p' are maxima: it is the cancellation of a maximum p and a saddle q, discussed in Section 2.2.4, and illustrated in Figure 2.8.

Another example is illustrated in Figure 4.1. Before the $remove_{1,2}(q, p, p')$, the 1-cell r_1 appears once in the immediate boundary of the 2-cells p and p' ($mult(r_1, \partial p) = mult(r_1, \partial p') = 1$). After the $remove_{1,2}(q, p, p')$, $mult'(r_1, \partial p')$ $= mult(r_1, \partial p') + mult(r_1, \partial p) \cdot mult(q, \partial p') = 1 + 1 \cdot 1 = 2$.

In 3D, there are two $remove_{i,i+1}$ operators: the $remove_{1,2}$ of a 1-saddle and a 2-saddle, and the $remove_{2,3}$ of a 2-saddle and a maximum. This latter is the same as the cancellation of a maximum p and a 2-saddle q, discussed in Section 2.2.4, and illustrated in Figure 2.9.

The $remove_{1,2}(q, p, p')$ in 3D is different from the cancellation of a 1-saddle q and a 2-saddle p, since it requires that the 1-saddle q is connected to exactly

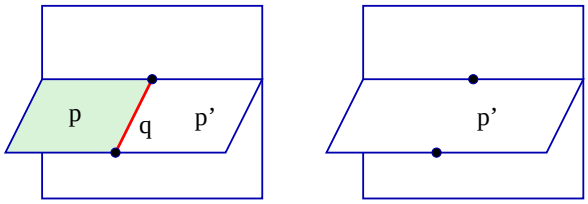


Figure 4.2: [26] A portion of a 3D descending Morse complex before (a) and after (b) the $remove_{1,2}(q, p, p')$.

two 2-saddles p and p', i.e., that the 1-cell q bounds exactly two 2-cells p and p' in the descending complex, and that the 1-cell q appears exactly once in the boundary of the 2-cell p. An example of the effect of the $remove_{1,2}(q, p, p')$ on a 3D descending Morse complex is illustrated in Figure 4.2. After the $remove_{1,2}$, in the simplified descending Morse complex Γ'_d, the 1-cell q is deleted, and the 2-cell p is merged with the unique 2-cell p' in the co-boundary of q and different from p. The boundary of p becomes part of the boundary of p'.

In the n-dimensional case, the $remove_{n-1,n}(q, p, p')$ is the same as the cancellation of an $(n-1)$-saddle q and a maximum p, while the $remove_{i,i+1}$, $1 \leq i < n-1$, is a special case of the cancellation, since it requires that the i-saddle is connected to exactly two $(i+1)$-saddles. Thus, the $remove_{i,i+1}(q, p, p')$ is always a merging of two cells sharing one common cell in both the descending and the ascending complex.

The $remove_{i,i+1}(q, p, \emptyset)$ of the second type of an i-saddle q and an $(i+1)$-saddle p transforms the set of integral lines converging to p or to q into a set of integral lines converging to critical points t of index at least $i+2$ that were the destination of the integral lines originating at q before the $remove_{i,i+1}(q, p, \emptyset)$. In the descending Morse complex Γ_d, the $remove_{i,i+1}$of the second type cannot be viewed as the merging of two $(i+1)$-cells into one, but rather as the merging of an $(i+1)$-cell (and an i-cell) into an empty cell, as it deletes the i-cell q and the $(i+1)$-cell p in Γ_d.

Definition 4 *The $remove_{i,i+1}(q, p, \emptyset)$ of the second type, $1 \leq i \leq n-1$, on the combinatorial (descending) complex Γ_d is feasible if the i-cell q is incident once to exactly one $(i+1)$-cell p in Γ_d ($mult(q, \partial p) = 1$). It transforms the complex Γ_d into the simplified complex Γ'_d by replacing the set $\{q, p\} \cup R \cup Z \cup S = \{q, p\} \cup C_{q,p}$ of cells in Γ_d with the set $R \cup Z \cup S = C_{q,p}$ of cells in Γ'_d. The*

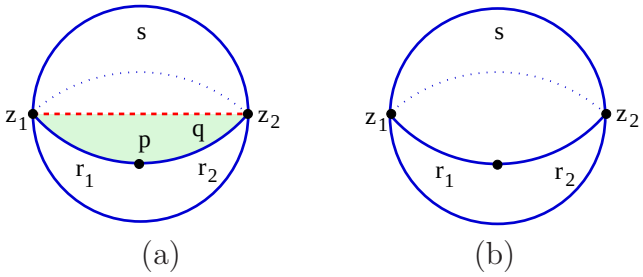


Figure 4.3: [26] A portion of a 3D descending complex before (a) and after (b) the $remove_{1,2}(q, p, \emptyset)$.

immediate boundary and co-boundary relation for the cells in Γ'_d *is the same as the immediate boundary and co-boundary relation for the corresponding cells in* Γ_d*, except that:*

1. *each instance of the* $(i+1)$*-cell* p *is removed from the immediate boundary of each* $(i+2)$*-cell* s_k *in* S*, and from the immediate co-boundary of each* i*-cell* r_j *in* R*, and*

2. *each instance of the* i*-cell* q *is removed from the immediate co-boundary of each* $(i-1)$*-cell* z_h *in* Z*.*

Each $(i+2)$*-cell* $s_k \in S$ *in* Γ'_d *is equal to the* $(i+2)$*-cell* s_k *in* Γ_d *plus a disjoint copy of the* i*-cell* q *and the* $(i+1)$*-cell* p*. Other cells in* Γ'_d *are the same as the corresponding cells in* Γ_d*.*

Dually, on the ascending complex Γ_a, the $remove_{i,i+1}(q, p, \emptyset)$ deletes the $(n-i)$-cell q and the $(n-i-1)$-cell p.

A 3D example of the effect of the $remove_{1,2}(q, p, \emptyset)$ is illustrated in Figure 4.3. The 1-cell q is incident once to exactly one 2-cell p. The 3-cell s is the only 3-cell in the co-boundary of the 2-cell p. After the $remove_{i,i+1}(q, p, p')$, the cells q and p are deleted. The 2-cell p is deleted from the immediate boundary of the 3-cell s and from the immediate co-boundary of the 1-cells r_1 and r_2, and the 1-cell q is deleted from the immediate co-boundary of the 0-cells z_1 and z_2.

The $remove_{i,i+1}(q, p, p')$ operator is a generalization of the $removal_i(q, p, p')$ operator, which is defined in the same way as the $remove_{i,i+1}(q, p, p')$ with an additional constraint that the i-cell q appears exactly once in the boundary of both the $(i+1)$-cell p and the $(i+1)$-cell p' ($mult(q, \partial p) = mult(q, \partial p') = 1$).

The (combinatorial) effect of the $removal_i(q, p, p')$ operator on the combinatorial descending Morse complex Γ_d is exactly the same as the effect of the $remove_{i,i+1}(q, p, p')$ operator, except that $mult'(r_j, \partial p') = mult(r_j, \partial p') + mult(r_j, \partial p)$ (since $mult(q, \partial p') = 1$). The (geometrical) effect of the $removal_i$ (q, p, p') operator on the combinatorial descending Morse complex Γ_d is exactly the same as the effect of the $remove_{i,i+1}(q, p, p')$ operator, except that (one copy of) the $(i + 1)$-cell p is merged in the $(i + 1)$-cell p' (since $mult(q, \partial p') = 1$). The $removal_i(q, p, \emptyset)$ operator is the same as the $remove_{i,i+1}(q, p, \emptyset)$ operator (see [26]).

4.1.2 Operator $remove_{i,i-1}$

Intuitively, the $remove_{i,i-1}$ operator collapses two $(i-1)$-saddles by deleting the i-saddle connected to both of them. It is feasible if the i-saddle has a correct neighborhood structure through integral lines.

Definition 5 *Let p and q be an $(i-1)$-saddle and an i-saddle respectively, $1 \leq i \leq n-1$, such that*

- *the i-saddle q is connected through an integral line (a separatrix) to the $(i-1)$-saddle p, and exactly one $(i-1)$-saddle p' different from p, and*
- *there is a unique integral line connecting the i-saddle q to the $(i-1)$-saddle p.*

Then, the cancellation of p and q is called the $remove_{i,i-1}$ (of the first type) of the i-saddle q and the $(i-1)$-saddle p, and it is denoted as $remove_{i,i-1}(q, p, p')$.

Definition 6 *Let p and q be an $(i-1)$-saddle and an i-saddle respectively, $1 \leq i \leq n-1$, such that*

- *the i-saddle q is connected through an integral line (a separatrix) to exactly one $(i-1)$-saddle p, and*
- *there is a unique integral line connecting the i-saddle q to the $(i-1)$-saddle p.*

Then, the cancellation of p and q is called the $remove_{i,i-1}$ (of the second type) of the i-saddle q and the $(i-1)$-saddle p, denoted as $remove_{i,i-1}(q, p, \emptyset)$.

The effect of the $remove_{i,i-1}(q,p,p')$ of the first type of the i-saddle q and the $(i-1)$-saddle p is as follows. The set of integral lines originating at p or q is transformed into a set of integral lines originating at p' (the unique $(i-1)$-saddle different from the $(i-1)$-saddle p, connected to the i-saddle q). Each critical point that was the destination of an integral line originating at p or at q becomes the destination of an integral line originating at p'.

The effect of the $remove_{i,i-1}(q,p,p')$ on the geometry of the cells in the descending complex Γ_d is the same as the effect of the $remove_{n-i,n-i+1}$ on the ascending complex Γ_a: the i-cell q in Γ_d is deleted and the $(i-1)$-cell p is merged into the $(i-1)$-cell p'. A disjoint copy of the higher-dimensional deleted cell (the i-cell q) is merged into each i-cell in the immediate co-boundary of the lower-dimensional deleted cell (the $(i-1)$-cell p) for each time the $(i-1)$-cell p' appears in the boundary of the i-cell q. More formally, each $(i+1)$-cell r in the co-boundary of the lower-dimensional deleted cell (the i-cell p) is merged with k disjoint copies of the higher-dimensional deleted cell (the $(i+1)$-cell q), where k is the multiplicity of the lower-dimensional deleted cell (the i-cell p) in the immediate boundary of the $(i+1)$-cell r (the number of times the i-cell q appears in the immediate boundary of the $(i+1)$-cell r, i.e., $mult(q,\partial r)$).

The combinatorial effect of the operator is that the immediate co-boundary of the i-cell p is merged in the immediate co-boundary of the i-cell p' with the appropriate adjustment of the multiplicities. We denote as $mult$ the multiplicities before the $remove_{i,i-1}$, and as $mult'$ the multiplicities after the $remove_{i,i-1}$. Each $(i+1)$-cell r that was in the immediate co-boundary of the i-cell p (with the exception of the $(i+1)$-cell q) with the multiplicity $mult(p,\partial r)$ belongs to the immediate co-boundary of the i-cell p' after the $remove_{i,i-1}(q,p,p')$ with the multiplicity increased by the product of the multiplicity of the $(i+1)$-cell r in the immediate co-boundary of the i-cell p and the multiplicity of the $(i+1)$-cell q in the immediate co-boundary of the i-cell p' ($mult'(p',\partial r) = mult(p',\partial r) + mult(p,\partial r) \cdot mult(p',\partial q)$).

We introduce the following notation for the cells in the combinatorial complex Γ_d (before the $remove_{i,i-1}(q,p,p')$ of the i-cell q and the $(i-1)$-cell p):

1. r_j, $j = 1,..,j_{max}$, are the i-cells (different from the i-cell q) in the immediate co-boundary of the $(i-1)$-cell p in Γ_d, $R = \{r_j, j = 1,..,j_{max}\}$,

2. z_h, $h = 1,..,h_{max}$, are the $(i+1)$-cells in the immediate co-boundary of the i-cell q in Γ_d, $Z = \{z_h, h = 1,..,h_{max}\}$, and

3. s_k, $k = 1, .., k_{max}$, are the $(i-2)$-cells in the immediate boundary of the $(i-1)$-cell p in Γ_d, $S = \{s_k, k = 1, .., k_{max}\}$.

We denote as $C_{q,p}$ the set of all the cells in the immediate boundary and the immediate co-boundary of the cancelled cells q and p ($C_{q,p} = \{p'\} \cup R \cup Z \cup S$).

Definition 7 *The $remove_{i,i-1}(q, p, p')$ of the first type, $1 \leq i \leq n-1$, on a (descending) combinatorial complex Γ_d is feasible if the i-cell q appears in the co-boundary of exactly two different $(i-1)$-cells p and p', and it appears exactly once in the co-boundary of the i-cell p ($mult(p, \partial q) = 1$). It replaces the set $\{q, p\} \cup \{p'\} \cup R \cup Z \cup S = \{q, p\} \cup C_{q,p}$ of cells in Γ_d with the set $\{p'\} \cup R \cup Z \cup S = C_{q,p}$, producing the simplified complex Γ'_d. The immediate boundary and co-boundary relations for the cells in the simplified complex Γ'_d are the same as the immediate boundary and co-boundary relations for the corresponding cells in Γ_d, except that:*

1. *each instance of the $(i-1)$-cell p is removed from the immediate co-boundary of each $(i-2)$-cell s_k in S, and from the immediate boundary of each i-cell r_j in R,*

2. *each instance of the i-cell q is removed from the immediate boundary of each $(i+1)$-cell z_h in Z, and from the immediate co-boundary of the new $(i-1)$-cell p', and*

3. *each instance of each i-cell r_j in R is added k times to the immediate co-boundary of the new $(i-1)$-cell p', where k is the multiplicity of the i-cell q in the immediate co-boundary of the $(i-1)$-cell p' ($mult'(p', \partial r_j) = mult(p', \partial r_j) + mult(p, \partial r_j) \cdot mult(p', \partial q)$).*

Each i-cell $r_j \in R$ in Γ'_d is the union of the i-cell r_j in Γ_d (before the $remove_{i,i-1}(q, p, p')$) and a disjoint copy of the i-cell q (and the $(i-1)$-cell p) for each time the $(i-1)$-cell p' appears on the boundary of the i-cell q. Other cells in Γ'_d are the same as the corresponding cells in Γ_d.

The effect of the $remove_{i,i-1}(q, p, p')$ on the ascending combinatorial complex Γ_a is the same as the effect of the $remove_{n-i,n-1+1}$ on the descending complex Γ_d.

In 2D, there is just one $remove_{i,i-1}$ operator (the $remove_{1,0}(q, p, p')$), where q is a saddle, and p and p' are minima. It is the same as the cancellation of

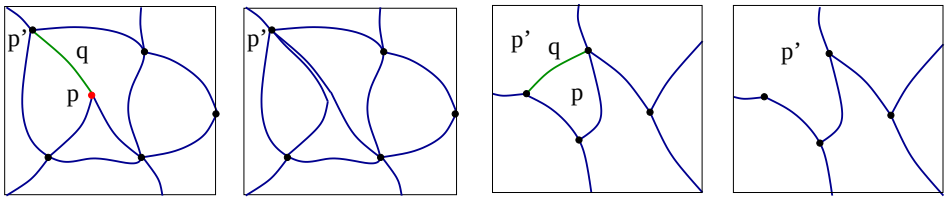


Figure 4.4: [26] A portion of the 2D descending (a) and ascending (b) Morse complex before and after the $remove_{1,2}(q, p, p')$.

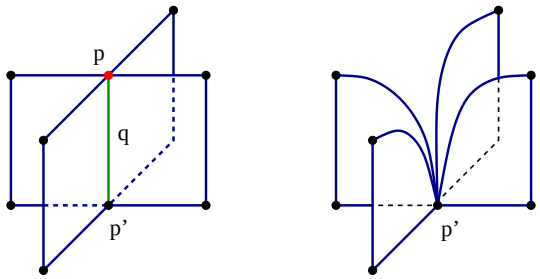
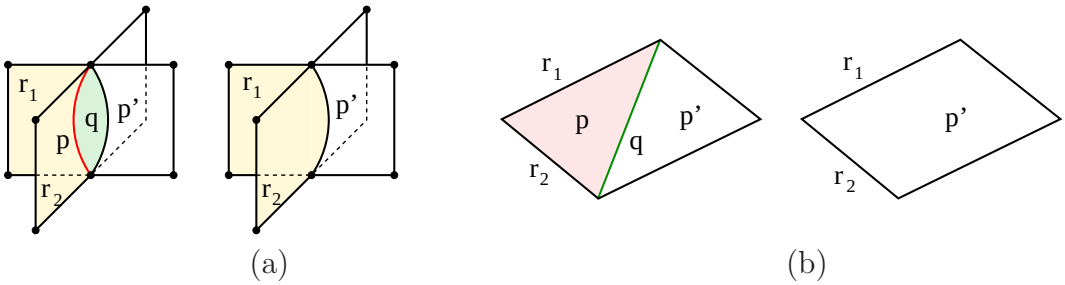


Figure 4.5: [26] A portion of a 3D descending Morse complex before and after the $remove_{1,0}(q, p, p')$. The co-boundary of the 0-cell p is merged into the co-boundary of the 0-cell p'.

the minimum p and the saddle q, discussed in Section 2.2.4. An example of the effect of the $remove_{1,0}(q, p, p')$ on the 2D descending Morse complex is illustrated in Figure 4.4 (a). It merges the co-boundary of the 0-cell p into the co-boundary of the 0-cell p'. The same operator merges the boundary of the 2-cell p into the boundary of the 2-cell p' in the dual ascending Morse complex, as shown in Figure 4.4 (b).

In 3D, there are two $remove_{i,i-1}$ operators: the $remove_{1,0}$ of a 1-saddle and a minimum, and the $remove_{2,1}$ of a 2-saddle and a 1-saddle. The former is the same as the cancellation of a minimum and a 1-saddle. An example of the effect of the $remove_{1,0}(q, p, p')$ on the 3D descending Morse complex is illustrated in Figure 4.5. It eliminates the 1-cell q and the 0-cell p. All cells in the co-boundary of p become part of the co-boundary of p' after the $remove_{1,0}(q, p, p')$.

The $remove_{2,1}(q, p, p')$ in 3D is a special case of the cancellation with the constraint that the 2-saddle q is connected to exactly two 1-saddles p and p',



(a) (b)

Figure 4.6: [26] (a) A portion of the 3D descending Morse complex before and after the $remove_{2,1}(q, p, p')$. The co-boundary of the 1-cell p, consisting of the 2-cells r_1 and r_2, is merged into the co-boundary of the 1-cell p'. (b) A portion of the corresponding 3D ascending Morse complex before and after the $remove_{2,1}(q, p, p')$. The boundary of the 2-cell p, consisting of the 1-cells r_1 and r_2, is merged into the boundary of the 2-cell p'.

i.e., that the 2-cell q is bounded by exactly two 1-cells p and p' in the descending complex, and that the 1-cell p appears exactly once in the boundary of the 2-cell q. An example of the effect of the $remove_{2,1}(q, p, p')$ on the 3D descending and ascending Morse complexes is illustrated in Figure 4.6 (a) and (b). In the descending complex Γ_d it merges the 1-cell p into the 1-cell p' eliminating the 2-cell q. All cells in the co-boundary of p different from q (such as the 2-cells r_1 and r_2), become part of the co-boundary of the 1-cell p'. In the dual ascending complex, the boundary of the 2-cell p (containing the 1-cells r_1 and r_2 different from the 1-cell q), is merged into the boundary of the 2-cell p'.

In the n-dimensional case, the $remove_{1,0}(q, p, p')$ is the same as the cancellation of a 1-saddle and a minimum, while the $remove_{i,i-1}(q, p, p')$, $1 < i \leq n-1$, is a special case of the cancellation, since it requires that the i-saddle is connected to exactly two $(i-1)$-saddles. The $remove_{i,i-1}(q, p, p')$ is always a collapse of two $(i-1)$ cells in the descending complex, and a merge of two $(n-i+1)$-cells in the ascending complex.

The $remove_{i,i-1}(q, p, \emptyset)$ of the second type of an i-saddle q and an $(i-1)$-saddle p transforms the set of integral lines originating at p or at q into a set of integral lines originating at the critical points r of index at most $i-2$ that were the origin of the integral lines converging to q. It is dual to the $remove_{n-i,n-i+1}(q, p, \emptyset)$ of the second type. In the descending Morse complex Γ_d, the $remove_{i,i-1}(q, p, \emptyset)$ deletes the i-cell q and the $(i-1)$-cell p, and in the ascending Morse complex Γ_a, it deletes the $(n-i)$-cell q and the $(n-i+1)$-cell

p.

Definition 8 *The* $remove_{i,i-1}(q, p, \emptyset)$ *of the second type,* $1 \leq i \leq n-1$*, on the combinatorial (descending) complex* Γ_d *is feasible if the* i*-cell* q *is incident once to exactly one* $(i-1)$*-cell* p *in* Γ_d *(*$mult(p, \partial q) = 1$*). It transforms the complex* Γ_d *into the simplified complex* Γ'_d *by replacing the set* $\{q, p\} \cup R \cup Z \cup S = \{q, p\} \cup C_{q,p}$ *of cells in* Γ_d *with the set* $R \cup Z \cup S = C_{q,p}$ *in* Γ'_d*. The immediate boundary and co-boundary relation for the cells in* Γ'_d *is the same as the immediate boundary and co-boundary relation for the corresponding cells in* Γ_d*, except that:*

1. *each instance of the* $(i-1)$*-cell* p *is removed from the immediate co-boundary of each* $(i-2)$*-cell* s_k *in* S*, and from the immediate boundary of each* i*-cell* r_j *in* R*, and*

2. *each instance of the* i*-cell* q *is removed from the immediate boundary of each* $(i+1)$*-cell* z_h *in* Z*.*

Each $(i+1)$*-cell* $z_h \in Z$ *in* Γ'_d *is equal to the* $(i+1)$*-cell* z_h *in* Γ_d *plus a disjoint copy of the* i*-cell* q *and the* $(i-1)$*-cell* p*. Other cells in* Γ'_d *are the same as the corresponding cells in* Γ_d*.*

Dually, on the ascending complex Γ_a, the $remove_{i,i-1}(q, p, \emptyset)$ deletes the $(n-i)$-cell q and the $(n-i+1)$-cell p.

The $remove_{i,i-1}(q, p, p')$ operator is a generalization of the $contraction_i(q, p, p')$ operator, which is defined in the same way as the $remove_{i,i-1}(q, p, p')$ with an additional constraint that the i-cell q appears exactly once in the co-boundary of both the $(i-1)$-cell p and the $(i-1)$-cell p' ($mult(p, \partial q) = mult(p', \partial q) = 1$). The (combinatorial) effect of the $contraction_i(q, p, p')$ operator on the combinatorial descending Morse complex Γ_d is exactly the same as the effect of the $remove_{i,i-1}(q, p, p')$ operator, except that $mult'(p', \partial r_j) = mult(p', \partial r_j) + mult(p, \partial r_j)$ (since $mult(p', \partial q) = 1$). The (geometrical) effect of the $contraction_i(q, p, p')$ operator on the combinatorial descending Morse complex Γ_d is exactly the same as the effect of the $remove_{i,i-1}(q, p, p')$ operator, except that (one copy of) the i-cell q is merged in each i-cell r_j (since $mult(p', \partial q) = 1$). The $contraction_i(q, p, \emptyset)$ operator is the same as the $remove_{i,i-1}$ $(q, p, \emptyset)$ operator (see [26]).

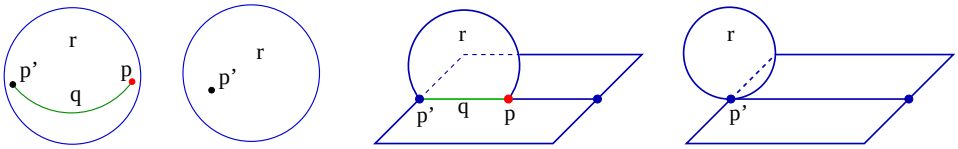


Figure 4.7: [26] A 2D (a) and a 3D (b) example of the $remove_{1,0}(q, p, p')$, which is not allowed as it would eliminate the only 1-cell q from the immediate boundary of the (bubble-like) 2-cell r.

4.1.3 Operator *remove* on Admissible Morse Complexes

The result of the $remove_{i,i+1}$ or the $remove_{i,i-1}$ may be a complex, which is not an admissible Morse complex, as stated in Section 2.2.3 (a complex in which each cell has a non-empty immediate boundary and a non-empty immediate co-boundary). We allow the $remove_{i,i+1}(q, p, p')$ of the first type, or a $remove_{i,i+1}(q, p, \emptyset)$ of the second type, $1 \leq i \leq n-1$, on a combinatorial complex Γ_d only if

1. each $(i+2)$-cell s_k in S in the immediate co-boundary of the $(i+1)$-cell p in Γ_d has an $(i+1)$-cell different from p in its immediate boundary,

2. each $(i-1)$-cell z_h in Z in the immediate boundary of the i-cell q in Γ_d has an i-cell different from q in its immediate co-boundary, and

3. (only for a $remove_{i,i+1}$ of the first type) p or p' have an i-cell different from the i-cell q in the immediate boundary,

4. (only for a $remove_{i,i+1}$ of the second type) each i-cell r_j in R in the boundary of the $(i+1)$-cell p in Γ_d has an $(i+1)$-cell different from p in its immediate co-boundary.

The conditions for the $remove_{i,i-1}$ are dual.

In Figures 4.7 (a) in 2D, and 4.7 (b) in 3D, we give examples of the $remove_{1,0}(q, p, p')$, which is not allowed as it eliminates the only 1-cell q in the immediate boundary of the 2-cell r.

In 3D, we consider the feasibility of the operators $remove_{1,2}$, $remove_{2,1}$, $remove_{1,0}$ and $remove_{2,3}$. The $remove_{1,2}$, which deletes a 1-cell q and a 2-cell p is feasible on the admissible Morse complex Γ if all the 2-cells bounded by the 1-cell q have at least one other 1-cell different from q in their immediate boundary. The $remove_{2,1}$, which eliminates the 1-cell q and the 2-cell p, is

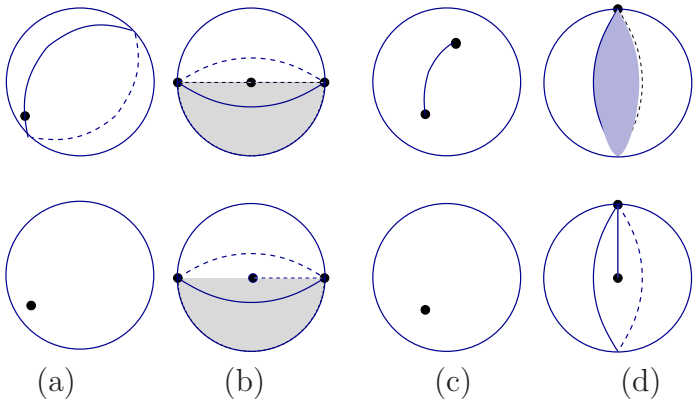


Figure 4.8: The 1-cell (loop-edge) bounds two 2-cells, which have no other 1-cell in their immediate boundary. The $remove_{1,2}(q, p, p')$ would leave the 2-cell p' with no 1-cell in its immediate boundary (a). Dually, the two 1-cells p and r in the immediate boundary of the shaded 2-cell q are in the immediate boundary of no other 2-cell different from the 2-cell q. The $remove(q, p, \emptyset)$ would leave a dangling 1-cell (the 1-cell which is not in the immediate boundary of any 2-cell) (b). The only 1-cell q in the immediate boundary of the 2-cell has two 1-cells p and p' in its immediate boundary. The $remove_{1,0}(q, p, p')$ would leave the 2-cell without a 1-cell in its immediate boundary (c). Dually, the 1-cell is in the immediate boundary of only one 2-cell, which separates two distinct 3-cells. The $remove_{2,3}$ would leave a dangling 1-cell (d).

feasible on the admissible Morse complex Γ if all the 1-cells in the immediate boundary of the 2-cell q are in the immediate boundary of at least one other 2-cell different from q. These situations are illustrated in Figure 4.8 (a) and (b).

The $remove_{1,0}$ deletes a 1-cell q and a 0-cell p. It is feasible if there is no 2-cell r such that the 1-cell q is the only 1-cell in the immediate boundary of the 2-cell r. Dually, the $remove_{2,3}$ deletes a 2-cell q and a 3-cell p. It is feasible if there is no 1-cell r such that r is in the immediate boundary of no other 2-cell but q. These situations are illustrated in Figure 4.8 (c) and (d).

