Force-free magnetospheres, Kerr-AdS black holes and holography

by

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B.Sc., Nankai University, 2003
M.Sc., University of Victoria, 2008

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ABSTRACT

In this thesis, we study the energy extraction from rotating black holes in anti-de Sitter (AdS) spacetime (Kerr-AdS black holes), via the Blandford-Znajek (BZ) process. The motivation is the anti-de Sitter/conformal field theory (AdS/CFT) correspondence which provides a duality between gravitational physics in asymptotically AdS spacetimes and lower dimensional boundary field theories. The BZ process operates via a force-free magnetosphere around black holes and the rotational energy of the black hole is extracted electromagnetically in the form of Poynting flux. The major part of the thesis is devoted to obtaining force-free solutions in the Kerr-AdS background, which generalize traditional BZ solutions in the asymptotically flat Kerr background. Given the solutions, we use the AdS/CFT to infer dual descriptions in terms of the boundary field theory, which hopefully will lead to a better understanding of the energy extraction for rotating black holes.

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Notations and conventions

AdS	anti-de Sitter
CFT	conformal field theory
BZ	Blandford-Znajek
BL	Boyer-Lindquist
KS	Kerr-Schild
ZAM(O)	zero angular momentum (observer)
DEC	dominant energy condition
BF	Breitenlohner-Freedman
NP	Newman-Penrose
KNAdS	Kerr-Newman-AdS
m, a, r_H, l	mass, rotation parameter, horizon radius, and AdS radius for the Kerr-AdS spacetime; m also the mass of scalar fields
Δ	factor in the Kerr metric; dimension of field theory operator
E, T, S, Ω, L	energy, temperature, entropy, angular velocity, and angular momentum in thermodynamics relations
ϕ	scalar field
φ	azimuthal angular coordinate
$\xi^\mu_{(t)}$, $\xi^\mu_{(\varphi)}$	temporal and azimuthal Killing vectors
$K^\mu_{\Omega'}$	linear combination of two Killing vectors: $K^\mu_{\Omega'} \equiv \xi^\mu_{(t)} + \Omega' \xi^\mu_{(\varphi)}$
Ω_H	angular velocity of the black hole
ω_B	Bardeen angular velocity, angular velocity of ZAMOs
Ω_∞	asymptotic value of Ω_B, angular velocity of the non-rotating frame at infinity

ω	mainly the angular velocity of the magnetic field lines in the BZ process; also the similar quantities in Penrose process and superradiance
$F_{\mu\nu}$, J^{μ}, $T^{\mu\nu}$	electromagnetic field, current, and energy-momentum tensor
A^*, $\bar{A}$	both the complex conjugate of a quantity A
${}^{\star}A_{\mu\nu}$	Hodge dual of a bivector $A_{\mu\nu}$
$\tilde{}$	duality rotation operation; Kerr-Schild coordinates

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I would like to thank:

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committee and departmental members, for inspiring communications and discussions.

DEDICATION

To my parents.

Chapter 1

Introduction

This thesis studies the energy extraction from rotating anti-de Sitter (AdS) black holes by means of a force-free magnetosphere, generalizing the same mechanism, known as the Blandford-Znajek (BZ) process, for black holes in flat spacetime. The results are then interpreted in terms of the conformal field theory (CFT) on the boundary, using the AdS/CFT correspondence.

In this chapter, we introduce the physics of energy extraction from rotating black holes and discuss the implications of putting the black hole in the AdS background.

1.1 Rotating black holes and energy extractions

Rotating black holes are important areas of focus in theoretical research. They possess an *ergosphere*, a region enclosing the event horizon and characterized by the non-existence of (timelike) static observers: all observers are forced to rotate in the same direction as the black hole (though not necessarily infalling). Through the ergosphere, rotating black holes exhibit the remarkable property that rotational energy can be extracted through purely classical means. The Penrose process, and super-radiance, represent the primary examples. The energy extraction relies on the fact that matter inside the ergosphere can have negative 'energy-at-infinity' (energy as seen by asymptotic observers). While local observers still see positive energies (for matter obeying appropriate energy conditions), static world lines inside the ergosphere lie outside light cones emanating from them. In other words, asymptotic observers typically represented by static world lines move faster than light relative to local matter flows and could see the flows as carrying negative energies. Then infall

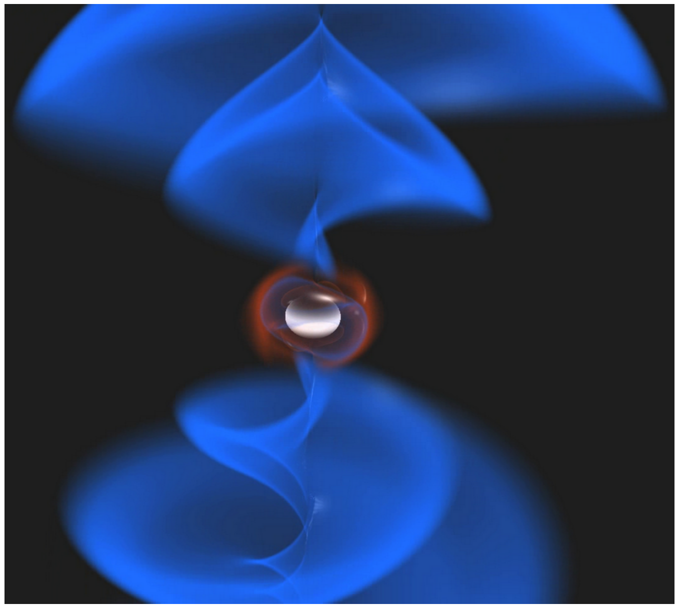


Figure 1.1: Snapshot of animation of superradiance. Red color represents the incident wave and blue color represents the scattered outgoing waves. Image from Frans Pretorius' and by Ralf Kahler, KIPAC [2].

of the matter into the horizon will reduce the total mass/energy of the black hole as seen at infinity and is reflected as a positive energy outflux of some sort. On the other hand, observers at the horizon necessarily see a positive local energy influx. This does not contradict the decrease in the black hole mass since the locally defined energy has incorporated the angular momentum contribution and in fact represents the *entropy* term as in the first law of black hole thermodynamics (to be discussed shortly).

As an example, in the Penrose process, a positive energy particle entering the ergosphere can split into a negative energy part which falls into the black hole and a part which carries more positive energy than the original particle and escapes to infinity. A similar idea underlies superradiance, where an incident wave scatters off the black hole producing an amplified outgoing wave and an ingoing wave carrying negative energy into the horizon (as can be computed e.g. for a scalar wave, as in [1]). A snapshot of an animation of superradiance is shown in fig. 1.1. In both processes, the dynamics (splitting or scattering of the particle or the wave, respectively) happens in the ergosphere, which is crucial for producing negative energy flux. The horizon provides the ingoing boundary condition to digest the negative energy matter. (In the presence of only an ergosphere but not the horizon, as outside a compact rotating star, the negative energy accumulated inside the ergosphere would lead to the so called "ergosphere instability" [3].) In addition, for superradiance one can impose reflective boundary conditions at some finite radius, e.g., by putting the black hole in a box or anti-de Sitter space, so that the wave travels back and forth between the horizon and the boundary, leading to an instability known as the "black hole bomb" [4].

To be more specific, the ergosphere is where the energy-defining timelike Killing vector becomes spacelike. This is a necessary condition for the energy flux vector of generic matter, defined from the contraction of the energy-momentum tensor and the energy-defining Killing vector, to be spacelike, provided that the energy-momentum tensor respects the dominant energy condition. This is in turn necessary for the energy flux across the horizon, defined from the contraction of the energy flux vector with the horizon generator, to be outgoing (as is true for the energy flux across any constant-radius surface outside the horizon). See Chapter 2 for a more rigorous derivation.

In terms of the energetics, we have gained a global picture of the continuous energy flow (the outcome of energy extraction) with consistent descriptions at the horizon and at infinity, though the dynamical details vary for different processes and are less understood. More insight can be gained from the fact that there is always an angular momentum extraction along with any energy extraction. In other words, the matter interacts with the black hole to "brake" its rotation. This is clear from the first law of black hole thermodynamics $\mathrm{d}E = T\,\mathrm{d}S + \Omega_H\,\mathrm{d}L$, where $\{E, T, S, L, \Omega_H\}$ are respectively the energy, temperature, entropy, angular momentum and angular velocity of the black hole. $\mathrm{d}L < 0$ then follows from $\mathrm{d}E < 0$ and $\mathrm{d}S \geq 0$. The non-decreasing of entropy $\mathrm{d}S \geq 0$ is associated with the fact that a local observer at the horizon (who necessarily rotates with angular velocity Ω_H) always sees ingoing positive energy. Defining the energy with his or her own 4-velocity, this observer finds a local first law $\delta E = T\delta S + 0 \cdot \delta L$ ("δ" indicating that the variations are not exact differentials) and would conclude that energy is not extracted but the angular momentum, defined the same way as for a static observer, keeps decreasing. Thus it seems that angular momentum extraction is a more robust feature. This point will be made more explicit once we have the solutions for the BZ process in chapter 5.

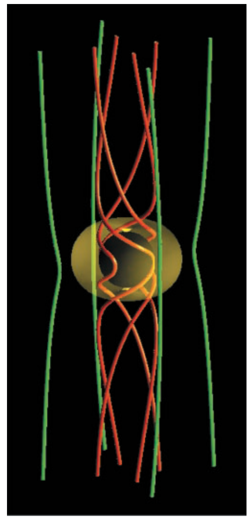


Figure 1.2: Twists of the magnetic field lines by the rotation of the black hole. Yellow area represents the ergosphere. From [11]. Reprinted with permission from AAAS.

1.2 BZ process and Kerr-AdS black holes

We now turn to the Blandford-Znajek (BZ) process [5] (see also [6, 7, 8, 9, 10]), which realizes energy extraction through an electromagnetic Poynting flux and is thought most likely to be the mechanism for power sources in astrophysics, e.g., in active galactic nuclei and quasars. Astrophysical black holes are usually immersed in external magnetic fields (e.g., supported by accretion discs) and surrounded by plasma and radiation. For strong magnetic fields, we can neglect matter contributions to the energy-momentum tensor and work in the force-free limit, i.e., vanishing Lorentz force by conservation of the electromagnetic energy-momentum tensor alone. Though the black hole does not source any electromagnetic field to interact with the magnetosphere, the spacetime vacuum acts like an electromagnetically active medium so that the BZ process operates in analogy to a unipolar inductor. In the magnetosphere, the rotation of the spacetime induces an electric field which, like an electromotive force, drives poloidal currents. The poloidal currents then produce toroidal magnetic fields which are needed to slow down the rotation and generate radial Poynting flux. A more intuitive picture, as shown in fig. 1.2, would be imagining the rotation twists the magnetic field lines and causes riddles propagating away along the field lines. BZ process offers a stationary and steady state scenario where energy is continuously extracted.

The BZ process was one of the triggers of the so called black hole "membrane paradigm" [12, 13], where the horizon is replaced by a fictitious membrane (the stretched horizon) endowed with transport properties such as conductivity. The electromotive force in the BZ process can be computed by treating the membrane as the unipolar conductor. The membrane paradigm is a realization of the holographic principle proposed during the studies of black hole thermodynamics (entropy of the black hole residing on the 2-D membrane). It has a modern and stricter version, the AdS/CFT (anti-de Sitter/conformal field theory) correspondence, where the physics of a black hole in the AdS space (the "bulk") is captured by that of a CFT on the boundary (a lower dimensional brane). The present work is motivated by possible applications of the AdS/CFT correspondence to the BZ process. As a powerful tool, AdS/CFT guarantees a mapping of bulk field states to observables on the CFT side. Our first step would be to translate BZ process to asymptotically AdS rotating black hole backgrounds, given by the Kerr-AdS family of solutions to Einstein's field equations, while the original BZ process is formulated in the asymptotically flat case (i.e Kerr black holes).

The implications of embedding the Kerr black hole in AdS depend on the relative size r_H/l of the black hole, with horizon radius r_H, and the AdS curvature scale l. For 'small' black holes with $r_H \ll l$, the near-horizon geometry is very similar to Kerr, and thus we expect the appearance of an ergosphere and a direct translation of the BZ process as observed in asymptotically flat space. In contrast, for large black holes with $r_H \geq l$, the AdS boundary conditions become important and modify the response to the force-free magnetosphere.

The Kerr geometry possesses a unique timelike Killing vector as $r \to \infty$, namely $\xi^\mu_{(t)}$ representing a static observer at infinity. As noted earlier, using this Killing vector to define energy, one finds an ergosphere outside the horizon, which allows for energy extraction. In contrast, 'large' Kerr-AdS black holes possess a family of asymptotically timelike Killing vectors, and thus there is no unique definition of energy for an asymptotic observer at $r \to \infty$. Amongst the family of asymptotically timelike Killing vectors for large black holes, the horizon generator $K^\mu_{\Omega_H} = \xi^\mu_{(t)} + \Omega_H \xi^\mu_{(\varphi)}$ is in fact globally timelike outside the horizon, where $\xi^\mu_{(\varphi)}$ is the axial Killing vector. In the conventional Boyer-Lindquist (BL) coordinate system for Kerr-AdS geometries (with rotation parameter a), the angular velocity of zero angular momentum observers (ZAMOs) Ω_B, which determines the horizon angular velocity Ω_H, is non-vanishing asymptotically where it takes the value $\Omega_\infty = -a/l^2$. Thus, the conformal boundary of Kerr-AdS spacetime is an Einstein universe rotating with angular velocity $\Omega_H - \Omega_\infty$

[14]. Moreover, one finds that the boundary Einstein universe rotates slower than the speed of light, provided that $\Omega_H - \Omega_\infty < 1/l \Leftrightarrow r_H^2 > al$, i.e., for sufficiently large black holes. As argued by Hawking and Reall [14], and discussed below in Section 5.3.4, this along with the dominant energy condition (DEC) implies stability of large Kerr-AdS geometries and ensures that energy cannot be extracted. On the contrary, small Kerr-AdS black holes could exhibit a genuine instability, e.g., via superradiance. The onset of the instability in AdS was identified, via the holographic AdS/CFT correspondence [15, 16, 17], with the limit in which the dual field theory is rotating at the speed of light [18]. In this thesis, we consider related questions about the BZ process for force-free magnetospheres around Kerr-AdS black holes

Recalling that only large black holes, with $r_H > l$, provide saddle points describing the thermodynamics of the holographic dual theory [18], it follows that the stability of the dual thermal state is a direct consequence of the existence of the globally defined timelike Killing vector in the bulk. Although this conclusion suggests the absence of a direct AdS dual of the BZ process, there are at least two interesting subtleties. The first is that stability actually relies on the DEC, which is known to be relatively easy to violate in AdS space, where the Breitenlohner-Freedman (BF) bound allows small negative masses for perturbing fields. Although there is no apparent need for the currents which source the BZ force-free magnetosphere to violate the DEC, this suggests a possible route around the above conclusion that energy extraction is not possible for large AdS black holes. The second subtlety is that the energy defined by the globally timelike Killing vector $K^\mu_{\Omega_H}$ is apparently not the one that naturally enters the thermodynamics of the dual field theory. It has been argued [19] that it is instead the Killing vector $K^\mu_{\Omega_\infty} = \xi^\mu_{(t)} + \Omega_\infty \xi^\mu_{(\varphi)}$ which should be used to define the energy E as use of the conserved charge $E = Q[K_\Omega]$ in the first law $\mathrm{d}E = T\,\mathrm{d}S + (\Omega_H - \Omega_\infty)\,\mathrm{d}L$ with $L = -Q[\xi_{(\varphi)}]$ ensures that the r.h.s. is an exact differential. The energy defined in this way does exhibit an ergosphere beyond the horizon even for large Kerr-AdS black holes. On the other hand, use of the globally timelike $K^\mu_{\Omega_H}$ may be just reflecting the fact that what an observer at the horizon sees (purely ingoing positive energy satisfying a local version of the first law) can be shared by a series of co-rotating observers all the way to infinity. This ambiguity again raises the question of what properties force-free magnetospheres may have for large black holes, given that they should be described in the dual field theory, and motivates finding an explicit bulk solution of this type.

Motivated by these arguments, in this thesis we obtain an analogue of the BZ (split) monopole force-free magnetosphere [5], with the goal of understanding how it evolves from small to large Kerr-AdS black holes. While this model is an abstraction compared to the physical case where the magnetosphere is induced by an accretion disc, it provides a concrete example in which the radial Poynting flux can be explicitly computed (see also [20, 21]). The solution to leading order in the Kerr rotation parameter turns out to be unique, with the axisymmetric magnetosphere co-rotating with a specific angular velocity, equal to half the angular velocity of the horizon. More recent numerical work has confirmed this basic picture (see, e.g., [6]). The primary goal of this thesis is to determine the corresponding solution with global AdS boundary conditions. We again work in the slow rotation limit, treating both $a \ll m$ & $a \ll l$, with m the black hole mass parameter. Away from the small black hole limit, which asymptotically approaches the Kerr case, we find that the field line angular velocity ω is not uniquely determined. For large black holes, we interpret these results within the holographic dual in terms of the properties of a fluid in a rotating magnetic field. We find a consistent picture of stable rotation, as the dual fluid is neutral at the corresponding order in the rotation parameter.

In the Kerr case, ω is fixed by considering the asymptotic behavior of the magnetic field and the force-free equation. Requiring the magnetic field fall off fast enough ($\sim 1/r$) leaves only one possible value for ω so as to make the force-free equation non-diverging at large r. In the Kerr-AdS case, the divergence of the equation is less severe and the way the magnetic field deforms affects ω, which is found to be related to the $\mathcal{O}(a^2/r^2)$ correction of the magnetic field. Put another way, the Kerr magnetosphere can be matched at infinity onto a unique configuration which is the rotating monopole field in flat spacetime found by Michel [22]. ω, of both the black hole magnetosphere and the monopole field, is then fixed from the matching. Analogous monopole field in AdS spacetime however can accommodate an additional arbitrary $\mathcal{O}(a^2/r^2)$ perturbation which affects ω as with the black hole magnetosphere.

There is even an ambiguity in defining the unperturbed rotating monopole. In the asymptotic region of the Kerr-AdS black hole one can choose two coordinate systems whose constant-radius surfaces deform from each other at $\mathcal{O}(a^2)$: one is the coordinate system for the standard global AdS metric, and the other is the zero-mass or large-radius limit of the BL coordinates for the Kerr-AdS metric. We can thus have two different monopole fields with radial field lines evenly distributed over the constant-radius surface in each coordinate system (at large radius). While the first system is natural in pure AdS space, the second (as used in BZ's original treatment) seems reasonable in the black hole case in view of the fact that the horizon is one of the constant-radius surfaces. However, it is also true that the horizon is a ZAM-equipotential surface which coincides at infinity with the constant-radius surfaces in the first (rather than the second) system. (The ZAM potential is defined as the normalization factor of ZAMO 4-velocity.) It would be interesting to find a monopole-like field adapted to the ZAM-equipotential surfaces.

The rest of this thesis is organized as follows. In Chapter 2, we give a more complete discussion of the energy extraction from rotating black holes, including brief derivations of the Penrose process and superradiance, demonstration of energy extraction in terms of the first law, and a general analysis of energy extraction in terms of the 3+1 formalism. We then review the formulation of the force-free magnetosphere dynamics in the BZ process as well as BZ's (split) monopole solution in the asymptotically flat spacetime.

Chapter 3 is devoted to a review of the AdS/CFT correspondence, beginning with the basic framework and implications of the correspondence, and followed by more concrete applications in aspects relevant to the present subject.

In Chapter 4, as a warm up, we solve the force-free equations for a rotating monopole in the pure AdS space. The solution serves as the asymptotic configuration of the full Kerr-AdS case and already captures the effect of the AdS boundary condition on the rotation of the magnetic fields.

Chapter 5 contains the main results with the force-free solution for a rotating monopole in the Kerr-AdS background. After some preliminaries on the Kerr-AdS geometry and the slow rotation limit, we present the detailed procedure of solving the force-free equations, including series and numerical solutions. We also obtain an analytic solution for the small Kerr-AdS black hole case, obtained as an expansion about the BZ solution in the Kerr limit. We primarily make use of Boyer-Lindquist (BL) coordinates, while Kerr-Schild (KS) coordinates which are nonsingular on the horizon are discussed in Appendix A. Some implications for the dual field theory are discussed and some comments given on the membrane paradigm interpretation of the BZ process [13, 23].

Chapter 6 includes a reformulation of the force-free equations in the Newman-Penrose (NP) formalism, where the equations are first-order. We then make use of the NP formulation to derive exact solutions in Kerr-AdS for the null current configuration, generalizing recent solutions by Brennan et al. [21] in the Kerr case. We also present some special new solutions with non-null currents and discuss other possible ways to find force-free solutions.

Conclusions and discussions are given in Chapter 7.

Chapter 2

Energy extraction from rotating black holes, BZ process and force-free magnetosphere

This chapter gives more detailed presentation of the energy extraction process from rotating black holes, with a review of the Penrose process and superradiance, and general analyses in terms of thermodynamics and spacetime geometries. We then concentrate on solving the conservation equations for the force-free magnetosphere.

2.1 Penrose process and superradiance

We give a brief derivation of the Penrose process and superradiance for rotating black holes. For concreteness, the coordinates $x^\mu = [t, r, \theta, \varphi]$ used below, and in the majority of this thesis, are implicitly assumed to be the Boyer-Lindquist (BL) coordinates of the Kerr(-AdS) metric, with the understanding that some general properties are coordinate-independent. Explicitly, the Kerr metric is

$$\mathrm{d}s^2 = -\frac{\Delta}{\Sigma}\Big[\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\Big]^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\Big[(r^2+a^2)\,\mathrm{d}\varphi - a\,\mathrm{d}t\Big]^2, \quad (2.1)$$

where $\Delta = r^2 + a^2 - 2mr$, $\Sigma = r^2 + a^2\cos^2\theta$, with m and a the black hole mass and rotation parameters. The spacetime, being stationary and axisymmetric, admits two Killing vectors: the time translational one $\xi^\mu_{(t)}$ and the azimuthal one $\xi^\mu_{(\varphi)}$. Rotating black holes have the peculiar property that static observers, represented by $\xi^\mu_{(t)}$, can not exist arbitrarily close to the horizon, since $\xi^\mu_{(t)}$ becomes spacelike in the ergosphere which encloses and is connected with the horizon [24]. For a 4-velocity u^μ to be timelike in the ergosphere, we have $g_{\mu\nu}u^\mu u^\nu < 0 \Rightarrow u^\varphi > 0$, as can be seen by noting that the only possible negative contribution is from the term $g_{\varphi t}u^\varphi u^t$ where $g_{\varphi t} < 0$. This is the frame-dragging effect. Especially, a zero angular momentum observer (ZAMO) is dragged to rotate with the Bardeen angular velocity

$$\omega_B \equiv -\frac{g_{\varphi t}}{g_{\varphi\varphi}}, \tag{2.2}$$

with the angular momentum defined by $L \equiv p_\mu \xi^\mu_{(\varphi)}$ for a 4-momentum p^μ. ω_B approaches Ω_H and Ω_∞ at the horizon and at infinity respectively, where Ω_H is the angular velocity of the black hole and $\Omega_\infty = 0$ for the Kerr case. ZAMO (for which we assume $u^r = u^\theta = 0$ from now on for simplicity) is the generalization of the static observer. To define the energy E, it is natural to use the asymptotic ZAMO 4-velocity, which is just $\xi^\mu_{(t)}$ for the Kerr case, and we have $E \equiv -p_\mu \xi^\mu_{(t)}$. In a coordinate system adapted to the Killing vectors (as in BL coordinates), $E = -p_t$ and $L = p_\varphi$.

Now consider a negative energy particle with 4-velocity u^μ in the ergosphere as in the Penrose process. Expanding the condition $u_\mu u^\mu < 0$ (for timelike u^μ) we have

$$u_t u^t + u_\varphi u^\varphi + g_{rr}(u^r)^2 < 0, \tag{2.3}$$

where $u^t > 0$ for future-pointing 4-velocities, $u_t > 0$ from the assumption $E \propto -u_\mu \xi^\mu_{(t)} = -u_t < 0$, and $u^\varphi > 0$ by frame-dragging as noted above. So for (2.3) to hold it is necessary that

$$u_\varphi = \Big(\frac{u^\varphi}{u^t} - \omega_B\Big) g_{\varphi\varphi} u^t \propto L < 0, \tag{2.4}$$

i.e., the particle has negative angular momentum with angular velocity

$$\omega = \frac{u^\varphi}{u^t} < \omega_B \leq \Omega_H. \tag{2.5}$$

Thus, in the Penrose process, the black hole also loses angular momentum; in other words, it is the rotational energy that is being extracted. One can also rewrite (2.3) as

$$(u_t + \Omega_H u_\varphi)u^t + (\omega - \Omega_H)u_\varphi u^t + g_{rr}(u^r)^2 < 0, \tag{2.6}$$

which when (2.4) is satisfied implies $-(u_t + \Omega_H u_\varphi) \propto E^{\mathcal{H}} > 0$, where $E^{\mathcal{H}}$ is the energy of the particle defined locally on the horizon using $K^\mu_{\Omega_H}$, so an observer just outside the horizon always sees an ingoing positive energy flux. Another way to see the positivity of $E^{\mathcal{H}}$ is by noting that $E^{\mathcal{H}} = E - \Omega_H L$ and that from (2.3) and (2.5), $0 < -E < -\omega L < -\Omega_H L$. Thus the phenomenon of energy extraction does not violate the ingoing condition for local energy on the horizon and indeed implies the latter if all constraints are properly considered as just shown.

For superradiance, consider a massless scalar field ϕ with the ansatz

$$\phi = R(r)S(\theta)\exp(-i\omega t + i\tilde{m}\varphi) \tag{2.7}$$

that solves the wave equation

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0 \tag{2.8}$$

in Kerr background (2.1). We quote the solution for the ingoing wave mode at the horizon [1]

$$\phi = S(\theta)\exp[-i(\omega - \tilde{m}\Omega_H)(t + r^*)]\exp[i\tilde{m}(\varphi - \Omega_H t)], \tag{2.9}$$

where r^* is the tortoise radial coordinate (see [1] for details). The ingoing energy flux is found to be

$$F^{\mathcal{H}}_E \propto F^r_t \propto \omega(\omega - \tilde{m}\Omega_H). \tag{2.10}$$

For energy extraction $F^{\mathcal{H}}_E < 0$, one has the constraint

$$\frac{\omega}{\tilde{m}} < \Omega_H \tag{2.11}$$

on the angular frequency, similar to that on the particle's angular velocity in the Penrose process (cf. (2.5)). Note that (2.11) involves only Ω_H rather than also ω_B since the energy flux is evaluated at the horizon. As will be shown in later chapters, there is also a similar constraint $\omega < \Omega_H$ for the BZ process, where ω is the angular velocity of the magnetic field lines.

2.2 Thermodynamics

The view on the energy and angular momentum extractions obtained from the above examples can also be demonstrated in terms of black hole thermodynamics. According to the first law, the black hole can only lose energy ($\delta E < 0$), with non-decreasing entropy ($\delta S \geq 0$), if it also possesses chemical potentials (e.g., electric potential or angular velocity). Correspondingly, the black hole mass m can be written as the sum of two parts: an irreducible mass $m_{\rm irr}$ related to the black hole entropy and a part $m_{\rm diss}$ related to the chemical potential. As the names suggest, it is $m_{\rm diss}$ that sources the energy outflux while at the same time compensating any increase in $m_{\rm irr}$.

We consider the example of an uncharged Kerr-AdS black hole whose explicit metric is not needed here (and will be given later). The relevant metric parameters are mass m or equivalently the horizon radius r_H, rotation parameter a and AdS curvature length l. Usually with l not treated as a thermodynamic quantity, all thermodynamic variables are given in terms of (r_H, a), and it is possible (at least for Kerr-AdS) to express (r_H, a) as functions of extensive variables (S, L), where L is the angular momentum conjugate to the angular velocity Ω. Then using (S, L) as independent variables we have for the energy E (cf. (3.111))

$$E(S, L) = \frac{m}{1 - a^2/l^2} = \sqrt{\frac{S}{4\pi}\Big(1 + \frac{S}{\pi l^2}\Big)^2 + \frac{\pi L^2}{S}\Big(1 + \frac{S}{\pi l^2}\Big)}. \tag{2.12}$$

The irreducible mass $m_{\rm irr} = m(L \sim a = 0)$ is indeed a monotonic function of S, while $m_{\rm diss}$ is non-zero only if $L \neq 0$, i.e., the extractable energy is the rotational energy of the black hole.

One can easily imagine (e.g., by plotting $E(S, L)$) a process in which a substantial amount of energy is extracted along a path of decreasing L in the E-S-L space. The path should be an integral curve of the first law

$$\mathrm{d}E(S, L) = T(S, L)\,\mathrm{d}S + \Omega(S, L)\,\mathrm{d}L, \tag{2.13}$$

where T is the temperature. To find an integral curve we need a relation $\mathrm{d}L \sim \mathrm{d}S$ as in some process. For demonstration purposes we choose

$$\mathrm{d}L = \frac{T}{\omega' - \Omega}\,\mathrm{d}S, \qquad \mathrm{d}E = \frac{\omega'}{\omega' - \Omega} T\,\mathrm{d}S, \tag{2.14}$$

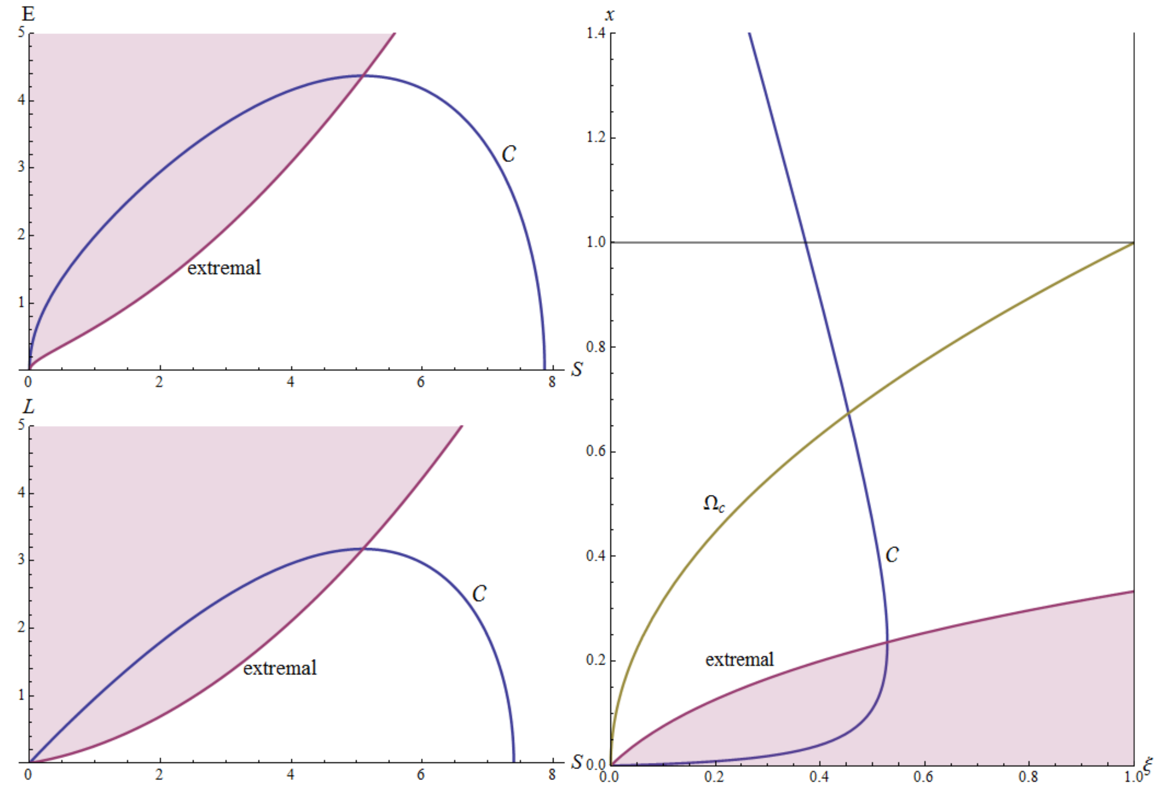


Figure 2.1: A sample integral curve $\mathcal{C}$ of the first law, as seen from E-S and L-S planes (left), and from the space of the parameters ($x \equiv r_H^2/l^2$, $\xi \equiv a^2/l^2$) (right). The unphysical shaded region in each plot is bounded by the curve representing extreme black hole solutions. Note that this curve passes exactly through the turning point of curve $\mathcal{C}$ in each plot. Thus in the physical region, both E and L decrease as S increases. Also, the black hole size r_H/l always increases.

where ω' is a constant. Energy and angular momentum extractions imply $0 < \omega' < \Omega$. For Kerr-AdS, $\Omega = \Omega_H - \Omega_\infty$ (cf. chapter 1), and we fix $\omega' = \Omega_H/2 - \Omega_\infty$. Such a value of ω' can be realized by a typical choice of 'ω' in superradiance (see e.g. [1]) or BZ process (cf. eqs. (5.73) to (5.75)), with the respective meanings of ω therein as discussed above, but otherwise just ad hoc. We then integrate the first equation in (2.14) and substitute the result into (2.12) to get (setting $l = 1$)

$$L(S) = S\sqrt{C - \frac{3S^2 + 8\pi S + 2\pi^2 \ln S}{4\pi^4}} \tag{2.15}$$

$$E(S) = \sqrt{S(S+\pi)}\sqrt{C - \frac{3S^2 + 7\pi S + 2\pi^2 \ln S - \pi^2}{4\pi^2}}, \tag{2.16}$$

where C is integration constant. Figure 2.1 shows a sample plot of $E(S)$ & $L(S)$ for

$C = 1$.

2.3 Energy extraction: general analysis

In this section, we analyze the energy extraction by examining properties of the geometry and matter near the horizon. We present these results in a general form that allows the usual treatment in the Kerr geometry to easily be extended to Kerr-AdS geometries with various coordinate choices.

2.3.1 Geometry: 3 + 1 formalism

We will use the $3+1$ formalism [25] which is convenient for presenting our results. Formally, a non-static spacetime metric can be written as

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + h_{ij}\big(\mathrm{d}x^i + \beta^i\,\mathrm{d}t\big)\big(\mathrm{d}x^j + \beta^j\,\mathrm{d}t\big), \qquad (i, j = \text{spatial directions}). \tag{2.17}$$

The 1-forms $\mathrm{d}x^i + \beta^i\,\mathrm{d}t$ are no longer exact; the condition that they vanish defines the 3-velocities of *fiducial observers* (FIDOs):

$$v^i_{\mathrm{FIDO}} = \frac{\mathrm{d}x^i}{\mathrm{d}t} = -\beta^i. \tag{2.18}$$

FIDOs generalize static observers and move orthogonally to the constant-t hypersurfaces Σ_t. In other words, the coordinate frame $\{x^i\}$ on Σ_t is drifting relative to the FIDO (at each point x) with 3-velocity $\beta^i(x)$, called the "shift vector". For example, in the rotating black hole case one has β^φ representing differential rotation of the coordinate system, though this may be more naturally thought of as FIDOs being dragged to rotate in the opposite direction. β^i can be promoted to a 4-vector

$$\beta^\mu = [0, \beta^i], \qquad \beta_\mu = [\beta_i\beta^i,\, \beta_i = h_{ij}\beta^j = g_{it}]. \tag{2.19}$$

The spatial metric h_{ij} is the projection of $g_{\mu\nu}$ onto Σ_t. Finally, α measures the "distance" in proper time between two adjacent hypersurfaces Σ_t and $\Sigma_{t+\delta t}$ so that $\delta\tau = \alpha\delta t$.

Dealing in fact with the stationary and axisymmetric Kerr(-AdS) spacetime, in this thesis we only consider cases with $\beta_\theta = g_{\theta t} = 0$, general enough to include both the Boyer-Lindquist (BL) and Kerr-Schild (KS) forms of the metric. For later

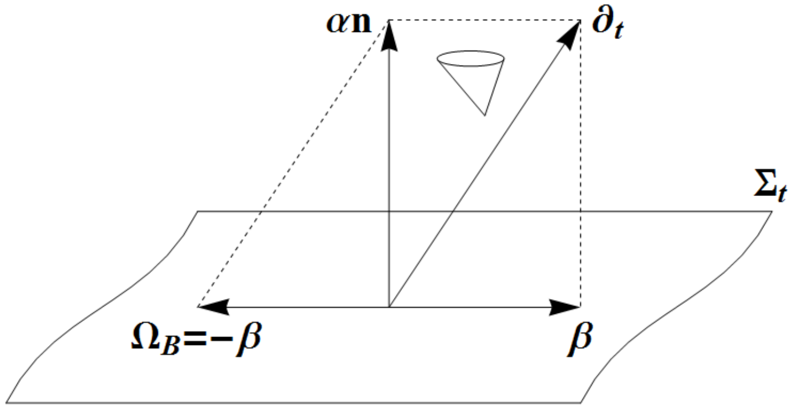


Figure 2.2: Sketch of the decomposition of the static Killing vector ∂_t, i.e. $\xi_{(t)}$, into the normal n to Σ_t and the shift vector β as given by (2.21). For BL coordinates, n can also be viewed as representing the ZAMO 4-velocity with angular velocity $\omega_B = -\beta^\varphi$ (cf. (2.23)).

convenience, we denote the linear combination of two Killing vectors by

$$K^\mu_{\Omega'} \equiv \xi^\mu_{(t)} + \Omega' \xi^\mu_{(\varphi)}, \tag{2.20}$$

for some constant (or function) Ω', so that $K^\mu_{\Omega'}$ is rotating with angular velocity Ω' relative to $\xi^\mu_{(t)}$. (We save the unprimed Ω for the angular velocity in thermodynamics.) If $\beta^i \neq 0$, $\xi^\mu_{(t)}$ fails to be orthogonal to Σ_t and can be decomposed as (see fig. 2.2)

$$\xi^\mu_{(t)} = \alpha n^\mu + \beta^\mu, \tag{2.21}$$

where n^μ is the future-pointing unit normal to Σ_t:

$$n_\mu \, \mathrm{d}x^\mu = -\alpha \, \mathrm{d}t, \qquad n^\mu \partial_\mu = \frac{1}{\alpha}(\partial_t - \beta^i \partial_i). \tag{2.22}$$

n^μ thus represents 4-velocities of FIDOs. Note also that the 1-form basis $[\alpha \, \mathrm{d}t, \, \mathrm{d}x^i + \beta^i \, \mathrm{d}t]$ is dual to the vector basis $[n^\mu \partial_\mu, \, \partial_i]$.

