Robust Model Predictive Control and Distributed Model Predictive Control: Feasibility and Stability

by

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B.Eng., Northwestern Polytechnical University, 2005
M.Sc., Northwestern Polytechnical University, 2008

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ABSTRACT

An increasing number of applications ranging from multi-vehicle systems, large-scale process control systems, transportation systems to smart grids call for the development of cooperative control theory. Meanwhile, when designing the cooperative controller, the state and control constraints, ubiquitously existing in the physical system, have to be respected. Model predictive control (MPC) is one of a few techniques that can explicitly and systematically handle the state and control constraints. This dissertation studies the robust MPC and distributed MPC strategies, respectively. Specifically, the problems we investigate are: the robust MPC for linear or nonlinear systems, distributed MPC for constrained decoupled systems and distributed MPC for constrained nonlinear systems with coupled system dynamics.

In the robust MPC controller design, three sub-problems are considered. Firstly, a computationally efficient multi-stage suboptimal MPC strategy is designed by exploiting the j-step admissible sets, where the j-step admissible set is the set of system states that can be steered to the maximum positively invariant set in j control steps.

Secondly, for nonlinear systems with control constraints and external disturbances, a novel robust constrained MPC strategy is designed, where the cost function is in a non-squared form. Sufficient conditions for the recursive feasibility and robust stability are established, respectively. Finally, by exploiting the contracting dynamics of a certain type of nonlinear systems, a less conservative robust constrained MPC method is designed. Compared to robust MPC strategies based on Lipschitz continuity, the strategy employed has the following advantages: 1) it can tolerate larger disturbances; and 2) it is feasible for a larger prediction horizon and enlarges the feasible region accordingly.

For the distributed MPC of constrained continuous-time nonlinear decoupled systems, the cooperation among each subsystems is realized by incorporating a coupling term in the cost function. To handle the effect of the disturbances, a robust control strategy is designed based on the two-layer invariant set. Provided that the initial state is feasible and the disturbance is bounded by a certain level, the recursive feasibility of the optimization is guaranteed by appropriately tuning the design parameters. Sufficient conditions are given ensuring that the states of each subsystem converge to the robust positively invariant set. Furthermore, a conceptually less conservative algorithm is proposed by exploiting $\kappa \circ \delta$ controllability set instead of the positively invariant set, which allows the adoption of a shorter prediction horizon and tolerates a larger disturbance level.

For the distributed MPC of a large-scale system that consists of several dynamically coupled nonlinear systems with decoupled control constraints and disturbances, the dynamic couplings and the disturbances are accommodated through imposing new robustness constraints in the local optimizations. Relationships among, and design procedures for the parameters involved in the proposed distributed MPC are derived to guarantee the recursive feasibility and the robust stability of the overall system. It is shown that, for a given bound on the disturbances, the recursive feasibility is guaranteed if the sampling interval is properly chosen.

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List of Abbreviations

MPC	Model Predictive Control
NCS	Networked Control System
UAV	Unmanned Aerial Vehicle

Chapter 1

Introduction

This chapter conducts the literature review on model predictive control (MPC) and distributed MPC. First, MPC, a control method widely adopted in cooperative control, is reviewed. Second, the state-of-the-art in distributed MPC, a specific form for MPC-based cooperative strategies, is recalled. A brief summary of the motivations and the main contributions of the dissertation ends the chapter.

MPC and distributed MPC have been extensively applied to cooperative control field. On the other hand, the increasingly expanding applications of cooperative control call for further studies on MPC and distributed MPC. In the following section, some background on cooperative control is presented.

1.1 Cooperative Control: Overview

Cooperative control is beneficial for large-scale systems in which the control objective is achieved by coordinating several subsystems. In recent years, the applications of cooperative control have increased steadily [5, 14, 94]: the formation control of flying UAVs each equipped with a sensor to form a synthetic aperture radar which can provide high resolution pictures [43, 115]; spatially distributed subsystems interacting with each other through heat, contact, etc. [14]. These promising applications have led to strong interest in the theoretical analysis of cooperative control methods.

Starting from the pioneering work in [114], a surge of research activities can be observed in the control community on cooperative control [24, 96, 97]. A first attempt is to apply centralized cooperative control to large-scale systems. However, when the number of subsystems becomes large, the implementation of centralized cooperative control becomes challenging because: 1) it is not easy to have access to system-wide information; 2) it is time-consuming (and prone to failure) to compute the control signal for all the subsystems at one computing unit; 3) the overall system scales poorly when the number of subsystems increases [100]; 4) it is more difficult to identify the dynamics of a large-scale system than it is to identify the dynamics of one subsystem [35]. Distributed cooperative control is a promising alternative because: 1) each subsystem needs only neighboring information and its own information; 2) the control strategy is designed locally, and thus the computational time is reduced. These attractive features make cooperative control a popular strategy for large-scale systems.

Because physical constraints, like maximum actuator torques, or like state constraints due to the safety consideration always exist, they have to be considered when designing the controller. MPC can handle control and state constraints explicitly. Therefore, cooperative control using MPC is studied in this thesis.

The following section presents the practical and theoretical issues of MPC. The section after it summarizes the current research on distributed MPC, the specific form of MPC implemented in cooperative control.

1.2 Model Predictive Control (MPC)

1.2.1 Design Strategy

This section presents the basic framework of MPC for systems subject to control and state constraints. Consider the following discrete-time system model:

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0, \tag{1.1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^m$ is the control input. The system state and the control signal are subject to the constraints

$$x \in \mathbb{X}, \quad u \in \mathbb{U}, \tag{1.2}$$

where $\mathbb{X}$, $\mathbb{U}$ are compact, convex sets which contain the origin.

Assume that the system state can be measured, and that the system is stabilizable. The control objective is to design an MPC controller to steer the system state to the origin and to satisfy the state and control constraints. The design procedure is as follows.

- □ At each successive time k, define the cost function

$$J_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} J_s(x(k+i|k), u(k+i|k)) + J_f(x(k+N|k)), \tag{1.3}$$

where $x(k+i|k)$ is the state at time $k+i$ predicted at time k; N is the prediction horizon; $J_s(\cdot, \cdot)$ is the stage cost; $J_f(\cdot)$ is the terminal cost; and $\mathbf{u} = (u(k|k), u(k+1|k), \cdots, u(k+N-1|k))$ is the control sequence computed at time k.

- □ Solve the optimization

$$\begin{aligned} &\min_u J_N(x, u), \\ \text{s.t.} \quad & x(k+1) = f(x(k), u(k)), x(0) = x_0, \\ & x \in \mathbb{X}, \\ & u \in \mathbb{U} \end{aligned} \tag{1.4}$$

to obtain the optimal control sequence $\mathbf{u}_0^0(x) = (u_0^0, u_1^0, \cdots, u_{N-1}^0)$.

- □ Define the implicit state feedback controller as

$$K_N(x) = u_0^0(x) \tag{1.5}$$

and apply the control signal in (1.5) to the system in (1.1).

The above MPC strategy can be implemented successfully if and only if: 1) the optimization in (1.4) is recursively feasible; and 2) the closed-loop system is stable. The following section reviews in detail the methods developed to fulfill these two conditions.

1.2.2 Literature Review

MPC has been successfully implemented in industry [70], and thus is an attractive control method in the control community. The advantages of MPC are that: 1) it incorporates the system model information; 2) it handles control and state constraints explicitly; 3) it facilitates the design of compensation strategies for networked control systems (NCSs) because it computes a control sequence at each time instant. Henceforth, the investigation of MPC has gained a lot of attention in academia [31].

Early work demonstrates the need for the theoretical analysis of MPC. The example in [6] shows that a closed-loop system may not be stable even if the state and control are unconstrained. The relationship between stability and the length of the prediction horizon is studied in [78, 83]. For a linear system with/without state and control constraints, the MPC control law with a large enough prediction horizon provably stabilizes the system. However, if the prediction horizon is very large, the computational complexity associated with the MPC strategy increases significantly. Thus, MPC strategies with tractable computational complexity are of great value.

Various techniques have been proposed for the design of stabilizing MPC controllers. We review some of the results in two categories: MPC controller design for linear systems and for nonlinear systems.

MPC controller design for linear systems. For deterministic linear systems, a first MPC strategy is designed by optimizing an infinite horizon cost function with finite decision variables [92, 106]. In [92], the stability of the closed-loop system is guaranteed by permitting violations of the state constraints for the first few time steps. An alternative, and time-consuming, MPC method is proposed in [106] by solving a set of finite-dimension quadratic programming problems. The typical approach to design stable constrained MPC is to use fixed horizon N and to modify the cost function of the optimization [105] and to introduce additional state constraints [13,18]. Modifications of the cost function include adding a terminal cost [12], a terminal equality constraint and/or a terminal constraint set [69, 74]. Introducing a terminal equality constraint degrade performance and the equality constraint can be satisfied only asymptotically [18]. Introducing additional state constraints is suitable only for controllable plants [13]. A distinct class of MPC strategies adopts the economic cost function [2,3,19]. The feasibility of an MPC for a deterministic linear system is trivial because the actual state at the next time instant is the same as the predicted state.

The design techniques for robust MPC for linear systems with disturbances can be classified into two categories.

- Tube-based MPC [49, 68, 71, 89]. Tube-based MPC methods compute the control action by solving an open-loop optimal control problem for the associated nominal system (i.e., the system without disturbances) with tightened state and control constraints. If the nominal system with tightened state and control constraints is stable, then the system with disturbances is also stable.

- Min-max MPC [42,104]. Min-max MPC strategies incorporate the disturbances explicitly into a min-max optimization and compute a control sequence which minimizes the maximum performance index due to all possible disturbance realizations [42]. Alternatively, feedback min-max MPC strategies include feedback into the min-max optimization [104] and derive a less conservative control law which selects the control action based on the disturbance realization. Feedback in the min-max optimization suppresses the effect of disturbances at the price of significantly increased computational complexity.

The existing robust MPC strategies for linear systems with disturbances are designed for the worst-case scenario. Therefore, the system state needs to be measured fast, and the control signal needs to be updated frequently. However, because the worst-case scenario does not always occur, the state updating and the re-computation of the control sequence at each time instant may not be necessary. Therefore, event-triggered MPC has been proposed recently to reduce the computational load [26, 34, 52]. Instead of updating the control periodically, the optimization which computes it is triggered by the violation of some pre-defined conditions. For systems with soft state constraints, i.e., state constraints which can be violated for a short period of time, soft constraint MPC [117] and stochastic MPC [47] have been introduce to reduce the conservativeness.

MPC controller design for nonlinear systems. In contrast to the well-developed MPC schemes for linear systems, research on MPC strategies for nonlinear systems remains challenging. Early work on MPC of nonlinear systems can be traced back to [69], where an equality constraint in the optimization guarantees the recursive feasibility of the optimization. The recursive feasibility further implies the nominal asymptotic stability of the closed-loop system. A general state equality constraint in [28] enlarges the feasible region for a fixed prediction horizon, but its robustness is difficult to analyze. A quasi-infinite horizon MPC scheme is proposed in [12], where an appropriate design of the terminal weighting matrix in the cost function leads to a cost which serves as a quasi-infinite horizon cost. Initial feasibility implies both recursive feasibility and asymptotic stability of the nominal closed-loop system. Because the disturbances and uncertainties are ubiquitous,, it is necessary to design the robust MPC for nonlinear systems.

In [32, 33], the inherent robustness of nonlinear MPC is analyzed, but the recursive feasibility of the optimization in the presence of disturbances, which is a key factor in successive implementation of MPC is not investigated. In [116], the inherent robustness properties of quasi-infinite horizon nonlinear MPC are established. The established recursive feasibility and robust stability results depend only on the persistent disturbances. The inherent stability of nonlinear MPC for discrete-time nonlinear systems is investigated in [67, 91]. The robustness results are established by showing that the cost function is continuous and, thus, in the presence of small disturbances, the state trajectory remains in a tube with respect to the reference trajectory pre-designed at the initial time instant. A recent review on min-max MPC explicitly takes into consideration the effect of the disturbances in [85], and the robustness of nonlinear MPC under the input-to-state stability (ISS) framework is reviewed in [17]. In special cases when a Lyapunov function and a pre-designed constrained control strategy are available, a Lyapunov-based MPC strategy [72, 73] is designed by taking advantage of the existing Lyapunov function, thus improving the system performance. For contracting systems, contractive MPC strategies are designed by imposing stability constraints on the magnitude of the first predicted state vector [13] and on the final predicted state vector [18], respectively. For general nonlinear systems, a dual-mode robust constrained MPC is designed in [74] whose stability is guaranteed by requiring that the prediction horizon at the next time instant be shorter than that of the current time instant. In [66], robust stability is analyzed for nonlinear discrete-time systems by introducing the ISS concept. However, since the conventional quadratic cost function does not satisfy the conditions proposed in [66], the stability is established based on the assumption that a control Lyapunov function can be constructed.

Table 1.1 summarizes the main centralized MPC strategies.

1.3 Distributed MPC

Centralized MPC becomes impractical due to communication needs, computational complexity, lack of scalability and the system identification issues mentioned in Section 1.1. A straightforward extension which overcomes these difficulties is decentralized MPC, where each subsystem solves its local optimization independently without communicating with other subsystems. Decentralized MPC has been implemented successfully when the coupling among subsystems is weak [65, 76, 86]. However, as pointed out in [16], neglecting the interaction results in severely degraded system

Linear MPC	

Without disturbance	Infinite horizon [92, 106]
	Finite horizon:
	  Terminal cost [12]
	  Terminal constraints [69, 74]
	  Economic MPC [2, 3, 19]

With disturbance	Tube-based MPC [49, 68, 71, 89] Min-max MPC [42, 104]
Nonlinear MPC	
Without disturbance	Equality constraint [69] Inequality constraint [12, 28]
With disturbance	Inherent robustness [32, 33, 67, 91, 116] ISS framework [17, 66] Lyapunov-based MPC [72, 73] Contractive MPC [13, 18] Dual-mode strategy [74]

Table 1.1: The main centralized MPC strategies.

performance or even instability when the coupling is strong.

Distributed MPC is a promising alternative which can take into account the interactions among subsystems and can have computational and communication requirements similar to decentralized MPC. In distributed MPC, the optimal control signal is computed locally by solving an optimization which takes into account the couplings among subsystems. However, the recursive feasibility of the optimization and the stability of the closed-loop system with distributed MPC are not trivial to guarantee [43]. Starting with the pioneering work in [114], increasing research effort has been dedicated to distributed MPC [22, 23, 75, 94, 103, 107–109, 115]. Comprehensive reviews of the state-of-the-art research on distributed MPC can be found in [1, 15, 77, 102]. Based on the interaction among subsystems, existing distributed MPC research results can be classified into: distributed MPC for dynamically coupled systems and distributed MPC for dynamically decoupled systems.

1.3.1 Distributed MPC of Dynamically Coupled Systems

Distributed MPC of dynamically coupled systems [20,108,109] can be found in numerous control scenarios. A typical example of coupled systems is the control of processes for which the overall plant is spatially distributed into a number of subsystems [35].

According to the information that can be acquired by each subsystem, the control strategies can be divided into cooperative distributed MPC and non-cooperative distributed MPC.

- **Cooperative distributed MPC:** In cooperative control strategies [108, 109, 112], each subsystem obtains and uses the state information of the overall system to compute its MPC signal. In [108, 112], the cooperation is achieved through iteratively and cooperatively optimizing the system-wide cost function. The strategy in [109] is extended to dynamically coupled nonlinear systems in [112]. The shared limitation is that each subsystem needs to communicate with all other subsystems. The communication requirement is relaxed in [107] through a hierarchical cooperative distributed MPC scheme. In low level, the subsystems communicate with their neighbors at each iteration, while the leaders of the low levels exchange information asynchronously in the high level. For systems with pre-designed Lyapunov-based controllers, Lyapunov-based distributed MPC strategies [37, 53–55] are designed to improve better performance.

- **Non-cooperative distributed MPC:** In non-cooperative distributed MPC, each subsystem can communicate only with its neighboring subsystems and, thus, computes its control based on only limited information. The interaction of the subsystem dynamics: 1) the dynamical interaction is treated as external disturbances [11,39,40,65]. In [40], the min-max distributed MPC is applied to discrete-time nonlinear systems by treating the effect of the system state interaction as additional disturbances. Similarly, by treating the state trajectory of neighboring subsystems as bounded disturbances, contractive based distributed MPC [65], stability constraint distributed MPC [11, 39] are investigated, respectively; 2) the feasibility and stability results established rely heavily on the consistency constraints that the predicted state trajectory at time instant $k + 1$ should not deviate too much from the state trajectory predicted at time k. In [29, 30], by restricting the difference between the future reference trajectory and the actual one in a certain bound, the distributed MPC of a group of dynamically coupled linear systems is investigated. Further extensions to the nonlinear counterpart are studied in [24].

1.3.2 Distributed MPC of Dynamically Decoupled Systems

Distributed MPC of dynamically decoupled subsystems finds applications in many practical problems, including the multi-vehicle formation problem. When the subsystems have decoupled dynamics, their cooperation is promoted through coupled state and/or control constraints and/or cost functions.

- **Distributed MPC of dynamically decoupled systems coordinated via coupled state and/or control constraints:** Guaranteeing the satisfaction of the couple constraints is the main problem in distributed MPC of decoupled subsystems with coupled constraints. In [95–97], the coupled constraints are guaranteed by requiring the subsystems to solve their local optimizations in sequence and then to send relevant information about their control action to all subsystems following them in the sequence. In [110], the coupled constraints are satisfied by designing robust tightening tubes around the ideal trajectory and by maintaining the subsystems in those tubes through local control. In [111], the approach in [110] is extended by including hypothetical state and control information about neighboring subsystems in the local optimizations.

- **Distributed MPC of dynamically decoupled systems coordinated via coupled cost function:** In this category, the cooperation among subsystems is promoted through coupling terms in the cost function [23, 24]. The distributed MPC strategy requires in [24] that each subsystem not deviate too much from its previously predicted state trajectory. The algorithm is implemented for stabilizing a leader-follower formation of unmanned aerial vehicles (UAVs) in [21]. The distributed MPC for dynamically decoupled systems with coupled state constraints and coupled cost function in [43] demands the prediction error be small enough and the updating frequency be fast. The distributed MPC scheme in [90] considers the delays with which the subsystems exchange information. For a class of systems satisfying the controllability conditions [113], an easily-verifiable constraint is imposed in the optimization solved by each subsystem. The practical distributed MPC for linear systems in [99] enables plug-and-play operations, i.e., only the controllers of the successor subsystems are re-designed when removing a subsystem, and only information from the predecessor subsystems is used by the controller of an added subsystem.

To reduce the communication burden and the computational complexity, recent studies have applied event-triggered strategies to the distributed MPC of large-scale systems [25, 27]. Table 1.2 summarizes the main strategies used in distributed MPC.

Distributed MPC of dynamically coupled systems	
Cooperative distributed MPC	Linear systems [108, 112] Nonlinear systems [109] Hierarchical structure [107] Lyapunov-based MPC [37, 53–55]
Non-cooperative control	Interaction treated as disturbances [11, 39, 40, 65] Consistency constraints [24, 29, 30]
Distributed MPC of dynamically decoupled systems	
Coupled through state/control constraints	Sequential updating [95–97] Tightening constraints [110, 111]
Coupled cost function	Consistency constraint [21, 24] Additional constraint [113] Plug-and-play [99]

Table 1.2: The main distributed MPC strategies.

1.3.3 Challenges and Motivations

MPC has been successfully implemented in a wide range of applications [84], and its theoretical analysis advanced significantly [70]. As stated in [47], *the developed theories on MPC have seldom been applied to the practical applications.* Motivated by this observation, this dissertation aims to reduce the existing gap between the theory and the industrial implementation of MPC by addressing several current challenges as follows.

- MPC strategies for linear systems need to have a large feasible region and limited computational complexity to be practical for implementation in industrial applications. A larger feasible region can be obtained by using a longer prediction horizon. However, a long prediction horizon increases the computational complexity of the optimization which yields the control signal. In other words, a larger feasible region is obtained at the cost of increasing the computational load. Chapter 2 seeks to overcome this current trade-off between the region of feasibility and the computational load of MPC for linear systems.

- While many research results exist for MPC of deterministic nonlinear systems, only a few guaranteed recursively feasible and robustly stable strategy have been reported on the robust MPC design of nonlinear systems to date. One obstacle facing the development of practical and provably feasible and robust stable MPC strategies is the conventional adoption of a quadratic integrand in the optimization which yields the control signal. Specifically, cross terms arise in the change of a cost with quadratic integrand at successive time instants and consistency constraints or robustness constraints are added to the optimization to bound the cross terms. The additional constraints increase the computational complexity of the optimization and make the MPC strategy conservative and, thus, impractical for applications. To avoid the need for consistency and/or robustness constraints in the optimization, Chapter 3 proposes a robust MPC strategy with non-quadratic integrand.

  Another obstacle to the development of practical robust MPC methods for nonlinear systems is the conventional reliance on only Lipschitz continuity in feasibility and continuity proofs. This reliance leads to general but conservative MPC strategies which seldom are practical for implementation. The Lipschitz continuous property makes the proofs conservative in two ways: 1) the value of Lipschitz constant used in the theoretical analysis is its maximum value over a certain state-space region; 2) in the evaluation of the discrepancy between the predicted and actual system state trajectories, we assume that the discrepancy is always expanding, which is not always the case. Incorporating some intrinsic properties of the nonlinear system into the design of the controller should yield less conservative robust MPC strategies. Chapter 4 investigates this conjecture for a class of nonlinear systems with contracting dynamics.

- For the large-scale systems, centralized MPC control strategies are impractical because their central processing unit requires the access of all the state information and solves an optimization with respect to a large number of decision variables. Alternatively, distributed MPC can reduce the computational and communication burden of centralized MPC and, thus, can potentially accommodate the practical requirements of a controller for large-scale systems. However, the recursive feasibility of the optimization and the stability of the closed-loop system are challenging to guarantee for distributed MPC. For large-scale systems with coupled cost function and disturbances, the few existing results are based on the robustness constraints and are conservative. To develop a less conservative distributed MPC for cooperating nonlinear systems with decoupled dynamics, Chapter 5 adopts a non-squared integrand in the coupling cost and takes advantage of a two-layer invariant set.

Furthermore, in the field of process control, large-scale systems usually consist of many dynamically coupled nonlinear systems. Therefore, the economic demand of distributed MPC strategies for large-scale systems with coupled dynamics and external disturbances is great. However, most research treats the dynamic couplings as external disturbances and, inevitably provides conservative results. Intuitively, distributed MPC methods are expected to be less conservative if they account for the dynamic couplings explicitly. Chapter 6 investigates this hypothesis.

1.4 Objectives and Contributions of the Dissertation

The objectives of this dissertation are two-fold: i) to design centralized MPC strategies which are less conservative than existing methods; and ii) to present novel distributed MPC strategies. In particular, for centralized MPC, the goals are to enlarge the feasible region, to reduce the computational demand of the optimization and to allow the closed-loop system to tolerate larger disturbances. For distributed MPC, the goal is to ensure cooperation through the coupling cost or in the presence of coupled dynamics. The main contributions of this dissertation are summarized in the following.

- **Design of a multi-stage MPC strategy with increased computational efficiency.** A multi-stage MPC strategy which has a larger feasible region with similar computational complexity to conventional MPC for a given horizon N is obtained. Equivalently, the new strategy has numerical efficiency similar to conventional MPC with a smaller horizon. Therefore, it can benefit applications that demand the control action to be derived on-line and with limited computational effort. The proposed multi-stage MPC requires a pre-computed sequence of j-step admissible sets, where the j-step admissible set is the set of system states that can be steered to the maximum positively invariant set in j control steps. Given the pre-computed admissible sets, multi-stage MPC first determines the minimum number of steps I required to drive the state to the terminal set. Then, it steers the state to the $(I - N)$-step admissible set if $I > N$, or to the terminal set otherwise. The off-line computation of the admissible sets is presented. The feasibility and stability of the multi-stage MPC for systems with and without disturbances are analyzed.

- **Design of novel robust MPC methods for constrained nonlinear systems.** First, a novel robust constrained MPC method for nonlinear systems with control constraints and external disturbances is proposed whereby the control signal results from optimizing an objective function with an integral non-squared stage cost and a non-squared terminal cost. The terminal weighting matrix is designed such that: i) the terminal cost serves as a control Lyapunov function; and ii) the resultant finite horizon cost can be treated as a quasi-infinite horizon cost. Provided that the Jacobian linearization of the system to be controlled is stabilizable and the optimization is initially feasible, sufficient conditions for the recursive feasibility of the optimization and for the robust stability of the closed-loop system are established. The sufficient conditions are shown to rely on the appropriate design of the sampling interval with respect to a certain given disturbance level. Second, a novel robust constrained MPC strategy that exploits the contracting dynamics of a nonlinear system is presented. The proposed technique can be applied to the class of nonlinear systems whose dynamics are contracting in a tube centered around the nominal state trajectory predicted at time t_0. Compared to robust MPC strategies based on Lipschitz continuity, the method proposed in this thesis: 1) can tolerate larger disturbances; and 2) is feasible for a larger prediction horizon and can potentially enlarge the feasible region accordingly. The maximum disturbance that can be tolerated by the proposed control strategy is explicitly evaluated. Sufficient conditions for its recursive feasibility and for its practical asymptotic stability are also derived.

- **Design of robust distributed MPC strategies handling coupling and disturbances.** First, a robust distributed MPC of constrained continuous-time nonlinear systems coupled by cost function is proposed whereby each subsystem communicates only with its neighbors exchanging the assumed system state trajectory. The cooperation among subsystems is achieved through incorporating a coupling term in the cost function. To handle the disturbances, the strategy is designed based on the two-layer invariant set. Provided that the initial state is feasible and the disturbance is bounded by a certain level, the recursive feasibility of the optimization is guaranteed through appropriate tuning of the design parameters. Sufficient conditions are derived for the states of each subsystem to converge to the robust positively invariant set. Second, a robust distributed MPC strategy is designed for a large-scale system which consists of several dynamically coupled nonlinear systems with decoupled control constraints and with disturbances. In the second strategy, all subsystems compute their control signals by solving local optimizations constrained by their nominal decoupled dynamics. The dynamic couplings and the disturbances are accommodated through new robustness constraints in the local optimizations. Relationships among, and design procedures for, the parameters involved in the proposed distributed MPC strategy are derived to guarantee its recursive feasibility and the robust stability of the overall system. For a given bound on the disturbances, the recursive feasibility is guaranteed by properly selecting the sampling interval.

1.5 Organizations of the Dissertation

The remainder of the dissertation is organized as follows. Chapters 2, 3 and 4 present novel robust MPC strategies which are less conservative than existing methods. Chapter 2 proposes a multi-stage MPC strategy suitable for both the deterministic and the robust cases. Chapter 3 designs a robust MPC controller for constrained continuous-time nonlinear systems with a non-squared integrand in the cost function. Chapter 4 studies the robust MPC strategy for contracting nonlinear systems.

Chapters 5 and 6 address the distributed MPC design problems. Chapter 5 introduces a robust distributed MPC strategy for dynamically decoupled nonlinear systems which handles disturbances based on the two-layer invariant set. Chapter 6 investigates the distributed MPC controller design for nonlinear systems with weakly coupled dynamics..

Finally, Chapter 7 summarizes the dissertation and discusses some future research directions.

Chapter 2

Multi-stage Suboptimal Model Predictive Control with Improved Computational Efficiency

2.1 Introduction

Existing research has proven the stability of unconstrained MPC [78], and of constrained MPC [83] with sufficiently large and fixed receding horizon N. However, a fixed horizon N is not trivial. A large N (long horizon) generally increases the computational complexity of the optimization and makes the implementation of constrained MPC impractical for applications which require the on-line computation of the control action. In contrast, a small limited N (short horizon) may not be able to generate a controller that can stabilize a plant in the absence of state and control constraints [6,106]. Therefore, the selection of a fixed horizon N remains a non-trivial problem.

For deterministic systems, guaranteed stable constrained MPC with fixed horizon N has been achieved through modifying the cost function of the open-loop optimization [105,106] and through introducing additional state constraints [13,18,69,74]. For systems with disturbances, tube-based MPC [49,68,71,89] and min-max MPC [42,104] strategies have been employed to guarantee the stability of constrained MPC with fixed horizon N.

Feasibility is another important issue for constrained MPC with fixed horizon N because the optimization which generates the control action may not have a solution for some system states for a given N. The set of feasible system states (i.e., the operating region of constrained MPC) can be enlarged by incorporating N as an additional decision variable in the open-loop optimization [71]. However, selecting N at each step is computationally demanding, and leads to an optimization with unpredictable computational time. Therefore, for applications that need to compute the control action on-line, optimizing N at each step may be impractical. In such applications, the operating region of constrained MPC with fixed horizon N needs to be enlarged without sacrificing its numerical performance.

2.1.1 Objective, Contributions and Chapter Organization

This chapter proposes a multi-stage constrained MPC strategy which, for a given horizon N, has larger feasible region than, but similar computational complexity to, conventional constrained MPC. Equivalently, the proposed strategy has numerical efficiency similar to conventional MPC with smaller horizon. Therefore, it can benefit applications which demand the control action to be derived on-line and with limited computational effort. The proposed multi-stage constrained MPC requires a pre-computed sequence of j-step admissible sets, where the j-step admissible set is the set of system states that can be steered to the maximum positively invariant set in j control steps. Given the pre-computed admissible sets, the proposed technique first determines the minimum number of steps I required to drive the state to the terminal set. Then, it steers the state to the $(I - N)$-step admissible set if $I > N$, or to the terminal set otherwise. This chapter presents the off-line computation of the admissible sets, and shows the feasibility and stability of multi-stage MPC for systems without and with disturbances.

In the remainder of this chapter, Section 2.2 summarizes preliminary results. Section 2.3 presents the off-line computation of the admissible sets. Sections 2.4 and 2.5 discuss the feasibility and stability of multi-stage constrained MPC for systems without and with disturbances, respectively. Section 2.6 validates the analysis in Sections 2.4 and 2.5 through a numerical example. Section 2.7 concludes the chapter with a discussion of the limitations of the technique.

Notation: Given the sets $\mathbb{G}_i$, $i = 1, \cdots, g$, $\mathbb{G}_1 \oplus \mathbb{G}_2 = \{g_1 + g_2 | g_1 \in \mathbb{G}_1,\ g_2 \in \mathbb{G}_2\}$ (set addition), $\mathbb{G}_1 \ominus \mathbb{G}_2 = \{g_1 | g_1 + g_2 \in \mathbb{G}_1, \forall g_2 \in \mathbb{G}_2\}$ (set subtraction), and $\oplus_{i=1}^{g} \mathbb{G}_i = \mathbb{G}_1 \oplus \mathbb{G}_2 \oplus \cdots \oplus \mathbb{G}_g$. The superscript "T" denotes matrix transposition.

2.2 Preliminaries

In this section, relevant definitions and results for MPC with fixed horizon N are presented for systems without disturbances and with disturbances, separately.

2.2.1 Deterministic MPC

Consider the system modeled by:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, k is the index of the current time step, and A and B are system matrices with appropriate dimensions. For simplicity, assume that the system state can be measured accurately. The state and the control input are subject to the constraints:

$$x \in \mathbb{X}, \quad u \in \mathbb{U}, \tag{2.2}$$

where $\mathbb{X}$, $\mathbb{U}$ are convex compact sets and each set contains the origin in its interior. The control constraints are due to physical limits of the actuator, while the state constraints arise from the physics of the system to be controlled and/or from safety considerations [83].

Without loss of generality, the control problem for the system in (2.1) is to steer the system state to the origin. Tracking problems can also be reduced to problems of steering the state to the origin through appropriate state transformation [48].

For a given receding horizon N, MPC solves the following optimization at each time step:

$$\begin{aligned} \min_{\mathbf{u}} J_N(x, \mathbf{u}, k) = & \sum_{i=0}^{N-1} J_s(x(k+i|k), u(k+i|k)) \\ & + J_f(x(k+N|k)), \\ \text{subject to: } & x \in \mathbb{X}, \\ & u \in \mathbb{U} \end{aligned} \tag{2.3}$$

where $x(k+i|k)$ is the state at time instant $k+i$ predicted at time instant k, $J_s(x(k+$

$i|k), u(k+i|k))$ is the stage cost, $i = 0, 1, \cdots, N-1$, $J_f(x(k+N|k))$ is the terminal cost, and $\mathbf{u}(k) = (u(k|k), u(k+1|k), \cdots, u(k+N-1|k))$ is a sequence of control inputs computed at time instant k. The solution of (2.3) yields the minimum cost $J_N^0(x,k)$ and the optimal control sequence $\mathbf{u}^0(k) = (u^0(k|k), u^0(k+1|k), \cdots, u^0(k+N-1|k))$, and only the first control action will be implemented:

$$K_N(x(k)) = u^0(k|k). \tag{2.4}$$

Equation (2.4) is an implicit state feedback control law.

Definition 2.1. *[7] A set $\mathbb{X}_f \subseteq \mathbb{X}$ is a controlled positively invariant set for the system in (2.1) if there exists a local state feedback $K_f x \subseteq \mathbb{U}$ such that $x(k+1) \in \mathbb{X}_f$ for all $x(k) \in \mathbb{X}_f$.*

Definition 2.2. *A set $\mathbb{X}_{fI} \subseteq \mathbb{X}$ is the maximum positively invariant set for the system in (2.1) if it is the union of all controlled positively invariant sets of the system in (2.1).*

If the state and control constraints in (2.2) are convex, then the maximum positively invariant set can be characterized by a convex polyhedron or by a convex ellipsoid [7], and can be computed using Algorithm 6.2 in [46].

If the local state feedback control law K_f is linear, and the stage and terminal costs are:

$$\begin{aligned} J_s(x(k+i|k), u(k+i|k)) =& (1/2)[x(k+i|k)^{\mathrm{T}} Q x(k+i|k) \qquad (2.5a)\\ &+ u(k+i|k)^{\mathrm{T}} R u(k+i|k)], \\ J_f(x(k+N|k)) =& (1/2) x(k+N|k)^{\mathrm{T}} P x(k+N|k), \qquad (2.5b) \end{aligned}$$

with Q, R and P positive definite, and if the following properties are satisfied [70]:

- A1: $(A + BK_f)\mathbb{X}_f \subset \mathbb{X}_f,\ \mathbb{X}_f \subset \mathbb{X}, K_f \mathbb{X}_f \subset \mathbb{U}$,
- A2: $J_f((A + BK_f)x) + J_s(x, K_f x) \leq J_f(x),\ \forall x \in \mathbb{X}_f$,

then, $J_N^0(x,k)$ is monotonically non-increasing and provides a Lyapunov function which shows that the control in (2.4) can stabilize the system in (2.1) with the constraints in (2.2).

With additional weak conditions that there exist constants $c_2 > c_1 > 0$ such that:

$$\begin{aligned} c_1||x(k)||^2 &\leq J_N^0(x,k), \forall x(k) \in \mathbb{I}_N, \\ J_N^0(x,k+1) &\leq J_N^0(x,k) - c_1||x(k)||^2, \forall x(k) \in \mathbb{I}_N, \\ J_N^0(x,k) &\leq c_2||x(k)||^2, \forall x(k) \in \mathbb{X}_f, \end{aligned}$$

the system is exponentially stable [71], as shown in [70, 71].

2.2.2 Robust MPC

Consider the system described by:

$$x(k+1) = Ax(k) + Bu(k) + w(k),\ x(0) = x_0, \tag{2.6}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, A, B are system matrices with appropriate dimensions, and $w(k) \in \mathbb{R}^w$ is the additive system disturbance. The state, the control input and the system disturbance are subject to the constraints:

$$x \in \mathbb{X}, \quad u \in \mathbb{U}, \quad w \in \mathbb{W}, \tag{2.7}$$

where $\mathbb{X}$, $\mathbb{U}$ are convex compact sets and each set contains the origin in its interior, and $\mathbb{W}$ is closed and bounded, and all three sets contain the respective origins in their interior.

The nominal model of the system in (2.6) is:

$$\bar{x}(k+1) = A\bar{x}(k) + B\bar{u}(k). \tag{2.8}$$

For the feedback control strategy $u(k) = \bar{u}(k) + K(x(k) - \bar{x}(k))$, where K is a static feedback gain, the error dynamics of the system in (2.6) become:

$$e(k+1) = (A + BK)e(k) + w(k), \tag{2.9}$$

where $e(k) = x(k) - \bar{x}(k)$.