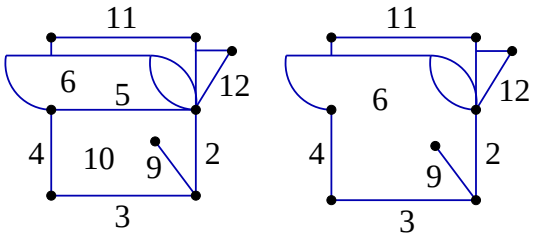


Figure 4.9: [26] A part of the 3D descending Morse complex before (a) and after (b) the cancellation of the 1-saddle with function value 5 and the 2-saddle with function value 10. The cells are labeled with the values of the scalar function f at the corresponding critical points. After the cancellation, the 1-cell labeled 9 becomes incident to the 2-cell labeled 6, and thus such cancellation is not feasible as regards the function values, although it is feasible as regards the connectivity of cells (the 1-cell labeled 5 appears exactly once in the immediate boundary of the 2-cell labeled 10, i.e., the multiplicity of the 1-cell labeled 5 in the immediate boundary of the 2-cell labeled 10 is equal to 1).

4.1.4 Feasibility of Simplification Operators as Regards the Scalar Field

Let us recall that if a cell p is in the boundary of a cell q in the descending Morse complex Γ_d of a Morse function f, then for the corresponding critical points p and q, $f(p) < f(q)$ (see Section 2.2.2). For the simplification *removal* operator we have defined, the consistency of the simplified complex Γ'_d from the point of view of function values can be guaranteed. For a (topologically) feasible $removal_i$ operator, from the two possible operators with middle cell q, $removal_i(q, p, p')$ and $removal_i(q, p', p)$, the first is performed if $f(p) < f(p')$, and the second is performed if $f(p) > f(p')$. In the first case, all the i-cells t that were in the immediate boundary of the $(i+1)$-cell p in Γ_d (before the $removal_i$), and that are in the immediate boundary of the $(i+1)$-cell p' in Γ'_d (after the $removal_i$), have a lower function value than p', i.e., $f(t) < f(p')$, and symmetrically for the second case.

The conditions for the *contraction* are dual: for the (topologically) feasible $contraction_i$, from the two possible operators with middle cell q, $contraction_i$ (q, p, p') and $contraction_i(q, p', p)$, we choose to perform the first one if $f(p) > f(p')$, and the second one if $f(p) < f(p')$. For the *removal* or the *contraction* of the second type, the consistency from the point of view of the function

values is trivially satisfied.

Such a guarantee for a cancellation (or a *remove*) cannot be given in general. In the example illustrated in Figure 4.9, the cancellation of the 1-saddle with the function value 5 and the 2-saddle with the function value 10 is topologically consistent, as the corresponding 1-cell labeled 5 appears exactly once in the immediate boundary of the 2-cell labeled 10 in the descending complex Γ_d. Nevertheless, this cancellation is not feasible, since it would make the 1-cell with function value 9 incident to the 2-cell with function value 6, and the descending complex obtained after such cancellation would not satisfy the condition stated in Section 2.2.2.

4.2 Refinement Operator *insert*

One of the aims of Morse theory is to relate the homotopy type of a manifold M to that of a cell complex with an n-cell for each i-saddle of a scalar function f defined over M. The cancellation operator decreases the number of critical points of f and provides a representation of the homotopy type of M through a cell complex with fewer cells. In Morse theory, an inverse operator of the cancellation in arbitrary dimensions has not been defined. Only in the two-dimensional case the inverse anticancellation operators of the cancellation of a minimum and a saddle and of a maximum and a saddle have been defined in [15].

In [26], we have defined two dimension independent refinement operators, that we have called *insertion* and *expansion*, inverse to the corresponding simplification operators, *removal* and *contraction*, respectively. In [32], we have defined a generalization of these refinement operators, the operator *insert*, inverse to the simplification operator *remove*. A refinement undoes the corresponding simplification by restoring the canceled cells and their immediate boundary and co-boundary in the combinatorial Morse complexes. The refinement *insert* operator has two instances, $insert_{i,i+1}$ and $insert_{i,i-1}$, inverse to the corresponding instances $remove_{i,i+1}$ and $remove_{i,i-1}$ of the simplification operator *remove*. The two instances of the refinement operator are dual to each other, just like the two instances of the simplification operator.

4.2.1 Operator $insert_{i,i+1}$

Intuitively, an $insert_{i,i+1}$ operator (of the first type) introduces two new cells of consecutive dimensions into Morse complexes by splitting one existing cell. It introduces an i-cell q and an $(i+1)$-cell p in the descending Morse complex, and an $(n-i)$-cell q and an $(n-i-1)$-cell p in the ascending complex. Namely,

1. in the descending complex Γ_d, the i-cell q is inserted into the interior of an existing $(i+1)$-cell p', splitting the $(i+1)$-cell p' into the $(i+1)$-cells p and p', and splitting the immediate boundary of the $(i+1)$-cell p' in the immediate boundaries of the new $(i+1)$-cells p and p'. The new i-cell q separates the $(i+1)$-cells p and p'.

2. in the ascending complex Γ_a, the $(n-1)$-cell q is expanded from an existing $(n-i-1)$-cell p', splitting the $(n-i-1)$-cell p' into the $(n-i-1)$-cells p and p', and splitting the immediate co-boundary of the $(n-i-1)$-cell p' in the immediate co-boundaries of the new $(n-i-1)$-cells p and p'. The new $(n-i)$-cell q is bounded by the $(n-i-1)$-cells p and p'.

The refinement is an undo operator of the corresponding simplification. It replaces one set of cells with another set of cells in the two dual Morse complexes, which are connected in the same way as they were connected before the corresponding simplification. It is specified by the cells in the immediate boundary and in the immediate co-boundary of the introduced cells, and by the multiplicities of these cells in the corresponding immediate boundaries and co-boundaries in the refined complex Γ'_d.

Let us consider the $remove_{i,i+1}(q,p,p')$ of the first type. We recall the following notation:

1. r_j, $j=1,..,j_{max}$, are the i-cells (different from the i-cell q) in the immediate boundary of the $(i+1)$-cell p before the $remove_{i,i+1}$, $R=\{r_j, j=1,..,j_{max}\}$,

2. z_h, $h=1,..,h_{max}$, are the $(i-1)$-cells in the immediate boundary of the i-cell q before the $remove_{i,i+1}$, $Z=\{z_h, h=1,..,h_{max}\}$, and

3. s_k, $k=1,..,k_{max}$, are the $(i+2)$-cells in the immediate co-boundary of the $(i+1)$-cell p before the $remove_{i,i+1}$, $S=\{s_k, k=1,..,k_{max}\}$.

We have denoted as $C_{q,p}$ the set of cells in the immediate boundary and in the immediate co-boundary of the cancelled cells q and p, i.e., $C_{q,p} = \{p'\} \cup R \cup Z \cup S$.

Before performing the refinement, we need to ensure that the situation around the two introduced cells p and q is the same as it was at the time of simplification, i.e., that: all the cells that were in the immediate boundary and in the immediate co-boundary of p and q at the time of the simplification are present in the complex at the time of the refinement; that the multiplicity $mult(r_j, \partial p')$ of each i-cell r_j in R in the immediate boundary of the $(i+1)$-cell p' in the complex Γ_d is greater than or equal to the product of the multiplicity $mult'(r_j, \partial p)$ of the i-cell r_j in the immediate boundary of the new $(i+1)$-cell p and the multiplicity $mult'(q, \partial p')$ of the new i-cell q in the immediate boundary of the $(i+1)$-cell p' in the refined complex Γ'. We have denoted as $mult$ the multiplicities in the complex Γ, and as $mult'$ the multiplicities in the refined complex Γ'.

Definition 9 *The refinement operator $insert_{i,i+1}(q, p, p')$, $1 \leq i \leq n-1$, inverse to the simplification operator $remove_{i,i+1}(q, p, p')$, is specified by the $(i+1)$-cell p', the i-cells $r_j \in R$, the $(i-1)$-cells $z_h \in Z$ and the $(i+2)$-cells $s_k \in S$, together with the corresponding multiplicities $mult(q, \partial p')$, $mult(r_j, \partial p)$, $mult(z_h, \partial q)$ and $mult(p, \partial s_k)$. It is feasible on the (descending) combinatorial complex Γ_d if the $(i+1)$-cell p', the i-cells $r_j \in R$, the $(i-1)$-cells $z_h \in Z$ and the $(i+2)$-cells $s_k \in S$ are in Γ_d and $mult(r_j, \partial p') \geq mult'(r_j, \partial p) \cdot mult'(q, \partial p')$.*

The operator replaces the set of cells $\{p'\} \cup R \cup Z \cup S = C_{q,p}$ in Γ_d with the set $\{q, p\} \cup \{p'\} \cup R \cup Z \cup S = \{q, p\} \cup C_{q,p}$ in the refined complex Γ'_d. The immediate boundary and co-boundary relation for the cells in Γ'_d is the same as the immediate boundary and co-boundary relation for the corresponding cells in Γ_d, except that:

1. *the immediate boundary of the $(i+1)$-cell p in Γ'_d consists of the i-cells r_j in R and the i-cell q, with the corresponding multiplicities $mult'(r_j, \partial p)$ and $mult'(q, \partial p) = 1$, respectively,*

2. *the immediate boundary of the i-cell q in Γ'_d consists of the $(i-1)$-cells z_h in Z, with the corresponding multiplicities $mult'(z_h, \partial q)$,*

3. *the immediate boundary of the new $(i+1)$-cell p' in Γ'_d consists of the same cells as its boundary in Γ_d plus the i-cell q with the multiplic-*

ity $mult'(q, \partial p')$ *and with* $mult'(r_j, \partial p') = mult(r_j, \partial p') - mult'(r_j, \partial p) \cdot mult'(q, \partial p')$ *for the* i*-cells* r_j *in* R*,*

4. *the immediate co-boundary of the* $(i+1)$*-cell* p *in* Γ'_d *consists of the* $(i+2)$*-cells* s_k *in* S*, with the corresponding multiplicities* $mult'(p, \partial s_k)$*,*

5. *the immediate co-boundary of the* i*-cell* q *in* Γ'_d *consists of the* $(i+1)$*-cells* p *and* p'*, with the corresponding multiplicities* $mult'(q, \partial p) = 1$ *and* $mult'(q, \partial p')$*, respectively, and*

6. *the immediate co-boundary of the new* $(i+1)$*-cell* p' *in* Γ'_d *is the same as the immediate co-boundary of the* $(i+1)$*-cell* p' *in* Γ_d *(before insert).*

The cells p *and* q *are the same as they were before the corresponding* $remove_{i,i+1}(q, p, p')$*. The new* $(i+1)$*-cell* p' *in* Γ'_d *is the same as the* $(i+1)$*-cell* p' *in* Γ_d *minus* $mult(q, \partial p')$ *disjoint copies of the* $(i+1)$*-cell* p *(and the* i*-cell* q*). Other cells in* Γ'_d *are the same as in* Γ_d*.*

Dually, in the ascending complex Γ_a, the $(n-i)$-cell q and the $(n-i-1)$-cell p are restored as they were before the $remove_{i,i+1}(q, p, p')$.

An example of the $insert_{1,2}(q, p, p')$ on the 2D descending complex is illustrated in Figure 4.10. It is specified by the new 2-cell p and the new 1-cell q, the 2-cell p' that will be split, the 1-cell r_1 in the immediate boundary of the 2-cell p' before *insert*, which is in the immediate boundary of the 2-cell p after the *insert*, and the 0-cells z_1 and z_2, which are in the immediate boundary of the new 1-cell q after the *insert*. All the multiplicities are equal to 1. There are no cells s_k in the co-boundary of the 2-cell p.

In 3D, there are two $insert_{i,i+1}$ operators: $insert_{1,2}$ of a 1-saddle and a 2-saddle, and $insert_{2,3}$ of a 2-saddle and a maximum. Figure 4.11 shows the effect of the $insert_{1,2}(q, p, p')$ on the 3D descending Morse complex. It is specified by the new cells p and q, the existing 2-cell p', the 1-cells r_1, r_2 and r_3 in the immediate boundary of the 2-cell p, the 0-cells z_1 and z_2 in the immediate boundary of the 1-cell q, the 3-cells s_1 and s_2 in the immediate co-boundary of the 2-cell p after the *insert* and the corresponding multiplicities that are all equal to 1.

Figure 4.12 (a) shows the effect of the $insert_{2,3}(q, p, p')$ on the 3D descending Morse complex. It is specified by the new cells q and p, the 3-cell p' to be split, the 2-cells r_1, r_2, r_3, r_4 and r_5 in the immediate boundary of the 3-cell

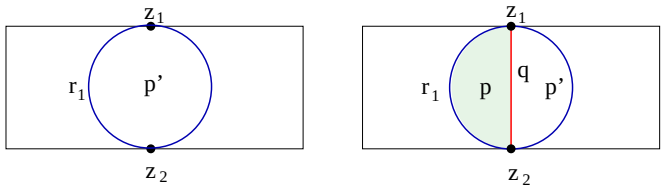


Figure 4.10: [26] An example of the $insert_{1,2}(q, p, p')$ on a portion of the 2D descending Morse complex Γ_d. It is specified by the cells p, q and p', the 1-cell r_1 in the immediate boundary of the 2-cell p, and the 0-cells z_1 and z_2 in the immediate boundary of the 1-cell q, and the corresponding multiplicities that are all equal to 1.

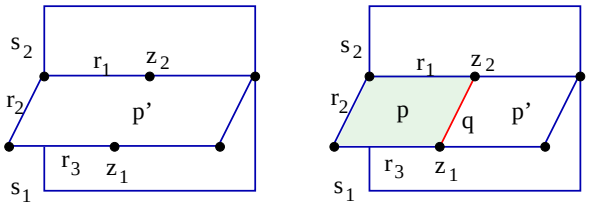


Figure 4.11: [26] An example of the $insert_{1,2}(q, p, p')$ on a portion of the descending Morse complex Γ_d in 3D. It is specified by the cells p, q and p', the 1-cells r_1, r_2 and r_3 in the immediate boundary of the 2-cell p, the 0-cells z_1 and z_2 in the immediate boundary of the 1-cell q, and the 3-cells s_1 and s_2 in the immediate co-boundary of the 2-cell p. All the multiplicities are equal to 1.

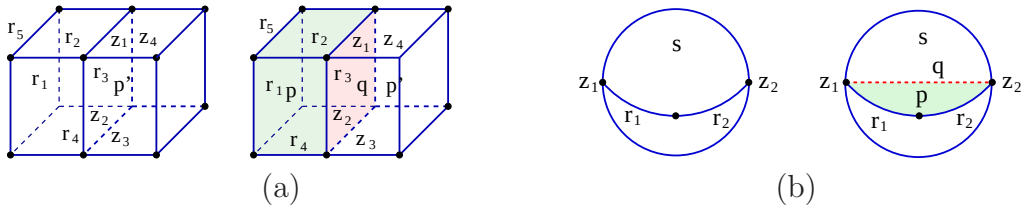


Figure 4.12: [26] (a) An example of the $insert_{2,3}(q, p, p')$ on a portion of the 3D descending Morse complex Γ_d. It is specified by the cells q, p, and p', the cells in the immediate boundary and in the immediate co-boundary of the introduced cells p and q and the corresponding multiplicities. (b) An example of the $insert_{1,2}(q, p, \emptyset)$ on a portion of the 3D descending Morse complex Γ_d.

p, and the 1-cells z_1, z_2, z_3 and z_4 in the immediate boundary of the 2-cell q. All the corresponding multiplicities are equal to 1.

The $insert_{i,i+1}(q, p, \emptyset)$ of the second type is inverse to the $remove_{i,i+1}(q, p, \emptyset)$ of the second type. It cannot be viewed as a split of an existing $(i+1)$-cell into two. It introduces the i-cell q and the $(i+1)$-cell p in the descending complex, and the $(n-i)$-cell q and the $(n-i-1)$-cell p in the ascending complex. In both complexes, it is specified by exactly the same cells (with the corresponding multiplicities) as the $insert_{i,i+1}$ of the first type, with the exception of the $(i+1)$-cell p' in Γ_d and the $(n-i-1)$-cell p' in Γ_a.

Definition 10 *The refinement operator $insert_{i,i+1}(q, p, \emptyset)$, $1 \leq i \leq n-1$, inverse to the simplification operator $remove_{i,i+1}(q, p, \emptyset)$, is specified by the i-cells $r_j \in R$, the $(i-1)$-cells $z_h \in Z$ and the $(i+2)$-cells $s_k \in S$, and the multiplicities $mult(r_j, \partial p)$, $mult(z_h, \partial q)$ and $mult(p, \partial s_k)$. It is feasible on the (descending) combinatorial complex Γ_d if the i-cells $r_j \in R$, the $(i-1)$-cells $z_h \in Z$ and the $(i+2)$-cells $s_k \in S$ are in Γ_d. The operator transforms the complex Γ_d into the refined complex Γ'_d by replacing the set of cells $R \cup Z \cup S = C_{q,p}$ in Γ_d with the set $\{q, p\} \cup R \cup Z \cup S = \{q, p\} \cup C_{q,p}$ in Γ'_d. The immediate boundary and co-boundary relation for the cells in Γ'_d is the same as the immediate boundary and co-boundary relation for the corresponding cells in Γ_d, except that:*

1. *the immediate boundary of the $(i+1)$-cell p in Γ'_d consists of the i-cells r_j in R and the i-cell q, with the multiplicities $mult'(r_j, \partial p)$ and $mult'(q, \partial p) = 1$, respectively,*

2. *the immediate boundary of the i-cell q in Γ'_d consists of the $(i-1)$-cells z_h in Z, with the multiplicities $mult'(z_h, \partial q)$,*

3. *the immediate co-boundary of the $(i+1)$-cell p in Γ'_d consists of the $(i+2)$-cells s_k in S, with the multiplicities $mult'(p, \partial s_k)$ and*

4. *the immediate co-boundary of the i-cell q in Γ'_d consists of the $(i+1)$-cell p with the multiplicity $mult'(q, \partial p) = 1$.*

The two inserted cells, the $(i+1)$-cell p and the i-cell q in Γ'_d are the same as they were before the corresponding $remove_{i,i+1}(q, p, \emptyset)$. Each $(i+2)$-cell $s_k \in S$ in Γ'_d is equal to the $(i+2)$-cell s_k in Γ_d minus a disjoint copy of the $(i+1)$-cell p and the i-cell q. Other cells in Γ'_d are the same as in Γ_d.

The operator $insert_{i,i+1}(q, p, \emptyset)$ is defined in a dual manner on the ascending complex Γ_a.

In Figure 4.12 (b), an example of the $insert_{i,i+1}(q, p, \emptyset)$ on the 3D descending complex is illustrated. It is specified by the 1-cell q and the 2-cell p, the 1-cells r_1 and r_2 in the immediate boundary of the 2-cell p, the 0-cells z_1 and z_2 in the immediate boundary of the 1-cell q, and the 3-cell s in the immediate co-boundary of the 2-cell p. All the corresponding multiplicities are equal to 1.

4.2.2 Operator $insert_{i,i-1}$

The operator $insert_{i,i-1}$ (of the first type), dual to the operator $insert_{i,i+1}$ and inverse to the operator $remove_{i,i-1}$, splits an existing $(i-1)$-cell in Γ_d by expanding a new i-cell, which splits its immediate co-boundary, and creates a new $(i-1)$-cell. The operator $insert_{i,i-1}(q, p, p')$ introduces the i-cell q and the $(i-1)$-cell p in the descending Morse complex Γ_d, and the $(n-i)$-cell q and the $(n-i+1)$-cell p in the ascending complex Γ_a. Namely,

1. in the descending complex Γ_d, the i-cell q is expanded from the existing $(i-1)$-cell p', splitting the $(i-1)$-cell p' into two $(i-1)$-cells p and p'. The new i-cell q is bounded by the $(i-1)$-cells p and p'.

2. in the ascending complex Γ_a, the $(n-i)$-cell q is inserted into the interior of the existing $(n-i+1)$-cell p', splitting the $(n-i+1)$-cell p' into the $(n-i+1)$-cells p and p'. The new $(n-i)$-cell q separates the $(n-i+1)$-cells p and p'.

The operator is specified by the cells in the immediate boundary and in the immediate co-boundary of the introduced cells, and by the multiplicities of these cells in the corresponding immediate boundaries and co-boundaries in the refined complex Γ'_d.

Let us consider the $remove_{i,i-1}(q, p, p')$ of the first type. We recall the following notation:

1. r_j, $j = 1, .., j_{max}$, are the i-cells (different from the i-cell q) in the immediate co-boundary of the $(i-1)$-cell p before the $remove_{i,i-1}$, $R = \{r_j, j = 1, .., j_{max}\}$,

2. z_h, $h = 1, .., h_{max}$, are the $(i+1)$-cells in the immediate co-boundary of the i-cell q before the $remove_{i,i-1}$, $Z = \{z_h, h = 1, .., h_{max}\}$, and

3. s_k, $k = 1, .., k_{max}$, are the $(i-2)$-cells in the immediate boundary of the $(i-1)$-cell p before the $remove_{i,i-1}$, $S = \{s_k, k = 1, .., k_{max}\}$.

We have denoted as $C_{q,p}$ the set of the cells in the immediate boundary and in the immediate co-boundary of the canceled cells q and p, i.e., $C_{q,p} = \{p'\} \cup R \cup Z \cup S$.

Before performing the refinement, we need to ensure that the situation around the two introduced cells p and q is the same as it was at the time of simplification, i.e., that: all the cells that were in the immediate boundary and in the immediate co-boundary of p and q at the time of the simplification are present in the complex at the time of the refinement; that the multiplicity $mult(p', \partial r_j)$ of the $(i-1)$-cell p' in the immediate boundary of each i-cell r_j in R in the complex Γ_d is greater than or equal to the product of the multiplicity $mult'(p, \partial r_j)$ of the new $(i-1)$-cell p in the immediate boundary of the i-cell r_j and the multiplicity $mult'(p', \partial q)$ of the $(i-1)$-cell p' in the immediate boundary of the new i-cell q in the refined complex Γ'. We have denoted as $mult$ the multiplicities in the complex Γ, and as $mult'$ the multiplicities in the refined complex Γ'.

Definition 11 *The refinement operator $insert_{i,i-1}(q, p, p')$, $1 \leq i \leq n-1$, inverse to the simplification operator $remove_{i,i-1}(q, p, p')$, is specified by the $(i-1)$-cell p', the i-cells $r_j \in R$, the $(i+1)$-cells $z_h \in Z$ and the $(i-2)$-cells $s_k \in S$, and the multiplicities $mult(p, \partial r_j)$, $mult(q, \partial z_h)$ and $mult(p', \partial q)$. It is feasible on the (descending) combinatorial complex Γ_d if the $(i-1)$-cell p', the i-cells $r_j \in R$, the $(i+1)$-cells $z_h \in Z$ and the $(i-2)$-cells $s_k \in S$ are in Γ_d and $mult(p', \partial r_j) \geq mult'(p, \partial r_j) \cdot mult(p', \partial q)$.*

The operator transforms the complex Γ_d into a refined complex Γ'_d by replacing the set $\{p'\} \cup R \cup Z \cup S = C_{q,p}$ of cells in Γ_d with the set $\{p, q, p'\} \cup R \cup Z \cup S = \{q, p\} \cup C_{q,p}$ in Γ'_d, so that

1. *the immediate co-boundary of the $(i-1)$-cell p in Γ'_d consists of the i-cells r_j in R and the i-cell q, with the given multiplicities $mult'(p, \partial r_j)$ and $mult'(p, \partial q) = 1$,*

2. *the immediate co-boundary of the i-cell q in Γ'_d consists of the $(i+1)$-cells z_h in Z, with the given multiplicities $mult'(q, \partial z_h)$,*

3. *the immediate co-boundary of the new $(i-1)$-cell p' in Γ'_d consists of the same i-cells as its co-boundary in Γ_d plus the i-cell q with the given mul-*

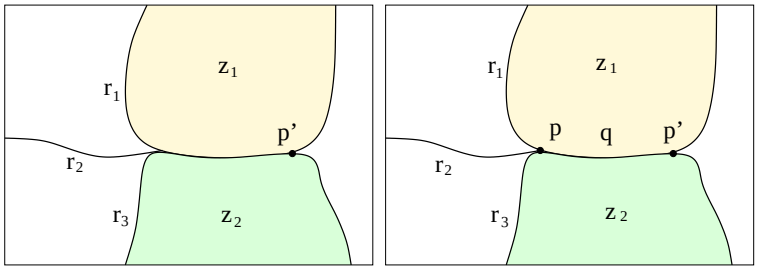


Figure 4.13: [26] A portion of the 2D descending Morse complex Γ_d before (a) and after (b) the $insert_{1,0}(q, p, p')$.

tiplicity $mult'(p', \partial q)$*, and* $mult'(p', \partial r_j) = mult(p', \partial r_j) - mult'(p, \partial r_j) \cdot mult'(p', \partial q)$ *for the* i*-cells* r_j *in* R*,*

4. *the immediate boundary of the* $(i-1)$*-cell* p *in* Γ'_d *consists of the* $(i-2)$*-cells* s_k *in* S*, with the given multiplicities* $mult(s_k, \partial p)$*,*

5. *the immediate boundary of the* i*-cell* q *in* Γ'_d *consists of the* $(i-1)$*-cells* p *and* p'*, with the given multiplicities* $mult'(p, \partial q) = 1$ *and* $mult'(p', \partial q)$*, respectively, and*

6. *the immediate boundary of the new* $(i-1)$*-cell* p' *in* Γ'_d *is the same as the immediate boundary of the* $(i-1)$*-cell* p' *in* Γ_d *(before the* $insert_{i,i-1}$*).*

The two introduced cells, the $(i-1)$*-cell* p *and the* i*-cell* q*, are the same after the* $insert_{i,i-1}(q, p, p')$ *as they were before the* $remove_{i,i-1}(q, p, p')$*. Each* i*-cell* r_j *in* Γ'_d *is equal to the* i*-cell* r_j *in* Γ_d *minus* $mult(p, \partial r_j)$ *disjoint copies of the* i*-cell* q *(and the* $(i-1)$*-cell* p*). Other cells in* Γ'_d *are the same as they were in* Γ_d*.*

In the ascending complex Γ_a, the $(n-i)$-cell q is restored and the $(n-i+1)$-cell p' is split into two $(n-i+1)$-cells p and p'. The $(n-i)$-cell q and the $(n-i+1)$-cell p are as they were before the corresponding simplification $remove_{i,i-1}(q, p, p')$. The $(n-i+1)$-cell p' in Γ'_d is the same as the $(n-i+1)$-cell p' in Γ_d minus the $(n-i+1)$-cell p and the $(n-i)$-cell q.

Figure 4.13 shows a 2D example of the $insert_{i,i-1}$ operator. In 2D there is only one $insert_{i,i-1}$ operator, namely $insert_{1,0}$ of a saddle q and a minimum p.

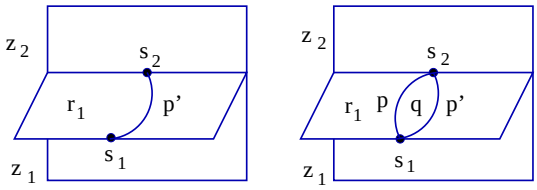


Figure 4.14: [26] An example of the $insert_{2,1}(q, p, p')$ on a portion of the 3D descending Morse complex.

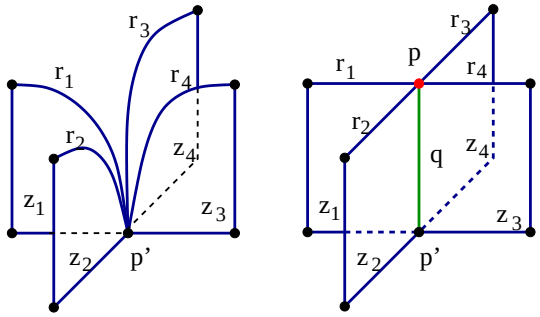


Figure 4.15: [26] A portion of the 3D descending Morse complex before and after the $insert_{1,0}(q, p, p')$.

It corresponds in the descending complex to an expansion of the 0-cell p' into two 0-cells p and p' through the 1-cell q corresponding to a saddle.