We briefly discuss BL and KS coordinates. In the former (cf. (2.1)),

$$\omega_B = -\beta^\varphi, \tag{2.23}$$

the only non-vanishing component of β^i, so a ZAMO with 4-velocity $K^\mu_{\omega_B}$ is a FIDO.

Indeed,

$$K^\mu_{\omega_B} = \xi^\mu_{(t)} + (-\beta^\varphi)\xi^\mu_{(\varphi)} = \alpha n^\mu, \tag{2.24}$$

by (2.21). The horizon is where $K^\mu_{\omega_B}$ becomes null; $K^\mu_{\omega_B}$ approaches $K^\mu_{\Omega_H}$, the horizon generator. The horizon locates at $\alpha = 0$ and can be viewed as the limiting case of the ZAM-equipotential surfaces $\alpha =$ const.. On the other hand, $\xi^\mu_{(t)}$ becomes null at $g_{tt} = \beta_i\beta^i - \alpha^2 = 0$ which is outside the horizon and defines the boundary of the ergosphere. (Generally, for the energy-defining Killing vector $K^\mu_{\Omega'}$, the ergosphere starts correspondingly at $K^\mu_{\Omega'}K_{\Omega'\mu} = 0$.)

BL coordinates are singular on the horizon ($g^{tt} = -\alpha^{-2}$, etc.), and thus it is also useful to consider KS coordinates (denoted with a tilde) which use a different foliation $\Sigma_{\tilde{t}}$ that is horizon penetrating. Thus $\tilde{n}^\mu$ is no longer aligned with $K^\mu_{\omega_B}$, i.e., FIDOs with respect to $\Sigma_{\tilde{t}}$ are no longer ZAMOs. We will make use of BL coordinates for much of the discussion below, as they are analytically more tractable, but the transformation $\{r, \theta, \tilde{\varphi}(\varphi, r), \tilde{t}(t, r)\}$ to KS coordinates is given in Appendix A, where we also translate a number of subsequent results for comparison.

2.3.2 Kinematics of generic matter fields

Making use of the above idea of spacetime foliation, we proceed to understand the energy extraction in general terms. For generic matter with energy-momentum tensor $T_{\mu\nu}$, define the conserved *energy-momentum flux vector* (see e.g. §6.4 of [26])

$$\mathcal{T}^\mu(\xi) \equiv -T^\mu_\nu \xi^\nu, \tag{2.25}$$

where ξ^μ is a Killing vector. We consider matter satisfying the dominant energy condition (DEC) which says that $\mathcal{T}^\mu(\xi)$ is non-spacelike and future-pointing if ξ^μ is time-like and future-pointing [24]. Applying Gauss' theorem in a spacetime domain $\mathcal{D}$:

$$\begin{aligned} 0 &= \int_{\mathcal{D}} \mathrm{d}^4x\sqrt{-g}\,\mathcal{T}^\mu_{\ ;\mu} = \int_{\partial\mathcal{D}} \mathrm{d}\mathcal{B}_\mu \mathcal{T}^\mu \\ &= \int_{\Sigma_{t_2}-\Sigma_{t_1}} \mathrm{d}^3x\sqrt{h}\,n_\mu \mathcal{T}^\mu + \int_{\mathcal{H}=\Sigma_{r_H}} \mathrm{d}\mathcal{B}_\mu \mathcal{T}^\mu + \int_{\Sigma_\infty} \mathrm{d}^3x\sqrt{-{}^3g}\,k_\mu \mathcal{T}^\mu, \end{aligned} \tag{2.26}$$

where $\mathrm{d}\mathcal{B}_\mu$ is the volume element restricted to the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. $\partial\mathcal{D}$ consists of two constant-t hypersurfaces Σ_{t_1} & Σ_{t_2} ($t_2 > t_1$) with future-pointing normal n_μ and

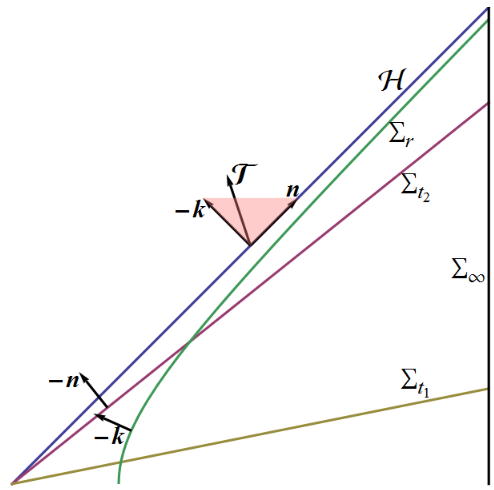


Figure 2.3: A spacetime region with horizon $\mathcal{H}$, AdS boundary Σ_∞ and constant-r(-t) slices Σ_{r,t_1,t_2} in BL coordinates. With $-n_\mu = \alpha\nabla_\mu t$ and $k_\mu = k_r\nabla_\mu r$, it holds that on $\mathcal{H}$ both $K_{\Omega_H\mu} \propto n_\mu \propto \nabla_\mu r$ and $-k_\mu$ are null normals and $[-k_\mu\, \mathrm{d}x^\mu,\, \mathrm{d}\theta,\, \mathrm{d}\varphi,\, \alpha n_\mu\, \mathrm{d}x^\mu]$ form a complete basis.

two constant-r hypersurfaces Σ_{r_H} & Σ_∞ (which are the horizon $\mathcal{H}$ and the boundary at infinity) with normal $k_\mu = [0, k_r, 0, 0]$. Figure 2.3 depicts $\mathcal{D}$ with various surfaces and vectors (discussed below). The integral on Σ_∞ can dropped if appropriate boundary conditions are chosen (e.g. in asymptotically AdS spacetime). Equation (2.26) then implies that

$$E(\Sigma_{t_2}) - E(\Sigma_{t_1}) + F_E^{\mathcal{H}} = 0, \tag{2.27}$$

where $E(\Sigma_t) \equiv -\int_{\Sigma_t} n_\mu \mathcal{T}^\mu(K_{\Omega'})$ and $F_E^{\mathcal{H}} \equiv -\int_{\mathcal{H}} \mathrm{d}\mathcal{B}_\mu \mathcal{T}^\mu(K_{\Omega'})$ are respectively the total energy on Σ_t and the *ingoing* energy flux across the horizon, and we have used $K^\mu_{\Omega'}$ as the energy-defining Killing vector.

To evaluate $F_E^{\mathcal{H}}$, following [14, 27], one makes use of the ingoing null vector

$$-k_\mu \propto -\nabla_\mu r \perp \mathcal{H} \tag{2.28}$$

and the null horizon generator

$$K_{\Omega_H\mu} \overset{\mathcal{H}}{=} \alpha n_\mu = -\alpha^2 \nabla_\mu t \tag{2.29}$$

normalized according to $(-k_\mu)K^\mu_{\Omega_H} = -1$, and the decomposition

$$\mathcal{T}_a \varpi^a = -(\mathcal{T}_\mu K^\mu_{\Omega_H})\varpi^1 - [\mathcal{T}_\mu(-k^\mu)]\varpi^4 + \mathcal{T}_2\varpi^2 + \mathcal{T}_3\varpi^3 \tag{2.30}$$

of $\mathcal{T}$ in the 1-form basis [$\varpi^1 = -\varpi^k = -k_r\,\mathrm{d}r$, $\varpi^2 = \varpi^\theta = \mathrm{d}\theta$, $\varpi^3 = \varpi^\varphi = \mathrm{d}\varphi$, $\varpi^4 = \varpi^{K_{\Omega_H}} = -\alpha^2\,\mathrm{d}t$]. One finds (using e.g. [28, 26, 29, 27]),

$$F_E^{\mathcal{H}} = -\int_{\mathcal{H}} \mathcal{T}_4{}^{\star}\varpi^1 = \int_{\mathcal{H}} (-\mathcal{T}_\mu K^\mu_{\Omega_H})(\varpi^2 \wedge \varpi^3 \wedge \varpi^4) = -(t_2 - t_1)\int_{\mathcal{H}\cap\Sigma_t} \mathrm{d}S\mathcal{T}_\mu K^\mu_{\Omega_H}, \tag{2.31}$$

where $\eta_{14} = \eta_{41} = -\eta_{22} = -\eta_{33} = -1$, $\epsilon_{r\theta\varphi t} = 1 = -\epsilon_{1234}$ and the last integral is on the 2-D spatial section of the horizon. Finally, note that on $\mathcal{H}$ [1],

$$n_\mu = \frac{1}{\alpha}K_{\Omega_H\,\mu} = -\frac{1}{2\kappa\alpha}\nabla_\mu(K^\nu_{\Omega_H}K_{\Omega_H\,\nu}) = \frac{1}{2\kappa\alpha}\nabla_\mu[f_g(r,\theta)\Delta_r] = \frac{f_g(r,\theta)}{2\kappa\alpha}\nabla_\mu\Delta_r \propto k_\mu. \tag{2.32}$$

It follows that $K_{\Omega_H\,\mu}$ is in the $\nabla_\mu r$ direction ($\parallel k_\mu$), which is also the $-\nabla_\mu t$ direction ($\parallel n_\mu$), where κ is the surface gravity and $f_g(r,\theta)$ is a function of metric components, so we have

$$F_E^{\mathcal{H}} \propto -\int_{\mathcal{H}} \mathcal{T}^r(K_{\Omega'}) = \int_{\mathcal{H}} T^r_t + \Omega' T^r_\varphi. \tag{2.33}$$

Energy extraction happens if $F_E^{\mathcal{H}} \propto -\int_{\mathcal{H}} \mathcal{T}_\mu(K_{\Omega'})K^\mu_{\Omega_H} < 0$, which implies, given that $K^\mu_{\Omega_H}$ is null on the horizon, that $\mathcal{T}^\mu(K_{\Omega'})$ must be space-like on (and, by continuity, just outside) the horizon. This in turn implies that the Killing vector $K^\mu_{\Omega'}$ with which $\mathcal{T}^\mu(K_{\Omega'})$ is defined fails to be time-like in the neighbourhood of the horizon (by the DEC), i.e., the existence of an ergosphere.

Arbitrarily close to the horizon, $K^\mu_{\Omega_H}$ is time-like, meaning that the following inequality always holds on the horizon

$$-\mathcal{T}^r(K_{\Omega_H}) \geq 0, \tag{2.34}$$

so a local observer co-rotating with $K^\mu_{\Omega_H}$ sees an ingoing energy flux. For an asymptotic observer, on the other hand, who defines energy with $K^\mu_{\Omega'}$, (2.34) implies

$$-\mathcal{T}^r(K_{\Omega'}) - (\Omega_H - \Omega')\mathcal{T}^r(\xi_{(\varphi)}) \geq 0 \tag{2.35}$$

$$\Rightarrow \quad F_E^{\mathcal{H}} - (\Omega_H - \Omega')L_E^{\mathcal{H}} \geq 0 \tag{2.36}$$

$$\Rightarrow \quad \delta E - (\Omega_H - \Omega')\delta L \equiv T\delta S \geq 0, \tag{2.37}$$

where

$$F_L^{\mathcal{H}} \equiv \int_{\mathcal{H}} \mathrm{d}\mathcal{B}_\mu \mathcal{T}^\mu(\xi_{(\varphi)}) \propto \int_{\mathcal{H}} \mathcal{T}^r(\xi_{(\varphi)}) = -\int_{\mathcal{H}} T^r_\varphi \tag{2.38}$$

is the total ingoing angular momentum flux across the horizon, and the last step shows that the derivation leads to the 1st and 2nd laws of black hole thermodynamics. To respect the 2nd law, there must also be an angular momentum extraction ($\delta L < 0$) accompanying any energy extraction from the black hole (for $\Omega_H - \Omega' > 0$, e.g., the Kerr case), as noted above. For large AdS black holes where we can choose $\Omega' = \Omega_H$ energy extraction is absent [14]. Similar conclusions follow for super-radiance [30].

2.4 Force-free magnetosphere and BZ process

In this section we give a basic description of the force-free magnetosphere, using the general metric (2.17) in BL coordinates. The problem of formulating energy extraction via a force-free magnetosphere (i.e. the BZ process) is summarized in the conservation equations for the energy-momentum tensor of the electromagnetic field, namely,

$$T^{\mu\nu}_{;\nu} = F^{\nu\mu} J_\nu = 0, \tag{2.39}$$

which we also refer to as the force-free equation/condition. In astrophysical situations we can have very strong magnetic field triggering pair creations so that the magnetosphere is filled with plasma, which screens the electric field in co-moving frames of the current to fulfill the force-free condition. However one neglects the inertia of the plasma and the currents and their contributions to the energy-momentum tensor. The current only serves as the medium for the BZ process to operate and its physical meaning is only through the electromagnetic field, as the rewriting of the derivative

$$J^\mu = F^{\mu\nu}_{;\nu}. \tag{2.40}$$

A constraint put on the electromagnetic field by the force-free equation with non-zero J^μ is vanishing of the invariant

$${}^\star F_{\mu\nu} F^{\mu\nu} = 0, \tag{2.41}$$

since ${}^\star F_{\mu\nu} F^{\mu\nu} \sim \det F_{\mu\nu}$ for antisymmetric $F_{\mu\nu}$, where ${}^\star F_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the Hodge dual. (2.41) is called the degeneracy condition. Further assuming stationarity and axisymmetry (so that $F_{\varphi t} = 0$), one deduces from the degeneracy condition the existence of the ratio

$$\omega \equiv -\frac{F_{tr}}{F_{\varphi r}} = -\frac{F_{t\theta}}{F_{\varphi\theta}}, \tag{2.42}$$

which is interpreted as the angular velocity of the magnetic field lines. The field lines are mainly poloidal (to thread the black hole) and are specified by the vector potential A_φ. One last quantity needed to completely specify the electromagnetic field is the toroidal magnetic field $B^\varphi = F_{r\theta}/\sqrt{-g}$, for which we more often use the equivalent quantity

$$B_T \equiv (g_{\varphi\varphi}g_{tt} - g_{\varphi t}^2)B^\varphi \overset{BL}{=\!=} -\alpha^2 h_{\varphi\varphi}B^\varphi. \tag{2.43}$$

Thus, the independent field quantities for the force-free magnetosphere are now

$$A_{\varphi,r} = -\sqrt{-g}B^\theta, \quad A_{\varphi,\theta} = \sqrt{-g}B^r, \quad \omega \quad \text{and} \quad B_T. \tag{2.44}$$

One can start with some initial configuration of A_φ representing an external magnetic field profile (e.g. that of a non-rotating monopole as in BZ's original paper), add to it a perturbation (e.g. due to rotation) and solve the force-free equations to obtain the complete profile (including ω and B_T) after the perturbation. It is also instructive to just formally manipulate the mathematical relations among various quantities in order to reveal some essential properties of the force-free magnetosphere. First, the relevant quantities for the radial energy and momentum fluxes are respectively

$$T_t^r = -\omega B_T \frac{A_{\varphi,\theta}}{\sqrt{-g}}, \qquad T_\varphi^r = -\frac{T_t^r}{\omega}. \tag{2.45}$$

They both depend on B_T (and the initial A_φ). An observer co-rotating with angular velocity ω will see a vanishing locally defined energy flux. Nevertheless, the angular momentum flux is more robust and not affected by such changes of frame. In turn, ω and B_T are related respectively to the poloidal electric field ($F_{tr} \sim F_{t\theta} \sim \omega$) and poloidal current $J_P \equiv (J^r, J^\theta)$ with

$$J^r = -\partial_\theta B_T, \qquad J^\theta = \partial_r B_T. \tag{2.46}$$

Again, the electric field can be made to vanish by going to the co-rotating frame with ω, but J_P generally does not as can be seen from the following argument. One can check the other invariant $F_{\mu\nu}F^{\mu\nu} \sim \vec{B}^2 - \vec{E}^2$ which should be positive in a magnetically dominant magnetosphere:

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = B_P^2\big[\alpha^2 - h_{\varphi\varphi}(\omega - \omega_B)^2\big] + \frac{B_T^2}{\alpha^2 h_{\varphi\varphi}} > 0 \tag{2.47}$$

$$\Leftrightarrow \quad c_\omega B_P^2 + (B^\varphi)^2 > 0, \tag{2.48}$$

where $B_P^2 \equiv h_{rr}(B^r)^2 + h_{\theta\theta}(B^\theta)^2 > 0$ and

$$c_\omega \equiv \frac{1}{h_{\varphi\varphi}} - \frac{(\omega - \omega_B)^2}{\alpha^2} \tag{2.49}$$

(which also appears later in (5.40)). In a (rotating) black hole spacetime where a horizon is present at $\alpha = 0$, (2.47) generally implies $B_T \neq 0$ and $J_P \neq 0$.

By actually solving the force-free equations in different spacetimes, one finds (using the example of a monopole field to be discussed in detail shortly) a relation

$$B_T^2 \propto (\omega - \Omega_H)^2, \qquad \Omega_H = \omega_B|_{r=r_H}, \tag{2.50}$$

from the horizon regularity condition for rotating black holes, and a second relation

$$B_T^2 \propto (\omega - \Omega_\infty)^2, \qquad \Omega_\infty \equiv \omega_B|_{r\to\infty}, \tag{2.51}$$

from *exact* solutions in the asymptotic static spacetimes (AdS or flat) which rotate with angular velocity Ω_∞ in the black hole coordinates ($\Omega_\infty = 0$ for Kerr).[1] It is natural to match the two solutions in the asymptotic region, i.e. equating B_T's in (2.50) and (2.51), so that B_T cannot vanish, tied to the fact that $\Omega_H \neq \Omega_\infty$.[2] In other words, B_T and J_P are generated due to the relative rotation between the horizon and the boundary, or the fact that K_{Ω_∞} becomes spacelike near the horizon, i.e. the existence of an ergosphere. One may develop the view that the spacetime "entity" rotates in the magnetic field like a Faraday disk. In the slow rotation limit, the coefficients of proportionality in (2.50) and (2.51) are the same, and one gets the celebrated relation $\omega = \Omega_H/2$ for the BZ process in Kerr, and an analogous $\omega = (\Omega_H + \Omega_\infty)/2$ in Kerr-AdS. Nevertheless, we will show in chapter 4 that the ambiguity in defining a monopole and the freedom to add perturbations to the monopole result in modified relations in place of (2.51) and lead to the non-uniqueness of ω.

That B_T and J_P are produced by the rotation of spacetime can be elucidated by finding the driving force of J_P. For this we turn to the 3+1 formalism, where the spacetime is effectively replaced by the traditional view of "absolute spaces" Σ_t evolving in global time t. The Maxwell's equations then take the familiar forms in

[1] Naively, the relation (2.51) also holds for Schwarzschild(-AdS) with $\Omega_\infty = 0$.

[2] We have implicitly assumed that ω and B_T are r-independent as for BZ's (slowly rotating) monopole solution, thus excluding $\omega = \omega_B(r, \theta)$, but otherwise we expect the same qualitative argument holds generally.

terms of 3-vectors

$$\check{B}^i = \alpha\, {}^\star F^{it}, \quad \check{E}_i = \frac{\alpha}{2}\epsilon_{ijk}\, {}^\star F^{jk}, \quad \check{D}^i = \alpha F^{ti}, \quad \check{H}_i = \frac{\alpha}{2}\epsilon_{ijk} F^{jk}, \quad \check{J}^\mu = \alpha J^\mu = [\check{\rho}, \check{J}^i] \tag{2.52}$$

where ϵ_{ijk} is the 3-d Levi-Civita tensor. The constitutive relations resemble those in a bi-anisotropic medium [31, 7]:

$$\check{\boldsymbol{E}} = \alpha \check{\boldsymbol{D}} + \boldsymbol{\beta} \times \check{\boldsymbol{B}} \tag{2.53}$$

$$\check{\boldsymbol{H}} = \alpha \check{\boldsymbol{B}} - \boldsymbol{\beta} \times \check{\boldsymbol{D}}, \tag{2.54}$$

where α and $\boldsymbol{\beta}$ are the lapse function and the shift vector. The force-free condition $F_{\mu\nu}J^\nu = 0$ reads

$$\check{\rho}\check{\boldsymbol{E}} + \check{\boldsymbol{J}} \times \check{\boldsymbol{B}} = 0, \qquad \check{\boldsymbol{E}} \cdot \check{\boldsymbol{J}} = 0, \tag{2.55}$$

implying

$$\check{\boldsymbol{E}} \cdot \check{\boldsymbol{B}} = \check{\boldsymbol{D}} \cdot \check{\boldsymbol{B}} = 0, \tag{2.56}$$

which is the degeneracy condition. The existence of ω is such that

$$\check{\boldsymbol{E}} = -\boldsymbol{\omega} \times \check{\boldsymbol{B}}, \tag{2.57}$$

where $\boldsymbol{\omega} = \omega\partial_\varphi$. The Poynting flux is $\check{\boldsymbol{S}} = \check{\boldsymbol{E}} \times \check{\boldsymbol{H}}$.

$\check{\boldsymbol{E}} = 0$ in the co-moving frame of the current by (2.55) or in a rotating frame with ω by (2.57). However there is still a non-zero $\check{\boldsymbol{D}} = -\alpha^{-1}\boldsymbol{\beta} \times \check{\boldsymbol{B}}$ by (2.53), indicating that the electric field can never be totally screened by the plasma. It is this residual electric field that serves as the driving force of the poloidal current, analogous to the electromotive force in a Faraday disk or the surface of a neutron star. (The non-vanishing of $\check{\boldsymbol{D}}$ can also be seen from its definition in (2.52) where one always has terms due to $g^{\varphi t} \neq 0$.) The poloidal current then sources the toroidal magnetic field $\check{H}^\varphi$ needed for the radial Poynting flux and the angular momentum flux. One sees again the crucial role of $\boldsymbol{\beta}$, associated with the ergosphere, in providing $\check{\boldsymbol{D}}$ which is referred as the "gravitationally induced" electric field. A similar argument for the sign of $F_{\mu\nu}F^{\mu\nu} \sim \check{\boldsymbol{B}}^2 - \check{\boldsymbol{D}}^2$ can be found in [31]. Note that in static spacetimes, although B_T and J_P can be non-zero, the driving electromotive force necessarily comes from ordinary rotating matter, e.g. the accretion disk.

The need to understand the "unipolar inductor" required to produce the electromotive force (EMF) was one of the triggers for the black hole membrane paradigm

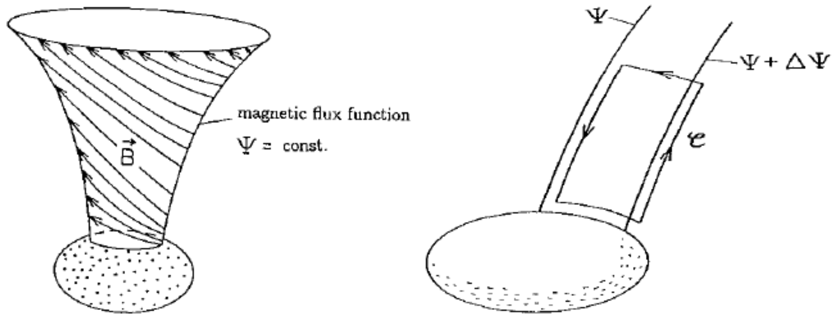


Figure 2.4: (From [32]. Reprinted with permission from Springer.) Left: a constant magnetic flux (Ψ) surface of the magnetosphere around a rotating black hole. Right: Integration path $\mathcal{C}$ for the EMF across two constant magnetic flux surfaces.

[12, 13], where the horizon is treated as a surface endowed with transport properties such as conductivity and is used to impose boundary conditions. Although as discussed above, the current driving force is not directly related to the horizon but to the ergosphere, the membrane paradigm still provides an effective way of calculating the EMF, given by the following integral along a circuit as shown in fig. 2.4 [32, 12]

$$\mathrm{EMF} \equiv \Delta V = \oint_{\mathcal{C}} \alpha \check{\boldsymbol{D}} \cdot \mathrm{d}\boldsymbol{l} = -\int_{\mathcal{C}_{\mathrm{H}}} \boldsymbol{\beta} \times \check{\boldsymbol{B}} \cdot \mathrm{d}\boldsymbol{l} = \frac{\Omega_{\mathrm{H}}}{2\pi} \Delta\Psi, \tag{2.58}$$

which shows the dependence on $\boldsymbol{\beta}$ except that only the path $\mathcal{C}_{\mathrm{H}}$ along the (stretched) horizon contributes ($\boldsymbol{\beta} \to 0$ at infinity), where Ψ is the magnetic flux as indicated in the figure.

As nowadays we understand it, the gravitationally induced electric field operates in the entire region of the ergosphere to drive the poloidal current. The associated toroidal magnetic field $\check{H}^{\varphi}$ is the one required to slow down the black hole by pushing plasma into orbits with negative mechanical energy-at-infinity, resulting in an outgoing flux of mechanical energy-at-infinity. The energy flux changes its nature from almost purely mechanical close to the horizon to almost purely electromagnetic far away from it in the form of a Poynting flux (twist of magnetic field lines propagates away). In a way, the ergospheric plasma and the magnetic field play roles similar to those of the negative and positive energy particles in the Penrose process.

2.5 BZ's monopole solution

In their original paper [5], Blandford and Znajek were able to provide a "proof of principle" by calculating the energy flux from a slowly rotating black hole with a (split) monopole magnetic field. The solution was found using an ansatz with a perturbative expansion in the rotation parameter a, the zeroth order term given by a known monopole solution in the Schwarzschild metric. The solution is required to match that of a rotating radial field in flat spacetime (Michel 1973 [22]) at infinity and satisfy a certain boundary (regularity) condition on the horizon. McKinney & Gammie [6] re-derived BZ's results using KS coordinates, which are free of the coordinate singularity on the horizon, and thus have the advantage of allowing regular boundary condition on the horizon and dispensing with the need to match to another solution at infinity. The transformation from BL to KS coordinates (the latter indicated by a tilde on quantities) involves defining new $(\tilde{\varphi}, \tilde{t})$ coordinates mixed with the r coordinate:

$$\mathrm{d}\tilde{\varphi} = \mathrm{d}\varphi + \frac{a}{\Delta}\,\mathrm{d}r \tag{2.59}$$

$$\mathrm{d}\tilde{t} = \mathrm{d}t + \frac{2mr}{\Delta}\,\mathrm{d}r. \tag{2.60}$$

The transformation is itself singular at the horizon, such that the constant-$\tilde{t}$ surfaces are now horizon penetrating. Physical quantities in KS coordinates are regular on the horizon. In particular, the energy and angular momentum fluxes out of the black hole are [6]

$$F_E = -T^r_t = -\omega \sin^2\theta \big[2(B^r)^2 r\big(\omega - \frac{a}{2mr}\big) - B^r \tilde{B}^\varphi \Delta\big], \qquad F_L = F_E/\omega \tag{2.61}$$

(where note that only $\tilde{B}^\varphi$ is different from the BL value). Evaluated on the horizon $\mathcal{H}$ (setting $\Delta = 0$),

$$F_E^{\mathcal{H}} = 2(B^r)^2 \omega r_H (\Omega_H - \omega) \sin^2\theta. \tag{2.62}$$

Energy extraction is possible if $0 < \omega < \Omega_H$. KS coordinates however result in additional off-diagonal metric components and we find that some general results (without specializing to the monopole field) take a more concise form in BL coordinates. Most importantly, the essential equation for A_φ is the same in both coordinates. We thus present the subsequent derivations in BL coordinates.

In the stationary and axisymmetric configuration, the force-free magnetosphere is effectively described by the following quantities: the angular velocity of the field lines ω, the toroidal magnetic field $B_T = (g_{\varphi\varphi}g_{tt} - g_{\varphi t}^2)B^\varphi$ and A_φ. The existence of ω follows from the degeneracy condition ${}^\star F^{\mu\nu}F_{\mu\nu} = 0$, along with stationarity and axisymmetry $\partial_t = \partial_\varphi$, and is given by (cf. (2.42))

$$\omega(\theta, \varphi) \equiv -\frac{A_{t,\theta}}{A_{\varphi,\theta}} = -\frac{A_{t,r}}{A_{\varphi,r}}. \tag{2.63}$$

The function A_φ = constant specifies poloidal field surfaces. From the definition (2.63) and the conservation equation (2.39), one has $\omega = \omega(A_\varphi)$ and $B_T = B_T(A_\varphi)$. All non-vanishing components of $F_{\mu\nu}$ are expressed in terms of $\{\omega,\ A_{\varphi,r} = -\sqrt{-g}B^\theta,\ A_{\varphi,\theta} = \sqrt{-g}B^r,\ B^\varphi = F_{r\theta}/\sqrt{-g}\}$.

The conservation equation (2.39) is now a second order differential equation for A_φ, with parameters ω & B_T. One can start with a solution found in the non-rotating limit $a = 0$ and perturb it by spinning up the black hole, treating $a/m \ll 1$. The conservation equations (2.39) are then solved perturbatively in a. It is convenient to set $m = 1$. In this slow rotation limit, we allow the following corrections to the field quantities:

$$A_\varphi = A_\varphi^{(0)} + a^2 A_\varphi^{(2)} \tag{2.64}$$

$$\omega = a\omega^{(1)} \tag{2.65}$$

$$B^\varphi = aB^\varphi_{(1)}, \qquad (B_T = aB_T^{(1)}), \tag{2.66}$$

keeping terms up to $\mathcal{O}(a^2)$. The orders of a above are from symmetry considerations.

BZ considered a split monopole model, which models the magnetic field produced by currents (with suitably chosen radial dependence) on the accretion disk around a *neutral* black hole. By symmetry, one can concentrate on the north hemisphere where the monopole field is initially given by $A_\varphi^{(0)} = -C\cos\theta$. In the perturbation scenario, one can deduce that ω is r-independent:

$$\partial_r[\omega(A_\varphi)] \sim \mathcal{O}(a^3), \tag{2.67}$$

which is specific to the monopole field. Similarly, B_T is also r-independent in BL coordinates, but the horizon regularity condition that fixes the relation $B_T(\omega)$ (cf. (2.50)) is more conveniently derived in the KS coordinates, simply by imposing $\tilde{B}_T(B_T, A_\varphi, \omega) = 0$ on the horizon from the regularity of $\tilde{B}^\varphi \sim \tilde{B}_T/\Delta$. (If we solely work in the KS coordinates, a relation $B^\varphi(\omega)$ can be found directly from the regularity of B^φ.) Now the equation for A_φ contains $\omega^{(1)}(\theta)$ as the only free parameter.

Applying separation of variable on $A_\varphi^{(2)}$, it is not difficult to show that it has the form

$$A_\varphi^{(2)} = Cf(r)\cos\theta\sin^2\theta, \tag{2.68}$$

and in addition, that $\omega^{(1)}(\theta) = \omega^{(1)}$. The equation for $f(r)$ is

$$f'' + \frac{2f'}{r(r-2)} - \frac{6f}{r(r-2)} + \Big[\frac{r+2}{r^3(r-2)} - \frac{[\omega^{(1)} - 1/8](r^2+2r+4)}{r(r-2)}\Big] = 0. \tag{2.69}$$

It has a regular singular point at the horizon $r = 2$ and an irregular one at infinity, more easily seen using the radial coordinate $z = 2/r$, so that near $z = 0$ the equation behaves like

$$f''(z) + \cdots + (\omega^{(1)} - 1/8)/z^4 = 0. \tag{2.70}$$

This suggests that we should choose $\omega = 1/8$ which turns out to be half of the horizon angular velocity. One can in fact analytically solve (2.69) and find

$$f(r) \sim \frac{1}{4r} + \mathcal{O}\Big(\frac{\ln r}{r^2}\Big) \tag{2.71}$$

at large r.

The value of ω seems quite robust as indicated from numerical studies. It is the value that ω eventually settles down to during a dynamic simulation. $F_E^{\mathcal{H}}$ as in (2.62) also reaches its maximum for this value. Numerical simulations allow one to study situations for finite a. For example [6], results for $a = 0.5$ show that in the force-free region, the feature $\omega/\Omega_H \approx 0.5$ persists. It is also found that ω is nearly constant for a large range of r. Recent numerical studies have lent increasing support to the ability of the BZ process to account for the high efficiency of observed relativistic jets [33, 27, 34, 35]. See e.g. fig. 2.5.

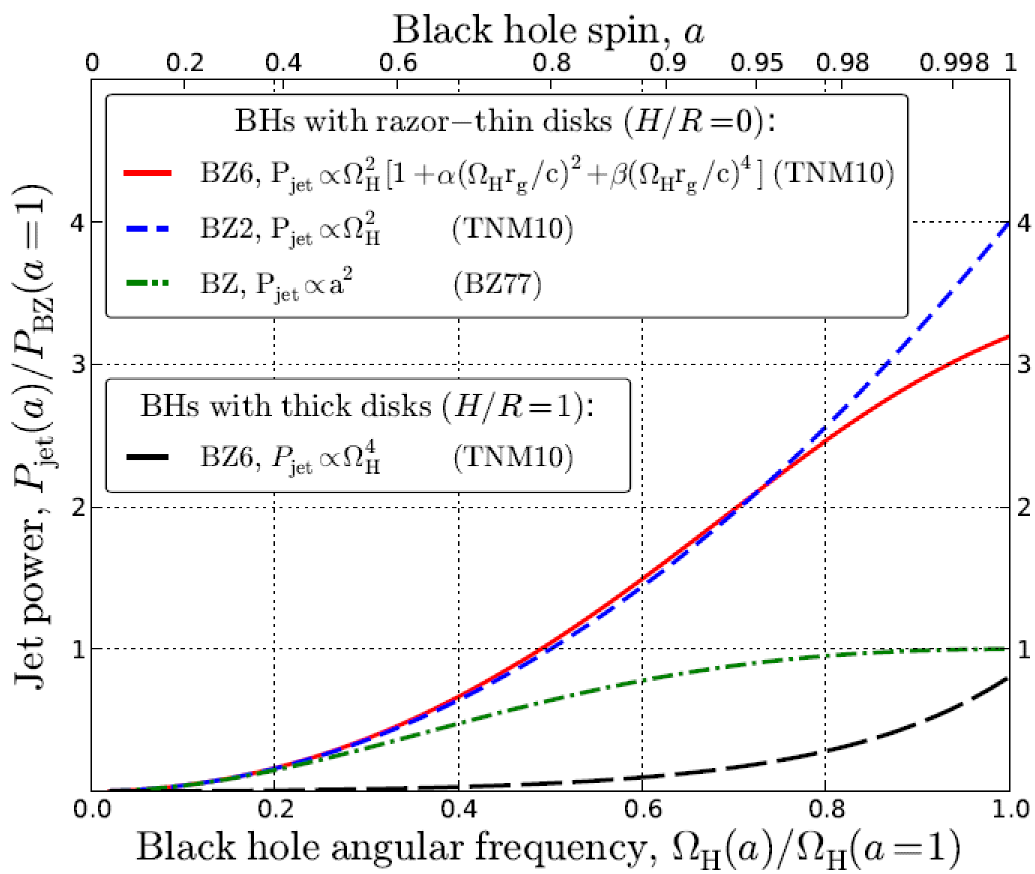


Figure 2.5: The dependence of the jet power of the BZ process on the black hole spin, reproduced from [33]. 'BZ' refers to the original BZ's formula valid for small a. 'BZ2' & 'BZ6' are improved estimations (expanded in Ω_H to the second and fourth orders respectively) which explain well the observed luminosities for different a.

Chapter 3

AdS/CFT

In this chapter we review aspects of the AdS/CFT correspondence, the possible application of which to the BZ process in the Kerr-AdS background is the main motivation and objective of this thesis.

3.1 AdS geometry

The AdS/CFT correspondence states that a gravitational theory in the higher dimensional asymptotically anti-de Sitter (AdS) spacetime is equivalent to a lower dimensional conformal field theory (CFT), usually considered as living on the boundary at infinity in some 'radial' direction of the asymptotically AdS spacetime. The higher dimensional spacetime is often referred as the "bulk". In this section we review the "AdS" part of the correspondence, while the "CFT" part will be reviewed later in section 3.5 when we consider symmetries.

