Global instability in the elliptic restricted three body problem

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Introduction

Celestial mechanics has been a continuous source of inspiration of the main problems arising in dynamical systems. Particularly, the phenomenon of chaos and the existence of homoclinic orbits appears for the first time in the memory of Poincaré about the stability of the three body problem [Hen90], where the first exponentially small splitting of separatrices was computed.

In this memory we deal with the elliptic restricted three body problem (ERTBP) which appears in a natural way to study global instability, since it has two and a half degrees of freedom. The global instability, commonly known as diffusion, or better said, Arnold diffusion, from the pioneer work of Arnold in 1964 [Arn64], has been studied in the ERTBP in several settings, or more precisely, in several zones of the phase space. Capinsky & Zgliczynski have studied the instability in this problem, close to librations points in [CZ11] following previous work of Llibre & Simó [LS80b] and Llibre, Martinez & Simó [LMS85] for the existence of transversal homoclinic orbits in the classical circular restricted three body problem (CRTBP), as well as Bolotin close to collision [Bol06], Xia has studied micro-diffusion in the ERTBP [Xia93]. Recently, diffusion along mean motion resonances in the ERTBP model for Sun-Jupiter-asteroid systems has been proven by Fejoz, Guardia, Kaloshin & Roldán in [FGKR14]

In the last years, there have bee several mechanisms used to prove diffusion, like variational and geometrical methods. This memory is based on the application of geometrical methods, which are based on the existence of a scattering map associated to a normally hyperbolic invariant manifold (NHIM) in the phase space [DdlLS08, DGdlLS08]. In such cases, both the inner dynamics inside the NHIM as well as the outer dynamics provided by the scattering map, are combined to design diffusing pseudo-trajectories, consisting on invariant tori plus their transversal heteroclinic connections, that is, transition chains in Arnold's language.

In this memory we deal with the existence of diffusion orbits of the ERTBP whose angular momentum increases. Those orbits correspond to motions where the comet moves far from the primaries and comes back many times, increasing at each turn its angular momentum by a small amount.

The ERTBP is a Hamiltonian system of two and a half degrees of freedom. The so-called manifold of infinity can be seen, after a suitable change of variables provided by McGehee in [McG73], as a three dimensional invariant manifold in the extended phase space which behaves topologically as a NHIM, although it is of parabolic type. This means that the rate of approach and departure from it along its invariant manifolds is polynomial in time, instead of exponential-like as happens in a standard NHIM. On the other hand, the inner dynamics is very simple, since it is formed by a two-parameter family of 2π-periodic orbits in the 5D extended phase space which correspond to constant solutions in the 4D phase space. As a consequence, the stable and unstable manifold of the infinity manifold are union of the stable and unstable manifolds of its periodic orbits, and as long as these manifolds intersect along transversal heteroclinic orbits, the scattering map can be defined. Unfortunately, since the inner dynamics of the infinity manifold is so simple, the classical mechanisms of diffusion, consisting of combining the inner and outer dynamics, do not work here. Instead, as a novelty, we will be able to find two different scattering maps which will be combined in a suitable way to provide orbits whose angular momentum increases.

The main difficulty of this work is the asymptotic computation, for the mass parameter and eccentricity small enough and big enough angular momentum, of two different scattering maps. This computation relies on the computation of a Melnikov function which is very complicated in this problem. The first computation of this Melnikov function was done for the CRTBP in [LS80a] which was later corrected and carried out formally for the ERTBP in [MP94]. In both works, one can realize the massive amount of computations required.

In Chapter 2 of this memory, we provide a rigorous computation of the so-called Melnikov potential, with asymptotics and rigorous bounds for the errors for some range of the parameters μ (mass parameter), e_0 (eccentricity of the primaries) and G_0 (angular momentum of the comet). More precisely, the results presented here, are valid for G_0 big enough, $e_0 G_0$ bounded and μ small enough.

In Chapter 1, the problem is introduced, as well as the main geometrical objects which play a role in the diffusion mechanism. Particularly, the infinity manifold, its stable and unstable manifolds and the two independent scattering maps as well as the asymptotic formulas for them. The combination of both of them lead to theorem 1.15 for $e_0 G_0 = \lambda$ small, and theorem 1.16 for $e_0 G_0 = \lambda$ finite, at the end of the chapter.

As a final comment, there are at least three remaining tasks that would complete or extend the research on this problem. First, to compute the Melnikov potential for fixed $0 < e_0 < 1$ and G_0 big enough. Secondly, analogously as it is done in [GMMS12], to prove that a similar formula for the scattering map holds for $0 < \mu \leq 1/2$. Finally, one needs a suitable shadowing lemma for the infinity manifold, which as already has been said, it is not a genuine NHIM.

Chapter 1

Main results

1.1 Preliminars

As in the classical setting of the restricted three body problem, consider a particle with zero mass that moves in the plane generated by the dynamics of two point masses called *primaries*. It is a well known fact that the primaries move over an ellipse with a focus in the center of mass and with certain eccentricity that we will call e_0. If we fix a coordinate reference system with the origin at the center of mass and call q_1 and q_2 the position of the primaries, then under the classical assumptions regarding time units, distance and masses normalization, the motion of the third particle whose position we will call q is given by

$$\frac{d^2q}{dt^2} = \frac{(1-\mu)(q_1(t,e_0)-q)}{|q_1(t,e_0)-q|^3} + \frac{\mu(q_2(t,e_0)-q)}{|q_2(t,e_0)-q|^3} \tag{1.1}$$

where $1-\mu$ is the mass of the particle at q_1 and μ the mass of the particle at q_2. If we introduce the conjugate momenta $p = dq/dt$ and the self-potential function (see [MHO09, p. 28])

$$U_\mu(q,t;e_0) = \frac{1-\mu}{|q-q_1(t,e_0)|} + \frac{\mu}{|q-q_2(t,e_0)|} \tag{1.2}$$

This equation can be rewritten as a Hamiltonian system with Hamiltonian

$$H_\mu(q,p,t;e_0) = \frac{p^2}{2} - U_\mu(q,t;e_0). \tag{1.3}$$

This is a time-periodic Hamiltonian of two and a half degrees of freedom.

By the first Kepler law the distance between the primaries (see [Win41, p. 194-195]) is given by

$$r_0(t) = \frac{1-e_0^2}{1+e_0\cos f} \tag{1.4}$$

where f is the so called *true anomaly* (see [Win41, p. 203], [MP94, p. 303]), which satisfies

$$\frac{df}{dt} = \frac{(1+e_0\cos f)^2}{(1-e_0^2)^{3/2}}. \tag{1.5}$$

or also by

$$r_0(t) = 1 - e_0\cos E \tag{1.6}$$

in terms of the *eccentric anomaly* E, given by the Kepler equation (see [Win41, p. 195])

$$t = E - e_0\sin E. \tag{1.7}$$

1.2 Changes of coordinates and setting of the problem

Because of the nature of the problem we are dealing with, it will be better to perform a polar-simplectic change of variables, to the Hamiltonian (1.3) say

$$X = r\cos\alpha \tag{1.8a}$$
$$Y = r\sin\alpha \tag{1.8b}$$

where $q = (X, Y)$. Now, we look for conjugate momenta P_r and P_α so that the change

$$(X, Y, P_X, P_Y) \quad \xrightarrow{\varphi} \quad (r, \alpha, P_r, P_\alpha) \tag{1.9}$$

is canonical. Following [Gol65], we get:

$$P_X = P_r\cos\alpha - \frac{P_\alpha}{r}\sin\alpha \tag{1.10a}$$
$$P_Y = P_r\sin\alpha - \frac{P_\alpha}{r}\cos\alpha. \tag{1.10b}$$

In this way, the change of variables (1.10) is simplectic and the equations of motion in the new coordinates are the associated to the Hamiltonian

$$H_\mu^* = H_\mu(\varphi(Q), t; e_0)$$

that we will write as

$$H_\mu^*(r, \alpha, P_r, P_\alpha, t; e_0) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} - U_\mu^*(r, \alpha, t; e_0) \tag{1.11}$$

and U_μ^* defined by

$$U_\mu^*(r, \alpha, t; e_0) = U_\mu(r\cos\alpha, r\sin\alpha, t; e_0). \tag{1.12}$$

In this notation, the primaries are

$$q_2 = q_2(t, e_0) = -r_J(t, e_0)(\cos f(t, e_0), \sin f(t, e_0)) \tag{1.13a}$$
$$q_1 = q_1(t, e_0) = \quad r_S(t, e_0)(\cos f(t, e_0), \sin f(t, e_0)). \tag{1.13b}$$

where

$$r_J(t, e_0) = (1-\mu)r_0(t), \qquad r_S(t, e_0) = \mu r_0(t)$$

and

$$|q - q_2|^2 = r^2 + 2(1-\mu)\, r_0(t)\, r\cos(\alpha - f) + (1-\mu)^2[r_0(t)]^2,$$
$$|q - q_1|^2 = r^2 - \ 2\mu\ [r_0(t)]\ r\cos(\alpha - f) + \mu^2[r_0(t)]^2.$$

From now on we will write

$$G = P_\alpha, \qquad y = P_r.$$

and then Hamiltonian (1.11) reads

$$H_\mu^*(r, \alpha, y, G, t; e_0) = \frac{y^2}{2} + \frac{G^2}{2r^2} - U_\mu^*(r, \alpha, t; e_0)$$

with U_μ^* defined in (1.12).

1.2.1 McGehee coordinates

To study the behavior of orbits near infinity, we make to the Hamiltonian equations of Hamiltonian $H^*_\mu(r, \alpha, y, G, t; e_0)$ the non-canonical change of variables:

$$r = \frac{2}{x^2} \tag{1.14}$$

to we get the so called *McGehee coordinates* (see [MP94] y [McG73]). Defining

$$\mathcal{U}_\mu(x, \alpha, t; e_0) = U^*_\mu(2/x^2, \alpha, t; e_0) \tag{1.15}$$

the equations associated to (1.11) become:

$$\dot{x} = -\frac{1}{4}x^3 y \qquad \dot{y} = \frac{1}{8}G^2 x^6 - \frac{x^3}{4}\frac{\partial \mathcal{U}_\mu}{\partial x} \tag{1.16a}$$

$$\dot{\alpha} = \frac{1}{4}x^4 G \qquad \dot{G} = \frac{\partial \mathcal{U}_\mu}{\partial \alpha} \tag{1.16b}$$

where

$$\mathcal{U}_\mu(x, \alpha, t; e_0) = \frac{x^2}{2}\left(\frac{1-\mu}{\sigma_1} + \frac{\mu}{\sigma_2}\right) \tag{1.17}$$

and

$$\sigma_1^2 = 1 - z_1 x^2 \cos(\alpha - f) + \frac{1}{4}z_1^2 x^4, \qquad z_1 = \mu r_0(t),$$

$$\sigma_2^2 = 1 + z_2 x^2 \cos(\alpha - f) + \frac{1}{4}z_2^2 x^4, \qquad z_2 = (1-\mu) r_0(t),$$

It is important to notice that f is present in these equations, and then, becomes necessary to add the equation for f given in (1.5) in order to have the complete description of the dynamics. Equations (1.16) were obtained in [MP94].

Hamiltonian structure

Proposition 1.1 (quasi-Hamiltonian structure)**.** If $\mathcal{H}_\mu$ is defined by

$$\mathcal{H}_\mu(x, \alpha, y, G, t; e_0) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_\mu(x, \alpha, t; e_0), \tag{1.18}$$

and $\mathcal{U}$ is given in (1.17), the equations (1.16) can be written as

$$\dot{x} = -\frac{x^3}{4}\left(\frac{\partial \mathcal{H}_\mu}{\partial y}\right) \qquad \dot{y} = -\frac{x^3}{4}\left(-\frac{\partial \mathcal{H}_\mu}{\partial x}\right) \tag{1.19a}$$

$$\dot{\alpha} = \frac{\partial \mathcal{H}_\mu}{\partial G} \qquad \dot{G} = -\frac{\partial \mathcal{H}_\mu}{\partial \alpha} \tag{1.19b}$$

Therefore, equations (1.19) are in fact Hamiltonian with symplectic (non-canonical) form associated:

$$\omega(\mathbf{z})(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \mathcal{J}^{-T} \mathbf{v} \tag{1.20}$$

where $\mathcal{J} = J\mathcal{D}$ with $\mathcal{D} = \mathrm{diag}(-x^3/4, 1, -x^3/4, 1)$ and J the symplectic matrix

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

As in the classical theory we can write equations (1.19) in terms of the Poisson bracket

$$\begin{aligned} \{\{f, g\}\}(\mathbf{z}) &= \nabla f(\mathbf{z})^T \mathcal{J} \nabla g(\mathbf{z}) \\ &= -\frac{x^3}{4}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial f}{\partial G}\frac{\partial G}{\partial \alpha} \end{aligned} \tag{1.21}$$

1.3 Invariant manifolds

In order to analyze the structure of system (1.19), we will write $\mathcal{H}_\mu$ given in (1.18) as

$$\mathcal{H}_\mu(x,\alpha,y,G,t;e_0) = \mathcal{H}_0(x,y,G) + \mu\mathcal{U}_\mu^*(x,\alpha,t;e_0) \tag{1.22}$$

or equivalently we write $\mathcal{U}_\mu$ in (1.18) as

$$\mathcal{U}_\mu(x,\alpha,t;e_0) = \mathcal{U}_0(x) + \mu\mathcal{U}_\mu^*(x,\alpha,t;e_0) = \frac{x^2}{2} + \mu\mathcal{U}_\mu^*(x,\alpha,t;e_0) \tag{1.23}$$

and then study the dynamics as a perturbation of the limit case $\mu = 0$. From (1.23),

$$\begin{aligned}\mathcal{U}_0^*(x,\alpha,t;e_0) &= \lim_{\mu\to 0}\frac{1}{\mu}\left(\mathcal{U}_\mu(x,\alpha,t;e_0) - \frac{x^2}{2}\right)\\ &= \frac{x^2}{\left[4 + x^4 r_0^2 + 4x^2 r_0\cos(\alpha - f)\right]^{1/2}} + \left(\frac{x^2}{2}\right)^2 r_0\cos(\alpha - f) - \frac{x^2}{2}\end{aligned} \tag{1.24}$$

1.3.1 The limit case $\mu = 0$

In this case, the Hamiltonian given in (1.18) becomes

$$\begin{aligned}\mathcal{H}_0(x,\alpha,y,G) &= \frac{y^2}{2} + \frac{x^4G^2}{8} - \mathcal{U}_0(x)\\ &= \frac{y^2}{2} + \frac{x^4G^2}{8} - \frac{x^2}{2}\end{aligned} \tag{1.25}$$

As the system is autonomous $\mathcal{H}_0$ is a first integral. Moreover, $\mathcal{H}_0$ is independent of e_0 and α. The equations associated to Hamiltonian (1.25) are

$$\dot{x} = -\frac{1}{4}x^3y \qquad \dot{y} = \frac{1}{8}G^2x^6 - \frac{1}{4}x^4 \tag{1.26a}$$

$$\dot{\alpha} = \frac{1}{4}x^4G \qquad \dot{G} = 0 \tag{1.26b}$$

where it is clear that G is a conserved quantity.

The level curves of $\mathcal{H}_0$ are represented in Figure 1.1 for G fixed and $\mathcal{H}_0 = h$.

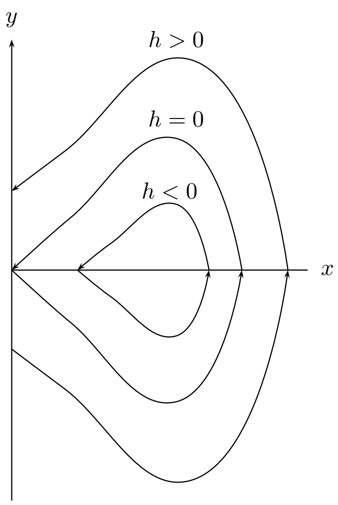


Figure 1.1: Level curves of $\mathcal{H}_0$

First, the phase space is given by

$$(x,\alpha,y,G) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}^2.$$

From equations (1.26) it is clear that

$$\mathcal{E} = \{z = (x,\alpha,y,G) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}\times\mathbb{R}_+;\ x = 0\} \tag{1.27}$$

is the set of equilibrium points of the system. Moreover, for any fixed $\alpha_0 \in \mathbb{T}, G_0 \in \mathbb{R}$,

$$\Lambda_{\alpha_0,G_0} = \{(0,\alpha_0,0,G_0)\}$$

is a parabolic critical point with stable and unstable 1-dimensional invariant manifolds:

$$\begin{aligned}\gamma_{\alpha_0,G_0} &= W^u(\Lambda_{\alpha_0,G_0}) = W^s(\Lambda_{\alpha_0,G_0})\\ &= \left\{z = (x,\alpha_0,y,G_0),\ \mathcal{H}_0(x,y,G_0) = 0, \alpha = \alpha_0 - G_0\int_{\mathcal{H}_0=0}\frac{x}{y}dx\right\}.\end{aligned}$$

In a natural way, we define the 2-dimensional manifold

$$\Lambda_\infty = \bigcup_{\alpha_0, G_0} \Lambda_{\alpha_0, G_0},$$

which is a "normally parabolic" invariant manifold with stable and unstable 3-dimensional invariant manifolds

$$\begin{aligned}\gamma &= W^u(\Lambda_\infty) = W^s(\Lambda_\infty) \\ &= \{z = (x, \alpha, y, G),\ \mathcal{H}_0(x, y, G) = 0\}.\end{aligned}$$

As we will deal with a periodic in time Hamiltonian, let us work in the extended phase space

$$\tilde{z} = (z, s) = (x, \alpha, y, G, s) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}^2 \times \mathbb{T}$$

just by adding the equation $\dot{s} = 1$ to the systems (1.19) and (1.26). Now we are going to write the extended version of the invariant sets we have defined so far. For any $\alpha_0 \in \mathbb{T}, G_0 \in \mathbb{R}$, the set

$$\tilde{\Lambda}_{\alpha_0, G_0} = \{\tilde{z} = (0, \alpha_0, 0, G_0, s_0), s_0 \in \mathbb{T}\} \tag{1.28}$$

is a 2π-periodic orbit with motion determined by $\dot{s} = 1$.

The manifold

$$\begin{aligned}\tilde{\gamma}_{\alpha_0, G_0} &= W^u(\tilde{\Lambda}_{\alpha_0, G_0}) = W^s(\tilde{\Lambda}_{\alpha_0, G_0}) \\ &= \left\{ z = (x, \alpha_0, y, G_0, s_0), s_0 \in \mathbb{T},\ \mathcal{H}_0(x, y, G_0) = 0, \alpha = \alpha_0 - G_0 \int_{\mathcal{H}_0 = 0} \frac{x}{y} dx \right\}.\end{aligned}$$

is a 2-dimensional homoclinic manifold to the periodic orbit $\tilde{\Lambda}_{\alpha_0, G_0}$. On the other hand we can construct the 3-dimensional invariant manifold

$$\tilde{\Lambda}_\infty = \bigcup_{\alpha_0, G_0} \tilde{\Lambda}_{\alpha_0, G_0} = \{(0, \alpha_0, 0, G_0, s_0),\ (\alpha_0, G_0, s_0) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}\}. \tag{1.29}$$

As the motion for points in $\tilde{\Lambda}_\infty$ is given by the dynamics on each $\tilde{\Lambda}_{\alpha_0, G_0}$, taking

$$\tilde{x}_0 = \tilde{x}_0(\alpha_0, G_0, s_0) = (0, \alpha_0, 0, G_0, s_0) \in \tilde{\Lambda}_\infty \simeq \mathbb{R} \times \mathbb{T}^2$$

the inner dynamics on $\tilde{\Lambda}_\infty$ (see [DdlLS06]) is trivial:

$$\tilde{\phi}_{t,0}(\tilde{x}_0) = (0, \alpha_0, 0, G_0, s_0 + t) = \tilde{x}_0(\alpha_0, G_0, s_0 + t) \in \tilde{\Lambda}_\infty. \tag{1.30}$$

The 4-dimensional stable and unstable manifolds of $\tilde{\Lambda}_\infty$ coincide along the 4-dimensional homoclinic invariant manifold

$$\begin{aligned}\tilde{\gamma} &= W^u(\tilde{\Lambda}_\infty) = W^s(\tilde{\Lambda}_\infty) \\ &= \{(x, \alpha, y, G, s), (\alpha, G, s) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T},\ \mathcal{H}_0(x, \alpha, y, G) = 0\}\end{aligned} \tag{1.31}$$

It is possible to parameterize $\tilde{\gamma}_{\alpha_0, G_0}$ by the solutions of the Hamiltonian flow contained in $\mathcal{H}_0 = 0$ in some time τ satisfying (see [MP94])

$$\frac{dt}{d\tau} = \frac{2G}{x^2}.$$

So that, the homoclinic solution to the periodic orbit $\tilde{\Lambda}_{\alpha_0, G_0}$ of the system (1.26) can be written as

$$x_h(t; G_0) = \tilde{x}_h(\tau; G_0) \tag{1.32a}$$

$$\alpha_h(t; \alpha_0, G_0) = \alpha_0 + \pi + \tilde{\alpha}_h(\tau; G_0) \tag{1.32b}$$

$$y_h(t;G_0) = \tilde{y}_h(\tau;G_0) \tag{1.32c}$$
$$G_h(t;G_0) = G_0 \tag{1.32d}$$

where α_0 and G_0 are free parameters and the relation

$$t = \frac{G_0^3}{2}\Big(\tau + \frac{\tau^3}{3}\Big) \tag{1.33}$$

holds. The equations (1.32) are explicitly, in τ-time, given by

$$\tilde{x}_h(\tau;G_0) = \frac{2}{G_0(1+\tau^2)^{1/2}}, \quad \tilde{\alpha}_h(\tau) = 2\arctan(\tau), \quad \tilde{y}_h(\tau;G_0) = \frac{2\tau}{G_0(1+\tau^2)}. \tag{1.34}$$

With this in mind, we have that taking

$$\begin{aligned}\tilde{\mathbf{z}}_0 &= \tilde{\mathbf{z}}_0(\nu,\alpha_0,G_0,s_0)\\ &= (\mathbf{z}_0(\nu,\alpha_0,G_0),s_0)\\ &= (x_h(\nu;G_0),\alpha_h(\nu;\alpha_0,G_0),y_h(\nu;G_0),G_0,s_0) \in \tilde{\gamma}\end{aligned} \tag{1.35}$$

we can write

$$\tilde{\gamma}_{\alpha_0,G_0} = \{\tilde{\mathbf{z}}_0 = (x_h(\nu;G_0),\alpha_h(\nu;\alpha_0,G_0),y_h(\nu;G_0),G_0,s_0),\ \nu\in\mathbb{R}, s_0\in\mathbb{T}\}.$$

Finally $\tilde{\gamma}$ can be seen as a union of homoclinic orbits to $\tilde{\Lambda}_\infty$ (homoclinic manifold).

$$\tilde{\gamma} = \bigcup_{\alpha_0,G_0} \tilde{\gamma}_{\alpha_0,G_0}$$

and then we can parameterize the 4-dimensional homoclinic manifold as

$$\tilde{\gamma} = W(\tilde{\Lambda}_\infty) = \{(x_h(\nu;G_0),\alpha_h(\nu;\alpha_0,G_0),y_h(\nu;G_0),G_0,s_0), \nu\in\mathbb{R}, G_0\in\mathbb{R}, (\alpha_0,s_0)\in\mathbb{T}^2\}. \tag{1.36}$$

and the motion in $\tilde{\gamma}$ is given by

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) = (x_h(\nu+t;G_0),\alpha_h(\nu+t;\alpha_0,G_0),y_h(\nu+t;G_0),G_0,s_0+t) \in \tilde{\gamma}.$$

1.3.2 The case $\mu \neq 0$

In the general case, we should note some things regarding the manifolds defined in section 1.3.1. First of all the set $\mathcal{E}$ remains invariant and, therefore, so does $\tilde{\Lambda}_\infty$, being again a "normally parabolic invariant manifold", and the periodic orbits $\tilde{\Lambda}_{\alpha_0,G_0}$ persist. The inner dynamics on $\tilde{\Lambda}_\infty$, that is the flow restricted to it is also trivial

$$(\alpha_0,G_0,s_0) \to (\alpha_0,G_0,s_0+t). \tag{1.37}$$

1.4 Melnikov potential for the parabolic orbits

From [McG73] we know that $W^s_\mu(\tilde{\Lambda}_\infty)$ and $W^u_\mu(\tilde{\Lambda}_\infty)$ exist for μ small enough and are 4-dimensional in the extended space. The classical geometric Melnikov method to find the first order approximation to the distance between the perturbed manifolds will work in this case because both manifolds have co-dimension one, and then, a normal vector to $\tilde{\gamma}$ will intersect $W^u_\mu(\tilde{\Lambda}_\infty)$ and $W^s_\mu(\tilde{\Lambda}_\infty)$ for μ small enough.

Let us take $\tilde{\mathbf{z}}_0 = (\mathbf{z}_0,s_0) = (\mathbf{z}_0(\nu,\alpha_0,G_0),s_0) \in \tilde{\gamma}$ as in (1.35). Now, we have to construct points in $W^s_\mu(\tilde{\Lambda}_\infty)$ and $W^u_\mu(\tilde{\Lambda}_\infty)$ to measure the distance between them. It is clear from the definition of $\tilde{\gamma}$ that

$$\mathbf{v} = (\nabla\mathcal{H}_0(\mathbf{z}_0),0)$$

is orthogonal to $\tilde{\gamma} = W^u(\tilde{\Lambda}_\infty) = W^s(\tilde{\Lambda}_\infty)$ and then if the normal bundle is defined by

$$N(\tilde{\mathbf{z}}_0) = \{\tilde{\mathbf{z}}_0 + \sigma\,\mathbf{v}, \sigma \in \mathbb{R}\}$$

we have that there exist unique points $\tilde{\mathbf{z}}_\mu^{u,s} = (z_\mu^{u,s}, s_0)$ such that

$$\{\tilde{\mathbf{z}}_\mu^{u,s}\} = W_\mu^{u,s}(\tilde{\Lambda}_\infty) \cap N(\tilde{\mathbf{z}}_0). \tag{1.38}$$

The distance we want to compute is the signed magnitude given by

$$d(\tilde{\mathbf{z}}_0, \mu) = \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^u) - \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^s). \tag{1.39}$$

Define (see [DdlLS06])

$$\mathcal{L}(\alpha_0, G_0, t_0; e_0) = \int_{-\infty}^{\infty} \mathcal{U}_0^*(x_h(s; G_0), \alpha_h(s; \alpha_0, G_0), s + t_0; e_0)\, ds \tag{1.40}$$

which is a convergent integral because of formulas (1.32)-(1.34), and

$$\mathcal{U}_0^*(x, \alpha, s; e_0) = O(x^2) \quad \text{as} \quad x \to \infty$$

Proposition 1.2. Given $(\alpha_0, G_0, s_0) \in \mathbb{T} \times \mathbb{R}^+ \times \mathbb{T}$ assume that the function

$$\nu \in \mathbb{R} \longrightarrow \mathcal{L}(\alpha_0, G_0, s_0 - \nu; e_0) \in \mathbb{R} \tag{1.41}$$

has a non-degenerate critical point $\nu^* = \nu^*(\alpha_0, G_0, s_0; e_0)$. Then for $0 < \mu$ small enough, there exists a locally unique point

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*(\nu^*, \alpha_0, G_0, s_0; \mu) \in W_\mu^s(\tilde{\Lambda}_\infty) \pitchfork W_\mu^u(\tilde{\Lambda}_\infty)$$

of the form

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}_0^* + O(\mu)$$

where $\tilde{\mathbf{z}}_0^* = (x_h(\nu^*; G_0), \alpha_h(\nu^*; \alpha_0, G_0), y_h(\nu^*; G_0), G_0, s_0) \in \tilde{\gamma}$. Also, there exist unique points $\tilde{\mathbf{z}}_\pm = (0, \alpha_\pm, 0, G_\pm, s_0) = (0, \alpha_0, 0, G_0, s_0) + O(\mu) \in \tilde{\Lambda}_\infty$ such that

$$\phi_{t,\mu}(\tilde{\mathbf{z}}^*) - \phi_{t,\mu}(\tilde{\mathbf{z}}_\pm) \to 0 \quad \text{for } t \to \pm\infty \tag{1.42}$$

Then, we have

$$G_+ - G_- = \mu \frac{\partial \mathcal{L}}{\partial \alpha_0}(\alpha_0, G_0, s_0 - \nu^*(\alpha_0, G_0, s_0)) + O(\mu^2).$$

Proof. From equation (1.35) we know that any point $\tilde{\mathbf{z}}_0 \in \tilde{\gamma}$ have the form

$$\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\nu, \alpha_0, G_0, s_0).$$

As we have seen in (1.38)

$$\tilde{\mathbf{z}}_\mu^{u,s} = (\tilde{z}_\mu^{u,s}, s_0) \in W_\mu^{u,s}(\tilde{\Lambda}_\infty) \cap N(\tilde{\mathbf{z}}_0).$$

We are looking for $\tilde{\mathbf{z}}_0$ such that $\tilde{\mathbf{z}}_\mu^s = \tilde{\mathbf{z}}_\mu^u$. From this, there must exist points $\tilde{\mathbf{z}}_\pm = (\tilde{z}_\pm, s_0) \in \tilde{\Lambda}_\infty$ such that

$$\phi_{t,\mu}(\tilde{\mathbf{z}}_\mu^{s,u}) - \phi_{t,\mu}(\tilde{\mathbf{z}}_\pm) \xrightarrow[t \to \pm\infty]{} 0, \tag{1.43}$$

moreover $\phi_{t,\mu}(\tilde{\mathbf{z}}_\mu^{s,u}) - \phi_{t,0}(\tilde{\mathbf{z}}_0) = O(\mu)$ (see [McG73]). Since $\mathcal{H}_0$ does not depend on time, by the chain rule we have that

$$\frac{d}{dt}\mathcal{H}_0(\phi_{t,\mu}(\tilde{\mathbf{z}}_\mu^{s,u})) = \{\{\mathcal{H}_0, \mathcal{H}_\mu\}\}(\phi_{t,\mu}(\tilde{\mathbf{z}}_\mu^{s,u})) = \mu\{\{\mathcal{H}_0, \mathcal{U}_\mu^*\}\}(\phi_{t,\mu}(\tilde{\mathbf{z}}_\mu^{s,u})).$$

Since $\mathcal{H}_0 = 0$ in $\tilde{\Lambda}_\infty$, using (1.43) and the trivial dynamics on $\tilde{\Lambda}_\infty$ we obtain

$$\mathcal{H}_0(\tilde{\mathbf{z}}_\mu^{s,u}) = -\mu \int_0^{\pm\infty} \{\{\mathcal{H}_0, \mathcal{U}_\mu^*\}\}(\phi_{s,\mu}(\tilde{\mathbf{z}}_\mu^{s,u}))\, ds.$$

Finally, using Taylor's series in μ,

$$\begin{aligned}\mathcal{H}_0(\tilde{\mathbf{z}}_\mu^u) - \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^s) &= \mu \int_{-\infty}^{\infty} \{\{\mathcal{H}_0, \mathcal{U}_0^*\}\}(\phi_{t,0}(\tilde{\mathbf{z}}_0))\, dt + O(\mu^2) \\ &= \mu \int_{-\infty}^{\infty} \{\{\mathcal{H}_0, \mathcal{U}_0^*\}\}(x_h(\nu + t; G_0), \alpha_h(\nu + t; \alpha_0, G_0), y_h(\nu + t; G_0), G_0, s_0 + t)\, dt + O(\mu^2)\end{aligned}$$

On the other hand, from (1.40),

$$\mathcal{L}(\alpha_0, G_0, t_0; e_0) = \int_{-\infty}^{\infty} \mathcal{U}_0^*(x_h(s - t_0; G_0), \alpha_h(s - t_0; \alpha_0, G_0), s; e_0)\, ds$$

and then

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial t_0}(\alpha_0, G_0, t_0; e_0) &= -\int_{-\infty}^{\infty} \{\{\mathcal{U}_0^*, \mathcal{H}_0\}\}(x_h(s - t_0; G_0), \alpha_h(s - t_0; \alpha_0, G_0), y_h(s - t_0; G_0), G_0, s)\, ds \\ &= \int_{-\infty}^{\infty} \{\{\mathcal{H}_0, \mathcal{U}_0^*\}\}(x_h(s - t_0; G_0), \alpha_h(s - t_0; \alpha_0, G_0), y_h(s - t_0; G_0), G_0, s)\, ds\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial t_0}(\alpha_0, G_0, s_0 - \nu; e_0) &= \int_{-\infty}^{\infty} \{\{\mathcal{H}_0, \mathcal{U}_0^*\}\}(x_h(s - s_0 + \nu; G_0), \alpha_h(s - s_0 + \nu; \alpha_0, G_0), y_h(s - s_0 + \nu; G_0), G_0, s)\, ds \\ &= \int_{-\infty}^{\infty} \{\{\mathcal{H}_0, \mathcal{U}_0^*\}\}(x_h(t + \nu; G_0), \alpha_h(t + \nu; \alpha_0, G_0), y_h(t + \nu; G_0), G_0, s_0 + t)\, dt\end{aligned}$$

and therefore

$$d(\tilde{\mathbf{z}}_0, \mu) = \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^u) - \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^s) = \mu \frac{\partial \mathcal{L}}{\partial t_0}(\alpha_0, G_0, s_0 - \nu; e_0) + O(\mu^2)$$

Now, it is clear by the implicit function theorem, for μ small enough, that a non degenerate critical value ν^* of the function (1.41) gives rise to homoclinic points to $\tilde{\Lambda}_\infty$ where the manifolds $W_\mu^s(\tilde{\Lambda}_\infty)$ and $W_\mu^u(\tilde{\Lambda}_\infty)$ intersect transversally that have the desired form $\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}_0^* + O(\mu)$.

Consider now the solution of the system (1.19) represented by $\phi_{t,\mu}(\tilde{\mathbf{z}}^*)$. Moreover, by the fundamental theorem of calculus and the definition (1.18) we have

$$\begin{aligned}G_+ - G_- &= -\int_{-\infty}^{\infty} \frac{\partial \mathcal{H}_\mu}{\partial \alpha}(\phi_{t,\mu}(\tilde{\mathbf{z}}^*))\, dt = \int_{-\infty}^{\infty} \frac{\partial \mathcal{U}_\mu}{\partial \alpha}(\phi_{t,\mu}(\tilde{\mathbf{z}}^*))\, dt \\ &= \mu \int_{-\infty}^{\infty} \frac{\partial \mathcal{U}_0^*}{\partial \alpha}(\phi_{t,0}(\tilde{\mathbf{z}}_0^*))\, dt + O(\mu^2) \\ &= \mu \int_{-\infty}^{\infty} \frac{\partial \mathcal{U}_0^*}{\partial \alpha}(x_h(\nu^* + t; G_0), \alpha_h(\nu^* + t; \alpha_0, G_0), y_h(\nu^* + t; G_0), G_0, s_0 + t)\, dt + O(\mu^2) \\ &= \mu \frac{\partial \mathcal{L}}{\partial \alpha}(\alpha_0, G_0, s_0 - \nu^*; e_0) + O(\mu^2).\end{aligned}$$

□

Once we have found a critical point ν^* of (1.41) we define the reduced Poincaré function (see [DdlLS06])

$$\mathcal{L}^*(\alpha_0, G_0; e_0) := \mathcal{L}(\alpha_0, G_0, s_0 - \nu^*; e_0) \tag{1.44}$$

The scattering map

The scattering map S is defined from the manifold $\tilde{\Lambda}_\infty$ (defined in (1.29)) to itself. Take $\tilde{\mathbf{z}}_-, \tilde{\mathbf{z}}_+ \in \tilde{\Lambda}_\infty$, then

$$S_\mu(\tilde{\mathbf{z}}_-) = \tilde{\mathbf{z}}_+$$

if there exist $\tilde{\mathbf{z}}^* \in W^u_\mu(\tilde{\Lambda}_\infty) \pitchfork W^s_\mu(\tilde{\Lambda}_\infty)$ such that

$$\phi_{t,\mu}(\tilde{\mathbf{z}}^*) - \phi_{t,\mu}(\tilde{\mathbf{z}}_\pm) \to 0 \quad \text{for } t \to \pm\infty. \tag{1.45}$$

In the case $\mu = 0$ we have that $\tilde{\gamma} = W^u(\tilde{\Lambda}_\infty) = W^s(\tilde{\Lambda}_\infty)$ implies that the scattering map S_0 is the identity. Indeed, for a generic point

$$\tilde{x}_0 = (0, \alpha_0, 0, G_0, s_0) \in \tilde{\Lambda}_\infty$$

we have $S_0(\tilde{x}_0) = \tilde{x}_0$. To see this, take

$$\tilde{\mathbf{z}}_0 = (x_h(\nu; G_0), \alpha_h(\nu; \alpha_0), y_h(\nu; G_0), G_0, s_0) \in \tilde{\gamma}$$

then by equations (1.32), (1.33) and (1.34)

$$\begin{aligned}&\phi_{t,0}(\tilde{\mathbf{z}}_0) - \phi_{t,0}(\tilde{x}_0) = \\ &\qquad (x_h(t+\nu; G_0), \alpha_h(t+\nu; \alpha_0), y_h(t+\nu; G_0), G_0, t+s_0) - (0, \alpha_0, 0, G_0, t+s_0) \xrightarrow[t\to\pm\infty]{} 0\end{aligned}$$

wich proves that $S_0 = Id$.

The next proposition gives an approximation of the scattering map in the general case $\mu \neq 0$

Proposition 1.3. The scattering map S_μ associated to the non degenerate critical point ν^* of the function defined in (1.41) is given by

$$(0, \alpha_-, 0, G_-, s_-) \longmapsto \Big(0, \alpha_- + \mu\frac{\partial \mathcal{L}^*}{\partial G}(\alpha_-, G_-; e_0) + O(\mu^2), 0, G_- - \mu\frac{\partial \mathcal{L}^*}{\partial \alpha}(\alpha_-, G_-; e_0) + O(\mu^2), s_-\Big)$$

where $\mathcal{L}^*$ is the Poincaré reduced function introduced in (1.44).

Proof. By hypothesis we have a non degenerate critical point ν^* of (1.41), by using definition (1.44) the proposition 1.2 gives the correspondence we look for between G_- and G_+. Finally, the equation

$$\alpha_+ - \alpha_- = -\mu\frac{\partial \mathcal{L}^*}{\partial G}(\alpha_0, G_0, t_0^*) + O(\mu^2)$$

is a direct consequence of the fact that $\mathcal{L}^*$ is symplectic, as shown in [DdlLS08].

Since we know that for all time,

$$\phi_{t,\mu}(\tilde{\mathbf{z}}^*) = \phi_{t,0}(\tilde{\mathbf{z}}_0^*) + O(\mu),$$

denoting $G(\phi_{t,\mu}(\tilde{\mathbf{z}}^*))$ and $\alpha(\phi_{t,\mu}(\tilde{\mathbf{z}}^*))$ the G and α coordinate of $\phi_{t,\mu}(\tilde{\mathbf{z}}^*)$ we have

$$\begin{aligned}G_\pm &= \lim_{t\to\pm\infty} G(\phi_{t,\mu}(\tilde{\mathbf{z}}^*)) = G_0 + O(\mu)\\ \alpha_\pm &= \lim_{t\to\pm\infty} \alpha(\phi_{t,\mu}(\tilde{\mathbf{z}}^*)) = \alpha_0 + O(\mu).\end{aligned}$$

Using that $G_- = G_0 + O(\mu)$ and $\alpha_- = \alpha_0 + O(\mu)$ we get the required formula which completes the proof. □

Next proposition concerns the circular case $e_0 = 0$

Proposition 1.4. If $e_0 = 0$ and $\nu^* = \nu^*(\alpha_0, G_0, s_0) \in \mathbb{R}$ is such that $\partial\mathcal{L}/\partial t_0\,(\alpha_0, G_0, s_0 - \nu^*; 0) = 0$ then

$$\frac{\partial \mathcal{L}}{\partial \alpha_0}(\alpha_0, G_0, s_0 - \nu^*; 0) = 0,$$

that is

$$\frac{\partial \mathcal{L}^*}{\partial \alpha_0}(\alpha_0, G_0; 0) = 0.$$

Proof. From the equation (1.40) we have that

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial t_0}(\alpha_0, G_0, s_0 - \nu^*; e_0) \\
&= \int_{-\infty}^{\infty} \frac{\partial \mathcal{U}_0^*}{\partial t}(x_h(s; G_0), \alpha_h(s; \alpha_0, G_0), y_h(s; G_0), G_0, s + s_0 - \nu^*; e_0)\, ds. \qquad (1.46)
\end{aligned}
$$

on the other hand, in the circular case $e_0 = 0$, formulas (1.4) and (1.6) give $r_0 = 1$ and $f = t$ in the self potential (1.17), so that $\mathcal{U}_\mu$ only depends on α and t through the combination $\alpha - t$ and therefore

$$
\frac{\partial \mathcal{U}_\mu^*}{\partial \alpha}(x, \alpha, t; 0) = -\frac{\partial \mathcal{U}_\mu^*}{\partial t}(x, \alpha, t; 0).
$$

Then equation (1.46) reads

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial t_0}(\alpha_0, G_0, s_0 - \nu^*; 0) \\
&= -\int_{-\infty}^{\infty} \frac{\partial \mathcal{U}_0^*}{\partial \alpha}(x_h(s; G_0), \alpha_h(s; \alpha_0, G_0), y_h(s; G_0), G_0, s + s_0 - \nu^*; 0)\, ds \\
&= -\frac{\partial \mathcal{L}}{\partial \alpha_0}(\alpha_0, G_0, s_0 - \nu^*; e_0) \quad \text{(by equation (1.32b) and the chain rule)}\ .
\end{aligned}
$$

□

By this proposition, if there exists a heteroclinc connection in the circular case, between two periodic orbits $\tilde{\Lambda}_{\alpha_-, G_-}$ and $\tilde{\Lambda}_{\alpha_+, G_+}$ in $\tilde{\Lambda}_\infty$ introduced in (1.28), $G_+ = G_- + O(\mu^2)$ by proposition 1.3. But indeed $G_+ = G_-$ in the circular case, since there exists the first integral provided by the Jacobi constant $C_J = \mathcal{H}_\mu + G$ and as $\mathcal{H}_\mu = 0$ on $\tilde{\Lambda}_{\alpha_-, G_-}$ and $\tilde{\Lambda}_{\alpha_+, G_+}$, $G_+ = G_-$. Therefore in the circular case there is no possibility to find diffusive orbits studying the intersection of $W_\mu^s(\tilde{\Lambda}_\infty)$ and $W_\mu^u(\tilde{\Lambda}_\infty)$ since by proposition 1.3 the angular momentum remains constant.

From the definition of $\mathcal{L}$ given in (1.40) and equation (1.24) we get

$$
\begin{aligned}
\mathcal{L}(\alpha_0, G_0, t_0; e_0) = \int_{-\infty}^{\infty} \Bigg[& \frac{x_h^2}{\left[4 + x_h^4 [r_0(t)]^2 + 4 x_h^2 [r_0(t)] \cos(\alpha_h - f)\right]^{1/2}} \\
& + \left(\frac{x_h^2}{2}\right)^2 [r_0(t)] \cos(\alpha_h - f) - \frac{x_h^2}{2} \Bigg] dt \qquad (1.47)
\end{aligned}
$$

where x_h and α_h are coordinates of the homoclinic orbit defined in (1.32) whereas r_0 and f defined in (1.4) and (1.5) and are evaluated at $t + t_0$.

