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Cohomologie entière et fibrations lagrangiennes sur certaines variétés holomorphiquement symplectiques singulières

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0.1 Introduction en français

0.1.1 Résumé

Les variétés symplectiques irréductibles sont définies comme des variétés käh-lériennes holomorphiquement symplectiques compactes avec un groupe fonda-mental trivial et telles que la structure symplectique soit unique à scalaire mul-tiplicatif près.

D'après le théorème de décomposition de Bogomolov [9], les variétés sym-plectiques irréductibles jouent (avec les variétés de Calabi-Yau et les tores com-plexes) un rôle central dans la classification des variétés kählériennes avec une première classe de Chern de torsion (voir la Section 1.1.2).

Très peu de classes de déformations de variétés symplectiques irréductibles sont connues (voir la Section 1.1.5). Etendre notre très courte liste de classes de déformations de variétés symplectiques irréductibles est un problème très difficile. On peut obtenir bien plus de classes de déformations en étendant notre champ d'étude aux variétés symplectiques irréductibles pouvant présenter des singularités (voir la Section 1.2). Ces variétés devraient apparaître comme des facteurs dans la conjecture de décomposition généralisée de Bogomolov (voir [28] et [45]).

Les variétés symplectiques irréductibles singulières sont des objets d'étude naturels. Toutes les classes de déformations de variétés irréductibles symplec-tiques lisses connues proviennent d'espaces de modules de faisceaux sur des surfaces K3 ou abéliennes. Comme c'est prouvé dans [24], [56], [70], si un tel espace de modules $\mathcal{M}$ est lisse, alors c'est une déformation d'un des exemples de Beauville (voir [7]) : l'espace de Hilbert de points sur une surfaces K3 ou la variété de Kummer généralisée d'une surface abélienne. Dans le cas où $\mathcal{M}$ est singulier, il y a trois alternatives. Le premier cas est le cas où les singularités de $\mathcal{M}$ peuvent être éliminées en changeant la polarisation; alors $\mathcal{M}$ est bira-tionnelle à un des exemples de Beauville. La seconde alternative correspond au cas où il est impossible d'éliminer les singularités en changeant la polarisation, mais $\mathcal{M}$ admet une résolution symplectique. Cela arrive si et seulement si $\mathcal{M}$ est l'un des deux exemples d'O'Grady (voir [53], [54], [30], [29], [32]). Et la troisième alternative fournit de nombreux espaces de modules singuliers pour lesquels les singularités ne peuvent pas être éliminées, en préservant la structure symplectique, ni par déformation, ni par résolution de singularités.

Les premiers exemples de variétés symplectiques irréductibles singulières ap-parurent, en même temps que les exemples lisses, dans [17] comme des quotients de produits de deux surfaces K3 ou de tores complexes compacts de dimension 4. Dans [36], Markushevich et Tikhomirov construisirent un exemple d'une var-iété symplectique irréductible avec une fibration lagrangienne de Prym comme une composante connexe du lieu fixe d'une involution symplectique sur un es-pace de modules de faisceaux semi-stables sur une surface K3 (voir la Section 1.2.3). Avec la même méthode deux autres exemples furent construits. Un par Arbarello, Saccà et Ferretti [1] et l'autre par Matteini [39].

Un outil important pour étudier les variétés symplectiques irréductibles est

la forme de Beauville–Bogomolov (voir la Section 1.1.3). Ainsi, en suivant
l'objectif de développer la théorie des variétés symplectiques irréductibles sin-
gulières, la question du calcul de la forme de Beauville–Bogomolov de la variété
de Markushevich–Tikhomirov apparut assez naturellement. Dans le cas lisse,
la forme de Beauville–Bogomolov fut calculée dans [7] pour les exemples de
Beauville et dans [62] et [63] pour les variétés d'O'Grady (voir la Section 1.1.5).

Dans notre démarche pour calculer la forme de Beauville–Bogomolov de la
variété de Markushevich–Tikhomirov, la principale difficulté rencontrée fut de
déterminer la cohomologie entière de variétés quotients. Pour cette raison, une
part importante de la thèse (Chapitre 3) est consacrée à l'élaboration d'outils
pour le calcul de la cohomologie entière de variétés quotients.

Notre étude de la cohomologie entière des variétés quotients nous a permis,
en particulier, de répondre à la question de départ. Mais aussi de calculer le
réseau de Beauville–Bogomolov et du cup produit pour d'autres exemples de
variétés quotients de dimension 4 et de surfaces quotients.

Un autre problème résolu dans cette thèse concerne la fibration lagrangienne
de la variété de Markushevich–Tikhomirov obtenue comme la variété de Prym
relative compactifiée de certaines familles de courbes munies d'une involution.
L'espace total de cette variété de Prym relative compactifiée est holomorphique-
ment symplectique et l'application de structure est une fibration lagrangienne
avec une surface de Prym de polarisation (1,2) comme fibre générique. Les var-
iétés symplectiques irréductibles munies d'une fibration lagrangienne sont d'un
intérêt particulier parce qu'elles généralisent d'une part les surfaces K3 munies
d'un pinceau elliptique et d'autre part les espaces de phases de systèmes com-
plètement algébriquement intégrables. Le problème de décrire le dual d'une
fibration lagrangienne est étudié dans [64], où un lien intéressant avec la trans-
formée de Fourier–Mukai tordue est découvert. Une autre raison d'examiner
le dual d'une fibration lagrangienne est que, d'après Strominger–Yau–Zaslow,
la symétrie miroir n'est rien d'autre que la dualité de fibrations lagrangiennes
particulières sur des espaces de Calabi–Yau. La dualité étant bien définie seule-
ment sur les fibres non-singulières, il est important d'obtenir des exemples de
fibrations lagrangiennes duales compactifiables. Dans notre cas, nous présentons
une description du dual de la fibration lagrangienne tel que sa compactification
soit une variété symplectique irréductible singulière du même type que celle de
départ.

0.1.2 Vue d'ensemble des résultats de la thèse

Notre but est de déterminer la cohomologie entière de X/G où X est une variété
complexe compacte et G est un groupe d'automorphismes d'ordre premier. Un
outil fondamental pour étudier ce problème est donné par la proposition suivante
(voir [65]).

Proposition 0.1.1. *Soit G un groupe fini d'ordre d agissant sur une variété
X tel que l'application quotient $\pi : X \to X/G$ soit un revêtement de degré d
ramifié. Alors il existe un morphisme naturel $\pi_* : H^*(X, \mathbb{Z}) \to H^*(X/G, \mathbb{Z})$ tel*

que

$$\pi_* \circ \pi^* = d\,\mathrm{id}_{H^*(X/G,\mathbb{Z})}, \qquad \pi^* \circ \pi_* = \sum_{g \in G} g^*.$$

Si X est une variété complexe compacte et G est un groupe d'automorphismes d'ordre premier p, nous verrons que cette proposition induit la suite exacte

$$0 \longrightarrow \pi_*(H^k(X,\mathbb{Z})) \longrightarrow H^k(X/G,\mathbb{Z})/\,\mathrm{tors} \longrightarrow (\mathbb{Z}/p\,\mathbb{Z})^{\alpha_k} \longrightarrow 0,$$

où $\pi : X \to X/G$ est l'application quotient et α_k est un entier non-négatif.

Il n'y a pas de méthode générale pour calculer α_k. Au chapitre 3, nous allons développer des critères d'annulation de α_k. Par la suite, la définition suivante va jouer le rôle central.

Définition 0.1.2. *Soient X une variété complexe compacte de dimension n, $G = \langle \varphi \rangle$ un groupe d'automorphismes d'ordre premier p et $0 \leq k \leq 2n$. On suppose que $H^k(X,\mathbb{Z})$ est sans torsion.*

Si l'application $\pi_* : H^k(X,\mathbb{Z}) \to H^k(X/G,\mathbb{Z})/\,\mathrm{tors}$ est surjective, on dira que (X,G) est H^k-normal.

Nous donnerons tout d'abord des résultats généraux sur cette notion qui expliqueront comment obtenir la H^k-normalité à partir de la H^{kt}-normalité (avec k et t entiers), comment obtenir le réseau du cup-produit de $H^n(X/G,\mathbb{Z})$ quand (X,G) est H^n-normal et quand cette propriété est invariante par un biméromorphisme. Nous prouverons aussi des propriétés simples qui s'appliquent dans de nombreux cas. Par exemple la Proposition 3.3.13 :

Proposition 0.1.3. *Soient X une variété complexe compacte de dimension n et G un groupe d'automorphismes d'ordre premier p agissant sur X. On suppose que $H^*(X,\mathbb{Z})$ est sans torsion et $2 \leq p \leq 19$. Soit $0 \leq k \leq 2n$.*

Si $a_G^k(X) = \mathrm{rk}\, H^k(X,\mathbb{Z})^G$, alors (X,G) est H^k-normal.

L'entier $a_G^k(X)$ est défini par l'isomorphisme de la Définition 5.5 de [11] :

$$\frac{H^k(X,\mathbb{Z})}{H^k(X,\mathbb{Z})^G \oplus S_G^k(X)} = (\mathbb{Z}/p\,\mathbb{Z})^{a_G^k(X)},$$

où $S_G^k(X)$ est un sous-$\mathbb{Z}$-module supplémentaire naturel de $H^k(X,\mathbb{Z})^G$; voir la Section 2.2 pour la définition précise.

Après ces énoncés généraux, on donnera des résultats plus précis dans le cas où l'action de G a un bon comportement local. D'après le Lemme 1 de Cartan dans [14], en chaque point fixe de G, on peut localement linéariser l'action de G. Ainsi en chaque point fixe $x \in X$, l'action de G sur X est localement équivalente à l'action de $G = \langle g \rangle$ sur $\mathbb{C}^n$ via

$$g = \mathrm{diag}(\xi_p^{k_1}, \ldots, \xi_p^{k_n}),$$

avec ξ_p une racine p-ième de l'unité. Sans perdre en généralité, on peut supposer que $k_1 \leq \cdots \leq k_n \leq p - 1$. Quand l'action locale de G en un point x est de la forme

$$g = \mathrm{diag}(1, \ldots, 1, \xi_p^\alpha, \ldots, \xi_p^\alpha),$$

$\alpha \in \{1, \dots, p-1\}$, on dit que x est un point de type 1. Nous verrons que lorsque tous les points de $\operatorname{Fix} G$ sont de type 1 et que le lieu fixe de G n'est pas trop grand (codim $\operatorname{Fix} G \geq \frac{\dim X}{2}$), la H^n-normalité est vérifiée si une certaine équation en lien avec l'action de G est vérifiée. Quand $\operatorname{Fix} G$ n'est pas trop grand, on dira que $\operatorname{Fix} G$ est négligeable ou presque négligeable (voir la Définition 3.5.1 pour l'énoncé exact).

L'idée principale est de travailler sur l'éclatement $\widetilde{X}$ de X en le lieu fixe de G. Soit $\widetilde{G}$ l'extension naturelle de G à $\widetilde{X}$. Quand tous les points fixes sont de type 1, le quotient $\widetilde{M} = \widetilde{X}/\widetilde{G}$ est lisse. On pose aussi $U = X \setminus \operatorname{Fix} G$. On utilisera l'unimodularité du réseau $H^n(\widetilde{M}, \mathbb{Z})$ pour établir le lien entre la cohomologie de U, $\widetilde{M}$, $\operatorname{Fix} G$ et la H^n-normalité de (X, G). Le théorème principal est le suivant:

Théorème 0.1.4. *Soit $G = \langle \varphi \rangle$ un groupe d'ordre premier p agissant par automorphismes sur une variété kählérienne X de dimension n. On suppose :*

i) $H^*(X, \mathbb{Z})$ est sans torsion,

ii) $\operatorname{Fix} G$ est négligeable ou presque négligeable,

iii) tous les points de $\operatorname{Fix} G$ sont de type 1.

Alors :

1) $\log_p(\operatorname{discr} \pi_*(H^n(X, \mathbb{Z}))) - h^{2*+\epsilon}(\operatorname{Fix} G, \mathbb{Z})$ est divisible par 2,

2) Les inégalités suivantes sont vérifiées:

$$\log_p(\operatorname{discr} \pi_*(H^n(X, \mathbb{Z}))) + 2 \operatorname{rktor} H^n(U, \mathbb{Z})$$
$$\geq h^{2*+\epsilon}(\operatorname{Fix} G, \mathbb{Z}) + 2 \operatorname{rktor} H^n(\widetilde{M}, \mathbb{Z})$$
$$\geq 2 \operatorname{rktor} H^n(U, \mathbb{Z}).$$

3) Si de plus

$$\log_p(\operatorname{discr} \pi_*(H^n(X, \mathbb{Z}))) + 2 \operatorname{rktor} H^n(U, \mathbb{Z})$$
$$= h^{2*+\epsilon}(\operatorname{Fix} G, \mathbb{Z}) + 2 \operatorname{rktor} H^n(\widetilde{M}, \mathbb{Z}),$$

alors (X, G) est H^n-normal.

Ici $\operatorname{rktor} H^n(U, \mathbb{Z})$ et $\operatorname{rktor} H^n(\widetilde{M}, \mathbb{Z})$ sont les rangs de la partie de torsion des groupes de cohomologie, le rang étant ici défini comme le plus petit nombre de générateurs. On définit

$$h^{2*+\epsilon}(\operatorname{Fix} G, \mathbb{Z}) = \sum_{k=0}^{\dim \operatorname{Fix} G} \dim H^{2k}(\operatorname{Fix} G, \mathbb{Z}),$$

quand n est pair et

$$h^{2*+\epsilon}(\operatorname{Fix} G, \mathbb{Z}) = \sum_{k=0}^{\dim \operatorname{Fix} G - 1} \dim H^{2k+1}(\operatorname{Fix} G, \mathbb{Z}),$$

quand n est impair.

Dans le cas où $2 \leq p \leq 19$, on calculera $\log_p(\operatorname{discr} \pi_*(H^n(X, \mathbb{Z})))$ et $\operatorname{rktor} H^n$ $(U, \mathbb{Z})$ à l'aide d'invariants introduits par Boissière, Nieper-Wisskirchen et Sarti dans [11]. Comme $H^*(X, \mathbb{Z})$ est sans torsion, on a $H^k(X, \mathbb{F}_p) = H^k(X, \mathbb{Z}) \otimes \mathbb{F}_p$ pour $0 \leq k \leq 2 \dim X$. Boissière, Nieper-Wisskirchen and Sarti définissent l'entier $l_q^k(X)$ comme le nombre de blocs de Jordan N_q de taille q dans la décomposition de Jordan du G-module $H^k(X, \mathbb{F}_p)$, de sorte que $H^k(X, \mathbb{F}_p) \simeq \oplus_{q=1}^p N_q^{\oplus l_q^k(X)}$.

Nous déduirons de nombreux corollaires du théorème précédent, par exemple le corollaire suivant (Corollaire 3.5.17) :

Corollaire 0.1.5. *Soit $G = \langle \varphi \rangle$ un groupe d'ordre premier $3 \leq p \leq 19$ agissant par automorphismes sur une variété kählérienne X de dimension $2n$. On suppose :*

i) *$H^*(X, \mathbb{Z})$ est sans torsion,*

ii) *$\operatorname{Fix} G$ est négligeable ou presque négligeable,*

iii) *tous les points de $\operatorname{Fix} G$ sont de type 1,*

iv) *$l_{p-1}^{2k}(X) = 0$ pour tout $1 \leq k \leq n$, et*

v) *$l_1^{2k+1}(X) = 0$ pour tout $0 \leq k \leq n-1$, quand $n > 1$.*

Alors :

1) *$l_1^{2n}(X) - h^{2*}(\operatorname{Fix} G, \mathbb{Z})$ est divisible par 2, et*

2) *on a :*

$$
l_1^{2n}(X) + 2 \left[\sum_{i=0}^{n-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{n-1} l_1^{2i}(X) \right]
$$
$$
\geq h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 2 \operatorname{rktor} H^{2n}(\widetilde{M}, \mathbb{Z})
$$
$$
\geq 2 \left[\sum_{i=0}^{n-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{n-1} l_1^{2i}(X) \right].
$$

3) *Si de plus*

$$
l_1^{2n}(X) + 2 \left[\sum_{i=0}^{n-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{n-1} l_1^{2i}(X) \right]
$$
$$
= h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 2 \operatorname{rktor} H^{2n}(\widetilde{M}, \mathbb{Z}),
$$

alors (X, G) est H^{2n}-normal.

Après l'étude des actions dont les points fixes sont de type 1, on sera en mesure de traiter des actions locales plus générales dans le cas $p = 3$. Nous verrons que l'on peut ramener ce problème au cas où les points fixes sont de type 1 en éclatant X en les points fixes d'un autre type, qui seront appelés points de type 2.

Comme on peut voir, la notion de H^k-normalité peut être généralisée dans de nombreuses directions. Si $H^k(X, \mathbb{Z})$ a de la torsion, on pourrait travailler avec $H^k(X, \mathbb{Z})/\text{tors}$. On pourrait aussi généraliser la notion quand X n'est pas lisse ou quand p n'est pas premier. Le théorème 0.3.4 peut aussi être généralisé à des actions locales de G plus générales en utilisant des éclatements toriques dans la preuve au lieu d'éclatements classiques. Par conséquent, il y a de nombreuses pistes pour généraliser notre travail.

Nous commençons par illustrer nos résultats sur la H^*-normalité en calculant le réseau du cup-produit d'une surface K3 quotientée par une involution symplectique et par deux automorphismes d'ordre 3, l'un symplectique et l'autre non-symplectique. Dans le tableau qui suit, on note ces quotients respectivement par $\overline{Y_2}$, $\overline{Y_3}$ et $\overline{Z_3}$. On calcule aussi le réseau du cup-produit d'un tore complexe de dimension 2 quotienté par $-\text{id}$; on note ce quotient $\overline{A}$. Puis, on continue avec l'application principale à nos outils, calculer le réseau de Beauville–Bogomolov de variétés symplectiques irréductibles de type $K3^{[2]}$ quotientées par certains automorphismes symplectiques. Le premier exemple est le quotient par un automorphisme symplectique d'ordre 3 numériquement standard (voir la Section 1.3.2 pour la définition de 'numériquement standard'). On fournit aussi la forme de Beauville–Bogomolov de résolutions partielles de variétés symplectiques irréductibles de type $K3^{[2]}$ quotientées par des involutions symplectiques. On note ces deux variétés symplectiques irréductibles singulières de dimension 4 respectivement par M_3 et M'. On résume les résultats de nos calculs dans le tableau suivant :

X/G	$H^2(X/G, \mathbb{Z})$
$\overline{Y}_2$	$E_8(-1) \oplus U(2)^3$
$\overline{Y}_3$	$U(3) \oplus U^2 \oplus A_2^2$
$\overline{Z}_3$	$U \oplus E_6$
$\overline{A}$	$U(2)$
M_3	$U(3) \oplus U^2 \oplus A_2^2 \oplus (-6)$
M'	$U(2)^3 \oplus E_8(-1) \oplus (-2)^2.$

Ici $H^2(X/G, \mathbb{Z})$ est muni du cup-produit pour les surfaces et de la forme de Beauville–Bogomolov pour les variétés de dimension 4. Voir la Section 4.7 pour plus de détails.

Le dernier chapitre est consacré aux variétés de Markushevich–Tikhomirov. La construction de la variété de Prym relative compactifiée $\mathcal{P}$ commence avec une paire de quartiques planes totalement tangentes $\overline{B_0}$ et $\overline{\Delta_0}$. La première est utilisée pour construire une surface X de del Pezzo de degré 2 et la seconde définit une surface K3, revêtement double de X. Par la suite, la famille de

courbes désirées est un système linéaire non-complet de courbes sur S, invariantes sous l'action de l'involution de Galois du revêtement double $S \to X$ et $\mathcal{P}$ est la variété de Prym relative compactifiée associée. En échangeant les rôles de $\overline{B_0}$, $\overline{\Delta_0}$, on obtient une autre surface K3 $\widetilde{S}$ et une autre variété de Prym correspondante $\widetilde{\mathcal{P}}$.

On va prouver que les fibrations lagrangiennes sur $\mathcal{P}$ et $\widetilde{\mathcal{P}}$ sont duales l'une de l'autre. De plus, on va prouver, pour S générique, que non seulement $S \not\simeq \widetilde{S}$ mais aussi que les catégories dérivées de $\widetilde{S}$, S ne sont pas équivalentes et que $\widetilde{S}^{[2]} \not\simeq S^{[2]}$. Cela nous permettra de conclure que les variétés de Prym compactifiées $\widetilde{\mathcal{P}}$, $\mathcal{P}$ ne sont pas isomorphes.

De plus, $\mathcal{P}$ est reliée à M' par un flop de Mukai, donc on obtient aussi le réseau de Beauville–Bogomolov de ces variétés de Prym compactifiées à partir de celui de M'.

0.1.3   Structure de la thèse

Le **Chapitre 1** donne une vue d'ensemble de résultats connus sur les variétés symplectiques irréductibles. Dans la Section 1.1, on rappelle les résultats principaux sur l'application des périodes pour les variétés symplectiques irréductibles. On rappelle le théorème de Torelli local de Beauville. Puis, on rappelle les résultats de Huybrechts et Verbitsky sur les points non-séparés de l'espace de modules et sur la surjectivité de l'application des périodes (Théorème de Torelli global). On fait aussi la liste des réseaux de Beauville–Bogomolov pour les variétés symplectiques irréductibles connues.

Ensuite, on introduit les variétés symplectiques irréductibles singulières dans la Section 1.2. En particulier, on rappelle la construction de Markushevich et Tikhomirov.

Dans la Section 1.3.2, on cite des résultats de Mongardi sur les automorphismes des variétés de type $K3^{[2]}$. Dans la Section 1.3.3, on fait des rappels sur la cohomologie du schéma de Hilbert de deux points sur une surface K3 en suivant des résultats de Markman, Verbitsky, Boissière–Nieper-Wisskirchen–Sarti et Qin–Wang.

Le **Chapitre 2** contient différents résultats sur les réseaux, la théorie de Smith et la cohomologie équivariante, nécessaires pour calculer la cohomologie entière de variétés quotients. Nous rappelons aussi des développements de Boissière, Nieper-Wisskirchen et Sarti [11] qui apportent des outils très utiles pour notre étude.

Dans le **Chapitre 3**, on considère une variété complexe X et un groupe d'automorphismes G d'ordre premier p agissant sur X. Ce chapitre contient des résultats techniques sur le calcul de la cohomologie entière de X/G. Dans la Section 3.2.2, on calcule la cohomologie entière d'un quotient quand l'action du groupe est libre. Puis, dans la Section 3.3, on introduit la H^*-normalité et on énonce plusieurs résultats généraux sur cette notion. Dans la Section 3.5, on donne des résultats plus précis dans le cas où les points fixes sont de type 1. Et finalement, la Section 3.6 est dédiée au cas $p = 3$.

Le **Chapitre 4** donne plusieurs applications aux résultats du Chapitre 3. On

calcule la forme du cup-produit et de Beauville–Bogomolov de $\overline{Y_2}$, $\overline{Y_3}$, $\overline{Z_3}$, $\overline{A}$, M_3 et M'.

Pour les cinq premières variétés, le calcul découle directement des résultats du Chapitre 3. Le cas de M' est plus compliqué. Soient S une surface K3 et i une involution symplectique sur S. On peut montrer facilement que $(S^{[2]}, i^{[2]})$ est H^4 et H^2-normal. Mais cela n'est pas suffisant pour calculer la forme de Beauville–Bogomolov de la variété M', résolution partielle de $S^{[2]}/i^{[2]}$. En fait, la plus grande partie de ce chapitre est consacrée au cas de M', pour lequel on est contraint de décrire explicitement l'action de $i^{[2]}$ dans une base particulière de $H^*(S^{[2]}, \mathbb{Z})$, obtenue à l'aide de la représentation de Nakajima de l'algèbre d'Heisenberg sur la cohomologie des variétés $S^{[n]}$.

Le **Chapitre 5** est consacré aux variétés de Markushevich–Tikhomirov. On donne le dual de la fibration lagrangienne dans la Section 5.1. Dans la Section 5.2 et la Section 5.3, on prouve que $S \not\simeq \widetilde{S}$ pour S générique, mais aussi que les catégories dérivées de $\widetilde{S}$, S ne sont pas équivalentes et que $\widetilde{S}^{[2]} \not\simeq S^{[2]}$. Pour conclure, on donne la forme de Beauville–Bogomolov de ces variétés.

0.2   Remerciements

0.3   Introduction in English

0.3.1   Abstract

Irreducible symplectic varieties are defined as compact holomorphically symplectic Kähler varieties with trivial fundamental group, whose symplectic structure is unique up to proportionality.

By the Bogomolov decomposition theorem [9], irreducible symplectic varieties play (together with Calabi–Yau manifolds and complex tori) a central role

in the classification of compact Kähler manifolds with torsion c_1 (see Section 1.1.2).

Very few deformation classes of irreducible symplectic varieties are known (see Section 1.1.5). The problem of extending the very short list of known deformation classes of irreducible symplectic varieties is very hard. One can get many more deformation classes by extending the scope to possibly singular irreducible symplectic varieties (see Section 1.2). Such varieties should appear as factors in the generalized Bogomolov decomposition conjecture (see [28] and [45]).

The singular irreducible symplectic varieties are natural objects of study. All the known deformation classes of irreducible symplectic manifolds come from moduli space of sheaves on K3 or abelian surfaces. As it is shown in [24], [56], [70], if such a moduli space $\mathcal{M}$ is smooth, then it is a deformation of one of the Beauville's examples (see [7]): the Hilbert scheme of points on a K3 surface or the generalized Kummer variety of an abelian surface. In the case when $\mathcal{M}$ is singular, there are three alternatives. The first is the case when the singularities of $\mathcal{M}$ can be removed by changing the polarization; then $\mathcal{M}$ is birational to a Beauville example. The second alternative is when it is impossible to remove the singularities by changing the polarization, but $\mathcal{M}$ admits a symplectic desingularization. This happens if and only if $\mathcal{M}$ is one of the two O'Grady examples (see [53], [54], [30], [29], [32]). And the third alternative provides many singular modular spaces that cannot been smoothed out to nonsingular holomorphically symplectic varieties, neither by deformations, nor by a resolution of singularities. These serve the main motivation for the study of singular holomorphically symplectic varieties.

The first examples of singular irreducible symplectic varieties appeared, simultaneously with nonsingular ones, in [17] as finite quotients of products of two K3 surfaces or of 4-dimensional compact complex tori. In [36], Markushevich and Tikhomirov provided an example of a singular irreducible symplectic variety with a Prym Lagrangian fibration, constructed as a connected component of the fixed locus of a symplectic involution on a moduli space of semi-stable sheaves on a K3 surface (see Section 1.2.3). By the same method, two other examples were constructed. One by Arbarello, Saccà and Ferretti [1] and the other by Matteini [39].

An important tool for the study of irreducible symplectic varieties is the Beauville–Bogomolov form (see Section 1.1.3). Therefore, following the idea to develop the theory of singular irreducible symplectic varieties, the question of calculating the Beauville–Bogomolov form of the Markushevich–Tikhomirov varieties appears quite natural. The Beauville–Bogomolov form and the Fujiki constant in smooth cases were calculated in [7] for Beauville's examples and in [62] and [63] for O'Grady varieties (see Section 1.1.5).

In our approach to calculating the Beauville–Bogomolov form of Markushevich–Tikhomirov varieties, the main difficulty is to determine integral cohomology of quotient varieties. For this reason, an important part of this thesis (Chapter 3) is devoted to the elaboration of tools for the calculation of the integral cohomology of quotient varieties.

Our study of integral cohomology of quotient varieties allowed us, in particular, to answer the original question. Moreover it naturally resulted in other examples of calculation of Beauville–Bogomolov and cup-product lattices on quotient fourfolds and quotient surfaces. We produce several examples of such lattices in this thesis.

Another problem about Markushevich–Tikhomirov variety solved in this thesis has to deal with its Lagrangian fibration, obtained as the relative compactified Prym variety of some family of curves with involution. The total space of this relative compactified Prymian is holomorphically symplectic, and the structure map is a Lagrangian fibration with a (1,2)-polarized Prym surface as generic fiber. The irreducible symplectic varieties with a Lagrangian fibration are of particular interest, as they generalize K3 surfaces with an elliptic pencil on the one hand, and the phase spaces of algebraically completely integrable systems on the other hand. The problem of describing the dual of a Lagrangian fibration is discussed in [64], where an interesting link to the twisted Fourier–Mukai transform is uncovered. Another reason to look at the dual Lagrangian fibration is the fact that, according to Strominger–Yau–Zaslow, the mirror symmetry is nothing but the duality of special Lagrangian fibrations on Calabi–Yau spaces. The duality being well-defined only on nonsingular fibers, it is important to obtain examples of compactifiable dual Lagrangian fibrations. We present a description of the dual Lagrangian fibration in our example, whose compactification turns out to be a singular irreducible symplectic variety of the same type as the original one.

0.3.2   Overview of the results

Our first goal is to determine the integral cohomology of X/G for a compact complex manifold X and an automorphism group G of prime order. A fundamental tool for studying this question is given by the following proposition [65].

Proposition 0.3.1. *Let G be a finite group of order d acting on a variety X with the orbit map $\pi : X \to X/G$, which is a d-fold ramified covering. Then there is a natural homomorphism $\pi_* : H^*(X, \mathbb{Z}) \to H^*(X/G, \mathbb{Z})$ such that*

$$\pi_* \circ \pi^* = d \operatorname{id}_{H^*(X/G,\mathbb{Z})}, \qquad \pi^* \circ \pi_* = \sum_{g \in G} g^*.$$

When X is a compact complex manifold and G is an automorphism group of prime order p, we will see that it induces an exact sequence

$$0 \longrightarrow \pi_*(H^k(X,\mathbb{Z})) \longrightarrow H^k(X/G,\mathbb{Z})/\operatorname{tors} \longrightarrow (\mathbb{Z}/p\mathbb{Z})^{\alpha_k} \longrightarrow 0,$$

where $\pi : X \to X/G$ is the quotient map and α_k is a non-negative integer.

There is no general recipe for computing α_k. In Chapter 3, we will provide some criteria for the vanishing of α_k. In the sequel, the following definition will play a central role.

Definition 0.3.2. *Let X be a compact complex manifold of dimension n, $G = \langle \varphi \rangle$ an automorphism group of prime order p and $0 \leq k \leq 2n$. We assume that $H^k(X, \mathbb{Z})$ is torsion-free.*

If the map $\pi_* : H^k(X, \mathbb{Z}) \to H^k(X/G, \mathbb{Z})/\operatorname{tors}$ is surjective, we will say that (X, G) is H^k-normal.

We first give some general results on this notion, explaining how to get the H^k-normality from the H^{kt}-normality for integers k and t, how to get the cup-product lattice of $H^n(X/G, \mathbb{Z})$ when (X, G) is H^n-normal, and how the notion can be transferred via a bimeromorphic map. We also prove some easy properties that apply in a large range of cases. For instance Proposition 3.3.13:

Proposition 0.3.3. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p acting on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $2 \leq p \leq 19$. Let $0 \leq k \leq 2n$.*

If $a_G^k(X) = \operatorname{rk} H^k(X, \mathbb{Z})^G$, then (X, G) is H^k-normal.

The integer $a_G^k(X)$ is defined by the isomorphism of Definition 5.5 of [11]:

$$\frac{H^k(X, \mathbb{Z})}{H^k(X, \mathbb{Z})^G \oplus S_G^k(X)} = (\mathbb{Z}/p\,\mathbb{Z})^{a_G^k(X)},$$

where $S_G^k(X)$ is a natural complementary $\mathbb{Z}$-submodule of $H^k(X, \mathbb{Z})^G$; see Section 2.2 for the precise definition.

After these general statements we give more precise results in particular cases of G-actions with good local behaviour. At each fixed point of G, by Cartan Lemma 1 of [14] we can locally linearize the action of G. Thus at a fixed point $x \in X$, the action of G on X is locally equivalent to the action of $G = \langle g \rangle$ on $\mathbb{C}^n$ via

$$g = \operatorname{diag}(\xi_p^{k_1}, \ldots, \xi_p^{k_n}),$$

where ξ_p is a p-th root of unity. Without loss of generality we can assume that $k_1 \leq \cdots \leq k_n \leq p - 1$. When the local action of G at a fixed point x is of the form

$$g = \operatorname{diag}(1, \ldots, 1, \xi_p^\alpha, \ldots, \xi_p^\alpha),$$

$\alpha \in \{1, \ldots, p - 1\}$, we say that x is a point of type 1. We will see that when all the points of $\operatorname{Fix} G$ are of type 1 and when the fixed locus of G is not too big ($\operatorname{codim} \operatorname{Fix} G \geq \frac{\dim X}{2}$), H^n-normality holds if some equation related to the action of G is verified. When $\operatorname{Fix} G$ is not too big, we will say that $\operatorname{Fix} G$ is negligible or almost negligible (see Definition 3.5.1 for the exact setting).

The idea is to work on the blowup $\widetilde{X}$ of X in the fixed locus of G. Let $\widetilde{G}$ be the natural lift of G to $\widetilde{X}$. When all the fixed points are of type 1, the quotient $\widetilde{M} = \widetilde{X}/\widetilde{G}$ is smooth. We also denote $U = X \setminus \operatorname{Fix} G$. We will use the unimodularity of the cohomology lattice $H^n(\widetilde{M}, \mathbb{Z})$ to establish a link between the cohomology of U, $\widetilde{M}$, $\operatorname{Fix} G$ and the H^n-normality of (X, G). The main theorem is the following:

Theorem 0.3.4. *Let $G = \langle \varphi \rangle$ be a group of prime order p acting by automorphisms on a Kähler manifold X of dimension n. We assume:*

  i) $H^*(X, \mathbb{Z})$ *is torsion-free,*

  ii) $\mathrm{Fix}\, G$ *is negligible or almost negligible,*

  iii) all the points of $\mathrm{Fix}\, G$ *are of type 1.*

Then:

  1) $\log_p(\mathrm{discr}\, \pi_*(H^n(X, \mathbb{Z}))) - h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z})$ *is divisible by 2,*

  2) The following inequalities are verified:

$$\log_p(\mathrm{discr}\, \pi_*(H^n(X, \mathbb{Z}))) + 2\,\mathrm{rktor}\, H^n(U, \mathbb{Z})$$
$$\geq h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z}) + 2\,\mathrm{rktor}\, H^n(\widetilde{M}, \mathbb{Z})$$
$$\geq 2\,\mathrm{rktor}\, H^n(U, \mathbb{Z}).$$

  3) If moreover

$$\log_p(\mathrm{discr}\, \pi_*(H^n(X, \mathbb{Z}))) + 2\,\mathrm{rktor}\, H^n(U, \mathbb{Z})$$
$$= h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z}) + 2\,\mathrm{rktor}\, H^n(\widetilde{M}, \mathbb{Z}),$$

  then (X, G) *is* H^n-*normal.*

Here $\mathrm{rktor}\, H^n(U, \mathbb{Z})$ and $\mathrm{rktor}\, H^n(\widetilde{M}, \mathbb{Z})$ are the ranks of the torsion parts of the cohomology, defined as the smallest number of generators. We define

$$h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z}) = \sum_{k=0}^{\dim \mathrm{Fix}\, G} \dim H^{2k}(\mathrm{Fix}\, G, \mathbb{Z}),$$

when n is even and

$$h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z}) = \sum_{k=0}^{\dim \mathrm{Fix}\, G - 1} \dim H^{2k+1}(\mathrm{Fix}\, G, \mathbb{Z}),$$

when n is odd.

In the case where $2 \leq p \leq 19$, we will calculate $\log_p(\mathrm{discr}\, \pi_*(H^n(X, \mathbb{Z})))$ and $\mathrm{rktor}\, H^n(U, \mathbb{Z})$ with the help of invariants introduced by Boissière, Nieper-Wisskirchen and Sarti in [11]. Since we assume $H^*(X, \mathbb{Z})$ is torsion-free, we have $H^k(X, \mathbb{F}_\mathrm{p}) = H^k(X, \mathbb{Z}) \otimes \mathbb{F}_\mathrm{p}$ for $0 \leq k \leq 2\dim X$. Boissière, Nieper-Wisskirchen and Sarti define the integer $l_q^k(X)$ as the number of Jordan blocks N_q of size q in the Jordan decomposition of the G-module $H^k(X, \mathbb{F}_\mathrm{p})$, so that $H^k(X, \mathbb{F}_\mathrm{p}) \simeq \bigoplus_{q=1}^p N_q^{\oplus l_q^k(X)}$.

We will deduce several corollaries from this theorem, for instance this one (Corollary 3.5.17):

Corollary 0.3.5. *Let $G = \langle \varphi \rangle$ be a group of prime order $3 \leq p \leq 19$ acting by automorphisms on a Kähler manifold X of dimension $2n$. We assume:*

  i) $H^*(X, \mathbb{Z})$ *is torsion-free,*

  ii) $\operatorname{Fix} G$ *is negligible or almost negligible,*

  iii) *all the points of* $\operatorname{Fix} G$ *are of type 1,*

  iv) $l_{p-1}^{2k}(X) = 0$ *for all* $1 \leq k \leq n$, *and*

  v) $l_1^{2k+1}(X) = 0$ *for all* $0 \leq k \leq n-1$, *when* $n > 1$.

Then:

  1) $l_1^{2n}(X) - h^{2*}(\operatorname{Fix} G, \mathbb{Z})$ *is divisible by 2, and*

  2) *we have:*

$$l_1^{2n}(X) + 2 \left[\sum_{i=0}^{n-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{n-1} l_1^{2i}(X) \right]$$
$$\geq h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 2 \operatorname{rktor} H^{2n}(\widetilde{M}, \mathbb{Z})$$
$$\geq 2 \left[\sum_{i=0}^{n-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{n-1} l_1^{2i}(X) \right].$$

  3) *If moreover*

$$l_1^{2n}(X) + 2 \left[\sum_{i=0}^{n-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{n-1} l_1^{2i}(X) \right]$$
$$= h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 2 \operatorname{rktor} H^{2n}(\widetilde{M}, \mathbb{Z}),$$

  then (X, G) *is* H^{2n}-*normal.*

After the study of actions with fixed points of type 1, we will be able to treat more general local actions in the case $p = 3$. We will see that we can reduce the problem to the case of fixed points of type 1 by a blowup of X in the fixed points of different type, which will be called points of type 2.

As we can see, the notion of H^k-normality could be generalized in many directions. If $H^k(X, \mathbb{Z})$ is not torsion-free, one might work with $H^k(X, \mathbb{Z})/\operatorname{tors}$. It could be also generalized when X is not smooth or when p is not a prime number. Theorem 0.3.4 can also be generalized to more general local actions of G using toric blowups in the proof instead of classical blowups. Thus there are many ways to generalize our work.

We first illustrate our results on H^*-normality by calculating the cup-product lattice of a K3 surface quotiented by a symplectic involution, by a symplectic and

non-symplectic automorphism of order 3. In the next table, we denote these quotients respectively by $\overline{Y_2}$, $\overline{Y_3}$ and $\overline{Z_3}$. We also calculate the cup-product lattice of a complex torus of dimension 2 quotiented by $-\operatorname{id}$; we denote this quotient by $\overline{A}$. Then we go on to the main application of our tools, providing the Beauville–Bogomolov lattice of irreducible symplectic varieties of $K3^{[2]}$-type quotiented by certain symplectic automorphisms. The first example is the quotient by numerically standard symplectic automorphisms of order 3 (see Section 1.3.2 for the definition of 'numerically standard'). We also provide the Beauville–Bogomolov form of partial resolutions of irreducible symplectic varieties of $K3^{[2]}$-type quotiented by symplectic involutions. We denote these two 4-dimensional singular irreducible symplectic varieties by M_3 and M' respectively. We summarize the results of our calculations in the following table:

X/G	$H^2(X/G, \mathbb{Z})$
$\overline{Y}_2$	$E_8(-1) \oplus U(2)^3$
$\overline{Y}_3$	$U(3) \oplus U^2 \oplus A_2^2$
$\overline{Z}_3$	$U \oplus E_6$
$\overline{A}$	$U(2)$
M_3	$U(3) \oplus U^2 \oplus A_2^2 \oplus (-6)$
M'	$U(2)^3 \oplus E_8(-1) \oplus (-2)^2$.

Here $H^2(X/G, \mathbb{Z})$ is endowed with the cup-product for the surfaces and with the Beauville–Bogomolov form for the fourfolds. See Section 4.7 for more details.

The last chapter is devoted to Markushevich–Tikhomirov varieties. The construction of the relative compactified Prymian $\mathcal{P}$ starts from a pair of totally tangent plane quartics $\overline{B_0}$ and $\overline{\Delta_0}$. The first is used to construct a degree 2 del Pezzo surface X, and the second determines a K3 double cover S of X. Then the wanted family of curves is a non-complete linear system of curves on S, invariant under the Galois involution of the double cover $S \to X$, and $\mathcal{P}$ is its relative compactified Prymian. Permuting the roles of $\overline{B_0}$, $\overline{\Delta_0}$, we obtain another K3 surface $\widetilde{S}$ and the corresponding Prymian $\widetilde{\mathcal{P}}$.

We will prove that the Lagrangian fibrations on $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are dual to each other. Moreover, we will prove that not only $S \not\cong \widetilde{S}$ for generic S, but also that the derived categories of $\widetilde{S}$, S are non-equivalent and $\widetilde{S}^{[2]} \not\cong S^{[2]}$. This will allow us to conclude that the associated compactified Prymians $\widetilde{\mathcal{P}}$, $\mathcal{P}$ are non-isomorphic.

Moreover, $\mathcal{P}$ is related to M' by a Mukai flop, hence we also get the Beauville–Bogomolov lattice of these compactified Prymians from that of M'.

0.3.3 Structure of the thesis

Chapter 1 provides a survey of known results on irreducible symplectic varieties. In Section 1.1, we recall the main results about the period map for smooth irreducible symplectic varieties. We recall the local Torelli Theorem of Beauville. Then, we recall the results on non-separated points of the moduli space and on the surjectivity of the period map (Global Torelli Theorem) after

Huybrechts and Verbitsky. We also list the Beauville–Bogomolov lattices for known irreducible symplectic manifolds.

Then we introduce singular irreducible symplectic varieties in Section 1.2. In particular, we recall the construction of Markushevich and Tikhomirov.

In Section 1.3.2, we cite results on the automorphisms of $K3^{[2]}$-type manifolds by Mongardi. In Section 1.3.3, we recall the description of the integral cohomology of the Hilbert scheme of two points on a K3 surface following Markman, Verbitsky, Boissière–Nieper-Wisskirchen–Sarti, and Qin–Wang.

Chapter 2 contains various results on lattices, Smith theory and equivariant cohomology needed for the calculation of the integral cohomology of quotient varieties. We also recall the setting of Boissière, Nieper-Wisskirchen and Sarti [11] which provides very useful tools for our study.

In **Chapter 3**, we consider a compact complex manifold X and an automorphism group G of prime order p acting on X. This chapter contains technical results on the calculation of the integral cohomology of X/G. In Section 3.2.2, we calculate the integral cohomology of a quotient when the action of the group is free. Then, in Section 3.3, we introduce the H^*-normality and we state all the general results on this notion. In Section 3.5, we provide more precise results in particular cases of fixed points of type 1. And finally, Section 3.6 is devoted to the case $p = 3$.

Chapter 4 provides several applications of the results of Chapter 3. We calculate the cup-product and Beauville–Bogomolov forms of $\overline{Y_2}$, $\overline{Y_3}$, $\overline{Z_3}$, $\overline{A}$, M_3 and M'.

For the first five of these six varieties, the calculation follows directly from the results of Chapter 3. The case of M' is more complicated. Let S be a K3 surface and i a symplectic involution on S. We can show easily that $(S^{[2]}, i^{[2]})$ is H^4 and H^2-normal. But it is not enough for calculating the Beauville–Bogomolov form of M', which is a partial resolution of $S^{[2]}/i^{[2]}$. In fact, the largest part of this chapter is devoted to the case of M', in which we have to write down the action of $i^{[2]}$ explicitly in a special basis of $H^*(S^{[2]}, \mathbb{Z})$, obtained with the help of Nakajima's representation of the Heisenberg algebra on the cohomology of the varieties $S^{[n]}$.

Chapter 5 is devoted to Markushevich–Tikhomirov varieties. We provide the dual of the Lagrangian fibration in Section 5.1. In Section 5.2 and Section 5.3, we prove that $S \not\simeq \widetilde{S}$ for generic S, and also that the derived categories of $\widetilde{S}$, S are non-equivalent and $\widetilde{S}^{[2]} \not\simeq S^{[2]}$. In conclusion, we provide the Beauville–Bogomolov form for these varieties.

Chapter 1

Irreducible symplectic varieties

1.1 Smooth irreducible symplectic varieties

We start by recalling some background on irreducible symplectic manifolds.

1.1.1 Definition

Definition 1.1.1. *An irreducible symplectic manifold is a compact, simply connected, holomorphically symplectic Kähler manifold, whose symplectic structure is unique up to proportionality.*

We know all the irreducible symplectic manifolds in dimension 2.

Theorem 1.1.2. *The irreducible symplectic manifolds of dimension 2 are the K3 surfaces.*

1.1.2 Importance of irreducible symplectic manifolds

There are two theorems which explain the importance of irreducible symplectic manifolds in algebraic geometry. Firstly, the Bogomolov's decomposition theorem:

Theorem 1.1.3. *Let M be a compact, Kähler manifold with trivial canonical bundle. Then there exists a finite covering $\widetilde{M}$ of M which is a product of Kaehler manifolds of the following form:*

$$\widetilde{M} = T \times M_1 \times \cdots \times M_i \times K_1 \times \cdots \times K_j,$$

where T is a torus, the M_k are irreducible symplectic manifolds, and the K_k are Calabi–Yau manifolds.

And secondly the Calabi–Yau Theorem:

Theorem 1.1.4. *The classes of compact simple hyperkähler manifolds and irreducible symplectic manifolds coincide.*

1.1.3 Beauville–Bogomolov form

There exists a very beautiful theory of moduli spaces of irreducible symplectic manifolds. A fundamental tool of this theory is the Beauville–Bogomolov form.

Definition 1.1.5. *Let X be a $2n$-dimensional irreducible symplectic manifold. We define the quadratic form q_X on $H^2(X, \mathbb{C})$ by*

$$q_X(\alpha) := \frac{n}{2} \int_X (\omega\overline{\omega})^{n-1}\alpha^2 + (1-n) \int_X \omega^{n-1}\overline{\omega}^n \alpha \cdot \int_X \omega^n\overline{\omega}^{n-1}\alpha,$$

where $\alpha \in H^2(X, \mathbb{C})$ and ω is a fixed generator of $H^0(X, \Omega_X^2)$ with $\int_X \omega^n\overline{\omega}^n = 1$.

Remark: We have $q_X(\omega + \overline{\omega}) = 1$.

This form allows us to construct a moduli space of irreducible symplectic manifolds with the following theorem from Beauville [7] (Theorem 5 of part 2).

Theorem 1.1.6. *Let X be an irreducible symplectic manifold.*

1) The quadratic form q_X is non-degenerate of signature $(3, b_X - 3)$ and there is a positive real number λ such that λq_X is integral on $H^2(X, \mathbb{Z})$.

2) Let Ω be the analytic subvariety of $\mathbb{P}(H^2(X, \mathbb{C}))$ defined by

$$\Omega = \mathbb{P}(\{\alpha \in H^2(X, \mathbb{C}) | \ q_X(\alpha) = 0, \ q_X(\alpha + \overline{\alpha}) > 0\}).$$

There exists a deformation family (the Kuranishi family) of X

$$f : \mathfrak{X} \to \mathcal{M}$$

such that $\mathcal{M}$ is smooth and for each $s \in \mathcal{M}$, $\mathfrak{X}_s$ is an irreducible symplectic manifold. Moreover, we have a diffeomorphism $u : X \times \mathcal{M} \to \mathfrak{X}$. Let $\mathscr{P}$ be the map:

$$\mathscr{P} : \mathcal{M} \to \Omega$$
$$s \mapsto u_s^*(\omega_s),$$

where ω_s is the holomorphic form of $\mathfrak{X}_s$. The map $\mathscr{P}$, which is called the period map, is a local isomorphism.

Definition 1.1.7. *Let X be an irreducible symplectic manifold. The primitive integral form proportional to q_X is called the Beauville–Bogomolov form.*

Moreover the Beauville–Bogomolov form is related to the cup product by the Fujiki relation (see [18]).

Theorem 1.1.8. *Let X be an irreducible symplectic manifold of dimension $2n$ and B_X the Beauville–Bogomolov bilinear form. There exists a unique positive constant $c_X \in \mathbb{Q}$, such that for any $\alpha \in H^2(X, \mathbb{C})$:*

$$\alpha^{2n} = c_X B_X(\alpha, \alpha)^n.$$

The constant c_X is called the Fujiki constant.

Remark: We have $B_X(\omega + \overline{\omega}, \omega + \overline{\omega}) > 0$.
The Beauville–Bogomolov form allows us to define the moduli space of irreducible symplectic manifolds.

Definition 1.1.9. *Let X be an irreducible symplectic manifold. The group $H^2(X, \mathbb{Z})$ endowed with the bilinear Beauville–Bogomolov form B_X is a lattice of signature $(3, b - 3)$ with $b \geq 3$.*

Let Γ be a lattice isometric to $H^2(X, \mathbb{Z})$. An isometry $\varphi : H^2(X, \mathbb{Z}) \to \Gamma$ is called a marking of X, and (X, φ) is called a marked irreducible symplectic manifold.

We define the moduli space $\mathscr{M}_\Gamma = \{(X, \varphi)\} / \sim$, where $(X, \varphi) \sim (X', \varphi')$ if and only if there exists an isomorphism $g : X \simeq X'$ such that $g^* = \pm(\varphi^{-1} \circ \varphi')$.