Definition 2.3. *[88] A set $\mathbb{S} \subseteq \mathbb{X}$ is a robust positively invariant set for the system in (2.9) if $(A + BK)\mathbb{S} \oplus \mathbb{W} \subseteq \mathbb{S}$.*

Definition 2.4. *A set $\mathbb{S}_f \subseteq \mathbb{X}$ is the maximum robust positively invariant set for the* *system in (2.9) if it is the union of all robust positively invariant sets of the system in (2.9).*

Definition 2.5. *[88] A set $\mathbb{Z} \subseteq \mathbb{X}$ is the minimum robust positively invariant set for the system in (2.9) if it is a robust positively invariant set that is contained in every robust positively invariant set of the system in (2.9).*

Solving the optimization:

$$\begin{aligned} \min_{\bar{\mathbf{u}}} J_N(\bar{x}, \bar{\mathbf{u}}, k) \;\; &= \;\; \sum_{i=0}^{N-1} [\bar{x}(k+i|k)^{\mathrm{T}} Q \bar{x}(k+i|k) \\ &\quad + \bar{u}(k+i|k)^{\mathrm{T}} R \bar{u}(k+i|k)] \\ &\quad + \bar{x}(k+N|k)^{\mathrm{T}} P \bar{x}(k+N|k), \\ \text{subject to :} \quad & \bar{x}(k+i|k) \in \mathbb{X} \ominus \mathbb{Z},\ i = 0, 1, \cdots, N-1, \\ & \bar{u}(k+i|k) \in \mathbb{U} \ominus K\mathbb{Z},\ i = 0, 1, \cdots, N-1, \\ & \bar{x}(k+N|k) \in \mathbb{X}_f \subseteq \mathbb{X} \ominus \mathbb{Z} \end{aligned} \tag{2.10}$$

yields the optimal control sequence $\bar{\mathbf{u}}^0(k) = (\bar{u}^0(k|k), \bar{u}^0(k+1|k), \cdots, \bar{u}^0(k+N-1|k))$. Then, the control strategy $K_N(x(k)) = \bar{u}^0(k|k) + K(x(k) - \bar{x}(k))$ can steer the state in (2.6) to the maximum robust positively invariant set $\mathbb{S}_f$ while satisfying the constraints in (2.7) [68].

2.3 The j-Step Admissible Sets

The implementation of the multi-stage MPC strategy proposed in this chapter requires a pre-computed sequence of j-step admissible sets. This section presents the admissible sets and their off-line computation.

Definition 2.6. *The j-step admissible set for the system in (2.1) is the set $\mathbb{I}_j \subseteq \mathbb{X}$ which contains all system states that can be steered to the maximum positively invariant set $\mathbb{X}_{fI}$ for the system in (2.1) in j steps while satisfying the constraints in (2.2).*

From this definition, it follows that MPC with horizon $N = j$ is feasible and can stabilize the system in (2.1) for any $x(0) \in \mathbb{I}_j$.

The j-step admissible set can be computed recursively, as shown conceptually in Algorithm 1. Algorithm 1 is similar to the computation of the admissible sets for the infinite horizon optimal control problem in [83], but starts from the maximum positively invariant set $\mathbb{X}_{fI}$ instead of starting from the origin.

Algorithm 1 Recursive Computation of the j-Step Admissible Set

procedure COMPUTING THE j-STEP ADMISSIBLE SET $(\{\mathbb{I}_1, \mathbb{I}_2, \cdots, \mathbb{I}_j\})$
    Compute the maximum positively invariant set $\mathbb{X}_{fI}$.
    Set $\mathbb{I}_0 = \mathbb{X}_{fI}$.
    for $i = 0$ to $j - 1$ **do**
        $\mathbb{I}_{i+1} = \{x : x \in \mathbb{X}, \exists u, u \in \mathbb{U}, Ax + Bu \in \mathbb{I}_i\}$.
    end for
end procedure

Remark 2.1. If the maximum positively invariant set $\mathbb{X}_{fI}$ and the state and control constraints in (2.2) are convex, then the j-step admissible sets are convex. In particular, if the terminal set and the state and control constraints are convex polytopes, then the j-step admissible sets are convex polytopes.

Remark 2.2. The maximum positively invariant set $\mathbb{X}_{fI}$ is a subset of any j-step admissible set, i.e., $\mathbb{X}_{fI} \subseteq \mathbb{I}_j$, $\forall j \geq 0$.

Theorem 2.1. *The sequence of j-step admissible sets of the system in (2.1) is monotonically nondecreasing, i.e., it obeys* $\mathbb{I}_1 \subseteq \mathbb{I}_2 \subseteq \cdots \subseteq \mathbb{I}_j \subseteq \cdots$.

Proof. The proof is through induction. From $\mathbb{I}_0 = \mathbb{X}_{fI}$, it follows that, for any $x(k) \in \mathbb{I}_0$, there exists a control action $u(k)$ such that $Ax(k) + Bu(k) \in \mathbb{I}_0$. Then, $\mathbb{I}_0 \subseteq \mathbb{I}_1$. Assume that $\mathbb{I}_{j-1} \subseteq \mathbb{I}_j$, and consider a state $x(k) \in \mathbb{I}_j$. Then there exists a feasible control $u(k)$ such that $Ax(k) + Bu(k) \in \mathbb{I}_j$. Then $\mathbb{I}_j \subseteq \mathbb{I}_{j+1}$. This completes the proof. □

Define

$$\mathbb{I}_\infty = \bigcup_{i=0}^{\infty} \mathbb{I}_i. \tag{2.11}$$

Similar to [83], the following theorem is given:

Theorem 2.2. *Let $J_\infty(x)$ denote the infinite horizon linear quadratic cost and assume that the system in (2.1) is stabilizable. Then $x \in \mathbb{I}_\infty \Leftrightarrow J_\infty(x) < \infty$.*

Proof. $x \in \mathbb{I}_\infty \Rightarrow J_\infty(x) < \infty$. The proof differs from [83] in that, after k steps, the state will enter the maximum positively invariant set $\mathbb{X}_{fI}$ instead of reaching the origin. In this set, by implementing a local feedback control law, the state will be steered to the origin asymptotically with a finite cost. This implies that the cost $J_\infty(x)$ is finite. This completes the sufficient part. The necessary part of the proof readily follows from [83]. □

Remark 2.3. The sequence of admissible sets is upper bounded, since the system state belongs to a compact set, i.e., $\mathbb{I}_\infty \subseteq \mathbb{X}$.

In this chapter, the conceptual Algorithm 1 is implemented as shown in Algorithm 2. Algorithm 2 is a modification of the algorithm given in Theorem 4.1 in [41]. Algorithm 2 computes the j-step admissible set accurately, but the number of inequalities in (2.13) grows large as the dimension of the system and j grow. A large number of inequalities leads to a computationally expensive optimization that is impractical for applications. To reduce the numerical complexity of the optimization, this chapter limits the number of inequalities in (2.13) and computes inner approximations of the admissible sets. Efficient algorithms for computing an inner approximation of a convex polygon or a convex polytope are presented in [62, 63].

The number $I_{\max}$ of admissible sets that need to be pre-computed is determined in two steps:

Step 1 : A shrinking factor $\alpha \in (0, 1)$ is computed such that $\forall x \in \mathbb{X}, \exists u \in \mathbb{U}$, $Ax + Bu \in \alpha\mathbb{X}$. This chapter uses the binary search with a pre-defined number of steps to determine a sufficiently accurate α. The shrinking factor α guarantees that $\forall x \in \alpha^i\mathbb{X}, \exists u \in \alpha^i\mathbb{U}$ such that $Ax + Bu \in \alpha^{i+1}\mathbb{X}$ for any positive integer i.

Step 2 : A factor β is determined such that $\beta\mathbb{X} \in \mathbb{X}_{fM}$. Then, the maximum number $I_{\max}$ of admissible sets that need to be pre-computed is $\log_\alpha \beta \leq I_{\max} \leq \log_\alpha \beta + 1$.

2.4 Multi-stage MPC

Given the fixed receding horizon N, the initial state $x(0)$, the maximum positively invariant set $\mathbb{X}_{fI}$ and the sequence of pre-computed j-step admissible sets $\{\mathbb{I}_1, \cdots, \mathbb{I}_{I_{\max}}\}$, multi-stage MPC drives the initial state to $\mathbb{X}_{fI}$ in two steps (see Algorithm 3):

Step 1 : Determine $\mathbb{I}_I$, the smallest j-step admissible set which contains the initial state.

Algorithm 2 Recursive Computation of the j-Step Admissible Set (Implementation)

procedure COMPUTING THE j-STEP ADMISSIBLE SET $(\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_j\})$

Write the system state and control constraints $\mathbb{X}$ and $\mathbb{U}$ in the form

$$Ex + Fu + \lambda \leq 0, \tag{2.12}$$

with E, F constant matrices and λ a constant vector, all with appropriate dimensions.

Set $\mathbb{I}_0 = \mathbb{X}_{fI}$.

for $i = 0$ to $j - 1$ **do**

Express $\mathbb{I}_i$ in the form

$$Px + \gamma \leq 0, \tag{2.13}$$

with P a constant matrix and γ a constant vector, both with appropriate dimensions.

Remove the redundant rows of $[P\ \gamma]$. If needed, use an inner approximation to limit the number of inequalities to a given maximum number.

Let

$$\hat{Z} = \{(x, u) : \begin{bmatrix} E \\ PA \end{bmatrix} x + \begin{bmatrix} F \\ PB \end{bmatrix} u + \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} \leq 0\}. \tag{2.14}$$

Solve (2.14) through Fourier-Motzkin elimination [41] to get

$$\mathbb{I}_{i+1} = \{x : (x, u) \in \hat{Z}\}. \tag{2.15}$$

end for

end procedure

Step 2 : Apply MPC to steer the state to $\mathbb{I}_{I-N}$ if $I > N$, or to $\mathbb{X}_{fI}$ otherwise.

Algorithm 3 Multi-stage MPC Strategy

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procedure MULTI-STAGE MPC
  Determine $\mathbb{I}_I$ s.t. $x(0) \in \mathbb{I}_I$ and $x(0) \notin \mathbb{I}_{I-1}$.
  if $I > N$ **then**
    for $k = 0$ to $I - N$ **do**
      Solve

$$\begin{aligned} \min_{\mathbf{u}} J_N(x, \mathbf{u}, k) = \ & \textstyle\sum_{i=0}^{N-1} \left[x(k+i|k)^{\mathrm{T}} Q x(k+i|k)\right. \\ & \left. + u(k+i|k)^{\mathrm{T}} R u(k+i|k)\right] \\ & + x(k+N|k)^{\mathrm{T}} P x(k+N|k), \\ \text{subject to}: \ & x(k+i|k) \in \mathbb{X},\ i = 0, 1, \cdots, N-1, \\ & u(k+i|k) \in \mathbb{U},\ i = 0, 1, \cdots, N-1, \\ & x(k+N|k) \in \mathbb{I}_{I-N-k} \end{aligned} \tag{2.16}$$

      Apply $u^0(k|k)$.
    end for
  end if
  Apply MPC with fixed horizon N to steer the state to $\mathbb{X}_I$.
end procedure

The smallest j-step admissible set that contains the state x is the set $\mathbb{I}_I$ such that $x \in \mathbb{I}_I$ and $x \notin \mathbb{I}_{I-1}$. This chapter considers admissible sets represented by convex polytopes:

$$\mathbb{I}_j = \{x | a_{i_j}^{\mathrm{T}} x \leq b_{i_j}, i_j = 1_j, 2_j, \cdots, k_j\},$$

and implements the computation of $\mathbb{I}_I$ through binary search and linear program [9]. Specifically, introducing one slack variable η leads to the linear programming:

$$\begin{aligned} & \min \eta, \\ \text{subject to}: \ & a_{1_j}^{\mathrm{T}} x + \eta \leq b_{1_j}, \\ & a_{2_j}^{\mathrm{T}} x + \eta \leq b_{2_j}, \\ & \vdots \\ & a_{k_j}^{\mathrm{T}} x + \eta \leq b_{k_j} \end{aligned} \tag{2.17}$$

whose solution indicates that x belongs to $\mathbb{I}_j$ if $\eta \leq 0$. Binary search yields I, the minimum j for which (2.17) admits a solution $\eta \leq 0$.

The computation of $\mathbb{I}_I$ guarantees that MPC with fixed horizon I is feasible and stabilizes the system in (2.1). As the following theorem shows, it also guarantees that multi-stage MPC with fixed horizon $N < I$ is feasible and can stabilize the system in (2.1).

Theorem 2.3. *If the initial state $x(0)$ belongs to an admissible set $\mathbb{I}_I$, then multi-stage MPC with fixed horizon N is feasible and can steer the state to the maximum robust positively invariant set $\mathbb{X}_{IM}$.*

Proof. **Feasibility**: For $x(0) \in \mathbb{I}_I$, $I \leq N$, multi-stage MPC reduces to MPC with fixed horizon N, whose feasibility and stability are ensured as in [70]. For $x(0) \in \mathbb{I}_I$, $I > N$, the proof is by induction. At the initial time step $k = 0$, according to the definition of $\mathbb{I}_I$, the optimization in (2.16) is feasible, and yields a control sequence $\mathbf{u}^0(0)$ to steer $x(0)$ to $\mathbb{I}_{I-N}$. Multi-stage MPC applies $u^0(0|0)$ and drives the state to $\mathbb{I}_{I-1}$. Now consider that multi-stage MPC has steered the state to $\mathbb{I}_{I-k}$ through a sequence of feasible control inputs $\{u^0(0|0), u^0(1|1), \ldots, u^0(k-1|k-1)\}, k < I - N$. According to the definition of $\mathbb{I}_{I-k}$, the optimization in (2.16) is feasible, and provides a control sequence $\mathbf{u}^0(k)$ to drive $x(k)$ to $\mathbb{I}_{I-k-N}$. Multi-stage MPC applies $u^0(k|k)$ and steers the state to $\mathbb{I}_{I-k-1}$. Hence, the optimization in (2.16) is feasible for all $k = 0, \ldots, I-N$, and the control sequence $\{u^0(0|0), u^0(1|1), \ldots, u^0(I-N-1|I-N-1)\}$ drives the state to $\mathbb{I}_N$.

Stability: The feasibility proof shows that the system state enters $\mathbb{I}_N$ in $I - N$ steps, that is, $x(I-N) \in \mathbb{I}_N$. Therefore, after $I - N$ steps, multi-stage MPC reduces to conventional MPC with fixed horizon N, whose stability is guaranteed [70]. □

Remark 2.4. The convexity of the admissible sets together with the convexity of the state constraints, of the control constraints and of the cost function guarantee the global minimum of the optimization in (2.16).

Remark 2.5. At each time step, multi-stage MPC solves a similar optimization as MPC with fixed horizon N, but uses a different terminal set. Because employing a different terminal set does not affect the numerical efficiency of the minimization problem compared with conventional MPC, the computational complexity of multi-stage MPC is comparable to that of conventional MPC with fixed horizon N. The only difference from the conventional MPC is the pre-computation required to evaluate the smallest admissible set $\mathbb{I}_I$ which contains the initial state. The cost of this computation is negligible given that $I < I_{\max}$ and that $I_{\max}$ has been pre-computed.

2.5 Multi-stage Tube-Based Robust MPC

This section extends the admissible sets and the multi-stage MPC to linear systems with disturbances. Consider the nominal system in (2.8), and the tightened constraints:

$$\bar{\mathbb{U}} = \mathbb{U} \ominus K\mathbb{Z},\ \bar{\mathbb{X}} = \mathbb{X} \ominus \mathbb{Z}, \tag{2.18}$$

where $\mathbb{Z}$ is the minimum robust positively invariant set with respect to the feedback controller K. Based on the tightened constraints in (2.18) and the local feedback controller K_f (to achieve the optimality, the unconstrained linear quadratic regulator (LQR) controller is usually adopted), the maximum robust positively invariant set $\mathbb{S}_f$ is calculated such that $\mathbb{S}_f \oplus \mathbb{Z} \subset \mathbb{X}$. Thereafter, the sequence of admissible sets $\{\mathbb{I}_1, \mathbb{I}_2, \cdots, \mathbb{I}_{I_{\max}}\}$ is computed using Algorithm 2 with the tightened constraints in (2.18) and with the maximum robust positively invariant set $\mathbb{S}_f$.

Now, given a receding horizon N, the initial state $x(0)$, the minimum robust positively invariant set $\mathbb{Z}$ and the sequence of pre-computed admissible sets $\{\mathbb{I}_1, \cdots, \mathbb{I}_{I_{\max}}\}$, robust multi-stage MPC drives the state to $\mathbb{S}_f$ in two steps (see Algorithm 4):

Step 1 : Compute $\mathbb{I}_I$, the smallest admissible set which contains the initial state;

Step 2 : Apply robust MPC to steer the state of the nominal system to $\mathbb{I}_{I-N}$ if $I > N$, or to steer the state of the uncertain system to $\mathbb{S}_f$ otherwise.

Theorem 2.4. *If the initial state $x(0)$ of the system with constraints and disturbances in (2.7) belongs to the admissible set associated with the nominal system in (2.8) with the tightened constraints in (2.18), the state $x(k)$ of the system in (2.6) with the robust controller in (2.20) will converge to the maximum robust positively invariant set $\mathbb{S}_f$ while obeying $x(k) \in \mathbb{X}$ and $K_N(x(k)) \in \mathbb{U}$ for all $k \geq 0$.*

Proof. For the nominal system, the state $\bar{x}(k)$ satisfies $\bar{x}(k+1) = A\bar{x}(k) + B\bar{u}(k)$. From Theorem 2.3, $\bar{x}(k)$ will converge to the origin as $k \to \infty$. As in [49, 68, 71], the states of the uncertain system and of the nominal system obey $x(k+1) - \bar{x}(k+1) = (A + BK)(x(k) - \bar{x}(k)) + w(k)$, so $x(k) \in \bar{x}(k) \oplus \mathbb{Z}$. The convergence of $\bar{x}(k)$ implies the convergence of $x(k)$ to $\mathbb{Z}$. Moreover, since $\bar{x}(k) \in \mathbb{X} \ominus \mathbb{Z}$ and $\bar{u}(k) \in \mathbb{U} \ominus K\mathbb{Z}$, then $x(k) \in \mathbb{X} \ominus \mathbb{Z} \oplus \mathbb{Z} \subseteq \mathbb{X}$ and $u(k) = \bar{u}(k) + K(x(k) - \bar{x}(k)) \in \mathbb{U} \ominus K\mathbb{Z} \oplus K\mathbb{Z} \subseteq \mathbb{U}$, respectively. □

Algorithm 4 Robust Multi-stage MPC Strategy

procedure ROBUST MULTI-STAGE MPC
  Determine $\mathbb{I}_I$ such that $x(0) \in \mathbb{I}_I$ and $x(0) \notin \mathbb{I}_{I-1}$.
  if $I > N$ **then**
    for $k = 0$ to $I - N$ **do**
      Solve

$$\begin{aligned} \min_{\bar{\mathbf{u}}} J_N(\bar{x}, \bar{\mathbf{u}}, k) = \ & \textstyle\sum_{i=0}^{N-1} \left[\bar{x}(k+i|k)^{\mathrm{T}} Q \bar{x}(k+i|k) \right. \\ & \left. +\bar{u}(k+i|k)^{\mathrm{T}} R \bar{u}(k+i|k)\right] \\ & +\bar{x}(k+N|k)^{\mathrm{T}} P \bar{x}(k+N|k), \\ \text{subject to}: \ & \bar{x}(k+i|k) \in \bar{\mathbb{X}}, \ i = 0, 1, \cdots, N-1, \\ & \bar{u}(k+i|k) \in \bar{\mathbb{U}}, \ i = 0, 1, \cdots, N-1, \\ & \bar{x}(k+N|k) \in \mathbb{I}_{I-N-k} \end{aligned} \tag{2.19}$$

      Apply

$$K_N(x(k)) = \bar{u}^0(k|k) + K(x(k) - \bar{x}(k)). \tag{2.20}$$

    end for
  end if
  Apply robust MPC with fixed horizon N to steer the state to $\mathbb{S}_f$.
end procedure

2.5.1 Modified Multi-stage Tube-based Robust MPC

The minimum robust positively invariant set $\mathbb{Z}$ for the error system in (2.9) is usually computed using:

$$\mathbb{Z} = \lim_{i\to\infty} \mathbb{Z}_i = \lim_{i\to\infty} \left(\oplus_{j=0}^{j=i} A_K^j \mathbb{W}\right). \tag{2.21}$$

Given that $\mathbb{Z}_i \subseteq \mathbb{Z}, \forall i \geq 0$, the state and control constraints in the optimization in (2.19) can be replaced with $\mathbb{X} \ominus \mathbb{Z}_{k+i}$ and $\mathbb{U} \ominus K\mathbb{Z}_{k+i}$, respectively, to obtain the modified multi-stage robust MPC in Algorithm 5. This modification allows the state and the control to take values in larger sets, and potentially improves performance. The sets $\mathbb{Z}_i$ are generated during the off-line computation of $\mathbb{Z}$, using (2.21) and $\mathbb{Z}_0 = 0$.

Algorithm 5 Modified Robust Multi-stage MPC Strategy

procedure MODIFIED ROBUST MULTI-STAGE MPC
  Determine $\mathbb{I}_M$ such that $x(0) \in \mathbb{I}_I$ and $x(0) \notin \mathbb{I}_{I-1}$.
  if $I > N$ **then**
    for $k = 0$ to $I - N$ **do**
      Solve

$$\begin{aligned} \min_{\bar{\mathbf{u}}} J_N(\bar{x}, \bar{\mathbf{u}}, k) = & \sum_{i=0}^{N-1} \left[\bar{x}(k+i|k)^{\mathrm{T}} Q \bar{x}(k+i|k) \right. \\ & \left. + \bar{u}(k+i|k)^{\mathrm{T}} R \bar{u}(k+i|k)\right] \\ & + \bar{x}(k+N|k)^{\mathrm{T}} P \bar{x}(k+N|k), \\ \text{subject to} : \bar{x}(k+ & i|k) \in \mathbb{X} \ominus \mathbb{Z}_{k+i},\ i = 0, 1, \cdots, N-1, \\ \bar{u}(k+ & i|k) \in \mathbb{U} \ominus K\mathbb{Z}_{k+i},\ i = 0, 1, \cdots, N-1, \\ \bar{x}(k+ & N|k) \in \mathbb{I}_{I-N-k} \end{aligned} \tag{2.22}$$

      Apply

$$K_N(x(k)) = \bar{u}^0(k|k) + K(x(k) - \bar{x}(k)). \tag{2.23}$$

    end for
  end if
  Apply robust MPC with fixed horizon N to steer the state to $\mathbb{S}_f$.
end procedure

Theorem 2.5. *If the optimization in (2.22) is feasible for the initial state $x(0)$, then it is feasible for all time instants $k \geq 1$.*

Proof. The proof is through induction. Assume that the initial state $x(0) \in \mathbb{I}_I$ is feasible. Then, there exists a control sequence $\bar{\mathbf{u}}^0(0)$, such that $\bar{u}^0(i|0) \in \mathbb{U} \ominus K\mathbb{Z}_i$,

$\bar{x}(i|0) \in \mathbb{X} \ominus \mathbb{Z}_i$, $\bar{x}(N|0) \in \mathbb{I}_{I-N}$. The nominal system obeys $\bar{x}(0|1) = \bar{x}(1|0)$. From $\bar{x}(N|0) \in \mathbb{I}_{I-N}$ and the definition of the $(I-N)$-step admissible set, it follows that there exists a control signal $u^* \in \mathbb{U} \ominus K\mathbb{Z}_N$ that can drive $\bar{x}(N|0)$ to $\mathbb{I}_{I-N-1}$. So $\bar{\mathbf{u}}(1) = (\bar{u}^0(1|0), \bar{u}^0(2|0), \cdots, \bar{u}^0(N-1|0), u^*)$ can be used to drive $\bar{x}(1) = \bar{x}(0|1)$ to $\mathbb{I}_{I-N-1}$ while obeying the state and control constraints. Similarly, it can be shown that all successive optimizations are feasible.

□

2.6 Illustrative Example

To illustrate the effectiveness of the proposed multi-stage MPC strategy, this section applies it to control the unmanned mini-hovercraft with system dynamics modeled in [38]. Because the dynamics of the vehicle are decoupled and are the same along the two directions of motion in the plane, this section considers the control of the motion along a single direction. The control of the motion along the other direction is similar to the one introduced here.

After zero-order hold discretization with the sampling period $T_s = 0.1$ s, the discrete-time state-space dynamics of the mini-hovercraft can be written as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.0995 \\ 0 & 0.9900 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0050 \\ 0.0995 \end{bmatrix} u(k) + w(k), \tag{2.24}$$

where x_1 and x_2 denote the vehicle position and velocity, respectively. The state and control constraints are:

$$|x_1| \leq 3,\ |x_2| \leq 2,\ |u| \leq 10.$$

The disturbance is assumed bounded by $w \in \mathbb{W}$, where $\mathbb{W} := \{w \in \mathbb{R}^2 | ||w||_\infty \leq 0.03\}$.

The control objective is to steer the mini-hovercraft from the initial state $x(0) = [-2.5 \ \ -1.5]^{\mathrm{T}}$ to the origin. For this problem, the control parameters are selected as follows: The local feedback gain is $K_f = -[54.3 \ 12.3]$ and is adopted when the state enters the terminal set; the gain in (2.9) is $K = [100.5 \ 15.0]$, and is chosen to stabilize the feedback matrix $A + BK$; the weighting matrices in the cost function in (2.5) are $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $R = 3$, respectively.

Given these control parameters, the terminal set $\mathbb{X}_{fI}$ is calculated as the maximum robust positively invariant set for the closed-loop system $x(k+1) = (A + BK_f)x(k)$; the minimum robust positively invariant set $\mathbb{Z}$ is computed using $A_K = A + BK$ in (2.21); the tightened state and control constraints are derived as $\mathbb{X} \ominus \mathbb{Z}$ and $\mathbb{U} \ominus K\mathbb{Z}$, respectively; and the maximum number of j-step admissible sets that need to be pre-computed is $I_{\max} = 30$.

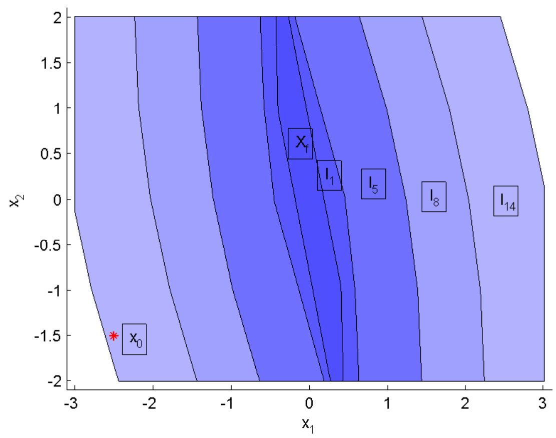

Figure 2.1: The initial state $x(0)$, the maximum positively invariant set $\mathbb{X}_{fm}$, and the admissible sets $\mathbb{I}_5$, $\mathbb{I}_8$, $\mathbb{I}_{14}$ for the mini-hovercraft without disturbances.

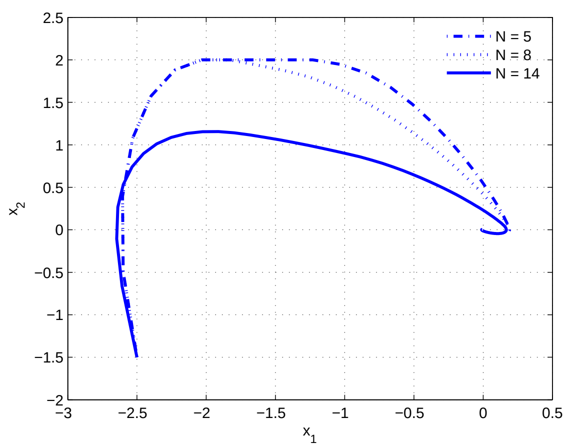

Figure 2.2: State evolution of mini-hovercraft without disturbances and controlled using: multi-stage MPC with $N = 5$; multi-stage MPC with $N = 8$; and multi-stage MPC with $N = 14$.

For the simulation of the mini-hovercraft without disturbances, MPC with minimum horizon $N = 14$ is needed to drive the initial state $[-2.5\ -1.5]^{\mathrm{T}}$ to the maximum

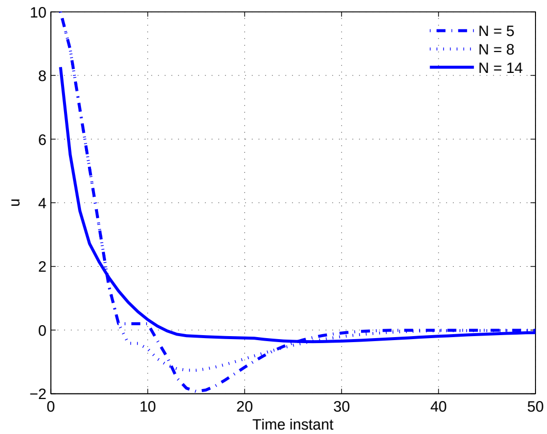

Figure 2.3: Control action of mini-hovercraft without disturbances and controlled using: multi-stage MPC with $N = 5$; multi-stage MPC with $N = 8$; and multi-stage MPC with $N = 14$.

positively invariant set $\mathbb{X}_{fI}$, and provides the benchmark for multi-stage MPC with $N = 8$ and with $N = 5$. The initial state $x(0)$, the maximum positively invariant set $\mathbb{X}_{fI}$ and the pre-computed j-step admissible sets $\mathbb{I}_5$, $\mathbb{I}_8$, $\mathbb{I}_{14}$ are depicted in Figure 2.1. As shown in Figure 2.1, the admissible sets are convex polygons. They are computed without using inner approximation because 14 inequalities are sufficient to characterize them. Note that $x(0)$ is in $\mathbb{I}_{14}$, but is neither in $\mathbb{I}_5$ nor in $\mathbb{I}_8$. Hence, $x(0)$ is feasible for MPC with $N = 14$, but infeasible for MPC both with $N = 8$ and with $N = 5$. Note also that the j-step admissible sets approach the set of state constraints as j increases.

Figure 2.2 and Figure 2.3 depict the state evolution and the corresponding control action for the deterministic mini-hovercraft controlled using the multi-stage MPC with $N = 14$; $N = 8$; and $N = 5$. The state trajectories validate that multi-stage MPC with limited horizon can steer the state of the deterministic system to the maximum positively invariant set. A limited horizon reduces the computational complexity of the optimization and makes multi-stage MPC practical for implementation in applications that demand the on-line computation of the control action. The price paid for enlarging the feasible region of MPC with fixed horizon N without increasing the cost of computing the control action is the larger overshoot (see Figure 2.2) and larger oscillation of the control signal (see Figure 2.3).

For the simulation of the mini-hovercraft with disturbances, tightened state and control constraints are required to guarantee robustness. Correspondingly, the ad-

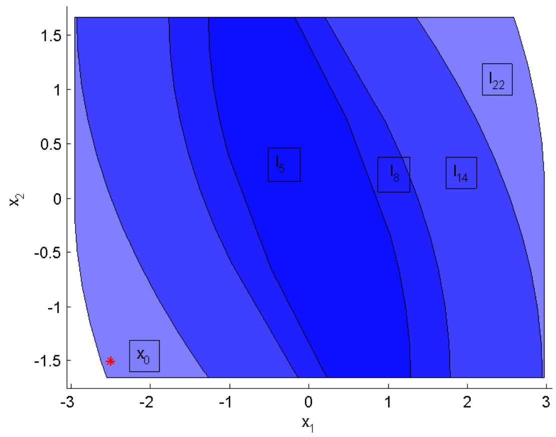

Figure 2.4: The initial state $x(0)$, the maximum robust positively invariant set $\mathbb{S}_f$, and the admissible sets $\mathbb{I}_5$, $\mathbb{I}_8$, $\mathbb{I}_{22}$ for the nominal mini-hovercraft with tightened constraints.

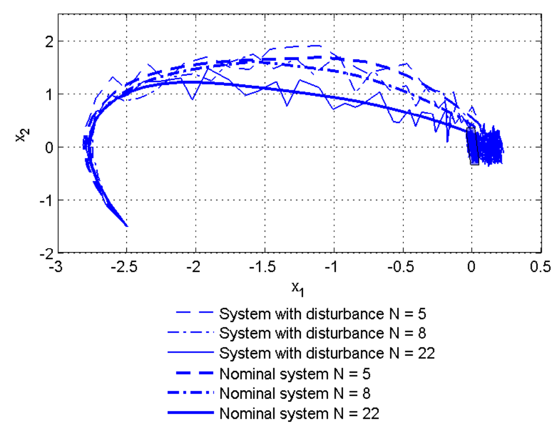

Figure 2.5: State evolution of the mini-hovercraft with disturbances and controlled using: multi-stage robust MPC with $N = 5$; multi-stage robust MPC with $N = 8$; and robust MPC with fixed horizon $N = 22$.

missible sets are smaller than for the simulation of the mini-hovercraft without disturbances. Therefore, robust MPC with minimum horizon $N = 22$ is needed to steer the initial state to the maximum robust positively invariant set $\mathbb{S}_f$, and provides the benchmark for multi-stage robust MPC with $N = 8$ and with $N = 5$. The initial state, the maximum robust positively invariant set $\mathbb{S}_f$ and the pre-computed admissible sets $\mathbb{I}_5$, $\mathbb{I}_8$, $\mathbb{I}_{22}$ for the nominal mini-hovercraft with tightened constraints are shown in Figure 2.4. As Figure 2.4 illustrates, the admissible sets are convex poly-

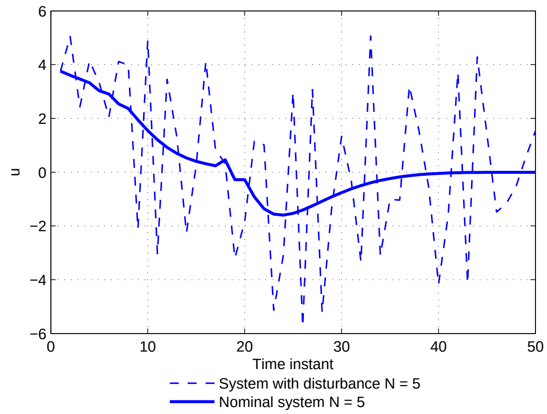

Figure 2.6: Control action for the mini-hovercraft with disturbances with $N = 5$.

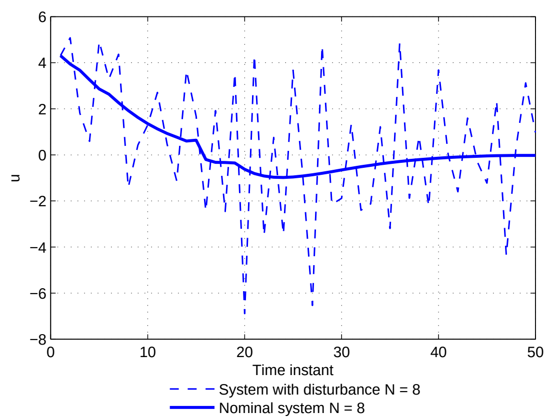

Figure 2.7: Control action for the mini-hovercraft with disturbances with $N = 8$.

gons. They are computed without using inner approximation because 23 inequalities are sufficient to characterize them.

Figure 2.5 depicts the state evolution of the mini-hovercraft affected by disturbances and controlled using: multi-stage robust MPC with $N = 22$; multi-stage robust MPC with $N = 8$; and multi-stage robust MPC with $N = 5$. The state trajectories confirm that multi-stage robust MPC with limited horizon can steer the state of the uncertain system to the maximum robust positively invariant set $\mathbb{S}_f$.

Figures 2.6, 2.7 and 2.8 plot the control actions that determine the state trajectories in Figure 2.5. Note that the control applied to the simulated mini-hovercraft affected by disturbances satisfies the control constraints regardless of the horizon

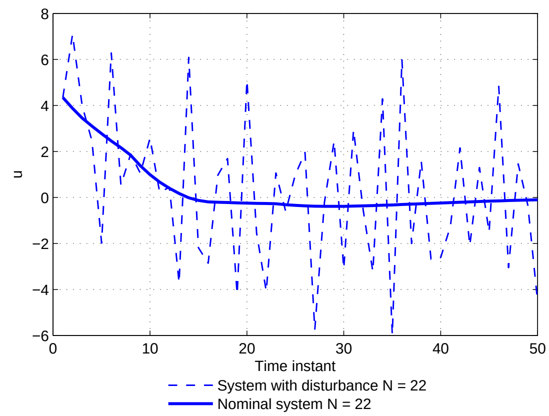

Figure 2.8: Control action for the mini-hovercraft with disturbances with $N = 22$.

selected for the multi-stage robust MPC strategy. The results in Figures 2.6, 2.7 and 2.8 validate that multi-stage robust MPC can enlarge the feasible region of conventional robust MPC with fixed horizon N without increasing the cost of computing the control action.

2.7 Conclusions

This chapter has proposed a multi-stage constrained MPC method which has larger operating region than, but similar computational complexity to, conventional MPC for a given horizon N. Equivalently, the proposed strategy has numerical efficiency similar to conventional MPC with smaller horizon. Therefore, it is advantageous in applications that demand the control action to be derived on-line and with limited computational effort. The proposed strategy relies on a pre-computed sequence of j-step admissible sets, where the j-step admissible set is the set of states from which the system can be driven to the terminal set in j steps. Given the pre-computed admissible sets, multi-stage constrained MPC first determines the minimum number I of steps required to drive the state to the terminal set. Then, it steers the state to the $(I-N)$-step admissible set if $I > N$, or to the terminal set otherwise. The off-line computation of the sequence of admissible sets has been presented, and the feasibility and stability of multi-stage MPC for systems without and with disturbances have been shown. A numerical example has validated that multi-stage MPC with $N = 5$ can stabilize a system that requires conventional MPC with $N \geq 14$ in the absence of disturbances, and requires MPC with $N \geq 22$ when affected by disturbances. The limitations of the proposed multi-stage method are that the computation of the admissible sets can be challenging, and that it cannot be extended to nonlinear systems directly.

