Linear methods for rational triangle decompositions

by

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Bachelor of Science, Simon Fraser University, 2004
Master of Science, Simon Fraser University, 2007

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ABSTRACT

Given a graph G, a K_3-*decomposition* of G, also called a *triangle decomposition*, is a set of subgraphs isomorphic to K_3 whose edges partition the edge set of G. Further, a *rational* K_3-*decomposition of* G is a non-negative rational weighting of the copies of K_3 in G such that the total weight on any edge of G equals one. In this thesis, we explore the problem of rational triangle decompositions of dense graphs.

We start by considering necessary conditions for a rational triangle decomposition, which can be represented by facets of a convex cone generated by a certain incidence matrix. We identify several infinite families of these facets that represent meaningful obstructions to rational triangle decomposability of a graph. Further, we classify all facets on up to 9 vertices and check all 8-vertex graphs of degree at least four for rational triangle decomposability. As the study of graph decompositions is closely related to design theory, we also prove the existence of certain types of designs.

We then explore sufficient conditions for rational triangle decomposability. A famous conjecture in the area due to Nash-Williams states that any sufficiently large graph (satisfying some divisibility conditions) with minimum degree at least $\frac{3}{4}v$ is K_3-decomposable; the same conjecture stands for rational K_3-decomposability (no divisibility conditions required). By perturbing and restricting the coverage matrix of a complete graph, we show that minimum degree of at least $\frac{22}{23}v$ is sufficient to guarantee that the given graph is rationally triangle decomposable. This density bound is a great improvement over the previously known results and is derived using estimates on the matrix norms and structures originating from association schemes.

We also consider applications of rational triangle decompositions. The method we develop in the search for sufficient conditions provides an efficient way to generate certain sampling plans in statistical experimental design. Furthermore, rational graph decompositions serve as building blocks within certain design-theoretic proofs and we use them to prove that it is possible to complete partial designs given certain constraints.

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Thanks to my past, you know who you are. Thanks for believing in me and making me stronger.

To my entire amazing family — without you I would not be here in every sense of the word.

DEDICATION

To my parents and grandparents, for their unwavering love and for trusting me when I was breaking all the rules.

Chapter 1

Introduction

The study of graph decompositions dates back to the 19th century and has since become one of the central problems in combinatorics. It has connections to design theory and the study of association schemes as well as applications to coding theory and design of efficient statistical experiments. The result of this multi-disciplinary popularity is the ability to approach the problem through many angles; amongst the most common ones are combinatorial design theory with its algebraic tools and graph theory with its well-developed algorithms.

Definition 1.1. Given a simple graph G, a K_3-*decomposition* of G, also called a *triangle decomposition* or *triangulation*, is a set of subgraphs isomorphic to K_3 whose edges partition the edge set of G. In this case, we say that G is K_3-*decomposable*.

Example 1.2. A triangle decomposition of $K_9 - C_9$, that is the complete graph on 9 vertices minus a cycle on 9 vertices (labelled 0 through 8), can be obtained by taking translates of the 3-subset $\{0, 2, 5\}$ modulo 9 (see Figure 1.1 on the following page).

Example 1.3. Reminiscent of tilings but covering only the edges, consider an infinite grid in $\mathbb{Z}^2$ defined by directions $(1, 0), (0, 1)$ and $(1, 1)$. It has a natural triangulation as seen in Figure 1.2 on the next page.

Figure 1.1: Triangle decomposition of $K_9 - C_9$.

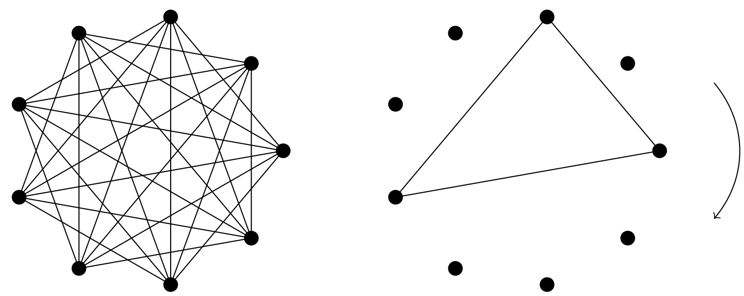
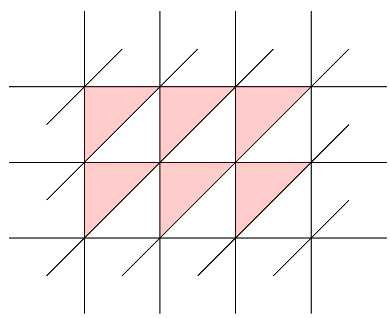


Figure 1.2: Triangulation of an infinite grid.



Example 1.4. Due to their connection to Latin squares, triangle decompositions of complete balanced tripartite graphs have been extensively studied in combinatorics. Let $[n] = \{1, \ldots, n\}$ and recall that a Latin square of order n is an $n \times n$ array whose cells are filled with n different symbols such that each symbol occurs exactly once in each row and exactly once in each column. Each entry of a Latin square L of order n can be written as a triple (i, j, k), $i, j, k \in [n]$, where i is the row, j is the column and k is the symbol in the cell $L(i, j)$. Then a triangulation of a complete tripartite graph $K_{n,n,n}$ is equivalent to a Latin square with parts representing rows, columns and symbols.

Example 1.5. We can easily find graphs which cannot be decomposed into triangles. Indeed, it is necessary that every edge of G belongs to a triangle and this rules out many graph families, including bipartite graphs and graphs of girth at least four. But even the presence of many triangles does not guarantee triangle decomposition; take, for instance, K_4 and K_5. Each of these graphs fails to be triangle decomposable: in K_4 each vertex has degree 3 while in K_5 the total number of edges is 10.

This example motivates the following necessary conditions for a graph G to be K_3-decomposable: every vertex of G must be of even degree and the total number of edges of G must be divisible by 3. We say that the graphs satisfying these necessary conditions are *K_3-divisible*. However, the necessary K_3-divisibility conditions are not sufficient for K_3-decomposition. In fact, the search for sufficient conditions for K_3-decomposability is ongoing. The first full result came in 1847 from design theory on the existence of so-called Steiner triple systems, which are equivalent to triangulations of the complete graph K_v.

Definition 1.6. A *Steiner triple system on v vertices* is a set V of v elements together with a set B of 3-subsets (also called *triples* or *blocks*) of V such that every 2-subset of V occurs in exactly one triple of B.

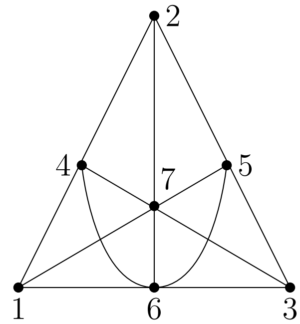
Example 1.7. The unique (up to isomorphism) Steiner triple system on 7 vertices is the famous *Fano plane* or *projective plane of order 2*. It has vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$ and block set

$$B = \{\{1, 2, 4\}, \{1, 3, 6\}, \{1, 5, 7\}, \{2, 3, 5\}, \{2, 6, 7\}, \{3, 4, 7\}, \{4, 5, 6\}\}$$

represented by the lines in Figure 1.3 on the following page.

Thought of as a K_3-decomposition of K_7, every pair of vertices in V corresponds to an edge in K_7 and every line above corresponds to a triangle in K_7. Then the property that every pair is contained in exactly one block of the Steiner triple system translates to the fact that every edge of K_7 is used exactly once in the decomposition.

Figure 1.3: Fano plane.



The question on the existence of Steiner triple systems was first raised by W. S. B. Woolhouse in 1844 in the Lady's and Gentlemen's Diary [51]. The solution to this problem was published by Reverend Thomas Kirkman in 1847: he showed, by construction, that a Steiner triple system on v vertices, and hence a K_3-decomposition of K_v, exists if and only if $v \equiv 1, 3 \pmod 6$ [32]. Therefore, for complete graphs the necessary K_3-divisibility conditions are also sufficient for K_3-decomposability. Independently, in 1853, Jacob Steiner introduced and studied triple systems and, as his work was better known at the time, these objects were named in his honour.

As such, the problem of triangulations of compete graphs has been settled. A natural question arises — how close to complete does a K_3-divisible graph have to be in order to be K_3-decomposable? This 'closeness' may be measured by the minimum degree and it is convenient to introduce the following definition: we say that a graph G is $(1-\epsilon)$-*dense* if $\delta(G) \geq (1-\epsilon)(v-1)$, where $\delta(G)$ denotes the minimum degree of G. That is, a graph is $(1-\epsilon)$-dense if the proportion of the missing edges at each vertex is at most ϵ. One of the largest and most interesting conjectures on triangle decompositions of non-complete graphs is due to Nash-Williams [35] stating that $\epsilon < 1/4$ suffices:

Conjecture 1.8 ([35]). *Any sufficiently large K_3-divisible $3/4$-dense graph is K_3-*

decomposable.

An interesting thing to note is that the conjectured density threshold is sharp. This is proved via a counting argument from a construction (presented in Theorem 2.8). The only existence result in the area of non-complete graph decompositions is asymptotic, due to Gustavsson [26] and, most recently, Keevash [31], but it requires a very small $\epsilon \sim 10^{-7}$.

Here, we are interested in a fractional relaxation of the problem.

Definition 1.9. Given a simple graph G, a *rational K_3-decomposition of G* is a non-negative rational weighting of the copies of K_3 in G such that the total weight on any edge of G equals 1. If G admits a rational K_3-decomposition, we say that G is *rationally triangle decomposable* or that G *has a rational triangulation.*

Clearly, the K_3-divisibility conditions from the integral case are no longer necessary for the rational decomposition to exist; however, due to the non-negativity requirement, for G to be rationally K_3-decomposable, we must still have that every edge of G belongs to a copy of K_3. In this vein, note that any complete graph on more than 3 vertices is rationally K_3-decomposable: take all possible embeddings of K_3 in K_v with weight $1/(v-2)$. In fact, a rational K_3-decomposition of a simple graph G is equivalent to an (integral) K_3-decomposition of a λ-fold graph G^λ, that is G where every edge appears λ times for some integer $\lambda > 0$.

We approach the problem of rational triangle decompositions from two directions, both through necessary (Chapter 3) and sufficient (Chapter 4) conditions. We often represent graphs as vectors, so the following notation is used throughout:

Definition 1.10. Let $\binom{V}{i}$ represent the set of all i-subsets of a v-set V. A *vector in* $\mathbb{R}^{\binom{V}{2}}$ is a $1 \times \binom{v}{2}$ row matrix indexed by the 2-subsets of V representing a weighting of the 2-subsets. For any subset $G \subseteq \binom{V}{2}$, the *characteristic vector of* G, denoted by $\mathbf{1}_G$, is a vector in $\mathbb{R}^{\binom{V}{2}}$ with 1 in thei^{th} coordinate if $i \in G$ and 0 otherwise.

In Chapter 3, we explore the following connection between vectors in $\mathbb{R}^{\binom{V}{2}}$ and K_3-decompositions of graphs on v vertices, which is a consequence of Farkas' Lemma (Theorem 2.22): a graph G on a v-set V of vertices has a K_3-decomposition if and only if $\langle \mathbf{y}, \mathbf{1}_G \rangle \geq 0$ whenever $\mathbf{y} \in \mathbb{R}^{\binom{V}{2}}$ is such that $\langle \mathbf{y}, \mathbf{1}_\triangle \rangle \geq 0$ for all triangles $\triangle$ in G. This connection motivates the study of such vectors $\mathbf{y} \in \mathbb{R}^{\binom{V}{2}}$ since each one of them provides a necessary condition for a (rational) K_3-decomposition of G. Furthermore, any such vector $\mathbf{y}$ is a positive linear combination of some finite set Y of extremal vectors in $\mathbb{R}^{\binom{V}{2}}$. While determining the full set of such vectors is believed to be very difficult, we initiate work on a partial classification. Our results in this direction include the classification of several infinite families in Y that present meaningful obstructions to triangle decompositions, an upper bound on $|Y|$ and the establishment of a relationship between Y and a metric polytope, all defined in Chapter 3. We also provide computer-generated data for small v and formulate some conjectures based on this data. As a related result, due to the connection between graph decomposition and designs, in Section 3.3, we also prove the existence of certain kinds of designs.

Interestingly, the conjectured Nash-Williams' bound for the integral triangle decomposition stands as the conjectured bound for the rational triangle decomposition.

Conjecture 1.11. *Any sufficiently large* $3/4$*-dense graph is rationally* K_3*-decomposable.*

Since integral triangle decomposition implies rational triangle decomposition, this bound is also sharp. The best bound on density known to date is due to Yuster [54]: using probabilistic methods he shows that $\epsilon < 1/90,000$ is sufficient for rational triangle decomposition. In Chapter 4, we improve this bound with the following result: any $(1-\epsilon)$-dense graph G has a rational triangle decomposition provided that $\epsilon < 1/23$. This bound is much closer to the conjectured $1/4$ than any of the previous results and is possibly stronger than is necessary since it guarantees a rational triangle decomposition of a specific type. We achieve this bound by exhibiting nonnegative vectors $\mathbf{x}$ as solutions to a certain system of linear equations by using tools from associations schemes defined in Chapter 2.

In Chapter 5, we highlight two applications of rational triangle decompositions. First, rational triangle decompositions provide constructions of certain statistical experimental designs. While our bound for sufficient conditions for the existence of these designs is worse than the previously established one, the method we develop in Chapter 4 provides an efficient way of generating these statistical designs even below the sufficiency threshold. Furthermore, rational graph decompositions serve as building blocks within certain design-theoretic proofs and we use them to prove that it is possible to complete a partial design given certain constraints.

Finally, the appendices include computer code for several computations, including characterizing all the obstructions for K_3-decomposability for 8-vertex graphs and then applying them to check rational triangle decomposability for all 8-vertex graphs of minimum degree at least 4.

Chapter 2

Background

We consider both the necessary and sufficient conditions for rational decomposability of dense graphs into triangles. Our approach to necessary conditions, to be developed in Chapter 3, consists of examining facets of a particular convex cone that represent meaningful obstructions to graph decompositions. In Chapter 4, we consider sufficient conditions for rational K_3-decomposability through studying association schemes and their linear algebraic structure. In this chapter, we provide the background needed for both approaches.

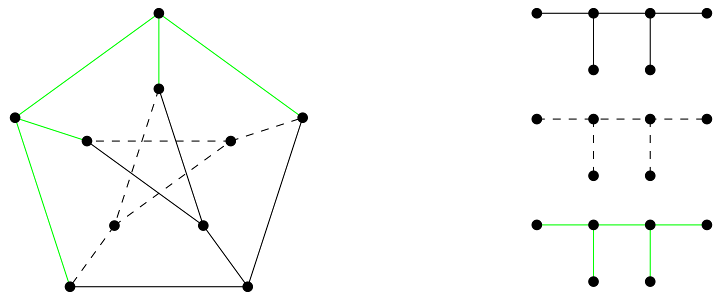
2.1 Preliminary definitions and results

While we are interested in K_3-decompositions, decompositions can be defined more generally.

Definition 2.1. Given two graphs G and H, a *decomposition of G into copies of H* or an *H-decomposition of G* is an edge-colouring of G such that every colour class induces a graph isomorphic to H.

Example 2.2. Figure 2.1 on the next page represents an interesting decomposition of the Petersen graph G into a certain tree H:

Figure 2.1: Decomposition of the Petersen graph into a tree.



For the simplest case of $H = K_2$, the decomposition of G into single edges is trivial; however, decomposing a graph into triangles is already a hard problem that has been well-studied [52]. In general, there are divisibility conditions on the number of edges and degrees of the two graphs involved.

Definition 2.3. We say that a graph G is *H-divisible* if it satisfies two conditions necessary for admitting an H-decomposition:

- the number of edges of G is divisible by the number of edges of H;
- every vertex degree of G is divisible by the greatest common divisor of all the vertex degrees of H.

Unfortunately, as discussed in Chapter 1, H-divisibility is not sufficient for H-decomposition, which motivates the search for nice sufficient conditions. In fact, the decomposition problem translates naturally into the design-theoretic setting: a K_k-decomposition of K_v is a 2-$(v, k, 1)$ design.

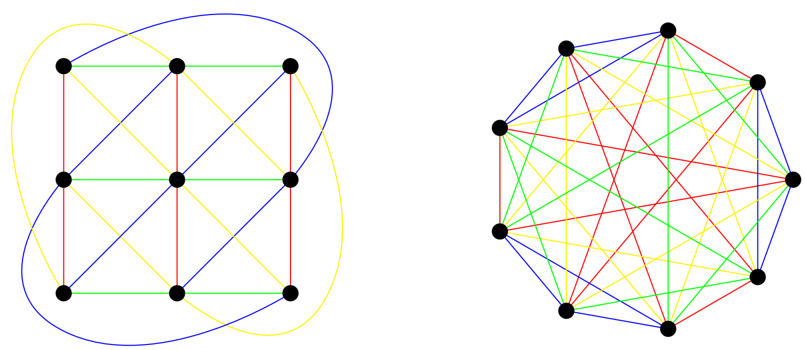
Definition 2.4. A t-(v, k, λ) *balanced incomplete block design (BIBD)*, or simply a *t-design*, is a pair (V, B), where V is a set of v elements called *points* and B is a collection of k-subsets of V, called *blocks*, such that every t-subset of the point set V is contained in exactly λ blocks.

Example 2.5. Another interesting example (besides the Fano plane) is that of the 2-$(9, 3, 1)$ design or the *affine plane of order 3*. The vertex set is $V = \{1, \ldots, 9\}$ and the block set is

$$B = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\},$$
$$\{1,5,9\}, \{2,6,7\}, \{3,4,8\}, \{1,6,8\}, \{2,4,9\}, \{3,5,7\}\},$$

which is graphically represented in Figure 2.2 on the left:

Figure 2.2: Affine plane of order 3.



When considered as a K_3-decomposition of K_9, each block of the affine plane (above left) corresponds to a triangle of K_9 (above right). While we generally do not concern ourselves with it, it is interesting to remark that this Steiner triple system has even more underlying structure: the block set can be partitioned into *parallel classes*, that is groups of 3 mutually disjoint blocks that partition the vertex set (or 3 mutually disjoint triangles in the triangulation of K_9), indicated above by colours.

We are mainly interested in 2-designs due to their connection to graph decompositions and will assume that $t = 2$ from now on, unless otherwise specified. In this case, simple counting of the parameters of the design provides immediate necessary divisibility conditions for their existence.

Lemma 2.6. *In a* 2-(v, k, λ) *BIBD, we have the following:*

$$\lambda v(v-1) = bk(k-1) \text{ and } bk = vr,$$

where b *is the number of blocks and* r *is the common number of blocks to which any point belongs (replication number).*

Therefore, the following divisibility conditions are necessary for the existence of a 2-design:

$$\lambda(v-1) \equiv 0 \pmod{(k-1)}, \tag{2.1}$$

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}. \tag{2.2}$$

Sufficiency for the smallest interesting family of cases with $k = 3$ and $\lambda = 1$ was solved by Kirkman in 1847 when he constructed Steiner triple system for each allowed v. Mathematicians then turned their attention to other designs. In 1979, Brouwer [7] proved that a 2-$(v, 4, 1)$ design exists if and only if $v \equiv 1, 4 \pmod{12}$. Therefore, K_4-divisibility is sufficient for K_4-decomposition to exist. Finally, Beth et al. [2] showed that a 2-$(v, 5, 1)$ design exists if and only if $v \equiv 1, 5 \pmod{20}$. This is the last fully completed case. For $H = K_k$, $k = 6, 7, 8, 9$, it has been shown that the above necessary conditions are sufficient except for a small list of undecided cases. Several other small cases have been settled; however, the constructions get more and more involved and often fail to generalize.

The major breakthrough came in an asymptotic form of Wilson's theorem, stating that the necessary conditions are sufficient for large enough complete graphs.

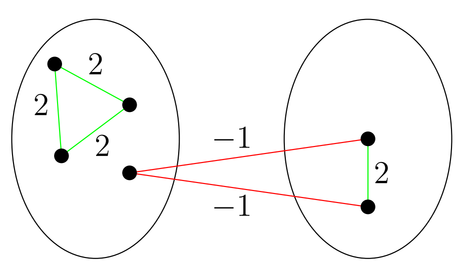
Theorem 2.7 ([46]). *For every fixed graph H, there exists an integer $N(H)$ so that for all $v > N(H)$, if K_v is H-divisible, then K_v is H-decomposable.*

In Wilson's result, "large enough" means truly astronomical, with $v > e^{e^{k^{k^2}}}$ necessary for the decomposition of K_v into K_k, this being a 2-$(v, k, 1)$ design. This asymptotic result, extended to higher values of t and hypergraphs, has been proved recently by Keevash by using randomized algorithms and probabilistic algebraic constructions [31]. With this in mind, in this thesis, we will concentrate on K_3-decompositions of non-complete graphs.

2.2 Nash-Williams' bound and motivation

Given a graph, consider the following weighting of its edges: arbitrarily partition the vertices of the graph into two parts, assign a weight of 2 to all edges within each part and a weight of -1 to all edges crossing between the parts as illustrated in Figure 2.3.

Figure 2.3: Weighted partition.