The Beauville theorem allows us to endow $\mathscr{M}_\Gamma$ with a structure of a non-separated complex manifold (the period maps from the Beauville theorem are the coordinate charts). Moreover, the period map can be considered as a holomorphic map on all of $\mathscr{M}_\Gamma$:

$$\mathscr{P} : \ \mathscr{M}_\Gamma \to \Omega = \mathbb{P}(\{\alpha \in \Gamma \otimes \mathbb{C} | \ \alpha^2 = 0, \ (\alpha + \overline{\alpha})^2 > 0\}) \subset \mathbb{P}(\Gamma \otimes \mathbb{C})$$
$$(X, \varphi) \mapsto \varphi(\omega_X),$$

where ω_X is the holomorphic form of X.

1.1.4 Properties of the moduli space of irreducible symplectic manifolds

The first two important results are from Huybrechts [25] (Theorem 4.3, Theorem 4.6' and Theorem 8.1). One on the non-separated points of $\mathscr{M}_\Gamma$ and the other on the surjectivity of the period map.

Theorem 1.1.10. *Let Γ be a lattice of signature $(3, b - 3)$ with $b \geq 3$. If $(X, \varphi), (X', \varphi') \in \mathscr{M}_\Gamma$ are non-separated points in the moduli space of marked irreducible symplectic manifolds, then X and X' are birational.*

Theorem 1.1.11. *Let X and X' be birational projective irreducible symplectic manifolds. Then there exist two markings $\varphi : H^2(X, \mathbb{Z}) \simeq \Gamma$ and $\varphi' : H^2(X', \mathbb{Z}) \simeq \Gamma$ such that $(X, \varphi), (X', \varphi') \in \mathscr{M}_\Gamma$ are non-separated points.*

The proof of this theorem uses the following lemma.

Lemma 1.1.12. *If X and X' are birational irreducible symplectic manifolds, then there exists a natural Hodge isometry between $H^2(X, \mathbb{Z})$ and $H^2(X', \mathbb{Z})$, (that is an isomorphism compatible with the Beauville–Bogomolov forms and the Hodge structures).*

Theorem 1.1.13. *Let Γ be a lattice of signature $(3, b-3)$ with $b \geq 3$. Let $\mathscr{M}_\Gamma^\circ$ be a non-empty connected component of the moduli space $\mathscr{M}_\Gamma$ of marked irreducible symplectic manifolds. Then the period map:*

$$\mathscr{P} : \mathscr{M}_\Gamma^\circ \to \Omega$$

is surjective.

Another important result from Huybrechts [26] (Theorem 2.1) is about the finiteness of connected components of $\mathscr{M}_\Gamma$.

Theorem 1.1.14. *The moduli space $\mathscr{M}_\Gamma$ has only a finite number of connected components.*

Quite an important problem was to obtain an analog of the global Torelli theorem for irreducible symplectic manifold. This was done by Verbitsky in [66] (see also [23] for our approach).

Theorem 1.1.15. *The period map $\mathscr{P} : \mathscr{M}_\Gamma \to \Omega$ factors through the 'Hausdorff reduction' $\overline{\mathscr{M}_\Gamma}$ of $\mathscr{M}_\Gamma$. More precisely, there exists a complex Hausdorff manifold $\overline{\mathscr{M}_\Gamma}$ and a locally biholomorphic map factoring the period map:*

$$\mathscr{P} : \mathscr{M}_\Gamma \twoheadrightarrow \overline{\mathscr{M}_\Gamma} \to \Omega,$$

such that $x = (X, \varphi), y = (X', \varphi') \in \mathscr{M}_\Gamma$ map to the same point in $\overline{\mathscr{M}_\Gamma}$ if and only if they are inseparable points of $\mathscr{M}_\Gamma$.

Theorem 1.1.16. *If $\overline{\mathscr{M}_\Gamma}^\circ$ is a connected component of $\overline{\mathscr{M}_\Gamma}$, then $\mathscr{P} : \overline{\mathscr{M}_\Gamma}^\circ \to \Omega$ is an isomorphism.*

1.1.5   Beauville–Bogomolov lattice

From the last theorems it follows that the Beauville–Bogomolov lattice $(H^2(X, \mathbb{Z}), B_X)$ encodes an important topological information on the irreducible symplectic manifold X. There are very few known deformation classes of irreducible symplectic manifolds, and for all of them the Beauville–Bogomolov lattices have been calculated:

X	$\dim X$	c_X	$b_2(X)$	$(H^2(X, \mathbb{Z}), B_X)$
$S^{[n]}$	$2n$	$\frac{(2n)!}{n!2^n}$	23	$U^3 \oplus E_8(-1)^2 \oplus (-2(n-1))$
$K_n(T)$	$2n$	$\frac{(2n)!(n+1)}{n!2^n}$	7	$U^3 \oplus (-2(n+1))$
$\widetilde{\mathscr{M}}$	6	60	8	$U^3 \oplus (-2)^2$
$\mathbb{M}$	10	945	24	$U^3 \oplus E_8(-1) \oplus \Lambda$

Here $\Lambda = \begin{pmatrix} -6 & 3 \\ 3 & -2 \end{pmatrix}$, the number c_X is the Fujiki constant, the manifold $S^{[n]}$ is the Hilbert scheme of n points on a K3 surface S, $K_n(T)$ is the generalized

Kummer manifold of dimension n for a torus T, and $\widetilde{\mathscr{M}}$, $\mathbb{M}$ are two families of sporadic irreducible symplectic manifolds of O'Grady.

A very difficult problem is to find other deformation classes of irreducible symplectic manifolds. There are more examples if we allow our varieties to be singular. Moreover, the singular irreducible symplectic varieties are a natural object of study, since they arise as moduli spaces of semistable sheaves (or twisted sheaves, or objects of the derived category endowed with a Bridgeland stability condition) on a K3 or abelian surface. Moreover, as follows from Namikawa, their deformation theory and period mapping behave similarly to the case of smooth ones.

1.2   Singular irreducible symplectic varieties

1.2.1   Definition

We adapt the definition of singular irreducible symplectic varieties given by Namikawa in [48].

Definition 1.2.1. *A normal compact Kähler variety Z is said to be symplectic if there is a nondegenerate holomorphic 2-form ω on the smooth locus U of Z which extends to a regular 2-form $\widetilde{\omega}$ on a desingularization $\widetilde{Z}$ of Z. If, moreover, $\dim H^1(Z, \mathcal{O}_Z) = 0$ and $\dim H^0(U, \Omega_U^2) = 1$, we say that Z is an irreducible symplectic variety.*

1.2.2   Beauville–Bogomolov form and local Torelli theorem

Namikawa [48] defines a Beauville–Bogomolov form on these varieties and provides a local Torelli theorem.

Definition 1.2.2. *Let Z be a $2n$-dimensional irreducible symplectic variety and $\nu : \widetilde{Z} \to Z$ a resolution of singularities of Z. Assume that*

- *the codimension of the singular locus of Z is ≥ 4;*

- *Z has only $\mathbb{Q}$-factorial singularities.*

We define the quadratic form q_Z on $H^2(Z, \mathbb{C})$ by

$$q_Z(\alpha) := \frac{n}{2} \int_{\widetilde{Z}} (\widetilde{\omega}\overline{\widetilde{\omega}})^{n-1} \widetilde{\alpha}^2 + (1-n) \int_{\widetilde{Z}} \widetilde{\omega}^{n-1} \overline{\widetilde{\omega}}^n \widetilde{\alpha} \cdot \int_{\widetilde{Z}} \widetilde{\omega}^n \overline{\widetilde{\omega}}^{n-1} \widetilde{\alpha},$$

where $\widetilde{\alpha} := \nu^*\alpha$, $\alpha \in H^2(Z, \mathbb{C})$ and $\int_{\widetilde{Z}} \widetilde{\omega}^n \cdot \overline{\widetilde{\omega}}^n = 1$.

We say that a normal variety has only $\mathbb{Q}$-factorial singularities if every Weil divisor is $\mathbb{Q}$-Cartier.

Let Z be a symplectic variety, $F := \mathrm{Sing}(Z)$ and $U := Z \setminus F$. Let $\overline{f} : \mathscr{Z} \to \mathcal{S}$ be the Kuranishi family of Z, which is, by definition, a semi-universal flat deformation of Z with $\overline{f}^{-1}(0) = Z$ for the reference point $0 \in \mathcal{S}$. When $\mathrm{codim}\, F \geq 4$, $\mathcal{S}$ is smooth by [47]. $\mathscr{Z}$ is not projective over $\mathcal{S}$, but one can show that every member of the Kuranishi family is a symplectic variety. Define $\mathcal{U}$ to be the locus in $\mathscr{Z}$ where $\overline{f}$ is a smooth map and let $f : \mathcal{U} \to \mathcal{S}$ be the restriction of $\overline{f}$ to $\mathcal{U}$. Then we have:

Theorem 1.2.3. *Let Z be a projective irreducible symplectic variety. Assume that*

- *the codimension of the singular locus of Z is ≥ 4;*

- *Z has only $\mathbb{Q}$-factorial singularities.*

Then the following holds.

(1) $R^2 f_*(f^{-1}\mathcal{O}_{\mathcal{S}})$ is a free $\mathcal{O}_{\mathcal{S}}$-module of finite rank. Let $\mathcal{H}$ be the image of the composite $R^2\overline{f}_*\mathbb{C} \to R^2 f_*\mathbb{C} \to R^2 f_*(f^{-1}\mathcal{O}_{\mathcal{S}})$. Then $\mathcal{H}$ is a local system on $\mathcal{S}$ with $\mathcal{H}_s = H^2(\mathcal{U}_s, \mathbb{C})$ for $s \in \mathcal{S}$.

(2) The form q_Z is independent of the choice of $\nu : \widetilde{Z} \to Z$.

(3) Put $H := H^2(U, \mathbb{C})$. Then there exists a trivialization of the local system $\mathcal{H} : \mathcal{H} \simeq H \times \mathcal{S}$. Let $\Omega := \{ x \in \mathbb{P}(H) |\ q_Z(x) = 0,\ q_Z(x + \overline{x}) > 0 \}$. Then one has a period map $p : \mathcal{S} \to \Omega$ and p is a local isomorphism.

Moreover Matsushita [38] (Theorem 1.2) showed the following theorem.

Theorem 1.2.4. *Let Z be a projective irreducible symplectic variety of dimension $2n$ with only $\mathbb{Q}$-factorial singularities, and $\mathrm{codim}\,\mathrm{Sing}\, Z \geq 4$. There exists a unique indivisible integral symmetric bilinear form $B_Z \in S^2(H^2(Z, \mathbb{Z}))^*$ and a unique positive constant $c_Z \in \mathbb{Q}$, such that for any $\alpha \in H^2(Z, \mathbb{C})$,*

$$\alpha^{2n} = c_Z B_Z(\alpha, \alpha)^n. \tag{1.1}$$

For $0 \neq \omega \in H^0(\Omega_U^2)$

$$B_Z(\omega + \overline{\omega}, \omega + \overline{\omega}) > 0. \tag{1.2}$$

Moreover the signature of B_Z is $(3, h^2(Z, \mathbb{C}) - 3)$.

The form B_Z is proportional to q_Z and is called the Beauville–Bogomolov form of Z.

Proof. The statement of the theorem in [38] does not say that the form is integral, but it follows from Lemma 2.2 of [38] using the proof of Theorem 5 a), c) of [7]. □

Remark: As proved in [31] (Theorems 3.3.18 and 3.5.11), Theorems 1.2.3 and 1.2.4 hold without the assumption of projectivity of Z. In our work, we will always deal with projective irreducible symplectic varieties.

We also give a very useful proposition which follows from this theorem.

Proposition 1.2.5. *Let Z be a projective irreducible symplectic variety of dimension $2n$ with only $\mathbb{Q}$-factorial singularities and such that* $\operatorname{codim} \operatorname{Sing} Z \geq 4$. *The equality (1) of Theorem 1.2.4 implies that*

$$\alpha_1 \cdot \ldots \cdot \alpha_{2n} = \frac{c_Z}{(2n)!} \sum_{\sigma \in S_{2n}} B_Z(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \cdots B_Z(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)}), \quad (1.3)$$

for all $\alpha_i \in H^2(Z, \mathbb{Z})$.

The equality (1) of Theorem 1.2.4 is called the Fujiki relation, and (1.3) is its polarized form.

Remark: For the moment, not much is known about the period mapping for singular irreducible symplectic varieties. There is the foundational result of Namikawa (Theorem 1.2.3), but no global Torelli, nor an analog of Huybrechts' description of the non-separated points of the moduli space.

1.2.3 Examples of singular irreducible symplectic varieties: the Markushevich–Tikhomirov varieties

Markushevich and Tikhomirov provide in [36] the first construction of an irreducible symplectic V-manifold in a way, other than quotienting a smooth irreducible symplectic variety by a symplectic action.

The idea of Markushevich and Tikhomirov is quite promising for constructing irreducible symplectic varieties, because there are many ways to generalize their construction.

In this construction, which we will recall here, (S, τ) is a 2-elementary K3 surface (a K3 surface with an anti-symplectic involution) with Mukai invariant $(8, 8, 1)$. The surface S is the double cover of a del Pezzo surface of degree 2. It is possible to apply the same construction to other kinds of 2-elementary K3 surfaces. This will provide new kinds of irreducible symplectic varieties.

For instance, Arbarello, Saccà and Ferretti in [1] have considered the case of K3 surfaces which are double covers of Enriques surfaces (2-elementary K3 surfaces with Mukai invariant $(10, 10, 0)$). Matteini in [39] provides a similar construction for a 2-elementary K3 surface with Mukai invariant $(7, 7, 1)$.

Nikulin has classified all 2-elementary K3 surfaces in [52]. There are 75 deformation classes of them.

Now we recall this construction, and it will be studied in greater detail in Chapter 5.

Let $\overline{B_0}$ be a smooth quartic curve in $\mathbb{P}^2$. Let $\mu : X \to \mathbb{P}^2$ be the double cover branched in $\overline{B_0}$. Then X is a Del Pezzo surface of degree 2. Let $\overline{\Delta_0}$ be a smooth quartic curve in $\mathbb{P}^2$ totally tangent to $\overline{B_0}$ at eight distinct points that lie on a conic. This is the case when the linear pencil $\langle \overline{B_0}, \overline{\Delta_0} \rangle$ contains a double conic. Let $B_0 = \mu^{-1}(\overline{B_0})$. We have $\mu^{-1}(\overline{\Delta_0}) = \Delta_0 + i(\Delta_0)$, where Δ_0 is a smooth curve. By Lemma 5.14 of [34], $\Delta_0 \in |-2K_X|$. Finally, let $\rho : S \to X$ be the double cover branched in Δ_0, $\Delta = \rho^{-1}(\Delta_0)$. Note that if we take a similar

double cover branched in $i(\Delta_0)$, $\rho' : S' \to X$, we get a surface S' isomorphic to S (indeed, $i \circ \rho'$ and ρ are two double covers branched in the same curve in X). Denote by τ the involution of S induced by ρ. We have the following diagram:

$$
\begin{array}{ccc}
B_0 & =\!=\!=\!=\!= & \overline{B_0} \\
\end{array}
$$

$$(1).$$

We also allow the case where $\overline{B_0}$ is a quartic and $\overline{\Delta_0}$ is equal to a double conic $2Q$ such that $\Delta_0 = \mu^{-1}(Q)$ is a smooth curve. In this case $i(\Delta_0) = \Delta_0$ and Δ_0 is in $|-2K_X|$. We will have some additional conditions for matching with [36]. Define also $H = \rho^*(-K_X)$.

The involution τ of the double cover $\rho : S \to X$ is H-linear and induces an involution on $|H| \simeq \mathbb{P}^3$, whose fixed locus consists of two components: a point and a plane. The plane parametrizes the curves of the form $\rho^{-1}\mu^{-1}(t)$, where t is a line in $\mathbb{P}^2$. Thus this plane is parametrized by the dual of $\mathbb{P}^2$, denoted $\mathbb{P}^{2\vee}$. Let $\epsilon : \mathcal{C} \to \mathbb{P}^{2\vee}$ be the linear subsystem of τ-invariant curves parametrized by $\mathbb{P}^{2\vee}$. The properties of this linear subsystem must be as in [36].

Consequently we require the following conditions for the couple $(\overline{B_0}, \overline{\Delta_0})$.

Definition 1.2.6. *A pair $(\overline{B_0}, \overline{\Delta_0})$ will be called sufficiently generic if the following conditions are satisfied:*

- *The quartic $\overline{B_0}$ must not have a tangent line with multiplicity 4 in a point. In this case $\overline{B_0}$ has exactly 28 bitangent lines $m_1, \dots, m_{28}$.*

  The curve $\mu^{-1}(m_i)$ is the union of two rational curves $l_i \cup l_i'$ meeting in 2 points. The 56 curves l_i, l_i' are all the lines on X, that is, curves of degree 1 with respect to $-K_X$. Further, the curves $C_i = \rho^{-1}(l_i)$, $C_i' = \rho^{-1}(l_i')$ are plane conics on S with respect to the injection $S \hookrightarrow \mathbb{P}^3$ defined by $|H|$.

- *A bitangent line of $\overline{B_0}$ tangent at $\overline{B_0}$ in a point p must not be tangent at $\overline{\Delta_0}$ in this same point p. In this case, the conics C_i, C_i' are not tangent, so meet in exactly 4 distinct points.*

- *The quartics $\overline{B_0}$ and $\overline{\Delta_0}$ must not have a common bitangent line. In this case the conics C_i and C_i' are irreducible. Moreover S contains no lines.*

We will denote the set of sufficiently generic pairs $(\overline{B_0}, \overline{\Delta_0})$ by $\mathfrak{L}$.

Assume for the rest of the section that $(\overline{B_0}, \overline{\Delta_0}) \in \mathfrak{L}$.

Let $\mathcal{M} = M_S^{H,ss}(0, H, -2)$ be the moduli space of semistable sheaves $\mathcal{F}$ on S with respect to the ample class H with Mukai vector $v(\mathcal{F}) = (0, H, -2)$. This moduli space has the following properties.

Proposition 1.2.7. *(i) $\mathcal{M}$ is an irreducible projective variety of dimension 6. The open part $\mathcal{M}^* = M_S^{H,s}(0, H, -2)$ corresponding to the stable sheaves is contained in the smooth locus of $\mathcal{M}$ and is a holomorphically symplectic manifold with symplectic form $\alpha \in H^0(\mathcal{M}^*, \Omega^2)$ induced by the Yoneda pairing*

$$\alpha_{[\mathcal{L}]} : \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \times \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \to \mathrm{Ext}^2(\mathcal{L}, \mathcal{L}) \xrightarrow{\mathrm{Tr}} H^2(S, \mathcal{O}_S) \simeq \mathbb{C},$$

where $[\mathcal{L}] \in \mathcal{M}^*$ is the class of a stable sheaf $\mathcal{L}$ and the tangent space $T_{[\mathcal{L}]}\mathcal{M}^*$ is identified with $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$.

(ii) $\mathcal{M}$ parametrizes the S-equivalence classes of pure 1-dimensional sheaves $\mathcal{L}$ whose supports are curves from the linear system $|H|$ and such that $\mathcal{L}|_C$ is a torsion-free $\mathcal{O}_C$-module of rank 1 with $\chi(\mathcal{L}) = -2$, where $C = \mathrm{Supp}\,\mathcal{L}$. In the case when $\mathcal{L}$ is invertible as a sheaf on its support, the condition $\chi(\mathcal{L}) = -2$ is equivalent to saying that $\deg \mathcal{L} = 0$.

(iii) The moduli space $\mathcal{M}$ contains exactly 28 S-equivalence classes of strictly semistable sheaves. Each of them is the class of the sheaf $\mathcal{O}_{C_i}(-2pt) \oplus \mathcal{O}_{C'_i}(-2pt)$, (i=1,...,28), where pt stands for the class of a point on either one of the conics C_i, C'_i.

Proof. See Proposition 1.2 of [36]. □

We define an involution on $\mathcal{M}$ by

$$\sigma : \mathcal{M} \to \mathcal{M}, \ [\mathcal{L}] \mapsto [\mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S(-H))],$$

and we set $\kappa = \tau^* \circ \sigma$. One can prove that κ is a regular involution on $\mathcal{M}$ and that its fixed locus has one 4-dimensional irreducible component plus 64 isolated points.

Definition 1.2.8. *We define the compactified Prymian $\mathcal{P}_{(S,\tau)}$ as the 4 - dimensional component of $Fix(\kappa)$.*

Theorem 1.2.9. *The variety $\mathcal{P}_{(S,\tau)}$ is an irreducible symplectic V-manifold of dimension 4 with only 28 singular points analytically equivalent to $(\mathbb{C}^4 / \{\pm 1\}, 0)$.*

Proof. See Theorem 3.4, Proposition 5.4 and Corollary 5.7 of [36]. □

Now, we will introduce the Lagrangian fibration. We consider the linear subsystem $\epsilon : \mathcal{C} \to \mathbb{P}^{2\vee}$. If $t \in \mathbb{P}^{2\vee}$ is not tangent to $\overline{B_0}$ neither to $\overline{\Delta_0}$, which is the generic case, then $C_t = \epsilon^{-1}(t) = \rho^{-1}\mu^{-1}(t)$ is a smooth genus-3 curve, and $E_t = C_t/\tau$ is elliptic. The double cover $\rho_t = \rho_{|C_t} : C_t \to E_t$ is branched at 4 points of the intersection $\Delta_0 \cap E_t$, and the double cover $\mu_t = \mu_{|E_t} : E_t \to t \simeq \mathbb{P}^1$ is branched at 4 points of the intersection $\overline{B_0} \cap t$. We denote also $\tau_t = \tau_{|C_t}$. Thus, we have the tower of double covers:

$$C_t \xrightarrow{2:1} E_t \xrightarrow{2:1} \mathbb{P}^1.$$

The following Lemma introduces the (1,2)-polarized Prym surfaces:

Lemma 1.2.10. *For a generic line* $t \in \mathbb{P}^2$, $\mathrm{Prym}(C_t, \tau_t) = ker(id + \tau_t)$ *is connected, and is thus an abelian surface. The restriction of the principal polarization from* $J(C_t)$ *to* $\mathrm{Prym}(C_t, \tau_t)$ *is a polarization of type* $(1,2)$.

Proof. See Lemma 3.2. in [36]. $\qquad\qquad\square$

Theorem 1.2.11. *Identifying, as above, the 2-dimensional linear subsystem of* τ*-invariant curves in* $|H|$ *with* $\mathbb{P}^{2\vee}$, *let* $f_{\mathcal{P}_{(S,\tau)}} : \mathcal{P}_{(S,\tau)} \to \mathbb{P}^{2\vee}$ *be the map sending each sheaf to its support. Then* $f_{\mathcal{P}_{(S,\tau)}}$ *is a Lagrangian fibration on* $\mathcal{P}_{(S,\tau)}$ *and the generic fiber* $f_{\mathcal{P}_{(S,\tau)}}^{-1}(t)$ *is the (1,2)-polarized Prym surface* $\mathrm{Prym}(C_t, \tau_t)$.

Proof. See Theorem 3.4 of [36]. $\qquad\qquad\square$

In fact, $\mathcal{P}_{(S,\tau)}$ is bimeromorphic to a partial resolution of a quotient of $S^{[2]}$. Consider Beauville's involution (see Section 6 of [6]):

$$\iota_{0,S} : S^{[2]} \to S^{[2]}, \xi \mapsto \xi' = (\langle \xi \rangle \cap S) - \xi.$$

Here S is taken in its embedding as a quartic surface in $\mathbb{P}^3$, given by the linear system $|H|$, $\langle \xi \rangle$ stands for the line in $\mathbb{P}^3$ spanned by ξ, and ξ' is the residual intersection of $\langle \xi \rangle$ with S. By [6], this involution is regular whenever S contains no lines, which is true in our case. Further, τ induces on $S^{[2]}$ an involution which we will denote by the same symbol. As τ on S is the restriction of a linear involution on $\mathbb{P}^3$, $\iota_{0,S}$ commutes with τ, and the composition $\iota_S = \iota_{0,S} \circ \tau$ is also an involution.

Proposition 1.2.12. *The fixed locus of* ι_S *is the union of a K3 surface* $\Sigma \subset S^{[2]}$ *and 28 isolated points.*

Proof. See Lemma 5.3 of [36]. In fact, as follows from the recent work of Mongardi, Theorem 4.1 of [43] (we will review the result of Mongardi in Section 1.3.8), the fixed locus of any symplectic involution on an irreducible symplectic variety deformation equivalent to the Hilbert square of a K3 surface is as in the statement of the proposition. $\qquad\qquad\square$

Let $M_{(S,\tau)} = S^{[2]}/\iota_S$ and $\overline{\Sigma}$ be the image of Σ in $M_{(S,\tau)}$. We also denote by $M'_{(S,\tau)}$ the partial resolution of singularities of $M_{(S,\tau)}$ obtained by blowing up $\overline{\Sigma}$, and by $\overline{\Sigma}'$ the exceptional divisor of the blowup.

Theorem 1.2.13. *The variety* $M'_{(S,\tau)}$ *is an irreducible symplectic V-manifold whose singularities are 28 points of analytic type* $(\mathbb{C}^4/\{\pm 1\}, 0)$. *Moreover there is a Mukai flop between* $M'_{(S,\tau)}$ *and* $\mathcal{P}_{(S,\tau)}$, *which is an isomorphism between* $M'_{(S,\tau)} \setminus \Pi'$ *and* $\mathcal{P}_{(S,\tau)} \setminus \Pi$, *where* Π' *and* Π *are Lagrangian subvarieties isomorphic to* $\mathbb{P}^2$.

Proof. See Corollary 5.7 of [36]. $\qquad\qquad\square$

1.3 On symplectic manifolds of $K3^{[n]}$-type

1.3.1 Beauville–Bogomolov form

We will also need to recall some properties of the Beauville–Bogomolov form on $H^2(S^{[2]}, \mathbb{Z})$ for a K3 surface S. We can find in [7] the following representation:

$$H^2(S^{[2]}, \mathbb{Z}) = j(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}\delta, \qquad (1.4)$$

where δ is half the diagonal of $S^{[2]}$. We are going to give the definition of j. Denote by $\omega : S^2 \to S^{(2)}$ and $\epsilon : S^{[2]} \to S^{(2)}$ the quotient map and the blowup in the diagonal respectively. Also denote by Pr_1 and Pr_2 the first and the second projections $S^2 \to S$. For $\alpha \in H^2(S, \mathbb{Z})$, we define $j(\alpha) = \epsilon^*(\beta)$, where β is the element of $H^2(S^{(2)}, \mathbb{Z})$ such that $\omega^*(\beta) = Pr_1^*(\alpha) + Pr_2^*(\alpha)$. The following theorem is proved in Section 9 of [7]:

Theorem 1.3.1. *We have:*

$$B_{S^{[2]}}(j(\alpha_1), j(\alpha_2)) = \alpha_1 \cdot \alpha_2, \qquad B_{S^{[2]}}(\delta, \delta) = -2.$$

Moreover, the Fujiki constant of $S^{[2]}$ is 3, and δ is orthogonal to $j(H^2(S, \mathbb{Z}))$.

Remark: The holomorphically symplectic form on $S^{[2]}$ is given by $j(\omega_S)$, where ω_S is the holomorphically symplectic form on S. Hence j is a Hodge isometry.

1.3.2 On symplectic automorphism groups of a manifold of $K3^{[n]}$-type

In this section, we will recall the principal result of [41] which will be useful for our applications.

Definition 1.3.2. *Let S be a K3 surface and let G be a group of symplectic automorphisms on S. The group G acts by symplectic automorphisms on $S^{[n]}$. We say that the pair $(S^{[n]}, G)$ is a natural pair or that all the automorphisms from G are natural. Any pair (X, H) deformation equivalent to a natural pair is called a standard pair.*

Definition 1.3.3. *Let X be a manifold of $K3^{[n]}$-type (that is deformation equivalent to the Hilbert scheme of n points on a K3 surface), and let G be a group of symplectic automorphisms of X (which act trivially on $H^{2,0}(X)$). We denote by B_X the Beauville–Bogomolov form on X. The group G is said to be numerically standard if there exist a K3 surface S and a finite group $\mathcal{G}$ acting on S by symplectic automorphisms with the following properties:*

- *$S_G^2(X) \simeq S_{\mathcal{G}}^2(S)$, where $S_G^2(X)$ is defined in Definition-Proposition 2.2.2.*

- *$H^2(X, \mathbb{Z})^G \simeq H^2(S, \mathbb{Z})^{\mathcal{G}} \oplus \langle t \rangle$,*

- *$B_X(t, t) = -2(n-1)$, $B_X(t, H^2(X, \mathbb{Z})) = 2(n-1)\mathbb{Z}$.*

- $\mathcal{G} \simeq G$.

We cite Theorem 2.5 of [41]:

Theorem 1.3.4. *Let X be a manifold of $K3^{[n]}$-type and let $n-1$ be a power of a prime. Let G be a finite group of numerically standard symplectic automorphisms of X. Then (X, G) is a standard pair.*

In the case of involutions in dimension four we have (see [43]):

Theorem 1.3.5. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and σ a symplectic involution on X. Then there exists a K3 surface S endowed with a symplectic involution i such that (X, σ) and $(S^{[2]}, i^{[2]})$ are deformation equivalent.*

This theorem implies the following description of the fixed locus.

Theorem 1.3.6. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and σ a symplectic involution on X. Then the fixed locus of σ consists of 28 isolated points and one K3 surface.*

Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and σ a symplectic involution on X. We need to understand the action of σ on $H^2(X, \mathbb{Z})$. By Theorem 1.3.5, there exists a K3 surface S endowed with a symplectic involution i such that (X, σ) and $(S^{[2]}, i^{[2]})$ are deformation equivalent. Hence to understand the action of σ on $H^2(X, \mathbb{Z})$, it is enough to understand the action of $i^{[2]}$ on $H^2(S^{[2]}, \mathbb{Z})$. We recall that the action of i on the second cohomology group of S is the following:

Proposition 1.3.7. *There is an isometry $H^2(X, \mathbb{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1)$ such that i^* acts as follows:*

$$i^* : H^2(X, \mathbb{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \to H^2(X, \mathbb{Z}), (u, x, y) \mapsto (u, y, x). \quad (1)$$

This implies that the invariant sublattice is

$$H^2(X, \mathbb{Z})^i \cong \left\{ (u, x, x) \in U^3 \oplus E_8(-1) \oplus E_8(-1) \right\} \cong U^3 \oplus E_8(-2). \quad (*)$$

The anti-invariant sublattice is the orthogonal complement to the invariant sublattice.

$$(H^2(X, \mathbb{Z})^i)^\perp \cong \left\{ (0, x, -x) \in U^3 \oplus E_8(-1) \oplus E_8(-1) \right\} \cong E_8(-2).$$

Proof. We can find this Proposition in [21] (it is a consequence of the proof of Theorem 5.7 of [45]). $\square$

And we deduce the action of $i^{[2]}$ on $H^2(S^{[2]}, \mathbb{Z})$.

Proposition 1.3.8. *There is an isometry $H^2(S^{[2]}, \mathbb{Z}) \cong U^3 \oplus (-2) \oplus E_8(-1) \oplus E_8(-1)$ such that ι^* acts as follows:*

$$\iota^* : H^2(S^{[2]}, \mathbb{Z}) \cong U^3 \oplus (-2) \oplus E_8(-1) \oplus E_8(-1) \to H^2(S^{[2]}, \mathbb{Z})$$
$$(u, \delta, x, y) \mapsto (u, \delta, y, x). \tag{1.5}$$

The invariant sublattice is

$$H^2(S^{[2]}, \mathbb{Z})^\iota \cong \left\{ (u, \delta, x, x) \in U^3 \oplus (-2) \oplus E_8(-1) \oplus E_8(-1) \right\}$$
$$\cong U^3 \oplus (-2) \oplus E_8(-2). \tag{1.6}$$

The anti-invariant sublattice, that is the orthogonal complement to the invariant sublattice, is

$$(H^2(S^{[2]}, \mathbb{Z})^\iota)^\perp \cong \left\{ (0, 0, x, -x) \in U^3 \oplus (-2) \oplus E_8(-1) \oplus E_8(-1) \right\} \cong E_8(-2).$$

Proof. We denote $\iota := i^{[2]}$. By Section 1.3.1, we have:

$$j \circ i = \iota \circ j. \tag{1.7}$$

Moreover, by Beauville [7]

$$B_{S^{[2]}}(\delta, \delta) = -2, \qquad B_{S^{[2]}}(j(\alpha_1), j(\alpha_2)) = \alpha_1 \cdot \alpha_2 \tag{1.8}$$

for all $(\alpha_1, \alpha_2) \in H^2(S, \mathbb{Z})^2$. Now, we consider the isometry $H^2(S, \mathbb{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1)$ of Proposition 1.3.7 with

$$i^* : H^2(S, \mathbb{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \to H^2(S, \mathbb{Z}), (u, x, y) \longmapsto (u, y, x).$$

Then, by (1.4) and (1.8), we get an isometry $H^2(S^{[2]}, \mathbb{Z}) \cong U^3 \oplus (-2) \oplus E_8(-1) \oplus E_8(-1)$, and (1.7) implies the wanted formula for ι^*. $\qquad\square$

1.3.3 Theorem of Qin–Wang on integral basis of cohomology groups

We start with the following theorem.

Theorem 1.3.9. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type.*

1) *We have $H^{odd}(X, \mathbb{Z}) = 0$ and $H^*(X, \mathbb{Z})$ is torsion-free.*

2) *The cup product map $\mathrm{Sym}^2 H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q})$ is an isomorphism.*

3) *Moreover we have:*

$$H^4(X, \mathbb{Z}) / \mathrm{Sym}^2 H^2(X, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{23} \oplus (\mathbb{Z}/5\mathbb{Z}).$$

Proof.   1) See Markman [35].

2) See Verbitsky [66].

3) See Proposition 6.6 of [11].

$\square$

Let S be a K3 surface. We will provide a more precise result and construct an integral basis of $H^4(S^{[2]}, \mathbb{Z})$ using Theorem 5.4 of [61] (Qin–Wang).

Let $(\alpha_k)_{k \in \{1,\ldots,22\}}$ be an integral basis of $H^2(S, \mathbb{Z})$. We denote $\gamma_k = j(\alpha_k)$. For $\alpha \in H^*(S, \mathbb{Z})$ and $l \in \mathbb{Z}$, we denote by $\mathfrak{q}_l(\alpha) \in \text{End}(H^*(S^{[2]}, \mathbb{Z}))$ the Nakajima operators [46] and by $|0\rangle \in H^*(S^{[0]}, \mathbb{Z})$ the unit. We also denote by 1 the unit in $H^0(S, \mathbb{Z})$ and by $x \in H^4(S, \mathbb{Z})$ the class of a point. We recall the definition of Nakajima operators. Let

$$Q^{[m+n,n]} = \left\{ (\xi, x, \eta) \in S^{[m+n]} \times S \times S^{[m]} \middle| \xi \supset \eta, \ \text{Supp}(I_\eta/I_\xi) = \{x\} \right\}.$$

We have

$$\mathfrak{q}_n(\alpha)(A) = \widetilde{p}_{1*} \left(\left[Q^{[m+n,m]} \right] \cdot \widetilde{\rho}^* \alpha \cdot \widetilde{p}_2^* A \right),$$

for $A \in H^*(S^{[m]})$ and $\alpha \in H^*(S)$, where $\widetilde{p}_1$, $\widetilde{\rho}$, $\widetilde{p}_2$ are the projections from $S^{[m+n]} \times S \times S^{[m]}$ to $S^{[m+n]}$, S, $S^{[m]}$ respectively.

We have the following theorem by Qin–Wang ([61] Theorem 5.4 and Remark 5.6):

Theorem 1.3.10. *The following elements form an integral basis of $H^4(S^{[2]}, \mathbb{Z})$:*

$$\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \quad \mathfrak{q}_2(\alpha_k)|0\rangle, \quad \mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_m)|0\rangle,$$

$$\mathfrak{m}_{1,1}(\alpha_k)|0\rangle = \frac{1}{2}(\mathfrak{q}_1(\alpha_k)^2 - \mathfrak{q}_2(\alpha_k))|0\rangle,$$

with $1 \leq k < m \leq 22$.

To get a better understanding of this theorem, we will give the following proposition, which is Remark 6.7 of [11].

Proposition 1.3.11.   • *For all $k \in \{1, \ldots, 22\}$,*

$$\mathfrak{q}_2(\alpha_k)|0\rangle = \delta \cdot \gamma_k.$$

• *For all $1 \leq k \leq m \leq 22$,*

$$\gamma_k \cdot \gamma_m = (\alpha_k \cdot \alpha_m)\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_m)|0\rangle.$$

• *For all $k \in \{1, \ldots, 22\}$,*

$$\mathfrak{m}_{1,1}(\alpha_k)|0\rangle = \frac{\gamma_k^2 - \delta \cdot \gamma_k}{2} - \frac{\alpha_k^2}{2}\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.$$

• *Denote by $d : S \to S \times S$ the diagonal embedding, and by $d_* : H^*(S, \mathbb{Z}) \to H^*(S, \mathbb{Z}) \otimes H^*(S, \mathbb{Z})$ the push-forward map followed by the Künneth isomorphism. Let $d_*(1) = \sum_{k,m} \mu_{k,m}\alpha_k \otimes \alpha_m + 1 \otimes x + x \otimes 1$, $\mu_{k,m} \in \mathbb{Z}$. Since $\mu_{k,m} = \mu_{m,k}$, one has:*

$$\delta^2 = \sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle + \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle + \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.$$

Proof. We have

$$\frac{1}{2}\mathfrak{q}_2(1)\ket{0} = \delta, \quad \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha_k)\ket{0} = j(\alpha_k) = \gamma_k,$$

for all $k \in \{1, \dots, 22\}$. The cup product map $\mathrm{Sym}^2 H^2(S^{[2]}, \mathbb{Q}) \to H^4(S^{[2]}, \mathbb{Q})$ can be computed explicitly by using the algebraic model constructed by Lehn-Sorger [33]:

1) for $\alpha \in H^2(S, \mathbb{Z})$, $\frac{1}{2}\mathfrak{q}_2(1)\ket{0} \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)\ket{0} = \mathfrak{q}_2(\alpha)\ket{0}$,

2) for $\alpha, \beta \in H^2(S, \mathbb{Z})$,

$$\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)\ket{0} \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(\beta)\ket{0} = (\alpha \cdot \beta)\mathfrak{q}_1(1)\mathfrak{q}_1(x)\ket{0} + \mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta)\ket{0},$$

This implies the Proposition. $\square$

We can also give the following proposition on the cup product with $\mathfrak{q}_1(1)\mathfrak{q}_1(x)\ket{0}$.

Proposition 1.3.12. *We have:*

$$\mathfrak{q}_1(1)\mathfrak{q}_1(x)\ket{0} \cdot \mathfrak{q}_2(\alpha_k)\ket{0} = \mathfrak{q}_1(1)\mathfrak{q}_1(x)\ket{0} \cdot \mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\ket{0} = 0$$

for all $(k, l) \in \{1, \dots, 22\}^2$, *and*

$$\mathfrak{q}_1(1)\mathfrak{q}_1(x)\ket{0} \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(x)\ket{0} = 1.$$

Proof. By definition of Nakajima's operators, we find that $\mathfrak{q}_1(1)\mathfrak{q}_1(x)\ket{0}$ corresponds to the cycle $\{\xi \in S^{[2]} \,|\, \mathrm{Supp}\,\xi \ni x\}$. The element $\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_m)\ket{0}$ corresponds to the cycle $\{\xi \in S^{[2]} \,|\, \mathrm{Supp}\,\xi = x + y, \ x \in \alpha_k, \ y \in \alpha_m\}$ and $\mathfrak{q}_2(\alpha_k)\ket{0}$ corresponds to the cycle $\{\xi \in S^{[2]} \,|\, \mathrm{Supp}\,\xi = \{x\}, \ x \in \alpha_k\}$. This implies the formula. $\square$

Chapter 2

Reminders

2.1 Lattices

2.1.1 Some general facts

We will call by a lattice a free $\mathbb{Z}$-module of finite rank endowed with a nondegenerate symmetric bilinear form with values in $\mathbb{Z}$. A lattice L is called *even* if x^2 is even for each $x \in L$. For a lattice L, we will denote by $L^\vee$ the *dual* of L and by $A_L = L^\vee/L$ the *discriminant group*. We will also denote its *rank* by $r(L)$ and its *signature* by $\operatorname{sign} L = (b^+(L), b^-(L))$. We denote by $\operatorname{discr} L$ the *discriminant* of L defined as the absolute value of the determinant of the bilinear form of L.

If $\operatorname{discr} L = 1$, we will say that L is *unimodular*. Let S be a sublattice of L. We will say that S is *primitive* if L/S is free. Let S be a sublattice of L and S' a primitive sublattice of L such that $S \subset S'$. We will call S' a *minimal primitive overlattice* of S if S'/S is a finite group. If L is an even lattice, the quadratic form on L induces a non-degenerate quadratic form q_L on A_L with values in $\mathbb{Q}/2\mathbb{Z}$. The form q_L is called the *discriminant form* of L. We recall the following properties.

Proposition 2.1.1. *Let S be a lattice and S' an overlattice of S such that S'/S is a finite group. Then*

$$\operatorname{discr} S' = (\operatorname{discr} S) \cdot [\#(S'/S)]^{-2},$$

We will also need the following property.

Proposition 2.1.2. *Let X be a compact, oriented 2n-manifold. Then $H^n(X, \mathbb{Z})$ endowed with the cup product is a unimodular lattice.*

For a primitive sublattice M of a lattice L, we denote by $M^\perp$ the orthogonal of M in L. Let

$$H_M = L/(M \oplus M^\perp).$$

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Since M is primitive, the projections $p_M : H_M \to A_M$ and $p_{M^\perp} : H_M \to A_{M^\perp}$ are injective. Hence we get an isomorphism $\gamma_M^L = p_{M^\perp} \circ p_M^{-1} : p_M(H_M) \simeq p_{M^\perp}(H_M)$. If L is an even lattice, we have $q_{M^\perp} \circ \gamma_M^L = -q_M$.

If L is unimodular, p_M is an isomorphism. Hence $\gamma_M^L : A_M \to A_{M^\perp}$ is an isomorphism. We get the following proposition.

Proposition 2.1.3. *Let L be a unimodular lattice and $M \subset L$ a primitive sublattice. Then $A_M \simeq A_{M^\perp}$ and $\operatorname{discr} M = \operatorname{discr} M^\perp$.*

In Chapter 5, we will also need the following Corollary (Corollary 1.5.2 of [50]).

Corollary 2.1.4. *Let L be an even lattice. Let M_1 and M_2 be two primitive sublattices of L and let $\varphi : M_1 \to M_2$ be an isometry. The isometry φ extends to an automorphism of L if and only if there exists an isometry $\psi : M_1^\perp \to M_2^\perp$ such that $\overline{\psi} \circ \gamma_{M_1}^L = \gamma_{M_2}^L \circ \overline{\varphi}$.*

2.1.2   On 2-elementary lattices

A lattice L is *Lorentzian* if $\operatorname{sign} L = (1, r(L) - 1)$. An even lattice L is 2-*elementary* if there is an integer a with $A_L \simeq (\mathbb{Z}/2\mathbb{Z})^a$; then we set $a(L) = \dim_{\mathbb{Z}/2\mathbb{Z}} A_L$. We also define $\delta(L) = 0$ if $x^2 \in \mathbb{Z}$ for all $x \in L^\vee$, otherwise $\delta(L) = 1$.

There are two important theorems of Nikulin [50] on 2-elementary lattices.

Theorem 2.1.5. *The isometry class of an indefinite even 2-elementary lattice L is determined by the invariants $(\operatorname{sign}(L), a(L), \delta(L))$.*

Proof. See Theorem 3.6.2 of [50]. $\square$

Theorem 2.1.6. *Let L be an indefinite even 2-elementary lattice. Then the homomorphism $\mathcal{O}(L) \to \mathcal{O}(A_L)$ is surjective, where $\mathcal{O}(L)$, $\mathcal{O}(A_L)$ are the isometry groups of L, A_L.*

Proof. See Theorem 3.6.3 of [50]. $\square$

2.2   The setting of Boissière–Nieper-Wisskirchen–Sarti

In this section we recall the notation and some results of [11] that will be necessary to study H^*-normality. We also extend definitions to the case $p = 2$ missing in [11].

Let p be a prime integer and $G = \langle \varphi \rangle$ a finite group of order p. We denote $\tau := \varphi - 1 \in \mathbb{Z}[G]$ and $\sigma := 1 + \varphi + \cdots + \varphi^{p-1} \in \mathbb{Z}[G]$. Let H be a finite-dimensional $\mathbb{F}_p$-vector space equipped with a linear action of G (a $\mathbb{F}_p[G]$-module). The minimal polynomial of φ, as an endomorphism of H, divides $X^p - 1 = (X - 1)^p \in \mathbb{F}_p[X]$, hence φ admits a Jordan normal form over $\mathbb{F}_p$.

Hence we can decompose H as a direct sum of some G-modules N_q of dimension q for $1 \leq q \leq p$, where φ acts on N_q in a suitable basis by a matrix of the following form:

$$
\begin{pmatrix}
1 & 1 & & & \\
 & \ddots & \ddots & & \mathbf{0} \\
 & & & \ddots & \ddots \\
\mathbf{0} & & & & \ddots & 1 \\
 & & & & & 1
\end{pmatrix}
$$

In all the thesis, the symbol N_q will always denote the $\mathbb{F}_\mathrm{p}[G]$-module defined by the above Jordan matrix of size q. We define the integer $l_q(H)$ as the number of blocks of size q in the Jordan decomposition of the G-module H, so that $H \simeq \bigoplus_{q=1}^p N_q^{\oplus l_q(H)}$. We will also write $\mathcal{N}_q = N_q^{\oplus l_q(H)}$. Let X be a complex manifold endowed with an action of G. We define the integer $l_q^k(X)$ for $1 \leq q \leq p$ and $0 \leq k \leq 2 \dim X$ as the number of blocks of size q in the Jordan decomposition of the G-module $H^k(X, \mathbb{F}_\mathrm{p})$, so that

$$
H^k(X, \mathbb{F}_\mathrm{p}) = \sum_{q=1}^p N_q^{\oplus l_q^k(X)} = \sum_{q=1}^p \mathcal{N}_q.
$$

We also define

$$
l_q^*(X) := \sum_{k=0}^{2 \dim X} l_q^k(X).
$$

Let ξ_p be a primitive p-th root of the unity, $K := \mathbb{Q}(\xi_p)$, and $\mathcal{O}_K := \mathbb{Z}[\xi_p]$ the ring of algebraic integers of K. By a classical theorem of Masley-Montgomery [37], $\mathcal{O}_K$ is a PID if and only if $p \leq 19$. The G-module structure of $\mathcal{O}_K$ is defined by $\varphi \cdot x = \xi_p x$ for $x \in \mathcal{O}_K$. For any $a \in \mathcal{O}_K$, we denote by $(\mathcal{O}_K, a)$ the module $\mathcal{O}_K + \mathbb{Z}$ whose G-module structure is defined by $\varphi \cdot (x, k) = (\xi_p x + ka, k)$.

In [11], we can find the following proposition (Proposition 5.1). We will give also the proof which will allow us to deduce Definition-Proposition 2.2.2 and Proposition 2.2.3.

Proposition 2.2.1. *Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $3 \leq p \leq 19$. Then for $0 \leq k \leq 2 \dim X$ we have:*

- $l_i^k(X) = 0$ *for* $2 \leq i \leq p - 2$.

- $\mathrm{rk}_{\mathbb{Z}} H^k(X, \mathbb{Z}) = p l_p^k(X) + (p-1) l_{p-1}^k(X) + l_1^k(X)$.

- $\dim_{\mathbb{F}_\mathrm{p}} H^k(X, \mathbb{F}_\mathrm{p})^G = l_p^k(X) + l_{p-1}^k(X) + l_1^k(X)$.

- $\mathrm{rk}_{\mathbb{Z}} H^k(X, \mathbb{Z})^G = l_p^k(X) + l_1^k(X)$.