The computation of Melnikov potential (1.47) will be done in chapter 2. Such computation will be done in two different ways, corresponding to whether the parameter $\lambda = e_0 G_0$ is small or not. The next two theorems correspond to these two cases.

Theorem 1.5. *If $G_0 \geq 32$, $e_0 G_0 < 1/8$, then there exists a positive constant K such that the Melnikov potential $\mathcal{L}$ given by* (1.47) *satisfies*

$$
\begin{aligned}
\mathcal{L} = L_0(\alpha_0) - \cos(t_0 - \alpha_0)\sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} \left(1 + \tilde{E}_1\right) - \cos(t_0 - 2\alpha_0) 3\sqrt{2\pi} e_0 G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}} \left(1 + \tilde{E}_2\right) \\
+ 2\Re\{\tilde{E}_3(\alpha_0) \mathrm{e}^{i t_0}\} + \tilde{E}_4(t_0, \alpha_0)
\end{aligned}
$$

with

$$
\begin{aligned}
|\tilde{E}_1| &\leq K(G_0^{-1} + e_0^2) \\
|\tilde{E}_2| &\leq K(G_0^{-1} + e_0)
\end{aligned}
$$

$$|\tilde{E}_3(\alpha_0)| \le K\mathrm{e}^{-\frac{G_0^3}{3}}\left[(1+e_0)^4 G_0^{-7/2} + e_0^2 G_0^{5/2} + e_0 G_0^{-3/2}\right]$$
$$|\tilde{E}_4(t_0,\alpha_0)| \le K G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}$$

for some positive constant K and

$$L_0(\alpha_0) - L_{0,0} = \frac{15}{8}\pi e_0 G_0^{-5}\cos(\alpha_0) + \tilde{F}_1 + \tilde{F}_2$$

where

$$|\tilde{F}_1| \le S e_0 G_0^{-9}$$
$$|\tilde{F}_2| \le S e_0^2 G_0^{-5}$$

for some positive constant S. And

$$L_{0,0} = \frac{\pi}{2} G_0^{-3} + F_1 + F_4$$

with

$$|F_1| \le S G_0^{-7}$$
$$|F_4| \le S G_0^{-3} e_0^2$$

Theorem 1.6. *Let λ be a real positive constant and $c \ge 1$. If*

$$G_0 \ge \max\{(3c)^{2/3}, 32, 8\lambda^{-1}, 3\lambda^{1/3}, \lambda^4\},$$

then there exists a positive constant K, depending on λ, such that if $e_0 G_0 = \lambda$, the Melnikov potential $\mathcal{L}$ given by (2.7) satisfy

$$\begin{aligned}\mathcal{L} = {} & L_0(\alpha_0) + \cos(t_0-\alpha_0)\sqrt{\frac{\pi}{8}} G_0^{-1/2}\mathrm{e}^{-G_0^3/3}(1+\tilde{E}_1) \\ & - \mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1} G_0^{1/2}\Re\Bigg\{\Bigg[\mathrm{e}^{-\lambda \mathrm{e}^{-i\alpha_0}}\frac{A}{1-A}\Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}} J_1(\pm 2i\sqrt{A(A-1)}) \\ & \qquad - J_0(\pm 2i\sqrt{A(A-1)})\Big] + A\Bigg]\mathrm{e}^{it_0}(1+\mathcal{R}_1(\alpha_0))\Bigg\} + \mathcal{R}_3(\alpha_0, G_0, t_0)\end{aligned}$$

with

$$A = \frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}$$
$$\frac{A}{A-1} = \frac{\lambda^2 - 2\lambda \mathrm{e}^{-i\alpha_0}}{\lambda^2 - 4\lambda\cos\alpha_0 + 4}$$
$$A(A-1) = -\frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}\Big(1 - \frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}\Big).$$

and

$$|\tilde{E}_1| \le K(G_0^{-1} + \lambda^2 G_0^{-2})$$
$$|\mathcal{R}_1(\alpha_0)| \le K G_0^{-1}$$
$$|\mathcal{R}_3(\alpha_0, G_0, t_0)| \le K G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}},$$

$$L_0(\alpha_0) - L_{0,0} = -\frac{15}{8}\pi e_0 G_0^{-5}\cos(\alpha_0) + \tilde{F}_1 + \tilde{F}_2$$

with

$$|\tilde{F}_1| \le K e_0 G_0^{-9}$$

$$|\tilde{F}_2| \leq K e_0^2 G_0^{-5},$$

and

$$L_{0,0} = \frac{\pi}{2} G_0^{-3} + F_1 + F_4$$

with

$$|F_1| \leq K G_0^{-7}$$
$$|F_4| \leq K G_0^{-3} e_0^2$$

The functions $J_0(z)$ and $J_1(z)$ are the Bessel's functions of the first kind [AS65] and whose expansion around $z = 0$ is given by

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{z}{2}\right)^{2m+n}$$

Corollary 1.7. If $\lambda = e_0 G_0$ is small, theorem 1.6 recovers the asymptotic expression found for the Melnikov potential $\mathcal{L}$ in theorem 1.5.

Proof. The first two terms in the expression for $\mathcal{L}$ in both theorems 1.5 and 1.6, coincides.

From the definition of A we have that $A = O(\lambda)$ and also $A(A-1) = O(\lambda)$, therefore looking for asymptotics for λ small is equivalent to look for asymptotics for A small. Using the asymptotics for the Bessel's functions J_0 and J_1, given in theorem 1.6 we have

$$\begin{aligned}
\mathrm{e}^{\lambda \mathrm{e}^{-i\alpha_0}} &= \mathrm{e}^{-2A} = 1 - 2A + O(A^2) \\
J_1(\pm 2i\sqrt{A(A-1)}) &= \pm i\sqrt{A(A-1)} + O([A(A-1)]^{3/2}) \\
J_0(\pm 2i\sqrt{A(A-1)}) &= 1 + A^2 - A + O([A(A-1)]^2) \\
\frac{A}{1-A} &= A + O(A^2)
\end{aligned}$$

therefore, we can write the third term as follows and get the asymptotic for λ small

$$\begin{aligned}
&- \mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1} G_0^{1/2} \Re\Bigg\{ \Bigg[\mathrm{e}^{-\lambda \mathrm{e}^{-i\alpha_0}} \frac{A}{1-A} \Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}} J_1(\pm 2i\sqrt{A(A-1)}) \\
&\qquad\qquad - J_0(\pm 2i\sqrt{A(A-1)}) \Big] + A \Bigg] \mathrm{e}^{it_0}(1 + \mathcal{R}_1(\alpha_0)) \Bigg\} \\
&= -\mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1} G_0^{1/2} \Re\Bigg\{ \mathrm{e}^{-2A} \frac{A}{1-A} \Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}} J_1(\pm 2i\sqrt{A(A-1)}) \\
&\qquad\qquad - J_0(\pm 2i\sqrt{A(A-1)}) + (1-A)\mathrm{e}^{2A} \Big] \mathrm{e}^{it_0}(1 + \mathcal{R}_1(\alpha_0)) \Bigg\} \\
&= -\mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1} G_0^{1/2} \Re\Bigg\{ \mathrm{e}^{-2A} \frac{A}{1-A} \Big[A + O(A^2) - 1 - A^2 + A + O(A^2) \\
&\qquad\qquad + (1-A)\mathrm{e}^{2A} \Big] \mathrm{e}^{it_0}(1 + \mathcal{R}_1(\alpha_0)) \Bigg\} = (*)
\end{aligned}$$

since

$$(1-A)\mathrm{e}^{2A} = 1 - A + (1-A)2A + (1-A)O(A^2) = 1 - A + 2A - 2A^2 + O(A^2) = 1 + A + O(A^2)$$

we have

$$(*) = -\mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1} G_0^{1/2} \Re\Bigg\{ \mathrm{e}^{-2A} \frac{A}{1-A} \Big[3A + O(A^2) \Big] \mathrm{e}^{it_0}(1 + \mathcal{R}_1(\alpha_0)) \Bigg\}$$

also, it is clear from the given asymptotics that

$$\mathrm{e}^{-2A}\frac{A}{1-A} = A + O(A^2)$$

and then

$$(*) = -\mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1}G_0^{1/2}\Re\left\{\left[3A^2 + O(A^3)\right]\mathrm{e}^{it_0}(1+\mathcal{R}_1(\alpha_0))\right\}$$

and, from its definition $A^2 = (\lambda^2/4)\mathrm{e}^{-2i\alpha_0}$ we have

$$\begin{aligned}
(*) &= -\mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1}G_0^{1/2}\Re\left\{\left[\frac{3}{4}\lambda^2\mathrm{e}^{i(t_0-2\alpha_0)} + O(A^3)\right](1+\mathcal{R}_1(\alpha_0))\right\} \\
&= -\mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1}G_0^{1/2}\Re\left\{\frac{3}{4}\lambda^2\mathrm{e}^{i(t_0-2\alpha_0)} + O(\lambda^2)(\lambda+\mathcal{R}_1(\alpha_0))\right\} \\
&= -\mathrm{e}^{-\frac{G_0^3}{3}} 3\sqrt{2\pi}e_0G_0^{3/2}\cos(t_0-2\alpha_0)\left[1+\Re\left\{O(\lambda)(G_0^{1/2}\lambda+\mathcal{R}_1(\alpha_0))\right\}\right] \\
&= -\mathrm{e}^{-\frac{G_0^3}{3}} 3\sqrt{2\pi}e_0G_0^{3/2}\cos(t_0-2\alpha_0)\left[1+O\left(e_0G_0^{3/2}(e_0G_0+G_0^{-3/2})\right)\right]
\end{aligned}$$

which is exactly the second term in the expression for the Melnikov potential $\mathcal{L}$ given in theorem 1.5. □

1.5 Global diffusion

1.5.1 $e_0G_0 = \lambda \ll 1$

To prove diffusion in the case $\lambda_0 = e_0G_0 \ll 1$ we will use propositions 1.2 and 1.3 to construct a suitable scattering map using the computation of the Melnikov potential given in theorem 1.5. From this theorem, we will introduce the next notation. The Melnikov potential is given by

$$\mathcal{L}(\alpha_0, G_0, t_0; e_0) = L_{0,0} + \mathcal{L}_0(\alpha_0, G_0; e_0) + \mathcal{L}_1(\alpha_0, G_0, t_0; e_0) + \mathcal{E}(\alpha_0, G_0, t_0; e_0) \tag{1.48}$$

where $L_{0,0}$ is given in theorem 1.5 as

$$L_{0,0} = \frac{\pi}{2}G_0^{-3} + \mathcal{F}_1 \tag{1.49}$$

with

$$\mathcal{F}_1 = \mathcal{F}_1(G_0; e_0) = O(e_0^2G_0^{-3} + G_0^{-7}) \tag{1.50}$$

and,

$$\mathcal{L}_0(\alpha_0, G_0; e_0) = -\frac{15}{8}\pi e_0 G_0^{-5}\cos(\alpha_0) + \mathcal{F}(\alpha_0, G_0; e_0) \tag{1.51}$$

$$\begin{aligned}
\mathcal{L}_1(\alpha_0, G_0, t_0; e_0) = \cos(t_0-\alpha_0)&\sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}\left(1+\tilde{E}_1\right) \\
&- \cos(t_0-2\alpha_0)3\sqrt{2\pi}e_0G_0^{3/2}\mathrm{e}^{-\frac{G_0^3}{3}}\left(1+\tilde{E}_2\right) + 2\Re\{\tilde{E}_3(\alpha_0)\mathrm{e}^{it_0}\}
\end{aligned} \tag{1.52}$$

$\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$ are bounded in theorem 1.5 and

$$\mathcal{E}(\alpha_0, G_0, t_0; e_0) = O(G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}) \tag{1.53}$$

$$\mathcal{F}(\alpha_0, G_0; e_0) = O(e_0^2G_0^{-5}, e_0G_0^{-9}) \tag{1.54}$$

Lemma 1.8. Let $\mathcal{L}_1$ be defined in (1.52) and $p = 12e_0G_0^2 = 12\lambda G_0$. If $P^2 = 1-2p\cos\alpha_0+p^2 \neq 0$, then

$$\mathcal{L}_1(\alpha_0, G_0, t_0; e_0) = \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}B\,\cos(t_0-\alpha_0-\theta)$$

where $\theta = \theta(\alpha_0, G_0; e_0) \in (-\pi, \pi]$ and $B = B(\alpha_0, G_0; e_0)$ satisfy

$$B^2 = P^2 + \tilde{B}$$
$$\tan\theta = \frac{p\sin\alpha_0 + \Im(\tilde{B}_1)}{1-p\cos\alpha_0}\Big(1 + O\big(\frac{\Re(\tilde{B}_1)}{1-p\cos\alpha_0}\big)\Big)$$

with

$$|\tilde{B}| \leq K[G_0^{-1} + e_0G_0(1+p+p^2)]$$
$$|\tilde{B}_1| \leq K[G_0^{-1} + e_0G_0(1+p)]$$

Proof. From the definition of $\mathcal{L}_1$ given in (1.52) we can write, defining $p = 12e_0G_0^2$

$$\mathcal{L}_1 = \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}\Big[\cos(t_0-\alpha_0)(1+\tilde{E}_1) - p\cos(t_0-2\alpha_0)(1+\tilde{E}_2) + \Re\{\widehat{E}_3(\alpha_0)\mathrm{e}^{it_0}\}\Big]$$

where

$$\widehat{E}_3(\alpha_0) = \sqrt{\frac{8}{\pi}}G_0^{1/2}\mathrm{e}^{\frac{G_0^3}{3}}2\tilde{E}_3(\alpha_0)$$

and then, by the bounds in theorem 1.5 we have

$$|\widehat{E}_3(\alpha_0)| \leq K\big[(1+e_0)^4G_0^{-3} + e_0^2G_0^3 + e_0G_0^{-1}\big],$$

so

$$\begin{aligned}\mathcal{L}_1 &= \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}\Big[\Re\big(\mathrm{e}^{i(t_0-\alpha_0)}\big)(1+\tilde{E}_1) - p\Re\big(\mathrm{e}^{i(t_0-2\alpha_0)}\big)(1+\tilde{E}_2) + \Re\{\widehat{E}_3(\alpha_0)\mathrm{e}^{it_0}\}\Big]\\ &= \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}\Re\Big(\mathrm{e}^{i(t_0-\alpha_0)}\big(1+\tilde{E}_1 - p(1+\tilde{E}_2)\mathrm{e}^{-i\alpha_0} + \widehat{E}_3(\alpha_0)\mathrm{e}^{i\alpha_0}\big)\Big)\end{aligned}$$

if we write now

$$1+\tilde{E}_1 - p(1+\tilde{E}_2)\mathrm{e}^{-i\alpha_0} + \widehat{E}_3(\alpha_0)\mathrm{e}^{i\alpha_0} = B\mathrm{e}^{-i\theta} \tag{1.55}$$

we have that

$$\begin{aligned}\mathcal{L}_1 &= \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}B\Re\Big(\mathrm{e}^{i(t_0-\alpha_0-\theta)}\Big)\\ &= \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}B\cos(t_0-\alpha_0-\theta).\end{aligned}$$

Let us find B and θ. From equation (1.55), we have

$$B\mathrm{e}^{-i\theta} = 1 - p\mathrm{e}^{-i\alpha_0} + \tilde{B}_1 \tag{1.56}$$

where

$$\tilde{B}_1 = \tilde{E}_1 - p\tilde{E}_2\mathrm{e}^{-i\alpha_0} + \widehat{E}_3(\alpha_0)\mathrm{e}^{i\alpha_0}$$

$|\tilde{B}_1| \leq K[G_0^{-1} + e_0G_0(1+p)]$. Therefore

$$B^2 = |1-p\mathrm{e}^{-i\alpha_0}|^2 + \tilde{B}$$

where $\tilde{B} = (1-p\mathrm{e}^{-i\alpha_0})\overline{\tilde{B}_1} + \tilde{B}_1(1-p\mathrm{e}^{i\alpha_0}) + |\tilde{B}_1|^2$, then using the bounds for $\tilde{B}_1$ and the definition of p, $|\tilde{B}| \leq K[G_0^{-1} + e_0G_0(1+p+p^2)]$.

$$
\begin{aligned}
B^2 &= 1 - p(\mathrm{e}^{-i\alpha_0} + \mathrm{e}^{i\alpha_0}) + p^2 + \tilde{B} \\
&= 1 - 2p\cos\alpha_0 + p^2 + \tilde{B} \\
&= P^2 + \tilde{B}
\end{aligned} \tag{1.57}
$$

assuming that $1 - p\cos\alpha_0 \neq 0$ we can see that $\tilde{B}_1/(1 - p\cos\alpha_0)$ is always small and therefore from (1.56)

$$
\begin{aligned}
\tan\theta &= \frac{p\sin\alpha_0 + \Im(\tilde{B}_1)}{1 - p\cos\alpha_0 + \Re(\tilde{B}_1)} \\
&= \frac{p\sin\alpha_0 + \Im(\tilde{B}_1)}{1 - p\cos\alpha_0}\Big(1 + O\big(\frac{\Re(\tilde{B}_1)}{1 - p\cos\alpha_0}\big)\Big)
\end{aligned}
$$

□

Remark 1.9. Under the assumptions of lemma 1.8, if $p = 1$ and $\cos\alpha_0 = -1$ the angle θ is not well defined, but this case corresponds to $B = 0$.

By proposition 1.2 we need to find critical points of the function $t_0 \longmapsto \mathcal{L}(\alpha_0, G_0, t_0; e_0)$, to this end we will check that $t_0 \longmapsto \mathcal{L}(\alpha_0, G_0, t_0; e_0)$ is a **cosine-like** function, that is, with a non-degenerate maximum (minimum) and no other critical points. By equation (1.48) and the bound (1.53), for G_0 big enough, the critical points in the variable t_0 are well approximated by the critical points of the function $\mathcal{L}$ and therefore will be close to $t_0 - \alpha_0 - \theta = 0, \pi(mod 2\pi)$ thanks to lemma 1.8. For this purpose, we introduce

$$
\mathcal{L}_1^* = \mathcal{L}_1^*(\alpha_0, G_0; e_0) = \sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} B \tag{1.58}
$$

where $B = B(\alpha_0, G_0; e_0)$ is given in lemma 1.8. With this notation the function $\mathcal{L}_1$ of lemma 1.8 can be written as

$$
\mathcal{L}_1(\alpha_0, G_0, t_0; e_0) = \mathcal{L}_1^*(\alpha_0, G_0; e_0)\, \cos(t_0 - \alpha_0 - \theta). \tag{1.59}
$$

First a technical lemma.

Lemma 1.10. Let $\mathcal{E}$ be the error function defined by (1.48) and $\mathcal{L}_1^*$ be defined in (1.58). If $G_0 \gg 1$, $e_0 G_0 \ll 1$ and

$$
16K(G_0^{-1} + e_0 G_0) < \kappa^2 < \Big(1 - \frac{\pi^2}{16}\Big)^2
$$

for $|p - 1| \geq 1$ or $\alpha_0 \in [\kappa, 2\pi - \kappa]$. Then

$$
\Big[\Big(\frac{\partial\mathcal{E}}{\partial t_0}\Big)^2 + \Big(\frac{\partial^2\mathcal{E}}{\partial t_0^2}\Big)^2\Big] / (\mathcal{L}_1^*)^2 < \frac{2K}{\kappa^2} G_0^4 \mathrm{e}^{-G_0^3 \frac{2}{9}}
$$

Proof. Since the Melnikov potential $\mathcal{L}$ defined in (1.47) and rewritten in (1.48) is 2π-periodic in t_0 we have that except for a constant $\partial\mathcal{E}/\partial t_0$ and $\partial^2\mathcal{E}/\partial t_0^2$ have similar bounds to the bound of $\mathcal{E}$, given indirectly in (1.53), therefore for some positive constant K

$$
\Big[\Big(\frac{\partial\mathcal{E}}{\partial t_0}\Big)^2 + \Big(\frac{\partial^2\mathcal{E}}{\partial t_0^2}\Big)^2\Big] / (\mathcal{L}_1^*)^2 < K \frac{G_0^3 \mathrm{e}^{-G_0^3 \frac{8}{9}}}{G_0^{-1} B^2 \mathrm{e}^{-G_0^3 \frac{2}{3}}} = \frac{K}{B^2} G_0^4 \mathrm{e}^{-G_0^3 \frac{2}{9}}
$$

it remains to show that $1/B^2$ can be bounded by $2/\kappa^2$. From the expression for B^2 given in (1.57) and the triangle inequality we have

$$
B^2 \geq P^2 - |\tilde{B}| \geq (p - 1)^2 - |\tilde{B}|. \tag{1.60}
$$

Let $\kappa \in (0,1)$ and $\omega = 1-\kappa$, $\sigma = \kappa + 1$. We have three different cases; if $p \geq \sigma > 1$, $p \leq \omega < 1$ and $\omega < p < \sigma$. We know from lemma 1.8 that

$$|\tilde{B}| \leq K[G_0^{-1} + e_0 G_0(1+p+p^2)] \tag{1.61}$$

then, when $p \geq \sigma = 1+\kappa > 1$ we have that

$$\frac{p}{p-1} \leq \frac{1+\kappa}{\kappa}$$

and then, from (1.61) that

$$|\tilde{B}| \leq 4Ke_0G_0p^2 \leq Ke_0G_0\frac{(p-1)^2}{2}2\Big(\frac{1+\kappa}{\kappa}\Big)^2$$

if we now choose

$$\kappa^2 \geq 16Ke_0G_0 \tag{1.62}$$

we have

$$|\tilde{B}| \leq \frac{(p-1)^2}{2}$$

and then from (1.60)

$$B^2 \geq \frac{(p-1)^2}{2}$$

or equivalently

$$\frac{1}{B^2} \leq \frac{2}{(p-1)^2} \leq \frac{2}{\kappa^2}.$$

When $p \leq \omega = 1-\kappa < 1$, from (1.61) that

$$|\tilde{B}| \leq 3K[G_0^{-1} + e_0G_0]$$

if we now choose

$$\kappa^2 \geq 6K(G_0^{-1} + e_0G_0) \tag{1.63}$$

we have

$$|\tilde{B}| \leq \frac{\kappa^2}{2}$$

and then from (1.60)

$$B^2 \geq (p-1)^2 - \frac{\kappa^2}{2} \geq \frac{\kappa^2}{2}$$

or equivalently

$$\frac{1}{B^2} \leq \frac{2}{\kappa^2}.$$

When $1-\kappa = \omega < p < \sigma = 1+\kappa$, from (1.61) we have

$$|\tilde{B}| \leq 7K[G_0^{-1} + e_0G_0]. \tag{1.64}$$

The function P^2 in (1.57) can be written as

$$P^2(p) = (p - \cos\alpha_0)^2 + (1 - cos^2\alpha_0) \geq 0$$

or more conveniently as

$$P^2 = (p-1)^2 + 2p(1-\cos\alpha_0) \geq 2p(1-\cos\alpha_0) \geq 2(1-\kappa)(1-\cos\alpha_0).$$

Restricting α_0 to the interval $[\kappa, 2\pi - \kappa]$ so that

$$1 - \cos\alpha_0 \geq 1 - \cos\kappa = 2\sin(\kappa^2/2) \geq 8(\kappa^2/\pi^2),$$

if we choose

$$16K(G_0^{-1} + e_0 G_0) < \kappa^2 < \Big(1 - \frac{\pi^2}{16}\Big)^2 \tag{1.65}$$

we have from (1.60) and (1.64) that

$$B^2 \geq 2\frac{\pi^2}{16}\frac{8}{\pi^2}\kappa^2 - \frac{\kappa^2}{2} = \frac{\kappa^2}{2}$$

or equivalently

$$\frac{1}{B^2} \leq \frac{2}{\kappa^2}.$$

Summarizing, for any κ verifying condition (1.65) one has that $1/B^2 \leq 2/\kappa^2$ if $|p-1| \geq 1$ or $\alpha_0 \in [\kappa, 2\pi - \kappa]$.

□

Proposition 1.11. Let $\mathcal{L}$ be the Melnikov potential given in (1.48) and G_0, e_0, α_0 and κ as in lemma 1.10. Then $t_0 \longmapsto \mathcal{L}(\alpha_0, G_0, t_0; e_0)$ is a **cosine-like** function, and its the critical point are given by

$$t_{0,\pm}^* = t_{0,\pm}^*(\alpha_0, G_0; e_0) = \varphi_\pm^* + \alpha_0 + \theta$$

and

$$\varphi_\pm^* = O(G_0^2 \mathrm{e}^{-G_0^3/9})$$

Proof. We look for critical points of $t_0 \longmapsto \mathcal{L}(\alpha_0, G_0, t_0; e_0)$

$$\frac{\partial \mathcal{L}}{\partial t_0} = \frac{\partial \mathcal{L}_1}{\partial t_0} + \frac{\partial \mathcal{E}}{\partial t_0} = 0 \tag{1.66}$$

or equivalently, using the formula given in equation (1.102),

$$\sin(\varphi) = f(\varphi) := \frac{1}{\mathcal{L}_1^*}\frac{\partial \mathcal{E}}{\partial t_0} \qquad (\varphi = t_0 - \alpha_0 - \theta). \tag{1.67}$$

By lemma (1.10), for G_0 large enough, we have that $|f| \ll 1$ and then $\varphi = \pm\pi/2$ are not solutions of $\sin\varphi = f(\varphi)$. So, on $(-\pi/2, 3\pi/2)$ we have

$$\begin{aligned} \varphi &= \arcsin f(\varphi) \qquad && \varphi \in (-\pi/2, \pi/2) \\ \varphi &= \pi - \arcsin f(\varphi) && \varphi \in (\pi/2, 3\pi/2) \end{aligned}$$

Since $|f| < 1$, $g(\varphi) = \arcsin f(\varphi)$ maps $[-\pi/2, \pi/2]$ into itself and

$$g'(\varphi) = \frac{f'(\varphi)}{\sqrt{1 - f(\varphi)^2}}$$

therefore $g'^2 < 1$ is equivalent to $f^2 + f'^2 < 1$ which is a direct consequence of lemma 1.10. So, g is a contraction and then there exists a unique $\varphi_-^* \in (-\pi/2, \pi/2)$ solution of $\varphi = g(\varphi)$. To prove that it is non degenerate we need to see that

$$\frac{\partial \mathcal{L}^2}{\partial t_0^2} = \frac{\partial^2 \mathcal{L}_1}{\partial t_0^2} + \frac{\partial^2 \mathcal{E}}{\partial t_0^2} \neq 0.$$

To see this we will see that

$$\Big(\frac{\partial^2 \mathcal{E}}{\partial t_0^2}\Big)^2 < \Big(\frac{\partial^2 \mathcal{L}_1}{\partial t_0^2}\Big)^2 = (\mathcal{L}_1^*)^2 \cos^2\varphi \tag{1.68}$$

but from (1.67)

$$\cos^2\varphi = 1 - \frac{1}{(\mathcal{L}_1^*)^2}\Big(\frac{\partial \mathcal{E}}{\partial t_0}\Big)^2$$

equation (1.68) is equivalent to

$$\Big(\frac{\partial \mathcal{E}}{\partial t_0}\Big)^2+\Big(\frac{\partial^2 \mathcal{E}}{\partial t_0^2}\Big)^2< (\mathcal{L}_1^*)^2,$$

which is true. By the same lemma 1.10,

$$|\varphi_-^*| = |\arcsin f(\varphi_-^*)| = O\Big(\frac{G_0^2}{\kappa}\mathrm{e}^{-G_0^3/9}\Big), \tag{1.69}$$

and consequently $t^*_{0,-} = \varphi^*_- + \alpha_0 + \theta$ is a non degenerate solution of (1.66). Analogously we can solve

$$\varphi = \pi - \arcsin f(\varphi) = \tilde{g}(\varphi)$$

showing that $\tilde{g}$ sends $(\pi/2, 3\pi/2)$ to itself and is a contraction proving the existence of a non degenerate fix point $\varphi^*_+ \in (\pi/2, 3\pi/2)$. Moreover

$$|\varphi_+^* - \pi| = |\arcsin \tilde{g}(\varphi_+^*)| = O\Big(\frac{G_0^2}{\kappa}\mathrm{e}^{-G_0^3/9}\Big). \tag{1.70}$$

Consequently $t^*_{0,+} = \varphi^*_+ + \alpha_0 + \theta$ is another non degenerate solution of (1.66). This concludes the proof. □

From proposition 1.11 we know that there exist $t^*_{0,-}$ and $t^*_{0,+}$, non degenerate critical points of $t_0 \longmapsto \mathcal{L}(\alpha_0, G_0, t_0; e_0)$. Therefore, we can define two different reduced Poincaré functions (1.44)

$$\begin{aligned}\mathcal{L}^*_\pm(\alpha_0, G_0; e_0) &= \mathcal{L}(\alpha_0, G_0, t^*_{0,\pm}; e_0)\\ &= L_{0,0}(G_0; e_0) + \mathcal{L}_0(\alpha_0, G_0; e_0) + \mathcal{L}_1^*(\alpha_0, G_0; e_0)\cos(t^*_{0,\pm} - \alpha_0 - \theta)\\ &\qquad + \mathcal{E}(\alpha_0, G_0, t^*_{0,\pm}; e_0).\end{aligned}$$

By Taylor's theorem

$$\begin{aligned}\cos(t^*_{0,-} - \alpha_0 - \theta) &= \cos(0) + O(|\varphi^*_-|^2) = 1 + O(G_0^4\mathrm{e}^{-G_0^3\frac{2}{9}})\\ \cos(t^*_{0,+} - \alpha_0 - \theta) &= \cos(\pi) + O(|\varphi^*_+ - \pi|^2) = -1 + O(G_0^4\mathrm{e}^{-G_0^3\frac{2}{9}})\end{aligned}$$

so that

$$\mathcal{L}^*_\pm(\alpha_0, G_0; e_0) = L_{0,0}(G_0; e_0) + \mathcal{L}_0(\alpha_0, G_0; e_0) \pm \mathcal{L}_1^*(\alpha_0, G_0; e_0)\cos(t^*_{0,\pm} - \alpha_0 - \theta) + E_\pm \tag{1.71}$$

where

$$E_\pm = \pm\mathcal{L}_1^* O(G_0^4\mathrm{e}^{-G_0^3\frac{2}{9}}) + \mathcal{E}, \tag{1.72}$$

and by the bound of $\mathcal{E}$ given in (1.53), the definition of $\mathcal{L}_1^*$ and the definition of B given in lemma 1.8 we have that

$$|E_\pm| \leq KG_0^{3/2}\mathrm{e}^{-4G_0^3/9}\Big(1 + \mathrm{e}^{-G_0^3/9}G_0^2(1+p)^2\Big) \leq KG_0^{7/2}\mathrm{e}^{-5G_0^3/9}(1+p)^2$$

the last inequality holds for G_0 large enough, but in any case is exponentially smaller.

$L_{0,0}$, $\mathcal{L}_0$ and $\mathcal{E}$ are given in (1.49), (1.51) and (1.53). Writing down $L_{0,0}$ and $\mathcal{L}_0$ we have

$$\mathcal{L}^*_\pm = \frac{\pi}{2}G_0^{-3} + \mathcal{F}_1(G_0, e_0) - \frac{15}{8}\pi e_0 G_0^{-5}\cos\alpha_0 + \mathcal{F}(\alpha_0, G_0; e_0) \pm \mathcal{L}_1^* + E_\pm \tag{1.73}$$

From the expression for the scattering map given in proposition 1.3 we can define two different scattering maps, given by

$$S_\pm(\alpha_0, G_0, s_0) = \Big(\alpha_0 + \mu\frac{\partial \mathcal{L}^*_\pm}{\partial G}(\alpha_0, G_0; e_0) + O(\mu^2), G_0 - \mu\frac{\partial \mathcal{L}^*_\pm}{\partial \alpha}(\alpha_0, G_0; e_0) + O(\mu^2), s_0\Big). \tag{1.74}$$

These two scattering maps are different since they depend on the two reduced Poincaré-Melnikov potentials $\mathcal{L}^*_\pm$. As it was proved in [DdlLS08] the scattering maps $S_\pm$ follow closely the level curves of the Hamiltonians $\mathcal{L}^*_\pm$. More precisely, up to $O(\mu^2)$ terms, it is given by the time $-\mu$ map of the Hamiltonian flow of Hamiltonians $\mathcal{L}^*_\pm$. Because of this, we want to show that the foliations of $\mathcal{L}^*_\pm = constant$ are different, since this will imply that the scattering maps $S_\pm$ are different. Even more, we will design a mechanism in which we will determine the places in the plane $\alpha_0 G_0$ where we will change from one scattering map to the other, obtaining trajectories with increasing angular momentum G.

In lemma 1.13 we will give the elements to construct a strategy to find a heteroclinc chain of periodic orbits in $\tilde{\Lambda}_\infty$ with increasing angular momentum, but first a technical lemma.

Lemma 1.12. Let $\mathcal{L}^*_\pm$ be defined by (1.71), B by lemma 1.8 and $p = 12e_0G_0^2$. If G_0, e_0, α_0 and κ are as in lemma 1.10. Then we have

$$\{\mathcal{L}^*_+, \mathcal{L}^*_-\} = \frac{-\mathcal{L}^*_1}{B^2}\frac{3\pi p\sin\alpha_0}{G_0^4}\left[1 - \frac{25}{4}\frac{e_0G_0}{G_0^3}\cos\alpha_0 - \frac{5}{48}\frac{P^2}{G_0}\Big[1 + \frac{1}{2G_0^3} - \frac{-\cos\alpha_0 + p}{P^2}\cdot\frac{24e_0G_0}{G_0^2}\Big]\right] + E_J$$

where

$$E_J = O\Big(G_0^{-5} + e_0G_0^{-3} + e_0^2G_0^3 + pe_0^2G_0^4\big(1 + p(e_0G_0 + G_0^{-6})\big)\Big)G_0^{-1/2}\mathrm{e}^{-G_0^3/3}$$
$$+ O\Big(\big(G_0(1+p)\mathrm{e}^{-G_0^3/9} + G_0^{-1}\big)\Big)G_0^{1/2}\mathrm{e}^{-G_0^3\frac{4}{9}}$$

Proof. Using expression (1.71) and using the properties of the Poisson brackets we have that

$$\{\mathcal{L}^*_+, \mathcal{L}^*_-\} = 2\{\mathcal{L}^*_1, L_{0,0} + \mathcal{L}_0\} + 2\{E, L_{0,0} + \mathcal{L}_0\}. \tag{1.75}$$

Where $\mathcal{L}^*_1$ is given in (1.58), $L_{0,0}$ in (1.49) and $\mathcal{L}_0$ in (1.51). From the definition of the Poisson bracket

$$\{\mathcal{L}^*_1, L_{0,0} + \mathcal{L}_0\} = \frac{\partial\mathcal{L}^*_1}{\partial\alpha_0}\frac{\partial}{\partial G_0}(L_{0,0} + \mathcal{L}_0) - \frac{\partial\mathcal{L}^*_1}{\partial G_0}\frac{\partial}{\partial\alpha_0}(L_{0,0} + \mathcal{L}_0) \tag{1.76}$$

to compute the partial derivatives in the above formula we will need to compute the partial derivatives with respect to α_0 and G_0 of B given in lemma 1.8 as

$$B^2 = P^2 + \tilde{B} = 1 - 2p\cos\alpha_0 + p^2 + \tilde{B}$$

where $p = 12e_0G_0^2$ and $\tilde{B} = O(G_0^{-1} + e_0G_0(1 + p + p^2))$, then

$$\frac{\partial B}{\partial G_0} = \frac{1}{2}\frac{1}{\sqrt{P^2 + \tilde{B}}}\Big(\frac{\partial P^2}{\partial G_0} + \frac{\partial\tilde{B}}{\partial G_0}\Big) \tag{1.77a}$$

$$\frac{\partial B}{\partial\alpha_0} = \frac{1}{2}\frac{1}{\sqrt{P^2 + \tilde{B}}}\Big(\frac{\partial P^2}{\partial\alpha_0} + \frac{\partial\tilde{B}}{\partial\alpha_0}\Big) \tag{1.77b}$$

Also,

$$\frac{\partial P^2}{\partial G_0} = -2\cos\alpha_0\frac{\partial p}{\partial G_0} + 2p\frac{\partial p}{\partial G_0}$$
$$= (-2\cos\alpha_0 + 2p)\frac{2p}{G_0} \tag{1.78a}$$

$$\frac{\partial P^2}{\partial\alpha_0} = 2p\sin\alpha_0, \tag{1.78b}$$

substituting equations (1.78) in equations (1.77) we can write

$$\frac{\partial\mathcal{L}^*_1}{\partial\alpha_0} = \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-G_0^3/3}\frac{\partial B}{\partial\alpha_0}$$

$$= \sqrt{\frac{\pi}{8}} G_0^{-1/2} e^{-G_0^3/3} \frac{2p\sin\alpha_0 + \frac{\partial\tilde{B}}{\partial\alpha_0}}{2\sqrt{P^2+\tilde{B}}}$$

$$= \frac{\mathcal{L}_1^*}{P^2+\tilde{B}}\Big(p\sin\alpha_0 + \frac{1}{2}\frac{\partial\tilde{B}}{\partial\alpha_0}\Big) \tag{1.79}$$

$$\frac{\partial\mathcal{L}_1^*}{\partial G_0} = \sqrt{\frac{\pi}{8}} G_0^{-1/2} e^{-G_0^3/3}\Big[-\Big(\frac{1}{2G_0}+G_0^2\Big)B + \frac{1}{2B}\Big((-2\cos\alpha_0+2p)\frac{2p}{G_0}+\frac{\partial\tilde{B}}{\partial G_0}\Big)\Big]$$

$$= \mathcal{L}_1^*\Big[-\Big(\frac{1}{2G_0}+G_0^2\Big)+\frac{1}{P^2+\tilde{B}}\Big((-\cos\alpha_0+p)\frac{2p}{G_0}+\frac{1}{2}\frac{\partial\tilde{B}}{\partial G_0}\Big)\Big]$$

$$= \frac{\mathcal{L}_1^*}{P^2+\tilde{B}}\Big[(-\cos\alpha_0+p)\frac{2p}{G_0} - \Big(\frac{1}{2G_0}+G_0^2\Big)P^2 - \Big(\frac{1}{2G_0}+G_0^2\Big)\tilde{B} + \frac{1}{2}\frac{\partial\tilde{B}}{\partial G_0}\Big] \tag{1.80}$$

and

$$\frac{\partial}{\partial\alpha_0}(L_{0,0}+\mathcal{L}_0) = \frac{15}{8}\pi e_0 G_0^{-5}\sin\alpha_0 + \frac{\partial\mathcal{F}}{\partial\alpha_0} \tag{1.81}$$

$$\frac{\partial}{\partial G_0}(L_{0,0}+\mathcal{L}_0) = -\frac{3}{2}\frac{\pi}{G_0^4} + \frac{75}{8}\frac{\pi}{G_0^6}e_0\cos\alpha_0 + \frac{\partial}{\partial G_0}(\mathcal{F}_1+\mathcal{F}) \tag{1.82}$$

substituting equations (1.79), (1.80), (1.81) and (1.82), in the expression for the Poisson bracket given in (1.76) we obtain

$$2\{\mathcal{L}_1^*, L_{0,0}+\mathcal{L}_0\} = \frac{\mathcal{L}_1^*}{P^2+\tilde{B}}(p\sin\alpha_0)\Big(-\frac{3\pi}{G_0^4}+\frac{75}{4}\frac{\pi}{G_0^6}e_0\cos\alpha_0\Big)+Q_1$$
$$-\Big(\frac{15}{4}\pi e_0 G_0^{-5}\sin\alpha_0\Big)\frac{\mathcal{L}_1^*}{P^2+\tilde{B}}\Big[(-\cos\alpha_0+p)\frac{2p}{G_0}-\Big(\frac{1}{2G_0}+G_0^2\Big)P^2\Big]+Q_2 \tag{1.83}$$

where

$$Q_1 = \frac{\mathcal{L}_1^*}{P^2+\tilde{B}}\Big[-\Big(\frac{3}{2}\frac{\pi}{G_0^4}-\frac{75}{8}\frac{\pi}{G_0^6}e_0\cos\alpha_0\Big)\frac{\partial\tilde{B}}{\partial\alpha_0}+2\frac{\partial}{\partial G_0}(\mathcal{F}_1+\mathcal{F})\Big(p\sin\alpha_0+\frac{1}{2}\frac{\partial\tilde{B}}{\partial\alpha_0}\Big)\Big] \tag{1.84}$$

$$Q_2 = \frac{\mathcal{L}_1^*}{P^2+\tilde{B}}\Bigg[\Big(\frac{15}{4}\pi e_0 G_0^{-5}\sin\alpha_0\Big)\Big[\Big(\frac{1}{2G_0}+G_0^2\Big)\tilde{B}-\frac{1}{2}\frac{\partial\tilde{B}}{\partial G_0}\Big]-2\frac{\partial\mathcal{F}}{\partial\alpha_0}$$
$$\cdot\Big[(-\cos\alpha_0+p)\frac{2p}{G_0}-\Big(\frac{1}{2G_0}+G_0^2\Big)P^2-\Big(\frac{1}{2G_0}+G_0^2\Big)\tilde{B}+\frac{1}{2}\frac{\partial\tilde{B}}{\partial G_0}\Big]\Bigg] \tag{1.85}$$

factorizing we have from (1.83)

$$2\{\mathcal{L}_1^*, L_{0,0}+\mathcal{L}_0\} = \frac{-\mathcal{L}_1^*}{P^2+\tilde{B}}\frac{3\pi p\sin\alpha_0}{G_0^4}\Bigg[1-\frac{25}{4}\frac{1}{G_0^2}e_0\cos\alpha_0$$
$$+\frac{5}{48}\frac{1}{G_0^3}\Big[(-\cos\alpha_0+p)\frac{2p}{G_0}-\Big(\frac{1}{2G_0}+G_0^2\Big)P^2\Big]\Bigg]+Q_1+Q_2$$
$$= \frac{-\mathcal{L}_1^*}{P^2+\tilde{B}}\frac{3\pi p\sin\alpha_0}{G_0^4}\Bigg[1-\frac{25}{4}\frac{e_0G_0}{G_0^3}\cos\alpha_0-\frac{5}{48}\frac{P^2}{G_0}$$
$$\cdot\Big[1+\frac{1}{2G_0^3}-\frac{-\cos\alpha_0+p}{P^2}\cdot\frac{24e_0G_0}{G_0^2}\Big]\Bigg]+Q_1+Q_2 \tag{1.86}$$

(1.87)

To find the size of Q_1 and Q_2 we have to bound

$$\frac{\mathcal{L}_1^*}{P^2+\tilde{B}} = \sqrt{\frac{\pi}{8}}\frac{1}{B}G_0^{-1/2}e^{-G_0^3/3}$$

so, we have to bound $1/B$ by a positive constant, or equivalently $1/B^2$, which has been done in the proof of lemma 1.10. Therefore

$$\frac{\mathcal{L}_1^*}{P^2+\tilde{B}} = O\Big(\frac{1}{\kappa}G_0^{-1/2}\mathrm{e}^{-G_0^3/3}\Big),$$

then using the bounds for $\mathcal{F}_1$ and $\mathcal{F}$ given in (1.50) and (1.54) and the bound for $\tilde{B}$ given in lemma 1.8 we have from equations (1.84) and (1.85) that

$$\begin{aligned} Q_1 &= O\Big((G_0^{-5}+e_0G_0^{-3}+e_0^2G_0^3+pe_0^2G_0^4(1+pe_0G_0))\Big)\frac{1}{\kappa}G_0^{-1/2}\mathrm{e}^{-G_0^3/3} \\ Q_2 &= O\Big((e_0G_0^{-4}+e_0^2G_0^{-2}(1+p+p^2))\Big)\frac{1}{\kappa}G_0^{-1/2}\mathrm{e}^{-G_0^3/3} \end{aligned}$$

and then

$$Q_1+Q_2 = O\Big(G_0^{-5}+e_0G_0^{-3}+e_0^2G_0^3+pe_0^2G_0^4(1+p(e_0G_0+G_0^{-6}))\Big)\frac{1}{\kappa}G_0^{-1/2}\mathrm{e}^{-G_0^3/3}. \qquad (1.88)$$

Now we want to know the size of $\{E, L_{0,0}+\mathcal{L}_0\}$. From the computations for the partial derivatives of $L_{0,0}+\mathcal{L}_0$ given in equations (1.81) and (1.82) and the size of the errors $\mathcal{F}_1$ and $\mathcal{F}$ given in (1.50) and (1.54) we have

$$\begin{aligned} \frac{\partial}{\partial\alpha_0}(L_{0,0}+\mathcal{L}_0) &= O(e_0G_0^{-5}) \\ \frac{\partial}{\partial G_0}(L_{0,0}+\mathcal{L}_0) &= O(G_0^{-4}). \end{aligned}$$

The size of E is computed in the proof of part (b) of lemma 1.13, and is given in equation (1.95), and therefore

$$\begin{aligned} \frac{\partial E}{\partial\alpha_0} &= O\Big(G_0^{1/2}\mathrm{e}^{-G_0^3\frac{4}{9}}\big(G_0^3(1+p)\mathrm{e}^{-G_0^3/9}+G_0\big)\Big) \\ \frac{\partial E}{\partial G_0} &= O\Big(G_0^{1/2}\mathrm{e}^{-G_0^3\frac{4}{9}}\big(G_0^5(1+p)\mathrm{e}^{-G_0^3/9}+G_0^3\big)\Big) \end{aligned}$$

with this sizes we can conclude that

$$\{E, L_{0,0}+\mathcal{L}_0\} = O\Big(G_0^{1/2}\mathrm{e}^{-G_0^3\frac{4}{9}}\big(G_0(1+p)\mathrm{e}^{-G_0^3/9}+G_0^{-1}\big)\Big)$$

substituting (1.86) in (1.75) and setting $E_J = 2\{E, L_{0,0}+\mathcal{L}_0\}+Q_1+Q_2$ we get the desired result. □

Lemma 1.13. Let $\mathcal{L}_\pm^*$ be defined in (1.73), and $p = 12e_0G_0^2$. Then

(a) Any curve $\mathcal{L}_\pm^*(\alpha_0, G_0; e_0) = l$ is a closed curve of the form $G = g_\pm(\alpha_0, l)$, $\alpha_0 \in [0, 2\pi]$, $g_\pm(0, l) = g_\pm(2\pi, l)$ which is cosine-like: it has a unique non-degenerate maximum for α_0 close to π and a non-degenerate minimum for α_0 close to 0.