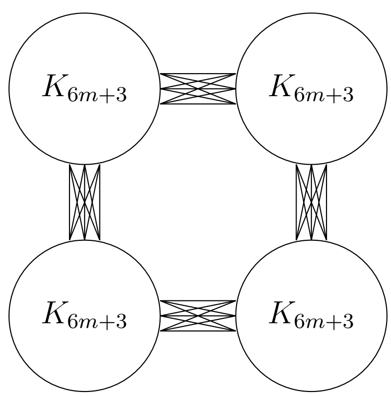
Then, in any K_3-decomposition of the graph, all triangles will have a non-negative inherited weight; more precisely, each triangle will have weight of six if it is contained within one of the two parts and weight of zero otherwise. This method provides a certificate of when a given graph cannot be decomposed into triangles — it is exactly when there exists a partition and an assignment of weights as described above such that each triangle receives nonnegative weight, but the total sum of all the weights on all the edges is negative. Alternatively, given a partition of the graph into two parts S_1 and S_2, we can have at most twice as many "cross-over" edges as we can have within S_1 and S_2. We shall refer to this as the *bipartition or cut test*. In fact, this is one of many tests that can be applied to check for triangle non-decomposability and these tests will be further explored in Chapter 3.

This idea and the cut test motivate the following construction that shows the tightness of the Nash-Williams' bound.

Theorem 2.8 ([35])**.** *There are infinitely many v for which there exist K_3-divisible graphs on v vertices with minimum degree at least $\frac{3}{4}v-1$ that are not K_3-decomposable.*

Proof. Let $v = 24m+12$ and consider the following graph $G = K_{6m+3} \boxtimes C_4$ consisting of 4 vertex-disjoint cliques on $6m+3$ vertices each connected as shown in Figure 2.4.

Figure 2.4: The graph $G = K_{6m+3} \boxtimes C_4$.



Assign weight 2 to each edge within each K_{6m+3} and weight -1 to all the edges with end vertices in distinct copies (so S_i is the union of two non-adjacent K_{6m+3}). Then all the triangles in this graph will have weight 0 or 6. However, the total weight on all the edges is

$$2 \cdot 4 \cdot \binom{6m+3}{2} + (-1) \cdot 4 \cdot (6m+3)^2 = 4\left((6m+3)(6m+2) - (6m+3)^2\right) < 0.$$

Since the total inherited weight of the graph is negative, G cannot be decomposed into triangles of non-negative weights. Moreover, notice that

- G has $\frac{1}{2}(18m+8)(24m+12) = 3(9m+4)(8m+4)$ edges and
- G is regular of degree $(6m+2) + 2(6m+3) = 18m+8 = \frac{3}{4}(24m+12) - 1$.

Therefore, G is K_3-divisible, but not K_3-decomposable. □

The construction in the proof above is generalized in a straightforward way: take a graph $G = (K_{k+1} - M) \boxtimes K_p$, where M is a matching and $p = k(k-1)m + k$. This G is regular of degree $\frac{k}{k+1}v - 1$, is K_k-divisible, but is not K_k-decomposable. This proves the following theorem, which in turn motivates the corresponding conjecture.

Theorem 2.9. *For each $k \geq 3$, there are infinitely many v for which there exist K_k-divisible graphs on v vertices with minimum degree at least $\frac{k}{k+1}v - 1$ that are not K_k-decomposable.*

Conjecture 2.10. *For each $k \geq 3$, there exists an integer $N(k)$ so that for all $v > N(k)$ any K_k-divisible graph G on v vertices with minimum degree $\delta \geq \frac{k}{k+1}v$ is also K_k-decomposable.*

In his Ph.D. thesis, Gustavsson proved the above conjecture for any H when the bound for δ is replaced by $\delta \geq (1 - \epsilon(H))v$ with $\epsilon \sim 10^{-7}$. Gustavsson uses the connection between triangulations and Latin squares (as discussed in Example 1.4): given a nearly complete tripartite graph, its tripartite complement is equivalent to a partially filled Latin square and filling it will correspond to triangulating the original nearly complete tripartite graph. Then Gustavsson relies on the following result by Chetwynd and Häggkvist [10]:

Theorem 2.11. *Any partial Latin square of order v in which each symbol, row, and column contains no more than $10^{-5}v$ nonblank cells can be completed, provided that v is even and $v > 10^7$.*

Recently, the above results have been improved by Bartlett [1], who strengthens Gustavsson's bound to $\epsilon = 1.197 \cdot 10^{-5}$. Using a different technique, Keevash generalizes Gustavsson's result to hypergraphs, but with another asymptotic bound. Using probabilistic methods, Yuster [54] brings the constant down to $1/9k^{10}$ for rational decompositions into K_k and down to $1/90,000$ for triangles in particular.

Our approach to necessary conditions for the existence of triangle decompositions will consist of examining the full set of tests (of which the cut test above is only one) that a graph must pass in order to be K_3-decomposable. Formally, we have the following definition.

Definition 2.12. A graph G of order v *passes a* $\mathbf{y}$*-test* if, for a $1 \times \binom{v}{2}$ edge weight vector $\mathbf{y}$ indexed by pairs of points, we have the following:

1. $\langle \mathbf{y}, \mathbf{1}_{\triangle} \rangle \geq 0$ for all triangles $\triangle$ in G,

2. $\langle \mathbf{y}, \mathbf{1}_G \rangle \geq 0$, where $\mathbf{1}_G$ denotes the characteristic vector of G.

In later sections, we will investigate the edge-weight vectors $\mathbf{y}$ which provide meaningful obstructions that prevent G from having a triangle decomposition. For now, notice that as the first condition runs over all triangles in G, it can be written as a matrix inequality $\mathbf{y}W \geq 0$, where W records all interactions between pairs and triangles. We shall closely study this matrix W.

2.3 Convex geometry of the inclusion matrix

Since we are interested in interactions between various subsets of points (pairs and triples in particular), it is useful to define a matrix that records all these interactions.

Definition 2.13. The *inclusion matrix* $W_{t,k}(v)$, or just $W_{t,k}$, is a $\binom{v}{t} \times \binom{v}{k}$ $(0,1)$-matrix with rows indexed by all t-subsets T of V and columns indexed by all k-subsets K of V such that $W_{t,k}(T,K) = 1$ if and only if $T \subset K$ for $|T| = t$ and $|K| = k$.

The ordering of the subsets used in indexing W or any vectors does not matter as long as it is consistent and will be specified when needed. We can also consider inclusion matrices indexed by a certain subset of all k-subsets; this will be further explored in Section 3.3.

Example 2.14. Let $V = \{1, \ldots, 5\}$ and consider its 3-subsets $\{\alpha_1, \ldots, \alpha_{10}\}$, where

$$\begin{array}{ll} \alpha_1 = \{1,2,3\} & \alpha_6 = \{1,4,5\} \\ \alpha_2 = \{1,2,4\} & \alpha_7 = \{2,3,4\} \\ \alpha_3 = \{1,2,5\} & \alpha_8 = \{2,3,5\} \\ \alpha_4 = \{1,3,4\} & \alpha_9 = \{2,4,5\} \\ \alpha_5 = \{1,3,5\} & \alpha_{10} = \{3,4,5\} \end{array}$$

Then we have:

$$
W_{2,3}(5) = \begin{array}{c} \\ \{1,2\} \\ \{1,3\} \\ \{1,4\} \\ \{1,5\} \\ \{2,3\} \\ \{2,4\} \\ \{2,5\} \\ \{3,4\} \\ \{3,5\} \\ \{4,5\} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 & \alpha_{10} \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}
$$

The following observation follows easily after one recalls Definition 2.4 of a design.

Observation 2.15. A t-(v, k, λ) design exists if and only if $W_{t,k}(v)\mathbf{x} = \lambda\mathbf{1}$ has a non-negative integer solution $\mathbf{x}$, where $\mathbf{1}$ is the all ones vector of the necessary dimension.

The vector $\mathbf{x}$ is called the *characteristic vector of the design* as it records the number of occurrences of each k-subset as a block of the corresponding design. In particular, the number of blocks of the design is equal to $|\mathbf{x}| = \lambda\binom{v}{t}/\binom{k}{t}$. The inclusion matrix $W_{t,k}(v)$ itself has several nice properties. It has constant row and column sum of $\binom{v-t}{k-t}$ and so $W\mathbf{1} = \binom{v-t}{k-t}\mathbf{1}$. Therefore, $W\mathbf{x} = \lambda\mathbf{1}$ always has a non-negative rational solution $\mathbf{x} = \lambda/\binom{v-t}{k-t}\mathbf{1}$.

Since we are looking for vectors $\mathbf{y}$ that are nonnegative on all the triangles, i.e. such that $\mathbf{y}W \geq 0$, we are led to the study of cones and their facets defined below.

Definition 2.16. Consider a finite-dimensional real vector space K. A *convex cone* in K is a subset of K closed under vector addition and non-negative scalar multiplication.

The *ray generated by* $\mathbf{x} \in K$ is the set $\{k\mathbf{x} \ : \ k \geq 0\}$. The *cone generated by a set of vectors* $\{\mathbf{x_i}\}_1^N \subset K$ is the set of non-negatively scaled sums $\{\sum_{i=1}^N k_i\mathbf{x_i} \ : \ k_i \geq 0\}$. The number n of linearly independent vectors in $\{\mathbf{x_i}\}_1^N$ is called the *dimension of the cone.*

Definition 2.17. A cone is called *full* if $n = \dim(K)$ and it is called *pointed* if the only vector contained in the cone together with its negative is the zero vector. A cone is called *polyhedral* if it is full, pointed and generated by a finite set of vectors.

The cones we consider will all be polyhedral cones in real Euclidean space.

Example 2.18. The cone in $\mathbb{R}^2$ generated by the vectors $(1, 0)$ and $(0, 1)$ is a polyhedral cone that consists of the entire non-negative orthant of $\mathbb{R}^2$, that is all the points $(x, y) \in \mathbb{R}^2$ such that $x, y \geq 0$.

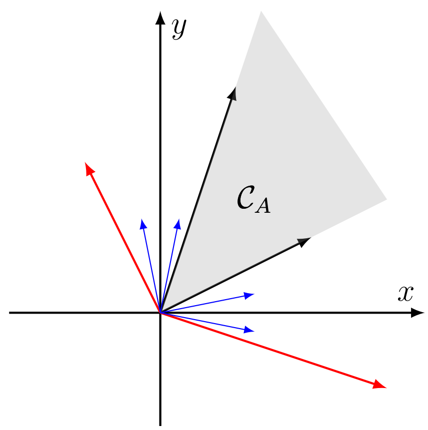
Definition 2.19. A *face* F of a cone $\mathcal{C} \in \mathbb{R}^m$ is a subcone of $\mathcal{C}$ such that for all $\mathbf{x} \in F$, $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C}$ implies that $\mathbf{x_1}, \mathbf{x_2} \in F$. A face of dimension 1 is called an *extremal ray* of $\mathcal{C}$. A face of codimension 1 (or dimension $m-1$) is called a *facet* of $\mathcal{C}$.

From now on, let $\langle \cdot, \cdot \rangle$ denote the inner product. Given an $m \times n$ matrix A, the set $\mathcal{C}_A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \geq 0\}$ is a polyhedral cone in $\mathbb{R}^m$ called *the cone of A*. We say that a vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}A \geq \mathbf{0}$ *supports* $\mathcal{C}_A$; that is because the space of vectors $\mathbf{b}$ such that $\langle \mathbf{y}, \mathbf{b} \rangle \geq \mathbf{0}$ contains the entire convex cone generated by the columns of A. Then the cone itself is the intersection of all half spaces described by its supporting vectors and the columns of A are, in fact, extreme rays of $\mathcal{C}_A$. Furthermore, we say that $\mathbf{y} \in \mathbb{R}^m$ *supports* $\mathcal{C}_A$ *on a facet* if $\mathbf{y}A \geq \mathbf{0}$ and the set of columns of A that are orthogonal to $\mathbf{y}$ spans a subspace of dimension $n-1$. Geometrically, this means that $\mathbf{y}$ is orthogonal to a facet of $\mathcal{C}_A$ and we shall refer to it as a *facet normal.* A facet normal of the cone cannot be written as a non-negative linear combination of other facet normals or supporting vectors [44], so facet normals are irreducible extreme supporting vectors of a cone. We also define the *dual cone* $\mathcal{C}_A^*$ as the cone generate by facet normals of $\mathcal{C}_A$.

Example 2.20. Given the $n \times n$ identity matrix I, the cone $\mathcal{C}_I$ is the non-negative orthant of $\mathbb{R}^n$, that is all the points $(x_1, \ldots, x_n) \in \mathbb{R}^N$ such that $x_i \geq 0$ for all $i = 1, \ldots, n$.

Example 2.21. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and consider the cone $\mathcal{C}_A$ in $\mathbb{R}^2$ as shown in Figure 2.5.

Figure 2.5: Cone of A.



Here, the shaded region represents the cone of A. Furthermore, the black vectors, which are the columns of A, represent the extremal rays, the red vectors represent facet normals and the blue vectors represent some of the supporting vectors of $\mathcal{C}_A$. As is clearly seen, the facet normals are the farthest possible supporting vectors. Alternatively, we can define this cone using half-spaces described by inequalities $3x - y \geq 0$ and $-x + 2y \geq 0$.

Note that there are two distinct ways to specify a cone. A *vertex representation* or a *V-representation* is equivalent to defining the cone as the convex hull of its extreme points (in our case, extreme rays). The minimal V-representation is also unique and is given by the set of extreme rays of the cone. A *half-space representation* or *an H-representation* is equivalent to defining the cone as the intersection of a finite number of half spaces. Then the minimal H-representation is also unique and is given by the half spaces defined by facets.

Motivated by graph decompositions, we are mainly interested in the cone of $W_{2,3}(v)$ as it records interactions between pairs and triangles. From now on we shall refer to the cone $\mathcal{C}_{W_{2,3}(v)}$ as the *triangulation cone* Tri_v. Notice that when considering triangle decompositions, instead of assigning weights to edges, we can consider assigning weights to triangles. Then a graph G being decomposable into triangles implies the existence of a nonnegative $\binom{v}{3} \times 1$ vector $\mathbf{x}$ that satisfies the matrix equation $W\mathbf{x} = \mathbf{1}_G$. Together with the conditions for passing a $\mathbf{y}$-test from Definition 2.12, we have that if $\mathbf{y}W \geq \mathbf{0}$, then

$$\langle \mathbf{y}, \mathbf{1}_G \rangle = \langle \mathbf{y}, W\mathbf{x} \rangle = \langle \mathbf{y}W, \mathbf{x} \rangle \geq 0,$$

which is one direction of the following important result.

Theorem 2.22 ([42]). *[Farkas' Lemma] Let A be an $m \times n$ matrix and $\mathbf{b}$ be an m-dimensional real vector. The equation $A\mathbf{x} = \mathbf{b}$ has a non-negative solution $\mathbf{x} \in \mathbb{R}^n$ (i.e. $\mathbf{b} \in \mathcal{C}_A$) if and only if $\langle \mathbf{y}, \mathbf{b} \rangle \geq \mathbf{0}$ for all $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}A \geq \mathbf{0}$.*

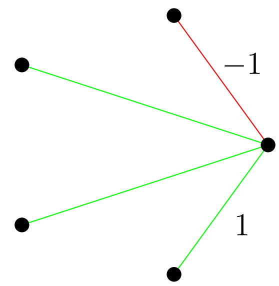
Geometrically, Farkas' Lemma can be interpreted as follows: given a convex cone and a vector, either the vector is in the cone or there is a hyperplane separating the vector and the cone. Now, according to the Krein-Milman theorem from functional analysis, we have that any compact convex subset of a finite-dimensional space is the closed convex hull of its extreme points. Here, it means that the set of extremal rays of a polyhedral cone $\mathcal{C}_A$ generates it and hence in Farkas' Lemma it is enough to check facets of $\mathcal{C}_A$ to ensure that $\mathbf{b} \in \mathcal{C}_A$.

In our triangle decomposition case, we are looking to decide whether or not $\mathbf{1}_G$ is in the cone Tri_v and so we consider the facets and facet normals of this cone as tests (in the sense of Definition 2.12) for rejecting or not rejecting the given graph G as having a rational triangle decomposition. As the first small example, let $\mathbf{y} = (-1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$ be the weight vector indexed by 2-subsets of a 5-set (unless otherwise specified, the indices occur in lexicographic order), where the edge $\{u, w\}$ receives weight $\mathbf{y}_{\{u,w\}}$. For this particular vector $\mathbf{y}$, that means that edge $\{1, 2\}$ receives weight -1, edges $\{1, 3\}$, $\{1, 4\}$ and $\{1, 5\}$ receive weight 1, while all other edges receive weight 0. Then $\mathbf{y}W_{2,3}(5) = (0\ 0\ 0\ 2\ 2\ 2\ 0\ 0\ 0\ 0)$ is similarly indexed by 3-subsets of a 5-set and represents their respective weights. Notice that $\mathbf{y}$ supports $W_{2,3}(5)$ since $\mathbf{y}W_{2,3}(5)$ is entry-wise non-negative, but it does not support it on a facet since there are not enough zero entries (namely, there are less than $9 = \binom{5}{2} - 1$). Quite naturally, to represent the supporting vectors and facets graphically, we consider the graph on 5 vertices, labelled 1 through 5, with edge weights given by $\mathbf{y}$. We shall further colour-code this in a straightforward way: red edges will correspond to negative weight, green ones to the positive weight and the missing edges correspond to weight 0. In our graphical representation, from now on we shall label the edges (usually on the boundary) with labels representing the weight of all the edges of the corresponding colour. Then, in this case we have the representation as in Fugure 2.6 on the following page.

While it is easy to check nonnegativity on all triangles, it is trickier to see whether some such edge-weighted graph corresponds to a facet (normal). For this, we need to check that the set of all zero-weight triangles spans a space of codimension 1 in $\mathbb{R}^{\binom{v}{2}}$.

The facet normals of $W_{2,3}(v)$ are not characterized and get increasingly complex for larger values of v. We generate them using the following algorithm [17].

Figure 2.6: Graphical representation of vector $\mathbf{y} = (-1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$.



Algorithm 2.23. Consider the following variation of the "dual simplex algorithm".

1. Given an $n \times m$ matrix A and start with a vector $\mathbf{y} \in \mathbb{R}^m$ supporting $\mathcal{C}_A$, that is such that $\mathbf{y}A \geq \mathbf{0}$.

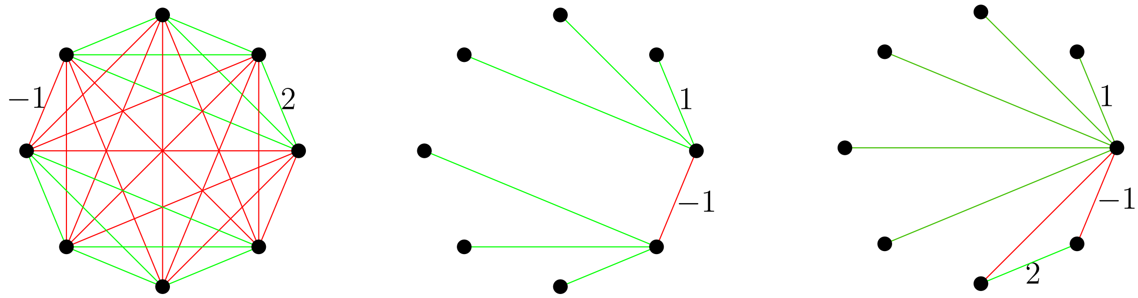
2. If the columns of A that are orthogonal to $\mathbf{y}$ span a subspace of dimension $m-1$, then $\mathbf{y}$ supports $\mathcal{C}_A$ on a facet and we are done. Otherwise, choose any $\mathbf{z} \in \mathbb{R}^m$ such that $\mathbf{z}A \geq \mathbf{0}$ (i.e. it supports $\mathcal{C}_A$) and $\mathbf{z}A$ vanishes on at least the same coordinates as $\mathbf{y}A$.

3. Let
$$\epsilon = \min_i \frac{(\mathbf{y}A)_i}{(\mathbf{z}A)_i},$$
where i ranges over all positive entries of $\mathbf{y}A$ and $\mathbf{z}A$.

4. Set $\mathbf{y} := \mathbf{y} - \epsilon\mathbf{z}$ and go to step 2.

Figure 2.7 on the next page shows some of the facet normals that occur for $v = 8$. Notice that the first one is exactly of the same form as the example in Section 2.2: the graph is partitioned into two parts with green edges having weight 2 and the red edges having weight -1. We shall call these facets the *bipartition or cut facets.* However, the other two facets present a new kind of obstruction — the last one even

Figure 2.7: Facet normals for $v = 8$.



has more than two weight values (recall that all edges of the same colour receive the same weight).

In Chapter 3, we will explore facets as representing obstructions to triangle decomposability of a graph. In Section 3.2, we prove the existence of various families of facets for all values of $v \geq 5$ and characterize all facets for up to $v = 8$. While a computer attack can enumerate all facets for $v \leq 8$, the number of isomorphism classes grows very fast.

2.3.1 The metric cone

Our cone Tri_v (that is, the cone $\mathcal{C}_{W_{2,3}(v)}$) appears in the context of metrics on a finite set.

Definition 2.24. A *metric* d on a set V is a function $d : V \times V \to \mathbb{R}$ that satisfies the following properties for all $x, y, z \in V$:

1. $d(x, y) \geq 0$,

2. $d(x, y) = 0$ if and only if $x = y$,

3. $d(x, y) = d(y, x)$,

4. $d(x, z) \leq d(x, y) + d(y, z)$.

A *semi-metric* is a function $d : V \times V \to \mathbb{R}$ that satisfies conditions 1, 3 and 4 and also that $d(x, x) = 0$.