Proof. By a theorem of Diederichsen and Reiner [15] (Theorem 74.3), $H^k(X, \mathbb{Z})$ is isomorphic as a $\mathbb{Z}[G]$-module to a direct sum:

$$(A_1, a_1) \oplus \cdots \oplus (A_r, a_r) \oplus A_{r+1} \oplus \cdots \oplus A_{r+s} \oplus Y$$

where the A_i are fractional ideals in K, $a_i \in A_i$ are such the $a_i \notin (\xi_p - 1)A_i$ and Y is a free $\mathbb{Z}$-module of finite rank on which G acts trivially. The G-module structure on A_i is defined by $\varphi \cdot x = \xi_p x$ for all $x \in A_i$, and (A_i, a_i) denotes the module $A_i \oplus \mathbb{Z}$ whose G-module structure is defined by $\varphi \cdot (x, k) = (\xi_p x + k a_i, k)$. Since $\mathcal{O}_K$ is a PID, there is only one ideal class in K so we have an isomorphism of $\mathbb{Z}[G]$-modules:

$$H^k(X, \mathbb{Z}) \simeq \oplus_{i=1}^r (\mathcal{O}_K, a_i) \oplus \mathcal{O}_K^{\oplus s} \oplus \mathbb{Z}^{\oplus t},$$

for some $a_i \notin (\xi_p - 1)\mathcal{O}_K$. The matrix of the action of φ on $\mathcal{O}_K$ is:

$$\begin{pmatrix} 0 & & & & -1 \\ 1 & \ddots & \mathbf{0} & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ \mathbf{0} & & & 1 & -1 \end{pmatrix},$$

so its minimal polynomial over $\mathbb{Q}$ is the cyclotomic polynomial Φ_p, hence $\mathcal{O}_K$ has no G-invariant element over $\mathbb{Z}$. Over $\mathbb{F}_p$, the minimal polynomial of $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{F}_p$ is $\Phi_p = (X - 1)^{p-1}$, so $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{F}_p$ is isomorphic to N_{p-1} as a $\mathbb{F}_p[G]$-module. The matrix of the action of φ on $(\mathcal{O}_K, a)$ is:

$$\begin{pmatrix} 0 & & & & -1 & * \\ 1 & \ddots & \mathbf{0} & & \vdots & \vdots \\ & \ddots & \ddots & & \vdots & \vdots \\ & & \ddots & 0 & \vdots & \vdots \\ \mathbf{0} & & & 1 & -1 & * \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

so its minimal polynomial over $\mathbb{Q}$ is $(X-1)\Phi_p(X) = X^p - 1$, hence the subspace of invariants $(\mathcal{O}_K, a)^G$ is a one-dimensional. Over $\mathbb{F}_p$, the minimal polynomial of $(\mathcal{O}_K, a) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is $(X - 1)^p$, so $(\mathcal{O}_K, a) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is isomorphic to $N_p \simeq \mathbb{F}_p[G]$ as a $\mathbb{F}_p[G]$-module. By reduction modulo p, the universal coefficient theorem implies:

$$H^k(X, \mathbb{F}_p) \simeq N_p^{\oplus r} \oplus N_{p-1}^{\oplus s} \oplus N_1^{\oplus t},$$

as $\mathbb{F}_p[G]$-modules, so $l_p^k(X) = r$, $l_{p-1}^k(X) = s$, $l_1^k(X) = t$ and $l_i^k(X) = 0$ for $2 \leq i \leq p-2$, this proves (1) and (2). Since each block contains a one-dimensional G-invariant subspace, this implies also that: $\dim_{\mathbb{F}_p} H^k(X, \mathbb{Z})^G = l_p^k(X) + l_{p-1}^k(X) +$

$l_1^k(X)$, this proves (3). Over $\mathbb{Z}$, only the trivial G-module in $H^k(X, \mathbb{Z})$ and the G-modules $(\mathcal{O}_K, a)$ contain a G-invariant subspace of dimension 1, so:

$$\mathrm{rk}_{\mathbb{Z}} H^k(X, \mathbb{Z})^G = r + t = l_p^k(X) + l_1^k(X),$$

this proves (4). □

Assume $p = 2$. We will need some additional notation. If we consider $x \in H^k(X, \mathbb{Z})$ such that $\overline{x} \in \mathcal{N}_1$ (with $\overline{x} = x \otimes 1 \in H^k(X, \mathbb{F}_2)$), then x could be invariant or anti-invariant. We want to distinguish these two cases. We add to the setting of Boissière–Nieper-Wisskirchen–Sarti the following definition-proposition in the case $p = 2$.

Definition-Proposition 2.2.2.

1) Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $p \leq 19$. Then for $0 \leq k \leq 2 \dim X$ we have the isomorphism of $\mathbb{Z}[G]$-modules:

$$H^k(X, \mathbb{Z}) \simeq \oplus_{i=1}^r (\mathcal{O}_K, a_i) \oplus \mathcal{O}_K^{\oplus s} \oplus \mathbb{Z}^{\oplus t}$$

for some $a_i \notin (\xi_p - 1)\mathcal{O}_K$. Hence, when $3 \leq p \leq 19$, $r = l_p^k(X)$; $s = l_{p-1}^k(X)$ and $t = l_1^k(X)$.

2) In the case $p = 2$, $\mathcal{O}_K$ is anti-invariant. For all $0 \leq k \leq \dim_{\mathbb{R}} X$, we denote $t := l_{1,+}^k(X)$ and $s := l_{1,-}^k(X)$. We have $l_1^k(X) = l_{1,+}^k(X) + l_{1,-}^k(X)$.

Proof. It follows from the proof of Proposition 2.2.1 (Proposition 5.1 of [11]). □

Remark: The invariants $l_i^k(X)$, $1 \leq i \leq p$ when $p > 2$ and $l_{1,+}^k$, $l_{1,-}^k$, $l_2^k(X)$ when $p = 2$ are uniquely determined by X, G and k.

Proposition 2.2.3. *Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $p = 2$. Then for $0 \leq k \leq 2 \dim X$ we have:*

- $\mathrm{rk}_{\mathbb{Z}} H^k(X, \mathbb{Z}) = 2l_2^k(X) + l_1^k(X)$.

- $\mathrm{rk}_{\mathbb{Z}} H^k(X, \mathbb{Z})^G = l_2^k(X) + l_{1,+}^k(X)$.

Proof. It follows from proof of Proposition 2.2.1. □

Definition-Proposition 2.2.4. *Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $2 \leq p \leq 19$. Let $0 \leq k \leq 2 \dim X$.*

1) Let $S_G^k(X) := \ker(\sigma) \cap H^k(X, \mathbb{Z})$. Then $H^k(X, \mathbb{Z})^G \cap S_G^k(X) = 0$.

2) $\frac{H^k(X, \mathbb{Z})}{H^k(X, \mathbb{Z})^G \oplus S_G^k(X)}$ is a p-torsion module. There is $a_G^k(X) \in \mathbb{N}$ such that

$$\frac{H^k(X, \mathbb{Z})}{H^k(X, \mathbb{Z})^G \oplus S_G^k(X)} = (\mathbb{Z}/p\mathbb{Z})^{a_G^k(X)}.$$

3) We have
$$a_G^k(X) = l_p^k(X).$$

4) rk $S_G^k(X)$ *is divisible by* $p-1$. *We define* $m_G^k(X) := \frac{\operatorname{rk} S_G^k(X)}{p-1}$.

Proof. See Lemma 5.3, Corollary 5.8 and Definition 5.9 of [11]. $\qquad\square$

Proposition 2.2.5. *Assume that* $H^*(X, \mathbb{Z})$ *is torsion-free and* $3 \leq p \leq 19$. *Then:*
$$m_G^k(X) = l_p^k(X) + l_{p-1}^k(X).$$

Proof. See Corollary 5.10 of [11]. $\qquad\square$

And in the case $p = 2$, we have:

Proposition 2.2.6. *Assume that* $H^*(X, \mathbb{Z})$ *is torsion-free and* $p = 2$. *Then:*

$$m_G^k(X) = l_2^k(X) + l_{1,-}^k(X).$$

Proof. Here $m_G^k(X) = \operatorname{rk} S_G^k(X)$, so this just follows from the fact that sign $i_k^* = (l_2^k(X) + l_{1,+}^k, l_2^k(X) + l_{1,-}^k)$, where $G = \langle i \rangle$ and i_k^* is the action induced by i on $H^k(X, \mathbb{Z})$. $\qquad\square$

We also recall the following useful lemma on irreducible symplectic manifolds of $K3^{[2]}$-type. It is Lemma 6.5 of [11]. It is an analog of Lemma 3.3.9 with cohomology groups endowed with the Beauville–Bogomolov form instead of the cup-product.

Lemma 2.2.7. *Assume that* X *is an irreducible symplectic manifold of* $K3^{[2]}$-*type and* G *is an order* p *group of automorphisms of* X *with* $3 \leq p \leq 19$. *The second cohomology group is endowed with the Beauville–Bogomolov form. Then the lattice* $S_G^2(X)$ *has discriminant group* $A_{S_G^2(X)} \simeq (\mathbb{Z}/p\mathbb{Z})^{a_G^2(X)}$. *The invariant lattice* $H^2(X, \mathbb{Z})^G$ *has discriminant group* $A_{H^2(X,\mathbb{Z})^G} \simeq (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})^{a_G^2(X)}$, *and* discr $H^2(X, \mathbb{Z})^G = 2p^{a_2^G(X)}$.

Proof. We recall the proof. By Proposition 2.1.1, we have:

$$[\#H^2(X, \mathbb{Z})/(H^2(X, \mathbb{Z})^G \oplus S_G^2(X))]^2$$
$$= \operatorname{discr}(H^2(X, \mathbb{Z})^G) \cdot \operatorname{discr}(S_G^2(X)) \cdot \operatorname{discr}(H^2(X, \mathbb{Z}))^{-1}.$$

By Definition-Proposition 2.2.4

$$H := H^2(X, \mathbb{Z})/(H^2(X, \mathbb{Z})^G \oplus S_G^2(X)) = (\mathbb{Z}/p\mathbb{Z})^{a_G^2(X)}.$$

Hence , we have discr$(H^2(X, \mathbb{Z})^G) \cdot \operatorname{discr}(S_G^2(X)) = 2p^{2a_G^2}$ (because discr $H^2(X, \mathbb{Z}) = 2$). Therefore discr$(H^2(X, \mathbb{Z})^G) = 2^\epsilon p^\alpha$ and discr$(S_G^2(X)) = 2^{1-\epsilon}p^\beta$ with $\epsilon \in \{0, 1\}$ since p is odd, with $\alpha + \beta = 2a_G^2(X)$. Let

$$H = H^2(X, \mathbb{Z})/(H^2(X, \mathbb{Z})^G \oplus S_G^2(X)).$$

Since $H^2(X,\mathbb{Z})^G$ is primitive, the projections $p_{H^2(X,\mathbb{Z})^G} : H \to A_{H^2(X,\mathbb{Z})^G}$ and $p_{S_G^2(X)} : H \to A_{S_G^2(X)}$ are injective (see Section 2.1.1). We deduce that $a_G^2(X) \leq \alpha$ and $a_G^2(X) \leq \beta$. This shows that $\alpha = \beta = a_G^2(X)$.

We will prove now that G acts trivially on $A_{S_G^2(X)}$. There are two possibilities:

(1) $H \simeq A_{H^2(X,\mathbb{Z})^G}$ and $A_{S_G^2(X)}/H \simeq \mathbb{Z}/2\,\mathbb{Z}$,

(2) $H \simeq A_{S_G^2(X)}$ and $A_{H^2(X,\mathbb{Z})^G}/H \simeq \mathbb{Z}/2\,\mathbb{Z}$.

By Remark 5.4 of [11], H is a trivial G-module so in case (2) the result is clear. In case (1) one has a G-equivariant inclusion:

$$H = (\mathbb{Z}/p\,\mathbb{Z})^{a_G^2(X)} \to (\mathbb{Z}/2\,\mathbb{Z}) \oplus (\mathbb{Z}/p\,\mathbb{Z})^{a_G^2(X)} = A_{S_G^2(X)}.$$

Since p is odd, this map is trivial on the first factor. Since H is a trivial G-module. This shows that G acts trivially on $A_{S_G^2(X)}$.

Since $S_G^2(X) = \operatorname{Ker}\sigma$ and G acts trivially on $A_{S_G^2(X)}$, it follows that $A_{S_G^2(X)}$ is a p-torsion module so $\epsilon = 0$. This shows that the case (1) cannot occur, so we have $H \simeq A_{S_G^2(X)} \simeq (\mathbb{Z}/p\,\mathbb{Z})^{a_G^2(X)}$ and $A_{H^2(X,\mathbb{Z})^G} \simeq (\mathbb{Z}/2\,\mathbb{Z}) \oplus (\mathbb{Z}/p\,\mathbb{Z})^{a_G^2(X)}$. $\qquad\square$

2.3 Reminder on equivariant cohomology

Let Y be a variety and G a group acting on Y. Let $EG \to BG$ be a universal G-bundle in the category of CW-complexes. Denote by $Y_G = EG \times_G Y$ the orbit space for the diagonal action of G on the product $EG \times Y$ and $f : Y_G \to BG$ the map induced by the projection onto the first factor. The map f is a locally trivial fibre bundle with typical fibre Y and structure group G. We define the G-equivariant cohomology of Y by $H_G^*(Y) := H^*(EG \times_G Y)$. We recall that when G acts freely,

$$H^*(Y/G) \simeq H_G^*(Y),$$

where the isomorphism is induced by the natural map $f : EG \times_G Y \to Y/G$, see for instance [4]. Moreover, the Leray–Serre spectral sequence associated to the map f gives a spectral sequence converging to the equivariant cohomology:

$$E_2^{p,q} := H^p(G; H^q(Y)) \Rightarrow H_G^{p+q}(Y).$$

2.4 Reminder on the basic tools of Smith theory

A use of Smith theory will be necessary in Section 4.6.4, hence we recall here its main result. Let T be a topological space and let G be a group of prime order p acting on T. We fix a generator g of G. Let $\tau := g - 1 \in \mathbb{F}_p[G]$ and $\sigma := 1 + g + \cdots + g^{p-1} \in \mathbb{F}_p[G]$. We consider the chain complex $C_*(T)$ of T with coefficients in $\mathbb{F}_p$ and its subcomplexes $\tau^i C_*(T)$ for $1 \leq i \leq p-1$ (we have $\sigma = \tau^{p-1}$). We denote also X^G the fixed locus of the action of G on T. We can find in [11], Section 7 the following proposition.

Proposition 2.4.1.   • *[12], Theorem 3.1. For $1 \leq i \leq p-1$ there is an exact sequence of complexes:*

$$0 \longrightarrow \tau^i C_*(T) \oplus C_*(T^G) \xrightarrow{\ f\ } C_*(T) \xrightarrow{\ \tau^{p-i}\ } \tau^{p-i} C_*(T) \longrightarrow 0 \ ,$$

where f denotes the sum of the inclusions.

• *[12], p.125. For $1 \leq i \leq p-1$ there is an exact sequence of complexes:*

$$0 \longrightarrow \sigma C_*(T) \xrightarrow{\ f\ } \tau^i C_*(T) \xrightarrow{\ \tau\ } \tau^{i+1} C_*(T) \longrightarrow 0 \ ,$$

where f denotes the inclusion.

• *[12], (3.4) p.124. There is an isomorphism of complexes:*

$$\sigma C_*(T) \simeq C_*(T/G, T^G),$$

where T^G is identified with its image in T/G.

2.5   Cohomology of the blowup

We recall Theorem 7.31 of Voisin [67] which will be used in Section 3.5.2.

Let X be a Kähler manifold, and let $Z \subset X$ be a submanifold. By proposition 3.24 of [67], the blowup $\widetilde{X_Z} \xrightarrow{\tau} X$ of X along Z is still a Kähler manifold. Let $E = r^{-1}(Z)$ be the exceptional divisor. E is a projective bundle of rank $r-1$, $r = \operatorname{codim} Z$. Moreover, $j : E \hookrightarrow \widetilde{X_Z}$ is a smooth hypersurface. The Hodge structure on $H^k(\widetilde{X_Z}, \mathbb{Z})$ is described as follows.

Theorem 2.5.1. *Let $h = c_1(\mathcal{O}_E(1) \in H^2(E, \mathbb{Z})$. Then we have an isomorphism of Hodge structures:*

$$H^k(X, \mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Z, \mathbb{Z}) \right) \xrightarrow{\ \tau^* + \sum_i j_* \circ h^i \circ \tau^*_{|E}\ } H^k(\widetilde{X_Z}, \mathbb{Z}).$$

Here, h^i is the morphism of Hodge structures given by the cup-product by $h^i \in H^{2i}(E, \mathbb{Z})$. On the components $H^{k-2i-2}(Z, \mathbb{Z})$ of the left-hand side, we consider the Hodge structure of Z with bidegree shifted by $(i+1, i+1)$, so that the left-hand side is a pure Hodge structure of weight k.

Chapter 3

On the integral cohomology of quotients of complex manifolds

3.1 Conventions and notation

Notation 3.1.1. *Let X be a complex variety of dimension n.*

- *We will always consider the singular cohomology $H^*(X, \mathbb{Z})$ endowed with the cup product as a graded ring.*

- *The cup product will be denoted by a dot.*

- *When X will be compact, the group $H^n(X, \mathbb{Z})$ endowed with the cup product will be considered as a lattice.*

- *We will denote by $\operatorname{tors} H^*(X, \mathbb{Z})$ the torsion part of $H^*(X, \mathbb{Z})$ and by $H^*(X, \mathbb{Z})/\operatorname{tors}$ the torsion-free part of $H^*(X, \mathbb{Z})$.*

- *We will denote by $\operatorname{rktor} H^*(X, \mathbb{Z}) := \operatorname{rk}(\operatorname{tors} H^*(X, \mathbb{Z}))$, the rank of the torsion part of the cohomology, defined as the smallest number of generators.*

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- *We set:*

$$h^*(X, \mathbb{Z}) = \sum_{k=0}^{2 \dim X} \dim H^k(X, \mathbb{Z}),$$

$$h^{2*}(X, \mathbb{Z}) = \sum_{k=0}^{\dim X} \dim H^{2k}(X, \mathbb{Z}),$$

$$h^{2*+1}(X, \mathbb{Z}) = \sum_{k=0}^{\dim X - 1} \dim H^{2k+1}(X, \mathbb{Z}),$$

  We also set $h^{2*+\epsilon}(X, \mathbb{Z}) = h^{2*}(X, \mathbb{Z})$ if n is even and $h^{2*+\epsilon}(X, \mathbb{Z}) = h^{2*+1}(X, \mathbb{Z})$ if n is odd.

- *Assume $H^*(X, \mathbb{Z})$ is torsion-free, then for all $0 \le k \le 2n$, $H^k(X, \mathbb{Z}) \otimes \mathbb{F}_p = H^k(X, \mathbb{F}_p)$. Let $x \in H^k(X, \mathbb{Z})$, we set $\overline{x} = x \otimes 1 \in H^k(X, \mathbb{Z}) \otimes \mathbb{F}_p = H^k(X, \mathbb{F}_p)$.*

- *In all the chapter, we will also use the notation of Section 2.2.*

Remark: In almost all the statements of this chapter, we will assume that X is a compact complex manifold and G is an automorphism group of prime order p. Some results could be stated in a more general setting, but we stick to this convention in order to avoid overloading the exposition with too many technical details.

Our goal will be to calculate the cohomology of the quotient X/G. In the case when G acts freely on X, the answer can be given in terms of the equivariant cohomology.

3.2   Use of equivariant cohomology

3.2.1   General facts

Let us consider a group $G = \langle \varphi \rangle$ of prime order p. We have the following projective resolution of $\mathbb{Z}$ considered as a G-module:

$$\cdots \xrightarrow{\tau} \mathbb{Z}[G] \xrightarrow{\sigma} \mathbb{Z}[G] \xrightarrow{\tau} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \,,$$

where ϵ is the summation map: $\epsilon(\sum_{j=0}^{p-1} \alpha_j g^j) = \sum_{j=0}^{p-1} \alpha_j$.

Let now H be a $\mathbb{F}_p[G]$-module of finite dimension over $\mathbb{F}_p$ as before. The cohomology of G with coefficients in H can be computed similarly as the cohomology of the complex:

$$0 \to H \xrightarrow{\overline{\tau}} H \xrightarrow{\overline{\sigma}} H \xrightarrow{\overline{\tau}} \cdots \,,$$

where $\overline{\tau}, \overline{\sigma} \in \mathbb{F}_p[G]$ denote respectively the reduction of τ and σ modulo p. To compute $H^*(G, H)$ as an F-vector space, it is enough to compute the groups $H^*(G, N_q)$. We will denote by $v_1, \ldots, v_q$ a basis of N_q such that $\varphi(v_1) = v_1$ and $\varphi(v_i) = v_i + v_{i-1}$ for all $i \geq 2$.

Proposition 3.2.1.   *1) We have* $\ker(\overline{\tau}) = \langle v_1 \rangle$ *and* $\mathrm{Im}(\overline{\tau}) = \langle v_1, \ldots, v_{q-1} \rangle$, *for all* $q \leq p$.

    2) We have: $\ker(\overline{\sigma}) = N_q$ *and* $\mathrm{Im}(\overline{\sigma}) = 0$, *for all* $q < p$. *We have:* $\ker(\overline{\sigma}) = \langle v_1, \ldots, v_{q-1} \rangle$ *and* $\mathrm{Im}(\overline{\sigma}) = \langle v_1 \rangle$, *if* $q = p$.

    3) If $q < p$ *then* $H^i(G, N_q) = \mathbb{F}_p$ *for all* $i \geq 0$.

    4) $H^0(G, N_p) = \mathbb{F}_p$ *and* $H^i(G, N_p) = 0$ *for all* $i \geq 1$.

We deduce the following lemma.

Lemma 3.2.2. *Let X be a compact complex manifold and G an automorphism group of prime order p acting on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free. For $x \in H^k(X, \mathbb{Z})^G$, $0 \leq k \leq 2 \dim X$, there exists $y \in H^k(X, \mathbb{Z})$ such that $x = y + \varphi(y) + \cdots + \varphi^{p-1}(y)$ if and only if $\overline{x} \in \mathcal{N}_p$.*

Proof. $\Rightarrow$ If $\overline{x} = 0$, then $\overline{x} \in \mathcal{N}_p$. Now we assume that $\overline{x} \neq 0$. Then $\overline{y} \notin \ker \overline{\sigma}$, so by Proposition 3.2.1, 2), $\overline{y} \in \mathcal{N}_p$. Hence $\overline{x} \in \mathcal{N}_p$.

    $\Leftarrow$ Since $\overline{x} \in \mathcal{N}_p$, we can write $\overline{x} = \sum_i \alpha_i v_{1,i}$, where $v_{1,i}$ are invariant elements of direct summands of $\mathcal{N}_p$, isomorphic to N_p (see Proposition 3.2.1, 1)). But, we have $v_{1,i} = v_{p,i} + \varphi(v_{p,i}) + \cdots + \varphi^{p-1}(v_{p,i})$ by Proposition 3.2.1, 2). The result follows.

$\square$

From Proposition 3.2.1, we can also deduce the following proposition for concrete calculation.

Proposition 3.2.3. *Let X be a compact complex manifold and G an automorphism group of prime order p acting on X. For $0 \leq k \leq 2 \dim X$ we have:*

- $H^0(G, H^k(X, \mathbb{F}_p)) = (\mathbb{Z}/p\mathbb{Z})^{\sum_{0 \leq q \leq p} l_q^k(X)}$,

- $H^i(G, H^k(X, \mathbb{Z})) = (\mathbb{Z}/p\mathbb{Z})^{\sum_{0 \leq q < p} l_q^k(X)}$, *for all* $i > 0$.

We can apply similar considerations to the cohomology with coefficients in $\mathbb{Z}$. If H is a $\mathbb{Z}[G]$-module of finite rank over $\mathbb{Z}$, the cohomology of G with coefficients in H is computed as the cohomology of the complex:

$$0 \rightarrow H \xrightarrow{\tau} H \xrightarrow{\sigma} H \xrightarrow{\tau} \cdots.$$

We have the following proposition.

Proposition 3.2.4. *Let X be a compact complex manifold and G an automorphism group of prime order p acting on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $3 \leq p \leq 19$. Then for $0 \leq k \leq 2 \dim X$ we have:*

- $H^0(G, H^k(X, \mathbb{Z})) = H^k(X, \mathbb{Z})^G$,

- $H^{2i-1}(G, H^k(X, \mathbb{Z})) = (\mathbb{Z}/p\mathbb{Z})^{l^k_{p-1}(X)}$,

- $H^{2i}(G, H^k(X, \mathbb{Z})) = (\mathbb{Z}/p\mathbb{Z})^{l^k_1(X)}$,

for all $i \in \mathbb{N}^*$.

Proof.   • By definition, $H^0(G, H^k(X, \mathbb{Z})) = \operatorname{Ker} \tau$.

- In odd degrees, $H^{2i-1}(G, H^k(X, \mathbb{Z})) = \operatorname{Ker} \sigma / \operatorname{Im} \tau$. In the proof of Theorem 74.3 of [15], it is shown that:

$$\operatorname{Ker} \sigma = \mathcal{O}_K b_1 \oplus \cdots \oplus \mathcal{O}_K b_{r+s-1} \oplus A b_{r+s},$$
$$\operatorname{Im} \tau = E_1 b_1 \oplus \cdots \oplus E_{n-1} b_{r+s-1} \oplus E_{r+s} A b_{r+s},$$

with $b_1, \ldots, b_n$, $\mathcal{O}_K$-free elements in $\operatorname{Ker} \sigma$, A an $\mathcal{O}_K$-ideal of K and

$$E_1 = \cdots = E_r = \mathcal{O}_K, \qquad E_{r+1} = \cdots = E_{r+s} = (\xi_p - 1)\mathcal{O}_K.$$

And the r and the s in the last equalities are the same as in the proof of Proposition 2.2.1 (Proposition 5.1 of [11]). Moreover, we find in the proof of Theorem 74.3 of [15] that $\mathcal{O}_K/(\xi_p - 1)\mathcal{O}_K = A/(\xi_p - 1)A = \mathbb{Z}/p\mathbb{Z}$. Hence, we get $\operatorname{Ker} \sigma / \operatorname{Im} \tau = (\mathbb{Z}/p\mathbb{Z})^{l^k_{p-1}(X)}$.

- For $i \geq 1$, $H^{2i}(G, H^k(X, \mathbb{Z})) = \operatorname{Ker} \tau / \operatorname{Im} \sigma$. We have seen in the proof of Proposition 2.2.1 that:

$$H^k(X, \mathbb{Z})^G \simeq \oplus_{i=1}^r (\mathcal{O}_K, a_i)^G \oplus \mathbb{Z}^{\oplus t}.$$

By Lemma 3.2.2 all the elements in $\oplus_{i=1}^r (\mathcal{O}_K, a_i)^G$ can be written $y + \varphi(y) + \cdots + \varphi^{p-1}(y)$ with $y \in H^k(X, \mathbb{Z})$. The result follows.
□

Now we state a similar result in the case $p = 2$.

Proposition 3.2.5. *Let X be a compact complex manifold and G an automorphism group of order 2 acting on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free. Then for $0 \leq k \leq 2 \dim X$ we have:*

- $H^0(G, H^k(X, \mathbb{Z})) = H^k(X, \mathbb{Z})^G$,

- $H^{2i-1}(G, H^k(X, \mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z})^{l^k_{1,-}(X)}$,

- $H^{2i}(G, H^k(X, \mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z})^{l^k_{1,+}(X)}$,

for all $i \in \mathbb{N}^*$.

Proof. The same proof as in the last proposition.   □

We can give more precise results on the cohomology of the quotient by imposing additional hypothesis on the degeneration of the spectral sequence.

Definition 3.2.6. *Let G be a group of prime order p acting by automorphisms on a complex manifold X. We will say that (X, G) is E_2-degenerate if the spectral sequence of equivariant cohomology with coefficients in $\mathbb{F}_\mathrm{p}$ degenerates at the E_2-term. We will say that (X, G) is E_2-degenerate over $\mathbb{Z}$ if the spectral sequence of equivariant cohomology with coefficients in $\mathbb{Z}$ degenerates at the E_2-term.*

3.2.2 Case where G acts freely

We can use the equivariant cohomology to calculate the integral cohomology of a quotient when the action of the group is free.

Proposition 3.2.7. *Let X be a compact complex manifold and G a group of prime order, acting freely on X in such a way that (X, G) is E_2-degenerate over $\mathbb{Z}$. Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $3 \leq p \leq 19$. Then for $0 \leq 2k \leq 2 \dim X$, we have*

$$H^{2k}(X/G, \mathbb{Z}) \simeq H^{2k}(X, \mathbb{Z})^G \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\,\mathbb{Z})^{l_{p-1}^{2i+1}(X)} \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\,\mathbb{Z})^{l_1^{2i}(X)},$$

and for $0 \leq 2k+1 \leq 2 \dim X$,

$$H^{2k+1}(X/G, \mathbb{Z}) \simeq H^{2k+1}(X, \mathbb{Z})^G \bigoplus_{i=0}^{k} (\mathbb{Z}/p\,\mathbb{Z})^{l_{p-1}^{2i}(X)} \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\,\mathbb{Z})^{l_1^{2i+1}(X)}.$$

Proof. We have $H^{2k}(X/G, \mathbb{Z}) \simeq H_G^{2k}(X)$ by Section 2.3. Moreover $E_2^{p,q} := H^p(G; H^q(X)) \Rightarrow H_G^{p+q}(X)$. Since the spectral sequence of equivariant cohomology degenerates at the E_2-page,

$$H^{2k}(X/G, \mathbb{Z}) \simeq \bigoplus_{i=0}^{2k} H^i(G; H^{2k-i}(X, \mathbb{Z})),$$

and by Proposition 3.2.4,

$$H^{2k}(X/G, \mathbb{Z}) \simeq H^{2k}(V, \mathbb{Z})^G \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\,\mathbb{Z})^{l_{p-1}^{2i+1}(X)} \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\,\mathbb{Z})^{l_1^{2i}(X)}.$$

$\square$

The same formula holds for $p = 2$.

Proposition 3.2.8. *Let X be a compact complex manifold and G a group order of order 2 acting freely on X in such a way that (X, G) is E_2-degenerate over $\mathbb{Z}$. Assume that $H^*(X, \mathbb{Z})$ is torsion-free. Then*

$$H^{2k}(X/G, \mathbb{Z}) \simeq H^{2k}(X, \mathbb{Z})^G \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\,\mathbb{Z})^{l_{1,-}^{2i+1}(X)} \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\,\mathbb{Z})^{l_{1,+}^{2i}(X)}$$

for $0 \le 2k \le 2 \dim X$, and

$$H^{2k+1}(X/G, \mathbb{Z}) \simeq H^{2k+1}(X, \mathbb{Z})^G \bigoplus_{i=0}^{k} (\mathbb{Z}/p\mathbb{Z})^{l_{1,-}^{2i}(X)} \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\mathbb{Z})^{l_{1,+}^{2i+1}(X)}$$

for $0 \le 2k+1 \le 2 \dim X$.

We can replace the condition of E_2-degeneration over $\mathbb{Z}$ by conditions on the $l_i^j(X)$.

Proposition 3.2.9. *Let X be a manifold and G a group of prime order acting freely on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $3 \le p \le 19$. For $0 \le 2k \le 2 \dim X$, assume:*

   i) $l_{p-1}^{2i}(X) = 0$ for all $1 \le i \le k$,

   ii) $l_1^{2i+1}(X) = 0$ for all $0 \le i \le k-1$ when $k > 1$.

Then we have:

$$H^{2k}(X/G, \mathbb{Z}) \simeq H^{2k}(X, \mathbb{Z})^G \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\mathbb{Z})^{l_{p-1}^{2i+1}(X)} \bigoplus_{i=0}^{k-1} (\mathbb{Z}/p\mathbb{Z})^{l_1^{2i}(X)}$$

Proof. It is enough to check that all the groups $H^i(G, H^{2k+1-i}(X, \mathbb{Z}))$, $1 \le i \le 2k-1$ and $H^i(G, H^{2k-i-1}(X, \mathbb{Z}))$, $1 \le i \le 2k-2$ are trivial. By Proposition 3.2.4, this is exactly the condition on the $l_i^j(X)$.

$\square$

We have a similar result for $p = 2$.

Proposition 3.2.10. *Let X be a manifold and G a group of order 2 acting freely on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free. For $0 \le 2k \le 2 \dim X$, assume:*

   i) $l_{1,-}^{2i}(X) = 0$ for all $1 \le i \le k$,

   ii) $l_{1,+}^{2i+1}(X) = 0$ for all $0 \le i \le k-1$, $k > 1$.

then we have:

$$H^{2k}(X/G, \mathbb{Z}) \simeq H^{2k}(X, \mathbb{Z})^G \bigoplus_{i=0}^{k-1} (\mathbb{Z}/2\mathbb{Z})^{l_{1,-}^{2i+1}(X)} \bigoplus_{i=0}^{k-1} (\mathbb{Z}/2\mathbb{Z})^{l_{1,+}^{2i}(X)}$$

Remark: It is also possible to calculate $H^k(X/G, \mathbb{F}_p)$ by the spectral sequence of equivariant cohomology with coefficients in $\mathbb{F}_p$ when (X, G) is E_2-degenerate and the action of G is free. We get similar formulas using Proposition 3.2.3. Then one can deduce $H^k(X/G, \mathbb{Z})$ by the universal coefficient theorem.

3.3 H^*-normality

3.3.1 Definition

Now we want to calculate the cohomology of X/G when the action of G is not free. A fundamental tool for studying this question is given by the following proposition, which follows from [65].

Proposition 3.3.1. *Let G be a finite group of order d acting on a variety X with orbit map $\pi : X \to X/G$, which is a d-fold covering (possibly ramified). Then there is a natural homomorphism $\pi_* : H^*(X, \mathbb{Z}) \to H^*(X/G, \mathbb{Z})$ such that*

$$\pi_* \circ \pi^* = d \operatorname{id}_{H^*(X/G,\mathbb{Z})}, \qquad \pi^* \circ \pi_* = \sum_{g \in G} g^*.$$

It easily implies the corollary:

Corollary 3.3.2. *Let G be a finite group of order d acting on a variety X with the orbit map $\pi : X \to X/G$, which is a d-fold ramified covering. Then:*

1) $\pi^*_{|H^*(X/G,\mathbb{Z})/\,\mathrm{tors}}$ *is injective,*

2) $\pi_{*|H^*(X,\mathbb{Z})^G} \circ \pi^* = d \operatorname{id}_{H^*(X/G,\mathbb{Z})}$ *and* $\pi^* \circ \pi_{*|H^*(X,\mathbb{Z})^G} = d \operatorname{id}_{H^*(X,\mathbb{Z})^G}$,

3) $H^*(X/G, \mathbb{Q}) \simeq H^*(X, \mathbb{Q})^G$.

Leaving aside the question of determining the torsion of $H^*(X/G, \mathbb{Z})$, we go on to the study of the image of π_* in $H^*(X/G, \mathbb{Z})/\,\mathrm{tors}$.

Proposition 3.3.3. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p. Let $0 \le k \le 2n$, and assume that $H^k(X, \mathbb{Z})$ is torsion-free. Then there is an exact sequence:*

$$0 \longrightarrow \pi_*(H^k(X, \mathbb{Z})) \longrightarrow H^k(X/G, \mathbb{Z})/\,\mathrm{tors} \longrightarrow (\mathbb{Z}/p\,\mathbb{Z})^{\alpha_k} \longrightarrow 0,$$

where $\pi : X \to X/G$ is the quotient map and α_k is a positive integer.

Proof. Let $x \in H^k(X/G, \mathbb{Z})/\,\mathrm{tors}$. Then $px = \pi_*(\pi^*(x))$ with $\pi^*(x) \in H^k(X, \mathbb{Z})$, which implies the result. $\qquad\qquad\square$

It remains to calculate α_k. In this section our goal will be to understand, in which cases $\alpha_k = 0$.

Definition-Proposition 3.3.4. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $0 \le k \le 2n$, and assume that $H^k(X, \mathbb{Z})$ is torsion-free.*

The integer α_k from Proposition 3.3.3 will be called the coefficient of normality. *The following statements are equivalent:*

- $\alpha_k = 0$,

- *the map $\pi_* : H^k(X, \mathbb{Z}) \to H^k(X/G, \mathbb{Z})/\operatorname{tors}$ is surjective.*

- *For $x \in H^k(X, \mathbb{Z})^G$, $\pi_*(x)$ is divisible by p if and only if there exists $y \in H^k(X, \mathbb{Z})$ such that $x = y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y)$.*

If one of these statements is verified, we will say that (X, G) is H^k-normal.

Proof. The two first statements are equivalent by Proposition 3.3.3. We show that the second one is equivalent to the third one.

$\Leftarrow$ Let $x \in H^k(X/G, \mathbb{Z})/\operatorname{tors}$. We have $\pi_*(\pi^*(x)) = px$. Hence, $\pi^*(x)$ can be written in the form $y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y)$ for $y \in \pi_*(H^k(X, \mathbb{Z}))$. Then $\pi_*(\pi^*(x)) = p\pi_*(y) = px$, so $\pi_*(y) = x$.

$\Rightarrow$ Let $x \in H^k(X, \mathbb{Z})^G$ such that p divides $\pi_*(x)$. Since $\pi_* : H^k(X, \mathbb{Z}) \to H^k(X/G, \mathbb{Z})/\operatorname{tors}$ is surjective, there is $z \in H^k(X, \mathbb{Z})$ such that $p\pi_*(z) = \pi_*(x)$. We apply π^* to this equality. By Proposition 3.3.1, we get $p(z + \varphi^*(z) + \cdots + (\varphi^*)^{p-1}(z)) = px$.

$\square$

Corollary 3.3.5. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $0 \leq k \leq 2n$. Let*

$$H^k_\sigma(X, \mathbb{Z}) = \left\{ x + \varphi^*(x) + \cdots + (\varphi^*)^{p-1}(x) \mid x \in H^k(X, \mathbb{Z}) \right\}.$$

Assume that $H^k(X, \mathbb{Z})$ is torsion-free. If the pair (X, G) is H^k-normal, then the map

$$\frac{1}{p}\pi_* : H^k_\sigma(X, \mathbb{Z}) \to H^k(X/G, \mathbb{Z})/\operatorname{tors}$$

is an isomorphism, and its inverse is

$$\pi^* : H^k(X/G, \mathbb{Z})/\operatorname{tors} \to H^k_\sigma(X, \mathbb{Z}).$$

Proof. This map is clearly well defined. Since (X, G) is H^k-normal, it is surjective. It remains to show that it is injective. If $\pi_*(x + \varphi^*(x) + \cdots + (\varphi^*)^{p-1}(x))$ is a torsion element in $H^k(X/G, \mathbb{Z})$, then by Corollary 3.3.2, $x + \varphi^*(x) + \cdots + (\varphi^*)^{p-1}(x)$ is also a torsion element. Since $H^k(X, \mathbb{Z})$ is torsion-free, $x + \varphi^*(x) + \cdots + (\varphi^*)^{p-1}(x) = 0$. $\square$

We will also need the following two lemmas.

Lemma 3.3.6. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $0 \leq k \leq 2n$. Assume that $H^k(X, \mathbb{Z})$ is torsion-free. Let $\mathcal{K}'_k$ be the overlattice of $\pi_*(H^k(X, \mathbb{Z})^G)$ obtained by dividing by p all the elements of the form $\pi_*(y + \varphi(y) + \cdots + \varphi^{p-1}(y))$, $y \in H^k(X, \mathbb{Z})$. Then:*

$$\mathcal{K}'_k = \pi_*(H^k(X, \mathbb{Z})).$$

Proof. Let $y \in H^k(X, \mathbb{Z})$, we have $\pi_*(y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y)) = p\pi_*(y)$. The result follows. $\square$

Lemma 3.3.7. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p.*

1) *Let $0 \le k \le 2\dim X$, q an integer such that $kq \le 2\dim X$ and $x \in H^k(X, \mathbb{Z})^G$. Then*

$$\pi_*(x)^q = p^{q-1}\pi_*(x^q) + \text{tors}.$$

If moreover $H^{kq}(X, \mathbb{Z})$ is torsion-free, then the property that $\pi_*(x)$ is divisible by p implies that $\pi_*(x^q)$ is divisible by p.

2) *Let $0 \le k \le 2\dim X$, q an integer such that $kq \le 2\dim X$, and let $(x_i)_{1 \le i \le q}$ be elements of $H^k(X, \mathbb{Z})^G$. Then*

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_q) = p^{q-1}\pi_*(x_1 \cdot \ldots \cdot x_q) + \text{tors}.$$

3) *Let $0 \le k \le 2\dim X$, q an integer such that $kq = 2n$ and let $(x_i)_{1 \le i \le q}$ be elements of $H^k(X, \mathbb{Z})^G$. Then*

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_q) = p^{q-1}x_1 \cdot \ldots \cdot x_q.$$

Proof. 1) By Corollary 3.3.2

$$\pi^*(\pi_*(x)^q) = p^q x^q = \pi^*(p^{q-1}\pi_*(x^q)).$$

The map π^* is injective on the torsion-free part, which implies the wanted equality. If moreover $\pi_*(x)$ is divisible by p, we can write $\pi_*(x) = py$ with $y \in H^{kq}(X/G, \mathbb{Z})$. This gives:

$$p^q y^q = p^{q-1}\pi_*(x^q) + \text{tors}.$$

We cannot divide by p^{q-1} because of the possible torsion of $H^{kq}(X/G, \mathbb{Z})$. We will use the fact that $H^{kq}(X, \mathbb{Z})$ is torsion-free to get round the problem. Applying π^* to this equality, we obtain:

$$p^q \pi^*(y^q) = p^q x^q + \pi^*(\text{tors}).$$

Since $H^{kq}(X, \mathbb{Z})$ is torsion-free, $\pi^*(\text{tors}) = 0$, and we have

$$\pi^*(y^q) = x^q.$$

Pushing down by π_*, we obtain:

$$py^q = \pi_*(x^q).$$

2) The proof is similar.

3) By 2), we have:

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_q) = p^{q-1}\pi_*(x_1 \cdot \ldots \cdot x_q) + \text{tors}.$$

But $H^{2n}(X, \mathbb{Z}) = \mathbb{Z}$ is torsion-free by Poincaré duality. Identifying $H^{2n}(X, \mathbb{Z})$ with $\mathbb{Z}$, we can write $\pi_*(x_1 \cdot \ldots \cdot x_q) = x_1 \cdot \ldots \cdot x_q$.

$\square$

3.3.2 H^n-normality and cup-product lattice

Under the assumption of the H^n-normality, we can determine the cup-product lattice.

Proposition 3.3.8. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p. Assume that $H^n(X, \mathbb{Z})$ is torsion-free and (X, G) is H^n-normal. Let us denote the sublattice $H^n(X, \mathbb{Z})^G$ by L. Then:*

1) *discr $L = p^{a_G^n(X)}$, with $a_G^n(X) \in \mathbb{N}$,*

2) *$H^n(X/G, \mathbb{Z})/\operatorname{tors} \simeq L^\vee(p)$,*

3) *discr $L^\vee(p) = p^{\operatorname{rk} L - a_G^n(X)}$.*

Proof. We need the following lemma.

Lemma 3.3.9. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order. Assume that $H^n(X, \mathbb{Z})$ is torsion-free. Then:*

1) *The projection*

$$\frac{H^n(X, \mathbb{Z})}{H^n(X, \mathbb{Z})^G \oplus (H^n(X, \mathbb{Z})^G)^\perp} \to A_{H^n(X, \mathbb{Z})^G}$$

is an isomorphism.

2) *$A_{H^n(X, \mathbb{Z})^G} \simeq (\mathbb{Z}/p\,\mathbb{Z})^{a_G^n}$, with $a_G^n \in \mathbb{N}$.*

3) *Moreover, let $x \in H^n(X, \mathbb{Z})^G$. Then $\frac{x}{p} \in (H^n(X, \mathbb{Z})^G)^\vee$ if and only if there is $z \in H^n(X, \mathbb{Z})$ such that $x = z + \varphi(z) + \cdots + \varphi^{p-1}(z)$.*

Proof.   1) The first assertion follows from the unimodularity of $H^n(X, \mathbb{Z})$.

2),3) We start by proving that

$$(H^n(X, \mathbb{Z})^G)^\perp = S_G(X).$$

First $(H^n(X, \mathbb{Z})^G)^\perp \supset S_G(X)$. Indeed, let $y \in S_G(X)$ and $z \in H^n(X, \mathbb{Z})^G$. Then $(\varphi^*)^k(y) \cdot z = (\varphi^*)^k(y) \cdot (\varphi^*)^k(z) = y \cdot z$ for all $0 \le k \le p$.

Now we prove $(H^n(X, \mathbb{Z})^G)^\perp \subset S_G(X)$. Let $y \in (H^n(X, \mathbb{Z})^G)^\perp$. Then $y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y) \in (H^n(X, \mathbb{Z})^G)^\perp \cap H^n(X, \mathbb{Z})^G$. Since the cup-product form is non-degenerate, $y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y) = 0$.

Now, let x be a primitive element of $H^n(X, \mathbb{Z})^G$ and $q \in \mathbb{N}^*$ such that $\frac{x}{q} \in (H^n(X, \mathbb{Z})^G)^\vee$. Then $\frac{\overline{x}}{q} \in A_{H^n(X, \mathbb{Z})^G}$. By the first assertion, there is $z \in H^2(X, \mathbb{Z})$ and $y \in S_G(X)$ such that $z = \frac{x+y}{q}$. Then $z + \varphi^*(z) + \cdots + (\varphi^*)^{p-1}(z) = \frac{p}{q}x + \frac{y+\varphi^*(y)+\cdots+(\varphi^*)^{p-1}(y)}{q}$. But $y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y) = 0$. Hence $z + \varphi^*(z) + \cdots + (\varphi^*)^{p-1}(z) = \frac{p}{q}x$. Since x is

primitive in $H^n(X,\mathbb{Z})^G$, q divides p. Hence $q = 1$ or $q = p$. If $q = p$, we get $z + \varphi^*(z) + \cdots + (\varphi^*)^{p-1}(z) = x$.

$\square$

Since (X, G) is H^n-normal, from the last lemma and Lemma 3.3.6, we see that $H^n(X/G, \mathbb{Z})/\operatorname{tors} = \pi_*(L^\vee)$. Hence by Lemma 3.3.7 3),

$$H^n(X/G, \mathbb{Z})/\operatorname{tors} = L^\vee(p).$$

By assertions 1) and 2) of the last lemma and by Proposition 2.1.3, discr $L \oplus L^\perp = p^{2a_G^n(X)}$. But, since $H^n(X, \mathbb{Z})$ is unimodular, discr $L = \operatorname{discr} L^\perp$. Hence discr $L = p^{a_G^n(X)}$.

Moreover by assertion 2) of the last lemma, $L^\vee/L = A_L \simeq (\mathbb{Z}/p\,\mathbb{Z})^{a_G^n(X)}$. Hence, by Proposition 2.1.1, we have discr $L = (\operatorname{discr} L^\vee) \cdot p^{2a_G^n(X)}$. It follows that discr $L^\vee = p^{-a_G^n(X)}$ and discr $L^\vee(p) = p^{\operatorname{rk} L - a_G^n(X)}$.

$\square$

Remark: This Proposition defines $a_G^n(X)$ for all prime numbers p although Proposition-Definition 2.2.4 defined it just for $2 \le p \le 19$.

Corollary 3.3.10. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p such that $H^n(X, \mathbb{Z})$ is torsion-free and (X, G) is H^n-normal. Let us denote the lattice $H^n(X/G, \mathbb{Z})/\operatorname{tors}$ by N. Then*

$$H^n(X, \mathbb{Z})^G \simeq N^\vee(p).$$

Proof. We denote the sublattice $H^n(X, \mathbb{Z})^G$ by L. By Proposition 3.3.8,

$$H^n(X/G, \mathbb{Z})/\operatorname{tors} \simeq L^\vee(p).$$

The result follows from the equality $(L^\vee(p))^\vee(p) = L$.

$\square$

We can also prove an upper bound for the coefficient of normality. We start with the following lemma.

Lemma 3.3.11. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p acting on X. We assume that $H^*(X, \mathbb{Z})$ is torsion-free. Then:*

1) discr $\pi_*(H^n(X, \mathbb{Z})) = p^b$, *with* $b \in \mathbb{N}$.

2) If moreover $2 \le p \le 19$, then

$$\operatorname{discr} \pi_*(H^n(X, \mathbb{Z})) = p^{l_1^n(X)}$$

for $p \ne 2$, and

$$\operatorname{discr} \pi_*(H^n(X, \mathbb{Z})) = 2^{l_{1,+}^n(X)}$$

for $p = 2$.

Proof.   1) By Lemma 3.3.6, $\pi_*(H^n(X,\mathbb{Z})) = \mathcal{K}'_n$. Hence

$$\mathcal{K}'_n \supset \pi_*(H^n(X,\mathbb{Z})^G).$$

Then, by Proposition 2.1.1, discr $\mathcal{K}'_n$ divides discr $\pi_*(H^n(X,\mathbb{Z})^G)$. Moreover, by Proposition 3.3.8 1), discr $H^n(X,\mathbb{Z})^G = p^{a^n_G}$, and by Lemma 3.3.7 3),

$$\operatorname{discr} \pi_*(H^n(X,\mathbb{Z})^G) = p^{a^n_G + \operatorname{rk} H^n(X,\mathbb{Z})^G}.$$

Hence discr $\pi_*(H^n(X,\mathbb{Z})) = $ discr $\mathcal{K}'_n$ divides $p^{a^n_G + \operatorname{rk} H^n(X,\mathbb{Z})^G}$.

2) We prove the proposition for $3 \leq p \leq 19$; the proof for $p = 2$ is exactly the same.

By Definition-Proposition 2.2.4 3) discr $H^n(X,\mathbb{Z})^G = p^{l^n_p(X)}$. Then by Lemma 3.3.7, discr $\pi_*(H^n(X,\mathbb{Z})^G) = p^{l^n_p(X) + \operatorname{rk} H^n(X,\mathbb{Z})^G}$. Hence by Proposition 2.2.1,

$$\operatorname{discr} \pi_*(H^n(X,\mathbb{Z})^G) = p^{2l^n_p(X) + l^n_1(X)}.$$

Moreover $\mathcal{K}'_n / \pi_*(H^n(X,\mathbb{Z})^G) = (\mathbb{Z}/p\mathbb{Z})^{l^n_p(X)}$. Indeed, we have seen in proof of Proposition 2.2.1 that

$$H^k(X,\mathbb{Z})^G \simeq \oplus^r_{i=1} (\mathcal{O}_K, a_i)^G \oplus \mathbb{Z}^{\oplus t}.$$

By Lemma 3.2.2, $(\mathcal{O}_K, a_i)^G$ is generated by an element $y + \varphi^*(y) + \cdots + (\varphi^*)^p(y)$ with $y \in (\mathcal{O}_K, a_i)$.

Therefore, by Proposition 2.1.1,

$$\operatorname{discr} \mathcal{K}'_n = p^{l^n_1(X)}.$$

$\square$

Corollary 3.3.12. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p acting on X. We assume that $H^*(X,\mathbb{Z})$ is torsion-free. Let α_n be the n-th coefficient of normality of (X,G). Then:*

1) $\alpha_n \leq \frac{\log_p \operatorname{discr} \pi_*(H^n(X,\mathbb{Z}))}{2}$.

2) If moreover $2 \leq p \leq 19$, then:

$$\alpha_n \leq \frac{l^n_1(X)}{2} \text{ for } p \neq 2, \text{ and}$$
$$\alpha_n \leq \frac{l^n_{1,+}(X)}{2} \text{ for } p = 2.$$

Proof. By Proposition 2.1.1 and Proposition 3.3.3,

$$\operatorname{discr}(H^n(X/G,\mathbb{Z})/\operatorname{tors}) = \operatorname{discr}(\pi_*(H^n(X,\mathbb{Z}))) \cdot p^{-2\alpha_n}.$$

Hence, the result follows from Lemma 3.3.11. $\square$

3.3.3 General results

We now can state some criteria for the H^k-normality.