(b) The **total variation** of $\mathcal{L}_\pm^*(\cdot, G_0; e_0)$ (i. e. the difference between its maximum value and its minimum value) is given by

$$\Delta\mathcal{L}_\pm^* = \Delta\mathcal{L}_0 \pm \Delta\mathcal{L}_1^* \pm \Delta E$$

where

$$\begin{aligned} \Delta\mathcal{L}_0 &= 2\frac{15\pi e_0}{8G_0^5}+\Delta\mathcal{F} \\ \Delta\mathcal{L}_1^* &= \sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-G_0^3/3}\Big[\sqrt{(p+1)^2+\tilde{B}_0}-\sqrt{(p-1)^2+\tilde{B}_\pi}\Big] \end{aligned}$$

$$\Delta E = O\Big(G_0^{1/2}\mathrm{e}^{-G_0^3\frac{4}{9}}\big(G_0^3(1+p)\mathrm{e}^{-G_0^3/9}+G_0\big)\Big)$$

and

$$\Delta\mathcal{F} = O(e_0^2G_0^{-5}, e_0G_0^{-9})$$
$$\tilde{B}_\pi, \tilde{B}_0 = O(G_0^{-1}+e_0G_0(1+p+p^2))$$

(c) $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$ are functionally independent except for three curves, two of them close to the straight lines $\alpha_0 = 0$ and $\alpha_0 = \pi$ and a third one cosine-like whenever $G_0 = O(e_0^{-2/3})$ and close to the curve

$$p = \sqrt{D(1-\cos\alpha_0)+\frac{3}{4}D^2} - 1 \tag{1.89}$$

where $D = 2\sqrt{48G_0/5}$.

Proof. (a) From the expression for $\mathcal{L}_\pm^*$ given in (1.73) we have that $l = \mathcal{L}_\pm^*(\alpha_0, G_0; e_0)$ is equivalent to

$$l = \frac{\pi}{2G_0^3}\Big(1+\tilde{\mathcal{F}}_1 - \frac{15e_0\cos\alpha_0}{4G_0^2} + \tilde{\mathcal{F}}(\alpha_0, G_0; e_0) \pm \sqrt{\frac{1}{2\pi}}G_0^{5/2}\mathrm{e}^{-G_0^3/3}B + \tilde{E}_\pm(\alpha_0, G_0, t_0; e_0)\Big) \tag{1.90}$$

where

$$\tilde{\mathcal{F}}_1 = \frac{2G_0^3}{\pi}\mathcal{F}_1 = O(e_0^2+G_0^{-4})$$
$$\tilde{\mathcal{F}}(\alpha_0, G_0; e_0) = \frac{2G_0^3}{\pi}\mathcal{F} = O(e_0^2G_0^{-2}+e_0G_0^{-6})$$
$$\tilde{E}_\pm(\alpha_0, G_0, t_0; e_0) = \frac{2G_0^3}{\pi}\Big(\mathcal{L}_1^*O(G_0^4\mathrm{e}^{-G_0^3\frac{2}{9}})+\mathcal{E}\Big) = \frac{2G_0^3}{\pi}\Big(\mathcal{L}_1^*O(G_0^4\mathrm{e}^{-G_0^3\frac{2}{9}})+O(G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}})\Big)$$

and $\mathcal{F}_1$, $\mathcal{F}$ and $\mathcal{E}$ are given in (1.50), (1.54) and (1.53), respectively. The actual size of $\tilde{E}_\pm$ will depend on the bound of B which in its turn depends on p. From lemma 1.8 it is not difficult to see that

$$|B| \le \begin{cases} K & \text{if } p \le 1 \\ Kp & \text{if } p > 1 \end{cases}$$

and by (1.58), and using that $p = 12e_0G_0^2$,

$$|\mathcal{L}_1^*| \le \begin{cases} KG_0^{-1/2}\mathrm{e}^{-G_0^3/3} & \text{if } p \le 1 \\ Ke_0G_0^{3/2}\mathrm{e}^{-G_0^3/3} & \text{if } p > 1 \end{cases} \tag{1.91}$$

with this we conclude that

$$\tilde{E} = \begin{cases} O(G_0^{13/2}\mathrm{e}^{-G_0^3\frac{5}{9}}) + O(G_0^{9/2}\mathrm{e}^{-G_0^3\frac{4}{9}}) & \text{if } p \le 1 \\ O(e_0G_0^{17/2}\mathrm{e}^{-G_0^3\frac{5}{9}}) + O(G_0^{9/2}\mathrm{e}^{-G_0^3\frac{4}{9}}) & \text{if } p > 1 \end{cases}$$

We can rewrite (1.90) as

$$\begin{aligned} G_0 &= \Big(\frac{\pi}{2l}\Big)^{1/3}\Big(\zeta - \frac{15e_0\cos\alpha_0}{4G_0^2} + \tilde{\mathcal{F}}(\alpha_0, G_0; e_0) \pm \sqrt{\frac{1}{2\pi}}G_0^{5/2}\mathrm{e}^{-G_0^3/3}B + \tilde{E}_\pm(\alpha_0, G_0, t_0; e_0)\Big)^{1/3} \\ &= g(\alpha_0, l) \end{aligned} \tag{1.92}$$

where $\zeta = 1 + \tilde{\mathcal{F}}_1 = O(1)$. This expression, implies

$$l = \frac{\zeta\pi}{2G_0^3}\Big(1 + O(\frac{e_0}{G_0^2})\Big)$$

or equivalently

$$G_0 = \left(\frac{\zeta\pi}{2l}\right)^{1/3}\Big(1+O(\frac{e_0}{G_0^2})\Big)^{1/3} = \left(\frac{\zeta\pi}{2l}\right)^{1/3}\Big(1+O(\frac{e_0}{G_0^2})\Big) = \left(\frac{\zeta\pi}{2l}\right)^{1/3}\Big(1+O(e_0 l^{2/3})\Big)$$

using this expression we can actually know a good estimation of the curve $G_0 = g(\alpha_0, l)$ substituting it in (1.92) Now, when $\tilde{E} = 0$, g is clearly a cosine-like with a non-degenerate maximum close to $\alpha_0 = 0$ and a non-degenerate minimum close to $\alpha_0 = \pi$ since its second term is larger than the third and fourth terms. When we also take into a account the fifth term, in the expression of g involving $\tilde{E}$, since this term is much smaller than the other ones, an argument very similar to the one used in the proof of proposition 1.11 implies that g is cosine-like with non-degenerate critical points close to 0 and π.

(b) From expression (1.73), we are going to analyze every term in α_0. The term

$$L_{0,0} = \frac{\pi}{2} G_0^{-3} + \mathcal{F}_1$$

gets canceled since its constant with respect to α_0. From its definition given in (1.51) we have

$$\mathcal{L}_0 = -\frac{15}{8}\pi e_0 G_0^{-5}\cos\alpha_0 + \mathcal{F}(\alpha_0, G_0; e_0),$$

the dominant term is a cosine in α_0, and then its maximum and minimum are $\alpha_0 = 0, \pi$, so

$$\Delta\mathcal{L}_0 = 2\frac{15\pi e_0}{8G_0^5} + \Delta\mathcal{F} \tag{1.93}$$

where

$$\Delta\mathcal{F} = \mathcal{F}(0, G_0; e_0) - \mathcal{F}(\pi, G_0; e_0) = O(e_0^2 G_0^{-5}, e_0 G_0^{-9}).$$

Now, from the definition of $\mathcal{L}_1^*$ given in (1.58), the maximum and minimum are determined by B. Since the square root is a monotone function, it is enough to analyze when B^2 have its critical points. From lemma 1.8 we know that $B^2 = P^2 + \tilde{B}$, and from the bound of $\tilde{B}$, since $e_0 G_0$ is small it is enough to look for the critical points of

$$P^2 = 1 - 2p\cos\alpha_0 + p^2$$

which again are attained whenever $\alpha_0 = 0, \pi$. Therefore

$$\Delta\mathcal{L}_1^* = \sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-G_0^3/3}\Big[\sqrt{(p+1)^2 + \tilde{B}_\pi} - \sqrt{(p-1)^2 + \tilde{B}_0}\Big] \tag{1.94}$$

where

$$\tilde{B}_\pi, \tilde{B}_0 = O(G_0^{-1} + e_0 G_0 (1 + p + p^2)).$$

Finally, from the definition of E given in (1.72) and the size of $\mathcal{L}_1^*$ given in (1.91) we have that

$$E = \begin{cases} O(G_0^{7/2}\mathrm{e}^{-G_0^3\frac{5}{9}}) + O(G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}) & \text{if } p \le 1 \\ O(e_0 G_0^{11/2}\mathrm{e}^{-G_0^3\frac{5}{9}}) + O(G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}) & \text{if } p > 1 \end{cases}$$

this can be written as

$$E = O\Big(G_0^{1/2}\mathrm{e}^{-G_0^3\frac{4}{9}}\big(G_0^3(1+p)\mathrm{e}^{-G_0^3/9} + G_0\big)\Big) \tag{1.95}$$

in any case, E is much more smaller that any term in $\mathcal{L}_\pm^*$ implying that the maximum and minimum of $\mathcal{L}_\pm^*$ are reached whenever $\alpha_0 = 0, \pi$, concluding then the desired result.

(c) To see that $\mathcal{L}_+^*(\alpha_0, G_0; e_0)$ and $\mathcal{L}_-^*(\alpha_0, G_0; e_0)$ are functionally independent we will analyze $\det J(\mathcal{L}_+^*, \mathcal{L}_-^*)$. Since

$$\det J(\mathcal{L}_+^*, \mathcal{L}_-^*) = \{\mathcal{L}_+^*, \mathcal{L}_-^*\}$$

we can use lemma 1.12, to conclude that, if the factor outside the brackets in the formula for $\{\mathcal{L}_+^*, \mathcal{L}_-^*\}$ is zero then $\det J(\mathcal{L}_+^*, \mathcal{L}_-^*)$ is close to zero or asymptotically is zero. This occurs when

$\alpha_0 = 0, \pi$. We have excluded this values of α_0 to bound the error in the formula of $\{\mathcal{L}_+^*, \mathcal{L}_-^*\}$ given in lemma 1.12, and before in lemma 1.10.

Using the dominant term inside the brackets of the formula for $\det J(\mathcal{L}_+^*, \mathcal{L}_-^*)$ we have

$$\det J(\mathcal{L}_+^*, \mathcal{L}_-^*) \sim \frac{\mathcal{L}_1^*}{B^2} \frac{-3\pi p \sin \alpha_0}{G_0^4} d$$

where

$$d = 1 - \frac{5}{48} \frac{P^2}{G_0}$$

this implies that, beside the curves $\alpha_0 = 0$ and $\alpha_0 = \pi$ the Jacobian can be asymptotically zero if $d = 0$. In what follows we will see that this gives a curve cosine-like in the plane $\alpha_0 G_0$. From the definition of d we have that $d = 0$ only if

$$P^2 = 1 - 2p \cos \alpha_0 + p^2 \sim G_0$$

this is not possible if $p \leq 1$, and if $p > 1$, we have that $P^2 \sim p^2$ and then, d will be equal to zero only if

$$p^2 \sim G_0$$

or equivalently if

$$G_0 \sim e_0^{-2/3}.$$

From the definition of P^2 is easy to see that,

$$(p-1)^2 \leq P^2 \leq (p+1)^2$$

this implies that

$$1 - \frac{5}{48 G_0}(p+1)^2 \leq d \leq 1 - \frac{5}{48 G_0}(p-1)^2$$

from this, is easy to see that if

$$0 < 1 - \frac{5}{48}(p+1)^2 \qquad \text{or} \qquad 1 - \frac{5}{48}(p-1)^2 < 0$$

then $d \neq 0$, or equivalently if

$$\left| p - \sqrt{\frac{48 G_0}{5}} \right| \geq 1$$

then $d \neq 0$. Therefore $d = 0$ in the region

$$\left| p - \sqrt{\frac{48 G_0}{5}} \right| \leq 1.$$

It is convenient then, to introduce

$$w = p - \sqrt{\frac{48 G_0}{5}}$$

which satisfies $|w| \leq 1$. Writing d in w we have

$$d = -\frac{5}{48} \frac{1}{G_0} \Big[(1+w)^2 + D(c+w) \Big]$$

where $c = \cos \alpha_0$ and $D = 2\sqrt{48 G_0/5}$. Then $d = 0$ if

$$c = -\frac{1}{D}(1+w)^2 - w \tag{1.96}$$

where $|w| \leq 1$ and $c = \cos \alpha_0$. Now

- When $c = 1$ so that $\alpha_0 = 0$ we have that equation (1.96) is equivalent to the quadratic equation
$$w^2 + w(2+D) + 1 + D = 0$$
whose solutions are $w = -1$ or $w = -1 - D$, since $|w| \leq 1$ only the first solution has sense.

- When $c = -1$ so that $\alpha_0 = \pi$ we have that equation (1.96) is equivalent to the quadratic equation
$$w^2 + (2+D)w + 1 - D = 0$$
whose solutions are
$$w_\pm = -1 - \frac{1}{2}\Big(D \mp \sqrt{D^2 + 8D}\Big).$$
Clearly $D + \sqrt{D^2 + 8D} \gg 1$, therefore $w_- < -1$ and then do not satisfy our condition $|w| \leq 1$. A straight forward computation shows that
$$-4 < D - \sqrt{D^2 + 8D} < 0$$
therefore $|w_+| < 1$.

- When $-1 < c < 1$ we have to analyze the behavior of c as a function of w. Taking derivative of (1.96) we have
$$c'(w) = -\frac{2}{D}(1+w) - 1$$
if we look for critical points of $c(w)$ and consider $c'(w) = 0$ we find that
$$w^* = -\frac{D}{2} - 1$$
is the only critical point and is smaller than -1. Actually this critical point is a maximum of $c(w)$ since
$$c''(w) = -\frac{2}{D} < 0.$$
Since $c(w)$ is a parabola we have that for $w \in [-1, 1]$, $c(w)$ is a decreasing function, and since $c(w_+) = -1$ if we consider $w > w_+$ we will have that $c(w) < -1$ and because there are no α_0 such that $\cos\alpha_0 < -1$ we conclude that in that case $d \neq 0$. Therefore, the only way to have $d = 0$ is to consider $w \in [-1, w_+]$. So, whenever $c \in (-1, 1)$ there exist an $w \in (-1, w_+)$ such that
$$\cos\alpha_0 = -\frac{1}{D}(1+w)^2 - w$$
which means that there are two different values of α_0 that make $d = 0$.

Coming back to p, we have seen that there is a curve contained in the region
$$\Big|p - \sqrt{\frac{48G_0}{5}}\Big| \leq 1$$
with equation given by (1.96). Rewriting this equation using that $c = \cos\alpha_0$ and $D = 2\sqrt{48G_0/5}$ we have
$$\cos\alpha_0 = \frac{1}{D}(1 + p - \frac{D}{2})^2 - p + \frac{D}{2}$$
or equivalently
$$p = \sqrt{D(1 - \cos\alpha_0) + \frac{3}{4}D^2} - 1$$
which is clearly a cosine-like function in terms of (α_0, p). In the variables (G_0, α_0) it is also a cosine-like function since $p = 12e_0G_0^2$ is an increasing function of G_0. □

Remark 1.14. The difference between total variation of $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$ is strictly positive but exponentially small. In fact
$$\Delta\mathcal{L}_+^* - \Delta\mathcal{L}_-^* = 2\Delta\mathcal{L}_1^* + 2\Delta E = O(G_0^{-1/2}\mathrm{e}^{-G_0^3/3})$$

1.5.2 Strategy for diffusion

We will describe the strategy to construct a chain of heteroclinic connections to the manifold $\tilde{\Lambda}_\infty$ defined in (1.29) using the results in lemma 1.13.

Let us take a point in $\tilde{\Lambda}_\infty$ in which the reduced Poincaré functions $\mathcal{L}^*_\pm$ are functionally independent. From part (c) of lemma 1.13 α_0 should be different from 0 or π and not in the curve given in (1.89). In figure 1.3 is shown schematically the curves in the plane $\alpha_0 G_0$ that we will avoid in the following procedure.

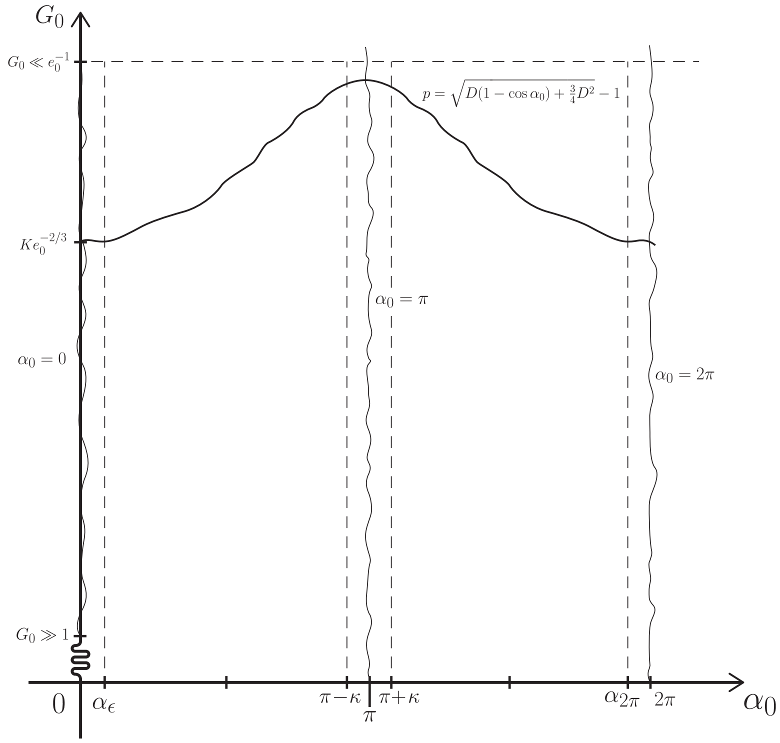


Figure 1.2: Zone of diffusion

Take $(\alpha_0 = \alpha_\epsilon, G_0 = G_1)$ and apply successively the scattering map S_- defined in (1.74), its trajectory will follow the level curve $l_- = \mathcal{L}^*_-(\alpha_\epsilon, G_1)$ up to certain $\alpha_0 = \alpha_\pi$ close to π where G_0 takes the value G^*. At this moment, we shift to the scattering map S_+ defined as well in (1.74). From applying S_+ successively, we will get points along the level curve $l_+ = \mathcal{L}^*_+(\alpha_\pi, G^*)$ up to $\alpha_0 = \alpha_{2\pi}$ close to $2\pi = 0(mod 2\pi)$ where G_0 takes the value G_2 with $G_2 > G_1$, by remark 1.14, we know that $G_2 - G_1 = O(\mathrm{e}^{-G_0^3/3})$. Continuing in this way, we can travel along all the allowed diffusion zone $G_1 < G_0 \ll 1/e_0$ avoiding always to shift from one scattering map to another, in a point of the curve given in (1.89) whenever $G_0 = O(e_0^{-2/3})$. Using part (a) and (c) of lemma 1.13 we get figure 1.3. The red arrows represent the trajectory that changes from one scattering map to the other.

Inside the domain $1 \ll G_0 \ll 1/e_0$ we can obtain diffusion orbits along arbitrary paths, except those which intersect the small regions described in lemma 1.10 and the curves given in lemma 1.13. This mechanism given by the application of scattering maps produce indeed pseudo-orbits, that is, heteroclinc connections between different periodic orbits $\tilde{\Lambda}_{\alpha_0, G_0}$ in $\tilde{\Lambda}_\infty$ which are commonly known as transition chains after Arnold's pioneering work [Arn64]. The existence of true orbits of the system which follow closely these transition chains relies on shadowing methods, which are standard for partially hyperbolic periodic orbits (the so-called whiskered tori in the literature) lying on a normally hyperbolic invariant manifold (NHIM). Such shadowing methods are equally applicable in our case, where we have an invariant manifold $\tilde{\Lambda}_\infty$ which is only topologically equivalent to a NHIM (see [Rob88], [Rob84], [Moe02], [Moe07], [GdlL06]).

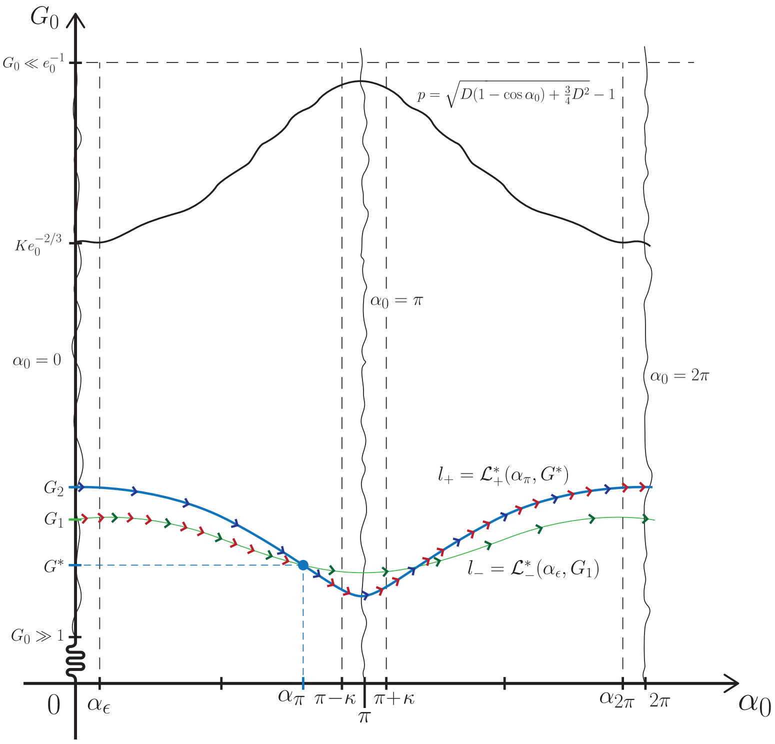


Figure 1.3: Mechanism for diffusion

With all these elements, we can now state our main results

Theorem 1.15. *Let $G_1^* < G_2^*$ large enough and e_0 small enough. More precisely $1 \ll G_1^* < G_2^* \ll 1/e_0$ and $\mu > 0$ small enough. Then, for any $G_1, G_2 \in (G_1^*, G_2^*)$ there exists a trajectory of the ERTBP such that $G(0) < G_1$, $G(T) > G_2$ for some $T > 0$.*

1.5.3 $e_0 G_0 = \lambda$, λ real positive

To prove diffusion in the case $e_0 G_0 = \lambda$, for λ a *fixed* positive number, we use propositions 1.2 and 1.3 as in section 1.5.1 to compute the scattering map. Nevertheless, we will use the computation of the Melnikov potential given in theorem 1.6, which gives a more involved expression of the scattering map in terms of the Bessel functions J_0 and J_1. Since the complete computations of the scattering maps are very cumbersome, it will not be possible to provide simple conditions, as in the case $\lambda \ll 1$, to guarantee the existence of diffusion on the complete zone $A/e_0 \leq G_0 \leq B/e_0$. Thus, in this section, we will see the same mechanism used in section 1.5.1 can be straight forwardly applied, up to some technical conditions that can be checked analytically or numerically.

The Melnikov potential is now given by the same formula (1.48), that is

$$\mathcal{L}(\alpha_0, G_0, t_0; e_0) = L_{0,0}(G_0) + \mathcal{L}_0(\alpha_0, G_0) + \mathcal{L}_1(\alpha_0, G_0, t_0) + \mathcal{E}(\alpha_0, G_0, t_0) \tag{1.97}$$

where $L_{0,0}$ is the same function as in equation (1.48) and is given by

$$L_{0,0}(G_0) = \frac{\pi}{2} G_0^{-3} + \mathcal{F}_1 \tag{1.98}$$

with

$$\mathcal{F}_1 = \mathcal{F}_1(G_0) = O(\lambda^2 G_0^{-5} + G_0^{-7}) = O(G_0^{-5}) \tag{1.99}$$

$\mathcal{L}_0(\alpha_0, G_0)$ is also the same function as in equation (1.48) and is given by

$$\mathcal{L}_0(\alpha_0, G_0) = -\frac{15}{8} \pi \lambda G_0^{-6} \cos(\alpha_0) + \mathcal{F} \tag{1.100}$$

with

$$\mathcal{F} = \mathcal{F}(\alpha_0, G_0) = O(\lambda^2 G_0^{-7}, \lambda G_0^{-10}) = O(G_0^{-7}). \tag{1.101}$$

Finally in this case, $\mathcal{E} = \mathcal{R}_3(\alpha_0, G_0, t_0)$ and therefore, as given in theorem 1.6 we have

$$\mathcal{E} = \mathcal{R}_3(\alpha_0, G_0, t_0) = O(G_0^{3/2} \mathrm{e}^{-G_0^3 \frac{4}{9}}).$$

Notice that we have omitted the dependence on e_0 by using that $e_0 = \lambda/G_0$ so that the functions depend on λ instead of e_0, although this dependence with respect to λ will not be written explicitly, since λ will be fixed along this section.

The expression for the function $\mathcal{L}_1$ differs from the one in equation (1.52). From theorem 1.6 we get now

$$\begin{aligned}\mathcal{L}_1(\alpha_0, G_0, t_0) = \cos(t_0 - \alpha_0)\sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-G_0^3/3}(1 + \tilde{E}_1) - \mathrm{e}^{-\frac{G_0^3}{3}} 4\sqrt{2\pi}\lambda^{-1} G_0^{1/2} \cdot \\ \cdot \Re\Bigg\{ \mathrm{e}^{it_0} \Big[\mathrm{e}^{-2A} \frac{A}{1-A} \Big[2f_1(A(A-1)) - f_0(A(A-1)) \Big] + A \Big] \\ (1 + \mathcal{R}_1(\alpha_0)) \Bigg\} + \mathcal{R}_3(\alpha_0, G_0, t_0)\end{aligned} \tag{1.102}$$

where $f_0(x) = J_0(2i\sqrt{x})$, $f_1(x) = J_1(2i\sqrt{x})/(2i\sqrt{x})$ (and both functions can be written in terms of the function $W(x) = \sum_{n\geq 0} x^n/(n!)^2$ introduced in section B.4), $A = (\lambda/2)\mathrm{e}^{-i\alpha_0}$ and the errors $\tilde{E}_1$, $\mathcal{R}_1$ and $\mathcal{R}_3$ satisfy

$$|\tilde{E}_1| \leq K(G_0^{-1} + \lambda^2 G_0^{-2}) = O(G_0^{-1}), \tag{1.103}$$

$$|\mathcal{R}_1| \leq K G_0^{-1}, \qquad |\mathcal{R}_3| \leq K G_0^{3/2} \mathrm{e}^{-4G_0^3/9}. \tag{1.104}$$

Analogously as lemma 1.8, we can write $\mathcal{L}_1$ in the form

$$\mathcal{L}_1(\alpha_0, G_0, t_0) = \sqrt{8\pi} G_0^{1/2} \mathrm{e}^{-G_0^3/3} B \cos(t_0 - \alpha_0 - \theta) \tag{1.105}$$

where

$$B\mathrm{e}^{-i\theta} = \frac{1 + \tilde{E}_1}{8G_0} - \Bigg[\frac{\mathrm{e}^{-2A}}{1-A} \Big[2Af_1(A(A-1)) - f_0(A(A-1)) \Big] + 1 \Bigg] (1 + \mathcal{R}_1(\alpha_0)) + \mathcal{R}_3(\alpha_0, G_0, t_0) \tag{1.106}$$

and $B = B(\alpha_0, \lambda_0) \geq 0$ and $\theta = \theta(\alpha_0, \lambda)$ is defined $mod(2\pi)$. As in lemma 1.8, θ is only well defined for those (α_0, G_0) such that $B > 0$. Notice that for G_0 big enough B will be positive as long as

$$\frac{\mathrm{e}^{-2A}}{1-A} \Big[2Af_1(A(A-1)) - f_0(A(A-1)) \Big] + 1 \neq 0$$

where we recall that $A = (\lambda/2)\mathrm{e}^{-i\alpha_0}$.

To check that the function $t_0 \longmapsto \mathcal{L}(\alpha_0, G_0, t_0)$ is a cosine-like function we just need a similar result to lemma 1.10 where now

$$\mathcal{L}_1^* = \sqrt{8\pi} G_0^{1/2} \mathrm{e}^{-G_0^3/3} B \tag{1.107}$$

and $\mathcal{L}_1^* = \mathcal{L}_1^*(\alpha_0, \lambda)$. Lemma 1.10 holds equally in this case since the size of the error term $\mathcal{E}$ in (1.97) is the same as the $\mathcal{E}$ in (1.48), in particular exponentially smaller than $\mathcal{L}_1^*$.

Analogously to proposition 1.11, we obtain two critical points $t_{0\pm}^* = t_{0\pm}^*(\alpha_0, \lambda)$ of the cosine-like function $t_0 \longmapsto \mathcal{L}(\alpha_0, G_0, t_0)$ which leads to two reduced Poincaré functions $\mathcal{L}_\pm^*$ which are given by

$$\mathcal{L}_\pm^*(\alpha_0, G_0) = L_{0,0}(G_0) + \mathcal{L}_0(\alpha_0, G_0) \pm \mathcal{L}_1^*(\alpha_0, \lambda) + E_\pm(\alpha_0, G_0) \tag{1.108}$$

with $E_\pm = O(\mathrm{e}^{-2G_0^3/9})$ which leads to an analogous formula to the one in (1.73).

We have now two scattering maps $S_\pm$ as the one given in (1.74). As before, it is essential to play with both of them, so we need them to be different, that is, we need some transversality condition like in lemma 1.13 which relies on the computation of the Poisson bracket $\{\mathcal{L}_+^*, \mathcal{L}_-^*\}$ performed in lemma 1.12. The computation of this Poisson bracket relies on a better knowledge of the function B given in (1.106).

We have now all the elements to use the same strategy of diffusion explained in section 1.5.2 which leads to the following diffusion theorem.

Theorem 1.16. *Fix $0 < \lambda_1 < \lambda_2$. Consider G_1^*, G_2^* large enough and $e_0 > 0$ small enough such that $\lambda_1/e_0 \leq G_1^* < G_2^* \leq \lambda_2/e_0$, and $\mu > 0$ small enough. Then for any $G_1, G_2 \in (G_1^*, G_2^*)$ in the zone where $\{\mathcal{L}_+^*, \mathcal{L}_-^*\} \neq 0$ one can find orbits of the ERTBP such that $G_0(0) < G_1$, $G(T) > G_2$ for some $T > 0$.*

To finish this memory, some words about the existence of diffusion for the case $e_0 G_0$ big, not studied here, are necessary. Most of the computations performed along this memory remain valid, for what concerns the computations of the Fourier coefficients $L_{q,k}$ of the Melnikov potential. The main difficulty relies on justify the validity of theorem 1.6 without the assumption $e_0 G_0 = \lambda$.

We believe that the error terms $\tilde{E}_1$, $\mathcal{R}_1$ and $\mathcal{R}_3$ in theorem 1.6 are still small in the general case $\lambda = e_0 G_0$ big, but the strategy to prove it has to be improved. In particular, the estimates for the error terms $\tilde{\mathcal{E}}_i$ of theorem 2.19 are not good enough in the case $\lambda = e_0 G_0$ big, and lemmas A.3, A.4, A.5 and A.6 need to be improved and as well as, and mainly, lemma A.7. On the other hand, the dominant part of the Melnikov potential which gives rise to the Poincaré reduced function $\mathcal{L}^*$ is easier in this case, since there are well known asymptotics for the Bessel functions $J_0(z)$ and $J_1(z)$ for $|z|$ large.

Chapter 2

Estimation of the Melnikov Potential

To prove theorems 1.5 and 1.6 we need to compute the Melnikov potential, whose formula is given by (1.47) in section 1.4 and reads

$$\mathcal{L}(\alpha_0, G_0, t_0; e_0) = \int_{-\infty}^{\infty} \left[\frac{x_h^2}{\left[4 + x_h^4 r_0^2 + 4x_h^2 r_0 \cos(\alpha_h - f)\right]^{1/2}} + \left(\frac{x_h^2}{2}\right)^2 r_0 \cos(\alpha_h - f) - \frac{x_h^2}{2} \right] dt \quad (2.1)$$

where x_h and α_h are coordinates of the homoclinic orbit which passes through the point $\tilde{\mathbf{z}}_0 \in \tilde{\gamma}$ defined in (1.32) we have chosen and are evaluated at t. f is the true anomaly defined in (1.5) and r_0 is defined in (1.4) and both are evaluated in $t + t_0$.

To estimate this Melnikov potential, we will follow different strategies, depending on the size of $e_0 G_0$. The main idea is to separate the periodic part from the one depending on the homoclinic. This will be done in the following way, if we rewrite equation (2.1) as

$$\mathcal{L}(\alpha_0, G_0, t_0; e_0) = \int_{-\infty}^{\infty} m(x_h(t), \alpha_h(t), t + t_0) dt$$

where $m(x, \alpha, s)$ is periodic in s. The classical way to compute these type of integrals is to use the Fourier expansion

$$m(x, \alpha, s) = \sum_{q \in \mathbb{Z}} m_q(x, \alpha) \mathrm{e}^{iqs}$$

to get

$$\mathcal{L}(\alpha_0, G_0, t_0; e_0) = \sum_{q \in \mathbb{Z}} L_q \mathrm{e}^{iqt_0}$$

where

$$L_q = \int_{-\infty}^{\infty} m_q(x_h(t), \alpha_h(t)) \mathrm{e}^{iqt} dt.$$

The main problem here is that we do not have an explicit expression for the Fourier coefficients m_q. Besides this, other problem is that we neither have explicit expressions for $x_h(t)$ and $\alpha_h(t)$, we only know these through a re-parametrization of time, given in equations (1.34).

To begin the computation of the Melnikov potential (2.1), first we present some results that will be useful. The Fourier expansion of the Melnikov potential is needed. Let us introduce some notation

$$L_{q,0} = \sum_{l \geq 1} \tilde{c}_q^{\ 2l,\ 0} N(q, l, l) \quad (2.2a)$$

$$L_{q,1} = \sum_{l\geq 2} \tilde{c}_q^{2l-1,-1} N(q, l-1, l) \tag{2.2b}$$

$$L_{q,-1} = \sum_{l\geq 2} \tilde{c}_q^{2l-1,\ \ 1} N(q, l, l-1) \tag{2.2c}$$

$$L_{q,k} = \sum_{l\geq k} \tilde{c}_q^{2l-k,-k} N(q, l-k, l) \quad \text{for } k \geq 2 \tag{2.2d}$$

$$L_{q,-k} = \sum_{l\geq k} \tilde{c}_q^{2l-k,\ \ k} N(q, l, l-k) \quad \text{for } k \geq 2 \tag{2.2e}$$

where $\tilde{c}_q^{n,m}$ is defined for $q, m, n \in \mathbb{Z}$ by the next Fourier expansion, given in [MP94, Win41]

$$[r_0(f(s))]^n \ \mathrm{e}^{imf(s)} = \sum_{q\in\mathbb{Z}} \tilde{c}_q^{n,m} \mathrm{e}^{iq\,(s)} \tag{2.3}$$

and

$$N(q,m,n) = \frac{2^{m+n}}{G_0^{2m+2n-1}} \binom{-1/2}{m} \binom{-1/2}{n} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq\frac{G_0^3}{2}\left(\tau+\frac{\tau^3}{3}\right)}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau \tag{2.4}$$

Because of equation (1.4), we know that $r_0(s)$ is a periodic function in s and because of equation (1.32b), $\cos(\alpha_h - f)$ is periodic in s and α_0 and therefore $\mathcal{L}$ is periodic in t_0 and α_0. With this in mind, the next proposition, whose proof is in appendix B makes sense

Proposition 2.1. The Melnikov potential given in (1.47) can be written as

$$\mathcal{L} = \sum_{q\in\mathbb{Z}} L_q \mathrm{e}^{iqt_0} \quad \text{with} \quad L_q = \sum_{k\in\mathbb{Z}} L_{q,k} \mathrm{e}^{ik\alpha_0}. \tag{2.5}$$

Then

$$\mathcal{L} = \sum_{q\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} L_{q,k} \mathrm{e}^{i(qt_0+k\alpha_0)} = 2\sum_{q\geq 0} \sum_{k\in\mathbb{Z}} L_{q,k} \cos(qt_0 + k\alpha_0). \tag{2.6}$$

where $L_{q,k}$ are given in (2.2).

Even more, since $\mathcal{L}$ is a real function even with respect to (α_0, G_0), and $\overline{L_{q,k}} = L_{-q,-k} = L_{q,k}$ and then $\overline{L_q} = L_{-q}$

$$\mathcal{L} = L_0 + 2\Re\Big\{ \sum_{q\geq 1} L_q \mathrm{e}^{iqt_0} \Big\} \tag{2.7}$$

where we can write

$$L_q = L_{q,0} + \sum_{k\geq 1} \big[L_{q,k} \mathrm{e}^{ik\alpha_0} + L_{q,-k} \mathrm{e}^{-ik\alpha_0} \big]$$

for $q \geq 0$.

In view of proposition (2.1) and formulas (2.2), to compute the dominant part of the Melnikov potential and obtain effective bounds of the errors we will need to estimate the constants $\tilde{c}_q^{n,m}$ defined in (2.3) and the integrals $N(q,m,n)$ defined in (2.4). This is done in the next three propositions.

Proposition 2.2. Let $n, m, q \in \mathbb{Z}$, $n, q \geq 0$, $n - m + 1 \geq 0$. Then the Fourier coefficients $\tilde{c}_q^{n,m}$ defined in (2.3) satisfy

$$|\tilde{c}_q^{n,m}| \leq \begin{cases} 2^{q+n+1} \mathrm{e}^{q\sqrt{1-e_0^2}} e_0^{|m-q|} & m \geq 0 \\ (1+e_0)^{n+1} & m < 0 \end{cases}$$

Also, the Fourier coefficients $\tilde{c}_q^{n,m}$ satisfy $\overline{\tilde{c}_q^{n,m}} = \tilde{c}_{-q}^{n,-m}$.

Proof. The integral formula for the Fourier coefficients reads

$$\tilde{c}_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} r_0(t)^n \mathrm{e}^{imf(t)}\mathrm{e}^{-iqt}dt \tag{2.8}$$

Changing the variable of integration t, using the identities (see [Win41, p. 194])

$$t(E) = E - e_0 \sin E \tag{2.9a}$$

$$\hat{r}(E) = r(t(E)) = 1 - e_0 \cos E \tag{2.9b}$$

$$\hat{r}(E)\mathrm{e}^{i\hat{f}(E)} = r(t(E))\mathrm{e}^{if(t(E))} = a^2\mathrm{e}^{iE} - e_0 + \frac{e_0^2}{4a^2}\mathrm{e}^{-iE} \tag{2.9c}$$

$$a^2 = \frac{1+\sqrt{1-e_0^2}}{2} = \frac{e_0^2}{2(1-\sqrt{1-e_0^2})} \tag{2.9d}$$

we have

$$\tilde{c}_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} [\hat{r}(E)\mathrm{e}^{i\hat{f}(E)}]^m \hat{r}(E)^{n-m+1}\mathrm{e}^{-iqt(E)}dE \tag{2.10}$$

To bound this integral we will consider two different cases for $m \geq 0$: $0 \leq q \leq m$ and $0 \leq m < q$. Let us first consider the case $0 \leq q \leq m$. By the analyticity and periodicity of the integral we change the path of integration from $\Im(E) = 0$ to $\Im E = \ln(2a^2/e_0)$, i. e.,

$$E = u + i\ln\Big(\frac{2a^2}{e_0}\Big) \qquad u \in [0, 2\pi]$$

we have

$$\mathrm{e}^{iE} = \mathrm{e}^{iu - \ln\left(\frac{2a^2}{e_0}\right)} = \frac{e_0}{2a^2}\mathrm{e}^{iu}$$

and then

$$\begin{aligned}\tilde{r}(u)\mathrm{e}^{i\tilde{f}(u)} = \hat{r}(E(u))\mathrm{e}^{i\hat{f}(E(u))} &= a^2\frac{e_0}{2a^2}\mathrm{e}^{iu} - e_0 + \frac{e_0^2}{4a^2}\frac{2a^2}{e_0}\mathrm{e}^{-iu} \\ &= \frac{e_0}{2}\mathrm{e}^{iu} - e_0 + \frac{e_0}{2}\mathrm{e}^{-iu} \\ &= e_0(\cos u - 1)\end{aligned} \tag{2.11}$$

$$\begin{aligned}\tilde{r}(u) = r(E(u)) &= 1 - \frac{e_0}{2}\left(\frac{e_0}{2a^2}\mathrm{e}^{iu} + \frac{2a^2}{e_0}\mathrm{e}^{-iu}\right) \\ &= 1 - \frac{e_0^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}\end{aligned} \tag{2.12}$$

$$\begin{aligned}\mathrm{e}^{-it(E)} = \mathrm{e}^{-i(E - e_0\sin E)} &= \frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0}{2}\left(\frac{e_0}{2a^2}\mathrm{e}^{iu} - \frac{2a^2}{e_0}\mathrm{e}^{-iu}\right)} \\ &= \frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}}\end{aligned} \tag{2.13}$$

therefore

$$\tilde{c}_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} [\tilde{r}(u)\mathrm{e}^{i\tilde{f}(u)}]^m \tilde{r}(u)^{n-m+1}\left[\frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}}\right]^q du \tag{2.14}$$

to bound this we just have to notice that thanks to (2.11), (2.12) and (2.9d)

$$|\tilde{r}(u)\mathrm{e}^{i\tilde{f}(u)}| \leq 2e_0 \tag{2.15}$$

$$|\tilde{r}(u)| = \sqrt{\Big(1 - \cos u\big(\frac{e_0^2}{4a^2} + a^2\big)\Big)^2 + \sin^2 u\big(a^2 - \frac{e_0^2}{4a^2}\big)^2}$$

$$
\begin{aligned}
&= \sqrt{\big(1-\cos u\big)^2 + \sin^2 u\big(1-e_0^2\big)} \\
&= \sqrt{2\big(1-\cos u\big) - e_0^2 \sin^2 u} \\
&\le 2
\end{aligned} \qquad (2.16)
$$

and using the definition of a^2 in (2.9)

$$
\begin{aligned}
\big|\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}}\big| &= \big|\mathrm{e}^{\left(\frac{e_0^2}{2a^2}-a^2\right)\cos u + i \sin u\left(\frac{e_0^2}{2a^2}+a^2\right)}\big| \\
&= \mathrm{e}^{\left(\frac{e_0^2}{2a^2}-a^2\right)\cos u} \\
&= \mathrm{e}^{-\sqrt{1-e_0^2}\cos u} \\
&\le \mathrm{e}^{\sqrt{1-e_0^2}}
\end{aligned} \qquad (2.17)
$$

substituting this bounds in the integral (2.14) and noticing that $a^2 \le 1$ we find directly the desired result for $0 \le q \le m$.