In 3D, there are two $insert_{i,i-1}$ operators: the $insert_{1,0}$ of a 1-saddle and a minimum, and the $insert_{2,1}$ of a 2-saddle and a 1-saddle. Figure 4.14 shows the effect of the $insert_{2,1}(q, p, p')$ on the 3D descending Morse complex. It is specified by the new cells q and p, the existing 1-cell p', the 2-cell r_1 in the immediate co-boundary of the 1-cell p, the two 3-cells z_1 and z_2 in the immediate co-boundary of the 2-cell q, and the two 0-cells s_1 and s_2 in the immediate boundary of the 1-cell p. All the corresponding multiplicities are equal to 1.

Figure 4.15 shows the effect of the $insert_{1,0}(q, p, p')$ on the 3D descending Morse complex. It is specified by the new cells q and p, the 0-cell p' to be split, the four 1-cells r_1, r_2, r_3 and r_4 in the immediate co-boundary of the 0-cell p, and the four 2-cells z_1, z_2, z_3 and z_4 in the immediate co-boundary of the 1-cell q. The boundary of the 0-cell p is empty. All the corresponding multiplicities are equal to 1.

The refinement operator $insert_{i,i-1}(q,p,\emptyset)$ of the second type is inverse to the simplification $remove_{i,i-1}(q,p,\emptyset)$ of the second type. It introduces the i-cell q and the $(i-1)$-cell p in the descending Morse complex, and the $(n-i)$-cell q and the $(n-i+1)$-cell p in the ascending complex. It is specified by the same cells as the $insert_{i,i-1}(q,p,p')$ of the first type, with the exception of the $(i-1)$-cell p' in the descending complex and the $(n-i+1)$-cell p' in the ascending complex.

Definition 12 *The refinement operator $insert_{i,i-1}(q,p,\emptyset)$, $1 \leq i \leq n-1$, inverse to the simplification $remove_{i,i-1}(q,p,\emptyset)$, is specified by the i-cells $r_j \in R$, the $(i+1)$-cells $z_h \in Z$ and the $(i-2)$-cells $s_k \in S$ (the cells in $C_{q,p)}$), with the multiplicities $mult(p,\partial r_j)$, $mult(q,\partial z_h)$ and $mult(s_k,\partial p)$. It is feasible on the (descending) combinatorial complex Γ_d if the i-cells $r_j \in R$, the $(i+1)$-cells $z_h \in Z$ and the $(i-2)$-cells $s_k \in S$ are in Γ_d. It transforms the complex Γ_d into the refined complex Γ'_d by replacing the set of cells $R \cup Z \cup S = C_{q,p}$ in Γ_d with the set $\{p,q\} \cup R \cup Z \cup S = \{q,p\} \cup C_{q,p}$ in Γ'_d, so that:*

1. *the immediate co-boundary of the $(i-1)$-cell p in Γ'_d consists of the i-cells r_j in R and the i-cell q, with the given multiplicities $mult'(p,\partial r_j)$ and $mult'(p,\partial q) = 1$, respectively,*

2. *the immediate co-boundary of the i-cell q in Γ'_d consists of the $(i+1)$-cells z_h in Z, with the given multiplicities $mult'(q,\partial z_h)$,*

3. *the immediate boundary of the $(i-1)$-cell p in Γ'_d consists of the $(i-2)$-cells s_k in S, with the given multiplicities $mult'(s_k,\partial p)$, and*

4. *the immediate boundary of the i-cell q in Γ'_d consists of the $(i-1)$-cell p, with the given multiplicity $mult'(p,\partial q) = 1$.*

The two introduced cells, the i-cell q and the $(i-1)$-cell p in Γ'_d, are the same as they were before the corresponding $remove_{i,i-1}(q,p,\emptyset)$. Each i-cell r_j in Γ'_d is equal to the i-cell r_j in Γ_d minus a disjoint copy of the i-cell q and the $(i-1)$-cell p. Other cells in Γ'_d are the same as they were in Γ_d.

4.3 Validity and Minimality of *remove* and *insert* Operators

We have shown in [26] that the *removal* and *contraction* simplification operators and the inverse *insertion* and *expansion* refinement ones are valid (i.e., if they are applied on a combinatorial Morse complex Γ_d on M they produce a combinatorial Morse complex Γ'_d on M), and that they form a basis for the set of operators that modify Morse complexes on M in a topologically consistent manner. In Sections 4.3.1 and 4.3.2, respectively, we show that the same two properties are satisfied for the *remove* and *insert* operators.

4.3.1 Validity

We show that simplification and refinement operators are valid, i.e., that if applied on a combinatorial complex Γ_d with manifold carrier M, they produce a combinatorial complex Γ'_d on M. Recall that in a combinatorial Morse complex cells are allowed to partially overlap (geometrically), while they are considered to be disjoint (combinatorially). We will show that if Γ_d is a combinatorial cell complex, and if the complex Γ'_d is obtained from the complex Γ_d after a simplification or a refinement, then

1. the cells in Γ'_d are disjoint,
2. the cells in Γ'_d cover M, and
3. the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

Proposition 1 *Each simplification remove operator is valid.*

Proof. We will show that each $remove_{i,i+1}(q, p, p')$ of the first type, $remove_{i,i+1}(q, p, \emptyset)$ of the second type, $remove_{i,i-1}(q, p, p')$ of the first type, and $remove_{i,i-1}(q, p, \emptyset)$ of the second type is valid.

Let us consider the $remove_{i,i+1}(q, p, p')$ of the first type. The effect of this operator on a combinatorial complex Γ_d is to merge the $(i+1)$-cell p and the i-cell q into the $(i+1)$-cell p' in the simplified complex Γ'_d, and to modify the immediate boundary and co-boundary relation by merging the boundary of p into the boundary of p', as explained in Section 4.1.1. We show that Γ'_d is a combinatorial cell complex by verifying the following three conditions:

1. The cells in Γ'_d are disjoint. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the $(i+1)$-cell p' in Γ'_d, which is the union of the $(i+1)$-cell p', and disjoint copies of the $(i+1)$-cell p and the i-cell q in Γ_d. If the cells in Γ_d are (combinatorially) disjoint, then the cells in Γ'_d are (combinatorially) disjoint.

2. The cells in Γ'_d cover M. If the cells in Γ_d cover M, then the cells in Γ'_d also cover M, since no part of M is deleted or created by the operator.

3. The boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d. The boundary relation is the transitive closure of the immediate boundary relation (without considering the multiplicities). The only cells with the changed immediate boundary relation are the $(i+2)$-cells $s_k \in S$ (the cells in the immediate co-boundary of the $(i+1)$-cell p) and the $(i+1)$-cell p' in Γ'_d.

   - The immediate boundary of each $(i+2)$-cell $s_k \in S$ in Γ'_d is the same as its immediate boundary in Γ_d minus (all instances of) the $(i+1)$-cell p. The boundary of each $(i+2)$-cell $s_k \in S$ in Γ'_d is the same as its boundary in Γ_d minus the $(i+1)$-cell p and the i-cell q.
   - The immediate boundary of the $(i+1)$-cell p' in Γ'_d is the same as its immediate boundary in Γ_d plus the i-cells $r_j \in R$ (cells on the immediate boundary of the $(i+1)$-cell p in Γ_d) with the multiplicities $mult(r_j, \partial p) \cdot mult(q, \partial p')$ and minus (all instances of) the i-cell q. The boundary of the $(i+1)$-cell p' in Γ'_d is the same as its boundary in Γ_d plus all the cells in the boundary of the $(i+1)$-cell p and minus the i-cell q.

   Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $remove_{i,i+1}(p, q, p')$ of the first type is valid.

Let us consider the $remove_{i,i+1}(q, p, \emptyset)$ of the second type. Its effect is to eliminate the $(i+1)$-cell p and the i-cell q. We verify that the simplified complex Γ'_d is a combinatorial cell complex:

1. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the $(i+2)$-cells $s_k \in S$ in Γ'_d (the cells in the immediate co-boundary of the $(i+1)$-cell p), which contain a disjoint copy of the $(i+1)$-cell p and the

i-cell q in Γ_d. If the cells in Γ_d are (combinatorially) disjoint, then the cells in Γ'_d are (combinatorially) disjoint.

2. The cells in Γ'_d cover M, since no part of M is deleted or created by the operator.

3. The only cells with the changed immediate boundary relation are the $(i+2)$-cells $s_k \in S$ (the cells in the immediate co-boundary of the $(i+1)$-cell p).

   - The immediate boundary of each $(i+2)$-cell $s_k \in S$ in Γ'_d consists of the same cells as its immediate boundary in Γ_d minus (all instances of) the $(i+1)$-cell p. The boundary of each $(i+2)$-cell $s_k \in S$ in Γ'_d consists of the same cells as its boundary in Γ_d minus the $(i+1)$-cell p and the i-cell q.

   Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $remove_{i,i+1}(q, p, \emptyset)$ of the second type is valid.

Let us consider the $remove_{i,i-1}(q, p, p')$ of the first type. The effect of this operator on a combinatorial complex Γ_d is to merge the $(i-1)$-cell p and the i-cell q into the $(i-1)$-cell p' in the simplified complex Γ'_d, and to modify the immediate boundary and co-boundary relation by merging the co-boundary of p into the co-boundary of p', as explained in Section 4.1.2. We show that the simplified complex Γ'_d is a combinatorial cell complex by verifying the conditions in the definition of a cell complex:

1. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the i-cells $r_j \in R$ in Γ'_d (which are the i-cells in the immediate co-boundary of the $(i-1)$-cell p). Each i-cell r_j in Γ'_d is a union of the i-cell r_j in Γ_d and $mult(p, \partial r_j)$ disjoint copies of the i-cell q and the $(i-1)$-cell p. If the cells in Γ_d are (combinatorially) disjoint, then the cells in Γ'_d are (combinatorially) disjoint.

2. The cells in Γ'_d cover M, since no part of M is deleted or created by the operator.

3. The only cells with the changed immediate boundary relation are the i-cells $r_j \in R$ (the cells in the immediate co-boundary of the $(i-1)$-cell

p) and the $(i+1)$-cells $z_h \in Z$ (the cells in the immediate co-boundary of the i-cell q) in Γ'_d.

- The immediate boundary of each i-cell $r_j \in R$ in Γ'_d is the same as its immediate boundary in Γ_d minus (all instances of) the $(i-1)$-cell p and plus $mult(p, \partial r_j) \cdot mult(p', \partial q)$ instances of the $(i-1)$-cell p'. The boundary of each i-cell $r_j \in R$ in Γ'_d is the same as its boundary in Γ_d minus the $(i-1)$-cell p plus the $(i-1)$-cell p' and all the cells in its boundary.
- The immediate boundary of each $(i+1)$-cell $z_h \in Z$ in Γ'_d is the same as its immediate boundary in Γ_d minus all instances of the i-cell q. Its boundary in Γ'_d is the same as its boundary in Γ_d minus the i-cell q and the $(i-1)$-cell p.

The cells with the changed boundary relation are the cells in the set U of cells of dimension at least $i+2$ in the co-boundary of the i-cell q and the cells in the set V of cells of dimension at least $i+1$ in the co-boundary of the $(i-1)$-cell p. For each cell u in U, the boundary of u in Γ'_d is the same as its boundary in Γ_d minus the i-cell q and the $(i-1)$-cell p. For each cell v in $V \backslash U$, the boundary of v in Γ'_d is the same as its boundary in Γ_d minus the $(i-1)$-cell p and plus the $(i-1)$-cell p' and all the cells in its boundary.

Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $remove_{i,i-1}(q, p, p')$ of the first type is valid.

Let us consider the $remove_{i,i-1}(q, p, \emptyset)$ of the second type. Its effect is to eliminate the $(i-1)$-cell p and the i-cell q. We verify that the simplified complex Γ'_d is a combinatorial cell complex:

1. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the i-cells $r_j \in R$ in Γ'_d (the cells in the immediate co-boundary of the $(i-1)$-cell p), which contain $mult(p, \partial r_j)$ disjoint copies of the i-cell q and the $(i-1)$-cell p in Γ_d. If the cells in Γ_d are (combinatorially) disjoint, then the cells in Γ'_d are (combinatorially) disjoint.

2. The cells in Γ'_d cover M, since no part of M is deleted or created by the operator.

3. The only cells with the changed immediate boundary relation are the i-cells $r_j \in R$ (the cells in the immediate co-boundary of the $(i-1)$-cell p) and the $(i+1)$-cells $z_h \in Z$ (the cells in the immediate co-boundary of the i-cell q) in Γ'_d.

   - The immediate boundary of each i-cell $r_j \in R$ in Γ'_d consists of the same cells as its immediate boundary in Γ_d minus $mult(p, \partial r_j)$ instances of the $(i-1)$-cell p. The boundary of i-cell r_j in Γ'_d consists of the same cells as its boundary in Γ_d minus the $(i-1)$-cell p.
   - The immediate boundary of each $(i+1)$-cell $z_h \in Z$ in Γ'_d consists of the same cells as its immediate boundary in Γ_d minus $mult(q, \partial z_h)$ instances of the i-cell q. The boundary of each $(i+1)$-cell z_h in Γ'_d consists of the same cells as its boundary in Γ_d minus the i-cell q and the $(i-1)$-cell p.

   The cells with the changed boundary relation are the cells in the set U of cells of dimension at least $i+2$ in the co-boundary of the i-cell q and the cells in the set V of cells of dimension at least $i+1$ in the co-boundary of the $(i-1)$-cell p. For each cell u in U, the boundary of u in Γ'_d is the same as its boundary in Γ_d minus the i-cell q and the $(i-1)$-cell p. For each cell v in $V \backslash U$, the boundary of v in Γ'_d is the same as its boundary in Γ_d minus the $(i-1)$-cell p.

   Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $remove_{i,i-1}(q, p, \emptyset)$ is valid. □

Proposition 2 *Each refinement insert operator is valid.*

Proof. We will show that each $insert_{i,i+1}$ (q, p, p') of the first type, $insert_{i,i+1}$ $(q, p, \emptyset)$ of the second type, $insert_{i,i-1}(q, p, p')$ of the first type, and $insert_{i,i-1}$ $(q, p, \emptyset)$ of the second type is valid.

Let us consider the $insert_{i,i+1}(q, p, p')$ of the first type. The effect of the operator on Γ_d is to split the $(i+1)$-cell p' in Γ_d into the $(i+1)$-cells p and p' by inserting the i-cell q in Γ'_d (see Section 4.2.1). We show that Γ'_d is a combinatorial cell complex by verifying the conditions in the definition of a cell complex:

1. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the $(i+1)$-cell p' in Γ_d, which is split into (disjoint) $(i+1)$-cells p and p' and i-cell q in Γ'_d. If the cells in Γ_d are (combinatorially) disjoint, then the cells in Γ'_d are also disjoint.

2. The cells in Γ'_d cover M, since no part of M is deleted or created by the operator.

3. The cells with the changed immediate boundary relation are the $(i+2)$-cells $s_k \in S$ (the cells in the immediate co-boundary of the $(i+1)$-cell p), the $(i+1)$-cell p', and the new cells p and q in Γ'_d.

   - The immediate boundary of the new i-cell q in Γ'_d consists of the $(i-1)$-cells $z_h \in Z$ with the given multiplicities $mult'(z_h, \partial q)$. The boundary of the i-cell q consists of the $(i-1)$-cells $z_h \in Z$, and all the cells on the boundary of the $(i-1)$-cells $z_h \in Z$ in Γ_d (and in Γ'_d).
   - The immediate boundary of the new $(i+1)$-cell p in Γ'_d consists of the i-cells $r_j \in R$ with the given multiplicities $mult'(r_j, \partial p)$ and the i-cell q with the multiplicity $mult'(q, \partial p)$ equal to 1. The boundary of the new $(i+1)$-cell p in Γ'_d consists of the i-cells $r_j \in R$ and the i-cell q, and all the cells in their boundary in Γ'_d.
   - The immediate boundary of each $(i+2)$-cell s_k in Γ'_d consists of the same $(i+1)$-cells as its immediate boundary in Γ_d plus the $(i+1)$-cell p with the given multiplicity $mult'(p, \partial s_k)$. The boundary of each $(i+2)$-cell s_k in Γ'_d consists of the same cells as its boundary in Γ_d plus the $(i+1)$-cell p and all the cells in its boundary in Γ'_d.
   - The immediate boundary of the $(i+1)$-cell p' in Γ'_d is the same as its immediate boundary in Γ_d minus $mult'(r_j, \partial p) \cdot mult'(q, \partial p')$ instances of the i-cells $r_j \in R$ (the cells in the immediate boundary of $(i+1)$-cell p in Γ'_d) and plus the i-cell q with the given multiplicity $mult'(q, \partial p')$. The boundary of the $(i+1)$-cell p' in Γ'_d is the same as its boundary in Γ_d minus the cells in the boundary of the $(i+1)$-cell p, and plus the i-cell q and all the cells in its boundary.

   The cells with the changed boundary relation are the cells in the set U of cells in the co-boundary of the $(i+2)$-cells $s_k \in S$ and the cells in

the set V of cells in the co-boundary of the $(i-1)$-cells $z_h \in Z$. The boundary of each cell $u \in U$ in Γ'_d is the same as its boundary in Γ_d plus the $(i+1)$-cell p and all the cells in its boundary (in Γ'_d). The boundary of each cell $v \in V \backslash U$ in Γ'_d is the same as its boundary in Γ_d plus the i-cell q and all the cells in its boundary (in Γ'_d).

Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $insert_{i,i+1}(q, p, p')$ is valid.

Let us consider the $insert_{i,i+1}(q, p, \emptyset)$ of the second type. Its effect is to introduce the $(i+1)$-cell p and the i-cell q. We verify that the refined complex Γ'_d is a combinatorial cell complex:

1. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the $(i+1)$-cell p and the i-cell q that are inserted in Γ'_d, and the $(i+2)$-cells $s_k \in S$ in Γ'_d (the cells in the co-boundary of the $(i+1)$-cell p), which are the same as the $(i+2)$-cells s_k in Γ_d minus a disjoint copy of the $(i+1)$-cell p and the i-cell q. If the cells in Γ_d are combinatorially disjoint, then so are the cells in Γ'_d.

2. The cells in Γ'_d cover M, since no part of M is created or deleted by the operator.

3. The cells with the changed immediate boundary relation are the $(i+2)$-cells $s_k \in S$ (the cells in the immediate co-boundary of the $(i+1)$-cell p), the $(i+1)$-cell p and the i-cell q in Γ'_d.

   - The immediate boundary of the i-cell q in Γ'_d consists of the $(i-1)$-cells $z_h \in Z$ with the given multiplicities $mult'(z_h, \partial q)$. The boundary of the i-cell q in Γ'_d consists of the $(i-1)$-cells $z_h \in Z$ and all the cells in their boundary.
   - The immediate boundary of the $(i+1)$-cell p in Γ'_d consists of the i-cells $r_j \in R$ and i-cell q with the given multiplicities $mult'(r_j, \partial p)$ and $mult'(q, \partial p) = 1$, respectively. The boundary of the $(i+1)$-cell p in Γ'_d consists of the i-cells $r_j \in R$ and the i-cell q, and all the cells in their boundary.
   - The immediate boundary of each $(i+2)$-cell s_k in Γ'_d consists of the same $(i+1)$-cells as its immediate boundary in Γ_d plus the $(i+1)$-cell p with the given multiplicity $mult'(p, \partial s_k)$. The boundary of

each $(i+2)$-cell s_k in Γ'_d consists of the same cells as its boundary in Γ_d plus the $(i+1)$-cell p and all the cells in its boundary.

The cells with the changed boundary relation are the cells in the set U of cells in the co-boundary of the $(i+2)$-cells $s_k \in S$ and the cells in the set V of cells in the co-boundary of the $(i-1)$-cells $z_h \in Z$. The boundary of each cell $u \in U$ in Γ'_d is the same as its boundary in Γ_d plus the $(i+1)$-cell p and all the cells in its boundary (in Γ'_d). The boundary of each cell $v \in V \backslash U$ in Γ'_d is the same as its boundary in Γ_d plus the i-cell q and all the cells in its boundary (in Γ'_d).

Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $insert_{i,i+1}(q, p, \emptyset)$ is valid.

Let us consider the $insert_{i,i-1}(q, p, p')$ of the first type. The effect of the operator on Γ_d is to split the $(i-1)$-cell p' in Γ_d into the $(i-1)$-cells p and p' by expanding the i-cell q in Γ'_d (see Section 4.2.2). We show that Γ'_d verifies the conditions in the definition of a combinatorial cell complex:

1. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the new cells p and q, and the i-cells $r_j \in R$ in Γ'_d (the cells in the immediate co-boundary of the $(i-1)$-cell p), which are equal to i-cells r_j in Γ_d minus $mult(p, \partial r_j)$ disjoint copies of the i-cell q and the $(i-1)$-cell p. If the cells in Γ_d are (combinatorially) disjoint, then so are the cells in Γ'_d.

2. The cells in Γ'_d cover M, since no part of M is deleted or created by the operator.

3. The cells with the changed immediate boundary relation are the i-cells $r_j \in R$ (the cells in the immediate co-boundary of the $(i-1)$-cell p), the $(i+1)$-cells $z_h \in Z$ (the cells in the immediate co-boundary of the i-cell r), and the new cells p and q in Γ'_d.

   - The immediate boundary of the new $(i-1)$-cell p in Γ'_d consists of the $(i-2)$-cells $s_k \in S$ with the given multiplicities $mult'(s_k, \partial p)$. The boundary of the $(i-1)$-cell p consists of the $(i-2)$-cells $s_k \in S$, plus all the cells in the boundary of each $(i-2)$-cell $s_k \in S$ in Γ_d (and in Γ'_d).

- The immediate boundary of the new i-cell q in Γ'_d consists of the $(i-1)$-cells p and p' with the given multiplicities $mult'(p, \partial q) = 1$ and $mult'(p', \partial q)$, respectively. The boundary of the new i-cell q in Γ'_d consists of the $(i-1)$-cells p and p', and all the cells on their boundary.
- The immediate boundary of each i-cell $r_j \in R$ in Γ'_d consists of the same $(i-1)$-cells as its immediate boundary in Γ_d minus $mult'(p, \partial r_j) \cdot mult'(p', \partial q)$ instances of the $(i-1)$-cell p' and plus the $(i-1)$-cell p with the given multiplicity $mult'(p, \partial r_j)$. If $mult(p', \partial r_j) - mult'(p, \partial r_j) \cdot mult'(p', \partial q) = 0$, then the boundary of the i-cell $r_j \in R$ in Γ'_d consists of the same cells as its boundary in Γ_d minus the $(i-1)$-cell p' and all the cells in its boundary plus the $(i-1)$-cell p and all the cells in its boundary. Otherwise, if $mult(p', \partial r_j) - mult'(p, \partial r_j) \cdot mult'(p', \partial q) > 0$, then the boundary of the i-cell $r_j \in R$ in Γ'_d consists of the same cells as its boundary in Γ_d plus the $(i-1)$-cell p and all the cells in its boundary.
- The immediate boundary of each $(i+1)$-cell $z_h \in Z$ in Γ'_d is the same as its immediate boundary in Γ_d plus the i-cell q with the given multiplicity $mult'(q, \partial z_h)$. The boundary of each $(i+1)$-cell $z_h \in Z$ in Γ'_d is the same as its boundary in Γ_d plus the i-cell q and all the cells in its boundary.

The cells with the changed boundary relation are the cells in the set U of cells in the co-boundary of the $(i+1)$-cells $z_h \in Z$ and the cells in the set V of cells in the co-boundary of the i-cells $r_j \in R$. The boundary of each cell $u \in U$ in Γ'_d is the same as its boundary in Γ_d plus the i-cell q and all the cells in its boundary (in Γ'_d). The boundary of each cell $v \in V \backslash U$ in Γ'_d is the same as its boundary in Γ_d plus the $(i-1)$-cell p and all the cells in its boundary (in Γ'_d).

Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $insert_{i,i-1}(q, p, p')$ is valid.

Let us consider the $insert_{i,i-1}(q, p, \emptyset)$ of the second type. It introduces the $(i-1)$-cell p and the i-cell q. We verify that the refined complex Γ'_d is a combinatorial cell complex:

1. The cells in Γ'_d are the same as the cells in Γ_d, with the exception of the $(i-1)$-cell p and the i-cell q that are inserted in Γ'_d, and the i-cells $r_j \in R$ in Γ'_d (the cells in the immediate co-boundary of the $(i-1)$-cell p), which are the same as the i-cells r_j in Γ_d minus $mult(p, \partial r_j)$ disjoint copies of the $(i-1)$-cell p and the i-cell q. If the cells in Γ_d are combinatorially disjoint, then so are the cells in Γ'_d.

2. The cells in Γ'_d cover M, since no part of M is deleted or created by the operator.

3. The cells with the changed immediate boundary relation are the i-cells $r_j \in R$ (the cells in the immediate co-boundary of the $(i-1)$-cell p), the $(i+1)$-cells $z_h \in Z$ (the cells in the immediate co-boundary of the i-cell q), the new $(i-1)$-cell p and the new i-cell q in Γ'_d.

   - The immediate boundary of the new $(i-1)$-cell p in Γ'_d consists of the $(i-2)$-cells $s_k \in S$ with the given multiplicities $mult'(s_k, \partial p)$. The boundary of the $(i-1)$-cell p consists of the $(i-2)$-cells $s_k \in S$, plus all the cells in the boundary of each $(i-2)$-cell $s_k \in S$ in Γ_d (and in Γ'_d).
   - The immediate boundary of the new i-cell q in Γ'_d consists of the $(i-1)$-cell p with the given multiplicity $mult'(p, \partial q) = 1$. The boundary of the new i-cell q in Γ'_d consists of the $(i-1)$-cell p and all the cells in the boundary of p in Γ'_d.
   - The immediate boundary of each i-cell $r_j \in R$ in Γ'_d consists of the same $(i-1)$-cells as its immediate boundary in Γ_d plus the $(i-1)$-cell p with the given multiplicity $mult'(p, \partial r_j)$. The boundary of each i-cell $r_j \in R$ in Γ'_d consists of the same cells as its boundary in Γ_d plus the $(i-1)$-cell p and all the cells in its boundary.
   - The immediate boundary of each $(i+1)$-cell $z_h \in Z$ in Γ'_d is the same as its immediate boundary in Γ_d plus the i-cell q with the given multiplicity $mult'(q, \partial z_h)$. The boundary of the $(i+1)$-cell $z_h \in Z$ in Γ'_d is the same as its boundary in Γ_d plus the i-cell q and all the cells in its boundary.

   The cells with the changed boundary relation are the cells in the set U of cells in the co-boundary of the $(i+1)$-cells $z_h \in Z$ and the cells in the set V of cells in the co-boundary of the i-cells $r_j \in R$. The boundary of

each cell $u \in U$ in Γ'_d is the same as its boundary in Γ_d plus the i-cell q and all the cells in its boundary (in Γ'_d). The boundary of each cell $v \in V \backslash U$ in Γ'_d is the same as its boundary in Γ_d plus the $(i-1)$-cell p and all the cells in its boundary (in Γ'_d).

Thus, the boundary of each cell in Γ'_d is a disjoint union of cells in Γ'_d.

The complex Γ'_d is a combinatorial cell complex, and each $insert_{i,i-1}(q, p, \emptyset)$ is valid.

□

4.3.2 Minimality

In the 2D case, the Euler operators have been defined as operators for modeling 2-complexes with manifold domain maintaining the Euler-Poincaré formula. This formula relates the number of cells in a decomposition of a surface with the topology of the surface [9]. It has been shown in [79] that the Euler operators form a complete set of operators for modeling manifold surfaces.

In this Section, we show that our simplification *remove* and refinement *insert* operators form a complete set of basis operators for modifying Morse complexes in arbitrary dimensions. We do this by using the Euler-Poincaré formula

$$\chi(M) = c_0 - c_1 + .. + (-1)^n c_n = \beta_0 - \beta_1 + .. + (-1)^n \beta_n.$$

Here, c_i is the number of i-cells in a cell complex Γ with manifold carrier M, and β_i is the ith Betti number of M.

Our simplification and refinement operators are examples of Euler operators in the sense that they affect a constant number of entities (cells) on the left hand side of the Euler-Poincaré formula. They affect only the decomposition of M without changing the topology of M, and thus its Betti numbers, or the Euler-Poincaré characteristic $\chi(M)$.