The AdS_{d+1} spacetime has a negative constant scalar curvature $R = -d(d+1)/l^2$, where the length scale l is referred as the AdS radius. It solves the Einstein's field equations with negative cosmological constant

$$\Lambda = -\frac{d(d-1)}{2l^2}. \tag{3.1}$$

The AdS spacetime is most easily visualized as a hyperboloid embedded in the $d+2$-dimensional flat space with Cartesian coordinates $[X^0, X^a, X^{d+1}]$ and signature

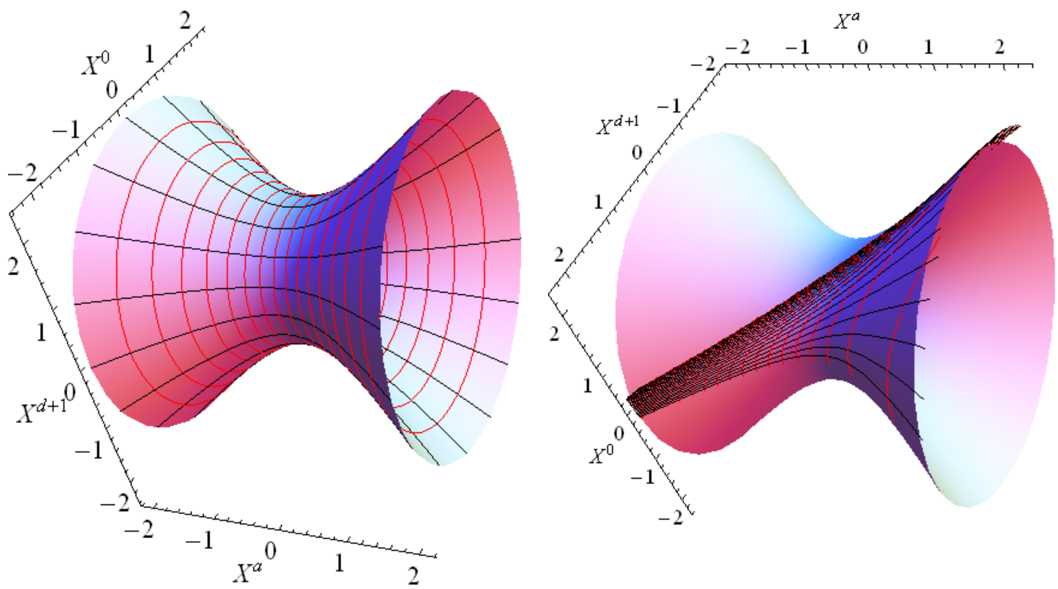


Figure 3.1: AdS_{d+1} space as represented by a hyperboloid in the d+2-dimensional flat space. The $\{X^0, X^{d+1}\}$ directions correspond to the two negative signs in the metric signature, and only one of the $\{X^a\}$ directions is drawn ($d = 1, a = 1$). Left: The grid of global coordinates covering the whole AdS space. Right: The Poincaré patch, covering half of the original hyperboloid (for clarity, not showing the whole range of the patch coordinates). On each graph, red lines mark constant radii and black lines mark constant times. For the global coordinates, the radial coordinate r is more intuitively related to the distance from the 'waist' of the hyperboloid (cf. (3.3)) and the time coordinate t goes around the hyperboloid. For the Poincaré patch the radial direction z point diagonally in the X^d-X^{d+1} plane from the origin to the increasing values of both coordinates (and thus cover half of the entire space) and the time coordinate is along X^0 of the embedding space but scaled by z. We have also set $l = 1$. To generate the grids we have used the definitions for the intrinsic coordinates given in [36].

$[-, + \ldots +, -]$, given by the equation [36]

$$-(X^0)^2 + \sum_{a=1}^{d} (X^a)^2 - (X^{d+1})^2 = -l^2. \tag{3.2}$$

Figure 3.1 depicts a hyperboloid for $d = 1$, manifesting the rotational symmetry in the X^0-X^2 plane with the rotation axis along the X^1 direction. The figure also shows the intrinsic coordinates discussed below.

Intrinsic coordinates $[r, t, \theta^i]$ ($i = 1, \ldots, d-1$), called the *global coordinates*, can be introduced on the hyperboloid such that the constant-X^a subspaces of the hyperboloid are circles of radii

$$\sqrt{(X^0)^2 + (X^{d+1})^2} = \sqrt{l^2 + r^2} \tag{3.3}$$

with angular coordinate t/l. These circles are located on a $(d-1)$-sphere of radius r in the $\{X^a\}$ subspace, i.e., $r^2 = \sum_{a=1}^{d}(X^a)^2$. The $(d-1)$-sphere is coordinatized by the angles $\{\theta^i\}$. Here r serves as the radial coordinate and t is the time coordinate which ranges from $-\infty$ to ∞ after unwrapping the circle.[1] The AdS metric then takes the form

$$\mathrm{d}s^2_{d+1} = -\left(1 + \frac{r^2}{l^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{d-1}, \tag{3.4}$$

where $\mathrm{d}\Omega^2_{d-1}$ is the line element on the unit $(d-1)$-sphere. (Note that in the $d = 1$ case in fig. 3.1 the $(d-1)$-sphere appears to be two separate points at equal distances from the origin of $X^a = X^1$ axis.)

Another useful set of the intrinsic coordinates $\{z, x^\mu\}$ covers a subregion, called the *Poincaré patch*, of the AdS space. The radial direction z is defined by

$$\frac{l^2}{z} = X^d + X^{d+1}, \qquad z < 0 < \infty \tag{3.5}$$

and we have a d-dimensional Minkowski spacetime at each constant z, with Cartesian coordinates:

$$x^\mu = \frac{z}{l}X^\mu, \qquad (\mu = 0, \ldots, d-1). \tag{3.6}$$

Finally, the other linear combination of X^{d+1} and X^d is not an independent coordinate but a function of (z, x^μ):

$$X^{d+1} - X^d = z + \frac{x^\mu x_\mu}{z}. \tag{3.7}$$

The metric for the Poincaré patch reads

$$\mathrm{d}s^2_{d+1} = \frac{l^2}{z^2}(\mathrm{d}z^2 + \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu), \tag{3.8}$$

where $z = 0$ is the boundary and $z = \infty$ is a degenerate horizon. The boundary at $z = 0$ corresponds to $r = \infty$ by noting that the two radial coordinates are related as

$$z^{-1} \propto \sqrt{1 + \frac{r^2}{l^2}}. \tag{3.9}$$

[1] One can do the unwrapping because the space is not simply connected, i.e., the time circle cannot be continuously shrunk to a point [37].

On the other hand, the horizon at $z = \infty$ does not correspond to a particular limit of r. In addition, as $x^0 \to \infty$, any constant-z surface will reach $r = \infty$ [36].

The Poincaré patch metric (3.8) is in the form of a Minkowski metric multiplied by a factor divergent at $z \to 0$, giving rise to a divergent surface area of the boundary and also an effective divergent potential which makes the AdS space act like a confining box for propagating modes. Similarly, the global metric (3.4) can also be written in the form of a diverging factor r^2/l^2 times a regular metric (i.e. that of the Einstein static universe) for $r \to \infty$ [38]. A massive particle traveling outwards cannot reach the boundary and will fall back to the interior. The timelike nature of the AdS boundary will also allow us to reflect back a massless particle so that it can meet an interior timelike trajectory (e.g. of an observer) twice.

3.2 AdS/CFT is a holographic principle

The radial direction of the AdS space plays a special role in the AdS/CFT correspondence. As will be reviewed later, it is the radial falloff of the bulk fields that provides the connection between the bulk and boundary quantities. Moreover, the radial direction itself can be viewed as emerging from the energy scale of the boundary CFT. AdS/CFT is thus a realization of the holographic principle [39], where a lower dimensional theory can encode all the degrees of freedom of a higher dimensional one. The correspondence is by no means trivial and its realization could be subtle; the two theories are "dual" to each other, i.e., there is a one-to-one and onto map between physical contents on the two sides.

In this respect, it should be emphasized that the correspondence is not in the sense that physics in one theory is merely the manifestation/results of that in another. For example, with a black hole in the bulk, the boundary theory is said to be at finite temperature given by the Hawking temperature T_H (as used in the black hole thermodynamics). However, this is not to be confused with the inference that the boundary is heated up by the Hawking radiation coming from the black hole. The local temperature due to thermal radiation is redshifted as $T_{\text{local}} = T_H/\sqrt{-g_{tt}} \sim T_H/r$ at large radius r, while AdS/CFT only deals with the conformal class of the boundary that is insensitive to this redshift factor (see e.g. [40, 41]). Moreover, for a CFT without other scales, all non-zero temperatures are equivalent (discussed in detail later). AdS/CFT is more nontrivial than the case of a usual hologram, where one is just replicating the information. Here we have two traditionally *different* theories being equivalent and treated on the equal footing.

This results in two ways of exploiting the duality: one can start on either side, certain aspects of which we understand better, and infer the dual descriptions on the other side. Usually, the two theories involved are best described in their own regimes of parameters. The duality then allows us to seek solutions for problems in one theory by doing calculations in the other. This often transforms a hard calculation (e.g., non-perturbative in nature) into an easier task, as well as helping to gain new insights for each theory from a dual point of view.

Relating two theories of different natures also makes the duality conceptually interesting. In one way, it can be thought of as parallel to the idea of finding a "gauge gravity" theory, the reformulation of gravity as a gauge theory. In another, the extra bulk dimension is emergent, corresponding to the energy scale in the boundary field theory; by going into the bulk, the field theory gets geometrized and covariantized. At last, it allows one to study quantum gravity using ordinary quantum field theories.

3.3 AdS/CFT is a strong/weak coupling correspondence: the example of $AdS_5 \times S^5/\mathcal{N} = 4$ SYM

The original and primary example [15] of AdS/CFT is the correspondence between the type IIB string theory in the $AdS_5 \times S^5$ background and the strongly coupled 4-dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills (SYM) theory, which is conformally invariant (unlike QCD). It was found in the studies of D3-branes, where $AdS_5 \times S^5$ arises as the near horizon geometry of the black 3-brane solution, corresponding to the low energy supergravity limit of the (set of N coincident) D3-branes, while SYM is the world volume theory on D3-branes, also in the low energy limit.

The 'derivation' of the correspondence can be made through the following discussion about the physical parameters on the two sides, compiling reviews in e.g. [42, 43, 44, 45, 46]. We start with a configuration of N parallel coincident D3-branes in 10-dimensional flat spacetime, with N large. There are open strings ending on the D3-branes and closed strings off the branes. The string length scale is l_s, related to the (closed) string coupling g_s as

$$2\sqrt{2}\pi^3 g_s l_s^4 = l_p^4 = G_{10}^{\frac{1}{2}}, \tag{3.10}$$

where l_p and G_{10} are the 10-dimensional Planck length and gravitational constant. For this system, one wishes to consider the quantity

$$4\pi g_s N \equiv \frac{l^4}{l_s^4}, \tag{3.11}$$

where l is a typical length scale which can be thought of as the Schwarzschild radius for the mass of the branes. Note then that

$$N = \frac{\pi^2 l^4}{\sqrt{2} l_p^4}. \tag{3.12}$$

By dialing up g_s, $g_s N$ can vary between two limits $g_s N \ll 1$ and $g_s N \gg 1$, at which the theory is tractable, in the following sense:

- $g_s N \ll 1 \quad \Leftrightarrow \quad l \ll l_s$. In this limit, the range of gravitational effect ($\sim l$) is much smaller than the string length and can be neglected. The D-branes are effectively in a flat spacetime background. This is also the regime for the string perturbation theory since the loop expansion parameter is $g_s N$.

- $g_s N \gg 1 \quad \Leftrightarrow \quad l \gg l_s$. The D-branes, carrying energy and charge, are strongly gravitating. The spacetime that results is an extremal black 3-brane solution with an asymptotically flat region connected in the interior to an infinitely long throat region, at the end of which is the horizon. Its metric is

$$\mathrm{d}s_{10}^2 = \left(1 + \frac{l^4}{r^4}\right)^{-\frac{1}{2}} \eta_{\mu\nu}^{(4)} \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu + \left(1 + \frac{l^4}{r^4}\right)^{\frac{1}{2}} (\mathrm{d}r^2 + r^2 \,\mathrm{d}\Omega_{S^5}^2). \tag{3.13}$$

  The planar horizon is at $r = 0$ and of radius l (= radius of $S^5 = r\sqrt{g_{rr}}$ as $r \to 0$).[2] It extends along the directions of the D3-branes (i.e. $\{x^\mu\}$). However, in this picture there are no D3-branes any more and we are just left with closed strings moving in the background (3.13).

We now take the *low energy* limits of both the above descriptions, resulting in

- $g_s N \ll 1$ at low energy. This tells us that we can only probe string excitations lower than the energy scale l_s^{-1}, which are massless open and closed strings. Equivalently, one can go to this limit by sending l_s itself to zero so that it

[2] By 'radius' of the horizon we mean its location in the radial coordinate $\rho^4 \equiv l^4 + r^4$ used to write down the general non-extremal black brane solution as in [42]. The analytical extension over $r = 0$ ($\rho = l$) is possible for the black 3-brane case.

is relatively small compared to all other length scales (or l_s^{-1} big compared to other energy scales) we are probing. Gravity appears weak at such long distances ($\gg l_s$), as can also be seen from the smallness of the gravitational constant $G_{10} \sim g_s^2 l_s^8$. The low energy limit is thus also the *decoupling limit* where the strings are free. Especially, the closed strings decouple from the open strings which turn out to be described by the $\mathcal{N} = 4$ $SU(N)$ SYM gauge theory on the D-brane world volume.

- $g_s N \gg 1$ at low energy. In the asymptotically flat region of the black brane geometry, the low energy limit is implemented in the same way as before, keeping only massless closed strings. In the throat region, any excitations will actually appear to have low energies due to the vanishing redshift factor $(1+l^4/r^4)^{-\frac{1}{4}} \to 0$ as $r \to 0$. Decoupling happens again between the two kinds of low energy excitations in the two regions. In particular, the gravitational potential prevent excitations near the horizon from climbing up the throat and reaching the asymptotic region. The near horizon geometry (the $r \ll l$ limit of (3.13)) is $AdS_5 \times S^5$:

$$\mathrm{d}s_{10}^2 = \frac{r^2}{l^2}\eta_{\mu\nu}^{(4)}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \frac{l^2}{r^2}\,\mathrm{d}r^2 + l^2\,\mathrm{d}\Omega_{S^5}^2 \tag{3.14}$$

$$= \frac{l^2}{z^2}(\mathrm{d}z^2 + \eta_{\mu\nu}^{(4)}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu) + l^2\,\mathrm{d}\Omega_{S^5}^2, \qquad z \equiv \frac{l^2}{r}, \tag{3.15}$$

  where in the second line one recognizes the Poincaré patch of AdS_5.

The result is illustrated in fig. 3.2. To establish the duality, one is led to identify the two low energy descriptions obtained, i.e., 4-d $\mathcal{N} = 4$ $SU(N)$ SYM theory (+ decoupled closed strings) and string theory on $AdS_5 \times S^5$ (+ decoupled closed strings), provided that one "pulls" them (in the parameter space) to the same regime of $g_s N$. Usually, this is done by taking the SYM theory to the $g_s N \gg 1$ regime where it is regarded as *strongly coupled* because its effective coupling is the 't Hooft coupling

$$\lambda \equiv g_{\mathrm{YM}}^2 N = 4\pi g_s N. \tag{3.16}$$

On the $AdS_5 \times S^5$ side, although $g_s N \gg 1$, the coupling g_s does not have to be large. In fact one assumes $g_s < 1$ for the gravitational description to be good (D-strings being heavy). This is often taken to the extreme case $N \to \infty$ while keeping $g_s N$

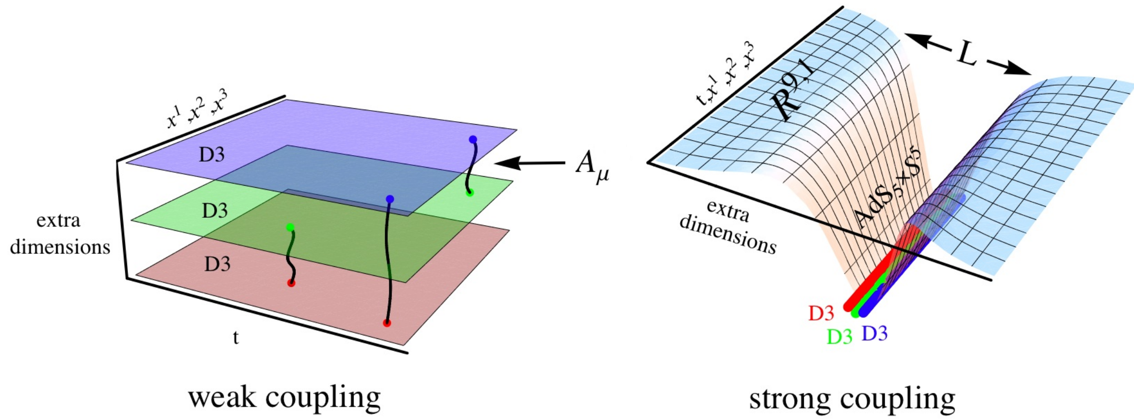


Figure 3.2: Two descriptions of D3-branes at weak ($\lambda \ll 1$) and strong ($\lambda \gg 1$) couplings. Figures from [47].

fixed. From (3.12), we then have

$$\frac{l}{l_p} = N^{\frac{1}{4}} \to \infty, \tag{3.17}$$

which means the quantum gravity effects can be neglected and the gravity theory is *weakly coupled*. (Still one has the full non-linear classical theory of general relativity.) On the field theory side, the $N \to \infty$ limit with fixed λ is the so called 't Hooft limit, which implies a large number of fields.

In summary, in this example of AdS/CFT one has a duality between strongly coupled gauge theory in 4-d and weakly coupled gravity theory on $AdS_5(\times S^5)$, as sketched in the following diagram:

$$\begin{array}{ccc} \text{D3-branes } (g_s N \ll 1) & \xrightarrow{g_s \uparrow} & \text{black 3-brane } (g_s N \gg 1) \\ \text{low energy (massless open strings)} \downarrow & & \downarrow \text{low energy (near horizon)} \\ SU(N) \text{ gauge theory} & \xrightarrow{g^2_{\text{YM}} \uparrow} & \text{AdS/CFT.} \end{array}$$

At last, it worth emphasizing that the duality is expected to hold at any coupling $\lambda = g_s N$ since both theories can be defined accordingly.

3.4 AdS/CFT is a UV/IR correspondence: matching degrees of freedom

Since it is advocated that the duality is an exact one-to-one correspondence, despite the different spacetime dimensions on the two sides, there arises the issue of a possible mismatch in the number of degrees of freedom. This can be resolved based on the following observations. The AdS side contains gravity and the formation of black holes is inevitable if the matter content is large enough. Since the black hole entropy scales as the area of the horizon, which is less than the area of the boundary, there is no problem for the boundary field theory to contain the same amount of entropy; indeed we are in the limit of a large number of boundary fields. The boundary area can serve as an upper bound (the Bekenstein bound) for the entropy in the bulk (containing gravity, with or without black holes).

Such considerations reveal an integrated feature of AdS/CFT, namely, it is a "UV/IR relation", as first pointed out in [48] in addressing the "information bound" problem. Consider the $AdS_5 \times S^5$ bulk geometry as in the last section. Since the surface area of the AdS boundary diverges, one replaces the boundary with a regulating surface at a cutoff radius $\hat{r} = 1 - \delta$ ($\delta \ll 1$), where $\hat{r}$ is the (dimensionless) radial coordinate in the following form the AdS_5 metric [49, 48]

$$\mathrm{d}s^2 = l^2\Big(\frac{1+\hat{r}^2}{1-\hat{r}^2}\Big)^2 \mathrm{d}\hat{t}^2 - l^2\Big(\frac{2}{1-\hat{r}^2}\Big)^2(\mathrm{d}\hat{r}^2 + \hat{r}^2\,\mathrm{d}\Omega_3^2), \tag{3.18}$$

as a product of a time axis and a unit 4 dimensional ball, $\hat{r} = 1$ being the boundary and $\hat{r} = 0$ the 'center' of the ball.[3] The cutoff surface (which is a 3-sphere in the coordinates of (3.18)) has area

$$A \propto l^3\delta^{-3}. \tag{3.19}$$

(The surface area of S^5 at the boundary contributes a constant factor.) This is an IR cutoff regulating large distance divergence from the AdS space perspective. From the CFT perspective, δ appears as a UV cutoff. Regarding the information bound, this can be seen from regulating the number of degrees of freedom of the SU(N) SYM theory for which

$$N_{\mathrm{dof}} \sim N^2\delta^{-3}, \tag{3.20}$$

[3]Near the boundary $\hat{r} = 1$, the metric (3.18) can be brought to the Poincaré form (3.8) upon identifying $z/l = 1 - \hat{r}$ and $t/l = \hat{t}$ [49].

if we think the 3-sphere is divided into cells of size $\sim \delta$ and each cell accommodates N^2 field degrees of freedom. The information density is then finite

$$N_{\rm dof}/A \sim N^2 l^{-3} \propto l^5 G_{10}^{-1} \propto G_5^{-1}, \tag{3.21}$$

using (3.10) and (3.12), where G_5^{-1} is the 5-d Newton constant.

3.5 Matching of symmetries

An important nontrivial test of the correspondence is that symmetries on the two side coincide. We start by reviewing basics of the symmetry transformations while introducing CFT, theories invariant under the conformal symmetry transformations.

We consider transformations of the "active" type, i.e., as mappings on both coordinates and fields [50]:

$$x \to x' = x'(x) \tag{3.22}$$

$$\Phi(x) \to \Phi'(x') = F[\Phi(x)], \tag{3.23}$$

where F is some function. Conformal transformations fall into the following types:

- translation

$$x'^\mu = x^\mu + a^\mu, \qquad \Phi'(x') = \Phi(x) \tag{3.24}$$

- rotation (Lorentz transformation)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \Phi'(x') = L_\Lambda \Phi(x) \tag{3.25}$$

- dilatation

$$x'^\mu = \lambda x^\mu, \qquad \Phi'(x') = \lambda^{-\Delta}\Phi(x) \tag{3.26}$$

- special conformal transformation (SCT)

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b\cdot x) + b^2 x^2} \qquad \Leftrightarrow \qquad \frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu. \tag{3.27}$$

Such transformations map the metric

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x)^2 g_{\mu\nu}(x), \tag{3.28}$$

	ϵ	$-G_x$	G_F
translation	a^μ	$P_\mu = -i\partial_\mu$	0
rotation	$\frac{1}{2}\omega_{\mu\nu}$	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$	$S_{\mu\nu}$
dilatation	α	$D = -ix^\mu\partial_\mu$	$-i\Delta$
SCT	b^μ	$K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$	$-2i\Delta x_\mu - x^\nu S_{\mu\nu} + \kappa_\mu$

Table 3.1: The infinitesimal parameters ϵ and the generators $G = G_F - F_x$ for the conformal transformations. Special symbols carrying spacetime indices have been used to denote G_x. Also, $S_{\mu\nu}$, Δ and κ_μ are values of G (or G_F) at $x = 0$.

for some function $\Omega(x)$. (Transformation rules for the connection, the Riemann and Ricci tensors are not surprisingly involve derivatives of $\Omega(x)$ [51].)

The infinitesimal changes of x and Φ induced by a small parameter ϵ (which could carry indices) are expressed as

$$\xi^\mu \equiv x'^\mu - x^\mu = \epsilon\frac{\delta x^\mu}{\delta\epsilon} \tag{3.29}$$

$$\delta\Phi \equiv \Phi'(x') - \Phi(x) = \epsilon\frac{\delta F}{\delta\epsilon}(x). \tag{3.30}$$

The *generator* G for an infinitesimal transformation is defined through the change of the functional form of Φ for the same coordinates:

$$\begin{aligned} -i\epsilon G\Phi(x') &\equiv \Phi'(x') - \Phi(x') = \delta\Phi - \xi^\mu\partial_\mu\Phi(x) \\ &= \epsilon\Big[\frac{\delta F(x)}{\delta\epsilon} - \frac{\delta x^\mu}{\delta\epsilon}\partial_\mu\Phi(x)\Big], \end{aligned} \tag{3.31}$$

which also equals to $-i\epsilon G\Phi(x)$ to the first order in ϵ. In this definition G includes two pieces, i.e. $G = G_F - G_x$ where G_F generates the change in the field via $-i\epsilon G_F\Phi = \delta\Phi$ and G_x is a "transport term" such that $-i\epsilon G_x = \xi^\mu\partial_\mu$. Explicit expressions of ϵ and G for conformal transformations are listed in table 3.1. For conformal transformations ξ^μ is called the "conformal Killing vector" and constrained to be at most quadratic in x, given as

$$\xi^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \alpha x^\mu + [2(b\cdot x)x^\mu - b^\mu x^2]. \tag{3.32}$$

The constant term corresponds to translation, the linear terms include rotation and dilatation, and the quadratic term corresponds to the special conformal transformation.

$\mathcal{N}=4$ $SU(N)$ SYM	$AdS_5\times S^5$
Conformal group: $SO(4,2)$	Isometry of AdS_5: $SO(4,2)$
R-symmetry: $SU(4)\sim SO(6)$	Isometry of S^5: $SO(6)$

Table 3.2: The symmetries of the two theories. The full isometry supergroup of the $AdS_5\times S^5$ background is isomorphic to the $\mathcal{N}=4$ superconformal symmetry.

For CFT$_d$, the generators G_x obey the following commutations relations

$$[J_{ab},J_{cd}]=i(\eta_{ad}J_{bc}+\eta_{bc}J_{ad}-\eta_{ac}J_{bd}-\eta_{bd}J_{ac}), \tag{3.33}$$

where

$$J_{\mu\nu}\equiv L_{\mu\nu},\qquad J_{-2,-1}\equiv D,\qquad J_{-2,\mu}\equiv\frac{1}{2}(P_\mu-K_\mu),\qquad J_{-1,\mu}\equiv\frac{1}{2}(P_\mu+K_\mu), \tag{3.34}$$

with $J_{ab}=-J_{ba}$. The indices $\mu,\nu=0,1,\ldots,d-1$ and $a,b=-2,-1,0,\ldots,d-1$. The metric $\eta_{ab}=\mathrm{diag}(-1,1,\eta_{\mu\nu})$. Depending on whether $\eta_{\mu\nu}$ is Euclidean or Lorentzian, the algebra (3.33) is that of $SO(d+1,1)$ or $SO(d,2)$. In the $d=4$ case, this matches perfectly with the isometry group of AdS_5. Another piece of the isometry, i.e. $O(6)$ of S^5, matches the R-symmetry of SYM theory. The symmetries are summarized in Table (3.2).

3.6 The dictionary

The observables in CFT are operators $\mathcal{O}_\Delta$ of scaling dimension Δ (same as in the generator of dilatation from the last section). They can be used to deform the field theory by adding to the action a term

$$\delta S=\int \mathrm{d}^d x\,\Phi_0\mathcal{O}_\Delta, \tag{3.35}$$

where Φ_0 stands for the source for $\mathcal{O}_\Delta$, a non-dynamical background field. We are interested in computing the correlation functions of $\mathcal{O}_\Delta$, usually via functional derivatives of the generating functional (the partition function) defined as

$$Z_{CFT}[\Phi_0]\equiv\Big\langle\exp\int \mathrm{d}^d x\,\Phi_0\mathcal{O}_\Delta\Big\rangle, \tag{3.36}$$

Φ	$\mathcal{O}_\Delta$
scalar ϕ	scalar operator
vector field A_μ	U(1) current J^μ
metric $g_{\mu\nu}$	energy-momentum $T^{\mu\nu}$

Table 3.3: Bulk fields Φ and field theory operators $\mathcal{O}_\Delta$.

so that the correlation function is

$$\langle \mathcal{O}_\Delta(x_1)\cdots\mathcal{O}_\Delta(x_n)\rangle = \frac{\delta}{\delta\Phi_0(x_1)}\cdots\frac{\delta}{\delta\Phi_0(x_n)} Z_{CFT}[\Phi_0]\Big|_{\phi_0=0}. \tag{3.37}$$

Under AdS/CFT, and moving to Euclidean signature, it is natural to conjecture a bulk dual of $Z_{CFT}[\Phi_0]$, which in fact is the partition function of the bulk theory

$$Z_{AdS}[\Phi_0] = \Big[\int \mathcal{D}\Phi\,\exp(-I_E[\Phi])\Big]_{\Phi|_{\partial_{AdS}}=\Phi_0}, \tag{3.38}$$

here represented by a functional integral over all possible bulk field configurations, subject to the boundary condition $\Phi|_{\partial_{AdS}} = \Phi_0$, where I_E is the bulk Euclidean action. Thus, the boundary values of bulk fields act as sources for CFT operators $\mathcal{O}_\Delta$. The precise statement of AdS/CFT is then the equality

$$Z_{CFT}[\Phi_0] = Z_{AdS}[\Phi_0] \tag{3.39}$$

between the two partition functions. In the classical (super)gravity limit, the functional integral in (3.38) can be replaced by a saddle point approximation. Namely, one only picks up the contributions that solve the classical Einstein's equation, so that

$$Z_{AdS}[\Phi_0] \approx \exp\big(-I_E[\Phi]\big)_{\Phi|_{\partial_{AdS}}=\Phi_0}, \tag{3.40}$$

where $I_E[\Phi]$ is evaluated on-shell. The duality then allows us to take an existing solution in classical general relativity and use AdS/CFT as a tool to obtain a dual field description, where the latter could be a familiar area of physics, e.g., condensed matter, fluid dynamics, QCD or quark-gluon plasma and there have been fruitful investigations in these areas using AdS/CFT.

More specifically, Φ may represent different fields, corresponding to different field operators. A few examples are given in Table (3.3). In practice, with a solution for

Φ one can examine its asymptotic falloff and extract information on the field theory side. We next illustrate this with concrete examples.

3.7 Scalar field example

We first consider a massive scalar field ϕ in Euclidean AdS with metric

$$\mathrm{d}s^2_{d+1} = \frac{l^2}{z^2}\big(\mathrm{d}z^2 + \sum_\mu \mathrm{d}x_\mu^2\big), \qquad (x_\mu = x^\mu), \tag{3.41}$$

which is just the Wick rotated Poincaré patch metric (3.8). The action is

$$I_E = \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}z \sqrt{g}\big[(\nabla\phi)^2 + m^2\phi^2\big], \tag{3.42}$$

and the equation of motion $(\Box - m^2)\phi = 0$ takes the form

$$\Big[z^{d+1}\frac{\partial}{\partial z}\Big(z^{-d+1}\frac{\partial}{\partial z}\Big) + z^2\frac{\partial}{\partial x^2} - m^2 l^2\Big]\phi(z,x) = 0. \tag{3.43}$$

Following [16] (also [36]), one recognizes the function

$$K_\Delta(z, x-x') = c_\Delta\Big[\frac{z}{z^2 + (x-x')^2}\Big]^\Delta, \qquad c_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})} \tag{3.44}$$

is a solution that vanishes on the boundary $z = 0$ except at $x = x'$. K_Δ is in fact a representation of δ function:

$$\lim_{z\to 0} K_\Delta = z^{d-\Delta}\delta(x - x'). \tag{3.45}$$

Thus the general solution can be written as

$$\phi(z,x) = \int \mathrm{d}^d x' K_\Delta(z, x-x')\phi_0(x') \tag{3.46}$$

with the asymptotic behaviour $\phi(z \to 0, x) \to z^{d-\Delta}\phi_0(x)$. K_Δ is called the bulk-to-boundary propagator.

A CFT operator $\mathcal{O}_\Delta$ of scaling dimension Δ transforms under $x \to \lambda x$ as

$$\mathcal{O}_\Delta(x) \to \mathcal{O}_\Delta(\lambda x) = \lambda^{-\Delta}\mathcal{O}_\Delta(x). \tag{3.47}$$

For the partition function (3.36) to be conformally invariant

$$\int \mathrm{d}^d x \phi_0(x) \mathcal{O}_\Delta(x) = \int \mathrm{d}^d(\lambda x) \phi_0(\lambda x) \mathcal{O}_\Delta(\lambda x) = \lambda^{d-\Delta} \int \mathrm{d}^d x \phi_0(\lambda x) \mathcal{O}_\Delta(x), \quad (3.48)$$

one needs to have

$$\phi_0(x) \to \phi_0(\lambda x) = \lambda^{\Delta - d} \phi_0(x) \quad (3.49)$$

under the same transformation, i.e. $\phi_0(x)$ has scaling dimension $d - \Delta$ [52].Thus the bulk field should behave near the boundary as

$$\phi(z, x) \sim z^{d-\Delta} \phi_0(x), \qquad z \to 0, \quad (3.50)$$

so that it is invariant under $(z, x) \to (\lambda z, \lambda x)$ which is an isometry of the AdS space [44]. This agrees with the solution found above. We are led to the AdS/CFT conjecture discussed earlier, that the (scaled) boundary value ϕ_0 of the bulk field acts as the source on the boundary for the field theory operator $\mathcal{O}_\Delta$.

In fact, the massive scalar solution has two modes and behaves near the boundary as [53]

$$\phi \to z^{d-\Delta_+} \phi_0(x) + z^{\Delta_+} A(x), \quad (3.51)$$

where

$$\Delta_+ = d/2 + \sqrt{d^2/4 + m^2 l^2} \quad (3.52)$$

is the larger root of

$$\Delta(\Delta - d) = m^2 l^2. \quad (3.53)$$

Note $\Delta_+ \geq d/2 \geq d - \Delta_+$. Usually, the finiteness of the action selects the mode $\phi \sim z^{\Delta_+}$, called the "normalizable mode", which falls off faster. For the slower falloff mode $\phi \sim z^{d-\Delta_+}$, called then "non-normalizable mode", the action has a diverging surface term from integration by parts. A refined statement of AdS/CFT is that the non-normalizable mode gives the source ϕ_0 for the boundary CFT operator $\mathcal{O}_{\Delta_+}$, while the expectation value of the operator can be read off as

$$\langle \mathcal{O}_{\Delta_+} \rangle = (2\Delta_+ - d) A \quad (3.54)$$

from the normalizable mode. In other words, AdS/CFT postulates the duality between a quantum fluctuating field propagating in the bulk (here the normalizable mode) and a boundary CFT whose operator is coupled to a classical non-fluctuating background field (here given by the non-normalizable mode).

A subtlety arises for

$$-d^2/4 \leq m^2 l^2 < -d^2/4 + 1. \tag{3.55}$$

(Negative mass squared is allowed without causing instability. The lower bound set by $m^2 \geq -d^2/4l^2$, for real Δ_+, is called the Breitenlohner-Freedman (BF) bound.) In this case, both modes are normalizable. The diverging surface term can be cancelled by adding a boundary term to the action. One is free to identify either ϕ_0 or A as the source, and then $(2\Delta - d)A$ or $(2\Delta - d)\phi_0$ as the expectation value, with $\Delta = \Delta_+$ or $d - \Delta_+$ respectively. Thus, for the mass range (3.55) we can have two CFTs with operators of different dimensions

$$\Delta = \Delta_+ \in [d/2, d/2 + 1) \qquad \text{or} \qquad \Delta = d - \Delta_+ \in (d/2 - 1, d/2]. \tag{3.56}$$

Δ is bounded below by $d/2 - 1$ (as opposed to $d/2$ in the single CFT case) which is precisely the unitarity bound for scalar operators. Naively, the Dirichlet prescription (3.39) only gives the $\mathcal{O}_{\Delta_+}$ theory, since ϕ_0 in (3.51) is the leading falloff. Nevertheless, the generating functionals of correlation functions in the two theories are in fact related by a Legendre transform, so that an analogue of (3.39) also holds for the $\mathcal{O}_{d-\Delta_+}$ theory. See [54] for further discussions and [55, 56] for the case of vector fields.

3.8 Introducing finite temperature and chemical potentials

We have been considered pure AdS geometry, which is a maximally symmetric spacetime, satisfying

$$R_{abcd} = \frac{R}{d(d+1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) = -l^{-2}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{3.57}$$

for AdS_{d+1}. One can break some of the symmetries by deforming the bulk geometry, but require that the full symmetry is recovered near the boundary, i.e., the spacetime is asymptotically AdS. One way of introducing the deformation is by putting a black hole in the bulk interior, breaking the dilatation symmetry. One could consider the simple case of a planar Schwarzschild-AdS solution

$$\mathrm{d}s_{d+1}^2 = \frac{l^2}{z^2}\Big[-f(z)\,\mathrm{d}t^2 + f(z)^{-1}\,\mathrm{d}z^2 + \mathrm{d}x^i\,\mathrm{d}x^i\Big], \tag{3.58}$$

different from (3.8) by the factor $f(z) = 1 - (z/z_H)^d$. On the boundary, $f(0) = 1$. In the interior, we have a horizon at $z = z_H$ and the solution is characterized by the black hole size z_H/l. Under the dilatation $\{z, t, x^i\} \to \lambda\{z, t, x^i\}$, we get a black hole of size $r_H/(\lambda l)$. The appearance of a horizon (a particular length scale) in the IR is equivalent to putting the boundary field in finite temperature, i.e. the Hawking temperature associated with the horizon, given by

$$T = \frac{d}{4\pi z_H}. \tag{3.59}$$

The above dilatation also maps the temperature from T to λT. Since the boundary field is scale invariant, all finite values of the temperature (the only scale in the theory) are equivalent.

The temperature also arises from considering the Euclidean metric, as the saddle point used to calculate the bulk partition function in the semiclassical limit. For Schwarzschild-AdS, one only needs to Wick rotate t to imaginary time $t \to -i\tau$ and use the resulting metric to evaluate the Euclidean action I_E. The (z, τ) plane of the bulk metric is like a disk and the absence of a conical singularity at $z = z_H$ (the origin of the disk in appropriate coordinates) requires τ to be periodic

$$\tau \sim \tau + \beta, \qquad \beta \equiv \frac{4\pi z_H}{d}. \tag{3.60}$$

The same identification on the boundary means that we are studying a field theory in a thermal ensemble at finite temperature $T = 1/\beta$, the same as the Hawking temperature of the bulk black hole:

$$\begin{aligned} \langle \mathcal{O} \rangle_T &= \int \mathcal{D}\Phi_0(x)\, \big\langle \Phi_0(x), t \,\big|\mathcal{O} e^{-\beta H}\big|\, \Phi_0(x), t \big\rangle \\ &= \int \mathcal{D}\Phi_0(x)\, \langle \Phi_0(x), t \,|\mathcal{O}|\, \Phi_0(x), t + i\beta \rangle\,, \end{aligned} \tag{3.61}$$

To compute the Euclidean action, one has to regulate the divergence due to the infinite volume of the spacetime. One way is to subtract the two actions computed for the Schwarzschild-AdS black hole and the empty AdS space (as the zero mass black hole limit). Explicitly [57],

$$I_E = -\frac{d}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{g}\Big[R + \frac{d(d-1)}{2l^2}\Big] = \frac{d}{8\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{g}. \tag{3.62}$$

For Schwarzschild-AdS, one integrates the radial direction from the horizon to some cutoff radius $r = R$ and the imaginary time from $\tau = 0 \sim \beta$. For the empty AdS, one starts from $r = 0$ but needs to rescale τ so that its period agrees with that of the black spacetime on the hypersurface $r = R$. Then send $R \to \infty$ and one finds [57]

$$I_E = \frac{\mathrm{Vol}(S^{d-1})(l^2 r_H^{d-1} - r_H^{d+1})}{4G_N[dr_H^2 + (d-2)l^2]}. \tag{3.63}$$

An alternative and more popular method to regulate the action is by adding to it a counterterm constructed solely from the intrinsic geometric invariants of the boundary. With the partition function $Z \approx e^{-I_E}$, one can obtain the free energy and entropy via $F = -T\ln Z = TI_E$ and $S = -\partial F/\partial T$, where κ is the gravitational constant.