Now consider the case $0 \le m < q$. From equation (2.10) we perform the change of the integration variable through

$$
E = v - i\ln\big(\frac{2a^2}{e_0}\big), \qquad \tau \in [0, 2\pi]
$$

we have

$$
\mathrm{e}^{iE} = \mathrm{e}^{iv + \ln\left(\frac{2a^2}{e_0}\right)} = \frac{2a^2}{e_0}\mathrm{e}^{iv}
$$

and using (2.9)

$$
\begin{aligned}
\tilde{\tilde{r}}(v)\mathrm{e}^{i\tilde{\tilde{f}}(v)} = r(E(v))\mathrm{e}^{if(E(v))} &= a^2\frac{2a^2}{e_0}\mathrm{e}^{iv} - e_0 + \frac{e_0^2}{4a^2}\frac{e_0}{2a^2}\mathrm{e}^{-iv} \\
&= \frac{2a^4}{e_0}\mathrm{e}^{iv} - e_0 + \frac{e_0^3}{8a^4}\mathrm{e}^{-iv} \\
&= \frac{1}{e_0}\big(2a^4\mathrm{e}^{iv} - e_0^2 + \frac{e_0^4}{8a^4}\mathrm{e}^{-iv}\big)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\tilde{r}}(v) &= 1 - \frac{e_0}{2}\left(\frac{2a^2}{e_0}\mathrm{e}^{iv} + \frac{e_0}{2a^2}\mathrm{e}^{-iv}\right) \\
&= 1 - a^2\mathrm{e}^{iv} - \frac{e_0^2}{4a^2}\mathrm{e}^{-iv}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{e}^{-i(E-e_0\sin E)} &= \frac{e_0}{2a^2}\mathrm{e}^{-iv}\mathrm{e}^{\frac{e_0}{2}\left(\frac{2a^2}{e_0}\mathrm{e}^{iv} - \frac{e_0}{2a^2}\mathrm{e}^{-iv}\right)} \\
&= \frac{e_0}{2a^2}\mathrm{e}^{-iv}\mathrm{e}^{a^2\mathrm{e}^{iv} - \frac{e_0^2}{4a^2}\mathrm{e}^{-iv}}
\end{aligned}
$$

therefore

$$
\tilde{c}_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} [\tilde{\tilde{r}}(v)\mathrm{e}^{i\tilde{\tilde{f}}(v)}]^m \tilde{\tilde{r}}(v)^{n-m+1}\left[\frac{e_0}{2a^2}\mathrm{e}^{-iv}\mathrm{e}^{a^2\mathrm{e}^{iv} - \frac{e_0^2}{4a^2}\mathrm{e}^{-iv}}\right]^q dv
$$

to bound this we just have to notice that, using the definition of a^2 given in (2.9), $a^2 \ge 1/2$

$$
|\tilde{\tilde{r}}(v)\mathrm{e}^{i\tilde{\tilde{f}}(v)}| \le \frac{1}{e_0}\Big(2 + e_0^2 + \frac{1}{2}e_0^4\Big) < \frac{7}{2}\frac{1}{e_0}
$$

$$
\begin{aligned}
|\tilde{\tilde{r}}(v)| &= \sqrt{\left(1-\cos v\left(\frac{e_0^2}{4a^2}+a^2\right)\right)^2+\sin^2 v\left(a^2-\frac{e_0^2}{4a^2}\right)^2} \\
&= \sqrt{\left(1-\cos v\right)^2+\sin^2 v\left(1-e_0^2\right)} \\
&= \sqrt{2\left(1-\cos v\right)-e_0^2\sin^2 v} \\
&\leq 2
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\mathrm{e}^{a^2\mathrm{e}^{iv}-\frac{e_0^2}{4a^2}\mathrm{e}^{-iv}}\right| &= \left|\mathrm{e}^{\left(a^2-\frac{e_0^2}{2a^2}\right)\cos v+i\sin v\left(\frac{e_0^2}{2a^2}+a^2\right)}\right| \\
&= \mathrm{e}^{\left(a^2-\frac{e_0^2}{2a^2}\right)\cos v} \\
&= \mathrm{e}^{\sqrt{1-e_0^2}\cos v} \\
&\leq \mathrm{e}^{\sqrt{1-e_0^2}}
\end{aligned}
$$

using that $a^2 \geq 1/2$ we conclude

$$
\begin{aligned}
|\tilde{c}_q^{n,m}| &\leq \left(\frac{7}{2}\right)^m 2^{n-m+1}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^{q-m} \\
&\leq \left(\frac{7}{4}\right)^m 2^{n+1}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^{q-m}
\end{aligned}
$$

and since $7/4 < 2$ and $0 \leq m < q$ we have that

$$
\left(\frac{7}{4}\right)^m < 2^q
$$

from where we get the desired result for this case too.

For $m < 0$ we bound directly over the equation (2.10). Since $|\mathrm{e}^{if}| = |\mathrm{e}^{-it}| = 1$ we have

$$
|\tilde{c}_q^{n,m}| \leq \frac{1}{2\pi}\int_0^{2\pi} |\hat{r}(E)|^{n+1} dE
$$

by noticing that $|r(E)| \leq (1+e_0)$ we conclude the proof of the bounds for the $\tilde{c}_q^{n,m}$. Now, define

$$
P_{n,m}(t) = [r_0(f(t))]^n\ \mathrm{e}^{imf(t)}
$$

then

$$
\overline{P}_{n,m}(t) = [r_0(f(t))]^n\ \mathrm{e}^{-imf(t)} = P_{n,-m}
$$

but by equation (2.3)

$$
\begin{aligned}
\overline{P}_{n,m}(t) &= \sum_{q\in\mathbb{Z}} \tilde{c}_q^{n,m}\mathrm{e}^{-iq\,t} \\
P_{n,-m}(t) &= \sum_{q\in\mathbb{Z}} \tilde{c}_q^{n,-m}\mathrm{e}^{iq\,t}
\end{aligned}
$$

from where $\tilde{c}_{-q}^{n,-m} = \tilde{c}_q^{n,m} = \overline{\tilde{c}}_q^{n,m}$. □

As we can see from equations (2.2) the Fourier coefficients of the Melnikov potential $\mathcal{L}$ depend on the function N defined in (2.4), so that the next result, proved in appendix B will be useful

Proposition 2.3. Let $q, m, n \in \mathbb{Z}, m, n \geq 0$ $m+n > 0$, $q > 0$, $c \geq 1$ and $G_0 \geq c^{2/3}$. Then,

$$
|N(q,m,n)| \leq K_2 \mathrm{e}^{-q\frac{G_0^3}{3}} \mathrm{e}^{qc^2} 2^{n+m} G_0^{m-2n-\frac{1}{2}}
$$

with $K_2 = 6\pi\mathrm{e}^{-1/2}$

As we want to compute an asymptotic formula for the Melnikov potential (1.47), propositions 2.2 and 2.3 allow us to easily bound a lot of Fourier coefficients $L_{q,k}$. Nevertheless, we need to compute the integral involved in $N(q,m,n)$ given in (2.4) for some values of m,n and q. Next proposition comes in that direction. But before, we need to introduce the constants $d_j^{n,m}$. Let us define

$$h(\tau) = i\Big(\frac{\tau^3}{3} + \tau\Big)$$

and

$$u(\tau) = h(i) - h(\tau) = -\frac{2}{3} - i\Big(\frac{\tau^3}{3} + \tau\Big) = (\tau - i)^2 - \frac{i}{3}(\tau - i)^3. \tag{2.18}$$

It is easy to see that u is an increasing real valued function in the direction of increasing imaginary part over the set $\Im(h(\tau)) = 0$ (see figure 2.1), moreover

$$u(\{\tau^+ : \Im(h(\tau^+)) = 0, \Re(\tau^+) > 0\}) = u(\{\tau^- : \Im(h(\tau^-)) = 0, \Re(\tau^-) < 0\}) \subset \mathbb{R}_0^+.$$

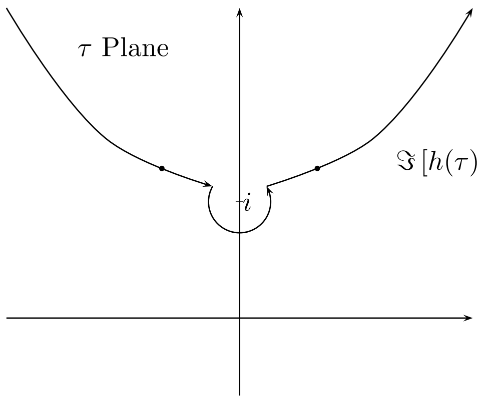


Figure 2.1: $\Im(h(\tau))$

Therefore u has two inverses inverses; τ^+ and τ^- with domain in $\mathbb{R}_0^+$. Now let

$$F_{m,n}^{\pm}(u) = \frac{1}{(\tau^{\pm}(u) - i)^{2m+1}(\tau^{\pm}(u) + i)^{2n+1}}.$$

whose expansion in $\sqrt{u}$ is given in lemma (A.4):

$$F_{m,n}^{\pm}(u) = (\pm\sqrt{u})^{-2m-1} \sum_{j=0}^{\infty} d_j^{n,m} (\pm\sqrt{u})^j. \tag{2.19}$$

for some coefficients $d_j^{n,m}$.

Let us call

$$d_{m,n} = i2^{m+n}\binom{-1/2}{n}\binom{-1/2}{m}. \tag{2.20}$$

Next proposition provides an asymptotic expression for $N(q,m,n)$ for big values of G_0. Its proof is given in section B.3.

Proposition 2.4. Let $n + m > 0$ and the constants $d_j^{n,m}$ be defined by equation (2.19) and $d_{n,m}$ by equation (2.20). If $q,n,m \in \mathbb{Z}, m,n \geq 0$, $q \geq 0$ then

$$N(q,m,n) = \frac{d_{m,n} \mathrm{e}^{-q\frac{G_0^3}{3}}}{G_0^{2m+2n-1}} \left[\sum_{s=0}^{m} (-1)^s \sqrt{\pi} \frac{2^{\frac{3}{2}} q^{s-\frac{1}{2}}}{(2s-1)!!} d_{2m-2s}^{n,m} G_0^{3s-\frac{3}{2}} + T_{m,n}^q + R_{m,n}^q \right]$$

where

$$|T_{m,n}^q| \leq K_{11} \gamma_4^m G_0^{-3} \qquad |R_{m,n}^q| \leq K_{12} q^{m-1} G_0^{3m-3}.$$

and

$$\beta = \left(-1 + \frac{\sqrt{11}}{4}\sqrt{3 + \frac{\sqrt{11}}{2}}\right)^{1/2}, \quad \gamma_4 = \frac{2}{\beta^2}, \quad K_{11} = 2^2\left(\sqrt{2} + \frac{2}{\beta(1-\beta)}\right), \quad K_{12} = 2\pi \mathrm{e}^{4/3}.$$

When $s = 0$ the factor $1/(2s-1)!!$ in the formula should be replaced by 1.

2.1 $e_0 G_0 \ll 1$

In view of proposition 2.4, the dominant part of the Melnikov potential $\mathcal{L}$ comes from the Fourier coefficient L_1, the main terms of this coefficients are computed using proposition 2.4, and the rest of the terms will be bounded using propositions 2.2 and 2.3.

In this section we will prove a much more quantitative version of theorem 1.5 wich will immediately imply it.

Theorem 2.5. *Let $c \geq 1$. If $G_0 \geq 32$, $e_0 G_0 < 1/8$, then the Melnikov potential $\mathcal{L}$ given by* (2.7) *satisfy*

$$\mathcal{L} = L_0(\alpha_0) + \cos(t_0 - \alpha_0)\left(\sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} + E_3 + E_5 + E_7\right) - \cos(t_0 - 2\alpha_0)\left(3\sqrt{2\pi} e_0 G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}} + E_4 + E_6 + E_8\right) \\ + 2\Re\{E_2(\alpha_0)\mathrm{e}^{it_0}\} + E_1(t_0, \alpha_0)$$

$$\begin{aligned}
|E_1(t_0, \alpha_0)| &\leq K_5 2^6 \mathrm{e}^{2\sqrt{1-e_0^2}} G_0^{3/2} \mathrm{e}^{-G_0^3 \frac{4}{9}} \\
|E_2(\alpha_0)| &\leq K_6 \mathrm{e}^{-\frac{G_0^3}{3}} \left[(1+e_0)^4 G_0^{-7/2} + 2^6 \mathrm{e}^{\sqrt{1-e_0^2}} (e_0^2 G_0^{5/2} + e_0 G_0^{-3/2})\right] \\
|E_3| &\leq K_7 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{-\frac{G_0^3}{3}} G_0^{-3/2} \\
|E_4| &\leq K_8 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{-\frac{G_0^3}{3}} e_0 G_0^{1/2} \\
|E_5| &\leq 2^5 \mathrm{e}^{\sqrt{1-e_0^2}} K_{13} G_0^{-2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|E_6| &\leq 2^4 \mathrm{e}^{\sqrt{1-e_0^2}} e_0 K_{13} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|E_7| &\leq \sqrt{\frac{\pi}{8}} 98 e_0^2 G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|E_8| &\leq \sqrt{2\pi} 50 e_0^2 G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}}
\end{aligned}$$

with

$$\begin{aligned}
K_5 &= 1152\pi \mathrm{e}^{-1/2} \\
K_6 &= 2^4 12\pi \mathrm{e}^{c^2 - 1/2} \\
K_7 &= 2^{14} \cdot 3\pi \mathrm{e}^{c^2 - 1/2} \\
K_8 &= 2^{12} \cdot 3\pi \mathrm{e}^{c^2 - 1/2} \\
\beta &= \left(-1 + \frac{\sqrt{11}}{4}\sqrt{3 + \frac{\sqrt{11}}{2}}\right)^{1/2} \\
K_{11} &= 2^2\left(\sqrt{2} + \frac{2}{\beta(1-\beta)}\right) \\
K_{12} &= 2\pi \mathrm{e}^{4/3} \\
\gamma_4 &= \frac{2}{\beta^2} \\
K_{13} &= 40\sqrt{\frac{\pi}{3}} + \frac{3}{2}\gamma_4^2 (K_{11} + K_{12})
\end{aligned}$$

and

$$L_0(\alpha_0) - L_{0,0} = -\frac{15}{8}\pi e_0 G_0^{-5} \cos(\alpha_0) + F_2 + F_3 + F_5$$

where

$$
\begin{aligned}
|F_2| &\le K_{22} 2^3 e_0 G_0^{-9} \\
|F_3| &\le K_{23} e_0^2 G_0^{-7} \\
|F_5| &\le K_{25} G_0^{-5} e_0^2
\end{aligned}
$$

with

$$
\begin{aligned}
K_{22} &= 2^{10} \mathrm{e}^{-1} \pi \\
K_{23} &= 2^7 \mathrm{e}^{-1} \pi \\
K_{25} &= 57\pi/4
\end{aligned}
$$

The proof of the theorem will be done constructively through the following series of lemmas and propositions

Let us first compute some coefficients $\tilde{c}_q^{n,m}$, more precisely $\tilde{c}_1^{3,1}$, $\tilde{c}_1^{2,2}$, $\tilde{c}_0^{2,0}$ and $\tilde{c}_0^{3,1}$

Lemma 2.6. Let $\tilde{c}_q^{n,m}$ be defined by (2.3). Then

$$
\begin{aligned}
\tilde{c}_1^{3,1} &= 1 + Q_1 \\
\tilde{c}_1^{2,2} &= -3e_0 + Q_2 \\
\tilde{c}_0^{2,0} &= 1 + Q_3 \\
\tilde{c}_0^{3,1} &= -\frac{5}{2} e_0 + Q_4
\end{aligned}
$$

with

$$
\begin{aligned}
|Q_1| &\le 98 e_0^2 \\
|Q_2| &\le 50 e_0^2 \\
|Q_3| &\le 4 e_0^2 \\
|Q_4| &\le 19 e_0^2
\end{aligned}
$$

Proof. From its definition given in (2.3) and using the change of variable $t = E - e_0 \sin E$ we have

$$
\begin{aligned}
\tilde{c}_1^{3,1} &= \frac{1}{2\pi} \int_0^{2\pi} [r(E) \mathrm{e}^{if(E)}] r(E)^3 \mathrm{e}^{-it(E)} dE \\
\tilde{c}_1^{2,2} &= \frac{1}{2\pi} \int_0^{2\pi} [r(E) \mathrm{e}^{if(E)}]^2 r(E) \mathrm{e}^{-it(E)} dE \\
\tilde{c}_0^{2,0} &= \frac{1}{2\pi} \int_0^{2\pi} r(E)^3 dE \\
\tilde{c}_0^{3,1} &= \frac{1}{2\pi} \int_0^{2\pi} [r(E) \mathrm{e}^{if(E)}] r(E)^3 dE
\end{aligned}
$$

From equations (2.9) we have

$$
\tilde{c}_1^{3,1} = \frac{1}{2\pi} \int_0^{2\pi} [a^2 \mathrm{e}^{iE} - e_0 + \frac{e_0^2}{4a^2} \mathrm{e}^{-iE}] (1 - e_0 \cos E)^3 \mathrm{e}^{-iE} \mathrm{e}^{ie_0 \sin E} dE \tag{2.21}
$$

$$
\tilde{c}_1^{2,2} = \frac{1}{2\pi} \int_0^{2\pi} [a^2 \mathrm{e}^{iE} - e_0 + \frac{e_0^2}{4a^2} \mathrm{e}^{-iE}]^2 (1 - e_0 \cos E) \mathrm{e}^{-iE} \mathrm{e}^{ie_0 \sin E} dE \tag{2.22}
$$

$$
\tilde{c}_0^{2,0} = \frac{1}{2\pi} \int_0^{2\pi} (1 - e_0 \cos E)^3 dE \tag{2.23}
$$

$$
\tilde{c}_0^{3,1} = \frac{1}{2\pi} \int_0^{2\pi} [a^2 \mathrm{e}^{iE} - e_0 + \frac{e_0^2}{4a^2} \mathrm{e}^{-iE}] (1 - e_0 \cos E)^3 dE \tag{2.24}
$$

To bound $\tilde{c}_1^{3,1}$ we use equation (2.21). It is easy to see that

$$a^2 e^{iE} - e_0 + \frac{e_0^2}{4a^2} e^{-iE} = e^{iE} - e_0 + \bar{E}_1 \tag{2.25a}$$
$$(1 - e_0 \cos E)^3 = 1 - 3e_0 \cos E + \bar{E}_2 \tag{2.25b}$$
$$e^{ie_0 \sin E} = 1 + ie_0 \sin E + \bar{E}_3 \tag{2.25c}$$

where

$$\bar{E}_1 = (a^2 - 1)e^{iE} + \frac{e_0^2}{4a^2} e^{-iE}$$
$$\bar{E}_2 = 3e_0^2 \cos^2 E - e_0^3 \cos^3 E$$
$$\bar{E}_3 = \frac{1}{2}(ie_0 \sin E)^2 \sum_{j=0}^{\infty} 2\frac{(ie_0 \sin E)^j}{(j+2)!}$$

Since

$$0 \le e_0 \le 1 \tag{2.26a}$$
$$\frac{1}{2} \le a^2 \le 1 \tag{2.26b}$$
$$|a^2 - 1| = |\frac{\sqrt{1-e_0^2} - 1}{2}| = |\frac{-e_0^2}{2(\sqrt{1-e_0^2}+1)}| \le \frac{e_0^2}{2} \tag{2.26c}$$

we have

$$|\bar{E}_1| \le \frac{e_0^2}{2} + \frac{e_0^2}{2} = e_0^2$$
$$|\bar{E}_2| \le 4e_0^2$$
$$|\bar{E}_3| \le \frac{e_0^2}{2} e^{e_0} \le e_0^2 \frac{e}{2} \le 2e_0^2$$

Using equations (2.25), we have from equation (2.21) that $\tilde{c}_1^{3,1}$ is the Fourier coefficient of order 1 of the function

$$(e^{iE} - e_0 + \bar{E}_1)(1 - 3e_0 \cos E + \bar{E}_2)(1 + ie_0 \sin E + \bar{E}_3) = \\ e^{iE} - e_0 - 3e_0 \cos E e^{iE} + ie_0 \sin E e^{iE} + \tilde{Q}_1(E)$$

where

$$\begin{aligned}\tilde{Q}_1(E) = {} & \bar{E}_1 - 3e_0^2 \cos E - 3e_0 \bar{E}_1 \cos E + \bar{E}_2(e^{iE} - e_0 + \bar{E}_1) \\ & - ie_0^2 \sin E - 3ie_0^2 \cos E \sin E e^{iE} - 3ie_0^3 \cos E \sin E - 3ie_0^2 \sin E \cos E \bar{E}_2 + ie_0 \sin E \bar{E}_2(e^{iE} - e_0 + \bar{E}_1) \\ & + \bar{E}_3(e^{iE} - e_0 + \bar{E}_1 - 3e_0 \cos E e^{iE} - 3e_0^2 \cos E - 3e_0 \bar{E}_1 \cos E + \bar{E}_2(e^{iE} - e_0 + \bar{E}_1))\end{aligned}$$

this implies that, up to order one in e_0 the Fourier coefficient $\tilde{c}_1^{3,1}$ is exactly one. From the bounds for $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ we find $|\tilde{Q}_1(E)| \le 98e_0^2$ wich implies the result for $\tilde{c}_1^{3,1}$.

From equation (2.22), it is easy to see that, using equation (2.25a)

$$\left[a^2 e^{iE} - e_0 + \frac{e_0^2}{4a^2} e^{-iE}\right]^2 = \left[e^{iE} - e_0 + \bar{E}_1\right]^2 = e^{2iE} - 2e_0 e^{iE} + \bar{E}_4$$

where

$$\bar{E}_4 = e_0^2 + 2\bar{E}_1(e^{iE} - e_0) + \bar{E}_1^2$$

in regard of equations (2.26) and the bound for $\tilde{E}_1$ we have

$$|\bar{E}_4| \le e_0^2 + 2e_0^2(1+e_0) + e_0^4 \le 6e_0^2.$$

Using equation (2.25c), we see from equation (2.22) that $\tilde{c}_1^{2,2}$ is the Fourier coefficient of order 1 of the function

$$(\mathrm{e}^{2iE} - 2e_0\mathrm{e}^{iE} + \bar{E}_4)(1 - e_0\cos E)(1 + ie_0\sin E + \bar{E}_3) = \\ \mathrm{e}^{2iE} - e_0\cos E\mathrm{e}^{2iE} - 2e_0\mathrm{e}^{iE} + ie_0\sin E\mathrm{e}^{2iE} + \tilde{Q}_2(E)$$

where

$$\begin{aligned}\tilde{Q}_2(E) &= 2e_0^2\cos E\mathrm{e}^{iE} + \bar{E}_4 - e_0\bar{E}_4\cos E \\ &= ie_0\sin E(-e_0\cos E\mathrm{e}^{2iE} - 2e_0\mathrm{e}^{iE} + 2e_0^2\cos E\mathrm{e}^{iE} + \bar{E}_4 - e_0\bar{E}_4\cos E) \\ &= \bar{E}_3(\mathrm{e}^{2iE} - e_0\cos E\mathrm{e}^{2iE} - 2e_0\mathrm{e}^{iE} + 2e_0^2\cos E\mathrm{e}^{iE} + \bar{E}_4 - e_0\bar{E}_4\cos E)\end{aligned}$$

with this expressions, we conclude that, up to order one in e_0, the Fourier coefficient $c_1^{2,2}$ is exactly $-3e_0$, and from the bounds for $\bar{E}_4$ and $\bar{E}_3$ we find that $|\tilde{Q}_2(E)| \le 50e_0^2$ wich implies the result for $\tilde{c}_1^{2,2}$.

Using equation (2.23) to compute $\tilde{c}_0^{2,0}$ we have using equation (2.25b)

$$\tilde{c}_0^{2,0} = \frac{1}{2\pi}\int_0^{2\pi}(1 - 3e_0\cos E + \bar{E}_2)dE = 1 + Q_3$$

then, by setting

$$Q_3 = \frac{1}{2\pi}\int_0^{2\pi}\bar{E}_2 dE$$

we have immediately, using the bound for $\bar{E}_2$, that $|Q_3| \le 4e_0^2$, the desired result for $\tilde{c}_0^{2,0}$.

Using equation (2.24) to compute $\tilde{c}_0^{3,1}$ using we have, (2.25a) and (2.25b)

$$\tilde{c}_0^{3,1} = \frac{1}{2\pi}\int_0^{2\pi}(\mathrm{e}^{iE} - e_0 + \bar{E}_1)(1 - 3e_0\cos E + \bar{E}_2)dE$$

Now, we want to find, up to order e_0 the Fourier coefficient of order zero of the function

$$(\mathrm{e}^{iE} - e_0 + \bar{E}_1)(1 - 3e_0\cos E + \bar{E}_2) = \mathrm{e}^{iE} - 3e_0\mathrm{e}^{iE}\cos E - e_0 + \bar{E}_5$$

where

$$\bar{E}_5 = \bar{E}_2\mathrm{e}^{iE} + 3e_0^2\cos E - e_0\bar{E}_2 + \bar{E}_1 - 3e_0\bar{E}_1\cos E + \bar{E}_2\bar{E}_1$$

from where we find

$$\tilde{c}_0^{3,1} = -\frac{5}{2}e_0 + Q_4$$

we can bound

$$Q_4 = \frac{1}{2\pi}\int_0^{2\pi}\bar{E}_5 dE$$

using the bounds for $\bar{E}_2$ and $\bar{E}_1$ to find $|Q_4| \le 19e_0^2$. □

Lemma 2.7. Let $c \ge 1$, $G_0 > 32$. If $q, k \in \mathbb{N}$, $k \ge 2$, then

$$\begin{aligned}|L_{q,\ 0}| &\le K_3 2^{q+5}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^q G_0^{-3/2}\mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} \\ |L_{q,\ 1}| &\le K_3 2^3(1+e_0)^4 G_0^{-7/2}\mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} \\ |L_{q,-1}| &\le K_3 2^{q+7}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^{|1-q|}G_0^{-1/2}\mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)}\end{aligned}$$

$$|L_{q,\ \ k}| \le K_3 2^k (1+e_0)^{k+1} G_0^{-2k-1/2} \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)}$$

$$|L_{q,-k}| \le K_3 2^{q+2k+1} \mathrm{e}^{q\sqrt{1-e_0^2}} e_0^{|k-q|} G_0^{k-1/2} \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)}$$

where $K_3 = 12\pi \mathrm{e}^{-1/2}$.

Proof. From equations (2.2) by propositions 2.2 and 2.3 we have

$$|L_{q,\ \ 0}| \le 2K_2 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} (2e_0 \mathrm{e}^{\sqrt{1-e_0^2}})^q G_0^{-1/2} \sum_{l\ge 1} (2^4 G_0^{-1})^l$$

$$|L_{q,\ \ 1}| \le 2^{-1} K_2 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} G_0^{-3/2} \sum_{l\ge 2} ((1+e_0)^2 2^2 G_0^{-1})^l$$

$$|L_{q,-1}| \le 2^{-1} K_2 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} \mathrm{e}^{q\sqrt{1-e_0^2}} 2^q e_0^{|1-q|} G_0^{3/2} \sum_{l\ge 2} (2^4 G_0^{-1})^l$$

$$|L_{q,\ \ k}| \le 2^{-k} K_2 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} (1+e_0)^{-k} G_0^{-k-1/2} \sum_{l\ge k} ((1+e_0)^2 2^2 G_0^{-1})^l$$

$$|L_{q,-k}| \le 2^{-2k+1} K_2 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} \mathrm{e}^{q\sqrt{1-e_0^2}} 2^q e_0^{|k-q|} G_0^{2k-1/2} \sum_{l\ge k} (2^4 G_0^{-1})^l$$

since by hypothesis $2^4/G_0 \le 1/2$ all these series converge and by setting

$$K_3 = 2K_2$$

we have proven the lemma. ◻

Lemma 2.8. If $q \in \mathbb{N}$, $q \ge 2$. Assume $G_0 \ge 32$, $e_0 G_0 < 1/8$ then

$$|L_q| \le \sum_{k\in\mathbb{Z}} |L_{q,k}| \le K_4 \mathrm{e}^{-qG_0^3\frac{2}{9}} \left[2^{3q} \mathrm{e}^{q\sqrt{1-e_0^2}} G_0^{q-1/2}\right] \tag{2.28}$$

where $K_4 = 288\pi \mathrm{e}^{-1/2}$

Proof. From lemma (2.7) we have for any $c \ge 1$

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |L_{q,k}| \le{}& |L_{q,0}| + |L_{q,1}| + |L_{q,-1}| + \sum_{k\ge 2} \big(|L_{q,k}| + |L_{q,-k}|\big) \\
\le{}& K_3 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} \Big[2^5 2^q e_0^q \mathrm{e}^{q\sqrt{1-e_0^2}} G_0^{-3/2} + 2^3 (1+e_0)^4 G_0^{-7/2} + 2^{7+q} e_0^{q-1} \mathrm{e}^{q\sqrt{1-e_0^2}} G_0^{-1/2} \\
&+ \sum_{k\ge 2} \Big(2^k (1+e_0)^{k+1} G_0^{-2k-1/2} + 2^{2k+q+1} \mathrm{e}^{q\sqrt{1-e_0^2}} e_0^{|k-q|} G_0^{k-1/2}\Big)\Big] \\
\le{}& K_3 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} \Big[2^7 2^q e_0^{q-1} \mathrm{e}^{q\sqrt{1-e_0^2}} G_0^{-1/2} + 2^3 (1+e_0)^4 G_0^{-7/2} + (1+e_0) G_0^{-1/2} \sum_{k=2}^{\infty} (2(1+e_0) G_0^{-2})^k \\
&+ \mathrm{e}^{q\sqrt{1-e_0^2}} G_0^{-1/2} 2^{q+1} e_0^q \sum_{k=2}^{q-1} (4 G_0 e_0^{-1})^k + \mathrm{e}^{q\sqrt{1-e_0^2}} G_0^{-1/2} e_0^{-q} 2^{q+1} \sum_{k=q}^{\infty} (4 e_0 G_0)^k\Big]
\end{aligned}$$

choosing $c = 1$, we have that $G_0 \ge (3c)^{2/3}$, and therefore using that $e_0 G_0 \le 1/8$

$$\sum_{k\in\mathbb{Z}} |L_{q,k}| \le K_3 \mathrm{e}^{-qG_0^3\frac{2}{9}} \Big[2^7 2^q e_0^{q-1} \mathrm{e}^{q\sqrt{1-e_0^2}} G_0^{-1/2} + 2^3 (1+e_0)^4 G_0^{-7/2} + 2^3 (1+e_0)^3 G_0^{-9/2}$$

$$+ 2^{3q}\mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{q-3/2}e_0 + \mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{q-1/2}2^{3q+2}\Big]$$

$$\le K_3\mathrm{e}^{-qG_0^3\frac{2}{9}}\Big[2^7 2^q e_0^{q-1}\mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{-1/2} + 2^4(1+e_0)^4 G_0^{-7/2}$$

$$+ 2^{3q}\mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{q-3/2}e_0 + \mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{q-1/2}2^{3q+2}\Big]$$

$$\le K_3\mathrm{e}^{-qG_0^3\frac{2}{9}}\Big[2^7 2^q e_0^{q-1}\mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{-1/2} + 2^4(1+e_0)^4 G_0^{-7/2}$$

$$+ 2^{3q+3}\mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{q-1/2}\Big]$$

$$\le K_3\mathrm{e}^{-qG_0^3\frac{2}{9}}\Big[3\cdot 2^{3q+3}\mathrm{e}^{q\sqrt{1-e_0^2}}G_0^{q-1/2}\Big]$$

setting $K_4 = 3\cdot 2^3 K_3$, we conclude the proof. □

Lemma 2.9. If $\mathcal{L}$ is given by (2.7), $G_0 \ge 32$, $e_0 G_0 < 1/8$. Then

$$\mathcal{L} = L_0 + 2\Re\Big\{\mathrm{e}^{it_0}L_1\Big\} + E_1(t_0,\alpha_0)$$

where

$$|E_1(t_0,\alpha_0)| \le K_5 2^6\mathrm{e}^{2\sqrt{1-e_0^2}}G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}$$

and $K_5 = 1152\pi\mathrm{e}^{-1/2}$.

Proof. From equation (2.7) we have that

$$E_1(t_0,\alpha_0) = 2\Re\Big\{\sum_{q\ge 2}\mathrm{e}^{iqt_0}L_q\Big\}$$

and then by lemma 2.8

$$|E_1(t_0,\alpha_0)| \le 2K_4 G_0^{-1/2}\sum_{q\ge 2}\Big[\mathrm{e}^{-G_0^3\frac{2}{9}}2^3\mathrm{e}^{\sqrt{1-e_0^2}}G_0\Big]^q$$

$$\le 4K_4 2^6\mathrm{e}^{2\sqrt{1-e_0^2}}G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}$$

where the last bound holds if

$$\mathrm{e}^{-G_0^3\frac{2}{9}}2^3\mathrm{e}^{\sqrt{1-e_0^2}}G_0 \le \frac{1}{2}. \tag{2.29}$$

wich is true for every $G_0 \ge 32$.

Letting $K_5 = 4K_4 = 1152\pi\mathrm{e}^{-1/2}$ we have proven the lemma. □

The next step is to compute an asymptotic formula for L_1.

Lemma 2.10. If L_1 is given by (2.7), $G_0 \ge 32$, $e_0 G_0 \le 1/8$ then

$$\Re\{\mathrm{e}^{it_0}L_1\} = \Re\big\{[(\tilde{c}_1^{3,1}N(1,2,1) + E_3)\mathrm{e}^{-i\alpha_0} + (\tilde{c}_1^{2,2}N(1,2,0) + E_4)\mathrm{e}^{-2i\alpha_0} + E_2(\alpha_0)]\mathrm{e}^{it_0}\big\}$$

where

$$|E_2(\alpha_0)| \le K_6\mathrm{e}^{-\frac{G_0^3}{3}}\Big[(1+e_0)^4 G_0^{-7/2} + 2^6\mathrm{e}^{\sqrt{1-e_0^2}}(e_0^2 G_0^{5/2} + e_0 G_0^{-3/2})\Big]$$

$$|E_3| \le K_7\mathrm{e}^{\sqrt{1-e_0^2}}\mathrm{e}^{-\frac{G_0^3}{3}}G_0^{-3/2}$$

$$|E_4| \le K_8\mathrm{e}^{\sqrt{1-e_0^2}}\mathrm{e}^{-\frac{G_0^3}{3}}e_0 G_0^{1/2}$$

and $K_6 = 2^4 12\pi\mathrm{e}^{c^2-1/2}$, $K_7 = 2^{14}\cdot 3\pi\mathrm{e}^{c^2-1/2}$, $K_8 = 2^{12}\cdot 3\pi\mathrm{e}^{c^2-1/2}$

Proof. From equation (2.5), we have that

$$L_1 = L_{1,0} + \sum_{k\geq 1}\left(L_{1,k}e^{ik\alpha_0} + L_{1,-k}e^{-ik\alpha_0}\right)$$
$$= L_{1,-1}e^{-i\alpha_0} + L_{1,-2}e^{-2i\alpha_0} + \sum_{k\geq 0} L_{1,k}e^{ik\alpha_0} + \sum_{k\geq 3} L_{1,-k}e^{-ik\alpha_0}$$

Now, setting

$$E_2(\alpha_0) = \sum_{k\geq 0} L_{1,k}e^{ik\alpha_0} + \sum_{k\geq 3} L_{1,-k}e^{-ik\alpha_0} \tag{2.30}$$

we can write

$$\Re\Big\{L_1 e^{it_0}\Big\} = \Re\Big\{\big(L_{1,-1}e^{-i\alpha_0} + L_{1,-2}e^{-2i\alpha_0} + \bar{E}_2(\alpha_0)\big)e^{it_0}\Big\} \tag{2.31}$$

By definitions (2.2) we have

$$L_{1,-1} = \tilde{c}_1^{3,1}N(1,2,1) + \sum_{l\geq 3}\tilde{c}_1^{2l-1,1}N(1,l,l-1) \tag{2.32a}$$
$$L_{1,-2} = \tilde{c}_1^{2,2}N(1,2,0) + \sum_{l\geq 3}\tilde{c}_1^{2l-2,2}N(1,l,l-2) \tag{2.32b}$$

If we set

$$E_3 = \sum_{l\geq 3}\tilde{c}_1^{2l-1,1}N(1,l,l-1) \tag{2.33a}$$
$$E_4 = \sum_{l\geq 3}\tilde{c}_1^{2l-2,2}N(1,l,l-2) \tag{2.33b}$$

we have from equations (2.32) and (2.31) that

$$\Re\{e^{it_0}L_1\} = \Re\big\{[(\tilde{c}_1^{3,1}N(1,2,1) + E_3)e^{-i\alpha_0} + (\tilde{c}_1^{2,2}N(1,2,0) + E_4)e^{-2i\alpha_0} + E_2(\alpha_0)]e^{it_0}\big\} \tag{2.34}$$

This is exactly the equation we want, it only remains to bound properly the $\bar{E}$'s.

Now from equation (2.30), by the triangle inequality and lemma 2.7 we have

$$|E_2(\alpha_0)| \leq |L_{1,0}| + |L_{1,1}| + \sum_{k\geq 2}|L_{1,k}| + \sum_{k\geq 3}|L_{1,-k}|$$
$$\leq K_3 e^{-\frac{G_0^3}{3}}e^{c^2}\Bigg[2^6 e_0 e^{\sqrt{1-e_0^2}}G_0^{-3/2} + 2^3(1+e_0)^4G_0^{-7/2} + \sum_{k\geq 2}2^k(1+e_0)^{k+1}G_0^{-2k-1/2}$$
$$+ \sum_{k\geq 3}2^{2k+2}e^{\sqrt{1-e_0^2}}e_0^{k-1}G_0^{k-1/2}\Bigg]$$
$$\leq K_3 e^{-\frac{G_0^3}{3}}e^{c^2}\Bigg[2^6 e_0 e^{\sqrt{1-e_0^2}}G_0^{-3/2} + 2^3(1+e_0)^4G_0^{-7/2} + (1+e_0)G_0^{-1/2}\sum_{k=2}^{\infty}(2(1+e_0)G_0^{-2})^k$$
$$+ 2^2 e^{\sqrt{1-e_0^2}}e_0^{-1}G_0^{-1/2}\sum_{k\geq 3}(2^2 e_0 G_0)^k\Bigg]$$
$$\leq K_3 e^{-\frac{G_0^3}{3}}e^{c^2}\Bigg[2^6 e_0 e^{\sqrt{1-e_0^2}}G_0^{-3/2} + 2^3(1+e_0)^4G_0^{-7/2} + 2^3(1+e_0)^3G_0^{-9/2}$$
$$+ 2^9 e^{\sqrt{1-e_0^2}}e_0^2G_0^{5/2}\Bigg]$$
$$\leq K_3 e^{-\frac{G_0^3}{3}}e^{c^2}\Bigg[2^4(1+e_0)^4G_0^{-7/2} + 2^{10}e^{\sqrt{1-e_0^2}}(e_0^2G_0^{5/2} + e_0G_0^{-3/2})\Bigg]$$
$$\leq K_6 e^{-\frac{G_0^3}{3}}\Bigg[(1+e_0)^4G_0^{-7/2} + 2^6 e^{\sqrt{1-e_0^2}}(e_0^2G_0^{5/2} + e_0G_0^{-3/2})\Bigg] \tag{2.35}$$

with

$$K_6 = 2^4 K_3 \mathrm{e}^{c^2} = 2^4 12\pi \mathrm{e}^{c^2-1/2} \tag{2.36}$$

Now E_3 and E_4. By propositions 2.2 and 2.3, we have from equations (2.33)

$$\begin{aligned}
|E_3| &\le \sum_{l\ge 3} |\tilde{c}_1^{2l-1,1} N(1,l,l-1)| \\
&\le K_2 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{c^2} \mathrm{e}^{-\frac{G_0^3}{3}} G_0^{3/2} \sum_{l\ge 3} (2^4 G_0^{-1})^l \\
&\le K_7 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{-\frac{G_0^3}{3}} G_0^{-3/2} \qquad (2.37)\\
|E_4| &\le \sum_{l\ge 3} |\tilde{c}_1^{2l-2,2} N(1,l,l-2)| \\
&\le K_2 2^{-2} e_0 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{c^2} \mathrm{e}^{-\frac{G_0^3}{3}} G_0^{7/2} \sum_{l\ge 3} (2^4 G_0^{-1})^l \\
&\le K_8 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{-\frac{G_0^3}{3}} e_0 G_0^{1/2} \qquad (2.38)
\end{aligned}$$

where

$$K_7 = 2^{13} K_2 \mathrm{e}^{c^2} = 2^{14}\cdot 3\pi \mathrm{e}^{c^2-1/2} \qquad K_8 = 2^{11} K_2 \mathrm{e}^{c^2} = 2^{12}\cdot 3\pi \mathrm{e}^{c^2-1/2}$$

In regard of equation (2.34) and the estimations (2.35), (2.37) and (2.38) we have proven this lemma. □

With the lemma 2.10, we have, from lemma 2.9 that

$$\mathcal{L} = L_0 + 2\Re\big\{[(\tilde{c}_1^{3,1} N(1,2,1) + E_3)\mathrm{e}^{-i\alpha_0} + (\tilde{c}_1^{2,2} N(1,2,0) + E_4)\mathrm{e}^{-2i\alpha_0} + E_2(\alpha_0)]\mathrm{e}^{it_0}\big\} + E_1(t_0,\alpha_0) \tag{2.39}$$

where

$$|E_1(t_0,\alpha_0)| \le K_5 2^6 \mathrm{e}^{2\sqrt{1-e_0^2}} G_0^{3/2} \mathrm{e}^{-G_0^3 \frac{4}{9}} \tag{2.40a}$$

$$|E_2(\alpha_0)| \le K_6 \mathrm{e}^{-\frac{G_0^3}{3}} \left[(1+e_0)^4 G_0^{-7/2} + 2^6 \mathrm{e}^{\sqrt{1-e_0^2}} (e_0^2 G_0^{5/2} + e_0 G_0^{-3/2})\right] \tag{2.40b}$$

$$|E_3| \le K_7 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{-\frac{G_0^3}{3}} G_0^{-3/2} \tag{2.40c}$$

$$|E_4| \le K_8 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{-\frac{G_0^3}{3}} e_0 G_0^{1/2} \tag{2.40d}$$

and

$$K_5 = 1152\pi \mathrm{e}^{-1/2}, \qquad K_6 = 2^4 12\pi \mathrm{e}^{c^2-1/2} \qquad K_7 = 2^{14}\cdot 3\pi \mathrm{e}^{c^2-1/2} \qquad K_8 = 2^{12}\cdot 3\pi \mathrm{e}^{c^2-1/2}$$

Lemma 2.11. Let N be defined by equations (2.4) then

$$\begin{aligned}
N(1,2,1) &= \frac{1}{4}\sqrt{\frac{\pi}{2}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} + {}^1E_{TT} \\
N(1,2,0) &= \sqrt{\frac{\pi}{2}} G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}} + {}^2E_{TT}
\end{aligned}$$

where

$$|{}^1E_{TT}| \le K_{13} G_0^{-2} \mathrm{e}^{-\frac{G_0^3}{3}} \qquad |{}^2E_{TT}| \le K_{13} \mathrm{e}^{-\frac{G_0^3}{3}}$$

with

$$K_{13} = 40\sqrt{\frac{\pi}{3}} + \frac{3}{2}\gamma_4^2 (K_{11} + K_{12})$$

with K_{11}, K_{12} and γ_4 are defined in proposition 2.4.