Both the $remove_{i,i+1}$, $1 \leq i \leq n-1$, and the $remove_{i+1,i}$, $0 \leq i \leq n-2$, can be interpreted as Euler operators, since they eliminate an i-cell and an $(i+1)$-cell from the descending Morse complex Γ_d, and an $(n-i)$-cell and an $(n-i-1)$-cell from the ascending Morse complex Γ_a. If c_i denotes the number of the i-cells in Γ_d before the simplification, and c'_i denotes the number of the i-cells in the simplified complex Γ'_d, then $c'_i = c_i - 1$, $c'_{i+1} = c_{i+1} - 1$, and $c'_j = c_j$ for $j \neq i, i+1$. Similarly, an $insert_{i,i+1}$, $1 \leq i \leq n-1$, and an

$insert_{i+1,i}$, $0 \leq i \leq n-2$, can be interpreted as Euler operators, since they introduce an i-cell and an $(i+1)$-cell in the descending Morse complex Γ_d, and an $(n-i)$-cell and an $(n-i-1)$-cell in the ascending Morse complex Γ_a. If c_i denotes the number of the i-cells in Γ_d before the refinement, and c_i'' denotes the number of the i-cells in the refined complex Γ_d', then $c_i'' = c_i + 1$, $c_{i+1}'' = c_{i+1} + 1$, and $c_j'' = c_j$ for $j \neq i, i+1$. As a consequence, we have that the Euler-Poincaré formula is satisfied after each simplification *remove* and refinement *insert* operator.

Proposition 3 *Operator remove (together with the inverse operator insert) forms a basis of the set U of topological operators that modify the Morse complexes subdividing a manifold M.*

Proof. The set of operators U is a subset of the set of operators W that modify Morse complexes by adding or removing cells from it. The set W is a free $\mathbb{Z}$-module over the ring of integers, and each operator in W can be expressed as an $(n+1)$-dimensional vector $(a_0, a_1, ..., a_n)_{\mathcal{B}}$, with integer coordinates ($a_i \in \mathbb{Z}$, $0 \leq i \leq n$) in the standard basis $\mathcal{B}$ of W. (The correspondence between the operators in W and the $(n+1)$-tuples over $\mathbb{Z}$ is not one-to-one. Each operator is represented as a unique tuple, but different operators may be represented by the same tuple.) The integer $|a_i|$ denotes the number of i-cells removed from the descending Morse complex (and the number of $(n-i)$-cells removed from the ascending Morse complex) if $a_i \geq 0$, and it denotes the number of i-cells introduced into the descending Morse complex (and the number of $(n-i)$-cells introduced into the ascending Morse complex) if $a_i < 0$.

Each operator u in U can be expressed as an $(n+1)$-dimensional vector $u = (a_0, a_1, .., a_n)_{\mathcal{B}}$ with integer coordinates a_i, $0 \leq i \leq n$, such that $a_0 - a_1 + ... + (-1)^n a_n = 0$. Thus, U is a submodule of W. Let us consider the $(n+1)$-dimensional vectors $d_0 = (1, 0, ..., 0, (-1)^{n-1})_{\mathcal{B}}$, $d_1 = (0, 1, 0, ..., 0, (-1)^{n-2})_{\mathcal{B}}$,..., $d_{n-1} = (0, ..., 1, (-1)^{n-(n-1)})_{\mathcal{B}} = (0, ..., 0, 1, -1)_{\mathcal{B}}$. They are in U, they are linearly independent, and they generate U. They form a basis of U, and thus each basis of U has n vectors.

The $remove_{i,i+1}$, $1 \leq i \leq n-1$, (and the $remove_{i+1,i}$, $0 \leq i \leq n-2$), which eliminates an i-cell p and an $(i+1)$-cell q, can be expressed as a vector b_i with coordinates (in the standard basis $\mathcal{B}$) $b_i = (a_0, a_1, .., a_n)_{\mathcal{B}}$, where $a_i = a_{i+1} = 1$, and $a_j = 0$, $j \neq i, i+1$. In other words, $b_0 = (1, 1, 0, ..., 0)_{\mathcal{B}}$, $b_1 = (0, 1, 1, 0, ..., 0)_{\mathcal{B}}$, ..., $b_{n-1} = (0, ..., 0, 1, 1)_{\mathcal{B}}$. We will show that these n vectors are linearly independent, and that they generate U.

To show that vectors b_0, b_1, ..., b_{n-1} are independent, let us consider the equation

$$\mu_0 b_0 + \mu_1 b_1 + ... + \mu_{n-1} b_{n-1} = 0, \text{ i.e.,}$$

$$\mu_0(1,1,0,..,0)_{\mathcal{B}} + \mu_1(0,1,1,0,..,0)_{\mathcal{B}} + ... + \mu_{n-1}(0,..,0,1,1)_{\mathcal{B}} = 0,$$

which is equivalent to the system of equations

$$\mu_0 = 0,\ \mu_0 + \mu_1 = 0,\ \mu_1 + \mu_2 = 0,\ ...,\ \mu_{n-2} + \mu_{n-1} = 0,\ \mu_{n-1} = 0.$$

The only solution to this system is

$$\mu_0 = \mu_1 = ... = \mu_{n-1} = 0,$$

and the vectors b_0, b_1, ..., b_{n-1} are linearly independent.

To show that the vectors b_0, b_1, ..., b_{n-1} generate U, let us consider the equation

$$u = \mu_0 b_0 + \mu_1 b_1 + ... + \mu_{n-1} b_{n-1},$$

where $u = (a_0, a_1, ..., a_n)_{\mathcal{B}}$ is an arbitrary vector in U. This equation is equivalent to the system of equations

$$a_0 = \mu_0,\ a_1 = \mu_0 + \mu_1,\ a_2 = \mu_1 + \mu_2\ , ...,\ a_{n-1} = \mu_{n-2} + \mu_{n-1},\ a_n = \mu_{n-1}.$$

The solution to this system is

$$\mu_0 = a_0,\ \mu_1 = a_1 - a_0,\ \mu_2 = a_2 - a_1 + a_0,\ ...,\ \mu_{n-1} = a_n.$$

Each vector u in U can be expressed as a linear combination of the vectors b_0, b_1, ..., b_{n-1}, and the vectors b_0, b_1, ..., b_{n-1} generate U.

The n vectors b_0, b_1, ..., b_{n-1} are linearly independent, they generate U, and thus they form a basis of U. □

Note that the above proof does not provide an algorithm for expressing an arbitrary operator u in U as a sequence of basis operators. The order in which the basis operators should be applied, and the cells affected by the basis operators are not specified in the proof. The vector b_i represents not only the simplification *remove* operator, but also (when multiplied by -1) the inverse refinement *insert* operator.

4.4 Representing a Sequence of *remove* Operators

We have designed a data structure for representing a sequence of simplification *removal* and *insertion* operators in [25]. We have called this data structure *augmented cancellation forest*. It is a generalization of the cancellation forest which represents a sequence of cancellations defined in [15] for the 2D case. Here, we recall the definition of the cancellation forest in [15], and we define the augmented cancellation forest for representing a sequence of *remove* operators.

4.4.1 Cancellation Forest in 2D

In 2D, a cancellation involves either a saddle q and a maximum p, or a saddle q and a minimum p, and each saddle q is connected to exactly two maxima, and exactly two minima. Thus, in 2D, a cancellation of a saddle q and a maximum p reduces to a $remove_{1,2}(q, p, p')$. In the descending Morse complex Γ_d, the $remove_{1,2}(q, p, p')$ merges the descending 2-cell p into the descending 2-cell p'. A cancellation of a saddle q and a minimum p reduces to a $remove_{1,0}(q, p, p')$. In Γ_d, it merges the descending 0-cell p into the descending 0-cell p'. A sequence of *remove* operators on the initial full-resolution Morse complex can be encoded in a data structure called cancellation forest, introduced in [15], in which each tree encodes either a set of $remove_{1,2}$ operators, or a set of $remove_{1,0}$ operators.

A $remove_{1,2}$ ($remove_{1,0}$) tree is a rooted tree. The nodes of the $remove_{1,2}$ ($remove_{1,0}$) tree correspond to maxima (minima) in the initial highest-resolution descending Morse complex. Each arc in the $remove_{1,2}$ ($remove_{1,0}$) tree represents a saddle, i.e., a 1-cell q, such that the $remove_{1,2}(q, p, p')$ ($remove_{1,0}(q, p, p')$) belongs to the set of simplifications on the Morse complex, and both nodes p and p' and arc q belong to the same tree in the forest. The root of a $remove_{1,2}$ ($remove_{1,0}$) tree T represents a 2-cell (a 0-cell), into which all cells corresponding to nodes of the tree are merged.

A cancellation forest encoding a sequence of simplifications on a descending Morse complex is built starting from a graph (forest), in which each tree has only one node, corresponding to a maximum or a minimum. Each simplification merges two trees into one tree, by introducing an arc in the forest, and the inverse refinement operator removes an arc from the forest.

Figure 4.16 shows a sequence of $remove_{1,2}$ operators on a 2D descending

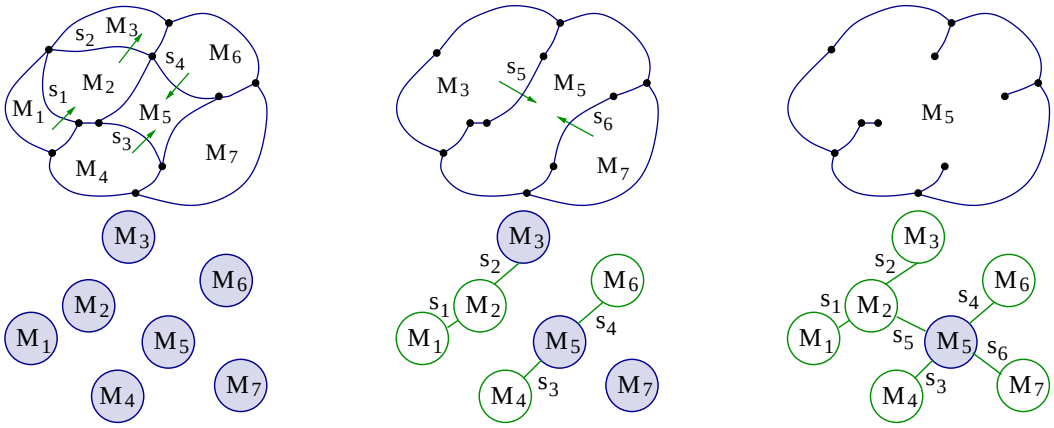


Figure 4.16: [25] A sequence of $remove_{1,2}$ on a portion of the 2D descending Morse complex, and the process of building the cancellation forest, consisting of $remove_{1,2}$ trees. The root of each tree created in the process is shaded.

Morse complex, and the construction of the corresponding $remove_{1,2}$ tree encoding that sequence. This example is taken from [15], but is expressed in terms of the descending Morse complex instead of the Morse-Smale complex. First, $remove_{1,2}(s_1, M_1, M_2)$ and $remove_{1,2}(s_2, M_2, M_3)$ merge the 2-cells M_1 and M_2 into M_3, and $remove_{1,2}(s_3, M_4, M_5)$ and $remove_{1,2}(s_4, M_6, M_5)$ merge the 2-cells M_4 and M_6 into the 2-cell M_5, as shown in Figure 4.16 (a). Then, $remove_{1,2}(s_6, M_7, M_5)$ and $remove_{1,2}(s_5, M_3, M_5)$ merge the 2-cells M_7 and M_3 into the 2-cell M_5. The node M_3 is not connected to the node M_5 in the cancellation forest, because the descending 2-cells M_3 and M_5 are not adjacent in the initial Morse complex. Instead, the tree rooted at the node M_3 is connected to the tree rooted at the node M_5 through an arc connecting the node M_2 to the node M_5 (since the descending 2-cells M_2 and M_5 are adjacent in the initial Morse complex), as shown in Figure 4.16 (b). The coarsest-resolution complex, and the corresponding cancellation forest (which in this example is a tree), are shown in Figure 4.16 (c).

4.4.2 Augmented Cancellation Forest

In this Section, we extend the notion of the cancellation forest to arbitrary dimensions. In Section 4.4.2, we discuss some issues related to the extension to

higher dimensions. In Section 4.4.2, we introduce the notion of an augmented cancellation forest. In Section 4.4.2, we give an algorithm for constructing an augmented cancellation forest, starting from a sequence of simplifications on an n-dimensional Morse complex.

Motivation

As we have seen, in 2D the cancellation reduces either to a $remove_{1,2}$ or to a $remove_{1,0}$, depending on index of the cancelled points, i.e., on the dimension of the corresponding descending cells. The cancellation of a 2-cell and a 1-cell reduces to a $remove_{1,2}$, and the cancellation of a 0-cell and a 1-cell reduces to a $remove_{1,0}$. In other words, the role of a critical point p in the cancellation forest is determined by the index of p: a maximum (minimum) may only correspond to a node in a $remove_{1,2}$ ($remove_{1,0}$) tree, and a saddle may only correspond to an arc in some tree.

The extension of the tree-based encoding of a sequence of simplification operators to arbitrary dimensions is non-trivial due to the following reasons.

1. In higher dimensions, a simplification is not uniquely determined by the index of the eliminated critical points, as a critical point p of index i and a critical point q of index $i+1$ may be involved either in a $remove_{i,i+1}(q, p, p')$, or in a $remove_{i+1,i}(p, q, p')$. Thus, a critical point of index i may correspond either to a node or to an arc in either a $remove_{i,i+1}$ or a $remove_{i,i-1}$ tree. For example, a simplification eliminating the 1-cell p and the 2-cell q may be a $remove_{1,2}(q, p, r)$, shown in Figure 4.17 (a), or a $remove_{2,1}(p, q, t)$, shown in Figure 4.17 (b).

2. The role of a critical point may change after simplification. An example is illustrated in Figure 4.18. Initially, the 2-cells p and q, and 1-cells a, b, and s, are the roots of trees, each having only one node. After the $remove_{1,2}(q, s, p)$, p and q are nodes, and s is an arc in a $remove_{1,2}$ tree rooted at p, as shown in Figure 4.18 (b). After the $remove_{2,1}(b, p, a)$, b and a are nodes, and p is an arc in a $remove_{2,1}$ tree, rooted at a, as shown in Figure 4.18 (c).

3. In higher dimensions, *remove* operators of the second kind are not ignored, as is the case in 2D. Thus, we will have two kinds of $remove_{i,i+1}$ ($remove_{i,i-1}$) trees, depending on the type of the simplification they encode.

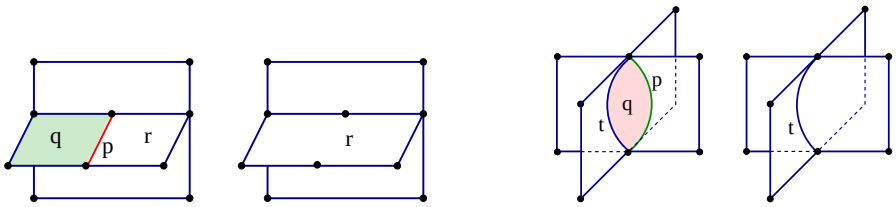


Figure 4.17: [25] The type of a simplification is not determined by the index of eliminated critical points. (a) The $remove_{1,2}(q, p, r)$ of the 1-cell p is feasible, while the $remove_{2,1}$ of the 2-cell q is not feasible. (b) The $remove_{2,1}(p, q, t)$ of the 2-cell q is feasible, while the $remove_{1,2}$ of the 1-cell p is not feasible.

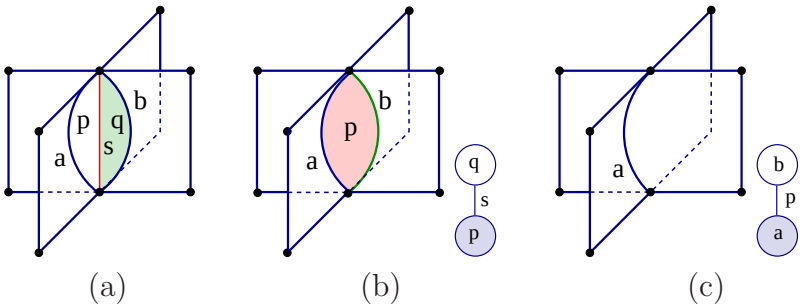


Figure 4.18: [25] (a) The initial 3D descending complex. (b) After the $remove_{1,2}(q, s, p)$. (c) After the $remove_{2,1}(b, p, a)$. The root of each tree is shaded.

4. We want a tree-based data structure which will be able to handle the inverse refinements efficiently, namely as deletion of arcs in the forest, like in the 2D case.

Definition

Informally, in an augmented tree each node and each arc may encode another augmented tree. An *augmented tree* is a tree which may have regular nodes and hyper-nodes, and regular arcs and hyper-arcs. A hyper-node (a hyper-arc) is a node (an arc) which encodes an augmented tree. Regular nodes (arcs) of augmented trees encoded by hyper-nodes (hyper-arcs) of T_p are called *hidden* nodes (arcs) of T_p. We will consider augmented trees with finite number of (regular and hidden) nodes.

An augmented cancellation forest is a forest of augmented trees such that all nodes and all arcs of each augmented tree are labeled consistently, as explained below. We assign a label $Type(T)$ to each tree T in an augmented cancellation forest, depending on the type of a sequence of simplifications it encodes ($remove_{i,i+1}$ or $remove_{i,i-1}$). We call such trees removal and contraction trees. All nodes in a tree correspond to critical points of the same index $i+1$, and all arcs correspond to critical points of index i if the tree is a removal tree. In a contraction tree, all nodes correspond to critical points of index $i-1$, and all arcs correspond to critical points of index i. Thus, we partition the set N of labels of nodes and arcs in the forest into $n+1$ subsets N_i, $0 \leq i \leq n$, such that labels in N_i correspond to critical points of index i. We introduce a hyper-node in the forest when a root of a removal tree labels a node in a contraction tree, and when a root of a contraction tree labels a node in a removal tree. We introduce a hyper-arc in the forest when a root of a removal or a contraction tree labels an arc of another tree in the cancellation forest. We also introduce a special tree T_{empty}, rooted at node *empty*, which encodes all $remove_{i,i+1}$ and $remove_{i,i-1}$ operators of the second kind (of the form $remove_{i,i+1}(q,p,\emptyset)$ and $remove_{i,i-1}(q,p,\emptyset)$). Tree T_{empty} does not have any label indicating its type.

More formally, an *augmented cancellation forest* ACF is a forest of rooted augmented trees, such that

1. all nodes and arcs in an ACF have a label l in the set $N = \{empty\} \cup N_0 \cup N_1 \cup .. \cup N_n$,

2. there is a bijection between the set N and the set of regular nodes and regular arcs (hidden or not) of ACF,

3. label *empty* labels the root of an augmented tree T_{empty}, which is not represented by any hyper-node or hyper-arc,

4. each augmented tree T_p has an associated label $Type(T_p)$, which is set to *void* (if T_p has only one node p, or if $T_p = T_{empty}$), or to r or c (if T_p has more than one node),

5. the labels l of nodes in each tree $T_p \neq T_{empty}$ in ACF belong to the same set N_i, $0 \leq i \leq n$,

6. the labels of arcs in each tree T_p rooted at node p with $l(p) \in N_i$, $0 \leq i \leq n$, all belong either to N_{i-1}, in which case $Type(T_p) = r$ (and T_p is

called a removal tree), or to N_{i+1}, in which case $Type(T_p) = c$ (and T_p is called a contraction tree),

7. if q_{new} is a hyper-node in an augmented tree T_p with $Type(T_p) = r$, representing an augmented tree T_q, then $Type(T_q) = c$, and $l(q_{new}) = l(q)$,

8. if q_{new} is a hyper-node in an augmented tree T_p with $Type(T_p) = c$, representing an augmented tree T_q, then $Type(T_q) = r$, and $l(q_{new}) = l(q)$,

9. if the root *empty* is deleted from T_{empty}, then the tree T_{empty} decomposes into subtrees T_q that all satisfy conditions 4. to 8.

Construction

An augmented cancellation forest ACF, representing a sequence of simplification operators on Morse complexes, is built bottom up, starting from the full-resolution Morse complex, over which a sequence of simplifications is applied. An ACF by itself is not sufficient for the recovery of all the information encoded in the sequence of simplified Morse complexes. Connectivity between the cells needs also to be stored. Specifically, for each i-cell p in the initial Morse complex, we need to store all $(i-1)$-cells in its immediate boundary, and all $(i+1)$-cells in its immediate co-boundary. This can be done using an *incidence graph* [49]. A regular node with label in N_i in an augmented tree represents an i-cell in the initial Morse complex. The root p of each removal or contraction tree T_p represents all the cells, corresponding to nodes of the tree, merged into it through cells, corresponding to arcs of T_p. Recall that a $remove_{i,i+1}(q, p, p')$ merges the boundary of p into the boundary of p', and does not change the co-boundary of p'. Dually, a $remove_{i,i-1}(q, p, p')$ merges the co-boundary of p into the co-boundary of p', and does not change the boundary of p'. This knowledge enables us to keep track of the immediate boundaries and co-boundaries of each intermediate cell p created in the simplification process, and represented by the root p of some tree T_p.

The construction process of an ACF actually describes the effect of a simplification applied on an ACF corresponding to the current Morse complex which is being simplified. Initially, when the descending Morse complex is at full resolution, each cell p (corresponding to a critical point p) defines a tree

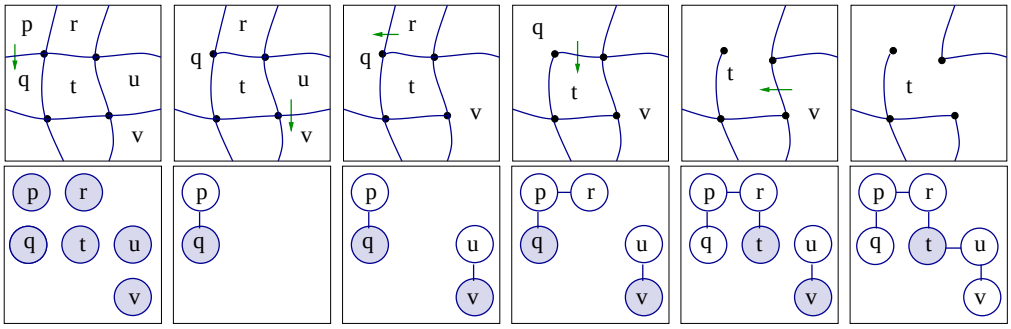


Figure 4.19: [25] A sequence of $remove_{1,2}$ operators on a 2D descending Morse complex, and the construction of the cancellation forest. The root of each tree is shaded.

T_p with only one regular node, labeled p, which is a root of T_p. Thus, labels in N_i correspond to i-saddles, i.e., to i-cells of the initial descending Morse complex. Each cell p is initially a representative only of itself. This is illustrated in 2D in Figure 4.19 (a), and in 3D in Figure 4.20 (a). For each tree T_p, the label $Active(T_p)$ is set to 1, meaning that cell p corresponding to the root p of tree T_p is present in the Morse complex, and the label $Type(T_p)$ is set to void, meaning that no simplification eliminating the cell p has yet been applied. A tree T_{empty}, rooted at node *empty*, without an *Active* or a *Type* label is initialized.

Let $sim(p, s, q)$ be a feasible simplification (a $remove_{i,i+1}$ or a $remove_{i,i-1}$) on a current Morse complex. Then p, s, and q are roots of trees T_p, T_s, and T_q, respectively. The label $Type(sim(p, s, q))$ is set to r or c if $sim(p, s, q)$ is a $remove_{i,i+1}$ or a $remove_{i,i-1}$, respectively. If $Type(sim(p, s, q)) = r$ ($Type(sim(p, s, q)) = c$), the set of all cells in the initial Morse complex which constitute the immediate boundary (the immediate co-boundary) of a current cell s, represented by the root s of tree T_s, will be denoted by S. The trees T_p and T_s rooted at nodes p and s, respectively, are deactivated from the forest, by changing the labels $Active(T_p)$ and $Active(T_s)$ from 1 to 0, indicating that cells p and s are no longer present in the Morse complex.

Let $sim(p, s, q)$ be a $remove_{i,i+1}$ (a $remove_{i,i-1}$). Then, a regular or a hidden node p' in tree T_p, and a regular or a hidden node q' in T_q are found, such that cells p' and q' are in the co-boundary (in the boundary) of cell s, i.e., such that $p', q' \in S$. Nodes p' and q' (if they are not hidden in T_p and T_q,

respectively), or the hyper-nodes containing them, are connected through an arc labeled s, if s is a cell in the initial Morse complex, or through a hyper-arc encoding augmented tree T_s otherwise. The $Type$ label of tree T_q depends on the $Type$ labels of trees T_p and T_q, and of simplification $sim(p, s, q)$. We describe in more detail how the nodes p' and q', and the label $Type(T_q)$, are determined.

- If $Type(T_p) = Type(T_q) = void$, as it happens at the beginning of the construction of the augmented cancellation forest, then nodes p and q are the only (regular) nodes of trees T_p and T_q, respectively. The two trees are merged into a new tree T_q through an arc labeled s (or a hyper-arc encoding T_s), connecting nodes p and q, as illustrated in Figure 4.19 (b) and (c), and in Figure 4.20 (b). The root of the new tree T_q with two regular nodes p and q is node q, and the label $Type(T_q)$ is set to be equal to the label $Type(sim(p, s, q))$.

- If $Type(T_p) = void$, and $Type(T_q) = Type(sim(p, s, q))$, then a node q' is found in tree T_q, such that cell q' is in S. If q' is a regular node, then trees T_p and T_q are merged into a new tree T_q through an arc labeled s or T_s, connecting nodes p and q', as illustrated in Figure 4.19 (d). If the node q' is hidden, then trees T_p and T_q are merged into a new tree T_q through an arc connecting node p to a hyper-node containing node q'. The root of the new tree T_q remains node q, and the label $Type(T_q)$ remains unchanged.

- If $Type(T_p) = Type(sim(p, s, q))$, and $Type(T_q) = void$, we proceed analogously to the previous case. A node p' is found in tree T_p, such that cell p' is in S. If p' is a regular node, then trees T_p and T_q are merged into a new tree T_q through an arc labeled s or T_s, connecting nodes p' and q, as illustrated in Figure 4.19 (e). If node p' is hidden, then trees T_p and T_q are merged into a new tree T_q through an arc connecting node q to a hyper-node containing node p'. The root of the new tree T_q remains node q, and the label $Type(T_q)$ is set to be equal to the label $Type(sim(p, s, q))$.

- If $Type(T_p) = Type(T_q) = Type(sim(p, s, q))$, then nodes p' and q' are found in trees T_p and T_q, respectively, such that cells p' and q' are in S. If p' and q' are regular nodes, then trees T_p and T_q are merged into

a new tree T_q through an arc labeled s or T_s, connecting nodes p' and q', as illustrated in Figure 4.19 (f). If p' or q' are hidden, then the corresponding hyper-nodes are connected, as explained above. The root of the new tree T_q remains the node q, and the label $Type(T_q)$ remains unchanged.

- If $Type(T_p) \neq Type(sim(p,s,q))$, $Type(T_p) \neq void$, and $Type(T_q) = Type\ (sim(p,s,q))$, then a new hyper-node p_{new}, associated to tree T_p is constructed. A node q' is found in tree T_q such that cell q' is in S. If node q' is regular, then trees $T_{p_{new}}$ (consisting of a single hyper-node p_{new}) and T_q are merged into a new tree T_q through an arc labeled s or T_s, connecting nodes p_{new} and q', as illustrated in Figure 4.20 (c). Otherwise, a hyper-node containing q' is connected to the hyper-node p_{new}. The root of the new tree T_q is node q, and the label $Type(T_q)$ remains equal to the label $Type(sim(p,s,q))$.

- If $Type(T_p) = Type(sim(p,s,q))$, $Type(T_q) \neq Type(sim(p,s,q))$ and $Type\ (T_q) \neq void$, then the tree T_q is deactivated from the forest, by setting its *Active* label to 0, and a new hyper-node q_{new}, associated to tree T_q is constructed. A node p' is found in tree T_p such that cell p' is in S. If node p' is regular, then trees T_p and $T_{q_{new}}$ (consisting of a single hyper-node q_{new}) are merged into a new tree $T_{q_{new}}$ through an arc labeled s or T_s, connecting nodes p' and q_{new}. Otherwise, a hyper-node containing p' is connected to hyper-node q_{new}, as illustrated in Figure 4.20 (d). The root of the new tree $T_{q_{new}}$ is node q_{new}, the label $Active(T_{q_{new}})$ is set to 1, and the label $Type(T_{q_{new}})$ is set to be equal to the label $Type(sim(p,s,q))$.