Another deformation in the bulk is the addition of a Maxwell field, whose gauged $U(1)$ symmetry corresponds to a global $U(1)$ symmetry in the boundary field theory. The asymptotic values of bulk fields become the background fields on the boundary and introduce new scales. For example, the temporal component of the vector potential introduces a chemical potential $\mu = A_{(0)t}(x)$ on the boundary, where the subscript "(0)" means the limit $z \to 0$. For the full Einstein-Maxwell action [58]

$$I = \int \mathrm{d}^{d+1}y\sqrt{-g}\Big[\frac{1}{2\kappa^2}\Big(R + \frac{d(d-1)}{l^2}\Big) - \frac{1}{4\mathsf{g}^2}F^2\Big] \tag{3.64}$$

we have the Reissner-Norström-AdS solution given by the metric (3.58) but with

$$f(z) = 1 - (1 + z_H^2\mu^2\gamma^{-2})(z/z_H)^d + z_H^2\mu^2\gamma^{-2}(z/z_H)^{2(d-1)} \tag{3.65}$$

in terms of the chemical potential μ, where $\gamma^2 = 2\mathsf{g}^2 l^2/\kappa^2$ is dimensionless. The temperature is again found by periodically identifying the Euclidean time:

$$T = \frac{1}{4\pi z_H}[d - (d-2)z_H^2\mu^2\gamma^{-2}]. \tag{3.66}$$

The dilatation $\{z, t, x^i\} \to \lambda\{z, t, x^i\}$ maps the solution to one with temperature

λT and chemical potential $\lambda\mu$, leaving the ratio T/μ unchanged. So unlike for the Schwarzschild-AdS case, boundary field theories are equivalent only if they have the same value of T/μ, which can be continuously sent to zero.

The free energy is be obtained from the Euclideanized action (3.64) as

$$F = TI_E = -\frac{l^{d-1}}{2\kappa^2 z_H^d}(1 + z_H^2\mu^2\gamma^{-2})V_{d-1} = f(T/\mu)V_{d-1}T^d, \tag{3.67}$$

where the function $f(T/\mu)$ is found by expressing $z_H\mu$ in terms of T/μ using (3.66). We are in the grand canonical ensemble and the charge density in the field theory is given by

$$\rho = -\frac{1}{V_{d-1}}\frac{\partial F}{\partial \mu} = \frac{l^{d-1}\mu}{\kappa^2\gamma^2 z_H^{d-2}} = \frac{(d-2)l^{d-3}\mu}{(d-1)\mathsf{g}^2 z_H^{d-2}} \tag{3.68}$$

Using AdS/CFT, $\mu = A_{(0)t}(x)$ and $\rho = \langle j^t(x)\rangle$ can also be read off from the asymptotic behaviour of the vector potential A_t, as the slower and faster falloffs respectively, where j^μ is the boundary current. Note that the vector potential includes a constant shift so that it vanishes on the horizon:

$$A_\mu \, \mathrm{d}x^\mu = \mu[1 - (z/z_H)^{d-2}] \, \mathrm{d}t. \tag{3.69}$$

Expanding this expression in z one finds consistent results for μ and ρ (as in (3.68), up to a numeric factor).

A further example with a bulk magnetic field can be given using the dyonic Reissner-Norström-AdS_4 metric having the same form as (3.58) but with

$$f(z) = 1 - \left[1 + \gamma^{-2}z_H^2(\mu^2 + z_H^2B^2)\right](z/z_H)^3 + \gamma^{-2}z_H^2(\mu^2 + z_H^2B^2)(z/z_H)^4, \tag{3.70}$$

where the dimensionless quantity $\gamma^2 = 2\mathsf{g}^2l^2/\kappa^2$ with g the gauge coupling. The gauge potential is

$$A = \mu(1 - z/z_H)\,\mathrm{d}t + Bx_1\,\mathrm{d}x_2. \tag{3.71}$$

Again, the scale invariance is broken but the rotational invariance is preserved. The thermodynamic quantities are calculated to be

$$T = \frac{1}{4\pi z_H}(3 - \gamma^{-2}z_H^2\mu^2 - \gamma^{-2}z_H^4B^2) \tag{3.72}$$

$$G = -T\ln Z = -\frac{l^2}{2\kappa^2 z_H^3}, \tag{3.73}$$

where G indicates that we are in the grand canonical ensemble with fixed μ.

3.9 Fluid/gravity correspondence

Another direction of application of the AdS/CFT correspondence is the fluid/gravity correspondence. Mathematically, it reflects a nontrivial relation between Einstein's equations and the relativistic Navier-Stokes equation which describes small perturbations of the above thermal states.

In the black brane case considered above, the scale introduced by finite temperature breaks the dilatation symmetry and we have a one parameter family of black brane solutions, i.e. (3.58) with (3.59), labelled by the temperature. Another symmetry that one can make nontrivial is the Lorentzian symmetry which can be used to generate a d parameter family of the so called *boosted black brane* solutions, labelled by the temperature T and a four-velocity $u_\mu u^\mu = -1$ in the boundary directions. These parameters are the starting point to construct the stress tensor of a d-dimensional conformal fluid and eventually towards a fluid/gravity correspondence. What we get is a one-to-one mapping between the boundary fluid states and a dynamical bulk geometry with regular horizons. Fluid dynamics describes long wavelength (compared with the scale set by the temperature[4]) fluctuations about the global thermal equilibrium. This implies slowly varying $T(x)$ and $u^\mu(x)$, as functions of the boundary coordinates: [59]

$$\frac{\partial u^\mu}{T}, \ \frac{\partial \ln T}{T} \ll 1. \tag{3.74}$$

The fluid is in *local* thermal equilibrium. Then it is studied in the derivative expansion in the boundary directions and the bulk solution to the Einstein's equations is constructed order-by-order.

To present more details, start with the static black brane metric (3.58) rewritten in coordinates with $r = l^2/z$:

$$\mathrm{d}s^2 = \frac{l^2\,\mathrm{d}r^2}{r^2 f(r)} + \frac{r^2}{l^2}[-f(r)\,\mathrm{d}t^2 + l^2\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j], \qquad f(r) = 1-(r_H/r)^d = \left(\frac{4\pi l^2 T}{dr}\right)^d, \tag{3.75}$$

Then go to ingoing Eddington coordinates $\mathrm{d}v = \mathrm{d}t + \frac{l^2\,\mathrm{d}r}{r^2 f(r)}$ regular on the horizon, followed by a boost with constant velocity u^μ: $v \to u_\mu x^\mu,\ x_i \to P_{i\mu}x^\mu$, with $P_{\mu\nu} =$

[4] This is the only scale if one does not consider chemical potentials and curved boundary manifold.

$\eta_{\mu\nu} + u_\mu u_\nu$. We arrive at [60]

$$ds^2 = -2u_\mu \, dx^\mu \, dr + r^2[-f(r)u_\mu u_\nu + P_{\mu\nu}] \, dx^\mu \, dx^\nu, \tag{3.76}$$

We can use the metric (3.76) to compute the boundary stress tensor and show that it agrees with that of a d-dimensional conformal fluid, given by

$$T^{\mu\nu} = \alpha T^d(\eta^{\mu\nu} + du^\mu u^\nu), \tag{3.77}$$

where the dimensionless normalization constant α is proportional to the central charge of the CFT. Note that this stress tensor is not sourcing the boundary background metric; it is read off from the normalizable mode of the bulk metric as in the standard AdS/CFT prescription.

After promoting $\{T, u^\mu\}$ to slowly varying functions $\{T(x), u(x)^\mu\}$, one can perform a derivative expansion and solve the conservation equation of the fluid to a given order in $\epsilon \sim \partial_x$. The next-to-leading order terms in $T^{\mu\nu}$ describe dissipations, with coefficients of $\partial T, \partial u^\mu$ being the transport coefficients. The bulk Einstein equations are also solved to a given order, and this yields a regular fluctuating horizon. It can be compared to the membrane paradigm in which the fluid dynamical properties are assigned to a timelike stretched horizon just outside the true horizon.

One difference is that the fluid in the AdS/CFT framework lives on the boundary and captures the whole bulk dynamics. The membrane paradigm deals with a surface in the IR region of the bulk. Nevertheless, this reflects the UV/IR connection: the fluid on the boundary is in the long wavelength and low energy limit and should correspond to the physics near the horizon in the bulk. It has been shown [61] that at the level of linear response the transport coefficients of the boundary fluid are indeed fully captured by geometric quantities evaluated at the horizon.

We have been considering the Poincaré patch of the AdS geometry. Compared to the global coordinates, it has the merit that the boundary is the flat geometry R_d or $R_{1,d-1}$, rather than S^d or $R \times S^{d-1}$. However, in global coordinates the boundary CFT displays a deconfinement-confinement phase transition when considered at finite temperature. The corresponding phenomenon in the bulk is the Hawking-Page transition between AdS black holes and thermal AdS space. The large AdS black hole phase has a dual description in terms of fluid dynamics in the Einstein universe, which we discuss in the next section.

3.10 Kerr-AdS black holes and ideal fluid mechanics

In this section we show that large Kerr-AdS black holes are dual to a rotating fluid on the boundary [62]. Especially, the partition functions from both sides have the same dependence on the angular velocity, and diverge as the rotation at the boundary approaches the speed of light. For large horizon radii, the spacetime can be 'tubewise' approximated by boosted black brane solutions, extending from the horizon to the boundary, and the dual fluid description applies. For small horizon radii, the validity of the fluid description (i.e., long wavelength, slow variation limit) is not guaranteed [62].

We will present calculations of the partition functions on the boundary side using fluid mechanics and a model of a conformally coupled scalar field, and compare them with large black hole results.

3.10.1 Metric

A rotating black hole in the AdS background is given by the Kerr-AdS metric which reads in 4-d [18]

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\Sigma}\Big[\mathrm{d}t - \frac{a}{\Xi}\sin^2\theta\,\mathrm{d}\varphi\Big]^2 + \frac{\Sigma}{\Delta_r}\,\mathrm{d}r^2 + \frac{\Sigma}{\Delta_\theta}\,\mathrm{d}\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\Big[\frac{r^2+a^2}{\Xi}\,\mathrm{d}\varphi - a\,\mathrm{d}t\Big]^2, \tag{3.78}$$

where

$$\Sigma \equiv r^2 + a^2\cos^2\theta, \quad \Xi \equiv 1 - \frac{a^2}{l^2}, \quad \Delta_\theta \equiv 1 - \frac{a^2}{l^2}\cos^2\theta, \quad \Delta_r \equiv (r^2+a^2)\big(1+\frac{r^2}{l^2}\big) - 2mr. \tag{3.79}$$

Introducing the orthonormal 1-form tetrad

$$\omega^{(0)} = \sqrt{\frac{\Delta_r}{\Sigma}}\Big[\mathrm{d}t - \frac{a}{\Xi}\sin^2\theta\,\mathrm{d}\varphi\Big], \qquad \omega^{(1)} = \sqrt{\frac{\Sigma}{\Delta_r}}\,\mathrm{d}r \tag{3.80}$$

$$\omega^{(2)} = \sqrt{\frac{\Sigma}{\Delta_\theta}}\,\mathrm{d}\theta, \qquad \omega^{(3)} = \sqrt{\frac{\Delta_\theta}{\Sigma}}\sin\theta\Big[\frac{r^2+a^2}{\Xi}\,\mathrm{d}\varphi - a\,\mathrm{d}t\Big], \tag{3.81}$$

the metric is just

$$\mathrm{d}s^2 = -[\omega^{(0)}]^2 + [\omega^{(1)}]^2 + [\omega^{(2)}]^2 + [\omega^{(3)}]^2. \tag{3.82}$$

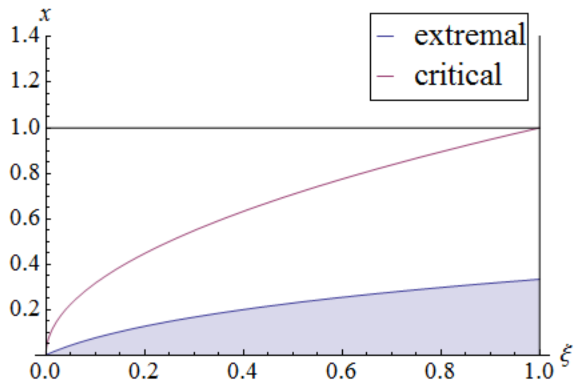


Figure 3.3: The plane of ($x = r_H^2/l^2$, $\xi = a^2/l^2$). The lower curve represents extreme black holes and the upper curve represents those rotating with the critical angular velocity (explained later). They are upper bounds on ξ for fixed x, or equivalently, lower bounds on x for fixed ξ. For $x > 1$, the critical angular velocity limit is represented by the line $\xi = 1$.

One has the constraint

$$\frac{a}{l} < 1 \tag{3.83}$$

in order to have a regular metric ($\Xi \neq 0$) and to preserve the signature ($\Delta_\theta > 0$). This is in addition to the cosmic censorship constraint which reads [63]

$$\frac{a^2}{l^2} \leq \frac{a_{\text{extr}}^2}{l^2} = \frac{(r_H/l)^2[1+3(r_H/l)^2]}{1-(r_H/l)^2}, \tag{3.84}$$

or equivalently $r_H > r_H^{\text{extr}}$, where r_H is the horizon radius and r_H^{extr} & a_{extr} are for extremal black holes. Measured in units of l, a black hole with parameters $\{r_H, a, l\}$ can be represented by a point in the $\{r_H/l, a/l\}$ plane, subject to the constraints (3.83) and (3.84). For large enough black holes with $r_H/l > 1/\sqrt{3}$, (3.83) is more stringent than (3.84) so we never reach extremality. Later in this section, we introduce a critical angular velocity bound a_c/l which changes dominance with 1 at $r_H/l = 1$. We sometimes use the shorthand notations $x \equiv r_H^2/l^2$, $\xi \equiv a^2/l^2$. Fig. (3.3) shows curves for $a/l = a_{\text{extr}}/l$, a_c/l, 1 on the x-ξ plane.

3.10.2 Comparing partitions from fluid and black hole sides

Under AdS/CFT, the thermodynamics of large Kerr-AdS black holes is shown [62] to be dual to that of a rotating conformal fluid on the boundary. The boundary is conformal to the Einstein universe $S^2 \times R$:

$$\mathrm{d}s^2 = -\,\mathrm{d}T^2 + \mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2, \tag{3.85}$$

where the coordinates are related to the bulk Boyer-Lindquist ones by

$$\cos\Theta = \sqrt{\frac{\Xi}{\Delta_\theta}}\cos\theta, \qquad \Phi = \varphi + al^{-2}t, \qquad T = tl^{-1} \tag{3.86}$$

for large radius. The boundary is taken to be the large constant-y hypersurface where $y = \sqrt{\frac{\Delta_\theta}{\Xi}}r$ for large radius. Following [62], to describe a stationary, rigidly rotating neutral fluid on the unit S^2 we need proper energy and entropy densities ρ and s, the pressure $\mathcal{P}$ and the temperature $\mathcal{T}$ measured in the local rest frame. The motion corresponds to the 3-velocity $u^\mu = \gamma[u^t = 1, u^\theta = 0, u^\varphi = \omega]$, where $\gamma = (1 - v^2)^{-1/2}$ and $v = \omega\sin\theta$. The stress-energy tensor is that of a perfect fluid, as a function $T^{\mu\nu} = T^{\mu\nu}(\rho, \mathcal{P}, \omega)$. In addition, $\mathcal{T} = \gamma\tau$ for constant τ. Thanks to conformal invariance, the thermodynamic potential $\Phi = \mathcal{E} - \mathcal{T}\mathcal{S}$ is of the specific form

$$\Phi = -\mathrm{Vol}(S^2)\mathcal{T}^3 h, \qquad h = \text{constant}, \tag{3.87}$$

from which one derives $\rho = 2\mathcal{P} = 2\mathcal{T}^3 h$ and $s = 3\mathcal{T}^2 h$, using the first law $\mathrm{d}\Phi = -\mathcal{S}\,\mathrm{d}\mathcal{T} - \mathcal{P}\,\mathrm{d}V$. Then the energy E_{fl}, angular momentum L_{fl} and entropy S_{fl} of the fluid, given by integration of $T^{\mu\nu}(\rho, \mathcal{P})$ and $J^\mu_S(\omega) = su^\mu$, are obtained as functions of (τ, h, ω), which in turn yield

$$T_{\text{fl}} = \frac{\partial E_{\text{fl}}}{\partial S_{\text{fl}}} = \tau, \qquad \Omega_{\text{fl}} = \frac{\partial E_{\text{fl}}}{\partial L_{\text{fl}}} = \omega. \tag{3.88}$$

Now the grand canonical partition function is evaluated using the above results as

$$\ln\mathcal{Z} = -\frac{E_{\text{fl}} - T_{\text{fl}}S_{\text{fl}} - \Omega L_{\text{fl}}}{T_{\text{fl}}} = \frac{\mathrm{Vol}(S^2)T_{\text{fl}}^2 h}{1 - \Omega_{\text{fl}}^2}. \tag{3.89}$$

On the gravity side, using the standard formulae for Kerr-AdS black hole thermodynamics [19], we find

$$\ln\mathcal{Z} = -\frac{E - TS - \Omega L}{T} = \frac{\pi(r_H^2 - l^2)(r_H^2 + a^2)^2}{\Xi[3r_H^4 + (l^2 + a^2)r_H^2 - a^2l^2]}, \tag{3.90}$$

Calculating the Euclidean action I_E relative to the zero-mass black hole background

[18] gives the same result: $I_E = -\ln \mathcal{Z}$. To leading order in the large black hole limit $r_H \gg l$ we have,

$$m = \frac{r_H^3}{2l^2}, \quad E = \frac{m}{\Xi^2}, \quad L = \frac{ma}{\Xi^2}, \quad S = \frac{\pi r_H^2}{\Xi}, \quad T = \frac{3r_H}{4\pi l^2}, \quad \Omega = \frac{a}{l^2}, \tag{3.91}$$

so that

$$\ln \mathcal{Z} \approx \frac{\pi r_H^2}{3\Xi} = \frac{16\pi^3 l^4 T^2}{27(1-\Omega^2 l^2)}, \tag{3.92}$$

which agrees with the fluid result (3.89) with the identification

$$\Omega = \Omega_{\rm fl}/l, \qquad T = T_{\rm fl}/l, \qquad h = \frac{4\pi^2 l^2}{27}. \tag{3.93}$$

The scaling between the two angular velocities is consistent with the scaling between the time coordinates in (3.86). Also, we are in the high temperature limit since $T \sim r_H$.

3.10.3 Critical angular velocity limit

The angular velocity entering the thermodynamics has a geometric meaning from the bulk side; it is defined as $\Omega = \Omega_H - \Omega_\infty$, the angular velocity at which the horizon is rotating relative to the boundary. This is also the angular velocity of the matter that rotates in the boundary Einstein universe [14]. The rotation is at the speed of light in the limit (cf. (5.12))

$$\Omega \to \Omega_c = 1/l, \qquad \Omega_{\rm fl} \to 1, \tag{3.94}$$

as can be found by requiring the corresponding 4-velocity be null at the boundary (and on the equator), and referred hereafter as the *critical angular velocity* limit. Below this limit, the horizon-co-rotating Killing vector is globally timelike, so that we can have thermal radiation co-rotating and in thermal equilibrium with the black hole all the way to infinity [18].

Note that

$$\Omega - \Omega_c = -\frac{(\frac{a}{l}-1)(\frac{a}{l}-\frac{r_H^2}{l^2})l}{\frac{r_H^2}{l^2}+\frac{a^2}{l^2}}, \tag{3.95}$$

so $\Omega \le \Omega_c$ is equivalent to

$$\frac{a}{l} \le \min\{\frac{r_H^2}{l^2}, 1\} = \begin{cases} \frac{r_H^2}{l^2} & \text{if } \frac{r_H}{l} \le 1 \\ 1 & \text{if } \frac{r_H}{l} > 1, \end{cases} \tag{3.96}$$

where the former case can be rephrased as 'globally timelike Killing vector exists for large black holes with $r_H^2 > al$', while the latter case already implies large black holes. In the context of the 'large black hole/fluid' correspondence discussed above, $\Omega = \Omega_c$ is at $a = l$ or $\Xi = 0$ and causes divergences in the extensive thermodynamic quantities E, S and L, as well as I_E of the black hole, since they are all proportional to inverse powers of Ξ. (It can also be shown that for the 4-velocity $u^\mu = [1, 0, 0, \Omega_c]$, the 4-acceleration $a_\mu a^\mu \sim \Xi^{-2}$.) Small black holes ($r_H/l \le 1$) are less interesting for which such divergences do not occur when $\Omega \to \Omega_c$.

To extract the divergent behavior at $\Omega_{\rm fl} \to \pm 1$ from the field theory side, [18] considered a conformally coupled scalar field ϕ in the 3-d Einstein universe (3.85), with the equation of motion

$$(\Box - \frac{1}{4})\phi = 0. \tag{3.97}$$

With the ansatz [64], $\phi \propto \exp(-i\omega T) Y_L^m(\Theta, \Phi)$, where $Y_L^m(\Theta, \Phi)$ are the spherical harmonics, the mode frequency ω and the angular momentum quantum number L are related by

$$\omega = L + \frac{1}{2}. \tag{3.98}$$

Then the partition function is given by

$$\ln \mathcal{Z} = -\sum_{L=0}^{\infty} \sum_{m=-L}^{L} \ln\big[1 - e^{-\beta(\omega - m\Omega_{\rm fl})}\big], \qquad \beta = \frac{1}{T}. \tag{3.99}$$

In the high temperature (small β) limit the summations can be approximated by integrals

$$\ln \mathcal{Z} \approx -\frac{1}{\beta^2} \int_0^\infty \mathrm{d}y \int_{-y}^{y} \mathrm{d}x \ln\big[1 - e^{-(y - \Omega_{\rm fl} x)}\big] = \frac{2\zeta(3)}{\beta^2(1 - \Omega_{\rm fl}^2)}. \tag{3.100}$$

[18] only sums over modes with $|x| = y$ which contribute most as $\Omega_{\rm fl} \to \pm 1$. The

result is

$$\ln \mathcal{Z} \approx -\frac{1}{\beta^2}\int_0^\infty \mathrm{d}y \sum_{x=\pm y} \ln\big[1-e^{-(y-\Omega_{\text{fl}}x)}\big] = \frac{\pi^2}{3\beta^2(1-\Omega_{\text{fl}}^2)}. \tag{3.101}$$

Thus, up to a numeric factor, the conformal scalar field calculation captures the same divergence at the critical angular velocity limit as the gravity result.

3.10.4 Kerr-Newman-AdS$_4$

We also discuss thermodynamics of charge rotating AdS black holes. We use the Kerr-Newman-AdS$_4$ metric in Boyer-Lindquist coordinates which is given by (3.78) or (3.82) with

$$\Delta_r = (r^2+a^2)\big(1+\frac{r^2}{l^2}\big) - 2mr + q_e^2 + q_m^2, \tag{3.102}$$

and q_e & q_m are electric and magnetic charge parameters. The Maxwell field is given by [65]

$$A = -\frac{q_e r}{\sqrt{\Delta_r \Sigma}}\omega^{(0)} - \frac{q_m \cos\theta}{\sqrt{\Sigma\Delta_\theta}\sin\theta}\omega^{(3)} \tag{3.103}$$

and

$$\begin{aligned} F = &-\Sigma^{-2}[q_e(r^2-a^2\cos^2\theta)+2q_m ra\cos\theta]\omega^{(0)}\wedge\omega^{(1)} \\ &- \Sigma^{-2}[q_m(r^2-a^2\cos^2\theta)-2q_e ra\cos\theta]\omega^{(2)}\wedge\omega^{(3)}. \end{aligned} \tag{3.104}$$

The black hole has mass M, angular momentum L, charges $\{Q_e, Q_m\}$ and entropy S given by

$$M = \frac{m}{\Xi^2}, \qquad L = \frac{ma}{\Xi^2} \qquad \text{(from Komar formula)} \tag{3.105}$$

$$Q_e = \frac{q_e}{\Xi}, \qquad Q_m = \frac{q_m}{\Xi} \qquad \text{(from total flux)} \tag{3.106}$$

$$S = \pi\frac{r_H^2+a^2}{\Xi} \qquad \text{(from horizon area)} \tag{3.107}$$

These serve as the extensive thermodynamic variables. For simplicity, we only consider $q_m = Q_m = 0$ and $q_e = q, Q_e = Q$.

The intensive variables can be found as follows. For the temperature, we go to the Euclidean sector by Wick rotating both the time coordinate and the rotation parameter: $t \to -i\tau$, $a \to i\alpha$. This ensures that the metric is real. Then the regularity at the horizon requires the identifications $\tau \sim \tau + \beta$, $\varphi \sim \varphi + i\beta\Omega_H$, with the inverse temperature

$$\beta = \frac{4\pi(r_H^2 + a^2)}{r_H\big[1 + a^2/l^2 + 3r_H^2/l^2 - (a^2+q^2)/r_H^2\big]}. \tag{3.108}$$

The angular velocity is that of the horizon relative to the non-rotating frame at infinity:

$$\Omega = \Omega_H - \Omega_\infty = a\frac{1 + r_H^2/l^2}{r_H^2 + a^2}. \tag{3.109}$$

Similarly, the chemical potential is also defined to be the difference between values measured at infinity and the horizon:

$$\mu = A_\mu K^\mu_{\Omega_H}\big|_{r\to\infty} - A_\mu K^\mu_{\Omega_H}\big|_{r=r_H} = q\frac{r_H}{r_H^2 + a^2}. \tag{3.110}$$

One can think of the metric parameters $\{r_H(\text{or } m), a, q\}$ as equivalent to either the set of extensive variables $\{S, L, Q\}$ or the intensive ones $\{T, \Omega, \mu\}$. (l is not treated as a thermodynamic variable.) Using the horizon condition $\Delta_r|_{r=r_H} = 0 \Rightarrow m = m(r_H, a, q_e)$, one can obtain the Smarr formula $M = M(S, L, Q)$:

$$M^2 = \frac{(S+\pi l^2)^2 S}{4\pi^3 l^4} + \frac{S+\pi l^2}{l^2 S}L^2 + \Big(\frac{S+\pi l^2}{2\pi l^2} + \frac{\pi Q^2}{4S}\Big)Q^2. \tag{3.111}$$

The conjugate intensive variables are computed as

$$T(S,L,Q) = \frac{\partial M}{\partial S} = \frac{1}{2M}\Big[\frac{(3S+\pi l^2)(S+\pi l^2)}{4\pi^3 l^4} - \frac{\pi L^2}{S^2} - \frac{\pi Q^4}{4S^2} + \frac{Q^2}{2\pi l^2}\Big] \tag{3.112}$$

$$\Omega(S,L,Q) = \frac{\partial M}{\partial L} = \frac{\pi L}{MS}\Big(1 + \frac{S}{\pi l^2}\Big) \tag{3.113}$$

$$\mu(S,L,Q) = \frac{\partial M}{\partial Q} = \frac{\pi Q}{2MS}\Big(Q^2 + \frac{S}{\pi} + \frac{S^2}{\pi^2 l^2}\Big). \tag{3.114}$$

One can check that they agree with those defined from the metric, i.e., (3.108), (3.109) and (3.110).

One can also start from the Gibbs potential

$$G(T, \Omega, \mu) = I_E/\beta, \tag{3.115}$$

with I_E the Euclidean action, and check that the extensive variables derived from

$$S = -\frac{\partial G}{\partial T}, \qquad J = -\frac{\partial G}{\partial \Omega}, \qquad Q = -\frac{\partial G}{\partial \mu} \tag{3.116}$$

agree with those defined from the metric, i.e., (3.105), (3.106) and (3.107). The Euclidean action used for the grand canonical ensemble is

$$I_E = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{g}(R + 6/l^2 - F^2) - \frac{1}{8\pi G}\int_{\partial} \mathrm{d}^3x\sqrt{h}K + I_{\mathrm{ct}}, \tag{3.117}$$

where I_{ct} is the counterterm built solely from the curvature invariants of the boundary to regulate the IR divergence. After the variation δI_E, the vanishing of the surface integral imposes $\delta A_\mu = 0$ so that we are in an ensemble with fixed chemical potential. The action evaluated on the KNAdS solution is

$$I_E(r_H, a, q) = \frac{\beta}{4G\Xi}\Big(-\frac{r_H^3}{l^2} + \Xi r_H + \frac{a^2 + q^2}{r_H} - 2\frac{q^2 r_H}{r_H^2 + a^2}\Big). \tag{3.118}$$

The evaluation of (3.116) is possible using (3.108), (3.109) and (3.110).

In summary, in this section we have discussed in detail aspects of the AdS/CFT correspondence for Kerr(-Newman)-AdS black holes. Many of these features will be revisited in chapter 5 where we derive and interpret our force-free solutions in Kerr-AdS.

Chapter 4

Monopole in AdS

As discussed in chapter 2, the force-free solutions in static spacetimes play important roles as the asymptotic configurations for solutions in rotating black hole backgrounds. In the present chapter we discuss solutions in flat and AdS spacetimes in more details. Novel features in the AdS case arise making the asymptotic matching less trivial.

4.1 Michel's rotating monopole solution in flat spacetime

We briefly review Michel's exact solution of a rotating monopole field in flat spacetime [22]. This solution was used by BZ as the asymptotic configuration for their rotating monopole in the Kerr background. This approach makes sense basically because the BL coordinates in which we write the Kerr metric will asymptote to usual spherical coordinates while the metric asymptotes to flat metric in these coordinates. A monopole in flat spacetime is given by the radial magnetic field and vector potential, as usual,

$$B^r = \frac{C}{r^2}, \qquad A_\varphi^{(0)} = -C\cos\theta. \tag{4.1}$$

For a monopole in the Kerr black hole background, when $a = 0$ (the spherically symmetric case), we can straightforwardly use the same expression as above, now regarding the coordinates as BL ones. When $a \neq 0$, we find for the same $A_\varphi^{(0)}$

$$B^r = \frac{C}{r^2 + a^2\cos^2\theta}, \tag{4.2}$$

which in the small a limit deviates from the non-rotating case at $\mathcal{O}(a^2)$: $B^r/C \approx r^{-2} - a^2\cos^2\theta$. Despite this deviation, we refer to (4.2) as the unperturbed monopole. In order to solve the force-free equation, BZ adds to $A_\varphi^{(0)}$ an $\mathcal{O}(a^2)$ term which falls off at least like $1/r$. At the end, we have a poloidal profile (B^r, B^θ) with $\mathcal{O}(a^2)$ deformations from that of $A_\varphi^{(0)}$. We will call it the perturbed monopole.

The rotation of the monopole fields in both the flat and Kerr backgrounds is characterized by a toroidal component $B_T \propto B^\varphi$ and an angular velocity ω, on which one performs the matching. Michel's solution imposes a relation by solving the conservation equation $T_{\mu\nu}^{;\nu} = 0$,

$$B_T(\theta) = -C\sin^2\theta\omega(\theta). \tag{4.3}$$

A second such relation is provided by the horizon regularity condition of BZ's solution. Thus B_T and ω are completely fixed by the matching. Luckily they are r-independent in both Michel's and BZ's solutions. It is also worth noting that the flat spacetime monopole can not accommodate an $\mathcal{O}(a^2)$ deviation in $A_\varphi^{(0)}$, which is no longer true in AdS space, as discussed in the next section.

The subtlety of the matching process in the AdS background lies in the fact that we have lost the natural mapping between BL coordinates for the Kerr-AdS metric and the standard global coordinates (y, Θ, Φ, T) for the AdS space. Especially, (y, Θ) differs from (r, θ) at $\mathcal{O}(a^2)$, which results in an $\mathcal{O}(a^2)$ difference even in $A_\varphi^{(0)}$ using (4.1), affecting ω.

Since Michel's solution is exact for arbitrary finite a, one may expect it to serve as the asymptotic configuration for some non-perturbative monopole-like solution of the black hole magnetosphere.

4.2 Rotating monopole(s) in AdS spacetime

We will proceed in this section to find an exact rotating monopole solution in AdS, an analogue of the Michel solution in flat space [22]. In practice, since the AdS boundary is only defined up to a Weyl scaling, the definition of the asymptotic monopole is ambiguous due to possible $\mathcal{O}(a^2)$ corrections associated with squashing consistent with axisymmetry. In addition, there is also the possibility to add further r-dependent $\mathcal{O}(a^2)$ corrections.

4.2.1 The unperturbed monopole

The standard AdS metric in global coordinates $\{y, U = \cos\Theta, \Phi, T\}$ reads

$$ds^2 = \left(1 + \frac{y^2}{l^2}\right)^{-1} dy^2 + \frac{y^2}{1 - U^2} dU^2 + y^2(1 - U^2) d\Phi^2 - \left(1 + \frac{y^2}{l^2}\right) dT^2. \tag{4.4}$$

It is related to the large-r (or zero-mass) Kerr-AdS metric in BL coordinates by non-trivial transformations [18] which take the form

$$y = r\sqrt{\frac{\Delta_\theta}{\Xi}}, \quad U = u\sqrt{\frac{\Xi}{\Delta_\theta}}, \quad \Phi = \varphi + \frac{a}{l^2}t, \quad T = t, \tag{4.5}$$

at large r (or y). When trying to define a monopole in AdS, a subtlety arises that, due to the non-trivial $U \leftrightarrow u$ transformation, what we called a monopole (i.e., $-Cu$) in BL coordinates is not quite the same object as $-CU$. When there is no rotation the two 'monopoles' are the same ($U = u$ for $a = 0$). When the black hole is spun up, an observer at the asymptotic AdS region may adopt one of the following two reference frames:

1. The asymptotic observer sees a rotating field given simply by $-CU$ as if the monopole is spun up in a fixed pure AdS background with the standard metric in (y, U) coordinates (apart from a constant shift in Φ by frame-dragging). This observer does not know that an interior observer would have switched to (r, u) coordinates by insisting on the horizon being a constant-r surface (in BL coordinates).

2. The asymptotic observer does account for the change in their local geometry caused by the black hole rotation and agrees with the interior observer who describes the monopole as $-Cu$. The asymptotic observer would then use new coordinates (y', U') to recognize the standard AdS geometry at the boundary, in accordance with the change in the shape of the horizon in the bulk.

The coordinate grids of (y, U) & (r, u) systems are sketched in Fig. 4.1 showing their relative deformation so that the monopole naturally defined in one system will appear to have non-uniformly distributed radial field lines as seen in another.

Denoting quantities in AdS metric (4.4) with a bar, the solution for $\bar{A}_\Phi(U) = -CU$ is exact, given by

$$\bar{B}_T(U)^2 = C^2(1 - U^2)^2 \bar{\omega}(U)^2, \tag{4.6}$$

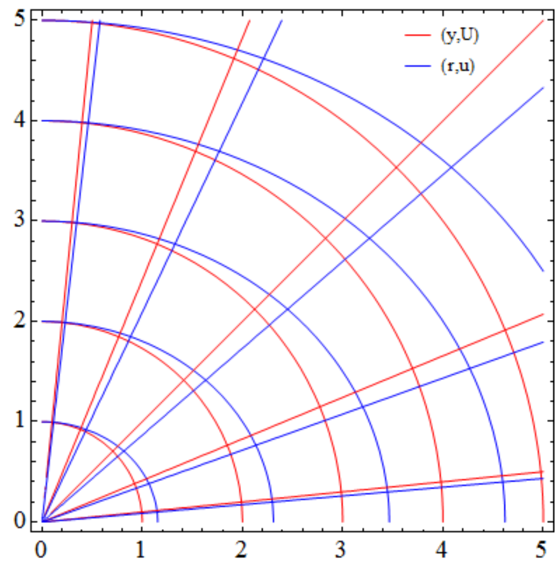


Figure 4.1: Poloidal coordinates for AdS are shown, indicating the asymptotic squashing of the 2-sphere in BL coordinates, with the vertical axis being the rotation axis. Red curves/lines are constant (y, U) grids and blue curves/lines are constant (r, u) grids using the same set of constants, e.g., $U = u = \cos\frac{\pi}{4}$ are shown together. The blue (BL) grid deforms away from those of a unit 2-sphere (times the radial direction).

where we have fixed an integration constant so that $\bar{B}_T(U = 1) = 0$. For

$$\bar{A}_\Phi(U) = -Cu(U) \approx -CU - CU(1 - U^2)\frac{a^2}{2l^2} + \mathcal{O}(a^4), \tag{4.7}$$

which is a monopole with an r-independent perturbation, we find in the small a limit

$$\bar{B}_T(U)^2 = C^2(1 - U^2)^2[\bar{\omega}(U)^2 - 3a^2/(2l^4)]. \tag{4.8}$$

4.2.2 perturbed monopole

We consider the ansatz with an r-dependent $\mathcal{O}(a^2)$ correction to the monopole, given by

$$\bar{A}_\Phi(y, U) = -CU + a^2 U(1 - U^2)\bar{f}(y), \quad \bar{B}_T(U) \equiv a(1 - U^2)\bar{B}^c_T. \tag{4.9}$$

The equation for $\bar{f}(y)$ for small a reads,[1]

$$f''(y) + \frac{2y}{y^2+l^2} f'(y) - \frac{6l^2}{y^2(y^2+l^2)} f(y) + 2Cl^4 \frac{(\bar{B}_T^c/C)^2 - \bar{\omega}^2}{(y^2+l^2)^2} = 0, \tag{4.10}$$

which has an analytic solution

$$\begin{aligned} f(y) = -\frac{\bar{c}_2}{4\pi y^2} \Big[& 2\pi\big(\frac{y^2}{l^2}+3\big) \arctan^2 \frac{y}{l} \\ & - \big[(\pi^2+12)\big(\frac{y^2}{l^2}+3\big) + 12\pi \frac{y}{l}\big] \arctan \frac{y}{l} + 3\frac{y}{l}(2\pi\frac{y}{l} + \pi^2 + 12)\Big], \end{aligned} \tag{4.11}$$

where

$$\bar{c}_2 = -Cl^4\big[(\bar{B}_T^c/C)^2 - \bar{\omega}^2\big] \tag{4.12}$$

is an overall free parameter which turns out to be the coefficient of the $\mathcal{O}(y^{-2})$ term in the expansion,

$$f(y) = \bar{c}_2\Big[\frac{\pi^2-12}{\pi l} y^{-1} + y^{-2} + \mathcal{O}(y^{-3})\Big]. \tag{4.13}$$

$\bar{c}_2$ renders ω arbitrary. In particular, one could have $B_T = 0$ corresponding to some configuration without poloidal currents. The $\mathcal{O}(a^2)$ perturbation is only significant around $y \sim l$.