Proof. From proposition 2.4 we have

$$N(1,2,1) = \frac{d_{2,1}}{G_0^5} e^{-\frac{G_0^3}{3}} \Big[d_4^{1,2} \sqrt{\pi} \Big(\frac{2}{G_0} \Big)^{3/2} - 2^2 d_2^{1,2} \sqrt{\pi} \sqrt{\frac{G_0^3}{2}} + \frac{2^3}{3} d_0^{1,2} \sqrt{\pi} (\sqrt{\frac{G_0^3}{2}})^3 + T^1_{2,1} + R^1_{2,1} \Big] \tag{2.41}$$

where

$$|T^1_{2,1}| \le K_{11} \gamma_4^2 G_0^{-3} \qquad |R^1_{2,1}| \le K_{12} G_0^3$$

and

$$N(1,2,0) = \frac{d_{2,0}}{G_0^3} e^{-\frac{G_0^3}{3}} \Big[2 d_4^{0,2} \sqrt{\pi} (\sqrt{\frac{G_0^3}{2}})^{-1} - 2^2 d_2^{0,2} \sqrt{\pi} \sqrt{\frac{G_0^3}{2}} + \frac{2^3}{3} d_0^{0,2} \sqrt{\pi} (\sqrt{\frac{G_0^3}{2}})^3 + T^1_{2,0} + R^1_{2,0} \Big] \tag{2.42}$$

where

$$|T^1_{2,0}| \le K_{11} \gamma_4^2 G_0^{-3} \qquad |R^1_{2,0}| \le K_{12} G_0^3$$

Computing explicitly the exponents of G_0 and taking the largest of the errors in equations (2.41) and (2.42) we can write them as

$$N(1,2,1) = d_{2,1} d_0^{1,2} \frac{2\sqrt{2}}{3} \sqrt{\pi} G_0^{-1/2} e^{-\frac{G_0^3}{3}} + {}^1E + {}^1E_{TR} \tag{2.43}$$

where

$${}^1E = 2^{\frac{3}{2}} d_{2,1} \sqrt{\pi} \big(d_4^{1,2} G_0^{-\frac{13}{2}} - d_2^{1,2} G_0^{-\frac{7}{2}} \big) e^{-\frac{G_0^3}{3}} \qquad {}^1E_{TR} = (T^1_{2,1} + R^1_{2,1}) d_{2,1} G_0^{-5} e^{-\frac{G_0^3}{3}}$$

with bounds

$$|{}^1E| \le 2^{\frac{3}{2}} |d_{2,1}| \sqrt{\pi} (|d_4^{1,2}| + |d_2^{1,2}|) G_0^{-\frac{7}{2}} e^{-\frac{G_0^3}{3}} \qquad |{}^1E_{TR}| \le |d_{2,1}| \gamma_4^2 (K_{11} + K_{12}) G_0^{-2} e^{-\frac{G_0^3}{3}}$$

and

$$N(1,2,0) = d_{2,0} d_0^{0,2} \frac{2\sqrt{2}}{3} \sqrt{\pi} G_0^{3/2} e^{-\frac{G_0^3}{3}} + {}^2E + {}^2E_{TR} \tag{2.44}$$

where

$${}^2E = 2^{\frac{3}{2}} d_{2,0} \sqrt{\pi} \big(d_4^{0,2} G_0^{-\frac{9}{2}} - d_2^{0,2} G_0^{-\frac{3}{2}} \big) e^{-\frac{G_0^3}{3}} \qquad {}^2E_{TR} = (T^1_{2,0} + R^1_{2,0}) d_{2,0} G_0^{-3} e^{-\frac{G_0^3}{3}}$$

with the bounds

$$|{}^2E| \le 2^{\frac{3}{2}} |d_{2,0}| \sqrt{\pi} (|d_4^{0,2}| + |d_2^{0,2}|) G_0^{-\frac{3}{2}} e^{-\frac{G_0^3}{3}} \qquad |{}^2E_{TR}| \le |d_{2,0}| \gamma_4^2 (K_{11} + K_{12}) e^{-\frac{G_0^3}{3}}$$

Using lemma A.4, $d_0^{n,m} = 1/(2i)^{2n+1}$ and by definition (2.20) for $d_{m,n}$ we have that

$$\begin{aligned} d_{2,1} d_0^{1,2} &= -i 2^3 \binom{-1/2}{2} \binom{-1/2}{1} \Big(\frac{i}{2^3} \Big) = -\frac{3}{2^4} \\ d_{2,0} d_0^{0,2} &= i 2^2 \binom{-1/2}{2} \Big(-\frac{i}{2} \Big) = \frac{3}{2^2}. \end{aligned}$$

We can then write equation (2.43) as

$$N(1,2,1) = \frac{1}{4} \sqrt{\frac{\pi}{2}} G_0^{-1/2} e^{-\frac{G_0^3}{3}} + {}^1E_{TT} \tag{2.45}$$

where

$${}^1E_{TT} = {}^1E + {}^1E_{TR},$$

using lemma A.7 to bound $d_j^{n,m}$ we find

$$
\begin{aligned}
|{}^1E_{TT}| &\leq \left[3\sqrt{2\pi}(|d_4^{1,2}| + |d_2^{1,2}|) + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] G_0^{-2}\mathrm{e}^{-\frac{G_0^3}{3}} \\
&\leq \left[3\sqrt{2\pi}\Big(|d_4^{1,2}| + |d_2^{1,2}|\Big) + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] G_0^{-2}\mathrm{e}^{-\frac{G_0^3}{3}} \\
&\leq \left[K_1 20\sqrt{2\pi} + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] G_0^{-2}\mathrm{e}^{-\frac{G_0^3}{3}} \\
&\leq \left[40\sqrt{\frac{\pi}{3}} + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] G_0^{-2}\mathrm{e}^{-\frac{G_0^3}{3}}
\end{aligned}
$$

so, defining

$$K_{13} = 40\sqrt{\frac{\pi}{3}} + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})$$

we have

$$|{}^1E_{TT}| \leq K_{13}G_0^{-2}\mathrm{e}^{-\frac{G_0^3}{3}}.$$

Analogously, equation (2.44) can be written as

$$N(1,2,0) = \sqrt{\frac{\pi}{2}}G_0^{3/2}\mathrm{e}^{-\frac{G_0^3}{3}} + {}^2E_{TT} \tag{2.46}$$

where

$${}^2E_{TT} = {}^2E + {}^2E_{TR}$$

with

$$
\begin{aligned}
|{}^1E_{TT}| &\leq \left[3\sqrt{2\pi}|d_4^{0,2} - d_2^{0,2}| + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] \mathrm{e}^{-\frac{G_0^3}{3}} \\
&\leq \left[3\sqrt{2\pi}\Big(|d_4^{0,2}| + |d_2^{0,2}|\Big) + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] \mathrm{e}^{-\frac{G_0^3}{3}} \\
&\leq \left[K_1 20\sqrt{2\pi} + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] \mathrm{e}^{-\frac{G_0^3}{3}} \\
&\leq \left[40\sqrt{\frac{\pi}{3}} + \frac{3}{2}\gamma_4^2(K_{11} + K_{12})\right] \mathrm{e}^{-\frac{G_0^3}{3}}
\end{aligned}
$$

then, we have

$$|{}^2E_{TT}| \leq K_{13}\mathrm{e}^{-\frac{G_0^3}{3}}$$

this proves the lemma. □

Using the approximations given in lemma 2.11 we have from lemmas 2.9 and 2.10

Lemma 2.12. If $\mathcal{L}$ is given by (2.7), $G_0 \geq 32$, $e_0G_0 \leq 1/8$. Then

$$
\begin{aligned}
\mathcal{L} = L_0 + \cos(t_0 - \alpha_0)\Big(\tilde{c}_1^{3,1}\sqrt{\frac{\pi}{8}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}} + E_3 + E_5\Big) + \cos(t_0 - 2\alpha_0)\Big(\tilde{c}_1^{2,2}\sqrt{2\pi}G_0^{3/2}\mathrm{e}^{-\frac{G_0^3}{3}} + E_4 + E_6\Big) \\
+ 2\Re\{E_2(\alpha_0)\mathrm{e}^{it_0}\} + E_1(t_0, \alpha_0)
\end{aligned}
$$

where E_k with $k = 1, ..., 4$ are given in equations (2.40) and

$$|E_5| \leq 2^5\mathrm{e}^{\sqrt{1-e_0^2}}K_{13}G_0^{-2}\mathrm{e}^{-\frac{G_0^3}{3}} \tag{2.47}$$

$$|E_6| \leq 2^4\mathrm{e}^{\sqrt{1-e_0^2}}e_0K_{13}\mathrm{e}^{-\frac{G_0^3}{3}} \tag{2.48}$$

where K_{13} is defined in lemma 2.11.

Proof. By lemma 2.11 we have that $N(1,2,1)$ and $N(1,2,0)$ are real and then coincide with their real part. Equation (2.39) gives the correct estimation of $\mathcal{L}$. To complete the proof is enough to take

$$E_5 = \tilde{c}_1^{3,1}\left({}^1E_{TT}\right) \qquad \text{and} \qquad E_6 = \tilde{c}_1^{2,2}\left({}^2E_{TT}\right)$$

where ${}^1E_{TT}$ and ${}^2E_{TT}$ are given in lemma 2.11. Therefore by proposition 2.2 we find directly the bounds of E_5 and E_6.

□

Lemma 2.13. If $\mathcal{L}$ is given by (2.7), $G_0 \geq 32$, $e_0 G_0 < 1/8$. Then

$$\mathcal{L} = L_0 + \cos(t_0 - \alpha_0)\Big(\sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} + E_3 + E_5 + E_7\Big) - \cos(t_0 - 2\alpha_0)\Big(3\sqrt{2\pi} e_0 G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}} + E_4 + E_6 + E_8\Big) + 2\Re\{E_2(\alpha_0)\mathrm{e}^{it_0}\} + E_1(t_0, \alpha_0)$$

where E_k with $k = 1, ..., 6$ are given in lemma 2.12 and

$$|E_7| \leq \sqrt{\frac{\pi}{8}} 98 e_0^2 G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}}$$
$$|E_8| \leq \sqrt{2\pi} 50 e_0^2 G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}}$$

Proof. From lemma 2.6 we have

$$\tilde{c}_1^{3,1} \sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} = \sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} + E_7$$
$$\tilde{c}_1^{2,2} \sqrt{2\pi} G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}} = -3\sqrt{2\pi} e_0 G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}} + E_8$$

with

$$E_7 = Q_1 \sqrt{\frac{\pi}{8}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}}$$
$$E_8 = Q_2 \sqrt{2\pi} G_0^{3/2} \mathrm{e}^{-\frac{G_0^3}{3}}$$

Therefore by lemma (2.12) and the bounds of Q_1 and Q_2 given in lemma 2.6 we conclude the proof. □

Now, we are going to study the term L_0, first a lemma

Lemma 2.14. Let N be defined by equations (2.4) then for $m, n \in \mathbb{N}$, $m + n > 0$

$$|N(0, m, n)| \leq K_{20} 2^{m+n} G_0^{-2m-2n+1}$$

where $K_{20} = \mathrm{e}^{-1}\pi$.

Proof. Since in the integral (2.4) $\tau \in \mathbb{R}$ then is easy to see that

$$\frac{1}{|\tau + i|}, \frac{1}{|\tau - i|} \leq 1$$

and then

$$\frac{1}{|\tau + i|^{2n}} \leq \frac{1}{|\tau + i|^2} \qquad \frac{1}{|\tau - i|^{2m}} \leq \frac{1}{|\tau - i|^2}$$

then using that $n, m > 0$, by equation (2.4) and lemma A.1 we have that

$$|N(0, m, n)| \leq 2^{m+n} G_0^{-2m-2n+1} \mathrm{e}^{-1} \int_{-\infty}^{\infty} \frac{d\tau}{1 + \tau^2}$$
$$= 2^{m+n} G_0^{-2m-2n+1} \mathrm{e}^{-1}\pi$$

naming $K_{20} = e^{-1}\pi$ we have proven this lemma. □

Lemma 2.15. Let $k \in \mathbb{N}$ and $L_{0,\pm k}$ defined by equations (2.2). Then

$$L_{0,\ k} = \sum_{l \geq k+1} \tilde{c}_0^{2l-k,-k} N(0, l-k, l)$$
$$L_{0,-k} = \sum_{l \geq k+1} \tilde{c}_0^{2l-k,\ k} N(0, l, l-k)$$

Proof. From equations (2.2), we have just to prove that for $k \geq 2$

$$N(0,0,k) = N(0,k,0) = 0.$$

By equations (2.4) this reduces to show that

$$\int_{-\infty}^{\infty} \frac{d\tau}{(\tau \pm i)^{2k}} = 0$$

where the positive sign in the denominator correspond to $I(0,0,k)$ and the negative to $I(0,k,0)$. Since the variable $\tau \in \mathbb{R}$ this integral is trivial

$$\int_{-\infty}^{\infty} \frac{d\tau}{(\tau \pm i)^{2k}} = -\frac{1}{2k-1} \frac{1}{(\tau \pm i)^{2k-1}} \Bigg|_{-\infty}^{\infty} = 0$$

this proves the lemma. □

Lemma 2.16. Let $L_{0,\pm k}$ be defined by equations (2.2) If $k \in \mathbb{N}$ and $G_0 \geq 4$ then

$$|L_{0,\pm k}| \leq K_{21} e_0^k 2^{2k} G_0^{-2k-3}$$

with $K_{21} = 2^6 \mathrm{e}^{-1} \pi$.

Proof. From lemma 2.15, we have

$$|L_{0,\ k}| \leq \sum_{l \geq k+1} |\tilde{c}_0^{2l-k,-k}||N(0, l-k, l)|$$
$$|L_{0,-k}| \leq \sum_{l \geq k+1} |\tilde{c}_0^{2l-k,\ k}||N(0, l, l-k)|$$

by proposition 2.2 we have that $\tilde{c}_0^{2l-k,-k} = \overline{\tilde{c}_0^{2l-k,k}}$ and by lemma 2.14 we can easily see that $N(0, l-k, l)$ and $N(0, l, l-k)$ have the same bound. Therefore

$$|L_{0,\pm k}| \leq K_{20} 2^{-2k+1} e_0^k G_0^{2k+1} \sum_{l \geq k+1} (2^4 G_0^{-4})^l$$
$$\leq K_{20} e_0^k 2^{2k+6} G_0^{-2k-3}$$

setting $K_{21} = 2^6 K_{20} = 2^6 \mathrm{e}^{-1} \pi$ the proof is completed. □

Lemma 2.17. Let L_0 be defined by equation (2.7). Then if $G_0 \geq 4$ we have

$$L_0 = L_{0,0} + \tilde{c}_0^{3,1} \frac{3}{4} \pi G_0^{-5} \cos(\alpha_0) + F_2 + F_3$$
$$L_{0,0} = \tilde{c}_0^{2,0} \frac{\pi}{2} G_0^{-3} + F_1$$

where

$$|F_1| \leq K_{22} G_0^{-7}$$
$$|F_2| \leq K_{22} 2^3 e_0 G_0^{-9}$$
$$|F_3| \leq K_{23} e_0^2 G_0^{-7}$$

where $K_{22} = 2^{10} \mathrm{e}^{-1} \pi$ and $K_{23} = 2^7 \mathrm{e}^{-1} \pi$

Proof. From proposition 2.1 is easy to deduce that

$$L_0 = L_{0,0} + 2\Re\Big\{\sum_{k\geq 1} L_{0,k} e^{ik\alpha_0}\Big\}$$

From lemma 2.15 we have that

$$L_{0,\ 0} = \tilde{c}_0^{2,\ 0} N(0,1,1) + \sum_{l\geq 2} \tilde{c}_0^{\ 2l,\ 0} N(0,l,l) \tag{2.49a}$$

$$L_{0,\ 1} = \tilde{c}_0^{3,-1} N(0,1,2) + \sum_{l\geq 3} \tilde{c}_0^{2l-1,-1} N(0,l-1,l) \tag{2.49b}$$

$$L_{0,\ k} = \sum_{l\geq k+1} \tilde{c}_0^{2l-k,-k} N(0,l-k,l) \qquad \text{for } k\geq 2 \tag{2.49c}$$

then, define

$$F_1 = \sum_{l\geq 2} \tilde{c}_0^{\ 2l,\ 0} N(0,l,l)$$
$$F_2 = 2\Re\Big\{e^{i\ \alpha_0} \sum_{l\geq 3} \tilde{c}_0^{2l-1,-1} N(0,l-1,l)\Big\}$$
$$F_3 = 2\Re\Big\{\sum_{k\geq 2} e^{ik\alpha_0} L_{0,k}\Big\}$$

by lemmas 2.14, 2.16 and proposition 2.2 , using the hypothesis on G_0 we have

$$\begin{aligned}
|F_1| &\leq K_{20} 2 G_0 \sum_{l\geq 2} (2^4 G_0^{-4})^l \\
&\leq K_{20} 2^{10} G_0^{-7} \\
|F_2| &\leq K_{20} e_0 G_0^3 \sum_{l\geq 3} (2^4 G_0^{-4})^l \\
&\leq K_{20} 2^{13} e_0 G_0^{-9} \\
|F_3| &\leq 2 \sum_{k\geq 2} |L_{0,k}| \\
&\leq K_{21} 2^6 e_0^2 G_0^{-7}
\end{aligned}$$

Now, from definition (2.4) we have that

$$N(0,1,1) = \frac{2^2}{G_0^3}\binom{-1/2}{1}\binom{-1/2}{1}\int_{-\infty}^{\infty} \frac{d\tau}{(\tau^2+1)^2} = 2^2\Big(-\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)G_0^{-3} = \frac{\pi}{2} G_0^{-3}$$
$$N(0,1,2) = \frac{2^3}{G_0^5}\binom{-1/2}{1}\binom{-1/2}{2}\int_{-\infty}^{\infty} \frac{d\tau}{(\tau-i)(\tau+i)^2} = 2^3\Big(-\frac{1}{2}\Big)\Big(\frac{3}{2^3}\Big)\Big(-\frac{\pi}{4}\Big)G_0^{-5} = \frac{3}{8}\pi G_0^{-5}$$

From these equations, substituting equations (2.49) in the definition of L_0 and the bounds given in equations (2.50) simply by setting

$$K_{22} = 2^{10} K_{20} = 2^{10} e^{-1}\pi$$
$$K_{23} = 2^6 K_{21} = 2^7 e^{-1}\pi$$

we have proven this lemma. □

A refinement of this lemma is

Lemma 2.18. Let L_0 be defined by equation (2.7). Then if $G_0 \geq 2^{3/2}$ we have

$$L_0 = L_{0,0} - \frac{15}{8}\pi e_0 G_0^{-5}\cos(\alpha_0) + F_2 + F_3 + F_5$$
$$L_{0,0} = \frac{\pi}{2}G_0^{-3} + F_1 + F_4$$

where F_1, F_2 and F_3 are given in lemma 2.17 and

$$|F_4| \leq K_{24}G_0^{-3}e_0^2$$
$$|F_5| \leq K_{25}G_0^{-5}e_0^2$$

with $K_{24} = 2\pi$ and $K_{25} = 57\pi/4$

Proof. In lemma 2.6 we have computed the constants $\tilde{c}_0^{2,0}$ and $\tilde{c}_0^{3,1}$, then by setting

$$F_4 = \frac{\pi}{2}Q_3 G_0^{-3}$$
$$F_5 = \frac{3}{4}\pi Q_4 G_0^{-5}\cos\alpha_0$$

and using the bounds for Q_3 and Q_4 we find the desired bound for F_4 and F_5. □

With this lemma we can rewrite lemma 2.13 exactly as theorem 2.5, and so we have proved it.

2.2 $e_0G_0 = \lambda$, λ real positive

In this section we will follow the same structure as in the case $e_0G_0 \ll 1$ except that some computations will be done in a different way. The main difference comes in the way we bound the module of the Fourier coefficient L_q given in (2.5). In section 2.1, the terms $|L_{q,-k}|$ were bounded in lemma 2.7, later on used to bound $|L_q|$ in lemma 2.8. To do this, since we sum over the index k, we have used the assumption $e_0G_0 \ll 1$. To overcome this assumption in this section, we will actually estimate the whole sum $\sum_{k\geq 0} L_{q,-k}$ by noticing that this sum can be computed as the Fourier coefficient of a suitable function, this will be done in proposition 2.22.

As we have done before, we will prove a much more quantitative version of theorem 1.6 wich will immediately imply it. We will prove the next theorem.

Theorem 2.19. *Let λ be a real positive constant, $\gamma_4 = 16/(3+\sqrt{11})$, $c \geq 1$. If*

$$G_0 \geq \max\{(3c)^{2/3}, 32, (24\lambda)^{1/3}, 8\lambda^{-1}, (2^4\lambda)^{1/3}, 2^{3/2}, 2(\gamma_4\lambda)^{1/3}, (2^5\gamma_4)^{1/4}\}.$$

then there exists a positive constant K, depending on λ, such that if $e_0G_0 = \lambda$, the Melnikov potential $\mathcal{L}$ given by (2.7) *satisfies*

$$\mathcal{L} = L_0(\alpha_0) + 2\Re\{L_1 e^{it_0} + \mathcal{E}\}$$

where

$$L_1 = \Big(+\frac{1}{4}\sqrt{\frac{\pi}{2}}G_0^{-1/2}e^{-\frac{G_0^3}{3}} + E_3 + E_5 + E_7\Big)e^{-i\alpha_0}$$
$$+ e^{-\frac{G_0^3}{3}}2\sqrt{2\pi}\lambda^{-1}G_0^{1/2}\Bigg[e^{-\lambda e^{-i\alpha_0}}\frac{A}{A-1}\Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}}J_1(\pm 2i\sqrt{A(A-1)})$$
$$- J_0(\pm 2i\sqrt{A(A-1)})\Big] - A\Bigg]$$
$$+ \tilde{\mathcal{E}}_3 + \tilde{\mathcal{E}}_2 + \tilde{\mathcal{E}}_1 + E_1$$

and

$$
\begin{aligned}
|E_3| &\le K_7 \mathrm{e}^{\sqrt{1-e_0^2}} G_0^{-3/2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|E_5| &\le 2^5 \mathrm{e}^{\sqrt{1-e_0^2}} K_{13} G_0^{-2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|E_7| &\le \sqrt{\frac{\pi}{8}} 98 e_0^2 G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|\tilde{\mathcal{E}}_1| &\le K \mathrm{e}^{(8/3)\lambda} \lambda \mathrm{e}^{\sqrt{1-e_0^2}} G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|\tilde{\mathcal{E}}_2| &\le K \lambda^3 \mathrm{e}^{2\lambda} G_0^{-3/2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|\tilde{\mathcal{E}}_3| &\le K e_0^2 G_0^{1/2} \mathrm{e}^{-\frac{G_0^3}{3}} \\
|E_1| &\le K_4' \mathrm{e}^{-G_0^3/3} G_0^{-3/2} (2 e_0 \mathrm{e}^{\sqrt{1-e_0^2}} + G_0^{-2}) \\
|\mathcal{E}| &\le K G_0^{3/2} \mathrm{e}^{-G_0^3 \frac{4}{9}}
\end{aligned}
$$

with

$$
\begin{aligned}
K_4' &= 2^{14} \cdot 3\pi \mathrm{e}^{-\frac{1}{2}} \mathrm{e}^{c^2} \\
K_7 &= 2^{14} \cdot 3\pi \mathrm{e}^{c^2 - 1/2} \\
\beta &= \left(-1 + \frac{\sqrt{11}}{4} \sqrt{3 + \frac{\sqrt{11}}{2}} \right)^{1/2} \\
K_{11} &= 2^2 \left(\sqrt{2} + \frac{2}{\beta(1-\beta)} \right) \\
K_{12} &= 2\pi \mathrm{e}^{4/3} \\
K_{13} &= 40 \sqrt{\frac{\pi}{3}} + \frac{3}{2} \gamma_4^2 (K_{11} + K_{12})
\end{aligned}
$$

and

$$
\begin{aligned}
A &= \frac{\lambda}{2} \mathrm{e}^{-i\alpha_0} \\
\frac{A}{A-1} &= \frac{\lambda^2 - 2\lambda \mathrm{e}^{-i\alpha_0}}{\lambda^2 - 4\lambda \cos\alpha_0 + 4} \\
A(A-1) &= -\frac{\lambda}{2} \mathrm{e}^{-i\alpha_0} \left(1 - \frac{\lambda}{2} \mathrm{e}^{-i\alpha_0} \right).
\end{aligned}
$$

Also,

$$
L_0(\alpha_0) - L_{0,0} = -\frac{15}{8} \pi e_0 G_0^{-5} \cos(\alpha_0) + F_2 + F_3 + F_5
$$

where

$$
\begin{aligned}
|F_2| &\le K_{22} 2^3 e_0 G_0^{-9} \\
|F_3| &\le K_{23} e_0^2 G_0^{-7} \\
|F_5| &\le K_{25} G_0^{-5} e_0^2
\end{aligned}
$$

with

$$
\begin{aligned}
K_{22} &= 2^{10} \mathrm{e}^{-1} \pi \\
K_{23} &= 2^7 \mathrm{e}^{-1} \pi \\
K_{25} &= 57\pi/4
\end{aligned}
$$

The functions $J_0(z)$ and $J_1(z)$ are the Bessel functions of the first kind [AS65] and whose expansion around $z=0$ is given by

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\frac{z}{2}\right)^{2m+n}.$$

The next series of lemmas and propositions go in the direction of proving this theorem. First let us bound the terms $L_{q,k}$ in L_q.

Proposition 2.20. Let $c \geq 1$, $q \geq 1$ and L_q be defined in (2.5) and $G_0 \geq \max\{(3c)^{2/3}, 32\}$. Then

$$L_q = \tilde{L}_q + E_q$$

where

$$\tilde{L}_q = \sum_{k\geq 1} L_{q,-k}\mathrm{e}^{-ik\alpha_0} \tag{2.51}$$

and

$$|E_q| \leq \begin{cases} K_4'\mathrm{e}^{-G_0^3/3}G_0^{-3/2}(2e_0\mathrm{e}^{\sqrt{1-e_0^2}} + G_0^{-2}) & q=1 \\ K_4'\mathrm{e}^{-qG_0^3\frac{2}{9}}G_0^{-3/2}\left[\left(2e_0\mathrm{e}^{\sqrt{1-e_0^2}}\right)^q + G_0^{-2}\right] & q \geq 2 \end{cases}$$

with $K_4' = 2^{14}\cdot 3\pi\mathrm{e}^{-\frac{1}{2}}\mathrm{e}^{c^2}$.

Proof. By equation (2.5) we have that

$$L_q = \sum_{k\geq 1} L_{q,-k}\mathrm{e}^{-ik\alpha_0} + \sum_{k\geq 0} L_{q,k}\mathrm{e}^{ik\alpha_0}$$

therefore, if we call

$$E_q = \sum_{k\geq 0} L_{q,k}\mathrm{e}^{ik\alpha_0}$$

it remains to bound it. By lemma 2.7, given $q \geq 1$, we have

$$\begin{aligned}
|E_q| &\leq 2^3 G_0^{-1/2} K_3 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)}\left[2^{q+2}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^q G_0^{-1} + (1+e_0)^4 G_0^{-3} + (1+e_0)2^{-3}\sum_{k\geq 2}(2(1+e_0)G_0^{-2})^k\right] \\
&\leq 2^3 G_0^{-1/2} K_3 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)}\left[2^{q+2}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^q G_0^{-1} + (1+e_0)^4 G_0^{-3} + (1+e_0)^3 G_0^{-4}\right] \\
&\leq 2^3 G_0^{-1/2} K_3 \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)}\left[2^{q+2}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^q G_0^{-1} + 2(1+e_0)^4 G_0^{-3}\right] \\
&\leq 2^8 K_3 G_0^{-3/2} \mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)}\left[\left(2e_0\mathrm{e}^{\sqrt{1-e_0^2}}\right)^q + G_0^{-2}\right]
\end{aligned} \tag{2.52}$$

by setting $K_4' = 2^8 K_3 \mathrm{e}^{c^2}$ we get the desired bound for $q=1$. To bound it for $q \geq 2$ we use our hypothesis that $G_0 \geq (3c)^{2/3}$ we have

$$\mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} \leq \mathrm{e}^{-qG_0^3\frac{2}{9}}$$

□

As we did in proposition 2.4, we need to compute the function N defined in equation (2.4), but now, we need an explicit computation of the residue $R_{m,n}^q$ of the function involved in the integral of N. The proof of the next proposition is found in appendix B.

Proposition 2.21. Let the constants $d_j^{n,m}$ be defined by equation (2.19) and $d_{n,m}$ by equation (2.20). If $q,n,m \in \mathbb{Z}, m,n \geq 0$, $m+n>0$, $q \geq 1$ then

$$N(q,m,n) = \frac{d_{m,n}\mathrm{e}^{-q\frac{G_0^3}{3}}}{G_0^{2m+2n-1}}\left[\sum_{s=0}^{m}(-1)^s\sqrt{\pi}\frac{2^{\frac{3}{2}}q^{s-\frac{1}{2}}}{(2s-1)!!}d_{2m-2s}^{n,m}G_0^{3s-\frac{3}{2}}+2\sum_{s=0}^{m-1}d_{2(m-s)-1}^{n,m}\left(-\frac{q}{2}G_0^3\right)^s\frac{1}{s!}+T_{m,n}^q\right]$$

where

$$|T_{m,n}^q| \leq K_{11}\gamma_4^m G_0^{-3}$$

and

$$\beta = \left(-1+\frac{\sqrt{11}}{4}\sqrt{3+\frac{\sqrt{11}}{2}}\right)^{1/2}, \quad \gamma_4 = \frac{2}{\beta^2}, \quad K_{11} = 2^2\left(\sqrt{2}+\frac{2}{\beta(1-\beta)}\right).$$

When $s=0$ the factor $1/(2s-1)!!$ in the formula should be replaced by 1.

Using proposition 2.21 to rewrite equations (2.2c) and (2.2e) we have that

$$L_{q,-1} = \mathrm{e}^{-q\frac{G_0^3}{3}}\Big[\sum_{l\geq 2}\sum_{s=0}^{l}\tilde{c}_q^{2l-1,1}g_{sl1}^q+\sum_{l\geq 2}\sum_{s=0}^{l-1}\tilde{c}_q^{2l-1,1}h_{sl1}^q+\sum_{l\geq 2}\tilde{c}_q^{2l-1,1}d_{l,l-1}G_0^{-4l+3}T_{l,l-1}^q\Big] \tag{2.53}$$

$$L_{q,-k} = \mathrm{e}^{-q\frac{G_0^3}{3}}\Big[\sum_{l\geq k}\sum_{s=0}^{l}\tilde{c}_q^{2l-k,k}g_{slk}^q+\sum_{l\geq k}\sum_{s=0}^{l-1}\tilde{c}_q^{2l-k,k}h_{slk}^q+\sum_{l\geq k}\tilde{c}_q^{2l-k,k}d_{l,l-k}G_0^{-4l+2k+1}T_{l,l-k}^q\Big] \tag{2.54}$$

where

$$g_{slk}^q = (-1)^s\sqrt{\pi}2^{\frac{3}{2}}q^{s-\frac{1}{2}}\frac{d_{l,l-k}d_{2(l-s)}^{l-k,l}}{(2s-1)!!}G_0^{-4l+2k+3s-\frac{1}{2}} \tag{2.55}$$

$$h_{slk}^q = (-1)^s 2q^s\frac{d_{l,l-k}d_{2(l-s)-1}^{l-k,l}}{2^s s!}G_0^{-4l+2k+3s+1} \tag{2.56}$$

where $d_{m,n}$ is defined in equation (2.20), $d_j^{n,m}$ in equation (2.19), $\tilde{c}_q^{n,m}$ in equation (2.3) and by proposition 2.21, for $k \geq 1$

$$|T_{l,l-k}^q| \leq K_{11}\gamma_4^l G_0^{-3},$$

also note that in formula (2.53) when $s=0$, the factor $1/(2s-1)!!$ in the formula should be replaced by 1.

From the definition of $\tilde{L}_q$ given in (2.51) we have that

$$\tilde{L}_q = L_{q,-1}\mathrm{e}^{-i\alpha_0}+\mathrm{e}^{-q\frac{G_0^3}{3}}\Bigg[\sum_{k\geq 2}\sum_{l\geq k}\sum_{s=0}^{l}\tilde{c}_q^{2l-k,k}g_{slk}^q\mathrm{e}^{-ik\alpha_0}+\sum_{k\geq 2}\sum_{l\geq k}\sum_{s=0}^{l-1}\tilde{c}_q^{2l-k,k}h_{slk}^q\mathrm{e}^{-ik\alpha_0}$$
$$+\sum_{k\geq 2}\sum_{l\geq k}\tilde{c}_q^{2l-k,k}d_{l,l-k}G_0^{-4l+2k+1}\mathrm{e}^{-ik\alpha_0}T_{l,l-k}^q\Bigg]$$

To bound $\tilde{L}_q$ we could use the bounds for the $\tilde{c}_q^{2l-k,k}$ given in proposition 2.2. Nevertheless, it is better to notice that, from the equation defining the constants $\tilde{c}_q^{n,m}$ given in (2.3), it can be seen that $\mathrm{e}^{q\frac{G_0^3}{3}}\left(\tilde{L}_q - L_{q,-1}\mathrm{e}^{-i\alpha_0}\right)$ is the q-th Fourier coefficient of the function

$$\tilde{M}_q(s) = \sum_{k\geq 2}\sum_{l\geq k}\sum_{s=0}^{l}r_0^{2l-k}(f(s))\mathrm{e}^{ikf(s)}g_{slk}^q\mathrm{e}^{-ik\alpha_0}+\sum_{k\geq 2}\sum_{l\geq k}\sum_{s=0}^{l-1}r_0^{2l-k}(f(s))\mathrm{e}^{ikf(s)}h_{slk}^q\mathrm{e}^{-ik\alpha_0}$$
$$+\sum_{k\geq 2}\sum_{l\geq k}r_0^{2l-k}(f(s))\mathrm{e}^{ikf(s)}d_{l,l-k}G_0^{-4l+2k+1}\mathrm{e}^{-ik\alpha_0}T_{l,l-k}^q$$

In fact, this function $\tilde{M}_q(s)$ depends on s through the true anomaly $f(s)$. For convenience, we will write

$$\begin{aligned}\tilde{M}_q(s) + \sum_{k\geq 2}\sum_{l\geq k}\sum_{s=0}^{l} r_0^{2(l-k)}[r_0 \mathrm{e}^{if}]^k g^q_{slk}\mathrm{e}^{-ik\alpha_0} + \sum_{k\geq 2}\sum_{l\geq k}\sum_{s=0}^{l-1} r_0^{2(l-k)}[r_0 \mathrm{e}^{if}]^k h^q_{slk}\mathrm{e}^{-ik\alpha_0}\\ + \sum_{k\geq 2}\sum_{l\geq k} r_0^{2(l-k)}[r_0\mathrm{e}^{if}]^k d_{l,l-k}G_0^{-4l+2k+1}\mathrm{e}^{-ik\alpha_0}T^q_{l,l-k}\end{aligned} \tag{2.57}$$

therefore

$$\mathrm{e}^{q\frac{G_0^3}{3}}\left(\tilde{L}_q - L_{q,-1}\mathrm{e}^{-i\alpha_0}\right) = \frac{1}{2\pi}\int_0^{2\pi}\tilde{M}_q(s)\mathrm{e}^{-iqs}ds.$$

Using (1.7) as a change of variables, that is $s = E - e_0\sin E$, and using the identity $r_0 = 1 - e_0\cos E$ already given in (1.6) we have

$$\mathrm{e}^{q\frac{G_0^3}{3}}\left(\tilde{L}_q - L_{q,-1}\mathrm{e}^{-i\alpha_0}\right) = \frac{1}{2\pi}\int_0^{2\pi} M_q(\hat{r}(E), \hat{r}(E)\mathrm{e}^{i\hat{f}(E)})\mathrm{e}^{-iqt(E)}\hat{r}(E)dE$$

where $\hat{r}(E)$, $\hat{r}(E)\mathrm{e}^{i\hat{f}(E)}$ and $t(E)$ are given in equations (2.9). If we now, using the periodicity of the eccentric anomaly E, change the path of integration to

$$E = u + i\ln\left(\frac{2a^2}{e_0}\right) \qquad u\in[0,2\pi]$$

we find that

$$\mathrm{e}^{q\frac{G_0^3}{3}}\left(\tilde{L}_q - L_{q,-1}\mathrm{e}^{-i\alpha_0}\right) = \frac{1}{2\pi}\int_0^{2\pi} M_q\left(\tilde{r}(u), \tilde{r}(u)\mathrm{e}^{i\tilde{f}(u)}\right)\left[\frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}}\right]^q\tilde{r}(u)du \tag{2.58}$$

where from equations (2.11), (2.12) and (2.13)

$$\begin{aligned}\tilde{r}(u)\mathrm{e}^{i\tilde{f}(u)} &= e_0(\cos u - 1)\\ \tilde{r}(u) &= 1 - \frac{e_0^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}.\end{aligned}$$

Let $\mathcal{M}_q(u)$ denote $\tilde{M}_q(s)$ after performing the changes of variables

$$s = E - e_0\sin E \tag{2.59}$$

$$E = u + i\ln\left(\frac{2a^2}{e_0}\right) \tag{2.60}$$

that is

$$\mathcal{M}_q(u) = \tilde{M}_q\left(\tilde{r}(u), \tilde{r}(u)\mathrm{e}^{i\tilde{f}(u)}\right), \tag{2.61}$$

and then (2.58) can be written as

$$\mathrm{e}^{q\frac{G_0^3}{3}}\left(\tilde{L}_q - L_{q,-1}\mathrm{e}^{-i\alpha_0}\right) = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{M}_q(u)\left[\frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}}\right]^q\tilde{r}(u)du \tag{2.62}$$

We will compute $\tilde{L}_1$ or equivalently $\mathcal{M}_1$ and show that the remaining terms are smaller, bound for $\mathcal{M}_q$ when $q > 1$ and use the bounds for $L_{q,-1}$ given in lemma 2.7. We have clearly computed the term $L_{1,-1}$ defined in equation (2.32a) in lemmas 2.10, 2.11 and 2.13, getting that

$$L_{1,-1} = \frac{1}{4}\sqrt{\frac{\pi}{2}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}} + E_3 + E_5 + E_7 \tag{2.63}$$

where

$$|E_3| \le K_7 \mathrm{e}^{\sqrt{1-e_0^2}} \mathrm{e}^{-\frac{G_0^3}{3}} G_0^{-3/2}$$
$$|E_5| \le 2^5 \mathrm{e}^{\sqrt{1-e_0^2}} K_{13} G_0^{-2} \mathrm{e}^{-\frac{G_0^3}{3}}$$
$$|E_7| \le \sqrt{\frac{\pi}{8}} 98 e_0^2 G_0^{-1/2} \mathrm{e}^{-\frac{G_0^3}{3}}$$

To bound L_q for $q \ge 2$ we will find good bounds for $\tilde{L}_q$ introduced in (2.51). Next proposition gives an asymptotic formula for $\mathcal{M}_1$ and bounds for $\mathcal{M}_q$ when $q \ge 2$.