Chapter 3

Robust Model Predictive Control of Constrained Nonlinear Systems – Adopting the Non-squared Integrand Objective Function

3.1 Introduction

Various existing approaches have addressed the applicability of the finite horizon optimization and the closed-loop stability of a system with MPC. For linear systems in particular, stabilizing MPC controllers and their synthesis have been studied extensively [42, 49, 68, 70, 71, 89, 92, 104]. In comparison to the well-developed MPC schemes for linear systems, research on MPC strategies for nonlinear systems remains challenging. For deterministic nonlinear systems, the recursive feasibility of the optimization and the closed-loop stability of the system with MPC can be guaranteed by introducing equality constraints [69] or terminal state constraints [12] in the optimization.

For nonlinear systems with disturbances, some existing works have investigated the inherent robustness of an MPC strategy which they first designed for the system without disturbances [32, 33, 67, 91, 116]. Other existing works have proposed various strategies for the design of robust nonlinear MPC, including min-max MPC [85], input-to-state-stability (ISS)-based MPC [17,66], Lyapunov-based MPC [72,73], contraction based MPC [13, 18]. However, those prior strategies need to introduce additional constraints to bound cross terms arising in the variation of the Lyapunov function between successive time instants. The introduction of additional constraints can be avoided by designing a Lyapunov function without cross terms.

3.1.1 Objective, Contributions and Chapter Organization

Inspired by [4, 21, 45], this chapter investigates the robust stability of nonlinear MPC for continuous-time systems with control constraints and external disturbances by adopting a non-squared cost function. Compared to [21] which uses a non-squared stage cost and a stability equality constraint difficult to satisfy in practice [74], the strategy proposed in this chapter uses an integral non-squared cost and a non-squared terminal cost. The contributions of this chapter are three-fold.

- The proposed non-squared cost function provides robustness to disturbances without imposing additional stability constraints on the state trajectory. Specifically, the non-squared cost function is sufficient for robustness because no cross terms appear when evaluating the cost variation between successive time instants.
- The weighting matrix of the terminal cost function is designed to serve as a local control Lyapunov-like function, and the invariant set is constructed accordingly. As a result, the non-squared cost can be treated as a quasi-infinite horizon cost.
- A recursive feasibility and robust stability analysis establishes that, for a given disturbance level, the appropriate design of the sampling interval can guarantee that the system state enters and remains in a robust positively invariant set.

In the remainder of this chapter, Section 3.2 formulates the problem, recalls some preliminary results and shows how to design the terminal weighting matrix associated with the terminal cost. Section 3.3 derives sufficient conditions for the recursive feasibility and for the robust stability of the proposed control strategy. Section 3.4 demonstrates the effectiveness of the proposed method through simulations.

Notation: The superscript “T” denotes the matrix transposition. For a matrix $A \in \mathbb{R}^{n\times n}$, $\bar{\lambda}(A)$ and $\underline{\lambda}(A)$ denote the maximum and minimum eigenvalues of A, respectively. For a vector $x \in \mathbb{R}^n$, $\|x\|_P$ denotes the P-weighted norm of the vector x and is defined as $\|x\|_P = \sqrt{x^{\mathrm{T}}Px}$.

3.2 Preliminaries

Consider the nonlinear system with additive disturbances

$$\dot{x}(t) = f(x(t), u(t)) + w(t),\ x(t_0) = x_0,\ t \geq 0, \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ represents the system state, $u(t) \in \mathbb{R}^m$ the control input, and $w(t) \in \mathbb{R}^n$ the additive disturbances. The disturbances are bounded in a compact set $\mathbb{W}$, and $0 \in \mathbb{W} \subset \mathbb{R}^n$. The control signal of the system in (3.1) is constrained as follows

$$u(t) \in \mathbb{U}, \quad t \geq 0, \tag{3.2}$$

where $\mathbb{U}$ is a compact set containing the origin in its interior.

The nominal dynamics associated with the system in (3.1) are

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t)). \tag{3.3}$$

The following fairly standard assumptions [12, 51] are imposed on the system in (3.1).

Assumption 3.1. *(a) The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in (3.1) is twice continuously differentiable and satisfies $f(0,0) = 0$; (b) For any initial condition $x_0 \in \mathbb{R}^n$ and any piecewise right-continuous $u : [0, \infty) \to \mathbb{U}$, the function in (3.1) admits a unique solution.*

Remark 3.1. *Assumption 3.1 implies that $f(x, u)$ is Lipschitz continuous with respect to x for x in a compact set. Specifically, given $x \in \mathbb{D}$, where $\mathbb{D}$ is a compact set, the following holds: $\|f(x, u) - f(x', u)\| \leq L\|x - x'\|$.*

Assumption 3.2. *The linearized dynamics of the nominal system in (3.3) around the origin are stabilizable, i.e., for $\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t)$, where $A = \frac{\partial f}{\partial x}(0, 0)$ and $B = \frac{\partial f}{\partial u}(0, 0)$, there exists a matrix K with an appropriate dimension such that $A_K = A + BK$ is stable.*

Before presenting the robust constrained MPC strategy, we recall some definitions from [7] that will be used throughout the chapter.

Definition 3.1. *A set $\Omega \subseteq \mathbb{R}^n$ is a positively invariant set for the system $\dot{x}(t) = f(x(t))$, if $x(t) \in \Omega$, $t \geq t_0$, $\forall x(t_0) \in \Omega$.*

Definition 3.2. *A set $\Omega \subseteq \mathbb{R}^n$ is said to be a robust positively invariant set for the system $\dot{x}(t) = f(x(t)) + w(t)$, if $x(t) \in \Omega$, $t \geq t_0$, $\forall x(t_0) \in \Omega$ and $\forall w(t) \in \mathbb{W}$.*

This chapter adopts the non-squared objective function

$$J(t_k) = \int_{t_k}^{t_k+T} \Big(\|\bar{x}(s;t_k)\|_Q + \|\bar{u}(s;t_k)\|_R \Big) ds + \|\bar{x}(t_k+T;t_k)\|_P, \tag{3.4}$$

where T is the prediction horizon and Q and R are weighting matrices satisfying $Q > 0$, $R > 0$. P is the terminal weighting matrix which is to be designed in Lemma 3.1. At time instant t_k, the following optimization is solved

$$\begin{aligned} \min_{\bar{u}(s;t_k), s\in[t_k,t_k+T]} J(t_k) = \min_{\bar{u}(s;t_k), s\in[t_k,t_k+T]} \int_{t_k}^{t_k+T} \Big(\|\bar{x}(s;t_k)\|_Q + \|\bar{u}(s;t_k)\|_R \Big) ds \\ + \|\bar{x}(t_k+T;t_k)\|_P, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \text{subject to}: \quad & \\ & \dot{\bar{x}}(s;t_k) = f(\bar{x}(s;t_k), \bar{u}(s;t_k)), \ s \in [t_k, t_k+T], \\ & \bar{u}(s;t_k) \in \mathbb{U}, \\ & \bar{x}(t_k+T;t_k) \in \Omega_{\alpha\varepsilon}. \end{aligned}$$

Here, Ω_ε is a positively invariant set which will be designed in Lemma 3.1, and α is a shrinking factor satisfying $\alpha \in (0,1)$ which will be determined in Section 3.3. After computing the optimal control signal $\bar{u}^*(s;t_k)$, $s \in [t_k, t_k+T]$, the control signal $\bar{u}^*(s;t_k)$, $s \in [t_k, t_{k+1})$ is applied to the system in (3.1). At the next time instant t_{k+1}, the system state is updated, and a new optimization is solved.

To launch the optimization in (3.5), we need to know *a priori* the positively invariant set Ω_ε and construct an appropriate terminal weighting matrix P. The lemma below provides the design procedure for P and shows that the terminal cost $\|\bar{x}(t_k+T;t_k)\|_P$ can be treated as a control Lyapunov-like function.

Lemma 3.1. *For the nominal system in (3.3), if Assumption 3.2 holds, then,*

(a) *there exists a unique positive definite matrix P^* for the following Lyapunov equation*

$$(A_K + \kappa I)^{\mathrm{T}} P^* + P^*(A_K + \kappa I) = -Q^*, \tag{3.6}$$

where $Q^* = 2(1+\alpha_2)\sqrt{\frac{1}{\underline{\lambda}(Q)}}Q$, $\alpha_2 = \sqrt{\frac{\bar{\lambda}(K^{\mathrm{T}}RK)}{\underline{\lambda}(Q)}}$; $\kappa \in [0, -\bar{\lambda}(A_K))$;

(b) *a neighborhood of the origin*

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n : \|x\|_P \leq \varepsilon\} \tag{3.7}$$

can be constructed with $P = \bar{\lambda}(P^*)P^*$ *such that*

1. *The state feedback controller* $u(t) = Kx(t)$ *within the set* Ω_ε *satisfies the control constraints, i.e.,* $\forall x(t) \in \Omega_\varepsilon,\ u(t) = Kx(t) \in \mathbb{U}$*;*
2. $\forall x(t_0) \in \Omega_\varepsilon$*, for the nominal system in (3.3) equipped with the state feedback control law* $u(t) = Kx(t)$*, we have* $x(t) \in \Omega_\varepsilon$*,* $\forall t \geq t_0$*;*
3. $\forall x(t_k) \in \Omega_\varepsilon$*, the terminal cost* $\|x(t_k)\|_P$ *can serve as a control Lyapunov-like function in the sense that*

$$\int_{t_k}^{t_k+\delta} (\|x(t)\|_Q + \|u(t)\|_R)dt \leq \|x(t_k)\|_P - \|x(t_k+\delta)\|_P, \tag{3.8}$$

or in the differential form

$$\frac{d}{dt}(\|x(t)\|_P) \leq -(\|x(t)\|_Q + \|u(t)\|_R). \tag{3.9}$$

Proof. **(a)** By definition, we have $Q^* > 0$. From the definition of κ, it follows that the matrix $A_K + \kappa I$ is stable, i.e., all the real parts of its eigenvalues are negative. Thus, the Lyapunov equation in (3.6) admits a unique positive definite solution P^*.

(b)

1. Since the control constraint $\mathbb{U}$ is a compact set containing the origin in its interior, and the state feedback control law $u(t) = Kx(t)$ is a linear mapping, we can always find a positive constant $\bar{\varepsilon}$ such that within the region specified by $\varepsilon \leq \bar{\varepsilon}$: $\Omega_\varepsilon = \{x \in \mathbb{R}^n : \|x\|_P \leq \varepsilon\}$, $Kx(t) \in \mathbb{U}$ for all $x(t) \in \Omega_\varepsilon$. This completes the proof.
2. The proofs of the remaining parts are provided hereafter. Differentiation of the terminal cost with respect to t gives

$$\frac{d}{dt}(\|x(t)\|_P) = \frac{x(t)^{\mathrm{T}}(A_K^{\mathrm{T}}P + PA_K)x(t) + 2x(t)^{\mathrm{T}}P\phi(x(t))}{2\|x(t)\|_P}, \tag{3.10}$$

where $\phi(x(t)) = f(x(t), Kx(t)) - A_K x(t)$. Denote

$$\begin{aligned} J_s &= \|x(t)\|_Q + \|u(t)\|_R \\ &= \|x(t)\|_Q + \|x(t)\|_{K^{\mathrm{T}}RK}. \end{aligned}$$

Note that

$$\|x(t)\|_{K^{\mathrm{T}}RK} \leq \sqrt{\frac{\bar{\lambda}(K^{\mathrm{T}}RK)}{\underline{\lambda}(Q)}}\|x(t)\|_Q, \tag{3.11}$$

and

$$\|x(t)\|_{K^{\mathrm{T}}RK} \geq \sqrt{\frac{\underline{\lambda}(K^{\mathrm{T}}RK)}{\bar{\lambda}(Q)}}\|x(t)\|_Q, \tag{3.12}$$

and combine (3.11) and (3.12) to yield

$$\left(1 + \sqrt{\frac{\underline{\lambda}(K^{\mathrm{T}}RK)}{\bar{\lambda}(Q)}}\right)\|x(t)\|_Q \leq J_s \leq \left(1 + \sqrt{\frac{\bar{\lambda}(K^{\mathrm{T}}RK)}{\underline{\lambda}(Q)}}\right)\|x(t)\|_Q.$$

For the simplicity of presentation, denote

$$\alpha_1 = \sqrt{\frac{\underline{\lambda}(K^{\mathrm{T}}RK)}{\bar{\lambda}(Q)}}, \quad \alpha_2 = \sqrt{\frac{\bar{\lambda}(K^{\mathrm{T}}RK)}{\underline{\lambda}(Q)}},$$

then,

$$(1 + \alpha_1)\|x(t)\|_Q \leq J_s \leq (1 + \alpha_2)\|x(t)\|_Q,$$

i.e.,

$$-(1 + \alpha_2)\|x(t)\|_Q \leq -J_s \leq -(1 + \alpha_1)\|x(t)\|_Q.$$

Thus, Inequality (3.9) holds if the following relatively conservative inequality holds

$$\frac{d}{dt}(\|x(t)\|_P) \leq -(1 + \alpha_2)\|x(t)\|_Q. \tag{3.13}$$

From (3.10), Inequality (3.13) can be further replaced by

$$\begin{aligned} &\frac{x(t)^{\mathrm{T}}(A_K^{\mathrm{T}}P + PA_K)x(t) + 2x(t)^{\mathrm{T}}P\phi(x(t))}{2\|x(t)\|_P} \\ &\leq -(1 + \alpha_2)\|x(t)\|_Q. \end{aligned} \tag{3.14}$$

Inequality (3.14) is equivalent to

$$\begin{aligned} &x(t)^{\mathrm{T}}(A_K^{\mathrm{T}}P + PA_K)x(t) + 2x(t)^{\mathrm{T}}P\phi(x(t)) \\ &\leq -2(1+\alpha_2)\|x(t)\|_Q\|x(t)\|_P. \end{aligned} \tag{3.15}$$

Considering that

$$\sqrt{\frac{\underline{\lambda}(P)}{\bar{\lambda}(Q)}}\|x(t)\|_Q \leq \|x(t)\|_P \leq \sqrt{\frac{\bar{\lambda}(P)}{\underline{\lambda}(Q)}}\|x(t)\|_Q,$$

the right-hand side of (3.15) is upper and lower bounded by

$$\begin{aligned} &-2(1+\alpha_2)\sqrt{\frac{\bar{\lambda}(P)}{\underline{\lambda}(Q)}}x(t)^{\mathrm{T}}Qx(t) \\ &\leq -2(1+\alpha_2)\|x(t)\|_Q\|x(t)\|_P \\ &\leq -2(1+\alpha_2)\sqrt{\frac{\underline{\lambda}(P)}{\bar{\lambda}(Q)}}x(t)^{\mathrm{T}}Qx(t). \end{aligned}$$

Therefore, Inequality (3.15) holds if the following inequality holds

$$\begin{aligned} &x(t)^{\mathrm{T}}(A_K^{\mathrm{T}}P + PA_K)x(t) + 2x(t)^{\mathrm{T}}P\phi(x(t)) \\ &\leq -2(1+\alpha_2)\sqrt{\frac{\bar{\lambda}(P)}{\underline{\lambda}(Q)}}x(t)^{\mathrm{T}}Qx(t). \end{aligned} \tag{3.16}$$

Inequality (3.16) can be rewritten as

$$\begin{aligned} &x(t)^{\mathrm{T}}\left(A_K^{\mathrm{T}}\frac{P}{\sqrt{\bar{\lambda}(P)}} + \frac{P}{\sqrt{\bar{\lambda}(P)}}A_K\right)x(t) + 2x(t)^{\mathrm{T}}\frac{P}{\sqrt{\bar{\lambda}(P)}}\phi(x(t)) \\ &\leq -2(1+\alpha_2)\sqrt{\frac{1}{\underline{\lambda}(Q)}}x(t)^{\mathrm{T}}Qx(t). \end{aligned} \tag{3.17}$$

Denote $P^* = \frac{P}{\sqrt{\bar{\lambda}(P)}}$ and $Q^* = 2(1+\alpha_2)\sqrt{\frac{1}{\underline{\lambda}(Q)}}Q$. Since $\frac{\|\phi(x(t))\|_{P^*}}{\|x(t)\|_{P^*}} \to 0$ as $\|x(t)\|_{P^*} \to 0$, there exists a neighborhood around the origin such that $\frac{\|\phi(x(t))\|_{P^*}}{\|x(t)\|_{P^*}} \leq \kappa$ holds.

Moreover, the left-hand side of Inequality (3.17) can be further written as

$$\begin{aligned}
&x(t)^{\mathrm{T}}(A_K^{\mathrm{T}}P^* + P^*A_K)x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)) \\
&= x(t)^{\mathrm{T}}[(A_K+\kappa I)^{\mathrm{T}}P^* + P^*(A_K+\kappa I)]x(t) \\
&\quad - 2x(t)^{\mathrm{T}}\kappa x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)) \\
&= -x(t)^{\mathrm{T}}Q^*x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)) - 2\kappa x(t)^{\mathrm{T}}P^*x(t). \qquad (3.18)
\end{aligned}$$

The last two terms in (3.18) are upper bounded as follows

$$\begin{aligned}
&2x(t)^{\mathrm{T}}P^*\phi(x(t)) - 2\kappa x(t)^{\mathrm{T}}P^*x(t) \\
&\leq 2[(P^*)^{\frac{1}{2}}x(t)]^{\mathrm{T}}[(P^*)^{\frac{1}{2}}\phi(x(t))] - 2\kappa x(t)^{\mathrm{T}}P^*x(t). \qquad (3.19)
\end{aligned}$$

Substituting $\frac{\|\phi(x(t))\|_{P^*}}{\|x(t)\|_{P^*}} \leq \kappa$ into (3.19) leads to

$$\begin{aligned}
&2[\|x(t)\|_{P^*}\|\phi(x(t))\|_{P^*} - \kappa x(t)^{\mathrm{T}}P^*x(t)] \\
&\leq 2[\kappa x(t)^{\mathrm{T}}P^*x(t) - \kappa x(t)^{\mathrm{T}}P^*x(t)] \\
&= 0.
\end{aligned}$$

Therefore, provided that the condition $\frac{\|\phi(x(t))\|_{P^*}}{\|x(t)\|_{P^*}} \leq \kappa$ holds, Inequality (3.17) holds. This completes the proof.

□

Remark 3.2. *Note from (3.8) that, $\forall x(t_k) \in \Omega_\varepsilon$, we have $\int_{t_k}^{\infty}(\|x(t)\|_Q + \|u(t)\|_R)dt \leq \|x(t_k)\|_P$, $\forall(t_k)$. In other words,when starting from a feasible initial point $x(t_k)$, the proposed state feedback controller effectively upper bounds the infinite horizon cost. Therefore, the cost function in the optimization in (3.5) can be regarded as a quasi-infinite horizon cost function.*

Remark 3.3. *The results in Lemma 3.1 are sufficient conditions, and thus, are most likely conservative. To make the conditions in Lemma 3.1 less conservative, the following procedure is applied to enlarge the invariant set.*

Step 1: *Given Q, compute Q^* as*

$$Q^* = 2(1+\alpha_2)\sqrt{\frac{1}{\underline{\lambda}(Q)}}Q, \text{ where } \alpha_2 = \sqrt{\frac{\bar{\lambda}(K^{\mathrm{T}}RK)}{\underline{\lambda}(Q)}}.$$

Step 2: *Choose κ from $[0,\ -\bar{\lambda}(A_K))$, then compute P^* using (3.6), and thus P. Introduce a shrinking factor γ. Decrease γ from 1, and at the same time, compute the corresponding ε as follows.*

Denote $\bar{P} = \gamma P$, and find the maximum ε such that the optimal value of the following optimization is negative

$$J_\gamma = \max\left\{x^{\mathrm{T}}\bar{P}f(x, Kx) + \|x\|_{\bar{P}}(\|x\|_Q + \|Kx\|_R)\right\}, \tag{3.20}$$
$$\text{subject to :}$$
$$\|x\|_P \le \varepsilon,$$

where Equation (3.20) is obtained using (3.9) and (3.10).

Through simulations, we observe that as γ decreases, ε also decreases. Thus, a tradeoff is required between γ and ε. The criteria we take are: 1) *ε should be as large as possible, to increase the disturbance level that can be tolerated;* 2) *the eigenvalues of γP should not be too large to make the results less conservative. We select an ε in the feasible region such that a change in ε induces a negligible change in the eigenvalues of γP.*

At each updating time instant t_k, the associated trajectories for the nonlinear system in (3.1) are defined as follows: $x(s; t_k)$ represents the actual state of the system at time s with the control signal $\bar{u}^*(s; t_k)$; $\bar{x}^*(s; t_k)$ is the optimal state of the nominal system at time s with the optimal control signal $\bar{u}^*(s; t_k)$; $\hat{u}^a(s; t_k)$ is the assumed control signal generated as

$$\hat{u}^a(s; t_k) = \begin{cases} \bar{u}^*(s; t_{k-1}),\ s \in [t_k, t_{k-1} + T), \\ K\hat{x}^a(s; t_k),\ s \in [t_{k-1} + T, t_k + T], \end{cases} \tag{3.21}$$

where $\hat{x}^a(s; t_k)$ is computed using $\dot{\hat{x}}^a(s; t_k) = f(\hat{x}^a(s; t_k), \hat{u}^a(s; t_k))$ with $\hat{x}^a(t_k; t_k) = x(t_k)$.

Due to the disturbances, the optimal nominal system state will deviate from the actual system state. The following lemma which is used in [51] provides an upper bound of the deviation.

Lemma 3.2. *1. The discrepancy between the optimal nominal system state and* *the actual system state is upper bounded by*

$$\|x(s;t_k) - \bar{x}^*(s;t_k)\|_P \leq \bar{\lambda}(P^{\frac{1}{2}})\rho(s-t_k)e^{L(s-t_k)}, \ s \in [t_k, t_k+T], \qquad (3.22)$$

where ρ *is the maximum disturbance level, i.e.,* $\max_{t\geq 0}\|w(t)\|_P \leq \rho$*, and* L *is the Lipschitz constant of* $f(x,u)$ *within the compact state space of interest.*

2. *The discrepancy between the assumed system state and the optimal nominal system state is upper bounded by*

$$\begin{aligned} &\|\hat{x}^a(s;t_{k+1}) - \bar{x}^*(s;t_k)\|_P \\ &\leq \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{L\delta}e^{L(s-t_{k+1})}, \end{aligned} \qquad (3.23)$$

where δ *is the sampling interval and* L *is the Lipschitz constant.*

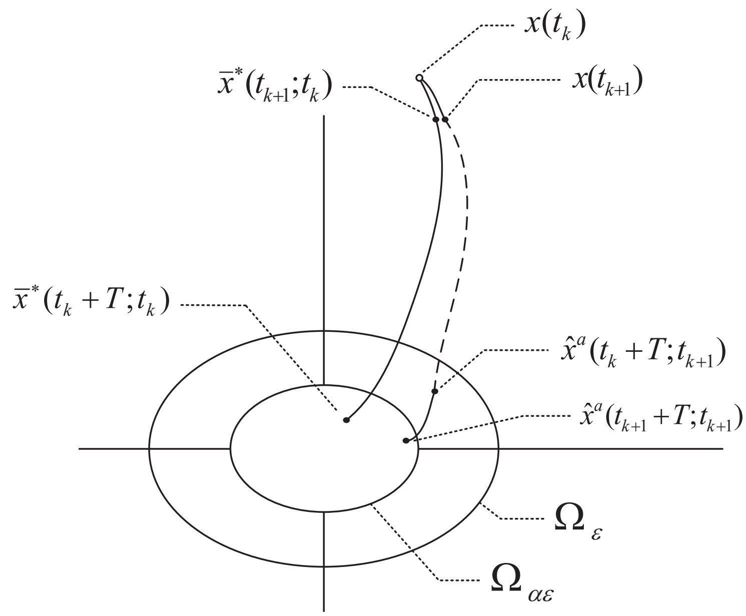

Figure 3.1: A robust MPC diagram.

Before presenting the main results, the main idea of the proposed robust MPC strategy is illustrated in Figure 3.1. Suppose that an optimal control signal $\bar{u}^*(s;t_k)$, $s \in [t_k, t_k+T]$, given at time instant t_k can steer the nominal system in (3.3) into the positively invariant set $\Omega_{\alpha\varepsilon}$. Due to the disturbances, the optimal control signal $\bar{u}^*(s;t_k)$, $s \in [t_k, t_k+\delta]$ will steer the state of the system in (3.1) $x(t_{k+1})$, instead of

$\bar{x}^*(t_{k+1}; t_k)$ at time t_{k+1}. An assumed control signal $\hat{u}^a(s; t_{k+1})$ as in (3.21) can be constructed. The assumed control signal will be feasible if the following conditions hold: 1) at time instant $t_k + T$, the assumed system state $\hat{x}^a(t_k + T; t_{k+1})$ enters the positively invariant set Ω_ε; 2) within the positively invariant set Ω_ε (contracting set), the system state enters the terminal set $\Omega_{\alpha\varepsilon}$ imposed in the optimization in (3.5) during the time interval $[t_k + T, t_{k+1} + T]$.

3.3 Main Results

Successful implementations of robust MPC require repeatedly solving the optimization in (3.5). This section derives sufficient conditions for the recursive feasibility of the proposed MPC with a non-squared cost function. It also shows that, under some mild conditions, the system state will enter and remain in the robust positively invariant set.

3.3.1 Recursive Feasibility

Lemma 3.3. *For the system in (3.1), if the optimization in (3.5) is feasible at time instant t_k, then it is also feasible at time instant t_{k+1}, provided that the sampling interval satisfies the following condition*

$$-\frac{1}{\beta}\ln\alpha \leq \delta \leq \frac{(1-\alpha)\varepsilon}{\rho e^{LT}\bar{\lambda}(P^{\frac{1}{2}})}, \tag{3.24}$$

where $\beta = \left(\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} + \sqrt{\frac{\underline{\lambda}(K^{\mathrm{T}}RK)}{\bar{\lambda}(P)}}\right)$, *and that the disturbances are upper bounded by*

$$\rho^{\max} = \frac{-(1-\alpha)\varepsilon\beta}{e^{LT}\bar{\lambda}(P^{\frac{1}{2}})\ln\alpha}. \tag{3.25}$$

Proof. In the constrained optimization in (3.5), the terminal state satisfies

$$\|\bar{x}^*(t_k + T; t_k)\|_P \leq \alpha\varepsilon. \tag{3.26}$$

According to Lemma 3.2, the discrepancy between the assumed system state and the optimal nominal system state is upper bounded as in (3.23). Substituting $s = t_k + T$

into (3.23) gives

$$\|\hat{x}^a(t_k+T;t_{k+1})-\bar{x}^*(t_k+T;t_k)\|_P \le \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{LT}. \tag{3.27}$$

Combining (3.26) and (3.27) leads to

$$\begin{aligned}
&\|\hat{x}^a(t_k+T;t_{k+1})\|_P \\
&\le \|\bar{x}^*(t_k+T;t_k)\|_P + \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{LT} \\
&\le \alpha\varepsilon + \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{LT}.
\end{aligned}$$

Considering the condition $\delta \le \frac{(1-\alpha)\varepsilon}{\rho e^{LT}\bar{\lambda}(P^{\frac{1}{2}})}$, it follows that

$$\bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{LT} + \alpha\varepsilon \le \varepsilon. \tag{3.28}$$

Inequality (3.28) implies that $\hat{x}^a(t_k+T;t_{k+1})$ enters the positively invariant set. Using Lemma 3.1 together with the Lyapunov function $V(\hat{x}^a(t)) = \|\hat{x}^a(t)\|_P$ yields

$$\begin{aligned}
\dot{V}(\hat{x}^a(t)) &= \frac{d}{dt}(\|\hat{x}^a(t)\|_P) \\
&\le -\Big(\|\hat{x}^a(t)\|_Q + \|\hat{u}^a(t)\|_R\Big) \\
&\le -\left(\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} + \sqrt{\frac{\underline{\lambda}(K^{\mathrm{T}}RK)}{\bar{\lambda}(P)}}\right)\|\hat{x}^a(t)\|_P.
\end{aligned} \tag{3.29}$$

For the simplicity of notation, let $\beta = \left(\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} + \sqrt{\frac{\underline{\lambda}(K^{\mathrm{T}}RK)}{\bar{\lambda}(P)}}\right)$, then $\dot{V}(\hat{x}^a(t)) \le -\beta V(\hat{x}^a(t))$.

From the comparison principle and (3.29) it follows that

$$V(\hat{x}^a(s;t_{k+1})) \le V(\hat{x}^a(t_k+T;t_{k+1}))e^{-\beta(s-t_k-T)},$$

and if $\delta \ge -\frac{1}{\beta}\ln\alpha$ holds, that

$$V(\hat{x}^a(t_{k+1}+T;t_{k+1})) \le \alpha\varepsilon. \tag{3.30}$$

Inequality (3.30) indicates that the terminal state of the assumed state trajectory satisfies the terminal inequality constraints in the optimization in (3.5). This completes the proof. □

Lemma 3.3 shows that the proposed MPC with non-squared cost is guaranteed recursively feasible if the sampling interval satisfies (3.24) and the disturbances are upper bounded as in (3.25).

3.3.2 Stability

As pointed out in [70], recursive feasibility does not necessarily ensure stability. This section analyzes the stability of the proposed control strategy. The following theorem states the conditions under which the state of the nonlinear system in (3.1) with the control scheme proposed in (3.5) converges to a positively invariant set.

Theorem 3.1. *Given the system in (3.1) with the control generated in (3.5), if Assumptions 3.1 and 3.2 hold, the sampling interval satisfies (3.24), the disturbance is upper bounded by $\rho^{\max}$ as in (3.25), and there exists a positive constant pair (ϵ, ζ) satisfying $\epsilon + \zeta < 1$ such that the following conditions hold*

$$\sqrt{\frac{\bar{\lambda}(Q)}{\underline{\lambda}(P)}}\frac{(1-\alpha)\varepsilon}{L} \leq (1-\epsilon-\zeta)\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}}\alpha\delta\varepsilon, \tag{3.31a}$$

$$\bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{LT} \leq \epsilon\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}}\alpha\delta\varepsilon, \tag{3.31b}$$

then, the system state converges to the positively invariant set Ω_ε in finite time.

Proof. The change of the cost function along the system trajectory between time instants t_k and t_{k+1} is evaluated as

$$
\begin{aligned}
J^*(t_{k+1}) - J^*(t_k) \leq{}& J(t_{k+1}) - J^*(t_k) \\
={}& \int_{t_{k+1}}^{t_{k+1}+T} \Big(\|\hat{x}^a(s;t_{k+1})\|_Q + \|\hat{u}^a(s;t_{k+1})\|_R \Big) ds \\
&\quad + \|\hat{x}^a(t_{k+1}+T;t_{k+1})\|_P \\
&- \int_{t_k}^{t_k+T} \Big(\|\bar{x}^*(s;t_k)\|_Q + \|\bar{u}^*(s;t_k)\|_R \Big) ds \\
&\quad + \|\bar{x}^*(t_k+T;t_k)\|_P \\
={}& \int_{t_{k+1}}^{t_k+T} \Big(\|\hat{x}^a(s;t_{k+1})\|_Q - \|\bar{x}^*(s;t_k)\|_Q \\
&\quad + \|\hat{u}^a(s;t_{k+1})\|_R - \|\bar{u}^*(s;t_k)\|_R \Big) ds \qquad (3.32a) \\
&+ \int_{t_k+T}^{t_{k+1}+T} \Big(\|\hat{x}^a(s;t_{k+1})\|_Q + \|\hat{u}^a(s;t_{k+1})\|_R \Big) ds \\
&\quad + \|\hat{x}^a(t_{k+1}+T;t_{k+1})\|_P - \|\bar{x}^*(t_k+T;t_k)\|_P \qquad (3.32b) \\
&- \int_{t_k}^{t_{k+1}} (\|\bar{x}^*(s;t_k)\|_Q + \|\bar{u}^*(s;t_k)\|_R) ds. \qquad (3.32c)
\end{aligned}
$$

Term (3.32a) is upper bounded as follows

$$
\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \Big(\|\hat{x}^a(s;t_{k+1})\|_Q - \|\bar{x}^*(s;t_k)\|_Q \\
&\quad + \|\hat{u}^a(s;t_{k+1})\|_R - \|\bar{u}^*(s;t_k)\|_R \Big) ds \\
={}& \int_{t_{k+1}}^{t_k+T} \Big(\|\hat{x}^a(s;t_{k+1})\|_Q - \|\bar{x}^*(s;t_k)\|_Q \Big) ds \\
\leq{}& \int_{t_{k+1}}^{t_k+T} \|\hat{x}^a(s;t_{k+1}) - \bar{x}^*(s;t_k)\|_Q ds. \qquad (3.33)
\end{aligned}
$$

According to Lemma 3.2, the right-hand side of (3.33) is upper bounded by

$$
\|\hat{x}^a(s;t_{k+1}) - \bar{x}^*(s;t_k)\|_P \leq \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{L\delta} e^{L(s-t_{k+1})}. \qquad (3.34)
$$

Substituting (3.34) into (3.33) yields

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \|\hat{x}^a(s;t_{k+1}) - \bar{x}^*(s;t_k)\|_Q ds \\
&\leq \int_{t_{k+1}}^{t_k+T} \sqrt{\frac{\bar{\lambda}(Q)}{\underline{\lambda}(P)}} \|\hat{x}^a(s;t_{k+1}) - \bar{x}^*(s;t_k)\|_P ds \\
&\leq \sqrt{\frac{\bar{\lambda}(Q)}{\underline{\lambda}(P)}} \int_{t_{k+1}}^{t_k+T} \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{L\delta} e^{L(s-t_{k+1})} ds.
\end{aligned} \tag{3.35}$$

The bound on (3.32a) can be simplified as follows

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{L(s+\delta-t_{k+1})} ds \\
&= \bar{\lambda}(P^{\frac{1}{2}})\rho\delta \int_{t_{k+1}}^{t_k+T} e^{L(s+\delta-t_{k+1})} ds \\
&= \bar{\lambda}(P^{\frac{1}{2}})\rho\delta \frac{e^{LT} - e^{L\delta}}{L} \\
&= \bar{\lambda}(P^{\frac{1}{2}})\rho\delta \frac{e^{LT}(1 - e^{-L(T-\delta)})}{L} \\
&\leq \frac{(1-\alpha)\varepsilon}{L}(1 - e^{-L(T-\delta)}) \\
&\leq \frac{(1-\alpha)\varepsilon}{L}.
\end{aligned} \tag{3.36}$$

Therefore, Term (3.32a) is upper bounded by

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \Big(\|\hat{x}^a(s;t_{k+1})\|_Q - \|\bar{x}^*(s;t_k)\|_Q \\
&\quad + \|\hat{u}^a(s;t_{k+1})\|_R - \|\bar{u}^*(s;t_k)\|_R \Big) ds \\
&\leq \sqrt{\frac{\bar{\lambda}(Q)}{\underline{\lambda}(P)}} \frac{(1-\alpha)\varepsilon}{L}.
\end{aligned} \tag{3.37}$$

For Term (3.32b), according to Lemma 3.1 and (3.8), we have

$$\begin{aligned}
&\int_{t_k+T}^{t_{k+1}+T}\Big(\|\hat{x}^a(s;t_{k+1})\|_Q+\|\hat{u}^a(s;t_{k+1})\|_R\Big)ds \\
&\quad+\|\hat{x}^a(t_{k+1}+T;t_{k+1})\|_P-\|\bar{x}^*(t_k+T;t_k)\|_P \\
&\le \|\hat{x}^a(t_k+T;t_{k+1})\|_P-\|\hat{x}^a(t_{k+1}+T;t_{k+1})\|_P \\
&\quad+\|\hat{x}^a(t_{k+1}+T;t_{k+1})\|_P-\|\bar{x}^*(t_k+T;t_k)\|_P \\
&=\|\hat{x}^a(t_k+T;t_{k+1})\|_P-\|\bar{x}^*(t_k+T;t_k)\|_P \\
&\le \|\hat{x}^a(t_k+T;t_{k+1})-\bar{x}^*(t_k+T;t_k)\|_P.
\end{aligned} \tag{3.38}$$

In light of Lemma 3.2, Term (3.32b) is bounded by

$$\begin{aligned}
&\|\hat{x}^a(t_k+T;t_{k+1})-\bar{x}^*(t_k+T;t_k)\|_P \\
&\le \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{LT}.
\end{aligned}$$

Term (3.32c) is upper bounded by

$$\begin{aligned}
&-\int_{t_k}^{t_{k+1}}\Big(\|\bar{x}^*(s;t_k)\|_Q+\|\bar{u}^*(s;t_k)\|_R\Big)ds \\
&\le -\int_{t_k}^{t_{k+1}}\|\bar{x}^*(s;t_k)\|_Q ds.
\end{aligned} \tag{3.39}$$

Considering

$$\|x(s;t_k)-\bar{x}^*(s;t_k)\|_P \le \bar{\lambda}(P^{\frac{1}{2}})\rho(s-t_k)e^{L(s-t_k)},$$

it follows that

$$\|\bar{x}^*(s;t_k)\|_P \ge \varepsilon-\bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{L\delta} \ \text{ for } s\in[t_k,t_{k+1}].$$

Therefore, Term (3.32c) is evaluated to be upper bounded as

$$-\int_{t_k}^{t_{k+1}} \|\bar{x}^*(s;t_k)\|_Q ds \leq -\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} \int_{t_k}^{t_{k+1}} \|\bar{x}^*(s;t_k)\|_P ds$$
$$\leq -\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} (\varepsilon - \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{L\delta})\delta$$
$$\leq -\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} \alpha\delta\varepsilon, \tag{3.40}$$

where the last inequality follows after using (3.28).