- If $Type(T_p) \neq void$, $Type(T_p) \neq Type(sim(p,s,q))$ $Type(T_q) \neq void$, and $Type(T_q) \neq Type(sim(p,s,q))$, then the tree T_q is deactivated from the forest by setting the label $Active(T_q)$ to 0. Two new hyper-nodes p_{new} and q_{new}, associated with trees T_p and T_q, respectively, are constructed. The trees $T_{p_{new}}$ and $T_{q_{new}}$ (consisting of a single hyper-node p_{new} and q_{new}, respectively) are merged into a new tree $T_{q_{new}}$ through an arc labeled s or T_s, connecting the hyper-nodes p_{new} and q_{new}. The root of the new tree $T_{q_{new}}$ is the node q_{new}, the label $Active(T_{q_{new}})$ is set to 1, and the label $Type(T_{q_{new}})$ is set to be equal to the label $Type(sim(p,s,q))$.

In Figure 4.20, a sequence of simplifications, consisting of the $remove_{1,2}$

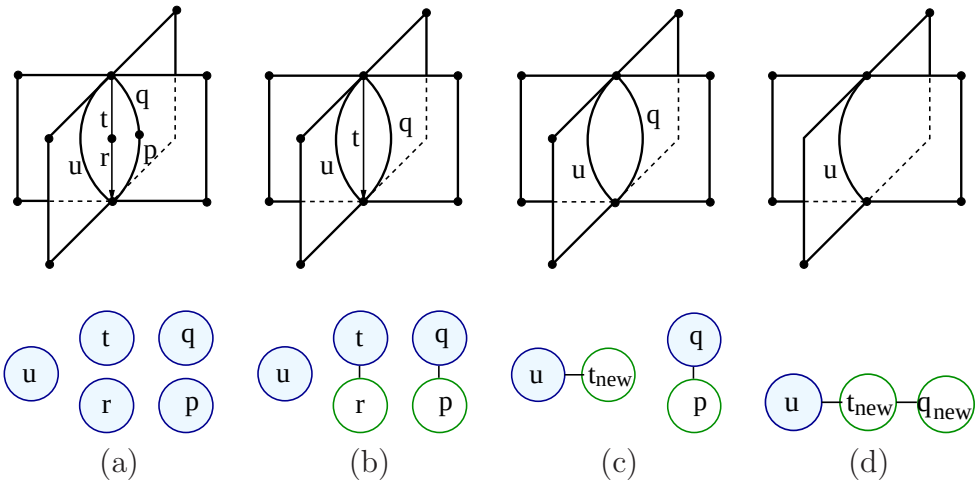


Figure 4.20: [25] The effect on the initial Morse complex and on the corresponding ACF (a) of a sequence of simplifications. It consists of two $remove_{1,2}$ operators, which merge 1-cells p and r into 1-cells q and t, respectively (b), followed by two $remove_{2,1}$ operators. The first merges 1-cell t into 1-cell u (c), and the second merges 1-cell q into 1-cell u (d). The root of each tree is shaded.

merging 1-cell p into 1-cell q, the $remove_{1,2}$ merging 1-cell r into 1-cell t, the $remove_{2,1}$ merging 1-cell t into 1-cell u, and the $remove_{2,1}$ merging 1-cell q into 1-cell u, is illustrated. After the first two simplifications, two removal trees, rooted at q and t are constructed. The third simplification is a $remove_{2,1}$, and T_t is a removal tree. Thus, a new hyper-node t_{new}, associated to tree T_t is created, and it is connected through an arc to node u. The fourth cancellation is a $remove_{2,1}$, and node q is a root of a removal tree. So again, a new hyper-node q_{new} is created, and it is connected through an arc to hyper-node t_{new} of tree T_u.

If a simplification is a $remove_{i.i+1}(p, s, \emptyset)$ or a $remove_{i,i-1}(p, s, \emptyset)$ of the second kind, then trees T_p and T_s are deactivated by setting their *Active* labels to 0. An arc labeled s or T_s, connecting node p to node *empty* is added to the set of arcs of the forest.

If we construct the augmented cancellation forest by applying a sequence of removal and contraction operators, then we can take care of the consistency of the simplified complexes with regard to function values when we construct an augmented cancellation forest. We can assign an f value to each hyper-node

p_{new} representing a tree T_p to be equal to the f value of the root p of T_p, i.e., $f(p_{new}) = f(p)$. Then the root of each removal tree T_p will be the regular node or the hyper-node of T_p with highest f value among all its nodes, and dually for contraction trees. This is important for two reasons. First, in this way, each feasible simplification is determined only by its type (removal or contraction) and its middle-cell s (the cell which is removed or contracted). Second, the inverse refinement operators are also determined only by the middle cell s, and can be implemented as deletion of arcs in the augmented cancellation forest. After the deletion, the root of each of the two new removal (contraction) trees will be its node or hyper-node with highest (lowest) f value.

4.5 Cancellation and *remove* in the 3D Case

The first attempts in the literature to the simplification of topological representations of 3D scalar fields used the cancellation operator and considered the effect of the cancellation on the Morse-Smale complexes of the field [62, 63]. It has been shown that the cancellation of a 1-saddle and a 2-saddle in general increases the size of the Morse-Smale complex, as it increases the number of the cells in the complex. This undesirable property of the cancellation has been the main obstacle to applying the cancellation for the simplification of scalar fields in dimensions higher than two.

In our first work on the simplification of topological representations of 3D scalar fields [22] we have considered the effect of the cancellation operator on the dual ascending and descending Morse complexes, and we have shown that the cancellation on the Morse complexes in general increases the incidence relation on the complexes. We have shown that the effect of the cancellation on the Morse complexes is easier to describe and more intuitive to understand than its effect on the Morse-Smale complexes.

In 3D, there are two types of a cancellation. The first type cancels a maximum and a 2-saddle, and it is the same as a $remove_{2,3}$, or dually it cancels a minimum and a 1-saddle, and it is the same as a $remove_{1,0}$. The second type cancels a 1-saddle p and a 2-saddle q. In general, it does not reduce directly to a *remove*. It increases the incidence relation in the Morse complexes, and it introduces new cells in the Morse-Smale complex.

Here, we will explain first:

1. the effect of the 1-saddle-2-saddle cancellation on the 3D descending

Morse complex (new incidences in the descending Morse complex created by the cancellation),

2. the effect of the 1-saddle-2-saddle cancellation on the 3D Morse-Smale complex (the new cells in the Morse-Smale complex created by the cancellation).

Then, we will show that the macro-operator defined in [62] on Morse-Smale complexes, consisting of a cancellation of a 1-saddle and a 2-saddle followed by cancellations involving extrema (which remove the new cells introduced by the 1-saddle-2-saddle cancellation) can be naturally expressed as a sequence of *remove* operators. This confirms the fact that the *remove* and the *insert* operators form a basis for the set of operators that modify Morse complexes on a manifold M. In [31], we have implemented a macro-operator defined in [62] based on cancellation, and a macro-operator based on *remove*, which eliminates first the minima connected to the 1-saddle, or the maxima connected to the 2-saddle, before performing the simplification (*remove*) of the 1-saddle and the 2-saddle. We have applied the macro-operators on the Morse Incidence Graph (MIG) representing the Morse complexes. In our experiments, we noticed that usually the operators involving the simplification of 1-saddles and 2-saddles are the first operators executed by the simplification algorithm. A high percentage (80% or even higher) of the *remove* operators involving extrema are triggered by the macro-operator involving saddles. This means that when using cancellation, the simplification algorithm introduces first many new arcs in the MIG that will be removed in subsequent steps of the algorithm. Simplification *remove* operator, on the contrary, reduces at each step the size of the MIG.

Finally, we will show how the transition matrix for the change of base can be used to infer some properties of a sequence of *remove* operators and to deduce the minimum number of *remove* operators needed to express any arbitrary operator on the Morse (and the Morse-Smale) complexes on a manifold M.

The material in this section has been presented in [22, 26].

4.5.1 1-saddle-2-saddle Cancellation on Morse and Morse-Smale Complexes

In a descending Morse complex, after the cancellation of the 1-cell p and the 2-cell q in 3D, each 1-cell p_i and each 0-cell r_j that was on the boundary

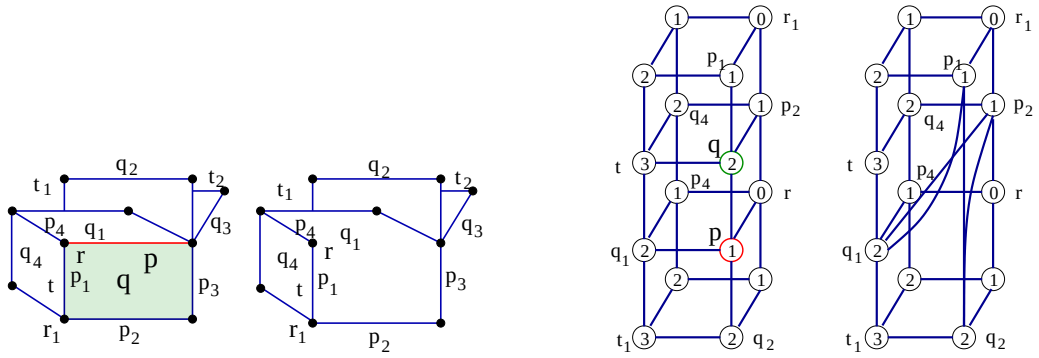


Figure 4.21: [26] The cancellation of a 1-saddle p and a 2-saddle q in 3D on the descending Morse complex (a), and on the Morse-Smale complex (b). Numbers indicate the index of the corresponding critical point.

of the 2-cell q becomes incident to each 2-cell q_k and each 3-cell t_l that was in the co-boundary of the 1-cell p. Thus, all pairs (p_i, q_k), (p_i, t_l), (r_j, q_k) and (r_j, t_l) become part of the incidence relation on the descending (and ascending) Morse complex. Each such pair of cells that become incident to each other in the descending Morse complex corresponds to a new cell of the appropriate dimension in the Morse-Smale complex. In particular, each pair consisting of a 0-cell r_j and a 3-cell t_l corresponds to a new 3-cell in the Morse-Smale complex. For example, a pair consisting of the 0-cell r_1 and the 3-cell t_1 in the descending Morse complex, illustrated in Figure 4.21 (a) right, corresponds to a new 3-cell determined by the 0-cell r_1 and the 0-cell t_1 in the Morse-Smale complex, illustrated in Figure 4.21 (b) right.

In a Morse-Smale complex, all cells that were incident to the 0-cells (vertices) p or q are affected by the cancellation of p and q. Specifically,

1. the 0-cells p and q, and the 1-cell connecting them, are deleted.

2. All the 3-cells incident to both p and q lose some faces on their boundary. For example, the middle 3-cell in Figure 4.21 (b) left, determined by the maximum t and the minimum r, is bounded by the 2-cells q, t, q_4, p_2; q, t, q_1, p; q, p, r, p_2; p_2, r, p_4, q_4; t, q_1, p_4, q_4; p, r, p_4, q_1 before the cancellation. It is transformed into a new 3-cell determined by t and r, whose boundary consists of the 2-cells r, p_2, q_4, p_4; p_2, q_4, t, q_1; q_4, t, q_1, p_4; r, p_2, q_1, p_4. In the descending Morse complex before the cancellation,

- the 1-cell p_1 is the unique 1-cell different from p incident to the 0-cell r and the 2-cell q,
- the 2-cell q_1 is the unique 2-cell different from q incident to the 3-cell t and the 1-cell p,
- the 1-cell p_4 is the unique 1-cell different from p incident to the 0-cell r and the 2-cell q_1, and
- the 2-cell q_4 is the unique 2-cell different from q incident to the 3-cell t and the 1-cell p_1.

3. Other 3-cells in the Morse-Smale complex incident to the 0-cell p are modified. For example, the lower 3-cell in Figure 4.21 (b) left, incident to the 0-cell p, is transformed to the lower 3-cell in Figure 4.21 (b) right, in which the 0-cell p is replaced with the 0-cell p_1. Other 3-cells incident to the 0-cell q are modified in a similar way. For example, the upper 3-cell in Figure 4.21 (b) left, incident to the 0-cell q, is transformed to the upper 3-cell in Figure 4.21 (b) right, in which the 0-cell q is replaced with the 0-cell q_1.

4. For each pair of 0-cells, consisting of

   - a 0-cell corresponding to a minimum, which together with the 0-cell q determines a 2-cell in the Morse-Smale complex before the cancellation, and
   - a 0-cell corresponding to a maximum, which together with the 0-cell p determines a 2-cell in the Morse-Smale complex before the cancellation,

   a new 3-cell in the Morse-Smale complex is created by the cancellation. Such new 3-cell is for example the 3-cell determined by r_1 and t_1, whose boundary consists of the 2-cells r_1, p_2, q_2, p_1; r_1, p_2, q_1, p_1; t_1, q_2, p_2, q_1; t_1, q_2, p_1, q_1.

The new 3-cells introduced in the Morse-Smale complex by the 1-saddle-2-saddle cancellation are undesirable. They are eliminated by applying further cancellations of minima and 1-saddles and of maxima and 2-saddles, as discussed in [62].

A new 3-cell determined by a maximum t_l and a minimum r_j can be eliminated by the cancellation of the maximum t_l and one of the (at most two)

2-saddles connected to both t_l and p before the cancellation of p and q, which are the only 0-cells corresponding to the 2-saddles in any new 3-cell incident to t_l. For example, the new 3-cell in the Morse-Smale complex, determined by the minimum r_1 and the maximum t_1, illustrated in Figure 4.21 (b), can be eliminated by the cancellation of the maximum t_1 and the 2-saddle q_1 or the 2-saddle q_2 (both cancellations are equivalent to a $remove_{2,3}$). A cancellation involving the maximum t_1 and the 2-saddle q_1 is feasible if the 2-saddle q_1 is connected to the maximum t_1, and to exactly one other maximum different from t_1.

In [62], the authors have defined a macro-operator, that we call here a *macro-1-saddle-2-saddle* operator. The effect of this macro-operator is to first introduce new cells in the Morse-Smale complex by applying the cancellation of a 1-saddle p and a 2-saddle q, and then to eliminate these new cells by applying cancellations involving extrema, as illustrated in Figure 4.22.

4.5.2 Macro-1-saddle-2-saddle Operator as a Sequence of *remove* Operators

Macro-1-saddle-2-saddle operator can be expressed as a sequence of *remove* simplification operators in such a way that each simplification in the sequence reduces the number of cells in the Morse-Smale complex. This sequence consists of $remove_{2,3}$ operators, which eliminate all but two 2-cells q and q' in the co-boundary of the 1-cell p, and of the $remove_{1,2}(p, q, q')$.

Recall that a cancellation only changes the connectivity of the cells incident in the cancelled 0-cells in the Morse-Smale complex, and the incidence relation of the cells on the immediate boundary and co-boundary of the cancelled cells in the Morse complexes. In particular, if a 0-cell q_k corresponding to a 2-saddle in the Morse-Smale complex (e.g. the 0-cells q_1 and q_2) was connected through a 1-cell to the 0-cell p before the cancellation of q and p, then only the 1-cells connecting q_k to 0-cells corresponding to 1-saddles are affected by the cancellation. The 1-cells connecting q_k to 0-cells corresponding to maxima are unchanged by the cancellation of q and p.

We give here one possible sequence of $remove_{i,i+1}$ operators through which the macro-1-saddle-2-saddle operator can be expressed. Another possible sequence would only consist of $remove_{i,i-1}$ operators. The problem here is that the 1-cell p is incident to k 2-cells, where $k > 2$, and the 2-cell q is incident to m 1-cells, where $m > 2$. To be able to apply a $remove_{1,2}$ operator, which will

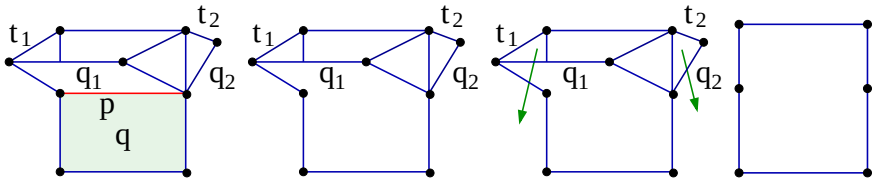


Figure 4.22: [26] A sequence consisting of the cancellation of the 1-saddle p and the 2-saddle q, followed by the $remove_{2,3}$ operators, which eliminate the 2-saddles and the 3-saddles connected to p, on a 3D descending Morse complex.

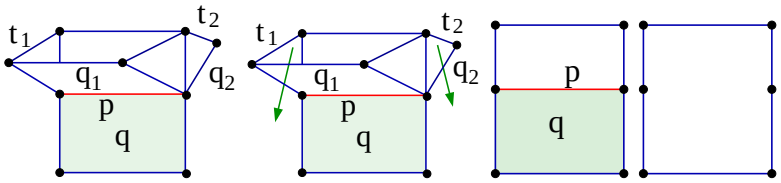


Figure 4.23: [26] A sequence consisting of the $remove_{2,3}$ operators, which eliminate the 2-saddles and the 3-saddles connected to p, followed by the $remove_{1,2}$ that eliminates the 1-saddle p and the 2-saddle q on a 3D descending Morse complex.

eliminate the 1-cell p and the 2-cell q, we need to have only two 2-cells q and q' incident to the 1-cell p. We can reduce the complex Γ_d to this situation by applying $k-1$ $remove_{2,3}$ operators, until all the 2-cells incident to the 1-cell p, with the exception of the 2-cell q and one other 2-cell q' are eliminated. Now, we can apply the $remove_{1,2}(q, p, q')$, which eliminates the 1-cell p and the 2-cell q, as shown in Figure 4.23. Such sequence of *remove* operators consists of the same number of operators as the macro-operator consisting of a sequence of cancellations (macro-1-saddle-2-saddle operator), and it maintains simpler Morse and Morse-Smale complexes at each step.

4.5.3 Transition Matrix in 3D

We show how we can infer the number and dimensions of cells introduced or removed by an operator u, given the coordinates of u in basis B, and vice versa, how we can infer the minimum number of our atomic operators needed to express an arbitrary operator u, given the number and dimensions of cells removed or introduced by u.

In 3D, there are three 4-tuples that form a basis of the set of 4-tuples

that correspond to topologically consistent operators for modifying Morse complexes. These three 4-tuples b_0, b_1 and b_2 have coordinates $(1,1,0,0)_{\mathcal{B}}$, $(0,1,1,0)_{\mathcal{B}}$ and $(0,0,1,1)_{\mathcal{B}}$, respectively, in the standard basis $\mathcal{B}$. Operator b_0 corresponds to the $remove_{1,0}$, operator b_1 corresponds to the $remove_{1,2}$, and to the $remove_{2,1}$, and operator b_2 corresponds to the $remove_{2,3}$. The vector $b_3 = (1,-1,1,-1)_{\mathcal{B}}$, together with vectors b_0, b_1 and b_2, completes the basis of the four-dimensional module over $\mathbb{Z}$.

The transition matrix T for the change of basis of $\mathbb{Z}^4$, from basis $B = \{b_0, b_1, b_2, b_3\}$ to the standard basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$, where $e_1 = (1,0,0,0)_{\mathcal{B}}$, $e_2 = (0,1,0,0)_{\mathcal{B}}$, $e_3 = (0,0,1,0)_{\mathcal{B}}$, $e_4 = (0,0,0,1)_{\mathcal{B}}$, is

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Using this matrix, it is possible to infer the characteristic of a macro-operator expressed as a sequence of *remove* operators. For example, a sequence consisting of one $remove_{1,0}$ and two $remove_{2,3}$ is a macro-operator u with coordinates $u_B = (1,0,2,0)_B$ in basis B. After the multiplication of T with u'_B (u' is the transpose of u), we get the coordinates of u in standard basis, $u_{\mathcal{B}} = Tu'_B = (a_0, a_1, a_2, a_3)_{\mathcal{B}} = (1,1,2,2)_{\mathcal{B}}$. This means that the operator u decreases the number of 0-cells and 1-cells by 1, and it decreases the number of 2-cells and 3-cells by 2.

The minimum number of basis operators needed to express an arbitrary operator $u \in U$ can be computed using the inverse matrix

$$T^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 & 1 \\ -2 & 2 & 2 & -2 \\ 1 & -1 & 1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

For example, an operator u, which decreases the number of 0-cells by 1, and increases the number of 2-cells by 1, has coordinates $u_{\mathcal{B}} = (a_0, a_1, a_2, a_3)_{\mathcal{B}}$ $= (1,0,-1,0)_{\mathcal{B}}$ in the standard basis $\mathcal{B}$. After multiplication by T^{-1}, we get $u_B = T^{-1}u'_{\mathcal{B}} = (1,-1,0,0)_B$. This means that u can be expressed as at least one $remove_{1,0}$, and at least one $insert_{1,2}$ (or one $insert_{2,1}$). The same minimal number of basis operators needed to express u could be obtained from

$$u = \mu_0 b_0 + \mu_1 b_1 + \mu_2 b_2 \text{ as}$$

$$\mu_0 = a_0 = 1,\ \mu_1 = a_1 - a_0 = -1,\ \mu_2 = a_2 - a_1 + a_0 = a_3 = 0.$$

4.6 Summary

In [24, 27], we have defined the simplification *removal* and *contraction* operators on the Morse functions and on the corresponding dual descending and ascending Morse complexes. In [26], we have defined their inverse refinement *insertion* and *expansion* operators, and we have shown that the defined simplification and refinement operators form a basis for the set of operators on Morse complexes on a manifold M. In [32], we have generalized these operators; the new generalized simplification operator is called *remove*, and its inverse refinement operator is called *insert*. Here, we have shown that the *remove* operator (together with the inverse *insert*) forms a basis for the set of operators that modify Morse complexes on a manifold M. As opposed to the simplification cancellation operator defined in Morse theory, the simplification *remove* operator has another important property: it does not increase the incidence relation on the Morse complexes, and it does not introduce new cells in the Morse-Smale complex. This property makes the *remove* operator viable for applications. In [25], we have presented a tree-based data structure for encoding a sequence of simplifications on Morse complexes.

In [22], we have compared the effect of the cancellation operator on 3D Morse and the Morse-Smale complexes of scalar fields. We have shown how a macro-operator, consisting of the cancellation of a 1-saddle and a 2-saddle followed by cancellations involving extrema which eliminate the spurious cells from the Morse-Smale complex introduced by the 1-saddle-2-saddle cancellation, can be expressed as a sequence of the *remove* operators. This work has been presented in [26].

Our more recent work, done in collaboration with Federico Iuricich, is on the experimental evaluation and comparison of the cancellation and the *remove* operators [35]. In [35], a simplification algorithm based on the cancellation operator and on the *remove* operator has been designed and implemented. The algorithm iteratively applies a sequence of simplification algorithms on an initial Morse-Smale complex at full resolution. It has been shown that, with the same number of the applied simplification operators, the number of 1-cells in the simplified Morse-Smale complex obtained through the cancellation operator always exceeds the number of 1-cells in the simplified Morse-Smale

complex obtained through the *remove* operator. The large number of 1-cells in the Morse-Smale complex influences not only the storage cost of the data structure for encoding it and the time required for performing the simplifications, but it also reduces the number of feasible simplification and thus the flexibility of the simplification algorithm.

Our next step has been to design and implement a multi-resolution model for the topology of the scalar fields and the corresponding Morse and Morse-Smale complexes [32]. We will describe this work in Section 5.

5

Multi-Resolution Models

In [29], we have defined a set of operators for updating cell complexes. We classify these operators as homology-preserving and homology-modifying ones. Homology-preserving operators are a generalization of the simplification *remove* operator and its inverse refinement *insert* operator defined on Morse complexes to simplification and refinement operators defined on arbitrary cell complexes.

The set of homology-preserving and homology-modifying operators forms a basis for the set of operators that modify cell complexes while maintaining the Euler-Poincaré formula. As a consequence, any operator that modifies cell complexes can be expressed as a suitable sequence of the operators we have defined. In [29], we have found such sequence for some widely used operators on cell complexes defined in the framework of geometric modeling, such as removal and contraction operators on nG-maps, Euler operators and handle operators.

In [36], we have defined a multi-resolution model for cell complexes in arbitrary dimensions based on the simplification and the refinement operators. We have implemented the multi-resolution model based on homology-preserving operators in collaboration with Federico Iuricich. The multi-resolution model based on homology-preserving operators is a generalization of the multi-resolution model for Morse complexes that we have defined and implemented in [32]. Here, we compare its 2D instance with the approaches proposed in the literature to the multi-resolution representation of 2D Morse or Morse-Smale complexes.

5.1 Topological Operators on Cell Complexes

There have been many proposals in the literature for manipulation operators on 2D and 3D cell complexes (see Subsection 3.4). In Subsection 5.1.1, we describe a minimal set of Euler operators on cell complexes in arbitrary dimensions that we have introduced in [29]. These operators subsume all the other Euler operators proposed in the literature. In Subsection 5.1.2, we show that these operators form a minimally complete basis for the set of operators that modify cell complexes in a topologically consistent manner. In Subsection 5.1.3, we show how some of the widely used update operators proposed in the literature can be expressed as suitable combinations of the ones presented here.

5.1.1 Topological Operators

We have defined both the simplification and the refinement topological operators on cell complexes. The topological operators can be classified as:

- homology-preserving operators:
  - simplification $KiC(i+1)C$ (*Kill i-Cell and (i+1)-Cell*) operators, which delete an i-cell and an $(i+1)$-cell, and
  - refinement $MiC(i+1)C$ (*Make i-Cell and (i+1)-Cell*) operators, which create an i-cell and an $(i+1)$-cell;
- homology-modifying operators:
  - simplification $KiCiCycle$ (*Kill i-Cell and i-Cycle*) operators, which delete an i-cell and an i-cycle, and
  - refinement $MiCiCycle$ (*Make i-Cell and i-Cycle*) operators, which create an i-cell and an i-cycle.

Homology-Preserving Simplification Operators

The homology-preserving simplification operator $KiC(i+1)C$ (*Kill i-Cell and (i+1)-Cell*) deletes an i-cell and an $(i+1)$-cell from a cell complex Γ, and it changes the immediate boundary and co-boundary relation of some cells in Γ. Specifically, the immediate co-boundary relation is changed for the cells in the immediate boundary of the deleted cells, and the immediate boundary relation

is changed for the cells in the immediate co-boundary of the deleted cells. The Euler characteristic and the Betti numbers of the complex remain unchanged. We have defined the $KiC(i+1)C$ operator so that it has the same effect on a cell complex Γ as the *remove* operator on a descending Morse complex Γ_d.

There are two types of the $KiC(i+1)C$ operator. The first type is feasible in the following two cases:

1. the $(i+1)$-cell q to be deleted is bounded by exactly two i-cells (the i-cell p to be deleted and the i-cell p' which will remain) and the i-cell p appears exactly once in the immediate boundary of the $(i+1)$-cell q (i.e., the multiplicity $mult(p, \partial q)$ of the i-cell p in the immediate boundary of the $(i+1)$-cell q is equal to 1);

2. the i-cell q to be deleted bounds exactly two $(i+1)$-cells (the $(i+1)$-cell p to be deleted and the $(i+1)$-cell p' which will remain) and the i-cell q appears exactly once in the immediate boundary of the $(i+1)$-cell p (i.e., the multiplicity $mult(q, \partial p)$ of the i-cell q in the immediate boundary of the $(i+1)$-cell p is equal to 1).

The first instance is denoted as $KiC(i+1)C_{co}(q, p, p')$ (contract). We denote as R, Z and S, respectively, the sets of cells in the immediate co-boundary of the i-cell p, in the immediate co-boundary of the $(i+1)$-cell q and in the immediate boundary of the i-cell p. We denote as $C_{q,p}$ the set $\{p'\} \cup R \cup Z \cup S$. The operator transforms a cell complex Γ in the simplified complex Γ' by replacing the set $\{q, p\} \cup C_{q,p}$ of cells with the set $C_{q,p}$ so that:

1. each instance of the $(i+1)$-cell q is deleted from the immediate boundary of each $(i+2)$-cell $z \in Z$, and from the immediate co-boundary of the i-cell p',

2. each instance of the i-cell p is deleted from the immediate co-boundary of each $(i-1)$-cell $s \in S$,

3. each instance of the i-cell p is replaced with k instances of the i-cell p' in the immediate boundary of each $(i+1)$-cell $r \in R$, where k is the multiplicity of the i-cell p' in the immediate boundary of the $(i+1)$-cell q ($k = mult(p', \partial q)$). In other words, the multiplicity $mult'$ in the simplified complex Γ' of the i-cell p' in the immediate boundary of each $(i+1)$-cell $r \in R$ is increased by the product of the multiplicity of

the i-cell p in the immediate boundary of the $(i+1)$-cell r and the multiplicity of the i-cell p' in the immediate boundary of the $(i+1)$-cell q ($mult'(p', \partial r) = mult(p', \partial r) + mult(p, \partial r) \cdot mult(p', \partial q)$).