Proposition 2.22. Let $\mathcal{M}_q$ be defined in equation (2.61) and (2.57), λ real positive constant and γ_4 be given in proposition 2.21, and

$$G_0 \ge \max\{8\lambda^{-1}, (2^4\lambda)^{1/3}, 2^{3/2}, 2(\gamma_4\lambda)^{1/3}, (2^5\gamma_4)^{1/4}\}.$$

Then there exists a positive constant K such that if $e_0 G_0 = \lambda$,

$$\mathcal{M}_1(u) = \sqrt{2\pi} G_0^{-1/2} \left[\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}} - 1 + \lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}\right] + \mathcal{E}_1$$

and

$$\mathcal{E}_1 \le K\lambda^2 \mathrm{e}^{(8/3)\lambda} G_0^{-3/2}.$$

Also, for $q \ge 1$

$$|\mathcal{M}_q(u)| \le K\lambda^2 \sqrt{q} \mathrm{e}^{(8/3)q\lambda} G_0^{-1/2}.$$

Proof. From the definition of $\tilde{M}_q(s)$ given in equation (2.57), we can see that is composed of three different sums and then so is $\mathcal{M}_q(u)$ in (2.61). The strategy is to change the order of the indexes to get bounds of each sum. Let us first begin by changing the order in the first sum of (2.57) To do so, we denote

$$G^q_{s,l,k} = \tilde{r}^{2(l-k)} [\tilde{r}\mathrm{e}^{i\tilde{f}}]^k g^q_{slk} \mathrm{e}^{-ik\alpha_0} \tag{2.64}$$

then

$$\begin{aligned}
\sum_{k\ge2}\sum_{l\ge k}\sum_{s=0}^{l} G_{s,l,k} &= \sum_{k\ge 2}\Big(\sum_{s=0}^{k-1}\sum_{l=k}^{\infty} G^q_{s,l,k} + \sum_{s=k}^{\infty}\sum_{l=s}^{\infty} G^q_{s,l,k}\Big) \\
&= \sum_{k\ge2}\sum_{s=0}^{k-1}\sum_{l=k}^{\infty} G^q_{s,l,k} + \sum_{k\ge2}\sum_{s=k}^{\infty}\sum_{l=s}^{\infty} G^q_{s,l,k} \\
&= \sum_{s=0}^{1}\sum_{k=2}^{\infty}\sum_{l=k}^{\infty} G^q_{s,l,k} + \sum_{s=2}^{\infty}\sum_{k=s+1}^{\infty}\sum_{l=k}^{\infty} G^q_{s,l,k} + \sum_{k\ge2}\sum_{s=k}^{\infty}\sum_{l=s}^{\infty} G^q_{s,l,k} \\
&= \sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=s+1}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty}\sum_{k=2}^{s}\sum_{l=s}^{\infty} G^q_{s,l,k} \\
&= \sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=s+1}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty}\sum_{l=s}^{\infty}\sum_{k=2}^{s} G^q_{s,l,k} \\
&= \sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty}\Big(\sum_{l=s+1}^{\infty}\sum_{k=s+1}^{l} G^q_{s,l,k} + \sum_{l=s}^{\infty}\sum_{k=2}^{s} G^q_{s,l,k}\Big) \\
&= \sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty}\sum_{l=s}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} \\
&= \sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty}\sum_{k=2}^{s} G^q_{s,s,k} + \sum_{s=2}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k}
\end{aligned}$$

$$= \sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} + \sum_{s=2}^{\infty} G^q_{s,s,s} + \sum_{s=3}^{\infty}\sum_{k=2}^{s-1} G^q_{s,s,k} + \sum_{s=2}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k}$$

$$= \sum_{s=2}^{\infty} G^q_{s,s,s} + E^q_G \tag{2.65}$$

where

$$E^q_G = \sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} + \sum_{s=3}^{\infty}\sum_{k=2}^{s-1} G^q_{s,s,k} + \sum_{s=2}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k} \tag{2.66}$$

We will bound E^q_G, then we will compute the sum involving $G^q_{s,s,s}$. From its definition given in (2.64) and the definition of g^q_{slk} given in equation (2.55) we have

$$G^q_{s,l,k} = \tilde{r}^{2(l-k)}[\tilde{r}e^{i\tilde{f}}]^k(-1)^s\sqrt{\pi}2^{\frac{3}{2}}q^{s-\frac{1}{2}}\frac{d_{l,l-k}d^{l-k,l}_{2(l-s)}}{(2s-1)!!}G_0^{-4l+2k+3s-\frac{1}{2}}e^{-ik\alpha_0} \tag{2.67a}$$

$$G^q_{s,s,k} = \tilde{r}^{2(s-k)}[\tilde{r}e^{i\tilde{f}}]^k(-1)^s\sqrt{\pi}2^{\frac{3}{2}}q^{s-\frac{1}{2}}\frac{d_{s,s-k}d^{s-k,s}_{0}}{(2s-1)!!}G_0^{-s+2k-\frac{1}{2}}e^{-ik\alpha_0} \tag{2.67b}$$

$$G^q_{s,s,s} = [\tilde{r}e^{i\tilde{f}}]^s(-1)^s\sqrt{\pi}2^{\frac{3}{2}}q^{s-\frac{1}{2}}\frac{d_{s,0}d^{0,s}_{0}}{(2s-1)!!}G_0^{s-\frac{1}{2}}e^{-is\alpha_0} \tag{2.67c}$$

From lemma A.4 and the bounds given in lemmas A.1 and A.7 we have

$$|d^{l-k,l}_{2(l-s)}| \le 2^l\Big(\frac{2}{3}\Big)^s$$
$$|d_{l,l-k}| \le e^{-1/2}2^{2l-k}$$
$$|d^{s-k,s}_0| = \frac{1}{2^{2(s-k)+1}}.$$

the bound for $|d^{l-k,l}_{2(l-s)}|$ in not optimized. We have that

$$(2s-1)!! = \prod_{k=0}^{s-1}(2s-1-2k) \ge \prod_{k=0}^{s-2}2(s-1-k) = 2^{s-1}(s-1)!$$

and then

$$\frac{1}{(2s-1)!!} \le \frac{2}{2^s(s-1)!}.$$

Recall, from proposition 2.21, that when $s=0$ the corresponding factor to $1/(2s-1)!!$ and its bound is exactly one. Therefore, using the bounds for $\tilde{r}$ and $\tilde{r}e^{i\tilde{f}}$ given in (2.16) and (2.15) we have from (2.67a) and (2.67b) that

$$|G^q_{s,l,k}| \le 2^{2(l-k)}[2e_0]^k\sqrt{\pi}2^{\frac{3}{2}}q^{s-\frac{1}{2}}\frac{1}{\sqrt{e}}\frac{2\cdot 2^{2l-k}}{2^s(s-1)!}2^l\Big(\frac{2}{3}\Big)^s G_0^{-4l+2k+3s-\frac{1}{2}} \tag{2.68}$$

$$|G^q_{s,s,k}| \le 2^{2(s-k)}[2e_0]^k\sqrt{\pi}2^{\frac{3}{2}}q^{s-\frac{1}{2}}\frac{1}{\sqrt{e}}\frac{2\cdot 2^{2s-k}}{2^s(s-1)!}\frac{1}{2^{2(s-k)+1}}G_0^{-s+2k-\frac{1}{2}}. \tag{2.69}$$

From inequality (2.68) and using that $e_0G_0=\lambda$ and the hypothesis on G_0 we have

$$\Big|\sum_{s=0}^{1}\sum_{l=2}^{\infty}\sum_{k=2}^{l} G^q_{s,l,k}\Big| \le \sqrt{\frac{2\pi}{e}}\frac{2}{\sqrt{qG_0}}\sum_{s=0}^{1}\Big(\frac{q}{3}G_0^3\Big)^s\frac{2}{(s-1)!}\sum_{l=2}^{\infty}\Big(\frac{2^5}{G_0^4}\Big)^l\sum_{k=2}^{l}\Big(\frac{G_0\lambda}{2^2}\Big)^k$$
$$\le \sqrt{\frac{2\pi}{e}}\frac{2^2}{\sqrt{qG_0}}\sum_{s=0}^{1}\Big(\frac{q}{3}G_0^3\Big)^s\frac{2}{(s-1)!}\sum_{l=2}^{\infty}\Big(\frac{2^3\lambda}{G_0^3}\Big)^l$$

$$
\begin{aligned}
&\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^3}{\sqrt{qG_0}}\sum_{s=0}^{1}\Big(\frac{q}{3}G_0^3\Big)^s\frac{2}{(s-1)!}\frac{2^6\lambda^2}{G_0^6}\\
&\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^9}{\sqrt{qG_0}}\frac{\lambda^2}{G_0^6}\Big(1+q\frac{2}{3}G_0^3\Big)\\
&\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^{11}}{3}\lambda^2\sqrt{q}G_0^{-7/2}
\end{aligned}
\tag{2.70}
$$

in the same way we bound the third term of E_G^q in (2.66)

$$
\begin{aligned}
\Big|\sum_{s=2}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{l}G_{s,l,k}^q\Big| &\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2}{\sqrt{qG_0}}\sum_{s=2}^{\infty}\Big(\frac{q}{3}G_0^3\Big)^s\frac{2}{(s-1)!}\sum_{l=s+1}^{\infty}\Big(\frac{2^5}{G_0^4}\Big)^l\sum_{k=2}^{l}\Big(\frac{G_0\lambda}{2^2}\Big)^k\\
&\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^2}{\sqrt{qG_0}}\sum_{s=2}^{\infty}\Big(\frac{q}{3}G_0^3\Big)^s\frac{2}{(s-1)!}\sum_{l=s+1}^{\infty}\Big(\frac{2^3\lambda}{G_0^3}\Big)^l\\
&\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^3}{\sqrt{qG_0}}\sum_{s=2}^{\infty}\Big(\frac{2^3}{3}q\lambda\Big)^s\frac{2}{(s-1)!}\cdot\frac{2^3\lambda}{G_0^3}\\
&= \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^4}{\sqrt{qG_0}}\sum_{s=2}^{\infty}\Big(\frac{2^3}{3}q\lambda\Big)^{s-1}\frac{1}{(s-1)!}\cdot\frac{2^3}{3}q\lambda\cdot\frac{2^3\lambda}{G_0^3}\\
&= \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^{10}}{3}\lambda^2\sqrt{q}G_0^{-7/2}\sum_{s=1}^{\infty}\Big(\frac{2^3}{3}q\lambda\Big)^s\frac{1}{s!}\\
&= \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^{10}}{3}\lambda^2\sqrt{q}G_0^{-7/2}\Big(\mathrm{e}^{\frac{2^3}{3}q\lambda}-1\Big)
\end{aligned}
\tag{2.71}
$$

From inequality (2.69) we bound the second term of E_G^q in (2.66)

$$
\begin{aligned}
\Big|\sum_{s=3}^{\infty}\sum_{k=2}^{s-1}G_{s,s,k}^q\Big| &\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2}{\sqrt{qG_0}}\sum_{s=3}^{\infty}\Big(\frac{2q}{G_0}\Big)^s\frac{1}{(s-1)!}\sum_{k=2}^{s-1}(G_0\lambda)^k\\
&\leq \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^2}{\sqrt{qG_0}}\sum_{s=3}^{\infty}\Big(2q\lambda\Big)^s\frac{1}{(s-1)!}\cdot\frac{1}{G_0\lambda}\\
&= \sqrt{\frac{2\pi}{\mathrm{e}}}\frac{2^2}{\sqrt{qG_0}}\sum_{s=3}^{\infty}\Big(2q\lambda\Big)^{s-1}\frac{1}{(s-1)!}\cdot\frac{2q\lambda}{G_0\lambda}\\
&= \sqrt{\frac{2\pi}{\mathrm{e}}}2^3\sqrt{q}G_0^{-3/2}\sum_{s=2}^{\infty}\Big(2q\lambda\Big)^s\frac{1}{s!}\\
&= \sqrt{\frac{2\pi}{\mathrm{e}}}2^3\sqrt{q}G_0^{-3/2}\Big(\mathrm{e}^{2q\lambda}-1-2q\lambda\Big).
\end{aligned}
\tag{2.72}
$$

From inequalities (2.70), (2.71) and (2.72), E_G^q defined in (2.66) can be bounded as

$$
|E_G^q| \leq K\lambda^2\sqrt{q}\big(G_0^{-7/2}+G_0^{-7/2}\mathrm{e}^{(8/3)q\lambda}+G_0^{-3/2}\mathrm{e}^{2q\lambda}\big). \tag{2.73}
$$

Now, we will bound the second term of in $\tilde{M}_q$ in (2.57). Analogously as we did with the first one, let us denote

$$
H_{s,l,k}^q = \tilde{r}^{2(l-k)}[\tilde{r}\mathrm{e}^{i\tilde{f}}]^k h_{slk}^q \mathrm{e}^{-ik\alpha_0}. \tag{2.74}
$$

Then, we can write the second term in the definition of $\tilde{M}_q$ given in equation (2.57) as

$$
E_H^q = \sum_{k\geq 2}\sum_{l\geq k}\sum_{s=0}^{l-1}H_{s,l,k}^q \tag{2.75}
$$

Now, changing the order of the indexes in E_H^q we have

$$
\begin{aligned}
E_H^q &= \sum_{k\geq 2}\Big(\sum_{s=0}^{k-2}\sum_{l=k}^{\infty} H_{s,l,k}^q + \sum_{s=k-1}^{\infty}\sum_{l=s+1}^{\infty} H_{s,l,k}^q\Big)\\
&= \sum_{k\geq 2}\sum_{s=0}^{k-2}\sum_{l=k}^{\infty} H_{s,l,k}^q + \sum_{k\geq 2}\sum_{s=k-1}^{\infty}\sum_{l=s+1}^{\infty} H_{s,l,k}^q\\
&= \sum_{s=0}^{\infty}\sum_{k=s+2}^{\infty}\sum_{l=k}^{\infty} H_{s,l,k}^q + \sum_{k\geq 2}\sum_{s=k-1}^{\infty}\sum_{l=s+1}^{\infty} H_{s,l,k}^q\\
&= \sum_{s=0}^{\infty}\sum_{k=s+2}^{\infty}\sum_{l=k}^{\infty} H_{s,l,k}^q + \sum_{s=1}^{\infty}\sum_{k=2}^{s+1}\sum_{l=s+1}^{\infty} H_{s,l,k}^q\\
&= \sum_{s=0}^{\infty}\sum_{l=s+2}^{\infty}\sum_{k=s+2}^{l} H_{s,l,k}^q + \sum_{s=1}^{\infty}\sum_{k=2}^{s+1}\sum_{l=s+1}^{\infty} H_{s,l,k}^q\\
&= \sum_{s=0}^{\infty}\sum_{l=s+2}^{\infty}\sum_{k=s+2}^{l} H_{s,l,k}^q + \sum_{s=1}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{s+1} H_{s,l,k}^q\\
&= \sum_{l=2}^{\infty}\sum_{k=2}^{l} H_{0,l,k}^q + \sum_{s=1}^{\infty}\sum_{l=s+2}^{\infty}\sum_{k=s+2}^{l} H_{s,l,k}^q + \sum_{s=1}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{s+1} H_{s,l,k}^q\\
&= \sum_{l=2}^{\infty}\sum_{k=2}^{l} H_{0,l,k}^q + \sum_{s=1}^{\infty}\Big(\sum_{l=s+2}^{\infty}\sum_{k=s+2}^{l} H_{s,l,k}^q + \sum_{l=s+1}^{\infty}\sum_{k=2}^{s+1} H_{s,l,k}^q\Big)\\
&= \sum_{l=2}^{\infty}\sum_{k=2}^{l} H_{0,l,k}^q + \sum_{s=1}^{\infty}\Big(\sum_{l=s+1}^{\infty}\sum_{k=2}^{l} H_{s,l,k}^q\Big)\\
&= \sum_{l=2}^{\infty}\sum_{k=2}^{l} H_{0,l,k}^q + \sum_{s=1}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{l} H_{s,l,k}^q
\end{aligned} \tag{2.76}
$$

From its definition given in (2.74) and the definition of h_{slk}^q given in equation (2.56) we have

$$H_{s,l,k}^q = \tilde{r}^{2(l-k)}[\tilde{r}\mathrm{e}^{i\tilde{f}}]^k(-1)^s 2q^s \frac{d_{l,l-k}d_{2(l-s)-1}^{l-k,l}}{2^s s!} G_0^{-4l+2k+3s+1}\mathrm{e}^{-ik\alpha_0} \tag{2.77}$$

$$H_{0,l,k}^q = \tilde{r}^{2(l-k)}[\tilde{r}\mathrm{e}^{i\tilde{f}}]^k 2d_{l,l-k}d_{2l-1}^{l-k,l}G_0^{-4l+2k+1}\mathrm{e}^{-ik\alpha_0} \tag{2.78}$$

From the bounds given in lemmas A.1 and A.7 we have

$$
\begin{aligned}
|d_{2(l-s)-1}^{l-k,l}| &\leq \sqrt{\frac{2}{3}}2^l\Big(\frac{2}{3}\Big)^s\\
|d_{2l-1}^{l-k,l}| &\leq \sqrt{\frac{2}{3}}2^l\\
|d_{l,l-k}| &\leq \mathrm{e}^{-1/2}2^{2l-k}
\end{aligned}
$$

Using the bounds for $\tilde{r}(u)$ and $\tilde{r}(u)\mathrm{e}^{i\tilde{f}(u)}$ given in (2.16) and (2.15) we have from (2.77) and (2.78) that

$$|H_{s,l,k}^q| \leq 2^{2(l-k)}[2e_0]^k 2q^s 2^{2l-k}\frac{1}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}2^l\Big(\frac{2}{3}\Big)^s\frac{1}{2^s s!}G_0^{-4l+2k+3s+1} \tag{2.79}$$

$$|H_{0,l,k}^q| \leq 2^{2(l-k)}[2e_0]^k 2 2^{2l-k}\frac{1}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}2^l G_0^{-4l+2k+1} \tag{2.80}$$

From inequality (2.79) and using that $e_0G_0=\lambda$ and the hypotheses on G_0 made on the statement of this proposition, we can bound the second term of E_H^q in (2.75)

$$
\begin{aligned}
\Big|\sum_{s=1}^{\infty}\sum_{l=s+1}^{\infty}\sum_{k=2}^{l}H^q_{s,l,k}\Big| &\le \frac{2}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}G_0\sum_{s=1}^{\infty}\Big(\frac{qG_0^3}{3}\Big)^s\frac{1}{s!}\sum_{l=s+1}^{\infty}\Big(\frac{2^5}{G_0^4}\Big)^l\sum_{k=2}^{l}\Big(\frac{\lambda G_0}{2^2}\Big)^k\\
&\le \frac{2^2}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}G_0\sum_{s=1}^{\infty}\Big(\frac{qG_0^3}{3}\Big)^s\frac{1}{s!}\sum_{l=s+1}^{\infty}\Big(\frac{2^3\lambda}{G_0^3}\Big)^l\\
&\le \frac{2^3}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}G_0\sum_{s=1}^{\infty}\Big(\frac{2^3q\lambda}{3}\Big)^s\frac{1}{s!}\cdot\frac{2^3\lambda}{G_0^3}\\
&= \frac{2^6}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}\lambda G_0^{-2}\sum_{s=1}^{\infty}\Big(\frac{2^3q\lambda}{3}\Big)^s\frac{1}{s!}\\
&= \frac{2^6}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}\lambda G_0^{-2}\Big(\mathrm{e}^{\frac{2^3}{3}q\lambda}-1\Big) \qquad (2.81)
\end{aligned}
$$

in the same way we can bound the first term of E_H^q in (2.75)

$$
\begin{aligned}
\Big|\sum_{l=2}^{\infty}\sum_{k=2}^{l}H^q_{0,l,k}\Big| &\le \frac{2}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}G_0\sum_{l=2}^{\infty}\Big(\frac{2^5}{G_0^4}\Big)^l\sum_{k=2}^{l}\Big(\frac{G_0\lambda}{2^2}\Big)^k\\
&\le \frac{2^2}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}G_0\sum_{l=2}^{\infty}\Big(\frac{2^3\lambda}{G_0^3}\Big)^l\\
&\le \frac{2^3}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}G_0\Big(\frac{2^3\lambda}{G_0^3}\Big)^2\\
&= \frac{2^9}{\sqrt{\mathrm{e}}}\sqrt{\frac{2}{3}}\lambda^2G_0^{-5} \qquad (2.82)
\end{aligned}
$$

therefore the second term of $\tilde{M}_q$ in (2.57) denoted by E_H^q and given in (2.75) and (2.76) can be bounded as

$$|E_H^q|\le K\lambda^2(G_0^{-5}+G_0^{-2}\mathrm{e}^{(8/3)q\lambda}). \qquad (2.83)$$

Finally, if we call E_T^q the last term of $\tilde{M}_q$ in (2.57):

$$E_T^q=\sum_{k\ge2}\sum_{l\ge k}\tilde{r}^{2(l-k)}[\tilde{r}\mathrm{e}^{i\tilde{f}}]^k d_{l,l-k}G_0^{-4l+2k+1}\mathrm{e}^{-ik\alpha_0}T^q_{l,l-k} \qquad (2.84)$$

we have, using the bound for $d_{l,l-k}$ given in lemma A.1 and the bounds for $\tilde{r}(u)$ and $\tilde{r}\mathrm{e}^{i\tilde{f}(u)}$ given in (2.16) and (2.15) and the bound for $T^q_{l,l-k}$ given in proposition 2.21 we have that

$$
\begin{aligned}
|E_T^q| &\le \sum_{k\ge2}\sum_{l\ge k}2^{2(l-k)}[2e_0]^k\frac{2^{2l-k}}{\sqrt{\mathrm{e}}}G_0^{-4l+2k+1}K_{11}\gamma_4^lG_0^{-3}\\
&\le \frac{K_{11}}{\sqrt{\mathrm{e}}}G_0^{-2}\sum_{k\ge2}\sum_{l\ge k}2^{2(l-k)}e_0^k2^{2l}G_0^{-4l+2k}\gamma_4^l\\
&\le \frac{K_{11}}{\sqrt{\mathrm{e}}}G_0^{-2}\sum_{k\ge2}\Big(\frac{\lambda G_0}{2^2}\Big)^k\sum_{l\ge k}\Big(\frac{2^4\gamma_4}{G_0^4}\Big)^l\\
&\le 2\frac{K_{11}}{\sqrt{\mathrm{e}}}G_0^{-2}\sum_{k\ge2}\Big(\frac{2^2\gamma_4\lambda}{G_0^3}\Big)^k\\
&\le 2^2\frac{K_{11}}{\sqrt{\mathrm{e}}}G_0^{-2}\Big(\frac{2^2\gamma_4\lambda}{G_0^3}\Big)^2
\end{aligned}
$$

$$= 2^6\frac{K_{11}}{\sqrt{\mathrm{e}}}\gamma_4^2\lambda^2 G_0^{-8}$$
$$\le K\lambda^2 G_0^{-8} \qquad (2.85)$$

From the definition of E_G^q, E_H^q and E_T^q given in (2.66), (2.75) and (2.84) we have, using equation (2.65), from the definition of $\mathcal{M}_q(u)$ given in (2.61) that

$$\mathcal{M}_q(u) = \sum_{s=2}^{\infty} G^q_{s,s,s} + E^q_G + E^q_H + E^q_T \qquad (2.86)$$

For $q \ge 2$ we will bound the sum of the $G^q_{s,s,s}$ and for $q = 1$ we will compute it. From the definition of $G^q_{s,s,s}$ given in (2.67c) we need to compute $d_{s,0}$ and $d_0^{0,s}$. It is not difficult to see that

$$\binom{-1/2}{s} = \frac{(-1)^s}{s!}\frac{1}{2^s}(2s-1)!!$$

and therefore by its definition given in (2.20)

$$d_{s,0} = \frac{i}{s!}(-1)^s(2s-1)!!.$$

From lemma A.4, we have that $d_0^{0,s} = 1/2i$, and then

$$d_{s,0}d_0^{0,s} = \frac{(-1)^s}{2}\frac{(2s-1)!!}{s!}$$

From this, we can rewrite (2.67c) as

$$G^q_{s,s,s} = [\tilde r(u)\mathrm{e}^{i\tilde f(u)}]^s\sqrt{2\pi}\frac{q^{s-\frac12}}{s!}G_0^{s-\frac12}\mathrm{e}^{-is\alpha_0}. \qquad (2.87)$$

Then, from the bound for $\tilde r(u)\mathrm{e}^{i\tilde f(u)}$ given in (2.15)

$$|G^q_{s,s,s}| \le [2e_0]^s\sqrt{2\pi}\frac{q^{s-\frac12}}{s!}G_0^{s-\frac12}$$

therefore, since $e_0G_0 = \lambda$,

$$\Big|\sum_{s=2}^{\infty} G^q_{s,s,s}\Big| \le \sqrt{\frac{2\pi}{qG_0}}\sum_{s=2}^{\infty}\frac{(2q\lambda)^s}{s!}$$
$$= \sqrt{\frac{2\pi}{qG_0}}(\mathrm{e}^{2q\lambda} - 1 - 2q\lambda)$$
$$\le K(qG_0)^{-1/2}\mathrm{e}^{2q\lambda} \qquad (2.88)$$

From the expression for $\mathcal{M}_q$ given in (2.86), we have that

$$|\mathcal{M}_q| \le K(qG_0)^{-1/2}\mathrm{e}^{2q\lambda} + |E^q_G| + |E^q_H| + |E^q_T|$$

and using the bounds for E_G^q, E_H^q and E_T^q given in (2.73), (2.83) and (2.85)

$$|\mathcal{M}_q| \le K\lambda^2(qG_0)^{-1/2}\mathrm{e}^{2q\lambda} + K\lambda^2\sqrt{q}\big(G_0^{-7/2} + G_0^{-7/2}\mathrm{e}^{(8/3)q\lambda} + G_0^{-3/2}\mathrm{e}^{2q\lambda}\big) + K\lambda^2(G_0^{-5} + G_0^{-2}\mathrm{e}^{(8/3)q\lambda}) + K\lambda^2G_0^{-8}$$
$$\le K\lambda^2\Big[\mathrm{e}^{2q\lambda}\big((qG_0)^{-1/2} + \sqrt{q}G_0^{-3/2}\big) + \mathrm{e}^{(8/3)q\lambda}\big(\sqrt{q}G_0^{-7/2} + G_0^{-2}\big) + (G_0^{-5} + G_0^{-8} + \sqrt{q}G_0^{-7/2})\Big]$$
$$\le K\lambda^2\Big[\mathrm{e}^{2q\lambda}\big(\sqrt{q}G_0^{-1/2} + \sqrt{q}G_0^{-3/2}\big) + \mathrm{e}^{(8/3)q\lambda}\big(\sqrt{q}G_0^{-7/2} + G_0^{-2}\big) + \sqrt{q}G_0^{-7/2}\Big]$$
$$\le K\lambda^2\Big[\mathrm{e}^{2q\lambda}\sqrt{q}G_0^{-1/2} + \mathrm{e}^{(8/3)q\lambda}\big(\sqrt{q}G_0^{-7/2} + G_0^{-2}\big) + \sqrt{q}G_0^{-7/2}\Big]$$

$$
\begin{aligned}
&\le K\lambda^2\Big[\mathrm{e}^{2q\lambda}\sqrt{q}G_0^{-1/2}+\mathrm{e}^{(8/3)q\lambda}\sqrt{q}G_0^{-2}+\sqrt{q}G_0^{-7/2}\Big]\\
&\le K\lambda^2\Big[\sqrt{q}\big(\mathrm{e}^{2q\lambda}G_0^{-1/2}+\mathrm{e}^{(8/3)q\lambda}G_0^{-2}+G_0^{-7/2}\big)\Big]\\
&\le K\lambda^2\Big[\sqrt{q}\mathrm{e}^{(8/3)q\lambda}\big(G_0^{-1/2}+G_0^{-2}+G_0^{-7/2}\big)\Big]\\
&\le K\lambda^2\sqrt{q}\mathrm{e}^{(8/3)q\lambda}G_0^{-1/2}
\end{aligned}
$$

this is the desired result for $q\ge 1$. For $q=1$ we have, using the expression (2.11) for $\tilde r(u)\mathrm{e}^{i\tilde f(u)}$ and (2.87)

$$
\begin{aligned}
\sum_{s=2}^{\infty}G^1_{s,s,s} &= \sum_{s=2}^{\infty}[\tilde r\mathrm{e}^{i\tilde f}]^s\sqrt{2\pi}\frac{1}{s!}G_0^{s-\frac12}\mathrm{e}^{-is\alpha_0}\\
&= \sqrt{2\pi}G_0^{-1/2}\sum_{s=2}^{\infty}[e_0(\cos u-1)]^s\frac{1}{s!}G_0^s\mathrm{e}^{-is\alpha_0}\\
&= \sqrt{2\pi}G_0^{-1/2}\sum_{s=2}^{\infty}[-e_0G_0(1-\cos u)\mathrm{e}^{-i\alpha_0}]^s\frac{1}{s!}\\
&= \sqrt{2\pi}G_0^{-1/2}\sum_{s=2}^{\infty}[-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}]^s\frac{1}{s!}\\
&= \sqrt{2\pi}G_0^{-1/2}\big[\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}}-1+\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}\big]
\end{aligned}
$$

Therefore using equation (2.86) we have

$$
\mathcal{M}_1=\sqrt{2\pi}G_0^{-1/2}\big[\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}}-1+\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}\big]+\mathcal{E}_1
$$

where

$$
\mathcal{E}_1=E^1_G+E^1_H+E^1_T.
$$

Analogously as we did to bound $\mathcal{M}_q$ we find that

$$
|\mathcal{E}_1|\le K\lambda^2\mathrm{e}^{(8/3)\lambda}G_0^{-3/2}.
$$

This concludes the proof of proposition 2.22

□

The proposition 2.22 allow us to bound L_q for $q\ge 2$

Proposition 2.23. Let $q\in\mathbb{N}$, $q\ge 2$, $c\ge 1$, γ_4 be given in proposition 2.21 and λ real positive constant. If

$$
G_0\ge\{32,(24\lambda)^{1/3},(3c)^{2/3},8\lambda^{-1},(2^4\lambda)^{1/3},2^{3/2},2(\gamma_4\lambda)^{1/3},(2^5\gamma_4)^{1/4}\}
$$

then, exist a positive constant K, such that if $e_0G_0=\lambda$

$$
|L_q|\le K\lambda^2G_0^{-1/2}\mathrm{e}^{q\sqrt{1-e_0^2}}(4G_0\lambda^{-1})^q\mathrm{e}^{-qG_0^3\frac{2}{9}}.
$$

Proof. Let $q\in\mathbb{N}$, $q\ge 2$. From expression (2.62) and using proposition 2.22 and the bounds (2.16) and (2.17), we have

$$
\begin{aligned}
\mathrm{e}^{q\frac{G_0^3}{3}}\big|\tilde L_q-L_{q,-1}\mathrm{e}^{-i\alpha_0}\big| &\le \frac{1}{2\pi}\int_0^{2\pi}\big|\mathcal{M}_q(u)\big|\Big|\frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu}-a^2\mathrm{e}^{-iu}}\Big|^q|\tilde r(u)|du\\
&\le \frac{1}{2\pi}\int_0^{2\pi}K\lambda^2\sqrt{q}\mathrm{e}^{(8/3)q\lambda}G_0^{-1/2}\Big(\frac{2a^2}{e_0}\Big)^q\mathrm{e}^{q\sqrt{1-e_0^2}}2du\\
&\le 2K\lambda^2\sqrt{q}\mathrm{e}^{q\left(\sqrt{1-e_0^2}+(8/3)\lambda\right)}\Big(\frac{2}{e_0}\Big)^qG_0^{-1/2}
\end{aligned}
$$

then

$$\left|\tilde{L}_q - L_{q,-1}\mathrm{e}^{-i\alpha_0}\right| \leq 2K\lambda^2 \mathrm{e}^{q\sqrt{1-e_0^2}}\mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{8}{G_0^3}\lambda\right)}\sqrt{q}\Big(\frac{2}{e_0}\Big)^q G_0^{-1/2}$$

using that $G_0 \geq (24\lambda)^{1/3}$ and $\sqrt{q} \leq 2^q$ for $q \geq 2$, we have

$$\begin{aligned}\left|\tilde{L}_q - L_{q,-1}\mathrm{e}^{-i\alpha_0}\right| &\leq 2K\lambda^2 \mathrm{e}^{q\sqrt{1-e_0^2}}\mathrm{e}^{-qG_0^3\frac{2}{9}}\sqrt{q}\Big(\frac{2}{e_0}\Big)^q G_0^{-1/2} \\ &\leq 2K\lambda^2 \mathrm{e}^{q\sqrt{1-e_0^2}}\mathrm{e}^{-qG_0^3\frac{2}{9}}\Big(\frac{4}{e_0}\Big)^q G_0^{-1/2}\end{aligned}$$

from this, using the triangle inequality we have

$$\left|\tilde{L}_q\right| \leq \left|L_{q,-1}\right| + 2K\lambda^2 \mathrm{e}^{q\sqrt{1-e_0^2}}\mathrm{e}^{-qG_0^3\frac{2}{9}}\Big(\frac{4}{e_0}\Big)^q G_0^{-1/2}$$

and by lemma 2.7

$$\left|\tilde{L}_q\right| \leq K_3 2^{q+7}\mathrm{e}^{q\sqrt{1-e_0^2}}e_0^{q-1}G_0^{-1/2}\mathrm{e}^{-q\frac{G_0^3}{3}\left(1-\frac{3}{G_0^3}c^2\right)} + 2K\lambda^2 \mathrm{e}^{q\sqrt{1-e_0^2}}\mathrm{e}^{-qG_0^3\frac{2}{9}}\Big(\frac{4}{e_0}\Big)^q G_0^{-1/2}$$

and using that $G_0 \geq (3c)^{2/3}$

$$\begin{aligned}\left|\tilde{L}_q\right| &\leq \mathrm{e}^{q\sqrt{1-e_0^2}}\mathrm{e}^{-qG_0^3\frac{2}{9}}G_0^{-1/2}\Big[K_3 2^{q+7}e_0^{q-1} + 2K\lambda^2\Big(\frac{4}{e_0}\Big)^q\Big] \\ &\leq K\lambda^2 G_0^{-1/2}\mathrm{e}^{q\sqrt{1-e_0^2}}\Big(\frac{4}{e_0}\Big)^q \mathrm{e}^{-qG_0^3\frac{2}{9}}\end{aligned}$$

From proposition 2.20 we have

$$\begin{aligned}|L_q| &\leq |\tilde{L}_q| + |E_q| \\ &\leq K\lambda^2 G_0^{-1/2}\mathrm{e}^{q\sqrt{1-e_0^2}}\Big(\frac{4}{e_0}\Big)^q \mathrm{e}^{-qG_0^3\frac{2}{9}} + K_4 \mathrm{e}^{-qG_0^3\frac{2}{9}}G_0^{-3/2}\big(2e_0\mathrm{e}^{\sqrt{1-e_0^2}}\big)^q \\ &\leq K\lambda^2 G_0^{-1/2}\mathrm{e}^{q\sqrt{1-e_0^2}}\Big(\frac{4}{e_0}\Big)^q \mathrm{e}^{-qG_0^3\frac{2}{9}}.\end{aligned}$$

□

Proposition 2.24. Let λ be a real positive constant and γ_4 be given in proposition 2.21, $G_0 \geq \max\{8\lambda^{-1}, (2^4\lambda)^{1/3}, 2^{3/2}, 2(\gamma_4\lambda)^{1/3}, (2^5\gamma_4)^{1/4}\}$. Then there exists a positive constant $K = K(\lambda)$ such that If $e_0 G_0 = \lambda$,

$$\begin{aligned}L_1 = &\Big(\frac{1}{4}\sqrt{\frac{\pi}{2}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}} + E_3 + E_5 + E_7\Big)\mathrm{e}^{-i\alpha_0} \\ &+ \mathrm{e}^{-\frac{G_0^3}{3}}2\sqrt{2\pi}\lambda^{-1}G_0^{1/2}\Bigg[\mathrm{e}^{-\lambda\mathrm{e}^{-i\alpha_0}}\frac{A}{A-1}\Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}}J_1(\pm 2i\sqrt{A(A-1)}) \\ &\qquad - J_0(\pm 2i\sqrt{A(A-1)})\Big] - A\Bigg] \\ &\qquad + \tilde{\mathcal{E}}_3 + \tilde{\mathcal{E}}_2 + \tilde{\mathcal{E}}_1 + E_1\end{aligned}$$

where

$$\begin{aligned}|E_3| &\leq K_7 \mathrm{e}^{\sqrt{1-e_0^2}}G_0^{-3/2}\mathrm{e}^{-\frac{G_0^3}{3}} \\ |E_5| &\leq 2^5 \mathrm{e}^{\sqrt{1-e_0^2}}K_{13}G_0^{-2}\mathrm{e}^{-\frac{G_0^3}{3}} \\ |E_7| &\leq \sqrt{\frac{\pi}{8}}98e_0^2 G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}\end{aligned}$$

$$
\begin{aligned}
|\tilde{\mathcal{E}}_1| &\le K\mathrm{e}^{(8/3)\lambda}\lambda\mathrm{e}^{\sqrt{1-e_0^2}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}\\
|\tilde{\mathcal{E}}_2| &\le K\lambda^3\mathrm{e}^{2\lambda}G_0^{-3/2}\mathrm{e}^{-\frac{G_0^3}{3}}\\
|\tilde{\mathcal{E}}_3| &\le Ke_0^2G_0^{1/2}\mathrm{e}^{-\frac{G_0^3}{3}}\\
|E_1| &\le K_4'\mathrm{e}^{-G_0^3/3}G_0^{-3/2}(2e_0\mathrm{e}^{\sqrt{1-e_0^2}}+G_0^{-2})
\end{aligned}
$$

and

$$
\begin{aligned}
A &= \frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}\\
\frac{A}{A-1} &= \frac{\lambda^2-2\lambda\mathrm{e}^{-i\alpha_0}}{\lambda^2-4\lambda\cos\alpha_0+4}\\
A(A-1) &= \frac{-\lambda}{2}\mathrm{e}^{-i\alpha_0}\Big(1-\frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}\Big).
\end{aligned}
$$

The functions $J_0(z)$ and $J_1(z)$ are the Bessel's functions of the first kind [AS65] and whose expansion around $z=0$ is given by

$$
J_n(z)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+n+1)}\Big(\frac{z}{2}\Big)^{2m+n}
$$

Proof. From expression (2.62) and proposition 2.22 we have that

$$
\begin{aligned}
&\tilde{L}_1-L_{1,-1}\mathrm{e}^{-i\alpha_0}\\
&=\mathrm{e}^{-\frac{G_0^3}{3}}\frac{\sqrt{2\pi}}{2\pi}G_0^{-1/2}\int_0^{2\pi}\big[\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}}-1+\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}\big]\left[\frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu}-a^2\mathrm{e}^{-iu}}\right]\tilde{r}(u)du+\tilde{\mathcal{E}}_1
\end{aligned}
\tag{2.89}
$$

where

$$
\tilde{\mathcal{E}}_1=\mathrm{e}^{-\frac{G_0^3}{3}}\frac{1}{2\pi}\int_0^{2\pi}\mathcal{E}_1\left[\frac{2a^2}{e_0}\mathrm{e}^{-iu}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu}-a^2\mathrm{e}^{-iu}}\right]\tilde{r}(u)du
$$

using the bounds for $\tilde{r}$ and $\tilde{r}\mathrm{e}^{i\tilde{f}}$ given in (2.16) and (2.15) and the bound for $\mathcal{E}_1$ given in proposition 2.22 we have that

$$
\begin{aligned}
|\tilde{\mathcal{E}}_1| &\le K\lambda^2\mathrm{e}^{(8/3)\lambda}G_0^{-3/2}\frac{2}{e_0}\mathrm{e}^{\sqrt{1-e_0^2}}2\mathrm{e}^{-\frac{G_0^3}{3}}\\
&\le K\mathrm{e}^{(8/3)\lambda}\lambda\mathrm{e}^{\sqrt{1-e_0^2}}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}.
\end{aligned}
\tag{2.90}
$$

To compute the integral in (2.89) we first notice, using the definition of a^2 given in (2.9d) and the definition of $\tilde{r}$ given in (2.12), that

$$
\begin{aligned}
\frac{2a^2}{e_0} &= \frac{1}{e_0}(1+\sqrt{1-e_0^2})=\frac{2}{e_0}(1+O(e_0^2))\\
\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu}} &= \mathrm{e}^{\frac{e_0^2}{4}(1+O(e_0^2))\mathrm{e}^{iu}}=(1+O(e_0^2))\\
\tilde{r}(u)=1-\frac{e_0^2}{4a^2}\mathrm{e}^{iu}-a^2\mathrm{e}^{-iu} &= 1-\frac{e_0^2}{4}(1+O(e_0^2))\mathrm{e}^{iu}-(1+O(e_0^2))\mathrm{e}^{-iu}=1-\mathrm{e}^{-iu}+O(e_0^2)\\
\mathrm{e}^{-a^2\mathrm{e}^{-iu}} &= \mathrm{e}^{-(1+O(e_0^2))\mathrm{e}^{-iu}}=\mathrm{e}^{-\mathrm{e}^{-iu}}(1+O(e_0^2))
\end{aligned}
$$

therefore

$$
\begin{aligned}
\left[\frac{2a^2}{e_0}\mathrm{e}^{\frac{e_0^2}{4a^2}\mathrm{e}^{iu}-a^2\mathrm{e}^{-iu}}\right]\tilde{r}(u) &= \frac{2}{e_0}(1+O(e_0^2))(1+O(e_0^2))\mathrm{e}^{-\mathrm{e}^{-iu}}(1+O(e_0^2))[1-\mathrm{e}^{-iu}+O(e_0^2)]\\
&= \frac{2}{e_0}\mathrm{e}^{-\mathrm{e}^{-iu}}[1-\mathrm{e}^{-iu}+O(e_0^2)](1+O(e_0^2))
\end{aligned}
$$

$$= \frac{2}{e_0} \mathrm{e}^{-\mathrm{e}^{-iu}}(1-\mathrm{e}^{-iu})(1+O(e_0^2)) + \mathrm{e}^{-\mathrm{e}^{-iu}} O(e_0)$$

From this, we can rewrite (2.89) as

$$\begin{aligned}&\tilde{L}_1 - L_{1,-1}\mathrm{e}^{-i\alpha_0}\\ &= \mathrm{e}^{-\frac{G_0^3}{3}} \frac{\sqrt{2\pi}}{2\pi} G_0^{-1/2}(1+O(e_0^2))\frac{2}{e_0}\int_0^{2\pi}\left[\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}} - 1 + \lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}\right]\mathrm{e}^{-\mathrm{e}^{-iu}}\mathrm{e}^{-iu}(1-\mathrm{e}^{-iu})du + \tilde{\mathcal{E}}_2 + \tilde{\mathcal{E}}_1\end{aligned} \tag{2.91}$$

with

$$\tilde{\mathcal{E}}_2 = \mathrm{e}^{-\frac{G_0^3}{3}} O(e_0)\frac{\sqrt{2\pi}}{2\pi} G_0^{-1/2}\int_0^{2\pi}\left[\mathrm{e}^{\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}} - 1 - \lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}\right]\mathrm{e}^{-iu}\mathrm{e}^{-\mathrm{e}^{-iu}}du$$

and since $|\mathrm{e}^{-iu}\mathrm{e}^{-\mathrm{e}^{-iu}}| \leq \mathrm{e}$, $\mathrm{e}^z - 1 - z \leq |z|^2\mathrm{e}^{|z|}$ and $e_0 G_0 = \lambda$ we have

$$|\tilde{\mathcal{E}}_2| \leq K\lambda^2\mathrm{e}^{2\lambda}G_0^{-1/2}\mathrm{e}^{-\frac{G_0^3}{3}}e_0 \leq K\lambda^3\mathrm{e}^{2\lambda}G_0^{-3/2}\mathrm{e}^{-\frac{G_0^3}{3}}. \tag{2.92}$$

It remains to compute the integral in (2.91). It can be expressed as the sum of three integrals as follows

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}&\left[\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}} - 1 + \lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}\right]\mathrm{e}^{-\mathrm{e}^{-iu}}\mathrm{e}^{-iu}(1-\mathrm{e}^{-iu})du\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}}\mathrm{e}^{-\mathrm{e}^{-iu}}\mathrm{e}^{-iu}(1-\mathrm{e}^{-iu})du\\ &- \frac{1}{2\pi}\int_0^{2\pi}(1-\lambda)\mathrm{e}^{-\mathrm{e}^{-iu}}\mathrm{e}^{-iu}(1-\mathrm{e}^{-iu})du\\ &- \frac{1}{2\pi}\int_0^{2\pi}\lambda(\cos u)\mathrm{e}^{-i\alpha_0}\mathrm{e}^{-\mathrm{e}^{-iu}}\mathrm{e}^{-iu}(1-\mathrm{e}^{-iu})du\end{aligned} \tag{2.93}$$

To compute the first of these three integrals we notice that

$$\begin{aligned}\mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}}\mathrm{e}^{-\mathrm{e}^{-iu}} &= \mathrm{e}^{-\lambda\mathrm{e}^{-i\alpha_0}}\mathrm{e}^{\lambda\cos u\mathrm{e}^{-i\alpha_0}}\mathrm{e}^{-\mathrm{e}^{-iu}}\\ &= \mathrm{e}^{-\lambda\mathrm{e}^{-i\alpha_0}}\mathrm{e}^{(\lambda/2)\mathrm{e}^{-i\alpha_0}(\mathrm{e}^{iu}+\mathrm{e}^{-iu})}\mathrm{e}^{-\mathrm{e}^{-iu}}\\ &= \mathrm{e}^{-\lambda\mathrm{e}^{-i\alpha_0}}\mathrm{e}^{A\mathrm{e}^{iu}}\mathrm{e}^{(A-1)\mathrm{e}^{-iu}}\end{aligned} \tag{2.94}$$

with

$$A = \frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}$$

Therefore, using expression (2.94) we see that, to compute the first integral in (2.93) is equivalent to

$$\mathrm{e}^{-\lambda\mathrm{e}^{-i\alpha_0}}(\mathcal{N}_1 - \mathcal{N}_2)$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the first two Fourier coefficients of the function

$$\mathcal{N}(u) = \mathrm{e}^{A\mathrm{e}^{iu}}\mathrm{e}^{(A-1)\mathrm{e}^{-iu}} = \sum_{q\in\mathbb{Z}}\mathcal{N}_q\mathrm{e}^{iqu}.$$