We define the metric cone $\mathrm{Met}_v \subseteq \mathbb{R}^{\binom{V}{2}}$ to consist of all semi-metrics on V, where $|V| = v$. If we further want to bound the allowed distances from above, we obtain a metric polytope met_v. Alternatively, we have the following definition:

Definition 2.25 ([12])**.** For a v-set V, let x_{ij} represent a coordinate of a point in $\mathbb{R}^{\binom{V}{2}}$. Then the *metric cone* Met_v is defined by the $3\binom{v}{3}$ halfspaces in the form of *triangle inequalities*

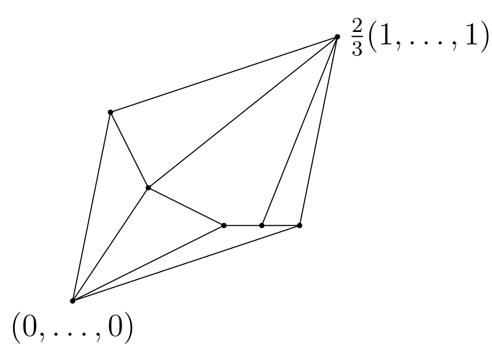
$$x_{ij} + x_{ik} - x_{jk} \geq 0,$$

where $\{i, j, k\} \in \binom{V}{3}$. The *metric polytope* met_v is defined by bounding Met_v by the $\binom{v}{3}$ *perimeter inequalities*

$$x_{ij} + x_{ik} + x_{jk} \leq 2.$$

From the above, we have that met_v has a total of $4\binom{v}{3}$ facets. It has two "extreme" vertices, $(0, \ldots, 0)$ and $\frac{2}{3}(1, \ldots, 1)$, and all of its extreme rays go through one of those vertices (see Figure 2.8).

Figure 2.8: Metric polytope.



There is an immediate connection between the metric cone and the triangulation cone as the former is a signed version of the latter. Recall that $\mathrm{Tri}_v = \{W\mathbf{x} : \mathbf{x} \in$

$\mathbb{R}^{\binom{V}{2}}, \mathbf{x} \geq 0\}$, where $W = W_{2,3}(v)$ is the $\binom{v}{2} \times \binom{v}{3}$ inclusion matrix (with three positive 1's in each column). Then $\mathrm{Met}_v = \{M\mathbf{x} : \mathbf{x} \in \mathbb{R}^{\binom{V}{2}}, \mathbf{x} \geq 0\}$ where M is the $\binom{v}{2} \times 3\binom{v}{3}$ whose columns are three copies of those in W with a different non-zero entry negated in each. So

$$M \begin{pmatrix} I \\ I \\ I \end{pmatrix} = W.$$

It is then easy to see that any supporting vector of Met_v is also a supporting vector of Tri_v since $\mathbf{y}M \geq 0$ implies $\mathbf{y}W \geq 0$. Moreover, if we add the three triangle inequalities for every unordered triple $\{i, j, k\} \in \binom{V}{3}$, we get inequalities $x_{ij} + x_{ik} + x_{jk} \geq 0$, which define the dual of Tri_v.

The metric cone and the metric polytope have been studied in particular for their applications in combinatorial optimization [12, 13] and through a more general setting. The metric cone is a relaxation of the cut cone that has more facets than those described by the triangular inequalities above.

Definition 2.26 ([12])**.** Given a subset S of a v-set V, the *cut of* S consists of pairs of points $(i, j) \in V \times V$ such that exactly one of i, j is in S. The cut is also defined by a vector $\delta(S) \in \mathbb{R}^{\binom{V}{2}}$ with $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise.

Now, define the following operation: given a cut $\delta(S)$, the *switching reflection* applied to a vector (or point) $\mathbf{x}$ produces a vector $\mathbf{y}$ with $y_{ij} = 1 - x_{ij}$ if (i, j) is in the cut $\delta(S)$ and $y_{ij} = x_{ij}$ otherwise. Note that the switching reflection switches the roles of inequalities in Definition 2.25 as long as not all x_{ij}, x_{ik}, x_{jk} are in the same part. The cuts themselves define both a cone and polytope.

Definition 2.27 ([12])**.** The *cut cone* Cut_v is the cone defined by all $2^{v-1} - 1$ nonzero cuts and the *cut polytope* cut_v is the cone generated by all 2^{v-1} cuts.

So $\mathrm{cut_v}$ is a $\binom{v}{2}$-dimensional polyhedron with 2^{v-1} vertices and met_v is a $\binom{v}{2}$-dimensional polytope containing cut_v and inscribed in the cube $[0,1]^{\binom{v}{2}}$.

The two polytopes are very close to each other. We have that $\mathrm{cut}_v \subset \mathrm{met}_v$ for $v \geq 5$; in particular, $\frac{2}{3}(1,\ldots,1)$ belongs to met_v but not to cut_v. All vertices of cut_v are vertices of met_v; in fact, the cuts are exactly the integral vertices of met_v [14]. For $v \geq 5$, the two polytopes share the symmetry group consisting of permutations and switching reflections. Therefore, the faces and the vertices of met_v are partitioned into orbits under permutations and switching reflections [14]. Due to the connection of Tri_v and the neighbourhood of the vertex $\frac{2}{3}(1,\ldots,1)$, we are interested in the orbit formed by 2^{v-1} so-called *anticuts* $\delta^*(S)_{ij} = \frac{2}{3}(1,\ldots,1) - \frac{1}{3}\delta(S)$. In general, $\mathbf{x} \mapsto \frac{2}{3}(1,\ldots,1) - \mathbf{x}$ maps the direction vectors in the neighbourhood of $\frac{2}{3}(1,\ldots,1)$ in met_v to facet normals of Tri_v.

In addition to generating the cone useful for studying obstructions for triangle decompositions, the matrix W is also connected to association schemes and the related Bose-Mesner algebra introduced in the next section.

2.4 Association schemes

In this section, we provide the background for proving the sufficient conditions for rational triangle graph decompositions. We start by introducing some notation.

2.4.1 Definitions and notation

The first definitions of an association scheme appear in the works of Bose and Nair [5] in 1939 and Bose and Shimamoto [6] in 1952, both in the context of statistical experimental designs. In 1959, Bose and Mesner [4] provided the algebraic setting for these objects, which was crucial in their further study.

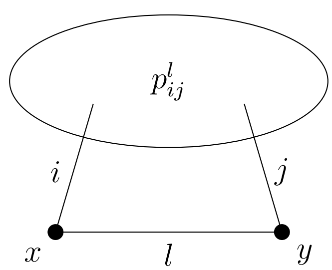
Given any two edges in a graph, they can interact in 3 different ways: they can coincide, intersect at one of the endpoints or be altogether disjoint. This idea of various interactions between k-subsets of a v-set is the essence behind the definition of an association scheme.

Definition 2.28 ([42])**.** A *d-class association scheme* on a set X of points is a set of $d+1$ non-empty symmetric binary relations $R_0, \dots R_d$ that partition $X \times X$ such that $R_0 = \{(x, x) : x \in X\}$ is the identity relation and the following holds:

> there exist nonnegative integers p^l_{ij} $(0 \leq i, j, l \leq d)$ such that given any $(x, y) \in R_l$, there are exactly p^l_{ij} elements $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$.

We say that x and y are i^{th} *associates* if $(x, y) \in R_i$. The integers p^l_{ij} are called *intersection numbers* (also *structure constants* or *parameters*) of the scheme. Figure 3.1 is a pictorial representation of intersection numbers with labels on each edge representing the relationship between the two points incident to that edge.

Figure 2.9: Intersection numbers.



Example 2.29 ([42])**.** A regular graph is *strongly regular* if every two adjacent vertices have λ common neighbours and every two non-adjacent vertices have μ common neighbours for some $\lambda, \mu \in \mathbb{Z}^+$. Any strongly regular graph G gives rise to a 2-class association scheme, where two distinct vertices are 1^{st} associates if they are adjacent in G and 2^{nd} associates if they are not. Here $p^1_{11} = \lambda$ and $p^2_{11} = \mu$.

Example 2.30. The *Hamming scheme*, denoted $H(k,q)$, is defined as follows: the points of $H(k,q)$ are the q^k ordered k-tuples over an alphabet of size q. Two k-tuples x and y are i^{th} associates if they disagree in exactly i coordinates.

Example 2.31. The *Johnson scheme*, denoted $J(v,k)$, is defined as follows: the points of $J(v,k)$ are the $\binom{v}{k}$ k-subsets of a v-set. Two k-subsets X and Y are i^{th} associates if they disagree in exactly i elements, i.e. if $|X \cap Y| = k - i$.

2.4.2 Bose-Mesner algebra

For an alternative view of association schemes, consider adjacency matrices of the relations of Definition 2.28. Then we immediately get that a symmetric association scheme on n points with d classes is a set $\mathcal{A} = \{A_0, \ldots, A_d\}$ of $(0,1)$-matrices such that:

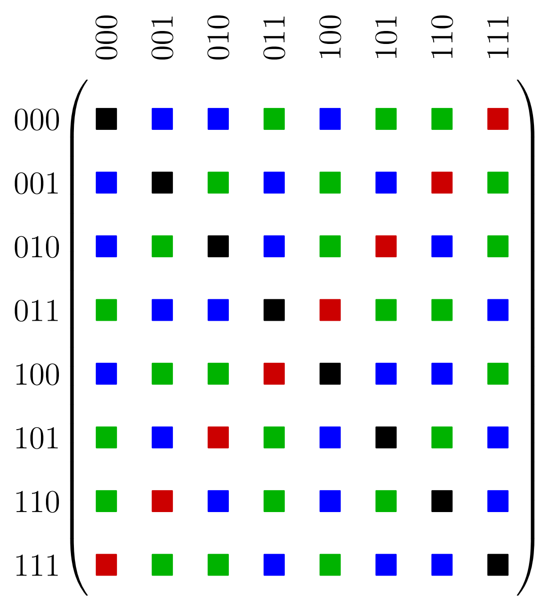
1. $A_0 = I$,
2. $\sum_{i=0}^{d} A_i = J$, where J is the all-ones $n \times n$ matrix,
3. $A_i^\top = A_i$ for each $i = 0, \ldots d$,
4. $A_i A_j = \sum_{l=0}^{d} p_{ij}^l A_l$, $i, j = 0, \ldots d$.

The A_i are linearly independent, since each of them has at least one 1 and in any position only one of the A_i is non-zero. Moreover, the A_i span a $(d+1)$-dimensional commutative algebra over $\mathbb{R}$, called the *Bose-Mesner algebra* as it was first introduced by Bose and Mesner in [4].

Example 2.32. The smallest example of an association scheme has just one class. Then $A_0 = I$ and $A_1 = J - I$. Note A_1^2 is a linear combination of I and J.

Example 2.33. To better understand the structure of the A_i, consider the 3-class Hamming scheme $H(3,2)$. The all-ones matrix below identifies the i^{th} associates as follows: black entry corresponds to an entry of 1 in A_0, blue to A_1, green to A_2 and red to A_3.

$J =$



Note also the following relations between the A_i of this Hamming scheme:

$$A_3^2 = A_0,$$
$$A_1 A_3 = A_2,$$
$$A_1 A_2 = 2A_1 + 3A_3.$$

An extension of the spectral theorem gives that a commutative algebra of real symmetric matrices has a basis of orthogonal idempotents with respect to regular matrix multiplication [42], therefore providing us with another basis for Bose-Mesner algebra.

Theorem 2.34 ([24])**.** *The algebra $\mathbb{R}[\mathcal{A}]$ has a basis $E_0, \ldots, E_d$ of orthogonal idempotents such that*

1. $E_iE_j = \begin{cases} E_i & \textit{if } i = j, \\ 0 & \textit{if } i \neq j, \end{cases}$ *and* $E_0 = \frac{1}{n}J$,

2. $\sum_{i=0}^{d} E_i = I$,

3. E_i *is symmetric for each* $i = 0, \ldots d$.

Geometrically, $\mathbb{R}^V$ can be written as a direct sum of mutually orthogonal eigenspaces and orthogonal idempotents are projections of $\mathbb{R}^V$ onto these eigenspaces.

Lemma 2.35. *Given the algebra* $\mathbb{R}[\mathcal{A}]$ *with a basis* $E_0, \ldots, E_d$ *of orthogonal idempotents, each column of* E_i *is an eigenvector of each matrix in* $\mathbb{R}[\mathcal{A}]$.

Proof. Let A be a matrix in $\mathbb{R}[\mathcal{A}]$. It can therefore be written as a linear combination of the E_i, so that

$$A = \sum_{i=0}^{d} a_i E_i$$

for some constants $a_i \in \mathbb{R}$. Then multiplying both sides of the above equation by some fixed E_i and invoking property 1 of Theorem 2.34, we get

$$AE_i = a_i E_i,$$

so every column of E_i is an eigenvector of A. □

In light of the above lemma, we have the following definition.

Definition 2.36. For $j = 0, \ldots d$, the basis change coefficients p_{ij} defined by $A_i = \sum_{j=0}^{d} p_{ij} E_j$ are called the *eigenvalues of the scheme*. Similarly, scalars q_{ij} defined by $E_i = \frac{1}{n} \sum_{j=0}^{d} q_{ij} A_j$ are called the *dual eigenvalues of the scheme*.

While the notation in the definition above is not ideal (p_{ij} of eigenvalues versus p_{ij}^l of intersection numbers), it has been historically used and we shall use it throughout.

Since $A_iE_j = p_{ij}E_j$, $i, j = 0, \ldots d$, so p_{ij} is an eigenvalue of A_i with multiplicity $m_j = \text{rank}(E_j)$. We define the eigenmatrix and the dual eigenmatrix of $\mathbb{R}[\mathcal{A}]$ by $P[i, j] = p_{ij}$ and $Q[i, j] = q_{ij}$, respectively.

Proposition 2.37. *We have $PQ = nI$.*

Proof. By Definition 2.36, we have:

$$\begin{pmatrix} A_0 \\ \vdots \\ A_d \end{pmatrix} = P \cdot \begin{pmatrix} E_0 \\ \vdots \\ E_d \end{pmatrix} = P \cdot \frac{1}{n} Q \begin{pmatrix} A_0 \\ \vdots \\ A_d \end{pmatrix}.$$

Therefore, $PQ = nI$. □

2.4.3 Johnson scheme

From now on, we will work with the Johnson association scheme with 2 classes. We shall build it from the line graph of K_v, so the vertex set consists of all 2-subsets of a v-set, where two vertices x and y are i^{th} associates if $|x \cap y| = 2 - i$ for $i = 0, 1, 2$. Then the A_i are $(0, 1)$-matrices indexed by $n = \binom{v}{2}$ pairs constructed as follows: any given row of A_i, $i = 0, 1, 2$, indexed by the edge $\{x, y\}$, records (with an entry of 1) the edges that intersect $\{x, y\}$ in 2, 1 and 0 points, respectively.

Example 2.38. Consider the complete graph on 4 vertices, labelled a, b, c, d, and construct the A_i from its line graph. As before, we have a partition of the all-ones matrix according to i^{th} associates: an entry of i corresponds to an entry of 1 in the matrix A_i, $i = 0, 1, 2$:

$$J = \begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array} \begin{array}{c} \begin{array}{cccccc} \rotatebox{90}{ab} & \rotatebox{90}{ac} & \rotatebox{90}{ad} & \rotatebox{90}{bc} & \rotatebox{90}{bd} & \rotatebox{90}{cd} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Proposition 2.39. *We can obtain all of the possible products A_iA_j for $i, j = 0, 1, 2$:*

$$\begin{aligned} A_1^2 = {} & 2(v-2)A_0 & +(v-2)A_1 & & +4A_2, \\ A_1A_2 = A_2A_1 = {} & & +(v-3)A_1 & & +2(v-4)A_2, \\ A_2^2 = {} & \binom{v-2}{2}A_0 & +\binom{v-3}{2}A_1 & & +\binom{v-4}{2}A_2. \end{aligned}$$

Proof. Given two matrices A_i and A_j, the coefficient of A_i in the expansion of the product A_iA_j is the number of i^{th} associates of one edge and j^{th} associates of another (not necessarily distinct) edge, simultaneously.

Consider A_1^2. In the product A_1A_1, consider the row in the first matrix indexed by the edge $\{x, y\}$ and the column in the second matrix indexed by the edge $\{e, f\}$. Recall that A_1 records edges that are incident with a given edge in exactly one vertex. Therefore, A_1^2 records the edges that are 1^{st} associates of each of $\{x, y\}$ and $\{e, f\}$, i.e. edges that touch each of $\{x, y\}$ and $\{e, f\}$ at exactly one vertex. We now have three cases corresponding to the coefficient of the A_i in the expansion of A_1^2 as recorded in Table 2.1 on the following page.

Therefore, $A_1^2 = 2(v-2)A_0 + (v-2)A_1 + 4A_2$.

Similarly, A_1A_2 records the edges that are 1^{st} associates of $\{x, y\}$ and 2^{nd} associates of $\{e, f\}$, i.e. edges that touch $\{x, y\}$ at exactly one vertex and are disjoint from

Table 2.1: Coefficients of A_i^2 in Johnson scheme.

Edges interaction	**Coefficient in expansion**
same edge	$2(v-2)$
incident edges	$v-3+1$
disjoint edges	4

$\{e, f\}$. Finally, A_2^2 records the edges that are disjoint from each of $\{x, y\}$ and $\{e, f\}$. Considering the three cases for A_1A_2 and A_2^2 produces the counts as stated above.

□

Proposition 2.40. *The eigenmatrix of* $\mathbb{R}[\mathcal{A}]$ *is given by* $P[i, j] = p_{ij}$ *with*

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 2(v-2) & v-4 & -2 \\ \binom{v-2}{2} & -v+3 & 1 \end{pmatrix}.$$

Proof. Recall that $A_i = \sum_{j=0}^{d} p_{ij}E_j = p_{i0}E_0 + p_{i1}E_1 + p_{i2}E_2$. We index the rows and the columns of matrix P by $0, 1, 2$.

Since the E_i are orthogonal idempotents and $E_0 = \frac{1}{\binom{v}{2}}J$, we have that $A_iJ = p_{i0}J$. Therefore, the p_{i0} are the common row sums of the A_i, given by Proposition 4.5.

Since $A_0 = I$ and $E_0 + E_1 + E_2 = I$, we get that $p_{i0} = 1$ for $i = 0, 1, 2$.

The derivation of the rest of the entries of P is not straightforward and we will rely on the use of the following combinatorial identity [25]:

$$p_{ij} = \sum_{r=0}^{i} (-1)^{i-r} \binom{k-r}{i-r} \binom{v-k+r-j}{r} \binom{k-j}{r}.$$

In our case, $k = 2$ and the corresponding values are computed in Appendix A.1.

□

With the necessary background in place, we can proceed to consider the necessary and sufficient conditions for rational triangle decompositions.

Chapter 3

Cone conditions and facet structure

In this chapter, we study necessary conditions for rational triangle decompositions: a graph G is rationally triangle decomposable if its characteristic vector $\mathbf{1}_G$ lies in the triangulation cone Tri_v. Since by Farkas' Lemma it is enough to check facet normals of Tri_v in order to decide whether or not $\mathbf{1}_G \in \mathrm{Tri}_v$, we concentrate our efforts on studying these objects. We classify several infinite families of facet normals of Tri_v as well as fully characterize and enumerate all facet normals for $v < 9$. We also run facet normal tests on all 8-vertex graphs with minimum degree four and consider some observations arising from the computational data. We produce several interesting examples of graphs that fail to be triangle decomposable but are not rejected by certain large families of facet normals. Finally, the study of Tri_v, and $W_{2,3}$ in particular, leads us into proving the existence of certain three-fold triple systems.

In this Chapter, we employ a graphical approach by representing facet normals as graphs on v vertices with weights attached to them. As such, for $t = 1$ we establish a correspondence between v-vertex weighted graphs and vectors in $\mathbb{R}^V$; for $t = 2$, the correspondence is between edge-weighted v-vertex graphs and vectors in $\mathbb{R}^{\binom{V}{2}}$. So a coordinate of the vector corresponds to a vertex (edge) in the graph and its value corresponds to the weight of that vertex (edge). Then, in a graphical setting, to *span a vertex* (*edge*) using a set $S \subset \binom{V}{k}$ means to be able to obtain the characteristic vector of that vertex (edge) as a linear combination of characteristic vectors of the k-subsets in S. We always consider facet normals up to isomorphism, that is up to scaling and permutations, by taking them to be in *standard form* in which all entries are integers and the greatest common divisor of all the entries is equal to 1.

3.1 Characterization of the facet normals of $\mathcal{C}_{W_{1,k}(v)}$

Before moving on to examining facet normals of $\mathrm{Tri}_v = \mathcal{C}_{W_{2,3}(v)}$, we characterize the facet normals of $\mathcal{C}_{W_{1,k}(v)}$. Recall that the matrix $W_{1,k}(v)$ records interactions between 1-subsets and k-subsets of a v-set V. As such, a row vector $\mathbf{y} \in \mathbb{R}^V$ supports $\mathcal{C}_{W_{1,k}(v)}$ if $\mathbf{y}W_{1,k}(v) \geq 0$ entry-wise, that is if every k-subset of coordinates of $\mathbf{y}$ has non-negative total weight. Furthermore, $\mathbf{y}$ is a facet normal if the columns of $W_{1,k}(v)$ orthogonal to $\mathbf{y}$ span a space of codimension 1 in $\mathbb{R}^V$. An interesting fact to notice here is that $W_{1,k}W_{1,k}^\top = \binom{v-1}{k-1}I + \binom{v-2}{k-2}(J - I)$, so it is always full rank.

Lemma 3.1. *Given a $(k+1)$-vertex graph with weights assigned to each vertex, every vertex in this graph is spanned by the $(k+1)$ k-subsets in it.*

Proof. To span any given vertex, take all k-subsets incident with it with coefficient $\frac{1}{k}$ (there are k of them) and the one k-subset disjoint from it with coefficient $-\frac{k-1}{k}$. □

Proposition 3.2. *The vectors $(1, 0, 0, \ldots 0)$ and $(-(k-1), 1, 1, \ldots 1)$ in $\mathbb{R}^V$ are facet normals of the $\mathcal{C}_{W_{1,k}(v)}$ for $k \geq 2$, $v \geq 5$ and $v > k + 1$. We refer to these facet normals as trivial facet normal and base facet normal, respectively.*

Proof. By Lemma 3.1, the vector $(1, 0, 0, \ldots 0)$ is a facet normal as we can span every coordinate except for the one with weight 1.