Proposition 3.3.13. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p acting on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $2 \leq p \leq 19$. Let $0 \leq k \leq 2n$.*

If $p = 2$ and $l^k_{1,+}(X) = 0$ then (X, G) is H^k-normal. If $p > 3$ and $l^k_p(X) = 0$ then (X, G) is H^k-normal. In other words, if $a^k_G(X) = \mathrm{rk}\, H^k(X, \mathbb{Z})^G$ then (X, G) is H^k-normal.

Proof. By the hypothesis, we can write:

$$H^k(X, \mathbb{Z}) \simeq \oplus^r_{i=1}(\mathcal{O}_K, a_i) \oplus \mathcal{O}^{\oplus s}_K.$$

Hence

$$H^k(X, \mathbb{Z})^G \simeq \oplus^r_{i=1}(\mathcal{O}_K, a_i)^G.$$

Let $x \in H^k(X, \mathbb{Z})^G$, then necessary $\bar{x} \in \mathcal{N}_p$. Hence by Lemma 3.2.2, $x = y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y)$. Therefore, by Proposition-Definition 3.3.4, (X, G) is H^k-normal. $\qquad \square$

Now consider the case when X/G is smooth.

Proposition 3.3.14. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order $2 \leq p \leq 19$ acting on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free and X/G is smooth. Then (X, G) is H^n-normal if and only if*

$$\mathrm{rk}\, H^n(X, \mathbb{Z})^G = a^n_G(X).$$

Proof. Assume that (X, G) is H^n-normal, then by Proposition 3.3.8,

$$H^n(X/G, \mathbb{Z})/\mathrm{tors} \simeq L^\vee(p)$$

with discr $L^\vee(p) = p^{\mathrm{rk}\, H^n(X,\mathbb{Z})^G - a^n_G(X)}$. Since X/G is smooth, $H^n(X/G, \mathbb{Z})/\mathrm{tors}$ is unimodular. The result follows.

$\square$

It is also possible to calculate the n-th coefficient of normality α_n in this case.

Proposition 3.3.15. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p acting on X. Assume that $H^*(X, \mathbb{Z})$ is torsion-free, $2 \leq p \leq 19$ and X/G is smooth. Then:*

1) $l^n_1(X)$ is even when $p > 2$, and $l^n_{1,+}(X)$ is even when $p = 2$.

2) $\alpha_n = \frac{l^n_1(X)}{2}$ when $p > 2$, and $\alpha_n = \frac{l^n_{1,+}(X)}{2}$ when $p = 2$.

Proof. By proposition 3.3.3, $(H^n(X/G, \mathbb{Z})/\mathrm{tors})/\pi_*(H^n(X, \mathbb{Z})) = (\mathbb{Z}/p\mathbb{Z})^{\alpha_n}$. Since $H^n(X/G, \mathbb{Z})/\mathrm{tors}$ is unimodular, the result follows by Proposition 2.1.1 and Lemma 3.3.11. $\qquad \square$

We can also deduce the H^k-normality from H^{kt}-normality.

Proposition 3.3.16. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order $2 \leq p \leq 19$ acting on X. Let $0 \leq k \leq 2 \dim X$, t an integer such that $kt \leq 2 \dim X$ and $H^*(X, \mathbb{Z})$ torsion-free (we have $H^*(X, \mathbb{Z}) \otimes \mathbb{F}_p = H^*(X, \mathbb{F}_p)$). Assume that (X, G) is H^{kt}-normal. If*

$$\mathscr{S} : \operatorname{Sym}^t H^k(X, \mathbb{F}_p) \to H^{kt}(X, \mathbb{F}_p)$$
$$\overline{x_1} \otimes \cdots \otimes \overline{x_t} \mapsto \overline{x_1 \cdot \ldots \cdot x_t}$$

is injective and $\mathscr{S}(\operatorname{Sym}^t H^k(X, \mathbb{F}_p))$ admits a complementary vector space, stable by the action of G, then (X, G) is H^k-normal.

Proof. We use the same notation for both Jordan decompositions of $H^k(X, \mathbb{Z})$ and of $H^{kt}(X, \mathbb{Z})$:

$$H^k(X, \mathbb{F}_p) = \sum_{q=1}^p N_q^{\oplus l_q^k(X)} = \sum_{q=1}^p \mathcal{N}_q.$$

Let $x \in H^k(X, \mathbb{Z})^G$. We assume that there is no $y \in H^k(X, \mathbb{Z})$ such that $x = y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y)$ and we show that $\pi_*(x)$ is not divisible by p.

Then, by Lemma 3.2.2, $\overline{x} \notin \mathcal{N}_p$. Since $\mathscr{S}$ is injective and $N_1^{\otimes t} = N_1$, we have $\overline{x^t} \notin \mathcal{N}_p$. By Lemma 3.2.2 there is no $z \in H^k(X, \mathbb{Z})$ such that $x^t = z + \varphi^*(z) + \cdots + (\varphi^*)^{p-1}(z)$. Since (X, G) is H^{kt}-normal $\pi_*(x^t)$ is not divisible by p. Now, since $H^{kt}(X, \mathbb{Z})$ is torsion-free, by Lemma 3.3.7 1), $\pi_*(x)$ is not divisible by p. $\qquad\square$

In particular, when $\mathscr{S}$ is an isomorphism, $\mathscr{S}(\operatorname{Sym}^t H^k(X, \mathbb{F}_p))$ admits a complementary vector space, stable by the action of G. Moreover, we can calculate the $l_q^{kt}(X)$ in terms of $l_q^k(X)$.

Proposition 3.3.17. *Let X be a topological space and G a group of prime order acting on X. Let t and k be integers. Assume that $H^*(X, \mathbb{Z})$ is torsion-free. If*

$$\mathscr{S} : \operatorname{Sym}^t H^k(X, \mathbb{F}_p) \to H^{kt}(X, \mathbb{F}_p)$$

is an isomorphism, then:

$$l_q(\operatorname{Sym}^t H^k(X, \mathbb{F}_p)) = l_q^{kt}(X),$$

where $1 \leq q \leq p$.

Under the hypotheses of the previous Proposition, we can use the following lemma (Lemma 6.14 from [11]):

Lemma 3.3.18. *Assume that* $3 \le p \le 19$, $G = \mathbb{Z}/p\mathbb{Z}$ *and let* M *be a finite* $\mathbb{F}_p[G]$-*module. Then:*

$$l_1(\mathrm{Sym}^2 M) = \frac{l_1(M) \cdot (l_1(M) + 1)}{2} + \frac{l_{p-1}(M) \cdot (l_{p-1}(M) - 1)}{2},$$

$$l_{p-1}(\mathrm{Sym}^2 M) = l_{p-1}(M) \cdot l_1(M),$$

$$l_p(\mathrm{Sym}^2 M) = \frac{p+1}{2} \cdot l_p(M) + p \cdot \frac{l_p(M) \cdot (l_p(M) - 1)}{2} + \frac{p-1}{2} \cdot l_{p-1}(M)$$
$$+ (p-1) \cdot l_p(M) \cdot l_{p-1}(M) + l_p(M) \cdot l_1(M)$$
$$+ (p-2) \cdot \frac{l_{p-1}(M) \cdot (l_{p-1}(M) - 1)}{2},$$

and $l_i(\mathrm{Sym}^2 M) = 0$ *for* $2 \le i \le p - 2$.

In some cases, one can guarantee the bijectivity of $\mathscr{S}$.

Proposition 3.3.19. *Let* X *be a topological space. Let* t *and* k *be integers and* p *a prime number. Assume that* $H^*(X, \mathbb{Z})$ *is torsion-free.*

If the cup product map $\mathrm{Sym}^t H^k(X, \mathbb{Q}) \to H^{kt}(X, \mathbb{Q})$ *is an isomorphism and* $H^{kt}(X, \mathbb{Z})/\mathrm{Sym}^t H^k(X, \mathbb{Z})$ *is* p-*torsion-free, then:*

$$\mathscr{S}: \ \mathrm{Sym}^t H^k(X, \mathbb{F}_p) \to H^{kt}(X, \mathbb{F}_p)$$

is an isomorphism.

Proof.   We prove the injectivity.

Let $\overline{x_1} \otimes \cdots \otimes \overline{x_t} \in \mathrm{Sym}^t H^k(X, \mathbb{F}_p)$ such that $\overline{x_1 \cdot \ldots \cdot x_t} = 0$. Then there exists $y \in H^{kt}(X, \mathbb{Z})$ such that $x_1 \cdot \ldots \cdot x_t = py$. Hence $\dot{y} \in H^{kt}(X, \mathbb{Z})/\mathrm{Sym}^t H^k(X, \mathbb{Z})$ is a p-torsion element (here $\dot{y}$ is the class of y modulo $\mathrm{Sym}^t H^k(X, \mathbb{Z})$). Hence by the hypothesis $\dot{y} = 0$. It follows that $y \in \mathrm{Sym}^t H^k(X, \mathbb{Z})$, so $y = y_1 \cdot \ldots \cdot y_t$ with $y_i \in H^k(X, \mathbb{Z})$. Since $\mathrm{Sym}^t H^k(X, \mathbb{Q}) \to H^{kt}(X, \mathbb{Q})$ is injective, $x_1 \otimes \cdots \otimes x_t = py_1 \otimes \cdots \otimes y_t$. So $\overline{x_1} \otimes \cdots \otimes \overline{x_t} = 0$.

We prove the surjectivity.

Let $\overline{y} \in H^{kt}(X, \mathbb{F}_p)$, with $y \in H^{kt}(X, \mathbb{Z})$. Since $\mathrm{Sym}^t H^k(X, \mathbb{Q}) \to H^{kt}(X, \mathbb{Q})$ is an isomorphism, there is $q \in \mathbb{N}$ and $x_1 \otimes \cdots \otimes x_t \in \mathrm{Sym}^t H^k(X, \mathbb{Z})$ such that $\frac{1}{q} x_1 \cdot \ldots \cdot x_t = y$. Hence $\dot{y} \in H^{kt}(X, \mathbb{Z})/\mathrm{Sym}^t H^k(X, \mathbb{Z})$ is a q-torsion element. But since $H^{kt}(X, \mathbb{Z})/\mathrm{Sym}^t H^k(X, \mathbb{Z})$ is p-torsion-free, p does not divide q. And $\mathscr{S}(\frac{1}{q}\overline{x_1} \otimes \cdots \otimes \overline{x_t}) = \overline{y}$.

$\square$

3.3.4 H^*-normality and commutative diagrams

Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $s : \widetilde{X} \to X$ be a morphism from a

compact complex manifold $\widetilde{X}$ of dimension n such that φ can be extended to an automorphism of $\widetilde{X}$. It means that there exists an automorphism $\widetilde{\varphi}$ of order p of $\widetilde{X}$ such that $s \circ \widetilde{\varphi} = \varphi \circ s$. We denote $\widetilde{G} = \langle \widetilde{\varphi} \rangle$. We can consider the quotients $M := X/G$ and $\widetilde{M} := \widetilde{X}/\widetilde{G}$. We get a Cartesian diagram

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\ r\ } & M \\
{\scriptstyle \widetilde{\pi}}\Big\uparrow & & \Big\uparrow{\scriptstyle \pi} \\
\widetilde{X} & \xrightarrow{\ s\ } & X.
\end{array}
$$

It induces a commutative diagram on cohomology:

$$
\begin{array}{ccc}
H^k(M,\mathbb{Z}) \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} H^k(X,\mathbb{Z}) & & (*) \\
{\scriptstyle r^*}\Big\downarrow \qquad\qquad \Big\downarrow{\scriptstyle s^*} & & \\
H^k(\widetilde{M},\mathbb{Z}) \underset{\widetilde{\pi}_*}{\overset{\widetilde{\pi}^*}{\rightleftarrows}} H^k(\widetilde{X},\mathbb{Z}). & &
\end{array}
$$

The idea is to find $(\widetilde{X}, \widetilde{G})$ whose H^*-normality descends to that of (X, G).

Definition 3.3.20. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $s : \widetilde{X} \to X$ be a morphism such that $\widetilde{X}$ is a compact complex manifold of dimension n and there is an automorphism $\widetilde{\varphi}$ of order p of $\widetilde{X}$ which verifies $s \circ \widetilde{\varphi} = \varphi \circ s$. We denote $\langle \widetilde{\varphi} \rangle$ by $\widetilde{G}$, and the induced map $\widetilde{X}/\widetilde{G} \to X/G$ by r.*

- *The quadruple $(\widetilde{X}, \widetilde{G}, r, s)$ will be called a* pullback *of (X, G).*

- *If moreover s is a bimeromorphic map, $s^* : H^k(X, \mathbb{F}_{\mathrm{p}}) \to H^k(\widetilde{X}, \mathbb{F}_{\mathrm{p}})$ is injective and $s^*(H^k(X, \mathbb{F}_{\mathrm{p}}))$ admits a complementary vector space, stable under the action of $\widetilde{G}$, the quadruple $(\widetilde{X}, \widetilde{G}, r, s)$ will be called a k-split* pullback *of (X, G).*

- *If $(\widetilde{X}, \widetilde{G}, r, s)$ is a k-split pullback of (X, G) and $\widetilde{M} = \widetilde{X}/\widetilde{G}$ is smooth, then $(\widetilde{X}, \widetilde{G}, r, s)$ will be called a* regular k-split pullback *of (X, G).*

- *If $(\widetilde{X}, \widetilde{G}, r, s)$ is a k-split pullback (resp. a regular k-split pullback) for all $0 \leq k \leq 2n$ of (X, G), we say that $(\widetilde{X}, \widetilde{G}, r, s)$ is a* split pullback *(resp. a* regular split pullback*) of (X, G).*

- *We will also write $(\widetilde{X}, \widetilde{G})$ for short, reserving the symbols r, s to denote the maps in the pullback $(\widetilde{X}, \widetilde{G}, r, s)$.*

We have the following lemma.

Lemma 3.3.21. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $(\widetilde{X}, \widetilde{G})$ be a pullback of (X, G). Let $0 \leq k \leq 2 \dim X$ and $x \in H^k(X, \mathbb{Z})^G$. Then:*

$$
\widetilde{\pi}_*(s^*(x)) = r^*(\pi_*(x)) + \mathrm{tors}.
$$

If moreover $H^k(\widetilde{X}, \mathbb{Z})$ is torsion-free, then the property that $r^*(\pi_*(x))$ is divisible by p implies that $\widetilde{\pi}_*(s^*(x))$ is divisible by p.

Proof. By Diagram (*), we have:

$$\widetilde{\pi}^*(r^*(\pi_*(x))) = s^*(\pi^*(\pi_*(x))) = p \cdot s^*(x) = \widetilde{\pi}^*(\widetilde{\pi}_*(s^*(x))).$$

The map $\widetilde{\pi}^*$ is injective on the torsion-free part, so we get the equality. If moreover $r^*(\pi_*(x))$ is divisible by p, we can write $r^*(\pi_*(x)) = py$ with $y \in H^k(\widetilde{M}, \mathbb{Z})$. This gives:

$$\widetilde{\pi}_*(s^*(x)) + \mathrm{tors} = py.$$

Applying $\widetilde{\pi}^*$ to this equality, we get:

$$ps^*(x) = p\widetilde{\pi}^*(y).$$

Since $H^k(\widetilde{X}, \mathbb{Z})$ is torsion-free, this is also the case for the group $s^*(H^k(X, \mathbb{Z}))$, hence:

$$\widetilde{\pi}^*(y) = s^*(x).$$

Hence by applying $\widetilde{\pi}_*$, we get:

$$\widetilde{\pi}_*(s^*(x)) = py.$$

$\square$

Lemma 3.3.22. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $(\widetilde{X}, \widetilde{G})$ be a n-split pullback of (X, G). Let $\mathcal{K}$ be the overlattice of $\widetilde{\pi}_*(s^*(H^n(X, \mathbb{Z})^G))$ obtained by dividing by p all the elements of the form $\widetilde{\pi}_*(s^*(y + \varphi(y) + \cdots + \varphi^{p-1}(y)))$, $y \in H^n(X, \mathbb{Z})$. We assume that $H^*(X, \mathbb{Z})$ and $H^*(\widetilde{X}, \mathbb{Z})$ are torsion-free. Then:*

1) $\mathrm{discr}\,\mathcal{K} = \mathrm{discr}\,\pi_*(H^n(X, \mathbb{Z}))$.

2) If moreover $2 \leq p \leq 19$, then

$$\mathrm{discr}\,\pi_*(H^n(X, \mathbb{Z})) = \mathrm{discr}\,\mathcal{K} = p^{l_1^2(X)}$$

for $p \neq 2$, and

$$\mathrm{discr}\,\pi_*(H^n(X, \mathbb{Z})) = \mathrm{discr}\,\mathcal{K} = 2^{l_{1,+}^2(X)}$$

for $p = 2$.

3) If $\mathcal{K}$ is primitive, then (X, G) is H^n-normal.

Proof.   1) By the last lemma, we have $r^*(\mathcal{K}'_n) + \mathrm{tors} = \mathcal{K}$. Hence

$$\mathrm{discr}\,r^*(\mathcal{K}'_n) = \mathrm{discr}\,\mathcal{K}.$$

Since r is a bimeromorphic map, we get $\mathrm{discr}\,\mathcal{K}'_n = \mathrm{discr}\,\mathcal{K}$. The result then follows from Lemma 3.3.6.

2) This follows by 1) and Lemma 3.3.11.

3) We use the same notation for the Jordan decomposition of $H^n(X, \mathbb{F}_\mathrm{p})$ and $H^n(\widetilde{X}, \mathbb{F}_\mathrm{p})$.

Let $x \in H^n(X, \mathbb{Z})^G$. We assume that there is no $y \in H^n(X, \mathbb{Z})$ such that $x = y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y)$ and we show that $\pi_*(x)$ is not divisible by p. By Lemma 3.2.2 $\overline{x} \notin \mathcal{N}_p$.

Since $s^* : H^n(X, \mathbb{F}_\mathrm{p}) \to H^n(\widetilde{X}, \mathbb{F}_\mathrm{p})$ is injective and $s \circ \widetilde{\varphi} = \varphi \circ s$, $\overline{s^*(x)} \notin \mathcal{N}_p$. Hence by Lemma 3.2.2, there is no $z \in H^n(\widetilde{X}, \mathbb{Z})$ such that $s^*(x) = z + \varphi(z) + \cdots + \varphi^{p-1}(z)$. Since $\mathcal{K}$ is primitive, $\widetilde{\pi}_*(s^*(x))$ is not divisible by p. Hence by Lemma 3.3.21, $r^*(\pi_*(x))$ is not divisible by p. It follows that $\pi_*(x)$ is not divisible by p. $\qquad\square$

Proposition 3.3.23. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $(\widetilde{X}, \widetilde{G})$ be a k-split pullback of (X, G). Assume $H^*(X, \mathbb{Z})$ and $H^*(\widetilde{X}, \mathbb{Z})$ are torsion-free. If $(\widetilde{X}, \widetilde{G})$ is H^k-normal then (X, G) is H^k-normal.*

Proof. The proof is almost the same as the proof of 3) of Lemma 3.3.22. We use the same notation for the Jordan decomposition of $H^k(X, \mathbb{F}_\mathrm{p})$ and $H^k(\widetilde{X}, \mathbb{F}_\mathrm{p})$.

Let $x \in H^k(X, \mathbb{Z})^G$. We assume that there is no $y \in H^k(X, \mathbb{Z})$ such that $x = y + \varphi^*(y) + \cdots + (\varphi^*)^{p-1}(y)$ and we show that $\pi_*(x)$ is not divisible by p. By Lemma 3.2.2 $\overline{x} \notin \mathcal{N}_p$.

Since $s^* : H^k(X, \mathbb{F}_\mathrm{p}) \to H^k(\widetilde{X}, \mathbb{F}_\mathrm{p})$ is injective and $s \circ \widetilde{\varphi} = \varphi \circ s$, $\overline{s^*(x)} \notin \mathcal{N}_p$. Hence by Lemma 3.2.2, there is no $z \in H^k(\widetilde{X}, \mathbb{Z})$ such that $s^*(x) = z + \varphi(z) + \cdots + \varphi^{p-1}(z)$. Since $(\widetilde{X}, \widetilde{G})$ is H^k-normal, $\widetilde{\pi}_*(s^*(x))$ is not divisible by p. Hence by Lemma 3.3.21, $r^*(\pi_*(x))$ is not divisible by p. It follows that $\pi_*(x)$ is not divisible by p. $\qquad\square$

The relation of being a pullback is transitive.

Proposition 3.3.24. *Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $0 \le k \le 2n$. Let (X_1, G_1, r_1, s_1) be a pullback (resp. a k-split pullback, a regular k-split pullback) of (X, G) and (X_2, G_2, r_2, s_2) be a pullback (resp. a k-split pullback, a regular k-split pullback) of (X_1, G_1). Then $(X_2, G_2, r_1 \circ r_2, s_1 \circ s_2)$ is a pullback (resp. a k-split pullback, a regular k-split pullback) of (X, G).*

We give an example of a split pullback.

Proposition 3.3.25. *Let X be a Kähler manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $F \subset \mathrm{Fix}\, G$ be a connected component. Assume that $H^*(X, \mathbb{Z})$ is torsion-free.*

Let $s : \widetilde{X} \to X$ be the blowup of X in F. Then G extends naturally to $\widetilde{X}$. Denote by $\widetilde{G}$ this extension. Then $(\widetilde{X}, \widetilde{G})$ is a split pullback of (X, G).

Proof. This follows from Theorem 7.31 of [67] (Theorem 2.5.1). $\qquad\square$

3.4 Resolution of the quotient

Let X be a Kähler manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. In the last section, we have seen that blowups of X in a connected component of $\operatorname{Fix} G$ provide regular split pullbacks. In this section we will find all the regular split pullbacks obtained from the blowup in connected components of $\operatorname{Fix} G$. Then in Section 3.5 and Section 3.6 we will use these regular split pullbacks and Lemma 3.3.22 3) to get some general theorems.

At each fixed point of G, by Cartan's Lemma 1 of [14] we can locally linearize the action of G. Thus at a fixed point $x \in X$, the action of G on X is locally equivalent to the action of $G = \langle g \rangle$ on $\mathbb{C}^n$ via

$$g = \operatorname{diag}(\xi_p^{k_1}, \ldots, \xi_p^{k_n}),$$

where ξ_p is a p-th root of unity. Without loss of generality, we can assume that $k_1 \leq \cdots \leq k_n \leq p - 1$.

Definition 3.4.1.
- We will say that a fixed point is of type 0 if $k_1 = \cdots = k_{n-1} = 0$.

- We will say that a fixed point is of type 1 if there is $i \in \{1, \ldots, n\}$ such that $k_1 = \cdots = k_i = 0$ and $k_{i+1} = \cdots = k_n$.

- We will say that a fixed point is of type 2 if $p = 3$ and if it is not a point of type 0 or 1.

When it is defined, we will denote $o(x)$ the type of a fixed point x.

Proposition 3.4.2. Let X be a complex manifold of dimension n and G an automorphism group of prime order p. Let $x \in \operatorname{Fix} G$.

1) The variety $M = X/G$ is smooth in $\pi(x)$ if and only if x is a point of type 0.

2) Let $\widetilde{X}$ be the blowup of X in the connected components of $\operatorname{Fix} G$ of codimension ≥ 2 and $\widetilde{M}$ the quotient of $\widetilde{X}$ by the natural action of G on $\widetilde{X}$. The variety $\widetilde{M}$ is smooth if and only if the points of $\operatorname{Fix} G$ are of type 0 or 1.

Proof. By Lemma 1 of [14], we can assume that $X = \mathbb{C}^n$ and

$$G = \left\langle \operatorname{diag}(\xi_p^{k_1}, \ldots, \xi_p^{k_n}) \right\rangle.$$

1) By the proof of Proposition 6 of [60], $\mathbb{C}^n/G$ is smooth if and only if $\operatorname{rk}(g - \operatorname{id}) = 1$. Hence we get the result.

2) If 0 is a point of type 0, then $s^*(0)$ is also of type 0 and $\widetilde{M}$ is smooth at $\widetilde{\pi}(s^*(0))$.

    Now assume that 0 is not of type 0. Let $G = \left\langle \operatorname{diag}(\xi_p^{k_1}, \ldots, \xi_p^{k_n}) \right\rangle$ acting on $\mathbb{C}^n$. If $k_1 = \cdots = k_i = 0$, then

    $$\mathbb{C}^n/G \simeq \mathbb{C}^i \times \left(\mathbb{C}^{n-i} / \left\langle \operatorname{diag}(\xi_p^{k_{i+i}}, \ldots, \xi_p^{k_n}) \right\rangle \right),$$

so without loss of generality, we can assume that all k_i are different from 0. Let $\widetilde{\mathbb{C}^n}$ be the blowup of $\mathbb{C}^n$ in 0 and $\widetilde{G}$ the automorphism group of $\widetilde{\mathbb{C}^n}$ induced by G. We will describe the action of $\widetilde{G}$ on

$$\widetilde{\mathbb{C}^n} = \{((x_1,\ldots,x_n),(a_1 : \cdots : a_n)) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid$$
$$\mathrm{rk}\,((x_1,\ldots,x_n),(a_1,\ldots,a_n)) = 1\}.$$

We denote by

$$\mathscr{O}_i = \left\{((x_1,\ldots,x_n),(a_1 : \cdots : a_n)) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid a_i \neq 0\right\}$$

the chart $a_i \neq 0$. We have

$$\widetilde{\mathbb{C}^n} \cap \mathscr{O}_i = \{((x_1,\ldots,x_n),(a_1 : \cdots : a_n)) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid$$
$$x_j = x_i a_j,\ j \in \{1,\ldots,n\}\}.$$

Hence we have an isomorphism:

$$f : \widetilde{\mathbb{C}^n} \cap \mathscr{O}_i \to \mathbb{C}^n$$
$$((x_1,\ldots,x_n),(a_1 : \cdots : a_n)) \to (a_1,\ldots,a_{i-1},x_i,a_{i+1},\ldots,a_n)$$

Thus the action of $\widetilde{G}$ on $\widetilde{\mathbb{C}^n} \cap \mathscr{O}_i$ is an action on $\mathbb{C}^n$ given by the diagonal matrix

$$\mathrm{diag}(\xi_p^{k_1-k_i},\ldots,\xi_p^{k_{i-1}-k_i},\xi_p^{k_i},\xi_p^{k_{i+1}-k_i},\ldots,\xi_p^{k_n-k_i}).$$

By assertion 1), $\widetilde{M}$ is smooth if and only if $k_1 = \cdots = k_n$.

$\square$

Let X be a complex manifold and G an automorphism group of prime order acting on X. Another idea to get regular split pullbacks of (X,G) is to consider a sequence of blowups:

$$\begin{array}{ccccccc}
M_k & \xrightarrow{r_k} & \cdots & \xrightarrow{r_2} & M_1 & \xrightarrow{r_1} & M \\
\pi_k \uparrow & & & & \pi_1 \uparrow & & \pi \uparrow \\
X_k & \xrightarrow{s_k} & \cdots & \xrightarrow{s_2} & X_1 & \xrightarrow{s_1} & X, \\
\circlearrowright & & & & \circlearrowright & & \circlearrowright \\
G_k & & & & G_1 & & G
\end{array}$$

where each s_{i+1} is the blowup of X_i in $\mathrm{Fix}\,G_i$ ($M = M_0$, $X = X_0$ and $G = G_0$). We can state the following proposition.

Proposition 3.4.3. *There exists $k \in \mathbb{N}$ such that M_k is smooth if and only if all the fixed points of G have type 0,1 or 2. Moreover in the case where $\mathrm{Fix}\,G$ has points of type 2, M_2 is smooth.*

Proof. By Lemma 1 of [14], we can assume that $X = \mathbb{C}^n$ and

$$G = \left\langle \operatorname{diag}(\xi_p^{k_1}, \ldots, \xi_p^{k_n}) \right\rangle.$$

1) If 0 is a point of type 0 or 1, by Proposition 3.4.2, M_1 is smooth. If 0 is a point of type 2, we will show that M_2 is smooth. Let $G = \left\langle \operatorname{diag}(\xi_3^{k_1}, \ldots, \xi_3^{k_n}) \right\rangle$ acting on $\mathbb{C}^n$. Without loss of generality, we can assume that all k_i are different from 0. Let $\widetilde{\mathbb{C}^n}$ be the blowup of $\mathbb{C}^n$ in 0. By the proof of Proposition 3.4.2, on the chart $a_i \neq 0$ the action of G_1 is given by the diagonal matrix

$$\operatorname{diag}(\xi_3^{k_1-k_i}, \ldots, \xi_3^{k_{i-1}-k_i}, \xi_3^{k_i}, \xi_3^{k_{i+1}-k_i}, \ldots, \xi_3^{k_n-k_i}).$$

As $p = 3$, there is a j such that $k_1 = \cdots = k_j = 1$ and $k_{j+1} = \cdots = k_n = 2$. So if $i \leq j$, by permuting the i-th and the j-th coordinates of the chart $\mathcal{O}_i$, we reduce the action to the form

$$\operatorname{diag}(1, \ldots, 1, \xi_3, \ldots, \xi_3).$$

If $i > j$, by a permutation of coordinates we obtain

$$\operatorname{diag}(\xi_3^2, \ldots, \xi_3^2, 1, \ldots, 1).$$

In both cases, these are points of type 1. Hence all the points of $\operatorname{Fix} G_1$ are points of type 0 or 1. Moreover, as there are no fixed points with both eigenvalues ξ_3, ξ_3^2 present in the diagonal matrix of the action, we can conclude that the components of $\operatorname{Fix} G_1$ with spectra $(1, \ldots, 1, \xi_3, \ldots, \xi_3)$ and $(\xi_3^2, \ldots, \xi_3^2, 1, \ldots, 1)$ are disjoint and can be blown up independently. Hence by Proposition 3.4.2 2), M_2 is smooth.

2) Now we will show that in the case of a point of type different from 0, 1 or 2, M_k will never be smooth. We start with $\dim X = 2$. By Lemma 1 of [14], we can assume that $X = \mathbb{C}^2$ and $G = \left\langle \operatorname{diag}(\xi_p, \xi_p^{\alpha}) \right\rangle$. Since 0 is of type different from 0, 1 or 2, $p > 3$ and α is not equal to 0 or 1.

For $x \in \operatorname{Fix} G_i$, we can write:

$$(X_i, G_i, x) \sim (\mathbb{C}^2, \left\langle \operatorname{diag}(\xi_p, \xi_p^{\beta}) \right\rangle, 0),$$

where ξ_p is a non-trivial p-th root of the unity. Hence, we can define a sequence as follows:

$$u_i = \left\{ \beta \in \mathbb{Z}/p\mathbb{Z} \mid \exists x \in X_i : \ (X_i, G_i, x) \sim (\mathbb{C}^2, \left\langle \operatorname{diag}(\xi_p, \xi_p^{\beta}) \right\rangle, 0) \right\}.$$

For instance, $u_0 = \left\{ \alpha, \frac{1}{\alpha} \right\}$, $u_1 = \left\{ \alpha - 1, \frac{1}{\alpha-1}, \frac{\alpha}{1-\alpha}, \frac{1-\alpha}{\alpha} \right\}, \ldots$ Now assume that there is $i \in \mathbb{N}$ such that M_i is smooth. Let i be the smallest integer such that M_i is smooth. Hence by Proposition 3.4.2 1), $u_i = \{0\}$ and we can write $u_{i-1} = \{\alpha_1, \ldots, \alpha_k\}$. Let $x \in \operatorname{Fix} G_{i-1}$ such that $(X_i, G_{i-1}, x) \sim$

$(\mathbb{C}^2, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_j}) \right\rangle, 0)$, with $\alpha_j \in u_{i-1} \setminus \{0\}$. Let $\widetilde{\mathbb{C}^2}$ be the blowup of $\mathbb{C}^2$ in 0. The action of $\left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_j}) \right\rangle$ on $\widetilde{\mathbb{C}^2}$ has 2 fixed points a_1 and a_2 with (see proof of Proposition 3.4.2):

$$(\widetilde{\mathbb{C}^2}, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_j}) \right\rangle, a_1') \sim (\mathbb{C}^2, \mathrm{diag}(\xi_p, \xi_p^{\alpha_j - 1}), 0)$$

and

$$(\widetilde{\mathbb{C}^2}, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_j}) \right\rangle, a_2') \sim (\mathbb{C}^2, \mathrm{diag}(\xi_p^{\alpha_j}, \xi_p^{1-\alpha_j}), 0).$$

Hence $\alpha_j - 1 \in u_i$, but $u_i = \{0\}$. Hence necessarily, $\alpha_j = 1$. Then $u_{i-1} = \{1\}$.

We do the same calculation with u_{i-2}. We can write $u_{i-2} = \{\alpha_1', \ldots, \alpha_k'\}$. Let $x \in \mathrm{Fix}\, G_{i-2}$ such that:

$$(X_{i-2}, G_{i-2}, x) \sim (\mathbb{C}^2, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_j'}) \right\rangle, 0).$$

with $\alpha_j' \in u_{i-2} \setminus \{0, 1\}$. We remark that $u_{i-2} \setminus \{0, 1\}$ is not empty because M_{i-1} is not smooth by definition of i. Let $\widetilde{\mathbb{C}^2}$ be the blowup of $\mathbb{C}^2$ in 0. The action of $\left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_{j'}'}) \right\rangle$ on $\widetilde{\mathbb{C}^2}$ has 2 fixed points a_1' and a_2' with (see proof of Proposition 3.4.2):

$$(\widetilde{\mathbb{C}^2}, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_j'}) \right\rangle, a_1) \sim (\mathbb{C}^2, \mathrm{diag}(\xi_p, \xi_p^{\alpha_j' - 1}), 0)$$

and

$$(\widetilde{\mathbb{C}^2}, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha_j'}) \right\rangle, a_1) \sim (\mathbb{C}^2, \mathrm{diag}(\xi_p^{\alpha_j'}, \xi_p^{1-\alpha_j'}), 0).$$

But

$$(\mathbb{C}^2, \mathrm{diag}(\xi_p^{\alpha_j'}, \xi_p^{1-\alpha_j'}), 0) \sim (\mathbb{C}^2, \mathrm{diag}(\xi_p, \xi_p^{\frac{1-\alpha_j'}{\alpha_j'}}), 0).$$

Hence necessarily, $\alpha_j' = 2$ and $\alpha_j' = \frac{1}{2} = \frac{p-1}{2}$. Hence $p = 3$ and we are done.

Now, we assume $n > 2$. By Lemma 1 of [14], we can assume that $X = \mathbb{C}^n$ and $G = \left\langle \mathrm{diag}(\xi_p^{k_1}, \xi_p^{k_2}, \ldots, \xi_p^{k_n}) \right\rangle$. Without loss of generality, we can assume that all the k_i are different from 0. Since 0 is of type different from 0,1 or 2, $p > 3$ and the k_i are not all equal. Without loss of generality, we can assume that $k_1 = 1$. Since not all the k_i are equal, there is $j \in \{1, \ldots, n\}$ such that $k_j \neq 1$. We also denote $\alpha = k_j$. We denote $X' = \mathbb{C}^2$ and $G' = \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha}) \right\rangle$. And we define as before the sequence:

$$u_i' = \left\{ \beta \in \mathbb{Z}/p\mathbb{Z} \mid \exists x \in X_i' : (X_i', G_i', x) \sim (\mathbb{C}^2, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\alpha}) \right\rangle, 0) \right\}.$$

We define also the following sequence:

$$U_i = \{ \beta \in \mathbb{Z}/p\mathbb{Z} \mid \exists x \in X_i :$$
$$(X_i, G_i, x) \sim (\mathbb{C}^n, \left\langle \mathrm{diag}(\xi_p, \xi_p^{\beta}, \xi_p^{t_3}, \ldots, \xi_p^{t_n}) \right\rangle, 0) \}.$$

We have to show that $U_i \neq \{0\}$ for all $i \in \mathbb{N}$. But $U_i \supset u_i'$. We have seen that $u_i' \neq \{0\}$ for all $i \in \mathbb{N}$. The result follows.

$\square$

Corollary 3.4.4. *Let X be a Kähler manifold and G an automorphism group of prime order acting on X. There exists a regular split pullback of (X, G) obtained as a sequence of blowups in connected components of fixed loci if and only if the points of $\mathrm{Fix}\, G$ are of types 0, 1 or 2.*

3.5   The case of fixed points of type 1

During all this section, we will use the following notation. Let X be a compact complex manifold of dimension n and $G = \langle \varphi \rangle$ an automorphism group of prime order p. Let $s : \widetilde{X} \to X$ be the blowup of X in $\mathrm{Fix}\, G$. We denote $\widetilde{G}$ the automorphism group induced by G on $\widetilde{X}$. We can consider the quotient $M := X/G$ and $\widetilde{M} := \widetilde{X}/\widetilde{G}$. In this section the fixed points will be of type 1, hence $\widetilde{M}$ will be always smooth. Hence $(\widetilde{X}, \widetilde{G}, r, s)$ will be a regular split pullback of (X, G), and the following diagram is Cartesian:

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\ r\ } & M \\
{\scriptstyle \widetilde{\pi}} \uparrow & & \uparrow {\scriptstyle \pi} \\
\widetilde{X} & \xrightarrow{\ s\ } & X,
\end{array}
$$

We also denote $V = X \setminus \mathrm{Fix}\, G$, $U = \pi(V)$, $F = s^{-1}(\mathrm{Fix}\, G)$, and we use the same symbol F for its image by $\widetilde{\pi}$.

3.5.1   The codimension of $\mathrm{Fix}\, G$

The technique of the proof of the main theorem of this section will be to use Lemma 3.3.22 3). To do this, we will have to understand $\mathcal{K}$ inside $H^n(\widetilde{M}, \mathbb{Z})$. For this, we will need the following exact sequence:

$$
0 \longrightarrow H^n(\widetilde{M}, U, \mathbb{Z}) \xrightarrow{\ g\ } H^n(\widetilde{M}, \mathbb{Z}) \longrightarrow H^n(U, \mathbb{Z}) \longrightarrow 0.
$$

So we need some conditions on $\mathrm{Fix}\, G$ which will guarantee that this sequence is exact.

Definition 3.5.1. *Let X be a compact complex manifold of dimension n and G an automorphism group of prime order p.*

1) We will say that $\mathrm{Fix}\, G$ is negligible if the following conditions are verified:

- *$H^*(\mathrm{Fix}\, G, \mathbb{Z})$ is torsion-free.*
- *$\mathrm{codim}\, \mathrm{Fix}\, G \geq \frac{n}{2} + 1$.*

2) *We will say that* $\mathrm{Fix}\,G$ *is almost negligible if the following conditions are verified:*

- $H^*(\mathrm{Fix}\,G, \mathbb{Z})$ *is torsion-free.*

- n *is even and* $n \geq 4$.

- $\mathrm{codim}\,\mathrm{Fix}\,G = \frac{n}{2}$, *and the purely* $\frac{n}{2}$*-dimensional part of* $\mathrm{Fix}\,G$ *is connected and simply connected. We denote the* $\frac{n}{2}$*-dimensional component by* Σ.

- *The cocycle* $[\Sigma]$ *associated to* Σ *is primitive in* $H^n(X, \mathbb{Z})$.

Remark: We might just assume that $[\Sigma]$ is not divisible by p, but this would imply technical complications.

3.5.2 The main theorem

Theorem 3.5.2. *Let* $G = \langle \varphi \rangle$ *be a group of prime order* p *acting by automorphisms on a Kähler manifold* X *of dimension* n. *We assume:*

i) $H^*(X, \mathbb{Z})$ *is torsion-free,*

ii) $\mathrm{Fix}\,G$ *is negligible or almost negligible,*

iii) *all the points of* $\mathrm{Fix}\,G$ *are of type 1.*

Then:

1) $\log_p(\mathrm{discr}\,\pi_*(H^n(X, \mathbb{Z}))) - h^{2*+\epsilon}(\mathrm{Fix}\,G, \mathbb{Z})$ *is divisible by 2,*

2) *The following inequalities are verified:*

$$\log_p(\mathrm{discr}\,\pi_*(H^n(X, \mathbb{Z}))) + 2\,\mathrm{rktor}\,H^n(U, \mathbb{Z})$$
$$\geq h^{2*+\epsilon}(\mathrm{Fix}\,G, \mathbb{Z}) + 2\,\mathrm{rktor}\,H^n(\widetilde{M}, \mathbb{Z})$$
$$\geq 2\,\mathrm{rktor}\,H^n(U, \mathbb{Z}).$$

3) *If moreover*

$$\log_p(\mathrm{discr}\,\pi_*(H^n(X, \mathbb{Z}))) + 2\,\mathrm{rktor}\,H^n(U, \mathbb{Z})$$
$$= h^{2*+\epsilon}(\mathrm{Fix}\,G, \mathbb{Z}) + 2\,\mathrm{rktor}\,H^n(\widetilde{M}, \mathbb{Z}),$$

then (X, G) *is* H^n*-normal.*

Proof. The idea of the proof is to compare $\mathcal{K}$ from Lemma 3.3.22 to its orthogonal complement in the unimodular lattice $H^n(\widetilde{M}, \mathbb{Z})$.

The proof is a little different if $\mathrm{Fix}\,G$ is negligible or almost negligible. Hence, we give a proof in both cases.

The case when $\operatorname{Fix} G$ *is negligible*

We consider the following commutative diagram:

$$H^n(\mathscr{N}_{\widetilde{M}/F}, \mathscr{N}_{\widetilde{M}/F} - 0, \mathbb{Z}) = H^n(\widetilde{M}, U, \mathbb{Z}) \xrightarrow{g} H^n(\widetilde{M}, \mathbb{Z}) \qquad (3.1)$$

$$\downarrow{\scriptstyle d\widetilde{\pi}^*} \qquad\qquad\qquad {\scriptstyle \widetilde{\pi}^*}\downarrow$$

$$H^n(\mathscr{N}_{\widetilde{X}/F}, \mathscr{N}_{\widetilde{X}/F} - 0, \mathbb{Z}) = H^n(\widetilde{X}, V, \mathbb{Z}) \xrightarrow{h} H^n(\widetilde{X}, \mathbb{Z}),$$

where $\mathscr{N}_{\widetilde{X}/F} - 0$ and $\mathscr{N}_{\widetilde{M}/F} - 0$ are vector bundles minus the zero section. We denote $T := h(H^n(\widetilde{X}, V, \mathbb{Z}))$. We will need the following lemmas about properties of T.

Lemma 3.5.3. *We have*

$$H^n(\widetilde{X}, \mathbb{Z}) = s^*(H^n(X, \mathbb{Z})) \oplus T.$$

Proof. The proof follows from Theorem 7.31 of [67] and its proof (Theorem 2.5.1).

By Thom isomorphism $H^n(\widetilde{X}, V, \mathbb{Z}) = H^{n-2}(F, \mathbb{Z})$, and the map h can be identified with the morphism $j_* : H^{n-2}(F, \mathbb{Z}) \to H^n(\widetilde{X}, \mathbb{Z})$, where j is the inclusion in $\widetilde{X}$. As in the proof of Theorem 7.31 of [67], the map

$$(s^*, j_*) : H^n(X, \mathbb{Z}) \oplus H^{n-2}(F, \mathbb{Z}) \to H^n(\widetilde{X}, \mathbb{Z})$$

is surjective, and its kernel coincides with the image of the map

$$\bigoplus_{S_m \subset \operatorname{Fix} G} H^{n-2r_m}(S_m, \mathbb{Z}) \to H^n(X, \mathbb{Z}) \oplus H^{n-2}(F, \mathbb{Z}),$$

where r_m is the codimension of the component S_m of $\operatorname{Fix} G$. But in our case $\bigoplus_{S_m \subset \operatorname{Fix} G} H^{n-2r_m}(S_m, \mathbb{Z}) = 0$. The result follows. $\qquad\square$

Lemma 3.5.4. *The sublattice T of $H^n(\widetilde{X}, \mathbb{Z})$ is unimodular.*

Proof. By Lemma 3.5.3, we have:

$$H^n(\widetilde{X}, \mathbb{Z}) = s^*(H^n(X, \mathbb{Z})) \oplus T.$$

Moreover, the sum is orthogonal with respect to the cup product. Indeed, let $x \in H^n(X, \mathbb{Z})$ and $y \in T$, then $s_*(s^*(x) \cdot y) = x \cdot s_*(y)$ by the projection formula. Since $s_*(y) = 0$, we have $s_*(s^*(x) \cdot y) = 0$, then $s^*(x) \cdot y = 0$. Hence, since $H^n(\widetilde{X}, \mathbb{Z})$ and $H^n(X, \mathbb{Z})$ endowed with the cup product are unimodular, T is also unimodular. $\qquad\square$

By the property of the Thom isomorphism,

$$d\widetilde{\pi}^*(H^n(\mathscr{N}_{\widetilde{M}/F}, \mathscr{N}_{\widetilde{M}/F} - 0, \mathbb{Z})) = pH^n(\mathscr{N}_{\widetilde{X}/F}, \mathscr{N}_{\widetilde{X}/F} - 0, \mathbb{Z}).$$

Then by commutativity of the diagram and Proposition 3.3.1, we have $g(H^n(\widetilde{M}, U, \mathbb{Z})) = \widetilde{\pi}_*(T)$.

We deduce the following lemma.

Lemma 3.5.5.   *1) We have the exact sequence:*

$$0 \longrightarrow \widetilde{\pi}_*(T) \longrightarrow H^n(\widetilde{M}, \mathbb{Z}) \longrightarrow H^n(U, \mathbb{Z}) \longrightarrow 0.$$

2) The torsion subgroups of $H^n(U, \mathbb{Z})$ and $H^n(\widetilde{M}, \mathbb{Z})$ are powers of $\mathbb{F}_p$.

Proof.   1) We have the following exact sequence:

$$0 \longrightarrow \widetilde{\pi}_*(T) \longrightarrow H^n(\widetilde{M}, \mathbb{Z}) \longrightarrow H^n(U, \mathbb{Z}) \longrightarrow H^{n+1}(\widetilde{M}, U, \mathbb{Z}).$$

Since $H^*(\text{Fix}\, G, \mathbb{Z})$ is torsion-free, by Thom's isomorphism $H^{n+1}(\widetilde{M}, U, \mathbb{Z})$ is torsion-free. Hence it is enough to show:

$$0 \longrightarrow \widetilde{\pi}_*(T \otimes \mathbb{C}) \longrightarrow H^n(\widetilde{M}, \mathbb{C}) \longrightarrow H^n(U, \mathbb{C}) \longrightarrow 0.$$

Hence, it is enough to show that $\dim H^n(\widetilde{M}, \mathbb{C}) = \dim H^n(U, \mathbb{C}) + \dim \widetilde{\pi}_*(T \otimes \mathbb{C})$. By Lemma 3.5.3

$$H^n(\widetilde{X}, \mathbb{C}) = s^*(H^n(X, \mathbb{C})) \oplus T \otimes \mathbb{C}.$$

Hence:
$$H^n(\widetilde{X}, \mathbb{C})^G = s^*(H^n(X, \mathbb{C})^G) \oplus T \otimes \mathbb{C}.$$

Since $\text{codim}\, \text{Fix}\, G \geq \frac{n}{2} + 1$, $H^n(V, \mathbb{Z}) = H^n(X, \mathbb{Z})$. Hence:

$$H^n(\widetilde{X}, \mathbb{C})^G = s^*(H^n(V, \mathbb{C})^G) \oplus T \otimes \mathbb{C}.$$

It follows:

$$\dim H^n(\widetilde{M}, \mathbb{C}) = \dim H^n(U, \mathbb{C}) \oplus \dim \widetilde{\pi}_*(T \otimes \mathbb{C}).$$

2) Since $\text{codim}\, \text{Fix}\, G \geq \frac{n}{2} + 1$, $H^n(V, \mathbb{Z}) = H^n(X, \mathbb{Z})$. Since $H^n(X, \mathbb{Z})$ is torsion-free, $H^n(V, \mathbb{Z})$ is torsion-free. Hence by Corollary 3.3.2, the torsion subgroup of $H^n(U, \mathbb{Z})$ is power of $\mathbb{F}_p$.

The proof is the same for $H^n(\widetilde{M}, \mathbb{Z})$. Indeed, $H^n(X, \mathbb{Z})$ and $H^*(\text{Fix}\, G, \mathbb{Z})$ are torsion-free. Hence by Theorem 7.31 of [67] (Theorem 2.5.1), $H^n(\widetilde{X}, \mathbb{Z})$ is torsion-free. Hence the result follows from Corollary 3.3.2.

$\square$

Let $\widetilde{T}$ be the minimal primitive overlattice of $\widetilde{\pi}_*(T)$ in $H^n(\widetilde{M},\mathbb{Z})$. We have $\widetilde{T} = \mathcal{K}^\perp$ by Lemma 3.5.3 (we recall that $\mathcal{K}$ is defined in Lemma 3.3.22). We will compare the discriminant of $\mathcal{K}$ and $\widetilde{T}$. Then by Proposition 2.1.3, we will be able to know whether $\mathcal{K}$ is primitive in $H^n(\widetilde{M},\mathbb{Z})$. We can state the following result.

Lemma 3.5.6. *We have:*

1) $\widetilde{T}/T = (\mathbb{F}_{\mathrm{p}})^{\mathrm{rktor}\, H^n(U,\mathbb{Z}) - \mathrm{rktor}\, H^n(\widetilde{M},\mathbb{Z})}$,

2) $\mathrm{discr}\, \widetilde{T} = p^{h^{2*+\epsilon}(X) + 2\left(\mathrm{rktor}\, H^n(\widetilde{M},\mathbb{Z}) - \mathrm{rktor}\, H^n(U,\mathbb{Z})\right)}$.