Expanding in Taylor series we have

$$\mathcal{N}(u) = \sum_{j=0}^{\infty}\frac{(A\mathrm{e}^{iu})^j}{j!}\sum_{k=0}^{\infty}\frac{[(A-1)\mathrm{e}^{-iu}]^k}{k!}$$

then the Fourier coefficient $\mathcal{N}_1$ is given by

$$\mathcal{N}_1 \mathrm{e}^{iu} = \sum_{j=1}^{\infty} \frac{(A\mathrm{e}^{iu})^j}{j!} \frac{[(A-1)\mathrm{e}^{-iu}]^{j-1}}{(j-1)!} = A \sum_{j=1}^{\infty} \frac{[A(A-1)]^{j-1}}{j!(j-1)!} \mathrm{e}^{iu} = AW'(A(A-1))\mathrm{e}^{iu}$$

and the Fourier coefficient $\mathcal{N}_2$ is given by

$$\mathcal{N}_2 \mathrm{e}^{2iu} = \sum_{j=2}^{\infty} \frac{(A\mathrm{e}^{iu})^j}{j!} \frac{[(A-1)\mathrm{e}^{-iu}]^{j-2}}{(j-2)!} = A^2 \sum_{j=2}^{\infty} \frac{[A(A-1)]^{j-2}}{j!(j-2)!} \mathrm{e}^{2iu} = A^2 W''(A(A-1))\mathrm{e}^{2iu}$$

where

$$W(w) = \sum_{n=0}^{\infty} \frac{w^n}{(w!)^2}$$

and its properties are detailed in appendix B. With these two Fourier coefficients we have that

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}} \mathrm{e}^{-\mathrm{e}^{-iu}} \mathrm{e}^{-iu}(1-\mathrm{e}^{-iu}) du = \\ \mathrm{e}^{-\lambda \mathrm{e}^{-i\alpha_0}} [AW'(A(A-1)) - A^2 W''(A(A-1))].$$

Using the expressions for W' and W'' given in (B.24) and (B.26) we have

$$\begin{aligned} AW'(A(A-1)) &= \frac{2A}{\pm 2i\sqrt{A(A-1)}} J_1(\pm 2i\sqrt{A(A-1)}) \\ A^2 W''(A(A-1)) &= \frac{A}{A-1}\Big(J_0(\pm 2i\sqrt{A(A-1)}) - \frac{2}{\pm 2i\sqrt{A(A-1)}} J_1(\pm 2i\sqrt{A(A-1)})\Big) \end{aligned}$$

then,

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\lambda(1-\cos u)\mathrm{e}^{-i\alpha_0}} \mathrm{e}^{-\mathrm{e}^{-iu}} \mathrm{e}^{-iu}(1-\mathrm{e}^{-iu}) du \\ &= \mathrm{e}^{-\lambda \mathrm{e}^{-i\alpha_0}} \Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}} J_1(\pm 2i\sqrt{A(A-1)})\Big(1 + \frac{1}{A-1}\Big) - \frac{A}{A-1} J_0(\pm 2i\sqrt{A(A-1)})\Big] \\ &= \mathrm{e}^{-\lambda \mathrm{e}^{-i\alpha_0}} \frac{A}{A-1}\Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}} J_1(\pm 2i\sqrt{A(A-1)}) - J_0(\pm 2i\sqrt{A(A-1)})\Big] \qquad (2.95) \end{aligned}$$

where

$$\frac{A}{A-1} = \frac{A}{A-1} \cdot \frac{\bar{A}-1}{\bar{A}-1} = \frac{|A|^2 - A}{|A-1|^2} = \frac{\lambda^2 - 2\lambda \mathrm{e}^{-i\alpha_0}}{\lambda^2 - 4\lambda\cos\alpha_0 + 4} \qquad (2.96)$$

$$A(A-1) = -\frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}\Big(1 - \frac{\lambda}{2}\mathrm{e}^{-i\alpha_0}\Big). \qquad (2.97)$$

Now, the second integral in (2.93) is clearly equal to zero, since the function $\mathrm{e}^{-\mathrm{e}^{-iu}}$ has no positive harmonics. The third integral, can be written as

$$\begin{aligned} -\frac{\lambda}{2}\mathrm{e}^{-i\alpha_0} \frac{1}{2\pi} \int_0^{2\pi} (\mathrm{e}^{iu} + \mathrm{e}^{-iu}) \mathrm{e}^{-\mathrm{e}^{-iu}} \mathrm{e}^{-iu}(1-\mathrm{e}^{-iu}) du &= A \frac{1}{2\pi} \int_0^{2\pi} (1 + \mathrm{e}^{-2iu}) \mathrm{e}^{-\mathrm{e}^{-iu}} (1-\mathrm{e}^{-iu}) du \\ &= A \frac{1}{2\pi} \int_0^{2\pi} (1 - \mathrm{e}^{-iu} + \mathrm{e}^{-2iu} - \mathrm{e}^{-3iu}) \mathrm{e}^{-\mathrm{e}^{-iu}} du \\ &= A. \end{aligned}$$

Substituting this and (2.95) in expression (2.93) we have by (2.91) that

$\tilde{L}_1 - L_{1,-1}\mathrm{e}^{-i\alpha_0}$

$$= \mathrm{e}^{-\frac{G_0^3}{3}} 2\sqrt{2\pi}\lambda^{-1}G_0^{1/2}(1+O(e_0^2))\left[\mathrm{e}^{-\lambda \mathrm{e}^{-i\alpha_0}}\frac{A}{A-1}\Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}}J_1(\pm 2i\sqrt{A(A-1)}) - J_0(\pm 2i\sqrt{A(A-1)})\Big]+A\right]$$
$$+\tilde{\mathcal{E}}_2+\tilde{\mathcal{E}}_1$$
$$= \mathrm{e}^{-\frac{G_0^3}{3}} 2\sqrt{2\pi}\lambda^{-1}G_0^{1/2}\left[\mathrm{e}^{-\lambda \mathrm{e}^{-i\alpha_0}}\frac{A}{A-1}\Big[\frac{2A}{\pm 2i\sqrt{A(A-1)}}J_1(\pm 2i\sqrt{A(A-1)}) - J_0(\pm 2i\sqrt{A(A-1)})\Big]+A\right]$$
$$+\tilde{\mathcal{E}}_3+\tilde{\mathcal{E}}_2+\tilde{\mathcal{E}}_1 \tag{2.98}$$

where

$$|\tilde{\mathcal{E}}_3| \leq K e_0^2 G_0^{1/2}\mathrm{e}^{-\frac{G_0^3}{3}},$$

with $K = K(\lambda)$. From the expression for $L_{1,-1}$ given in (2.63) and using equation (2.98) along with proposition 2.20 we get the proof of proposition 2.24 □

Analogously as we did in lemma 2.9 we have the next lemma

Lemma 2.25. Let $\mathcal{L}$ be given by (2.7) and $q \in \mathbb{N}$, $q \geq 2$, $c \geq 1$, γ_4 be given in proposition 2.21 and λ real positive constant. If

$$G_0 \geq \{32, (24\lambda)^{1/3}, (3c)^{2/3}, 8\lambda^{-1}, (2^4\lambda)^{1/3}, 2^{3/2}, 2(\gamma_4\lambda)^{1/3}, (2^5\gamma_4)^{1/4}\}$$

Then, there exist a positive constant K depending on λ, such that if $e_0 G_0 = \lambda$,

$$\mathcal{L} = L_0(\alpha_0) + 2\Re\Big\{\mathrm{e}^{it_0}L_1(\alpha_0) + \mathcal{E}(t_0,\alpha_0)\Big\} \tag{2.99}$$

where

$$|\mathcal{E}(t_0,\alpha_0)| \leq K G_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}.$$

Proof. From the formula for the Melnikov potential (2.7), we can write directly equation (2.99) by defining

$$\mathcal{E}(t_0,\alpha_0) = \sum_{q\geq 2} L_q \mathrm{e}^{iqt_0}.$$

Then, by proposition 2.23 we have

$$\begin{aligned}
|\mathcal{E}(t_0,\alpha_0)| &\leq \sum_{q\geq 2} K\lambda^2 G_0^{-1/2}\mathrm{e}^{q\sqrt{1-e_0^2}}(4G_0\lambda^{-1})^q \mathrm{e}^{-qG_0^3\frac{2}{9}} \\
&\leq K\lambda^2 G_0^{-1/2}\sum_{q\geq 2}\Big(\mathrm{e}^{\sqrt{1-e_0^2}}4G_0\lambda^{-1}\mathrm{e}^{-G_0^3\frac{2}{9}}\Big)^q \\
&\leq 2KG_0^{-1/2}\Big(\mathrm{e}^{\sqrt{1-e_0^2}}4G_0\lambda^{-1}\mathrm{e}^{-G_0^3\frac{2}{9}}\Big)^2 \\
&\leq 2KG_0^{-1/2}\mathrm{e}^{2\sqrt{1-e_0^2}}16G_0^2\lambda^{-2}\mathrm{e}^{-G_0^3\frac{4}{9}} \\
&\leq KG_0^{3/2}\mathrm{e}^{-G_0^3\frac{4}{9}}
\end{aligned}$$

the third bound is possible since for $G_0 \geq 32$ we have $\mathrm{e}^{\sqrt{1-e_0^2}}4G_0\lambda^{-1}\mathrm{e}^{-G_0^3\frac{2}{9}} < 1/2$. This concludes the proof of lemma 2.25. □

Finally, theorem 2.19 and therefore theorem 1.6 is a direct consequence of proposition 2.24 and lemma 2.25. The condition imposed on G_0 in theorem 1.6 is obtained simply by noticing that $\gamma_4 < 3$, from its definition given in proposition 2.21.

Appendices

Appendix A

Change to complex integral

The aim of this appendix is to show how a change of path in the integral (2.4) can be made to pursue a good estimation. So, let us call that integral

$$I(q,m,n) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq\frac{G_0^3}{2}\left(\tau+\frac{\tau^3}{3}\right)}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau \tag{A.1}$$

. We will write

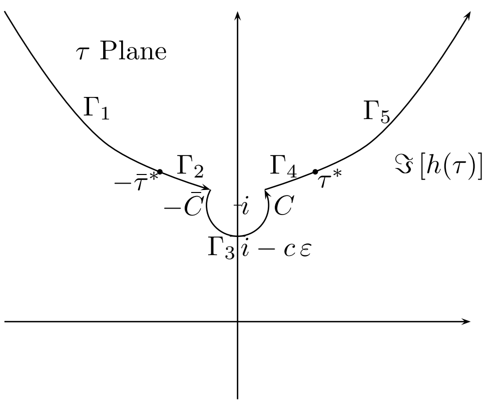
$$h(\tau) = i\Big(\frac{\tau^3}{3} + \tau\Big) \tag{A.2}$$

Since the integral I involves an exponential, it will be useful a Laplace type method (see [Erd56]) of integration. In particular when $\Im(h(\tau)) = 0$. So, let us define the path

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \tag{A.3}$$

where $\varepsilon > 0$ and c is taken such that $c \geq 1$ and $c\varepsilon < 1$:

$$\begin{aligned}
\Gamma_1 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | \Re(\tau) \leq \Re(-\bar{\tau}^*)\} \\
\Gamma_5 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | \Re(\tau) \geq \Re(\ \ \tau^*)\} \\
\Gamma_2 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | \Re(-\bar{\tau}^*) \leq \Re(\tau) \leq 0\} \cap \{\tau \in \mathbb{C} || \tau - i| \geq c\varepsilon\} \\
\Gamma_4 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | 0 \leq \Re(\tau) \leq \Re(\ \ \tau^*)\} \cap \{\tau \in \mathbb{C} || \tau - i| \geq c\varepsilon\} \\
\Gamma_3 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) \leq 0\} \cap \{\tau \in \mathbb{C} || \tau - i| = c\varepsilon\}
\end{aligned} \tag{A.4}$$



By means of the Cauchy-Goursat theorem and a limit argument, it can be shown that the integral $I(q,m,n)$, defined in (A.1), which is taken over the real axis, is equal to the one taken over the path Γ thinking of τ as a complex number (see [LS80a]). In fact, by the same argument, its value does not depend on ε.

The positive branch of the hyperbola defined by $\Im(h(\tau)) = 0$ intersects the circumference of radius ε in two points that can be expressed as C and $-\bar{C}$ and rigorously are defined in the following way

$$\{\ \ C\} = \Gamma_3 \cap \Gamma_4 \tag{A.5}$$
$$\{-\bar{C}\} = \Gamma_3 \cap \Gamma_2 \tag{A.6}$$

Since the integral over Γ does not depend on ε, we will choose a particular value of ε to bound $I(q,m,n)$ and consequently $N(q,m,n)$ defined in (2.4). Later on, in proposition 2.4 we will just compute the ε-independent terms.

It is not difficult to see that if we define the function

$$u(\tau) = h(i) - h(\tau) = -\frac{2}{3} - i\Big(\frac{\tau^3}{3} + \tau\Big) = (\tau - i)^2 - \frac{i}{3}(\tau - i)^3, \tag{A.7}$$

then

$$u(\Gamma_1 \cup \Gamma_2),\ u(\Gamma_4 \cup \Gamma_5) \subset \mathbb{R}_0^+.$$

Moreover, if $\tau^- \in \Gamma_1 \cup \Gamma_2$ then $\tau^+ = -\bar{\tau}^- \in \Gamma_4 \cup \Gamma_5$ and

$$u(\tau^-) = u(\tau^+).$$

On the other hand one can see that u is an increasing function while moving along $\Gamma_1 \cup \Gamma_2$ or $\Gamma_4 \cup \Gamma_5$ in the direction of increasing imaginary part. Therefore in $\mathbb{R}_0^+$ u has two inverses; τ^+ and τ_-. Before writing them down let us notice that the point C defined in (A.5) can be written as

$$C = i + \varepsilon\, c\mathrm{e}^{i\theta_\varepsilon} \quad \text{with} \quad \theta_\varepsilon \in (0, \pi/2) \tag{A.8}$$

and in coordinates u, defined in (A.7), has the expression

$$u(C) = \varepsilon^2\, c^2 \mathrm{e}^{2i\theta_\varepsilon} - \frac{\varepsilon^3\, c^3}{3} i\mathrm{e}^{3i\theta_\varepsilon} = O(\varepsilon^2\, c^2) \tag{A.9}$$

Moreover

$$u(C) = |u(C)| = \varepsilon^2\, c^2 \big|1 - \frac{\varepsilon}{3} i\mathrm{e}^{i\theta_\varepsilon}\big| = \varepsilon^2\, c^2\ k_\varepsilon \tag{A.10}$$

with $1 \le k_\varepsilon$. To see this, just consider

$$\begin{aligned} k_\varepsilon = \big|1 - \frac{\varepsilon\, c}{3} i\mathrm{e}^{i\theta_\varepsilon}\big| &= \big|1 - \frac{\varepsilon\, c}{3} i(\cos\theta_\varepsilon + i\sin\theta_\varepsilon)\big| \\ &= \big|1 - \frac{\varepsilon\, c}{3}(i\cos\theta_\varepsilon - \sin\theta_\varepsilon)\big| \\ &= \big|1 + \frac{\varepsilon\, c}{3}\sin\theta_\varepsilon - i\frac{\varepsilon\, c}{3}\cos\theta_\varepsilon\big| \\ &= \sqrt{\Big(1 + \frac{\varepsilon\, c}{3}\sin\theta_\varepsilon\Big)^2 + \Big(\frac{\varepsilon\, c}{3}\cos\theta_\varepsilon\Big)^2} \ge 1 \end{aligned}$$

since by construction, $\theta_\varepsilon \in (0, \pi/2)$ and then $0 < \sin\theta_\varepsilon$.

Now, we can write the inverses of the function u

$$\begin{aligned} \tau^+ :&[u(C), +\infty) \to \Gamma_4 \cup \Gamma_5 & \qquad \tau^- :&[u(C), +\infty) \to \Gamma_1 \cup \Gamma_2 \\ &u \longrightarrow \xi(u) + i\eta(u) & &u \longrightarrow -\xi(u) + i\eta(u) \end{aligned} \tag{A.11}$$

The change (A.7) is useful over $\Gamma_1 \cup \Gamma_2$ and $\Gamma_4 \cup \Gamma_5$, thus performing this change in (2.4), we have that for any $\varepsilon > 0$

$$N(q,m,n) = \frac{d_{m,n}\mathrm{e}^{-q\frac{G_0^3}{3}}}{G_0^{2m+2n-1}} \left[\int_{u(C)}^{\infty} [F_{m,n}^+(u) - F_{m,n}^-(u)]\mathrm{e}^{-q\frac{G_0^3}{2}u} du + (-i)\mathrm{e}^{q\frac{G_0^3}{3}} \int_{\Gamma_3} f_{m,n}^q(\tau) d\tau \right] \tag{A.12}$$

where

$$d_{m,n} = i2^{m+n}\binom{-1/2}{n}\binom{-1/2}{m} \tag{A.13}$$

$$F_{m,n}^{\pm}(u) = \frac{1}{(\tau^{\pm}(u) - i)^{2m+1}(\tau^{\pm}(u) + i)^{2n+1}} \tag{A.14}$$

$$f_{m,n}^{q}(\tau) = \frac{\mathrm{e}^{q\frac{G_0^3}{2}h(\tau)}}{(\tau - i)^{2m}(\tau + i)^{2n}} \tag{A.15}$$

Now a series of lemmas that will be useful.

Lemma A.1. Let $m, n \in \mathbb{Z}$, $m, n \geq 0$ and $d_{m,n}$ be defined by equation (A.13). Then

$$|d_{m,n}| \leq \mathrm{e}^{-1/2}2^{m+n} \qquad \text{if } m + n > 0$$
$$|d_{m,n}| \leq \mathrm{e}^{-1}2^{m+n} \qquad \text{if } m, n > 0$$

Proof. Let $s \in \mathbb{N}$, then

$$\begin{aligned}\left|\binom{-1/2}{s}\right| &= \left|\frac{(-1)^s}{s!}\Big(\frac{1}{2}\Big)\Big(\frac{1}{2}+1\Big)\cdots\Big(\frac{1}{2}+s-1\Big)\right| \\ &= \frac{1}{2^s}\Big[1 \cdot \frac{3}{2} \cdots \frac{2(s-1)}{s}]\Big] \\ &\leq \frac{1}{2^s}\Big(2 - \frac{1}{s}\Big)^s \\ &= \Big(1 - \frac{1}{2s}\Big)^s \\ &\leq \lim_{s\to\infty}\Big(1 - \frac{1}{2s}\Big)^s \\ &= \mathrm{e}^{-1/2}\end{aligned}$$

Using this inequality and equation (A.13) we have that, in the case $n + m > 0$, n and m cannot be simultaneously zero and therefore the product of combinatorial is at most $\mathrm{e}^{-1/2}$ if neither of m and n is zero, then clearly, that product is bounded by e^{-1}. ☐

The next lemma, found in [Erd56], gives information of $\tau^{\pm}(u)$ when $u \in \mathbb{C}$

Lemma A.2. A local expression for the inverses $\tau^{\pm}$ given in (A.11) is

$$\tau^{\pm}(u) - i = \sum_{n=1}^{\infty} A_n(\pm\sqrt{u})^n, \tag{A.16}$$

where

$$A_n = \frac{i^{n-1}\Gamma(3n/2 - 1)}{n!\Gamma(n/2)3^{n-1}} \tag{A.17}$$

the series (A.16) is convergent whenever $|\sqrt{u}| < 2/\sqrt{3}$.

Lemma A.3. Let $\tau^{\pm}$ be defined by equations (A.11) and $u^* = (3 + \sqrt{13})/6$. Then, for $u \in \mathbb{R}$ and $0 < u < u^*$ we have that

$$|\tau^{\pm}(u) - i| < 1$$

and $|\tau^{\pm}(u^*) - i| = 1$. Moreover, for $u \in \mathbb{C}$ with $|\sqrt{u}| \leq \sqrt{2/3}$ we have that

$$|\tau^{\pm}(u) - i| \leq 1$$

and the curve $|\tau^{\pm}(u) - i| = 1$ is contained in the ring

$$\sqrt{\frac{2}{3}} \leq |\sqrt{u}| \leq \frac{2}{\sqrt{3}}.$$

Therefore the region $|\tau^{\pm}(u) - i| \leq 1$ is contained in the disk $|\sqrt{u}| \leq 2/\sqrt{3}$.

Proof. First we will consider the case where $u \in \mathbb{R}$, that is when $\Im(h(\tau(u))) = 0$. Let us write

$$\tau = \xi + i\eta, \tag{A.18}$$

from this we have

$$h(\tau) = i\Big(\frac{\tau^3}{3} + \tau\Big) = -\eta(1 + \xi^2 - \frac{\eta^2}{3}) + i\Big(\xi + \frac{\xi^3}{3} - \eta^2\xi\Big)$$

and then

$$\Im(h(\tau)) = \xi + \frac{\xi^3}{3} - \eta^2\xi.$$

from this equation, $\Im(h(\tau)) = 0$ if

$$\eta = \pm\sqrt{1 + \frac{\xi^2}{3}} \tag{A.19}$$

The positive sign in the last equation correspond clearly to the positive branch of a hyperbola. Having this in mind, equation (A.7) can be expressed as

$$u = -\frac{2}{3} + \eta(1 + \xi^2 - \frac{\eta^2}{3}) = -\frac{2}{3} + \frac{\eta}{3}(8\eta^2 - 6). \tag{A.20}$$

Let $\tau_*^{\pm} = \pm\xi_* + i\eta_*$ be such that $\Im(h(\tau_*^{\pm})) = 0$ and

$$|\tau_*^{\pm} - i| = 1$$

or, using equation (A.19) (we are only interested in the positive branch of the hyperbola)

$$\xi_*^2 + \left(\sqrt{1 + \frac{\xi_*^2}{3}} - 1\right)^2 - 1 = 0$$

from where

$$\xi_*^2 = \frac{3}{8}(\sqrt{13} - 1)$$

and then by equation (A.19), since we are only interested in the positive branch of the hyperbola

$$\eta_* = \sqrt{1 + \frac{1}{8}(\sqrt{13} - 1)} = \frac{1}{4}\big(1 + \sqrt{13}\big)$$

with this, we define, by means of equation (A.20)

$$u^* = u(\tau_*^{\pm}) = -\frac{2}{3} + \frac{1}{24}(1 + \sqrt{13})^3 - \frac{1}{2}(1 + \sqrt{13}) = \frac{1}{6}(3 + \sqrt{13}) \approx 1.1009$$

By construction $|\tau^{\pm}(u^*) - i| = 1$. Also, we have, as expected that $\sqrt{u^*} \approx 1.0492 < 2/\sqrt{3}$.

It is clear from equation (A.20) that u is a monotone increasing function of η with $\eta \in [1, +\infty)$ therefore its inverse $\tau^{\pm}(u)$ is a monotone increasing function of u and then

$$|\tau^{\pm}(u) - i| = \sqrt{\xi(u)^2 + (\eta(u) - 1)^2} = \sqrt{4\eta(u)^2 - 2(1 + \eta(u))}$$

is a monotone increasing function of u. This completes the case $u \in \mathbb{R}$.

Now, let $u \in \mathbb{C}$. If we fix

$$\tau^{\pm} - i = s e^{i\theta} \qquad \theta \in [0, 2\pi) \quad 0 < s \leq 1 \tag{A.21}$$

From equation (A.7) we have

$$u(\tau) = (\tau - i)^2 - \frac{i}{3}(\tau - i)^3,$$

and using (A.21)

$$|u(\tau)| = s^2\Big|1 - \frac{is}{3}\mathrm{e}^{i\theta}\Big| = s^2\sqrt{1 + \frac{s^2}{9} + \frac{2s}{3}\sin\theta}.$$

From where, since $-1 \le \sin\theta \le 1$, we have

$$\sqrt{u_{\max}} = \max_{|\tau-i|=s} |\sqrt{u(\tau)}| = s\left(1 + \frac{s^2}{9} + \frac{2s}{3}\right)^{1/4}$$
$$\sqrt{u_{\min}} = \min_{|\tau-i|=s} |\sqrt{u(\tau)}| = s\left(1 + \frac{s^2}{9} - \frac{2s}{3}\right)^{1/4}$$

and since the functions between brackets are increasing in $[0, 1]$ we have that

$$\sqrt{u_{\max}} \le \sqrt{u^*_{\max}} = \max_{|\tau-i|=1} |\sqrt{u(\tau)}| = \frac{2}{\sqrt{3}}$$
$$\sqrt{u_{\min}} \le \sqrt{u^*_{\min}} = \min_{|\tau-i|=1} |\sqrt{u(\tau)}| = \sqrt{\frac{2}{3}}$$

From these equations we conclude two things. First, that the interior of the curve defined by

$$\tau^{\pm}(u) - i = \mathrm{e}^{i\theta} \qquad \theta \in [0, 2\pi)$$

contains all the points such that $|\tau^{\pm}(u) - i| \le 1$. Second, that the circle $|\sqrt{u}| = \sqrt{u^*_{\min}}$ lies entirely in the interior of that curve. From these two points it is clear now that if $|\sqrt{u}| \le \sqrt{u^*_{\min}}$ then $|\tau^{\pm}(u) - i| \le 1$ and that the curve

$$|\tau^{\pm}(u) - i| = 1$$

is contained in the ring $\sqrt{u^*_{\min}} \le |\sqrt{u}| \le \sqrt{u^*_{\max}}$, wich conclude the proof.

□

The next lemma is a straightforward observation from lemma A.2 and $\tau^{\pm}(0) = i$

Lemma A.4. Let $F^{\pm}_{m,n}(u)$ defined by (A.14), then

$$F^{\pm}_{m,n}(u) = (\pm\sqrt{u})^{-2m-1} \sum_{j=0}^{\infty} d_j^{n,m} (\pm\sqrt{u})^j. \tag{A.22}$$

This series is convergent for $|\sqrt{u}| < \sqrt{2/3}$. Equation (A.22) defines the constants $d_j^{n,m}$, in particular $d_0^{n,m} = 1/(2i)^{2n+1}$.

Proof. From equation (A.16) we have that

$$(\tau^{\pm}(u) - i)^{2m+1} = \Big[\sum_{k=1}^{\infty} A_k (\pm\sqrt{u})^k\Big]^{2m+1}$$
$$(\tau^{\pm}(u) + i)^{2n+1} = \Big[2i + \sum_{k=1}^{\infty} A_k (\pm\sqrt{u})^k\Big]^{2n+1}$$

from these equations we have

$$(\tau^{\pm}(u) - i)^{2m+1}(\tau^{\pm}(u) + i)^{2n+1} = (\pm\sqrt{u})^{2m+1} \sum_{j=0}^{\infty} B_j (\pm\sqrt{u})^j$$

for some coefficients B_j. It is easy to see that

$$B_0 = (2i)^{2n+1}(A_1)^{2m+1} = (2i)^{2n+1}$$

And therefore is possible to solve the equation

$$\sum_{j=0}^{\infty} d_j^{n,m}(\pm\sqrt{u})^j = \Big[\sum_{j=0}^{\infty} B_j(\pm\sqrt{u})^j\Big]^{-1} = (\pm\sqrt{u})^{2m+1}F^{\pm}_{m,n}(u) \tag{A.23}$$

for $d_j^{n,m}$. In particular

$$d_0^{n,m} = \frac{1}{B_0} = \frac{1}{(2i)^{2n+1}}.$$

The series in equation (A.22) can be written as

$$T_{\pm}(x) := x^{2m+1}F^{\pm}_{m,n}(x^2) = \sum_{j=0}^{\infty} d_j^{n,m}x^j. \tag{A.24}$$

We have already seen that $T_{\pm}(0) = 1/(2i)^{2n+1}$. To find a radius where $T_{\pm}$ is analytic we look at the definition of $F^{\pm}_{m,n}(x^2)$ given in equation (A.14) and notice that if $|\tau^{\pm}(x^2) - i| \leq 1$, then by the triangle inequality we have that $1 \leq |\tau^{\pm}(x^2) + i|$ and therefore $T_{\pm}$ would be analytic. By lemma A.3, we know that $|\tau^{\pm}(x^2) - i| \leq 1$ whenever $x \leq \sqrt{2/3}$ or in other words, the series is convergent when $\sqrt{u} \leq \sqrt{2/3}$. This conclude the proof. □

From equation (A.22) we have

$$F^{\pm}_{m,n}(u) = (\pm\sqrt{u})^{-2m-1}\sum_{j=0}^{2m} d_j^{n,m}(\pm\sqrt{u})^j + g^{\pm}_{m,n}(\pm\sqrt{u}), \tag{A.25}$$

where the regular part of the function $F^{\pm}_{m,n}(u)$ is given by

$$g^{\pm}_{m,n}(\pm\sqrt{u}) = (\pm\sqrt{u})^{-2m-1}\sum_{j=2m+1}^{\infty} d_j^{n,m}(\pm\sqrt{u})^j \tag{A.26}$$

and $d_j^{n,m}$ are defined by equation (A.22).

Lemma A.5. Let $F^{\pm}_{m,n}(u)$ defined by (A.14). Then for $u \in \mathbb{R}$ such that $0 < u \leq u^*$, with u^* defined in lemma A.3. Then

$$(\sqrt{u})^{2m+1}|F^{\pm}_{m,n}(u)| \leq K_1\gamma_1^{\min\{m,n\}} < \gamma_1^{\min\{m,n\}}$$

with $K_1 = \sqrt{2/3}$ and $\gamma_1 = 2/3$.

Proof. From lemma A.3 and the triangle inequality is easy to deduce that

$$1 \leq |\tau^{\pm}(u) + i| \leq 3. \tag{A.27}$$

However the upper bound can be refined. Since $\tau^{\pm}(u)$ is a point over the hyperbola $\Im(h(\tau)) = 0$ its largest norm is reached when the hyperbola intersects the circle centered in i with radius 1. These intersection points are

$$z^{\pm} = \pm\frac{1}{2}\sqrt{\frac{3}{2}\big(\sqrt{13}-1\big)} + \frac{i}{4}\big(1+\sqrt{13}\big)$$

and then

$$1 \leq |\tau^{\pm}(u) + i| \leq |z^{\pm} + i| = \sqrt{\frac{1}{2}\big(1+\sqrt{13}\big)} < \frac{16}{10} < 2. \tag{A.28}$$

From these we have

$$\left|\frac{\sqrt{\tau^{\pm}+2i}}{\tau^{\pm}+i}\right| = \left|\sqrt{\frac{\tau^{\pm}+2i}{(\tau^{\pm}+i)^2}}\right| = \left|\left(\frac{1}{\tau^{\pm}+i} + \frac{i}{(\tau^{\pm}+i)^2}\right)^{1/2}\right| \leq (1+1)^{1/2} = 2^{1/2} \tag{A.29}$$

Now, from equation (A.7) we have

$$u = (\tau^{\pm} - i)^2(\tau^{\pm} + 2i)\frac{1}{3i}$$

and then from equation (A.14),

$$(\sqrt{u})^{2m+1}F^{\pm}_{m,n}(u) = \frac{1}{(\sqrt{3i})^{2m+1}}\frac{(\sqrt{\tau^{\pm}(u) + 2i})^{2m+1}}{(\tau^{\pm}(u) + i)^{2n+1}}.$$

From this equation we write down two different expressions depending on n and m. These expressions are

$$(\sqrt{u})^{2m+1}F^{\pm}_{m,n}(u) = \frac{1}{(\sqrt{3i})^{2m+1}}\left(\frac{\sqrt{\tau^{\pm} + 2i}}{\tau^{\pm} + i}\right)^{2m+1}(\tau^{\pm} + i)^{2(m-n)} \quad \text{if} \quad m \leq n$$
$$(\sqrt{u})^{2m+1}F^{\pm}_{m,n}(u) = \frac{1}{(\sqrt{3i})^{2m+1}}\left(\frac{\sqrt{\tau^{\pm} + 2i}}{\tau^{\pm} + i}\right)^{2n+1}\left(\sqrt{\tau^{\pm} + 2i}\right)^{2(m-n)} \quad \text{if} \quad m > n$$

naturally, from (A.28) and (A.29) we have

$$(\sqrt{u})^{2m+1}|F^{\pm}_{m,n}(u)| \leq \frac{1}{3^{m+\frac{1}{2}}}2^{m+\frac{1}{2}} \qquad \text{if} \quad m \leq n$$
$$(\sqrt{u})^{2m+1}|F^{\pm}_{m,n}(u)| \leq \frac{1}{3^{m+\frac{1}{2}}}2^{n+\frac{1}{2}}3^{m-n} \quad \text{if} \quad m > n \tag{A.30}$$

in this way, we have that

$$(\sqrt{u})^{2m+1}|F^{\pm}_{m,n}(u)| \leq \left(\frac{2}{3}\right)^{1/2}\left(\frac{2}{3}\right)^{m} \quad \text{if} \quad m \leq n \tag{A.31}$$
$$(\sqrt{u})^{2m+1}|F^{\pm}_{m,n}(u)| \leq \left(\frac{2}{3}\right)^{1/2}\left(\frac{2}{3}\right)^{n} \quad \text{if} \quad m > n$$

So, by letting $\gamma_1 = 2/3$ and $K_1 = \sqrt{2/3}$ we have proved the lemma. □

From the proof of this lemma, we can actually prove another one, very similar, that will be useful in the proof of the proposition 2.4

Lemma A.6. Let $F^{\pm}_{m,n}(u)$ defined by (A.14). Then, for $u \in \mathbb{C}$ such that $|\sqrt{u}| \leq \sqrt{2/3}$. Then

$$(\sqrt{u})^{2m+1}|F^{\pm}_{m,n}(u)| \leq K_1\gamma_2^m < \gamma_2^m$$

with K_1 given in lemma A.5 and $\gamma_2 = 4/3$.

Proof. Since $|\sqrt{u}| \leq \sqrt{2/3}$, by lemma A.3 we have that $|\tau^{\pm}(u) - i| \leq 1$ then by the triangle inequality, (A.27) is still valid, and therefore equation (A.30) becomes

$$(\sqrt{u})^{2m+1}|F^{\pm}_{m,n}(u)| \leq \frac{1}{3^{m+\frac{1}{2}}}2^{n+\frac{1}{2}}4^{m-n} = \left(\frac{2}{3}\right)^{1/2}\left(\frac{4}{3}\right)^{m}\left(\frac{1}{2}\right)^{n} \quad \text{if} \quad m > n.$$

Then, in regard of inequality (A.31), we conclude the desired result. □

The next lemma give us information about the coefficients $d_j^{n,m}$ defined in equation (A.22).

Lemma A.7. Let $d_j^{n,m}$ be defined by equation (A.22) and $u^{**} = 2/3$. Then

$$|d_j^{n,m}| \leq \frac{1}{(\sqrt{u^{**}})^j}K_1\gamma_2^m < \frac{1}{(\sqrt{u^{**}})^j}\gamma_2^m$$

where K_1 in lemma A.5 and γ_2 in lemma A.6.

Proof. The function $T_\pm$ defined in equation (A.24), by lemma A.4, is analytic for $\sqrt{u} \le \sqrt{2/3}$.

Therefore, if $|x| \le x^*_{\min} = \sqrt{u^{**}}$ we can use the lemma A.6 to get that $|T_\pm(x)| \le K_1 \gamma_2^m$, then using Cauchy estimates we have

$$|d_j^{n,m}| \le \frac{1}{(\sqrt{u^{**}})^j} K_1 \gamma_2^m.$$

□

With this lemma is possible to prove the next one.

Lemma A.8. Let $g^\pm_{m,n}(\pm\sqrt{u})$ as in equation (A.26), $0 < \beta < 1$ and $0 < \sqrt{u} < \beta\sqrt{u^{**}} < \sqrt{u^{**}}$. Then

$$|g^\pm_{m,n}(\pm\sqrt{u})| < \frac{K_1}{1-\beta} \gamma_2^m (\sqrt{u^{**}})^{-2m-1}.$$

where u^{**} is given in lemma A.7, K_1 in lemma A.5 and γ_2 in lemma A.6.

Proof. It is clear from equation (A.26) that

$$g^\pm_{m,n}(\pm\sqrt{u}) = \sum_{s=0}^{\infty} d^{n,m}_{s+2m+1} (\pm\sqrt{u})^s$$

by hypothesis $0 < \sqrt{u} < \beta\sqrt{u^{**}} < \sqrt{u^{**}}$, then by lemma A.7

$$\begin{aligned}
|g^\pm_{m,n}(\pm\sqrt{u})| &\le K_1 \gamma_2^m \frac{1}{(\sqrt{u^{**}})^{2m+1}} \sum_{s=0}^{\infty} \frac{1}{(\sqrt{u^{**}})^s} (\sqrt{u})^s \\
&\le K_1 \gamma_2^m \frac{1}{(\sqrt{u^{**}})^{2m+1}} \sum_{s=0}^{\infty} \frac{1}{(\sqrt{u^{**}})^s} (\beta\sqrt{u^{**}})^s \\
&= K_1 \gamma_2^m \frac{1}{(\sqrt{u^{**}})^{2m+1}} \sum_{s=0}^{\infty} \beta^s \\
&= K_1 \gamma_2^m \frac{1}{(\sqrt{u^{**}})^{2m+1}} \frac{1}{1-\beta}
\end{aligned}$$

which proves the lemma. □

Appendix B

Proofs of Propositions 2.1, 2.3, 2.4, 2.21 W function

B.1 Proof of Proposition 2.1

Lemma B.1. Let $q, n, m \in \mathbb{Z}$, $q, n, m \geq 0$ and

$$I(q,m,n) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq\frac{G_0^3}{2}\left(\tau+\frac{\tau^3}{3}\right)}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau \tag{B.1a}$$

$$N(q,m,n) = \frac{2^{m+n}}{G_0^{2m+2n-1}} \binom{-1/2}{m} \binom{-1/2}{n} I(q,m,n) \tag{B.1b}$$

Let $k, l \in \mathbb{N}$ and $\tilde{L}$, $\tilde{S}$ be defined by

$$\tilde{L}(q, \quad k, \quad l) = \tilde{c}_q^{2l-k,-k} N(q, l-k, \qquad l) \tag{B.2a}$$

$$\tilde{S}(q,-k,-l) = \tilde{c}_q^{2l-k, \quad k} N(q, \qquad l, l-k) \tag{B.2b}$$

where the constants appearing in (B.2) are defined by (2.3).