If the i-cell p appears $mult(p, \partial r)$ times in the immediate boundary of the $(i+1)$-cell r, then $mult(p, \partial r)$ disjoint copies of the $(i+1)$-cell q are merged into each $(i+1)$-cell r.

The second instance, denoted as $KiC(i+1)C_{re}(q, p, p')$ (remove), is dual. We denote as R, Z and S, respectively, the sets of cells in the immediate boundary of the $(i+1)$-cell p, in the immediate boundary of the i-cell q and in the immediate co-boundary of the $(i+1)$-cell p. We denote as $C_{q,p}$ the set $\{p'\} \cup R \cup Z \cup S$. The operator transforms a cell complex Γ in the simplified complex Γ' by replacing the set $\{q, p\} \cup C_{q,p}$ of cells with the set $C_{q,p}$ so that:

1. each instance of the i-cell q is deleted from the immediate co-boundary of each $(i-1)$-cell $z \in Z$ and from the immediate boundary of the $(i+1)$-cell p',

2. each instance of the $(i+1)$-cell p is deleted from the immediate boundary of each $(i+2)$-cell $s \in S$,

3. each instance of the $(i+1)$-cell p is replaced with $mult(q, \partial p')$ instances of the $(i+1)$-cell p' in the immediate co-boundary of each i-cell r. In other words, the multiplicity $mult'$ in the simplified complex Γ' of each i-cell $r \in R$ in the immediate boundary of the $(i+1)$-cell p' is increased by the product of the multiplicity of the i-cell r in the immediate boundary of the $(i+1)$-cell p and the multiplicity of the i-cell q in the immediate boundary of the $(i+1)$-cell p' ($mult'(r, \partial p') = mult(r, \partial p') + mult(r, \partial p) \cdot mult(q, \partial p')$).

Figure 5.1 shows the effect of the homology-preserving $K1C2C_{re}(q, p, p')$ operator on a cell 2-complex.

The second type of the $KiC(i+1)C$ operator is feasible in the following two cases:

1. the $(i+1)$-cell q to be deleted is bounded only by the i-cell p which will be deleted as well, and the multiplicity of the i-cell p in the immediate boundary of the $(i+1)$-cell q is equal to 1 ($mult(p, \partial q) = 1$);

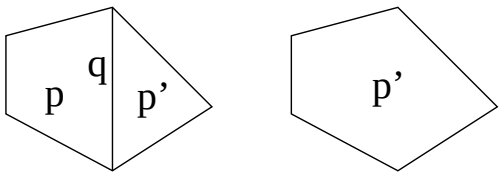


Figure 5.1: [21] The $K1C2C_{re}(q, p, p')$ operator on a 2D cell complex.

2. the i-cell q to be deleted bounds only the $(i + 1)$-cell p which will be deleted as well, and the multiplicity of the i-cell q in the immediate boundary of the $(i + 1)$-cell p is equal to 1 ($mult(q, \partial p) = 1$).

The operator is denoted as $KiC(i+1)C_{co}(q, p)$ and $KiC(i+1)C_{re}(q, p)$, respectively. The sets R, Z and S are defined in the same way as for the corresponding operators of the first type. The set $C_{q,p}$ is equal to $R \cup Z \cup S$. The effect of the operator is to delete both cells from the complex, and to delete each instance of the cells q and p from the immediate boundary and the immediate co-boundary of each cell in $C_{q,p}$. The $KiC(i + 1)C_{co}(q, p)$ merges $mult(p, \partial r)$ disjoint copies of the $(i + 1)$-cell q and i-cell p into each $(i + 1)$-cell $r \in R$. The $KiC(i + 1)C_{re}(q, p)$ merges $mult(p, \partial s)$ disjoint copies of the $(i + 1)$-cell q and i-cell p into each $(i + 2)$-cell $s \in S$.

Homology-Preserving Refinement Operators

The homology-preserving refinement operator $MiC(i + 1)C$ (*Make i-Cell and (i+1)-Cell*) creates an i-cell and an $(i+1)$-cell, thus transforming a cell complex Γ in the refined complex Γ'. It changes the immediate boundary relation for the cells in the immediate co-boundary of the created cells and the immediate co-boundary relation for the cells in the immediate boundary of the created cells. The Euler characteristic and the Betti numbers of the complex remain unchanged. We have defined the $MiC(i + 1)C$ operator as an undo of the corresponding $KiC(i+1)C$ operator on the cell complexes, in the same way as the *insert* operator has been defined as an undo of the corresponding *remove* operator on the Morse complexes.

The homology-preserving refinement operator $MiC(i + 1)C$ can create the two new cells from an existing i- or $(i + 1)$-cell, or insert the two new cells in the complex. It is feasible on the complex Γ if all the cells in the immediate boundary and in the immediate co-boundary of the two created cells are present

in Γ, and it is specified by these cells. There are two types of homology-preserving refinement operators, where the operators of the first type create the two new cells by splitting an existing cell.

The first type of $MiC(i+1)C$ operator either splits the existing $(i+1)$-cell into two by splitting its boundary, thus creating the i-cell separating the two $(i+1)$-cells, or dually, it splits the existing i-cell in two by splitting its co-boundary, thus creating the $(i+1)$-cell bounded by the two i-cells.

The first type of $MiC(i+1)C$ operator has two instances, as follows:

1. the first instance of the operator, that we denote as $MiC(i{+}1)C_{ex}(q,p,p')$ (expand), splits the existing i-cell p' into two i-cells p and p' by splitting the immediate co-boundary of p', and it creates the $(i{+}1)$-cell q bounded by the two i-cells p and p';

2. the second instance of the operator, that we denote as $MiC(i{+}1)C_{in}(q,p,p')$ (insert), splits the existing $(i+1)$-cell p' into two $(i+1)$-cells p and p' by splitting the immediate boundary of p', and it creates the i-cell q separating the two $(i+1)$-cells p and p'.

In both cases, the created i-cell appears exactly once in the immediate boundary of the created $(i+1)$-cell.

The $MiC(i+1)C_{ex}(q,p,p')$ operator is specified by

1. the i-cell p' that is the only i-cell (with the exception of the i-cell p) that will be in the immediate boundary of the $(i+1)$-cell q, and the multiplicity $mult'(p', \partial q)$ of the i-cell p' in the immediate boundary of the $(i+1)$-cell q in the refined complex Γ';

2. the set Z of the $(i+2)$-cells that will be in the immediate co-boundary of the $(i+1)$-cell q, and the multiplicity $mult'(q, \partial z)$ of the $(i+1)$-cell q in the immediate boundary of each $(i+2)$-cell $z \in Z$ in the refined complex Γ';

3. the set S of the $(i{-}1)$-cells and the set R of the $(i{+}1)$-cells that will be in the immediate boundary and the immediate co-boundary, respectively, of the i-cell p, together with the multiplicities $mult'(s, \partial p)$ and $mult'(p, \partial r)$ of each boundary (and co-boundary) relation.

We denote as $C_{q,p}$ the set $\{p'\} \cup Z \cup R \cup S$.

The $MiC(i+1)C_{ex}(q,p,p')$ operator is feasible on the cell complex Γ if all the specified cells are in Γ, and the multiplicity $mult(p',\partial r)$ of the i-cell p' in the immediate boundary of each $(i+1)$-cell r in R in Γ is greater than or equal to the product of the multiplicity $mult'(q,\partial p')$ of the i-cell q in the immediate boundary of the $(i+1)$-cell p' and the multiplicity $mult'(p,\partial r)$ of the i-cell p in the immediate boundary of the same $(i+1)$-cell r in Γ' ($mult(p',\partial r) \geq mult'(p',\partial q) \cdot mult'(p,\partial r)$).

The effect of the $MiC(i+1)C_{ex}(q,p,p')$ operator is to insert the i-cell q and the $(i+1)$-cell p in Γ, by replacing the set $C_{q,p}$ of cells with the set $\{q,p\} \cup C_{q,p}$, and to adjust the immediate boundary relation of the cells q, p and the cells in $C_{q,p}$. For each $(i+1)$-cell r in R (in the immediate boundary of the $(i+1)$-cell r), the multiplicity of the i-cell p' in the immediate boundary of the $(i+1)$-cell r is diminished by the product of the multiplicity of the i-cell p in the immediate boundary of the $(i+1)$-cell r and the multiplicity of the i-cell p' in the immediate boundary of the $(i+1)$-cell q ($mult'(p',\partial r) = mult(p',\partial r) - mult'(p',\partial q) \cdot mult'(p,\partial r)$). Each $(i+1)$-cell $r \in R$ in the refined complex Γ' is equal to the same $(i+1)$-cell r in Γ minus $mult'(p,\partial r)$ disjoint copies of the $(i+1)$-cell q and the i-cell p.

The second instance $MiC(i+1)C_{in}(q,p,p')$ has a dual effect.

For a 2-complex Γ embedded in $\mathbb{R}^3$, the homology-preserving operators are also called: MVE (*Make Vertex and Edge*) and MEF (*Make Edge and Face*), which correspond to $M0C1C$ and $M1C2C$, respectively. These operators are illustrated in Figures 3.2 and 3.3, respectively. For a 3-complex Γ embedded in $\mathbb{R}^3$, there is an additional homology-preserving operator, $MFVl$ (*Make Face and Volume (3-Cell)*) which creates a new face (2-cell) and a new three-dimensional (volumetric) cell. It is the same as the $M2C3C$ operator.

The second type of the $MiC(i+1)C$ operator either creates an i-cell and an $(i+1)$-cell bounded only by the i-cell, or dually, it creates an $(i+1)$-cell and an i-cell bounding only the $(i+1)$-cell. The created i-cell appears exactly once in the boundary of the created $(i+1)$-cell. We will denote the first instance of the operator as $MiC(i+1)C_{ex}(q,p)$ and the second one as $MiC(i+1)C_{in}(q,p)$. The operator is specified by the set $C_{q,p}$ of cells in the immediate boundary and co-boundary of the created cells, and is feasible on a cell complex Γ if all the cells in $C_{q,p}$ are in Γ. Its effect is to update the immediate boundary and co-boundary relation of the cells in $\{q,p\} \cup C_{q,p}$. After the $MiC(i+1)C_{ex}(q,p)$, each $(i+1)$-cell $r \in R$ in Γ' is equal to the same $(i+1)$-cell r in Γ minus $mult'(p,\partial r)$ disjoint copies of the $(i+1)$-cell q and the i-cell p. After the

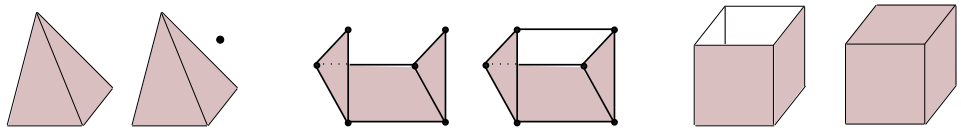
$MiC(i+1)C_{in}(q,p)$, each $(i+2)$-cell $s \in S$ in Γ' is equal to the same $(i+2)$-cell s in Γ minus $mult'(p, \partial s)$ disjoint copies of the $(i+1)$-cell p and the i-cell q.

Homology-Modifying Operators

Homology-modifying operators change both the number of cells in the complex Γ and its Betti numbers, and thus the Euler characteristic of Γ.

The homology-modifying refinement operator $MiCiCycle$ (*Make i-Cell and i-Cycle*) increases the number n_i of i-cells and the number β_i of non-bounding i-cycles by one. It is specified by the set $C_p = R$ of the $(i-1)$-cells in the immediate boundary of the created i-cell p (together with the corresponding multiplicities $mult'(r, \partial p)$, $r \in R$). It is feasible on a cell complex Γ if all the cells in the set C_p (in the immediate boundary of the created cell) are in Γ. Its effect is to introduce the created cell in the immediate co-boundary of each cell in the set $C_p = R$ with the appropriate multiplicity. The co-boundary of the created i-cell p is empty.

The inverse homology-modifying simplification operator $KiCiCycle$ (*Kill i-Cell and i-Cycle*) deletes an i-cell and destroys an i-cycle, thus decreasing the numbers n_i and β_i by one. It is feasible on a cell complex Γ if the immediate co-boundary of the cell to be deleted is empty. It changes the immediate co-boundary relation of each $(i-1)$-cell $r \in R = C_p$ in the immediate boundary of the deleted i-cell. Its effect is to remove all instances of the deleted cell from the immediate co-boundary of each $(i-1)$-cell $r \in R$.



(a) (b) (c)

Figure 5.2: [29] Homology-modifying operators on a portion of the 2-complex in $\mathbb{E}^3$: $MV0Cycle$ (*Make Vertex and 0-Cycle*) (a); $ME1Cycle$ (*Make Edge and 1-Cycle*) (b); $MF2Cycle$ (*Make Face and 2-Cycle*) (c).

For a 2-complex Γ embedded in $\mathbb{R}^3$, the homology-modifying operators $M0C0Cycle$, $M1C1Cycle$ and $M2C2Cycle$ (illustrated in Figure 5.2) are also called $MV0Cycle$ (*Make Vertex and 0-Cycle*), $ME1Cycle$ (*Make Edge and 1-Cycle*) and $MF2Cycle$ (*Make Face and 2-Cycle*), respectively.

The operator $MV0Cycle$ creates a new vertex and a new connected component, it increases by one the number of vertices (0-cells) and the zeroth Betti number β_0. It is also an initialization operator, which creates a new complex Γ. The operator $ME1Cycle$ creates a new edge and a new 1-cycle, thus increasing by one the number of edges (1-cells) and the first Betti number β_1. The operator $MF2Cycle$ creates a new face and a new 2-cycle, thus increasing by one the number of faces (2-cells) and the second Betti number β_2.

For a 3-complex Γ embedded in $\mathbb{R}^3$, the homology-modifying operators will be the same as for 2-complexes, since in this case the third Betti number β_3 is null.

We note here that the Betti numbers of a cell complex are determined by the immediate boundary relation ∂_i on the complex, which relates i-cells with $(i-1)$-cells. The creation of an i-cell affects relations ∂_{i+1} and ∂_i, and thus it either increases β_i by one, or it decreases β_{i-1} by one, to maintain the validity of the Euler-Poincaré formula. In the first case, we obtain our operators $MiCiCycle$ (Make i-Cell and i-Cycle). The operators $MiCK(i-1)Cycle$ (Make i-Cell, Kill $(i-1)$-Cycle), $i \geq 2$, obtained in the second case, can be expressed through the proposed ones as: $K(i-1)C(i-1)Cycle$ (Kill $(i-1)$-Cell and $(i-1)$-Cycle) applied on one $(i-1)$-cell in the boundary of the i-cell to be created (possibly preceded by some $K(i-1)CiC$, which delete all the i-cells in the co-boundary of the $(i-1)$-cell), followed by $M(i-1)CiC$ (Make $(i-1)$-Cell and i-Cell), which re-creates the deleted $(i-1)$-cell and creates the i-cell (followed by $M(i-1)CiC$, inverse to $K(i-1)CiC$, which restore the deleted cells).

$M1CK0Cycle$ (Make 1-cell, Kill 0-Cycle) can be obtained by destroying one of the merged 0-cycles (components), applying one $M0C1C$, and re-creating the destroyed component.

5.1.2 Minimality and Completeness

The topological operators described in Section 5.1 form a set of basis operators for creating and updating cell complexes of dimension at most d. This can be shown by interpreting these operators as ordered $(2d+2)$-tuples $(c_0, c_1, .., c_d, x_0, x_1, .., x_d)$ in an integer grid, belonging to the hyperplane Π: $\sum_{i=0}^{d}(-1)^i c_i = \sum_{i=0}^{d}(-1)^i x_i$ defined by the Euler-Poincaré formula. The first $d+1$ coordinates

denote the number of i-cells created or deleted by the operator, depending on the sign of the coordinate, and the last $d+1$ coordinates denote the change in the Betti numbers of the complex induced by the operator. The operator $MiC(i+1)C$, $0 \leq i \leq d-1$, has coordinates $c_i = c_{i+1} = 1$, $c_j = 0$, $j \in \{0,1,...,d\}\backslash\{i, i+1\}$, $x_j = 0$, $j \in \{0,1,..,d\}$. The operator $MiCiCycle$, $0 \leq i \leq d$, has coordinates $c_i = x_i = 1$, $c_j = x_j = 0$, $j \in \{0,1,...,d\}\backslash\{i\}$.

We will show that

(i) the $2d+1$ $(2d+2)$-tuples corresponding to our operators are linearly independent, and

(ii) any $(2d+2)$-tuple in the hyperplane Π can be expressed as a linear combination of $2d+1$ $(2d+2)$-tuples corresponding to our operators,

which will imply the claim.

A linear combination $\sum_{i=0}^{d-1} \mu_i MiC(i+1)C + \sum_{i=0}^{d} \nu_i MiCiCycle$ vanishes if and only if $(\mu_0, \mu_0, 0, .., 0) + (0, \mu_1, \mu_1, .., 0) + .. + (0, ..0, \mu_{d-1}, \mu_{d-1}, 0, .., 0) + (\nu_0, 0, .., 0, \nu_0, 0, .., 0) + (0, \nu_1, 0, .., 0, \nu_1, 0, .., 0) + .. + (0, .., 0, \nu_d, 0, .., 0, \nu_d) = 0$, which is equivalent to $(\mu_0 + \nu_0, \mu_0 + \mu_1 + \nu_1, \mu_1 + \mu_2 + \nu_2, .., \mu_{d-2} + \mu_{d-1} + \nu_{d-1}, \mu_{d-1} + \nu_d, \nu_0, \nu_1, .., \nu_d) = 0$. It follows that $\mu_i = 0$, $0 \leq i \leq d-1$, $\nu_i = 0$, $0 \leq i \leq d$, implying that the tuples corresponding to our operators are linearly independent.

A tuple $(a_0, a_1, .., a_d, b_0, b_1, .., b_d)$ in the hyperplane Π can be expressed through the $2d+1$ independent $(2d+2)$-tuples corresponding to our operators as $\sum_{i=0}^{d-1} \mu_i MiC(i+1)C + \sum_{i=0}^{d} \nu_i MiCiCycle$ if $(\mu_0 + \nu_0, \mu_0 + \mu_1 + \nu_1, \mu_1 + \mu_2 + \nu_2, .., \mu_{d-2} + \mu_{d-1} + \nu_{d-1}, \mu_{d-1} + \nu_d, \nu_0, \nu_1, .., \nu_d) = (a_0, a_1, .., a_d, b_0, b_1, .., b_d)$. It follows that $\nu_i = b_i$, $0 \leq i \leq d$, and $\mu_0 = a_0 - b_0$, $\mu_1 = a_1 - b_1 - \mu_0 = (a_1 - a_0) - (b_1 - b_0)$, $\mu_2 = a_2 - b_2 - \mu_1 = (a_2 - a_1 + a_0) - (b_2 - b_1 + b_0)$,..,$\mu_{d-1} = (a_{d-1} - a_{d-2} + .. + (-1)^d a_0) - (b_{d-1} - b_{d-2} + .. + (-1)^d b_0) = a_d - b_d$. In short, $\mu_i = \sum_{j=0}^{i} (-1)^{i-j} a_j - \sum_{j=0}^{i} (-1)^{i-j} b_j$, $0 \leq i \leq d-1$, and $\nu_i = b_i$, $0 \leq i \leq d$. Thus, each operator satisfying the Euler-Poincaré formula on a cell complex Γ can be expressed as a linear combination of the topological operators.

In the space (hyperplane) of dimension $(2d+1)$, a generating set consisting of $(2d+1)$ independent tuples forms a basis for the hyperplane.

5.1.3 Comparison with other Operators on Cell Complexes

We compare the operators proposed here with other operators on cell complexes proposed in the literature, in particular with removal and contraction operators defined in the framework of n-maps and nG-maps, and with Euler operators and handle operators, defined in the framework of geometric modeling (see Section 3.4). We show how these operators can be expressed through the topological $MiC(i+1)C$ and $MiCiCycle$ operators. We have presented this work in [29].

Comparison with Removal and Contraction Operators

Removal and contraction operators have been introduced in digital geometry literature as simplification operators on n-G-maps [40]. An i-cell q, $0 \leq i \leq n-1$, can be removed in two cases: if it bounds exactly two different $(i+1)$-cells p and p' and it appears exactly once in the immediate boundary of both p and p'; or if it bounds exactly one $(i+1)$-cell p and it appears exactly twice in the immediate boundary of p. The contraction operator is dual.

The first instance of the removal operator is a special case of the $KiC(i+1)C_{re}(q,p,p')$ operator, as it requires that the i-cell q appears exactly once not only in the immediate boundary of the $(i+1)$-cell p but also in the immediate boundary of the $(i+1)$-cell p'.

The second instance of the removal operator may, but is not guaranteed to, preserve the topological characteristics of the complex (it may produce cells that are not topological cells, or it may disconnect the complex). Thus, it is not an operator on cell complexes and it cannot be classified neither as homology-preserving nor as homology-modifying.

In [39], homology generators of a cell complex are computed using two homology-preserving simplification operators: the removal of a degree-two cell (which is the same as $KiC(i+1)C_{re}(q,p,p')$ and as the first instance of the removal operator in [40]) and the removal of a dangling cell (which is the same as $KiC(i+1)C_{re}(q,p)$). The inverse (refinement) insertion and expansion operators have been introduced in [5]. They are the same as $MiC(i+1)C_{in}(q,p,p')$ and $MiC(i+1)C_{ex}(q,p,p')$, respectively.

Comparison with other Euler Operators

We show that various Euler operators proposed in the literature for 2D and 3D cell complexes are either instances of our operators, or can be expressed through them.

Virtually all the proposed sets of basis Euler operators on 2D and 3D cell complexes contain MEV (Make Edge and Vertex) and MEF (Make Edge and Face) operators, which are the same as our $M0C1C$ (Make 0-cell and 1-cell) and $M1C2C$ (Make 1-cell and 2-cell) homology-preserving operators, respectively.

Several sets of basis operators have been proposed for manifold 2-complexes bounding a solid in $\mathbb{R}^3$ (boundary models) [12, 48, 79, 80].

The *glue* operator in [48] merges two faces and deletes both of them. Two faces may be glued if they have the same number of vertices, and they have no edges in common. The glue operator deletes not only the two faces, but it deletes also all the edges and vertices on the boundary of one of the deleted faces. If the two glued faces belong to two different shells, one shell is deleted (β_0 is decreased by one), and the operator is called KFS (*Kill Face and Shell*). If the two glued faces belong to the same shell, a handle (genus) is created (β_1 is increased by two), and the operator is called $KFMH$ (*Kill Face, Make Hole*).

Let $v_1, e_1, v_2, e_2, .., v_k, e_k$ and $v'_1, e'_1, v'_2, e'_2, .., v'_k, e'_k$ be the cyclical lists of vertices and edges of the two glued faces f and f', listed in the order in which they are identified. Both instances of the glue operator can be expressed through our operators as follows:

- $M1CK0Cycle$ for KFS and $M1C1Cycle$ for $KFMH$ creates an edge connecting v_1 and v'_1,

- $K0C1C$ contracts the edge (v_1, v'_1), and identifies v'_1 with v_1 (vertex v_1 is the current vertex),

- $M1C2C$ makes a triangular face with vertices v_i, v_{i+1}, v'_{i+1} for current vertex v_i, $K0C1C$ and $K1C2C$ identify vertex v'_{i+1} with vertex v_{i+1} and edge e'_{i+1} with edge e_{i+1}, respectively (v_{i+1} is the current vertex),

- $M2C2Cycle$ and $K1C2C$ identify edge e'_k with edge e_k,

- $K2C2Cycle$ deletes face f' and the 2-cycle formed by faces f and f',

- $K2C2Cycle$ for KFS merges the two solids bounded by shells containing f and f' into one, and $K2CM1Cycle$ for $KFMH$ deletes face f and creates a 1-cycle determined by edges e_1,..,e_k.

In [12, 79, 80], the topology-modifying operator is called $MRKF$ (*Make Ring, Kill Face*). It glues the face f' to the face f, and deletes the face f'. $MRKF$ can be expressed through our operators as follows (see Figure 5.3):

- l $M0C1C$ operators, where l is the number of the edges and of the vertices of f',
- l $M1C2C$ operators, which create a copy of f' in f,
- a sequence defining the glue operator in [48],
- $(l-1)$ $K1C2C$ operators (we leave one edge joining a vertex of f to a vertex of f' to maintain the topological validity of the face f).

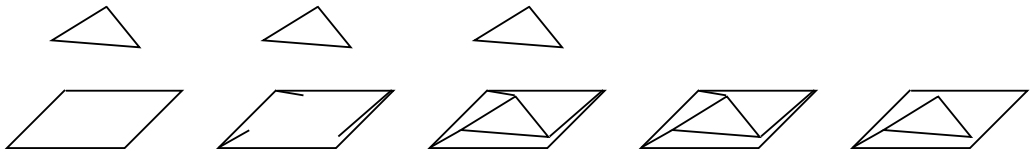


Figure 5.3: [29] $MRKF$ operator expressed through our operators: triangular face to be glued to the quadrangular face (a), three $M0C1C$ operators (b), three $M1C2C$ operators (c), *glue* (d), two $K1C2C$ operators (e).

Homology-modifying operators defined for non-manifold 2-complexes in $\mathbb{R}^3$ [71] are called $MECh$ (*Make Edge and Complex Hole*), $MFKCh$ (*Make Face, Kill Complex Hole*) and $MFCc$ (*Make Face and Complex Cavity*). They are the same as our operators $M1C1Cycle$, $M2CK1Cycle$ (*Make 2-Cell Kill 1-Cycle*) and $M2C2Cycle$, respectively. For 3-complexes in $\mathbb{R}^3$ [82, 83], an additional homology-modifying operator is defined, called $MVlKCc$ (*Make Volume, Kill Complex Cavity*). It is the same as $M3CK2Cycle$ (*Make 3-Cell, Kill 2-Cycle*).

In [58], homology-preserving operators are called *cell subdividers* and homology-modifying ones are called *global hole shapers*.

A cell subdivider subdivides an i-cell by inserting into it an $(i-1)$-cell. This operator is equal to the $M(i-1)CiC$ operator.

A global hole shaper either attaches or detaches a cell, thus creating a hole. There are two instances of this operator: the attached topological i-cell creates an i-hole or the detached topological i-cell creates an $(i-1)$-hole. The first instance of this operator is the same as $MiCiCycle$. The second instance is the same as $KiCM(i-1)Cycle$ (*Kill i-Cell, Make (i-1)-Cycle*).

The inverse homology-modifying operators attach or detach a cell, thus deleting a hole. They are the same as $KiCiCycle$ and $MiCK(i-1)Cycle$ (inverse to $KiCM(i-1)Cycle$), respectively.

Comparison with Handle Operators

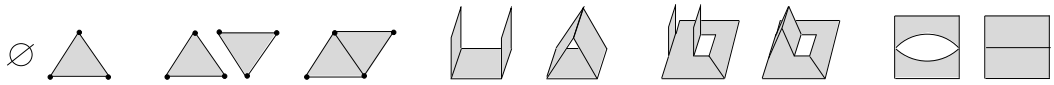


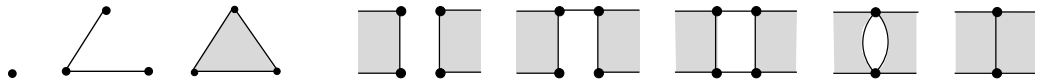
Figure 5.4: [29] Handle operators in 2D: the attachment of a 0-handle (a); the attachment of a 1-hande (b), (c) and (d); the attachment of a 2-handle (e).

Handle operators on a manifold cell 2-complex Γ triangulating a surface S, introduced in [74], can be classified as homology-modifying operators. They are illustrated in Figure 5.4, and can be expressed through our operators as discussed below:

1. The attachment of a 0-handle corresponds to creating an initial triangle (a 2-ball). It can be expressed as $M0C0Cycle$ operator, two $M0C1C$ operators and one $M1C2C$ operator (see Figure 5.5 (a)).

2. The attachment of a 1-handle identifies two boundary edges e_1 and e_2 with no vertices in common. It can be expressed through one $M1C$ $K0Cycle$ and one $M1C1Cycle$ operator if e_1 and e_2 belong to different components, or two $M1C1Cycle$ operators if they belong to the same component (the created edges connect the endpoints of e_1 to the corresponding endpoints of e_2), two $K0C1C$ operators (they contract the two created edges and identify the corresponding endpoints), one $M2CK1Cycle$ operator (it creates a face that fills the ring and deletes the cycle formed by e_1 and e_2), and finally one $K1C2C$ operator (it contracts the created face and identifies e_1 with e_2) (see Figure 5.5 (b)).