To prove that $(-(k-1), 1, 1, \ldots 1)$ is a facet normal, let $\mathcal{K}$ denote the set of all zero-weight k-subsets in this vector's graphical representation. We will prove that all zero-weight k-subsets together with the addition of another k-subset span $\mathbb{R}^V$. Without loss of generality, let $S = \{i_1, i_2, \ldots, i_{k-2}, j_1, j_2\} \subset V$, $1 \notin S$, be the additional k-subset and set $\mathcal{K}' = \mathcal{K} \cup \{S\}$. By Lemma 3.1, the vertices in $S \cup \{1\}$ are all spanned by $\mathcal{K}'$ and it suffices to show that any vertex outside of $S \cup \{1\}$, say x, can also be written as a linear combination of the sets in $\mathcal{K}'$. This is done using the coefficients as in Table 3.1.

Table 3.1: Coefficients of k-subsets used to span the vertex x.

k-subset	**Coefficient**
$\{1, i_1, i_2, \ldots, i_{k-2}, j_1\}$	$-\frac{k-1}{k}$
$\{1, i_1, i_2, \ldots, i_{k-2}, j_2\}$	$-\frac{k-1}{k}$
$\{1, i_1, i_2, \ldots, i_{k-2}, x\}$	1
$\{1, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_{k-2}, j_1, j_2\}$	$\frac{1}{k}$
$\{i_1, i_2, \ldots, i_{k-2}, j_1, j_2\}$	$\frac{1}{k}$

While most of the k-subsets above appear only once, note that there are $k-2$ k-subsets of the form $\{1, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_{k-2}, j_1, j_2\}$. After checking all the points, we can see that point x will receive a contribution of 1 and the rest of the points will receive the total contribution of 0. $\square$

Observation 3.3. In a graphical representation of a facet normal, every vertex belongs to a zero-weight k-subset; alternatively, every coordinate of a facet normal belongs to a zero-weight k-subset of coordinates. Otherwise, if a coordinate belongs to only positive k-subsets, the corresponding vector can be decomposed into a (scalar multiple of) the trivial facet and another supporting vector, meaning it is not a facet normal.

Proposition 3.4. *The only facet normals, up to isomorphism, of the cone* $\mathcal{C}_{W_{1,k}(v)}$ ($k \geq 2$*,* $v \geq 5$ *and* $v > k+1$) *are the trivial facet normal* $(1, 0, 0, \ldots, 0)$ *and the base facet normal* $(-(k-1), 1, 1, \ldots, 1)$*.*

Proof. Proposition 3.2 guarantees that $(1, 0, 0, \ldots, 0)$ and $(-(k-1), 1, 1, \ldots, 1)$ are indeed facet normals of $\mathcal{C}_{W_{1,k}(v)}$. Next, we show that there are no other facet normals.

A facet normal with no negative coordinates can only be the trivial facet normal since it decomposes into a positive combination of them. Now consider a facet normal with exactly one negative coordinate. By Observation 3.3, all positive-valued coordinates have to be of the same value. Therefore, any facet normal with exactly one negative coordinate is a scalar multiple of the base facet normal.

Suppose now that a vector $\mathbf{y}$ is a facet normal with at least two negative coordinates. By Observation 3.3, the weights of all negative-valued coordinate must be equal (otherwise, there exists a negatively weighted k-subset) and, similarly, all the positive-valued coordinates must have the same weight. Therefore, $\mathbf{y}$ must be of the form $(\underbrace{-a, \ldots, -a}_{r}, 1, \ldots, 1)$ (up to isomorphism), where $r < k$ and $a \geq \frac{k-r}{r}$. With some calculations, this vector can be written as non-negative linear combination of r base facet normals and trivial facet normals with coefficient $a - \frac{k-r}{r}$. It therefore cannot be a facet normal. □

All facet normals of $\mathcal{C}_{W_{1,k}(v)}$ are now characterized. It is easy to see that a facet normal of $\mathcal{C}_{W_{1,3}(v)}$ induces a supporting vector of $\mathcal{C}_{W_{2,3}(v)}$. However, the converse is not true. In fact, the structure of facet normals of $\mathcal{C}_{W_{2,3}(v)}$ (not even general k) is much more complex.

3.2 Structural properties of facet normals of Tri_v

We now return to the rational triangle decompositions and to the study of the facets of $\mathcal{C}_{W_{2,3}(v)}$, or simply Tri_v. Recall that $\mathbf{y} \in \mathbb{R}^{\binom{V}{2}}$ supports Tri_v on a facet if $\mathbf{y}W \geq \mathbf{0}$ and the set of all columns in $W_{2,3}(v)$ orthogonal to $\mathbf{y}$ spans the space of dimension $\binom{v}{2} - 1$; that is, the set of all zero-weight triangles spans the space of the edges of dimension $\binom{v}{2} - 1$. As such, facet normals can be thought of as critically non-spanning structures and, as in Section 3.1, we will employ the following technique for checking something is a facet normal: add in one additional triangle to the space of all zero-weight triangles and ensure that the resulting set of triangles spans $\mathbb{R}^{\binom{V}{2}}$.

Some properties of the facet normals are immediate:

1. Since zero-weight triangles have to span a space of codimension 1 in $\mathbb{R}^{\binom{V}{2}}$, for every facet normal $\mathbf{y}$, we have that $\mathbf{y}W$ vanishes on at least $\binom{v}{2} - 1$ coordinates. Equivalently, $\mathbf{y}$ induces at least $\binom{v}{2} - 1$ zero-weight triangles.

2. Since every facet normal is orthogonal to a space of dimension $\binom{v}{2} - 1$, any two facet normals that vanish on the same triangles are scalar multiples of each other.

3. In a facet normal, every pair of coordinates (except for possibly one) has to appear together in at least one zero-weight triangle, since otherwise zero-weight triangles have deficient span.

Moreover, from the definition and property 2, we also get an easy upper bound for the number of facet normals:

Lemma 3.5. *The number of facets of the cone* Tri_v *is at most* $\binom{\binom{v}{3}}{\binom{v}{2}-1}$.

Proof. Given any facet normal, by property 2 above, there is a unique set of $\binom{v}{2}-1$ columns that are orthogonal to it. At worst, every set of $\binom{v}{2}-1$ columns of $W_{2,3}(v)$ corresponds to a different facet normal. □

For more structural results, we shall investigate some particular facet normals closer in the next section.

3.2.1 General properties of facet normals

To start off with, consider minimal spanning configurations of pairs and zero-weight triangles.

Lemma 3.6. *Every pair in a 5-set is spanned by the* $\binom{5}{2}$ *triples in that set.*

Proof. The statement is equivalent to proving that the matrix $W_{2,3}(5)$ is full rank. We have $W_{2,3}(5)W_{2,3}(5)^\top = 3I + A$, where A is the adjacency matrix of the line graph of K_5. Since the eigenvalues of A are known to be $(-2)^5$, 1^4 and 6^1, [36], it follows that $W_{2,3}(5)$ has full rank. □

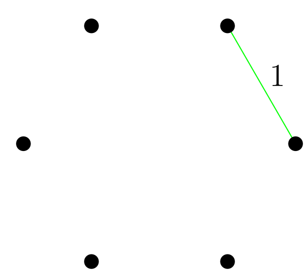
The lemma above can also be proven using graphical representation of facet normals: take a 5-vertex graph with vertices labelled 1 through 5 and consider the space spanned by the triangles in this graph. We would like to show that the characteristic vector of any edge can be obtained as a linear combination of characteristic vectors of triangles. This can be done by taking all the triangles through the desired edge and the one triangle disjoint from it with weight $2/6$, while taking the remaining 6 triangles with weight $-1/6$.

We will now consider some infinite families of vectors and prove that they are facet normals of Tri_v. Note that the first reasonable case is $v = 5$ since for smaller values of v there are simply not enough total triangles available to span the dimension of the needed size. Even for $v = 5$ there is only one possible isomorphism type of facet normal. In fact, we will prove that some structures give rise to whole families of facet normals starting at some small value of v.

Proposition 3.7. *For any $v \geq 6$, the $1 \times \binom{v}{2}$ vector $\mathbf{y} = (1, 0, \ldots, 0)$ is a facet normal of Tri_v, called the trivial facet normal.*

Proof. Graphically, we have the representation in Figure 3.1.

Figure 3.1: Trivial facet normal.



The vector $\mathbf{y} = (1, 0, \ldots, 0)$ is clearly a supporting vector of Tri_v as all of the triangles have non-negative total weight. Note that by repeated applications of Lemma 3.6 to any subset of 5 vertices, every edge but one (of weight one) is spanned by triangles of zero-weight. □

Roughly speaking, the following proposition shows how to extend a facet normal into an infinite family by "gluing" copies of it along a triangle.

Proposition 3.8. *Let $\mathbf{y}_0$ be a facet normal of Tri_v such that $\langle \mathbf{y}_0, \mathbf{1}_{\{1,2,3\}} \rangle > 0$. Let $i > v$ and suppose $\mathbf{y}$ supports Tri_i. Suppose further that every element of $\binom{[i]}{2}$ is contained in some v-set $S \supset \{1, 2, 3\}$ such that the restriction $\mathbf{y}|_S$ of $\mathbf{y}$ to $\mathbb{R}^{\binom{S}{2}}$ agrees with $\mathbf{y}_0$ (upon a relabelling of points fixing $\{1, 2, 3\}$). Then $\mathbf{y}$ is a facet of Tri_i.*

Proof. All that is needed to show is that $\mathbf{y}$ vanishes maximally on $\binom{V}{3}$. Here, it suffices to show that every pair in $\binom{[i]}{2}$ is a linear combination of zero-weight triangles with $\{1,2,3\}$ added in. More formally, we want to show that

$$\mathbf{1}_T \in \langle \mathbf{1}_\triangle : \mathbf{y}W(\triangle) = 0 \text{ or } \triangle = \{1,2,3\}\rangle$$

for every $T \in \binom{[i]}{2}$. By assumption, $T \subset S$ where $\mathbf{y}|_S$ is a copy of $\mathbf{y}_0$ containing $\{1,2,3\}$. Since this is a facet normal of lower dimension, it follows that $\mathbf{1}_T$ is indeed a non-negative linear combination as claimed. □

Let us call a family of facets which arise from 'copies of $\mathbf{y}_0$' in this way a *seeded family*. We shall say that $\mathbf{y}_0$ is the *seed* and $\{1,2,3\}$ is the *anchor* of the family. For example, the family of trivial facet normals is seeded for $v \geq 6$ with the seed being the trivial facet normal on 6 vertices. For more complex facet families, we will also have to specify the anchor to allow it to span all the edges in the configuration.

Observe that if a facet normal has no negative coordinates, it can have only one positive coordinate, since otherwise it can be decomposed into a combination of trivial facet normals. Therefore, the trivial facet normal is the only facet normal with no negative coordinates.

To construct infinite seeded families of facet normals, we now only need to consider their seeds. Note that for every case below, the seed is the smallest possible element of the corresponding family that is itself a facet normal. As such, different facet normal families emerge starting from different values of v.

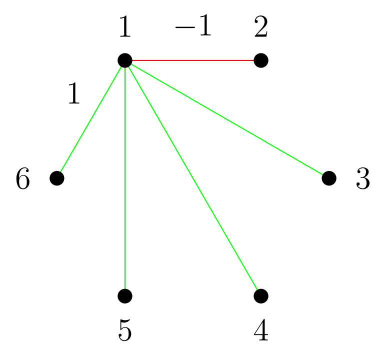
In what follows, for sets $X, Y \subset V = \{1, 2, \ldots, v\}$, let $\binom{X}{2}$ denote the set of all possible pairs of points in X and let $X \cdot Y$ denote the unordered version of the cartesian product, that is the set of pairs of points with exactly one point in each set. In a facet normal description, braces underneath specify the pairs of points receiving the corresponding weight. The ordering, unless specified otherwise, is lexicographical.

Proposition 3.9. *The* $1 \times \binom{v}{2}$ *vector* $\mathbf{y} = (\underbrace{-1,}_{\{1,2\}} \underbrace{1, 1, \ldots, 1,}_{\{1\}\cdot\{3,\ldots,v\}} 0, \ldots, 0)$ *is a facet normal of* Tri_v*,* $v \geq 6$*, called the star facet normal.*

Proof. The weighted star configuration (pictured below) is supporting and is minimal in a sense that it cannot be decomposed into supporting vectors of Tri_v.

By Proposition 3.8, it suffices to show that the seed of this seeded family, the star on 6 vertices, is indeed a facet normal. Without loss of generality, consider the labelling as in Figure 3.2.

Figure 3.2: Star on 6 vertices.

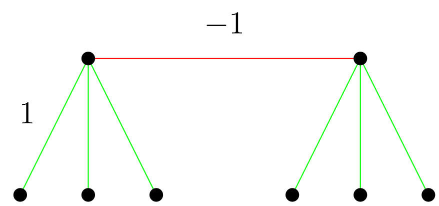


Note that the only non-zero-weight triangles here are of the form $\{1, i, j\}$, $i, j \neq 2$. Now, consider the space spanned by the zero-weight triangles with the addition of one more triangle, the anchor; without loss of generality, let the anchor be the triangle $\{1, 5, 6\}$. It is sufficient to show that the 15×15 $(0, 1)$-incidence matrix A of all the possible pairs versus all the available triples (that is, the triples with total weight zero together with the anchor) is invertible. This implies that a matrix system $A\mathbf{x} = \mathbf{b}$ has a solution for a characteristic vector $\mathbf{b}$ of any edge. In this case, the matrix can be easily constructed and its invertability is checked using a computer. $\square$

The technique of finding a seed and proving (via a computer-assisted method) that it is a facet normal will be employed in many proofs to follow.

A generalization of the star facet normal is a *binary star* vector, consisting of a negative edge of weight -1 and positive edges of weight 1 attached to each end of it as shown in Figure 3.3.

Figure 3.3: Binary star.



An (a, b)-binary star has a positive edges on one side and b positive edges on the other side of the negative edge.

Proposition 3.10. *Let* $\{A, B\}$ *be a partition of* $V \setminus \{1, 2\}$, $|A| = a, |B| = b$, $a, b \geq 3$. *Then the* $1 \times \binom{v}{2}$ (a, b)*-binary star vector* $\mathbf{y} = (\underbrace{-1,}_{\{1,2\}} \underbrace{1, \ldots, 1,}_{\{1\}\cdot A} \underbrace{1, \ldots, 1,}_{\{2\}\cdot B} 0, \ldots, 0)$ *is a facet normal of* Tri_v *for* $v \geq 8$.

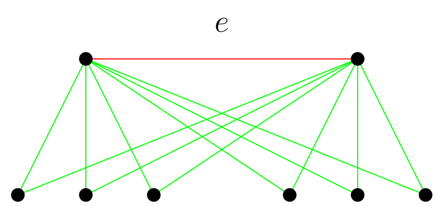
Proof. The binary star is clearly a supporting vector of Tri_v, so by Proposition 3.8 it suffices to show that the seed $(3, 3)$-binary star is a facet. This is accomplished by adding an anchor and employing the help of the computer to show that every edge in this configuration is spanned. □

The star and binary star facets together form a special family of facets.

Proposition 3.11. *The star and binary star facet normals are the only facet normals, up to isomorphism, with exactly one negative coordinate.*

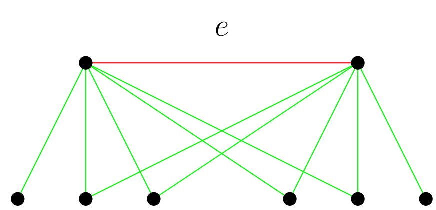
Proof. Towards contradiction, assume that there is different facet normal with exactly one negative coordinate. In graphical representation, it implies the existence of the negative edge, say e. Since every edge has to lie in a zero-weight triangle, there cannot be any positive edges disjoint from e. Furthermore, since all the triangles have to have non-negative weight and this graph cannot be a star or a binary star, e must lie in at least one triangle with two other positive edges (see Figure 3.4).

Figure 3.4: Edge e in triangles with 2 positive edges.



Moreover, we can assume there are no positive edges disjoint from e. There is at least one zero-weight edge incident with each of the endpoints of e: if not, subtract from this vector an appropriately scaled star facet normal rooted at that vertex and obtain another supporting vector. Therefore, the structure looks as in Figure 3.5.

Figure 3.5: Modified structure.

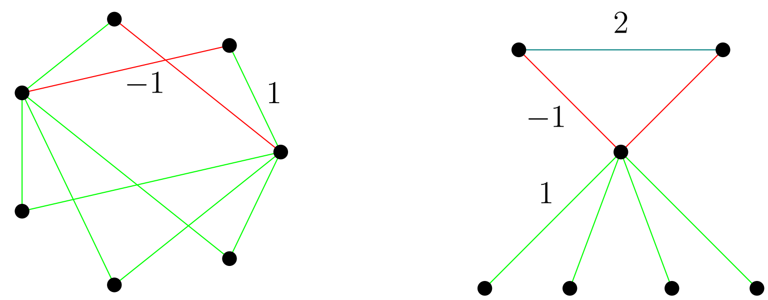


Repeatedly applying the process as described above to the remaining non-zero edges forming triangles with e, we obtain a (binary) star facet. □

The star and binary star facet families provide the smallest examples to illustrate that the structure of the *negative graph*, that is the subgraph formed by the negatively weighted edges of a vector, does not uniquely determine the facet normal. This can also be seen with other examples for larger values of v (for list of facets of Tri_8, please see Appendix B).

With two negative edges, we see two emerging structures for $v \geq 7$ that we call *negative fan* and *octopus* (see Figure 3.6).

Figure 3.6: Negative fan and octopus.



Using proofs similar to above, we obtain the following result:

Proposition 3.12. *Let* $S = (\{1,2\} \cdot \{5,6,\ldots,v\}) \cup \{\{1,2\},\{3,4\}\}$. *Then the* $1 \times \binom{v}{2}$ *vectors*

$$(\underbrace{-1,}_{\{1,3\}}\ \underbrace{-1,}_{\{3,4\}}\ \underbrace{1,\ldots,1,}_{S}\ 0,\ldots,0),$$

$$(\underbrace{-1,}_{\{1,2\}}\ \underbrace{-1,}_{\{1,3\}}\ \underbrace{2,}_{\{2,3\}}\ \underbrace{1,\ldots,1,}_{\{1\}\cdot\{4,\ldots,v\}}\ 0,\ldots,0),$$

are facet normals of Tri_v *for* $v \geq 7$.

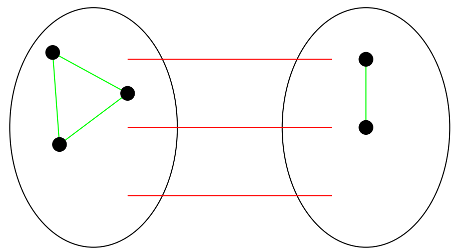
So far, we have considered sparse, in the sense of non-zero coordinates, facet normals. We now study dense facet normals with no zero-weight edges.

Proposition 3.13. *Let* $v \geq 5$ *and consider any partition of the vertex set into two parts* A *and* B, *each of size at least 2, with* $|A| = a$ *and* $|B| = b$. *Then the* $1 \times \binom{v}{2}$ *vector* $\mathbf{y} = (\underbrace{2,\ldots,2}_{\binom{A}{2}},\underbrace{-1,\ldots,-1}_{A\cdot B},\underbrace{2,\ldots,2}_{\binom{B}{2}})$ *is a facet normal of* Tri_v, *called the* (a,b)*-cut facet.*

Proof. The above vector $\mathbf{y}$ is a supporting vector of Tri_v as all of the triangles formed by it have weight 0 or 6.

In a graphical representation, since $v \geq 5$, one part contains at least 3 vertices and we have the following 5-vertex structure containing only one non-zero triangle (Figure 3.7).

Figure 3.7: Cut partition.



By adding this one non-zero triangle into the space of zero-weight triangles and applying Lemma 3.6, we see that $(3, 2)$-cut is indeed a facet normal. Now, use it as the seed and apply Proposition 3.8. □

Proposition 3.14. *For any $v \geq 6$, Table 3.2 shows the counts of zero-weight triangles in seeded facet normal families.*

Table 3.2: Counts of zero-weight triangles.

Facet normal	**Number of zero-weight triangles**
Trivial	$2(v-2) + \binom{v-2}{3}$
Star	$v - 2 + \binom{v-1}{3}$
(a, b)-binary star	$\binom{a+b}{3} + \binom{a}{2} + \binom{b}{2} + a + b$
Negative fan	$\binom{v-2}{3} + 2(v-2)$
Octopus	$\binom{v-3}{3} + 2\binom{v-3}{2} + 2v - 5$
(a, b)-cut	$a\binom{b}{2} + b\binom{a}{2}$

Proof. The counts below are based on graphical representation of the facet normals.

In the trivial facet normal, zero-weight triangles are $v - 2$ triangles through each endpoint of the positive edge as well as the $\binom{v-2}{3}$ triangles disjoint from the positive edge.

In the star facet normal, zero-weight triangles are $v - 2$ triangles through the negative edge and $\binom{v-1}{3}$ triangles not through the "root" vertex.

In the binary star facet normal, zero-weight triangles are $a + b$ triangles through the negative edge, $\binom{a}{2} + \binom{b}{2}$ triangles containing only one endpoint of the negative edge and $\binom{a+b}{3}$ triangles disjoint from the negative edge.