Finally, the difference of the cost function between $J^*(t_{k+1})$ and $J^*(t_k)$ satisfies

$$J^*(t_{k+1}) - J^*(t_k) \leq J(t_{k+1}) - J^*(t_k)$$
$$\leq -\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} \alpha\delta\varepsilon + \sqrt{\frac{\bar{\lambda}(Q)}{\underline{\lambda}(P)}} \frac{(1-\alpha)\varepsilon}{L} + \bar{\lambda}(P^{\frac{1}{2}})\rho\delta e^{LT}. \tag{3.41}$$

Substituting (3.31a) and (3.31b) into (3.41) gives

$$J^*(t_{k+1}) - J^*(t_k) \leq -\zeta\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}} \alpha\delta\varepsilon. \tag{3.42}$$

Using $\zeta\sqrt{\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}}\alpha\delta\varepsilon > 0$, it follows that any trajectory of the system in (3.1) starting from a feasible initial point will enter the set Ω_ε in finite time. This completes the proof. □

Remark 3.4. *The positively invariant set Ω_ε characterized by ε plays an important role in proving stability, recursive feasibility and convergence to a set. However, an extremely small positively invariant set (as in the case of [92]) could result in a small domain of attraction, which would restrict the applicability of the proposed control strategy. A possible remedy is to exploit the idea introduced in [36]. In [36], in order to steer the system with initial state x_0 to the target domain $\mathbb{D}_0$, the authors developed the expansion sets $\mathbb{E}_f^1(\mathbb{D}_0, \epsilon)$, $\mathbb{E}_f^2(\mathbb{D}_0, \epsilon)$, $\cdots$, (See Sections 3.1 and 3.2 in [36]), and derived the controller by solving a set of inequalities obtained directly from the system equation. Using the expansion sets constructed in [36], the terminal constraint Ω_ε in the optimization in (3.5) can be replaced with $\mathbb{E}_f^i(\Omega_\varepsilon, \epsilon)$ at time t_0. For the successive optimizations, the terminal set can be updated continuously with $\mathbb{E}_f^{i-1}(\Omega_\varepsilon, \epsilon)$,*

$\mathbb{E}_f^{i-2}(\Omega_\varepsilon, \epsilon)$ *until* $\mathbb{E}_f^0(\Omega_\varepsilon, \epsilon) = \Omega_\varepsilon$. *Recursive feasibility, stability, and convergence of this strategy deserves further research.*

After the system state enters the positively invariant set Ω_ε, the fixed state feedback control law $u(t) = Kx(t)$ will be adopted. The following theorem guarantees that there exists a robust positively invariant set where the state will stay if the disturbance is bounded in a certain level.

Theorem 3.2. *For the system in (3.1), assume that the system state has entered the positively invariant set* Ω_ε*, and the state feedback control law* $u(t) = Kx(t)$ *is adopted within the positively invariant set.. Then, for any positive constant* $\beta \in (0, 1)$*, if the disturbance level is further bounded by* $\bar{\rho} \leq \frac{\underline{\lambda}(Q^*)\beta\varepsilon}{2\bar{\lambda}(P)\bar{\lambda}(P^{\frac{1}{2}})}$*, using the robust MPC strategy proposed in this chapter, the set* $\Omega_{\beta\varepsilon} = \{x \in \mathbb{R}^n : \|x\|_P \leq \beta\varepsilon\}$ *will be a robust positively invariant set for the system in (3.1).*

Proof. Based on (3.6), the differential form in (3.10) is

$$\frac{d}{dt}(\|x(t)\|_P) \leq \frac{-x(t)^{\mathrm{T}}Q^*x(t) + 2x(t)^{\mathrm{T}}Pw(t)}{2\|x(t)\|_P}. \tag{3.43}$$

For any $\beta \in (0, 1)$, if $x(t)$ is not in the set $\Omega_{\beta\varepsilon} = \{x \in \mathbb{R}^n : \|x\|_P \leq \beta\varepsilon\}$, in view of $\|w(t)\| \leq \bar{\rho} = \frac{\underline{\lambda}(Q^*)\beta\varepsilon}{2\bar{\lambda}(P)\bar{\lambda}(P^{\frac{1}{2}})}$, the numerator in the right-hand side of (3.43) is upper bounded by

$$\begin{aligned}
&- x(t)^{\mathrm{T}}Q^*x(t) + 2x(t)^{\mathrm{T}}Pw(t) \\
&\leq -\frac{\underline{\lambda}(Q^*)}{\bar{\lambda}(P)}\|x(t)\|_P^2 + 2\|x(t)P^{\frac{1}{2}}\| \cdot \|p^{\frac{1}{2}}\| \cdot \|w(t)\| \\
&\leq 0.
\end{aligned}$$

Therefore, the system state will asymptotically converge to the set $\Omega_{\beta\varepsilon}$.

Once the state enters the set $\Omega_{\beta\varepsilon}$, the line of thought in [74] can be followed to show that, the system state will stay within this set. This completes the proof. □

Algorithm 6 summarizes the implementation procedure for the proposed robust MPC strategy.

Remark 3.5. *When the system state enters the positively invariant set* Ω_ε*, it is still valid to implement the MPC strategy, instead of using the dual-mode controller* $u(t) = Kx(t)$*. The robustness to disturbances of the MPC strategy relies on the*

Algorithm 6 Robust MPC-Adopting the non-squared integrand objective function

Initialization:

The system dynamics in (3.1), the control constraints $u(t) \in \mathbb{U}$, and the disturbance bound $w(t) \in \mathbb{W}$; the weighting matrices Q, R in the optimization in (3.5), and the prediction horizon T.

Step 1:

Compute the terminal weighting matrix P and the associated positively invariant set Ω_ε according to Lemma 3.1.

Step 2:

Use (3.24) to compute the values that the sampling interval δ can take; furthermore, use Theorem 3.1 to choose a sampling interval δ compatible with (3.24) and with (3.31). In this process, determine a shrinking factor α.

Step 3:

At each time instant t_k, solve the optimization in (3.5) with the P, α, Ω_ε and δ computed above. Apply the resulting optimal control signal to the system in (3.1) from t_k to $t_k + \delta$.

Step 4:

Repeat Step 3 until the system state enters the positively invariant set Ω_ε. Then, apply the state feedback control law $u(t) = Kx(t)$.

---

continuity property illustrated in [32], where it is shown that small disturbances can be tolerated.

Remark 3.6. *When the continuous-time MPC controller proposed in this chapter is implemented in a sampled-data fashion, several stability issues may arise as pointed out in [79]. These issues will be addressed in further research work.*

3.4 Illustrative Example

To demonstrate the effectiveness of the proposed robust constrained MPC scheme, the results developed in Section 3.3 are applied to the numerical example adopted in [12, 69]. The system dynamics are

$$\dot{x}_1(t) = x_2(t) + u(t)(\mu + (1-\mu)x_1(t)), \tag{3.44a}$$

$$\dot{x}_2(t) = x_1(t) + u(t)(\mu - 4(1-\mu)x_2(t)) + w(t). \tag{3.44b}$$

It can be verified that, for $\mu \in (0, 1)$, the linearization of the nominal system in (3.44) around the origin is stabilizable, which satisfies Assumption 3.2. The control constraints are

$$\mathbb{U} = \{u(t) \in \mathbb{R}^1 : -2.0 \leq u(t) \leq 2.0\}. \tag{3.45}$$

The weighting matrices Q and R in the cost function of the optimization in (3.5) are set as

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad R = 1.0. \tag{3.46}$$

In the following simulation, μ is chosen as 0.9, and then the corresponding linearized nominal model of the system in (3.44) is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} u(t). \tag{3.47}$$

The stabilizing state feedback control law is obtained by solving the linear quadratic regulator problem for the system in (3.47) with the weighting matrices in (3.46)

$$K = \begin{bmatrix} -1.3030 & -1.3030 \end{bmatrix}.$$

Then, the eigenvalues of the matrix $A_K = A + BK$ are $\lambda_1 = -1.0000,\ \lambda_2 = -1.3454$. Choosing $\kappa = 0.95$ ensures that the Lyapunov equation in (3.6) admits a unique positive definite solution

$$P = \begin{bmatrix} 4.9633 & -0.0367 \\ -0.0367 & 4.9633 \end{bmatrix}.$$

According to Lemma 3.1, the positively invariant set is a ball with radius $\varepsilon = 0.6$, i.e.,

$$\Omega_\varepsilon = \{x \in \mathbb{R}^2 : \|x\|_P \leq 0.6\}. \tag{3.48}$$

Following the method in [44], the Lipschitz constant is calculated as $L = 1.4181$. Based on (3.25), the maximum disturbance level that the system can tolerate is $w(t) \leq \rho^{\max} = 0.004$, and the sampling interval should obey $0.3332 \leq \delta \leq 1.2234$ for guaranteed recursive feasibility. The remaining parameters are selected as $T = 1.2$ s, $\delta = 0.34$ s and $\alpha = 0.9$, to satisfy Lemma 3.3 and Theorem 3.1 with $\epsilon = 0.2870$, $\zeta = 0.4876$.

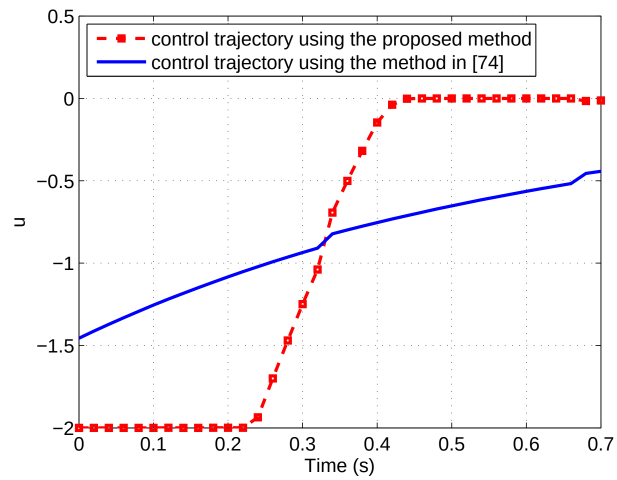

Figure 3.2: The control trajectory with initial point (0.4 0.55).

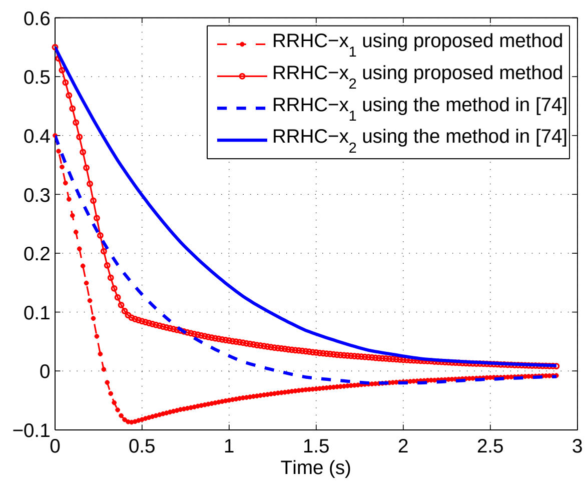

Figure 3.3: The state trajectory with initial point (0.4 0.55).

The effectiveness of the proposed MPC with non-squared cost is demonstrated by applying Algorithm 6 to two initial points: (0.4 0.55) and (0.9 0.55). The simulated optimal control trajectories associated with the two points, depicted in Figures 3.2 and 3.4, show that the control constraints in (3.45) are satisfied. The corresponding simulated state trajectories, shown in Figures 3.3 and 3.5, illustrate the convergence to the origin of the nonlinear system in closed loop with the MPC with non-squared cost.

Figures 3.3 and 3.5 also contrast the proposed nonlinear MPC strategy to the strategy in [74] which guarantees stability by requiring that the prediction horizon

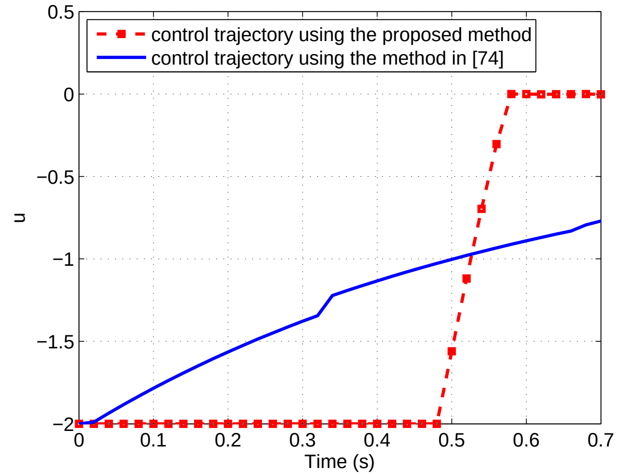

Figure 3.4: The control trajectory with initial point (0.9 0.55).

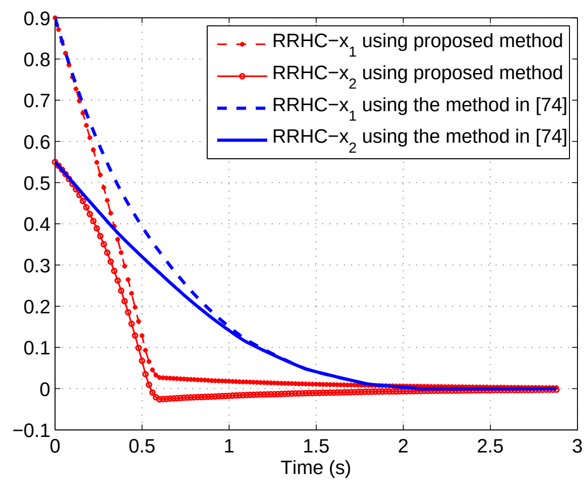

Figure 3.5: The state trajectory with initial point (0.9 0.55).

be shorter at time t_k than at time t_{k-1}. These figures illustrate that the algorithm proposed in this chapter converges faster than the algorithm in [74] because it implements an MPC strategy that exploit a control effort which is initially high and then rapidly decreases to zero.

For the deterministic case, Figures 3.6 and 3.7 compare the MPC strategy proposed in this chapter to the strategy introduced in [12]. These figures show that MPC with non-squared cost generates a higher initial control effort and therefore, converges faster than the MPC in [12].

Since the derivations in Lemma 3.3 and Theorem 3.1 rely on the Lipschitz continu-

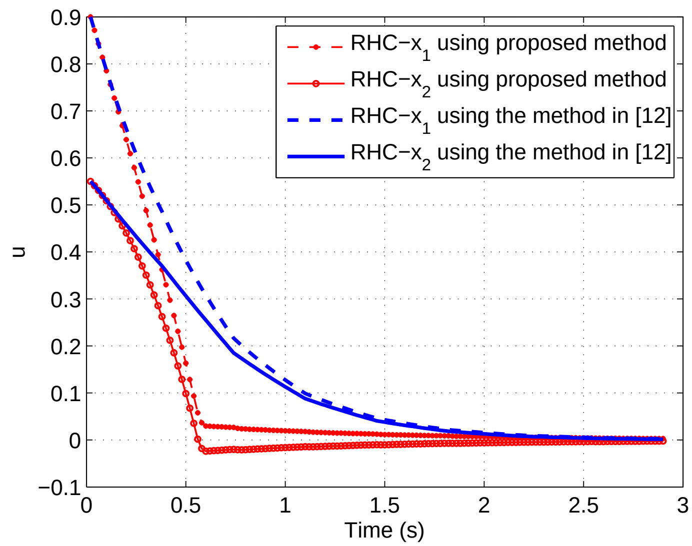

Figure 3.6: The state trajectory with initial point (0.9 0.55).

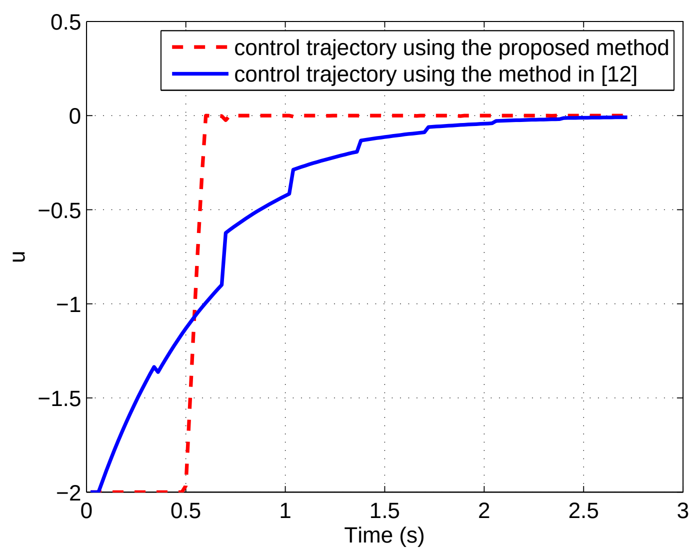

Figure 3.7: The control trajectory with initial point (0.9 0.55).

ity of the system dynamics, the results are likely conservative. The conservativeness of the proposed method can be shown by setting the disturbance level $\rho = 0.05$. Simulation results show that the state trajectory of the closed-loop system converges to the origin although δ is larger than the upper bound computed in Lemma (3.3).

Remark 3.7. *Although the Lipschitz condition is commonly used in [24, 32, 33, 67, 91, 116] [74], the results established are conservative mainly due to the following reasons:*

1) the Lipschitz constant is estimated to be the maximum value over a certain state-space region; the analysis always uses the maximum value; 2) the derivations assume that the discrepancy between the predicted and the actual state trajectories is always expanding which is not always the case.

Some promising directions that may lead to effective algorithms without exploiting the Lipschitz continuous condition are as follows.

- *For contractive nonlinear dynamics [58, 101], a robust MPC strategy can be designed exploiting this contractive property.*

- *The optimization associated with the MPC strategy is solved in an open-loop fashion, although the implementation of the strategy is essentially in a closed-loop form. Therefore, for (locally) constrained controllable nonlinear systems, their controllability can possibly be used to design a closed-loop form of MPC as in the case of [104]."*

3.5 Conclusions

This chapter has proposed a new robust MPC strategy for general constrained nonlinear systems with control constraints and additive disturbances. The cost function in the optimization associated with the proposed MPC strategy includes an integral non-squared stage cost and a non-squared terminal cost. Provided that the Jacobian linearization of the nonlinear system around the origin is stabilizable, proper design of the terminal weighting matrix ensures that the terminal cost is a control Lyapunov function. Moreover, the resultant cost function serves as a quasi-infinite horizon cost. The chapter has established sufficient conditions for the recursive feasibility of the optimization and for the robust stability of the closed-loop system with the proposed MPC strategy. It has shown that, for a given disturbance level, appropriate design of the sampling interval guarantees that the system state enters and remains in a robust positively invariant set. The effectiveness of the proposed MPC scheme has been demonstrated through a simulation example. Because the derivations use the Lipschitz continuity property of the nonlinear system dynamics extensively, the established results are conservative. Strategies to reduce the conservativeness of the proposed method will be pursued in future work.

Chapter 4

Robust Constrained Model Predictive Control Using Contraction Theory

4.1 Introduction

The MPC theory for linear systems has been developed systematically. However, the MPC methods for nonlinear systems are still ad hoc. In [69], the closed-loop stability of a deterministic nonlinear system has been guaranteed by adding an equality constraint to the optimization that yields the control signal. This equality constraint is difficult to satisfy given the iterative nature of the optimization, and has been replaced with an inequality constraint in [74]. In [12], the closed-loop stability has been ensured through a terminal weight matrix design which transforms the resulting cost of the optimization into a quasi-infinite horizon cost. The strategies in [12, 69, 74] are not suitable for nonlinear systems with disturbances. For such systems with an available control Lyapunov function, the existing control Lyapunov function has been incorporated into the optimization that yields the control signal [72, 73]. A common assumption in nonlinear MPC is that the system has Lipschitz continuous dynamics [50, 57, 74]. The Lipschitz continuity condition property is then used to upper bound the deviation between state trajectories with different initial states, and, thus, to ensure feasibility and stability. However, the upper bound on the trajectory deviation grows exponentially with the receding horizon. Therefore, conventional robust nonlinear MPC based on Lipschitz continuity is applicable to systems with small disturbances and short prediction horizon and, thus, may have limited the feasibility region.

4.1.1 Objective, Contributions and Chapter Organization

This chapter proposes a novel robust MPC which exploits the contractive property of a class of nonlinear systems. The new strategy aims to overcome the drawbacks of the general Lipschitz continuity property by taking advantage of the contracting system dynamics. Motivated by fluid mechanics, the contraction theory of nonlinear systems has been introduced in [58], and has been extended to nonlinear distributed systems in [61] and to resetting hybrid systems in [98]. Contraction has also been used to analyze the stochastic incremental stability [80]. Applications of contraction theory have been illustrated for mechanical systems in [59], and for networked systems in [101]. This chapter assumes that there exists a tube-like region along the nominal state trajectory predicted at time t_0 and designs a robust MPC strategy which relies on the contracting system dynamics within this tube. The contributions of this chapter are three-fold:

- It develops a novel robust MPC method for nonlinear systems contracting in a tube-like region along the nominal state trajectory predicted at t_0. It also upper bounds the deviation between the nominal and the actual state trajectories, when they are both within the contraction region.

- It establishes two conditions which together are sufficient for the recursive feasibility of the proposed MPC: (i) the nominal state trajectory and the assumed state trajectory should be within the contraction region; (ii) the optimization at time instant t_{k+1} should be feasible given that it is feasible at time instant t_k.

- It shows that the recursive feasibility of the control signal is sufficient for guaranteed stability of the contracting nonlinear system with MPC.

The remainder of this chapter is organized as follows. Section 4.2 introduces the proposed robust MPC strategy, and upper bounds the deviation between the actual and the predicted state trajectories. Section 4.3 establishes sufficient conditions for the recursive feasibility of the optimization that yields the MPC signal. Section 4.4

shows that the proposed robust MPC strategy asymptotically stabilizes the closed-loop nonlinear system with disturbances. Section 4.5 illustrates the proposed method through a simulation example. Section 4.6 draws the conclusions of this work.

In this chapter, the superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; for a matrix P, $P > 0$ ($P \geq 0$) means that P is real symmetric positive definite (positive semidefinite); for a vector $x \in \mathbb{R}^n$, $\|x\|_2$ is its Euclidean norm and $\|x\|_P = \sqrt{x^{\mathrm{T}} P x}$ is its P-weighted norm.

4.2 Robust MPC for Contracting Systems

Consider the nonlinear system with additive disturbances

$$\dot{x}(t) = f(x(t), u(t)) + w(t),\ x(t_0) = x_0,\ t \geq 0, \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{U} \subset \mathbb{R}^m$ is the constrained control input and $w(t) \in \mathbb{W} \subset \mathbb{R}^n$ is the bounded additive disturbance satisfying $\rho_{\max} = \max_t \|w(t)\|_M$. The sets $\mathbb{U}$ and $\mathbb{W}$ are both compact and contain the origin in their interiors. The nominal dynamics of the system in (4.1) are

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t)), \tag{4.2}$$

and obey two assumptions in this work.

Assumption 4.1. *The linearization around the origin of the nominal dynamics in (4.2) is stabilizable, i.e., a matrix K with appropriate dimensions exists such that $(A + BK)$ is stable, where $A = \frac{\partial f}{\partial x}|_{(0,0)}$ and $B = \frac{\partial f}{\partial u}|_{(0,0)}$.*

Assumption 4.1 is fairly common in nonlinear MPC [12, 20], and the following lemma follows if Assumption 4.1 holds for the nominal dynamics in (4.2).

Lemma 4.1. *A control positively invariant set $\Omega_\alpha = \{x : \|x\|_M \leq \alpha\}$ with respect to a matrix $M > 0$ exists for the nominal nonlinear dynamics in (4.2) with the static feedback control $\bar{u}(t) = K\bar{x}(t)$.*

Assumption 4.2. *An MPC signal exists that, over the prediction horizon T, steers the nominal dynamics in (4.2) to Ω_α along a trajectory $\dot{\bar{x}}^*(t) = f(\bar{x}^*(t), \bar{u}^*(t)), \bar{x}^*(t_0) = x_0$, such that*

$$\frac{\partial f}{\partial \bar{x}}^{\mathrm{T}} M + M \frac{\partial f}{\partial \bar{x}} \leq -\beta M, \quad \forall \bar{x} \in \Theta_l \tag{4.3}$$

with constant $\beta > 0$, constant $l > 0$, $\Theta_l =: \{\bar{x} : \|\bar{x} - \bar{x}^*(s;t_0)\|_M \leq l, \forall s \in [t_0, t_0+T]\}$, and Ω_α and M given in Lemma 4.1.

Assumption 4.2 replaces the typical nonlinear MPC requirement that $f(x,u)$ be Lipschitz continuous with respect to x and u. This assumption guarantees the existence of an MPC strategy that, over the prediction horizon T, can steer the nominal system in (4.2) to Ω_α along a trajectory $\bar{x}^*$ which is the medial axis of a tube-like region Θ_l of the state space of (4.2). Given Assumption 4.2, Theorem 2 in [58] guarantees that any nominal state trajectory which starts in the tube Θ_l remains in Θ_l and converges exponentially to $\bar{x}^*$. Assumption 4.2 uses a particular type of contraction region, obtained from the generalized contraction region defined in [58] for a constant metric M. The extension of the robust MPC strategy introduced here to nonlinear systems contracting with respect to a uniformly positive definite metric $M(x,t)$ [58] is a topic of future work.

Given the system in (4.1) with Assumptions 4.1 and 4.2, this chapter proposes a robust MPC control strategy whereby:

1. At the initial time instant t_0, the control signal is obtained by solving

$$
\begin{aligned}
&u^*(t;t_0) \\
&= \arg \min_{\bar{u}(s;t_0),\, s\in[t_0,t_0+T]} \int_{t_0}^{t_0+T} \Big(\|\bar{x}(s;t_0)\|_Q \\
&\quad + \|\bar{u}(s;t_0)\|_R \Big) ds + \|\bar{x}(t_0+T;t_0)\|_P, \qquad (4.4\text{a})
\end{aligned}
$$

subject to:

$$
\begin{aligned}
&\dot{\bar{x}}(s;t_0) = f(\bar{x}(s;t_0), \bar{u}(s;t_0)),\ \bar{x}(t_0;t_0) = x(t_0), \\
&\qquad\qquad s \in [t_0, t_0+T], \qquad (4.4\text{b}) \\
&\bar{u}(s;t_0) \in \mathbb{U},\ s \in [t_0, t_0+T], \\
&\frac{\partial f}{\partial \bar{x}}^{\mathrm{T}} M + M \frac{\partial f}{\partial \bar{x}} \leq -\beta M,\ \bar{x} \in \Theta_l, \qquad (4.4\text{c}) \\
&\bar{x}(t_0+T;t_0) \in \Omega_\alpha, \qquad (4.4\text{d})
\end{aligned}
$$

where T is the prediction horizon, $\bar{x}(s;t_0)$, $s \in [t_0, t_0+T]$ is the nominal state trajectory predicted at time t_0, $\bar{u}(s;t_0)$, $s \in [t_0, t_0+T]$ is the control trajectory predicted at time t_0, $Q > 0$ and $R \geq 0$ are the stage cost weighting matrices, and $P > 0$ is the terminal cost weighting matrix;

2. At the k-th time step t_k, $k > 0$, the control signal is obtained by solving

$$
\begin{aligned}
&u^*(t;t_k) \\
&= \arg \min_{\bar{u}(s;t_k),\, s\in[t_k,t_k+T]} \int_{t_k}^{t_k+T} \Big(\|\bar{x}(s;t_k)\|_Q \\
&\quad + \|\bar{u}(s;t_k)\|_R \Big) ds + \|\bar{x}(t_k+T;t_k)\|_P, && \text{(4.5a)}
\end{aligned}
$$

subject to:

$$
\begin{aligned}
&\dot{\bar{x}}(s;t_k) = f(\bar{x}(s;t_k), \bar{u}(s;t_k)), \bar{x}(t_k;t_k) = x(t_k), \\
&\qquad s \in [t_k, t_k+T], && \text{(4.5b)} \\
&\bar{u}(s;t_k) \in \mathbb{U},\ s \in [t_k, t_k+T], \\
&\|\bar{x}(t_k+\tau;t_k) - \bar{x}(t_k+\tau;t_0)\|_M \leq \frac{1}{2}(l_1 + l), \\
&\qquad \tau \in [0,\delta], && \text{(4.5c)} \\
&\|\bar{x}(t_k+\tau;t_k) - \bar{x}(t_k+\tau;t_0)\|_M \leq l_1,\ \tau \in [\delta, T], && \text{(4.5d)}
\end{aligned}
$$

where the constant $l_1 > 0$ is designed in Proposition 4.2, and the sampling interval δ is designed in Proposition 4.2 and Theorem 4.1.

Compared to existing robust MPC strategies, the optimizations in (4.4) and (4.5) include new constraints in (4.4c), (4.5c) and (4.5d). The constraint in (4.4c) guarantees that the optimal nominal state trajectory predicted at time instant t_0 is in a tube-like region Θ_l, which is a contraction region with respect to M for the nominal system in (4.2) and is centered at this trajectory. The constraints in (4.5c) and (4.5d) bound the deviation of the nominal state trajectory predicted at time t_k from the nominal state trajectory predicted at time t_0 and, thus, provide robustness for the proposed MPC strategy.

Because the optimizations in (4.4) and (4.5) rely on the nominal dynamics and ignore disturbances, the state $x(t_{k+1})$ of the nonlinear system in (4.1) will deviate from the predicted state $\bar{x}(t_{k+1};t_k)$. This deviation is upper bounded in the following proposition.

Proposition 4.1. *Let the optimizations in (4.4) and (4.5) be feasible with $x(t_k)$, $k \geq 0$, and let the system state $x(t)$ be in the contraction tube Θ_l for all $t \in [t_k, t_k+T]$.*

Then, the deviation between $x(t)$ and $\bar{x}(t;t_k)$ satisfies

$$\|x(t)-\bar{x}(t;t_k)\|_M^2 \le \frac{2l}{\beta}\rho_{\max}(1-e^{-\beta(t-t_k)}). \tag{4.6}$$

Proof. Let $V(t)=(x(t)-\bar{x}(t;t_k))^{\mathrm{T}}M(x(t)-\bar{x}(t;t_k))$. Given that

$$\begin{cases} \dot{\bar{x}}(t;t_k) &= f(\bar{x}(t;t_k),\bar{u}(t;t_k)), \\ \dot{x}(t) &= f(x(t),\bar{u}(t;t_k))+w(t), \end{cases} \quad \bar{x}(t_k;t_k)=x(t_k),$$

it follows that $V(t_k)=0$, and

$$\begin{aligned} \dot{V}(t) =&(f(x(t),\bar{u}(t;t_k))-f(\bar{x}(t;t_k),\bar{u}(t;t_k))+w(t))^{\mathrm{T}} \\ &M(x(t)-\bar{x}(t;t_k))+(x(t)-\bar{x}(t;t_k))^{\mathrm{T}}M \\ &(f(x(t),\bar{u}(t;t_k))-f(\bar{x}(t;t_k),\bar{u}(t;t_k))+w(t)). \end{aligned}$$

According to [58], a state $\widetilde{x}\in[\bar{x}(t;t_k),x(t)]$ can be found such that

$$\begin{aligned} \dot{V}(t) = (x(t)-\bar{x}(t;t_k))^{\mathrm{T}} &\left(\frac{\partial f}{\partial x}^{\mathrm{T}}|_{\widetilde{x}}M + M\frac{\partial f}{\partial x}|_{\widetilde{x}}\right) \\ &(x(t)-\bar{x}(t;t_k))+2w(t)^{\mathrm{T}}M\left(x(t)-\bar{x}(t;t_k)\right). \end{aligned} \tag{4.7}$$

Since both trajectories lie in the contraction region Θ_l, they obey

$$\begin{aligned} \|x(t)-\bar{x}(t;t_0)\|_M &\le l, \\ \|\bar{x}(t;t_k)-\bar{x}(t;t_0)\|_M &\le \frac{1}{2}(l_1+l)\le l, \end{aligned}$$

where the last inequality is obtained after using (4.5c) and (4.5d). It then follows that

$$\begin{aligned} &\|x(t)-\bar{x}(t;t_k)\|_M \\ &\le \|x(t)-\bar{x}(t;t_0)\|_M+\|\bar{x}(t;t_k)-\bar{x}(t;t_0)\|_M \le 2l, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} & w(t)^{\mathrm{T}} M(x(t) - \bar{x}(t;t_k)) \\ & = w(t)^{\mathrm{T}} \sqrt{M}\sqrt{M}(x(t) - \bar{x}(t;t_k)) \\ & \leq \|\sqrt{M} w(t)\|_2 \cdot \|\sqrt{M}(x(t) - \bar{x}(t;t_k))\|_2 \\ & \leq \|w(t)\|_M \cdot \|x(t) - \bar{x}(t;t_k)\|_M \\ & \leq 2l \cdot \|w(t)\|_M, \end{aligned} \tag{4.9}$$

where (4.9) is obtained after using (4.8). Combining (4.4c) and (4.7) leads to

$$\begin{aligned} \dot{V}(t) & \leq -\beta V(t) + 2l \cdot \|w(t)\|_M \\ & \leq -\beta V(t) + 2l\rho_{\max}. \end{aligned}$$

From the comparison principle and $V(t_k) = 0$, it follows that

$$V(t) \leq \frac{2l}{\beta}\rho_{\max}(1 - e^{-\beta(t-t_k)}),$$

and

$$\|x(t) - \bar{x}(t;t_k)\|_M \leq \sqrt{\frac{2l\rho_{\max}}{\beta}(1 - e^{\beta(t-t_k)})}.$$

This completes the proof. □

4.3 Feasibility Analysis

This section presents sufficient conditions for the recursive feasibility of the optimization in (4.5) given that the optimization in (4.4) is feasible at the initial time. The section first establishes sufficient conditions to bound the nominal and the actual state trajectories in the tube-like region Θ_l in Proposition 4.2. Then, it exploits Θ_l as a contraction region of the nominal closed-loop system with the MPC in (4.4) and (4.5) to derive sufficient conditions for recursive feasibility in Theorem 4.1. The analysis in this section requires the assumed control signal $\hat{u}(\tau;t_k)$, $\tau \in [t_{k+1}, t_{k+1} + T]$, which is defined by

$$\hat{u}(\tau;t_{k+1}) = \begin{cases} u^*(\tau;t_k),\ \tau \in [t_{k+1}, t_k+T], \\ K\hat{x}(\tau;t_{k+1}),\ \tau \in [t_k+T, t_{k+1}+T], \end{cases} \tag{4.10}$$

where $u^*(\tau;t_k)$, $\tau \in [t_{k+1}, t_k+T]$ is the optimal control signal computed at time t_k, and $\hat{x}(\tau;t_{k+1})$ is the assumed nominal state, given by $\dot{\hat{x}}(\tau;t_{k+1}) = f(\hat{x}(\tau;t_{k+1}), \hat{u}(\tau;t_{k+1}))$, $\tau \in [t_k+T, t_{k+1}+T]$ with $\hat{x}(t_k+T;t_{k+1})$ the state of the system in (4.2) with initial state $x(t_k)$ by applying $\hat{u}(\tau;t_{k+1})$, $\tau \in [t_{k+1}, t_k+T]$.

Proposition 4.2. *Let the system in (4.1) start from a state $x(t_k)$ for which the optimizations in (4.4) for $k=0$ and in (4.5) for $k>0$ are feasible. If the sampling interval δ satisfies*

$$\delta \leq -\frac{1}{\beta}\ln(1-\frac{\beta(l-l_1)^2}{8l\rho_{\max}}), \tag{4.11}$$

with $l_1 < l$ a positive constant, then both the state trajectory $x(t_k+\tau)$ and the nominal state trajectory $\bar{x}(t_k+\tau;t_k)$ lie in the contraction region Θ_l for all $\tau \in [0,\delta]$.