In conclusion, we have found that for the rotating monopole in AdS, the relation between $\bar{\omega}$ and $\bar{B}_T$ is not unique, and especially becomes arbitrary when including a radial $\mathcal{O}(a^2)$ perturbation. In the next chapter wee will find that matching the black hole solution onto the above cases will result in different values of ω.

[1] It is worth noting that the equation takes a simpler form $f''(x) - 6f(x)/\sin^2 x + 2Cl^2[(\bar{B}_T^c/C)^2 - \bar{\omega}^2] = 0$ if we define the new radial coordinate $x \equiv \arctan(y/l)$.

Chapter 5

BZ in Kerr-AdS background

5.1 Overview

In this chapter we present in detail the solution of the force-free equations for the analog BZ monopole ansatz in the Kerr-AdS background. We work with BL coordinates (with KS results given in Appendix A) and the small a limit (a being the rotation parameter in the metric). Series and numerical solutions are shown for general sizes of black holes and an analytic solution is found for small black holes. We then use the AdS/CFT dictionary to interpret our results in terms of the dual field theory.

5.2 Kerr-AdS and the slow rotation limit

In this section we do some preliminary setup, discussing properties of the Kerr-AdS metric and clarifying notions of 'slow rotation' and black hole size.

5.2.1 Kerr-AdS solution

The Kerr-AdS metric in BL coordinates is explicitly given by (3.78) which we repeat here:

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\Sigma}\left[\mathrm{d}t - \frac{a}{\Xi}\sin^2\theta\,\mathrm{d}\varphi\right]^2 + \frac{\Sigma}{\Delta_r}\,\mathrm{d}r^2 + \frac{\Sigma}{\Delta_\theta}\,\mathrm{d}\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\Sigma}\left[\frac{r^2+a^2}{\Xi}\,\mathrm{d}\varphi - a\,\mathrm{d}t\right]^2, \tag{5.1}$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2} \tag{5.2}$$

$$\Delta_r = (r^2 + a^2)\big(1 + \frac{r^2}{l^2}\big) - 2mr, \qquad \Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta. \tag{5.3}$$

To make connection with the 3+1 form of the metric (2.17), we note the following relations

$$h_{rr} = \frac{\Sigma}{\Delta_r}, \qquad h_{\theta\theta} = \frac{\Sigma}{\Delta_\theta}, \qquad h_{\varphi\varphi} = \frac{\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2 \sin^2\theta}{\Xi^2\Sigma}\sin^2\theta \tag{5.4}$$

$$\alpha^2 = \frac{\Sigma\Delta_r\Delta_\theta}{\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta}, \qquad \beta^i = \left[0, 0, -a\Xi\frac{\Delta_\theta(r^2+a^2) - \Delta_r}{\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta}\right]. \tag{5.5}$$

We use the same shorthand notation as in section 3.10 for the dimensionless ratios

$$\xi \equiv \frac{a^2}{l^2} \leq 1, \qquad x \equiv \frac{r_H^2}{l^2}, \tag{5.6}$$

where the horizon radius r_H is the largest root of $\Delta_r = 0$, which can be written in the form

$$\Delta_r(r_H) = 0 \quad \Leftrightarrow \quad \frac{l}{2m}(x+\xi)(x+1) = \sqrt{x}. \tag{5.7}$$

This relation is useful for analyzing the various limiting cases as discussed below. The angular velocity $\beta^\varphi = -\Omega_B$ in (5.5) varies from the horizon,

$$\Omega_H = -\beta^\varphi|_{r_H} = \frac{a\Xi}{r_H^2 + a^2}, \tag{5.8}$$

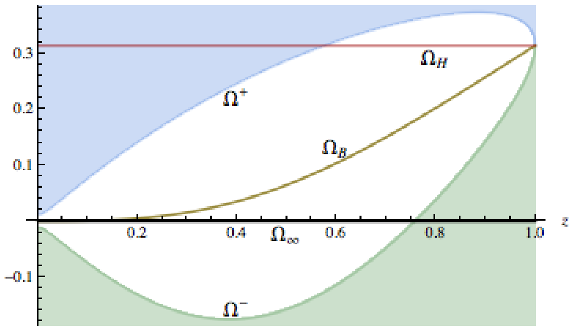
to the boundary,

$$\Omega_\infty = -\beta^\varphi|_\infty = -\frac{a}{l^2}, \tag{5.9}$$

and this feature of BL coordinates has to be taken into account in considering the reference frame of the holographic dual theory. For the discussion of slow rotation below, it is also convenient to define the critical mass parameter [66],

$$m_{\text{ext}}(a) = l\frac{\left[\left((1+\xi)^2 + 12\xi\right)^{\frac{1}{2}} + 2(1+\xi)\right]\left[\left((1+\xi)^2 + 12\xi\right)^{\frac{1}{2}} - (1+\xi)\right]^{\frac{1}{2}}}{3\sqrt{6}}, \tag{5.10}$$

and the constraint $m \geq m_{\text{ext}}$ ensures the absence of naked singularities, with $m = m_{\text{ext}}$



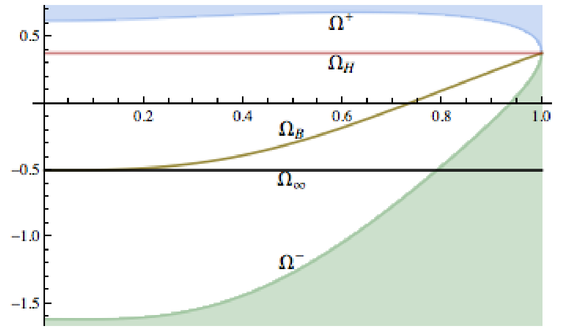


Figure 5.1: The condition $\Omega^- \leq \Omega' \leq \Omega^+$ for $K^\mu_{\Omega'}$ to be non-space-like is shown for small (left) and large (right) Kerr-AdS black holes, where $z \equiv \frac{r_H}{r} = 1$ and 0 are the horizon and boundary at infinity. The ergosphere for $K^\mu_{\Omega'}$ is where the horizontal line $\Omega' =$ const. is within the shaded regions. The lines show specific values of angular velocity, $\Omega' = \Omega_H$ and $\Omega' = \Omega_\infty$, plus the axis $\Omega' = 0$. $\Omega_B(z = 1) = \Omega_H$ and $\Omega_B(z = 0) = \Omega_\infty = -\frac{a}{l^2}$.
Left: $\{m = 1, a = 0.9, l = 100, \cos\theta = 0.5\}$ for a faster rotating and smaller black hole, close to the Kerr limit. Ω_∞ and the axis are indistinguishable. There is no globally time-like Killing vector K^μ_Ω.
Right: $\{m = 1, a = 0.5, l = 1, \cos\theta = 0.5\}$ for a slower rotating and larger black hole. $K^\mu_{\Omega_H}$ is now globally time-like.

corresponding to extremal black holes for which Δ_r has a double root at the degenerate horizon.

The Killing vectors $\xi^\mu_{(t)}$ and $\xi^\mu_{(\varphi)}$ can be used to define mass and angular momentum (e.g., through Komar formulae in asymptotic flat spacetimes and the conformal definition [19] in asymptotic AdS spacetimes), but there is an ambiguity in choosing the asymptotic timelike Killing vector and this implies that the definition of energy is not unique. This is significant for the analysis of energy extraction, and we discuss the range of asymptotically timelike Killing vectors in more detail below. The general condition for a Killing vector $K^\mu_\Omega \equiv \xi^\mu_{(t)} + \Omega' \xi^\mu_{(\varphi)}$ (cf. (2.20)) to be non-spacelike requires $\Omega^- \leq \Omega' \leq \Omega^+$ where

$$\Omega^\pm = \Omega_B \pm \sqrt{-\frac{K^\mu_{\Omega_B} K_{\Omega_B \mu}}{h_{\varphi\varphi}}} \overset{\text{BL}}{=\!=} \Omega_B \pm \frac{\alpha}{\sqrt{h_{\varphi\varphi}}}. \tag{5.11}$$

These bounding contours are shown for two cases in fig. 5.1.

The conformal boundary of Kerr-AdS spacetime is an Einstein universe rotating with angular velocity $\Omega = \Omega_H - \Omega_\infty$, where $\Omega_\infty = -a/l^2$ is the angular velocity of the non-rotating frame at infinity. A feature of the Kerr-AdS geometry is that, as seen from the plot, the Killing vector $K^\mu_{\Omega_H}$ is globally timelike, and the Einstein universe rotates slower than the speed of light, provided

$$\Omega_H - \Omega_\infty < \frac{1}{l} \quad \Leftrightarrow \quad r_H^2 > al, \tag{5.12}$$

i.e., for large black holes [18, 65, 67]. The critical angular velocity for the Einstein universe to rotate at the speed of light corresponds to $\Omega_H = \Omega^+(r \to \infty, \theta = \pi/2) = \Omega_\infty + 1/l$. Usually the ergosphere for $K^\mu_{\Omega'}$ starts at $\Omega' = \Omega^-$ and extends to the horizon [24]. The existence of an ergosphere is of course essential for any energy extraction mechanism from the black hole. However, the plot reveals that there is no unique time-like Killing vector $K^\mu_{\Omega'}$ at infinity, hence the ambiguity in defining energy. As argued by [14] and shown explicitly for the BZ process in Section 5.3.4, with energy defined with the globally time-like $K^\mu_{\Omega_H}$ there is no energy extraction and the black hole is perturbatively stable. On the other hand, consideration of the thermodynamics of the dual field theory [19] suggests that $K^\mu_{\Omega_\infty}$ is the appropriate choice of Killing vector to use in defining energy; namely the unique choice that yields the first law $\mathrm{d}E = T\,\mathrm{d}S + (\Omega_H - \Omega_\infty)\,\mathrm{d}L$ with the r.h.s. an exact differential. Here we are adopting the definitions of energy and angular momentum as conserved charges associated with Killing vectors, denoted $E = Q[K_{\Omega'}]$ and $L = -Q[\xi_{(\varphi)}]$ [19], with

$$Q[\xi_{(t)}] = \frac{m}{\Xi}, \qquad L = \frac{ma}{\Xi^2} \tag{5.13}$$

for Kerr-AdS.[1] The ambiguity in the definition of energy motivates a more detailed investigation of the BZ process and force-free magnetospheres even for large Kerr-AdS black holes. We will turn to this topic in the next section, after describing some useful features of the slow rotation limit.

5.2.2 Slow rotation

The Kerr-AdS solution is characterized by three parameters $\{m, a, l\}$ or equivalently $\{r_H, a, l\}$. Slow rotation generically implies a regime far from extremality, set by

[1] Note that, without spoiling the exactness of the r.h.s. of the first law, one can in principal choose a different $K^\mu_{\Omega'}$ with $\Delta\Omega = \Omega' - \Omega_\infty$ independent of $\{L, S\}$, so that $\mathrm{d}(E - \Delta\Omega L) = T\,\mathrm{d}S + (\Omega_H - \Omega_\infty - \Delta\Omega)\,\mathrm{d}L$. In terms of the independent variables $\{L, S\}$, we have, $r_H^2 = \frac{S}{\pi}\left[\frac{4L^2}{(S/\pi)^2(S/(l^2\pi)+1)} + 1\right]^{-1}$ and $a^2 = \frac{S}{\pi}\left[\frac{(S/\pi)^2(S/(l^2\pi)+1)^2}{4L^2} + \frac{S}{l^2\pi} + 1\right]^{-1}$.

$m \gg m_{\rm ext}$ (see (5.10)) or alternatively $r_H \geqslant r_H^{\rm ext}$, where

$$r_H^{\rm ext} = l\left[\frac{-(1+\xi) + \left((1+\xi)^2 + 12\xi\right)^{1/2}}{6}\right]^{1/2}. \tag{5.14}$$

$m_{\rm ext}$ & $r_H^{\rm ext}$ are the extremal limits of m & r_H (see e.g. [63]). For Kerr ($l \to \infty, \xi \to 0$), $m_{\rm ext} = r_H^{\rm ext} = a$ and the condition $\frac{a}{m} \ll 1$ used in the perturbative solution guarantees $m \gg m_{\rm ext}$. For Kerr-AdS, $m \gg m_{\rm ext}$ implies,

$$\frac{a}{m} \ll \frac{a}{m_{\rm ext}} \sim \mathcal{O}(1) \tag{5.15}$$

where the latter condition holds for $\xi \in [0,1]$, and thus $\frac{a}{m} \ll 1$ is still a good criterion for "far from extremality".

The AdS length scale l enables us to talk about black hole sizes in terms of $x \equiv \frac{r_H^2}{l^2}$. To gain some intuition, we plot r_H, l and $\frac{r_H}{l}$ for various $\frac{a}{m}$ in fig. 5.2, which shows that large black holes ($x > 1$) are only possible for $\frac{a}{m} < \frac{1}{2}$. Indeed, rearranging $\Delta_r(r_H) = 0$ formally into a quadratic equation for r_H with fixed x: $r_H^2 - \frac{2m}{1+x} r_H + a^2 = 0$, the condition that r_H be real is $\frac{a}{m} < \frac{1}{1+x^2}$. As discussed above, large black holes satisfying $r_H^2 > al \Leftrightarrow x > \sqrt{\xi}$ are generically stable [14], so combining the two inequalities we have,

$$\frac{a}{m} < \frac{1}{1+x^2} < \frac{1}{1+\xi}, \tag{5.16}$$

with $\frac{1}{1+x^2} \in [0,1]$ and $\frac{1}{1+\xi} \in [\frac{1}{2}, 1]$, for $l \in (0,\infty)$ keeping $\xi \leq 1$. It is then clear that large black holes ($\frac{1}{1+x^2} < \frac{1}{2}$) imply slow rotation and ensure stability, while fast rotation ($\frac{a}{m} > \frac{1}{2}$) allows for an instability (violating the second inequality by insisting on the first one).

In addition to the slow rotation condition $\frac{a}{m} \ll 1$, our small 'a' expansion also treats $\frac{a}{l} \ll 1$ and includes the following three regimes categorized according to the relative scale between m and l (and also that between r_H and l):

1. small black holes

$$a \ll m \sim \frac{r_H}{2} \ll l, \tag{5.17}$$

2. intermediate black holes

$$a \ll m \sim r_H \sim l, \tag{5.18}$$

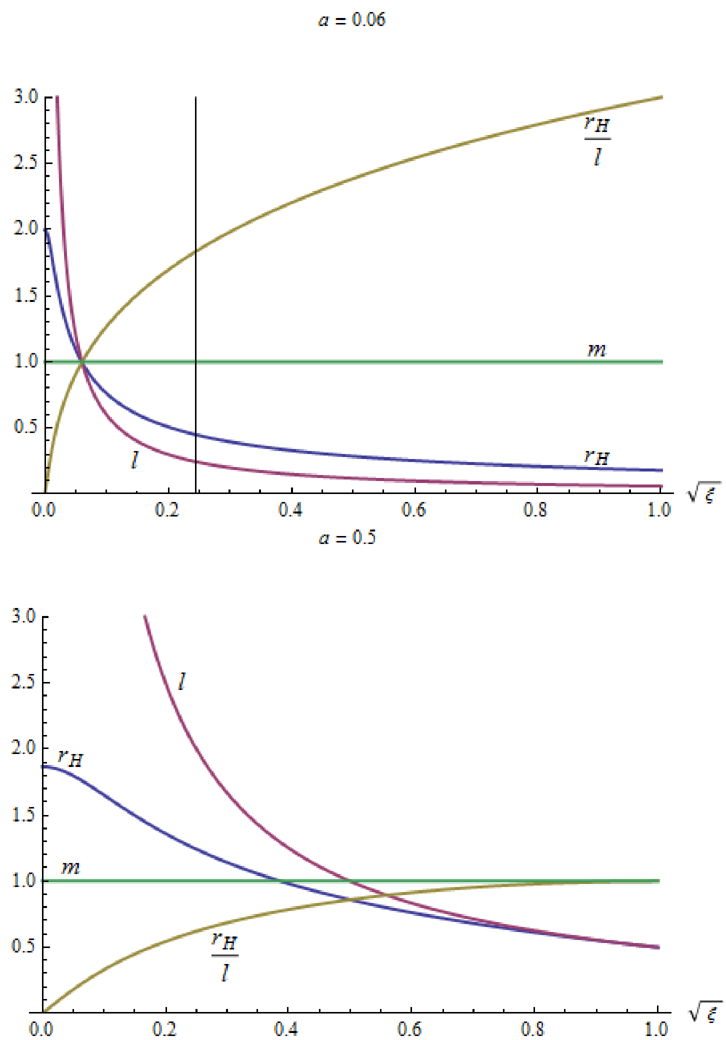


Figure 5.2: r_H, l and $\frac{r_H}{l}$ as functions of ξ, in units $m = 1$. Real solutions for r_H do not exist for all ξ when $\frac{a}{m} \to 1$, as expected. Our small 'a' expansion is valid for small values of ξ up to the vertical line in the first graph.

3. large black holes

$$a \ll l \ll r_H \ll m. \tag{5.19}$$

Note that the criterion $r_H^2 > al$ for globally time-like $K^\mu_{\Omega_H}$ is met in regimes 2 and 3, but may or may not be met in regime 1. In the first graph of fig. 5.2, regime 1 corresponds to the leftmost region of the graph, regime 2 is around the transition point from small to large black holes where the curves meet and regime 3 is further to the right up to the vertical line. To be more precise, in terms of the small parameter $\frac{a}{m} \equiv \epsilon$, the transition point is at $\sqrt{\xi} \sim \epsilon$ and the vertical line bounding regime 3 would be e.g. at $\sqrt{\xi} \sim \epsilon^{\frac{1}{2}}$. The regime to the right of the vertical line is where $\frac{a}{l} \sim \mathcal{O}(1)$ and is not covered by our small 'a' expansion. It describes very large black hole with sizes diverging as, e.g., $\frac{r_H}{l} \sim \epsilon^{-\frac{1}{3}}$ at $\sqrt{\xi} = 1$.

5.2.3 Small 'a' expansion

To implement the small a limit, it is useful to define

$$\Delta_0 \equiv \Delta_r(a=0) = \frac{r}{l^2}(r^3 + l^2 r - 2ml^2) = \frac{r}{l^2}(r - r_1)(r^2 + r_1 r + r_1^2 + l^2), \qquad (5.20)$$

where the second relation identifies the (only real) root $r = r_1$, namely the Schwarzschild-AdS horizon radius which satisfies

$$r_1^3 + l^2(r_1 - 2m) = 0. \qquad (5.21)$$

One can check that $r_1 = r_H(\xi = 0) > r_H(\xi \neq 0)$, e.g. by considering the form of $\Delta_r(r_H) = 0$ as given in (5.7). Of most relevance here, one can show that in the small a expansion

$$r_1 - r_H \sim \mathcal{O}(a^2), \qquad (5.22)$$

so the regularity condition for quantities diverging like $\Delta_r^{-n} \sim (r - r_H)^{-n}$ on the horizon can be translated to that at $r = r_1$ at each order of the a expansion, i.e. $(r - r_H)^{-1} = (r - r_1)^{-1} + (r - r_1)^{-2}\mathcal{O}(a^2)$, so we will still refer to the latter as the "horizon regularity condition". To leading order in a, the horizon angular velocity approximates to

$$\Omega_H = \frac{a}{r_H^2} + \mathcal{O}(a^3) = \frac{a}{r_1^2} + \mathcal{O}(a^3). \qquad (5.23)$$

We will also find it useful to treat all quantities as dimensionless by working in $m = 1$ units, and $\Delta_0(m = 1) = 0$ provides the relation

$$l^2 = \frac{r_1^3}{2 - r_1}, \quad (0 \leq r_1 \leq 2), \qquad (5.24)$$

which allows us to eliminate l in each order of the small a expansion, leaving r_1 as the only free metric parameter. With these conventions, r_1 encodes both the AdS curvature length and black hole sizes: $r_1 = 2$ ($l \to \infty$, $\frac{r_1}{l} = 0$) is the Kerr and small black hole limit, while $r_1 \to 0$ ($l \to 0 \sim \mathcal{O}(a)$, $\frac{r_1}{l} \to \infty$) is the highly curved AdS and large black hole limit.

5.3 The AdS analogue of the Blandford-Znajek split monopole

We turn now to the main task of obtaining an explicit solution for a force-free magnetosphere in a Kerr-AdS background. We will work in the probe approximation, ignoring the back-reaction of the magnetosphere on the geometry.[2] As discussed in chapter 2, the force-free magnetosphere is described by the independent field quantities $\{A_\varphi, B_T, \omega\}$, which are all implicit functions of (r, θ). Moreover, as shown by Blandford and Znajek [5], the function $A_\varphi =$ constant specifies poloidal field surfaces, and thus $B_T = B_T(A_\varphi)$ and $\omega = \omega(A_\varphi)$.

To obtain an explicit solution below, following BZ we will work in the small 'a' expansion outlined above, starting from an initial radial magnetic field in the Schwarzschild limit. The physical situation assumes that the magnetic field is produced by currents in an accretion disk. A split monopole field is a crude approximation to this with opposite charge in the north and south hemispheres, allowing for a discontinuity on the equator, associated with the accretion disk. In a more realistic situation, the accretion disk would produce poloidal magnetic field threading the equator. For simplicity, in the discussion below, we will not explicitly split the monopole across the equator. As it turns out, the sign of the magnetic charge does not affect the direction of the energy flux. We also ignore all dissipative effects in the accretion disk and concentrate on solving the magnetosphere configurations of the (non-split) monopole under small 'a' perturbation.

5.3.1 General form of the equations in the 3+1 formalism

Rather than solve the force-free equations $F_{\mu\nu}J^\nu = 0$ directly, following [6], we will consider the conservation equations $T^{\mu\nu}_{;\nu} = 0$. Before invoking the explicit Kerr-AdS metric, we use the general metric (2.17) to derive some important results in concise forms. Defining the shorthand notations,

$$dT^\mu \equiv T^{\mu\nu}{}_{;\nu}, \tag{5.25}$$

$$\{X, Y\} \equiv X_{,r}Y_{,\theta} - X_{,\theta}Y_{,r}, \tag{5.26}$$

[2] Relaxing this condition may change the asymptotic form of the geometry, as discussed recently in [67].

the definition of ω (cf. (2.42)) implies

$$\{A_\varphi, \omega\} \equiv 0, \tag{5.27}$$

which means that ω is solely a function of A_φ as noted above. For T^μ_t & T^μ_φ that are relevant for the energy and angular momentum fluxes we have

$$T^r_t = -\omega B_T \frac{A_{\varphi,\theta}}{\sqrt{-g}}, \qquad T^\theta_t = \omega B_T \frac{A_{\varphi,r}}{\sqrt{-g}}, \tag{5.28}$$

$$T^\varphi_t = \omega\Big[\frac{\beta^\varphi(\omega+\beta^\varphi)}{\alpha^2} - \frac{1}{h_{\varphi\varphi}}\Big] h^{MN} A_{\varphi,M} A_{\varphi,N}, \tag{5.29}$$

$$2T^t_t = \Big[\frac{(\beta^\varphi)^2 - \omega^2}{\alpha^2} - \frac{1}{h_{\varphi\varphi}}\Big] h^{MN} A_{\varphi,M} A_{\varphi,N} - \frac{B_T^2}{h_{\varphi\varphi}\alpha^2}, \tag{5.30}$$

and

$$T^r_\varphi = -\frac{T^r_t}{\omega}, \qquad T^\theta_\varphi = -\frac{T^\theta_t}{\omega}, \tag{5.31}$$

$$T^\varphi_\varphi = -T^t_t - \frac{B_T^2}{h_{\varphi\varphi}\alpha^2}, \tag{5.32}$$

$$T^t_\varphi = \frac{\beta^\varphi + \omega}{\alpha^2} h^{MN} A_{\varphi,M} A_{\varphi,N}. \tag{5.33}$$

For the conservation equation $dT_\mu = 0$ we have[3]

$$dT_r = \frac{A_{\varphi,r}}{\sqrt{-g}}\big(c_\omega\sqrt{-g}h^{MN}A_{\varphi,N}\big)_{,M} + \frac{(h^{MN}A_{\varphi,M}A_{\varphi,N})h_{\varphi\varphi}(\beta^\varphi+\omega)\omega_{,r} + \frac{1}{2}(B_T^2)_{,r}}{h_{\varphi\varphi}\alpha^2}, \tag{5.34}$$

$$dT_\theta = \frac{A_{\varphi,\theta}}{\sqrt{-g}}\big(c_\omega\sqrt{-g}h^{MN}A_{\varphi,N}\big)_{,M} + \frac{(h^{MN}A_{\varphi,M}A_{\varphi,N})h_{\varphi\varphi}(\beta^\varphi+\omega)\omega_{,\theta} + \frac{1}{2}(B_T^2)_{,\theta}}{h_{\varphi\varphi}\alpha^2}, \tag{5.35}$$

$$dT_\varphi = -\{A_\varphi, B_T\}/\sqrt{-g}, \tag{5.36}$$

$$dT_t = \{A_\varphi, B_T\omega\}/\sqrt{-g}, \tag{5.37}$$

together with the following relations

$$dT_t + \omega dT_\varphi = \frac{1}{\sqrt{-g}} B_T\{A_\varphi, \omega\} \overset{(5.27)}{=\!=} 0, \tag{5.38}$$

[3] We find that the covariant components are more suitable for presenting subsequent results.

$$B^i dT_i = -\frac{1}{\sqrt{-g}\alpha^2} h^{MN} A_{\varphi,M} A_{\varphi,N} \{A_\varphi, \omega\}(\beta^\varphi + \omega) \overset{(5.27)}{=\!=} 0, \tag{5.39}$$

where the indices $M, N = r, \theta$; $i, j = r, \theta, \varphi$ and

$$c_\omega \equiv \frac{1}{h_{\varphi\varphi}} - \frac{(\beta^\varphi + \omega)^2}{\alpha^2}, \tag{5.40}$$

as in (2.49). We can write the second order derivative terms in dT_r & dT_θ more concisely using a 4-d d'Alembertian:

$$\bar{\Box} A_\varphi = \frac{1}{\sqrt{-\bar{g}}}(\sqrt{-\bar{g}}\bar{g}^{\mu\nu} A_{\varphi,\nu})_{,\mu} = \frac{1}{c_\omega^2\sqrt{-g}}(c_\omega\sqrt{-g} g^{\mu\nu} A_{\varphi,\nu})_{,\mu} = \frac{1}{c_\omega^2\sqrt{-g}}\left(c_\omega\sqrt{-g} h^{MN} A_{\varphi,N}\right)_{,M}, \tag{5.41}$$

owing to the conditions $\partial_t(\ldots) = \partial_\varphi(\ldots) = 0$, where $\bar{\Box}$ is associated with the metric $\bar{g}_{\mu\nu}$ obtained by Weyl transforming $g_{\mu\nu}$,

$$\bar{g}_{\mu\nu} = c_\omega g_{\mu\nu}, \quad \bar{g}^{\mu\nu} = c_\omega^{-1} g^{\mu\nu}, \quad \sqrt{-\bar{g}} = c_\omega^2\sqrt{-g}. \tag{5.42}$$

The equations (5.38) & (5.39) are components of the identity,

$$^\star F^{\mu\nu} dT_\nu (= {}^\star F^{\mu\nu} F_{\rho\nu} J^\rho = \frac{1}{4}\, {}^\star F^{\alpha\beta} F_{\alpha\beta} J^\mu) = 0, \tag{5.43}$$

and constrain the number of independent equations in $dT_\mu = 0$ from four to two. The fact that a single condition (5.27) gives two constraints follows from the following general argument. Namely, the existence of non-trivial solutions (i.e., $J^\mu \neq 0$) to $F_{\nu\mu} J^\nu (= dT_\mu) = 0$ implies $\det F_{\mu\nu} = 0$ which is equivalent to the degeneracy condition (and to (5.27)) by the identity $\det F_{\mu\nu} = (F_{\mu\nu}{}^\star F^{\mu\nu})^2/16$ and thus the matrix $F_{\mu\nu}$ cannot have full rank 4. $F_{\mu\nu}$ thus has rank 2 since antisymmetric matrices can only have even ranks. We choose one of the independent equations to be $dT_\varphi = 0$ or $dT_t = 0$ which just gives the condition (cf. (5.36), (5.37) and (5.27))

$$-\sqrt{-g}\, dT_\varphi = \{A_\varphi, B_T\} = 0. \tag{5.44}$$

Then (5.39) determines the remaining equation to be either $dT_r = (5.34) = 0$ or

$dT_\theta = (5.35) = 0$ which we focus on from now on.[4] In the next subsection, we will make use of the slow rotation expansion to obtain a solution.

5.3.2 Solving equations in the small 'a' expansion

Starting from a simple monopole solution in the Schwarzschild-AdS limit, we employ the following ansatz expanding field quantities about $a = 0$ (or more precisely an expansion in a/m), keeping terms up to $\mathcal{O}(a^2)$,

$$A_\varphi = -Cu + a^2 A_\varphi^{(2)}, \tag{5.45}$$

$$\omega = a\omega^{(1)}, \tag{5.46}$$

$$B_T = aB_T^{(1)}, \qquad (B^\varphi = aB^\varphi_{(1)}), \tag{5.47}$$

where C is proportional to the magnetic charge (if we don't 'split' the monopole), and $u = \cos\theta$. Applying this ansatz to the conditions $\{A_\varphi, \omega\} = 0$ and $\{A_\varphi, B_T\} = 0$ yields

$$aC\omega^{(1)}_{,r} + a^3\{A_\varphi^{(2)}, \omega^{(1)}\} = 0, \tag{5.48}$$

$$aC(B_T^{(1)})_{,r} + a^3\{A_\varphi^{(2)}, B_T^{(1)}\} = 0. \tag{5.49}$$

Consistently dropping the $\mathcal{O}(a^3)$ terms,[5] we arrive at the constraints

$$\omega_{,r} = (B_T)_{,r} = 0. \tag{5.50}$$

Counting powers of 'a' in dT_r and dT_u we find

$$dT_r = \underbrace{A_{\varphi,r}}_{\mathcal{O}(a^2)}\, c_\omega^2\, \underbrace{\bar{\Box} A_\varphi}_{\mathcal{O}(a^2)} + \Big[\underbrace{\mathcal{O}(1)\,(\beta^\varphi + \omega)\,\overbrace{\omega_{,r}}^{=0}}_{\mathcal{O}(a^2)} + \underbrace{\mathcal{O}(1)\,\overbrace{(B_T^2)_{,r}}^{=0}/2}_{\mathcal{O}(a^2)}\Big] \quad \sim \quad \mathcal{O}(a^4) \tag{5.51}$$

$$dT_u = \underbrace{A_{\varphi,u}}_{\mathcal{O}(1)}\, c_\omega^2\, \underbrace{\bar{\Box} A_\varphi}_{\mathcal{O}(a^2)} + \Big[\underbrace{\mathcal{O}(1)\,(\beta^\varphi + \omega)\omega_{,u} + \mathcal{O}(1)(B_T^2)_{,u}/2}_{\mathcal{O}(a^2)}\Big] \quad \sim \quad \mathcal{O}(a^2) \tag{5.52}$$

---

[4] Equations (5.34) and (5.35) should correspond to the *force-free Grad-Shafranov equation* as presented in, e.g., [8], if we interchange Ψ, ϖ^2, $\delta\Omega$ and I there with A_φ, $g_{\varphi\varphi}$, $\beta^\varphi + \omega$ and B_T respectively.

[5] Although a appears as an overall factor here, the subleading terms do indeed contribute at $\mathcal{O}(a^3)$ to dT_φ and at $\mathcal{O}(a^4)$ to dT_t and should be dropped.

(where it is important to note that $\bar{\Box} A_\varphi$ has a vanishing $\mathcal{O}(1)$ term which is specific to the monopole field). Thus $dT_r = 0$ is automatically solved to the desired order. The non-trivial equation $dT_u = 0$ is a second order partial differential equation (PDE) for $A^{(2)}_\varphi$ and reads explicitly in the Kerr-AdS metric,[6]

$$\begin{aligned} dT_u = &A^{(2)}_{\varphi,rr} + \frac{(1-u^2)}{\Delta_0} A^{(2)}_{\varphi,uu} + 2\frac{r^3+ml^2}{l^2\Delta_0} A^{(2)}_{\varphi,r} \\ &+ \frac{1}{2C\Delta_0^2}\Big[C^2(1-u^2)^2\big[r^4(\omega^{(1)})^2 + (\frac{r^2}{l^2} - \frac{2m}{r})(2r^2\omega^{(1)}-1)\big] - r^4(B^{(1)}_T)^2\Big]_{,u}, \end{aligned} \tag{5.53}$$

where Δ_0 is given in (5.20).

To solve (5.53), we employ a separation of variables $A^{(2)}_\varphi = f(r)g(u)$. For terms in the inhomogeneous part with different powers of r to have the same u-dependence, namely $u(1-u^2)$, and for $\omega^{(1)}(u)$ & $B^\varphi_{(1)}(u) \sim B^{(1)}_T(u)/(1-u^2)$ to be regular at $u=1$, we require

$$\omega^{(1)}(u) = \omega^{(1)}, \tag{5.54}$$

$$B^{(1)}_T(u) = B^c_T(1-u^2), \tag{5.55}$$

where $\omega^{(1)}$ and B^c_T are constants. Meanwhile, we can fix

$$g(u) = u(1-u^2). \tag{5.56}$$

The remaining radial equation is (using (5.21) to replace m with r_1)

$$\begin{aligned} &f''(r) + \frac{2r^3 + r_1(r_1^2+l^2)}{r(r-r_1)(r^2+r_1 r+r_1^2+l^2)} f'(r) - \frac{6l^2 f(r)}{r(r-r_1)(r^2+r_1 r+r_1^2+l^2)} \\ &\quad + 2Cl^2\frac{\{[C^{-2}(B^c_T)^2 - (\omega^{(1)})^2]l^2 - 2\omega^{(1)}\}r^5 + r^3 + (2\omega^{(1)}r^2-1)r_1(r_1^2+l^2)}{r^3(r-r_1)^2\big(r^2+r_1 r+r_1^2+l^2\big)^2} = 0, \end{aligned} \tag{5.57}$$

with prime denoting the radial derivative. At this point, we can eliminate one of the

[6]It is interesting to note that eq. (5.53) does not involve r-derivatives of $\omega^{(1)}$ and $B^{(1)}_T$ even if we do not impose $\omega^{(1)}_{,r} = B^{(1)}_{T,r} = 0$. In addition, the inhomogeneous part is a total u-derivative, which is not obvious from the original form (5.35) of the equation.

factors $(r - r_1)^{-2}$ in the inhomogeneous part by choosing

$$(B_T^c)^2 = C^2\Big(\omega^{(1)} - \frac{1}{r_1^2}\Big)^2. \tag{5.58}$$

Though this may appear ad hoc, it is actually equivalent to BZ's horizon regularity condition [5] (a more direct way is to impose the regularity of the invariant (2.47)). The same relation can be obtained more rigorously from the regularity of $\tilde{B}^\varphi$ in KS coordinates, as presented in Appendix A, where the sign ambiguity of B_T^c in (5.58) is also fixed, given by (A.0.13) & (A.0.19):

$$B_T^c = -\big(\omega^{(1)} - \frac{\Omega_H}{a}\big)A_{\varphi,u} + \mathcal{O}(a^3) = C\Big(\omega^{(1)} - \frac{1}{r_1^2}\Big) + \mathcal{O}(a^3). \tag{5.59}$$

We proceed with B_T^c fixed via (5.58). Transforming to a dimensionless radial coordinate

$$z \equiv \frac{r_1}{r}, \tag{5.60}$$

and using $m = 1$ units to eliminate l, we arrive at

$$\begin{aligned} f''(z) + \frac{2z(3z - r_1)}{(z-1)[2z^2 - (r_1 - 2)(z+1)]}f'(z) + \frac{6r_1 f(z)}{(z-1)[2z^2 - (r_1 - 2)(z+1)]} \\ - C\frac{4(z^2 - 2\omega^{(1)}r_1^2)(z^2 + z + 1) + 2r_1(z+1)}{(z-1)[2z^2 - (r_1 - 2)(z+1)]^2 r_1} = 0. \end{aligned} \tag{5.61}$$

The horizon is at $z = 1$ and spatial infinity at $z = 0$.