In the negative fan, let a and b denote the vertices that are incident to the $v - 3$ edges of weight one. Then there are $\binom{v-2}{3}$ zero-weight triangles avoiding a and b and $2(v - 2)$ containing a or b.

In the octopus, there is one zero-weight triangle containing both negative edges, $2(v - 3)$ containing exactly one negative edge, $2\binom{v-3}{2}$ zero-weight triangles incident with exactly one negative edge and $\binom{v-3}{2}$ zero-weight triangles disjoint from the negative edges.

In the cut facet normal, the only zero-weight triangles are the triangles crossing between the two parts.

□

Since any facet normal induces at least $\binom{v}{2} - 1$ zero-weight triangles, the above values provide necessary lower bounds for the existence of the seed facet normal and hence for the entire family of facet normals. For example, for $v = 5$, there is not enough zero-weight triangles in trivial or star facet normals to span the space of the necessary dimensions and the binary star is only a facet normal for $v \geq 8$. This count also proves that the $(2, 4)$-binary star is not a facet normal for $v = 8$.

We can use the above families and a built-in cone package in Sage to characterize the facet normals for $v = 5, 6, 7, 8$. Unfortunately, the package exhaustively enumerates all facet normals with no regard for isomorphisms, and, since the number of facet normals grows fast, it quickly reaches the limit of its computational efficiency. However, we can compute the automorphism groups of the known families of facet normals (through the lemmas above as well as by analyzing the output of the simplex algorithm searching for facet normals) and hence determine all isomorphism types of facet normals for small values of v as summarized in Tables 3.3 on the next page and Table 3.4 on page 51 as well as Appendix A.2. The number of facet normals is now Sequence A246427 in The On-Line Encyclopedia of Integer Sequences [37].

Recall from Section 2.3.1 that in met_v the points adjacent (along the extreme rays) to the point $\frac{2}{3}(1, \ldots, 1)$ correspond to the facet normals of Tri_v. This connection with the metric cone provides another algorithm for enumeration of facet normals of Tri_v and the numbers produced by Deza (see `http://www.cas.mcmaster.ca/~deza/metric.html`) confirm our counts for $v < 9$.

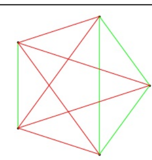
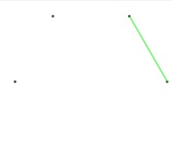
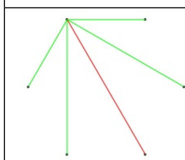
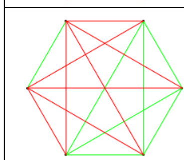
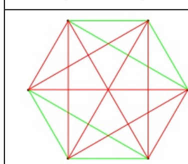
v	Number of facet normals	Isomorphism class	Class counts
5	10	 $(2, 3)$-cut	$\binom{5}{2} = 10$
6	70	 trivial	$\binom{6}{2} = 15$
		 star	$2\binom{6}{2} = 30$
		 $(2, 4)$-cut	$\binom{6}{2} = 15$
		 $(3, 3)$-cut	$\binom{6}{3}/2 = 10$

Table 3.3: Classification of facets of Tri_v for $v = 5, 6$.

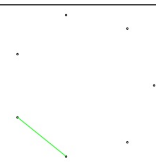
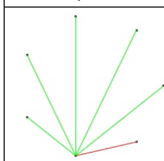
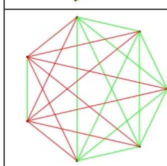
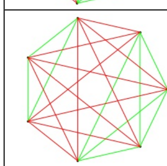
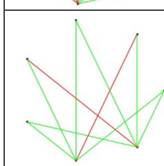
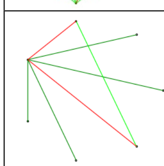
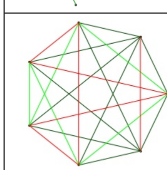
v	Number of facet normals	Isomorphism classes	Class counts
7	896	 trivial	$\binom{7}{2} = 21$
		 star	$2\binom{7}{2} = 42$
		 $(2,5)$-cut	$\binom{7}{2} = 21$
		 $(3,4)$-cut	$\binom{7}{3} = 35$
		 disjoint red edges	$2\binom{7}{2}\binom{5}{2} = 420$
		 red P_2	$3\binom{7}{3} = 105$
		 C_5+red edge	$12\binom{7}{2} = 252$

Table 3.4: Classification of facets of Tri_v for $v = 7$.

3.2.2 Facets of Tri_v and triangle decompositions

In this section, we study facet normals of Tri_v as obstructions to rational triangle decomposability of a fixed graph. We are interested in understanding the properties that a fixed graph G must possess in order to not be rationally triangle decomposable. Since the set of all facets of Tri_v represents all needed tests a graph of v vertices must pass to be rationally triangle decomposable, we examine particular structures rejected or not rejected by various facet normals. We shall say that a graph *passes a facet* $\mathbf{y}$ if, for all permutations of that facet's normal, its inner product with the graph's characteristic vector is non-negative.

Trivial facet normals do not reject any graphs. Star facet normals represent a better test as they reject graphs with pendant vertices, that is vertices of degree 1. As it is clear from the motivating example of Section 2.2, a graph passes a cut test if it has at most twice as many "cross-over" edges as "inside" ones. Recall the Nash-Williams' density conjecture that minimum degree of $3/4v$ (and hence at least $3v^2/8$ edges) is enough for triangle decomposition. The following lemma highlights that this edge density bound is also the bound at which the cut facets no longer represent an obstruction to rational triangle decompositions.

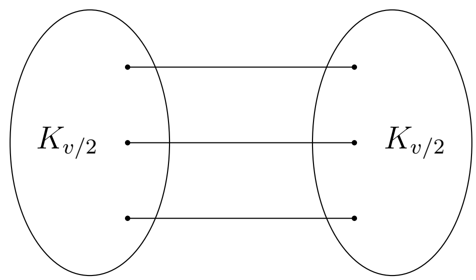
Proposition 3.15. *Any graph with more than* $3v^2/8$ *edges passes all the cut facet normals.*

Proof. Given a graph on v vertices, the cut facet normal with the most negative coordinates is $(\lfloor \frac{v}{2} \rfloor, \lceil \frac{v}{2} \rceil)$-cut facet normal. Take the inner product of this cut facet normal and the graph's characteristic vector. There are at most $v^2/4$ edges that will receive weight of -1, leaving $v^2/8$ edges to receive weight 2, resulting in a non-negative total weight. □

Using Sage, we run facet normal tests on all graphs with up to 8 vertices (see

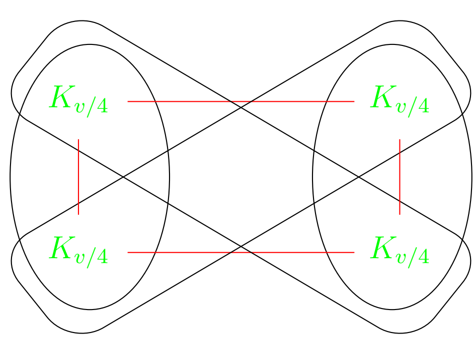
Appendix A.2) and every graph that fails to be rationally triangle decomposable fails the $(\lfloor \frac{v}{2} \rfloor, \lceil \frac{v}{2} \rceil)$-cut facet normal (among others). It is therefore tempting to believe that it is enough to pass cut facets to guarantee rational triangle decomposition. However, if we drop the density requirement of Proposition 3.15, then we can construct examples of graphs that pass all the cut tests but are not rational triangle decomposable. Consider $G = 2K_{v/2} + M$, where M is a matching and v is even (Figure 3.8).

Figure 3.8: Graph $G = 2K_{v/2} + M$.



Clearly, G is not triangle decomposable, rationally or otherwise, since the edges of the matching belong to no triangles. This graph is regular of degree $v/2$, has $v^2/4$ edges and is not rejected by any cut facet. To see the latter, note that applying the $(\lfloor \frac{v}{2} \rfloor, \lceil \frac{v}{2} \rceil)$-cut facet diagonally across the matching produces the most crossing (i.e. negative) edges as shown in Figure 3.9.

Figure 3.9: Cut facet applied to $G = 2K_{v/2} + M$.

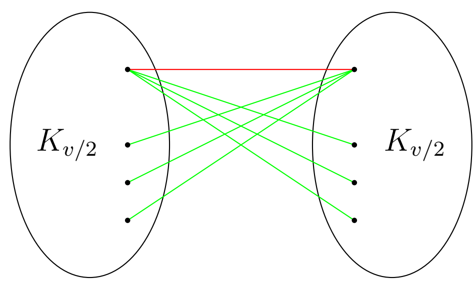


The total weight (that is the inner product of the facet with the characteristic vector of the graph) is twice the number of inside edges minus the number of edges crossing between parts. In this case, it amounts to

$$2 \cdot 4\binom{v/4}{2} - 1 \cdot \left(2 \cdot \left(\frac{v}{4}\right)^2 + 2 \cdot \frac{v}{4}\right) = \frac{v^2 - 12v}{8},$$

which is non-negative for all $v \geq 12$. This graph is rejected not by cut facet normals, but by a binary star applied as shown in Figure 3.10.

Figure 3.10: Binary star applied to $G = 2K_{v/2} + M$.



The total weight here is -1 and therefore G is rejected by the binary star as having a (rational) triangle decomposition.

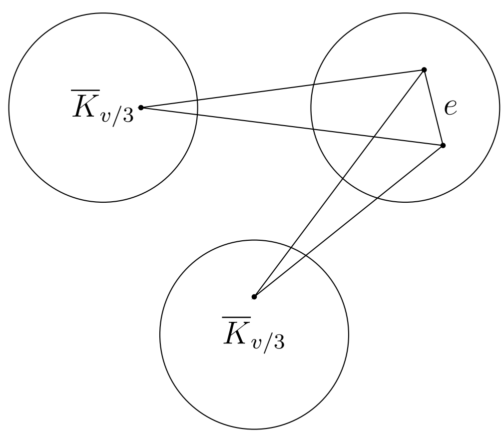
Recall that in the rational decomposition case, the divisibility conditions necessary for integral decomposition are no longer required; however, in order to be rationally triangle decomposable, the given graph must have K_3 as a subgraph (if fact, every edge of G must belong to a copy of K_3). Note here that if a graph G on v vertices is triangle-free, then $\delta(G) < \frac{v}{2}$: simply consider two adjacent vertices in G and require their neighbourhoods to be disjoint. Moreover, by Turán's Theorem [42] we have that the maximum number of edges in a triangle-free graph is $\lfloor \frac{v^2}{4} \rfloor$. Of course we require not just one triangle, but many of them, so these conditions are weak, but they do justify our search for good examples in specific graph families.

Interestingly, rational triangle decomposability is not preserved under edge addition. In the integral case, this is proven in [39] with an addition of a 6-cycle (which preserves K_3-divisibility); in the rational case, addition of a single edge is enough.

Proposition 3.16. *There exists an arbitrary large graph G with minimum degree at least $\frac{2}{3}|E(G)|$ such that $G + e$ is not rationally triangle decomposable for some additional edge e.*

Proof. Let G be a complete tripartite graph on v vertices. Clearly, it is regular of degree $\frac{2}{3}v$ and it is rationally triangle decomposable. Now, add an edge e to one of the parts. Towards contradiction, assume that $G+e$ has a rational triangle decomposition. In this decomposition, edge e has to use a combination of the two types of triangles illustrated in Figure 3.11.

Figure 3.11: Triangle decomposition of the complete tripartite graph.



Considering the graph $G + e$ without the contribution of these triangles leaves us with a misbalanced tripartite graph. Every triangle in this graph uses three edges in between the parts; however, there is no longer the same number of edges in between each pair of parts. Therefore, $G + e$ cannot be rationally decomposed into triangles.

□

The graph $G+e$ constructed in the proof above, while not rationally triangle decomposable, passes all the cut facets. The cut facet normal with the most crossing edges has one $\overline{K}_{v/3}$ as one part and the two other $\overline{K}_{v/3}$ as the other part. Then the total inherited weight is

$$2 \cdot \left(\left(\frac{v}{3} \right)^2 + 1 \right) - 1 \cdot \left(2 \left(\frac{v}{3} \right)^2 \right) = 2.$$

In fact, this graph passes all infinite families of facet normals identified in the previous section (trivial, star, binary star, octopus and negative fan). Moreover, it is not hard to persuade oneself that the facet normal that rejects this graph will have to be dense and hence comes from one of yet to be studied complex families.

In view of Farkas' Lemma (Theorem 2.22), we know that every hyperplane separates some vectors from being in the triangulation cone, which means that every facet does reject some vectors that other facets do not reject. However, this need not be the case here when the only vectors we consider are $(0,1)$-characteristic vectors of graphs. In other words, there might be a cone C wrapping Tri_v so closely that if a graph passes facets of C, then it passed facets of Tri_v and hence is rationally triangle decomposable.

Motivated by the above example and Proposition 3.15, we have a conjecture that for all sufficiently dense graphs the triangular cut cone (that is, the cone generated by all the cut facets) can replace the tests of the triangulation cone.

Conjecture 3.17. *Suppose G is a graph on v vertices with $\delta(G) > \frac{2}{3}v$. Then G admits a rational triangle decomposition if and only if G passes all cut facets.*

3.3 Three-fold triple systems of full rank

In the sections before, we are interested in the dimension of the space spanned by certain columns of $W_{2,3}(v)$. Here, we consider the inclusion matrices of designs (where columns are indexed only by blocks of the design rather than all possible k-subsets) and prove the existence of designs with full rank inclusion matrices. The contents of this section appear in [22].

Definition 3.18 ([20])**.** Let $\mathcal{D} = (V, \mathcal{B})$ be an incidence structure with points $V = \{x_1, \ldots, x_v\}$ and blocks $\mathcal{B} = \{B_1, \ldots, B_b\}$. An *incidence matrix* N for $\mathcal{D}$ is a $v \times b$ matrix defined by $N(i,j) = 1$ if x_i is incident with B_j and $N(i,j) = 0$ otherwise. A *higher incidence matrix* N_t for $\mathcal{D}$ is a $\binom{v}{t} \times b$ matrix with $N_t(i,j) = 1$ if and only if the i^{th} t-subset is contained in the j^{th} k-subset. If $\mathcal{D}$ consists of all k-subsets of V, then $N_t = W_{t,k}(v)$.

Recall the definition of a design (Definition 2.4). Here, we shall work with 2-$(v, 3, 3)$ designs, called *three-fold triple systems* and abbreviated by $TS_3(v)$. First of all, we note the following:

Lemma 3.19. *For any triple system $TS_3(v, \lambda)$, we have*

$$r = \frac{\lambda(v-1)}{2} \ \textit{and} \ \ b = \frac{\lambda}{3}\binom{v}{2},$$

where r is the number of times any given point occurs within a block (replication number) and b is the number of blocks.

Proof. Use Lemma 2.6 with $k = 3$. □

A classical result in design theory, Fisher's inequality, states that $v \leq b$ for any design with $k < v$. Therefore, the $v \times b$ incidence matrix N of a design is often of full rank. On the other hand, by the above lemma, any three-fold triple system has a property that its higher incidence matrix N_2 is square since the number of blocks of the design equals the number of 2-subsets on v points. Therefore, the problem of determining when N_2 is full rank is more interesting.

We can rule out certain types of designs that fail to have a full rank N_2 for any v. For example, any design that has repeated blocks also has repeated columns in N_2 and therefore its N_2 does not have full rank. Furthermore, full rank N_2 designs have to be *trade-free*, where a *trade* is defined as a pair of subsets of blocks that cover the same 2-subsets of points, since the sum of columns corresponding to those subsets will be the same. Designs containing trades include the so-called *resolvable designs* with their parallel classes (such as the affine plane of order 3 in Example 2.5), which do not have full rank N_2.

Let us now consider the rank of N_2 matrices for three-fold triple systems, starting with the smallest case of $v = 5$.

Lemma 3.20. *There exists only one* $TS_3(5)$ *(up to an isomorphism) and it has full rank* N_2*.*

Proof. By Lemma 3.19, $TS_3(5)$ has 10 blocks and hence contains all possible triples on 5 points. So in this case, $N_2 = W_{2,3}(5)$ and it has full rank. □

The following example illustrates how using this small three-fold triple system together with a larger design produces a full rank square N_2 design.

Example 3.21. Start with a 2-$(v, 5, 1)$ design (they exist for all $v \equiv 1, 5 \pmod{20}$ [2]) and replace all blocks with $TS_3(5)$; that is, replace each block with all possible triples on those 5 points. The resulting larger structure is a $TS_3(v)$: every pair in the original 2-$(v, 5, 1)$ design occurs together in exactly once block and it occurs together in exactly 3 blocks of the $TS_3(5)$. As for its N_2, it is a block-diagonal matrix with

$W_{2,3}(5)$ on the diagonal and zeros everywhere else:

$$N_2 = \begin{pmatrix} W_{2,3} & & & \\ & W_{2,3} & & \\ & & \ddots & \\ & & & W_{2,3} \end{pmatrix}$$

Since $W_{2,3}(5)$ is a full rank square matrix, the resulting N_2 is also full rank and square.

For $v = 7$ and $v = 9$, there are more non-isomorphic three-fold triple systems. The following lemma is proved by using computer software to check the ranks of the N_2 of the non-isomorphic $TS_3(7)$ and the non-isomorphic $TS_3(9)$.

Lemma 3.22. *There exists a unique $TS_3(7)$ and at least one $TS_3(9)$ with a full rank square N_2.*

Proof. Of the 10 non-isomorphic $TS_3(7)$ designs, there is exactly one $TS_3(7)$ with full rank N_2:

$$\{\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,2,3\},\{0,2,5\},\{0,3,6\},\{0,4,5\},$$
$$\{0,4,6\},\{0,5,6\},\{1,2,4\},\{1,2,6\},\{1,3,5\},\{1,3,6\},\{1,4,5\},$$
$$\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}.$$

The following is one of exactly 27 non-isomorphic $TS_3(9)$ with full rank N_2:

$$\{\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,2,3\},\{0,2,5\},\{0,3,6\},\{0,4,6\},\{0,4,7\},\{0,5,7\},$$
$$\{0,5,8\},\{0,6,8\},\{0,7,8\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,3,8\},\{1,4,7\},\{1,5,6\},$$
$$\{1,5,8\},\{1,6,7\},\{1,7,8\},\{2,3,4\},\{2,3,7\},\{2,4,8\},\{2,5,6\},\{2,6,7\},\{2,6,8\},$$
$$\{2,7,8\},\{3,4,5\},\{3,4,8\},\{3,5,7\},\{3,5,8\},\{3,6,7\},\{4,5,6\},\{4,5,7\},\{4,6,8\}\}.$$

□

Now we can use the small three-fold triple systems as ingredients to create larger full rank square N_2 designs. However, we need the designs to put them into, so we require some background and existence results from classical design theory.

Definition 3.23. Let $K \subset \mathbb{N} \setminus \{0, 1\}$ be a set of block sizes and let λ be a positive integer. A *pairwise balanced design* $PBD(v, K)$ is a set V of v elements together with a collection $\mathcal{B}$ of k-subsets of V, where $k \in K$, such that every 2-subset of V occurs in exactly one block of $\mathcal{B}$.

This is one of the generalizations of a 2-design as we are now allowed blocks of various pre-specified sizes. The necessary conditions for the existence of a $PBD(v, K)$ are, not surprisingly, the generalization of the necessary conditions for the existence of a t-design as in equations (2.1) and (2.2).

Lemma 3.24 ([48])**.** *The necessary conditions for the existence of a* $PBD(v, K)$ *are*

$$v - 1 = 0 \pmod{\alpha(K)},$$
$$v(v - 1) = 0 \pmod{\beta(K)},$$

where $\alpha(K) = \gcd\{k - 1 : k \in K\}$, $\beta(K) = \gcd\{k(k - 1) : k \in K\}$.

In [49], Wilson proves that the above necessary conditions are asymptotically sufficient for the existence of a $PBD(v, K)$. This can further be restated in terms of the PBD-closure.

Definition 3.25. For a set $K \subset \{1, 2, \ldots\}$, its *PBD-closure* is defined as $\mathbb{B}(K) = \{v : PBD(v, K) \text{ exists}\}$. A set K such that $\mathbb{B}(K) = K$ is called *PBD-closed*.

For example, the necessary and sufficient conditions for Steiner triple systems to exist now translate to $\mathbb{B}(\{3\}) = \{v : v \equiv 1, 3 \pmod 6\}$. The following theorem by

Wilson ties it all together and provides an avenue for using PBD-closure to prove the existence of designs.

Theorem 3.26 ([49]). *If K is a PBD-closed set, then K is eventually periodic with period $\beta(K)$. That is, there exists a constant $c(K)$ such that for every $k \in K$ and $v \geq c(K)$, we have $\{v : v \equiv k \pmod{\beta(K)}\} \subset K$.*

This theorem implies that to prove the existence of a certain set of designs, one only needs a rich enough set of small examples and PBD-closure will imply the rest.

Theorem 3.27. *Full rank square N_2 designs exist for all odd $v \geq 181$.*

Proof. PBD-closure guarantees the existence of a $PBD(v, \{5, 7, 9\})$ for all odd $v \geq 181$ (see tables in [38]). Now, similar to the construction in Example 3.21, we can use the full rank square N_2 three-fold triple systems on 5, 7 and 9 points from Lemmas 3.20 and 3.22 to replace the blocks of the given $PBD(v, \{5, 7, 9\})$. Then, the corresponding N_2 is a block-diagonal matrix consisting of ingredient full rank square N_2 matrices of the $TS_3(5)$, $TS_3(7)$ and $TS_3(9)$ and it is, therefore, itself full rank. □

It is also interesting to consider a *p-rank*, that is a rank over $\mathbb{F}_p$, of N_2. In general, if a matrix does not have a full rank in a field of characteristic zero, then it does not have a full p-rank. However, full rank over rationals does not imply full p-rank.