Proof. Applying the predicted control trajectory $\bar{u}(t;t_0)$ to the system in (4.1) with initial state $x(t_k)$ during the time interval $t \in [t_k, t_k+\delta]$ leads to

$$\begin{aligned} &\|x(t_k+\tau) - \bar{x}(t_k+\tau;t_0)\|_M \\ &= \|x(t_k+\tau) - \bar{x}(t_k+\tau;t_k) \\ &\qquad\qquad + \bar{x}(t_k+\tau;t_k) - \bar{x}(t_k+\tau;t_0)\|_M \\ &\leq \|x(t_k+\tau) - \bar{x}(t_k+\tau;t_k)\|_M \\ &\qquad\qquad + \|\bar{x}(t_k+\tau;t_k) - \bar{x}(t_k+\tau;t_0)\|_M. \end{aligned}$$

From Proposition 4.1 and the constraint in (4.5c), it follows that

$$\begin{aligned} &\|x(t_k+\tau) - \bar{x}(t_k+\tau;t_0)\|_M \\ &\leq \sqrt{\frac{2l\cdot\|w(t)\|_M}{\beta}\cdot(1-e^{-\beta\tau})} + \frac{1}{2}(l+l_1),\ \forall\tau\in[0,\delta], \end{aligned} \tag{4.12}$$

where the right-hand side increases with τ and, thus, attains its maximum at $\tau = \delta$ over the time interval $[0,\delta]$. Then, the substitution from (4.11) into (4.12) yields

$$\|x(t_k+\delta) - \bar{x}(t_k+\delta;t_0)\|_M \leq l,$$

and

$$
\begin{aligned}
&\|x(t_k+\tau)-\bar{x}(t_k+\tau;t_0)\|_M \\
&\leq \|x(t_k+\delta)-\bar{x}(t_k+\delta;t_0)\|_M \\
&\leq l,\ \forall \tau \in [0,\delta].
\end{aligned}
$$

This completes the proof. □

Theorem 4.1. *Let the system in (4.1) start from an initial state x_0 for which the optimization in (4.4) is feasible and the sampling interval δ satisfy (4.11). Then, the optimization in (4.5) is feasible for all t_k with $k \geq 1$ if the following conditions hold*

$$\delta \geq \frac{2}{\beta}\ln\frac{l+l_1}{2l_1}, \tag{4.13}$$

$$\rho_{\max} \leq \frac{\beta(l-l_1)^2 \cdot l_1^2}{8l(l^2-l_1^2)}. \tag{4.14}$$

Proof. The proof is divided into two steps. The first step shows that, if the optimization in (4.4) is feasible at time t_0 starting from x_0 and the conditions in (4.11), (4.13) and (4.14) hold, then $\|x(t_1)-\bar{x}(t_1;t_0)\|_M \leq \frac{1}{2}(l+l_1)$.

From (4.6), it follows that

$$\|x(t_1)-\bar{x}(t_1;t_0)\|_M \leq \sqrt{\frac{2l}{\beta}\|w(t)\|_M \cdot (1-e^{-\beta\delta})},$$

which together with (4.11) leads to

$$\|x(t_1)-\bar{x}(t_1;t_0)\|_M \leq \frac{1}{2}(l-l_1) < \frac{1}{2}(l+l_1). \tag{4.15}$$

Equation (4.15) shows that the results established in Proposition 4.1 can be used. The second step of the proof aims to show that for any state $x(t_k) \in \Omega_{\frac{1}{2}(l+l_1)} = \{x : \|x-\bar{x}(t_k;t_0)\|_M \leq \frac{1}{2}(l+l_1)\}$, the optimization in (4.5) is feasible and the successive state $x(t_{k+1}) \in \Omega_{\frac{1}{2}(l+l_1)} = \{x : \|x-\bar{x}(t_{k+1};t_0)\|_M \leq \frac{1}{2}(l+l_1)\}$.

Since $x(t_k) \in \Omega_{\frac{1}{2}(l+l_1)} = \{x : \|x-\bar{x}(t_k;t_0)\|_M \leq \frac{1}{2}(l+l_1)\}$, the contraction property holds by applying the control signal predicted at time instant t_0, so the constraint in (4.5c) always holds. Define $V(t) = (\bar{x}(t;t_k)-\bar{x}(t;t_0))^{\mathrm{T}}M(\bar{x}(t;t_k)-\bar{x}(t;t_0))$ over the interval $t \in [t_k, t_k+T]$ with $V(t_k) = (\bar{x}(t_k;t_k)-\bar{x}(t_k;t_0))^{\mathrm{T}}M(\bar{x}(t_k;t_k)-\bar{x}(t_k;t_0)) \leq [\frac{1}{2}(l+l_1)]^2$.

Similar to the derivation in Proposition 4.1, it follows that

$$\dot{V}(t) \leq -\beta V(t_k), \quad \text{with } V(t_k) \leq [\frac{1}{2}(l+l_1)]^2.$$

Using the comparison principle, it follows that

$$\begin{aligned} V(t) &\leq e^{-\beta(t-t_k)} V(t_k) \\ &\leq e^{-\beta(t-t_k)} [\frac{1}{2}(l+l_1)]^2, \end{aligned}$$

and

$$\begin{aligned} V(t_k+\delta) &= \|\bar{x}(t_k+\delta;t_k) - \bar{x}(t_k+\delta;t_0)\|_M^2 \\ &\leq e^{-\beta\delta} [\frac{1}{2}(l+l_1)]^2. \end{aligned} \tag{4.16}$$

Together with (4.13), Equation (4.16) implies that

$$\|\bar{x}(t_k+\delta;t_k) - \bar{x}(t_k+\delta;t_0)\|_M \leq l_1, \tag{4.17}$$

i.e., the predicted state $\bar{x}(t_k+\delta;t_k)$ is in the region $\Omega_{l_1} = \{x : \|x-\bar{x}(t_k+\delta;t_0)\|_M \leq l_1\}$. After implementing the assumed control signal in (4.10) and using the contraction property in (4.4c), the predicted state satisfies the constraint in (4.5d).

In view of Proposition 4.1 and the condition in (4.11), it follows that

$$\begin{aligned} &\|x(t_{k+1}) - \bar{x}(t_{k+1};t_0)\|_M \\ &\leq \|x(t_{k+1}) - \bar{x}(t_{k+1};t_k)\|_M \\ &\qquad + \|\bar{x}(t_{k+1};t_k) - \bar{x}(t_{k+1};t_0)\|_M \\ &\leq \frac{1}{2}(l+l_1). \end{aligned} \tag{4.18}$$

Equation (4.18) shows that the successive system state $x(t_{k+1})$ falls in the region $\Omega_{\frac{1}{2}(l+l_1)} = \{x : \|x - \bar{x}(t_{k+1};t_0)\|_M \leq \frac{1}{2}(l+l_1)\}$. To satisfy (4.11) and (4.13) simultaneously, the maximum disturbance needs to satisfy (4.14). This completes the proof. □

4.4 Stability Analysis

This section proves the stability of the system in (4.1) with the robust MPC obtained by solving the optimization in (4.4) at time t_0, and obtained by solving the optimization in (4.5) at all consecutive time steps $t_k, k > 0$.

Theorem 4.2. *Let the sampling interval δ satisfy the conditions in (4.11) and (4.13) and the disturbance be upper bounded by (4.14) for the system in (4.1) with Assumption 1 and Assumption 2. Then, for any given initial state x_0 for which the optimization in (4.4) is feasible, the state of the closed-loop system with the proposed MPC strategy will enter a neighborhood of the origin $\Omega_{l+\epsilon} = \{x : \|x\|_M \leq l + \epsilon\}$ for any infinitesimal value $\epsilon > 0$ in finite time and then remain in this neighborhood thereafter.*

Proof. Since $\bar{x}(t_0 + T; t_0) \in \Omega_\alpha$ and the control $\bar{u}(t; t_0) = K\bar{x}(t; t_0)$ stabilizes the nominal system to the origin for any state $\bar{x} \in \Omega_\alpha$, any such $\bar{x} \in \Omega_\alpha$ will shrink to the set Ω_ϵ in finite time. Without loss of generality, assume that, at time t^*, the system state enters and remains within the set Ω_ϵ for $t \geq t^*$, that is $\|\bar{x}(t; t_0)\|_M \leq \epsilon$, $\forall t \geq t^*$. At time $t \geq t^*$, the constraint (4.5c) guarantees that

$$\|\bar{x}(t; t_k) - \bar{x}(t; t_0)\|_M \leq \frac{1}{2}(l + l_1),$$

which, combined with Proposition 4.1, leads to

$$\|x(t) - \bar{x}(t; t_0)\|_M \leq l.$$

It then follows that $\|x(t)\|_M \leq \|x(t) - \bar{x}(t; t_0)\|_M + \|\bar{x}(t; t_0)\|_M \leq l + \epsilon$, which means that, from $t \geq t^*$, the state of the system in (4.1) remains in the neighborhood of the origin $\Omega_{l+\epsilon} = \{x : \|x\|_M \leq l + \epsilon\}$. In fact, as $t \to \infty$, $\|\bar{x}(t; t_0)\|_M \to 0$, the closed-loop system state asymptotically approaches the set Ω_l. This completes the proof. □

4.5 Simulation Example

In order to demonstrate its effectiveness, the proposed algorithm is applied to a numerical example used in [60]. The system dynamics are

$$\dot{x}_1(t) = k_1 x_1(t) + k_2 x_1^2(t) + k_3 x_2(t) + u + w, \tag{4.19a}$$

$$\dot{x}_2(t) = -q x_2(t) + n_1 x_1(t) - n_2 x_2^2(t). \tag{4.19b}$$

The Jacobian matrix of the dynamics in (4.19) is

$$\begin{aligned} J = \frac{\partial f}{\partial x} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} k_1 + 2k_2 x_1(t) & k_3 \\ n_1 & -q - 2n_2 x_2(t) \end{bmatrix}. \end{aligned} \tag{4.20}$$

For $M = \begin{bmatrix} n_1 & 0 \\ 0 & -k_3 \end{bmatrix}$, the contractive constraint in (4.4c) becomes

$$\begin{aligned} &\frac{\partial f}{\partial x}^{\mathrm{T}} M + M \frac{\partial f}{\partial x} \\ &= \begin{bmatrix} 2n_1(k_1 + 2k_2 x_1(t)) & 0 \\ 0 & -2k_3(-q - 2n_2 x_2(t)) \end{bmatrix}. \end{aligned} \tag{4.21}$$

The numerical values of all parameters are chosen as follows: $k_1 = -0.5$, $k_2 = 0.05$, $k_3 = -1$, $q = 0.5$, $n_1 = 1$, $n_2 = 0.05$. The weighting matrices in the objective function are chosen as $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$, and $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since the Jacobian matrix in (4.20) is stable, the state-feedback controller is selected as $K = 0$ and the control positively invariant set Ω_α is characterized by $\alpha = 1$. The radius l and the contraction rate β of the contraction tube Ω_l are $l = 0.3$ and $\beta = 0.5740$. The design parameters are chosen as $l_1 = 0.25$, and $\rho_{\max} = 0.0032$. Then the upper bound for the sampling interval is $\delta \leq 0.3603$ s from Proposition 4.2, and the lower bound for the sampling interval is $\delta \geq 0.3321$ s from Theorem 4.1. In the simulation, the sampling interval is chosen as $\delta = 0.36$ s, and the receding horizon is $T = 4$ s. The state trajectory of the closed-loop system in (4.19), with the proposed robust MPC in (4.4) and (4.5) and starting from the initial state $(1.90, 1.55)$, is shown in Figure 4.1. Also shown in Figure 4.1 is the nominal state trajectory predicted at t_0 using the optimization in (4.4). The control signal applied to the system and the nominal control signal generated by the optimization in (4.4) at time t_0 are depicted in Figure 4.2. Note in these figures that the control signal satisfies the control constraints and that the closed-loop system converges to the origin.

Remark 4.1. *The system in (4.19) can be steered to the origin using a robust MPC strategy based on the Lipschitz continuous condition instead of contracting systems. For the same initial state and the same prediction horizon, the Lipschitz constant is* $L = 2.018$*, and the maximum disturbance that can be tolerated is* 0.00029.

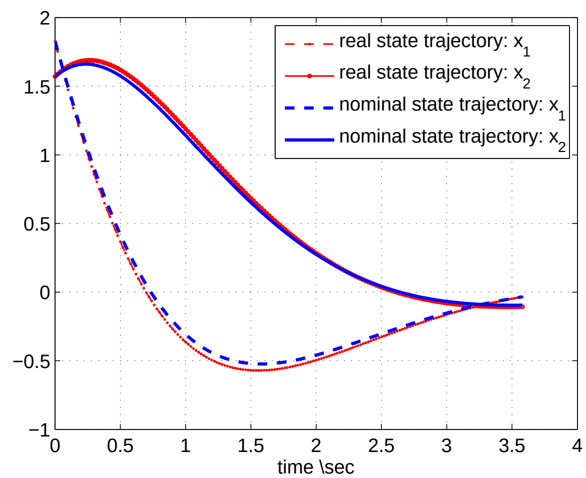

Figure 4.1: State trajectories starting from the initial state (1.90 1.55).

Remark 4.2. *The system in (4.19) with the robust MPC strategy designed via the Lipschitz condition can tolerate the same maximum disturbance* $\rho_{\max} = 0.0032$ *as the system in (4.19) with the robust MPC proposed in this chapter if the prediction horizon is selected as* $T = 2.82$ *s (this value results from the feasibility and stability conditions in [57]). Typically, a larger receding horizon leads to a larger feasible region. Since the feasible region of a nonlinear system is not trivial to compute, this work has evaluated the feasibility of robust MPC with* $T = 4$ *s heuristically, for different initial states of the system in (4.19). The investigation has shown that the state* $(4.95, -5.05)$ *is feasible for the system in (4.19) with the proposed robust MPC, but it is not in the feasible region of the system in (4.19) with the robust MPC derived based on the Lipschitz continuous condition. This result indicates that exploiting the contracting*

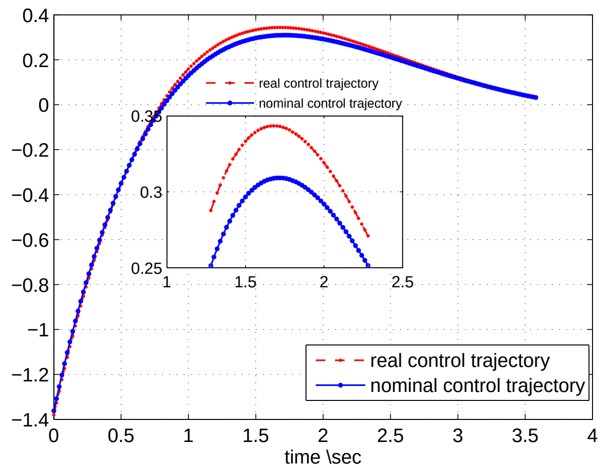

Figure 4.2: Control trajectories.

dynamics of a system as proposed in this chapter could enlarge the feasible region of the robust MPC.

4.6 Conclusion

This chapter has introduced a novel robust constrained MPC strategy for nonlinear systems whose dynamics are contracting in a tube centered around the nominal state trajectory predicted at t_0. The proposed strategy exploits the contracting dynamics instead of the common Lipschitz continuity. The chapter has derived sufficient conditions for the recursive feasibility of the optimization which generates the proposed control signal, and has also shown that the recursive feasibility of the proposed robust MPC guarantees that the state of the closed-loop system converges asymptotically to a neighborhood of the origin. Simulation results indicate that, compared to the robust MPC strategy based on Lipschitz continuity, the new technique: 1) can practically stabilize nonlinear systems with larger disturbances; and 2) could enlarge the feasible region of the closed-loop system.

Chapter 5

Robust Distributed Model Predictive Control of Continuous-Time Constrained Nonlinear Systems Using A Two-Layer Invariant Set

5.1 Introduction

Cooperative control has a wide range of applications in multi-vehicle systems [94], large-scale chemical processes [14], transportation systems [5], and so on. MPC strategies are a natural selection because they provide a control signal which satisfies the state and control constraints that arise in such applications, for example due to the physical limitations of actuators. For cooperative large-scale systems, distributed MPC strategies are preferred to centralized MPC schemes for several reasons: (i) they have smaller computational complexity because each subsystem solves a local optimization with a small number of decision variables; (ii) they require less communication bandwidth because each subsystem exchanges information only with a subset of the other subsystems; and (iii) they are more robust because the large-scale system may achieve its task even when one of its subsystems malfunctions.

Large-scale systems composed of subsystems with decoupled dynamics, like multi-vehicle formations, cooperate either through state and/or control coupling constraints, or through coupled cost functions. When the subsystems are coupled through constraints, the recursive feasibility and stability are ensured by sequential algorithms for solving the optimization [95–97] or by tightening methods [110, 111]. When the cooperation is achieved through coupled cost functions, additional consistency constraints [21] or robustness constraints [50] are introduced in the optimization.

5.1.1 Objective, Contributions and Chapter Organization

This chapter presents a robust distributed MPC strategy for constrained continuous-time nonlinear systems coupled by the cost function. In the proposed strategy, the subsystems communicate only with their neighbors exchanging their assumed system state trajectories. The cooperation among the subsystems is achieved by incorporating a coupling term in the cost function. The disturbances are handled by designing a robust control strategy based on the two-layer invariant set. Provided that the initial state is feasible and the disturbance is bounded by a certain level, the appropriate selection of the design parameters guarantees the recursive feasibility of the optimization. The chapter also derives sufficient conditions for robust stability. A conceptually less conservative algorithm is finally proposed which exploits $\kappa \circ \delta$ controllability set [81] rather than the positively invariant set and thus, allows a shorter prediction horizon and tolerates a larger disturbance level.

The contributions of this chapter are three-fold.

- In contrast to [50], the subsystems are coupled through a non-squared integrand cost function. As a result, no cross terms appear when evaluating the Lyapunov function and therefore, no additional constraints are needed to bound them.

- The disturbances are addressed through a novel control strategy based on the two-layer invariant set. The chapter also analyzes the recursive feasibility of the optimization and the robust stability of the distributed system in closed-loop with the proposed MPC strategy. As the simulation results illustrate, the method proposed in this chapter leads to a stronger cooperation than the technique in [50].

- The conservativeness of the initial robust distributed MPC strategy is reduced by taking advantage of the $\kappa \circ \delta$ controllability set. The less conservative distributed MPC algorithm uses a shorter prediction horizon to stabilize a continuous-time constrained nonlinear system affected by larger disturbances.

In the remainder of this chapter, Section 5.2 formulates the problem and presents some preliminaries. Section 5.3 establishes the main recursive feasibility and robust stability results. Section 5.4 presents the conceptually less conservative strategy. Section 5.5 illustrates the developed algorithm through a simulation example. Section 5.6 summarizes this chapter.

Notation: The superscript "T" stands for the matrix transposition. For a matrix P, $P > 0$ denotes that the matrix P is positive definite. For a matrix A, $\bar{\lambda}(A)$ represents the largest eigenvalue of A; $\underline{\lambda}(A)$ represents the smallest eigenvalue of matrix A; $\|A\|_P$ represents the P-weighted norm of the matrix A, defined by $\|A\|_P = \sqrt{A^{\mathrm{T}}PA}$. For a given $c > 0$, $\mathcal{B}(c)$ denotes a closed ball centered at the origin and of radius c.

5.2 Preliminaries

Consider a nonlinear system consisting of S decoupled subsystems $\mathcal{A}_i,\ i = 1, 2, \cdots, S$. The dynamics of the subsystem $\mathcal{A}_i$ are

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t)) + \omega_i(t),\ t \geq 0,\ x_i(0) = x_i^0, \tag{5.1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ and $\omega_i(t) \in \mathbb{R}^n$ are the system state, the control signal and the additive disturbances, respectively. Assume that the control input and the external disturbances are bounded as follows

$$u_i(t) \in \mathbb{U}_i,\ \ \omega_i(t) \in \mathbb{W}_i, \tag{5.2}$$

where $\mathbb{U}_i$ and $\mathbb{W}_i$ are compact sets containing the origin in their interior.

By ignoring the disturbances, the nominal dynamics of the system in (5.1) are

$$\dot{\hat{x}}_i(t) = f_i(\hat{x}_i(t), \hat{u}_i(t)). \tag{5.3}$$

The following two fairly standard assumptions [12, 24, 50] are imposed on the system in (5.1).

Assumption 5.1. *For each subsystem $\mathcal{A}_i$, $f_i(x, u)$ is Lipschitz continuous with respect to x with Lipschitz constant L_i, i.e.,*

$$\|f_i(x_1, u) - f_i(x_2, u)\|_{P_i} \leq L_i\|x_1 - x_2\|_{P_i},\ L_i > 0,\ P_i > 0. \tag{5.4}$$

Assumption 5.2. *For each subsystem $\mathcal{A}_i$, the linearized system of the associated nominal subsystem in (5.3) of each subsystem $\mathcal{A}_i$ is stabilizable, i.e., for $\dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + B_i\hat{u}(t)$, where $A_i = \frac{\partial f_i}{\partial x_i}(0,0)$ and $B_i = \frac{\partial f_i}{\partial u_i}(0,0)$, there exists K_i such that $A_i + B_iK_i$ is stable.*

Instead of adopting a quadratic integrand in the cost function [12,24,29], inspired by the work in [21,66], this chapter adopts the non-squared integrand

$$
\begin{aligned}
&J_i(\hat{x}_i(s;t_k), \hat{u}_i(s;t_k), \hat{x}^a_{-i}(s;t_k)) \\
&= \int_{t_k}^{t_k+T} \left(\|\hat{x}_i(s;t_k)\|_{Q_i} + \|\hat{u}_i(s;t_k)\|_{R_i} + \sum_{j\in N_i} r_{ij}\|\hat{x}_i(s;t_k) - \hat{x}^a_j(s;t_k)\|_{Q_{ij}} \right) ds, \quad (5.5)
\end{aligned}
$$

where Q_i, R_i and Q_{ij} are positive definite matrices with appropriate dimensions, T is the prediction horizon, and r_{ij} is a design parameter which decides the cooperation strength among subsystems; N_i is the set of indices of all neighbors of the subsystem $\mathcal{A}_i$; $\hat{x}^a_{-i}(s;t_k)$ is a compact notation for the assumed state trajectories of all subsystems who are neighbors of the subsystem $\mathcal{A}_i$ and whose indices are in N_i, i.e., $\hat{x}^a_{-i}(s;t_k) = (\cdots, \hat{x}^a_j(s;t_k), \cdots)$, $j \in N_i$ with $\hat{x}^a_j(s;t_k)$ being the assumed state trajectory obtained by

$$
\dot{\hat{x}}^a_j(s;t_k) = f_j(\hat{x}^a_j(s;t_k), \hat{u}^a_j(s;t_k)),\ \hat{x}^a_j(t_k;t_k) = x_j(t_k), \quad (5.6)
$$

where $\hat{u}^a_j(s;t_k)$ is generated using (5.9); $\hat{x}_i(s;t_k)$ is the predicted nominal state trajectory of the subsystem $\mathcal{A}_i$

$$
\dot{\hat{x}}_i(s;t_k) = f_i(\hat{x}_i(s;t_k), \hat{u}_i(s;t_k)),\ \hat{x}_i(t_k;t_k) = x_i(t_k), \quad (5.7)
$$

and $x_i(t_k)$ is the actual state of the subsystem $\mathcal{A}_i$ at time t_k.

Remark 5.1. *The stability of the strategy proposed in this chapter is based on Lyapunov theory. However, unlike typical distributed MPC strategies for decoupled large-scale nonlinear systems [12, 24, 29] the strategy proposed in this chapter uses a non-squared integrand in the coupling cost. Compared to a conventional quadratic cost function, the cost function with non-squared integrand does not introduce cross terms in the variation of the Lyapunov function from one time step to the next. Thus, it sidesteps the need to compensate for such cross terms [20].*

For the subsystem $\mathcal{A}_i$ in (5.1) with Assumption 5.2, the following lemmas holds.

Lemma 5.1. *[12, 50] If the linearization around the origin of (5.3), then a constant* $\varepsilon_i > 0$ *and a matrix* $P_i > 0$ *exist, such that: (1). The set* $\Omega_i(\varepsilon_i) = \{\hat{x}_i(t) : V_i(\hat{x}_i(t)) = \|\hat{x}_i(t)\|^2_{P_i} \le \varepsilon_i^2\}$ *is a control invariant set for the nominal subsystem in (5.3) with the static state feedback control law* $\hat{u}_i(t) = K_i\hat{x}_i(t)$*; (2). For any* $\hat{x}_i(t) \in \Omega_i(\varepsilon_i)$*, the inequality* $\dot{V}_i(\hat{x}_i(t)) \le -\|\hat{x}_i(t)\|^2_{Q_i^*}$ *holds, where* $Q_i^* = Q_i + K_i^{\mathrm{T}} R_i K_i$*.*

Lemma 5.2. *Given that* $\Omega_i(\varepsilon_i)$ *is a control invariant set for the nominal subsystem in (5.3), then, for any constant* $1 > \alpha > 0$*,* $\Omega_i(\alpha\varepsilon_i)$ *is also a control invariant set for the nominal subsystem in (5.3) with the static state feedback control law* $\hat{u}_i(t) = K_i\hat{x}_i(t)$*.*

Considering the subsystem, at each sampling instant t_k, each subsystem $\mathcal{A}_i$ solves the constrained optimization

$$\hat{u}_i(s;t_k) = \arg\min_{\hat{u}_i(t;t_k)} J_i(\hat{x}_i(s;t_k), \hat{u}_i(s;t_k), \hat{x}^a_{-i}(s;t_k)), \tag{5.8}$$

subject to :

$$\begin{aligned}
&\dot{\hat{x}}_i(s;t_k) = f_i(\hat{x}_i(s;t_k), \hat{u}_i(s;t_k)),\ s \in [t_k, t_k + T],\\
&\dot{\hat{x}}^a_j(s;t_k) = f_j(\hat{x}^a_j(s;t_k), \hat{u}^a_j(s;t_k)),\ s \in [t_k, t_k + T],\\
&\hat{u}_i(s;t_k) \in \mathbb{U}_i,\ s \in [t_k, t_k + T],\\
&\|\hat{x}_i(s;t_k) - \hat{x}^a_i(s;t_k)\|_{\bar{Q}_i} \le \delta\beta,\\
&\|\hat{x}_i(t_k + T;t_k)\|_{P_i} \le \alpha_{1i}\varepsilon_i,
\end{aligned}$$

where β and α_{1i} are parameters designed in Section 5.3; ε_i specifies the positively invariant set in Lemma 5.1 for the subsystem $\mathcal{A}_i$; δ is the sampling interval. The constrained optimization in (5.8) yields the optimal control sequence $\hat{u}_i^*(s;t_k)$, $s \in [t_k, t_k + T]$, and the associated optimal cost $J_i^*(t_k)$. The assumed control sequence is obtained as follows

$$\hat{u}^a_i(s;t_{k+1}) = \begin{cases} \hat{u}^*_i(s;t_k),\ \text{if } s \in [t_{k+1}, t_k + T],\\ K_i\hat{x}^a_i(s;t_{k+1}),\ s \in [t_k + T, t_{k+1} + T], \end{cases} \tag{5.9}$$

where $\hat{x}^a_i(t_{k+1};t_{k+1}) = x_i(t_{k+1})$.

Let the system state trajectory obtained by applying the optimal control trajectory $\hat{u}_i(s;t_k)$ to the actual system in (5.1) be $x_i(s;t_k)$. Due to the disturbances, the optimal predicted state trajectory differs from the system state trajectory $x_i(s;t_k)$. The following lemma upper bounds the difference between two trajectories.

Lemma 5.3. *[50] The discrepancy between the optimal predicted state trajectory and the state trajectory $x_i(s;t_k)$ is upper bounded by*

$$\|x_i(s;t_k) - \hat{x}_i(s;t_k)\|_{P_i} \le \bar{\lambda}(P_i^{\frac{1}{2}})\rho_i(s-t_k)e^{L_i(s-t_k)},\ s \in [t_k, t_k+T], \tag{5.10}$$

where ρ_i satisfies $\max_{t\ge 0}\|\omega_i(t)\|_{P_i} \le \rho_i$.

5.3 Main Results

The proposed distributed MPC strategy relies on the feasibility of the optimization (5.8). The following lemma shows that the recursive feasibility of the optimization depends on the disturbance level and on the sampling interval.

Lemma 5.4. *For the subsystem $\mathcal{A}_i$ in (5.1), Let the constrained optimization in (5.8) be feasible at time t_k, and the sampling period δ for $1 > \alpha_{2i} > \alpha_{1i} > 0$ and the disturbance bound satisfy*

$$-2\frac{\bar{\lambda}(P_i)}{\underline{\lambda}(Q_i^*)}\ln\frac{\alpha_{1i}}{\alpha_{2i}} \le \delta \le \frac{(\alpha_{2i}-\alpha_{1i})\varepsilon_i}{\rho_i e^{L_iT}\bar{\lambda}(P_i^{\frac{1}{2}})}, \tag{5.11}$$

$$\rho_i \le \rho_i^{\max} = -2(\alpha_{2i}-\alpha_{1i})\varepsilon_i e^{-L_iT}\bar{\lambda}(P_i^{\frac{1}{2}})^{-1}\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i^*)}/(\ln\frac{\alpha_{2i}}{\alpha_{1i}}), \tag{5.12}$$

then the constrained optimization in (5.8) is feasible at time t_{k+1}.

Proof. At time t_k, it follows from (5.8) that the terminal state satisfies

$$\|\hat{x}_i(t_k+T;t_k)\|_{P_i} \le \alpha_{1i}\varepsilon_i, \tag{5.13}$$

and that

$$\begin{aligned}
&\|\hat{x}_i^a(s;t_{k+1}) - \hat{x}_i^*(s;t_k)\|_{P_i} \\
&= \|x_i(t_{k+1};t_{k+1}) + \int_{t_{k+1}}^{s} f_i(x_i(\tau;t_{k+1}), \hat{u}^*(\tau;t_k))d\tau \\
&\quad - \hat{x}_i^*(t_{k+1};t_k) - \int_{t_{k+1}}^{s} f_i(\hat{x}_i^*(\tau;t_k), \hat{u}_i^*(\tau;t_k))d\tau\|_{P_i} \\
&\le \|x_i(t_{k+1};t_{k+1}) - \hat{x}_i^*(t_{k+1};t_k)\|_{P_i} + L_i\int_{t_{k+1}}^{s}\|x_i(\tau;t_{k+1}) - \hat{x}_i^*(\tau;t_k)\|_{P_i}d\tau.
\end{aligned} \tag{5.14}$$

From Lemma 5.3,

$$\|\hat{x}_i(s;t_{k+1}) - \hat{x}_i^*(s;t_k)\|_{P_i} \leq \bar{\lambda}(P_i^{\frac{1}{2}})\rho_i \delta e^{L_i\delta} e^{L_i(s-t_{k+1})}, \tag{5.15}$$

so

$$\|x_i(t_k+T;t_{k+1}) - \hat{x}_i^*(s;t_k)\|_{P_i} \leq \bar{\lambda}(P_i^{\frac{1}{2}})\rho_i \delta e^{L_iT}.$$

From

$$\|\hat{x}_i^*(t_k+T;t_k)\|_{P_i} \leq \alpha_{1i}\varepsilon_i,$$

it follows that

$$\|x_i(t_k+T;t_k)\|_{P_i} \leq \alpha_{1i}\varepsilon_i + \bar{\lambda}(P_i^{\frac{1}{2}})\rho_i \delta e^{L_iT}.$$

After requiring

$$\bar{\lambda}(P_i^{\frac{1}{2}})\rho_i \delta e^{L_iT} + \alpha_{1i}\varepsilon_i \leq \alpha_{2i}\varepsilon_i,$$

it can be shown that

$$\delta \leq \frac{(\alpha_{2i}-\alpha_{1i})\varepsilon_i}{\rho_i e^{L_iT}\bar{\lambda}(P_i^{\frac{1}{2}})}.$$

Thus $\|x_i^*(t_k+T;t_{k+1})\|_{P_i} \leq \alpha_{2i}\varepsilon_i < \varepsilon_i$ is in the control invariant set of the nominal system in (5.3) with the state feedback control law $\hat{u}_i(s;t_{k+1}) = K_i\hat{x}_i(s;t_{k+1}),\ s \in [t_k+T, t_{k+1}+T]$. The associated Lyapunov function satisfies

$$\begin{aligned}\dot{V}(\hat{x}_i(s;t_{k+1})) &\leq -\|\hat{x}_i(s;t_{k+1})\|_{Q_i^*}^2 \\ &\leq -\frac{\underline{\lambda}(Q_i^*)}{\bar{\lambda}(P_i)} V_i(\hat{x}_i(s;t_{k+1})).\end{aligned}$$

From the comparison principle, it follows that

$$\begin{aligned}V_i(\hat{x}_i(s;t_{k+1})) &\leq V_i(\hat{x}_i(t_k+T;t_{k+1}))e^{-\frac{\underline{\lambda}(Q_i^*)}{\bar{\lambda}(P_i)}(s-t_k-T)} \\ &\leq \alpha_{2i}^2\varepsilon_i^2 e^{-\frac{\underline{\lambda}(Q_i^*)}{\bar{\lambda}(P_i)}(s-t_k-T)}.\end{aligned} \tag{5.16}$$

After

$$V_i(\hat{x}_i(s;t_{k+1})) \leq \alpha_{2i}^2\varepsilon_i^2 e^{-\frac{\underline{\lambda}(Q_i^*)}{\bar{\lambda}(P_i)}(s-t_k-T)} \leq \alpha_{1i}^2\varepsilon_i^2,$$

it can be proven that

$$\begin{aligned}
&\alpha_{2i}^2 e^{-\frac{\underline{\lambda}(Q_i^*)}{\bar{\lambda}(P_i)}\delta} \leq \alpha_{1i}^2 \\
&\Leftrightarrow e^{-\frac{\underline{\lambda}(Q_i^*)}{\bar{\lambda}(P_i)}\delta} \leq \frac{\alpha_{1i}^2}{\alpha_{2i}^2} \\
&\Leftrightarrow \delta \geq -2\frac{\bar{\lambda}(P_i)}{\underline{\lambda}(Q_i^*)} \ln \frac{\alpha_{1i}}{\alpha_{2i}}.
\end{aligned}$$

This completes the proof. □

The following proposition will be useful for evaluating the variation of the cost function from time t_k to time t_{k+1}.

Proposition 5.1. *For each subsystem $\mathcal{A}_i$ satisfying Assumptions 1 & 2, let the conditions in (5.11) and (5.12) hold ensuring the recursive feasibility of the optimization in (5.8), then the following inequality holds*

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij}\Big(\|\hat{x}_i(s;t_{k+1}) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} - \|\hat{x}_i^*(s;t_k) - \hat{x}_j^a(s;t_k)\|_{Q_{ij}} \Big) ds \\
&\leq \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}} \sum_{j\in N_i} r_{ij}(T-\delta)(\alpha_{2i}-\alpha_{1i})\varepsilon_i + \sum_{j\in N_i} r_{ij}\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(\bar{Q}_i)}}\beta\delta(T-\delta) \\
&\quad + \sum_{j\in N_i} r_{ij}\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}(T-\delta)(\alpha_{2j}-\alpha_{1j})\varepsilon_j. \qquad (5.17)
\end{aligned}$$

Proof. Algebraic manipulations using the triangle inequality lead to

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij}\Big(\|\hat{x}_i(s;t_{k+1}) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} - \|\hat{x}_i^*(s;t_k) - \hat{x}_j^a(s;t_k)\|_{Q_{ij}} \Big) ds \\
&\leq \int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij}\|\hat{x}_i(s;t_{k+1}) - \hat{x}_i^*(s;t_k) + \hat{x}_j^a(s;t_k) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} ds \\
&\leq \int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij}\|x_i(s;t_{k+1}) - x_i^*(s;t_k)\|_{Q_{ij}} ds \qquad (5.18a) \\
&\quad + \int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij}\|\hat{x}_j^a(s;t_k) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} ds. \qquad (5.18b)
\end{aligned}$$

After substitutions from (5.15), Term (5.18a) becomes

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij} \|\hat{x}_i(s;t_{k+1}) - \hat{x}_i^*(s,t_k)\|_{Q_{ij}} ds \\
&\le \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}} \int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij} \bar{\lambda}(P_i^{\frac{1}{2}}) \rho_i \delta e^{L_i\delta} e^{L_i(s-t_{k+1})} ds \\
&= \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}} \sum_{j\in N_i} r_{ij} \int_{t_{k+1}}^{t_k+T} \bar{\lambda}(P_i^{\frac{1}{2}}) \rho_i \delta e^{L_i(s+\delta-t_{k+1})} ds \\
&\le \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}} \sum_{j\in N_i} r_{ij} (T-\delta)(\alpha_{2i} - \alpha_{1i})\varepsilon_i. \qquad (5.19)
\end{aligned}$$

Term (5.18b) can be transformed to

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij} \|\hat{x}_j^a(s;t_k) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} ds \\
&= \int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij} \|\hat{x}_j^a(s;t_k) - \hat{x}_j^*(s;t_k) + \hat{x}_j^*(s;t_k) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} ds \\
&\le \int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij} \|\hat{x}_j^a(s;t_k) - \hat{x}_j^*(s;t_k)\|_{Q_{ij}} ds \qquad (5.20a) \\
&\quad + \int_{t_{k+1}}^{t_k+T} \sum_{j\in N_i} r_{ij} \|\hat{x}_j^*(s;t_k) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} ds. \qquad (5.20b)
\end{aligned}$$

From the constraints imposed in the optimization in (5.8), it follows that the integrand in (5.20b) obeys

$$\begin{aligned}
&\|\hat{x}_j^*(s;t_k) - \hat{x}_j^a(s;t_k)\|_{Q_{ij}} \\
&\le \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(\bar{Q}_i)}} \|\hat{x}_j^*(s;t_k) - \hat{x}_j^a(s;t_k)\|_{\bar{Q}_i} \\
&\le \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(\bar{Q}_i)}} \beta\delta.
\end{aligned}$$

Term (5.20a) obeys

$$\begin{aligned}&\int_{t_{k+1}}^{t_k+T}\sum_{j\in N_i} r_{ij}\|\hat{x}_j^a(s;t_k)-\hat{x}_j^*(s;t_k)\|_{Q_{ij}}ds\\&\leq\sum_{j\in N_i} r_{ij}\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(\bar{Q}_i)}}\beta\delta(T-\delta),\end{aligned}\tag{5.21}$$

while Term (5.20b) obeys

$$\begin{aligned}&\int_{t_{k+1}}^{t_k+T}\sum_{j\in N_i} r_{ij}\|\hat{x}_j^*(s;t_k)-\hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}}ds\\&\leq\int_{t_{k+1}}^{t_k+T}\sum_{j\in N_i} r_{ij}\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}\|\hat{x}_j^*(s;t_k)-\hat{x}_j^a(s;t_{k+1})\|_{P_j}ds\\&\leq\sum_{j\in N_i} r_{ij}\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}\int_{t_{k+1}}^{t_k+T}\bar{\lambda}(P_j^{\frac{1}{2}})\rho_j\delta e^{L_j\delta}e^{L_j(s-t_{k+1})}ds\\&\leq\sum_{j\in N_i} r_{ij}\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}(T-\delta)(\alpha_{2j}-\alpha_{1j})\varepsilon_j.\end{aligned}\tag{5.22}$$

Combining (5.19), (5.21) and (5.22) yields (5.17). This completes the proof. □

The following theorem provides the conditions under which the subsystem $\mathcal{A}_i$ with the MPC in (5.8) enters the positively invariant set $\Omega_i(\varepsilon_i)$ in finite time.