A comparison with the Kerr case is in order. In the Kerr limit ($r_1 = 2$), equation (5.61) develops a second singular point $z = 0$ (besides $z = 1$) near which it behaves like

$$\begin{aligned} f''(z) + [2z^{-1} + \mathcal{O}(1)]f'(z) - [6z^{-2} + \mathcal{O}(z^{-1})]f(z) \\ - \frac{C}{2}z^{-4}(8\omega^{(1)} - 1)(1 + 2z + 3z^2) + \mathcal{O}(z^{-1}) = 0. \end{aligned} \tag{5.62}$$

The leading $\mathcal{O}(z^{-4})$ divergence of the inhomogeneous part can only be removed by choosing $\omega^{(1)} = 1/8$, a fixed rotation frequency which is half the horizon angular velocity. For the general Kerr-AdS case ($r_1 \neq 2$), the equation is well-behaved at $z = 0$ and no obvious constraint on $\omega^{(1)}$ is needed. This is the first, and perhaps most significant, difference we observe in the properties of the force-free magnetosphere in the Kerr-AdS background.

5.3.3 Series and numerical solutions

For the Kerr geometry, the indicial equation implies integer asymptotics $f(z) \sim z^0, z^1$ and due to the singular point there is logarithmic scaling. Requiring the boundary condition $f(0) = 0$, the expansion for the homogeneous equation has the form

$$f(z)_{\text{Kerr}} = \sum_{n=1}^{\infty} \big[c_n + c'_n \ln(z)\big] z^n, \qquad (c'_1 = 0). \tag{5.63}$$

It turns out that the inhomogeneous term is only consistent with this regular scaling at infinity with the unique choice of $\omega^{(1)} = 1/(8m^2)$ noted above. An analytic solution for $f(z)$ can then be obtained in terms of dilogarithms. Blandford and Znajek used a matching condition at infinity that we will return to later to obtain the same result [5].

For the general Kerr-AdS case, although an analytic solution does not appear possible - barring a perturbative expansion about the Kerr limit discussed in section 5.4 - we can proceed in the same way since the indicial equation has the same form. The lack of an additional singular point in this case implies the existence of a regular series solution about $z = 0$, and we again fix the boundary condition $f(0) = 0$ corresponding to the normalizable mode in AdS,

$$f_c(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{5.64}$$

In this case, the inhomogeneous term is nonsingular away from the Kerr limit, and thus we do not obtain a unique constraint on the field angular velocity ω. Substituting (5.64) into (5.61) and expanding in z, we obtain recursion relations expressing $\omega^{(1)}$ and c_n $(n \geq 3)$ in terms of c_1 and c_2:

$$\omega^{(1)} = \frac{1}{4r_1} + \frac{c_2}{C}\frac{(r_1 - 2)^2}{4r_1}, \tag{5.65}$$

$$c_3 = \frac{2c_1 r_1}{3(2 - r_1)}, \tag{5.66}$$

$$c_n = c_n(c_1, c_2), \quad (n > 3). \tag{5.67}$$

Note that $\omega^{(1)}$ depends only on c_2 (concavity of $f(z)$ at $z = 0$) and c_3 only on c_1.

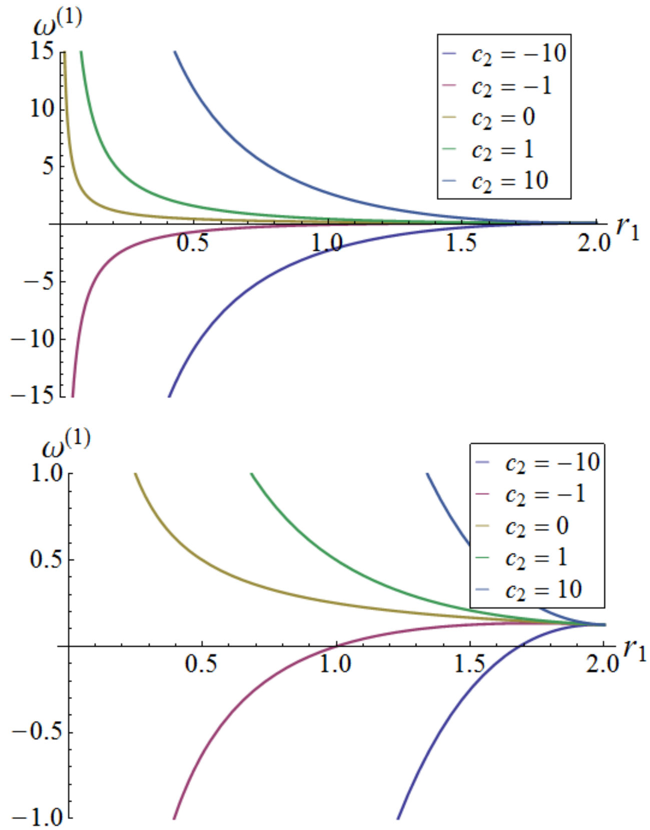


Figure 5.3: Plots of $\omega^{(1)}$ as functions of r_1 for various values of c_2, based on (5.65). The second graph shows the region close to the r_1-axis.

We will therefore use $\omega^{(1)}$ and c_2 interchangeably below, and the explicit relation is shown for various parameters in Fig. 5.3.

For comparison with the numerical solution to be discussed below, it is also useful to consider a second series solution constructed about the singular point $z = 1$, i.e., the horizon, where we explicitly demand the absence of the logarithmic term:

$$f_b(z) = \sum_{n=0}^{\infty} b_n (z-1)^n, \tag{5.68}$$

with

$$\omega^{(1)} = \frac{b_0}{C}\frac{r_1 - 3}{2} - \frac{b_1}{C}\frac{(r_1-3)^2}{6r_1} + \frac{r_1+3}{6r_1^2}, \tag{5.69}$$

$$b_n = b_n(b_0, b_1), \quad n \geq 2. \tag{5.70}$$

The free parameters $\{\omega^{(1)}, c_1\}$ in $f_c(z)$ and $\{\omega^{(1)}, b_0\}$ in $f_b(z)$ are related through two boundary conditions $f_b(0) = 0$ & $f_b'(0) = c_1$ imposed on the series $f_b(z)$ whose radius of convergence covers $z = 0$.[7] This leaves only one free parameter which we take to be $\omega^{(1)}$ (or equivalently c_2), with

$$c_1(r_1, c_2) \;\text{ or }\; b_0(r_1, c_2) = p(r_1) - q(r_1)c_2 \tag{5.71}$$

where $p(r_1)$ & $q(r_1)$ are ratios of polynomials in r_1. Matching the expansion e.g. at $z = 0$ reproduces the numerical results discussed below to high precision.

The equation for $f(z)$ can be solved numerically by shooting from the boundary to the horizon for each value of $\omega^{(1)}$ (or equivalently c_2), by tuning the value of c_1 (or b_0) until we get a regular solution near $z = 1$. This fixes the final integration constant and, as noted above, the values of c_1 and b_0 are numerically close to those determined through direct analysis of the series solution using (5.71). Plots of these solutions are shown in Fig. 5.4 for a range of different r_1 values, and for each r_1 we show a set of curves labelled by $\omega^{(1)}$ (or more conveniently c_2). As the plots show, for each arbitrarily picked c_2, a unique solution curve can be found that is regular at $z = 1$ and satisfies $f(0) = 0$. This agrees with the above analysis using series solutions, namely that the boundary conditions alone do not put any constraints on ω. Note that the curve with $c_2 = 0$, that asymptotes to the Kerr solution in the small black hole limit, has $\omega = a/(4r_1) \leq \Omega_H = a/r_1^2$ for $0 < r_1 \leq 2$.

5.3.4 Energy-momentum flux in the BZ process

We can now evaluate the relevant radial energy and angular momentum flux densities in eqs. (5.28) and (5.31)

$$T_t^r = -\omega T_\varphi^r = r^{-2}\omega(\omega - \Omega_H)A_{\varphi,\theta}^2, \tag{5.72}$$

using the solution (5.59) for B_T (after accounting for the differences due to switching between u and θ coordinates which do not affect the final result, cf. comments below (A.0.13)). Relating (5.72) to the energy and angular momentum changes in the

[7] Among the boundary conditions $f_b(0) = 0$, $f_b'(0) = c_1$, $f_c(1) = b_0$ & $f_c'(1) = b_1$, only the first two are consistent with the value of c_1 obtained by BZ in the Kerr limit.

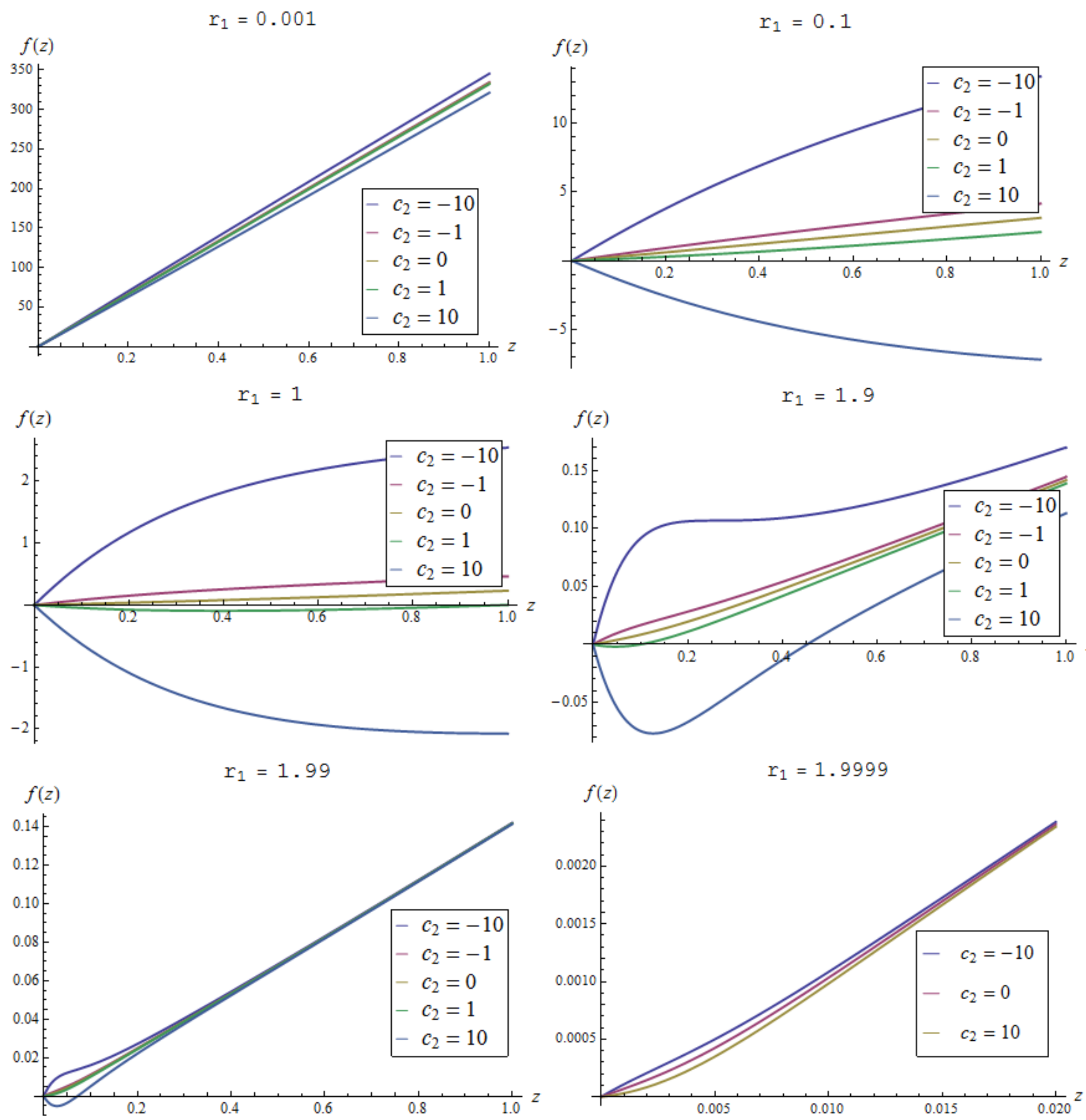


Figure 5.4: In each graph (with fixed r_1), a set of solution curves corresponding to various choices of $\omega^{(1)} = \frac{1}{4r_1} + \frac{c_2}{C}\frac{(r_1-2)^2}{4r_1}$ are shown, by varying c_2 from -10 to 10 (setting $C = 1$). Note that for a large black hole with $r_1 = 0.001$, $\omega^{(1)}$ varies by a relative factor of 100 as c_2 is varied, while for the small black hole with $r_1 = 1.9999$, $\omega^{(1)}$ only varies by a relative factor of 10^{-7}. Thus the curves effectively zoom in to the "middle curve" with $c_2 = 0$ as r_1 increases, and $c_2 = 0$ is indeed the only solution in the Kerr limit. The "middle curve" is plotted for various r_1's in Fig. 5.5.

thermodynamic relations, we get from (2.37) & (2.34)

$$T\delta S \propto T^r_t + \Omega_H T^r_\varphi = r^{-2}(\omega - \Omega_H)^2 A^2_{\varphi,\theta} \geq 0, \tag{5.73}$$

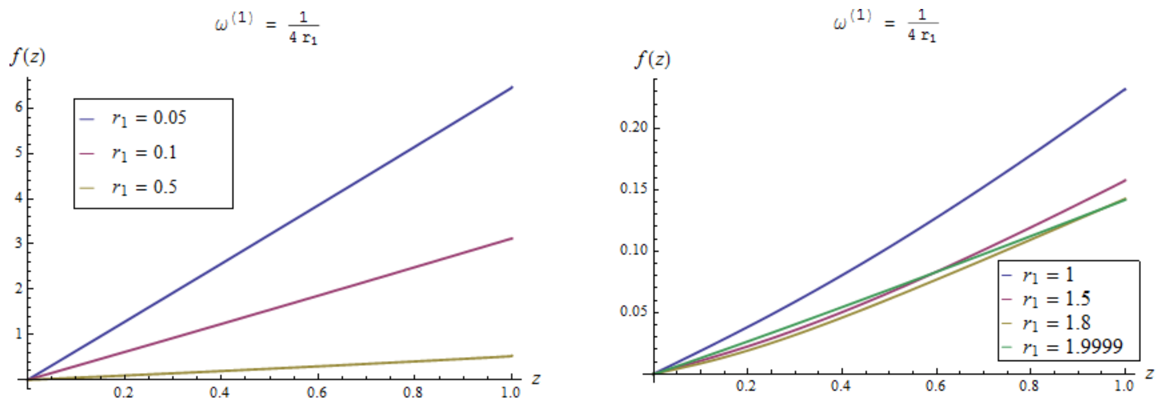


Figure 5.5: Solution curves with $\omega^{(1)} = 1/(4r_1)$ (the middle curve in each graph of Fig. 5.4) for various r_1's. Note both $c_1 = f'(0)$ and $b_0 = f(1)$ decrease monotonically with increasing r_1, while the middle part of the curve bounces back when $r_1 \to 2$.

which explicitly checks the second law. Similarly, from (2.35),

$$\delta E \propto T_t^r + \Omega' T_\varphi^r = r^{-2}(\omega - \Omega_H)(\omega - \Omega')A_{\varphi,\theta}^2, \tag{5.74}$$

$$\delta L \propto -T_\varphi^r = r^{-2}(\omega - \Omega_H)A_{\varphi,\theta}^2, \tag{5.75}$$

from which we see that the ambiguities in ω and Ω' affect the signs of δE and δL. Various possibilities (six of them) according to the relative magnitudes of $\{\omega, \Omega', \Omega_H\}$ are summarized in the following list and fig. 5.6 (on the plane of $\{\omega/\Omega_H, \Omega'/\Omega_H\}$)

1. $\Omega' < \omega < \Omega_H, \quad \delta E < 0, \quad \delta L < 0$
2. $\omega < \Omega' < \Omega_H, \quad \delta E > 0, \quad \delta L < 0$
3. $\omega < \Omega_H < \Omega', \quad \delta E > 0, \quad \delta L < 0$
4. $\Omega_H < \omega < \Omega', \quad \delta E < 0, \quad \delta L > 0$
5. $\Omega_H < \Omega' < \omega, \quad \delta E > 0, \quad \delta L > 0$
6. $\Omega' < \Omega_H < \omega, \quad \delta E > 0, \quad \delta L > 0.$

So for energy extraction $\delta E < 0$ we have either case 1 or 4 above.

The energy-defining Killing vector with $\Omega' = \Omega_H$ results in a first law without the δL term so that $\delta E = T\delta S \propto (\omega - \Omega_H)^2 \geq 0$, and thus there is no energy extraction, regardless of the value of ω. This is of course consistent with the stability arguments discussed in earlier sections. However, if we define energy using the Killing

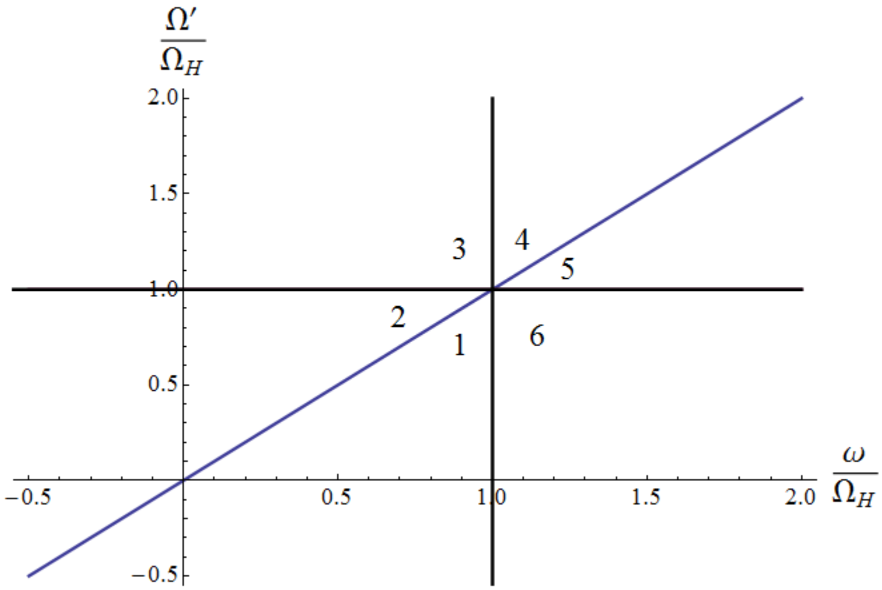


Figure 5.6: Six possibilities for the relative magnitudes of $\{\omega, \Omega', \Omega_H\}$ (giving different signs of δE and δL) as represented by the numbered regions on the $\{\omega/\Omega_H, \Omega'/\Omega_H\}$ plane. The numbers correspond to the enumerated cases in the main text.

vector with $\Omega' = \Omega_\infty < \Omega_H$, we find that 'energy' can be extracted if ω falls in the range in case 1 above, which is usually expected as in the Kerr case, and we have $\delta E \propto (\omega + a/l^2)(\omega - \Omega_H) < 0$. If ω is outside the range in case 1 we have field lines rotating either backwards (even as seen in the non-rotating frame at infinity) (case 2) or faster than the black hole (case 6), and there is no energy extraction. These two possibilities are in fact ruled out, as discussed in section 2.4, if we match the rotating black hole magnetosphere at infinity to that of the asymptotic static spacetime.

In conclusion, the stability condition apparently implies that any choice of energy-defining Killing vector, other than via Ω_H, leads to a rather benign form of 'ergosphere' and BZ process. An outgoing radial Poynting flux is possible, but does not reflect an instability of the black hole, as it can be turned off by switching to an alternate definition of energy for an asymptotic observer. In effect, there is still a net ingoing flux when one properly accounts for both energy and angular momentum. Nonetheless, the fact that the AdS/CFT correspondence points to a specific definition of 'energy' which apparently exhibits this benign ergosphere in the bulk raises the question of how it is reflected (if at all) in the dual field theory. Moreover, small black holes do not have a globally timelike Killing vector and there is energy extraction which does not have an AdS dual. We will turn to dual interpretation shortly, after first considering in more detail the matching to AdS monopole solutions at large radius and also presenting an analytic solution for small black holes.

5.3.5 Matching black hole and asymptotic static spacetime force-free solutions

As have been seen above, whether we can have energy extraction depends both on the choice of Ω' and the actual rotational frequency ω of the magnetosphere. One requires ω of the monopole in Kerr-AdS background match that in the asymptotic AdS background. If one only considers an unperturbed monopole in the latter, ω would be fixed as a function $\omega = \omega(\Omega_H, \Omega_\infty)$, up to an ambiguity in the definition of the monopole. On the other hand, a monopole with a perturbation in A_φ can rotate with arbitrary ω, as revealed by the free parameter c_2 in the Kerr-AdS case (cf. (5.65)) and $\bar{c}_2$ in the AdS case (cf. (4.12)). c_2 and $\bar{c}_2$ are related to the subleading radial falloff of the perturbation and do not affect the boundary condition of A_φ. One only needs to match B_T and ω, which we now describe and consider as a "consistency check" that the ambiguity in the black hole magnetosphere solution perfectly matches the ambiguity in the asymptotic monopole.

We require using the transformations (4.5)

$$\bar{B}_T^c(\bar{\omega}^{(1)}) = B_T^c(\omega^{(1)}), \qquad \bar{\omega}^{(1)} = \omega^{(1)} + \frac{1}{l^2}, \tag{5.76}$$

where the first relation holds to leading order in a (note $B_T^c \sim B_T/a$) and the second relation is merely due to the constant shift in the Φ-φ transformation. $B_T^c(\omega^{(1)})$ is given by (5.59):

$$B_T^c = C\Big(\omega^{(1)} - \frac{1}{r_1^2}\Big), \tag{5.77}$$

and $\bar{B}_T^c(\bar{\omega}^{(1)})$ has different forms in the following situations, where the values of $\omega^{(1)}$ from matching are also given:

- exact monopole solution (4.6)

$$(\bar{B}_T^c)^2 = C^2(\bar{\omega}^{(1)})^2 \tag{5.78}$$

$$\omega^{(1)} = \frac{l^2 - r_1^2}{2l^2 r_1^2} \overset{m=1}{=\!=\!=} \frac{r_1 - 1}{r_1^3}, \qquad c_2 = -\frac{C}{r_1^2}. \tag{5.79}$$

- alternative definition of monopole (4.7)

$$(\bar{B}_T^c)^2 = C^2[(\bar{\omega}^{(1)})^2 - 3/(2l^4)] \tag{5.80}$$

$$\omega^{(1)} = \frac{2l^4 + r_1^4}{4l^2 r_1^2 (l^2 + r_1^2)} \overset{m=1}{=\!=} \frac{3r_1^2 - 4r_1 + 4}{8r_1^3}, \qquad c_2 = \frac{C}{2r_1^2}. \tag{5.81}$$

- perturbative monopole (4.12)

$$(\bar{B}_T^c)^2 = C^2(\bar{\omega}^{(1)})^2 - \bar{c}_2 C/l^4 \tag{5.82}$$

$$\omega^{(1)} = \frac{l^2 - r_1^2}{2l^2 r_1^2} + \frac{\bar{c}_2}{C}\frac{r_1^2}{2l^2(l^2 + r_1^2)} \overset{m=1}{=\!=} \frac{r_1 - 1}{r_1^3} + \frac{\bar{c}_2}{C}\frac{(r_1 - 2)^2}{4r_1^3}, \qquad c_2 = \frac{\bar{c}_2 - C}{r_1^2} \tag{5.83}$$

We have used (5.65) for c_2 in each case. The relation $\omega = \omega(\Omega_H, \Omega_\infty)$ can be found for each of the above cases to be

$$(5.79) \qquad \Rightarrow \qquad \omega = \frac{\Omega_H + \Omega_\infty}{2} \tag{5.84}$$

$$(5.81) \qquad \Rightarrow \qquad \omega = \frac{2\Omega_H^2 + \Omega_\infty^2}{4(\Omega_H - \Omega_\infty)} \tag{5.85}$$

$$(5.83) \qquad \Rightarrow \qquad \omega = \frac{\Omega_H + \Omega_\infty}{2} + \frac{\bar{c}_2}{C}\frac{\Omega_\infty^2}{2(\Omega_H - \Omega_\infty)}, \tag{5.86}$$

which all reduce to $\omega = \Omega_H/2$ in the Kerr case ($\Omega_\infty = 0$), and (except for the last $\bar{c}_2$-dependent one) satisfy the condition $\Omega_\infty < \omega < \Omega_H$ for energy extraction if we use $K^\mu_{\Omega_\infty}$ as the energy-defining Killing vector. One can also use the black hole size parameter $x = r_1^2/l^2$ to rewrite the above relations as

$$(5.79) \qquad \Rightarrow \qquad \frac{\omega}{\Omega_H} = \frac{1 - x}{2} \tag{5.87}$$

$$(5.81) \qquad \Rightarrow \qquad \frac{\omega}{\Omega_H} = \frac{x^2 + 2}{4(x + 1)} \tag{5.88}$$

$$(5.83) \qquad \Rightarrow \qquad \frac{\omega}{\Omega_H} = \frac{1 - x}{2} + \frac{\bar{c}_2}{C}\frac{x^2}{2(x + 1)}, \tag{5.89}$$

where one notices the sign difference of the first two results when $x > 1$.

Naively the rotating monopole solutions in AdS found in chapter 4 are able to produce energy and momentum fluxes: $T^y_T = -\bar{\omega} T^y_\Phi = C\bar{\omega}\bar{B}_T/y^2$. This is not surprising if we expect them to serve as the asymptotic limits of the interior BZ process.

Note that the fluxes are singular at $y = 0$, but this solution needs to be interpreted with a physical cut-off such as the surface of a star.

For completeness, we also present the currents for the perturbed AdS monopole solution, assuming $\bar{\omega}$ given by (5.83) from the matching:

$$J^y = -\frac{aU}{l^2y^2}\Big(C\frac{r_1^2+l^2}{r_1^2} - \bar{c}_2\frac{r_1^2}{r_1^2+l^2}\Big) \overset{m=1}{=\!=\!=} -\frac{aU}{2r_1^3y^2}\big[4C - \bar{c}_2(r_1-2)^2\big], \tag{5.90}$$

$$J^U = 0, \tag{5.91}$$

$$J^\Phi = -2a^2U\frac{\bar{c}_2}{l^2y^2(l^2+y^2)} \overset{m=1}{=\!=\!=} -2a^2U\frac{\bar{c}_2(r_1-2)^2}{r_1^3y^2[y^2(2-r_1)+r_1^3]}, \tag{5.92}$$

$$\begin{aligned} J^T &= -\frac{aU}{y^2(l^2+y^2)}\Big(C\frac{r_1^2+l^2}{r_1^2} + \bar{c}_2\frac{r_1^2}{r_1^2+l^2}\Big) \\ &\overset{m=1}{=\!=\!=} -\frac{aU}{2y^2[y^2(2-r_1)+r_1^3]}\big[4C + \bar{c}_2(r_1-2)^2\big], \end{aligned} \tag{5.93}$$

where we have separated contributions from the monopole ($\sim C$) and the $\mathcal{O}(a^2)$ correction ($\sim \bar{c}_2$). These can be viewed as the asymptotic currents of the BZ process. Note that J^Φ only contains $\bar{c}_2$.

As another example, one can also consider the gauge potential A_Φ in the exact KNAdS 'vacuum' solution, expanded for small 'a'

$$A_\Phi = -Cu\Big[1 + a^2\Big(\frac{1}{l^2} + \frac{1-u^2}{r^2}\Big)\Big] = -CU\Big[1 + a^2\Big(\frac{3-U^2}{2l^2} + \frac{1-U^2}{y^2}\Big)\Big]. \tag{5.94}$$

One obtains a similar configuration that has energy and momentum fluxes.

5.3.6 Summary

To summarize, we have obtained the analog of BZ's monopole solution in the Kerr-AdS background. Ambiguities in the configuration of the force-free magnetosphere arise due to the boundary conditions of the AdS geometry. This is reflected in the main force-free equation as the lack of regularity constraints at infinity, namely, the equation is less divergent due to the additional AdS length scale. This can also be seen from the asymptotic matching to a rotating monopole in pure AdS space. Closely associated with it, the arbitrariness in choosing the energy-defining Killing vector makes the energy extraction an observer dependent phenomenon; even with the ambiguity in the angular velocity of the magnetosphere, there are always frames in which the ingoing energy flux is non-negative. Nevertheless, we have been able to obtain series and numerical solutions which are unique once the free parameters are fixed.

5.4 Analytic force-free magnetosphere for small Kerr-AdS black holes

In this section, we determine an analytic solution for the force-free magnetosphere about a 'small' Kerr-AdS black hole, which belongs to the regime (5.17). More precisely, in the slow rotation, small 'a', limit we also expand in r_H/l and consider the leading correction of $\mathcal{O}(r_H^2/l^2)$. For simplicity below, we will only keep track of the order in $1/l$, and refer to this as the $1/l$ expansion.

We employ the ansatz

$$A_\varphi^{(2)}(r,u) = \left[f_{[0]}(r) + f_{[2]}(r)l^{-2}\right]Cu(1-u^2), \tag{5.95}$$

$$\omega^{(1)} = \frac{1}{8m^2} + \omega_{[2]}^{(1)} l^{-2}, \tag{5.96}$$

$$\frac{B_T^c}{C} = \omega^{(1)} - \frac{1}{r_1^2} = -\frac{1}{8m^2} + (\omega_{[2]}^{(1)} - 2)l^{-2}, \tag{5.97}$$

where $\omega_{[2]}^{(1)}$ is constant and sub(super)scripts in square brackets indicate the order in the $1/l$ expansion. Also, we use m instead of r_1 in this section. The ansatz solves the $dT_{\varphi(t)} = 0$ equation and the $dT_u = 0$ equation (5.53) has the expansion $dT_u^{[0]} + dT_u^{[2]} l^{-2} + \cdots = 0$. The leading order equation $dT_u^{[0]} = 0$ is solved by the known BZ solution in the Kerr geometry [5, 6]

$$f_{[0]}(r) = \frac{2r-3m}{m^4}\left[\frac{r^2}{16m}\left(2\text{dilog}\frac{r}{2m} + \ln^2\frac{r}{2m} + \frac{\pi^2}{3}\right) - \frac{r}{4} - \frac{m}{8} - \frac{m^2}{9r}\right] - \frac{1}{m^4}\left(\frac{r^2}{2} - \frac{mr}{4} - \frac{m^2}{12}\right)\ln\frac{r}{2m}. \tag{5.98}$$

The next-to-leading-order equation $dT_u^{[2]} = 0$ has a solution of the form

$$f_{[2]}(x) = \frac{1}{36}\left[3C_1 h_1(x) + C_2 h_2(x) + h_2(x)\int h_1(x)h_3(x)\,\mathrm{d}x - h_1(x)\int h_2(x)h_3(x)\,\mathrm{d}x\right], \tag{5.99}$$

where $x \equiv \frac{r}{2m}$ and

$$h_1(x) = -4x^2(4x-3), \tag{5.100}$$

$$h_2(x) = -12x^2(4x-3)\ln\left(1-\frac{1}{x}\right) - 2(24x^2-6x-1), \tag{5.101}$$

$$h_3(x) = -\frac{9}{2}x^2(8x-3)(2\text{dilog}x + \ln^2 x) + \frac{3x(48x^3-90x^2+43x-2)\ln x}{2(x-1)^2} - \frac{48\pi^2x^6 - (66\pi^2+288)x^5 + (14\pi^2+324)x^4 - (35+12\omega_{[2]}^{(1)})x^3 - 7x^2 + 12\omega_{[2]}^{(1)}}{4x^2(x-1)}. \tag{5.102}$$

The function $h_3(x)$ can be written in a slightly different form

$$h_3(x) = -9x^2(8x-3)\Big(\overbrace{\text{Li}_2\frac{1}{x} - \ln(x-1)\ln x + \ln^2 x}^{=\text{Li}_2\frac{1}{x}+\text{Li}_1\frac{1}{x}\ln x}\Big) + \frac{3x(48x^3-90x^2+43x-2)\ln x}{2(x-1)^2} + \frac{288x^5 - 324x^4 + (35+12\omega_{[2]}^{(1)})x^3 + 7x^2 - 12\omega_{[2]}^{(1)}}{4x^2(x-1)}, \tag{5.103}$$

using the dilogarithm identities

$$\begin{aligned}\text{dilog}x = \text{Li}_2(1-x) &= \text{Li}_2\frac{1}{x} + \frac{1}{2}\ln x \ln\frac{x}{(x-1)^2} - \frac{\pi^2}{6} \\ &= \text{Li}_2\frac{1}{x} - \ln(x-1)\ln x + \frac{1}{2}\ln^2 x - \frac{\pi^2}{6} \\ &= \text{Li}_2\frac{1}{x} + \text{Li}_1\frac{1}{x}\ln x - \frac{1}{2}\ln^2 x - \frac{\pi^2}{6}, \qquad (x>1).\end{aligned} \tag{5.104}$$

With some manipulation, the integrals in (5.99) can be evaluated using *Maple*. In particular, the second integral can be simplified using integration by parts, as sketched below,

$$\int h_2(x)h_3(x)\,\mathrm{d}x = \int \text{part}_1\,\mathrm{d}x + \underbrace{\int \ln\left(\frac{x-1}{x}\right)\text{part}_2\,\mathrm{d}x}_{=\text{part}_{1a}+\left(\int \text{part}_{1b}\,\mathrm{d}x + \underbrace{\int \ln^2 x\,\text{part}_{2b}\,\mathrm{d}x}_{\text{integration by parts}}\right)}. \tag{5.105}$$

Before presenting the explicit result, we note that certain terms in (5.105) appear to be complex. For example,

$$\frac{4212}{35}\left[\mathrm{Li}_3 x-\mathrm{Li}_2 x\ln x-\frac{1}{2}\ln(1-x)\ln^2 x\right], \tag{5.106}$$

since it contains $\mathrm{Li}_s x$, is only real for $x\le 1$ ($\ln(1-x)=-\mathrm{Li}_1 x$) while our x lies in the range $[1,\infty)$. However, all the imaginary parts actually cancel out as can be shown using the following identities

$$\begin{aligned}
\mathrm{Li}_1 x &= \mathrm{Li}_1\frac{1}{x}-\ln x-i\pi \quad \big(\Leftrightarrow \ln(1-x)=\ln(x-1)+i\pi\big),\\
\mathrm{Li}_2 x &= -\mathrm{Li}_2\frac{1}{x}-\frac{1}{2}\ln^2 x+\frac{\pi^2}{3}-i\pi\ln x \qquad (x>1),\\
\mathrm{Li}_3 x &= \mathrm{Li}_3\frac{1}{x}-\frac{1}{6}\ln^3 x+\frac{1}{3}\pi^2\ln x-\frac{1}{2}i\pi\ln^2 x,
\end{aligned}$$

or more generally

$$\mathrm{Li}_s x+(-1)^s\mathrm{Li}_s\frac{1}{x}=2\sum_{k=0}^{\lfloor s/2\rfloor}\frac{\ln(-x)^{(s-2k)}\mathrm{Li}_{2k}(-1)}{(s-2k)!}. \tag{5.107}$$

The final solution is given by

$$\begin{aligned}
f_{[2]}(x) &= \frac{78x^2(4x-3)}{35}\left(6\mathrm{Li}_3\frac{1}{x}+4\mathrm{Li}_2\frac{1}{x}\ln x+\mathrm{Li}_1\frac{1}{x}\ln^2 x\right)\\
&\quad -\frac{2(120x^5+195x^4-234x^3-312x^2+78x+13)}{35}\left(\mathrm{Li}_2\frac{1}{x}+\mathrm{Li}_1\frac{1}{x}\ln x\right)\\
&\quad -\frac{13(24x^2-6x-1)\ln^2 x}{35}+\frac{x(240x^4+270x^3-1951x^2+1397x-26)\ln x}{35(x-1)}\\
&\quad +\frac{48x^4}{7}+\frac{90x^3}{7}-\frac{2659x^2}{42}+\frac{1427x}{105},
\end{aligned} \tag{5.108}$$

where we have set $\omega_{[2]}^{(1)}=1/2$ to remove the $\mathcal{O}(x^3,x^2)$ divergences at large x, while $\mathcal{O}(x)$ and $\ln x$ divergences remain (implying however finite B^r and B^θ); in the two limits,

$$f_{[2]}(x\to\infty)=-\frac{x}{3}+\frac{\ln x}{30}-\frac{833}{1800}, \tag{5.109}$$

$$f_{[2]}(x=1)=\frac{468\zeta(3)}{35}+\frac{4\pi^2}{3}-\frac{9434}{315}. \tag{5.110}$$

Thus we have

$$\omega = a\Big(\frac{1}{8m^2} + \frac{1}{2l^2}\Big) = \frac{\Omega_H + \Omega_\infty}{2}, \tag{5.111}$$

$$B_T = C(1-u^2)(\omega - \Omega_H) = -Ca(1-u^2)\big(\frac{1}{8m^2} + \frac{3}{2l^2}\big), \tag{5.112}$$

consistent with matching to the unperturbed monopole (cf. (5.84)). As noted in section 5.2.2, the criterion $r_H^2 > al$ for a globally timelike Killing vector may or may not be met in this 'small' black hole limit, and an outgoing energy flux is possible.

5.5 Aspects of the dual field theory

As reviewed in section 3.10, the near-equilibrium behaviour of 'large' AdS black holes, satisfying $r_H > l$, has a dual holographic description in terms of the grand canonical ensemble for a field theory on the 2+1-dimensional boundary geometry. For rotating black holes, this system is characterized by a fluid at finite temperature on a rotating two-sphere, and the force-free magnetosphere we have studied translates to an electromagnetic perturbation of this rotating fluid. We will focus attention on large black holes with $r_H > l$ for the rest of this section.