Proposition 3.28. *The 3-rank of N_2 for a $TS_3(v)$ is at most $\binom{v}{2} - 1$ for all $v \geq 5$.*

Proof. Consider the matrix $N_2 N_2^\top$ that records interactions between pairs of points in the design. Since every 2-subset lies in exactly 3 blocks, the matrix $N_2 N_2^\top$ has 3's on the diagonal and six ones in off-diagonal positions. Therefore, the matrix has a constant row sum of nine. Hence, the all-ones vector is in the kernel of $N_2 N_2^\top$ over $\mathbb{F}_3$ and therefore N_2 does not have a full 3-rank. □

Chapter 4

Association schemes and perturbation matrices

In this chapter, we study sufficient conditions for rational triangle decomposition. Recall that the conjectured density bound that guarantees rational triangle decomposability of a graph is $\epsilon < 1/4$; that is, the conjecture states that if a graph is missing at most 1/4 of all the edges at each vertex, it can still be rationally decomposed into triangles. The best bound on ϵ known to date is due to Yuster [54], who uses probabilistic methods to show that $\epsilon < 1/90,000$ is sufficient for rational triangle decomposability. In this chapter, we show that $\epsilon < 1/23$ is sufficient. We do so by examining the magnitude of the perturbation that we are allowed to perform on every vertex of a complete graph and still keep it rationally triangle decomposable. We explore the related matrices through the Johnson association scheme, which allows us to derive accurate estimates.

4.1 Motivation and notation

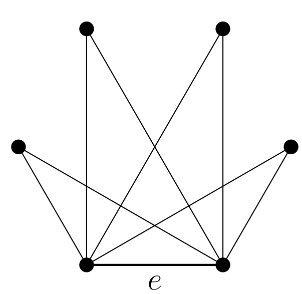
We are interested in studying how far a v-vertex graph G can be from K_v and still remain rationally triangle decomposable. Before we define the notion of the "difference" between G and K_v, we introduce the following matrices associated with both graphs.

Definition 4.1. Let A denote the $\binom{v}{2} \times \binom{v}{2}$ matrix indexed by the edges of K_v such that $A(i,j)$ represents the number of triangles in K_v containing $i \cup j$. Furthermore, given a non-complete graph G of order $|V| = v$, let $\widehat{A}$ denote the matrix indexed by the edges of a fixed graph G such that $\widehat{A}(i,j)$ represents the number of triangles in G containing $i \cup j$. Finally, the matrix $A|_G$ is the matrix A with rows and columns corresponding to non-edges of G being deleted.

Note the immediate connection between the matrix A and the inclusion matrix $W_{2,3}(v)$: we have that $A = W_{2,3}W_{2,3}^{\top}$. An interesting thing to note here is that since the matrix A is full rank, so is the matrix $W_{2,3}$.

From now on, let $\mathbf{1}$ denote the all-ones column vector of the necessary dimensions understood from the context (it will be either of full dimension $\binom{v}{2}$ or of dimension $|E(G)|$). Define a *fan of an edge e in a graph G* to be the set of all triangles in G that use the edge e as illustrated in Figure 4.1.

Figure 4.1: Fan of edge e.



Then a non-negative solution to $A\mathbf{x}_0 = \mathbf{1}$ corresponds to a rational decomposition of a complete graph into fans, where the coordinates of $\mathbf{x}_0$ represent the weights assigned to fans. A rational fan decomposition, in turn, implies a rational triangle decomposition, where the weight of a triangle $\{x, y, z\}$ is the sum of the weights of the fans of edges $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$. In fact, given a row vector $\mathbf{f} \in \mathbb{R}^{\binom{V}{2}}$ representing weights of fans, we can obtain the row vector $x \in \mathbb{R}^{\binom{V}{3}}$ representing the weights of triangles by $\mathbf{x} = \mathbf{f}W$. This is illustrated with a particular graph together with both its rational fan and its rational triangle decomposition in Example 4.2.

However, since the matrix A has constant row sums of $3(v-2)$, the equation $A\mathbf{x}_0 = \mathbf{1}$ always has a unique rational non-negative solution, so rational fan decomposition of complete graphs is resolved. For a non-complete graph G, we consider the system $\widehat{A}\widehat{\mathbf{x}} = \mathbf{1}$ whose non-negative solution, if it exists, corresponds to a rational fan, and hence triangle, decomposition of G. The general idea is that, for dense graphs, the perturbation that transforms A into $\widehat{A}$ is small and hence the non-negative solution to $\widehat{A}\widehat{\mathbf{x}} = \mathbf{1}$ indeed exists. This will require showing that $\widehat{A}$ is non-singular and that the corresponding $\widehat{\mathbf{x}}$ is entrywise non-negative, which is done in Sections 4.3 and 4.4.

We index matrices A, $\widehat{A}$, $A|_G$ first by edges and then non-edges of G. This gives the following structure of the matrix A:

$$A = \left(\begin{array}{c|c} A|_G & C_1 \\ \hline C_2 & C_3 \end{array}\right).$$

Since we are interested in perturbations occurring between the matrices A, $\widehat{A}$ and $A|_G$, it is convenient to let $\widehat{B} = A|_G - \widehat{A}$ denote the perturbation matrix and let B be defined as follows:

$$B = \left(\begin{array}{c|c} \widehat{B} & C_1 \\ \hline 0 & C_3 - (v-2)I \end{array}\right) = \underbrace{\left(\begin{array}{c|c} A|_G & C_1 \\ \hline C_2 & C_3 \end{array}\right)}_{=A} - \underbrace{\left(\begin{array}{c|c} \widehat{A} & 0 \\ \hline C_2 & (v-2)I \end{array}\right)}_{=A-B}.$$

For illustration purposes, let us consider one particular example.

Example 4.2. Suppose $v = 6$ and the graph G is nearly complete, namely $G = K_6 - \{\{5, 6\}\}$. As $\widehat{A}$ is a perturbation of $A|_G$, below bold entries in the $A|_G$ portion of A indicate the entries that will be reduced by 1 in $\widehat{A}$ and hence determine the pattern for $\widehat{B}$ (indices on rows and columns below represent unordered pairs):

$$A = \begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 23 \\ 24 \\ 25 \\ 26 \\ 34 \\ 35 \\ 36 \\ 45 \\ 46 \\ 56 \end{array} \begin{array}{cccccccccccccc|c}
12 & 13 & 14 & 15 & 16 & 23 & 24 & 25 & 26 & 34 & 35 & 36 & 45 & 46 & 56 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 4 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & \mathbf{4} & \mathbf{1} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & \mathbf{1} & \mathbf{4} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 4 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & \mathbf{4} & \mathbf{1} & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & \mathbf{1} & \mathbf{4} & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 4 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \mathbf{4} & \mathbf{1} & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \mathbf{1} & \mathbf{4} & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \mathbf{4} & \mathbf{1} & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \mathbf{1} & \mathbf{4} & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 4
\end{array}$$

The solution to the corresponding system $\widehat{A}\widehat{\mathbf{x}} = \mathbf{1}$ is

$$\widehat{\mathbf{x}} = \frac{1}{36} \begin{array}{cccccccccccccc} 12 & 13 & 14 & 15 & 16 & 23 & 24 & 25 & 26 & 34 & 35 & 36 & 45 & 46 \\ 2 & 2 & 2 & 5 & 5 & 2 & 2 & 5 & 5 & 2 & 5 & 5 & 5 & 5 \end{array}$$

The vector $\widehat{\mathbf{x}}$ provides a fractional decomposition of G into fans, where the weights of the fans in the rational decomposition are given by the entries of $\widehat{\mathbf{x}}$. So here, for instance, each triangle in the fan of $\{1, 2\}$ receives weight $2/36$ and each triangle in the fan of $\{1, 6\}$ receives weight $5/36$. A rational decomposition of G into fans implies a rational decomposition of G into triangles, with each triangle receiving weight equalling the sum of the weights of the fans supported by its edges. Therefore, the following vector represents the weights of the triangles in the decomposition of G:

$$\begin{array}{c} \\ \frac{1}{36} \end{array} \begin{array}{cccccccccccccccc} 123 & 124 & 125 & 126 & 134 & 135 & 136 & 145 & 146 & 234 & 235 & 236 & 245 & 246 & 345 & 346 \\ 6 & 6 & 12 & 12 & 6 & 12 & 12 & 12 & 12 & 6 & 12 & 12 & 12 & 12 & 12 & 12 \end{array}$$

It is not hard to check that each edge will receive the total weight of 1. Take, for instance, edge $\{1, 2\}$: it will get contributions from the triangles $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$ and $\{1, 2, 6\}$ for a total of $(6 + 6 + 12 + 12)/36 = 1$.

Observation 4.3. As illustrated above, a rational fan decomposition of a graph implies a rational triangle decomposition. The converse, however, is not true. Out of 46 graphs on 8 vertices with minimum degree at least 5 (up to an isomorphism), 22 graphs pass the facets tests, meaning they are rationally triangle decomposable. Out of those 22, only 14 graphs are rationally fan decomposable. Among those that are rationally triangle but not rationally fan decomposable are 8-vertex graphs such as the complement of a cycle on 6 vertices and the complement of a path on 6 vertices. See Appendix A.2 for corresponding Sage code.

4.2 Properties of the matrix A and its inverse

In this section, we derive some useful properties of A and its inverse. From the Johnson association scheme, we have the following result:

Proposition 4.4. *The matrix A lies in the Bose-Mesner algebra with $A = (v-2)A_0 + A_1$.*

Proof. By definition, matrix A counts triangles in K_v and matrices A_i from the Johnson scheme record the edge interactions within the graph itself. Given any edge (or two coinciding edges), there are $v-2$ triangles containing it, which corresponds to the coefficient of A_0. Given any two incident edges, there is exactly one triangle containing both of them corresponding to the coefficient of A_1. Finally, given any two disjoint edges, there are no triangles containing them both rendering no contribution from A_2. □

The all-ones matrix J commutes with each A_i in the Johnson scheme (since J is the sum of all the A_i) meaning that each A_i has constant column and row sum as we calculate in the following proposition. For convenience, we shall use the following norm notation throughout: let the *infinity norm of a matrix* be its maximum absolute row sum, that is $\|A\|_\infty = \max_i \sum_j |a_{ij}|$. Clearly, the infinity norm of a matrix A is always non-negative and is equal to zero only if and only if A is the all-zero matrix. Moreover, the infinity norm satisfies the triangle inequality and sub-multiplicativity (for square matrices), that is $\|A+B\|_\infty \le \|A\|_\infty + \|B\|_\infty$ and $\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty$.

Proposition 4.5. *$\|A_0\|_\infty = 1$, $\|A_1\|_\infty = 2(v-2)$ and $\|A_2\|_\infty = \binom{v-2}{2}$.*

Proof. First consider the matrix A_0. Since there is only one edge that intersects any given edge in 2 points, we have that $A_0 = I$.

For matrix A_1, we record edges that intersect any given edge in exactly one point. Disregarding the fixed edge itself, there are $v-2$ edges incident with each of its endpoints. Therefore, $\|A_1\|_\infty = 2(v-2)$.

Finally, A_2 records edges that are disjoint from a given edge and there are $\binom{v-2}{2}$ of those. Hence, $\|A_2\|_\infty = \binom{v-2}{2}$. □

As an immediate consequence of the Propositions 4.4 and 4.5, we have:

Lemma 4.6. *The matrix A is symmetric with constant row (column) sum of $3(v-2)$ and diagonal entries of $v-2$.*

The next natural thing is to write A in the basis of orthogonal idempotents of Bose-Mesner algebra.

Proposition 4.7. *We have:*

$$A = 3(v-2)E_0 + 2(v-3)E_1 + (v-4)E_2.$$

Proof. Using Lemma 4.4 and Proposition 2.40, we get:

$$\begin{aligned} A &= (v-2)A_0 + A_1 \\ &= (v-2)(p_{00}E_0 + p_{01}E_1 + p_{02}E_2) + p_{10}E_0 + p_{11}E_1 + p_{12}E_2 \\ &= (v-2)(E_0 + E_1 + E_2) + 2(v-2)E_0 + (v-4)E_1 - 2E_2 \\ &= 3(v-2)E_0 + 2(v-3)E_1 + (v-4)E_2. \end{aligned}$$

□

Corollary 4.8. *The eigenvalues of A are $3(v-2)$, $2(v-3)$ and $v-4$.*

Since we are ultimately interested in solving the perturbed variation of the system $A\mathbf{x}_0 = \mathbf{1}$, we wish to examine the structure of A^{-1} and estimate its infinity norm.

Proposition 4.9. *We have that $A^{-1} = c_0A_0 + c_1A_1 + c_2A_2$, where*

$$\begin{aligned} c_0 &= \frac{3v^2 - 18v + 26}{3(v-2)(v-3)(v-4)}, \\ c_1 &= -\frac{3v-10}{6(v-2)(v-3)(v-4)}, \\ c_2 &= \frac{2}{3(v-2)(v-3)(v-4)}. \end{aligned}$$

So, asymptotically, $c_0 \sim \frac{1}{v}$, $c_1 \sim -\frac{1}{2v^2}$ and $c_2 \sim \frac{2}{3v^3}$.

Proof. First of all, since A lies in Bose-Mesner algebra, so does A^{-1}, i.e. $A^{-1} = c_0A_0 + c_1A_1 + c_2A_2$ or $A^{-1} = c[A_0\ A_1\ A_2]^\top$ for some $c = [c_0\ c_1\ c_2]$, $c_0, c_1, c_2 \in \mathbb{R}$.

For convenience, write $A = \theta_0E_0 + \theta_1E_1 + \theta_2E_2$, where $\theta = [\theta_0\ \theta_1\ \theta_2]$ is determined by Proposition 4.7.

Then by Definition 2.36 and Proposition 2.37, we have:

$$A = \theta[E_0\ E_1\ E_2]^\top = \theta P^{-1}[A_0\ A_1\ A_2]^\top,$$

where P is the eigenvalue matrix as computed in Proposition 2.40. Then, since the E_i are orthogonal idempotents, we have:

$$A^{-1} = [1/\theta_0\ 1/\theta_1\ 1/\theta_2][E_0\ E_1\ E_2]^\top = ([1/\theta_0\ 1/\theta_1\ 1/\theta_2]P^{-1})[A_0\ A_1\ A_2]^\top.$$

We can now compute the coefficient vector c:

$$\begin{aligned} c &= [1/\theta_0\ 1/\theta_1\ 1/\theta_2]P^{-1} \\ &= \begin{pmatrix} \dfrac{1}{3(v-2)} & \dfrac{1}{2(v-3)} & \dfrac{1}{v-4} \end{pmatrix} \begin{pmatrix} 1/\binom{v}{2} & 1/\binom{v}{2} & 1/\binom{v}{2} \\ \dfrac{2}{v} & \dfrac{v-4}{v(v-2)} & \dfrac{v-4}{v(v-2)} \\ \dfrac{v-3}{v-1} & -\dfrac{v-3}{(v-1)(v-2)} & \dfrac{2}{(v-1)(v-2)} \end{pmatrix}. \end{aligned}$$

Completing the computations (see Appendix A.1), we get the expressions for the c_i as in the statement. □

Corollary 4.10. *Asymptotically,* $\|A^{-1}\|_\infty \sim \dfrac{7}{3v}$.

Proof. Combining Propositions 4.5 and 4.9, we get:

$$\begin{aligned}\|A^{-1}\|_\infty &\le |c_0|\cdot\|A_0\|_\infty + |c_1|\cdot\|A_1\|_\infty + |c_2|\cdot\|A_2\|_\infty \\ &= |c_0\cdot 1| + |c_1\cdot 2(v-2)| + \left|c_2\cdot\binom{v-2}{2}\right| \\ &= \frac{7v^2-39v+52}{3(v-2)(v-3)(v-4)} \sim \frac{7}{3v}.\end{aligned}$$

□

4.3 Perturbation matrix B and its properties

In this section, we derive bounds on the perturbation matrix B in terms of the missing degree of the graph. Recall that a graph G is $(1-\epsilon)$-dense if $\delta(G) \ge (1-\epsilon)(v-1)$, where $\delta(G)$ denotes the minimum degree of G.

Lemma 4.11. *The infinity norm of the matrix $\widehat{B}$ satisfies $\|\widehat{B}\|_\infty \le 4(v-\delta(G))$.*

Proof. Fix one row of $\widehat{B}$ indexed by the edge $e = \{x, y\}$. Consider the diagonal entry $\widehat{B}(e,e) = A|_G(e,e) - \widehat{A}(e,e)$. Recall $A|_G(e,e)$ and $\widehat{A}(e,e)$ count the number of triangles containing e in K_v and G, respectively. Therefore, $\widehat{B}(e,e)$ counts the number of vertices z such that $\{x,y,z\}$ is not a triangle in G. The vertex x has at most $v-2-(\delta(G)-2)$ non-neighbours, similarly for the vertex y. Therefore $\widehat{B}(e,e) \le 2(v-\delta(G))$.

Consider the off-diagonal entries in the row indexed by e. The only entries affected (reduced from 1 to 0) correspond to one of the two scenarios:

Therefore, the non-diagonal entries also count the number of vertices z such that $\{x, y, z\}$ is not a triangle in G. Hence, the row sum of $\widehat{B}$ is calculated as follows:

$$\sum_{f \in G} \widehat{B}(e, f) \leq 4(v - \delta(G)).$$

□

Using the density notation, the above result translates to $\|\widehat{B}\|_\infty \leq 4v\epsilon$.

Furthermore, we have:

Lemma 4.12. *The infinity norm of the matrix B satisfies the bound $\|B\|_\infty \leq 6v\epsilon$.*

Proof. Similar to the proof of Lemma 4.11, we have that $\|C_1\|_\infty \leq 2v\epsilon$ and $\|C_3 - (v-2)I\|_\infty \leq 2v\epsilon$. Then, together with the fact that $\|\widehat{B}\|_\infty \leq 4v\epsilon$, we obtain the result. □

Before we proceed, recall the following classical result.

Theorem 4.13 ([27])**.** *[Cauchy's interlacing theorem] Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Suppose that a symmetric matrix B is obtained from A by removing row and column i for some $i \in \{1, \ldots n\}$. Then the eigenvalues of B, say $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$, satisfy the interlacing property $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \ldots \leq \mu_{n-1} \leq \lambda_n$.*

Lemma 4.14. *If $\delta(G) > \frac{3}{4}v + 1$, then the corresponding matrix $\widehat{A}$ is positive definite and hence non-singular.*

Proof. Invoking Lemma 4.11, the largest eigenvalue of $\widehat{B}$ is bounded above by

$$\|\widehat{B}\|_2 \leq \|\widehat{B}\|_\infty \leq 4(v - \delta(G)) < 4\left(\frac{1}{4}v - 1\right) = v - 4 = \lambda_{min}(A),$$

which, by Cauchy's interlacing theorem, is bounded above by $\lambda_{min}(A|_G)$. Since the maximum eigenvalue of $\widehat{B}$ is strictly smaller than the smallest eigenvalue of $A|_G$, we have that all eigenvalues of $\widehat{A} = A|_G - \widehat{B}$ are positive. Therefore, $\widehat{A}$ is positive definite and invertible. □

Lemma 4.15. *If $\delta(G) > \frac{3}{4}v+1$, then the corresponding matrix $A-B$ is non-singular.*

Proof. Recall that

$$A - B = \left(\begin{array}{c|c} \widehat{A} & 0 \\ \hline C_2 & (v-2)I \end{array}\right).$$

By Lemma 4.14, matrix $\widehat{A}$ is non-singular and hence $A-B$ has non-singular diagonal blocks and is therefore itself nonsingular. In fact, we have that

$$(A - B)^{-1} = \left(\begin{array}{c|c} \widehat{A}^{-1} & 0 \\ \hline -\frac{1}{v-2}C_2\widehat{A}^{-1} & \frac{1}{v-2}I \end{array}\right).$$

□

Since $A\mathbf{1} = 3(v-2)\mathbf{1}$ and we will consider perturbations of this system, from now on, to avoid cumbersome fractions, we consider the solutions to the matrix systems where the right-hand side is a multiple of $\theta_0 = 3(v-2)$ as defined in Proposition 4.9.

Lemma 4.16. *The system* $\widehat{A}\widehat{\mathbf{x}} = \theta_0\mathbf{1}$ *has a solution if and only if the system* $(A - B)\mathbf{x} = \theta_0\mathbf{1}$ *has a solution. Furthermore, if the solution to* $(A-B)\mathbf{x} = \theta_0\mathbf{1}$ *is non-negative, then the solution to* $\widehat{A}\widehat{\mathbf{x}} = \theta_0\mathbf{1}$ *is also non-negative.*

Proof. Given the form of $A - B$ as above, the solution $\widehat{\mathbf{x}}$ to $\widehat{A}\widehat{\mathbf{x}} = \theta_0\mathbf{1}$ implies that the system $(A - B)\mathbf{x} = \theta_0\mathbf{1}$ has a solution of the form $\mathbf{x} = [\widehat{\mathbf{x}}\ \mathbf{y}]$, where $\mathbf{y} = 3 \cdot \mathbf{1} - C_2\widehat{\mathbf{x}}/(v-2)$. Moreover, if the system $(A - B)\mathbf{x} = \theta_0\mathbf{1}$ has a non-negative solution, it implies that the system $\widehat{A}\widehat{\mathbf{x}} = \theta_0\mathbf{1}$ also has a non-negative solution. □

By Lemma 4.14, we now know that the system $\widehat{A}\widehat{\mathbf{x}} = \theta_0\mathbf{1}$ has a unique solution. All that remains to show is that the corresponding $\widehat{\mathbf{x}}$ is non-negative.