Theorem 5.1. *For the subsystem $\mathcal{A}_i$ in (5.1) with Assumptions 1 & 2, let conditions in (5.11) and (5.12) hold. Then, the subsystem $\mathcal{A}_i$ with the optimization in (5.8) will enter the positively invariant set $\Omega_i(\varepsilon_i)$ if the following three conditions are satisfied*

1.

$$\sum_{j\in N_i} r_{ij}\Xi_{ij}\leq(1+\alpha_{1i}-\alpha_{2i}-\varrho_i-\varsigma_i-\zeta_i)\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}p\varepsilon_i T,\tag{5.23}$$

where ϱ_i, ς_i, ζ_i are positive constants satisfying $\varrho_i+\varsigma_i+\zeta_i\in(0,1+\alpha_{1i}-\alpha_{2i})$ and $p=\frac{\delta}{T}$;

2.

$$\left[\left(\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}+\sqrt{\frac{\bar{\lambda}(K_i^{\mathrm{T}}R_iK_i)}{\underline{\lambda}(P_i)}}p\right)\alpha_{2i}-\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}(1-p)\alpha_{1i}\right]\leq\varrho_i\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}p;\tag{5.24}$$

3. There exists $\beta > 0$ *such that*

$$\sum_{j\in N_i} r_{ij}\frac{\sqrt{\bar{\lambda}(Q_{ij})}\sqrt{\bar{\lambda}(P_i)}}{\sqrt{\underline{\lambda}(\bar{Q}_i)}\sqrt{\underline{\lambda}(Q_i)}}\beta(1-p)T \leq \varsigma_i\varepsilon_i \tag{5.25}$$

holds, where

$$\begin{aligned}\Xi_{ij} =&\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}}(1-p)(\alpha_{2i}-\alpha_{1i})\varepsilon_i T + \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}(1-p)(\alpha_{2j}-\alpha_{1j})\varepsilon_j T\\ &+ (\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}}\alpha_{2i}\varepsilon_i + \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}\alpha_{2j}\varepsilon_j)pT.\end{aligned}$$

Proof. At time instant t_k, each subsystem will independently solve a constrained optimization based on its nominal system model, and obtain the optimal control sequence and its optimal nominal state trajectory. At time instant t_{k+1}, the cost function by employing the assumed control signal is

$$\begin{aligned}&J_i(\hat{x}_i, \hat{u}_i(s;t_{k+1}), \hat{x}^a_{-i}(s;t_{k+1}))\\ &= \int_{t_{k+1}}^{t_{k+1}+T}\left(\|\hat{x}_i(s;t_{k+1})\|_{Q_i} + \|\hat{u}_i(s;t_{k+1})\|_{R_i} + \sum_{j\in N_i} r_{ij}\|\hat{x}_i(s;t_{k+1}) - \hat{x}^a_j(s;t_{k+1})\|_{Q_{ij}}\right)ds.\end{aligned} \tag{5.26}$$

Since $J_i^*(t_{k+1}) \leq J_i(t_{k+1})$, it follows that $J_i^*(t_{k+1}) - J_i^*(t_k) \leq J_i(t_{k+1}) - J_i^*(t_k)$. The change in the cost function between the successive time instants t_k and t_{k+1} is

$$J_i(t_{k+1}) - J_i^*(t_k)$$
$$= \int_{t_{k+1}}^{t_k+T} \Big(\|\hat{x}_i(s;t_{k+1})\|_{Q_i} + \|\hat{u}_i(s;t_{k+1})\|_{R_i} - \|\hat{x}_i^*(s;t_k)\|_{Q_i} - \|\hat{u}_i^*(s,t_{k+1})\|_{R_i} \Big) ds \tag{5.27a}$$
$$+ \int_{t_k+T}^{t_{k+1}+T} \Big(\|\hat{x}_i(s;t_{k+1})\|_{Q_i} + \|\hat{u}_i(s;t_{k+1})\|_{R_i} \Big) ds \tag{5.27b}$$
$$- \int_{t_k}^{t_{k+1}} \Big(\|\hat{x}_i^*(s;t_k)\|_{Q_i} + \|\hat{u}_i^*(s;t_k)\|_{R_i} \Big) ds \tag{5.27c}$$
$$+ \int_{t_{k+1}}^{t_k+T} \sum_{j \in N_i} r_{ij} \Big(\|\hat{x}_i(s;t_{k+1}) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} - \|\hat{x}_i^*(s;t_k) - \hat{x}_j^a(s;t_k)\|_{Q_{ij}} \Big) ds \tag{5.27d}$$
$$+ \int_{t_k+T}^{t_{k+1}+T} \sum_{j \in N_i} r_{ij} \|\hat{x}_i(s;t_{k+1}) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} ds \tag{5.27e}$$
$$- \int_{t_k}^{t_{k+1}} \sum_{j \in N_i} r_{ij} \|\hat{x}_i^*(s;t_k) - \hat{x}_j^a(s;t_k)\|_{Q_{ij}} ds. \tag{5.27f}$$

Because $J_i(t_{k+1})$ is computed using the assumed state trajectory, combining the triangle inequality with Term (5.27a) leads to

$$\int_{t_{k+1}}^{t_k+T} \Big(\|\hat{x}_i(s;t_{k+1})\|_{Q_i} + \|\hat{u}_i(s;t_{k+1})\|_{R_i} - \|\hat{x}_i^*(s;t_k)\|_{Q_i} - \|\hat{u}_i^*(s,t_{k+1})\|_{R_i} \Big) ds$$
$$\leq \int_{t_{k+1}}^{t_k+T} \|\hat{x}_i(s;t_{k+1}) - \hat{x}_i^*(s;t_k)\|_{Q_i} ds.$$

After substitution from (5.15), Term (5.27a) is upper bounded by

$$\begin{aligned}
&\int_{t_{k+1}}^{t_k+T}\bigg(\|\hat{x}_i(s;t_{k+1})\|_{Q_i}+\|\hat{u}_i(s;t_{k+1})|_{R_i}-\|\hat{x}_i^*(s;t_k)\|_{Q_i}-\|\hat{u}_i^*(s,t_{k+1})\|_{R_i}\bigg)ds\\
&\le\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}\int_{t_{k+1}}^{t_k+T}\bar{\lambda}(P_i^{\frac{1}{2}})\rho_i\delta e^{L_i\delta}e^{L_i(s-t_{k+1})}ds\\
&\le\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}\int_{t_{k+1}}^{t_k+T}\bar{\lambda}(P_i^{\frac{1}{2}})\rho_i\delta e^{L_iT}ds\\
&=\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}(T-\delta)\bar{\lambda}(P_i^{\frac{1}{2}})\rho_i\delta e^{L_iT}\\
&\le\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}(T-\delta)(\alpha_{2i}-\alpha_{1i})\varepsilon_i. \qquad (5.28)
\end{aligned}$$

Lemma 5.2 provides sufficient conditions for the assumed states $\hat{x}_i^a(t_k+T;t_{k+1})$ and $\hat{x}_i^a(t_{k+1}+T;t_{k+1})$ to be in $\Omega_i(\alpha_{2i}\varepsilon_i)$ and in $\Omega_i(\alpha_{1i}\varepsilon)$, respectively. Because the state feedback control law $\hat{u}_i(s;t_{k+1})=K_i\hat{x}_i(s;t_{k+1})$ keeps the nominal state within the control invariant set, $\hat{x}_i^a(s;t_{k+1})$ will always be in the control invariant set $\Omega_i(\alpha_{2i}\varepsilon_i)$ for $s\in[t_k+T,t_{k+1}+T]$.

Using $\|x\|_Q^2\le\bar{\lambda}(Q)\|x\|^2\le\frac{\bar{\lambda}(Q)}{\underline{\lambda}(P)}\|x\|_P^2$, Term (5.27b) is upper bounded by

$$\begin{aligned}
&\int_{t_k+T}^{t_{k+1}+T}\bigg(\|\hat{x}_i(s;t_{k+1})\|_{Q_i}+\|\hat{u}_i(s;t_{k+1})\|_{R_i}\bigg)ds\\
&\le\int_{t_k+T}^{t_{k+1}+T}\left(\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}\|\hat{x}_i(s;t_{k+1})\|_{P_i}+\sqrt{\frac{\bar{\lambda}(K_i^{\mathrm{T}}R_iK_i)}{\underline{\lambda}(P_i)}}\|\hat{x}_i(s;t_{k+1})\|_{P_i}\right)ds\\
&\le\left(\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}+\sqrt{\frac{\bar{\lambda}(K_i^{\mathrm{T}}R_iK_i)}{\underline{\lambda}(P_i)}}\right)\alpha_{2i}\varepsilon_i\delta. \qquad (5.29)
\end{aligned}$$

Term (5.27c) is bounded by ignoring the second term in the integrant:

$$-\int_{t_k}^{t_{k+1}}\bigg(\|\hat{x}_i^*(s;t_k)\|_{Q_i}+\|\hat{u}_i^*(s;t_k)\|_{R_i}\bigg)ds\le-\int_{t_k}^{t_{k+1}}\|\hat{x}_i^*(s;t_k)\|_{Q_i}. \qquad (5.30)$$

If the actual system state $x_i(s;t_k) \notin \Omega_i(\varepsilon)$, from Lemma 5.3, it follows that

$$\begin{aligned}\|\hat{x}_i^*(s;t_k)\|_{P_i} &\geq \|\hat{x}_i(s;t_k)\|_{P_i} - \bar{\lambda}(P_i^{\frac{1}{2}})\rho_i(s-t_k)e^{L_i(s-t_k)} \\ &\geq \varepsilon_i - \bar{\lambda}(P_i^{\frac{1}{2}})\rho_i(s-t_k)e^{L_i(s-t_k)},\end{aligned}$$

and that Term (5.27c) is upper bounded as

$$\begin{aligned}&- \int_{t_k}^{t_{k+1}} \|\hat{x}_i^*(s;t_k)\|_{Q_i} ds \\ &\leq -\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}} \int_{t_k}^{t_{k+1}} (\varepsilon_i - \bar{\lambda}(P_i^{\frac{1}{2}})\rho_i(s-t_k)e^{L_i(s-t_k)})ds \\ &\leq \sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}} \int_{t_k}^{t_{k+1}} [\varepsilon_i - \bar{\lambda}(P_i^{\frac{1}{2}}\rho_i\delta e^{L_i\delta})]ds \\ &\leq -\delta\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}[\varepsilon_i - (\alpha_{2i}-\alpha_{1i})\varepsilon_i] \\ &= -(1+\alpha_{1i}-\alpha_{2i})\delta\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}\varepsilon_i. \end{aligned} \tag{5.31}$$

Term (5.27d) has been addressed in Proposition 5.1.

Term (5.27e) is bounded by

$$\begin{aligned}&\int_{t_k+T}^{t_{k+1}+T} \sum_{j\in N_i} r_{ij}\|\hat{x}_i(s;t_{k+1}) - \hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} ds \\ &\leq \int_{t_k+T}^{t_{k+1}+T} \sum_{j\in N_i} r_{ij}(\|\hat{x}_i(s;t_{k+1})\|_{Q_{ij}} + \|\hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}})ds.\end{aligned}$$

Lemma 5.2 leads to

$$\begin{aligned}&\|\hat{x}_i^a(s;t_{k+1})\|_{P_i} \leq \alpha_{2i}\varepsilon_i,\ for\ s \in [t_k+T, t_{k+1}+T]; \\ &\|\hat{x}_j^a(s;t_{k+1})\|_{P_j} \leq \alpha_{2j}\varepsilon_j,\ for\ s \in [t_k+T, t_{k+1}+T].\end{aligned}$$

Therefore, $\|\hat{x}_i^a(s;t_{k+1})\|_{Q_{ij}} \leq \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}}\|\hat{x}_i^a(s;t_{k+1})\|_{P_i} \leq \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}}\alpha_{2i}\varepsilon_i$ and $\|\hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}} \leq \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}\|\hat{x}_j(s;t_{k+1})\|_{P_j} \leq \sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}\alpha_{2j}\varepsilon_j$ hold, and Term (5.27e) is finally upper bounded by

$$\begin{aligned}&\int_{t_k+T}^{t_{k+1}+T}\sum_{j\in N_i} r_{ij}\|\hat{x}_i(s;t_{k+1})-\hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}}ds\\ &\leq \sum_{j\in N_i} r_{ij}[\int_{t_k+T}^{t_{k+1}+T}\|\hat{x}_i(s;t_{k+1})\|_{Q_{ij}}ds+\int_{t_k+T}^{t_{k+1}+T}\|\hat{x}_j^a(s;t_{k+1})\|_{Q_{ij}}ds]\\ &\leq \sum_{j\in N_i} r_{ij}[\delta\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}}\alpha_{2i}\varepsilon_i+\delta\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}\alpha_{2j}\varepsilon_j].\end{aligned} \tag{5.32}$$

Term (5.27f) is non-positive and can be ignored.

Letting $\delta = pT$, the change in the cost function between time instants t_k and t_{k+1} is

$$\begin{aligned}J_i(t_{k+1})-J_i^*(t_k)=&-(1+\alpha_{1i}-\alpha_{2i})\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}p\varepsilon_i T\\ &+[\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}(1-p)(\alpha_{2i}-\alpha_{1i})+(\sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}+\sqrt{\frac{\bar{\lambda}(K_i^{\mathrm{T}}R_iK_i)}{\underline{\lambda}(P_i)}})\alpha_{2i}p]\varepsilon_i T\\ &+\sum_{j\in N_i} r_{ij}[\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}}(1-p)(\alpha_{2i}-\alpha_{1i})\varepsilon_i T\\ &+\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}(1-p)(\alpha_{2j}-\alpha_{1j})\varepsilon_j T\\ &+(\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_i)}}\alpha_{2i}\varepsilon_i+\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(P_j)}}\alpha_{2j}\varepsilon_j)pT]\\ &+\sum_{j\in N_i} r_{ij}\sqrt{\frac{\bar{\lambda}(Q_{ij})}{\underline{\lambda}(\bar{Q}_i)}}\beta(1-p)pT^2.\end{aligned}$$

Substituting of (5.23), (5.24) and (5.25) to the above inequality yields

$$J_i(t_{k+1})-J_i^*(t_k)\leq -\zeta_i\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}p\varepsilon_i T. \tag{5.33}$$

Thus, the state of each subsystem $\mathcal{A}_i$ will converge to the positively invariant set $\Omega_i(\varepsilon_i)$ in finite time. □

Once the state of the subsystem $\mathcal{A}_i$ enters the positively invariant set, the state feedback controller will be implemented. The following theorem shows that if the disturbance level satisfies (5.34), the state of each subsystem $\mathcal{A}_i$ converges to a robust positively invariant set.

Theorem 5.2. *If all conditions in Lemma 5.1 are satisfied, the sampling interval satisfies (5.11) and the disturbance bound satisfies* $\rho_i \leq \bar{\rho}_i \leq \rho_i^{\max}$*, where*

$$\bar{\rho}_i = \frac{\beta_i^{'}\underline{\lambda}(Q_i^*)\varepsilon_i}{2\bar{\lambda}(P_i^{\frac{1}{2}})\bar{\lambda}(P_i)}, \quad \beta_i^{'} \in (0,1), \tag{5.34}$$

then the system state asymptotically converges to the set $\Omega_1(\sqrt{\beta_1^{'}}\varepsilon_1)\times\cdots\times\Omega_S(\sqrt{\beta_S^{'}}\varepsilon_S)$*.*

Proof. The proof is similar to the Theorem 2 in [50]. □

Remark 5.2. *The maximum disturbance that can be tolerated in Theorem 5.2 can be relaxed through the following procedure*

Step 1: *Set* $\rho_i^1 = \rho_i^{\max}$, $\rho_i^2 = 0$, *set* $\bar{\rho}_i = \frac{1}{2}(\rho_i^1 + \rho_i^2)$, *and set a tolerance* $\chi = 0.001$*;*

Step 2: *Solve the optimization*

$$\begin{aligned} \max \dot{V}_i(x_i(t)) = &- x_i^{\mathrm{T}} Q_i^* x_i(t) + 2x_i(t)^{\mathrm{T}} P_i \omega_i(t) \\ &\text{subject to}: \\ &\qquad \|x_i(t)\|_{P_i} \leq \varepsilon_i, \\ &\qquad \|\omega_i(t)\|_2 \leq \bar{\rho}_i; \end{aligned} \tag{5.35}$$

Step 3: *If the solution of (5.35) is less than* 0*, then set* $\rho_i^2 = \bar{\rho}_i$*, else set* $\rho_i^1 = \bar{\rho}_i$*;*

Step 4: *If* $|\rho_i^1 - \rho_i^2| \leq \chi$*, then set* $\bar{\rho}_i = \rho_i^2$*, terminate, else, set* $\bar{\rho}_i = \frac{1}{2}(\rho_i^1 + \rho_i^2)$*, go to Step 2.*

Remark 5.3. *Given a disturbance level that is less than* $\bar{\rho}_i$ *calculated in Remark 5.2, Remark 5.2 can be used to determine a less conservative robust positively invariant set* $\|x_i(t)\|_{P_i} \leq \varepsilon_i^{'}$*. Furthermore,* $\beta_i^{'}$ *and* $\bar{\rho}_i$ *in (5.34) can be calculated using binary search.*

5.4 A Conceptually Less Conservative Distributed MPC Strategy

For some nonlinear systems as in [74], the robust positively invariant set is very small, and algorithms in Section 5.3 is conservative, in the sense that it computes a small allowable disturbance level and small feasible region. A possible remedy, suggested by [36] and [81], is to replace the terminal set by the $\kappa \circ \delta$ controllability set defined in [81].

Definition 5.1. *($\kappa \circ \delta$ Controllability Set to Ξ) For the nominal subsystem in (5.3) with the control constraints $u_i(t) \in \mathbb{U}_i$, a positively invariant set Ξ and a time interval δ, the $\kappa \circ \delta$ controllability set to Ξ, ($\bar{\mathcal{C}_\kappa}(\Xi)$) is*

$$\bar{\mathcal{C}_\kappa}(\Xi) = \{x_i(0) \in \mathbb{R}^n, \exists \mathbf{u}_i[0, \kappa - 1] \in \mathbb{U}^i \times \mathbb{U}^i \times \cdots \times \mathbb{U}^i, \text{ such that } \hat{x}_i(\kappa \circ \delta | x_i(0)) \in \Xi\}.$$

Thus, $\bar{\mathcal{C}_\kappa}(\Xi)$ includes all nominal states that can be steered to Ξ in $\kappa \circ \delta$ time. Note that Ξ is a positively invariant set, i.e., for any state $x \in \Xi$, there exists a state feedback control $u(t) = Kx(t)$ which keeps the successive state of the nominal system in (5.3) in Ξ. Then, it follows that $\bar{\mathcal{C}}_{\kappa-1}(\Xi) \subseteq \bar{\mathcal{C}_\kappa}(\Xi)$, for $\kappa \geq 1$, where $(\bar{\mathcal{C}_0}(\Xi)) = \Xi$.

Remark 5.4. *The procedure to compute the $\kappa \circ \delta$ controllability set for a specific class of nonlinear systems illustrated in [87]. A numerical approximation algorithm for the same class of systems is proposed in [10]. However, a systematic way to compute the $\kappa \circ \delta$ controllability set is not available yet for an arbitrary nonlinear system.*

Due to disturbance, the actual state trajectory deviates from the predicted state trajectory, as characterized in (5.10). In the sequel, the robust $\kappa \circ \delta$ controllability set $\mathcal{C}_\kappa(\Xi)$ is defined.

Definition 5.2. *(Robust $\kappa \circ \delta$ Controllability Set to Ξ) For the nonlinear subsystem in (5.1) with the control constraints $u_i(t) \in \mathbb{U}_i$, a positively invariant set Ξ, and a time interval δ, the robust $\kappa \circ \delta$ controllability set to Ξ, ($\mathcal{C}_\kappa(\Xi)$) is given by:*

$$\mathcal{C}_\kappa(\Xi) = \Big\{x_i(0) \in \mathbb{R}^n, \exists \mathbf{u}_i[0, \kappa - 1] \in \mathbb{U}^i \times \mathbb{U}^i \times \cdots \times \mathbb{U}^i, \text{such that } x_i(\kappa \circ \delta | x_i(0)) \in \Xi\Big\}. \tag{5.36}$$

According to Lemma 5.2, $\alpha_{1i} X_f$ is a positively invariant set has been found. Then, the robust $\kappa \circ \delta$ controllability set $\mathcal{C}_\kappa(\alpha_{1i} X_f)$ can be determined as follows.

- Let $\mathcal{C}_0(\alpha_{1i}X_f) = \alpha_{1i}X_f$. Suppose a positive scalar $c > 0$ such that $\mathcal{C}_\kappa(\alpha_{1i}X_f)$ is non-empty.

- Compute $\bar{\mathcal{C}}_1(\alpha_{1i}X_f)$ using Definition 5.1. Then set $\mathcal{C}_1(\alpha_{1i}X_f) = \bar{\mathcal{C}}_1(\alpha_{1i}X_f) \ominus \mathcal{B}(c)$.

- For $\kappa \geq 2$, compute $\mathcal{C}_\kappa(\alpha_{1i}X_f)$ iteratively using

$$\mathcal{C}_\kappa(\alpha_{1i}X_f) = \mathcal{C}_1(\mathcal{C}_{\kappa-1}(\alpha_{1i}X_f)).$$

Assumption 5.3. *For each subsystem in (5.1) with the weighting matrix P_i, the disturbance level ρ_i and the prediction horizon T, the following inequality holds*

$$\bar{\lambda}(P_i^{\frac{1}{2}})\rho_i T e^{L_i T} \leq c. \tag{5.37}$$

Given Definition 5.2 and Assumption 5.3 above, the following modified distributed MPC algorithm can be defined.

Remark 5.5. *Note that the optimization in (5.38) is implemented iteratively, in the sense that at time instant t_k, and for $\kappa > 1$, the constraint in (5.38c) is $\mathcal{C}_\kappa(\alpha_{1i}X_f)$. At time t_k+1, the robust $(\kappa-1)\circ\delta$ controllability set $\mathcal{C}_{\kappa-1}(\alpha_{1i}X_f)$ provides the constraint in (5.38).*

Because the robust $\kappa \circ \delta$ controllability set is computed using the constraint tightening technique, the initial feasibility of the optimization is sufficient to guarantee the recursive feasibility of the optimization in (5.38). The following theorem states the robust stability property when implementing the proposed algorithm.

Theorem 5.3. *For the nonlinear subsystem in (5.1) with the control constraints $u_i(t) \in \mathbb{U}_i$ and the initial state $x_i(0)$, let there exist a robust $\kappa \circ \delta$ controllability set $\mathcal{C}_\kappa(\alpha_{1i}X_f)$ such that the optimization in (5.38) is feasible. Then, by implementing Algorithm 1, the system state will finally converge to a robust positively invariant set in finite time.*

Remark 5.6. *The distributed MPC in Algorithm 7 is less conservative than the strategy in Section 5.3. A state of the subsystem $\mathcal{A}_i$ in (5.1) can be steered to the robust positively invariant set by the proposed distributed MPC using a prediction horizon T, and by the optimization in (5.38) with a prediction horizon $T-\kappa\delta$. Because the largest*

Algorithm 7 Modified distributed MPC

Initialization:

For the nonlinear subsystem $\mathcal{A}_i$ in (5.1), the weighting matrices Q_i, R_i, Q_{ij}, the robust $\kappa \circ \delta$ controllability set $\mathcal{C}_\kappa(\alpha_{1i} X_f)$, and the given prediction horizon T, let the optimization in (5.38) is initially feasible.

Step 1:

Solve

$$\hat{u}_i(s;t_k) = \arg \min_{\hat{u}_i(t;t_k)} J_i(\hat{x}_i(s;t_k), \hat{u}_i(s;t_k), \hat{x}^a_{-i}(s;t_k)), \tag{5.38a}$$

subject to :

$$\begin{aligned} &\dot{\hat{x}}_i(s;t_k) = f_i(\hat{x}_i(s;t_k), \hat{u}_i(s;t_k)),\ s \in [t_k, t_k+T]; \\ &\hat{x}^a_j(s;t_k) = f_j(\hat{x}^a_j(s;t_k), \hat{u}^a_j(s;t_k)),\ s \in [t_k, t_k+T]; \\ &\hat{u}_i(s;t_k) \in \mathbb{U}_i,\ s \in [t_k, t_k+T]; && (5.38b) \\ &\hat{x}_i(t_k+T;t_k) \in \mathcal{C}_\kappa(\alpha_{1i} X_f). && (5.38c) \end{aligned}$$

Step 2:

At the next time instant, measure the updated state information.

If $\kappa > 1$, update $\kappa \triangleq \kappa - 1$.

Solve the optimization in (5.8) with the updated system state and apply the first control signal to subsystem $\mathcal{A}_i$. Only the first control signal is applied to the system. Go to **Step 1**.

else

Go to **Step 2**.

Step 3:

Update the terminal constraints to $\mathcal{C}_0(\alpha_{1i} X_f)$. Apply the distributed MPC strategy proposed in Section 5.3.

---

possible deviation between the predicted state trajectory and the actual state trajectory increases exponentially, a shorter prediction horizon guarantees stability for a tolerate larger disturbance level, which is especially useful for the cases when the disturbance is state dependent as in [82].

5.5 Illustrative Example

In this section, the distributed MPC in (5.8) is applied to the constrained nonlinear system from [50], which consists of three cart-spring-damper subsystems. The dynamics of subsystem i, $i = 1, 2, 3$, are

$$\dot{x}_{i1}(t) = x_{i2}(t), \tag{5.39a}$$

$$\dot{x}_{i2}(t) = -\frac{k}{m}e^{-x_{i1}(t)}x_{i1}(t) - \frac{h}{m}x_{i2}(t) + \frac{u_i(t)}{m} + \frac{\omega_i(t)}{m}, \tag{5.39b}$$

where x_{i1} and x_{i2} are the cart displacement and the velocity, respectively; $u_i(t)$ and $\omega_i(t)$ are the control signal and the external disturbance, respectively; k, h and m are the cart stiffness, damping and mass, respectively. The cart parameters are constant and their values are selected as: $k = 1.05$ N/m, $h = 0.3$ Ns/m, $m = 1.5$ kg. The control signal is assumed to be restricted to $u_i(t) \in [-4\ 4]$. The parameters in the optimization in (5.8) are: $Q_i = \text{diag}([1.5\ 1.5])$, $R_i = 0.1$, $Q_{ij} = 0.1I$. Placing the eigenvalues of $A_i + B_iK_i$ at $[-1\ \ -0.95]$, the corresponding control law is $K_i = [-0.3750\ \ -2.6250]$. The associated $P_i = \begin{bmatrix} 3.1639 & -0.9084 \\ -0.9084 & 1.1161 \end{bmatrix}$ is computed by using the LQR method. The Lipschitz constant is calculated as $L_i = 2.1982$.

The receding horizon is chosen as $T = 0.8$ s. Then, to ensure recursive feasibility and robust stability, the following parameters are computed according to Lemma 5.4 and Theorem 5.1. $\varrho_i = 0.74$, $\varsigma_i = 0.1340$, $\zeta_i = 0.014$, $\alpha_{1i} = 0.18$, $\alpha_{2i} = 0.20$, $r_{ij} = 0.20$, and $\beta = 1.3561$; we take $\delta = 0.6$ s , because its computed bounds are 0.4929 s and 1.6431 s; the maximum disturbance level from (5.12) is 0.0050.

The initial states of the three subsystems are $(0.5\ -0.6)$, $(-0.6\ 0.5)$ and $(0.65\ -0.4)$, respectively.

The linearized cart dynamics around the origin are

$$\begin{aligned} \dot{x}_i(t) &= \begin{bmatrix} \frac{\partial f_{i1}}{\partial x_{i1}} & \frac{\partial f_{i1}}{\partial x_{i2}} \\ \frac{\partial f_{i2}}{\partial x_{i1}} & \frac{\partial f_{i2}}{\partial x_{i2}} \end{bmatrix} x_i(t) + \begin{bmatrix} \frac{\partial f_{i1}}{\partial u_i} \\ \frac{\partial f_{i2}}{\partial u_i} \end{bmatrix} u_i(t) \\ &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{h}{m} \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u_i(t) \\ &= \begin{bmatrix} 0 & 1.00 \\ -0.70 & -0.20 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 0.6667 \end{bmatrix} u_i(t). \end{aligned} \tag{5.40}$$

The relationship between the parameters β and the parameter r_{ij}, for all $i, j = 1, 2, 3$, is illustrated in Figure 5.1. Note that a stronger cooperation, a larger r_{ij} is required to strengthen the cooperation, and, a larger β permits the state trajectory predicted at the current time step to deviate more from the state trajectory predicted

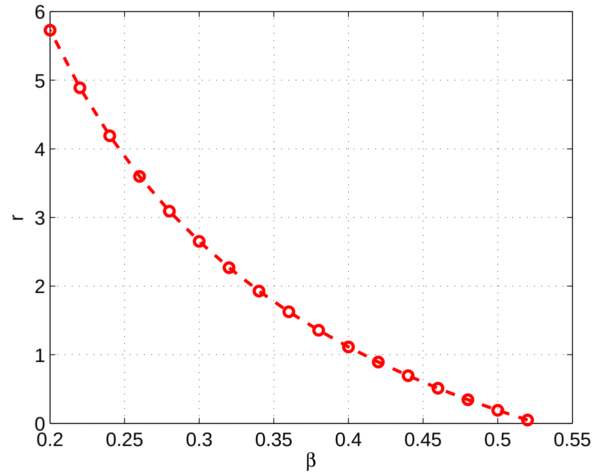

Figure 5.1: Relationship between r and β.

at the precedent time step. A larger deviation between successive predicted state trajectories, i.e., a larger β, can be tolerated if the monotonic decrease of the cost function is ensured through selecting a smaller $r = r_{ij}, \forall i, j \in 1, 2, 3, i \neq j$. Figure 5.1 shows the tradeoff when selecting β and r.

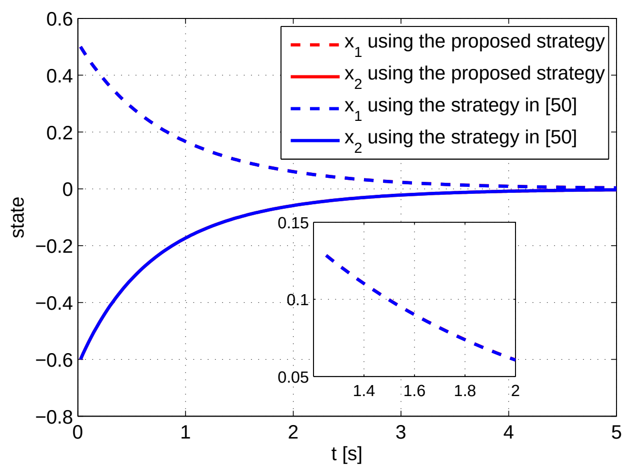

Figure 5.2: The state trajectories of cart 1 controlled using the distributed MPC strategies proposed in this chapter and in [50].

The state trajectories of the three carts coupled by the distributed MPC proposed in Section 5.3 are shown in Figures 5.2, 5.3 and 5.4 together with their state trajectories when coupled through the algorithm proposed in [50]. The corresponding control trajectories are plotted in Figures 5.5, 5.6 and 5.7. Lastly, the difference between the

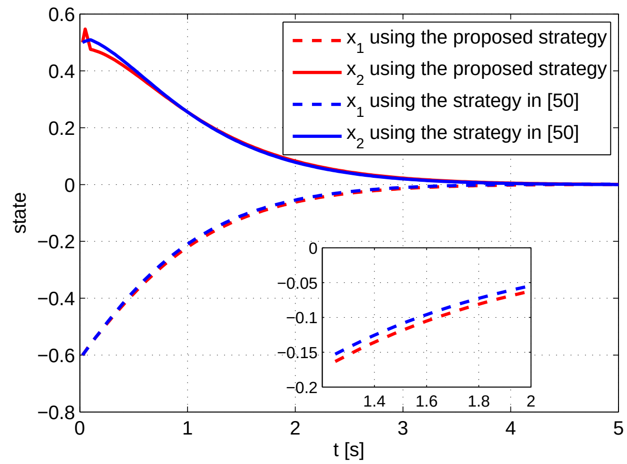

Figure 5.3: The state trajectories of cart 2 controlled using the distributed MPC strategies proposed in this chapter and in [50].

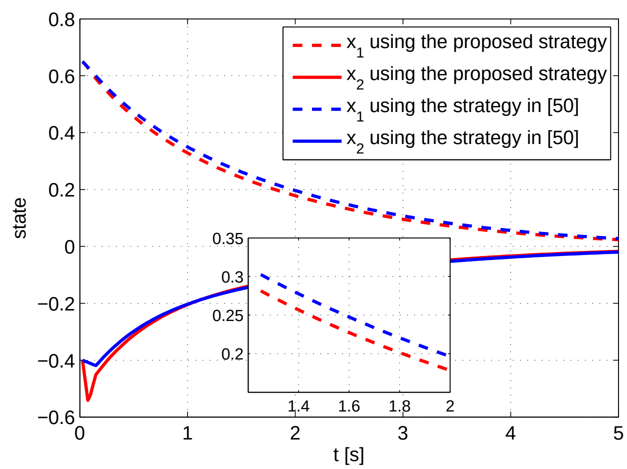

Figure 5.4: The state trajectories of cart 3 controlled using the distributed MPC strategies proposed in this chapter and in [50].

trajectories of two carts under decentralized MPC and under the distributed MPC in this chapter and in [50] are depicted in Figures 5.8 and 5.9. As can be seen that compared to the method in [50], the strategy proposed in this chapter allows a stronger cooperation. The stronger cooperation is enforced by larger initial control efforts applied to carts 2 and 3, as shown in Figures 5.6 and 5.7.

Remark 5.7. *According to the recursive feasibility conditions in (5.11) and (5.12), and the robust stability conditions in Theorem 5.1 and Theorem 5.2, if* ε_i *which speci-*

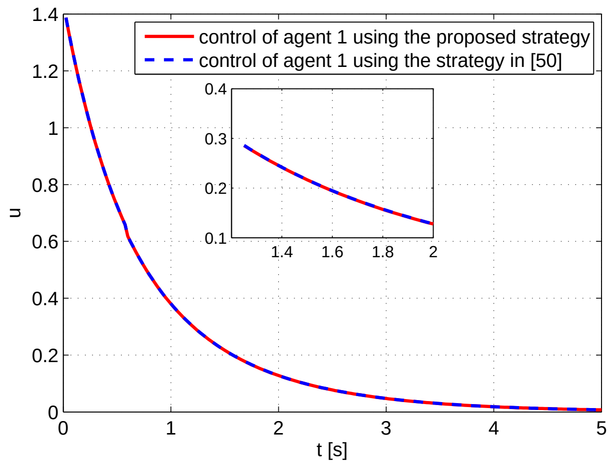

Figure 5.5: The distributed MPC signal for cart 1, computed using the strategies proposed in this chapter and in [50].

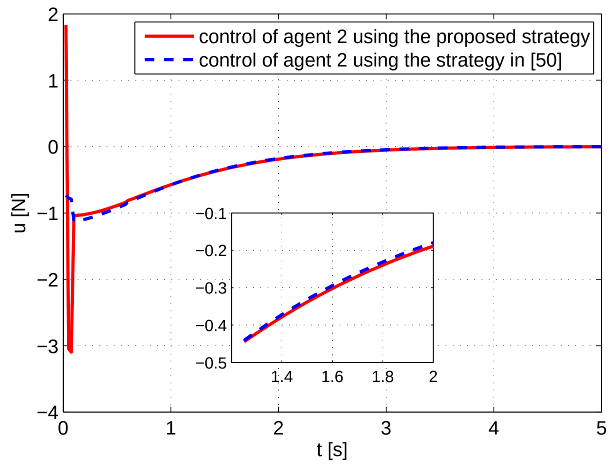

Figure 5.6: The distributed MPC signal for cart 2, computed using the strategies proposed in this chapter and in [50].

fies the positively invariant set is enlarged, then the sampling interval δ can be chosen larger and a larger disturbance level can be tolerated. This indicates that by adopting the modified distributed MPC in Section 5.3 potentially yields less conservative results. However, a systematic way to compute the $\kappa \circ \delta$ controllability set is not available yet. The development of a technique for determining this set will be pursued in the future.