To gain insight into the properties of this system, it is useful to compare with the corresponding Kerr-Newman-AdS (KNAdS) vacuum solution of Einstein-Maxwell theory revealed in chapter 3. In the slow-rotation regime, and using BL coordinates, the KNAdS geometry with magnetic charge C admits the expansion

$$\mathrm{d}s^2_{\mathrm{KNAdS}} = \mathrm{d}s^2_{\mathrm{KAdS}} + \mathcal{O}(C^2) \tag{5.113}$$

$$A_{\mathrm{KNAdS}} = -C\cos\theta\mathrm{d}\varphi + \mathcal{O}(a). \tag{5.114}$$

Thus, by working to linear order in C (and thus ignoring magnetic back-reaction on the Kerr-AdS geometry), we can equivalently consider the force-free solution as a particular $\mathcal{O}(a/m)$ perturbation of the KNAdS background. The first difference emerges in the $\mathcal{O}(a)$ correction to A_t. For KNAdS, $A_t = Ca\cos\theta/r^2 + \mathcal{O}(a^3)$ and this completes the solution up to $\mathcal{O}(a^3)$. In contrast, the force-free solution necessarily satisfies $A_t = -\omega A_\varphi = C\omega^{(1)}a\cos\theta + \mathcal{O}(a^2)$, up to a possible constant shift in A_t that, as we will see below, can be gauged away. With ω constant, this produces a non-vanishing boundary limit for A_t. The holographic dictionary (cf. chapter 3) generally allows us to isolate the chemical potential and charge density from the asymptotics of $A_t \sim \mu + \rho/r + \cdots$ (constant coefficients will be absorbed in this identification, since the bulk gauge coupling is arbitrary in the limit that we ignore back-reaction on the metric). In the present case, this would lead to the odd conclusion that the fluid on the sphere had a θ-dependent chemical potential, but no charge density to $\mathcal{O}(a^2)$. However, the full definition of the chemical potential [68],

$$\mu l = A_\mu K^\mu_{\Omega_H}|_{r\to\infty} - A_\mu K^\mu_{\Omega_H}|_{r\to r_H} = 0 + \mathcal{O}(a^3), \tag{5.115}$$

does in fact vanish to this order, as expected (cf. (3.69)).

This discussion suggests that the distinction between the holographic dual of the force-free magnetosphere and KNAdS may in effect be rather minor up to $\mathcal{O}(a)$. There is no azimuthal current at this order, and thus the boundary fluid rotates in the leading-order monopole magnetic field. The absence of a charge density at this order appears consistent with the conclusion that the angular velocity ω of the electromagnetic field was not uniquely fixed, at least to $\mathcal{O}(a)$, in the solution. I.e. there is no restriction to rotating a magnetic field in a neutral fluid. Another viewpoint follows from noting that for an electromagnetic field strength $f_{\mu\nu}$ in 2+1 dimensions, $\det(f_{\mu\nu}) = 0$ identically, and thus the force-free condition for the dual fluid $f_{\mu\nu}j^\nu = 0$ can always be solved for a specific current configuration independent of the background field. There is no analog of the constraint $\mathrm{tr}({}^\star FF) = 0$ required in 3+1 dimensions. This opens the possibility that some of the above conclusions may actually extend to higher orders in the expansion in the rotation parameter a, where the vanishing of the charge density need no longer hold. This also suggests that repeating the calculation in one higher dimension, where the boundary force-free condition is less trivial, may lead to somewhat different conclusions.

The arguments above imply that, at least to $\mathcal{O}(a)$, we can directly translate various results from the equilibrium thermodynamics of KNAdS duals to the force-free solution. In fact, to linear order in the magnetic charge, we can adopt KAdS relations for the field theory temperature,

$$T \sim \frac{r_1}{4\pi l^2}\left(1 + \frac{l^2}{r_1^2}\right) + \mathcal{O}(a^2) \xrightarrow{m=1} \frac{1}{2\pi r_1^2} + \mathcal{O}(a^2), \tag{5.116}$$

and the angular velocity

$$\Omega = \Omega_H - \Omega_\infty \sim \frac{a}{l^2}\left(1 + \frac{l^2}{r_1^2}\right) + \mathcal{O}(a^3) \xrightarrow{m=1} \frac{2a}{r_1^3} + \mathcal{O}(a^3). \tag{5.117}$$

The two quantities here Ω_H and Ω_∞ are the bulk angular velocities at $r = r_H$ and $r = \infty$ respectively. The dual field theory is identified as a neutral fluid with temperature T in a rotating Einstein universe, with angular velocity Ω. These quantities along with suitable definitions of mass and angular momentum then satisfy the first law, as discussed earlier [19]. Indeed the partition function of an ideal gas in this background can be computed and reproduces the structure of the bulk partition function [18, 65, 62], which for KNAdS has the form [68] (cf. (3.67) and (3.89)),

$$\frac{1}{V}\ln Z = T^2 \frac{h\left(\mu/T, B(1-\Omega^2 l^2)/T^2\right)}{1-\Omega^2 l^2} \sim T^2 h\left(\mu/T, B/T^2\right) + \mathcal{O}(a^2). \tag{5.118}$$

The quantity $h(\mu/T, B/T^2)$ specifies the partition function of the static charged black hole. Note that the free energy diverges, and the rotation velocity exceeds the speed of light, unless $\Omega < 1/l$ [18]. In the slow rotation limit, this condition is always satisfied for $a < r_1/2$ given that we require $r_1 > l$ to have a dual description in field theory.

5.5.1 Currents at $\mathcal{O}(a^2)$

The corrections to the free energy arise at $\mathcal{O}(a^2)$, and thus are only fully calculable on accounting for the back-reaction to the metric which starts at this order. However, we can look again at the electromagnetic field, and ask whether this picture of a neutral dual fluid persists to higher orders in a. In fact, since $A_t = -\omega A_\varphi$ is generic for axisymmetric force-free solutions, the $\mathcal{O}(a^2)$ correction to A_φ characterized by $f(r) \to \mathcal{O}(1/r)$ has the right falloff to produce a contribution to the charge density at $\mathcal{O}(a^3)$. However, we would need to compute the full solution at this order to test whether this remains or is cancelled by other terms.

Nonetheless, since the bulk solution is valid at $\mathcal{O}(a^2)$, we can read off the corresponding boundary currents from the asymptotics of the gauge field. In particular, with $f(z) \to c_1 z$ we can (up to normalization), identify the the azimuthal boundary current,

$$j_\varphi = a^2 c_1, \quad (\text{given } r_H > l), \tag{5.119}$$

with the results plotted in Fig. 5.7 for various choices of r_1 and with $\frac{c_2}{C} = -10, -1, 0, 1, 10$. As noted above, the field theory charge density $j_t = -\omega j_\varphi$ (taking the asymptotic limit of $\partial_r A_t = -\omega \partial_r A_\varphi$) is of higher order, $\mathcal{O}(a^3)$, and thus we cannot perform a nontrivial test of the putative 'boundary force-free condition' that is hinted

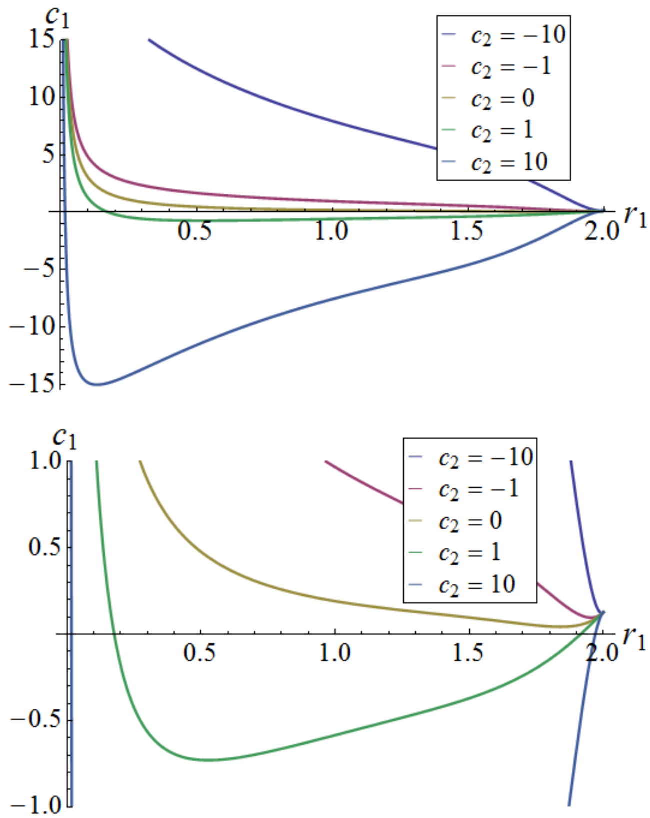


Figure 5.7: Plots of the azimuthal current c_1 as functions of r_1 for various values of c_2, with c_1 obtained by imposing the boundary conditions $f_b(0) = 0$ and $f_b'(0) = c_1$ for the series solution $f_b(z)$ constructed up to $\mathcal{O}((z-1)^{10})$. Note that the boundary field theory interpretation only holds for large black holes with $r_1 < 1$. The second graph shows the region close to the r_1-axis. For c_1, good agreement is found between the numerical and series results, except for the non-monotonic behaviour of c_1 near $r_1 \approx 2$.

at by the results at $\mathcal{O}(a)$.

Although we are primarily concerned with large black holes in this section, we note more generally that the condition for energy extraction constrains ω (and thus c_2) to the range $\Omega_\infty < \omega < \Omega_H$ (or equivalently $-\frac{(r_1-2)^2+4}{(r_1-2)^2 r_1^2} < c_2 < \frac{4-r_1}{(r_1-2)^2 r_1}$). Correspondingly, we find constraints on $c_1^{\min}(r_1) < c_1 < c_1^{\max}(r_1)$, which we plot in fig. 5.8. Of course, c_1 only has a holographic interpretation in terms of the dual current for large black holes with $r_H > l$.

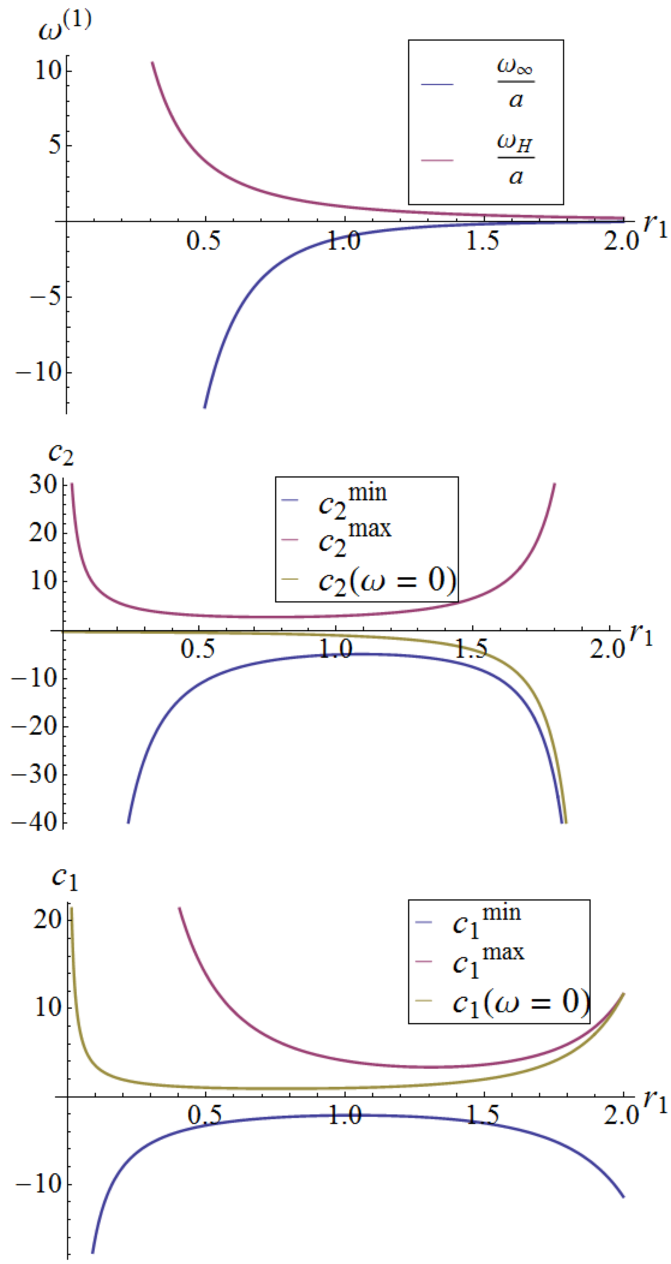


Figure 5.8: To have energy extraction for small black holes, $\omega^{(1)}$, c_2 and c_1 must lie between the top and bottom curves in each plot. The middle curves correspond to $\omega = 0$. Note that $\omega^{(1)}_{\min/\max}$ and $c_2^{\min/\max}$ correspond to $c_1^{\max/\min}$.

5.5.2 Stability

As discussed in sections 2.3 and 5.2, and briefly reviewed again here, Kerr and Kerr-AdS geometries have important differences concerning timelike Killing vectors and ergospheres. For Kerr black holes in asymptotically flat space, there is a unique normalized Killing vector which is timelike at infinity, $K^\mu_{\Omega=0} = \xi^\mu_{(t)}$. This Killing vector becomes spacelike inside the ergosphere, allowing for the possibility of energy extraction from the black hole via super-radiance or the BZ process. In AdS, super-radiant modes would be reflected back off the boundary, leading to a genuine instability. However, as reviewed above, this situation changes in large Kerr-AdS geometries, where a family of Killing vectors remain timelike at infinity. Among this class, the horizon generator $K^\mu_{\Omega_H} = \xi^\mu_{(t)} + \Omega_H \xi^\mu_{(\varphi)}$ is globally timelike, becoming null on the horizon itself. Thus, for "large" black holes there is no ergoregion for the energy flux vector defined by $\mathcal{T}^\mu = -T^\mu_\nu K^\nu_{\Omega_H}$, which is itself timelike outside the horizon if the dominant energy condition (DEC) is satisfied. Hawking and Reall have argued that the existence of this global timelike Killing vector, along with the DEC, ensures stability of Kerr-AdS if $\Omega < 1/l$ [14]. This argument is apparent in the discussion of section 2.3, and implies that energy cannot be extracted from large Kerr-AdS black holes by super-radiance or indeed by the BZ process. This argument of course breaks down for small black holes, which then behave in a similar manner to Kerr geometries.

This stability argument for large black holes ultimately appears consistent with the above conclusions that the boundary rotating fluid is neutral, albeit to low order in the rotation parameter. Nonetheless, an important caveat is that the dominant energy condition needs to be satisfied. In AdS, violating the DEC is not as dramatic as it would be in flat space since the Breitenlohner-Freedman (BF) bound allows a range of negative mass e.g. for scalar fields. This loophole was noted by Gubser & Mitra [69] as a way to realize a Gregory-Laflamme-type instability for large black holes. In the present case, it is not clear that the currents which produce a force-free magnetosphere satisfy the DEC,[8] but we can try to check this by looking at the asymptotics of the bulk current, given in terms of the bulk solution as $J^\mu = F^{\mu\nu}{}_{;\nu}$. For the boundary field theory directions $\{t, \theta, \varphi\}$, we find the covariant current components,

$$J_t \sim 2Cau \frac{l^2\omega^{(1)} + 1}{l^2 r^2} + \mathcal{O}(r^{-5}), \tag{5.120}$$

$$J_u \propto \partial_r B_T(u) = 0, \tag{5.121}$$

$$J_\varphi \sim 2Ca^2 u(1-u^2) \frac{(r_1^2 + l^2)(l^2 - r_1^2 - 2l^2 r_1^2 \omega^{(1)})}{l^2 r_1^4 r^2} + \mathcal{O}(r^{-4}). \tag{5.122}$$

[8]The DEC can be verified for the force-free electromagnetic field configuration to $\mathcal{O}(a^2)$. We find that $-T^\mu_\nu K^\nu_{\Omega_H}$ is indeed a future-directed timelike vector, with the norm scaling as $1/r^2$ as $r \to \infty$. The $\mathcal{O}(a^2)$ correction is actually negative, but is necessarily subleading in the slow rotation limit.

We would like to interpret these falloff conditions in terms of the conformal dimension Δ of the dual vector operator according to the boundary coupling $\mathcal{O}_\mu J^\mu$, and compare with the BF bound. To do this, we can consider modelling the current with a specific bulk field, and for simplicity consider the case of a charged scalar ϕ in the bulk, with the current $J_\mu = \phi^* \overleftrightarrow{D}_\mu \phi$. Then $J_t = i\phi^*\phi A_t$, and $J_\varphi = i\phi^*\phi A_\varphi$. Recall that the BF bound for a scalar field in (3+1)-D is $m^2 l^2 > -9/4$, where for scalars $\Delta(d-\Delta) = -m^2 l^2$. The falloff conditions for the (covariant) current components $J_t \sim J_\varphi \sim 1/r^2 + \mathcal{O}(1/r^4)$ then imply $\Delta = 1$, i.e. $m^2 = -2/l^2$ which is above the BF bound. This allows for two possible normalizable falloff conditions, $\phi \sim \alpha_1/r + \alpha_2/r^2 + \cdots$, and those above suggest $\alpha_1 \neq 0$ and $\alpha_2 = 0$, which is a consistent choice.[9] The result is of course consistent with the stability of the solution.

The radial component of the current is given by

$$J_r \sim 2Cl^2 au \frac{\omega^{(1)} r_1^2 - 1}{r^4 r_1^2} + \mathcal{O}(r^{-6}), \tag{5.123}$$

which in principle sources another scalar operator, independent of $\mathcal{O}_\mu$. Indeed, expressing D_μ in BL coordinates, the simple scalar model above would imply $J_r \sim i\phi^* \partial_r \phi \sim 1/r^3$, which is not consistent with the $1/r^4$ scaling above, suggesting instead a higher dimensional operator.

To conclude this section, we also point out that the azimuthal current J^φ can change sign from the horizon to the boundary: in $m = 1$ units and with $z = \frac{r_1}{r}$, $J^\varphi_{\rm min} = J^\varphi(z=0) \sim 1 - r_1^2 \omega^{(1)} - 2r_1\omega^{(1)}$ while $J^\varphi_{\rm max} = J^\varphi(z=1) \sim 1 - r_1^2\omega^{(1)}$. For example, with $\omega^{(1)} = \Omega_H/2 = 1/(2r_1^2)$, $J^\varphi_{\rm min} = 1/2 - 1/r_1 < 0$ while $J^\varphi_{\rm max} = 1/2$ and the sign change happens closer to the horizon for smaller r_1 (i.e., larger black holes). This is consistent with the sign change in the ZAMO frame, suggesting a dominant effect of frame dragging. However, the sign change actually persists for $J^\Phi = J^\varphi + \frac{a}{l^2} J^t$ as measured with respect to the non-rotating frame at infinity.

Another (perhaps related) observation about the current is that in the Kerr case it is everywhere space-like (i.e. 'magnetostatic') outside the horizon, while for Kerr-AdS $J_\mu J^\mu$ can change sign. In the presence of both positive and negative charges, a spacelike 'magnetostatic' current is perfectly physical, and indeed is likely the most stable configuration under the assumption that local electric fields are fully screened. An example from our solution is the current $J^\mu(\omega = \Omega_H) \propto \xi^\mu_{(t)} + \Omega_H \xi^\mu_{(\varphi)}$ which satisfies

[9] Note that for near-extremal black holes, this dual operator dimension is also in the range for which condensation is possible producing a holographic superfluid.

the force-free condition by noting $F_{\mu\nu}\left[\xi^{\nu}_{(t)}+\omega\xi^{\nu}_{(\varphi)}\right]\equiv 0$ and can certainly be space-like for small black holes from the previous discussions. The question of whether timelike (i.e. electrically dominant) current domains actually imply instabilities of some sort deserves further investigation.

5.5.3 Summary

In this chapter we have completed the main task of the present thesis, obtaining force-free solutions for a rotating monopole in Kerr-AdS background, in the slow rotation limit up to $\mathcal{O}(a^2)$. The AdS boundary condition render the rotation frequency of the magnetosphere arbitrary. Drawing an analogy to the field of a KNAdS black hole in the probe limit, we interpret our results on the boundary side as describing a freely roting magnetic field in a neutral fluid, up to $\mathcal{O}(a)$. Nevertheless, at $\mathcal{O}(a^2)$ we identify a dual azimuthal current from our non-trivial radial solutions. Whether a non-zero charge density appears at $\mathcal{O}(a^3)$ would require higher order solutions. The dual description at leading orders is also consistent with the stability arguments for large black holes.

The question of finding general exact solutions for finite 'a' still remains open. We next present some efforts that have been made in this direction, building on the remarks about the properties of the current source in the last section. A more thorough conclusion to the thesis will be given in chapter 7.

Chapter 6

Kerr-AdS Magnetospheres in Newman-Penrose formalism

The Newman-Penrose (NP) formalism [70] has successfully been applied in black hole perturbation theory, and has the merit that the Maxwell equations are first-order. Especially, exact solutions for finite 'a' with null currents have recently been constructed [21] using the NP formalism. In this chapter we rewrite the Kerr-AdS force-free equations in the NP formalism and search for new solutions for finite 'a' (necessary to understand the dual field theory beyond the limit of chapter 5).

6.1 Review of the NP formalism

The NP formalism [70, 71, 51], makes use of the null tetrad $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ normalized according to

$$l^\mu n_\mu = -m^\mu \bar{m}_\mu = 1, \tag{6.1}$$

so that the metric is $g_{\mu\nu} = 2l_{(\mu}n_{\nu)} - 2m_{(\mu}\bar{m}_{\nu)}$, where l^μ and n^μ are real, m^μ is complex and a bar denotes complex conjugate. The null tetrad is related to a real orthonormal basis $\{e_0^\mu, e_1^\mu, e_2^\mu, e_3^\mu\}$ by

$$l^\mu = \frac{e_0^\mu + e_1^\mu}{\sqrt{2}}, \quad n^\mu = \frac{e_0^\mu - e_1^\mu}{\sqrt{2}}, \quad m^\mu = \frac{e_2^\mu - ie_3^\mu}{\sqrt{2}} \tag{6.2}$$

$$e_0^\mu = \frac{l^\mu + n^\mu}{\sqrt{2}}, \quad e_1^\mu = \frac{l^\mu - n^\mu}{\sqrt{2}}, \quad e_2^\mu = \frac{m^\mu + \bar{m}^\mu}{\sqrt{2}}, \quad e_3^\mu = \frac{i(m^\mu - \bar{m}^\mu)}{\sqrt{2}} \tag{6.3}$$

We use indices in parentheses $\{(1),(2),(3),(4)\}$, interchangeably with $\{l,n,m,\bar{m}\}$ (as indices), to indicate tensor components from contractions with tetrad vectors $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ respectively. E.g., the metric with respect to the tetrad appears flat

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{6.4}$$

The contractions of the electromagnetic field $F_{\mu\nu}$ with tetrad vectors are specified by three complex NP variables $\{\phi_0, \phi_1, \phi_2\}$, which are conveniently defined as the coefficients in the expansion of the anti-self-dual part $F^-_{\mu\nu}$ of $F_{\mu\nu}$ [72]:

$$F^-_{\mu\nu} = \phi_0 U_{\mu\nu} + \phi_1 W_{\mu\nu} + \phi_2 V_{\mu\nu}, \tag{6.5}$$

where $F^-_{\mu\nu} \equiv \frac{1}{2}(F_{\mu\nu} + i\,{}^\star F_{\mu\nu})$ and the basis for anti-self-dual bivectors (i.e. antisymmetric tensors) is formed using the NP tetrad as,

$$U_{\alpha\beta} \equiv 2\bar{m}_{[\alpha} n_{\beta]}, \qquad W_{\alpha\beta} \equiv 2(n_{[\alpha} l_{\beta]} + m_{[\alpha}\bar{m}_{\beta]}), \qquad V_{\alpha\beta} \equiv 2 l_{[\alpha} m_{\beta]}. \tag{6.6}$$

The anti-self-duality of some bivector $A_{\mu\nu}$ is defined through the Hodge dual operation '$\star$' as ${}^\star A_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} A^{\alpha\beta} = -iA_{\mu\nu}$. Similarly, the self-duality is defined by the condition ${}^\star B_{\mu\nu} = iB_{\mu\nu}$, for some other bivector $B_{\mu\nu}$. The anti-self-duality of the basis bivectors (6.6) implies that their complex conjugates are self-dual, e.g., ${}^\star\bar{U}_{\mu\nu} = \overline{{}^\star U}_{\mu\nu} = \overline{-iU}_{\mu\nu} = i\bar{U}_{\mu\nu}$, and form the basis for self-dual bivectors. Now we can decompose the electromagnetic bivector as the sum of the anti-self-dual and self-dual parts:

$$F_{\mu\nu} = (\phi_0 U_{\mu\nu} + \phi_1 W_{\mu\nu} + \phi_2 V_{\mu\nu}) + (\bar{\phi}_0 \bar{U}_{\mu\nu} + \bar{\phi}_1 \bar{W}_{\mu\nu} + \bar{\phi}_2 \bar{V}_{\mu\nu}). \tag{6.7}$$

Such a decomposition is consistent with the fact that the double Hodge operations on bivectors give identically ${}^{\star\star} = -1 = (\pm i)^2$. By noting that the only non-vanishing contractions between any two of $U_{\mu\nu}$, $V_{\mu\nu}$, $W_{\mu\nu}$ and their conjugates are [51]

$$U_{\mu\nu}V^{\mu\nu} = 2, \quad W_{\mu\nu}W^{\mu\nu} = \bar{W}_{\mu\nu}\bar{W}^{\mu\nu} = -4, \tag{6.8}$$

we can express the NP variables as projections onto the anti-self-dual bivector basis and equivalently as (the more commonly seen) contractions with tetrad vectors:

$$\phi_0 = \frac{1}{2}F_{\mu\nu}V^{\mu\nu} = F_{\mu\nu}l^\mu m^\nu \tag{6.9}$$

$$\phi_1 = -\frac{1}{4}F_{\mu\nu}W^{\mu\nu} = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) \tag{6.10}$$

$$\phi_2 = \frac{1}{2}F_{\mu\nu}U^{\mu\nu} = F_{\mu\nu}\bar{m}^\mu n^\nu. \tag{6.11}$$

Using (6.5) and (6.8), we also have

$$F^{-\mu\nu}F^-_{\mu\nu} = \frac{1}{2}(I_1 + iI_2) = 4(\phi_0\phi_2 - \phi_1^2), \tag{6.12}$$

where

$$I_1 \equiv F_{\mu\nu}F^{\mu\nu} = -{}^\star F_{\mu\nu}{}^\star F^{\mu\nu}, \qquad I_2 \equiv F_{\mu\nu}{}^\star F^{\mu\nu} = {}^\star F_{\mu\nu}F^{\mu\nu} \tag{6.13}$$

are the two invariants of the electromagnetic field. So the degeneracy condition becomes

$$I_2 = 8\Im(\phi_0\phi_2 - \phi_1^2) = 0. \tag{6.14}$$

6.2 Force-free equations in the original NP variables $\phi_{0,1,2}$

The force-free equations $F_{(a)(b)}J^{(b)} = 0$ are [21]

$$\Re(\phi_1 J_n - \phi_2 J_m) = 0, \qquad \Re(\phi_1 J_l - \bar{\phi}_0 J_m) = 0, \qquad 2i\Im\phi_1 J_{\bar{m}} + \bar{\phi}_0 J_n - \phi_2 J_l = 0. \tag{6.15}$$

We specify to the Kerr-AdS metric with the following Kinnersley-like tetrad in BL coordinates [73],[1]

$$l^\mu = \Big[\frac{r^2+a^2}{\Delta_r}, 1, 0, \frac{a\Xi}{\Delta_r}\Big] \tag{6.16}$$

$$n^\mu = \frac{\Delta_r}{2\Sigma}\Big[\frac{r^2+a^2}{\Delta_r}, -1, 0, \frac{a\Xi}{\Delta_r}\Big] \tag{6.17}$$

$$m^\mu = -\bar{\rho}\sqrt{\frac{\Delta_\theta}{2}}\Big[\frac{ia\sin\theta}{\Delta_\theta}, 0, 1, \frac{i\Xi}{\Delta_\theta\sin\theta}\Big], \tag{6.18}$$

[1] This tetrad corresponds to the metric signature $[+,-,-,-]$. Also, the t-components differ from those in [73] by a factor of Ξ.

listing components in the order $[t, r, \theta, \varphi]$. We have for the current (with $J_{(a)}$ defined without the 2π factor)

$$J_l = \rho^2 \nabla_l \phi_1' - \frac{1}{\rho \Delta_r \sqrt{\Delta_\theta} \sin\theta} \nabla_{\bar{m}} \phi_0' \tag{6.19}$$

$$J_n = -\rho^2 \nabla_n \phi_1' + \frac{\rho}{\sqrt{\Delta_\theta} \sin\theta} \nabla_m \phi_2' \tag{6.20}$$

$$J_m = \rho^2 \nabla_m \phi_1' - \frac{1}{\rho \Delta_r \sqrt{\Delta_\theta} \sin\theta} \nabla_n \phi_0' \tag{6.21}$$

$$J_{\bar{m}} = -\rho^2 \nabla_{\bar{m}} \phi_1' + \frac{\rho}{\sqrt{\Delta_\theta} \sin\theta} \nabla_l \phi_2', \tag{6.22}$$

where the rescaled NP variables are

$$\phi_0' \equiv \rho \Delta_r \sqrt{\Delta_\theta} \sin\theta \phi_0, \qquad \phi_1' \equiv \rho^{-2} \phi_1, \qquad \phi_2' \equiv \rho^{-1} \sqrt{\Delta_\theta} \sin\theta \phi_2, \tag{6.23}$$

and

$$\rho \equiv -\frac{1}{r - ia\cos\theta}. \tag{6.24}$$

Note that $\rho\bar{\rho} = 1/\Sigma$. The above 'definitions' (6.19)–(6.22) of the currents have incorporated the homogeneous Maxwell equations $\mathrm{d}F = 0$ which read

$$J_l = \bar{J}_l, \qquad J_n = \bar{J}_n, \qquad J_m = \bar{J}_{\bar{m}}, \qquad J_{\bar{m}} = \bar{J}_m. \tag{6.25}$$

Later (in section 6.6) when we present special solutions for $\phi_{0,1,2}$, we will need to verify (6.25) separately.

6.3 Formulation in modified NP variables associated with an orthonormal frame

In this section we introduce a set of modified NP variables, in terms of which we find physical quantities take more concise forms. We first make manifest that only two of the $F^{(a)(b)} J_{(b)} = 0$ equations are independent, as noted in section 5.3.1, which we choose to be

$$\phi_0 J_{(2)} = \bar{\phi}_2 J_{(1)} + (\phi_1 - \bar{\phi}_1) J_{(3)}, \tag{6.26}$$

$$\phi_0 J_{(4)} = (\phi_1 + \bar{\phi}_1) J_{(1)} - \bar{\phi}_0 J_{(3)}. \tag{6.27}$$

Now define the modified NP variables (with the same ϕ_1)

$$\Phi_1 \equiv \Delta_r \rho \phi_0 + \frac{2\phi_2}{\rho}, \qquad \Phi_2 \equiv \Delta_r \rho \phi_0 - \frac{2\phi_2}{\rho}, \qquad \Phi_{1,2} = \frac{\phi_0' \pm 2\phi_2'}{\sqrt{\Delta_\theta} \sin\theta}, \tag{6.28}$$

and split the current[2]

$$2J^T_{(2)} \equiv J_{(2)} + \frac{J_{(1)}}{k_1}, \qquad 2J^P_{(2)} \equiv J_{(2)} - \frac{J_{(1)}}{k_1}, \tag{6.29}$$

$$2J^T_{(4)} \equiv J_{(4)} + \frac{J_{(3)}}{k_2}, \qquad 2J^P_{(4)} \equiv J_{(4)} - \frac{J_{(3)}}{k_2}, \tag{6.30}$$

where $k_1 \equiv 2\Sigma/\Delta_r$, $k_2 \equiv -\bar{\rho}/\rho$. Linear combinations of the force-free equations (6.26) and (6.27) give the following equivalent set in terms of the above newly define quantities:

$$(\Im\Phi_1 + \Im\Phi_2) J^P_{(2)} - 2\bar{\rho}\Delta_r \Im\phi_1 J^P_{(4)} = 0 \tag{6.31}$$

$$(\Re\Phi_2 + i\Im\Phi_1) J^T_{(2)} + \bar{\Phi}_1 J^P_{(2)} + 2i\bar{\rho}\Delta_r \Im\phi_1 J^T_{(4)} = 0. \tag{6.32}$$

The degeneracy condition (6.14) becomes

$$\Im(\Phi_1^2 - \Phi_2^2 - 8\Delta_r \phi_1^2) = 0. \tag{6.33}$$

The newly defined currents read explicitly

$$J^T_{(2)} = \frac{\partial_\theta(\Phi_2 \sin\theta\sqrt{\Delta_\theta})}{4\sqrt{2}\Sigma\sin\theta} + \frac{\Delta_r\rho^2}{2\Sigma}\partial_r\frac{\phi_1}{\rho^2} - i\frac{\Xi\partial_\varphi + a\sin^2\theta\partial_t}{4\sqrt{2}\Sigma\sin\theta\sqrt{\Delta_\theta}}\Phi_1 \tag{6.34}$$

---

[2]The superscript "T/P" indicates that, e.g., $J^{T/P}_{(2)}$ only involves the contractions of J_μ with the toroidal/poloidal components of n^μ. Note that in the Kerr limit, Φ_1 & Φ_2 are proportional to the electric and magnetic field components in the orthonormal frame associated with the Carter tetrad, e.g., $\Phi_1 \sim (E_3 + iB_3)_{\text{Carter}}$ etc., and $J^{T/P}_{(2)/(4)}$ are proportional to components of the current in this orthonormal frame [74].

$$J^P_{(2)} = -\frac{\partial_\theta(\Phi_1 \sin\theta\sqrt{\Delta_\theta})}{4\sqrt{2}\Sigma\sin\theta} - \frac{a\Xi\partial_\varphi + (r^2+a^2)\partial_t}{2\Sigma}\phi_1 + i\frac{\Xi\partial_\varphi + a\sin^2\theta\partial_t}{4\sqrt{2}\Sigma\sin\theta\sqrt{\Delta_\theta}}\Phi_2 \tag{6.35}$$

$$J^T_{(4)} = -\frac{\rho\partial_r\Phi_2}{4} + \frac{\sqrt{\Delta_\theta}\rho^3}{\sqrt{2}}\partial_\theta\frac{\phi_1}{\rho^2} + \rho\frac{a\Xi\partial_\varphi + (r^2+a^2)\partial_t}{4\Delta_r}\Phi_1 \tag{6.36}$$

$$J^P_{(4)} = \frac{\rho\partial_r\Phi_1}{4} - \rho\frac{a\Xi\partial_\varphi + (r^2+a^2)\partial_t}{4\Delta_r}\Phi_2 - i\rho\frac{\Xi\partial_\varphi + a\sin^2\theta\partial_t}{\sqrt{2}\sin\theta\sqrt{\Delta_\theta}}\phi_1. \tag{6.37}$$

Note that $J^{T/P}_{(2)}$ are real and $J^{T/P}_{(4)}$ are complex. The real electromagnetic field takes relatively more compact forms:

$$F_{r\varphi} = -2\frac{a\sin^2\theta}{\Xi}\Re\phi_1 - \frac{(r^2+a^2)\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Xi\Delta_r}\Im\Phi_2 \tag{6.38}$$

$$F_{rt} = 2\Re\phi_1 + \frac{a\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Delta_r}\Im\Phi_2 \tag{6.39}$$

$$F_{\theta\varphi} = -\frac{a\sin^2\theta}{\sqrt{2}\Xi\sqrt{\Delta_\theta}}\Re\Phi_2 + 2\frac{(r^2+a^2)\sin\theta}{\Xi}\Im\phi_1 \tag{6.40}$$

$$F_{\theta t} = -2a\sin\theta\Im\phi_1 + \frac{\Re\Phi_2}{\sqrt{2}\sqrt{\Delta_\theta}} \tag{6.41}$$

$$B_T + iF_{\varphi t} = \frac{\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Xi}\Phi_1. \tag{6.42}$$

One sees again that the non-vanishing of the poloidal currents J^P depends crucially on the toroidal magnetic field $B_T \sim \Re\Phi_1$. The equations (6.31) & (6.32) form a useful starting point for further study of force-free solutions in Kerr-AdS away from the slow-rotation ansatz. We next employ the NP formalism to present some existing and new solutions.

6.4 BZ's monopole solution

We first recast BZ's monopole ansatz using the modified NP variables. This is the stationary and axisymmetric case with $F_{\varphi t} \propto \Im\Phi_1 = 0$ and all quantities being (φ, t)-independent. The degeneracy condition (6.33) becomes

$$\Re\Phi_2\Im\Phi_2 = -8\Delta_r\Re\phi_1\Im\phi_1. \tag{6.43}$$

The two force-free equations (6.31) and (6.32) read explicitly (with Φ_1 now real)

$$(2\sqrt{2}\Delta_r \Im\phi_1 \partial_r + \sqrt{\Delta_\theta}\Im\Phi_2\partial_\theta)(\Phi_1 \sin\theta\sqrt{\Delta_\theta}) = 0 \tag{6.44}$$

$$\begin{aligned}\Big(\frac{\Re\Phi_2}{2\sqrt{2}\Delta_r}\partial_\theta - i\frac{\Im\phi_1}{\sqrt{\Delta_\theta}}\partial_r\Big)(\Phi_2\sin\theta\sqrt{\Delta_\theta}) - \frac{\Phi_1\partial_\theta(\Phi_1\sin\theta\sqrt{\Delta_\theta})}{2\sqrt{2}\Delta_r}\\ +\rho^2\sin\theta(\Re\Phi_2\partial_r + i2\sqrt{2}\sqrt{\Delta_\theta}\Im\phi_1\partial_\theta)\frac{\phi_1}{\rho^2} = 0.\end{aligned} \tag{6.45}$$

We also have

$$\omega = \Xi\frac{\Re\Phi_2 - 2\sqrt{2}a\sin\theta\sqrt{\Delta_\theta}\Im\phi_1}{a\sin^2\theta\Re\Phi_2 - 2\sqrt{2}(r^2+a^2)\sin\theta\sqrt{\Delta_\theta}\Im\phi_1} \tag{6.46}$$

$$T^r_t = \frac{\Phi_1}{2\Sigma}(\Re\Phi_2 - 2\sqrt{2}a\sin\theta\sqrt{\Delta_\theta}\Im\phi_1). \tag{6.47}$$

It can be checked that (6.45) indeed reduces to (5.57) for the slowly rotating monopole ansatz, while (6.44) is equivalent to $\{A_\varphi, B_T\} = 0$.