Then the Fourier coefficients defined in (2.5) satisfy

$$L_{q,0} = \sum_{l\geq 1} \tilde{L}(q,0,l) \tag{B.3a}$$

$$L_{q,1} = \sum_{l\geq 2} \tilde{L}(q,1,l) \tag{B.3b}$$

$$L_{q,-1} = \sum_{l\geq 2} \tilde{S}(q,-1,-l) \tag{B.3c}$$

$$L_{q,k} = \sum_{l\geq k} \tilde{L}(q,k.l) \qquad \text{for} \quad k \geq 2 \tag{B.3d}$$

$$L_{q,-k} = \sum_{l\geq k} \tilde{S}(q,-k,-l) \quad \text{for} \quad k \geq 2 \tag{B.3e}$$

Proof. We have from equation (1.47) that

$$\mathcal{L} = \tilde{\mathcal{L}}_1 + \int_{-\infty}^{\infty} \left[\left(\frac{x_h^2}{2}\right)^2 r_0 \cos(\alpha_h - f) - \frac{x_h^2}{2} \right] dt \tag{B.4}$$

where

$$\tilde{\mathcal{L}}_1 = \int_{-\infty}^{\infty} \frac{x_h^2}{\left[4 + x_h^4 r_0^2 + 4x_h^2 r_0 \cos(\alpha_h - f)\right]^{1/2}}\, dt$$

this can be written as

$$\tilde{\mathcal{L}}_1 = \int_{-\infty}^{\infty} \frac{x_h^2}{2} \left(1 + \frac{x_h^2}{2} r_0(f(t+t_0))\mathrm{e}^{i(\alpha_h - f(t+t_0))}\right)^{-1/2} \left(1 + \frac{x_h^2}{2} r_0(f(t+t_0))\mathrm{e}^{-i(\alpha_h - f(t+t_0))}\right)^{-1/2} dt \tag{B.5}$$

using that

$$(1+z)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} z^k$$

we get that

$$\tilde{\mathcal{L}}_1 = \sum_{k\geq 0} \sum_{l\geq k} \tilde{L}_k^l + \sum_{k<0} \sum_{l\leq k} \tilde{S}_k^l$$

where

$$\tilde{L}_k^l = \frac{1}{2^{2l-k+1}} \binom{-1/2}{l-k} \binom{-1/2}{l} \int_{-\infty}^{\infty} x_h^{4l-2k+2} \, [r_0(f(t+t_0))]^{2l-k} \, \mathrm{e}^{ik\alpha_h} \mathrm{e}^{-ikf(t+t_0)}\, dt; \quad 0 \leq k \leq l$$

$$\tilde{S}_k^l = \frac{1}{2^{-2l+k+1}} \binom{-1/2}{-l+k} \binom{-1/2}{-l} \int_{-\infty}^{\infty} x_h^{-4l+2k+2} [r_0(f(t+t_0))]^{-2l+k} \mathrm{e}^{ik\alpha_h} \mathrm{e}^{-ikf(t+t_0)} dt; \quad l \leq k < 0.$$

With these expressions is easy to see that $\tilde{L}_0^0$ cancels out the last term in the integral (B.4) and that $\tilde{L}_1^1 + \tilde{S}_{-1}^{-1}$ cancels the cosine term, so

$$\mathcal{L} = \sum_{l\geq 1} \tilde{L}_0^l + \sum_{l\geq 2} \tilde{L}_1^l + \sum_{l\leq -2} \tilde{S}_{-1}^l + \sum_{k>1} \sum_{l\geq k} \tilde{L}_k^l + \sum_{k<-1} \sum_{l\leq k} \tilde{S}_k^l \tag{B.6}$$

Equation (2.3) allow us to expand in Fourier series the function

$$[r_0(f(t+t_0))]^n \; \mathrm{e}^{imf(t+t_0)}.$$

Considering this and performing the change of variable

$$t = \frac{G_0^3}{2}\left(\tau + \frac{\tau^3}{3}\right)$$

introduced in (1.33) one gets from the equation for x_h given in (1.32a) and the function for α_h given in (1.32b) which implies

$$\mathrm{e}^{i\alpha_h} = \frac{\tau - i}{\tau + i} \mathrm{e}^{i\alpha_0}$$

that

$$\tilde{L}_k^l = \mathrm{e}^{ik\alpha_0} \frac{2^{2l-k}}{G_0^{4l-2k-1}} \binom{-1/2}{l-k} \binom{-1/2}{l} \sum_{q\in\mathbb{Z}} \mathrm{e}^{iq\,t_0} \tilde{c}_q^{2l-k,-k} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq\frac{G_0^3}{2}\left(\tau+\frac{\tau^3}{3}\right)}}{(\tau+i)^{2l}(\tau-i)^{2l-2k}} d\tau; \qquad 0 \leq k \leq l \tag{B.7a}$$

$$\tilde{S}_k^l = \mathrm{e}^{ik\alpha_0} \frac{2^{-2l+k}}{G_0^{-4l+2k-1}} \binom{-1/2}{-l+k} \binom{-1/2}{-l} \sum_{q\in\mathbb{Z}} \mathrm{e}^{iq\,t_0} \tilde{c}_q^{-2l+k,-k} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq\frac{G_0^3}{2}\left(\tau+\frac{\tau^3}{3}\right)}}{(\tau-i)^{-2l}(\tau+i)^{-2l+2k}} d\tau; \quad l \leq k < 0 \tag{B.7b}$$

substituting now equations (B.7) in (B.6) we get in terms of the definitions (B.1) and (B.2) that

$$\mathcal{L} = \sum_{q\in\mathbb{Z}} \sum_{l\geq 1} \mathrm{e}^{iqt_0} \tilde{L}(q, 0, l)$$

$$
\begin{aligned}
&+\sum_{q\in\mathbb{Z}}\sum_{l\geq 2}\mathrm{e}^{i(qt_0+\alpha_0)}\tilde{L}(q,1,l)+\sum_{q\in\mathbb{Z}}\sum_{l\leq -2}\mathrm{e}^{i(qt_0-\alpha_0)}\tilde{S}(q,-1,l)\\
&+\sum_{q\in\mathbb{Z}}\sum_{k>1}\sum_{l\geq k}\mathrm{e}^{i(qt_0+k\alpha_0)}\tilde{L}(q,k.l)+\sum_{q\in\mathbb{Z}}\sum_{k<-1}\sum_{l\leq k}\mathrm{e}^{i(qt_0+k\alpha_0)}\tilde{S}(q,k,l) \qquad \text{(B.8)}
\end{aligned}
$$

If we use the equations for $L_{q,k}$ given in (B.3), the lemma becomes clear. □

B.2 Proof of proposition 2.3

We will bound $N(q,m,n)$ by means of the expression (A.12) separating the terms in it. First take u^* as in lemma A.3 and choose

$$\varepsilon = G_0^{-3/2}$$

with $G_0 > c^{2/3}$. We write down then

$$\int_{u(C)}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du = \int_{u^*}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du + \int_{u(C)}^{u^*} F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du$$

and bound independently the two terms in the right. We have

$$
\begin{aligned}
\left|\int_{u(C)}^{u^*} F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du\right| &\leq \int_{G_0^{-3}c^2k_\epsilon}^{u^*} |F^{\pm}_{m,n}(u)|\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du \quad \text{(by means of (A.10))}\\
&\leq \int_{G_0^{-3}c^2}^{u*} |F^{\pm}_{m,n}(u)|\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du \quad (k_\epsilon \geq 1)\\
&\leq K_1\gamma_1^{\min\{m,n\}}\int_{G_0^{-3}c^2}^{u*} u^{-m-\frac{1}{2}}\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du \quad \text{(by lemma A.5)}\\
&\leq \gamma_1^{\min\{m,n\}}\frac{G_0^{3m+\frac{3}{2}}}{c^{2m+1}}\frac{2}{qG_0^3}\left[\mathrm{e}^{-q\frac{c^2}{2}}-\mathrm{e}^{-q\frac{G_0^3}{2}u^*}\right] \quad (c\geq 1)\\
&\leq 2\gamma_1^{\min\{m,n\}}G_0^{3m-\frac{3}{2}} \qquad \text{(B.9)}
\end{aligned}
$$

Now, using the definitions of $F^{\pm}_{m,n}(u)$ given in (A.14) and u^* in lemma A.3,

$$
\begin{aligned}
\left|\int_{u^*}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G_0^3}{2}u}\,du\right| &\leq \int_{u^*}^{\infty}\frac{\mathrm{e}^{-q\frac{G_0^3}{2}u}}{|(\tau^{\pm}(u)-i)^{2m+1}(\tau^{\pm}(u)+i)^{2n+1}|}\,du\\
&\leq \frac{2\mathrm{e}^{-q\frac{G_0^3}{2}u^*}}{qG_0^3}\frac{1}{|\tau^{\pm}(u^*)-i|^{2m+1}}\frac{1}{|\tau^{\pm}(u^*)+i|^{2n+1}}\\
&\leq 2G_0^{-3}\mathrm{e}^{-q\frac{G_0^3}{2}u^*} \quad \text{(by lemma A.3)}\\
&\leq 2G_0^{-3} \qquad \text{(B.10)}
\end{aligned}
$$

It remains only the last integral of (A.12) where the integrand is given in (A.15) and the domain Γ_3 in (A.4). The path Γ_3 can be parametrized by

$$\tau = i + c\,G_0^{-\frac{3}{2}}\mathrm{e}^{\mathrm{i}\theta} \quad \text{with } \theta\in[\theta_1,\theta_2]=[\pi-\theta_\epsilon,\theta_\epsilon]. \qquad \text{(B.11)}$$

If we define

$$\tilde{h}(\theta) = h(\tau(\theta)) = i\Big(\frac{\tau(\theta)^3}{3}+\tau(\theta)\Big),$$

a straightforward computation using (A.7) shows that

$$\tilde{h}(\theta) = -\frac{2}{3} - G_0^{-3}\Big(c^2\mathrm{e}^{2i\theta}+\frac{1}{3i}c^3G_0^{-\frac{3}{2}}\mathrm{e}^{3i\theta}\Big)$$

and then

$$\begin{aligned}\left|\mathrm{e}^{q\frac{G_0^3}{2}\tilde{h}(\theta)}\right| &= \mathrm{e}^{-\frac{q}{3}G_0^3}\mathrm{e}^{-\frac{q}{2}c^2(\cos 2\theta+\frac{c}{3}G_0^{-\frac{3}{2}}\sin 3\theta)}\\ &\leq \mathrm{e}^{-\frac{q}{3}G_0^3}\mathrm{e}^{\frac{q}{2}c^2(1+\frac{c}{3}G_0^{-\frac{3}{2}})}\\ &\leq \mathrm{e}^{-\frac{q}{3}G_0^3}\mathrm{e}^{qc^2} \quad (\text{for } G_0\geq (c^2/9)^{1/3}\ ). \end{aligned} \tag{B.12}$$

Note that over Γ_3 we have, for $G_0 \geq c^{2/3}$, that

$$\left|1+\frac{\tau-i}{2i}\right|\geq 1-\left|\frac{\tau-i}{2i}\right|= 1-\frac{c}{2}G_0^{-3/2}\geq\frac{1}{2}$$

and therefore

$$|\tau+i|^{2n} = |2i|^{2n}\left|1+\frac{\tau-i}{2i}\right|^{2n}\geq 2^{2n}\frac{1}{2^{2n}} = 1 \tag{B.13}$$

Now, we can bound the last integral of (A.12)

$$\begin{aligned}\left|\int_{\Gamma_3}\frac{\mathrm{e}^{q\frac{G_0^3}{2}h(\tau)}}{(\tau-i)^{2m}(\tau+i)^{2n}}\,d\tau\right| &\leq \left|\int_{\theta_1}^{\theta_2}\frac{\mathrm{e}^{q\frac{G_0^3}{2}\tilde{h}(\theta)}}{(\tau(\theta)-i)^{2m}(\tau(\theta)+i)^{2n}}ic\,G_0^{-\frac{3}{2}}\mathrm{e}^{i\theta}\,d\theta\right|\\ &\leq \int_{\theta_1}^{\theta_2}\frac{|\mathrm{e}^{q\frac{G_0^3}{2}\tilde{h}(\theta)}|}{|c\,G_0^{-3/2}\mathrm{e}^{i\theta}|^{2m}}cG_0^{-\frac{3}{2}}\,d\theta \quad (\text{by (B.11) and (B.13)})\\ &\leq \int_{\theta_1}^{\theta_2}\frac{\mathrm{e}^{-\frac{q}{3}G_0^3}\mathrm{e}^{qc^2}}{c^{2m}\,G_0^{-3m}}cG_0^{-\frac{3}{2}}\,d\theta \quad (\text{by (B.12)})\\ &\leq \frac{2\pi}{c^{2m-1}}G_0^{3m-3/2}\mathrm{e}^{-\frac{q}{3}G_0^3}\mathrm{e}^{qc^2}\\ &\leq 2\pi G_0^{3m-3/2}\mathrm{e}^{-\frac{q}{3}G_0^3}\mathrm{e}^{qc^2}. \end{aligned} \tag{B.14}$$

From lemma A.1 and the bounds (B.9), (B.10) and (B.14), we can bound $N(q,m,n)$ by equation (A.12) as follows

$$\begin{aligned}|N(q,m,n)| &\leq \mathrm{e}^{-1/2}2^{m+n}\mathrm{e}^{-q\frac{G_0^3}{3}}G_0^{-2m-2n+1}\big(4\gamma_1^{\min\{m,n\}}G_0^{3m-3/2}+4G_0^{-3}+2\pi G_0^{3m-3/2}\mathrm{e}^{qc^2}\big)\\ &\leq 2\pi\mathrm{e}^{-1/2}\mathrm{e}^{qc^2}2^{m+n}\mathrm{e}^{-q\frac{G_0^3}{3}}G_0^{-2m-2n+1}\big(\gamma_1^{\min\{m,n\}}G_0^{3m-3/2}+G_0^{-3}+G_0^{3m-3/2}\big)\\ &\leq 2\pi\mathrm{e}^{-1/2}\mathrm{e}^{qc^2}2^{m+n}\mathrm{e}^{-q\frac{G_0^3}{3}}G_0^{m-2n-\frac{1}{2}}\big(\gamma_1^{\min\{m,n\}}+1+1\big)\\ &\leq K_2 2^{m+n}\mathrm{e}^{qc^2}\mathrm{e}^{-q\frac{G_0^3}{3}}G_0^{m-2n-\frac{1}{2}}\end{aligned}$$

with

$$K_2 = 6\pi\mathrm{e}^{-1/2}.$$

This proves proposition 2.3.

B.3 Proof of propositions 2.4 and 2.21

The only difference in proving these two propositions consists in the treatment of the residue $R^q_{m,n}$ of the function $f^q_{m,n}$ given in (A.15). At the end of this section we point out the difference and conclude the proof of either case.

To prove the statement we will proceed as in the proof of proposition 2.3 changing the path of integration to the path Γ defined in (A.3) leading to equation (A.12). The important fact to notice is that the integral (A.12) does not depend on ε. So, we will compute only the ε-independent terms of that integral. We will follow a series of lemmas leading to the proof of the statement.

Lemma B.2. Let $0 < \varepsilon < 1/c \leq 1$ and $u(C)$ be as in equation (A.9), $F^{\pm}_{m,n}$ defined by (A.14) and $0 < \beta < 1$, then if $u^{**} = 2/3$ as given in lemma A.7 and $0 < \sqrt{u} < \sqrt{u_*} = \beta\sqrt{u^{**}} < \sqrt{u^{**}}$ for any $\varepsilon > 0$ small enough we have

$$\int_{u(C)}^{\infty} F^{\pm}_{m,n}(u) \mathrm{e}^{-q\frac{G_0^3}{2}u} du = \sum_{j=0}^{2m} \int_{u(C)}^{u_*} \mathrm{e}^{-q\frac{G_0^3}{2}u} d_j^{n,m} (\pm\sqrt{u})^{-2m-1+j} du + \widehat{E}_1 + \widehat{E}_2$$

where

$$|\widehat{E}_1| \leq 2 \cdot \rho^{-2m-1} G_0^{-3}, \quad |\widehat{E}_2| \leq \frac{2}{(1-\beta)} \gamma_3^m G_0^{-3}$$

with $\rho = |\tau^{\pm}(u_*) - i| \leq 1$ and $\gamma_3 = 2$.

Proof. By definition, for $\varepsilon > 0$ small enough we have that $0 < u(C) < u_* < \sqrt{u_*} < 2/\sqrt{3}$, then

$$\int_{u(C)}^{\infty} F^{\pm}_{m,n}(u) \mathrm{e}^{-q\frac{G_0^3}{2}u} du = \int_{u(C)}^{u_*} F^{\pm}_{m,n}(u) \mathrm{e}^{-q\frac{G_0^3}{2}u} du + \widehat{E}_1$$

with

$$\widehat{E}_1 = \int_{u_*}^{\infty} F^{\pm}_{m,n}(u) \mathrm{e}^{-q\frac{G_0^3}{2}u} du$$

now, since $0 < u_* < u^*$ by lemma A.3 and the triangle inequality

$$\begin{aligned}
|\widehat{E}_1| = \left| \int_{u_*}^{\infty} F^{\pm}_{m,n}(u) \mathrm{e}^{-q\frac{G_0^3}{2}u} \, du \right| &\leq \int_{u_*}^{\infty} \frac{\mathrm{e}^{-q\frac{G_0^3}{2}u}}{|(\tau^{\pm}(u) - i)^{2m+1} (\tau^{\pm}(u) + i)^{2n+1}|} \, du \\
&\leq \frac{2\mathrm{e}^{-q\frac{G_0^3}{2}u_*}}{qG_0^3} \frac{1}{|\tau^{\pm}(u_*) - i|^{2m+1}} \frac{1}{|\tau^{\pm}(u_*) + i|^{2n+1}} \\
&\leq 2G_0^{-3} \mathrm{e}^{-q\frac{G_0^3}{2}u_*} \left(\frac{1}{\rho}\right)^{2m+1} \\
&\leq 2G_0^{-3} \rho^{-2m-1}. \qquad \text{(B.15)}
\end{aligned}$$

By lemma A.4 and equation (A.25) we have

$$\int_{u(C)}^{u_*} F^{\pm}_{m,n}(u) \mathrm{e}^{-q\frac{G_0^3}{2}u} du = \sum_{j=0}^{2m} \int_{u(C)}^{u_*} d_j^{n,m} \mathrm{e}^{-q\frac{G_0^3}{2}u} (\pm\sqrt{u})^{-2m-1+j} du + \widehat{E}_2$$

where

$$\widehat{E}_2 = \int_{u(C)}^{u_*} g^{\pm}_{m,n}(\pm\sqrt{u}) \mathrm{e}^{-q\frac{G_0^3}{2}u} du$$

then by lemma A.8 we have that for any $\varepsilon > 0$ small enough

$$\begin{aligned}
|\widehat{E}_2| &\leq \int_{u(C)}^{u_*} |g^{\pm}_{m,n}(\pm\sqrt{u})| \mathrm{e}^{-q\frac{G_0^3}{2}u} du \\
&\leq \frac{K_1}{1-\beta} \gamma_2^m (\sqrt{u^{**}})^{-2m-1} \int_0^{\infty} \mathrm{e}^{-q\frac{G_0^3}{2}u} du \\
&\leq \frac{2K_1}{q(1-\beta)} \gamma_2^m (\sqrt{u^{**}})^{-2m-1} G_0^{-3} \\
&\leq \frac{2K_1}{(1-\beta)} \gamma_2^m (\sqrt{u^{**}})^{-2m-1} G_0^{-3} \\
&= \frac{2}{(1-\beta)} \gamma_3^m G_0^{-3}
\end{aligned}$$

now the lemma is proven. □

The next lemma is a straight forward application of the last one.

Lemma B.3. Let $0 < \varepsilon < 1/c \leq 1$ and $u(C)$ be as in equation (A.12), $F^{\pm}_{m,n}$ defined by (A.14) and $0 < \beta < 1$, then if u^{**} is given by lemma A.7 and $0 < \sqrt{u} < \sqrt{u_*} = \beta\sqrt{u^{**}} < \sqrt{u^{**}}$ for any $\varepsilon > 0$ small enough we have

$$\int_{u(C)}^{\infty} \left[F^{+}_{m,n}(u) - F^{-}_{m,n}(u)\right] \mathrm{e}^{-q\frac{G_0^3}{2}u} du = 2\sum_{s=0}^{m} \int_{u(C)}^{u_*} \mathrm{e}^{-q\frac{G_0^3}{2}u} d^{n,m}_{2m-2s}(\sqrt{u})^{-2s-1} du + 2\widehat{E}_1 + 2\widehat{E}_2$$

where $\widehat{E}_1$ and $\widehat{E}_2$ are the same as in lemma B.2.

Proof. By lemma B.2 we have

$$\int_{u(C)}^{\infty} \left[F^{+}_{m,n}(u) - F^{-}_{m,n}(u)\right] \mathrm{e}^{-q\frac{G_0^3}{2}u} du = \sum_{j=0}^{2m} \int_{u(C)}^{u_*} \mathrm{e}^{-q\frac{G_0^3}{2}u} d^{n,m}_{j}[1-(-1)^{-2m-1+j}](\sqrt{u})^{-2m-1+j} du + 2\widehat{E}_1 + 2\widehat{E}_2$$

then the non trivial terms in the sum are given when $-2m-1+j = -2s-1$ with $s = 0, .., m$. This observation proves the lemma. □

Lemma B.4. Let $0 < \varepsilon < 1/c \leq 1$ and $u(C)$ be as in equation (A.12), $0 < \beta < 1$, then if $u^{**} = 2/3$ as given in lemma A.7 and $\sqrt{u_*} = \beta\sqrt{u^{**}} < \sqrt{u^{**}}$, then the ε-independent term of

$$\int_{u(C)}^{u_*} \mathrm{e}^{-q\frac{G_0^3}{2}u} d^{n,m}_{2m-2s}(\sqrt{u})^{-2s-1} du$$

is

$$(-1)^s 2^{s+\frac{3}{2}}(2s+1)\frac{(s+1)!}{(2s+2)!} d^{n,m}_{2m-2s} q^{s-\frac{1}{2}} G_0^{3s-\frac{3}{2}} \Gamma\left(\frac{1}{2}\right) + \widehat{E}_3(m,s)$$

with

$$|\widehat{E}_3(m,s)| \leq 2\gamma_3^m \beta^{-2s-1} G_0^{-3}.$$

and $\gamma_3 = 2$.

Proof. First we write down

$$\int_{u(C)}^{u_*} \mathrm{e}^{-q\frac{G_0^3}{2}u} d^{n,m}_{2m-2s}(\sqrt{u})^{-2s-1} du = \int_{u(C)}^{\infty} \mathrm{e}^{-q\frac{G_0^3}{2}u} d^{n,m}_{2m-2s}(\sqrt{u})^{-2s-1} du + \widehat{E}_3 \tag{B.16}$$

where

$$\widehat{E}_3(m,s) = -\int_{u_*}^{\infty} \mathrm{e}^{-q\frac{G_0^3}{2}u} d^{n,m}_{2m-2s}(\sqrt{u})^{-2s-1} du$$

observe that $E_3(m,s)$ is independent of ε. Let us bound $\widehat{E}_3(m,s)$

$$\begin{aligned}
|\widehat{E}_3(m,s)| &\leq |d^{n,m}_{2m-2s}|(\sqrt{u_*})^{-2s-1} \int_{u_*}^{\infty} \mathrm{e}^{-q\frac{G_0^3}{2}u} du \\
&\leq |d^{n,m}_{2m-2s}|(\sqrt{u_*})^{-2s-1} 2\mathrm{e}^{-q\frac{G_0^3}{2}u_*} \frac{G_0^{-3}}{q} \\
&\leq 2|d^{n,m}_{2m-2s}|(\beta\sqrt{u^{**}})^{-2s-1} G_0^{-3} \\
&\leq 2\frac{1}{(\sqrt{u^{**}})^{2m-2s}} K_1 \gamma_2^m (\beta\sqrt{u^{**}})^{-2s-1} G_0^{-3} \qquad \text{(by lemma A.7)} \\
&\leq 2\frac{1}{(\sqrt{u^{**}})^{2m}} K_1 \gamma_2^m \beta^{-2s-1} (\sqrt{u^{**}})^{-1} G_0^{-3} \\
&= 2\gamma_3^m \beta^{-2s-1} G_0^{-3},
\end{aligned}$$

where $\gamma_3 = 2$.

By equation (A.9) we know that $u(C) = O(c\,\varepsilon^2)$ and then the following definitions make sense

$$\begin{aligned} I_{p,s}(\varepsilon) &= \int_{u(C)}^{\infty} \mathrm{e}^{-q\delta u} u^{\frac{1}{2}(-2s-1)+p}\, du \\ f_{p,s}(\varepsilon) &= (u(C))^{\frac{1}{2}(-2s-1)+p} \mathrm{e}^{-q\delta u(C)} \\ \delta &= \frac{G_0^3}{2} \end{aligned}$$

using this notation and integrating by parts we have

$$\begin{aligned} I_{p-1,s}(\varepsilon) &= \frac{q\delta}{-s-\frac{1}{2}+p} \int_{u(C)}^{\infty} \mathrm{e}^{-q\delta u} u^{\frac{1}{2}(-2s-1)+p}\, du - \frac{1}{-s-\frac{1}{2}+p} (u(C))^{\frac{1}{2}(-2s-1)+p} \mathrm{e}^{-q\delta u(C)} \\ &= \frac{1}{-s-\frac{1}{2}+p} [q\delta I_{p,s}(\varepsilon) - f_{p,s}(\varepsilon)] \end{aligned} \tag{B.17}$$

also

$$\int_{u(C)}^{\infty} \mathrm{e}^{-q\frac{G_0^3}{2}u} d_{2m-2s}^{n,m} (\sqrt{u})^{-2s-1} du = d_{2m-2s}^{n,m} I_{0,s}(\varepsilon) \tag{B.18}$$

Now, in the case where $s > 0$, using equation (B.17) s-times we get

$$I_{0,s}(\varepsilon) = \frac{(q\delta)^s}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} I_{s,s}(\varepsilon) - \sum_{p=1}^{s} \frac{(q\delta)^{p-1} f_{p,s}(\varepsilon)}{(-s-\frac{1}{2}+1)\cdots(-s-\frac{1}{2}+p)}$$

The ε-independent term of $I_{0,s}(\varepsilon)$ is given by

$$\begin{aligned} \frac{(q\delta)^s}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} \lim_{\varepsilon\to 0} I_{s,s}(\varepsilon) &= \frac{(q\delta)^s}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} \frac{1}{\sqrt{q\delta}} \Gamma\big(\frac{1}{2}\big) \\ &= \frac{(\sqrt{q\delta})^{2s-1}}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} \Gamma\big(\frac{1}{2}\big). \end{aligned}$$

then the ε-independent term of the integral in equation (B.18) is

$$\frac{d_{2m-2s}^{n,m} (\sqrt{q\delta})^{2s-1}}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} \Gamma\big(\frac{1}{2}\big)$$

when $s > 0$.

In the same way, we have, that the ε-independent term of

$$I_{0,0}(\varepsilon) = \int_{u(C)}^{\infty} \mathrm{e}^{-q\frac{G_0^3}{2}u} d_{2m}^{n,m} (\sqrt{u})^{-1} du$$

is $d_{2m}^{n,m}(\sqrt{q\delta})^{-1}\Gamma\big(\frac{1}{2}\big)$ and by equation (B.16) and the bound on E_3 the lemma is proved if we notice that

$$\begin{aligned} \big(-s-\frac{1}{2}+1\big)\big(-s-\frac{1}{2}+2\big)\cdots\big(-\frac{1}{2}\big) &= \frac{1}{2^s}(-2s+1)(-2s+3)\cdots(-1) \\ &= \frac{(-1)^s}{2^s}(2s-1)(2s-3)\cdots(1) \\ &= \frac{(-1)^s}{2^s}\frac{(2s+1)!!}{2s+1} \end{aligned}$$

and using that

$$(2s+1)!! = \frac{(2s+2)!}{2^s(s+1)!}$$

we get

$$\big(-s-\frac{1}{2}+1\big)\big(-s-\frac{1}{2}+2\big)\cdots\big(-\frac{1}{2}\big) = \frac{(-1)^s}{2^{2s+1}(2s+1)}\frac{(2s+2)!}{(s+1)!}$$

This expression allow us to write the cases $s > 0$ and $s = 0$ in one equation which completes the proof. □

Lemma B.5. Let $u(C)$ given in equation (A.9) and $F^{\pm}_{m,n}$ defined by (A.14), then the ε-independent terms of

$$\int_{u(C)}^{\infty} \left[F^{+}_{m,n}(u) - F^{-}_{m,n}(u)\right] \mathrm{e}^{-q\frac{G_0^3}{2}u} du$$

are given by

$$\sum_{s=0}^{m} (-1)^s 2^{s+\frac{5}{2}}(2s+1)\frac{(s+1)!}{(2s+2)!} d^{n,m}_{2m-2s} q^{s-\frac{1}{2}} G_0^{3s-\frac{3}{2}} \Gamma\big(\frac{1}{2}\big) + T^q_{m,n}$$

where

$$|T^q_{m,n}| \leq K_{11}\gamma_4^m G_0^{-3}$$

with

$$\beta = \left(-1 + \frac{\sqrt{11}}{4}\sqrt{3+\frac{\sqrt{11}}{2}}\right)^{1/2}, \quad \gamma_4 = \frac{2}{\beta^2}, \quad K_{11} = 2^2\left(\sqrt{2} + \frac{2}{\beta(1-\beta)}\right).$$

Proof. A straight forward application of lemmas B.3 and B.4 gives the correct prediction on the value of integral, it only remains to show that the errors behave as stated. Let

$$T^q_{m,n} = 2\widehat{E}_1 + 2\widehat{E}_2 + 2E'_3$$

with $\widehat{E}_1$ and $\widehat{E}_2$ are given by lemma B.3 and

$$E'_3 = \sum_{s=0}^{m} \widehat{E}_3(m,s)$$

where $\widehat{E}_3(m,s)$ is given in lemma B.4. In this way

$$\begin{aligned}
|T^q_{m,n}| &\leq 2|\widehat{E}_1| + 2|\widehat{E}_2| + 2\sum_{s=0}^{m}|\widehat{E}_3(m,s)| \\
&\leq 2^2\rho^{-2m-1}G_0^{-3} + \frac{2^2}{(1-\beta)}\gamma_3^m G_0^{-3} + 2^2\gamma_3^m G_0^{-3}\sum_{s=0}^{m}\beta^{-2s-1} \\
&= 2^2 G_0^{-3}\left(\rho^{-2m-1} + \frac{\gamma_3^m}{1-\beta} + \frac{\gamma_3^m}{\beta}\sum_{s=0}^{m}\beta^{-2s}\right)
\end{aligned}$$

since $\gamma_3 = 2$ and

$$\sum_{s=0}^{m}\beta^{-2s} = \frac{\beta^{-2m-2}-1}{\beta^{-2}-1} \leq \frac{\beta^{-2m}}{1-\beta^2} \leq \frac{\beta^{-2m}}{1-\beta}$$

choosing $\rho = 1/\sqrt{2}$ we have

$$\begin{aligned}
|T^q_{m,n}| &\leq 2^2 G_0^{-3}\left(\gamma_3^m\sqrt{2} + \frac{\gamma_3^m}{1-\beta} + \frac{1}{\beta}\left(\frac{\gamma_3}{\beta^2}\right)^m\frac{1}{1-\beta}\right) \\
&\leq 2^2 G_0^{-3}\left(\frac{\gamma_3}{\beta^2}\right)^m\left(\sqrt{2} + \frac{2}{\beta(1-\beta)}\right).
\end{aligned}$$

As we have proceeded in the proof of lemma A.3, from fixing $\rho^2 = |\tau^{\pm}(u_*) - i|^2 = 1/2$, using equations (A.18), (A.19) and (A.20) we can find u_*, and therefore, by its definition given in lemma B.2 we have that

$$\beta^2 = \frac{u_*}{u^**} = -1 + \frac{\sqrt{11}}{4}\sqrt{3+\frac{\sqrt{11}}{2}} \approx 0.79,$$

and then by setting

$$K_{11} = 2^2\left(\sqrt{2} + \frac{2}{\beta(1-\beta)}\right) \qquad \gamma_4 = \frac{2}{\beta^2} \approx 2.53$$

we have

$$|T^q_{m,n}| \leq K_{11}\gamma_4^m G_0^{-3}.$$

□

Lemma B.6. Let $f^q_{m,n}$ be defined in equation (A.15), then

$$Res(f^q_{m,n}(\tau), i) = \frac{e^{-qG_0^3/3}}{(2i)^{2n}} \sum_{\substack{2j+3r+s=2m-1 \\ j,r,s\geq 0}} \binom{-2n}{s} \frac{(-i)^{r+s}(-qG_0^3)^{r+j}}{3^r 2^{s+r+j} j! r!} \tag{B.19}$$

Proof. From the definition of $f^q_{m,n}$ given in (A.15), and substituting the expression

$$h(\tau) = -2/3 - (\tau - i)^2 + i(\tau - i)^3/3$$

we have

$$f^q_{m,n}(\tau) = \frac{e^{-q\delta\frac{2}{3}} e^{-q\delta(\tau-i)^2} e^{q\delta\frac{i}{3}(\tau-i)^3}}{(\tau - i)^{2m}(\tau + i)^{2n}}, \qquad \delta = \frac{G_0^3}{2}$$

if we use

$$(\tau + i)^{-2n} = (2i)^{-2n}(1 + (\tau - i)/2i)^{-2n}$$

and expand in Taylor series around i we get

$$f^q_{m,n}(\tau) = \frac{1}{(\tau - i)^{2m}} \frac{e^{-q\delta\frac{2}{3}}}{(2i)^{2n}} \sum_{s=0}^{\infty} \binom{-2n}{s} (\frac{\tau - i}{2i})^s \sum_{j=0}^{\infty} \frac{(-q\delta)^j(\tau - i)^{2j}}{j!} \sum_{r=0}^{\infty} \frac{1}{r!} (\frac{q\delta i}{3})^r (\tau - i)^{3r}$$

by taking the coefficient of the -1 degree term we obtain the desired result. □

Remark B.7. If j, r and s represent the indexes in (B.22) and $m \geq 2$ then the choice $j = m - 1$, $r = 0$, $s = 1$ satisfy

$$0 \leq r, j, s \qquad 2j + 3r + s = 2m - 1.$$

and also $j + r = m - 1$.

Lemma B.8. Let $f^q_{m,n}$ be defined in equation (A.15) and $\delta = G_0^3/2$, then

$$Res(f^q_{m,n}(\tau), i) = O(\delta^{m-1}) = O(G_0^{3m-3})$$

Proof. By induction on m. Is easy to see that for $m = 1, 2$ the result is true. Assume, then that for $\tilde{m}$ we have

$$\max\{r + j\} = \tilde{m} - 1$$

under the constraints

$$0 \leq r, j, s \qquad 2j + 3r + s = 2\tilde{m} - 1$$

We will show that

$$\max\{r + j\} = \tilde{m}$$

under the constraints

$$0 \leq r, j, s \qquad 2j + 3r + s = 2(\tilde{m} + 1) - 1 = 2\tilde{m} + 1.$$

let (j, r, s) such that $2j + 3r + s = 2\tilde{m} + 1$, then

$$2(j - 1) + 3r + s = 2\tilde{m} - 1, \tag{B.20}$$

here we have two different cases

- $j \geq 1$
- $j = 0$

When $j \geq 1$ we have that, by equation (B.20), $(j-1, r, s)$ satisfy the induction hypothesis, that is $j-1+r \leq \tilde{m}-1$ and then

$$j + r \leq \tilde{m}$$

When $j = 0$ the equation (B.20) reads

$$3r + s - 2 = 2\tilde{m} - 1 \tag{B.21}$$

here we have three different cases

- $s \geq 2$
- $s = 0$
- $s = 1$

When $s \geq 2$ we have that, by equation (B.21), $(0, r, s-2)$ satisfy the induction hypothesis, that is $j + r = r \leq \tilde{m} - 1$ and then

$$j + r \leq \tilde{m}$$

When $s = 0$ we have that, by equation (B.21), $3r - 2 = 2\tilde{m} - 1$ or $3r = 2\tilde{m} + 1$ from where $\tilde{m} = 3s' + 1$ and then $r = 2s' + 1$ with $s' = 0, 1, \ldots$. This imply

$$r + j = r = 2s' + 1 = \tilde{m} - s' \leq \tilde{m}$$

When $s = 1$ we have that, by equation (B.21), $3r = 2\tilde{m}$, from where $\tilde{m} = 3s'$ and then $r = 2s'$ with $s' = 1, 2, \ldots$. This imply

$$r + j = r = 2s' = \tilde{m} - s' \leq \tilde{m}$$

In this way we have seen that in every case we get $r + j \leq \tilde{m}$ and by remark B.7 there exist a configuration such that $r + j = \tilde{m}$. Therefore the maximum value of $r + j$ is exactly $\tilde{m}$. This completes the induction.

□

Lemma B.9. Let Γ_3 be the path defined in (A.4), and $f^q_{m,n}$ be defined in equation (A.15), then the ε-independent terms of

$$\mathrm{e}^{q\frac{G_0^3}{3}} \int_{\Gamma_3} f^q_{m,n}(\tau) d\tau$$

are bounded by $K_{12} q^{m-1} G_0^{3m-3}$ where $K_{12} = 2\pi \mathrm{e}^{4/3}$.

Proof. If Γ_3 is the path defined in (A.4) we can parameterize it by

$$\tau = i + c\,\varepsilon \mathrm{e}^{i\theta} \qquad \theta \in (\theta_1, \theta_2) = [\pi - \theta_\varepsilon, \theta_\varepsilon]$$

where θ_ε is given in (A.8). Then by using a Taylor series argument is not difficult to see that the ε-independent term of

$$\int_{\Gamma_3} f^q_{m,n}(\tau) d\tau$$

is exactly $\pi i Res(f^q_{m,n}, i)$. By lemmas B.6 and B.8 we have, naming $\delta = G_0^3/2$, that

$$\begin{aligned}
|\mathrm{e}^{q\frac{G_0^3}{3}} Res(f^q_{m,n}(\tau), i)| &\leq \frac{(q\delta)^{m-1}}{2^{2n}} \sum_{2j+3r+s=2m-1} \left| \binom{-2n}{s} \right| \frac{1}{2^s} \cdot \frac{1}{3^r} \frac{1}{r!} \cdot \frac{1}{j!} \\
&\leq \frac{(q\delta)^{m-1}}{2^{2n}} \sum_{j,r,s \in \mathbb{N} \cup \{0\}} \binom{2n+s-1}{s} \frac{1}{2^s} \cdot \frac{1}{3^r} \frac{1}{r!} \cdot \frac{1}{j!} \\
&\leq \frac{(q\delta)^{m-1}}{2^{2n}} \sum_{s=0}^{\infty} \binom{2n+s-1}{s} \frac{1}{2^s} \sum_{r=0}^{\infty} \frac{1}{3^r} \frac{1}{r!} \sum_{j=0}^{\infty} \frac{1}{j!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(q\delta)^{-l-1}}{2^{2n}} 2^{2n} \mathrm{e}^{1/3}\mathrm{e} \\
&= (q\delta)^{m-1}\mathrm{e}^{4/3} \\
&= 2\mathrm{e}^{4/3}(qG_0^3)^{m-1}\left(\frac{1}{2}\right)^m \\
&\leq 2\mathrm{e}^{4/3}q^{m-1}G_0^{3m-3}
\end{aligned}
$$

So, by setting

$$K_{12} = 2\pi\mathrm{e}^{4/3}$$

this lemma is proved. □

Now we can prove proposition 2.4. $N(q,m,n)$ is given in (A.12), and since it does not depend on ε we can apply lemmas B.5 and B.6 to obtain

$$N(q,m,n) = \frac{d_{m,n}\mathrm{e}^{-q\frac{G_0^3}{3}}}{G_0^{2m+2n-1}}\left[\sum_{s=0}^{m}(-1)^s 2^{s+\frac{5}{2}}(2s+1)\frac{(s+1)!}{(2s+2)!}d^{n,m}_{2m-2s}q^{s-\frac{1}{2}}G_0^{3s-\frac{3}{2}}\Gamma\left(\frac{1}{2}\right)+T^q_{m,n}+R^q_{m,n}\right]$$

where

$$R^q_{m,n} = (-i)\mathrm{e}^{q\frac{G_0^3}{3}}\int_{\Gamma_3} f^q_{m,n}(\tau)d\tau$$

and by lemma B.5

$$|T^q_{m,n}| \leq K_{11}\gamma_4^m G_0^{-3}.$$

By lemma B.9

$$|R^q_{m,n}| \leq K_{12}q^{m-1}G_0^{3m-3}.$$

Using that $2^{s+1}(s+1)!(2s+1)!! = (2s+2)!$ to show that

$$\frac{(2s+1)(s+1)!}{(2s+2)!} = \frac{1}{2^{s+1}(2s-1)!!}.$$

completes the proof of the proposition 2.4. Due to the fact that the right hand side of this last expression is not defined when $s=0$ but the left hand side is and is equal to one, we need to point out that when $s=0$, the term $1/(2s-1)!!$ in the final formula should be replaced by 1.

To prove proposition 2.21, instead of using lemmas B.6, B.8 and B.9, we use the next lemma, which also implies lemmas B.8 and B.9.

Lemma B.10. Let $f^q_{m,n}$ be defined in equation (A.15), then

$$Res(f^q_{m,n}(\tau),i) = 2i\mathrm{e}^{-q\frac{G_0^3}{3}}\sum_{s=0}^{m-1}d^{n,m}_{2(m-s)-1}\left(-\frac{q}{2}G_0^3\right)^s\frac{1}{s!} \tag{B.22}$$

where the constants $d^{n,m}_{2(m-s)-1}$ were introduced in lemma (A.4).

Proof. From the definition of $f^q_{m,n}$ given in (A.15), and substituting the expression

$$h(\tau) = -2/3 - (\tau-i)^2 + i(\tau-i)^3/3$$

we have for any $\rho < 1$

$$
\begin{aligned}
Res(f^q_{m,n}(\tau),i) &= \frac{1}{2\pi i}\int_{|\tau-i|=\rho}\frac{\mathrm{e}^{q\frac{G_0^3}{2}(-\frac{2}{3}-(\tau-i)^2+\frac{i}{3}(\tau-i)^3)}}{(\tau-i)^{2m}(\tau+i)^{2n}}d\tau \\
&= \frac{1}{2\pi i}\int_{|w|=\rho}\frac{\mathrm{e}^{-q\frac{G_0^3}{3}}\mathrm{e}^{q\frac{G_0^3}{2}(-w^2+\frac{i}{3}w^3)}}{w^{2m}(w+2i)^{2n}}dw.
\end{aligned}
$$

We now make the change of variables in the integral $z = w(1 - iw/3)^{1/2}$ which satisfies

$$z^2 = w^2 - \frac{i}{3}w^3.$$

This change is exactly the change (A.7) just noticing that $z^2 = u$, then as we discussed, it has inverse, and then it is well defined.

$$\begin{aligned} Res(f^q_{m,n}(\tau), i) &= \frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{\mathrm{e}^{-q\frac{G_0^3}{3}}\mathrm{e}^{-q\frac{G_0^3}{2}z^2}}{-iw(z^2)^{2m+1}(w(z^2)+2i)^{2n+1}} 2zdz \\ &= \frac{1}{2\pi i}\frac{2}{-i}\mathrm{e}^{-q\frac{G_0^3}{3}}\int_{|z|=\epsilon} \mathrm{e}^{-q\frac{G_0^3}{2}z^2} F^+_{m,n}(z^2)dz \end{aligned}$$

whenever the circle with radius ϵ is contained within the curve $\{z = w(1-i\frac{w}{3})^{1/2} : |w| = \rho\}$. Then, as we have seen from the change of variables (A.7) and the formula (A.22), we can write

$$\begin{aligned} Res(f^q_{m,n}(\tau), i) &= 2i\mathrm{e}^{-q\frac{G_0^3}{3}} Res(\mathrm{e}^{-q\frac{G_0^3}{2}z^2} zF^+_{m,n}(z^2)) \\ &= 2i\mathrm{e}^{-q\frac{G_0^3}{3}} Res(\mathrm{e}^{-q\frac{G_0^3}{2}z^2} \sum_{j=0}^{\infty} d_j^{n,m} z^{j-2m}) \end{aligned}$$

since

$$\mathrm{e}^{-q\frac{G_0^3}{2}z^2} = \sum_{s=0}^{\infty}\left(-q\frac{G_0^3}{2}\right)^s \frac{z^{2s}}{s!}$$

by taking the coefficient of degree -1, we need that $2s+j-2m = -1$ or equivalently $j = 2(m-s)-1$, which leads to

$$Res(f^q_{m,n}(\tau), i) = 2i\mathrm{e}^{-q\frac{G_0^3}{3}} \sum_{s=0}^{m-1} d^{n,m}_{2(m-s)-1}\left(-\frac{q}{2}G_0^3\right)^s \frac{1}{s!}$$

□

this concludes the proof of proposition 2.21.

B.4 Function W

Define

$$\begin{aligned} W(w) &= \sum_{n=0}^{\infty} \frac{w^n}{(n!)^2} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pm 2i\sqrt{w}}{2}\right)^{2n}}{(n!)^2} \\ &= J_0(\pm 2i\sqrt{w}) \end{aligned} \tag{B.23}$$

also

$$W'(w) = \sum_{n=1}^{\infty} n\frac{w^{n-1}}{(n!)^2} = \sum_{n=1}^{\infty} \frac{w^{n-1}}{n!(n-1)!}, \qquad W''(w) = \sum_{n=2}^{\infty}(n-1)\frac{w^{n-2}}{n!(n-1)!} = \sum_{n=2}^{\infty}\frac{w^{n-2}}{n!(n-2)!}$$

using that

$$J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z)$$

we have

$$J_0'(z) = -J_1(z)$$

$$J_0''(z) = \frac{1}{2}(J_2(z) - J_0(z))$$

and by the chain rule

$$\begin{aligned} W'(w) &= J_0'(\pm 2i\sqrt{w})\frac{(\pm 2i)}{2\sqrt{w}} \\ &= J_0'(\pm 2i\sqrt{w})\frac{(\mp 2)}{2i\sqrt{w}} \\ &= \frac{2}{\pm 2i\sqrt{w}}J_1(\pm 2i\sqrt{w}) \end{aligned} \tag{B.24}$$

$$\begin{aligned} W''(w) &= \pm i\left[\frac{1}{\sqrt{w}}J_0''(\pm 2i\sqrt{w})\frac{(\pm 2i)}{2\sqrt{w}} - \frac{1}{2w^{3/2}}J_0'(\pm 2i\sqrt{w})\right] \\ &= \pm i\left[\frac{\pm i}{w}J_0''(\pm 2i\sqrt{w}) - \frac{1}{2w^{3/2}}J_0'(\pm 2i\sqrt{w})\right] \\ &= \pm i\left[\frac{\pm i}{2w}(J_2(\pm 2i\sqrt{w}) - J_0(\pm 2i\sqrt{w})) + \frac{1}{2w^{3/2}}J_1(\pm 2i\sqrt{w})\right] \\ &= \frac{1}{2w}\left[J_0(\pm 2i\sqrt{w}) - J_2(\pm 2i\sqrt{w}) \pm \frac{i}{\sqrt{w}}J_1(\pm 2i\sqrt{w})\right] \\ &= \frac{1}{2w}\left[J_0(\pm 2i\sqrt{w}) - J_2(\pm 2i\sqrt{w}) - \frac{2}{\pm 2i\sqrt{w}}J_1(\pm 2i\sqrt{w})\right] \end{aligned} \tag{B.25}$$

from equation (B.25), if we set

$$z = \pm 2i\sqrt{w}$$

we have

$$wW''(w) = \frac{1}{2}\left[J_0(z) - J_2(z) - \frac{2}{z}J_1(z)\right]$$

and using that $J_2(z) = (2/z)J_1(z) - J_0(z)$

$$wW''(w) = J_0(z) - \frac{2}{z}J_1(z) \tag{B.26}$$

also, from equation (B.23), we have

$$W(w) = J_0(z) \tag{B.27}$$

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