Proof. By 3) of Lemma 3.3.7, we have $\mathrm{discr}\,\pi_*(T) = p^{\mathrm{rk}\, T}$. But by Theorem 7.31 of [67],

$$\mathrm{rk}\, T = \mathrm{rk} \bigoplus_{S_m \subset \mathrm{Fix}\, G} \bigoplus_{k=0}^{r_m-2} H^{n-2k-2}(S_m, \mathbb{Z}).$$

And since $\mathrm{codim}\, \mathrm{Fix}\, G \leq \frac{n}{2} + 1$, we get $\mathrm{rk}\, T = h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z})$. So

$$\mathrm{discr}\,\pi_*(T) = p^{h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z})}.$$

Moreover, by the exact sequence of Lemma 3.5.5, we have:

$$\widetilde{T}/T = (\mathbb{F}_{\mathrm{p}})^{\mathrm{rktor}\, H^n(U,\mathbb{Z}) - \mathrm{rktor}\, H^n(\widetilde{M},\mathbb{Z})}.$$

Hence, by Proposition 2.1.1,

$$\mathrm{discr}\,\widetilde{T} = \frac{p^{h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z})}}{p^{2\left(\mathrm{rktor}\, H^n(U,\mathbb{Z}) - \mathrm{rktor}\, H^n(\widetilde{M},\mathbb{Z})\right)}}.$$

$\square$

Conclusion

The unimodularity of $H^n(\widetilde{M},\mathbb{Z})$ will allow us to conclude. Let K be the primitive overlattice of $\mathcal{K}$ in $H^n(\widetilde{M},\mathbb{Z})$. We have $K^\perp = \widetilde{T}$. Hence by Proposition 2.1.3,

$$\mathrm{discr}\, K = \mathrm{discr}\, \widetilde{T} = p^{h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z}) + 2\left(\mathrm{rktor}\, H^n(\widetilde{M},\mathbb{Z}) - \mathrm{rktor}\, H^n(U,\mathbb{Z})\right)}.$$

By Lemma 3.3.22, we know that $\mathrm{discr}\,\mathcal{K} = \mathrm{discr}\,\pi_*(H^n(X,\mathbb{Z}))$. Then

$$\mathrm{discr}\,\mathcal{K} = \mathrm{discr}\,\pi_*(H^n(X,\mathbb{Z})) \geq \mathrm{discr}\, K \qquad and \qquad \mathrm{discr}\,\widetilde{T} \geq 1$$

and we get part 2) of the Theorem. By Proposition 2.1.1,

$$K/\mathcal{K} = (\mathbb{Z}/p\,\mathbb{Z})^{\frac{\log_p(\mathrm{discr}\,\pi_*(H^n(X,\mathbb{Z}))) - h^{2*+\epsilon}(\mathrm{Fix}\, G, \mathbb{Z}) - 2\left(\mathrm{rktor}\, H^n(\widetilde{M},\mathbb{Z}) - \mathrm{rktor}\, H^n(U,\mathbb{Z})\right)}{2}}.$$

We have proved statement 1) of the Theorem.

Now if

$$\log_p(\operatorname{discr} \pi_*(H^n(X,\mathbb{Z}))) - h^{2*+\epsilon}(\operatorname{Fix} G, \mathbb{Z})$$
$$- 2\left(\operatorname{rktor} H^n(\widetilde{M}, \mathbb{Z}) - \operatorname{rktor} H^n(U, \mathbb{Z})\right) = 0,$$

$K = \mathcal{K}$. Hence, $\mathcal{K}$ is primitive in $H^n(\widetilde{M}, \mathbb{Z})$. And we finish the proof by an application of Lemma 3.3.22 3).

The case when $\operatorname{Fix} G$ *is almost negligible*

In this case, n is even, so we can write $n = 2m$.

We consider the following commutative diagram:

$$H^{2m}(\mathscr{N}_{\widetilde{M}/F}, \mathscr{N}_{\widetilde{M}/F} - 0, \mathbb{Z}) = H^{2m}(\widetilde{M}, U, \mathbb{Z}) \overset{g}{\Longrightarrow} H^{2m}(\widetilde{M}, \mathbb{Z}) \qquad (3.2)$$

$$\left\downarrow{\scriptstyle d\widetilde{\pi}^*} \qquad\qquad\qquad\qquad\qquad {\scriptstyle \widetilde{\pi}^*}\right\downarrow$$

$$H^{2m}(\mathscr{N}_{\widetilde{X}/F}, \mathscr{N}_{\widetilde{X}/F} - 0, \mathbb{Z}) = H^{2m}(\widetilde{X}, V, \mathbb{Z}) \overset{h}{\longrightarrow} H^{2m}(\widetilde{X}, \mathbb{Z}),$$

where $\mathscr{N}_{\widetilde{X}/F} - 0$ and $\mathscr{N}_{\widetilde{M}/F} - 0$ are vector bundles minus the zero section. We denote $R := h(H^{2m}(\widetilde{X}, V, \mathbb{Z}))$. The following lemma follows Theorem 7.31 of [67] (Theorem 2.5.1) and its proof.

Lemma 3.5.7. *We can write:*

$$H^{2m}(\widetilde{X}, \mathbb{Z}) = s^*(H^{2m}(X, \mathbb{Z})) \oplus T,$$

with $R = T \oplus \mathbb{Z}\Sigma$.

Proof. The proof is very similar to that of Lemma 3.5.3.

By Thom isomorphism $H^{2m}(\widetilde{X}, V, \mathbb{Z}) = H^{2m-2}(F, \mathbb{Z})$, and the map h can be identified with the morphism $j_* : H^{2m-2}(F, \mathbb{Z}) \to H^{2m}(\widetilde{X}, \mathbb{Z})$, where j is the inclusion in $\widetilde{X}$. As in the proof of Theorem 7.31 of [67], the map

$$(s^*, j_*) : H^{2m}(X, \mathbb{Z}) \oplus H^{2m-2}(F, \mathbb{Z}) \to H^{2m}(\widetilde{X}, \mathbb{Z})$$

is surjective and its kernel is the image of the map

$$\bigoplus_{S_k \subset \operatorname{Fix} G} H^{2m-2r_k}(S_k, \mathbb{Z}) \to H^{2m}(X, \mathbb{Z}) \oplus H^{2m-2}(F, \mathbb{Z}),$$

where r_k is the codimension of the component S_k of $\operatorname{Fix} G$. But in our case $\bigoplus_{S_k \subset \operatorname{Fix} G} H^{2m-2r_k}(S_k, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z})$. The result follows.

$\square$

Lemma 3.5.8. *The sublattice T of $H^{2m}(\widetilde{X}, \mathbb{Z})$ is unimodular.*

Proof. The same proof as in Lemma 3.5.4. $\qquad\qquad\qquad\square$

By the property of the Thom isomorphism, $d\widetilde{\pi}^*(H^{2m}(\mathscr{N}_{\widetilde{M}/F}, \mathscr{N}_{\widetilde{M}/F} - 0, \mathbb{Z})) = pH^{2m}(\mathscr{N}_{\widetilde{X}/F}, \mathscr{N}_{\widetilde{X}/F} - 0, \mathbb{Z})$. Then by the commutativity of the diagram and Proposition 3.3.1, we have $g(H^{2m}(\widetilde{M}, U, \mathbb{Z})) = \widetilde{\pi}_*(R)$.

Lemma 3.5.9. * 1) $H^{2m-1}(X, \mathbb{Z}) = H^{2m-1}(V, \mathbb{Z})$.*

* 2) $H^{2m}(V, \mathbb{Z}) = H^{2m}(X, \mathbb{Z})/\mathbb{Z}\Sigma$.*

* 3) $H^{2m}(V, \mathbb{Z})$ is torsion-free, and the torsion subgroups of $H^{2m}(U, \mathbb{Z})$, $H^{2m}(\widetilde{M}, \mathbb{Z})$ are powers of $\mathbb{F}_{\mathrm{p}}$.*

* 4) We have the exact sequence*

$$0 \longrightarrow \widetilde{\pi}_*(R) \longrightarrow H^{2m}(\widetilde{M}, \mathbb{Z}) \longrightarrow H^{2m}(U, \mathbb{Z}) \longrightarrow 0.$$

Proof. 1) We have the following exact sequence:

$$H^{2m-1}(X, V, \mathbb{Z}) \twoheadrightarrow H^{2m-1}(X, \mathbb{Z}) \overset{f}{\twoheadrightarrow} H^{2m-1}(V, \mathbb{Z}) \longrightarrow$$
$$H^{2m}(X, V, \mathbb{Z}) \overset{\rho}{\longrightarrow} H^{2m}(X, \mathbb{Z}) \longrightarrow H^{2m}(V, \mathbb{Z}) \longrightarrow H^{2m+1}(X, V, \mathbb{Z}).$$

By Thom's isomorphism, $H^{2m-1}(X, V, \mathbb{Z}) = 0$, $H^{2m+1}(X, V, \mathbb{Z}) = H^1(\Sigma, \mathbb{Z}) = 0$ and $H^{2m}(X, V, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z})$. The image of ρ is not trivial in $H^{2m}(X, \mathbb{Z})$ (see Section 11.1.2 of [67]). Hence the cokernel of f is a torsion group, but $H^0(\Sigma, \mathbb{Z})$ is torsion-free. Hence, ρ is an isomorphism and

$$H^{2m-1}(X, \mathbb{Z}) = H^{2m-1}(V, \mathbb{Z}).$$

  2) In view of 1), the exact sequence becomes:

$$0 \twoheadrightarrow H^0(\Sigma, \mathbb{Z}) \twoheadrightarrow H^{2m}(X, \mathbb{Z}) \twoheadrightarrow H^{2m}(V, \mathbb{Z}) \twoheadrightarrow 0,$$

which implies the result.

  3) The group $H^{2m}(V, \mathbb{Z})$ is torsion-free, because

$$H^{2m}(V, \mathbb{Z}) = H^{2m}(X, \mathbb{Z})/\mathbb{Z}\Sigma$$

and $\mathbb{Z}\Sigma$ is primitive inside $H^{2m}(X, \mathbb{Z})$. Hence by Corollary 3.3.2, the torsion subgroup of $H^{2m}(U, \mathbb{Z})$ is a power of $\mathbb{F}_{\mathrm{p}}$.

The proof is the same for $H^{2m}(\widetilde{M}, \mathbb{Z})$. Indeed, $H^{2m}(X, \mathbb{Z})$ and $H^*(\mathrm{Fix}\, G, \mathbb{Z})$ are torsion-free. Hence by Theorem 7.31 of [67] (Theorem 2.5.1), $H^{2m}(\widetilde{X}, \mathbb{Z})$ is torsion-free. Hence the result follows from Corollary 3.3.2.

4) We have the following exact sequence:

$$0 \longrightarrow \widetilde{\pi}_*(R) \longrightarrow H^{2m}(\widetilde{M},\mathbb{Z}) \longrightarrow H^{2m}(U,\mathbb{Z}) \longrightarrow H^{2m+1}(\widetilde{M},U,\mathbb{Z}).$$

Since $H^*(\operatorname{Fix} G,\mathbb{Z})$ is torsion-free, by Thom's isomorphism $H^{2m+1}(\widetilde{M}, U,\mathbb{Z})$, is torsion-free. Hence it is enough to show:

$$0 \longrightarrow \widetilde{\pi}_*(R \otimes \mathbb{C}) \longrightarrow H^{2m}(\widetilde{M},\mathbb{C}) \longrightarrow H^{2m}(U,\mathbb{C}) \longrightarrow 0.$$

Hence, it is enough to show that $\dim H^n(\widetilde{M},\mathbb{C}) = \dim H^n(U,\mathbb{C}) + \dim \widetilde{\pi}_*(R \otimes \mathbb{C})$. By Lemma 3.5.4,

$$H^{2m}(\widetilde{X},\mathbb{C}) = s^*(H^{2m}(X,\mathbb{C})) \oplus T \otimes \mathbb{C}.$$

Hence

$$H^{2m}(\widetilde{X},\mathbb{C})^G = s^*(H^{2m}(X,\mathbb{C})^G) \oplus T \otimes \mathbb{C}.$$

By 2),

$$H^{2m}(\widetilde{X},\mathbb{C})^G = s^*(H^{2m}(V,\mathbb{C})^G) \oplus \mathbb{C}\Sigma \oplus T \otimes \mathbb{C}.$$

Then, by Lemma 3.5.4,

$$H^{2m}(\widetilde{X},\mathbb{C})^G = s^*(H^{2m}(V,\mathbb{C})^G) \oplus R \otimes \mathbb{C}.$$

It follows:

$$\dim H^{2m}(\widetilde{M},\mathbb{C}) = \dim H^{2m}(U,\mathbb{C}) + \dim \widetilde{\pi}_*(R \otimes \mathbb{C}).$$

$$\square$$

Let $\widetilde{R}$ be the minimal primitive overlattice of $\widetilde{\pi}_*(R)$ in $H^{2m}(\widetilde{M},\mathbb{Z})$ and $\widetilde{T}$ the minimal primitive overlattice of $\widetilde{\pi}_*(T)$ in $H^{2m}(\widetilde{M},\mathbb{Z})$. As before, we need to calculate its discriminant. We start with the following lemma.

Lemma 3.5.10. *There exists $x \in \widetilde{\pi}_*(T)$ such that $\frac{x+(-1)^{n-1}\widetilde{\pi}_*(s^*(\Sigma))}{p} \in H^{2m}(\widetilde{M},\mathbb{Z})$.*

Proof. Let $s_1 : Y \to X$ be the blowup of X in Σ and Σ_1 the exceptional divisor, and $s_2 : \widetilde{X} \to Y$ the blowup in the other components of F such that $s = s_2 \circ s_1$. We denote $\Sigma_2 = s_2^*(\Sigma_1)$. Consider the following diagram:

$$
\begin{array}{ccc}
\Sigma_2 & \stackrel{l_2}{\hookrightarrow} & \widetilde{X} \\
\downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle s_2} \\
\Sigma_1 & \stackrel{l_1}{\hookrightarrow} & Y \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle s_1} \\
\Sigma & \stackrel{l_0}{\hookrightarrow} & X,
\end{array}
$$

where l_0, l_1 and l_2 are the inclusions and $g_i := s_{i|\Sigma_i}$, $i \in \{1,2\}$.

We have $\widetilde{\pi}_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_{\widetilde{M}} \oplus \mathcal{L}$, with $\mathcal{L}^p = \mathcal{O}_{\widetilde{M}} \left(- \left(\sum_{S_k \subset F \setminus \Sigma} \widetilde{S_k} \right) - \widetilde{\Sigma} \right)$, where each $\widetilde{S_k}$ is the exceptional divisor associated to the irreducible component $S_k \neq \Sigma$ of F. Thus

$$\frac{\sum_{S_k \subset F \setminus \Sigma} \widetilde{S_k} + \widetilde{\Sigma}}{p} \in H^2(\widetilde{M}, \mathbb{Z}).$$

It follows that

$$\left(\frac{\sum_{S_k \subset F \setminus \Sigma} \widetilde{S_k} + \widetilde{\Sigma}}{p} \right)^m \in H^{2m}(\widetilde{M}, \mathbb{Z}).$$

By Lemma 3.3.7 1), we get

$$\frac{x + \widetilde{\pi}_*(\Sigma_2^m)}{p} \in H^{2m}(\widetilde{M}, \mathbb{Z}), \tag{3.3}$$

with $x \in \widetilde{\pi}_*(T)$.

Now, it remains to calculate Σ_2^m. By Proposition 6.7 of [19], we have

$$s_1^* l_{0*}(\Sigma) = l_{1*}(c_{m-1}(E)),$$

where $E := g_1^*(\mathscr{N}_{\Sigma/X})/\mathscr{N}_{\Sigma_1/Y}$. Calculating, we find:

$$
\begin{aligned}
s_1^* l_{0*}(\Sigma) &= l_{1*} \left(\sum_{i=0}^{m-1} (-1)^{m-1-i} c_i(g_1^*(\mathscr{N}_{\Sigma/X})) \cdot c_1(\mathscr{N}_{\Sigma_1/Y})^{m-1-i} \right) \\
&= l_{1*} \left(\sum_{i=1}^{m-1} (-1)^{m-1-i} c_i(g_1^*(\mathscr{N}_{\Sigma/X})) \cdot c_1(\mathscr{N}_{\Sigma_1/Y})^{m-1-i} \right) \\
&\quad + (-1)^{m-1} l_{1*} \left(c_1(\mathscr{N}_{\Sigma_1/Y})^{m-1} \right) \\
&= l_{1*} \left(\sum_{i=1}^{m-1} (-1)^{m-1-i} c_i(g_1^*(\mathscr{N}_{\Sigma/X})) \cdot c_1(\mathscr{N}_{\Sigma_1/Y})^{m-1-i} \right) \\
&\quad + (-1)^{m-1} \Sigma_1^m.
\end{aligned}
$$

By applying s_2^*, we get:

$$\Sigma_2^m = (-1)^{m-1} \left(s^*(\Sigma) - s_2^* l_{1*}(a) \right),$$

where $a = \sum_{i=1}^{m-1} (-1)^{m-1-i} c_i(g_1^*(\mathscr{N}_{\Sigma/X})) \cdot c_1(\mathscr{N}_{\Sigma_1/Y})^{m-1-i} \in T$. And pushing forward via $\widetilde{\pi}_*$, we get:

$$\widetilde{\pi}_*(\Sigma_2^m) = (-1)^{m-1} \left(\widetilde{\pi}_*(s^*(\Sigma)) - \widetilde{\pi}_* \left(s_2^* l_{1*}(a) \right) \right).$$

The result follows from (3.3). $\qquad \square$

Lemma 3.5.11. *We have:*

1) $\widetilde{T}/\widetilde{\pi}_*(T) = (\mathbb{Z}/p\,\mathbb{Z})^{\mathrm{rktor}\,H^{2m}(U,\mathbb{Z})-\mathrm{rktor}\,H^{2m}(\widetilde{M},\mathbb{Z})-1}$,

2) $\mathrm{discr}\,\widetilde{T} = p^{h^{2*+\epsilon}(\mathrm{Fix}\,G,\mathbb{Z})-2\left[\mathrm{rktor}\,H^{2m}(U,\mathbb{Z})-\mathrm{rktor}\,H^{2m}(\widetilde{M},\mathbb{Z})\right]}$.

Proof. By 3) of Lemma 3.3.7, $\mathrm{discr}\,\widetilde{\pi}_*(T) = p^{\mathrm{rk}\,T}$, By Theorem 7.31 of [67],

$$
\begin{aligned}
\mathrm{rk}\,T = \mathrm{rk} \bigoplus_{S_k \subset \mathrm{Fix}\,G} \bigoplus_{i=0}^{r_k-2} H^{2m-2i-2}(S_k,\mathbb{Z}) \\
= h^{2*+\epsilon}(\mathrm{Fix}\,G,\mathbb{Z}) - \mathrm{rk}\,H^0(\Sigma,\mathbb{Z}) - \mathrm{rk}\,H^{2m}(\Sigma,\mathbb{Z}) \\
= h^{2*+\epsilon}(\mathrm{Fix}\,G,\mathbb{Z}) - 2,
\end{aligned}
$$

where r_k is the codimension of the component S_k. So

$$\mathrm{discr}\,\widetilde{\pi}_*(T) = p^{h^{2*+\epsilon}(\mathrm{Fix}\,G,\mathbb{Z})-2}.$$

Moreover, by the exact sequence of Lemma 3.5.9, we have:

$$\widetilde{R}/\widetilde{\pi}_*(R) = (\mathbb{F}_{\mathrm{p}})^{\mathrm{rktor}\,H^{2m}(U,\mathbb{Z})-\mathrm{rktor}\,H^{2m}(\widetilde{M},\mathbb{Z})}.$$

But by Lemma 3.5.10, we already know that there exists $x \in \widetilde{\pi}_*(T)$ such that $\frac{x+(-1)^{m-1}\widetilde{\pi}_*(s^*(\Sigma))}{p} \in H^{2m}(\widetilde{M},\mathbb{Z})$. We are going to deduce $\widetilde{T}/\widetilde{\pi}_*(T)$. If $\widetilde{\pi}_*(s^*(\Sigma))$ is divisible by p in $H^{2m}(\widetilde{M},\mathbb{Z})$, then $\overline{\frac{\widetilde{\pi}_*(s^*(\Sigma))}{p}} \in (\widetilde{R}/\widetilde{\pi}_*(R)) \setminus (\widetilde{T}/\widetilde{\pi}_*(T))$, if not $\overline{\frac{x+(-1)^{m-1}\widetilde{\pi}_*(s^*(\Sigma))}{p}} \in (\widetilde{R}/\widetilde{\pi}_*(R)) \setminus (\widetilde{T}/\widetilde{\pi}_*(T))$. Then in both cases

$$\widetilde{T}/\widetilde{\pi}_*(T) = (\mathbb{Z}/p\,\mathbb{Z})^{\mathrm{rktor}\,H^{2m}(U,\mathbb{Z})-\mathrm{rktor}\,H^{2m}(\widetilde{M},\mathbb{Z})-1}.$$

Hence by Proposition 2.1.1,

$$\mathrm{discr}\,\widetilde{T} = p^{h^{2*+\epsilon}(\mathrm{Fix}\,G,\mathbb{Z})-2-2\left[\mathrm{rktor}\,H^{2n}(U,\mathbb{Z})-\mathrm{rktor}\,H^{2n}(\widetilde{M},\mathbb{Z})-1\right]}.$$

$\square$

Conclusion

We use the unimodularity of $H^{2m}(\widetilde{M},\mathbb{Z})$. Let K be the primitive overlattice of $\mathcal{K}$ in $H^{2m}(\widetilde{M},\mathbb{Z})$ (We recall that $\mathrm{discr}\,\mathcal{K} = \mathrm{discr}\,\pi_*(H^{2m}(X,\mathbb{Z}))$ by Lemma 3.3.22). We have $K^\perp = \widetilde{T}$. Hence by Proposition 2.1.3,

$$\mathrm{discr}\,K = \mathrm{discr}\,\widetilde{T} = p^{h^{2*+\epsilon}(\mathrm{Fix}\,G,\mathbb{Z})-2\left[\mathrm{rktor}\,H^{2m}(U,\mathbb{Z})-\mathrm{rktor}\,H^{2n}(\widetilde{M},\mathbb{Z})\right]}.$$

Since

$$\mathrm{discr}\,\mathcal{K} = \mathrm{discr}\,\pi_*(H^{2m}(X,\mathbb{Z})) \geq \mathrm{discr}\,K \qquad and \qquad \mathrm{discr}\,\widetilde{T} \geq 1,$$

we get the statement 2) of the Theorem. By Proposition 2.1.1

$$K/\mathcal{K} = (\mathbb{Z}/p\mathbb{Z})^{\frac{\log_p(\operatorname{discr}\pi_*(H^{2m}(X,\mathbb{Z}))) - h^{2*+\epsilon}(\operatorname{Fix}G,\mathbb{Z}) + 2\left[\operatorname{rktor}H^{2m}(U,\mathbb{Z}) - \operatorname{rktor}H^{2m}(\widetilde{M},\mathbb{Z})\right]}{2}}.$$

Hence, we get the statement 1) of the Theorem.

Now if

$$\log_p(\operatorname{discr}\pi_*(H^{2m}(X,\mathbb{Z}))) + 2\operatorname{rktor}H^{2m}(U,\mathbb{Z})$$
$$= h^{2*+\epsilon}(\operatorname{Fix}G,\mathbb{Z}) + 2\operatorname{rktor}H^{2m}(\widetilde{M},\mathbb{Z}),$$

$K = \mathcal{K}$. Hence, $\mathcal{K}$ is primitive in $H^{2m}(\widetilde{M},\mathbb{Z})$. And we finish the proof by Lemma 3.3.22 3).

$\square$

3.5.3   Calculation of $\operatorname{rktor}H^n(\widetilde{M},\mathbb{Z})$

In the applications of the above theorem, we will almost never calculate rktor $H^n(\widetilde{M},\mathbb{Z})$. We will have:

$$\log_p(\operatorname{discr}\pi_*(H^n(X,\mathbb{Z}))) + 2\operatorname{rktor}H^n(U,\mathbb{Z}) = h^{2*+\epsilon}(\operatorname{Fix}G,\mathbb{Z}).$$

We give a corollary in the case when the above equality is satisfied.

Corollary 3.5.12. *Let $G = \langle\varphi\rangle$ be a group of prime order $2 \leq p \leq 19$ acting by automorphisms on a Kähler manifold X of dimension n. We assume:*

i) $H^*(X,\mathbb{Z})$ is torsion-free,

ii) $\operatorname{Fix}G$ is negligible or almost negligible,

iii) all the points of $\operatorname{Fix}G$ are of type 1, and

iv) $\log_p(\operatorname{discr}\pi_*(H^n(X,\mathbb{Z}))) + 2\operatorname{rktor}H^n(U,\mathbb{Z}) = h^{2*+\epsilon}(\operatorname{Fix}G,\mathbb{Z}).$

Then:

1) $H^n(\widetilde{M},\mathbb{Z})$ is torsion-free, and

2) (X,G) is H^n-normal.

Proof. Statement 1) follows from 2) of Theorem 3.5.2. Then we conclude by 3) of Theorem 3.5.2. $\square$

3.5.4 Calculation of $\mathrm{rktor}\, H^n(U,\mathbb{Z})$

It is possible to calculate $\mathrm{rktor}\, H^n(U,\mathbb{Z})$ with the spectral sequence of equivariant cohomology (see Section 3.2.2).

Proposition 3.5.13. *Let X be a compact complex manifold of dimension $2m$ and G an automorphism group of prime order acting on X. Let $U := (X \smallsetminus \mathrm{Fix}\, G)/G$. Assume that $H^*(X,\mathbb{Z})$ is torsion-free, $3 \leq p \leq 19$ and $\mathrm{Fix}\, G$ is negligible or almost negligible. Assume that (X,G) is E_2-degenerate over $\mathbb{Z}$, or that*

i) $l^{2i}_{p-1}(X) = 0$ for all $1 \leq i \leq m$, and

ii) $l^{2i+1}_1(X) = 0$ for all $0 \leq i \leq m-1$ when $m > 1$.

Then we have:

$$\mathrm{rktor}\, H^{2m}(U,\mathbb{Z}) = \sum_{i=0}^{m-1} l^{2i+1}_{p-1}(X) + \sum_{i=0}^{m-1} l^{2i}_1(X).$$

Proof. We use the equivariant cohomology. When $\mathrm{Fix}\, G$ is negligible, we have $H^k(V,\mathbb{Z}) = H^k(X,\mathbb{Z})$ for all $k \leq 2m$. Hence we can exchange V by X in the calculation of $H^{2m}(U,\mathbb{Z})$, so we get the result by Proposition 3.2.7 and Proposition 3.2.9.

When $\mathrm{Fix}\, G$ is almost negligible, we have $H^k(V,\mathbb{Z}) = H^k(X,\mathbb{Z})$ for all $k \leq 2m - 2$. Moreover, by Lemma 3.5.9 $H^{2m-1}(V,\mathbb{Z}) = H^{2m-1}(X,\mathbb{Z})$ and $H^{2m}(V,\mathbb{Z}) = H^{2m}(X,\mathbb{Z})/\mathbb{Z}\Sigma$, where Σ is the component of codimension m in $\mathrm{Fix}\, G$. Since Σ is primitive in $H^{2m}(X,\mathbb{Z})$, $H^{2m}(V,\mathbb{Z})$ is torsion-free. Hence, we can replace V by X in the calculation. Then we get the result by Proposition 3.2.7 and Proposition 3.2.9. $\qquad\square$

We have a similar proposition for $p = 2$.

Proposition 3.5.14. *Let X be a compact complex manifold of dimension $2m$ and G an automorphism group of order 2 acting on X. Let $U := (X \smallsetminus \mathrm{Fix}\, G)/G$. Assume that $H^*(X,\mathbb{Z})$ is torsion-free and $\mathrm{Fix}\, G$ is negligible or almost negligible. Assume that (X,G) is E_2-degenerate over $\mathbb{Z}$, or that*

i) $l^{2i}_{1,-}(X) = 0$ for all $1 \leq i \leq m$, and

ii) $l^{2i+1}_{1,+}(X) = 0$ for all $0 \leq i \leq m-1$.

Then we have:

$$\mathrm{rktor}\, H^{2m}(U,\mathbb{Z}) = \sum_{i=0}^{m-1} l^{2i+1}_{1,-}(X) + \sum_{i=0}^{m-1} l^{2i}_{1,+}(X).$$

We can also give a similar result when n is odd.

Proposition 3.5.15. *Let X be a compact complex manifold of dimension $2m+1$ and G an automorphism group of prime order acting on X. Let $U := (X \smallsetminus \mathrm{Fix}\, G)/G$. Assume that $H^*(X, \mathbb{Z})$ is torsion-free, $3 \le p \le 19$ and $\mathrm{Fix}\, G$ is negligible. Assume (X, G) is E_2-degenerate over $\mathbb{Z}$. Then we have:*

$$\mathrm{rktor}\, H^{2m+1}(U, \mathbb{Z}) = \sum_{i=0}^{m} l^{2i}_{p-1}(X) + \sum_{i=0}^{m-1} l^{2i+1}_{1}(X).$$

Proof. The same proof using Proposition 3.2.7. $\qquad\qquad\qquad\square$

Proposition 3.5.16. *Let X be a compact complex manifold of dimension $2m+1$ and G a group of order 2 acting on X. Let $U := (X \smallsetminus \mathrm{Fix}\, G)/G$. Assume that $H^*(X, \mathbb{Z})$ is torsion-free and $\mathrm{Fix}\, G$ is negligible. Assume (X, G) is E_2-degenerate over $\mathbb{Z}$. Then we have:*

$$\mathrm{rktor}\, H^{2m+1}(U, \mathbb{Z}) = \sum_{i=0}^{m} l^{2i}_{1,-}(X) + \sum_{i=0}^{m-1} l^{2i+1}_{1,+}(X).$$

Remark: Similar results hold over $\mathbb{F}_\mathrm{p}$ when (X, G) is E_2-degenerate.

3.5.5   Corollaries

The calculation of $\mathrm{rktor}\, H^n(U, \mathbb{Z})$ with the spectral sequence of equivariant cohomology and Section 3.3 implies a lot of corollaries. We will just give one example using Lemma 3.3.11 2).

Corollary 3.5.17. *Let $G = \langle \varphi \rangle$ be a group of prime order $3 \le p \le 19$ acting by automorphisms on a Kähler manifold X of dimension $2n$. We assume:*

  i) $H^*(X, \mathbb{Z})$ *is torsion-free,*

  ii) $\mathrm{Fix}\, G$ *is negligible or almost negligible,*

  iii) *all the points of* $\mathrm{Fix}\, G$ *are of type 1,*

  iv) $l^{2k}_{p-1}(X) = 0$ *for all* $1 \le k \le n$*, and*

  v) $l^{2k+1}_{1}(X) = 0$ *for all* $0 \le k \le n-1$*, when* $n > 1$*.*

Then:

  1) $l^{2n}_{1}(X) - h^{2*}(\mathrm{Fix}\, G, \mathbb{Z})$ *is divisible by 2, and*

  2) *we have:*

$$l^{2n}_{1}(X) + 2 \left[\sum_{i=0}^{n-1} l^{2i+1}_{p-1}(X) + \sum_{i=0}^{n-1} l^{2i}_{1}(X) \right]$$
$$\ge h^{2*}(\mathrm{Fix}\, G, \mathbb{Z}) + 2\, \mathrm{rktor}\, H^{2n}(\widetilde{M}, \mathbb{Z})$$
$$\ge 2 \left[\sum_{i=0}^{n-1} l^{2i+1}_{p-1}(X) + \sum_{i=0}^{n-1} l^{2i}_{1}(X) \right].$$

3) If moreover

$$l_1^{2n}(X) + 2\left[\sum_{i=0}^{n-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{n-1} l_1^{2i}(X)\right]$$
$$= h^{2*}(\mathrm{Fix}\,G, \mathbb{Z}) + 2\,\mathrm{rktor}\,H^{2n}(\widetilde{M}, \mathbb{Z}),$$

then (X, G) is H^{2n}-normal.

Proof. In Theorem 3.5.2, we replace rktor $H^{2m}(U, \mathbb{Z})$ by $\sum_{i=0}^{m-1} l_{p-1}^{2i+1}(X) + \sum_{i=0}^{m-1} l_1^{2i}(X)$ with Proposition 3.5.13 and $\log_p(\mathrm{discr}\,\pi_*(H^n(X, \mathbb{Z})))$ by l_1^{2n} with Lemma 3.3.11 2). $\qquad\square$

There is the same corollary when $p = 2$. Moreover, when $p = 2$ all the fixed points are of type 1.

Corollary 3.5.18. *Let $G = \langle \varphi \rangle$ be a group of prime order $p = 2$ acting by automorphisms on a Kähler manifold X of dimension $2n$. We assume:*

i) $H^*(X, \mathbb{Z})$ is torsion-free,

ii) Fix G is negligible or almost negligible,

iii) $l_{1,-}^{2k}(X) = 0$ for all $1 \leq k \leq n$, and

iv) $l_{1,+}^{2k+1}(X) = 0$ for all $0 \leq k \leq n - 1$, when $n > 1$.

Then:

1) $l_{1,+}^{2n}(X) - h^{2*}(\mathrm{Fix}\,G, \mathbb{Z})$ is divisible by 2.

2) We have:

$$l_{1,+}^{2n}(X) + 2\left[\sum_{i=0}^{n-1} l_{1,-}^{2i+1}(X) + \sum_{i=0}^{n-1} l_{1,+}^{2i}(X)\right]$$
$$\geq h^{2*}(\mathrm{Fix}\,G, \mathbb{Z}) + 2\,\mathrm{rktor}\,H^{2n}(\widetilde{M}, \mathbb{Z})$$
$$\geq 2\left[\sum_{i=0}^{n-1} l_{1,-}^{2i+1}(X) + \sum_{i=0}^{n-1} l_{1,+}^{2i}(X)\right].$$

3) If moreover

$$l_{1,+}^{2n}(X) + 2\left[\sum_{i=0}^{n-1} l_{1,-}^{2i+1}(X) + \sum_{i=0}^{n-1} l_{1,+}^{2i}(X)\right]$$
$$= h^{2*}(\mathrm{Fix}\,G, \mathbb{Z}) + 2\,\mathrm{rktor}\,H^{2n}(\widetilde{M}, \mathbb{Z}),$$

then (X, G) is H^{2n}-normal.

We can also provide a variant of the criterion for H^{2n}-normality which does not involve rktor $H^{2n}(\widetilde{M}, \mathbb{Z})$. We give a corollary in the case when the above equality is satisfied.

Corollary 3.5.19. *Let $G = \langle \varphi \rangle$ be a group of prime order $p = 2$ acting by automorphisms on a Kähler manifold X of dimension $2n$. We assume:*

   i) $H^*(X, \mathbb{Z})$ is torsion-free,

  ii) $\mathrm{Fix}\, G$ is negligible or almost negligible.

 iii) $l^{2k}_{1,-}(X) = 0$ for all $1 \le k \le n$,

 iv) $l^{2k+1}_{1,+}(X) = 0$ for all $0 \le k \le n-1$, when $n > 1$,

  v) and

$$l^{2n}_{1,+}(X) + 2 \left[\sum_{i=0}^{n-1} l^{2i+1}_{1,-}(X) + \sum_{i=0}^{n-1} l^{2i}_{1,+}(X) \right]$$
$$= h^{2*}(\mathrm{Fix}\, G, \mathbb{Z}).$$

Then:

 1) $H^{2n}(\widetilde{M}, \mathbb{Z})$ is torsion-free,

 2) (X, G) is H^{2n}-normal.

Proposition 3.5.20. *Let $G = \langle \varphi \rangle$ be a group of order 2 acting by automorphisms on a Kähler manifold X of dimension n. We assume:*

   i) $H^*(X, \mathbb{Z})$ is torsion-free,

  ii) $\mathrm{Fix}\, G$ is negligible or almost negligible,

 iii) and

$$l^{2n}_{1,+}(X) + 2 \left[\sum_{i=0}^{n-1} l^{2i+1}_{1,-}(X) + \sum_{i=0}^{n-1} l^{2i}_{1,+}(X) \right]$$
$$= h^{2*}(\mathrm{Fix}\, G, \mathbb{Z}).$$

Then $\widetilde{\pi}_*(s^*(H^n(X, \mathbb{Z})))$ is primitive in $H^n(\widetilde{M}, \mathbb{Z})$.

Proof. In our case the equality:

$$l^{2n}_{1,+}(X) + 2 \left[\sum_{i=0}^{n-1} l^{2i+1}_{1,-}(X) + \sum_{i=0}^{n-1} l^{2i}_{1,+}(X) \right]$$
$$= h^{2*}(\mathrm{Fix}\, G, \mathbb{Z})$$

is equivalent to the equality:

$$\log_p(\operatorname{discr} \pi_*(H^n(X,\mathbb{Z}))) + 2\operatorname{rktor} H^n(U,\mathbb{Z})$$
$$= h^{2*+\epsilon}(\operatorname{Fix} G, \mathbb{Z}) + 2\operatorname{rktor} H^n(\widetilde{M}, \mathbb{Z}).$$

Hence, by the conclusions of proof of Theorem 3.5.2, $\mathcal{K}$ is primitive in $H^n(\widetilde{M}, \mathbb{Z})$. Since $\widetilde{\pi}_*(s^*(H^n(X,\mathbb{Z}))) = \mathcal{K}$, the result follows. $\square$

Remark: Corollary 3.5.19 and Proposition 3.5.20 are also true for $3 \leq p \leq 19$ if we replace $l^i_{1,-}(X)$ by $l^i_{p-1}(X)$ and $l^i_{1,+}(X)$ by $l^i_1(X)$.

3.5.6   The case of simply connected surfaces

We can also give a practical application in the case of surfaces.

Corollary 3.5.21. *Let $G = \langle \varphi \rangle$ be a group of prime order $2 \leq p \leq 19$ acting by automorphisms on a simply connected Kähler surface X. We assume:*

   i) $\operatorname{Fix} G$ is finite, non empty and contains just points of type 1.

   ii) $l^2_{p-1}(X) = 0$ when $p > 2$ and $l^2_{1,-}(X) = 0$ when $p = 2$.

Then (X, G) is H^2-normal.

Proof. The cohomology graded group $H^*(X,\mathbb{Z})$ is torsion-free. Since X is simply connected, $H_1(X,\mathbb{Z}) = 0$. Hence by Poincaré duality $H^3(X,\mathbb{Z}) = 0$. Moreover, by universal coefficient theorem $H^1(X,\mathbb{Z}) = 0$ and $H^2(X,\mathbb{Z})$ is torsion-free.

By Corollary 3.5.17 and Corollary 3.5.19, it is enough to prove that $l^2_1(X) + 2 = \#\operatorname{Fix} G$ when $p > 2$ and $l^2_{1,+}(X) + 2 = \#\operatorname{Fix} G$ when $p = 2$. By Proposition 4.5 and Corollary 4.4 of [11], we have:

$$\#\operatorname{Fix} G = 2 + l^2_1(X) + l^2_{p-1}(X), \tag{3.4}$$

if $p > 2$ and

$$\#\operatorname{Fix} G = 2 + l^2_1(X), \tag{3.5}$$

if $p = 2$. Since we have assumed that $l^2_{p-1}(X) = 0$ when $p > 2$ and $l^2_{1,-}(X) = 0$ when $p = 2$, the equalities (3.4) and (3.5) become

$$\#\operatorname{Fix} = 2 + l^2_1(X), \tag{3.6}$$

if $p > 2$ and

$$\#\operatorname{Fix} = 2 + l^2_{1,+}(X), \tag{3.7}$$

what we wanted to prove. $\square$

3.6 The case of fixed points of type 2

3.6.1 Notation

Let X be a Kähler manifold of dimension n and G an automorphism group of order 3. We assume that $\operatorname{codim} \operatorname{Fix} G \geq 2$ (so that we have no points of type 0 in $\operatorname{Fix} G$). We consider the diagram:

$$
\begin{array}{ccccc}
M_2 & \xrightarrow{r_2} & M_1 & \xrightarrow{r_1} & M \\
{\scriptstyle\pi_2}\Big\uparrow & & {\scriptstyle\pi_1}\Big\uparrow & & {\scriptstyle\pi}\Big\uparrow \\
X_2 & \xrightarrow{s_2} & X_1 & \xrightarrow{s_1} & X \\
\circlearrowright & & \circlearrowright & & \circlearrowright \\
G_2 & & G_1 & & G
\end{array} \quad ,
$$

where s_1 is the blowup of X in the fixed points of type 2 and s_2 is the blowup of X_1 in $\operatorname{Fix} G_1$. By Proposition 3.4.3 and its proof, $\operatorname{Fix} G_1$ has only points of type 1 and M_2 is smooth. We also denote $V_1 := X_1 \setminus \operatorname{Fix} G_1$ and $U_1 := \pi_1(V_1)$. We have also $U_1 = M_2 \setminus \pi_2(s_2^{-1}(\operatorname{Fix} G_1))$.

3.6.2 2-dimensional case

We will state a result in the case when X is a surface. When $\operatorname{Fix} G$ is finite, we denote by $n_i(G)$ the number of fixed points of type $i = 1, 2$.

Theorem 3.6.1. *Let X be a Kähler surface and G an automorphism group of order 3. We assume:*

   i) $H^2(X, \mathbb{Z})$ is torsion-free,

  ii) $\operatorname{Fix} G$ is finite,

 iii) $l_2^2(X) = 0$.

Then:

   1) the number $\# \operatorname{Fix} G - l_1^2(X)$ is divisible by 2,

  2) $l_1^2(X) + 2 + 2l_2^1(X) \geq \# \operatorname{Fix} G + 2\operatorname{rktor} H^2(M_2, \mathbb{Z}) \geq 2 + 2l_2^1(X) - n_2(G)$,

  3) if moreover $l_1^2(X) + 2 + 2l_2^1(X) = \# \operatorname{Fix} G + 2\operatorname{rktor} H^2(M_2, \mathbb{Z})$, then (X, G) is H^2-normal.

Proof.

Lemma 3.6.2.   *1) $\# \operatorname{Fix} G_1 = \# \operatorname{Fix} G + n_2(G)$.*

  2) $H^2(X_1, \mathbb{Z}) = H^2(X, \mathbb{Z}) \bigoplus (\oplus_{i=1}^{n_2(G)} E_i)$, where E_i are the exceptional (-1)-curves.

  3) $H^1(X_2, \mathbb{Z}) = H^1(X_1, \mathbb{Z}) = H^1(X, \mathbb{Z})$,

4) $l_i^1(X_1) = l_i^1(X)$ *for all* $i \in \{1, 2, 3\}$,

5) $l_2^2(X_1) = l_2^2(X)$ *and* $l_3^2(X_1) = l_3^2(X)$,

6) $l_1^2(X_1) = l_1^2(X) + n_2(G)$.

Proof. 1) Now consider $x \in \operatorname{Fix} G$ of type 2. We have

$$(X, G, x) \sim (\mathbb{C}^2, \langle \operatorname{diag}(\xi_3, \xi_3^2) \rangle, 0).$$

Let $\widetilde{\mathbb{C}^2}$ be the blowup of $\mathbb{C}^2$ in 0,

$$\widetilde{\mathbb{C}^2} = \left\{ ((x_1, x_2), (a_1 : a_2)) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid x_1 a_2 = x_2 a_1 \right\}.$$

On $\widetilde{\mathbb{C}^2}$, $\langle \operatorname{diag}(\xi_3, \xi_3^2) \rangle$ acts as follows:

$$\operatorname{diag}(\xi_3, \xi_3^2) \cdot ((x_1, x_2), (a_1 : a_2)) = \left((\xi_3 x_1, \xi_3^2 x_2), (a_1 : \xi_3 a_2) \right).$$

Hence there are two fixed points: $((0,0), (0,1))$ and $((0,0), (1,0))$. The result follows.

2) Consequence of Theorem 7.31 of [67] (Theorem 2.5.1).

3) Consequence of Theorem 7.31 of [67] (Theorem 2.5.1).

4) Consequence of 3).

5),6) Since all the E_i are invariant by G, we get the result from Definition-Proposition 2.2.2.

$\square$

Now we apply Corollary 3.5.17 to (X_1, G_1). The group $H^*(X_1, \mathbb{Z})$ is torsion-free because $H^*(X, \mathbb{Z})$ and $H^*(\operatorname{Fix} G, \mathbb{Z})$ are torsion-free. The set $\operatorname{Fix} G_1$ is negligible and $l_2^2(X_1) = 0$. Hence $\#\operatorname{Fix} G_1 - l_1^2(X_1)$ is divisible by 2. But

$$\#\operatorname{Fix} G_1 - l_1^2(X_1) = \#\operatorname{Fix} G + n_2(G) - l_1^2(X) - n_2(G) \quad = \#\operatorname{Fix} G - l_1^2(X).$$

Hence $\#\operatorname{Fix} G - l_1^2(X)$ is divisible by 2.

We also have:

$$l_1^2(X_1) + 2 + 2l_2^1(X_1) \geq \#\operatorname{Fix} G_1 + 2\operatorname{rktor} H^2(M_2, \mathbb{Z}) \geq 2 + 2l_2^1(X_1).$$

Hence,

$$l_1^2(X) + n_2(G) + 2 + 2l_2^1(X) \geq \#\operatorname{Fix} G + n_2(G) + 2\operatorname{rktor} H^2(M_2, \mathbb{Z}) \geq 2 + 2l_2^1(X).$$

By subtracting $n_2(G)$ on both sides of the inequality, we get 2). If moreover

$$l_1^2(X) + 2 + 2l_2^1(X) = \#\operatorname{Fix} G + 2\operatorname{rktor} H^2(M_2, \mathbb{Z}),$$

then

$$l_1^2(X) + n_2(G) + 2 + 2l_2^1(X) = \#\operatorname{Fix} G + n_2(G) + 2\operatorname{rktor} H^2(M_2, \mathbb{Z}),$$

hence
$$l_1^2(X_1) + 2 + 2l_2^1(X_1) = \#\operatorname{Fix} G_1 + 2\operatorname{rktor} H^2(M_2, \mathbb{Z}).$$

Therefore (X_1, G_1) is H^2-normal by Corollary 3.5.17; so by Proposition 3.3.25 and Proposition 3.3.23, (X, G) is H^2-normal. $\qquad\square$

We can give a corollary similar to Corollary 3.5.21.

Corollary 3.6.3. *Let $G = \langle \varphi \rangle$ be a group of order 3 acting by automorphisms on a simply connected Kähler surface X. We assume:*

i) $\operatorname{Fix} G$ is finite,

ii) $l_2^2(X) = 0$,

Then (X, G) is H^2-normal.

Proof. The same proof as in Corollary 3.5.21. We use Proposition 4.5, Corollary 4.4 of [11] and Theorem 3.6.1. $\qquad\square$

3.6.3   Result in dimension 4

Let X be a Kähler manifold of dimension 4 and G an automorphism group of order 3. We denote
$$\mathscr{F}_1 := \{ x \in \operatorname{Fix} G \,|\, o(x) = 1 \} .$$

We have different kinds of points of type 2. There are four kinds of local actions of G: $\frac{1}{3}(0, 0, 1, 2)$, $\frac{1}{3}(0, 1, 1, 2)$, $\frac{1}{3}(1, 1, 1, 2)$ and $\frac{1}{3}(1, 1, 2, 2)$, where we denote by $\frac{1}{r}(a_1, \dots, a_n)$ the action of the cyclic group of order r by $\operatorname{diag}(\xi_p^{a_1}, \dots, \xi_p^{a_n})$.

Points with action $\frac{1}{3}(0, 0, 1, 2)$ fill in surfaces of type-2 fixed points of G. The points with action $\frac{1}{3}(0, 1, 1, 2)$ fill in fixed curves. The points with action $\frac{1}{3}(1, 1, 1, 2)$ or $\frac{1}{3}(1, 1, 2, 2)$ are isolated. We will state our result in the case when points of type 2 are all of the form $\frac{1}{3}(1, 1, 2, 2)$.

The cases of the points $\frac{1}{3}(0, 0, 1, 2)$, $\frac{1}{3}(0, 1, 1, 2)$ and $\frac{1}{3}(1, 1, 1, 2)$ are more complicated and will not be treated here, for they are not needed in the case of a symplectic action on a symplectic 4-fold.

Theorem 3.6.4. *Let X be a Kähler manifold of dimension 4 and G an automorphism group of order 3. We assume:*

i) $H^*(X, \mathbb{Z})$ is torsion-free,

ii) $\operatorname{Fix} G$ is negligible or almost negligible and the points of type 2 are all of the form $\frac{1}{3}(1, 1, 2, 2)$.

iii) $l_1^3(X) = l_2^4(X) = l_1^1(X) = l_2^2(X) = 0$.

Then:

1) $\# \operatorname{Fix} G - l_1^2(X)$ *is divisible by 2,*

2) *we have:*

$$l_1^4(X) + 2\left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X)\right]$$
$$\geq h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 2\operatorname{rktor} H^4(M_2, \mathbb{Z})$$
$$\geq 2\left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X)\right] - n_{1,1,2,2},$$

where $n_{1,1,2,2}$ *is the number of points of type 2 (of the form $\frac{1}{3}(1,1,2,2)$).*

3) *If moreover*

$$l_1^4(X) + 2\left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X)\right] = h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 2\operatorname{rktor} H^4(M_2, \mathbb{Z}),$$

then (X, G) is H^4-normal.

Proof. As in the proof of Lemma 3.6.2 1), for each fixed point of type $\frac{1}{3}(1,1,2,2)$, there are two lines fixed by G_1. We denote these lines by $(l_i)_{1,\ldots,2n_{1,1,2,2}}$.
 The following lemma holds:

Lemma 3.6.5.   *1)* $\operatorname{Fix} G_1 = \mathscr{F}_1 \sqcup (\bigsqcup_{i=1}^{2n_{1,1,2,2}} l_i)$. *Hence* $\operatorname{Fix} G_1$ *and* $\operatorname{Fix} G$ *have the same property of negligibility.*

2) $h^{2*}(\operatorname{Fix} G_1, \mathbb{Z}) = h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 3n_{1,1,2,2}$,

3) $l_i^1(X_1) = l_i^1(X)$ *for all* $i \in \{1, 2, 3\}$,

4) $l_2^2(X_1) = l_2^2(X)$ *and* $l_3^2(X_1) = l_3^2(X)$,

5) $l_1^2(X_1) = l_1^2(X) + n_{1,1,2,2}$.