3. The attachment of a 2-handle identifies two edges with both vertices in common. It can be expressed as $M2CK1Cycle$, followed by $K1C2C$.



(a) (b)

Figure 5.5: [29] Attachment of a 0-handle in 2D can be expressed as one $M0C0Cycle$, two $M0C1C$ and one $M1C2C$ (a). Attachment of a 1-hande in 2D can be expressed as one $M1CK0Cycle$ or one $M1C1Cycle$, one $M1C1Cycle$, two $K0C1C$, one $M2CK1Cycle$ and one $K1C2C$ (b).

Handle operators have been extended to 3D in [75]. The handle operators in 3D generalize the glue operator in [48], since the two faces identified by a handle operator may have none, some, or all edges in common. They can be expressed in terms of our operators in a similar manner.

5.2 Multi-Resolution Cell Complexes

In this Section, we introduce a hierarchy of cell complexes, that we call a *Multi-Resolution Cell Complex (MRCC)*. We construct the $MRCC$ by iteratively applying simplification operators, and we define it in terms of the inverse refinement operators.

A *Multi-Resolution Cell Complex (MRCC)* is generated from a d-complex Γ at full resolution by iteratively applying simplification $KiC(i+1)C$ and $KiCiCycle$ operators. Each simplification $KiC(i+1)C$ operator deletes two cells p and q from the complex Γ. This affects the immediate boundary or co-boundary relation of the cells in the set $C_{q,p}$ (see Subsection 5.1.1). Each simplification $KiCiCycle$ operator deletes one cell q from the complex Γ. This affects the immediate co-boundary relation of the cells in the set C_q.

Each simplification $KiC(i+1)C$ ($KiCiCycle$) can be seen as a replacement of the set $\{q,p\} \cup C_{q,p}$ ($\{q\} \cup C_q$) of cells with the set $C_{q,p}$ (the set C_q) of cells (and an update of the immediate boundary and co-boundary relation). We denote such simplification as $(\{q,p\} \cup C_{q,p}, C_{q,p})$ $((\{q\} \cup C_q, C_q))$

We can apply first the homology-preserving operators, to obtain a complex Γ' (with fewer cells) having the same homology as the initial complex Γ, and

such that no homology-preserving operator is feasible on Γ'. Next, we apply homology-modifying operators to iteratively remove the cells of Γ', each time affecting the homology of the complex. After each application of a homology-modifying operator, we apply all feasible homology-preserving ones. We repeat this until a complex is obtained that has one i-cell for each $0 \leq i \leq n$. At each step when a homology-modifying operator is applied, we remove a top cell from the complex. We denote the set of the applied simplification operators as $\mathcal{S}$.

The complex obtained as a result of the simplification sequence is the coarsest representation of the cell complex. We denote such coarse complex as Γ_B, and we call it the *base complex*. It is the first component of the $MRCC$.

The second component of the $MRCC$ is the set $\mathcal{M}$ of the *inverse refinements* to the simplifications in $\mathcal{S}$, that have produced Γ_B from Γ.

The third component is the dependency relation between the refinements in $\mathcal{M}$. Let us consider the set $\mathcal{M}$ of all refinements. We consider, for simplicity, the creation of the base complex Γ_B as a dummy refinement that we denote as μ_0 (μ_0 generates Γ_B). We define the dependency relation between the refinements in $\mathcal{M}$ as follows:

- Homology-preserving refinement $\mu = MiC(i+1)C$ that creates the cells p and q and is defined by the cells in the set $C_{q,p}$ that will be in the immediate boundary or co-boundary of either p or q (together with the corresponding multiplicities), *directly depends* on the refinement μ^*, if μ^* creates a cell in the set $C_{q,p}$ (that will be in the immediate boundary or co-boundary of either p or q).

- Homology-modifying refinement $\mu = MiCiCycle$ that creates the i-cell q and is defined by the $(i-1)$-cells in the set C_q that will be in the immediate boundary of q (together with the corresponding multiplicities), *directly depends* on the refinement μ^*, if μ^* creates a cell in the set C_q (that will be in the immediate boundary of the i-cell q).

Thus, the homology-preserving refinement $\mu = MiC(i+1)C(q,p,p')$, creating the cells p and q and defined by the cells in the set $C_{q,p}$, directly depends on the refinement $\mu^* = MiC(i+1)C(q^*,p^*,p'^*)$, if $\{q^*,p^*\} \cap C_{q,p} \neq \emptyset$. It directly depends on the refinement $\mu^* = MiCiCycle(q^*)$ if $\{q^*\} \cap C_{q,p} \neq \emptyset$.

The homology-modifying refinement $\mu = MiCiCycle(q)$, creating the i-cell q and defined by the $(i-1)$-cells in the set C_q, directly depends on the

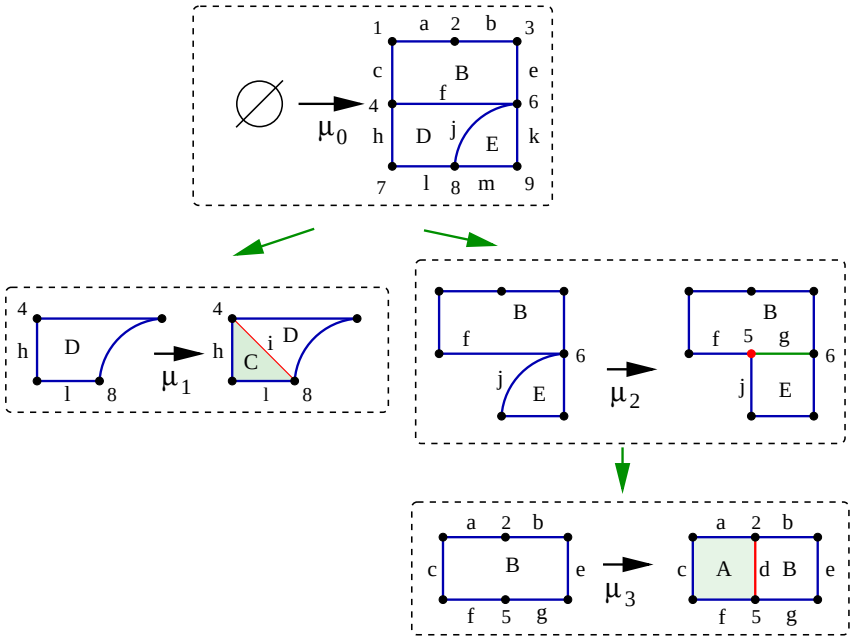


Figure 5.6: [32] An example of the $MRCC$ in 2D.

refinement $\mu^* = MiC(i+1)C(q^*, p^*, p'^*)$, if $\{q^*, p^*\} \cap C_q \neq \emptyset$. It directly depends on a refinement $\mu^* = MiCiCycle(q^*)$ if $\{q^*\} \cap C_q \neq \emptyset$.

The $MRCC$ is defined as the triple $(\Gamma_B, \mathcal{M}, \mathcal{R})$, where $\mathcal{R}$ denotes the direct dependency relation defined above. The dependency relation between refinements is the transitive closure of the direct dependency relation. It is a partial order relation, since a cell is never introduced twice by the modifications in $\mathcal{M}$.

A 2D example of the $MRCC$ is illustrated in Figure 5.6. The refinement μ_0 creates the base complex Γ_B at the coarsest resolution. The refinement μ_1 inserts 1-cell i and 2-cell C in 2-cell D, i.e., $\mu_1 = M1C2C_{in}(C, i, D)$. For μ_1, $p'_1 = D$, $R_1 = \{h, l\}$, $Z_1 = \{4, 8\}$, $S_1 = \emptyset$, and the set $C_{C,i} = \{D, h, l, 4, 8\}$. The refinement μ_2 inserts the 0-cell 5 and the 1-cell g in the 0-cell 6, i.e., $\mu_2 = M0C1C_{ex}(5, g, 6)$. For μ_2, $p'_2 = 6$, $R_2 = \{f, j\}$, $Z_2 = \{B, E\}$, $S_2 = \emptyset$, and the set $C_{g,5} = \{6, f, j, B, E\}$. The refinement μ_3 inserts the 1-cell d and the 2-cell A in the 2-cell B, i.e., $\mu_3 = M1C2C_{in}(A, d, B)$. For μ_3, $p'_3 = B$, $R_3 = \{a, c, f\}$, $Z_3 = \{2, 5\}$, $S_3 = \emptyset$, and the set $C_{d,A} = \{B, a, c, f, 2, 5\}$.

The refinements μ_1 and μ_2 are independent. The refinement μ_3 depends on the refinement μ_2, as $p_2 = 5 \in \{2, 5\} \subseteq C_{p_3,q_3} = C_{A,d}$, and it does not depend

on μ_1.

5.3 Extracting Adaptive Representations from the MRCC

We discuss how to extract a large number of adaptive representations of the topological structure of cell complexes from the $MRCC$, by providing the basic definitions and results and briefly discussing the algorithmic aspects.

Definition 13 *A sequence $U = (\mu_0, \mu_1, \mu_2, ..., \mu_m)$ of refinements in $\mathcal{M}$ is* feasible *if μ_1 is feasible on the base complex Γ_B and each refinement μ_i, $2 \leq i \leq m$ is feasible on the complex Γ_{i-1} obtained from the base complex Γ_B by applying on it the sequence $(\mu_1, .., \mu_{i-1})$.*

Definition 14 *Let $U = (\mu_0, \mu_1, \mu_2, ..., \mu_m)$ be a feasible sequence of refinements in $\mathcal{M}$. The* front complex Γ_U *associated with the sequence U is the complex obtained from the base complex Γ_B by applying on it the sequence of refinements $(\mu_1, \mu_2, ..., \mu_m)$.*

The front complex represents the topological structure of the cell complex Γ at an intermediate level of detail. If the feasible sequence U contains all the refinements in $\mathcal{M}$, then the front complex Γ_U associated with U is the same as the complex Γ at full resolution. Figure 5.7 shows the front complex obtained by applying refinements μ_0, μ_1, μ_2 and μ_3 on the $MRCC$ illustrated in Figure 5.6.

Let Γ_U be the cell complex obtained by applying the sequence U of refinements on the base complex Γ_B. We will show that a refinement μ is feasible on the complex Γ_U if the sequence U contains all the refinements in $\mathcal{M}$ on which μ depends. In more detail: the homology-preserving refinement μ, which creates the cells q and p, is feasible on the cell complex Γ_U if and only if the sequence U contains all the refinements μ_i, $1 \leq i \leq k$, which create cells in the immediate boundary and co-boundary of q and p; the homology-modifying refinement μ, which creates the cell q, is feasible on Γ_U if and only if the sequence U contains all the refinements μ_i, $1 \leq i \leq k$, which create cells in the immediate boundary of q, and it does not contain any refinement ν which creates a cell in the immediate co-boundary of q.

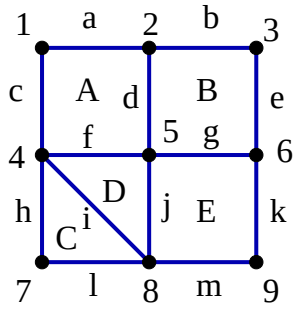


Figure 5.7: [32] The complex at full resolution, obtained from the $MRCC$ in Figure 5.6 by applying the refinements μ_0, μ_1, μ_2 and μ_3.

Proposition 4 *The homology-preserving refinement $\mu = (C_{q,p}, \{q,p\} \cup C_{q,p})$, where $C_{q,p}$ is the set of cells in the immediate boundary or co-boundary of the introduced cells q and p, is feasible on a complex $\Gamma = \Gamma_U$ (at some intermediate resolution) if and only if the sequence U of refinements that produces the complex Γ from the base complex Γ_B contains all refinements μ_i, $1 \leq i \leq k$, such that $\{q_i, p_i\} \cap C_{q,p} \neq \emptyset$ if $\mu_i = (C_{q_i,p_i}, \{q_i, p_i\} \cup C_{q_i,p_i})$ is homology-preserving, i.e., such that $\{q_i\} \cap C_{q,p} \neq \emptyset$ if $\mu_i = (C_{q_i}, \{q_i\} \cup C_{q_i})$ is homology-modifying.*

Proof. Let us denote as $\mathcal{V}$ the set of refinements in $\mathcal{M}$ that create at least one cell in $C_{q,p}$.

We show first that the refinement μ is feasible on the complex $\Gamma = \Gamma_U$ if all refinements in $\mathcal{V}$ are also in U. The only refinements (beside refinement μ) that affect the immediate boundary and co-boundary relation of the cells in $C_{q,p}$ are refinements in $\mathcal{V}$. Since a refinement is defined as an undo of a simplification, refinements μ_i restore those relations as they were before the corresponding inverse simplifications τ_i. No other refinements, different from the refinements in $\mathcal{V}$, create the cells in $C_{q,p}$. Thus, the cells in $C_{q,p}$ are in the complex Γ, with the multiplicity that is greater than or equal to their multiplicity needed for the feasibility of the refinement μ, and the refinement μ is feasible on $\Gamma = \Gamma_U$.

Next, we show that if the refinement μ is feasible on the complex $\Gamma = \Gamma_U$, then all the refinements in $\mathcal{V}$ are contained in the sequence U of refinements that produce the complex Γ from the base complex Γ_B. The refinement μ is feasible on Γ if all the cells in $C_{q,p}$ are in Γ (with sufficiently large multiplicities in the corresponding immediate boundaries and co-boundaries). The only refinements that introduce cells in $C_{q,p}$, and that affect the immediate boundary

and co-boundary relation of these cells, are the refinements μ_i in $\mathcal{V}$. Thus, those refinements must be in the sequence U. □

Proposition 5 *The homology-modifying refinement $\mu = (C_q, \{q\} \cup C_q)$, where C_q is the set of cells in the immediate boundary of the introduced cell q, is feasible on the complex $\Gamma = \Gamma_U$ (at some intermediate resolution) if and only if the sequence U of refinements that produces the complex Γ from the base complex Γ_B contains all the refinements μ_i, $1 \leq i \leq k$, such that $\{q_i, p_i\} \cap C_q \neq \emptyset$ if $\mu_i = (C_{q_i,p_i}, \{q_i, p_i\} \cup C_{q_i,p_i})$ is homology-preserving, and such that $\{q_i\} \cap C_q \neq \emptyset$ if $\mu_i = (C_{q_i}, \{q_i\} \cup C_{q_i})$ is homology-modifying.*

Proof. Let us denote as $\mathcal{V}$ the set of refinements in $\mathcal{M}$ that create at least one cell in C_q.

We show first that the refinement μ is feasible on Γ if all the refinements in $\mathcal{V}$ are in U. The only refinements (beside refinement μ) that affect the immediate boundary relation of the cells in C_q are refinements in $\mathcal{V}$. Since a refinement is defined as an undo of a simplification, refinements μ_i restore those relations as they were before the corresponding inverse simplifications τ_i. No other refinements, different from the refinements in $\mathcal{V}$, create the cells in C_q. Thus, the cells in C_q are in the complex Γ, with the multiplicity that is greater than or equal to their multiplicity needed for the feasibility of the refinement μ. The sequence U cannot contain any refinement that creates a cell in the co-boundary of q, since any such refinement is feasible only if the cell q has already been created. Thus, the refinement μ is feasible on $\Gamma = \Gamma_U$ if U contains all refinements in $\mathcal{V}$.

Next, we show that if the refinement μ is feasible on a complex $\Gamma = \Gamma_U$, then all the refinements in $\mathcal{V}$ are contained in a sequence U of refinements that produce the complex Γ from the base complex Γ_B. The refinement μ is feasible on Γ if all the cells in C_q are in Γ, with the corresponding multiplicities. The only refinements that introduce cells in C_q, and that affect the immediate boundary relation of these cells, are the refinements in $\mathcal{V}$. Thus, those refinements must be in the sequence U. □

This means that the homology-preserving refinement μ is feasible on the complex Γ if and only if all the cells in $C_{q,p}$ are contained in Γ ($C_{q,p} \subseteq \Gamma$). If the cells in $C_{q,p}$ are in Γ, then the set $C_{q,p}$ of cells that defines the feasibility of the refinement μ is a subset of the set of cells in the complex $\Gamma = \Gamma_U$. Conversely, if $C_{q,p} \subseteq \Gamma_U$, then all the refinements that create at least one cell in $C_{q,p}$ are in U.

A large number of adaptive morphological representations can be extracted from the $MRCC$ defined by the triple $(\Gamma_B, \mathcal{M}, \mathcal{R})$ by considering the closed sets of refinements in $\mathcal{M} \cup \mu_0$ under the dependency relation $\mathcal{R}$. Recall that the dependency relation $\mathcal{R}$ is a partial order relation, and thus it defines a closure operator on the set $\mathcal{M}$ of refinements. We denote a closed set of such refinements as $\mathcal{U}$. The set $\mathcal{U}$ implicitly defines a complex representing an approximation of the original full-resolution complex.

Definition 15 *Let $(\Gamma_B, \mathcal{M}, \mathcal{R})$ be an $MRCC$, and let $\mathcal{U} = \{\mu_0, \mu_1, \mu_2, ..., \mu_m\}$ be a set of refinements in $\mathcal{M}$. The set $\mathcal{U}$ is* closed *with respect to the dependency relation $\mathcal{R}$ if for each i, $1 \leq i \leq m$, each refinement on which the refinement μ_i depends is in $\mathcal{U}$.*

Proposition 6 *If the sequence $U = (\mu_0, \mu_1, \mu_2, ..., \mu_m)$ of refinements in $\mathcal{M}$ is feasible, then the set $\mathcal{U} = \{\mu_0, \mu_1, \mu_2, ..., \mu_m\}$ is closed with respect to the dependency relation $\mathcal{R}$.*

Proof. Let $U = (\mu_0, \mu_1, \mu_2, ..., \mu_m)$ be a feasible sequence of refinements in $\mathcal{M}$, and let $\mu_i \in U$. Since the sequence U is feasible, the refinement μ_i is feasible on the complex $\Gamma_{U_{i-1}}$, where $U_{i-1} = (\mu_0, \mu_1, .., \mu_{i-1})$. By previous proposition, this means that the set U_{i-1}, and thus the set U, contains all refinements on which the refinement μ_i depends. Thus, the set $\mathcal{U} = \{\mu_0, \mu_1, \mu_2, ..., \mu_m\}$ is closed with respect to the dependency relation $\mathcal{R}$. □

Definition 16 *Let μ_1 and μ_2 be two feasible refinements on the complex Γ. We say that the refinements μ_1 and μ_2 are* interchangeable *if the sequence (μ_1, μ_2) of refinements (consisting of μ_1 followed by μ_2) on the complex Γ produces the same refined complex Γ' as the sequence (μ_2, μ_1) (consisting of μ_2 followed by μ_1).*

Proposition 7 *Two homology-preserving independent refinements μ_1 and μ_2 are interchangeable.*

Proof. Let μ_1 and μ_2 be two independent homology-preserving refinements of the first type feasible on a complex Γ. Let us first consider the sequence (μ_1, μ_2) applied on Γ. The effect of $\mu_1 = (C_{q_1,p_1}, \{q_1, p_1\} \cup C_{q_1,p_1})$, $C_{q_1,p_1} = \{p_1'\} \cup R_1 \cup Z_1 \cup S_1$, is to delete the cells $r_{1,j}$ from the set R_1, to create the cells q_1 and p_1 and to adjust the immediate boundary and co-boundary of the cells in $\{q_1, p_1\} \cup C_{q_1,p_1}$, thus creating the complex Γ_1.

The effect of μ_2 on the complex Γ_1 is to remove the appropriate number of instances of the cells $r_{2,j}$ from the set R_2 (this is possible since μ_2 is feasible on Γ, and p_1 and q_1 are not in C_{q_2,p_2}), to create the cells q_2 and p_2 and adjust the immediate boundary and co-boundary relation of the cells in $\{q_2, p_2\} \cup C_{q_2,p_2}$, thus creating the refined complex $\Gamma_{1,2}$.

Similarly, the sequence (μ_2, μ_1) of refinement creates the refined complex $\Gamma_{2,1}$. We will show that $\Gamma_{1,2} = \Gamma_{2,1}$.

Each refinement $\mu = (C_{q,p}, \{q, p\} \cup C_{q,p})$ introduces some cells in the immediate boundary and co-boundary of the cells in $\{q, p\} \cup C_{q,p}$ with the corresponding multiplicities. For each refinement μ in $\mathcal{M}$, we can label the corresponding instances of the cells introduced in the immediate boundary and co-boundary of the cells in $\{q, p\} \cup C_{q,p}$ by μ (or, equivalently, by the inverse simplification τ). Similarly, we label each instance of each cell in the immediate boundary or co-boundary of all cells in Γ_B by μ_0. Then it is obvious that the two complexes $\Gamma_{1,2}$ and $\Gamma_{2,1}$ are equal: they consist of the same cells, the corresponding cells in the two complexes have the same immediate boundary and co-boundary relations, and all the corresponding cells in the immediate boundaries and co-boundaries have the same labels. □

Proposition 8 *Two homology-modifying independent refinements μ_1 and μ_2 are interchangeable.*

Proof. Let μ_1 and μ_2 be two independent homology-modifying refinements feasible on a complex Γ. Let us first consider the sequence (μ_1, μ_2) applied on Γ. The effect of $\mu_1 = (C_{q_1}, \{q_1\} \cup C_{q_1})$, $C_{q_1} = R_1$, is to create the cell q_1 and to adjust the immediate boundary of the cell q_1 and the immediate co-boundary of the cells in $C_{q_1} = R_1$, thus creating the complex Γ_1.

The effect of $\mu_2 = (C_{q_2}, \{q_2\} \cup C_{q_2})$ on the complex Γ_1 is to create the cell q_2 and to adjust the immediate boundary of the cell q_2 and the immediate co-boundary of the cells in $C_{q_2} = R_2$, thus creating the refined complex $\Gamma_{1,2}$.

Similarly, the sequence (μ_2, μ_1) of refinement creates the refined complex $\Gamma_{2,1}$. We will show that $\Gamma_{1,2} = \Gamma_{2,1}$.

Since the refinement μ_2 is feasible on the complex Γ, all the cells in C_{q_2} are in Γ and no cell in the co-boundary of q_2 is in Γ. The refinement μ_1 deletes no cells. Thus, all the cells in C_{q_2} are in Γ_1. If the cell q_1 is in the immediate co-boundary of the cell q_2 in Γ_1, then the refinement μ_1 is not feasible on Γ (the cell q_2 is in C_{q_1} and is not in Γ). If the cell q_1 is in the co-boundary of the cell q_2 in Γ_1, then after the refinement μ_2, the cell q_1 is not a topological cell

(homeomorphic to a disc). The co-boundary of q_2 is empty in Γ and the cell q_1 created by the refinement μ_1 is not in the co-boundary of the cell q_2. Thus, the co-boundary of the cell q_2 is empty in Γ_1, and the refinement μ_2 is feasible on Γ_1. Similarly, the refinement μ_1 is feasible on Γ_2. It is obvious that the two complexes $\Gamma_{1,2}$ and $\Gamma_{2,1}$ are equal: they consist of the same cells and the corresponding cells in the two complexes have the same immediate boundary and co-boundary relations. □

Proposition 9 *Let μ_1 be a homology-preserving refinement and let μ_2 be a homology-modifying refinement on a complex Γ. If μ_1 and μ_2 are independent, then they are interchangeable.*

Proof. Let us denote as Γ_1 and Γ_2, respectively, the complex obtained from the complex Γ by applying the refinement μ_1 and μ_2. We will show first that the refinement μ_2 is feasible on Γ_1 and that μ_1 is feasible on Γ_2.

The refinement μ_2 is feasible on the complex Γ. Thus, all the cells in C_{q_2} are in Γ and no cell in the co-boundary of q_2 is in Γ. The refinement μ_1 deletes no cells. Thus, all the cells in C_{q_2} are in Γ_1. If either the cell q_1 or the cell p_1 is in the immediate co-boundary of the cell q_2 in Γ_1, then the refinement μ_1 is not feasible on Γ (the cell q_2 is in C_{q_1,p_1} and is not in Γ). If the cell q_1 (the cell p_1) is in the co-boundary of the cell q_2 in Γ_1, then after the refinement μ_2, the cell q_1 (the cell p_1) is not a topological cell (homeomorphic to a disc). The co-boundary of q_2 is empty in Γ and the cells q_1 and p_1 created by the refinement μ_1 are not in the co-boundary of the cell q_2. Thus, the co-boundary of the cell q_2 is empty in Γ_1, and the refinement μ_2 is feasible on Γ_1.

The refinement μ_1 is feasible on the complex Γ. Thus, all the cells in the set C_{q_1,p_1} are in Γ, and the multiplicities of the cells $r \in R_1$ are sufficiently large for the feasibility of the refinement μ_1. The refinement μ_2 does not delete any cells from Γ, and does not remove any cells from any immediate boundary or co-boundary. Thus, the refinement μ_1 is feasible on the complex Γ_2.

It is obvious that the two complexes $\Gamma_{1,2}$ (obtained from Γ_1 by applying the refinement μ_2) and $\Gamma_{2,1}$ (obtained from Γ_2 by applying the refinement μ_1) are equal: they consist of the same cells and the corresponding cells in the two complexes have the same immediate boundary and co-boundary relations. □

Proposition 10 *Two interchangeable refinement modifications μ_1 and μ_2 are independent.*

Proof. Let us denote as C_{μ}^{new} the set of cells introduced by the refinement μ. It consists of two cells if μ is homology-preserving, and of one cell if μ is homology-modifying.

If the refinements μ_1 and μ_2 are interchangeable, then they are both feasible on the complex Γ, i.e., all the cells in both C_{μ_1} and C_{μ_2} are in Γ, and the cells in $C_{\mu_1}^{new} \cup C_{\mu_2}^{new}$ are not in Γ. This implies that the cells in $C_{\mu_1}^{new}$ are not in C_{μ_2}, i.e., $C_{\mu_1}^{new} \cap C_{\mu_2} = \emptyset$, and similarly $C_{\mu_2}^{new} \cap C_{\mu_1} = \emptyset$. Thus, refinements μ_1 and μ_2 are independent. □

Definition 17 *Let $(G_B, \mathcal{M}, \mathcal{R})$ be an MRCC, and let $U = (\mu_0, \mu_1, \mu_2, ..., \mu_m)$ be a feasible sequence of refinements in $\mathcal{M}$. A permutation $\mu_0, \mu_{i1}, \mu_{i2}, ..., \mu_{im}$ of the refinements in U is* consistent *if the sequence $V = (\mu_0, \mu_{i1}, \mu_{i2}, .., \mu_{im})$ is a feasible sequence of refinements in $\mathcal{M}$.*

Proposition 11 *Let $U = (\mu_0, \mu_1, \mu_2, ..., \mu_m)$ be a feasible sequence of refinement modifications in $\mathcal{M}$, and let the sequence $V = (\mu_0, \mu_{i1}, \mu_{i2}, .., \mu_{im})$ be obtained from U through a consistent permutation of refinements in U. Then, the front complex G_U associated with the sequence U is the same as the front complex G_V associated with the sequence V.*

Proof. A permutation that defines V starting from U is consistent if each refinement μ_{ij} is feasible in sequence V. This means that each refinement μ_{ik} on which μ_{ij} depends has a position $ik < ij$ in V. The permutation defining V from U is a composition of adjacent transpositions of two independent refinements (composition of permutations obtained by reversing the order of two consecutive refinements). For each such transposition, the associated front complex before and after the transposition remains unchanged. Thus, the front complex G_V associated with the sequence V is the same as the front complex G_U associated with the sequence U. □

A closed set $\mathcal{U}$ of refinements can be applied to the base complex Γ_B in any total order U that extends the partial order, producing the complex Γ_U at an intermediate resolution. An $MRCC$ encodes a collection of all representations of the cell complex Γ at intermediate levels of detail which can be obtained from the base representation Γ_B by applying a closed set of modifications on Γ_B. From an $MRCC$ it is thus possible to dynamically extract representations of the topology of the cell complex at *uniform* and *variable* resolutions. The basic query for extracting a single-resolution representation from a multi-resolution model is known as *selective refinement*.