4.4 Non-negativity of the solution

We consider the system $(A-B)\mathbf{x} = \theta_0\mathbf{1}$ as a perturbation of the system $A\mathbf{x}_0 = \theta_0\mathbf{1}$, where $\mathbf{x}_0 = \mathbf{1}$. For the first estimate on the density required for $\widehat{\mathbf{x}}$ to be non-negative, we will use the following result from linear algebra.

Lemma 4.17 ([9])**.** *Let* $A\mathbf{x} = \mathbf{b}$ *and let* δA *and* $\delta\mathbf{b}$ *be perturbations of* A *and* $\mathbf{b}$*, respectively. If* $\|A^{-1}\delta A\|_\infty < 1$*, then* $A + \delta A$ *is nonsingular and there is a unique solution* $\mathbf{x} + \delta\mathbf{x}$ *to the equation* $(A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} - \delta\mathbf{b}$ *where*

$$\frac{\|\delta\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \leq \frac{\|A\|_\infty\|A^{-1}\|_\infty}{1 - \|A^{-1}\delta A\|_\infty}\left(\frac{\|\delta A\|_\infty}{\|A\|_\infty} + \frac{\|\delta\mathbf{b}\|_\infty}{\|\mathbf{b}\|_\infty}\right).$$

By applying the above lemma to the system $(A - B)\mathbf{x} = \theta_0\mathbf{1}$, we obtain the first bound.

Theorem 4.18. *The system* $\widehat{A}\widehat{\mathbf{x}} = \theta_0\mathbf{1}$ *has a unique solution* $\widehat{\mathbf{x}} \geq \mathbf{0}$ *for* $\epsilon \leq \dfrac{1}{28}$.

Proof. Consider the system $(A - B)(\mathbf{1} - \delta\mathbf{x}) = \theta_0\mathbf{1}$. By Proposition 4.10, Lemma 4.12 and the submiltiplicativity of matrix norms, we have the following inequality:

$$\|A^{-1}B\|_\infty \leq \|A^{-1}\|_\infty \cdot \|B\|_\infty \leq \frac{7}{3v} \cdot 6v\epsilon = 14\epsilon,$$

which is less than 1 (as required by Lemma 4.17) for $\epsilon < 1/14$. Then a non-negative solution $\mathbf{x} = \mathbf{1} - \delta\mathbf{x}$ is implied by $\|\delta\mathbf{x}\|_\infty \leq 1$. By applying Lemma 4.17 with $\delta A = B$, $\mathbf{x} = \mathbf{1}$, $\mathbf{b} = 3(v-2)\mathbf{1}$ and $\delta\mathbf{b} = \mathbf{0}$ we have:

$$\frac{\|\delta\mathbf{x}\|_\infty}{\|\mathbf{1}\|_\infty} \leq \frac{\|A^{-1}\|_\infty \|B\|_\infty}{1 - \|A^{-1}B\|_\infty} \leq \frac{\frac{7}{3v} \cdot 6v\epsilon}{1 - 14\epsilon} = \frac{14\epsilon}{1 - 14\epsilon},$$

which is at most 1 for $\epsilon \leq \dfrac{1}{28}$. □

With a little more finesse and knowledge about the structure of A^{-1} and B within the Johnson association scheme, we can improve the above bound.

Observation 4.19. Since B is a perturbation matrix of A, we can decompose it within the structure of Bose-Mesner algebra as follows: $B = B_0 + B_1 + B_2$, where each B_i is supported by A_i in the sense that all non-zero entries of B_i are contained in the non-zero entries of A_i, $i = 0, 1, 2$. Notice the slight change in notation here as the coefficients of the B_i in this decomposition are all ones, which is not the case for the decomposition of the matrix A into the A_i. Given the structure of B, we know that $0 \leq B_0 \leq 2v\epsilon A_0$ and $B_2 = 0$. However, the structure of B_1 is a little more complicated, since the entries of it are not just scaled entries of A_1, rather at most $2\epsilon v$ of ones in A_1 (per row) remain in B_1 and even fewer in its border. We shall use this observation to sharpen our analysis.

Theorem 4.20. *The system* $\widehat{A}\widehat{\mathbf{x}} = \theta_0\mathbf{1}$ *has solution* $\widehat{\mathbf{x}} \geq \mathbf{0}$ *for* $\epsilon < \dfrac{1}{22.6}$.

Proof. By Lemma 4.15, matrix $A - B$ is invertible, so consider the following:

$$(A-B)^{-1}A = (I - A^{-1}B)^{-1} = \sum_{r\geq 0}(A^{-1}B)^r.$$

Therefore, we have: $(A-B)^{-1} = \left[\sum_{r\geq 0}(A^{-1}B)^r\right] A^{-1}$. Using Observation 4.19 and Proposition 4.9, consider the following product:

$$\begin{aligned} A^{-1}B &= (c_0A_0 + c_1A_1 + c_2A_2)(B_0 + B_1) \\ &= c_0A_0B_0 + c_1A_1B_0 + c_2A_2B_0 + c_0A_0B_1 + c_1A_1B_1 + c_2A_2B_1. \end{aligned}$$

Recall from Propositions 4.5 and 4.9, we have: $\|A_0\|_\infty = 1$, $\|A_1\|_\infty = 2(v-2)$, $\|A_2\|_\infty = \binom{v-2}{2}$ and, asymptotically, $c_0 \sim \dfrac{1}{v}$, $c_1 \sim -\dfrac{1}{2v^2}$, $c_2 \sim \dfrac{2}{3v^3}$.

<u>Term $c_0A_0B_0$.</u> Since $B_0 \leq 2v\epsilon A_0$ entry-wise, we have $A_0B_0 \leq 2v\epsilon A_0$. Therefore,

$$\|c_0A_0B_0\|_\infty \leq \frac{1}{v}\cdot 2v\epsilon = 2\epsilon.$$

<u>Term $c_1A_1B_0$.</u> We have $A_1B_0 \leq A_1 \cdot 2v\epsilon A_0 \leq 2v\epsilon A_1$ entry-wise and therefore

$$\|c_1A_1B_0\|_\infty \leq \left|-\frac{1}{2v^2}\right| \cdot 2v\epsilon \cdot \|A_1\|_\infty \leq \epsilon.$$

<u>Term $c_2A_2B_0$.</u> Here, $A_2B_0 \leq A_2 \cdot 2v\epsilon A_0 = 2v\epsilon A_2$. Hence

$$\|c_2A_2B_0\|_\infty \leq \frac{2}{3v^3}\cdot 2v\epsilon \cdot \|A_2\|_\infty \leq \frac{2}{3}\epsilon.$$

<u>Term $c_0A_0B_1$.</u> We have that $A_0B_1 = B_1$. Here, recall that $\|B_1\|_\infty \leq 4v\epsilon$ as we have a contribution of $2v\epsilon$ coming from the off-diagonal entries of $\widehat{B}$ and $2v\epsilon$ coming

from the border C_1. We have:

$$\|c_0 A_0 B_1\|_\infty \leq \frac{1}{v} \cdot 4v\epsilon = 4\epsilon.$$

Term $c_1 A_1 B_1$. Using submultiplicativity of the infinity norm, we get:

$$\|c_1 A_1 B_1\|_\infty \leq \left| -\frac{1}{2v^2} \right| \cdot \|A_1\|_\infty \|B_1\|_\infty \leq \frac{1}{2v^2} \cdot 2(v-2) \cdot 4v\epsilon \leq 4\epsilon.$$

Term $c_2 A_2 B_1$. Similarly to the $A_1 B_1$ term, we get:

$$\|c_2 A_2 B_1\|_\infty \leq \frac{2}{3v^3} \|A_2\|_\infty \|B_1\|_\infty \leq \frac{2}{3v^3} \cdot \binom{v-2}{2} \cdot 4v\epsilon \leq \frac{4}{3}\epsilon.$$

Furthermore, using the triangle inequality for infinity norms, we have:

$$\|A^{-1}B\|_\infty \leq 2\epsilon + \epsilon + \frac{2}{3}\epsilon + 4\epsilon + 4\epsilon + \frac{4}{3}\epsilon = 13\epsilon.$$

Note that the terms involving c_1 provide the only negative contribution to $A^{-1}B$, which is at most 5ϵ. Now, ignoring the positive contribution from $\|A^{-1}B\|_\infty$ in the first term, we have:

$$\begin{aligned} \mathbf{x} = (A-B)^{-1} \cdot \theta_0 \mathbf{1} &= \mathbf{1} + (A^{-1}B) \cdot \mathbf{1} + \left(\sum_{r \geq 2} (A^{-1}B)^r \right) \cdot \mathbf{1} \\ &\geq \left(1 - 5\epsilon - \frac{(13\epsilon)^2}{1 - 13\epsilon} \right) \cdot \mathbf{1} \geq 0. \end{aligned}$$

Solving the last inequality, we get $\mathbf{x} \geq \mathbf{0}$ when $\epsilon < \dfrac{\sqrt{185} - 9}{104} \lesssim 1/22.6$. □

Interpreting the solution to the above system as a rational fan and then triangle decomposition, we immediately obtain the following results:

Corollary 4.21. *Any* $(1-\epsilon)$*-dense graph* G *has a rational fan decomposition provided that* $\epsilon < \dfrac{1}{23}$.

Corollary 4.22. *Any* $(1-\epsilon)$*-dense graph* G *has a rational triangle decomposition provided that* $\epsilon < \dfrac{1}{23}$.

As mentioned in Observation 4.3, a rational triangle decomposition does not imply a rational fan decomposition and hence the above bound for the latter is probably stronger than it needs to be.

Chapter 5

Applications of rational graph decompositions

Rational graph decompositions have several applications that are quite different from those of integral graph decompositions and are interesting in their own right. In this chapter, we highlight the use of rational graph decompositions in the statistical design of experiments. We prove the existence of certain types of statistical experimental designs for a large enough number of points. Our bound on the number of points required for the existence of these designs is worse than the bound in [16]; however, our method provides an easy and fast way of generating these statistical designs even below the sufficiency threshold.

Furthermore, rational decompositions (together with signed decompositions) provide one key step in a large index embedding theorem for partial designs and, with our improved bound on rational triangle decompositions, we prove that it is possible to complete a partial 2-$(v, 3, 1)$ design with certain constraints.

5.1 Balanced sampling plans

In statistical studies, particularly in survey methodology, setting up a system to obtain a random yet suitable sample is an important consideration. Depending on the nature of the experiment, it might not be desirable for each possible subset of the population to occur as a sample with equal probability. Since resources are usually limited, researchers have to specify certain constraints on their samples while keeping them random and well distributed. For example, in an environmental setting, neighbouring units (of land) provide similar information and therefore should be avoided in any sample. This leads to a study of sampling avoiding adjacent units, the focus of this section.

The construction of sampling plans avoiding adjacent units was first proposed in 1988 [29, 30]. While in some situations there exists a natural ordering of units (whether in time or space), in general the notion of adjacency depends on the context and can be adjusted as needed. Note here that this ordering can be linear, when the last and first unit are not adjacent, or cyclic, when they are. We will generally work with the one-dimensional version of the latter situation.

Let us now define the terms more rigorously.

Definition 5.1 ([16, 29]). Let X be a population of size N. Then *the sampling plan* is defined as

$$\{(s_i, p_i) : i = 1, 2, \ldots t\},$$

where the s_i are k-subsets of units and $p_i > 0$ is the probability of selecting the subset s_i.

Clearly, $\sum_{i=1}^{t} p_i = 1$.

Definition 5.2 ([16, 29]). The *first-order inclusion probability* π_x is the probability of selecting a particular unit $x \in X$. The *second-order inclusion probability* π_{xy}, $x, y \in X$ is the probability of selecting a sample containing both x and y.

In the situations discussed above, we want first-order probabilities to be constant over all $x \in X$ giving every unit equal probability of being considered, while second-order probabilities will have to vanish on neighbouring units and be constant otherwise.

Definition 5.3 ([16]). Identify the population X with integers modulo N and declare two units x and y adjacent if $|x - y| \leq m$ for some fixed positive integer m. A *circular balanced sampling plan avoiding adjacent units*, or simply a $BSA(N, k, m)$, is a sampling plan such that the second-order inclusion probabilities are zero for all adjacent pairs and constant, say λ, for all non-adjacent pairs.

Example 5.4. Here is an example of a $BSA(9, 3, 1)$ that can be obtained by taking translates of the 3-subset $\{0, 2, 5\}$ modulo 9:

$$\{0,2,5\},\{1,3,6\},\{2,4,7\},\{3,5,8\},\{4,6,0\},\{5,7,1\},\{6,8,2\},\{7,0,3\},\{8,1,4\}.$$

Here, all first-order inclusion probabilities are equal to $\frac{1}{3}$, while the second-order inclusion probabilities are $\frac{1}{9}$ for non-adjacent units and 0 for adjacent ones. This revisits our first example of a graph decomposition, Example 1.2.

Interestingly, $BSA(N, k, m)$ are equivalent to rational graph decompositions of certain classes of graphs.

Definition 5.5. An undirected simple graph $G = (\mathbb{Z}_N, E)$ is a *circulant* if $\{u, v\} \in E$ implies $\{u + x, v + x\} \in E$ for all $x \in V$, where arithmetic is done modulo N.

Examples of circulant graphs include complete graphs and cycles. An equivalent definition of a circulant graph is as follows:

Definition 5.6. Let $D \subset \mathbb{Z}_N \setminus \{0\}$ be a set nonzero allowable distances closed under subtraction. An undirected simple graph $G = (\mathbb{Z}_N, E)$ is *circulant* if $\{u, v\} \in E$ if and only if $u - v \in D$.

For a fixed set $D = \{\pm(m+1), \pm(m+2), \dots, \pm\lfloor N/2 \rfloor\}$ of distances, denote the circulant graph associated with these distances by $G(N, m)$. With the requirement that non-adjacent units appear together in a constant number of samples, say λ, the existence of $BSA(N, k, m)$ is equivalent to an integral decomposition of λ-fold $G(N, m)$ into copies of K_k or a fractional decomposition of a simple $G(N, m)$ into copies of K_k. In the fractional decomposition, the rational weights of k-subsets correspond to the probability of that sample being selected. With these connections in mind, we will concern ourselves with the case of $BSA(N, 3, m)$.

Example 5.7. The $BSA(9, 3, 1)$ from Example 5.4 can be obtained as a triangle (integral) decomposition of $G(9, 1)$ as illustrated in Figure 1.1 on page 2.

Necessary lower bounds and existence results for certain values of parameters (particularly for small values of m and k) have been previously studied in numerous settings, including design theory since $BSA(N, k, m)$ is equivalent to a so-called polygonal design. Here we cite the relevant cases.

Proposition 5.8 ([40])**.** *Suppose there exists a $BSA(N, k, m)$. If $k \geq 3$, then $N \geq (2m+1)k$.*

On the asymptotic side, in [16], it is proven (via a linear algebraic and combinatorial construction) that $BSA(N, k, m)$ exist for all $k \geq 2$, $m \geq 1$ and sufficiently large N. In the case of $k = 3$, they require $N > 12m + 7$. We can provide a similar asymptotic existence of $BSA(N, 3, m)$, albeit with a worse bound on N. Using Corollary 4.21 on rational triangle graph decompositions, we obtain the following existence result.

Proposition 5.9. *$BSA(N, 3, m)$ exist for all $N > 45.2m + 1$.*

Proof. Given a graph G on N vertices with minimum degree $\delta(G) \geq (1 - \epsilon)(N - 1)$, Corollary 4.21 guarantees a rational triangle decomposition of G for $\epsilon > 1/22.6$. The circulant graph $G(N, m)$ is regular of degree $N - 1 - 2m$ and hence we only need to find a suitable lower bound for N. We have:

$$N - 1 - 2m > (1 - 1/22.6)(N - 1) \iff N > 45.2m + 1.$$

□

Note that Corollary 4.21 and hence Proposition 5.9 guarantee the existence of $BSA(N, 3, m)$ for large values of N, but it does not rule out their existence for smaller values of N. In fact, the method developed in Chapter 4 provides the means of constructing $BSA(N, 3, m)$: construct the matrix $\widehat{A}$ corresponding to some circulant graph $G(N, m)$ and solve the matrix system $\widehat{A}\widehat{\mathbf{x}} = \mathbf{1}$. This allows us to create balanced sampling plans avoiding adjacent units for smaller values of N than those guaranteed by Proposition 5.9.

Example 5.10. Rational triangle graph decompositions give solutions for sampling plans as small as $BSA(11, 3, 1)$ with the triples and probabilities of their as shown in Table 5.1 on the following page.

Table 5.1: Example of $BSA(11,3,1)$.

Probability	**Triple**
5/22	{0, 3, 7}, {0, 4, 7}, {0, 4, 8}, {1, 4, 8}, {1, 5, 8}, {1, 5, 9}, {3, 6, 10}, {3, 7, 10}, {2, 5, 9}, {2, 6, 9}, {2, 6, 10}
7/33	{0, 3, 6}, {0, 3, 8}, {0, 5, 8}, {1, 4, 7}, {1, 4, 9}, {1, 6, 9}, {2, 5, 8}, {2, 5, 10}, {2, 7, 10}, {3, 6, 9}, {4, 7, 10}
7/66	{0, 2, 4}, {0, 2, 9}, {0, 7, 9}, {1, 3, 5}, {1, 3, 10}, {1, 8, 10}, {2, 4, 6}, {3, 5, 7}, {4, 6, 8}, {5, 7, 9},{6, 8, 10}
29/132	{0, 2, 6}, {0, 2, 7}, {0, 4, 6}, {0, 4, 9}, {0, 5, 7}, {0, 5, 9}, {1, 3, 7}, {1, 3, 8}, {1, 5, 7}, {1, 5, 10}, {1, 6, 8}, {1, 6, 10}, {2, 4, 8}, {2, 4, 9}, {2, 6, 8}, {2, 7, 9}, {3, 5, 9}, {3, 5, 10}, {3, 7, 9}, {3, 8, 10}, {4, 6, 10}, {4, 8, 10}
23/132	{1, 7, 9}, {1, 7, 10}, {1, 3, 6}, {1, 3, 9}, {0, 6, 8}, {0, 6, 9}, {0, 2, 8}, {0, 3, 5}, {0, 3, 9}, {0, 2, 5}, {1, 4, 6}, {1, 4, 10}, {2, 4, 7}, {2, 4, 10}, {2, 5, 7}, {2, 8, 10}, {3, 5, 8}, {3, 6, 8}, {4, 6, 9}, {4, 7, 9}, {5, 7, 10}, {5, 8, 10}

5.2 Large index embeddings of partial designs

Given a problem of integral graph decompositions, several relaxations of the problem arise. One of them is rational graph decomposition, equivalent to an integral decomposition of the λ-fold version of the same graph (that is a graph where each edge occurs λ times). Another one is signed graph decomposition, where negative edge-weights are allowed. By combining these two variations, we can obtain certain results for the integral case.

In this section, we consider the problem of completing partial designs. A *partial* t-(v,k,λ) *design* is a set of v elements (points) with a collection of k-subsets (blocks) such that every t-subset of the point set is contained in *at most* λ blocks. A natural question arises of when a given partial t-(v, k, λ) design can be completed or embedded into some t-(w, k, λ) design, where $w \geq v$ and the blocks of the partial design are contained within the blocks of the t-(w, k, λ) design. For $\lambda = 1$, this question is hard even for $t = 2$. It has been recently settled for Steiner triple systems [8]; other full results are also known for various combinations of small values of t and k. The solutions range from various constructions, many coming from finite fields, to asymptotic and probabilistic asymptotic results.

Here, we showcase the use of rational decompositions for embedding a partial t-$(v, k, 1)$ design into a t-(v, k, λ) design. The idea is as follows: given a partial design P, first obtain a rational decomposition of its leave L, that is the set of all t-subsets that do not occur together in any block. Then, consider the rational weights of all the blocks and take an appropriate multiple of this decomposition by taking the least common multiple of all rational weights therefore transforming a rational decomposition of a $\lambda = 1$ design into an integral decomposition of a higher index design. Next, use a signed decomposition to adjust that index to the desired value. Finally, by taking the corresponding number of copies of P itself, obtain the embedding of P into a design with a higher index. The general case is considered in [21]; we will consider the particular case of $t = 2$ and $k = 3$ for which we have improved bounds for such a construction.

Recall that for a fixed graph G, we let the $1 \times \binom{|G|}{2}$ vector $\mathbf{1}_G$ denote the characteristic vector of G. Then, given the $(0, 1)$-inclusion matrix $W_{2,3}$, a decomposition of G into triangles is equivalent to a $(0, 1)$ solution $\mathbf{x}$ to $W_{2,3}\mathbf{x} = \mathbf{1}_G$. We will use Corollary 4.21 for the existence of rational triangle decompositions and the following modified result by Wilson for the existence of signed decompositions:

Lemma 5.11 ([21, 50])**.** *Given positive integers $t \leq k$ and $v \geq t + k$, the necessary* *and sufficient conditions for the existence of an integral solution* $\mathbf{x} \in \mathbb{Z}^{\binom{v}{k}}$ *of* $W_{t,k}\mathbf{x} = \mathbf{f}$ *are*

$$W_{i,t}\mathbf{f} \equiv \mathbf{0} \pmod{\binom{k-i}{t-i}}$$

for each $i = 0, \ldots, t$.