6) $l_i^3(X_1) = l_i^3(X)$ *for all* $i \in \{1, 2, 3\}$,

7) $l_2^4(X_1) = l_2^4(X)$,

8) $l_1^4(X_1) = l_1^4(X) + n_{1,1,2,2}$.

Proof.   1) Obvious.

2)

$$h^{2*}(\operatorname{Fix} G_1, \mathbb{Z}) = h^{2*}(\mathscr{F}_1) + 2n_{1,1,2,2}h^{2*}(\mathbb{P}^1)$$
$$= (h^{2*}(\operatorname{Fix} G, \mathbb{Z}) - n_{1,1,2,2}) + 4n_{1,1,2,2}$$
$$= h^{2*}(\operatorname{Fix} G, \mathbb{Z}) + 3n_{1,1,2,2}.$$

3) By Theorem 7.31 of [67] (Theorem 2.5.1),

$$H^1(X_1, \mathbb{Z}) = H^1(X, \mathbb{Z}),$$

which implies the result.

4),5) By Theorem 7.31 of [67],

$$H^2(X_1, \mathbb{Z}) = H^2(X, \mathbb{Z}) \bigoplus (\oplus_{i=1}^{n_{1,1,2,2}} D_i),$$

where D_i are the exceptional divisors. Since the D_i are fixed by the action of G_1, the result follows from Definition-Proposition 2.2.2.

6) By Theorem 7.31 of [67],

$$H^3(X_1, \mathbb{Z}) = H^3(X, \mathbb{Z}),$$

which implies the result.

7),8) By Theorem 7.31 of [67],

$$H^4(X_1, \mathbb{Z}) = H^4(X, \mathbb{Z}) \bigoplus (\oplus_{i=1}^{n_{1,1,2,2}} h_i),$$

where $h_i = c_1(\mathcal{O}_{D_i}(1))^2$. Since h_i are fixed by the action of G_1, the result follows.

$\square$

Now we apply Corollary 3.5.17 to (X_1, G_1). The group $H^*(X_1, \mathbb{Z})$ is torsion-free because $H^*(X, \mathbb{Z})$ and $H^*(\text{Fix}\,G, \mathbb{Z})$ are torsion-free. The set $\text{Fix}\,G_1$ is negligible and

$$l_1^3(X_1) = l_2^4(X_1) = l_1^1(X_1) = 0.$$

Hence $h^{2*}(\text{Fix}\,G_1, \mathbb{Z}) - l_1^2(X_1) = h^{2*}(\text{Fix}\,G, \mathbb{Z}) - l_1^2(X) + 2n_{1,1,2,2}$ is divisible by 2. Hence $h^{2*}(\text{Fix}\,G, \mathbb{Z}) - l_1^2(X)$ is divisible by 2. We have:

$$l_1^4(X_1) + 2 \left[1 + l_2^1(X_1) + l_2^3(X_1) + l_1^2(X_1) \right]$$
$$\geq h^{2*}(\text{Fix}\,G_1, \mathbb{Z}) + 2\,\text{rktor}\,H^4(M_2, \mathbb{Z})$$
$$\geq 2 \left[1 + l_2^1(X_1) + l_2^3(X_1) + l_1^2(X_1) \right].$$

Hence:

$$l_1^4(X_1) + n_{1,1,2,2} + 2 \left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X) + n_{1,1,2,2} \right]$$
$$\geq h^{2*}(\text{Fix}\,G, \mathbb{Z}) + 3n_{1,1,2,2} + 2\,\text{rktor}\,H^4(M_2, \mathbb{Z})$$
$$\geq 2 \left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X) + n_{1,1,2,2} \right].$$

By subtracting $3n_{1,1,2,2}$ on each side of the inequality, we get 2). If moreover

$$l_1^4(X) + 2 \left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X) \right] = h^{2*}(\text{Fix}\,G, \mathbb{Z}) + 2\,\text{rktor}\,H^4(M_2, \mathbb{Z}),$$

then

$$l_1^4(X) + n_{1,1,2,2} + 2 \left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X) + n_{1,1,2,2} \right]$$
$$= h^{2*}(\text{Fix}\,G, \mathbb{Z}) + 3n_{1,1,2,2} + 2\,\text{rktor}\,H^4(M_2, \mathbb{Z}),$$

so that

$$l_1^4(X_1) + 2\left[1 + l_2^1(X_1) + l_2^3(X_1) + l_1^2(X_1)\right] = h^{2*}(\mathrm{Fix}\,G_1, \mathbb{Z}) + 2\,\mathrm{rktor}\,H^4(M_2, \mathbb{Z}).$$

Hence (X_1, G_1) is H^4-normal by Corollary 3.5.17. By Propositions 3.3.23 and 3.3.25, (X, G) is H^4-normal. $\qquad\square$

When the equality

$$l_1^4(X) + 2\left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X)\right] = h^{2*}(\mathrm{Fix}\,G, \mathbb{Z})$$

is satisfied (this will be the case in the applications), by statements 1) and 2) of the last theorem, we deduce the following corollary.

Corollary 3.6.6. *Let X be a Kähler manifold of dimension 4 and G an automorphism group of order 3. We assume:*

   i) $H^*(X, \mathbb{Z})$ is torsion-free,

  ii) $\mathrm{Fix}\,G$ is negligible or almost negligible and the points of type 2 are all of the form $\frac{1}{3}(1, 1, 2, 2)$,

 iii) $l_1^3(X) = l_2^4(X) = l_1^1(X) = l_2^2(X) = 0$,

 iv) $l_1^4(X) + 2\left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X)\right] = h^{2*}(\mathrm{Fix}\,G, \mathbb{Z})$.

Then:

  1) $H^4(M_2, \mathbb{Z})$ is torsion-free,

  2) (X, G) is H^4-normal.

Chapter 4

Application to cup-product and Beauville–Bogomolov lattices

In this chapter, we give some examples of applications of the results of the last chapter.

4.1 Quotient of a K3 surface by a symplectic involution

Corollary 4.1.1. *Let S be a K3 surface and i a Nikulin involution (i.e. a symplectic involution) on S. The couple $(S, \langle i \rangle)$ is H^2-normal. Let $\overline{Y}_2 := X/i$ be the quotient. The lattice $\left(H^2(\overline{Y}_2, \mathbb{Z}), \cdot \right)$ is isometric to $E_8(-1) \oplus U(2)^3$.*

Proof. We know that $H^2(S, \mathbb{Z})^i \simeq U^3 \oplus E_8(-2)$ (see for instance [21]). Hence by Proposition 3.3.8 1) and Definition-Proposition 2.2.4 3), $l_2^2(S) = 8$. Since $\operatorname{rk} H^2(S, \mathbb{Z}) = 22$ and $\operatorname{rk} H^2(S, \mathbb{Z})^i = 14$, by Proposition 2.2.3:

$$\operatorname{rk} H^2(S, \mathbb{Z}) - \operatorname{rk} H^2(S, \mathbb{Z})^{i^*} = l_2^2(S) + l_{1,-}^2(S) = 22 - 14 = 8.$$

Hence $l_{1,-}^2(S) = 0$. Moreover $\operatorname{Fix} i$ contains exactly 8 points (see for instance [21]), S is simply connected and $H^*(S, \mathbb{Z})$ is torsion-free. Hence by Corollary 3.5.21, $(S, \langle i \rangle)$ is H^2-normal.

We finish the proof by Proposition 3.3.8 2). $\qquad\square$

4.2 Quotient of a K3 surface by a symplectic automorphism of order 3

Corollary 4.2.1. *Let S be a K3 surface and G a symplectic automorphism group of order 3 acting on S. Then (S, G) is H^2-normal. Let $\overline{Y}_3 := S/G$ be the*

quotient. The lattice $\left(H^2(\overline{Y}_3, \mathbb{Z}), \cdot\right)$ is isometric to $U(3) \oplus U^2 \oplus A_2^2$.

Proof. By Theorem 4.1 of [20], $H^2(S, \mathbb{Z})^G \simeq U \oplus U(3)^2 \oplus A_2^2$. Hence by Proposition 3.3.8 1) and Definition-Proposition 2.2.4 3), $l_3^2(S) = 6$. Since $\operatorname{rk} H^2(S, \mathbb{Z}) = 22$ and $\operatorname{rk} H^2(S, \mathbb{Z})^G = 10$, by Proposition 2.2.1,

$$\operatorname{rk} H^2(S, \mathbb{Z}) - \operatorname{rk} H^2(S, \mathbb{Z})^G = 2l_3^2(S) + l_2^2(S) = 22 - 10 = 12.$$

Hence $l_2^2(S) = 0$. Moreover the action is symplectic, hence the fixed points are all isolated. Furthermore S is simply connected and $H^*(S, \mathbb{Z})$ is torsion-free. Hence by Corollary 3.6.3, (S, G) is H^2-normal.

We finish the proof by Proposition 3.3.8 2) (here $A_2^\vee(3) = A_2$). $\qquad\square$

4.3   Quotient of a K3 surface by a non-symplectic automorphism of order 3

There exist K3 surfaces with non-symplectic automorphisms of order 3 such that the fixed locus contains only isolated points (see Theorem 3.3 and Table 2 of [2]).

Corollary 4.3.1. *Let S be a K3 surface and G a non-symplectic group of prime order 3 acting on S. We assume that $\operatorname{Fix} G$ is finite. Then (S, G) is H^2-normal. Let $\overline{Z}_3 := S/G$ be the quotient. The lattice $\left(H^2(\overline{Z}_3, \mathbb{Z}), \cdot\right)$ is isometric to $U \oplus E_6$.*

Proof. By Table 2 of [2], $H^2(S, \mathbb{Z})^G \simeq U(3) \oplus E_6^\vee(3)$. We have $\operatorname{discr} E_6 = 3$, hence $\operatorname{discr} E_6^\vee(3) = 3^5$. Hence by Proposition 3.3.8 1) and Definition-Proposition 2.2.4 3), $l_3^2(S) = 7$. Since $\operatorname{rk} H^2(S, \mathbb{Z}) = 22$ and $\operatorname{rk} H^2(S, \mathbb{Z})^G = 8$, by Proposition 2.2.1,

$$\operatorname{rk} H^2(S, \mathbb{Z}) - \operatorname{rk} H^2(S, \mathbb{Z})^G = 2l_3^2(S) + l_2^2(S) = 22 - 8 = 14.$$

Hence $l_2^2(S) = 0$. Since S is simply connected and $H^2(S, \mathbb{Z})$ is torsion-free, by Corollary 3.6.3, (S, G) is H^2-normal (in fact, here all the fixed points are of type 1, so we can also use Corollary 3.5.21).

We finish the proof by Proposition 3.3.8 2). $\qquad\square$

4.4   Quotient of a complex torus of dimension 2 by $-\operatorname{id}$

Corollary 4.4.1. *Let A be a complex torus of dimension 2. We denote $\overline{A} = A/-\operatorname{id}$. Then $H^2(\overline{A}, \mathbb{Z})$ endowed with the cup product is isometric to $U(2)^3$.*

Proof. The ring $H^*(A, \mathbb{Z})$ is torsion-free and $\operatorname{Fix} G$ contains 16 isolated points. In this case $\widetilde{M}$ (blowup of $\overline{A}$ in the 16 points) is a K3 surface. Hence $\operatorname{rktor} H^2(\widetilde{M}, \mathbb{Z}) = 0$. The map $(-\operatorname{id})^*$ acts on $H^1(A, \mathbb{Z})$ as $-\operatorname{id}$ and acts on $H^2(A, \mathbb{Z})$ as id. Hence $l_{1,-}^1(A) = 4$, $l_{1,+}^2(A) = 6$ and $l_{1,-}^2(A) = 0$. Therefore the equality of Corollary 3.5.18 3) is verified: $6 + 2 + 2 \times 4 = 16$. Hence $(\overline{A}, -\operatorname{id})$ is H^2-normal.

We finish the proof by Proposition 3.3.8 2). $\qquad\square$

4.5 Quotient of a $K3^{[2]}$-type manifold by an automorphism of order 3

4.5.1 Symplectic groups and Beauville–Bogomolov forms

Here, we study the quotient of a manifold of $K3^{[2]}$-type X by a symplectic group G of prime order. Hence $M = X/G$ will be a singular irreducible symplectic variety. Moreover, in the case where the non-free locus of G is finite, the codimension of the singular locus of $M = X/G$ will be 4. Hence, in this situation, we can use Theorem 1.2.4 and also the Torelli Theorem of Namikawa (Theorem 1.2.3). Therefore, we can calculate the Beauville–Bogomolov form in this case.

We will need the following proposition.

Proposition 4.5.1. *Let X be a manifold of $K3^{[2]}$-type, G a symplectic group of prime order p with $\mathrm{Fix}\,G$ finite. Then*

$$B_{X/G}(\pi_*(\alpha), \pi_*(\beta)) = \sqrt{\frac{3p^3}{C_{X/G}}} B_X(\alpha, \beta),$$

where $C_{X/G}$ is the Fujiki constant of X/G and α, β are in $H^2(X, \mathbb{Z})^G$.

Proof. By (1) of Theorem 1.2.4, we have

$$(\pi_*(\alpha))^4 = C_{X/G} B_{X/G}(\pi_*(\alpha), \pi_*(\alpha))^2.$$

By Theorem 1.3.1, the Fujiki constant of X is 3. Hence we also have:

$$\alpha^4 = 3B_X(\alpha, \alpha)^2.$$

Moreover, by Lemma 3.3.7, 3),

$$(\pi_*(\alpha))^4 = p^3 \pi_*(\alpha^4).$$

By (2) of Theorem 1.2.4, we get the result. $\qquad\square$

4.5.2 Beauville–Bogomolov lattice

We study the case $p = 3$. We have the following corollary.

Corollary 4.5.2. *Let X be a manifold of $K3^{[2]}$-type. Let G be an order 3 group of numerically standard symplectic automorphisms of X. Then (X, G) is H^2-normal and H^4-normal.*

Proof. By Proposition 3.3.16, Proposition 3.3.19 and Theorem 1.3.9, if (X, G) is H^4-normal then (X, G) is H^2-normal. Hence we have just to show the H^4-normality. We will apply Corollary 3.6.6.

By Theorem 2.5 of [41] and Example 4.2.1 of [40], we know that $\mathrm{Fix}\,G$ consists of 27 isolated points. Moreover the action of G is symplectic on X,

hence all the fixed points are of type $(1, 1, 2, 2)$. By Theorem 1.3.9, $H^*(X, \mathbb{Z})$ is torsion-free. It remains to show that $l_2^2(X) = l_2^4(X) = 0$.

By definition of numerically standard and Theorem 4.1 of [20], we know that

$$H^2(X, \mathbb{Z})^G \simeq U \oplus U(3)^2 \oplus A_2^2 \oplus (-2),$$

where $H^2(X, \mathbb{Z})$ is endowed with the Beauville–Bogomolov form. Hence by Lemma 2.2.7 and Definition-Proposition 2.2.4, $l_3^2(X) = 6$. Since $\operatorname{rk} H^2(X, \mathbb{Z}) = 23$ and $\operatorname{rk} H^2(X, \mathbb{Z})^G = 11$, by Proposition 2.2.1,

$$\operatorname{rk} H^2(X, \mathbb{Z}) - \operatorname{rk} H^2(X, \mathbb{Z})^G = 2l_3^2(X) + l_2^2(X) = 23 - 11 = 12.$$

Hence $l_2^2(X) = 0$. Then, by Proposition 3.3.17, Proposition 3.3.19, Theorem 1.3.9 and Lemma 3.3.18, $l_2^4(X) = 0$.

Now, we show that

$$l_1^4(X) + 2\left[1 + l_2^1(X) + l_2^3(X) + l_1^2(X)\right] = h^{2*}(\operatorname{Fix} G, \mathbb{Z}).$$

Since $H^{odd}(X, \mathbb{Z}) = 0$ (Theorem 1.3.9) and $\operatorname{Fix} G$ consists of 27 isolated points, we have jsute to show:

$$l_1^4(X) + 2(1 + l_1^2(X)) = 27. \tag{4.1}$$

It remains to calculate $l_1^4(X)$ and $l_1^2(X)$. By Proposition 2.2.1, $\operatorname{rk} H^2(X, \mathbb{Z})^G = l_1^2(X) + l_3^2(X)$. Since $\operatorname{rk} H^2(X, \mathbb{Z})^G = 11$ and $l_3^2(X) = 6$, we get $l_1^2(X) = 5$. Now by Proposition 3.3.17, Proposition 3.3.19, Theorem 1.3.9 and Lemma 3.3.18, $l_1^4(X) = \frac{5 \times 6}{2} = 15$. Hence we get (4.1). We conclude by Corollary 3.6.6. □

We deduce the following theorem.

Theorem 4.5.3. *Let X be a manifold of $S^{[2]}$-type. Let G be an order 3 group of numerically standard symplectic automorphisms of X. We denote $M_3 = X/G$. Then the Beauville–Bogomolov lattice $H^2(M_3, \mathbb{Z})$ is isomorphic to $U(3) \oplus U^2 \oplus A_2^2 \oplus (-6)$, and the Fujiki constant of M_3 is 9.*

Proof. By definition of numerically standard and Theorem 4.1 of [20], there is an isometry of lattices $H^2(S^{[2]}, \mathbb{Z})^G \simeq U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2 \oplus (-2)$. Now, we need a lemma.

Lemma 4.5.4. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and G a symplectic automorphism group of order $3 \leq p \leq 19$. We have $A_{H^2(X,\mathbb{Z})^G} = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})^{a_G(X)}$. We denote $A_{H^2(X,\mathbb{Z})^G, p} := (\mathbb{Z}/p\mathbb{Z})^{a_G(X)}$. Then the projection*

$$\frac{H^2(X, \mathbb{Z})}{H^2(X, \mathbb{Z})^G \oplus S_G^2(X)} \to A_{H^2(X,\mathbb{Z})^G, p}$$

is an isomorphism. Moreover, let $x \in H^2(X, \mathbb{Z})^G$ and assume $\frac{x}{p} \in (H^2(X, \mathbb{Z})^G)^\vee$. If $\frac{x}{p} \in A_{H^2(X,\mathbb{Z})^G, p}$ then there is $z \in H^2(X, \mathbb{Z})$ such that $x = z + \varphi(z) + \cdots + \varphi^{p-1}(z)$.

Proof. The first assertion follows from Lemma 2.2.7 and its proof.

Now let $x \in H^2(X, \mathbb{Z})^G$ such that $\frac{\overline{x}}{p} \in A_{H^2(X,\mathbb{Z})^G, p}$. By the first assertion, there is $z \in H^2(X, \mathbb{Z})$ and $y \in S_G(X)$ such that $z = \frac{x+y}{p}$. Then $z + \varphi(z) + \cdots + \varphi^{p-1}(z) = x + \frac{y + \varphi(y) + \cdots + \varphi^{p-1}(y)}{p}$. But $y + \varphi(y) + \cdots + \varphi^{p-1}(y) = 0$, so $z + \varphi(z) + \cdots + \varphi^{p-1}(z) = x$. □

Here we have $A_{H^2(X,\mathbb{Z})^G, 3} = A_{U(3)} \oplus A_{U(3)} \oplus A_{A_2} \oplus A_{A_2}$. Hence by Lemma 4.5.4 and Corollary 4.5.2, we have $\frac{1}{3}\pi_*(U(3)) \subset H^2(M_3, \mathbb{Z})$. And if we denote by $\widetilde{A_2}$ the minimal primitive overgroup of $\pi_*(A_2)$ in $H^2(M_3, \mathbb{Z})$, we will have $\widetilde{A_2}/\pi_*(A_2) = \mathbb{Z}/3\mathbb{Z}$. We denote (a, b) an integral basis of A_2, with $B_X(a, a) = B_X(b, b) = -2$ and $B_X(a, b) = 1$. Hence $\frac{\overline{a-b}}{3} \in A_{A_2}$. Then by Lemma 4.5.4 and Corollary 4.5.2, $\pi_*(a) - \pi_*(b)$ is divisible by 3 in $H^2(M_3, \mathbb{Z})$, and $\widetilde{A_2}/\pi_*(A_2)$ is generated by $\frac{\overline{\pi_*(a)-\pi_*(b)}}{3}$. Hence we can choose $\frac{\pi_*(a)-\pi_*(b)}{3}$ and $\frac{\pi_*(a)-\pi_*(b)}{3} + \pi_*(b) = \frac{\pi_*(a)+2\pi_*(b)}{3}$ as a basis of $\widetilde{A_2}$. The matrix of the sublattice generated by $a - b$ and $a + 2b$ in A_2 is

$$A_2(3) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}.$$

By Corollary 4.5.2 and Lemma 4.5.4, we have

$$H^2(M_3, \mathbb{Z}) = \pi_*(U) \oplus \frac{1}{3}\pi_*(U(3))^2 \oplus \frac{1}{3}\pi_*(A_2(3))^2 \oplus \pi_*(-2).$$

Then by Proposition 4.5.1, the Beauville–Bogomolov form of $H^2(M_3, \mathbb{Z})$ gives the lattice

$$U\left(\sqrt{\frac{81}{C_{M_3}}}\right) \oplus \frac{1}{3}U^2\left(3\sqrt{\frac{81}{C_{M_3}}}\right) \oplus \frac{1}{3}A_2^2\left(3\sqrt{\frac{81}{C_{M_3}}}\right) \oplus \left(-2\sqrt{\frac{81}{C_{M_3}}}\right)$$

$$= U\left(3\sqrt{\frac{9}{C_{M_3}}}\right) \oplus U^2\left(\sqrt{\frac{9}{C_{M_3}}}\right) \oplus A_2^2\left(\sqrt{\frac{9}{C_{M_3}}}\right) \oplus \left(-6\sqrt{\frac{9}{C_{M_3}}}\right).$$

It follows that $C_{M_3} = 9$ and we get the lattice

$$U(3) \oplus U^2 \oplus A_2^2 \oplus (-6).$$

□

4.6 Quotient of a $K3^{[2]}$-type manifold by a symplectic involution

4.6.1 The H^4-normality

In all this section, we will consider an irreducible symplectic manifold X of $K3^{[2]}$-type and a symplectic involution ι on X. By Theorem 1.3.5 (Mongardi),

we can assume in all the proofs that $X = S^{[2]}$ for a K3 surface S and $\iota = i^{[2]}$, where i is a symplectic involution on S. Moreover, by Theorem 4.1 of [43] (Theorem 1.3.6), the fixed locus of σ is always the union of 28 points and a K3 surface Σ. We start with the following corollary of Corollary 3.5.19.

Corollary 4.6.1. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and ι a symplectic involution on X. Then $(X, \langle \iota \rangle)$ is H^4-normal.*

Proof. We use Corollary 3.5.19. By Theorem 1.3.9, $H^*(S^{[2]}, \mathbb{Z})$ is torsion-free. Since $\mathrm{Fix}\, G$ consists of a K3 surface and 28 isolated point, $H^*(\mathrm{Fix}\, G, \mathbb{Z})$ is torsion-free. Let Σ be the K3 surface fixed by ι. The following lemma says that the class of Σ is primitive in $H^4(S^{[2]}, \mathbb{Z})$.

Lemma 4.6.2. *We have:*

$$\Sigma \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle = 1, \quad \Sigma \cdot \mathfrak{q}_2(\alpha_k)\,|0\rangle = 0,$$

and

$$\Sigma \cdot \mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\,|0\rangle = \alpha_k \cdot i^*\alpha_l,$$

for all $(k, l) \in \{1, \ldots, 22\}^2$.

Proof. By the definition of the Nakajima operators, $\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle$ corresponds to the cycle $\left\{ \xi \in S^{[2]} \,\middle|\, \mathrm{Supp}\,\xi \ni x \right\}$. The element $\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_m)\,|0\rangle$ corresponds to the cycle $\left\{ \xi \in S^{[2]} \,\middle|\, \mathrm{Supp}\,\xi = x + y,\ x \in \alpha_k,\ y \in \alpha_m \right\}$. And $\mathfrak{q}_2\,(\alpha_k)\,|0\rangle$ corresponds to the cycle $\left\{ \xi \in S^{[2]} \,\middle|\, \mathrm{Supp}\,\xi = \{x\},\ x \in \alpha_k \right\}$. The result follows from the fact that

$$\Sigma = \left\{ \xi \in S^{[2]} \,\middle|\, \mathrm{Supp}\,\xi = x + i(x) \right\}.$$

$\square$

Hence $\mathrm{Fix}\, G$ is almost negligible. Then it remains to check that $l^2_{1,-}(X) = l^4_{1,-}(X) = 0$ and that

$$l^4_{1,+}(X) + 2 \left[l^1_{1,-}(X) + l^3_{1,-}(X) + l^0_{1,+}(X) + l^2_{1,+}(X) \right] = h^*(\mathrm{Fix}\, G, \mathbb{Z}),$$

so that

$$l^4_{1,+}(X) + 2 \left(1 + l^2_{1,+}(X) \right) = 28 + 1 + 22 + 1 = 52. \qquad (4.2)$$

By Proposition 1.3.8 and Definition-Proposition 2.2.2, we know that $l^2_{1,+}(X) = 7$ and $l^2_{1,-}(X) = 0$. The most difficult part is to calculate $l^4_{1,+}(X)$ and $l^4_{1,-}(X)$. To do this, we will use the integral basis of Qin–Wang of Section 1.3.3 (here it is not possible to apply Proposition 3.3.19 because of Theorem 1.3.9 3)).

We will start by introducing some notation.

Consider a fixed isometry $H^2(S^{[2]}, \mathbb{Z}) \cong U^3 \oplus (-2) \oplus E_8(-1) \oplus E_8(-1)$ given by Proposition 1.3.8. In all the section we will write $U^3 \oplus (-2) \oplus E_8(-1) \oplus E_8(-1)$ for the lattice $H^2(S^{[2]}, \mathbb{Z})$. Let $(u_{k,l})_{k \in \{1,2,3\}, l \in \{1,2\}}$ be a basis of U^3 and $(e_{k,l})_{k \in \{1,\ldots,8\},\ l \in \{1,2\}}$ a basis of $E_8(-1) \oplus E_8(-1)$. To simplify, we will also use the notation

$$(\gamma_k)_{k \in \{1,\ldots,22\}} = (u_{a,b})_{a \in \{1,2,3\}, b \in \{1,2\}} \cup (e_{l,p})_{l \in \{1,\ldots,8\}, p \in \{1,2\}}.$$

By the proof of Proposition 1.3.8, we can write $\gamma_k = j(\alpha_k)$ for all $k \in \{1, \ldots, 22\}$ where $(\alpha_k)_{k \in \{1,\ldots,22\}}$ is the corresponding basis of $H^2(S, \mathbb{Z})$ and j is defined in Section 1.3.1. Also denote $j(v_{k,l}) = u_{k,l}$, for all $k \in \{1, 2, 3\}$, $l \in \{1, 2\}$.

Lemma 4.6.3. *We have*
$$l_{1,-}^4(X) = 0.$$

Proof. With the above notation,

$$\iota^*(\mathfrak{q}_2(\alpha_k)\,|0\rangle) = \mathfrak{q}_2(i^*\alpha_k)\,|0\rangle\,, \quad \iota^*(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle) = \mathfrak{q}_1(i^*\alpha_k)\mathfrak{q}_1(i^*\alpha_j)\,|0\rangle\,,$$

$$\iota^*(\mathfrak{m}_{1,1}(\alpha_k)\,|0\rangle) = \mathfrak{m}_{1,1}(i^*\alpha_k)\,|0\rangle\,, \quad \iota^*(\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle) = \mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle\,.$$

Let $x \in H^4(S^{[2]}, \mathbb{Z})$ such that $\iota^*(x) = -x$. By Theorem 1.3.10,

$$
\begin{aligned}
x = \ &\sum_{0 \le k < j \le 22} \lambda_{k,j}\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle \\
&+ \sum_{0 \le k \le 22} \eta_k\mathfrak{q}_2(\alpha_k)\,|0\rangle \\
&+ \sum_{0 \le k \le 22} \nu_k\mathfrak{m}_{1,1}(\alpha_k)\,|0\rangle \\
&+ y\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle\,,
\end{aligned}
$$

where the $\lambda_{k,j}$, η_k, ν_k are in $\mathbb{Z}$. By Proposition 1.3.8,

$$
\begin{aligned}
\iota^* &\left(\sum_{0 \le k < j \le 22} \lambda_{k,j}\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle \right) \\
&= \sum_{0 \le k < j \le 6} \lambda_{k,j}\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle + \sum_{0 \le k \le 6 < j \le 14} \lambda_{k,j}\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_{8+j})\,|0\rangle \\
&+ \sum_{0 \le k \le 6, 15 \le j \le 22} \lambda_{k,j}\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_{j-8})\,|0\rangle + \sum_{7 \le k < j \le 14} \lambda_{k,j}\mathfrak{q}_1(\alpha_{k+8})\mathfrak{q}_1(\alpha_{j+8})\,|0\rangle \\
&+ \sum_{7 \le k \le 14 < j \le 22} \lambda_{k,j}\mathfrak{q}_1(\alpha_{k+8})\mathfrak{q}_1(\alpha_{j-8})\,|0\rangle + \sum_{15 \le k < j \le 22} \lambda_{k,j}\mathfrak{q}_1(\alpha_{k-8})\mathfrak{q}_1(\alpha_{j-8})\,|0\rangle
\end{aligned}
$$

Since x is anti-invariant and by Theorem 1.3.10,

$$\iota^* \left(\sum_{0 \le k < j \le 22} \lambda_{k,j}\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle \right) = - \sum_{0 \le k < j \le 22} \lambda_{k,j}\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle.$$

Hence

$$\sum_{0 \le k < j \le 6} \lambda_{k,j} \mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_j) \, |0\rangle + \sum_{0 \le k \le 6 < j \le 14} \lambda_{k,j} \mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_{8+j}) \, |0\rangle$$

$$+ \sum_{0 \le k \le 6, 15 \le j \le 22} \lambda_{k,j} \mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_{j-8}) \, |0\rangle + \sum_{7 \le k < j \le 14} \lambda_{k,j} \mathfrak{q}_1(\alpha_{k+8}) \mathfrak{q}_1(\alpha_{j+8}) \, |0\rangle$$

$$+ \sum_{7 \le k \le 14 < j \le 22} \lambda_{k,j} \mathfrak{q}_1(\alpha_{k+8}) \mathfrak{q}_1(\alpha_{j-8}) \, |0\rangle + \sum_{15 \le k < j \le 22} \lambda_{k,j} \mathfrak{q}_1(\alpha_{k-8}) \mathfrak{q}_1(\alpha_{j-8}) \, |0\rangle$$

$$= - \sum_{0 \le k < j \le 22} \lambda_{k,j} \mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_j) \, |0\rangle.$$

Hence

- $\lambda_{k,j} = 0$ for $0 \le k \le j \le 6$.

- $\lambda_{k,j} = -\lambda_{k,j+8}$ for $0 \le k \le 6 < j \le 14$.

- $\lambda_{k,j} = -\lambda_{k+8,j+8}$ for $7 \le k \le j \le 14$.

- $\lambda_{k,j} = -\lambda_{j-8,k+8}$ for $7 \le k \le 14 < j \le 22$.

Therefore after a similar calculation for η_k and ν_k, we get:

$$x = \sum_{1 \le k \le 6 < j \le 14} \lambda_{k,j} (\mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_j) \, |0\rangle - \mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(i^* \alpha_j) \, |0\rangle)$$

$$+ \sum_{7 \le k \le 14} \eta_k (\mathfrak{q}_2(\alpha_k) \, |0\rangle - \mathfrak{q}_2(i^* \alpha_k) \, |0\rangle)$$

$$+ \sum_{7 \le k \le j \le 14} \lambda_{k,j} (\mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_j) \, |0\rangle - \mathfrak{q}_1(i^* \alpha_k) \mathfrak{q}_1(i^* \alpha_j) \, |0\rangle)$$

$$+ \sum_{7 \le k \le 14} \nu_k (\mathfrak{m}_{1,1}(\alpha_k) \, |0\rangle - \mathfrak{m}_{1,1}(i^* \alpha_k) \, |0\rangle)$$

$$+ \sum_{k=8}^{14} \sum_{j=15}^{7+k} \lambda_{k,j} (\mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_j) \, |0\rangle - \mathfrak{q}_1(i^* \alpha_k) \mathfrak{q}_1(i^* \alpha_j) \, |0\rangle),$$

By Definition-Proposition 2.2.2, this implies $l_{1,-}^4(X) = 0$. □

Lemma 4.6.4. *1) The following elements form an integral basis of $H^4(S^{[2]}, \mathbb{Z})^\iota$:*

a)

$$\mathfrak{q}_2(\alpha_k) \, |0\rangle, \quad \mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_l) \, |0\rangle, \quad \mathfrak{m}_{1,1}(\alpha_k) \, |0\rangle,$$

for $1 \le k < l \le 6$;

b)

$$\mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(\alpha_l) \, |0\rangle + \mathfrak{q}_1(\alpha_k) \mathfrak{q}_1(i^* \alpha_l) \, |0\rangle), \quad \mathfrak{q}_2(\alpha_l) \, |0\rangle + \mathfrak{q}_2(i^* \alpha_l) \, |0\rangle,$$

for $k \in \{1, \dots, 6\}$ and $l \in \{7, \dots, 14\}$;

c)

$$\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\,|0\rangle + \mathfrak{q}_1(i^*\alpha_k)\mathfrak{q}_1(i^*\alpha_l)\,|0\rangle\,, \qquad \mathfrak{m}_{1,1}(\alpha_k)\,|0\rangle + \mathfrak{m}_{1,1}(i^*\alpha_k)\,|0\rangle\,,$$

for $7 \le k < l \le 14$;

d)

$$\mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(i^*\alpha_l)\,|0\rangle\,,$$

for $l \in \{7,\ldots,14\}$;

e)

$$\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\,|0\rangle + \mathfrak{q}_1(i^*\alpha_k)\mathfrak{q}_1(i^*\alpha_l)\,|0\rangle),$$

for $k \in \{8,\ldots,14\}$ and $l \in \{15,\ldots,7+k\}$;

f)

$$\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle\,.$$

2) $l^4_{1,+}(X) = 36$.

Remark: Considering the isometry $H^2(S^{[2]},\mathbb{Z}) \simeq U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus (-2)$ from Proposition 1.3.8,

(a) the elements of type (a) are products of elements of $U^3 \oplus (-2)$

(b) the elements of type (b) are products of one element of $U^3 \oplus (-2)$ and one element of $E_8(-2)$,

(c) the elements of type (c) are sums $x \cdot y + \iota^*(x) \cdot \iota^*(y)$ with x and y in $E_8(-1)$,

(d) the elements of type (d) are products of one element of $E_8(-1)$ with its image by ι^*,

(e) the elements of type (e) are sums $x \cdot y + \iota^*(x) \cdot \iota^*(y)$ with x in $E_8(-1)$ and y in $\iota^*(E_8(-1))$, $y \neq \iota^*(x)$.

Proof.   1) Let $x \in H^4(S^{[2]},\mathbb{Z})^\iota$. As follows from Theorem 1.3.10, by the

same method as in the proof of Lemma 4.6.3, we can write:

$$
\begin{aligned}
x = &\sum_{1 \le k < j \le 6} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle) \\
&+ \sum_{1 \le k \le 6} \eta_k(\mathfrak{q}_2(\alpha_k)\,|0\rangle) + \nu_k(\mathfrak{m}_{1,1}(\alpha_k)\,|0\rangle) \\
&+ \sum_{1 \le k \le 6 < j \le 14} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle + \mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(i^*\alpha_j)\,|0\rangle) \\
&+ \sum_{7 \le k \le 14} \eta_k(\mathfrak{q}_2(\alpha_k)\,|0\rangle + \mathfrak{q}_2(i^*\alpha_k)\,|0\rangle) \\
&+ \sum_{7 \le k \le j \le 14} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle + \mathfrak{q}_1(i^*\alpha_k)\mathfrak{q}_1(i^*\alpha_j)\,|0\rangle) \\
&+ \sum_{7 \le k \le 14} \nu_k(\mathfrak{m}_{1,1}(\alpha_k)\,|0\rangle + \mathfrak{m}_{1,1}(i^*\alpha_k)\,|0\rangle) \\
&+ \sum_{j=7}^{14} \lambda_{j,j+8}(\mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(i^*\alpha_j)\,|0\rangle) \\
&+ \sum_{k=8}^{14}\sum_{j=15}^{7+k} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j)\,|0\rangle + \mathfrak{q}_1(i^*\alpha_k)\mathfrak{q}_1(i^*\alpha_j)\,|0\rangle) + y\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle,
\end{aligned}
$$

with the $\lambda_{k,j}, \mu_k, \nu_k$ in $\mathbb{Z}$.

2) Hence the element which are in the part $\mathbb{Z}^t$ of the decomposition of Definition-Proposition 2.2.2 are the elements of type a), d) and f). Their numbers are respectively $\frac{6 \times 5}{2} + 6 + 6 = 27$, 8 and 1. Hence $l_1^4(X) = 27 + 8 + 1 = 36$.

$\square$

Hence (4.2) is verified and we get the result by Corollary 3.5.19. $\square$

We also deduce the following Lemma from Lemma 4.6.4 and Corollary 4.6.1.

Lemma 4.6.5. *Let $M = S^{[2]}/\iota$. The following elements form an integral basis of $H^4(M,\mathbb{Z})$:*

a)
$$
\pi_*(\mathfrak{q}_2(\alpha_k)\,|0\rangle), \quad \pi_*(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\,|0\rangle), \quad \pi_*(\mathfrak{m}_{1,1}(\alpha_k)\,|0\rangle),
$$
for $1 \le k < l \le 6$;

b)
$$
\pi_*(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\,|0\rangle), \quad \pi_*(\mathfrak{q}_2(\alpha_l)\,|0\rangle),
$$
for $k \in \{1,\ldots,6\}$ and $l \in \{7,\ldots,14\}$;

c)

$$\pi_*(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\,|0\rangle), \qquad \pi_*(\mathfrak{m}_{1,1}(\alpha_k)\,|0\rangle),$$

for $7 \leq k < l \leq 14$;

d)

$$\pi_*(\mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(i^*\alpha_l)\,|0\rangle),$$

for $l \in \{7,\ldots,14\}$;

e)

$$\pi_*(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_l)\,|0\rangle),$$

for $k \in \{8,\ldots,14\}$ and $l \in \{15,\ldots,7+k\}$;

f)

$$\pi_*(\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle).$$

We also deduce the following numerical values.

Proposition 4.6.6. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and ι a symplectic involution on X.*

1) *$\mathrm{Fix}\,G$ is almost negligible,*

2) *$l^2_{1,+}(X) = 7$,*

2) *$l^2_{1,-}(X) = l^4_{1,-}(X) = 0$,*

3) *$l^4_{1,+}(X) = 36$,*

4) *$l^4_{1,+}(X) + 2\left[l^1_{1,-}(X) + l^3_{1,-}(X) + l^0_{1,+}(X) + l^2_{1,+}(X)\right] = h^*(\mathrm{Fix}\,G, \mathbb{Z})$.*

4.6.2 The H^2-normality

From Corollary 4.6.1, we deduce the following statement:

Corollary 4.6.7. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and ι a symplectic involution on X. Then $(X, \langle \iota \rangle)$ is H^2-normal.*

Proof. We cannot use Proposition 3.3.16 because of Theorem 1.3.9 3). By Proposition 1.3.8, our isometry $H^2(S^{[2]}, \mathbb{Z})^2 \simeq U^3 \oplus E_8(-1)^2 \oplus (-2)$ implies $H^2(S^{[2]}, \mathbb{Z})^\iota \simeq U^3 \oplus E_8(-2) \oplus (-2)$. Throughout this section we will write $U^3 \oplus E_8(-2) \oplus (-2)$ for referring to the sublattice $H^2(S^{[2]}, \mathbb{Z})^\iota$. By Proposition 1.3.8, all the elements $\pi_*(z)$ with $z \in E_8(-2)$ are divisible by 2. Hence we have to prove that the elements $\pi_*(z)$ with $z \in U^3 \oplus (-2)$ are divisible by 2 if and only if z is divisible by 2.

Let $x = \pi_*(z)$, $z \in U^3 \oplus (-2)$. Assuming x divisible by 2, we will show that z is divisible by 2.

We can write

$$x = \sum_{(k,j)\in\{1,2,3\}\times\{1,2\}} a_{k,j}\pi_*(u_{k,j}) + y(\pi_*(\delta)),$$

where $a_{k,j}$ and y are integers, $(u_{i,j})_{1 \leq i \leq 3, 1 \leq j \leq 2}$ is a basis of U^3 and δ is half the diagonal. Then by Lemma 3.3.7 1),

$$\epsilon = \pi_* \left[\left(\sum_{(k,j) \in \{1,2,3\} \times \{1,2\}} a_{k,j} u_{k,j} + y\delta \right)^2 \right]$$

is divisible by 2. Hence:

$$\epsilon' = \pi_* \left[\sum_{(k,j) \in \{1,2,3\} \times \{1,2\}} a_{k,j}^2 u_{k,j}^2 + y^2 \delta^2 \right]$$

is divisible by 2 .

We also have by Proposition 1.3.11:

$$u_{k,j}^2 = \mathfrak{q}_1(v_{k,j})^2 \ket{0} = 2\mathfrak{m}_{1,1}(v_{k,j}) \ket{0} + \mathfrak{q}_2(v_{k,j}) \ket{0} \, ,$$

where $j(v_{k,j}) = u_{k,j}$, and

$$\delta^2 = \sum_{i<j} \mu_{i,j} \mathfrak{q}_1(\alpha_i) \mathfrak{q}_1(\alpha_j) \ket{0} + \frac{1}{2} \sum_i \mu_{i,i} \mathfrak{q}_1(\alpha_i)^2 \ket{0} + \mathfrak{q}_1(1)\mathfrak{q}_1(x) \ket{0} \, .$$

Then

$$\begin{aligned}
\epsilon' = \pi_* \Bigg[&\sum_{(k,j) \in \{1,2,3\} \times \{1,2\}} a_{k,j}^2 \mathfrak{q}_2(v_{k,j}) \ket{0} \\
&+ y^2 \left(\sum_{i<j} \mu_{i,j} \mathfrak{q}_1(\alpha_i) \mathfrak{q}_1(\alpha_j) \ket{0} + \frac{1}{2} \sum_i \mu_{i,i} \mathfrak{q}_2(\alpha_i) \ket{0} \right) + y^2 \mathfrak{q}_1(1)\mathfrak{q}_1(x) \ket{0} \Bigg] \\
&+ A'
\end{aligned}$$

with A' divisible by 2.

Since ϵ' is divisible by 2, by Lemma 4.6.5, the coefficient of $\mathfrak{q}_1(1)\mathfrak{q}_1(x) \ket{0}$ must be even. So y is even.

Now, since

$$\epsilon'' = \pi_* \left[\sum_{(k,j) \in \{1,2,3\} \times \{1,2\}} a_{k,j}^2 \mathfrak{q}_2(v_{k,j}) \ket{0} \right]$$

is divisible by 2, by Lemma 4.6.5, all the coefficients $a_{k,j}$ must be even. Hence z is divisible by 2. $\qquad\square$

4.6.3 Beauville–Bogomolov lattice of a partial resolution of the quotient of a $K3^{[2]}$-type manifold by a symplectic involution

By Theorem 4.1 of [43] (Theorem 1.3.6), the fixed locus of σ is the union of 28 isolated points and a K3 surface Σ. Then the singular locus of $M := S^{[2]}/\sigma$ is the union of a K3 and 28 isolated points. The singular locus is not of codimension four. We lift to a partial resolution M' of singularities of M, obtained by blowing up the image of Σ. By Section 2.3 and Lemma 1.2 of [17], M' is an irreducible symplectic V-manifold which has singular locus of codimension four. Hence by Section 1.2.2, we can endow M' with its Beauville–Bogomolov form. We will prove the following Theorem.

Theorem 4.6.8. *Let X be an irreducible symplectic manifold of $K3^{[2]}$-type and ι a symplectic involution on X. Let Σ be the K3 surface in the fixed locus of ι. We denote $M = X/\iota$ and M' the partial resolution of singularities of M obtained by blowing up the image of Σ. Then the Beauville–Bogomolov lattice $H^2(M',\mathbb{Z})$ is isomorphic to $E_8(-1) \oplus U(2)^3 \oplus (-2)^2$, and the Fujiki constant is equal to 6.*

4.6.4 Proof of Theorem 4.6.8

Notation

Let $r_1 : M' \to M$ be the partial resolution of singularities obtained by blowing up $\overline{\Sigma} := \pi(\Sigma)$, where $\pi : S^{[2]} \to M$ is the quotient map. Denote by $\overline{\Sigma}'$ the exceptional divisor. Let $s_1 : X' \to S^{[2]}$ be the blowup of $S^{[2]}$ in Σ, and denote by Σ_1 the exceptional divisor in X'. Denote by ι_1 the involution on X' induced by ι. We have $M' \simeq X'/\iota_1$, and we denote by $\pi_1 : X' \to M'$ the quotient map.

Let also $s_2 : \widetilde{X} \to X'$ be the blowup in the 28 points fixed by ι_1. We denote by $(E_k)_{1 \leq k \leq 28}$ the exceptional divisors and $\Sigma_2 = s_2^{-1}(\Sigma_1)$. Let $r_2 : \widetilde{M} \to M'$ be the blowup in the 28 singular points of M'. We denote by $(D_k)_{1 \leq k \leq 28}$ the exceptional divisors and $\widetilde{\Sigma} = r_2^{-1}(\overline{\Sigma}')$. We denote by ι_2 the involution on $\widetilde{X}$ induced by ι_1. We have $\widetilde{M} = \widetilde{X}/\iota_2$ and we denote $\pi_2 : \widetilde{X} \to \widetilde{M}$ the quotient map.

We also denote $s = s_1 \circ s_2$, $r = r_1 \circ r_2$, $V = X \backslash \mathrm{Fix}\,\iota = X' \backslash \mathrm{Fix}\,\iota_1 = \widetilde{X} \backslash \mathrm{Fix}\,\iota_2$ and $U = \pi(V) = \pi_1(V) = \pi_2(V)$ as in Section 3.5. We sum up the notation in the diagram:

$$
\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{\ r_2\ } & M' & \xrightarrow{\ r_1\ } & M \\
\uparrow^{\pi_2} & & \uparrow^{\pi_1} & & \uparrow^{\pi} \\
\widetilde{X} & \xrightarrow{\ s_2\ } & X' & \xrightarrow{\ s_1\ } & S^{[2]}. \\
\circlearrowleft_{\iota_2} & & \circlearrowleft_{\iota_1} & & \circlearrowleft_{\iota}
\end{array}
$$

We have

$$H^2(X',\mathbb{Z}) \cong H^2(S^{[2]},\mathbb{Z}) \oplus \mathbb{Z}\Sigma',$$

We also write $\delta' = s_1^*(\delta)$, where δ is half the diagonal of $S^{[2]}$. We need also the following lemma.

Lemma 4.6.9. *The cohomology group $H^4(\widetilde{M}, \mathbb{Z})$ is torsion-free.*

Proof. This follows from Proposition 4.6.6 and Corollary 3.5.19.   □

Lemmas on the Beauville–Bogomolov form on M'

Proposition 4.6.10. *We have the formula*

$$B_{M'}(\pi_{1*}(s_1^*(\alpha)), \pi_{1*}(s_1^*(\beta))) = \sqrt{\frac{24}{C_{M'}}} B_{S^{[2]}}(\alpha, \beta),$$

where $C_{M'}$ is the Fujiki constant of M' and α, β are in $H^2(S^{[2]}, \mathbb{Z})^\iota$.

Proof. By (1) of Theorem 1.2.4, we have

$$(\pi_{1*}(s_1^*(\alpha)))^4 = C_{M'} B_{M'}(\pi_{1*}(s_1^*(\alpha)), \pi_{1*}(s_1^*(\alpha)))^2.$$

And

$$\alpha^4 = 3 B_{S^{[2]}}(\alpha, \alpha)^2.$$

Moreover, by Lemma 3.3.7 3),

$$(\pi_{1*}(s^*(\alpha)))^4 = 8 s^*(\alpha)^4 = 8\alpha^4.$$

By (2) of Theorem 1.2.4, we get the result.   □

Proposition 4.6.11. *We have*

$$B_{M'}(\overline{\Sigma}', \overline{\Sigma}') = B_{M'}(\pi_{1*}(s_1^*(\delta)), \pi_{1*}(s_1^*(\delta))) = -2\sqrt{\frac{24}{C_{M'}}}.$$

Proof. First, by Theorem 1.3.1, we have $B_{S^{[2]}}(\delta, \delta) = -2$, hence

$$B_{M'}(\pi_{1*}(s_1^*(\delta)), \pi_{1*}(s_1^*(\delta))) = -2\sqrt{\frac{24}{C_{M'}}}$$

by the last proposition.

We need the following lemma similar to Lemma 3.5.10.