A *selective refinement query* on an $MRCC$ consists of extracting from it a complex with the minimum number of cells, satisfying some application-dependent criterion. This criterion can be formalized by defining a Boolean function τ over all nodes of an $MRCC$, such that the value of τ is $true$ on the nodes which satisfy the criterion, and $false$ otherwise. A complex Γ is said to satisfy the *criterion* τ if function τ assumes the value $true$ on all cells in Γ. Thus, a selective refinement query consists of extracting from the $MRCC$ an intermediate complex of minimum size that satisfies τ. Equivalently, it consists of extracting a minimal closed set $\mathcal{U}$ of modifications from $\mathcal{M}$ such that the corresponding complex satisfies τ. Such closed set of modifications uniquely determines a front complex, which is obtained from the base complex Γ by applying to it all modifications from $\mathcal{U}$ in any order that is consistent with the partial order defined by the dependency relation.

The selective refinement algorithm is independent of the criterion τ, which may be expressed in terms on the size of the created cell or cells, e.g., the diameter of the bounding box or the maximum distance between the vertices, or in terms of the portion of space covered by the complex where the resolution should be maximal, while it can be arbitrary otherwise.

When the complex is the descending (or ascending) Morse complex and the $MRCC$ is built based only on the homology-preserving operators (that is, on *remove* and *insert* operators), the $MRCC$ is called the *Multi-Resolution Morse complex*. In this context, the Boolean criterion τ is defined based on persistence, which is a measure of importance of a feature defined by the pair (p, q) of critical points of the scalar field f, i.e., by the pair (p, q) of cells in the Morse complexes of f. Thus, persistence value is assigned to the refinement μ that introduces a given pair of cells. We assign the persistence value of refinement μ to the cells p and q, and we say that cells p and q satisfy the Boolean criterion τ if the persistence value associated with p and q is greater than some prescribed value P.

We can have query at *uniform resolution*, when we extract a topological representation in which all nodes have persistence value greater than a pre-defined threshold value, or at *variable resolution*, when we request a value of persistence which varies in different parts of the domain. In collaboration with Federico Iuricich, we have implemented a depth-first algorithm for the selective refinement query. The algorithm starts from the coarse complex Γ_B and recursively applies to it all refinements μ_i which are required to satisfy the error criterion. In order that a new modification μ be applied, all its ancestor

modifications need to be applied before μ to maintain the partial order. It can easily be proven that the result of a selective refinement algorithm is the complex Γ with minimal number of cells among the ones encoded in the $MRCC$, which contains all cells satisfying criterion τ.

5.4 Comparison with Other Approaches

To the extent of our knowledge, the only approaches proposed in the literature to the multi-resolution representation of Morse or Morse-Smale complexes are for 2D scalar fields. We compare the hierarchical representation for Morse-Smale complexes presented in [14] and the multi-resolution representation proposed in [15] with the 2D instance of the $MRMC$ (Multi-Resolution Morse Complex), which is essentially the same as a $MRCC$ (Multi-Resolution Cell Complex) based on homology-preserving operators (i.e., based on *remove* and *insert* operators).

The two approaches are based on the cancellation operator [52, 84] which in 2D reduces to the *remove* operator. It eliminates a saddle and a maximum ($remove_{1,2}$), or a saddle and a minimum ($remove_{1,0}$).

In all the approaches, including the $MRMC$, each simplification (cancellation, i.e., *remove*), and the inverse refinement, can be seen as a replacement of one set of cells in some cell complex with another set of cells, with the suitable adjustment of the immediate boundary and co-boundary relations. Each refinement that introduces the critical points q and p (or equivalently, the cells q and p in the Morse complexes, or the vertices q and p in the Morse-Smale complex) replaces a set of cells $S_{q,p}$ with the set $\{q, p\} \cup S_{q,p}$. We will show that in all the approaches, the dependency relation is defined based on the cells in the two sets. A refinement is feasible on a complex Γ if all the cells in the set $S_{q,p}$ are in Γ. The dependency relation in all the approaches is defined in the same way: each refinement directly depends on all the refinements that create a cell in $S_{q,p}$, and the dependency relation is defined as the transitive closure of the direct dependency relation. In other words, each refinement depends on all the refinements that make it feasible.

We will describe for each approach: the complex on which the simplifications and refinements are defined; the set $S_{q,p}$; and the direct dependency relation $\mathcal{R}$.

The basic component of the hierarchical data structures described in [14]

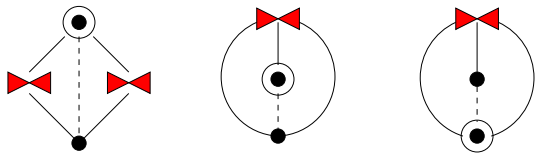


Figure 5.8: Splitting of a quadrangular 2-cell (slope district) in the Morse-Smale complex of a scalar field f, by an integral line inside that 2-cell, in two triangles. (a) A generic slope district, (b) an isolated mountain and (c) a crater.

and [15] is a diamond, which consists of a saddle s, its adjacent minima and maxima, and the integral lines connecting them. The lines that connect the saddle s to the extremal points are the separatrix lines incident in s, while the lines that connect the extremal points are chosen arbitrarily among the integral lines in the corresponding quadrangular 2-cell in the Morse-Smale complex (slope district). In other words, each quadrangular 2-cell in the Morse-Smale complex is split in two triangles, by an arbitrary integral line which connects the vertex of the 2-cell corresponding to a minimum to the vertex corresponding to a maximum, as illustrated in Figure 5.8. The four triangles incident in the same saddle are grouped together in the diamond. The possible types of the diamonds for a Morse-Smale function f are illustrated in Figure 5.9. More formally, the diamond associated with the simplification $remove_{1,2}(q, p, p')$ (and with the inverse refinement $insert_{1,2}(q, p, p')$) is the quadrangle z_1, p, z_2, p', where z_1 and z_2 are the two (not necessarily distinct) minima connected to the 1-saddle q (see Figure 5.10 (a)). Dually, in the diamond z_1, p, z_2, p' associated with the simplification $remove_{1,0}(q, p, p')$ (and with the inverse refinement $insert_{1,0}(q, p, p')$), z_1 and z_2 are the two maxima connected to q.

Thus, the cell complex on which the simplification and refinement operators are defined in [14] and [15] is a complex in which each 2-cell is a quadrangle (a diamond), each 1-cell corresponds to an integral line connecting a minimum and a maximum that define a 2-cell in the Morse-Smale complex of f (i.e., each 1-cell corresponds to a slope district of f), and each 0-cell corresponds to a minimum or to a maximum of f.

The cancellation deletes a cell (a diamond) from the complex, and reconnects the adjacent cells. The adjacency relation of the cells that share a 1-cell

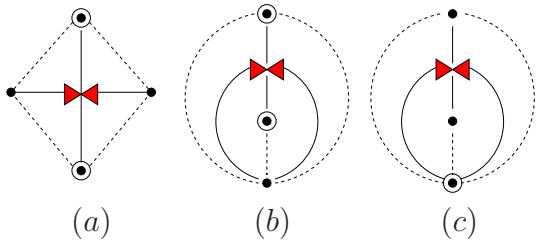


Figure 5.9: The possible types of the diamonds. (a) All four 2-cells incident in the saddle in the Morse-Smale complex are generic. (b) One of the 2-cells incident in the saddle is an isolated mountain. (c) One of the 2-cells incident in the saddle is a crater.

with the deleted diamond is affected by the cancellation. The diamonds that share only a vertex (and not an edge) with the deleted diamond notice no (topological) change in the adjacency relation.

In both [14] and [15], the dependency relation between the refinements is defined in terms of *diamonds*. The dependency relation is defined as follows: two refinements (anticancellations) are dependent if the associated diamonds have at least one vertex in common [14], or if they have an edge in common [15].

Thus, in [14], $S_{q,p}$ is the set of diamonds that share at least one vertex with the diamond centered at q, and in [15], $S_{q,p}$ is the set of (at most four) diamonds that share an edge with the diamond centered at q.

The refinement introducing a diamond Q centered at q replaces the set $S_{q,p}$ of cells in the complex of diamonds with the set $Q \cup S_{q,p}$. It is feasible if the cells in the replaced set $S_{q,p}$ are present in the complex, and it directly depends on all the refinements that create a cell in $S_{q,p}$.

The dependency relation in [15] is clearly less restrictive than the one in [14] (the set of cells in $S_{q,p}$ for the approach in [15] is contained in the set of cells $S_{q,p}$ for the approach in [15]).

We compare the two approaches with the 2D instance of the $MRMC$. To this aim, we interpret the dependency relation expressed in terms of the diamonds as a dependency relation expressed in terms of the cells in the descending Morse complex. Let $remove_{1,2}(q, p, p')$ be the simplification deleting a diamond Q (associated with the saddle q) in [14] and in [15], and let $insert_{1,2}(q, p, p')$ be the inverse refinement.

In [14], the refinement $insert_{1,2}(q, p, p')$ directly depends on all refinements that introduce either the 2-cell p' or at least one of the following cells: a cell

in the boundary of the 2-cell p, a cell in the boundary of the 2-cell p' or a cell in the co-boundary of the 0-cell $z \in Z$ (a cell in the boundary of the 1-cell q in the descending Morse complex). These cells constitute the set $S_{q,p}$. The direct dependency relation is dual for the $remove_{1,0}(q, p, p')$.

In [15], the refinement $insert_{1,2}(q, p, p')$ directly depends on all refinements that introduce one of at most four 1-cells incident both in a 0-cell $z \in Z$ and in 2-cell p or p'. We denote this set of cells as $S_{q,p}$. The direct dependency relation is dual for the $remove_{1,0}(q, p, p')$.

The dependency relation defining the $MRMC$ is less restrictive than the one in [14]. If a refinement $insert(q, p, p')$ depends on a refinement $insert(s, t, t')$ in the $MRMC$, then the associated diamonds share a vertex. In other words, the set $S_{q,p}$ of cells for the $MRMC$ is the subset of the set $S_{q,p}$ for the approach in [14].

There is no containment relation between the dependency relation in the $MRMC$ and the one in [15] in the sense that we cannot say which one is more restrictive than the other. In other words, there is no containment relation between the sets $S_{q,p}$ for the $MRMC$ and for the approach in [15].

Figure 5.10 shows a sequence of simplifications consisting of $remove_{1,2}(s, M_1, M_2)$, $remove_{1,0}(s_5, m_5, m_2)$ and $remove_{1,0}(s_2, m_3, m_4)$. Let $\mu_1 = insert_{1,0}(s_5, m_5, m_2)$, $\mu_2 = insert_{1,0}(s_2, m_3, m_4)$ and $\mu_3 = insert_{1,2}\ (s, M_1, M_2)$ be the inverse refinements. The diamonds associated with the saddles s (refinement μ_3) and s_2 (refinement μ_2) have one common vertex M_1, and the diamonds associated with the saddle s (refinement μ_3) and s_5 (refinement μ_1) have two common vertices, m_2 and M_2. The refinement μ_3 directly depends on μ_1, and does not depend on μ_2, in the approach in [15].

In the $MRMC$, the refinement μ_3 depends on μ_2, since the 1-cell s_2 is in the immediate boundary of the 2-cell M_1, while μ_3 does not depend on μ_1, since none of the cells m_5 and s_5 is in the immediate boundary or the immediate co-boundary of M_1 or s.

To conclude, in all three approaches to the hierarchical or multi-resolution representation of the topology of 2D scalar fields (the 2D instance of the $MRMC$, the hierarchical approach in [14] and the multi-resolution approach in [15]), the same set of simplification operators is used: the cancellation operator, which in 2D is equal either to a $remove_{1,2}$ or to a $remove_{1,0}$. In each of the three approaches, the simplification, and the inverse refinement, can be interpreted as a replacement of one set $S_{q,p}$ of cells in some cell complex with another set of cells. The dependency relation is defined in the same way

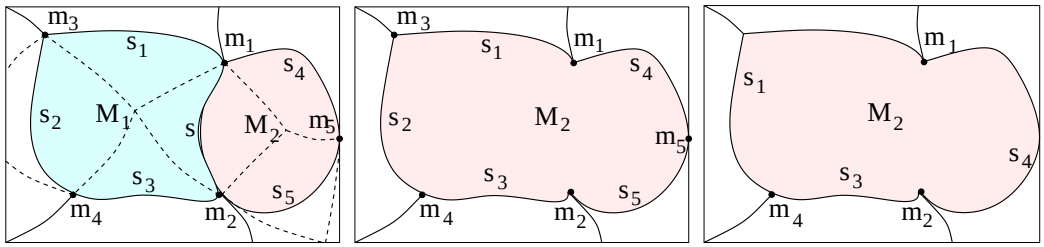


Figure 5.10: A portion of the 2D descending Morse complex. Dotted lines delineate diamonds associated with 1-saddles (a). After $remove_{1,2}(s, M_1, M_2)$ (b). After $remove_{1,0}(s_5, m_5, m_2)$ and $remove_{1,0}(s_2, m_3, m_4)$.

in all three approaches: a refinement directly depends on all the refinements that create a cell in the set $S_{q,p}$. The difference between the three approaches lies in the way the set $S_{q,p}$ is defined. The smaller the set $S_{q,p}$, the smaller the dependency relation between the refinements and the larger the number of complexes at intermediate resolution that can be extracted from the corresponding $MRMC$.

We propose here another multi-resolution model for the topology of scalar fields represented in the form of descending and ascending Morse complexes, using the notion of the *cancellation forest* [15] (see Section 4.4).

We propose to build a cancellation forest starting from the two dual full-resolution Morse complexes by applying all the feasible simplifications in the order determined by persistence. If the domain M of the scalar field f is homeomorphic to the plane (or the 2-sphere), then the cancellation forest consists of exactly two trees, one $remove_{1,2}$ tree and one $remove_{1,0}$ tree. Any refinement *insert* operator introducing a 1-cell q is independent of all the other *insert* operators and can be performed at any step of the refinement process. In other words, the set $S_{q,p}$ for each refinement is the empty set. The effect of the refinement on the cancellation forest is to split one tree in the forest in two trees by deleting the arc corresponding to the 1-cell q. The root of each of the two new trees is the node corresponding to the highest 2-cell (maximum) in the $remove_{1,2}$ trees, and the node corresponding to the lowest 0-cell (minimum) in the $remove_{1,0}$ trees. The effect of the $insert_{1,2}$ operator on the complex is to create the two cells p and q in the complex, such that the immediate boundary and the immediate co-boundary of the two inserted cells consists of the roots of the trees to which the cells in the set $\{p'\} \cup R \cup Z \cup S$ belong. After each

refinement $insert_{1,2}$ introducing the 1-cell q, the boundaries of the 2-cells p and p' separated by the 1-cell q need to be updated, by checking to which of the two new $remove_{1,2}$ trees the corresponding 1-nodes belong. The effect of the $insert_{1,2}$ operator on the complex is completely dual.

The proposed multi-resolution model offers full flexibility in modeling the Morse complexes at intermediate resolutions, but it is more time and space consuming than the present three approaches.

5.5 Implementation and Experimental Evaluation

We have performed various sets of experiments that validate our operators, both on Morse and on cell complexes, and that show the versatility and flexibility of the multi-resolution model in extracting representations at uniform and variable resolutions.

In [35], we have implemented a simplification algorithm on Morse complexes, and we have compared its version based on the operator *remove* with the one based on cancellation for various 2D and 3D Morse functions. In the implementation, we have encoded the Morse complexes in the form of the Morse Incidence Graph (MIG) [23,24], see Section 2.2.3. We have shown that the number of arcs in the graph simplified with cancellation always exceeds the number of arcs in the graph simplified with the same number of *remove* operators. Recall that the arcs of the MIG encode the incidence relation between cells in Morse complexes. Such behavior influences the efficiency of the whole algorithm, doubling the time needed to manage and enqueue a larger number of arcs (and thus, a larger number of possible simplifications) for large data sets. When the data set is small and the number of simplifications is high compared to the total number of nodes the two simplification operators perform quite similar. With the increase of size of the data set the two operators start to differ: by using *remove* we can get a 20% more compressed MIG in about half the time than by using *cancellation*. The *remove* operator is particularly useful in the first simplifications performed on a data set (simplifications that can be interpreted as noise removal). On many data sets we have noticed that by using *cancellation* the number of arcs remains approximately the same while by using *remove* their number immediately decreases. In general, the cost of the MIG is reduced by 10% to 20% by using *remove* instead

of *cancellation* and the same number of simplifications can be performed in half the time.

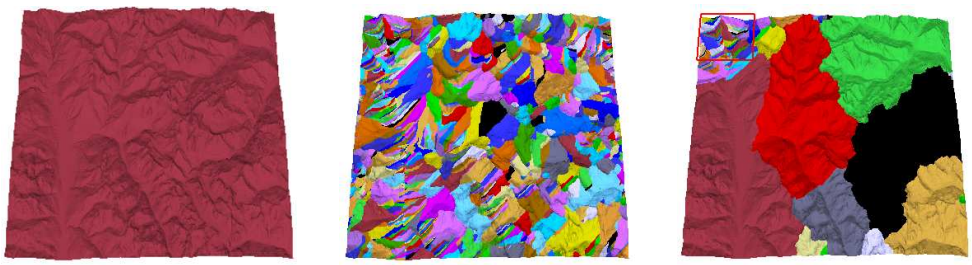
We have implemented a version of the $MRCC$ based on homology-preserving operators. We have used the encoding of the initial full-resolution complex as an incidence graph. Note however that our definition of the $MRCC$ is independent of the specific implementation data structure. We have chosen the incidence graph as the basis of our implementation since it can effectively and efficiently encode arbitrary cell complexes in any dimension.

The $MRCC$ based on homology-preserving operators and the IG-based representation of cell complexes is essentially the same as the multi-resolution representation of Morse complexes given in the form of the Morse Incidence Graph (MIG): Morse complexes are cell complexes, and the effect of the *remove* and *insert* operators on Morse complexes (and on the MIG) is the same as the effect of the homology-preserving $KiC(i{+}1)C$ and $MiC(i{+}1)C$ operators on cell complexes (represented as the IG). We have called such multi-resolution model the Multi-Resolution Morse Incidence Graph ($MMIG$). In both the $MRMC$ and the $MRCC$, each simplification and refinement operator is interpreted as a modification of the corresponding graph, i.e., it can be seen as a replacement of one subgraph with another subgraph. The dependency relation between modifications is expressed in terms of the nodes of the graph. The results of the $MMIG$ and $MRCC$ implementation, done in collaboration with Federico Iuricich, have been presented in [32] and [34, 36], respectively.

We have performed some experiments to show the versatility of the $MMIG$ in extracting adaptive representations of scalar fields at uniform and variable resolutions. We illustrate the results of applying the selective refinement algorithm on *Matterhorn* 2D data set in Figure 5.11, and on *Hydrogen* 3D data set in Figure 5.12.

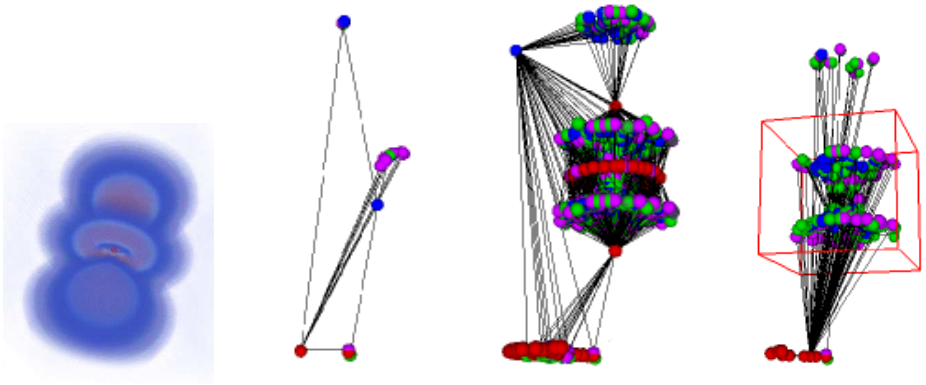
In Figure 5.13 we show examples of extractions from the $MRCC$ at uniform resolution performed on the *Fertility* data set. We have also conducted experiments with extractions at variable resolution from the $MRCC$, with a desired percentage of refinement modifications performed inside a query box chosen by hand and with size between 15 and 30 percent of the whole data set. In Figure 5.14, we show examples of refinement queries at uniform and variable resolution performed on the *VaseLion* data set. The holes that seem to appear in the crown of the lion are rendering artifacts.

Note that extracting at variable resolution is a distinctive feature of the $MRCC$ which cannot be performed on other hierarchical representations, like



(a) (b) (c)

Figure 5.11: The descending Morse complexes at the coarsest resolution (there is only one 2-cell in the descending Morse complex) (a), the full resolution complex (b), and the complex at intermediate resolution (full resolution inside a query box) (c) for the *Matterhorn* 2D terrain data set.

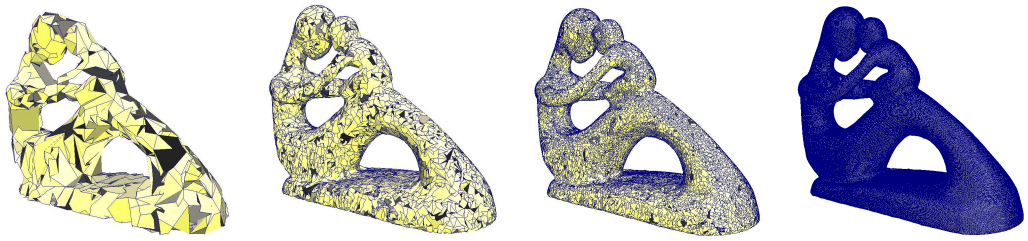


(a) (b) (c) (d)

Figure 5.12: The original scalar field (a), the coarsest resolution MIG (b), the full resolution MIG (c), and the MIG at intermediate resolution (full resolution inside a query box) (d) for the *Hydrogen* 3D data set.

pyramids.

The $MRCC$ based only on homology-preserving operators has been used to compute the homology of a cell complex. Homology is computed on the base complex at the coarsest resolution using known techniques, and homology generators are propagated to complexes at intermediate resolution using the $MRCC$ [36]. Homology computation has been done in collaboration with Ulderico Fugacci.



(a) (b) (c) (d)

Figure 5.13: The base complex (a) of the *Fertility* data set, and complexes obtained from it after 300K (b), 600K (c) and 700K (d) refinements at uniform resolution.

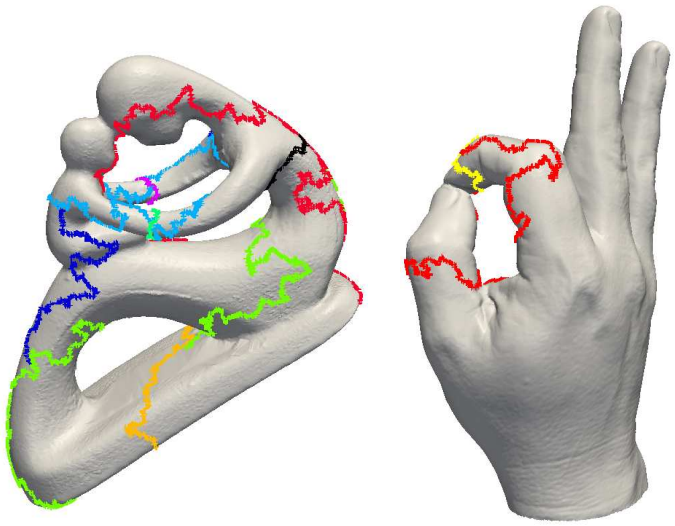
(a) (b) (c) (d) (e)

Figure 5.14: [34] In (a), (b) and (c) the representations obtained from the $MRCC$ after 10000, 50000 and 2000000 refinements, respectively. In (d), the complex at full resolution of the *VaseLion* data set. In (e) the representation obtained with a query at variable resolution.

In Figure 5.15, we show the H_1 generators computed on two 2D shapes: *Fertility* and *Hand* data sets.

5.6 Summary

In [29], we have defined a set of homology-preserving and homology-modifying simplification and refinement operators on cell complexes and we have shown that they form a basis for the set of update operators on cell complexes. We have compared these operators with other known update operators on cell complexes. The defined operators enabled us to define a multi-resolution model



(a) (b)

Figure 5.15: [36] The H_1 generators computed on the *Fertility* data set (a) and on the *Hand* data set (b) by fully refining the cell complex.

for the topology of cell complexes, that we have called the *Multi-Resolution Cell Complex* ($MRCC$). The $MRCC$ encodes a large number of topological representations at uniform and variable resolutions. It is constructed based on the simplification operators, and defined in terms of the refinement ones. In collaboration with Federico Iuricich, we have implemented a version of the $MRCC$ based only on homology-preserving operators, and we have presented the results in [34] and [36]. This version of the $MRCC$ is basically the same as the multi-resolution model of Morse complexes based on the *remove* and *insert* operators. We have called such model the *Multi-Resolution Morse Complex* ($MRMC$). We have defined it in [32] and implemented it in collaboration with Federico Iuricich.

We have compared the 2D instance of the $MRMC$ with the two known approaches to the hierarchical and multi-resolution representation of the topology of 2D scalar fields. We have shown that in each approach, a simplification (and a refinement) can be seen as a replacement of one set S of cells in some complex with another set of cells. The dependency relation between refinements is defined in the same way: each refinement directly depends on all refinements that create a cell in the set S. The smaller the set S, the larger the number of

complexes at intermediate resolutions encoded in the multi-resolution model.

The $MRCC$ based only on homology-preserving operators has been used to compute the homology of a cell complex. Homology is computed on the base complex at the coarsest resolution using known techniques, and homology generators are propagated to complexes at intermediate resolution using the $MRCC$. This work has been done in collaboration with Federico Iuricich and Ulderico Fugacci, and has been presented in [36].

6

Concluding Remarks and Future Work

Morse complexes are a widely used representation for the topology of scalar fields in many application domains. We have defined operators for simplification and refinement of Morse complexes in 3D [23] and in arbitrary dimensions [24, 27]. We have shown their effect on the ascending and descending Morse complexes, we have shown that these operators are valid, and that they form a minimally complete set of basis operators for creating and modifying Morse complexes on a manifold M [26]. We have proven this result by interpreting our operators as Euler operators, that is, as operators that affect a constant number of cells in the Euler-Poincaré formula without changing the topology of M. As a consequence, any macro-operator that modifies Morse or Morse-Smale complexes can be expressed as a sequence of our operators. Simplification operators reduce at each step the incidence relation on the Morse complexes and they reduce the number of cells in the Morse-Smale complexes [35]. We have defined a graph-based data structure, called the *extended cancellation forest*, for encoding a sequence of simplification operators on Morse complexes [25].

We have defined homology-preserving and homology-modifying operators on arbitrary (not necessarily Morse) cell complexes [29]. Homology-preserving operators affect the cell complex in the same way as the simplification and refinement operators affect the Morse complexes. We have shown that the defined operators form a basis for the set of operators on cell complexes that maintain the Euler-Poincaré formula, and we have shown how various existing

operators on cell complexes can be expressed through the proposed ones.

We have defined a multi-resolution model for the topological structure of 3D [23] and nD scalar fields [32]. We have also defined a multi-resolution model for cell complexes, based on homology preserving [34] and on homology-modifying operators [36], and we have implemented it in collaboration with Federico Iuricich. We have shown how a large number of complexes at intermediate resolution can be extracted from the multi-resolution model. The multi-resolution model based only on homology-preserving operators is essentially the same as the multi-resolution model for Morse complexes [32]. We have compared the 2D instance of such model with the existing hierarchical representations for 2D scalar fields.

We have used the multi-resolution model for cell complexes based on homology preserving operators to compute homology generators of the complex [36] in collaboration with Federico Iuricich and Ulderico Fugacci. The idea is to compute homology generators on the base complex at the coarsest resolution using standard techniques [2], and then to propagate those generators to complexes at intermediate resolution using the multi-resolution model. The advantages of our approach are that a homology-preserving $MRCC$ is dimension-independent, can be applied to general cell complexes and enables the extraction of homology generators at variable resolutions.

In our current and future work, we first plan to extend the previous approach to the computation of homology and homology generators with coefficients in $\mathbb{Z}$. We also plan to adapt the $MRCC$ framework to simplicial complexes. We plan to consider two simplification operators for generating an $MRCC$: *simplex collapse* [68], which is an instance of simplification operator $KiC(i+1)C_{re}(q,p)$, and *edge contraction*, a widely used operator in mesh processing which has been proven to be homology-preserving [4].

Another possible research direction is to use the extended cancellation forest [25] to define a smaller dependency relation between the refinements and thus to obtain a more flexible multi-resolution model for Morse and cell complexes.

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