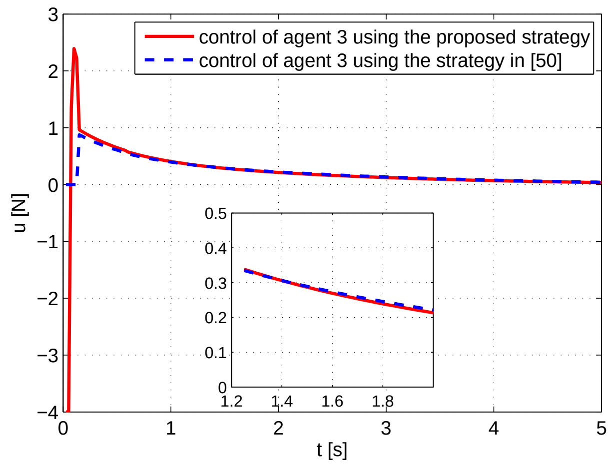

Figure 5.7: The distributed MPC signal for cart 3, computed using the strategies proposed in this chapter and in [50].

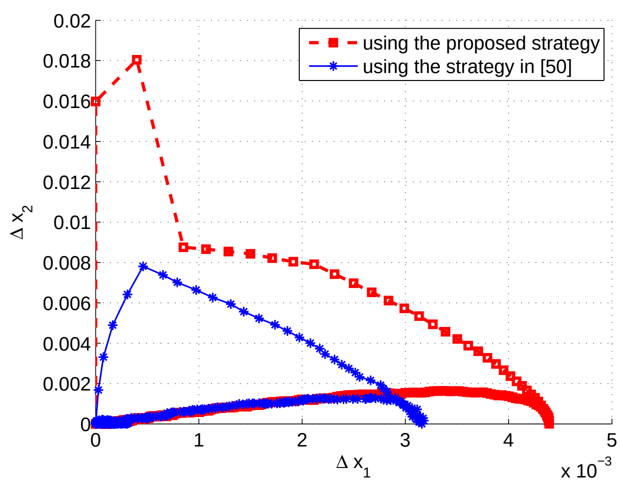

Figure 5.8: Difference between the states of cart 2 under decentralized MPC and under distributed MPC computed via: (i) the strategy proposed in this chapter; and (ii) via the strategy in [50].

5.6 Conclusions

This chapter has proposed a distributed MPC for constrained continuous-time nonlinear systems whose subsystems have decoupled dynamics and cooperate with each other through a coupling term in the cost function. Because each subsystem solves a local optimization using the state information from its neighboring subsystems only, the proposed strategy has reduced the communication burden compared to the cen-

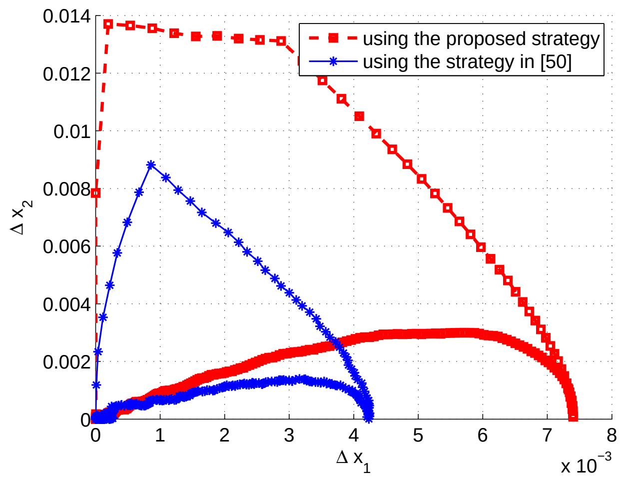

Figure 5.9: Difference between the states of cart 3 under decentralized MPC and under distributed MPC computed via: (i) the strategy proposed in this chapter; and (ii) via the strategy in [50].

tralized control. The recursive feasibility of the optimization and the robust stability of the proposed strategy are analyzed. Sufficient conditions for the recursive feasibility and robust stability have been derived which rely on the appropriate tuning of several design parameters. The required tuning leads to a small terminal set and, thus, makes the proposed strategy relatively conservative. A conceptually less conservative strategy can be designed by exploiting the $\kappa \circ \delta$ controllability set. Its potential advantages are that its prediction horizon can be smaller and that it can tolerate a larger disturbance. The proposed distributed MPC has been illustrated through a numerical example. In this numerical example, the fact that the simulated subsystems converge to their invariant sets even without terminal constraints indicates that the proposed distributed MPC is relatively conservative. Future research will seek to make the proposed method less conservative.

Chapter 6

Distributed Model Predictive Control of Constrained Weakly Coupled Nonlinear Systems

6.1 Introduction

For nonlinear systems, the existing distributed MPC approaches [20, 109] can be classified into cooperative and non-cooperative strategies, based on the topology of the communication among subsystems.

In cooperative distributed MPC, all subsystems use the state information of the overall system and solve system-wide optimizations constrained by the nonlinear dynamics [109]. The requirement that all subsystems communicate with each other imposes a heavy communication burden. One approach to relaxing the communication burden has been presented in [107], through a hierarchical cooperative distributed MPC scheme that collects all subsystems into several groups, each with its own leader, and divides the communication into two layers. In the lower layer, the subsystems communicate with other subsystems from the same group at the rate of their local control. In the higher layer, the group leaders communicate with other group leaders at an asynchronous and slower rate. Another Lyapunov-based approach to relaxing the computational burden of cooperative distributed MPC has been introduced in [56]. The Lyapunov-based approach accounts for disturbances and time-delayed and asynchronous measurements, but assumes that a pre-designed Lyapunov controller is available which makes the origin of the nominal closed-loop system asymptotically stable.

In non-cooperative distributed MPC, all subsystems have access only to their neighbors' information and they use this information in two different ways. In the first approach, all subsystems treat their dynamic interactions with their neighbors as disturbances and compute their local control signals using a min-max strategy [40], a contractive-based strategy [65], a stability constraint strategy [11, 39], or an input-to-state stability (ISS) strategy [86]. In the second approach, subsystems with decoupled dynamics use their neighbors' information to add constraints [20, 24, 90] or cost terms [50] to their local optimizations. The role of the added constraints is to limit the deviation between the trajectories planned by the subsystems and their trajectories assumed by their neighbors [20] and, thus to ensure global stability [90] and global string stability [21]. The role of the added terms in the local cost is to distribute the global control objective to the local controllers [50]. The additions of a terminal constraint in the local optimizations offers robustness by guaranteeing decreasing local costs regardless of disturbances [50].

6.1.1 Objective, Contributions and Chapter Organization

This chapter proposes a distributed model predictive control (MPC) strategy for large-scale systems whose subsystems have weakly coupled nonlinear dynamics and decoupled control constraints, and are affected by additive disturbances. In the proposed strategy, all subsystems compute their control signals by solving local optimizations constrained by their nominal decoupled dynamics. The dynamic couplings and the disturbances are accommodated through new robustness constraints in the local optimizations. The contributions of this chapter are three-fold.

- It proposes a new constraint to provide robustness to dynamic couplings among neighbors and to disturbances, and computes an upper bound on the discrepancy between the actual and predicted state trajectories of each subsystem.

- It analyzes the recursive feasibility of the optimization for each subsystem, and shows that, for a given disturbance level, recursive feasibility can be guaranteed through appropriate design of the sampling interval.

- Sufficient conditions for the state of the large-scale system to converge to a robust positively invariant set are derived.

In the remainder of this chapter, Section 6.2 formulates the problem and upper bounds the discrepancy between the actual and predicted state trajectories of each subsystem. Section 6.3 analyzes the recursive feasibility of the local optimizations which yield the control signal for each subsystem. Section 6.4 establishes sufficient conditions for the convergence of the overall system state to a robust positively invariant set. Section 6.5 illustrates the proposed MPC strategy through applying it to a system which consists of three weakly coupled cart-(nonlinear) spring-damper subsystems.

Notation: The superscript "T" indicates matrix transposition. Given a matrix A: $\bar{\lambda}(A)$ and $\underline{\lambda}(A)$ are its maximum and minimum eigenvalues, respectively; $A > 0$ ($A \geq 0$) shows that A is real symmetric and positive definite (positive semidefinite). Given a vector x: $\|x\|_2$ is its Euclidean norm; and $\|x\|_A = \sqrt{x^{\mathrm{T}}Ax}$ is its A-weighted norm. For a set $\mathbb{N}$, $\mathrm{Card}\{\mathbb{N}\}$ denotes its cardinality. $\mathrm{diag}\{\cdots\}$ indicates a block-diagonal matrix.

6.2 Problem Formulation and Preliminaries

Consider a large-scale system consisting of S interconnected nonlinear subsystems with decoupled control constraints and affected by additive disturbances. The dynamics of the i-th subsystem $\mathcal{A}_i$ are

$$\dot{x}_i(t) = f_i(x_i(t), x_{-i}(t), u_i(t)) + \omega_i(t),\ x_i(t_0) = x_{i0},\ t \geq t_0, \tag{6.1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state of the subsystem $\mathcal{A}_i$; $u_i(t) \in \mathbb{U}_i$ is its constrained control signal; $\omega_i(t)$ is the additive disturbance; and $x_{-i}(t)$ concatenates the states of all subsystems $\mathcal{A}_j$ that are neighbors of the subsystem $\mathcal{A}_i$, i.e.,

$$x_{-i}(t) = (\cdots, x_j(t), \cdots) \in \mathbb{R}^{\sum_{j\in\mathbb{N}_i} n_j}.$$

A subsystem $\mathcal{A}_j$ is a neighbor of $\mathcal{A}_i$, i.e., $\mathcal{A}_j \in \mathbb{N}_i$, if the dynamics of subsystem $\mathcal{A}_i$ depend on the state of subsystem $\mathcal{A}_j$. The disturbance $\omega_i(t)$ is assumed to be bounded in a compact set $\mathbb{W}$, $\mathbb{W} = \{\omega : \|\omega\|_2 \leq \rho\}$.

The linearized dynamics of the subsystem $\mathcal{A}_i$ around the origin are

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j\in\mathbb{N}_i} A_{ij}x_j(t) + B_iu_i(t) + \omega_i(t), \tag{6.2}$$

where $A_{ii} = (\partial f_i/\partial x_i)(0,0)$, $A_{ij} = (\partial f_i/\partial x_j)(0,0)$ for $j \in \mathbb{N}_i$, and $B_i = (\partial f_i/\partial u_i)(0,0)$.

The following assumptions [12] are imposed on the subsystem dynamics in (6.1) and on their linearization around the origin in (6.2).

Assumption 6.1. *a). For each subsystem $\mathcal{A}_i$ in (6.1), the vector field f_i is twice continuously differentiable and satisfies $f_i(0,0,0) = 0$.*
b). Each subsystem $\mathcal{A}_i$ in (6.1) has a unique absolutely continuous solution for any initial condition $(x_1(t_0), x_2(t_0), \cdots, x_S(t_0))$ and any piecewise control $u_i : [t_0, \infty) \to \mathbb{U}_i$.
c). The control constraints $\mathbb{U}_i$ are a compact subset of $\mathbb{R}^{m_i}$ that contains the origin in its interior.

Assumption 6.2. *For each linearized subsystem in (6.2), the pair (A_{ii}, B_i) is stabilizable, i.e., there exists a matrix K_i such that $A_{di} = A_{ii} + B_i K_i$ is Hurwitz.*

By definition, the nominal decoupled dynamics of subsystem $\mathcal{A}_i$ are

$$\dot{\bar{x}}_i(t) = f_i(\bar{x}_i(t), \bar{u}_i(t)),\ \bar{x}_i(t_0) = x_{i0},\ t \geq t_0. \tag{6.3}$$

The dynamics of the overall constrained coupled nonlinear system are

$$\dot{x}(t) = f(x(t), u(t)) + \omega(t), \tag{6.4}$$

where $x(t)^{\mathrm{T}} = \begin{bmatrix} x_1(t)^{\mathrm{T}} & \cdots & x_S(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\omega(t)^{\mathrm{T}} = \begin{bmatrix} \omega_1(t)^{\mathrm{T}} & \cdots & \omega_S(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, and $f(x(t), u(t)) = \begin{bmatrix} f_1(x_1(t), x_{-1}(t), u_1(t))^{\mathrm{T}} \cdots f_S(x_S(t), x_{-S}(t), u_S(t))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, and the linearized dynamics of the overall system around the origin are

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega(t), \tag{6.5}$$

where $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1S} \\ A_{21} & A_{22} & \cdots & A_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ A_{S1} & A_{S2} & \cdots & A_{SS} \end{bmatrix}$, $B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_S \end{bmatrix}$.

By definition, the nominal decoupled dynamics of the overall system are

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t)),\ \bar{x}(t_0) = x_0,\ t \geq t_0. \tag{6.6}$$

Instead of steering the state of the constrained coupled nonlinear system, which consists of all subsystems in (6.1), to the origin through a centralized controller, this chapter adopts a distributed MPC strategy which computes the control for each subsystem $\mathcal{A}_i$ by solving

$$\min_{\bar{u}_i(\tau;t_k),\tau\in[t_k,t_k+T]} J_i(t_k, x_i(t_k))$$
$$= \min_{\bar{u}_i(\tau;t_k),\tau\in[t_k,t_k+T]} \int_{t_k}^{t_k+T} \Big(\|\bar{x}_i(\tau;t_k)\|_{Q_i} + \|\bar{u}_i(\tau;t_k)\|_{R_i} \Big) d\tau$$
$$+ \|\bar{x}_i(t_k+T;t_k)\|_{P_i}$$

subject to:

$$\dot{\bar{x}}_i(\tau;t_k) = f_i(\bar{x}_i(\tau;t_k), \bar{u}_i(\tau;t_k)),\ \bar{x}_i(t_k;t_k) = x_i(t_k),\ \tau \in [t_k, t_k+T], \tag{6.7a}$$
$$\bar{u}_i(\tau;t_k) \in \mathbb{U}_i,\ \tau \in [t_k, t_k+T],$$
$$\|\bar{x}_i(\tau;t_k)\|_{P_i} \leq \frac{T}{\tau - t_{k-1}} \alpha_i \varepsilon_i^{'},\ \tau \in [t_k, t_k+T], \tag{6.7b}$$

where $Q_i > 0$ and $R_i \geq 0$ are given state stage and control stage weighting matrices, respectively; P_i is the terminal weighting matrix designed as in Lemma 1 [57]; T is the prediction horizon; $\varepsilon_i^{'}$ is a positive constant that characterizes the positively invariant set $\Omega_{\varepsilon_i^{'}}$ that will be designed in (6.14); and α_i is a shrinking factor that will be designed in Theorem 6.1. In the distributed MPC strategy in (6.7), all subsystems predict their trajectories based on their decoupled nominal dynamics, and the constraint (6.7b) imposes a monotonically decreasing bound on the predicted system trajectory. This bound is needed for the recursive feasibility of the distributed MPC controller in (6.7), as well as for the stability of the constrained weakly coupled nonlinear system with this controller.

Remark 6.1. *The constraint in (6.7b) is inspired by the robustness constraint* $\|\bar{x}_i(\tau;t_k)\|_{P_i} \leq \frac{T}{\tau - t_k} \alpha_i \varepsilon_i^{'}$, $\tau \in [t_{k+1}, t_k+T]$ *in [50], but serves a different purpose and therefore, has a different form. The robustness constraint in [50] accommodates disturbances, whereas the constraint in (6.7b) provides robustness to dynamic couplings and to disturbances simultaneously. Therefore, the constraint in (6.7b) upper bounds the predicted state trajectory by* $\frac{T}{\tau - t_{k-1}} \alpha_i \varepsilon_i^{'}$ *over* $\tau \in [t_k, t_k+T]$, *as opposed to the robust constraint in [50], which upper bounds the predicted trajectory by* $\frac{T}{\tau - t_k} \alpha_i \varepsilon_i^{'}$ *over* $\tau \in [t_{k+1}, t_k+T]$.

For the reader's convenience, Lemma 1 in [57] is presented below.

Lemma 6.1. *[57] For the nominal decoupled subsystem in (6.3) with Assumption 6.2:*

(a) *there exists a unique positive definite matrix* P_i^* *which solves the following Lyapunov equation* $(A_{di}+\kappa_i I)^{\mathrm{T}}P_i^* + P_i^*(A_{di}+\kappa_i I) = -Q_i^*$*, where* $Q_i^* = 2(1+\alpha_{2i})\sqrt{\frac{1}{\underline{\lambda}(Q_i)}}Q_i$*,* $\alpha_{2i} = \sqrt{\frac{\bar{\lambda}(K_i^{\mathrm{T}}R_iK_i)}{\underline{\lambda}(Q_i)}}$*; and* $\kappa_i \in [0, -\bar{\lambda}(A_{di}))$*;*

(b) *a neighborhood of the origin* $\Omega_{\varepsilon_i} = \{x \in \mathbb{R}^{n_i} : \|x\|_{P_i} \le \varepsilon_i\}$ *exists with* $P_i = \bar{\lambda}(P_i^*)P_i^*$ *such that*

1. *the state feedback controller* $\bar{u}_i(t) = K_i\bar{x}_i(t)$ *satisfies the control constraints* $\mathbb{U}_i$ *for any state* $\bar{x}_i(t) \in \Omega_{\varepsilon_i}$*, i.e.,* $\bar{u}_i(t) = K_i\bar{x}_i(t) \in \mathbb{U}_i$*,* $\forall \bar{x}_i(t) \in \Omega_{\varepsilon_i}$*;*
2. Ω_{ε_i} *is a control positively invariant set for the nominal decoupled subsystem in (6.3) with the state feedback control* $\bar{u}_i(t) = K_i\bar{x}_i(t)$*, i.e.,* $\bar{x}_i(t) \in \Omega_{\varepsilon_i}$*,* $\forall \bar{x}_i(t_0) \in \Omega_{\varepsilon_i}$*,* $\forall t \ge t_0$*;*
3. *the terminal cost* $\|\bar{x}_i(t_k)\|_{P_i}$ *serves as a control Lyapunov-like function for any state* $\bar{x}_i(t_k) \in \Omega_{\varepsilon_i}$ *in the sense that*

$$\int_{t_k}^{t_k+\delta} \left(\|\bar{x}_i(\tau)\|_{Q_i} + \|\bar{u}_i(\tau)\|_{R_i}\right) d\tau \le \|\bar{x}_i(t_k)\|_{P_i} - \|\bar{x}_i(t_k+\delta)\|_{P_i}, \tag{6.8}$$

or, in differential form

$$\frac{d}{dt}\left(\|\bar{x}_i(t)\|_{P_i}\right) \le -\left(\|\bar{x}_i(t)\|_{Q_i} + \|\bar{u}_i(t)\|_{R_i}\right). \tag{6.9}$$

Let

$$\begin{aligned} Q^* &= \mathrm{diag}\{Q_1^*\ Q_2^*\ \cdots\ Q_S^*\}, \\ P^* &= \mathrm{diag}\{P_1^*\ P_2^*\ \cdots\ P_S^*\}, \\ D_\kappa &= \mathrm{diag}\{\kappa_1 I\ \kappa_2 I\ \cdots\ \kappa_S I\}, \end{aligned}$$

where Q_i^*, P_i^* and κ_i, $i = 1, 2, \cdots, S$ are defined in Lemma 6.1. Then the following Lyapunov equation holds for the overall nominal decoupled system in (6.6)

$$(A_d + D_\kappa)^{\mathrm{T}}P^* + P^*(A_d + D_\kappa) = -Q^*. \tag{6.10}$$

Lemma 6.1 has provided the method to determine both the terminal weighting matrix P_i for the optimization in (6.7) and the positively invariant set Ω_{ε_i} for the nominal decoupled subsystem in (6.3). Lemma 6.2 will establish the existence of a positively invariant set Ω_ε for the constrained weakly coupled nonlinear system in (6.4) with the state feedback control $u(t) = Kx(t)$, $K = \text{diag}\{K_1\ K_2\ \cdots\ K_S\}$. Lemma 6.2 and the derivations in Sections 6.3 and 6.4 use the following standard assumptions [20].

Assumption 6.3. *The inequality* $A_0^{\mathrm{T}} P^* + P^* A_0 \leq \frac{1}{2} Q^*$ *holds, where* $A_0 = A_c - A_d$, *with* $A_d = \text{diag}\{A_{d1}\ A_{d2}\ \cdots\ A_{dS}\}$, $A_c = A + BK$.

Assumption 6.3 limits the strength of interconnection among subsystems.

Assumption 6.4. *For each subsystem* $\mathcal{A}_i$ *in (6.1), the vector field* f_i *satisfies the Lipschitz condition*

$$\begin{aligned}&\|f_i(x_i(\tau;t_k), x_{-i}(\tau;t_k), \bar{u}_i(\tau;t_k)) - f_i(\bar{x}_i(\tau;t_k), \bar{u}_i(\tau;t_k))\|_2 \\ &\leq L_{f_i}\|x_i(\tau;t_k) - \bar{x}_i(\tau;t_k)\|_2 + \sum_{j\in\mathbb{N}_i} \beta_{ij}\|x_j(\tau;t_k)\|_2\end{aligned}$$

where $L_{f_i} > 0$ *and* $\beta_{ij} > 0$.

Lemma 6.2. *For the constrained weakly coupled nonlinear system in (6.4), with Assumptions 6.1 and 6.3 and with the state feedback control* $u(t) = Kx(t)$, *there exists a neighborhood of the origin* Ω_ε

$$\Omega_\varepsilon = \{x \in \mathbb{R}^{\sum_{i=1}^{S} n_i} : \|x\|_{P^*} \leq \varepsilon\}$$

such that:

- Ω_ε *is a control positively invariant set for the system in (6.4);*
- $u(t)$ *is admissible everywhere in* Ω_ε, *i.e.,* $u(t) = Kx(t) \in \mathbb{U}, \forall x(t) \in \Omega_\varepsilon$.

Proof. In view of (6.9), Ω_ε can be shown to be a positively invariant set of (6.5) with the feedback control $u(t) = Kx(t)$ by showing that

$$\frac{d}{dt}(\|x(t)\|_{P^*}) = \frac{x(t)^{\mathrm{T}}(A_c^{\mathrm{T}} P^* + P^* A_c)x(t) + 2x(t)^{\mathrm{T}} P^* \phi(x(t))}{2\|x(t)\|_{P^*}} \leq 0 \qquad (6.11)$$

holds for $\forall x(t) \in \Omega_\varepsilon$, where $\phi(x(t)) = f(x(t), Kx(t)) - A_c x(t)$. Using $A_c = A_d + A_0$, the numerator in (6.11) can be written as

$$\begin{aligned}
&x(t)^{\mathrm{T}}(A_d^{\mathrm{T}}P^* + P^*A_d)x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)) + x(t)^{\mathrm{T}}(A_0^{\mathrm{T}}P^* + P^*A_0)x(t) \\
&= -x(t)^{\mathrm{T}}Q^*x(t) - 2x(t)^{\mathrm{T}}D_\kappa P^*x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)) \\
&\qquad + x(t)^{\mathrm{T}}(A_0^{\mathrm{T}}P^* + P^*A_0)x(t) \\
&\le -\frac{1}{2}x(t)^{\mathrm{T}}Q^*x(t) - 2x(t)^{\mathrm{T}}D_\kappa P^*x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)),
\end{aligned} \tag{6.12}$$

where Equation (6.12) follows after applying Assumption 6.3.

Let $\kappa = \min_i(\kappa_i)$, then

$$-2x(t)^{\mathrm{T}}D_\kappa P^*x(t) \le -2x(t)^{\mathrm{T}}\kappa P^*x(t). \tag{6.13}$$

Since $\frac{\|\phi(x(t))\|_{P^*}}{\|x(t)\|_{P^*}} \to 0$ as $\|x(t)\|_{P^*} \to 0$, an $\varepsilon > 0$ can always be found such that $\frac{\|\phi(x(t))\|_{P^*}}{\|x(t)\|_{P^*}} \le \kappa$ holds when $\|x(t)\|_{P^*} \le \varepsilon$. After substitution from (6.13), Equation (6.12) yields

$$\begin{aligned}
&-\frac{1}{2}x(t)^{\mathrm{T}}Q^*x(t) - 2x(t)^{\mathrm{T}}D_\kappa P^*x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)) \\
&\le -\frac{1}{2}x(t)^{\mathrm{T}}Q^*x(t) - 2x(t)^{\mathrm{T}}\kappa P^*x(t) + 2\kappa\|x(t)\|_{P^*}\|x(t)\|_{P^*} \\
&= -\frac{1}{2}x(t)^{\mathrm{T}}Q^*x(t) \le 0.
\end{aligned}$$

This completes the proof. □

The control positively invariant set Ω_ε can be computed as follows:

Step 0: Set $\bar{\varepsilon}$ a large enough value to include the feasible ε, $\underline{\varepsilon} = 0$ and the tolerance '*tol*' to satisfy the system performance;

Step 1: Set $\varepsilon = \frac{1}{2}(\bar{\varepsilon} + \underline{\varepsilon})$, and compute

$$\begin{aligned}
f_{\min} &= \min_{x(t)\in\Omega_\varepsilon} (-f(x(t))) \\
&= \min_{x(t)\in\Omega_\varepsilon} \left(\frac{1}{2}x(t)^{\mathrm{T}}Q^*x(t) + 2x(t)^{\mathrm{T}}D_\kappa P^*x(t) - 2x(t)^{\mathrm{T}}P^*\phi(x(t))\right);
\end{aligned}$$

Step 2: If $f_{\min} > 0$, set $\underline{\varepsilon} = \varepsilon$; otherwise, set $\bar{\varepsilon} = \varepsilon$;

Step 3: If $\bar{\varepsilon} - \underline{\varepsilon} > tol$, go to Step 1. Otherwise, terminate and output $\varepsilon = \underline{\varepsilon}$.

Now, the positively invariant set $\Omega_{\varepsilon_i'} = \{x \in \mathbb{R}^{n_i} : \|x\|_{P_i} \leq \varepsilon_i'\}$ required in (6.7b) can be computed using [20]

$$\varepsilon_i' = \min\left(\varepsilon_i, \frac{\varepsilon}{\sqrt{S}}\sqrt{\bar{\lambda}(P_i^*)}\right), \tag{6.14}$$

and $\Omega_{\varepsilon'} = \Omega_{\varepsilon_1'} \times \Omega_{\varepsilon_2'} \times \cdots \times \Omega_{\varepsilon_S'}$ becomes a positively invariant set for the overall nominal decoupled nonlinear system in (6.6) with the control $u(t) = Kx(t)$.

The distributed MPC signal in (6.7) is generated using the nominal decoupled subsystem dynamics in (6.3) as the prediction model. However, the couplings among the subsystems and the external disturbances lead to deviations of the state trajectories of the subsystems $\mathcal{A}_i$ from their predicted state trajectories $\bar{x}_i(t;t_k)$ employed in (6.7). Lemma 6.3 bounds these deviations.

Lemma 6.3. *For each subsystem $\mathcal{A}_i$ in (6.1) with Assumption 6.4, the deviation of its state trajectory from the predicted state trajectory of its nominal and decoupled dynamics in (6.6) is upper bounded by*

$$\|x_i(t;t_k) - \bar{x}_i(t;t_k)\|_2 \leq \lambda(t-t_k)e^{\mu(t-t_k)}, \tag{6.15}$$

where

$$\mu = L + \beta\bar{N}, \quad \lambda = \left(\beta\bar{\gamma}\bar{N}\frac{T\alpha\varepsilon'}{\delta} + \rho\right), \tag{6.16}$$

and $L = \max_i\{L_{f_i}\}$, $\beta = \max_{i,j}\{\beta_{ij}\}$, $\alpha = \max_j\{\alpha_j\}$, $\varepsilon' = \max_i\{\varepsilon_i'\}$, $\bar{\gamma} = \max_i\{\sqrt{\frac{1}{\underline{\lambda}(P_i)}}\}$. *$\bar{N}$ is the number of neighbors that each subsystem has in the symmetric super-graph G' [8] of the graph G which encodes the topology of the inter-subsystem couplings, and, $\bar{N} \geq \max_i \mathrm{Card}\{\mathbb{N}_i\}$.*

Proof. Let: $\bar{u}_i(t;t_k)$ be a feasible control trajectory; $\bar{u}_i^*(t;t_k)$ be the optimal control trajectory computed in (6.7); $\bar{x}_i(t;t_k)$ be the state trajectory predicted by applying $\bar{u}_i(t;t_k)$; and $\bar{x}_i^*(t;t_k)$ be the state trajectory predicted by applying $\bar{u}_i^*(t;t_k)$. After substitution from (6.1) and (6.3), (6.15) yields

$$\begin{aligned}
&\|x_i(t;t_k) - \bar{x}_i(t;t_k)\|_2 \\
&\leq \int_{t_k}^{t} \Bigg(L_{f_i}\|x_i(\tau;t_k) - \bar{x}_i(\tau;t_k)\|_2 + \sum_{j\in\mathbb{N}_i} \beta_{ij}\|x_j(\tau;t_k) - \bar{x}_j(\tau;t_k)\|_2 \\
&\quad + \sum_{j\in\mathbb{N}_i} \beta_{ij}\|\bar{x}_j(\tau;t_k)\|_2 + \|\omega_i(\tau)\|_2 \Bigg) d\tau, \qquad (6.17)
\end{aligned}$$

and after substitution from (6.7b), Equation (6.17) becomes

$$\begin{aligned}
&\|x_i(t;t_k) - \bar{x}_i(t;t_k)\|_2 \\
&\leq \int_{t_k}^{t} \Bigg(L\|x_i(\tau;t_k) - \bar{x}_i(\tau;t_k)\|_2 + \beta\bar{N}\|x_i(\tau;t_k) - \bar{x}_i(\tau;t_k)\|_2 \\
&\qquad + \beta\bar{\gamma}\bar{N}\frac{T\alpha\varepsilon^{'}}{\tau - t_{k-1}} + \rho \Bigg) d\tau \\
&= \int_{t_k}^{t} \left((L + \beta\bar{N})\|x_i(\tau;t_k) - \bar{x}_i(\tau;t_k)\|_2 + \beta\bar{\gamma}\bar{N}\frac{T\alpha\varepsilon^{'}}{\tau - t_{k-1}} + \rho \right) d\tau. \qquad (6.18)
\end{aligned}$$

After substitution from (6.16), Equation (6.18) yields

$$\begin{aligned}
&\sum_{i=1}^{S} \|x_i(t;t_k) - \bar{x}_i(t;t_k)\|_2 \\
&\leq \int_{t_k}^{t} (L + \beta\bar{N}) \sum_{i=1}^{S} \|x_i(\tau;t_k) - \bar{x}_i(\tau;t_k)\|_2 d\tau + \left(\beta\bar{\gamma}\bar{N}ST\alpha\varepsilon^{'}/\delta + S\rho\right)(t - t_k) \\
&= \int_{t_k}^{t} \mu \sum_{i=1}^{S} \|x_i(\tau;t_k) - \bar{x}_i(\tau;t_k)\|_2 d\tau + S\lambda(t - t_k). \qquad (6.19)
\end{aligned}$$

Applying the Gronwall-Bellman inequality to (6.19) leads to

$$\sum_{i=1}^{S} \|x_i(t;t_k) - \bar{x}_i(t;t_k)\|_2 \leq S\lambda(t - t_k)e^{\mu(t-t_k)}. \qquad (6.20)$$

From (6.20), (6.15) follows based on symmetry. This completes the proof. □

6.3 Recursive Feasibility

To successfully implement the proposed distributed MPC strategy, the optimization in (6.7) should be recursively feasible, i.e., the feasibility of the optimization in (6.7) at time t_k should indicate its feasibility at time t_{k+1}. This section establishes sufficient conditions for the recursive feasibility of (6.7) after first determining sufficient conditions for the satisfaction of the terminal constraints in (6.7b). The derivations use the assumed control trajectory $\hat{u}_i(t;t_k)$ which is defined by

$$\hat{u}_i(t;t_k) = \begin{cases} \bar{u}_i^*(t;t_{k-1}), \ t \in [t_k, t_{k-1}+T), \\ K_i\hat{x}_i(t;t_k), \ t \in [t_{k-1}+T, t_k+T], \end{cases} \tag{6.21}$$

where $\dot{\hat{x}}_i(t;t_k) = f_i(\hat{x}_i(t;t_k), \hat{u}(t;t_k))$ with $\hat{x}_i(t_k;t_k) = x_i(t_k)$.

Theorem 6.1. *For the subsystem in (6.1) with the control in (6.7), if the sampling interval δ satisfies*

$$\frac{\alpha_i\varepsilon_i^{'}}{T+\delta} \geq \sqrt{\bar{\lambda}(P_i)}\Big(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta + \rho\Big)e^{\mu T}, \tag{6.22a}$$

$$\frac{T}{T+\delta} \geq e^{-\eta_i\delta}, \tag{6.22b}$$

where $\eta_i = \left(\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}} + \sqrt{\frac{\underline{\lambda}(K_i^{\mathrm{T}}R_iK_i)}{\bar{\lambda}(P_i)}}\right)$, *then the assumed control signal $\hat{u}_i(t;t_k)$ in (6.21) steers the assumed state trajectory $\hat{x}_i(t;t_k)$ of the nominal decoupled dynamics of the subsystem $\mathcal{A}_i$ to the positively invariant set $\Omega_{\frac{T}{T+\delta}\alpha_i\varepsilon_i^{'}}$ within the prediction horizon T.*

Proof. From (6.15), it follows that

$$\begin{aligned} \|x_i(t;t_k) - \bar{x}_i(t;t_k)\|_{P_i} &\leq \sqrt{\bar{\lambda}(P_i)}\|x_i(t;t_k) - \bar{x}_i(t;t_k)\|_2 \\ &\leq \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta + \rho)(t-t_k)e^{\mu(t-t_k)}. \end{aligned} \tag{6.23}$$

The substitution $t = t_{k+1}$ in (6.23) leads to

$$\|x_i(t_{k+1};t_k) - \bar{x}_i(t_{k+1};t_k)\|_{P_i} \leq \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta + \rho)\delta e^{\mu\delta}, \tag{6.24}$$

and further to

$$
\begin{aligned}
&\|\hat{x}_i(\tau;t_{k+1}) - \bar{x}_i^*(\tau;t_k)\|_{P_i} \le \sqrt{\bar{\lambda}(P_i)}\|\hat{x}_i(\tau;t_{k+1}) - \bar{x}_i^*(\tau;t_k)\|_2 \\
&\le \sqrt{\bar{\lambda}(P_i)}\Big(\|\hat{x}_i(t_{k+1};t_{k+1}) - \bar{x}_i^*(t_{k+1};t_k)\|_2 \\
&\qquad + L_{f_i}\int_{t_{k+1}}^{\tau} \|\hat{x}_i(s;t_{k+1}) - \bar{x}_i^*(s;t_k)\|_2 ds\Big). \qquad (6.25)
\end{aligned}
$$

After applying the Gronwall-Bellman inequality and using (6.24), Equation (6.25) yields

$$
\|\hat{x}_i(\tau;t_{k+1}) - \bar{x}_i^*(\tau;t_k)\|_{P_i} \le \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta + \rho)\delta e^{\mu\delta}e^{L_{f_i}(\tau - t_{k+1})}. \qquad (6.26)
$$

The constraints in (6.7b) together with (6.26) lead to $\|\bar{x}_i^*(t_k+T;t_k)\|_{P_i} \le \frac{T\alpha_i\varepsilon_i^{'}}{T+\delta}$, and further to

$$
\begin{aligned}
&\|\hat{x}_i(t_k+T;t_{k+1})\|_{P_i} \le \|\bar{x}_i^*(t_k+T;t_k)\|_{P_i} + \|\hat{x}_i(t_k+T;t_{k+1}) - \bar{x}_i^*(t_k+T;t_k)\|_{P_i} \\
&\le \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta + \rho)\delta e^{\mu\delta}e^{L_{f_i}(T-\delta)} + \frac{T}{T+\delta}\alpha_i\varepsilon_i^{'} \\
&\le \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta + \rho)\delta e^{\mu T} + \frac{T}{T+\delta}\alpha_i\varepsilon_i^{'}. \qquad (6.27)
\end{aligned}
$$

If (6.22a) holds, then

$$
\|\hat{x}_i(t_k+T;t_{k+1})\|_{P_i} \le \alpha_i\varepsilon_i^{'}, \qquad (6.28)
$$

which shows that the assumed state $\hat{x}_i(t_k+T;t_{k+1})$ of the nominal and decoupled dynamics of the subsystem $\mathcal{A}_i$ are steered to the positively invariant set $\Omega_{\alpha_i\varepsilon_i^{'}}$. When $\hat{x}_i(t_k+T;t_{k+1}) \in \Omega_{\alpha_i\varepsilon_i^{'}}$, the state feedback control $u_i(t) = K_i x_i(t)$ is implemented.