Lemma 4.6.12. *We have*
$$\Sigma_1^2 = -s_1^*(\Sigma).$$

Proof. Consider the following diagram:

$$
\begin{array}{ccc}
\Sigma_1 & \overset{l_1}{\hookrightarrow} & X' \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s_1} \\
\Sigma & \overset{l_0}{\hookrightarrow} & S^{[2]},
\end{array}
$$

where l_0 and l_1 are the inclusions and $g := s_{1|\Sigma_1}$. By Proposition 6.7 of [19], we have:

$$s_1^* l_{0*}(\Sigma) = l_{1*}(c_1(E)),$$

where $E := g^*(\mathscr{N}_{\Sigma/S^{[2]}})/\mathscr{N}_{\Sigma_1/N_1}$. Hence

$$s_1^* l_{0*}(\Sigma) = c_1(g^*(\mathscr{N}_{\Sigma/S^{[2]}})) - \Sigma_1^2.$$

It remains to calculate $c_1(g^*(\mathscr{N}_{\Sigma/S^{[2]}}))$. We consider the diagram

$$
\begin{array}{ccc}
\Sigma_0 & \overset{l_0'}{\hookrightarrow} & S \times S \\
{\scriptstyle r_0}\big\uparrow & & \big\uparrow{\scriptstyle r} \\
\widetilde{\Sigma_0} & \overset{\widetilde{l_0}}{\hookrightarrow} & \mathscr{S} \\
{\scriptstyle p_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\Sigma & \overset{l_0}{\hookrightarrow} & S^{[2]},
\end{array}
$$

where $\Sigma_0 = \{(x, i(x)) \,|\, x \in S\}$, $r : \mathscr{S} \to S \times S$ is the blowup in the diagonal of $S \times S$: Δ_0, $r_0 : \widetilde{\Sigma_0} \to \Sigma_0$ is the blowup in $\Delta_0 \cap \Sigma_0 = 8pt$, and p, p_0 are the quotient maps. Since Δ_0 and Σ_0 intersect properly in $S \times S$, $\widetilde{\Sigma_0}$ is equal to the total transform of Σ_0 by r in $\mathscr{S}$. Hence

$$c_1(\mathscr{N}_{\widetilde{\Sigma_0}/\mathscr{S}}) = c_1(r_0^*(\mathscr{N}_{\Sigma_0/S \times S})).$$

But since $\Sigma_0 \simeq S$, we have $\mathscr{N}_{\Sigma_0/S \times S} \simeq \mathscr{T}_S$. Hence $c_1(\mathscr{N}_{\widetilde{\Sigma_0}/\mathscr{S}}) = 0$. Since $\widetilde{\Sigma_0}$ is also the total transform of Σ by p in $\mathscr{S}$, we have

$$c_1(\mathscr{N}_{\widetilde{\Sigma_0}/\mathscr{S}}) = c_1(p^*(\mathscr{N}_{\Sigma/S^{[2]}})).$$

Hence $c_1(\mathscr{N}_{\Sigma/S^{[2]}}) = 0$. So

$$\Sigma_1^2 = -s_1^* l_{0*}(\Sigma).$$

$\square$

Now, we can calculate $B_{M'}(\overline{\Sigma}', \overline{\Sigma}')$ from the cup product (Proposition 1.2.5).

$$
\begin{aligned}
\overline{\Sigma}'^2 \cdot \pi_{1*}(s_1^*(\delta))^2 &= \frac{C_{M'}}{3} B_{M'}(\overline{\Sigma}', \overline{\Sigma}') \times B_{M'}(\pi_{1*}(s_1^*(\delta)), \pi_{1*}(s_1^*(\delta))) \\
&= \frac{C_{M'}}{3} B_{M'}(\overline{\Sigma}', \overline{\Sigma}') \times \left(-2\sqrt{\frac{24}{C_{M'}}} \right) \\
&= -4\sqrt{\frac{2C_{M'}}{3}} B_{M'}(\overline{\Sigma}', \overline{\Sigma}')
\end{aligned}
$$

By Lemma 3.3.7 3), we have $\overline{\Sigma}'^2 \cdot \pi_{1*}(s_1^*(\delta))^2 = 8\Sigma_1^2 \cdot (s_1^*(\delta))^2$. By the projection formula, $\Sigma_1^2 \cdot (s_1^*(\delta))^2 = s_{1*}(\Sigma_1^2) \cdot \delta^2$. Moreover by the last lemma, $s_{1*}(\Sigma_1^2) = -\Sigma$. Hence,

$$-8\Sigma \cdot \delta^2 = -4\sqrt{\frac{2C_{M'}}{3}} B_{M'}(\overline{\Sigma}', \overline{\Sigma}').$$

It is possible to understand geometrically the intersection $\Sigma \cdot \Delta^2$, where $\Delta = 2\delta$ is the diagonal in $S^{[2]}$. Since $\iota = i^{[2]}$ with i a symplectic involution on S, we have $\Sigma = \left\{ \xi \in S^{[2]} \,\middle|\, \operatorname{Supp} \xi = x + i(x), \ x \in S \right\}$ and $\Delta \to \Delta_0$ is a $\mathbb{P}^1$-bundle over the diagonal Δ_0 in $S^{(2)}$. We recall that i has 8 fixed points on S: $x_1, \dots, x_8$. Then we see that $\Delta \cdot \Sigma = \cup_{j=1}^8 \left\{ \xi \in S^{[2]} \,\middle|\, \operatorname{Supp} \xi = \{x_j\} \right\}$, the union of 8 lines. Therefore $\Delta^2 \cdot \Sigma$ is the self-intersection of 8 lines in the K3 surface Σ. So $\Delta^2 \cdot \Sigma = -2 \times 8$. We get:

$$-8 \times \frac{-2 \times 8}{4} = -4\sqrt{\frac{2C_{M'}}{3}} B_{M'}(\overline{\Sigma}', \overline{\Sigma}'),$$

and so

$$B_{M'}(\overline{\Sigma}', \overline{\Sigma}') = -2\sqrt{\frac{24}{C_{M'}}}.$$

$\square$

Proposition 4.6.13.
$$B_{M'}(\pi_{1*}(s_1^*(\alpha)), \overline{\Sigma}') = 0,$$

for all $\alpha \in H^2(S^{[2]}, \mathbb{Z})^\iota$.

Proof. We have $\pi_{1*}(s_1^*(\alpha))^3 \cdot \overline{\Sigma}' = 8s_1^*(\alpha)^3 \cdot \Sigma_1$ by Lemma 3.3.7 3), and $s_{1*}(s_1^*(\alpha)^3 \cdot \Sigma_1) = \alpha^3 \cdot s_{1*}(\Sigma_1) = 0$ by the projection formula. We conclude by Proposition 1.2.5. $\square$

The element $\overline{\Sigma}'$

We will now show that the class of $\overline{\Sigma}'$ is not divisible by 2 in $H^2(M', \mathbb{Z})$.

Lemma 4.6.14. *The element $\overline{\Sigma}'$ is not divisible by 2 in $H^2(M', \mathbb{Z})$.*

Proof. Since $\widetilde{\Sigma} = \widetilde{r}^*(\overline{\Sigma}')$, it is enough to show that $\widetilde{\Sigma}$ is not divisible by 2 in $H^2(\widetilde{M}, \mathbb{Z})$. We have the following exact sequence:

$$H^1(U, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}, U, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}, \mathbb{Z}) \longrightarrow H^2(U, \mathbb{Z}) \longrightarrow H^3(\widetilde{M}, U, \mathbb{Z}) \,.$$

By the universal coefficient theorem, $H^1(U, \mathbb{Z})$ is torsion-free. Since $H^1(V, \mathbb{Z}) = 0$, we have $H^1(U, \mathbb{Z}) = 0$. Moreover, by Thom's isomorphism, we have $H^3(\widetilde{M}, U, \mathbb{Z}) \simeq H^1(\widetilde{\Sigma}, \mathbb{Z}) \oplus (\oplus_{i=1}^{28} H^1(D_i, \mathbb{Z})) = 0$. We have also $H^2(\widetilde{M}, U, \mathbb{Z}) \simeq H^0(\widetilde{\Sigma}, \mathbb{Z}) \oplus (\oplus_{i=1}^{28} H^0(D_i, \mathbb{Z}))$. Then the exact sequence gives:

$$H^2(U, \mathbb{Z}) \simeq H^2(\widetilde{M}, \mathbb{Z}) / \left\langle \widetilde{\Sigma}, D_1, \dots, D_{28} \right\rangle.$$

With equivariant cohomology, we calculate that the torsion of $H^2(U, \mathbb{Z})$ is equal to $\mathbb{Z}/2\mathbb{Z}$. This means that if $\widetilde{\mathcal{D}}$ is the minimal primitive overgroup of $\left\langle \widetilde{\Sigma}, D_1, \ldots, D_{28} \right\rangle$ in $H^2(\widetilde{M}, \mathbb{Z})$, then $\widetilde{\mathcal{D}} / \left\langle \widetilde{\Sigma}, D_1, \ldots, D_{28} \right\rangle = \mathbb{Z}/2\mathbb{Z}$. But, we have $\widetilde{\pi}_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_{\widetilde{M}} \oplus \mathcal{L}$, with $\mathcal{L}^2 = \mathcal{O}_{\widetilde{M}}\left(-D_1 - \cdots - D_{28} - \widetilde{\Sigma}\right)$. Hence, we know that $\widetilde{\Sigma} + D_1 + \cdots + D_{28}$ is divisible by 2. So

$$\widetilde{\mathcal{D}} = \left\langle \widetilde{\Sigma}, D_1, \ldots, D_{28}, \frac{\widetilde{\Sigma} + D_1 + \cdots + D_{28}}{2} \right\rangle.$$

Hence $\widetilde{\Sigma}$ is not divisible by 2, and neither is $\overline{\Sigma}'$. □

What it remains to prove

Lemma 4.6.15. *The sublattice*

$$\frac{1}{2}\pi_{1*}(s_1^*(E_8(-2))) \oplus \pi_{1*}(s_1^*(U^3)) \oplus \mathbb{Z}\,\pi_{1*}(s_1^*(\delta))$$

is primitive in $H^2(M', \mathbb{Z})$.

Proof. This follows from our results on the H^2-normality of $(S^{[2]}, \iota)$.

By Proposition 3.5.20 and Proposition 4.6.6, $\pi_{2*}(s^*(H^4(S^{[2]}, \mathbb{Z})))$ is primitive in $H^4(\widetilde{M}, \mathbb{Z})$. Then by the proof of Corollary 4.6.7, $\frac{1}{2}\pi_{2*}(s^*(E_8(-2))) \oplus \pi_{2*}(s^*(U^3)) \oplus \mathbb{Z}\,\pi_{2*}(s^*(\delta))$ is primitive in $H^2(\widetilde{M}, \mathbb{Z})$.

The quadruple $(\widetilde{X}, \iota_2, r_2, s_2)$ is a pullback of (X', ι_1), and the cohomology groups $H^2(\widetilde{M}, \mathbb{Z})$ and $H^2(M', \mathbb{Z})$ are torsion-free. Hence by Lemma 3.3.21, we have

$$\frac{1}{2}\pi_{2*}(s^*(E_8(-2))) \oplus \pi_{2*}(s^*(U^3)) \oplus \mathbb{Z}\,\pi_{2*}(s^*(\delta))$$
$$= \frac{1}{2}r_2^*(\pi_{1*}(s_1^*(E_8(-2)))) \oplus r_2^*(\pi_{1*}(s_1^*(U^3))) \oplus \mathbb{Z}\,r_2^*(\pi_{1*}(s_1^*(\delta))).$$

Hence the primitivity of $\frac{1}{2}\pi_{2*}(s^*(E_8(-2))) \oplus \pi_{2*}(s^*(U^3)) \oplus \mathbb{Z}\,\pi_{2*}(s^*(\delta))$ implies that of $\frac{1}{2}\pi_{1*}(s_1^*(E_8(-2))) \oplus \pi_{1*}(s_1^*(U^3)) \oplus \mathbb{Z}\,\pi_{1*}(s_1^*(\delta))$. □

It remains to find an answer to one more divisibility question.

The lattices $\frac{1}{2}\pi_{1*}(s_1^*(E_8(-2))) \oplus \pi_{1*}(s_1^*(U^3)) \oplus \mathbb{Z}\,\pi_{1*}(s_1^*(\delta))$ and $\mathbb{Z}\,\overline{\Sigma}'$ are primitive in $H^2(M', \mathbb{Z})$, but it might turn out that an element of the form

$$x = \frac{y \pm \overline{\Sigma}'}{2},$$

with $y \in \frac{1}{2}\pi_{1*}(s_1^*(E_8(-2))) \oplus \pi_{1*}(s_1^*(U^3)) \oplus \mathbb{Z}\,\pi_{1*}(s_1^*(\delta))$ is integral. To simplify the formulas, we will use the following notation:

$$\widetilde{\delta} := \pi_{2*}(s^*(\delta)), \quad \widetilde{\delta^2} := \pi_{2*}(s^*(\delta^2)), \quad \overline{\delta}' := \pi_{1*}(s_1^*(\delta)),$$

$$\widetilde{u_{k,l}} := \pi_{2*}(s^*(u_{k,l})), \qquad \overline{u_{k,l}}' := \pi_{1*}(s_1^*(u_{k,l})),$$

for all $k \in \{1, 2, 3\}$ and $l \in \{1, 2\}$. We will show that $\frac{\overline{\delta}' + \overline{\Sigma}'}{2} \in H^2(M', \mathbb{Z})$.

We will need the following propositions. Let $(a_k)_{1 \leq k \leq 22}$ be an integral basis of $H^2(\Sigma, \mathbb{Z})$. Denote $\theta_k = s_{|\Sigma_2}^*(a_k)$ for all $1 \leq k \leq 22$.

Proposition 4.6.16. *We have:*

1) *$E_l \cdot E_k = 0$ if $l \neq k$, $E_l^2 = -f_{l*}(h_l)$, $E_l^4 = -1$ and $E_l \cdot z = 0$ for all $(l, k) \in \{1, \ldots, 28\}^2$ and for all $z \in s^*(H^4(S^{[2]}, \mathbb{Z}))$;*

2) *$\Sigma_2 \cdot E_k = 0$ for all $k \in \{1, \ldots, 28\}$ and $\Sigma_2^2 = -s^*(\Sigma)$;*

3) *$\theta_k \cdot z = 0$ for all $k \in \{1, \ldots, 22\}$ and $z \in s^*(H^4(S^{[2]}, \mathbb{Z}))$.*

4) *Denote $\sigma_x := \Sigma_2 \cdot s^*(x)$ for all $x \in H^2(S^{[2]}, \mathbb{Z})$. We have*

$$\Sigma_2^2 \cdot s^*(x) \cdot s^*(y) = -2B_{S^{[2]}}(x, y)$$

for all $(x, y) \in H^2(S^{[2]}, \mathbb{Z})^\iota$. Hence

$$\mathrm{rk} \left\langle \left\{ \sigma_x \,|\, x \in H^2(S^{[2]}, \mathbb{Z})^\iota \right\} \right\rangle = \mathrm{rk}\, H^2(S^{[2]}, \mathbb{Z})^\iota = 15.$$

Proof. 1) Let $k \in \{1, \ldots, 28\}$. We have

$$\omega_{E_k} = \omega_{N_2} \otimes \mathcal{N}_{E_k/N_2}.$$

Since $E_k \simeq \mathbb{P}^3$,

$$\mathcal{O}_{E_k}(-4) = \mathcal{O}_{N_2}(3 \sum_{j=1}^{28} E_j + \Sigma_2) \otimes \mathcal{O}_{N_2}(E_k) \otimes \mathcal{O}_{E_k}$$

$$= \mathcal{O}_{N_2}(4E_k) \otimes \mathcal{O}_{E_k}.$$

We get $E_k^2 = -f_{k*}(h_k)$, where h_k is the class of a hyperplane in $E_k \simeq \mathbb{P}^3$. Hence $E_k^4 = c_1(\mathcal{N}_{E_k/N_2})^3 = (-h_k)^3 = -1$.

2) We have $\Sigma_2^2 = s_2^*(\Sigma_1^2)$, so the result follows from Lemma 4.6.12.

3) If we take $z = s^*(y)$, then $s_*(z \cdot \theta_i) = y \cdot s_*(\theta_i)$ by the projection formula. Since $s_*(\theta_i) = 0$, we have $s_*(z \cdot \theta_i) = 0$, and then $z \cdot \theta_i = 0$.

4) We have
$$\Sigma_2^2 \cdot s^*(x) \cdot s^*(y) = \Sigma_1^2 \cdot s_1^*(x) \cdot s_1^*(y).$$

By Lemma 3.3.7 3),

$$\Sigma_1^2 \cdot s_1^*(x) \cdot s_1^*(y) = \frac{1}{8} \overline{\Sigma}'^2 \cdot \pi_{1*}(s_1^*(x)) \cdot \pi_{1*}(s_1^*(y)).$$

By Proposition 1.2.5 and Proposition 4.6.13,

$$\Sigma_1^2 \cdot s_1^*(x) \cdot s_1^*(y) = \frac{C_{M'}}{24} B_{M'}(\overline{\Sigma}', \overline{\Sigma}') \times B_{M'}(\pi_{1*}(s_1^*(x)), \pi_{1*}(s_1^*(y))).$$

Hence, by Proposition 4.6.10 and Proposition 4.6.11,

$$\Sigma_1^2 \cdot s_1^*(x) \cdot s_1^*(y) = \frac{C_{M'}}{24} \times \left[-2\sqrt{\frac{24}{C_{M'}}}\right] \times \sqrt{\frac{24}{C_{M'}}} B_{S^{[2]}}(x,y)$$
$$= -2 B_{S^{[2]}}(x,y).$$

$\square$

From this proposition and Lemma 3.3.7, we deduce a similar proposition on the cohomology of $\widetilde{M}$. We will denote

$$h_k = -E_k^2, \quad \widetilde{\Sigma} := \pi_{2*}(\Sigma_2), \quad D_k := \pi_{2*}(E_k),$$

$$\widetilde{h_k} := \pi_{2*}(h_k), \quad \widetilde{\sigma_x} := \pi_{2*}(\sigma_x), \quad \widetilde{\theta_l} := \pi_{2*}(\theta_l),$$

for $k \in \{1, \ldots, 28\}$, $l \in \{1, \ldots, 22\}$ and $x \in H^2(S^{[2]}, \mathbb{Z})$.

Proposition 4.6.17. *We have:*

1) *$D_l \cdot D_k = 0$ if $l \neq k$, $D_l^2 = -2\widetilde{h_l}$, and $D_l \cdot z = 0$ for all $(l,k) \in \{1, \ldots, 28\}^2$ and for all $z \in \pi_{2*}(s^*(H^4(S^{[2]}, \mathbb{Z})^\iota))$;*

2) *$\widetilde{\Sigma} \cdot D_k = 0$ for all $k \in \{1, \ldots, 28\}$ and $\widetilde{\Sigma}^2 = -2\pi_{2*}(s^*(\Sigma))$;*

3) *$\widetilde{\theta_k} \cdot z = 0$ for all $k \in \{1, \ldots, 22\}$ and $z \in \pi_{2*}(s^*(H^4(S^{[2]}, \mathbb{Z})^\iota))$,*

4) *$\widetilde{\Sigma} \cdot \pi_{2*}(s^*(x)) = 2\widetilde{\sigma_x}$ for all $x \in H^2(S^{[2]}, \mathbb{Z})^\iota$. Moreover*

$$\widetilde{\Sigma}^2 \cdot \pi_{2*}(s^*(x)) \cdot \pi_{2*}(s^*(y)) = -16 B_{S^{[2]}}(x,y),$$

for all $(x,y) \in H^2(S^{[2]}, \mathbb{Z})^\iota$. Hence

$$\mathrm{rk} \left\langle \left\{ \widetilde{\sigma_x} \,|\, x \in H^2(S^{[2]}, \mathbb{Z})^\iota \right\} \right\rangle = \mathrm{rk}\, H^2(S^{[2]}, \mathbb{Z})^\iota = 15.$$

5) *Let T be the sublattice of $(H^4(\widetilde{X}, \mathbb{Z}), \cdot)$ generated by the set*

$$\{\theta_i \,|\, i \in \{1, \ldots, 22\}\} \cup \left\{ E_k^2 \,\big|\, k \in \{1, \ldots, 28\} \right\}.$$

Let $\widetilde{T}$ be the minimal primitive overlattice of $\pi_{2*}(T)$ in $H^4(\widetilde{M}, \mathbb{Z})$, then $\widetilde{T}/\pi_{2*}(T) = (\mathbb{Z}/2\mathbb{Z})^7$.

Proof. We have just to show 5). By Theorem 7.31 of [67] (Theorem 2.5.1), we have

$$H^4(\widetilde{X}, \mathbb{Z}) = s^*(H^4(S^{[2]}, \mathbb{Z}) \oplus T.$$

Hence by Lemma 3.5.11,

$$\widetilde{T}/\pi_{2*}(T) = (\mathbb{Z}/2\mathbb{Z})^{\operatorname{rktor} H^n(U,\mathbb{Z}) - \operatorname{rktor} H^n(\widetilde{M},\mathbb{Z}) - 1}.$$

By Lemma 4.6.9, $H^4(\widetilde{M}, \mathbb{Z})$ is torsion-free. And by Proposition 3.5.14 and Proposition 4.6.6, $\operatorname{rktor} H^4(U, \mathbb{Z}) = 8$. $\qquad\square$

Now, the plan of the proof will be the following. We will show that

$$\frac{\widetilde{\delta^2} - \pi_{2*}(s^*(\Sigma))}{2} = \left(\frac{\widetilde{\delta} + \widetilde{\Sigma}}{2}\right)^2 - \widetilde{\sigma_\delta} \in H^4(\widetilde{M}, \mathbb{Z});$$

next we will deduce that

$$\frac{\widetilde{\delta} + \widetilde{\Sigma}}{2} \in H^2(\widetilde{M}, \mathbb{Z}),$$

and finally we will be able to prove that

$$\frac{\overline{\delta}' + \overline{\Sigma}'}{2} \in H^2(M', \mathbb{Z}).$$

The element $\widetilde{\delta^2} - \pi_{2*}(s^*(\Sigma))$

Lemma 4.6.18. *The element $\widetilde{\delta^2} - \pi_{2*}(s^*(\Sigma))$ is divisible by 2 in $H^4(\widetilde{M}, \mathbb{Z})$.*

Proof. We need to show that $\pi_{2*}(s^*(\delta^2 - \Sigma))$ is divisible by 2. So, look at

$\delta^2 - \Sigma \in H^4(S^{[2]}, \mathbb{Z})$. By (1) of Lemma 4.6.4, we can write:

$$
\begin{aligned}
\delta^2 - \Sigma = & \sum_{1 \leq k < j \leq 6} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j) \,|0\rangle) \\
& + \sum_{1 \leq k \leq 6} \eta_k(\mathfrak{q}_2(\alpha_k) \,|0\rangle) + \nu_k(\mathfrak{m}_{1,1}(\alpha_k) \,|0\rangle) \\
& + \sum_{1 \leq k \leq 6 < j \leq 14} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j) \,|0\rangle + \mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(i^*\alpha_j) \,|0\rangle) \\
& + \sum_{7 \leq k \leq 14} \eta_k(\mathfrak{q}_2(\alpha_k) \,|0\rangle + \mathfrak{q}_2(i^*\alpha_k) \,|0\rangle) \\
& + \sum_{7 \leq k \leq j \leq 14} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j) \,|0\rangle + \mathfrak{q}_1(i^*\alpha_k)\mathfrak{q}_1(i^*\alpha_j) \,|0\rangle) \\
& + \sum_{7 \leq k \leq 14} \nu_k(\mathfrak{m}_{1,1}(\alpha_k) \,|0\rangle + \mathfrak{m}_{1,1}(i^*\alpha_k) \,|0\rangle) \\
& + \sum_{j=7}^{14} \lambda_{j,j+8}(\mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(i^*\alpha_j) \,|0\rangle) \\
& + \sum_{k=8}^{14} \sum_{j=15}^{7+k} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j) \,|0\rangle + \mathfrak{q}_1(i^*\alpha_k)\mathfrak{q}_1(i^*\alpha_j) \,|0\rangle) + y\mathfrak{q}_1(1)\mathfrak{q}_1(x) \,|0\rangle \,,
\end{aligned}
$$

with $\lambda_{k,j}, \mu_k, \nu_k$ in $\mathbb{Z}$. To see that $\pi_{2*}(s^*(\delta^2)) + \pi_{2*}(\Sigma_2^2)$ is divisible by 2, we need to show that the coefficients of the basis elements of type a), d) and f) are even. Then we rewrite:

$$
\begin{aligned}
\delta^2 - \Sigma = & \sum_{1 \leq k < j \leq 6} \lambda_{k,j}(\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(\alpha_j) \,|0\rangle) + \sum_{1 \leq k \leq 6} \eta_k(\mathfrak{q}_2(\alpha_k) \,|0\rangle) + \nu_k(\mathfrak{m}_{1,1}(\alpha_k) \,|0\rangle) \\
& + \sum_{j=7}^{14} \lambda_{j,j+8}(\mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(i^*\alpha_j) \,|0\rangle) \\
& + y\mathfrak{q}_1(1)\mathfrak{q}_1(x) \,|0\rangle + Z,
\end{aligned}
$$

where Z is a sum of elements of type b), c), e). Now we re-arrange the sums as follows:

$$
\begin{aligned}
\delta^2 - \Sigma = & \sum_{(i,j) \neq (l,k) \in \{1,2,3\} \times \{1,2\}} \lambda_{i,j,k,l}(\mathfrak{q}_1(v_{i,j})\mathfrak{q}_1(v_{k,l}) \,|0\rangle) \\
& + \sum_{(l,k) \in \{1,2,3\} \times \{1,2\}} \eta_{l,k}(\mathfrak{q}_2(v_{k,l}) \,|0\rangle) + \nu_{l,k}(\mathfrak{m}_{1,1}(v_{k,l}) \,|0\rangle) \\
& + \sum_{j=7}^{14} \lambda_{j,j+8}(\mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(i^*\alpha_j) \,|0\rangle) \\
& + y\mathfrak{q}_1(1)\mathfrak{q}_1(x) \,|0\rangle + Z,
\end{aligned}
$$

Making the cup product of the two sides of the equality by $\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle$ and using Propositions 1.3.11 and 1.3.12, we obtain $y = 0$. Now, again by Proposition 1.3.11, we can rewrite:

$$\delta^2 - \Sigma = \sum_{(i,j)\neq(l,k)\in\{1,2,3\}\times\{1,2\}} \lambda_{i,j,k,l}(u_{i,j}\cdot u_{k,l} - \Delta_{i=k}\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle)$$
$$+ \sum_{(l,k)\in\{1,2,3\}\times\{1,2\}} \eta_{k,l}(u_{k,l}\cdot\delta) + \nu_{k,l}\left(\frac{u_{k,l}^2 - u_{k,l}\cdot\delta}{2}\right)$$
$$+ \sum_{j=7}^{14} \lambda_{j,j+8}(\gamma_j\cdot\iota^*\gamma_j)$$
$$+ Z,$$

where $\Delta_{i=k}$ is the Kronecker symbol. Now making the cup product with $u_{k,l}^2$ and using Propositions 1.2.5, 1.3.11, we get $\nu_{k,l} = 0$ for all $(k,l)\in\{1,2,3\}\times\{1,2\}$. Next, we get $\eta_{k,l} = 0$ by taking the cup product with $u_{k,l}\cdot\delta$. Now we take the cup product with $u_{i,j}\cdot u_{k,l}$, $i\neq k$, and we obtain that all the $\lambda_{i,j,k,l}$ with $i\neq k$ vanish. Then it remains:

$$\delta^2 - \Sigma = \sum_{i=1}^{3} \lambda_{i,i,1,2}(u_{i,1}\cdot u_{i,2} - \mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle) + \sum_{j=7}^{14} \lambda_{j,j+8}(\gamma_j\cdot\iota^*\gamma_j) + Z$$

Now by taking the cup product with the $u_{i,1}\cdot u_{i,2}$ we get the three equations

$$-4 = -\lambda_{2,2,1,2} - \lambda_{3,3,1,2}$$
$$-4 = -\lambda_{1,1,1,2} - \lambda_{3,3,1,2}$$
$$-4 = -\lambda_{2,2,1,2} - \lambda_{1,1,1,2}.$$

Hence $\lambda_{1,1,1,2} = \lambda_{2,2,1,2} = \lambda_{3,3,1,2} = 2$. So

$$\delta^2 - \Sigma = \sum_{j=7}^{14} \lambda_{j,j+8}(\gamma_j\cdot\iota^*\gamma_j) + Z',$$

with $Z' = Z + 2(u_{1,1}\cdot u_{1,2} + u_{2,1}\cdot u_{2,2} + u_{3,1}\cdot u_{3,2} - 3\mathfrak{q}_1(1)\mathfrak{q}_1(x)\,|0\rangle)$.

Now, it remains to handle the cup-products $\gamma_j\cdot\iota^*\gamma_j$. We recall that the lattice E_8 can be embedded in $\mathbb{R}^8$ with its canonical scalar product as the lattice freely generated by the columns of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

With this identification, we use the columns of this matrix as the basis of $E_8(-1) \subset H^2(S, \mathbb{Z})$.

Then making the cup product of $\delta^2 - \Sigma$ with $\gamma_{14} \cdot \iota^* \gamma_{14}$, we get: $2 = -2\lambda_{14,14+8} - \lambda_{7,7+8} + \gamma_{14} \cdot \iota^* \gamma_{14} \cdot Z'$. Since $\gamma_{14} \cdot \iota^* \gamma_{14} \cdot Z'$ is necessarily even, we see that $\lambda_{7,7+8}$ is even. Next, we take the cup product with $\gamma_7 \cdot \iota^* \gamma_7$, and we get that $\lambda_{14,14+8}$ is even; we go on with the cup products with $\gamma_{13} \cdot \iota^* \gamma_{13}, \ldots,$ $\gamma_8 \cdot \iota^* \gamma_8$, and we get that all the $\lambda_{j,j+8}$ are even. $\qquad \square$

We will deduce that $\widetilde{\delta} + \widetilde{\Sigma}$ is divisible by 2 in $H^2(\widetilde{M}, \mathbb{Z})$. To this end, we will use Smith theory (see Section 2.4).

The element $\widetilde{\delta} + \widetilde{\Sigma}$

To apply Smith theory, we need the following lemma.

Lemma 4.6.19. *We have:*

1) $H^3(\widetilde{M}, \mathbb{Z}) = 0$

2) $H^*(\widetilde{M}, \mathbb{Z})$ *is torsion-free.*

Proof.   1) We have the following exact sequence:

$$H^3(S^{[2]}, V, \mathbb{Z}) \to H^3(S^{[2]}, \mathbb{Z}) \xrightarrow{f} H^3(V, \mathbb{Z}) \to H^4(S^{[2]}, V, \mathbb{Z}) \xrightarrow{\rho} H^4(S^{[2]}, \mathbb{Z}).$$

By Thom isomorphism, $H^3(S^{[2]}, V, \mathbb{Z}) = 0$ and $H^4(S^{[2]}, V, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z})$. Moreover ρ is injective, so $H^3(V, \mathbb{Z}) = H^3(S^{[2]}, \mathbb{Z}) = 0$.

Hence, using the spectral sequence of equivariant cohomology, we find that $H^3(U, \mathbb{Z}) = 0$. Since $H^3(\widetilde{X}, \mathbb{Z}) = 0$, $H^3(\widetilde{M}, \mathbb{Z})$ is a torsion group. Hence the result follows from the exact sequence

$$H^3(\widetilde{M}, U, \mathbb{Z}) \twoheadrightarrow H^3(\widetilde{M}, \mathbb{Z}) \twoheadrightarrow H^3(U, \mathbb{Z})$$

and from the fact that $H^3(\widetilde{M}, U, \mathbb{Z}) = 0$ by Thom isomorphism.

2) By 1), $H^3(\widetilde{M}, \mathbb{Z})$ is torsion-free. Since $\widetilde{M}$ is simply connected, $H_1(\widetilde{M}, \mathbb{Z}) = 0$. Hence, the group $H^2(\widetilde{M}, \mathbb{Z})$ is torsion-free. The group $H^4(\widetilde{M}, \mathbb{Z})$ is torsion-free by Lemma 4.6.9. By 1) and the universal coefficient theorem, $H_2(\widetilde{M}, \mathbb{Z})$ is torsion-free. Hence by Poincaré duality, $H^6(\widetilde{M}, \mathbb{Z})$ is torsion-free. Finally, by Poincaré duality, $H^7(\widetilde{M}, \mathbb{Z}) \simeq H_1(\widetilde{M}, \mathbb{Z}) = 0$ is torsion-free.

$\qquad \square$

Look at the following exact sequence:

$$0 \twoheadrightarrow H^2(\widetilde{M}, \widetilde{\Sigma} \cup (\cup_{k=1}^8 D_k), \mathbb{F}_2)) \twoheadrightarrow H^2(\widetilde{M}, \mathbb{F}_2) \twoheadrightarrow H^2(\widetilde{\Sigma} \cup (\cup_{k=1}^8 D_k, \mathbb{F}_2))$$

$$\twoheadrightarrow H^3(\widetilde{M}, \widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2) \longrightarrow 0.$$

First, we will calculate the vector spaces $H^2(\widetilde{M}, \widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2)$ and $H^3(\widetilde{M}, \widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2)$. By 3) of Proposition 2.4.1, we have

$$H^*(\widetilde{M}, \widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2) \simeq H_\sigma^*(\widetilde{X}).$$

We denote $h_\sigma^i(\widetilde{X}) := \dim H_\sigma^i(\widetilde{X})$.

Lemma 4.6.20. *We have:*

$$h_\sigma^2(\widetilde{X}) = 36, \qquad h_\sigma^3(\widetilde{X}) = 43.$$

Proof. The previous exact sequence gives us the following equation:

$$h_\sigma^2(\widetilde{X}) - h^2(\widetilde{M}, \mathbb{F}_2) + h^2(\widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2) - h_\sigma^3(\widetilde{X}) = 0.$$

As $h^2(\widetilde{M}, \mathbb{F}_2) = 16 + 28 = 44$ and $h^2(\widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2) = 23 + 28 = 51$, we obtain:
$$h_\sigma^2(\widetilde{X}) - h_\sigma^3(\widetilde{X}) = 7.$$

Moreover by 2) of Proposition 2.4.1, we have the exact sequence

$$0 \rightarrow H_\sigma^1(\widetilde{X}) \rightarrow H_\sigma^2(\widetilde{X}) \rightarrow H^2(\widetilde{X}, \mathbb{F}_2) \rightarrow H_\sigma^2(\widetilde{X}) \oplus H^2(\Sigma_2 \cup (\cup_{k=1}^{28} E_k), \mathbb{F}_2)$$
$$\rightarrow H_\sigma^3(\widetilde{X}) \longrightarrow 0.$$

By Lemma 7.4 of [11], $h_\sigma^1(\widetilde{X}) = h^0(\Sigma_2 \cup (\cup_{k=1}^{28} E_k), \mathbb{F}_2) - 1$. Then we get the equation

$$h^0(\Sigma_2 \cup (\cup_{k=1}^{28} E_k), \mathbb{F}_2) - 1 - h_\sigma^2(\widetilde{X}) + h^2(\widetilde{X}, \mathbb{F}_2)$$
$$- h_\sigma^2(\widetilde{X}) - h^2(\Sigma_2 \cup (\cup_{k=1}^{28} E_k), \mathbb{F}_2) + h_\sigma^3(\widetilde{X}) = 0,$$

or

$$29 - 2h_\sigma^2(\widetilde{X}) + h_\sigma^3(\widetilde{X}) = 0.$$

From the two equations, we deduce that

$$h_\sigma^2(\widetilde{X}) = 36, \qquad h_\sigma^3(\widetilde{X}) = 43.$$

$\square$

Lemma 4.6.21. *The following seven elements belong to* $H^2(\widetilde{M}, \mathbb{Z})$: $\frac{\widetilde{u_{k,l}} + d_{k,l}}{2}$
$(k, l) \in \{1, 2, 3\} \times \{1, 2\}$ *and* $\frac{\widetilde{\delta} + d_\delta}{2}$. *Moreover,* $\mathrm{Vect}_{\mathbb{F}_2}((d_{i,j})_{(i,j) \in \{1,2,3\} \times \{1,2\}}, d_\delta)$
is a subspace of $\mathrm{Vect}_{\mathbb{F}_2}(D_1, \ldots, D_{28})$ *of dimension 7.*

Proof. Come back to the exact sequence

$$0 \longrightarrow H^2(\widetilde{M}, \widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2) \longrightarrow H^2(\widetilde{M}, \mathbb{F}_2) \xrightarrow{\varsigma^*} H^2(\widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2),$$

where $\varsigma : \widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k) \hookrightarrow \widetilde{M}$ is the inclusion. Since $h^2(\widetilde{M}, \widetilde{\Sigma} \cup (\cup_{k=1}^{28} D_k), \mathbb{F}_2) = h_\sigma^2(\widetilde{X}) = 36$, we have $\dim_{\mathbb{F}_2} \varsigma^*(H^2(\widetilde{M}, \mathbb{F}_2)) = (16 + 28) - 36 = 8$. We can interpret this as follows. Consider the homomorphism

$$\varsigma_{\mathbb{Z}}^* : H^2(\widetilde{M}, \mathbb{Z}) \to H^2(\widetilde{\Sigma}, \mathbb{Z}) \oplus (\oplus_{k=1}^{28} H^2(D_k, \mathbb{Z}))$$

$$u \to (u \cdot \widetilde{\Sigma}, u \cdot D_1, \ldots, u \cdot D_{28}).$$

Since this is a map of torsion-free $\mathbb{Z}$-modules (by Lemma 4.6.19), we can tensor by $\mathbb{F}_2$,

$$\varsigma^* = \varsigma_{\mathbb{Z}}^* \otimes \mathrm{id}_{\mathbb{F}_2} : H^2(\widetilde{M}, \mathbb{Z}) \otimes \mathbb{F}_2 \to H^2(\widetilde{\Sigma}, \mathbb{Z}) \oplus (\oplus_{k=1}^{28} H^2(D_k, \mathbb{Z})) \otimes \mathbb{F}_2,$$

and we have 8 independent elements such that the intersection with the D_k $k \in \{1, \ldots, 8\}$ and $\widetilde{\Sigma}$ are not all even. But $\varsigma^*(\pi_{2*}(s^*(U^3 \oplus (-2))) \oplus \frac{1}{2}\pi_{2*}(s^*(E_8(-2))) \oplus \left\langle D_1, \ldots, D_{28}, \widetilde{\Sigma} \right\rangle) = 0$, (it follows from Lemma 3.3.7, 2)). Hence, there are 8 more independent elements in $H^2(\widetilde{M}, \mathbb{Z})$. Moreover, these elements must be of the form $\frac{u+d}{2}$ with $u \in \pi_{2*}(s^*(U^3 \oplus (-2)))$ and $d \in \left\langle D_1, \ldots, D_{28}, \widetilde{\Sigma} \right\rangle$. Indeed, applying π_{2*} to an element of the form $\frac{e_k + u_k + n_k}{2}$ with $e_k \in \frac{1}{2}\pi_{2*}(s^*(E_8(-1)))$, we see that e_k is divisible by 2.

By Proposition 4.6.17, we know that $\varsigma^*(\frac{\widetilde{\Sigma} + D_1 + \cdots + D_{28}}{2}) \neq 0$. Let

$$x_1 := \varsigma_{\mathbb{Z}}^*(\frac{\widetilde{\Sigma} + D_1 + \cdots + D_{28}}{2}) \in H^2(\widetilde{\Sigma}, \mathbb{Z}) \oplus (\oplus_{k=1}^{28} H^2(D_k, \mathbb{Z}))$$

and

$$\overline{x_1} := x_1 \otimes 1 \in H^2(\widetilde{\Sigma}, \mathbb{F}_2) \oplus (\oplus_{k=1}^{28} H^2(D_k, \mathbb{F}_2)).$$

We complete the family $(\overline{x_1})$ to a basis $(\overline{x_k} := x_k \otimes 1)_{1 \leq k \leq 8}$ of $\varsigma^*(H^2(\widetilde{M}, \mathbb{F}_2))$. For all $1 \leq k \leq 8$, we have

$$x_k = \varsigma_{\mathbb{Z}}^*(\frac{u_k + d_k}{2}),$$

where $u_k \in \pi_{2*}(s^*(U^3 \oplus (-2)))$ and d_k is an integral combination of the D_l, $1 \leq l \leq 28$, and $\widetilde{\Sigma}$. In particular, $u_1 = 0$ and $d_1 = \widetilde{\Sigma} + D_1 + \cdots + D_{28}$.

Now, let $\mathcal{D}$ be the vector subspace of $\mathrm{Vect}_{\mathbb{F}_2}(\widetilde{\Sigma}, D_1, \ldots, D_{28})$ generated by the d_k, $1 \leq k \leq 8$. We have

$$\dim_{\mathbb{F}_2} \mathcal{D} = 8.$$

The fact that the family $(\overline{x_k} := x_k \otimes 1)_{1 \leq k \leq 8}$ of $\varsigma^*(H^2(\widetilde{M}, \mathbb{F}_2))$ is a basis, will imply that the d_k are $\mathbb{F}_2$-linearly independent.

To show this, we just need to see that $(d_k)_{1 \leq k \leq 8}$ is free. Assume $\sum_{k=1}^8 \overline{\epsilon_k}\overline{d_k} = 0$, then $\sum_{k=1}^8 \epsilon_k d_k = 2d$, where d is an integral combination of the D_k and $\widetilde{\Sigma}$ by the definition of $\mathcal{D}$. Now we have $\frac{\sum_{k=1}^8 \epsilon_k (u_k + d_k)}{2} = \frac{\sum_{k=1}^8 \epsilon_k u_k}{2} + d$. The

primitivity of $\pi_{2*}(s^*(U^3 \oplus (-2)))$ (Proposition 3.5.20 and Proposition 4.6.6) implies that $\frac{\sum_{k=1}^{8} \epsilon_k u_k}{2} \in \pi_{2*}(s^*(U^3 \oplus (-2)))$, so $\varsigma^*\left(\frac{\sum_{k=1}^{8} \epsilon_k u_k}{2}\right) = 0$. Then

$$\sum_{k=1}^{8} \epsilon_k x_k = \sum_{k=1}^{8} \epsilon_k \varsigma^*_{\mathbb{Z}}\left(\frac{u_k + d_k}{2}\right)$$
$$= \varsigma^*_{\mathbb{Z}}\left(\frac{\sum_{k=1}^{8} \epsilon_k(u_k + d_k)}{2}\right)$$
$$= \varsigma^*_{\mathbb{Z}}\left(\frac{\sum_{k=1}^{8} \epsilon_k u_k}{2}\right) + \varsigma^*_{\mathbb{Z}}(d)$$

and $\sum_{k=1}^{8} \overline{\epsilon_k x_k} = \varsigma^*(d) = 0$.

Now let $\mathcal{U}$ be the subspace of $\mathrm{Vect}_{\mathbb{F}_2}((\widetilde{u_{l,m}})_{1 \leq l \leq 3, 1 \leq m \leq 2}, \widetilde{\delta})$ generated by the u_k, $2 \leq k \leq 8$. We will show that

$$\mathcal{U} = \mathrm{Vect}_{\mathbb{F}_2}((\widetilde{u_{l,m}})_{1 \leq l \leq 3, 1 \leq m \leq 2}, \widetilde{\delta}).$$

To do this, we just need to show that the family $(u_k)_{2 \leq k \leq 8}$ is free. We consider $\sum_{k=2}^{8} \overline{\epsilon_k} u_k = 0$, hence $\sum_{k=2}^{8} \epsilon_k u_k = 2u$, where u is an integral combination of the $\widetilde{u_{l,m}}$ and $\widetilde{\delta}$ by the definition of $\mathcal{U}$. Then $d = \frac{\sum_{k=2}^{8} \epsilon_k d_k}{2}$ is integral. By the proof of Lemma 4.6.14, there are just two possibilities: $d \in \left\langle (D_k)_{k \in \{1,\ldots,28\}}, \widetilde{\Sigma} \right\rangle$ or $d = \frac{\widetilde{\Sigma} + D_1 + \cdots + D_{28}}{2}$. In the first case we get $\sum_{k=2}^{8} \overline{\epsilon_k x_k} = \varsigma^*(d) = 0$, so that the ϵ_k are even for all k. In the second case, we get $\sum_{k=2}^{8} \overline{\epsilon_k} d_k = d_1$, which is impossible.

This proves the existence of elements $\frac{\widetilde{u_{k,l}} + d_{k,l}}{2}$, $(k,l) \in \{1,2,3\} \times \{1,2\}$ and $\frac{\widetilde{\delta} + d_\delta}{2}$ in $H^2(\widetilde{M}, \mathbb{Z})$, for which $d_{k,l}$ and d_δ integral combination of the D_l, $1 \leq l \leq 28$ and $\widetilde{\Sigma}$. We can suppose that they are only combinations of the D_l. If this is not the case, we just have to add $\frac{\widetilde{\Sigma} + D_1 + \cdots + D_{28}}{2}$. And finally, by the primitivity of $\pi_{2*}(s^*(U^3 \oplus (-2)))$, $\mathrm{Vect}_{\mathbb{F}_2}((d_{i,j})_{(i,j) \in \{1,2,3\} \times \{1,2\}}, d_\delta)$ is a subspace of $\mathrm{Vect}_{\mathbb{F}_2}(D_1, \ldots, D_{28})$ of dimension 7. $\qquad \square$

Lemma 4.6.22. *The element $\widetilde{\delta} + \widetilde{\Sigma}$ is divisible by 2 in $H^2(\widetilde{M}, \mathbb{Z})$.*

Proof. Multiplying by $\frac{\widetilde{\Sigma} + D_1 + \cdots + D_{28}}{2}$ the 7 elements of the Lemma 4.6.21, we get seven independent elements of the form $\frac{\widetilde{\sigma_{u_{k,l}}} + h_{k,l}}{2}$, $(k,l) \in \{1,2,3\} \times \{1,2\}$ and $\frac{\widetilde{\sigma_\delta} + h}{2}$; the elements $h_{k,l}$, and h are integral combinations of the $(\widetilde{h_k})_{k \in \{1,\ldots,28\}}$. By Proposition 4.6.17 5),we know that $\widetilde{T}/\pi_{2*}(T) = (\mathbb{Z}/2\mathbb{Z})^7$, hence

$$\widetilde{T} = \left\langle \pi_{2*}(T), (\frac{\widetilde{\sigma_{u_{k,l}}} + h_{k,l}}{2})_{(k,l) \in \{1,2,3\} \times \{1,2\}}, \frac{\widetilde{\sigma_\delta} + h}{2} \right\rangle. \qquad (*)$$

We have shown that $\frac{\widetilde{\delta}+d_\delta}{2} \in H^2(\widetilde{M},\mathbb{Z})$; now we will show that $d_\delta = D_1 + \cdots + D_{28}$. It will follow that $\widetilde{\delta} + \widetilde{\Sigma}$ is divisible by 2. We can write $d_\delta = \epsilon_1 D_1 + \cdots + \epsilon_{28} D_{28}$ with $\epsilon_k \in \{0,1\}$. We have:

$$\left(\frac{\widetilde{\delta} + d_\delta}{2}\right)^2 = \frac{\widetilde{\delta^2} + \sum_{k=1}^{28} \epsilon_k \widetilde{h_k}}{2}.$$

We have also:

$$\left(\frac{\widetilde{\Sigma} + D_1 + \cdots + D_{28}}{2}\right)^2 = \frac{-\pi_{2*}(s^*(\Sigma)) + \widetilde{h_1} + \cdots + \widetilde{h_{28}}}{2}.$$

We sum up these two elements and we get the element

$$\frac{\widetilde{\delta^2} - \pi_{2*}(s^*(\Sigma)) + \sum_{k=1}^{28} (\epsilon_k + 1)\widetilde{h_k}}{2}.$$

Since $\widetilde{\delta^2} - \pi_{2*}(s^*(\Sigma))$ is divisible by 2, we see that $\frac{\sum_{k=1}^{28}(\epsilon_k+1)\widetilde{h_k}}{2}$ is integral. Then by (*) and Proposition 4.6.17 (4), the unique possibility is $\epsilon_k = 1$ for all $k \in \{1,\ldots,28\}$. $\qquad\square$

The end of the proof

Lemma 4.6.23. *The element $\overline{\delta}' + \overline{\Sigma}'$ is divisible by 2 in $H^2(M',\mathbb{Z})$.*

Proof. We can find a Cartier divisor on $\widetilde{M}$ which corresponds to $\frac{\pi_{2*}(s^*(\delta))+\widetilde{\Sigma}}{2}$ and which does not meet $\cup_{k=1}^{28}\widetilde{r}^{-1}(p_k)$. Then this Cartier divisor induces a Cartier divisor on M' which necessarily corresponds to half the cocycle $\pi_{1*}(s_1^*(\delta))$ $+\overline{\Sigma}'$. $\qquad\square$

Finally, we get the following theorem.

Theorem 4.6.24. *We have*

$$H^2(M',\mathbb{Z}) = \frac{1}{2}\pi_{1*}(s_1^*(E_8(-2))) \oplus \pi_{1*}(s_1^*(U^3)) \oplus \mathbb{Z}(\frac{\overline{\delta}' + \overline{\Sigma}'}{2}) \oplus \mathbb{Z}(\frac{\overline{\delta}' - \overline{\Sigma}'}{2}).$$

Now we are able to finish the calculation of the Beauville–Bogomolov form on $H^2(M',\mathbb{Z})$. By Propositions 1.3.8, 4.6.10, 4.6.11, 4.6.13 and Theorem 4.6.24, the Beauville–Bogomolov form on $H^2(M',\mathbb{Z})$ gives the lattice:

$$\frac{1}{2}E_8\left(-2\sqrt{\frac{24}{C_{M'}}}\right) \oplus U^3\left(\sqrt{\frac{24}{C_{M'}}}\right) \oplus \left(-\sqrt{\frac{24}{C_{M'}}}\right)^2$$

$$= E_8\left(-\sqrt{\frac{6}{C_{M'}}}\right) \oplus U^3\left(2\sqrt{\frac{6}{C_{M'}}}\right) \oplus \left(-2\sqrt{\frac{6}{C_{M'}}}\right)^2$$

It follows that $C_{M'} = 6$, and we get Theorem 4.6.8.