Theorem 5.12. *Let* P *be a partial* 2-$(v, 3, 1)$ *design where every point belongs to at most* $\frac{1}{46}(v-1)$ *blocks. Then for any large enough* Λ *satisfying necessary conditions for existence of a* 2*-design (equations (2.1) and (2.2)), there exists a* 2-$(v, 3, \Lambda)$ *design containing* Λ *copies of* P*.*

Proof. Let L be the graph representing the leave of P, that is the 2-subsets (edges) that do not occur together in any triangle (block) of P. First note that any Λ can be written as $\Lambda = m \cdot \lambda + \lambda_0$, where $\lambda_0 \leq \lambda$. We will use the rational decompositions to construct decomposition of index λ (and hence any multiple of it) and signed decompositions to adjust it by index λ_0, therefore obtaining all possible large enough values.

Since L has minimum degree at least $\frac{22}{23}(v-1)$, by Corollary 4.21 it has a rational triangle decomposition. Therefore, the system $W_{2,3}\mathbf{x} = \lambda \mathbf{1}_L$ has a non-negative solution $\mathbf{x}$ for some λ.

Next, by Lemma 5.11, the system $W_{2,3}\mathbf{x}_0 = \lambda_0 \mathbf{1}_L$ has an integral solution $\mathbf{x}_0$. While $\mathbf{x}_0$ can have negative entries, for a large enough multiple m, the vector $\mathbf{x}^* = m\mathbf{x} + \mathbf{x}_0$ is entry-wise non-negative. Then we have:

$$W_{2,3}\mathbf{x}^* = W_{2,3}m\mathbf{x} + W_{2,3}\mathbf{x}_0 = m\lambda\mathbf{1}_L + \lambda_0\mathbf{1}_L = \Lambda\mathbf{1}_L. \tag{5.1}$$

This is equivalent to an integral non-negative decomposition of the leave L of index Λ and, together with taking Λ copies of P itself, we have the embedding of P into a design with index Λ. $\square$

Chapter 6

Further questions and open problems

In this chapter, we briefly summarize the results of Chapters 3 and 4, pose several questions and highlight several possible future directions in the study of rational triangle decompositions.

6.1 Necessary conditions

6.1.1 Facet enumeration and structure

In this thesis, we have studied necessary conditions for triangle decomposition by considering facets of the triangulation cone. In Chapter 3, we identified several infinite families of facets, which has allowed us to better understand certain obstructions when it comes to triangle decomposability. We also classified and enumerated the facets for up to $v = 8$. Since knowing the full set of facets on v vertices allows us to check all v-vertex graphs for their triangle decomposability (which is done for $v = 8$ in Appendix A.2), it is desirable to extend this result for higher values of v.

Problem 6.1. Enumerate and identify the isomorphism classes for facets of Tri_v for $v \geq 9$.

Moreover, due to the connection between the facets of Tri_v and the vertices of met_v, the enumeration of isomorphism classes is of interest in the study of metric polytopes (the current count due to Deza also stops at $v = 8$). Moreover, of interest would be a closer connection between the cut cone Cut_v and the cone generated by cut facets of $W_{2,3}(v)$.

As the computational data shows, the number of isomorphism classes of facets grows very fast, so classifying or enumerating facets for large values of v appears to be hard. However, an asymptotic result may be possible. Recall that Lemma 3.5 provides a rough bound for the number of facets of Tri_v of order 2^{v^3}. Recall also that in met_v the points adjacent (along the extreme rays) to the point $\frac{2}{3}(1, \ldots, 1)$ correspond to the facet normals of Tri_v. The best known applicable upper bound is the upper bound on the number of extreme rays of Met_v or, alternatively, the size of the neighbourhood of the point $(0, \ldots, 0)$ is given by $2^{2.72v^2}$ [23]. A conjecture due to Laurent and Poljak states that any vertex of the metric polytope is adjacent to a cut, in which case the size of the neighbourhood of $\frac{2}{3}(1, \ldots, 1)$ is bounded above by $2^{2.72v^2} \cdot 2^{v-1}$, which motivates the following problem:

Problem 6.2. Prove that the number of facet of Tri_v or, alternatively, the number of vertices in met_v adjacent to the point $\frac{2}{3}(1, \ldots, 1)$ is at most c^{v^2} for some constant $c \in \mathbb{R}$.

Similarly, we can consider the problem on the bound for the number of isomorphism classes of facets for Tri_v.

Enumeration aside, by studying the structural properties of facets, we get closer to understanding families of important obstructions to triangle graph decompositions.

Some important questions remain to be investigated. Proposition 3.11 is the first result in examining the structure of the negative subgraph as it classifies all facets with exactly one negative edge. This result also shows that the structure of the negative subgraph does not uniquely define a facet. With two negative edges, two negative subgraphs are possible: negative edges are adjacent or negative edges are disjoint and we see both of the possibilities with the existence of negative fan and octopus. However, without proof, we cannot guarantee that those are the only two structures with two negative edges. With three negative edges, there are more structures still (see Appendix B), but not all interactions between the negative edges are possible: clearly we cannot have an all-negative triangle. This raises the following question.

Question 6.3. What are (some of) the impossible negative subgraphs in a facet of Tri_v?

Another possible direction is to look at the magnitude of the coordinates in facet normals. Recall that we consider facets in their standard form, that is when all the entries are integers and their greatest common divisor is 1. Then, for example, in any facet with no zero entries, the absolute value of the most negative weight can be at most the sum of the two most positive weights. This fact, however, does not allow us to estimate the largest or smallest possible entry in a facet or the ratio between the two.

Problem 6.4. Find an upper and a lower bound on the magnitude of the entries of standardized facets of Tri_v.

6.1.2 Structure of the W matrix

There are many directions that can be taken when continuing the study of the cone of the W matrix and the matrix itself. One interesting approach is to decompose the support of a vector $\mathbf{y}$ not directly onto W, but onto E_iW ($i = 0, 1, 2$), where the E_i are orthogonal idempotents of the Bose-Mesner algebra of the Johnson scheme. Consider the following:

$$\mathbf{y}W = \mathbf{y}IW = \mathbf{y}(E_0 + E_1 + E_2)W = \mathbf{y}E_0W + \mathbf{y}E_1W + \mathbf{y}E_2W \geq 0.$$

Question 6.5. Given a supporting vector $\mathbf{y}$ of W, does it support the E_i? Which of the terms $\mathbf{y}E_iW$ are non-negative?

Another approach is to consider a *singular value decomposition* of W, a factorization of the matrix as follows: $W = U\Sigma V^\top$, where U consists of eigenvectors of $WW^\top$, V consists of eigenvectors of $W^\top W$ and Σ is a diagonal matrix containing non-negative real numbers on the diagonal. Consider $\mathbf{y}W = \mathbf{y}U\Sigma V^\top$ and let $\mathbf{z} = \mathbf{y}U\Sigma$, then $\mathbf{y}W = \mathbf{z}V^\top$. Therefore, $\mathbf{y}$ being a supporting vector of W implies that $\mathbf{z}$ is a supporting vector of $V^\top$, so the question is whether we can describe supporting vectors or facets of $V^\top$. What supporting vector or facets of $V^\top$ have zeros in the last $\binom{v}{3} - \binom{v}{2}$ positions? By a more careful analysis, it might be possible to study the facets of W by studying facets of $V^\top$.

6.1.3 Structure of the zero hypergraph

Let us start with some definitions. A *hypergraph* is a generalization of a graph, where edges (also called *hyperedges*) may connect more than two vertices. A hypergraph is said to be k-*uniform* if all of its hyperedges have size k. Then regular graphs, that is graphs where all the vertices have the same degree, are 2-uniform hypergraphs. A hypergraph is k-*chromatic* if its vertex set can be partitioned into k classes such that no edge is a subset of any of them; further, a graph is *critically* k-*chromatic* if it is k-chromatic and a deletion of any edge decreases the chromatic number.

We can consider facets in the setting of hypergraphs: given a facet $\mathbf{y}$, define the *zero hypergraph of* $\mathbf{y}$ to be a 3-uniform hypergraph whose edges are zero-weight triangles in $\mathbf{y}$. In a cut facet, for example, the zero hypergraph is bipartite and 2-chromatic. It would be interesting to investigate the properties of the zero hypergraphs of various facets.

Consider the incidence matrix for a (hyper)graph: let $B_{k-1}(H)$ be the incidence matrix of $(k-1)$-subsets of V versus the edges of H. In Section 3.3, we examined higher incidence matrices of certain designs. Here, some interesting results on incidence matrices exist for both ordinary graphs and hypergraphs.

Theorem 6.6 ([43]). *The rows of the incidence matrix of a graph are linearly independent over the reals if and only if no component is bipartite.*

Lemma 6.7 ([33]). *If H is a critically 3-chromatic k-uniform hypergraph, then the columns of $B_{k-1}(H)$ are linearly independent.*

Recall that the set of all zero-weight triangles in a facet spans the space of dimension $\binom{v}{2} - 1$ and it is a maximally non-spanning structure since the addition of an extra zero-weight triangle implies the spanning of all the edges. One could look for an analogue of the above lemma that applies to critically non-spanning hypergraphs. It would also be interesting to see whether there exists a connection between critically 3-chromatic hypergraphs and zero hypergraphs of certain facets.

6.2 Sufficient conditions

In Chapter 4, we considered sufficient conditions for rational triangle decomposition and improved the bound on the density threshold to $e < 1/22$. Our analysis can be potentially sharpened in several places.

Question 6.8. Can we use the non negativity of $A - B$ to estimate $\|(A^{-1}B)^k\|_\infty$ for $k > 1$?

Question 6.9. Is there a proof of Theorem 4.20 that avoids using a series approximation of $(I - A^{-1}B)^{-1}$?

Now, Conjecture 1.11 on the $\frac{3}{4}$ density still stands. However, an interesting question to consider is whether the requirement on the minimum degree alone is too strict and can be modified to include an average degree.

Question 6.10. Given a graph G with minimum degree at least $\frac{1}{6}v$ and an average degree at least $\frac{3}{4}v$, does G have to be K_3-decomposable?

The minimum degree bound of $\frac{1}{6}v$ is proven to be necessary via a counterexample [34].

When it comes to generalizations, can a similar technique be applied to consider rational decompositions of graphs into subgraphs other than triangles? Finally, the decomposition question can also be considered for a more general setting of hypergraphs.

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Appendix A

Sage code

In this Appendix, we include computer code (in Sage software) used for several computations in this thesis.

Appendix A.1 includes computations corresponding to association schemes; specifically, it computes the eigenvalue matrix P for the Johnson scheme with two classes (Proposition 2.40), the coefficients of matrix A^{-1} (the inverse of the matrix recording edge interactions in K_v) in the Bose-Mesner algebra and its infinity norm (Proposition 4.9).

The first part of the code in Appendix A.2 uses facets on 8 vertices to check all 8-vertex graphs with minimum degree at least 4 for rational triangle decomposability. We do so by first using a built-in graphs package to create a collection of all graphs on 8 vertices of minimum degree 4. Then for each graph G in this collection, we create a matrix where each row represents a permutation of the characteristic vector of G. We then multiply this matrix by each facet on 8 vertices and output facets that produce negative entries in the product, that is those that reject G, if any. In the process, we also record which graphs are rational triangle decomposable (passed84) and which ones are not (rejects84).

The second part of the code in Appendix A.2 checks which 8-vertex graphs with minimum degree at least 5 are rationally triangle decomposable but not fan decomposable. Here, we also first build the collection of graphs under consideration and, by running these graphs against the facets on 8 vertices, create a list of graphs that are rational triangle decomposable (passed85). Then, for each graph that passed all facet tests, we construct the matrix $\widehat{A}$, check its rank and output the result: if it is of full rank, then the corresponding graph has a fan-decomposition, otherwise it does not.

A.1 Code for association schemes

```
sage: v,k,r = var('v,k,r'); R = PolynomialRing(QQ, 'v')


def p(i,j):
...s=sum((-1)^(i-r)*binomial(k-r,i-r)*
                    binomial(v-k+r-j,r)*binomial(k-j,r),r,0,i)
...return s


sage: P=matrix(R,3,3,lambda i,j:p(i,j).substitute(k=2));
sage: print P.str()
[                 1                    1                   1]
[           2*v - 4                v - 4                  -2]
[1/2*v^2 - 5/2*v + 3               -v + 3                   1]


sage: print P.inverse().simplify_rational().str()
[     2/(v^2 - v)                2/(v^2 - v)            2/(v^2 - v)]
[             2/v          (v - 4)/(v^2 - 2*v)        -4/(v^2 - 2*v)]
[ (v - 3)/(v - 1)   -(v - 3)/(v^2 - 3*v + 2)     2/(v^2 - 3*v + 2)]


sage: theta=matrix([[v-2,1,0]])*P; theta
[3*v - 6 2*v - 6   v - 4]


sage: Pi=P.inverse();
sage: coeff=matrix([1/theta[0,0], 1/theta[0,1], 1/theta[0,2]])*Pi


sage: c0=coeff[0,0].simplify_rational().factor(); c0
```

```
1/3*(3*v^2 - 18*v + 26)/((v - 2)*(v - 3)*(v - 4))

sage: c1=coeff[0,1].simplify_rational().factor(); c1
-1/6*(3*v - 10)/((v - 2)*(v - 3)*(v - 4))

sage: c2=coeff[0,2].simplify_rational().factor(); c2
2/3/((v - 2)*(v - 3)*(v - 4))

sage: Ainv=(c0-c1*2*(v-2)+c2*binomial(v-2,2)).simplify_rational();

sage: Ainv
1/3*(7*v^2 - 39*v + 52)/((v - 2)*(v - 3)*(v - 4))
```

A.2 Code for facets tests on dense 8-vertex graphs

```
sage: n=8; T=[[i,j] for j in range(n) for i in range(j)]

sage: def permuted_char_vector(g,p):
...       ans=[0 for t in T]
...       for edge in g.edges():
...           for i in range(len(T)):
...               if [p[edge[0]], p[edge[1]]] == T[i] or
                     [p[edge[1]], p[edge[0]]] == T[i]:
...                   ans[i] = 1
...                   break
...       return ans
```

```
sage: def densegraphs(n,i):
...       G=[]
...       for g in graphs.nauty_geng(n):
...           if min(g.degree_sequence()) > i:
...               G.append(g)
...       return G

sage: perms=Permutations([0..n-1])

sage: D=densegraphs(8,3); rejects84=[]; passed84=[];

sage: for i in range(len(D)):
...       gperms=[];
...       rejecters=[];
...       for p in perms:
...           gperms.append(permuted_char_vector(D[i],p));
...       Aperms=matrix(gperms);
...       ind=0;
...       for j in range(len(f8)):
...           b=Aperms*(matrix([f8[j]]).transpose());
...           for k in range(len(gperms)):
...               if b[k][0] < 0:
...                   ind=1;
...                   rejecters.append(j)
...                   break
```

```
...       if ind==0:
...           passed84.append(D[i])
...       if ind==1:
...           print 'Graph',i,'is rejected by facets', rejecters
...           rejects84.append(D[i])
Graph 0 is rejected by facets [0, 3, 5, 6]
Graph 1 is rejected by facets [0, 1, 3, 5, 6, 11, 16]
Graph 2 is rejected by facets [0, 1, 3, 5, 6, 16]
Output truncated

sage: #similarly to above, build rejects85 and passes85

sage: len(densegraphs(8,4)); len(rejects85); len(passed85);
46
24
22

sage: def edges(g):
...       edges=[];
...       for i in range(len(g.edges())):
...           edges.append([g.edges()[i][0],g.edges()[i][1]])
...       return edges
...

sage: def Ahat_values(x,y):
...       k=0; z=Set(x).union(Set(y))
```

```
...        for i in range(len(triangles)):
...            if z.issubset(Set(triangles[i])):
...                k=k+1
...        return k

sage: D=passed85;
...        for ii in range(len(D)):
...        triangles=[]; T=edges(D[ii]);
...        Ts=[Set(T[i]) for i in range(len(T))];
...        Adj=matrix(ZZ,n,n,lambda i,j:Set([i,j]) in Set(Ts));
...        for i in range(n):
...            for j in [i+1..n-1]:
...                if Adj[i,j]==1:
...                    for k in[j+1..n-1]:
...                        if Adj[i,k]==1 and Adj[j,k]==1:
...                            triangles.append([i,j,k])
...        Ahat=matrix(ZZ,len(T),len(T),lambda
                     i,j:Ahat_values(Set(T[i]), Set(T[j])));
...        Y = vector([1 for i in range(len(T))]);
...        if (Ahat.augment(Y)).rank() <= Ahat.rank():
...            X = Ahat.solve_right(Y);
...            for j in range(len(X)):
...                ind=0;
...                if X[j]<0:
...                    ind=1;
...                    print 'Graph', ii,'is not fan-decomposable'
```

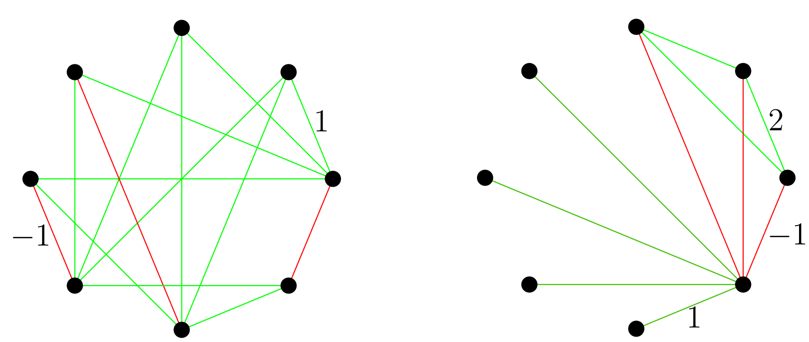
```
...                   break
...         else:
...             print 'Graph', ii, 'is not fan-decomposable at all'
Graph 0 is not fan-decomposable
Graph 5 is not fan-decomposable
Graph 7 is not fan-decomposable
Graph 8 is not fan-decomposable
Graph 10 is not fan-decomposable
Graph 13 is not fan-decomposable
Graph 15 is not fan-decomposable
Graph 16 is not fan-decomposable
```

Appendix B

Facet normals of Tri_v for $v = 8$

In this Appendix, we present the 19 families of facet normals of Tri_v for $v = 8$. In the table below, they are represented by their characteristic vectors indexed in colexicographical ordering of the pairs. We label all the known families and note several new seeds that clearly extend for larger values of v, such as the two structures in Figure B.1 that arise for $v = 8$.

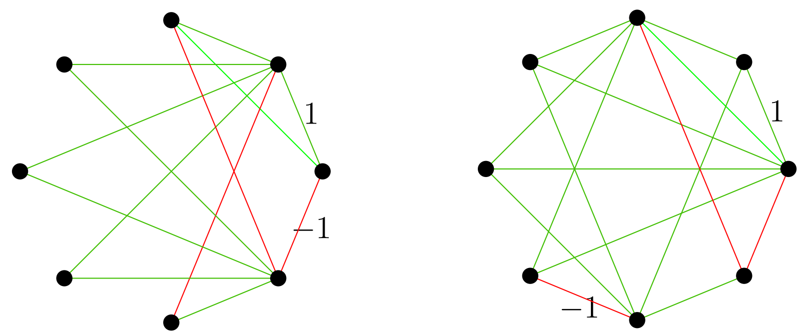
Figure B.1: Facet normals for $v = 8$ with 3 negative edges.



Moreover, in the facets with 3 negative edges, we see various interactions between negative edges. In particular, in Figure B.2 on the next page, we can once again see that the structure of the negative subgraph does not uniquely determine the facet:

For $v = 9$, the number of facets is over 130 and the currently computed list can

Figure B.2: Facet normals for $v = 8$ with the same negative subgraph.



be found at `http://www.math.uvic.ca/~dukes/facets-tri9.txt`.

Table B.1: All facet normals for $v = 8$.

Facet family (if known)	**Facet normal**
trivial	$[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$
star	$[1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0]$
$(3, 3)$-binary star	$[1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 1, 1, 1]$
negative fan	$[1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, -1, -1, 0, 0, 0, 0, 0, 1]$
octopus	$[1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 2]$
$(2, 6)$-cut	$[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 2]$
$(3, 5)$-cut	$[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, -1, -1, 2, 2]$
$(4, 4)$-cut	$[2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, -1, 2, 2, -1, -1, -1, -1, 2, 2, 2]$
	$[1, 2, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, -1, 1, 1, 1, 1]$
	$[-1, -4, 5, -1, 2, 5, 2, -1, 2, -1, 2, -1, 2, -1, 2, 5, 2, -1, 2, -1, -1, 5, 2, -1, 2, -1, -1, 2]$
	$[2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, 1, 1, 1, 1]$
	$[1, 4, 1, 4, 1, -2, 1, 4, 1, 1, 1, -2, 1, 1, -2, -2, 1, 4, -2, 1, 1, -2, 1, -2, 4, 1, 1, 4]$
	$[1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, -1, 0, 1, 1, -1, 1, 0, -1, 0, 0, 0, 0, 1, 1]$
	$[1, 2, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, -1, -1, 0, -1, 0, 0, 0, 1]$
	$[1, 4, 1, 4, 1, 4, 4, 1, 4, -2, 1, -2, 1, 1, 1, -2, 1, -2, 4, -2, 1, -2, 1, -2, -2, 4, 1, 4]$
	$[1, 2, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, -1, 0, -1, 0, 0, 0, -1, 0, -1, 0, 0, 0, 2]$
	$[1, 7, 4, 7, -2, 4, 4, 1, 1, 1, 4, 1, 1, 1, -2, 1, -2, -2, 4, 1, 1, -5, 4, -2, -2, 1, 1, 4]$
	$[1, 3, 2, 1, 0, 2, 1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, -1, 0, -2, 0, 2, 2, 2]$
	$[2, 8, 2, 5, 5, -1, 5, 5, -1, 2, 2, -4, 2, -1, -1, -4, 2, 2, -1, -1, 2, -4, 2, -4, 5, 5, 2, 8]$