From (6.9), it follows that

$$
\begin{aligned}
\frac{d}{dt}(\|\bar{x}_i(t)\|_{P_i}) &\le -(\|\bar{x}_i(t)\|_{Q_i} + \|\bar{u}_i(t)\|_R) \\
&\le -\left(\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}} + \sqrt{\frac{\underline{\lambda}(K_i^{\mathrm{T}}R_iK_i)}{\bar{\lambda}(P_i)}}\right)\sqrt{\bar{x}_i^{\mathrm{T}}(t)P_i\bar{x}_i(t)} \\
&= -\eta_i\|\bar{x}_i(t)\|_{P_i}, \qquad (6.29)
\end{aligned}
$$

where $\eta_i = \left(\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}} + \sqrt{\frac{\underline{\lambda}(K_i^{\mathrm{T}} R_i K_i)}{\bar{\lambda}(P_i)}}\right)$. Applying the comparison principle [44] to (6.29) yields

$$\|\hat{x}_i(t;t_{k+1})\|_{P_i} \leq \|\hat{x}_i(t_k+T;t_{k+1})\|_{P_i} e^{-\eta_i(t-t_k-T)}, t \in [t_k+T, t_{k+1}+T]. \tag{6.30}$$

Combining (6.28), (6.30) and (6.22b) leads to

$$\|\hat{x}_i(t_{k+1}+T;t_{k+1})\|_{P_i} \leq \alpha_i \varepsilon_i^{'} e^{-\eta_i \delta} \leq \frac{T}{T+\delta}\alpha_i \varepsilon_i^{'},$$

which shows that the terminal state $\hat{x}_i(t_{k+1}+T;t_{k+1})$ of the assumed state trajectory $\hat{x}_i(t;t_{k+1})$ enters the positively invariant set $\Omega_{\frac{T}{T+\delta}\alpha_i\varepsilon_i^{'}}$. This completes the proof. □

Theorem 6.1 establishes sufficient conditions for the assumed control $\hat{u}_i(t;t_k)$ in (6.21) to steer the assumed state of the nominal decoupled dynamics of the subsystem $\mathcal{A}_i$ to the positively invariant set within the prediction horizon T. Sufficient conditions for the feasibility of the assumed control $\hat{u}_i(t;t_k)$ are provided next.

Theorem 6.2. *Given a feasible initial state for the subsystem $\mathcal{A}_i$ in (6.1), the optimization in (6.7) is recursively feasible if the sampling interval δ satisfies (6.22a) and (6.22b), and if*

$$T\eta_i - 1 \geq 0. \tag{6.31}$$

Proof. Assume the optimization in (6.7) is feasible at time t_k. To determine its feasibility at time t_{k+1}, the assumed state trajectory $\hat{x}_i(t;t_{k+1})$ will be investigated over two time intervals, $[t_{k+1}, t_k+T]$ and $[t_k+T, t_{k+1}+T]$.

Consider the time interval $[t_{k+1}, t_k+T]$. From (6.26), it follows that

$$\begin{aligned}\|\hat{x}_i(\tau;t_{k+1})\|_{P_i} &\leq \|\bar{x}_i^*(\tau;t_k)\|_{P_i} + \|\hat{x}_i(\tau;t_{k+1}) - \bar{x}_i^*(\tau;t_k)\|_{P_i} \\ &\leq \frac{T\alpha_i\varepsilon_i^{'}}{\tau - t_{k-1}} + \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta+\rho)\delta e^{\mu\delta} e^{L_{f_i}(\tau-t_{k+1})}.\end{aligned} \tag{6.32}$$

In view of (6.27) and (6.28), when $\tau = t_k + T$, the last term on the right-hand side of (6.32) is bounded by

$$\sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta+\rho)\delta e^{\mu\delta} e^{L_{f_i}(T-\delta)} \leq \frac{\delta\alpha_i}{T+\delta}\varepsilon_i^{'}. \tag{6.33}$$

After substitution from (6.33), Equation (6.32) yields

$$\begin{aligned}\|\hat{x}_i(\tau;t_{k+1})\|_{P_i} &\leq \frac{T\alpha_i\varepsilon_i^{'}}{\tau-t_{k-1}} + \frac{\delta\varepsilon_i^{'}\alpha_i}{(T+\delta)e^{L_{f_i}(T-\delta)}}e^{L_{f_i}(\tau-t_{k+1})}\\ &= \frac{T\alpha_i\varepsilon_i^{'}}{\tau-t_{k-1}} + \frac{\delta\varepsilon_i^{'}\alpha_i}{T+\delta}e^{L_{f_i}(\tau-t_{k+1}+\delta-T)}.\end{aligned}$$

Considering $\frac{T\alpha_i\varepsilon_i^{'}}{\tau-t_k} - \frac{T\alpha_i\varepsilon_i^{'}}{\tau-t_{k-1}} = \frac{\delta T\alpha_i\varepsilon_i^{'}}{(\tau-t_k)(\tau-t_{k-1})}$, the recursive feasibility of (6.7) is guaranteed if

$$\frac{\delta\alpha_i\varepsilon_i^{'}}{(T+\delta)}e^{L_{f_i}(\tau-t_{k+1}+\delta-T)} \leq \frac{\delta T\alpha_i\varepsilon_i^{'}}{(\tau-t_k)(\tau-t_{k-1})}. \tag{6.34}$$

From $\tau-t_k \leq T$ and $\tau-t_{k-1} \leq T+\delta$ over the interval $[t_{k+1}, t_k+T]$, it follows that

$$\frac{\delta T\alpha_i\varepsilon_i^{'}}{(\tau-t_k)(\tau-t_{k-1})} \geq \frac{\delta T\alpha_i\varepsilon_i^{'}}{T(T+\delta)} = \frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta}, \tag{6.35}$$

and that (6.34) holds if

$$\frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta}e^{L_{f_i}(\tau-t_{k+1}+\delta-T)} \leq \frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta}. \tag{6.36}$$

In summary, Equation (6.36) holds because $e^{L_{f_i}(\tau-t_{k+1}+\delta-T)} \leq 1$ for $\forall\tau \in [t_{k+1}, t_k+T]$ and the feasibility of $\hat{u}(t;t_{k+1})$ over the interval $[t_{k+1}, t_k+T]$ is ensured.

Now consider the time interval $[t_k+T, t_{k+1}+T]$. Define

$$T_i(\tau) = \alpha_i e^{-\eta_i(\tau-t_k-T)}, \quad T_i^o(\tau) = \frac{T\alpha_i}{\tau-t_k}.$$

It follows that $T_i^o(\tau) - T_i(\tau) = \frac{T\alpha_i}{\tau-t_k} - \alpha_i e^{-\eta_i(\tau-t_k-T)} = \frac{T\alpha_i e^{\eta_i(\tau-t_k-T)} - \alpha_i(\tau-t_k)}{(\tau-t_k)e^{\eta_i(\tau-t_k-T)}}$, and the substitution of $\tau = t_k+T$ leads to $T_i^o(t_k+T) - T_i(t_k+T) = 0$. After defining $\Delta T(t) = T\alpha_i e^{\eta_i t} - \alpha_i(t+T)$, $t \in [0,\delta]$, it follows that $\frac{d\Delta T(t)}{dt} = T\alpha_i\eta_i e^{\eta_i t} - \alpha_i$. Then, if $\frac{d\Delta T(t)}{dt} = T\alpha_i\eta_i e^{\eta_i t} - \alpha_i \geq 0$ over the interval $t \in [0,\delta]$, that is, if $T\eta_i - 1 \geq 0$, then $\Delta T(t)$ is increasing along $[0,t]$.

From (6.31), it follows that $T_i^o(\tau) - T_i(\tau) \geq 0$, and using (6.30) that $\|\hat{x}_i(\tau;t_{k+1})\|_{P_i} \leq \varepsilon_i^{'}\alpha_i e^{-\eta_i(\tau-t_k-T)} \leq \frac{T\alpha_i\varepsilon_i^{'}}{\tau-t_k}$. The feasibility over the interval $[t_k+T, t_{k+1}+T]$ is also guaranteed. This completes the proof. $\square$

6.4 Stability

This section establishes sufficient conditions for the distributed MPC signal in (6.7) to steer the state of the overall constrained weakly coupled nonlinear system in (6.4) to the origin.

The optimal state cost for subsystem $\mathcal{A}_i$ is

$$J_i^*(t_k, x_i(t_k)) = \int_{t_k}^{t_k+T} (\|\bar{x}_i^*(\tau; t_k)\|_{Q_i} + \|\bar{u}_i^*(\tau; t_k)\|_{R_i})\, d\tau + \|\bar{x}_i^*(t_k + T; t_k)\|_{P_i}$$

at time t_k. Using the assumed control trajectory $\hat{u}_i(t; t_{k+1})$, the optimal cost at time t_{k+1} can be upper bounded by

$$\begin{aligned} & J_i^*(t_{k+1}, x_i(t_{k+1})) \leq J_i(t_{k+1}, x_i(t_{k+1})) \\ & = \int_{t_{k+1}}^{t_{k+1}+T} (\|\hat{x}_i(\tau; t_{k+1})\|_{Q_i} + \|\hat{u}_i(\tau; t_{k+1})\|_{R_i})\, d\tau + \|\hat{x}_i(t_{k+1} + T; t_{k+1})\|_{P_i}, \end{aligned}$$

and the difference between the optimal cost $J_i^*(t_{k+1}, x_i(t_{k+1}))$ and $J_i^*(t_k, x_i(t_k))$ can be upper bounded by

$$\begin{aligned} & J_i^*(t_{k+1}, x_i(t_{k+1})) - J_i^*(t_k, x_i(t_k)) \leq J_i(t_{k+1}, x_i(t_{k+1})) - J_i^*(t_k, x_i(t_k)) \\ & = \int_{t_{k+1}}^{t_k+T} (\|\hat{x}_i(\tau; t_{k+1})\|_{Q_i} - \|\bar{x}_i^*(\tau; t_k)\|_{Q_i} + \|\hat{u}_i(\tau; t_{k+1})\|_{R_i} - \|\bar{u}_i^*(\tau; t_k)\|_{R_i})\, d\tau \qquad (6.37a) \\ & - \int_{t_k}^{t_{k+1}} (\|\bar{x}_i^*(\tau; t_k)\|_{Q_i} + \|\bar{u}_i^*(\tau; t_k)\|_{R_i})\, d\tau \qquad (6.37b) \\ & + \int_{t_k+T}^{t_{k+1}+T} (\|\hat{x}_i(\tau; t_{k+1})\|_{Q_i} + \|\hat{u}_i(\tau; t_{k+1})\|_{R_i})\, d\tau \\ & + \|\hat{x}_i(t_{k+1} + T; t_{k+1})\|_{P_i} - \|\bar{x}_i^*(t_k + T; t_k)\|_{P_i}. \qquad (6.37c) \end{aligned}$$

Upper bounds on (6.37a), (6.37b) and (6.37c) are derived in the following proposition.

Proposition 6.1. *For the subsystem $\mathcal{A}_i$ in (6.1), with Assumptions 6.1 and 6.2, with a sampling interval δ that satisfies the conditions in (6.22a) and (6.22b), and with* *the condition in (6.31), the terms in (6.37a), (6.37b) and (6.37c) satisfy*

$$\int_{t_{k+1}}^{t_k+T} (\|\hat{x}_i(\tau;t_{k+1})\|_{Q_i} - \|\bar{x}_i^*(\tau;t_k)\|_{Q_i} + \|\hat{u}_i(\tau;t_{k+1})\|_{R_i} - \|\bar{u}_i^*(\tau;t_k)\|_{R_i})\, d\tau$$

$$\leq \sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}} \frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta} \frac{1-e^{L_{f_i}(\delta-T)}}{L_{f_i}}; \tag{6.38a}$$

$$-\int_{t_k}^{t_{k+1}} (\|\bar{x}_i^*(\tau;t_k)\|_{Q_i} + \|\bar{u}_i^*(\tau;t_k)\|_{R_i})\, d\tau$$

$$\leq -\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}\varepsilon_i^{'}\delta + \sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}} \frac{\delta^2\alpha_i}{T+\delta}\varepsilon_i^{'} e^{L_{f_i}(\delta-T)}; \tag{6.38b}$$

$$\int_{t_k+T}^{t_{k+1}+T} (\|\hat{x}_i(\tau;t_{k+1})\|_{Q_i} + \|\hat{u}_i(\tau;t_{k+1})\|_{R_i})\, d\tau$$

$$+\|\hat{x}_i(t_{k+1}+T;t_{k+1})\|_{P_i} - \|\bar{x}_i^*(t_k+T;t_k)\|_{P_i} \leq \frac{\delta}{T+\delta}\alpha_i\varepsilon_i^{'}. \tag{6.38c}$$

Proof. The integral term in (6.37a) is upper bounded by

$$\int_{t_{k+1}}^{t_k+T} (\|\hat{x}_i(\tau;t_{k+1})\|_{Q_i} - \|\bar{x}_i^*(\tau;t_k)\|_{Q_i} + \|\hat{u}_i(\tau;t_{k+1})\|_{R_i} - \|\bar{u}_i^*(\tau;t_k)\|_{R_i})\, d\tau$$

$$\leq \int_{t_{k+1}}^{t_k+T} \|\hat{x}_i(\tau;t_{k+1}) - \bar{x}_i^*(\tau;t_k)\|_{Q_i} d\tau$$

$$\leq \sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}} \int_{t_{k+1}}^{t_k+T} \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta+\rho)\delta e^{\mu\delta} e^{L_{f_i}(\tau-t_{k+1})} d\tau. \tag{6.39}$$

In view of (6.27) and (6.28), it follows that

$$\sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon^{'}/\delta+\rho)\delta e^{\mu\delta} \leq \frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta} e^{-L_{f_i}(T-\delta)}. \tag{6.40}$$

Together, Equations (6.39) and (6.40) yield

$$\sqrt{\frac{\underline{\lambda}(P_i)}{\bar{\lambda}(Q_i)}} \int_{t_{k+1}}^{t_k+T} \|\hat{x}_i(\tau;t_{k+1}) - \bar{x}_i^*(\tau;t_k)\|_{Q_i} d\tau$$

$$\leq \frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta} \int_{t_{k+1}}^{t_k+T} e^{L_{f_i}(\tau-t_{k+1}+\delta-T)} d\tau \leq \frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta} \frac{1-e^{L_{f_i}(\delta-T)}}{L_{f_i}}, \tag{6.41}$$

and Equation (6.41) together with (6.39) leads to (6.38a).

The integral term in (6.37b) is upper bounded by

$$-\int_{t_k}^{t_{k+1}} (\|\bar{x}_i^*(\tau;t_k)\|_{Q_i} + \|\bar{u}_i^*(\tau;t_k)\|_{R_i})\, d\tau \leq -\int_{t_k}^{t_{k+1}} \|\bar{x}_i^*(\tau;t_k)\|_{Q_i} d\tau. \tag{6.42}$$

From (6.20), it follows that

$$\|x_i(\tau;t_k) - \bar{x}_i^*(\tau;t_k)\|_{P_i} \leq \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon'/\delta + \rho)(\tau - t_k)e^{\mu(\tau - t_k)}. \tag{6.43}$$

If $x_i(\delta, t_k)$ does not enter the positively invariant set $\Omega_{\varepsilon_i'}$ during the sampling interval $[t_k, t_{k+1}]$, then (6.43) guarantees that

$$\|\bar{x}_i^*(\tau;t_k)\|_{P_i} \geq \varepsilon_i' - \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon'/\delta + \rho)\delta e^{\mu\delta}. \tag{6.44}$$

After substitution from (6.44), Equation (6.42) leads to

$$\begin{aligned} &-\int_{t_k}^{t_{k+1}} \|\bar{x}_i^*(\tau;t_k)\|_{Q_i} d\tau \\ &\leq -\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}\varepsilon_i'\delta + \sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}} \int_{t_k}^{t_{k+1}} \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon'/\delta + \rho)\delta e^{\mu\delta} d\tau. \end{aligned} \tag{6.45}$$

Together, Equations (6.27) and (6.28) imply that

$$\sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon'/\delta + \rho)\delta e^{\mu\delta} \leq \frac{\delta\alpha_i}{T+\delta}\varepsilon_i' e^{L_{f_i}(\delta - T)}, \tag{6.46}$$

and, after integration over $[t_k, t_{k+1}]$, that

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \sqrt{\bar{\lambda}(P_i)}(\beta\bar{\gamma}\bar{N}T\alpha\varepsilon'/\delta + \rho)\delta e^{\mu\delta} d\tau \\ &\leq \int_{t_k}^{t_{k+1}} \frac{\delta\alpha_i}{T+\delta}\varepsilon_i' e^{L_{f_i}(\delta - T)} d\tau = \frac{\delta^2\alpha_i}{T+\delta}\varepsilon_i' e^{L_{f_i}(\delta - T)}. \end{aligned} \tag{6.47}$$

Together, Equations (6.42), (6.45), (6.46) and (6.47) imply (6.38b).

Equation (6.38c) can be shown to hold starting from Lemma 6.1, which guarantees that

$$\int_{t_k+T}^{t_{k+1}+T} \left(\|\hat{x}_i(\tau;t_{k+1})\|_{Q_i} + \|\hat{u}_i(\tau;t_{k+1})\|_{R_i}\right) d\tau$$
$$+ \|\hat{x}_i(t_{k+1}+T;t_{k+1})\|_{P_i} - \|\bar{x}_i^*(t_k+T;t_k)\|_{P_i}$$
$$\leq \|\hat{x}_i(t_k+T;t_{k+1}) - \bar{x}_i^*(t_k+T;t_k)\|_{P_i}. \tag{6.48}$$

Together, Equations (6.26), (6.27) and (6.28) yield

$$\|\hat{x}_i(t_k+T;t_{k+1}) - \bar{x}_i^*(t_k+T;t_k)\|_{P_i} \leq \frac{\delta}{T+\delta}\alpha_i\varepsilon_i^{'},$$

which implies (6.38c) when combined with (6.48). □

Theorem 6.3. *Let the overall constrained weakly coupled nonlinear system in (6.4) with the distributed MPC controller in (6.7), with a sampling interval δ that satisfies (6.22a) and (6.22b), and with the condition in (6.31) start from a feasible initial state. If there exist $\vartheta > 0$, $\xi > 0$ satisfying $\vartheta + \xi \leq 1$ and if*

$$\frac{1}{1+p}\alpha_i\varepsilon_i^{'}\left(1 + \sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}pTe^{-L_{f_i}(1-p)T} + \sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}\frac{1-e^{-L_{f_i}(1-p)T}}{L_{f_i}}\right)$$
$$\leq \vartheta\varepsilon_i^{'}T\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}, \quad \delta = pT \;\; \text{with} \;\; p \in (0,1) \tag{6.49}$$

holds, then the state of subsystem $\mathcal{A}_i$ converges to the positively invariant set $\Omega_{\varepsilon_i^{'}}$.

Proof. Together, Equations (6.37) and (6.38) lead to

$$J_i^*(t_{k+1}, x_i(t_{k+1})) - J_i^*(t_k, x_i(t_k))$$
$$\leq \sqrt{\frac{\bar{\lambda}(Q_i)}{\underline{\lambda}(P_i)}}\frac{\delta\alpha_i\varepsilon_i^{'}}{T+\delta}\frac{1-e^{L_{f_i}(\delta-T)}}{L_{f_i}} - \sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}\varepsilon_i^{'}\delta$$
$$+ \sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}\frac{\delta^2\alpha_i}{T+\delta}\varepsilon_i^{'}e^{L_{f_i}(\delta-T)} + \frac{\delta}{T+\delta}\alpha_i\varepsilon_i^{'}. \tag{6.50}$$

After substitution from (6.49), Equation (6.50) yields

$$J_i^*(t_{k+1}, x_i(t_{k+1})) - J_i^*(t_k, x_i(t_k)) \leq -(1-\vartheta)\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}\varepsilon_i^{'}\delta \leq -\xi\sqrt{\frac{\underline{\lambda}(Q_i)}{\bar{\lambda}(P_i)}}\varepsilon_i^{'}\delta.$$

Hence, the system state $x_i(t)$ will enter the positively invariant set $\Omega_{\varepsilon_i'}$ in finite time. This completes the proof. □

Within the positively invariant set $\Omega_{\varepsilon_i'}$, the state feedback controller K_i is implemented. The following theorem establishes sufficient conditions for the state of the constrained weakly coupled nonlinear system in (6.4) to enter a robust positively invariant set.

Theorem 6.4. *The state of the weakly coupled nonlinear system in (6.4) with Assumptions 6.1 and 6.3, with the control (6.7) and with the disturbance*

$$\|\omega(t)\|_2 \leq \frac{\underline{\lambda}(Q^*)\varrho\varepsilon}{4\bar{\lambda}(P^*)^{1/2}\bar{\lambda}(P^*)}, \quad \varrho \in (0,1), \tag{6.51}$$

converges to the robust positively invariant set $\Omega_{\sqrt{\varrho}\varepsilon}$.

Proof. From Lemma 6.2, it follows that

$$\begin{aligned}\dot{V}(x(t)) &= x(t)^{\mathrm{T}}(A_c^{\mathrm{T}}P^* + P^*A_c)x(t) + 2x(t)^{\mathrm{T}}P^*\phi(x(t)) + 2x(t)^{\mathrm{T}}P^*\omega(t)\\ &\leq -\frac{1}{2}x(t)^{\mathrm{T}}Q^*x(t) + 2\|(P^*)^{1/2}x(t)\|_2 \cdot \|(P^*)^{1/2}\|_2 \cdot \|\omega(t)\|_2.\end{aligned} \tag{6.52}$$

After substitution from (6.51), Equation (6.52) leads to $\dot{V}(x(t)) \leq 0$. The remaining of the proof is similar to the proof in [50]. □

Remark 6.2. *A distributed MPC strategy for a fully coupled nonlinear system has already been proposed in [55]. If a Lyapunov-based controller exists, the strategy in [55] has been designed by incorporating a contractive constraint in the optimization. In [53], the results in [55] have been extended to sequential and iterative architectures for distributed MPC for nonlinear systems in which several distinct sets of inputs are used to regulate the process. The strategy proposed in this paper does not directly lend itself to sequential or to iterative implementations like those introduced in [53]. The investigation would require the appropriate reconfiguration of the communication topology, and the redesign of the optimization associated with each subsystem.*

Remark 6.3. *For general coupled nonlinear systems, particularly in the presence of strong couplings, the distributed MPC strategy proposed in this chapter would be conservative because the recursive feasibility condition depends on how close the state trajectory of the coupled nonlinear system stays to the trajectories of the linearized* *and of the decoupled systems. Nonetheless, the proposed approach is useful for weakly coupled nonlinear systems, for which it distributes the centralized optimization to several smaller optimizations with reduced computational burden.*

6.5 Simulation Example

This section verifies the feasibility and the stability of the proposed distributed MPC strategy through applying it to steer to the origin a simulated system that consists of three similar cart-(nonlinear) spring-damper subsystems connected to each other with similar linear springs [50, 64]. The schematic diagram of the system is shown in Figure 6.1. The dynamics of the three carts are

$$\begin{aligned}
\dot{x}_{11}(t) &= x_{12}(t), \;\; \dot{x}_{21}(t) = x_{22}(t), \;\; \dot{x}_{31}(t) = x_{32}(t), \\
m\dot{x}_{12}(t) &= u_1 - k_0 e^{-x_{11}(t)} x_{11}(t) - h_d x_{12}(t) - k_c(x_{11}(t) - x_{21}(t)) + \omega_1(t), \\
m\dot{x}_{22}(t) &= u_2 - k_0 e^{-x_{21}(t)} x_{21}(t) - h_d x_{22}(t) - k_c(x_{21}(t) - x_{11}(t)) \\
&\qquad - k_c(x_{21}(t) - x_{31}(t)) + \omega_2(t), \\
m\dot{x}_{32}(t) &= u_3 - k_0 e^{-x_{31}(t)} x_{31}(t) - h_d x_{32}(t) - k_c(x_{31}(t) - x_{21}(t)) + \omega_3(t).
\end{aligned}$$

Here, x_{i1} and x_{i2} are the displacement and the velocity of cart i, $i = 1, 2, 3$, respectively; k_0 is the steady-state stiffness of the local nonlinear spring of each cart; k_c is the stiffness of the interconnecting linear springs; h_d is the local damping of each cart; m is the mass of each cart; $u_i(t)$ and $\omega_i(t)$ are the control signal and the disturbance for cart i, respectively. The numerical values selected for these system parameters are: $k_0 = 1.05$ N/m, $k_c = 0.01$ N/m, $h_d = 0.3$ Ns/m, $m = 1.5$ kg, $u_i(t) \in [-2\ 2]$ N. The initial states of the three carts are $x_1(0) = (0.5 \text{ m } 0 \text{ m/s})$, $x_2(0) = (-0.6 \text{ m } 0 \text{ m/s})$ and $x_3(0) = (0.65 \text{ m } 0 \text{ m/s})$.

The numerical values selected for the control parameters are: $Q_i = \text{Diag}([1.5\ 1.5])$, $R_i = 0.1$; the eigenvalues of the matrix $A_i + B_iK_i$ are $\lambda_1(A_i + B_iK_i) = -1$ and $\lambda_2(A_i + B_iK_i) = -0.95$, respectively; $\kappa_i = 0.90$ by Lemma 6.1; $T = 3$ s and $\delta = 0.6$ s by Theorem 6.3; $\alpha_i = 0.90$ in (6.7b). Using these selections, the remaining control parameters become: $K_i = [-0.3750\ \ -2.6250]$, $P_i = \begin{bmatrix} 3.0321 & -0.8706 \\ -0.8706 & 1.0696 \end{bmatrix}$, $L_{f_i} = 2.5329$, for $i = 1, 2, 3$. $\beta = 0.01$, $\|\omega_{\max}\|_2 = 0.005$, $\vartheta = 0.2$, $\xi = 0.75$.

The displacements, velocities and control signals of the three simulated carts are depicted in Figures 6.2, 6.3 and 6.4, respectively. Figures 6.2 and 6.3 illustrate that

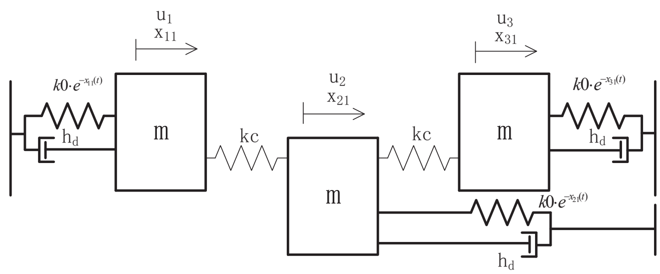

Figure 6.1: The schematic diagram of the simulated system.

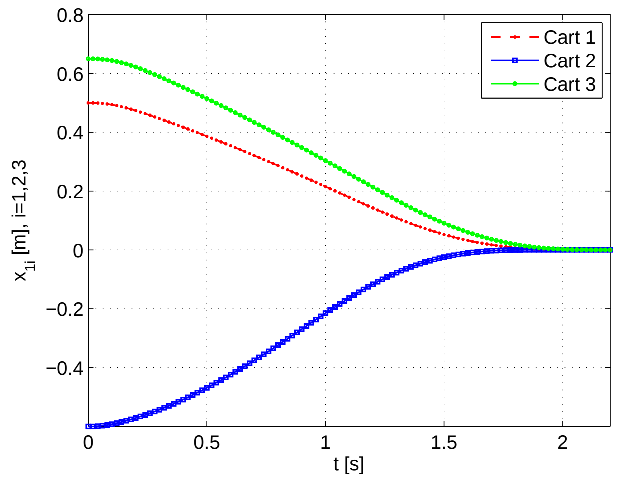

Figure 6.2: The displacements of the three carts.

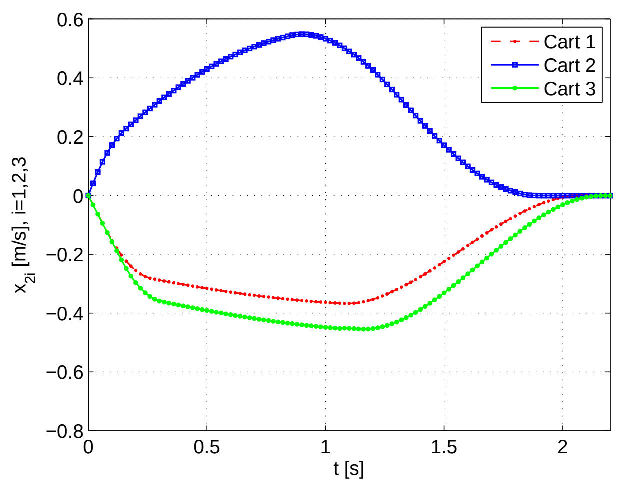

Figure 6.3: The velocities of the three carts.

the proposed distributed MPC drives the states of all carts to the origin despite the dynamic couplings among them. Figure 6.4 confirms the feasibility of the proposed distributed MPC controller. It also shows that the control effort for each cart decreases as the cart state approaches the neighborhood of the origin.

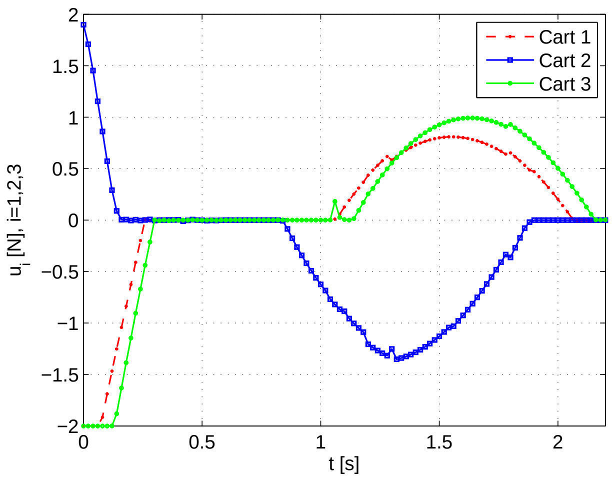

Figure 6.4: The control signals of the three carts.

6.6 Conclusion

This chapter has proposed a distributed MPC strategy for a group of dynamically weakly coupled subsystems with decoupled control constraints and external disturbances. In the proposed strategy, all subsystems need to compute their control signals by solving local optimizations constrained by their nominal decoupled dynamics. The dynamic couplings and the disturbances are accommodated through new robustness constraints in the local optimizations. The bounds on the deviation between the actual and predicted system state trajectories have been computed and analyzed. It has shown that, for a given bound on the disturbances, the recursive feasibility of the proposed distributed MPC strategy depends on the appropriate choice of the sampling interval. Sufficient conditions were established for the robust stability of the overall system. The effectiveness of the proposed control strategy has been verified by applying it to three weakly coupled cart-(nonlinear) spring-damper systems.

Chapter 7

Conclusions and Future Work

The work concerned in this thesis can be summarized into two parts. The first part of this thesis has investigated the MPC theory. Specifically, a few strategies aiming to reduce the conservativeness of MPC is proposed. After the theoretical study of MPC in the first part, the second part of the thesis has investigated distributed MPC as a means to address the heavy computational and/or communication loads of centralized MPC for large-scale systems.

7.1 Conclusions

The dissertation has proposed a computationally efficient multi-stage MPC strategy in Chapter 2. Conventionally, a larger feasible region of an MPC strategy is enlarged by lengthening the prediction horizon. However, a long prediction horizon indicates a heavy computational load. The proposed multi-stage MPC enlarges a larger feasible region without increasing the computational complexity for a given prediction horizon. The recursive feasibility of the optimization and the stability of the closed-loop system have also been analyzed in Chapter 2.

To guarantee the recursive feasibility and stability, robust MPC strategies for nonlinear systems typically require additional constraints to be added to the optimization. In turn, the added constraints increase the computational complexity of the controller and lead to conservative feasibility and stability results. Chapter 3 has sidestepped the need for additional constraints in the optimization by proposing a novel robust MPC for constrained nonlinear systems whose cost function consists of an integral non-squared stage cost function and a non-squared terminal cost. The proposed strategy has the advantage that its cost function serves as a quasi-infinite horizon cost.

Most existing research results on nonlinear MPC assume only that the nonlinear system to be controlled is Lipschitz continuous and ignores any additional properties which the nonlinear dynamics might have. This approach leads to general but conservative results. Taking advantage of specific system properties can yield more practical MPC strategies for certain classes of nonlinear systems. Following this reasoning, Chapter 4 has proposed a robust MPC strategy for contracting nonlinear systems. Compared to a Lipschitz continuity-based MPC, the proposed strategy maintains closed-loop stability for larger levels of disturbance and has larger feasible region.

For cooperating systems, distributed MPC strategies are preferred to centralized strategies because the computational and communication demands of centralized MPC schemes make them impractical for implementation. Therefore, for nonlinear systems with decoupled dynamics, Chapter 5 has developed a robust distributed MPC which ensures their cooperation through coupling terms in the optimizations that yield their control signals. The new technique exploits the two-layer invariant set theory to handle the disturbances. Compared to existing results, the proposed strategy imposes stronger cooperation among subsystems.

For large-scale systems with coupled dynamics, most prior work treats the dynamical interactions as disturbances and therefore, presents results which are conservative. Chapter 6 has proposed a novel robust distributed MPC strategy by imposing a robustness constraint in the local optimizations. Suitable for weakly coupled nonlinear systems, the new strategy uses the robustness constraint to explicitly evaluate the effect of the dynamical couplings. Its recursive feasibility of the optimization and the closed-loop stability have also been investigated.

7.2 Future Work

The work presented in this dissertation has addressed some theoretical problems facing distributed MPC strategies and has developed algorithms for implementing such algorithms. However, the results have been illustrated only through simulation examples and their experimental validation needs to be carried out in future work. Furthermore, there are many open problems ahead in the implementation of distributed MPC strategies. Specifically, the following topics are possible directions for future work.

1. As pointed out in [47], theoretically developed MPC strategies have seldom been implemented in practice. This is mainly because the theoretical algorithms yield results which are often conservative and impractical. For example, the computational requirement or the ability to tolerate only very limited disturbances are two key factors which typically hinder the use of theoretically developed MPC algorithms in applications. Future research effort will be dedicated to understanding the sources of conservativeness and to designing less conservative MPC algorithms.

2. Another research direction is the decomposition of a centralized optimization into several small-sized optimizations. Centralized MPC for large-scale systems is not proper to implement either because of its the computational load or because of its communication burden. Existing work typically proposes distributed MPC as a remedy for centralized MPC. However, a distributed MPC strategy steers the overall system to a Nash Equilibrium [93] rather than to the equilibrium of the centralized MPC strategy. It would be interesting to consider how a centralized optimization, formulated for the large-scale system according to the performance requirement, could be decomposed into several small-sized optimizations whose solutions collectively yield a good enough approximation to the solution of the centralized optimization. Such an approach: 1) would have reduced the computational time since all subsystems would solve small-sized optimizations in parallel; 2) would steer the large-scale system to the same equilibrium as the centralized controller.

Appendix A

Publications

- **Refereed journal papers that have been accepted**

J1. **X. Liu**, D. Constantinescu, and Y. Shi, "Multi-stage suboptimal model predictive control with improved computational efficiency," *ASME Journal of Dynamic Systems, Measurement, and Control,* vol. 136, no. 3, Article Number: 031026, May 2014.

J2. **X. Liu**, D. Constantinescu, and Y. Shi, "Robust model predictive control of constrained nonlinear systems adopting the non-squared integrand objective function," *IET Control Theory* & *Applications,* DOI: 10.1049/iet-cta.2013.1078, 2014. (in press)

J3. **X. Liu**, Y. Shi, and D. Constantinescu, "Distributed model predictive control of constrained coupled nonlinear systems," *Systems* & *Control Letters,* vol. 74, no. 12, pp. 41-49, December 2014.

J4. M. Liu, **X. Liu**, Y. Shi, and S. Wang, "T-S fuzzy-model-based H_2 and H_∞ filtering for networked control systems with two-channel Markovian random delays," *Digital Signal Processing,* vol. 27, no. 1, pp. 167-174, April 2014.

J5. H. Li, **X. Liu**, and Y. Shi, "Output feedback $\mathcal{H}_\infty$ control of stochastic nonlinear time-delay systems with state and disturbance-dependent noises," *Nonlinear Dynamics,* vol. 77, no. 3, pp. 529-544, August 2014.

J6. H. Zhang, **X. Liu**, J. Wang, and H. Karimi, "Robust $\mathcal{H}_\infty$ sliding-mode control with pole placement for a fluid power electrohydraulic actuator (EHA) system," *International Journal of Advanced Manufacturing Technology,* vol. 73, no. 5-8, pp. 1095-1104, July 2014.

- **Refereed conference papers that have appeared or been accepted**

C1. **X. Liu**, D. Constantinescu, and Y. Shi, "H_2 controller design of networked bilateral teleoperation system with Markovian time delays," *The 2012 Haptics Symposium,* Vancouver, BC, Canada, March 4-7, 2012.

C2. **X. Liu**, Y. Shi, and D. Constantinescu, "Robust distributed model predictive control of constrained continuous-time nonlinear systems using two-layer invariant set," *The 2014 American Control Conference,* Portland, Oregon, USA, June 4-6, 2014.

C3. **X. Liu**, Y. Shi, and D. Constantinescu, "Robust constrained model predictive control using contraction theory," *The 53rd IEEE Conference on Decision and Control,* accepted, Los Angeles, California, USA, December 15-17, 2014.

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