Remark: Theorem 4.6.24 shows that (X', ι_1) is not H^2-normal. Indeed $\pi_{1*}(\Sigma_1 + \delta')$ is divisible by 2 in $H^2(M', \mathbb{Z})$, though $\Sigma_1 + \delta'$ cannot be written in the form $y + \iota_1^*(y)$ with $y \in H^2(X', \mathbb{Z})$. Moreover, we find the coefficient of normality: $\alpha_2(X') = 1$.

4.7 Summary

4.7.1 Singular symplectic surfaces

The surfaces $\overline{Y}_2$ and $\overline{Y}_3$ of Section 4.1 and Section 4.2 are simply connected by Lemma 1.2 of [17]. Hence they are singular irreducible symplectic surfaces.

Proposition 4.7.1. *We have* $H^1(\overline{A}, \mathbb{C}) = H^3(\overline{A}, \mathbb{C}) = 0$.

Proof. Indeed $(-\operatorname{id})^*$ acts as $-\operatorname{id}$ on $H^1(X, \mathbb{C})$ and on $H^3(X, \mathbb{C})$. $\qquad\square$

The following table summarizes our results:

X/G	b_2	χ	$(H^2(X/G, \mathbb{Z}), \cdot)$
$\overline{Y}_2$	14	16	$E_8(-1) \oplus U(2)^3$
$\overline{Y}_3$	10	12	$U(3) \oplus U^2 \oplus A_2^2$
$\overline{A}$	6	8	$U(2)^3$

4.7.2 Singular irreducible symplectic fourfolds

Proposition 4.7.2. *We have:*

1) $b_4(M') = 178$ *and* $\chi(M') = 212$,

2) $b_4(M_3) = 102$ *and* $\chi(M_3) = 126$.

Proof.   1) We have $\dim H^k(M', \mathbb{C}) = \dim H^k(X', \mathbb{C})^{\iota_1}$ for all $0 \leq k \leq 8$. Hence we have to calculate $\dim H^k(X', \mathbb{C})^{\iota_1}$. By Theorem 7.31 of [67] (Theorem 2.5.1), we have:

$$H^4(X', \mathbb{Z}) = s_1^*(H^4(S^{[2]}, \mathbb{Z})) \oplus l_{1*}(s_{1|\Sigma_1}^*(H^2(\Sigma, \mathbb{Z}))),$$

where $l_1 : \Sigma_1 \hookrightarrow X'$. Then

$$H^4(X', \mathbb{Z})^{\iota_1} = s_1^*(H^4(S^{[2]}, \mathbb{Z})^\iota) \oplus l_{1*}(s_{1|\Sigma_1}^*(H^2(\Sigma, \mathbb{Z}))).$$

Hence

$$\dim H^4(X', \mathbb{C})^{\iota_1} = \dim H^4(S^{[2]}, \mathbb{C})^\iota + \dim H^2(\Sigma, \mathbb{C}).$$

Since $\dim H^2(\Sigma, \mathbb{C}) = 22$, it remains to calculate $\dim H^4(S^{[2]}, \mathbb{C})^\iota$.

By Theorem 1.3.9, we know that the cup-product map $\mathrm{Sym}^2 H^2(S^{[2]}, \mathbb{C}) \to H^4(S^{[2]}, \mathbb{C})$ is an isomorphism. Hence

$$\dim H^4(S^{[2]}, \mathbb{C})^\iota = \dim(\mathrm{Sym}^2 H^2(S^{[2]}, \mathbb{C}))^\iota.$$

By Proposition 1.3.8, the signature of ι^* on $H^2(S^{[2]}, \mathbb{C})$ is $(15, 8)$. Hence $\dim(\mathrm{Sym}^2 H^2(S^{[2]}, \mathbb{C}))^\iota = \frac{16 \times 15}{2} + \frac{9 \times 8}{2} = 156$. Then

$$b_4(M') = \dim H^4(M', \mathbb{C}) = \dim H^4(X', \mathbb{C})^{\iota_1} = 156 + 22 = 178.$$

We have also $b_3(M') = 0$ and $b_2(M') = 16$ by Theorem 7.31 of [67] (Theorem 2.5.1) and Proposition 1.3.8.

By Poincaré duality, we have $\mathrm{rk}\, H^5(X', \mathbb{Z})^{\iota_1} = \mathrm{rk}\, H^7(X', \mathbb{Z})^{\iota_1} = 0$, $\mathrm{rk}\, H^6(X', \mathbb{Z})^{\iota_1} = 16$ and $\mathrm{rk}\, H^8(X', \mathbb{Z})^{\iota_1} = 1$. Finally $\chi(M') = 1 - 0 + 16 - 0 + 178 - 0 + 16 - 0 + 1 = 212$.

Remark: We can also calculate $b_4(M')$ by counting the number of elements of the basis of $H^4(M, \mathbb{Z})$ given in Lemma 4.6.5.

2) Let (X, G) be as in the statement of Corollary 4.5.2. In the proof of Corollary 4.5.2, we have found: $l_3^2(X) = 6$, $l_1^2(X) = 5$ and $l_2^2(X) = 0$. Hence by Propositions 3.3.17, 3.3.19, Theorem 1.3.9 and Lemma 3.3.18, $l_1^4(X) = \frac{5 \times 6}{2} = 15$, $l_2^4(X) = 0$ and $l_3^4(X) = 2 \times 6 + 3 \times \frac{6 \times 5}{2} + 6 \times 5 = 87$. Hence by Proposition 2.2.1, $\mathrm{rk}\, H^4(X, \mathbb{Z})^G = 87 + 15 = 102$. It follows that $b_4(M_3) = 102$.

And $\chi(M_3) = 126$ follows from Poincaré duality as in the proof of point 1).

$\square$

The following table summarizes our results:

X/G	b_2	b_4	χ	$C_{X/G}$	$B_{X/G}$
M'	16	178	212	6	$E_8(-1) \oplus U(2)^3 \oplus (-2)^2$
M_3	11	102	126	9	$U(3) \oplus U^2 \oplus A_2^2 \oplus (-6)$

Chapter 5

The Markushevich–Tikhomirov variety

In this Chapter, we use the notation of Section 1.2.3.

5.1 Dual of the (1,2)-polarized Lagrangian fibration

We start by calculating the dual of the Lagrangian fibration from Theorem 1.2.11.

In this section, $\overline{B_0}$ and $\overline{\Delta_0}$ are smooth quartics tangent to each other at eight points lying on a conic; we will denote by U the set of such pairs. Moreover, we assume that the pairs $(\overline{B_0}, \overline{\Delta_0})$ and $(\overline{\Delta_0}, \overline{B_0})$ are in $\mathfrak{L}$. We can permute the roles of $\overline{\Delta_0}$ and $\overline{B_0}$ in the construction of Section 1.2.3. Namely, consider the double cover $\widetilde{\mu} : \widetilde{X} \to \mathbb{P}^2$ branched in $\overline{\Delta_0}$. Let $\widetilde{\Delta_0} = \widetilde{\mu}^{-1}(\overline{\Delta_0})$, and let $\widetilde{B_0}$ and $\widetilde{B_0'}$ be the two curves mapped to $\overline{B_0}$ by $\widetilde{\mu}$. We denote by $\widetilde{i}$ the involution of $\widetilde{X}$ induced by $\widetilde{\mu}$ which exchanges $\widetilde{B_0}$ and $\widetilde{B_0'}$. Consider the double cover $\widetilde{\rho} : \widetilde{S} \to \widetilde{X}$ branched in $\widetilde{B_0}$ and set $\widetilde{B} = \rho^{-1}(\widetilde{B_0})$. Finally, denote by $\widetilde{\tau}$ the involution of S induced by $\widetilde{\rho}$. We have the diagram

$$\begin{array}{ccc}
\widetilde{\Delta_0} & \!\!=\!\!\!=\!\!\!=\!\! & \overline{\Delta_0} \\
\end{array}$$

$$\widetilde\tau \,\subset\, \widetilde{S} \xrightarrow[\widetilde\rho]{2:1} \widetilde{X} \xrightarrow[\widetilde\mu]{2:1} \mathbb{P}^2$$

$$B \!\!=\!\!\!=\!\!\!=\!\! B_0$$

similar to the one of Section 1.2.3.

Like in Section 1.2.3, for a generic line in $\mathbb{P}^2$, we denote $\widetilde{E_t} = \widetilde\mu^{-1}(t)$, $\widetilde{C_t} = \widetilde\rho^{-1}(\widetilde{E_t})$, $\widetilde\mu_t = \widetilde\mu_{|E_t}$, $\widetilde\rho_t = \widetilde\rho_{|C_t}$ and $\widetilde\tau_t = \widetilde\tau_{|C_t}$. The generic curves $\widetilde{E_t}$ are elliptic and the curves $\widetilde{C_t}$ are of genus 3. Thus, we have the tower of double covers:

$$\widetilde{C_t} \xrightarrow{2:1} \widetilde{E_t} \xrightarrow{2:1} \mathbb{P}^1.$$

By Lemma 1.2.10, $\mathrm{Prym}(\widetilde{C_t}, \widetilde\tau_t)$ is also a (1,2)-polarized Prym surface. We denote by $\mathrm{Prym}(C_t, \tau_t)^\vee$ the dual of the polarized abelian variety $\mathrm{Prym}(C_t, \tau_t)$. The answer to our questions is given by the following proposition.

Proposition 5.1.1. *For a generic line $t \in \mathbb{P}^2$, we have an isomorphism*

$$\mathrm{Prym}(C_t, \tau_t)^\vee \simeq \mathrm{Prym}(\widetilde{C_t}, \widetilde\tau_t).$$

Proof.

- Step 1: *The curve bigonally related to C_t*

  Starting with the tower $C_t \to E_t \to \mathbb{P}^1$, we will construct a curve $C_t^\vee$ whose points correspond to the different ways to lift the pairs $\mu_t^{-1}(p)$, for $p \in \mathbb{P}^1$, to a pair in C_t, i.e.,

  $$C_t^\vee = \left\{ p + q \in \mathrm{Div}^{(2)}(C_t) \,\middle|\, [\rho_t(p) + \rho_t(q)] = [\mu_t^* \mathcal{O}_{\mathbb{P}^1}(1)] \right\}.$$

  We denote by $\tau_t^\vee$ the involution $C_t^\vee \to C_t^\vee$ sending a lift to its complement. We have $\tau_t^\vee = \tau_{t|\,\mathrm{Div}^{(2)}(C_t)}^*$. Let $E_t^\vee = C_t^\vee / \tau_t^\vee$. We also define the map $\mu_t^\vee : E_t^\vee \to \mathbb{P}^1$ which sends a lift of $\mu_t^{-1}(p)$ ($p \in \mathbb{P}^1$) to p.

  For a better understanding we draw a diagram. Let α be a generic point in $\mathbb{P}^1$ (a point which is not a branch point of μ_t nor the image of a branch point of ρ_t), β_i, ($i = 1, 2$) its preimages under μ_t, and $\gamma_{i,j}$, $(i,j) \in \{1,2\}^2$, the preimages of the β_i under ρ_t, as shown in the diagram:

  $$\begin{array}{ccc}
  C_t & \xrightarrow{\rho_t} E_t & \xrightarrow{\mu_t} \mathbb{P}^1 \\
  \gamma_{1,1} \rightrightarrows \beta_1 & \rightarrow \alpha. \\
  \gamma_{1,2} & \nearrow \\
  \gamma_{2,1} \rightrightarrows \beta_2 & \nearrow \\
  \gamma_{2,2} &
  \end{array}$$

This gives the following diagram for points in $C_t^\vee$ and $E_t^\vee$:

$$
\begin{array}{ccccc}
C_t^\vee & \xrightarrow{\;\rho_t^\vee\;} & E_t^\vee & \xrightarrow{\;\mu_t^\vee\;} & \mathbb{P}^1 \\
\gamma_{1,1}+\gamma_{2,1} \longrightarrow & \overline{\gamma_{1,1}+\gamma_{2,1}} = \overline{\gamma_{1,2}+\gamma_{2,2}} & & \longrightarrow & \alpha. \\
\gamma_{1,2}+\gamma_{2,2} & & & & \\
\gamma_{1,1}+\gamma_{2,2} \longrightarrow & \overline{\gamma_{1,1}+\gamma_{2,2}} = \overline{\gamma_{1,2}+\gamma_{2,1}} & & & \\
\gamma_{1,2}+\gamma_{2,1} & & & &
\end{array}
$$

Like ρ_t, μ_t, the maps $\rho_t^\vee$, $\mu_t^\vee$ are double covers:

$$
C_t^\vee \xrightarrow{2:1} E_t^\vee \xrightarrow{2:1} \mathbb{P}^1. \qquad (*)
$$

Barth [5] calls this way to obtain $C_t^\vee$ Pantazis's bigonal construction (see [58], p. 304).

Proposition 5.1.2. *The abelian varieties* $\mathrm{Prym}(C_t,\tau_t)$ *and* $\mathrm{Prym}(C_t^\vee,\tau_t^\vee)$ *are dual to each other in such a way that* $C_t^\vee$ *(resp.* C_t*) embeds in* $\mathrm{Prym}(C_t,\tau_t)$ *(resp.* $\mathrm{Prym}(C_t^\vee,\tau_t^\vee)$*) as a theta-divisor of a polarization of type (1,2).*

Proof. See [58] Proposition 1 Section 3 page 307. $\qquad\qquad\square$

Now, we will show that $\mathrm{Prym}(C_t^\vee,\tau_t^\vee)$ and $\mathrm{Prym}(\widetilde{C_t},\widetilde{\tau}_t)$ are isomorphic. To this end, we will look what happens when α is a branch point.

- Step 2: *The ramification of the double covers of the diagram (*)*

We will denote by $(a_i)_{1\le i\le 4}$ the branch points of μ_t, $(b_i)_{1\le i\le 4}$ their preimages in E_t, $(e_i)_{1\le i\le 4}$ the branch points of ρ_t, $(p_i)_{1\le i\le 4}$ their images in $\mathbb{P}^1$, $(e_i')_{1\le i\le 4}$ the other preimages of the p_i in E_t, $(c_i)_{1\le i\le 4}$ the preimages of the $(e_i)_{1\le i\le 4}$ in C_t, $(b_{i,j})_{1\le i\le 4,1\le j\le 2}$ the preimages of the $(b_i)_{1\le i\le 4}$ in C_t and $(c_{ij}')_{1\le i\le 4,1\le j\le 2}$ the preimages of the $(e_i')_{1\le i\le 4}$ in C_t, as in the following diagram:

$$
\begin{array}{ccccc}
C_t & \xrightarrow{\rho_t} & E_t & \xrightarrow{\mu_t} & \mathbb{P}^1 \\
b_{i,j} & \longrightarrow & b_i & \longrightarrow & a_i \\
c_i & \longrightarrow & e_i & \longrightarrow & p_i. \\
c_{i,j}' & \longrightarrow & e_i' & \nearrow &
\end{array}
$$

We have $\tau_t(b_{i,1})=b_{i,2}$, $\tau_t(c_{i,1}')=c_{i,2}'$ for $\mu_t^{-1}(a_i)=\{b_i\}$ and $\rho_t^{-1}(\mu_t^{-1}(a_i)) = \{b_{i,1},b_{i,2}\}$, for all $i\in\{1,2,3,4\}$. So we see that the ramification points of $\rho_t^\vee$ are the pairs $b_{i,1}+b_{i,2}$ $1\le i\le 4$. We have also $\mu_t^{-1}(p_i)=\{e_i,e_i'\}$ and $\rho_t^{-1}(\mu_t^{-1}(p_i)) = \{c_i,c_{i,1}',c_{i,2}'\}$. The involution $\tau_t^\vee$ exchanges $c_i+c_{i,1}'$ for $c_i+c_{i,2}'$. And the classes $c_i^\vee$ of the pairs $c_i+c_{i,1}'$ and $c_i+c_{i,2}'$ in $E_t^\vee$, $1\le i\le 4$, are the ramification points of $\mu_t^\vee$. We show this in the diagram:

$$
\begin{array}{ccc}
C_t^{\vee} \xrightarrow{\ \ \rho_t^{\vee}\ \ } & E_t^{\vee} \xrightarrow{\ \ \mu_t^{\vee}\ \ } & \mathbb{P}^1 \\
c_i + c_{i,j}' \longrightarrow \overline{c_i + c_{i,1}'} = \overline{c_i + c_{i,2}'} \longrightarrow & p_i \\
b_{i,1} + b_{i,2} \longrightarrow \overline{b_{i,1} + b_{i,2}} \\
2b_{i,j} \longrightarrow \overline{2b_{i,1}} = \overline{2b_{i,2}}
\end{array}
\; a_i.
$$

- Step 3: *Conclusion*

  We see that the maps $\mu_t^{\vee}$ and $\widetilde{\mu}_t$ have the same branch points in $\mathbb{P}^1$ by Step 2. This gives an isomorphism φ_t between $E_t^{\vee}$ and $\widetilde{E}_t$. Now, we want that φ_t sends the branch points of $\rho_t^{\vee}$ to the branch points of $\widetilde{\rho}_t$ to build an isomorphism between $\widetilde{C}_t$ and $C_t^{\vee}$.

  To show this, we map $E_t^{\vee}$ into $\widetilde{X}$ by φ_t. By step 2, $\widetilde{\mu}$ sends the branch points of $\rho_t^{\vee}$ on $t \cap \overline{B_0} = \{a_1, \dots, a_4\}$ for all $t \in \mathcal{U} := \mathbb{P}^{2\vee} \setminus \overline{B_0}^{\vee}$. The group $\pi_1(\mathcal{U})$ acts by monodromy on the four points $\{a_1, \dots, a_4\}$. This action is transitive because of the irreducibility of $\overline{B_0}$. If we assume for a moment that k of the 4 branch points of $\rho_t^{\vee}$ are on $\widetilde{B_0}$, and $4 - k$ on $\widetilde{B_0'}$, $1 \le k \le 3$, then we see that the image of $\pi_1(\mathcal{U})$ is contained in $\mathscr{S}_k \times \mathscr{S}_{4-k} \subset \mathscr{S}_4$, which contradicts the transitivity. Hence the branch points of $\rho_t^{\vee}$ are all on $\widetilde{B_0}$ or all on $\widetilde{B_0'}$. If they are on $\widetilde{B_0'}$, we just need to compose φ_t with $\widetilde{i}$ (the involution on $\widetilde{X}$ we have defined in the very beginning) to obtain an isomorphism between $E_t^{\vee}$ and $\widetilde{E}_t$ which sends the branch points of $\rho_t^{\vee}$ to the branch points of $\widetilde{\rho}_t$. Denote this isomorphism by φ_t. Then we obtain the commutative diagram

$$
\begin{array}{ccc}
\widetilde{C}_t & \xrightarrow{\ \widetilde{\rho}_t\ } & \widetilde{E}_t & \xrightarrow{\ \widetilde{\mu}_t\ } & \mathbb{P}^1 \ , \\
\Big\| & & \varphi_t \Big\| & \raise2pt{\mu_t^{\vee}} \\
C_t^{\vee} & \xrightarrow{\ \rho_t^{\vee}\ } & E_t^{\vee}
\end{array}
$$

  which implies $\mathrm{Prym}(C_t^{\vee}, \tau_t^{\vee}) \simeq \mathrm{Prym}(\widetilde{C}_t, \widetilde{\tau}_t)$.

  $\square$

5.2   Relation between S and $\widetilde{S}$

It is a natural question to know whether the two K3 surfaces S, $\widetilde{S}$ are isomorphic or not. We are going to prove that the answer is no for generic S.

To explain the meaning of "generic", we need to recall the definition of the moduli space of 2-elementary K3 surfaces $\mathfrak{M}_{r,a,\delta}$.

5.2.1 Definition of $\mathfrak{M}_{r,a,\delta}$

Let S be a K3 surface equipped with an antisymplectic involution $\tau : S \to S$. Let $P = \mathrm{Pic}(S)^\tau$. Then P is a primitive 2-elementary Lorentzian sublattice of $H^2(S, \mathbb{Z})$ endowed with the cup product (see for instance Lemma 1.3 of [68]). Let (r, a, δ) be a triple of integers. A couple (S, τ) is called a *2-elementary K3 surface of type* (r, a, δ) if $(r(P), a(P), \delta(P)) = (r, a, \delta)$. We denote by $\mathfrak{M}_{r,a,\delta}$ the moduli space of isomorphism classes of 2-elementary K3 surfaces of type (r, a, δ).

For a K3 surface S, $H^2(S, \mathbb{Z})$ endowed with the cup-product pairing is isometric to the K3 lattice $L = E_8(-1)^2 \oplus U^3$. An isometry of lattices $\alpha : H^2(S, \mathbb{Z}) \cong L$ is called a *marking* of S. The pair (S, α) is called a *marked K3 surface*. Let $M \subset L$ be a primitive 2-elementary Lorentzian sublattice. Let I_M be the involution on $M \oplus M^\perp$ defined by

$$I_M(x, y) = (x, -y), \qquad (x, y) \in M \oplus M^\perp.$$

Then I_M extends uniquely to an involution on L by Corollary 2.1.4. A K3 surface equipped with an anti-symplectic holomorphic involution $\tau : S \to S$ is called a *2-elementary K3 surface of type M* if there exists a marking α of S satisfying

$$\tau^* = \alpha^{-1} \circ I_M \circ \alpha.$$

Such a marking will be called a *M-marking* of (S, τ). We note that $\alpha((\mathrm{Pic}(S))^\tau) = M$. Now we will show that a 2-elementary K3 surface of type (r, a, δ) and a 2-elementary K3 surface of type M where $(r(M), a(M), \delta(M)) = (r, a, \delta)$, are equivalent notions.

Lemma 5.2.1. *Let $\varphi : M_1 \simeq M_2$ be an isometry between two 2-elementary sublattices of L. We assume that $\mathrm{sign}\, M_1 = \mathrm{sign}\, M_2 = (2, x)$ where x is an integer. Then we can extend φ to an isometry of L.*

Proof. We will use Corollary 2.1.4. We start by constructing an isometry between $M_1^\perp$ and $M_2^\perp$. We have $\mathrm{sign}(M_1^\perp) = \mathrm{sign}(M_2^\perp) = (1, 19 - x)$ (because $\mathrm{sign}(L) = (3, 19)$). Since L is unimodular, we have an isomorphism $\gamma_{M_1} : A_{M_1} \to A_{M_1^\perp}$ with $q_{M_1^\perp} \circ \gamma_{M_1} = -q_{M_1}$ and an isomorphism $\gamma_{M_2} : A_{M_2} \to A_{M_2^\perp}$ with $q_{M_2^\perp} \circ \gamma_{M_2} = -q_{M_2}$ (see Section 2.1.1). This implies $(a(M_1^\perp), \delta(M_1^\perp)) = (a(M_2^\perp), \delta(M_2^\perp))$. Then, by Theorem 2.1.5 there is an isometry $\psi : M_1^\perp \to M_2^\perp$.

On the other hand, φ (resp. ψ) induces an isometry $\overline{\varphi} : A_{M_1} \to A_{M_2}$ (resp. $\overline{\psi} : A_{M_1^\perp} \to A_{M_2^\perp}$). Now we consider the composition

$$\gamma_{M_2} \circ \overline{\varphi} \circ \gamma_{M_1}^{-1} \circ \overline{\psi^{-1}},$$

which is an isometry of $A_{M_2^\perp}$. Since $M_2^\perp$ is a 2-elementary Lorentzian sublattice, Theorem 2.1.6 gives us an isometry $\chi \in \mathcal{O}(M_2^\perp)$ with $\overline{\chi} = \gamma_{M_2} \circ \overline{\varphi} \circ \gamma_{M_1}^{-1} \circ \overline{\psi^{-1}}$. Hence

$$\gamma_{M_2} \circ \overline{\varphi} = \overline{\chi \circ \psi} \circ \gamma_{M_1}.$$

By Corollary 2.1.4, φ extends to an isometry of L. $\qquad\square$

Remark: The same result holds if $\operatorname{sign} M_i = (1, x)$, as we are going to see in the proof of the next proposition.

Proposition 5.2.2. *A K3 surface is a 2-elementary K3 surface of type (r, a, δ) if and only if it is a 2-elementary K3 surface of type M for some primitive 2-elementary Lorentzian sublattice M with $(r(M), a(M), \delta(M)) = (r, a, \delta)$.*

Proof. It is obvious that a 2-elementary K3 surface (S, τ) of type M with $(r(M), a(M), \delta(M)) = (r, a, \delta)$ belongs to $\mathfrak{M}_{r,a,\delta}$. So, we will show the other implication. Let $M \subset L$ with $(r(M), a(M), \delta(M)) = (r, a, \delta)$ and let $(S, \tau) \in \mathfrak{M}_{r,a,\delta}$. We will find a M-marking of S. Let $P = \operatorname{Pic}(S)^\tau$. We can assume that P is a sublattice of L; it suffices to take its image by some marking of S. We have $(r(P), a(P), \delta(P)) = (r, a, \delta)$. Then by Theorem 2.1.5, we have an isometry $\psi : P \to M$. Moreover, $\operatorname{sign}(P^\perp) = \operatorname{sign}(M^\perp)$, and since L is unimodular, we have $(a(P^\perp), \delta(P^\perp)) = (a(M^\perp), \delta(M^\perp))$. Once more, it follows by Theorem 2.1.5 that there is an isometry $\varphi : P^\perp \to M^\perp$. By Lemma 5.2.1, this isometry extends to an isometry of L. Denoting this isometry by $\widetilde{\varphi}$, we have $\tau^* = \widetilde{\varphi}^{-1} \circ I_M \circ \widetilde{\varphi}$. Indeed, let $x \in H^2(S, \mathbb{Z})$, $x = \frac{p+t}{2}$, where $p \in P$ and $t \in P^\perp$. Then

$$\widetilde{\varphi}^{-1} \circ I_M \circ \widetilde{\varphi}\left(\frac{p+t}{2}\right) = \widetilde{\varphi}^{-1} \circ I_M\left(\frac{\widetilde{\varphi}(p) + \widetilde{\varphi}(t)}{2}\right) = \widetilde{\varphi}^{-1}\left(\frac{\widetilde{\varphi}(p) - \widetilde{\varphi}(t)}{2}\right) = \frac{p-t}{2}.$$

$\square$

The moduli space of 2-elementary K3 surface was introduced by Nikulin in [51], see also [52] and [68] Section 1 for more details. We can also regard [16] Section 11 for a similar construction in a more general case.

5.2.2 A Torelli Theorem for 2-elementary K3 surfaces

For a better understanding of this moduli space we will give a kind of Torelli theorem for it (see [68] and [69] for more details). We need some more notation.

Let (S, α) be a marked K3 surface. Recall the definition of the period map for marked K3 surfaces: the period of (S, α) is defined to be

$$\pi(S, \alpha) := [\alpha(\eta)] \in \mathbb{P}(L \otimes \mathbb{C}), \qquad \eta \in H^0(S, \omega_S) \setminus \{0\}.$$

Let Λ be a lattice of signature $(2, n)$. We define

$$\Omega_\Lambda := \{[x] \in \mathbb{P}(\Lambda \otimes \mathbb{C}); \langle x, x \rangle = 0, \langle x, \overline{x} \rangle > 0\}.$$

Let $\Delta_\Lambda := \{x \in \Lambda; \langle x, x \rangle = -2\}$.
For $\lambda \in \Lambda \otimes \mathbb{R}$, set $H_\lambda := \{[x] \in \Omega_\Lambda; \langle x, \lambda \rangle = 0\}$. We define the discriminant locus of Ω_Λ by

$$\mathcal{D}_\Lambda := \sum_{d \in \Delta_\Lambda / \pm 1} H_d.$$

Assume that Λ is a primitive 2-elementary sublattice of L with $\Lambda^\perp$ Lorentzian. Then we set

$$\Gamma(\Lambda) := \{g \in \mathcal{O}(L), \ I_{\Lambda^\perp} g = g I_{\Lambda^\perp}\},$$

$$\Gamma_\Lambda := \{g_{|\Lambda} \in \mathcal{O}(\Lambda); \ g \in \Gamma(\Lambda)\},$$

$$\Omega_\Lambda^\circ := \Omega_\Lambda \setminus \mathcal{D}_\Lambda, \qquad \mathcal{M}_\Lambda^\circ := \Omega_\Lambda^\circ / \Gamma_\Lambda.$$

The following theorem, due to Yoshikawa ([68] Theorem 1.8) can be thought of as a Torelli Theorem for 2-elementary K3 surface:

Theorem 5.2.3. *Via the period map, the analytic space $\mathcal{M}_{M^\perp}^\circ$ is a coarse moduli space of 2-elementary K3 surfaces of type M.*

Proof. The proof uses the classical Torelli Theorem for K3 surfaces (see [59] and [10]) and results of Nikulin [51]. $\qquad\square$

Next, Yoshikawa improves this result in [69], proving the following proposition (Proposition 11.2 in [69]).

Proposition 5.2.4. *The following equality holds:*

$$\Gamma_{M^\perp} = \mathcal{O}(M^\perp).$$

Proof. The proof uses Theorem 2.1.6 and Corollary 2.1.4. The idea is the same as in the proof of Lemma 5.2.1 and Proposition 5.2.2. $\qquad\square$

We thus obtain the following result. Let $M \subset L$ be a primitive 2-elementary Lorentzian sublattice with $(r(M), a(M), \delta(M)) = (r, a, \delta)$. Define the map:

$$\overline{\varpi}_{r,a,\delta}: \begin{array}{ccc} \mathfrak{M}_{r,a,\delta} & \to & \Omega_{M^\perp}^\circ / \mathcal{O}(M^\perp) \\[2mm] (S, \tau) & \mapsto & \mathcal{O}(M^\perp) \cdot \pi(S, \alpha), \end{array}$$

where α is a M-marking of S.

Corollary 5.2.5. *The map $\overline{\varpi}_{r,a,\delta}$ is an isomorphism.*

Proof. See [69] page 8. $\qquad\square$

We define

$$\mathfrak{D}_{r,a,\delta} = \overline{\varpi}_{r,a,\delta}^{-1} \left(\left\{ \mathcal{O}(M^\perp) \cdot \eta \in \Omega_{M^\perp}^\circ / \mathcal{O}(M^\perp) \,\middle|\, \langle \eta, x \rangle \neq 0, \ \forall x \in M^\perp \setminus \{0\} \right\} \right).$$

It is a dense subset of $\mathfrak{M}_{r,a,\delta}$, and "generic 2-elementary K3 surface" will mean for us a 2-elementary K3 surface with modulus in $\mathfrak{D}_{r,a,\delta}$. The following important property is verified.

Proposition 5.2.6. *Let $M \subset L$ be a 2-elementary Lorentzian sublattice with $(r(M), a(M), \delta(M)) = (r, a, \delta)$. If $(S, \tau) \in \mathfrak{D}_{r,a,\delta}$, then a M-marking of (S, τ) induces an isometry between $\operatorname{Pic} S$ and M. In particular, all elements of $\operatorname{Pic} S$ are invariant under τ^*.*

Proof. Indeed, let $x \in \operatorname{Pic} S$ then $\langle x, \eta \rangle = 0$, where $\eta \in H^0(S, \omega_S) \setminus \{0\}$. Let α be a M-marking of S. Then we can write $\alpha(x) = \frac{a+b}{2}$, where $a \in M$ and $b \in M^\perp$. Since $\langle a, \alpha(\eta) \rangle = 0$, we have $\langle b, \alpha(\eta) \rangle = 0$, so by hypothesis $b = 0$. $\square$

Corollary 5.2.7. *Let (S, τ) and (S', τ') be in $\mathfrak{D}_{r,a,\delta}$. If S and S' are isomorphic, then (S, τ) and (S', τ') are isomorphic.*

Proof. Let $M \subset L$ be a sublattice with $(r(M), a(M), \delta(M)) = (r, a, \delta)$ and α_τ, $\alpha_{\tau'}$ M-markings of (S, τ), (S', τ') respectively. Let $\eta \in H^0(S, \omega_S) \setminus \{0\}$ and $\eta' \in H^0(S', \omega_{S'}) \setminus \{0\}$. We have the following diagram:

$$
\begin{array}{ccc}
H^2(S, \mathbb{Z}) & & \\
\Big\downarrow{\scriptstyle\varphi} & \searrow{\scriptstyle\alpha_\tau} & \\
 & & L, \\
 & \nearrow{\scriptstyle\alpha_{\tau'}} & \\
H^2(S', \mathbb{Z}) & &
\end{array}
$$

where φ is a Hodge isometry. Then we have

$$\alpha_\tau(\eta) = (\alpha_\tau \circ \varphi^{-1} \circ \alpha_{\tau'}^{-1})_{|M^\perp}(\alpha_{\tau'}(\eta')).$$

Since $(\alpha_\tau \circ \varphi^{-1} \circ \alpha_{\tau'}^{-1})_{|M^\perp} \in \mathcal{O}(M^\perp)$, we have

$$\overline{\varpi}_M(S, \tau) = \overline{\varpi}_M(S', \tau').$$

$\square$

5.2.3   Applications

Now, we will work with the moduli space $\mathfrak{M}_{8,8,1}$.

Remark: We have $M = I_{1,7}(2)$ for the associated Lorentzian sublattice of L, where $I_{p,q}$ stands for the lattice $\mathbb{Z}^{p+q}$ with quadratic form given by the diagonal matrix

$$\operatorname{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})$$

and $\Lambda(d)$ denotes Λ with quadratic form multiplied by d for any lattice Λ and any integer d.

We recall that U is the locus of pairs $(\overline{B_0}, \overline{\Delta_0})$ in $|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(4)|$ such that $\overline{B_0}$ and $\overline{\Delta_0}$ are smooth quartics, tangent to each other at eight points lying on a conic. We will denote by $\mu_{\overline{B_0}} : X_{\overline{B_0}} \to \mathbb{P}^2$ the double cover of $\mathbb{P}^2$ branched over $\overline{B_0}$. We define $\mathcal{Q} \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(4)|$ to be the set of pairs $(\overline{B_0}, 2Q)$, where $\overline{B_0}$ is a smooth quartic and Q is a conic such that $\mu_{\overline{B_0}}^*(Q)$ is smooth.

Proposition 5.2.8. *There is an isomorphism between* $U/PGL_3 \cup \mathcal{Q}/PGL_3$ *and* $\mathfrak{M}_{8,8,1}$.

Proof.

- Step 1: *The map* $\mathscr{P} : U/PGL_3 \cup \mathcal{Q}/PGL_3 \to \mathfrak{M}_{8,8,1}$

  We will construct the map $U/PGL_3 \to \mathfrak{M}_{8,8,1}$; the construction of $\mathcal{Q}/PGL_3 \to \mathfrak{M}_{8,8,1}$ is similar.

  First, we have the map $U \to \mathfrak{M}_{8,8,1}$. Remember that diagram (1) of Section 1 gave a K3 surface S with an involution τ. By page 663 of [52] $(S, \tau) \in \mathfrak{M}_{8,8,1}$. So the diagram (1) gives us the map $U \to \mathfrak{M}_{8,8,1}$. Now, let $(\overline{B_0}, \overline{\Delta_0})$ and $(\overline{B_0}', \overline{\Delta_0}')$ be in $|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(4)|$ such that $f(\overline{B_0}, \overline{\Delta_0}) = (\overline{B_0}', \overline{\Delta_0}')$, where $f \in PGL_3$. We can draw the commutative diagram

$$
\begin{array}{ccc}
B_0 & =\!=\!= & \overline{B_0} \\
\cap\downarrow & & \downarrow \\
i\,\subset\, X & \xrightarrow{\ \mu\ } & \mathbb{P}^2 \\
\wr\!\downarrow\phi & & \wr\!\downarrow f \\
i'\,\subset\, X' & \xrightarrow{\ \mu'\ } & \mathbb{P}^2 \\
\uparrow & & \uparrow \\
B_0' & =\!=\!= & \overline{B_0}',
\end{array}
$$

  where ϕ is induced by f (the other symbols are the same as in the diagram (1) of Section 1). The map ϕ sends $\mu^{-1}(\overline{\Delta_0}) = \Delta_0 + i(\Delta_0)$ on $\mu'^{-1}(\overline{\Delta_0}') = \Delta_0' + i'(\Delta_0')$. The curves $\overline{\Delta_0}$ and $\overline{\Delta_0}'$ are smooth, so they are irreducible. Then all the curves Δ_0, $i(\Delta_0)$, Δ_0' and $i'(\Delta_0')$ are irreducible. Therefore ϕ sends Δ_0 on Δ_0' or on $i'(\Delta_0')$. If ϕ sends Δ_0 on $i'(\Delta_0')$, we replace ϕ by $i' \circ \phi$. Now, let $\rho : S \to X$ and $\rho' : S' \to X'$ be the double covers branched in Δ_0 and Δ_0' respectively. We get the commutative diagram

$$
\begin{array}{ccc}
\Delta & =\!=\!= & \Delta_0 \\
\cap\downarrow & & \downarrow \\
\tau\,\subset\, S & \xrightarrow{\ \rho\ } & X \\
\wr\!\downarrow\varphi & & \wr\!\downarrow\phi \\
\tau'\,\subset\, S' & \xrightarrow{\ \rho'\ } & X' \\
\uparrow & & \uparrow \\
\Delta' & =\!=\!= & \Delta_0',
\end{array}
$$

  where φ is induced by ϕ. This implies $(S, \tau) \simeq (S', \tau')$, and we get the map

$$U/PGL_3 \to \mathfrak{M}_{8,8,1}.$$

- Step 2: *The inverse map* $\mathscr{G} : \mathfrak{M}_{8,8,1} \to U/PGL_3 \cup \mathcal{Q}/PGL_3$

  Let (S, τ) be in $\mathfrak{M}_{8,8,1}$. By [52], $\rho : S \to X = S/\tau$ is a double cover ramified in a smooth curve Δ of genus 3, and X is a del Pezzo surface.

Moreover the linear system $|-K_X|$ defines a double cover $\mu : X \to \mathbb{P}^2$ branched in a smooth quartic of $\overline{B}_0 \subset \mathbb{P}^2$. We have $\rho(\Delta) \in |-2K_X|$, and by Lemma 5.14 of [34], $(\overline{B}_0, \mu(\rho(\Delta))) \in U$ or $(\overline{B}_0, \mu(\rho(\Delta))) \in \mathcal{Q}$. Now let (S, τ) and (S', τ') be two isomorphic objects from $\mathfrak{M}_{8,8,1}$. We denote by $(\overline{B}_0, \overline{\Delta}_0)$ and $(\overline{B}_0', \overline{\Delta}_0')$ the two pairs corresponding to (S, τ) and (S', τ') respectively (here $\overline{\Delta}_0$ and $\overline{\Delta}_0'$ may be double conics). To have a well defined map from $\mathfrak{M}_{8,8,1}$ to $U/PGL_3 \cup \mathcal{Q}/PGL_3$, we must verify that $(\overline{B}_0, \overline{\Delta}_0)$ and $(\overline{B}_0', \overline{\Delta}_0')$ are exchanged by an automorphism of $\mathbb{P}^2$. We have an isomorphism $f : S \simeq S'$ with $f \circ \tau = \tau' \circ f$. It induces a commutative diagram

$$
\begin{array}{ccccc}
S & \xrightarrow{\;\rho\;} & X & \longrightarrow & |-K_X| \simeq \mathbb{P}^2 \\
f \Big\| & & g \Big\| & & (g^{-1})^* \Big\| \\
S' & \xrightarrow{\;\rho'\;} & X' & \longrightarrow & |-K_{X'}| \simeq \mathbb{P}^2,
\end{array}
$$

which implies the result.

To finish, we see easily that the composition of $\mathscr{G}$ and $\mathscr{P}$ is the identity. $\qquad\square$

Corollary 5.2.9. *The involution on U/PGL_3 given by $(\overline{B}_0, \overline{\Delta}_0) \to (\overline{\Delta}_0, \overline{B}_0)$ induces a rational involution of $\mathfrak{M}_{8,8,1}$ with indeterminacy on $\mathscr{P}(\mathcal{Q}/PGL_3)$, which exchanges the two 2-elementary K3 surfaces $\mathscr{P}(\overline{B}_0, \overline{\Delta}_0) = (S, \tau)$ and $\mathscr{P}(\overline{\Delta}_0, \overline{B}_0) = (\widetilde{S}, \widetilde{\tau})$. Moreover (S, τ) and $(\widetilde{S}, \widetilde{\tau})$ are isomorphic if and only if there exists an automorphism f of $\mathbb{P}^2$ such that $f(\overline{B}_0, \overline{\Delta}_0) = (\overline{\Delta}_0, \overline{B}_0)$.*

We define an open subset of U/PGL_3 by

$$
\mathscr{O} = \left\{ PGL_3 \cdot (\overline{B}_0, \overline{\Delta}_0) \in U/PGL_3 \,\middle|\, PGL_3 \cdot (\overline{B}_0, \overline{\Delta}_0) \neq PGL_3 \cdot (\overline{\Delta}_0, \overline{B}_0) \right\}.
$$

Now we are able to answer to the question we asked in the beginning of the section:

Corollary 5.2.10. *Let $(S, \tau) \in \mathfrak{O}_{8,8,1} \cap \mathscr{P}(\mathscr{O})$. Then S and $\widetilde{S}$ are not isomorphic.*

Proof. Since $(S, \tau) \in \mathscr{P}(\mathscr{O})$, (S, τ) and $(\widetilde{S}, \widetilde{\tau})$ are not isomorphic. Moreover, $(S, \tau) \in \mathfrak{O}_{8,8,1}$, therefore, by Corollary 5.2.7, S and $\widetilde{S}$ are not isomorphic either.
$\qquad\square$

Remarks:

1) The dimension of $\mathfrak{M}_{8,8,1}$ is 12.

2) Let $\mathcal{B} := \{PGL_3 \cdot (\Gamma, \Gamma) \in U/PGL_3\}$; we have $\mathcal{B} \subset (U/PGL_3) \setminus \mathscr{O}$. Moreover $\dim \mathcal{B} = 6$, and $\mathscr{P}(\mathcal{B})$ parametrized the quadruple covers of $\mathbb{P}^2$ branched in smooth quartics. We also have $\mathcal{B} \subset (U/PGL_3) \setminus (\mathfrak{L}/PGL_3)$, where $\mathfrak{L}$ is the set of sufficiently generic pairs $(\overline{B}_0, \overline{\Delta}_0)$, see Definition 1.2.6.

3) The quotient variety $\mathcal{Q}/PGL_3$ has dimension 11.

5.2.4 Derived categories

In fact, we can say even more: S and $\widetilde{S}$ are not even derived equivalent. We will denote by $D^b(S)$ the derived category of coherent sheaves on S. Let T_S be the transcendental lattice of S, that is the orthogonal complement to $\operatorname{Pic} S$ in $H^2(S, \mathbb{Z})$. By Theorem 4.2.4. of [57], the categories $D^b(S)$ and $D^b(S')$ are equivalent as triangulated categories if and only if there exists a Hodge isometry between T_S and $T_{S'}$. We have the following theorem.

Theorem 5.2.11. *Let S and S' be K3 surfaces such that $D^b(S)$ and $D^b(S')$ are equivalent. If T_S is a 2-elementary sublattice of $H^2(S, \mathbb{Z})$, then S and S' are isomorphic.*

Proof. Let S and S' be two K3 surfaces such that $D^b(S)$ and $D^b(S')$ are equivalent. By Theorem 4.2.4. of [57] we have a Hodge isometry $\rho : T_S \to T_{S'}$. Let $\alpha : H^2(S, \mathbb{Z}) \simeq L$ and $\beta : H^2(S', \mathbb{Z}) \simeq L$ be markings of S and S' respectively. The lattices $\alpha(T_S)$ and $\beta(T_{S'})$ are 2-elementary sublattices of L of signature $(2, x)$. So by Lemma 5.2.1, $\beta \circ \rho \circ \alpha^{-1}_{|\alpha(T_S)}$ extends to an isometry of L, that we will denote by ν. Then $\beta^{-1} \circ \nu \circ \alpha : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ is a Hodge isometry, therefore by the Global Torelli Theorem for K3 surfaces (see for instance Chapter 10, Theorem 5.3. of [27]), S and S' are isomorphic. $\square$

Remark: For all 2-elementary K3 surfaces $(S, \tau) \in \mathfrak{O}_{r,a,\delta}$, T_S is a 2-elementary sublattice of $H^2(S, \mathbb{Z})$.

Corollary 5.2.12. *Let $(S, \tau) \in \mathfrak{O}_{8,8,1} \cap \mathscr{P}(\mathscr{O})$, then $D^b(S)$ and $D^b(\widetilde{S})$ are not equivalent.*

Proof. Indeed, if $(S, \tau) \in \mathfrak{O}_{8,8,1}$ then T_S is a 2-elementary lattice. Then if $D^b(S)$ and $D^b(\widetilde{S})$ were equivalent, then S and $\widetilde{S}$ would be isomorphic, which is false by Corollary 5.2.10 $\square$

5.3 Non-equivalence of dual Relative Compactified Prymians

We will need the following proposition:

Proposition 5.3.1. *Let S and S' be two complex K3 surfaces. If $S^{[2]}$ and $S'^{[2]}$ are bimeromorphic, then $D^b(S) \sim D^b(S')$.*

Proof. By Lemma 1.1.12, if $S^{[2]}$ and $S'^{[2]}$ are birational, there is a Hodge isometry Φ between $H^2(S^{[2]}, \mathbb{Z})$ and $H^2(S'^{[2]}, \mathbb{Z})$, where $H^2(S^{[2]}, \mathbb{Z})$ and $H^2(S'^{[2]}, \mathbb{Z})$ are endowed with the Beauville–Bogomolov form. Moreover, by Section 1.3.1, we have

$$H^2(S^{[2]}, \mathbb{Z}) = j(H^2(S, \mathbb{Z})) \oplus^{\perp} \mathbb{Z}\delta_S,$$

where $j : H^2(S, \mathbb{Z}) \to H^2(S^{[2]}, \mathbb{Z})$ and δ_S are defined in Section 1.3.1. This implies:

$$\left\{ a \in H^2\left(S^{[2]}, \mathbb{Z}\right) \middle| \ B_S(a, i(\eta_S)) \neq 0 \right\} = \left\{ i(b) \in H^2(S, \mathbb{Z}) \middle| b \in T_S \right\},$$

where $\eta_S \in H^0(S, \omega_S) \setminus \{0\}$, B_S is the Beauville–Bogomolov form of $H^2(S^{[2]}, \mathbb{Z})$ and T_S is the transcendental lattice of S. We have the same results for S', so Φ induces a Hodge isometry between T_S and $T_{S'}$. Then by Theorem 4.2.4 of [57], S and S' are derived equivalent. $\qquad\square$

We will denote by $\mathcal{P}_{(S,\tau)}$ the relative compactified Prymian built from the pair $(S, \tau) \in \mathscr{P}(\mathfrak{L}/PGL_3)$, (see Definition 1.2.8). If (S, τ) and (S', τ') are two isomorphic 2-elementary K3 surfaces, then $\mathcal{P}_{(S,\tau)}$ and $\mathcal{P}_{(S',\tau')}$ are naturally isomorphic. Now, we can prove the following theorem:

Theorem 5.3.2. *Let $(S, \tau) \in \mathfrak{O}_{8,8,1} \cap \mathscr{P}(\mathfrak{L}/PGL_3)$ and $(S', \tau') \in \mathscr{P}(\mathfrak{L}/PGL_3)$ be such that $\mathcal{P}_{(S,\tau)}$ and $\mathcal{P}_{(S',\tau')}$ are isomorphic. Then (S, τ) and (S', τ) are isomorphic.*

Proof. We will denote by $M_{(S,\tau)}$ and $M'_{(S,\tau)}$ the varieties defined in Section 1.2.3, which are the quotients of $S^{[2]}$ by the involution ι_S and the partial resolution of singularities of $M_{(S,\tau)}$ respectively. We denote by $M_{(S',\tau')}$ and $M'_{(S',\tau')}$ the same varieties with (S', τ') instead of (S, τ). By Theorem 1.2.13, $M'_{(S,\tau)}$ is bimeromorphic to $\mathcal{P}_{(S,\tau)}$ and $M'_{(S'\tau')}$ is bimeromorphic to $\mathcal{P}_{(S'\tau')}$. Therefore $M'_{(S,\tau)}$ and $M'_{(S'\tau')}$ are bimeromorphic, then $M_{(S,\tau)}$ and $M_{(S',\tau')}$ are bimeromorphic, hence also $M_{(S,\tau)} \setminus \operatorname{Sing} M_{(S,\tau)}$ and $M_{(S',\tau')} \setminus \operatorname{Sing} M_{(S',\tau')}$, so $S^{[2]}$ and $S'^{[2]}$ are bimeromorphic. By Proposition 5.3.1 we have $D^b(S) \sim D^b(S')$, so by Theorem 5.2.11, S and S' are isomorphic, and by Corollary 5.2.7 we have (S, τ) and (S', τ') isomorphic. $\qquad\square$

Corollary 5.3.3. *The dense set $\mathfrak{O}_{8,8,1} \cap \mathscr{P}(\mathfrak{L}/PGL_3)$ of $\mathfrak{M}_{8,8,1}$ provides a 1-to-1 parametrization of the relative compactified Prymians defined in Definition 1.2.8, and the non trivial rational involution on $\mathfrak{M}_{8,8,1}$ defined in Section 5.2.3 induces a non trivial involution on the set of the relative compactified Prymians.*

5.4   Beauville–Bogomolov form

As a consequence of Theorem 1.2.13 and Theorem 4.6.8, we have the following corollary.

Corollary 5.4.1. *The Beauville–Bogomolov lattice $H^2(\mathcal{P}, \mathbb{Z})$ is isomorphic to $E_8(-1) \oplus U(2)^3 \oplus (-2)^2$. Moreover, the Fujiki constant $C_\mathcal{P}$ is 6.*

As a consequence of Theorem 1.2.13 and Proposition 4.7.2 1), we can state the following proposition.

Proposition 5.4.2. *The variety $\mathcal{P}$ has the following numerical invariants: $b_2(\mathcal{P}) = 16$, $b_3(\mathcal{P}) = 0$, $b_4(\mathcal{P}) = 178$ and $\chi(\mathcal{P}) = 212$.*

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