6.5 Brennan et al's solution in Kerr-AdS

The analytic solutions for the BZ process have been restricted to the perturbatively slow rotation regime. The only exceptions known at present are exact solutions with null currents [20, 75, 76, 21]. A recent thorough discussion has been given in [21] using the NP formalism for the Kerr case, which we now generalize to Kerr-AdS.

6.5.1 Derivation of the equations

We consider a null current J^μ along the principle null direction n^μ of the NP tetrad so that only $J_l = n_\mu l^\mu \equiv \mathcal{J}$ is non-zero, which is real. Then the force-free equations (6.15) or (6.26) & (6.27) impose

$$\phi_2 = 0 = \Re\phi_1, \tag{6.48}$$

which already makes the degeneracy condition (6.14) hold and is stronger than the latter. In terms of the new variables defined in (6.28) this implies

$$\Phi_1 = \Phi_2 = \Delta_r\rho\phi_0. \tag{6.49}$$

In addition, $J_n = 0$ gives $\nabla_n \phi_1' = 0$ from (6.20), where ∇_n becomes simply ∂_r if one transforms to the (Kerr-AdS analogue of) ingoing Kerr coordinates

$$\mathrm{d}\psi = \mathrm{d}\varphi + \frac{a\Xi}{\Delta_r}\,\mathrm{d}r \tag{6.50}$$

$$\mathrm{d}v = \mathrm{d}t + \frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r. \tag{6.51}$$

Then $\partial_r \phi_1' = 0$ implies $\Im\phi_1 = 0$ for $a \neq 0$, so $\phi_1 = 0$. Similarly, $J_m = 0$ implies $\Phi_1(r,\theta,\psi,v) = \Phi_1(\theta,\psi,v)$ in these new coordinates. The whole problem now reduces to solving one of Maxwell equations $J_l = \mathcal{J}$ for some prescribed function $\mathcal{J}$, subject to the reality of J_l:

$$\sqrt{2}\mathcal{J} = \sqrt{2}\Re J_l = \frac{1}{\Delta_r\sqrt{\Delta_\theta}\sin\theta}\left(\Xi\partial_\psi\Im\Phi_1 + a\sin^2\theta\partial_v\Im\Phi_1\right) + \frac{\partial_\theta(\Re\Phi_1\sqrt{\Delta_\theta}\sin\theta)}{\Delta_r\sin\theta} \tag{6.52}$$

$$0 = \sqrt{2}\Im J_l = -\frac{1}{\Delta_r\sqrt{\Delta_\theta}\sin\theta}\left(\Xi\partial_\psi\Re\Phi_1 + a\sin^2\theta\partial_v\Re\Phi_1\right) + \frac{\partial_\theta(\Im\Phi_1\sqrt{\Delta_\theta}\sin\theta)}{\Delta_r\sin\theta}, \tag{6.53}$$

which reduce to Brennan et al's equations [21] in the Kerr case. In conclusion, the requirement of the current being null reduces the force-free equations to a single Maxwell equation supplemented by a series of constraints on the (NP) field variables ((6.48), etc.).

6.5.2 Relation to real electromagnetic field components

For the above null current configuration, the real field components (in the original BL coordinates) are

$$F_{r\theta} = -\frac{\Sigma}{\sqrt{2}\Delta_r\sqrt{\Delta_\theta}}\Re\Phi_1, \quad F_{\theta\varphi} = -\frac{a\sin^2\theta}{\sqrt{2}\Xi\sqrt{\Delta_\theta}}\Re\Phi_1, \quad F_{\theta t} = \frac{1}{\sqrt{2}\sqrt{\Delta_\theta}}\Re\Phi_1,$$

$$F_{r\varphi} = -\frac{(r^2+a^2)\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Xi\Delta_r}\Im\Phi_1, \quad F_{rt} = \frac{a\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Delta_r}\Im\Phi_1, \quad F_{\varphi t} = \frac{\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Xi}\Im\Phi_1. \tag{6.54}$$

(In the ingoing Kerr coordinates: $F^{\rm IK}_{r\theta} = 2F_{r\theta}$, $F^{\rm IK}_{r\psi} = 2F_{r\varphi}$, $F^{\rm IK}_{rv} = 2F_{rt}$.) Also,

$$B_T = \frac{\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Xi}\Re\Phi_1. \tag{6.55}$$

The definition of ω only exists for $F_{\varphi t} \propto \Im\Phi_1 = 0$, with

$$\omega = -\frac{F_{\theta t}}{F_{\theta\varphi}} = \frac{\Xi}{a\sin^2\theta}, \tag{6.56}$$

and in this case the degeneracy condition ${}^\star F_{\mu\nu}F^{\mu\nu} = 0$ is satisfied trivially with $F_{r\varphi} = F_{rt} = F_{\varphi t} = 0$.

6.5.3 Some solutions with null currents

Stationary (and axisymmetric) case

Analogously to the analysis in [21], assuming stationarity ($\partial_v \to 0$) and with (6.53) reducing to

$$\partial_\theta\left[\sin\theta\sqrt{\Delta_\theta}\Im\Phi_1(\theta,\psi)\right] - \partial_\psi\left[\frac{\Xi}{\sqrt{\Delta_\theta}}\Re\Phi_1(\theta,\psi)\right] = 0, \tag{6.57}$$

one deduces the existence of an exact 1-form $\partial_\psi S\,\mathrm{d}\psi + \partial_\theta S\,\mathrm{d}\theta$ with

$$\partial_\psi S(\theta,\psi) = \sin\theta\sqrt{\Delta_\theta}\Im\Phi_1(\theta,\psi), \qquad \partial_\theta S(\theta,\psi) = \frac{\Xi}{\sqrt{\Delta_\theta}}\Re\Phi_1(\theta,\psi). \tag{6.58}$$

Using (6.58) one can turn (6.52) into an equation for S.

To further impose axisymmetry, [21] assumes a ψ-independent potential $S(\theta)$ so that

$$\Im\Phi_1 = 0, \tag{6.59}$$

and (6.52) becomes

$$[\sin\theta\Delta_\theta S'(\theta)]' = \sqrt{2}\Xi\Delta_r\sin\theta\mathcal{J}(r,\theta), \tag{6.60}$$

where a prime denotes θ-derivative. In the Kerr limit, this reduces to the solution first found by Menon and Dermer [76].

However, one can impose axisymmetry only on the field Φ_1 and allow a linear

ψ-dependence of S, namely, we have the solution

$$S(\theta,\psi) = C_1\psi + S_1(\theta), \qquad \Im\Phi_1(\theta) = \frac{C_1}{\sin\theta\sqrt{\Delta_\theta}}, \qquad \Re\Phi_1(\theta) = \frac{\sqrt{\Delta_\theta}}{\Xi}S_1'(\theta), \quad (6.61)$$

with $S_1(\theta)$ satisfying the same equation (6.60) as $S(\theta)$. Setting $C_1 = 0$ gives back the above case considered in [21].

We can try to solve (6.54) for vector potentials using (6.61). For $C_1 = 0$, and in addition assuming A_μ to be (φ, t)-independent, we find

$$A_t(\theta) = \frac{S_1(\theta)}{\sqrt{2}\Xi} + \text{const.} \qquad (6.62)$$

$$A_\varphi(\theta) = -\frac{a}{\sqrt{2}\Xi^2}\int S_1'(\theta)\sin^2\theta\,\mathrm{d}\theta + \text{const.} \qquad (6.63)$$

$$\partial_\theta A_r(r,\theta) - \partial_r A_\theta(r,\theta) = \frac{\Sigma}{\sqrt{2}\Xi\Delta_r}S_1'(\theta), \qquad (6.64)$$

with $A_{r(\theta)}$ undetermined. For $C_1 \neq 0$, we only allow a φ-dependence through a linear $\psi(\varphi, r)$-dependence of A_t (with other A_μ remaining (φ, t)-independent) given by

$$A_t = \frac{S[\theta,\psi(\varphi,r)]}{\sqrt{2}\Xi} = \frac{1}{\sqrt{2}\Xi}\big[C_1(\varphi + \int\frac{a\Xi}{\Delta_r}\,\mathrm{d}r) + S_1(\theta)\big], \qquad (6.65)$$

and we find

$$A_\varphi = -\frac{C_1}{\sqrt{2}\Xi}\int\frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r - \frac{a}{\sqrt{2}\Xi^2}\int S_1'(\theta)\sin^2\theta\,\mathrm{d}\theta, \qquad (6.66)$$

and (6.64) still holds.

General non-stationary, non-axisymmetric case

An example non-stationary, non-axisymmetric solution presented in [21] can also be generalized to Kerr-AdS, solving equations (6.52) & (6.53):

$$\Re\Phi_1 = 15\sqrt{\Delta_\theta}X(\Xi v)\cos\theta\sin^2\theta\cos\psi \qquad (6.67)$$

$$\Im\Phi_1 = \frac{\Xi}{\sqrt{\Delta_\theta}}\big[-5X(\Xi v)\sin^2\theta\sin\psi + 3aX'(\Xi v)\sin^4\theta\cos\psi + \frac{F_1(\psi,v)}{\sin\theta}\big], \qquad (6.68)$$

where $X(\Xi v)$ is a real function and we have kept the integration constant $F_1(\psi, v)$ from solving (6.53).

Again we try to find the vector potential for this solution.

$$\partial_r A_\theta - \partial_\theta A_r = -15\sqrt{2}X(\Xi v)\frac{\Sigma}{\Delta_r}\cos\theta\sin^2\theta\cos\psi \tag{6.69}$$

$$\partial_r A_\psi - \partial_\psi A_r = \sqrt{2}\frac{r^2+a^2}{\Delta_r}\big[5X(\Xi v)\sin^3\theta\sin\psi - 3aX'(\Xi v)\sin^5\theta\cos\psi - F_1(\psi,v)\big] \tag{6.70}$$

$$\partial_r A_v - \partial_v A_r = -\sqrt{2}\frac{a\Xi}{\Delta_r}\big[5X(\Xi v)\sin^3\theta\sin\psi - 3aX'(\Xi v)\sin^5\theta\cos\psi - F_1(\psi,v)\big] \tag{6.71}$$

$$\partial_\theta A_\psi - \partial_\psi A_\theta = -\frac{15a}{\sqrt{2}\Xi}X(\Xi v)\cos\theta\sin^4\theta\cos\psi \tag{6.72}$$

$$\partial_\theta A_v - \partial_v A_\theta = \frac{15}{\sqrt{2}}X(\Xi v)\cos\theta\sin^2\theta\cos\psi \tag{6.73}$$

$$\partial_\psi A_v - \partial_v A_\psi = -\frac{1}{\sqrt{2}}\big[5X(\Xi v)\sin^3\theta\sin\psi - 3aX'(\Xi v)\sin^5\theta\cos\psi - F_1(\psi,v)\big]. \tag{6.74}$$

We consider a simple case $X(\Xi v) = 0$, finding

$$F_1(\psi,v) = -\sqrt{2}F_1 \tag{6.75}$$

$$A_v = \partial_v\int A_\theta\,\mathrm{d}\theta + F_2(r,\psi,v) \tag{6.76}$$

$$A_\psi = \partial_\psi\Big[\int A_\theta\,\mathrm{d}\theta + \int F_2(r,\psi,v)\,\mathrm{d}v\Big] + F_1 v + F_3(r,\psi) \tag{6.77}$$

$$A_r = \partial_r\Big[\int A_\theta\,\mathrm{d}\theta + \int F_2(r,\psi,v)\,\mathrm{d}v + \int F_3(r,\psi)\,\mathrm{d}\psi\Big] + 2F_1\Big(\frac{a\Xi}{\Delta_r}\int \mathrm{d}v - \frac{r^2+a^2}{\Delta_r}\int\mathrm{d}\psi\Big) + F_4(r), \tag{6.78}$$

with $A_\theta(r,\theta,\psi,v)$ remaining arbitrary. One notices that

$$\mathcal{A} \equiv \int A_\theta\,\mathrm{d}\theta + \int F_2(r,\psi,v)\,\mathrm{d}v + \int F_3(r,\psi)\,\mathrm{d}\psi \tag{6.79}$$

appears to be pure gauge, so $A_\mu - \partial_\mu\mathcal{A}$ reads

$$A_\theta = A_v = 0, \qquad A_\psi(v) = F_1 v, \tag{6.80}$$

$$A_r(r,\psi,v) = 2F_1\Big(\frac{a\Xi}{\Delta_r}\int \mathrm{d}v - \frac{r^2+a^2}{\Delta_r}\int \mathrm{d}\psi\Big) + F_4(r). \tag{6.81}$$

As an alternative, we try the ansatz

$$A_v = k_1\psi + f_1(r) + g_1(\theta), \qquad A_\psi = k_2 v + f_2(r) + g_2(\theta), \tag{6.82}$$

and find

$$A_\theta = g_1'(\theta)v + g_2'(\theta)\psi + K_1(r,\theta) \tag{6.83}$$

$$\begin{aligned} A_r = \big[f_1'(r) - 2\frac{a\Xi}{\Delta_r}(k_1-k_2)\big]v + \big[f_2'(r) + 2\frac{(r^2+a^2)}{\Delta_r}(k_1-k_2)\big]\psi \\ + \partial_r\int K_1(r,\theta)\,\mathrm{d}\theta + K_2(r), \end{aligned} \tag{6.84}$$

where $K_1(r,\theta)$ and $K_2(r)$ are integration constants.

Energy densities of null current solutions

The energy density for the null current case takes a simple form

$$-T_{tt} = \frac{|\Phi_1|^2}{2\Sigma}, \qquad -T^t_t = \frac{r^2+a^2}{2\Delta_r\Sigma}|\Phi_1|^2. \tag{6.85}$$

Then for the stationary and axisymmetric solution (6.61) the energy density can be an arbitrary function of θ, while for the solution given by (6.67) and (6.68) the energy density can further have (ψ, v)-dependences.

6.6 Looking for new solutions ($\phi_1 = 0$)

The null current solutions presented above have the common constraints $\phi_1 = \phi_2 = 0$ (or $\phi_1 = \phi_0 = 0$ if we assume $J_{(a)} = J_n$ rather than $J_{(a)} = J_l$), and the electromagnetic field is necessarily "null" (by which one means $I_1 = I_2 = 0$.). In this section we try to construct some solutions with weaker constraints on the field (so that it is not null) and then infer the causal nature of the currents *a posteriori*. The search is not exhaustive but the solutions we found have non-null currents and provide some interesting alternatives.

We proceed by first manipulating two special cases:

1. $\Re\phi_1 = 0$.

   The first two force-free equations in (6.15) become $\Re(\phi_2 J_m) = \Re(\bar{\phi}_0 J_m) = 0$ which implies

$$\phi_2 = c(r, \theta, \varphi, t)\bar{\phi}_0, \tag{6.86}$$

   for some real function $c(r, \theta, \varphi, t)$, and the last equation in (6.15) becomes $2i\Im\phi_1 J_{\bar{m}} + \bar{\phi}_0[J_n - c(r, \theta, \varphi, t)J_l] = 0$.

2. $\Im\phi_1 = 0$.

   The last equation in (6.15) implies

$$\frac{\phi_2}{\bar{\phi}_0} = \frac{J_n}{J_l}, \tag{6.87}$$

   which is real, and the first two equations in (6.15) are both equivalent to $\Re(\phi_1 J_l - \bar{\phi}_0 J_m) = 0$.

In either case it is still not easy to find a solution, so we simply assume $\phi_1 = 0$. This would be the condition implied by (6.15) if $J_l = J_n = 0$ which is of interest here (as an alternative to the $J_m = J_{\bar{m}} = 0$ solutions in [21]). Now the field variables are

$$\phi_1 \equiv 0, \qquad \phi_2 = c(r, \theta, \varphi, t)\bar{\phi}_0 \quad \Leftrightarrow \quad \phi'_2 = c(r, \theta, \varphi, t)\frac{\Sigma}{\Delta_r}\bar{\phi}'_0, \tag{6.88}$$

and the force-free equations (6.15) reduce to

$$J_n = c(r, \theta, \varphi, t)J_l, \qquad \Re(\bar{\phi}_0 J_m) = 0, \tag{6.89}$$

with $J_l = J_n = 0$ as a special case. In addition, one needs to check that the identity $\bar{J}_{\bar{m}} = J_m$ (by definition) and the condition J_n (or J_l) being real hold for the field variables constrained as in (6.88). This configuration is degenerate but not null, i.e., $\phi_0\phi_2 - \phi_1^2 \propto I_1 + iI_2$ is real and non-zero. Explicitly, we have for the two force-free equations (6.89) and the two constraints on the currents

$$\bar{J}_n = J_n = cJ_l \qquad \Rightarrow \qquad \nabla_{\bar{m}}(\phi'_0\bar{\phi}'_2) = 0 \tag{6.90}$$

$$\Re(\bar{\phi}_0 J_m) = 0 \qquad \Rightarrow \qquad 2\Re(\bar{\phi}'_0\nabla_n\phi'_0) = \nabla_n(\bar{\phi}'_0\phi'_0) = 0 \tag{6.91}$$

$$\bar{J}_{\bar{m}} = J_m \qquad \Rightarrow \qquad \frac{\Sigma}{\Delta_r}\nabla_n\phi_0' + \nabla_l\bar{\phi}_2' = 0 \tag{6.92}$$

$$J_n = \bar{J}_n \qquad \Rightarrow \qquad \sin\theta\Delta_\theta\partial_\theta\Im\phi_2' + (\Xi\partial_\varphi + a\sin^2\theta\partial_t)\Re\phi_2' = 0, \tag{6.93}$$

where we have eliminated c using (the conjugate of) (6.88).

We also impose stationarity and axisymmetry through $F_{\varphi t} \propto \Im\Phi_1 = 0$, which by (6.28), (6.23) and (6.88) implies[3]

$$\Im\big[\phi_0' + \frac{2\Sigma}{\Delta_r}c(r,\theta,\varphi,t)\bar{\phi}_0'\big] = 0, \tag{6.94}$$

which breaks up into two cases

$$\Im\phi_0' = 0 \qquad \text{and} \qquad c(r,\theta,\varphi,t) = \frac{\Delta_r}{2\Sigma}. \tag{6.95}$$

We next construct solutions for each case.

6.6.1 Case 1: $\Im\phi_0' = 0$

In this case we additionally have $\Im\phi_2' = 0$ by (6.88). Then (6.90), (6.91), (6.92) and (6.93) reduce respectively to

$$\Re(6.90) \qquad \Leftrightarrow \qquad \partial_\theta(\phi_0'\phi_2') = 0 \tag{6.96}$$

$$\Im(6.90) \qquad \Leftrightarrow \qquad (\Xi\partial_\varphi + a\sin^2\theta\partial_t)(\phi_0'\phi_2') = 0 \tag{6.97}$$

$$\nabla_n\phi_0' = 0 \tag{6.98}$$

$$\nabla_l\phi_2' = 0 \tag{6.99}$$

$$(\Xi\partial_\varphi + a\sin^2\theta\partial_t)\phi_2' = 0. \tag{6.100}$$

It is easy to deduce from (6.96) and (6.97) that

$$\phi_0'\phi_2' = p(r), \tag{6.101}$$

for some function $p(r)$. We again transform to the ingoing Kerr coordinates $\{v, r, \theta, \psi\}$ as in section 6.5. Then (6.98) is simply $\partial_r\phi_0' = 0$ so $\phi_0' = \phi_0'(v,\theta,\psi)$, and (6.99) and

---

[3] The condition $F_{\varphi t} = 0$ usually corresponds to $\partial_\varphi, \partial_t \to 0$. Nevertheless for the moment we keep the full coordinate dependence of the unknowns $\phi_0(r,\theta,\varphi,t)$, $\phi_2(r,\theta,\varphi,t)$ and $c(r,\theta,\varphi,t)$.

(6.100) become, using (6.101),

$$\Delta_r \partial_r \ln p(r) - 2[a\Xi\partial_\psi + (r^2+a^2)\partial_v]\phi_0'(v,\theta,\psi) = 0 \tag{6.102}$$

$$(\Xi\partial_\psi + a\sin^2\theta\partial_v)\phi_0'(v,\theta,\psi) = 0. \tag{6.103}$$

The solution formally takes the form

$$\phi_0'(v,\theta,\psi) = q(\theta)\exp\Big[\frac{\Delta_r\partial_r \ln p(r)}{\Sigma}\big(v - \frac{a\sin^2\theta}{\Xi}\psi\big)\Big], \tag{6.104}$$

where $q(\theta)$ is an arbitrary function of θ. For the r.h.s to be r-independent, one therefore must have $p(r) = p = \text{const.}$,[4] and the solutions are (ψ, v)-independent:

$$\phi_0' = q(\theta), \qquad \phi_2' = \frac{p}{q(\theta)}, \tag{6.105}$$

or for the modified NP variables,

$$\Phi_{1,2} = \frac{q(\theta) \pm 2\frac{p}{q(\theta)}}{\sqrt{\Delta_\theta}\sin\theta}. \tag{6.106}$$

One can check that $I_2 = 0$ trivially for real $\phi_{0,2}'$ and $\Phi_{1,2}$, while the other invariant is

$$I_1 = \frac{\Phi_1^2 - \Phi_2^2}{\Delta_r} = \frac{8p}{\Delta_r\Delta_\theta\sin^2\theta}, \tag{6.107}$$

with the sign depending on that of p. In fact from (6.107) one finds

$$0 = \partial_\theta p \propto \Phi_1\partial_\theta(\Phi_1\sqrt{\Delta_\theta}\sin\theta) - \Phi_2\partial_\theta(\Phi_2\sqrt{\Delta_\theta}\sin\theta) \propto \Phi_1 J^P_{(2)} + \Phi_2 J^T_{(2)}, \tag{6.108}$$

which is just one of the force-free equations (6.32) for the current case. (The other one (6.31) is satisfied trivially due to vanishing coefficients.) Also, the real field quantities are

$$B_T = \frac{\sin\theta\sqrt{\Delta_\theta}}{\sqrt{2}\Xi}\Phi_1, \qquad \omega = \frac{\Xi}{a\sin^2\theta} \tag{6.109}$$

$$F_{\theta\varphi} = -\frac{a\sin^2\theta}{\sqrt{2}\Xi\sqrt{\Delta_\theta}}\Phi_2, \qquad F_{\theta t} = \frac{\Phi_2}{\sqrt{2}\sqrt{\Delta_\theta}} = -\omega F_{\theta\varphi}, \qquad F_{r\varphi} = F_{rt} = 0. \tag{6.110}$$

[4] One may also restrict oneself to the orbits $\psi = v\Xi/(a\sin^2\theta)$, or to the constant-$\theta$ surfaces on which the r-dependent coefficient on the exponent can be made constant: $\Sigma^{-1}\Delta_r\partial_r \ln p(r) = \text{const.}$.

One sees that ω is fixed (and coincides with (6.56) from the null current case), and there are essentially two independent quantities: the toroidal and poloidal magnetic fields, B_T and $F_{\theta\varphi} \propto B^r$, which are related as (by rewriting (6.107))

$$B_T^2 - (F'_{\theta\varphi})^2 = \frac{4p}{\Xi^2}, \qquad (F'_{\theta\varphi} \equiv \frac{\Delta_\theta}{a\sin\theta} F_{\theta\varphi}). \tag{6.111}$$

We also write the currents in terms of these more conventional quantities (instead of NP variables):

$$J^T_{(2)} = -\frac{\Xi}{4\Sigma\sin\theta}\partial_\theta F'_{\theta\varphi}, \qquad J^P_{(2)} = -\frac{\Xi}{4\Sigma\sin\theta}\partial_\theta B_T, \qquad J^T_{(4)} = J^P_{(4)} = 0. \tag{6.112}$$

One can check the regularities at $\theta \to 0$ for the above quantities, against the known physical configuration of the BZ monopole, for which

$$B_T \sim \mathcal{O}(\theta^2), \qquad F'_{\theta\varphi} \sim B^r \sim \mathcal{O}(1), \qquad J^\mu \sim \mathcal{O}(1). \tag{6.113}$$

Taking a 'monopole' field $F_{\theta\varphi} = Ca\sin\theta$ (with charge Ca), then

$$F'_{\theta\varphi} = C\Delta_\theta \qquad B_T^2 = C^2\Delta_\theta^2 + \frac{4p}{\Xi^2} \tag{6.114}$$

$$J^T_{(2)} = -Ca^2\cos\theta\frac{\Xi}{2l^2\Sigma}, \qquad J^P_{(2)} = C^2a^2\cos\theta\frac{\Xi\Delta_\theta}{2l^2\Sigma B_T}. \tag{6.115}$$

All of the above quantities are $\sim \mathcal{O}(\theta^0)$, which however means a divergent $B^\varphi \propto B_T/\sin^2\theta$. The best one can do is set $p = p_0 \equiv -C^2\Xi^4/4$ so that

$$B_T = C\frac{a}{l}\sin\theta\sqrt{\Delta_\theta + \Xi}, \tag{6.116}$$

but at the price of a divergent $J^P_{(2)} \propto 1/B_T \sim \mathcal{O}(\theta^{-1})$. For other values of p, B_T may have a zero at some $\theta = \theta_0$ with $\Delta^2_{\theta_0} = \Xi^2 p/p_0$ resulting in a divergent $J^P_{(2)}$. This can only be avoided if $p/p_0 < 1$.

We therefore need to look for other field configurations than the monopole if the scaling (6.113) is desired in the current solution. One simple possibility is as follows

$$(F'_{\theta\varphi})^2 = -\frac{4p}{\Xi^2} + \sin^4\theta, \qquad B_T = \sin^2\theta \tag{6.117}$$

$$J^T_{(2)} = -\frac{\Xi\cos\theta\sin^2\theta}{2\Sigma F'_{\theta\varphi}}, \qquad J^P_{(2)} = -\frac{\Xi\cos\theta}{2\Sigma}. \tag{6.118}$$

Similarly, being careful about the possible divergence of $J^T_{(2)} \propto 1/F'_{\theta\varphi}$ at finite θ, one finds the constraint $p < -\Xi^2/4$. Thus at least for the above two cases, the constraints on p make the solutions electrically dominant, i.e. $I_1 \propto p < 0$.

To further infer the causal nature of the current, we evaluate

$$J_{(a)}J^{(a)} = 2J_l J_n = 4\frac{\Sigma}{\Delta_r}[(J^T_{(2)})^2 - (J^P_{(2)})^2] = \frac{[\partial_\theta \ln q(\theta)]^2}{\Delta_r \Sigma \sin^2\theta} p, \tag{6.119}$$

so p also determines if the current is timelike, spacelike or null. Again for the above two cases, the currents are always spacelike (for metric signature $[+,-,-,-]$), as in the BZ monopole solution (cf. discussions in section 5.5.2).

It is also interesting to evaluate the energy density

$$-T_{tt} = -\frac{2a^2 p}{\Delta_r \Sigma} - \frac{\Xi^2 B_T^2 - 2p}{\Sigma \Delta_\theta \sin^2\theta} \tag{6.120}$$

$$-T^t_t = \left[-2a^2 p + \frac{r^2 + a^2}{\sin^2\theta}(\Xi^2 B_T^2 - 2p)\right]\frac{1}{\Delta_r \Delta_\theta \Sigma}. \tag{6.121}$$

The divergences at $\sin\theta = 0$ are removed if we choose, e.g.,

$$\Xi^2 B_T^2 - 2p = \Xi^2 (F'_{\theta\varphi})^2 + 2p = C \sin^2\theta, \tag{6.122}$$

where (6.111) is used and C is constant. In this case the currents are also regular.

Finally, note that B_T, $F_{\theta\varphi}$ and $J^{T,P}_{(2)}$ are each only determined up to a sign as is clear from (6.111). Or equivalently from (6.106), the sign flip $\{\Phi_1, \Phi_2\} \to \{\Phi_1, -\Phi_2\}$ corresponds to the substitution $q(\theta) \to p/q(\theta)$, $\{\Phi_1, \Phi_2\} \to \{-\Phi_1, \Phi_2\}$ corresponds to $q(\theta) \to -p/q(\theta)$, and $\{\Phi_1, \Phi_2\} \to \{-\Phi_1, -\Phi_2\}$ to $q(\theta) \to -q(\theta)$.

6.6.2 Case 2: $c(r, \theta, \varphi, t) = \Delta_r/(2\Sigma)$

We now turn to the second case in (6.95), which implies

$$\phi'_2 = \frac{\bar{\phi}'_0}{2}. \tag{6.123}$$

We rewrite, again in the ingoing Kerr coordinates,

$$\phi'_0 = \exp[R(r,\theta,\psi,v) + i\Theta(r,\theta,\psi,v)], \qquad \phi'_2 = \frac{\exp[2R(r,\theta,\psi,v)]}{2\phi'_0}, \tag{6.124}$$

for arbitrary functions R and Θ. Then (6.90), (6.91) and (6.92) reduce respectively to

$$\Re(6.90) \quad \Leftrightarrow \quad \sin\theta\Delta_\theta\partial_\theta R + a\sin^2\theta\partial_v\Theta - \Xi\partial_\psi\Theta = 0 \tag{6.125}$$

$$\Im(6.90) \quad \Leftrightarrow \quad \sin\theta\Delta_\theta\partial_\theta\Theta + a\sin^2\theta\partial_v R - \Xi\partial_\psi R = 0 \tag{6.126}$$

$$(6.91) \quad \Leftrightarrow \quad \partial_r R = 0 \tag{6.127}$$

$$\Re(6.92) \quad \Leftrightarrow \quad (r^2+a^2)\partial_v R + a\Xi\partial_\psi R = 0 \tag{6.128}$$

$$\Im(6.92) \quad \Leftrightarrow \quad (r^2+a^2)\partial_v\Theta + a\Xi\partial_\psi\Theta = 0, \tag{6.129}$$

and (6.93) turns out to be equivalent to (6.90). (6.127) and (6.128) imply $R = R(\theta)$, and then (6.126) implies $\Theta = \Theta(r,\psi,v)$. The remaining two equations (6.125) and (6.129) give the solutions

$$R(\theta) = R = \text{const.}, \qquad \Theta(r,\psi,v) = \Theta(r). \tag{6.130}$$

Then the real field quantities are

$$B_T = \sqrt{2}\frac{e^R\cos\Theta(r)}{\Xi}, \qquad \omega = \frac{a\Xi}{r^2+a^2} \tag{6.131}$$

$$F_{r\varphi} = -\sqrt{2}\frac{r^2+a^2}{\Xi\Delta_r}e^R\sin\Theta(r), \qquad F_{rt} = \sqrt{2}\frac{a}{\Delta_r}e^R\sin\Theta(r), \qquad F_{\theta\varphi} = F_{\theta t} = 0 \tag{6.132}$$

$$J^T_{(4)} = -i\rho\frac{e^R\partial_r\sin\Theta(r)}{2\sqrt{\Delta_\theta}\sin\theta}, \qquad J^P_{(4)} = \rho\frac{e^R\partial_r\cos\Theta(r)}{2\sqrt{\Delta_\theta}\sin\theta}, \qquad J^T_{(2)} = J^P_{(2)} = 0. \tag{6.133}$$

These expressions contain periodic functions of the radial coordinate r, but since $\Theta(r) = \arg\phi_0'$, we can simply choose

$$\Theta(r) = \arctan\Big(\frac{r}{r_H} - 1\Big). \tag{6.134}$$

This way $\Theta(r)$ ranges from 0 to $\pi/2$ for $r \geq r_H$, and we remove the periodicities. This choice of $\Theta(r)$ also makes $F_{r\varphi}$ & F_{rt} regular at $r = r_H$, which should be the case even in BL coordinates. Unfortunately, the divergences at $\sin\theta = 0$ for B^φ and $J^{T,P}_{(4)}$ remain. We plot the magnetic field (B^θ, B^φ) at a constant-r surface in fig. 6.1. For $J_aJ^{(a)}$ we have

$$J_aJ^{(a)} = -2J_mJ_{\bar{m}} = -\frac{e^{2R}}{2\Sigma\Delta_\theta\sin^2\theta}[\partial_r\Theta(r)]^2, \tag{6.135}$$

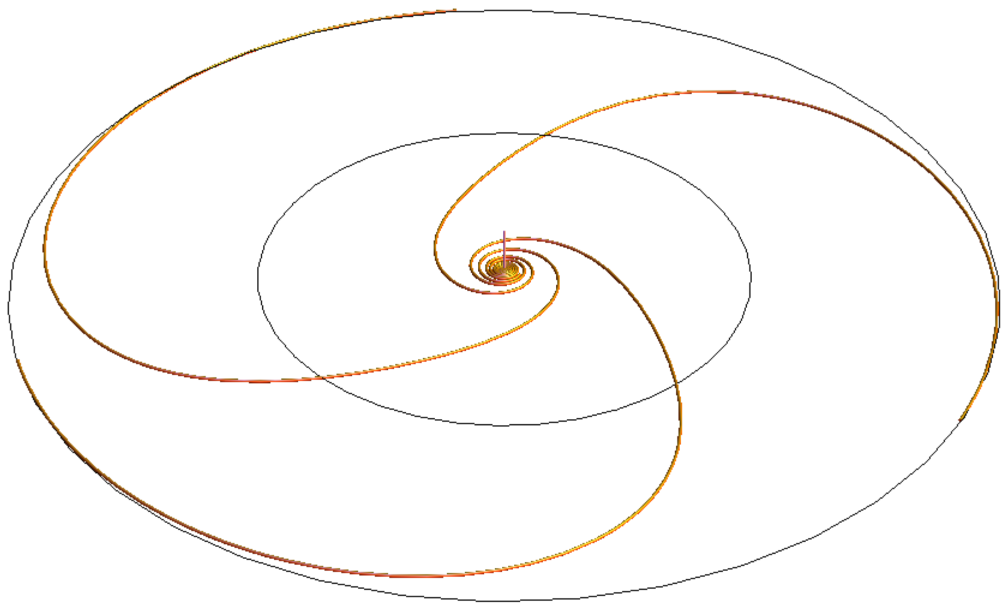


Figure 6.1: Magnetic field lines as given by (6.131) and (6.132), with $B^r = 0$. The disk region represents the sphere at $r = 2r_H$ deformed such that the 'radial' direction of the disk corresponding to the θ coordinate on the sphere ($\theta = 0$ at the center and $\theta = \pi$ at the rim of the disk) and the azimuthal direction corresponding to the φ coordinate. The circle in the middle is at $\theta = \pi/2$, i.e. the equator. The divergences of $B^\theta \sim 1/\sin\theta$ and $B^\varphi \sim 1/\sin^2\theta$ at $\sin\theta = 0$ can be seen from the behaviors of the field lines at the center and the rim.

so the current is always spacelike. The invariant is

$$I_1 = \frac{4e^{2R}}{\Delta_r \Delta_\theta \sin^2\theta}, \tag{6.136}$$

so the solution is magnetically dominant.

6.7 Other possible ways to construct force-free magnetosphere

In this section we present other possible ways to describe/understand the force-free magnetosphere and some efforts to construct new solutions. Obtaining force-free magnetospheres involves essentially dealing with two equations:

$$I_2 = {}^\star F_{\mu\nu} F^{\mu\nu} = 0 \tag{6.137}$$

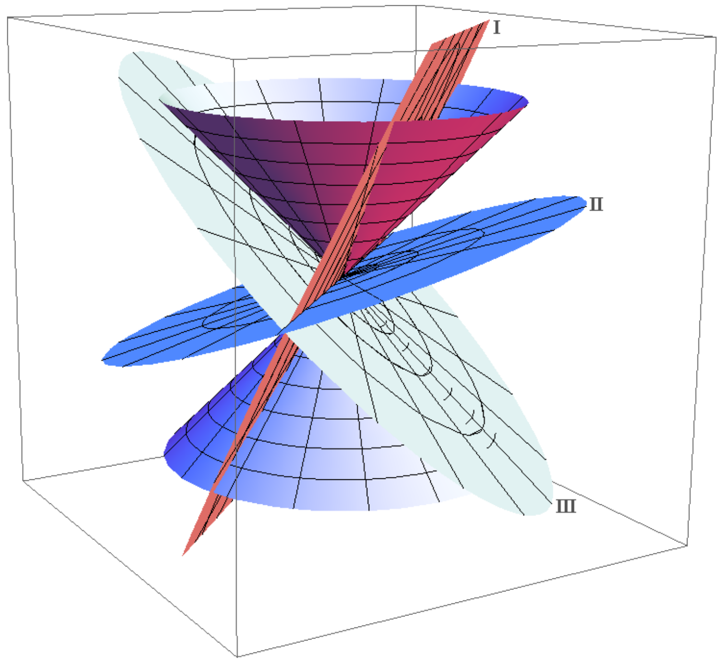


Figure 6.2: Blades represented by the planes I, II and III are time-like, space-like and light-like respectively.

$$T^{\mu\nu}_{;\nu} = 0, \tag{6.138}$$

namely, the degeneracy condition and the conservation equation respectively. Mathematically, an antisymmetric tensor or bivector $F_{\mu\nu}$ satisfying $I_2 = 0$ is called *simple*, and we denote it with the lower case $f_{\mu\nu}$. Simple bivectors have the property that they can be written as the wedge product of two 1-forms [77]:

$$f_{\mu\nu} = p_{[\mu}q_{\nu]}. \tag{6.139}$$

(6.139) is equivalent to

$$f_{\mu[\alpha}f_{\nu\beta]}(= f_{[\mu\alpha}f_{\nu\beta]}) = f_{\mu\rho}f_{\sigma\tau}\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\nu}\delta^{\tau}_{\beta]} = -\frac{I_2}{3}\varepsilon_{\mu\alpha\nu\beta} = 0, \tag{6.140}$$

using the identity $\varepsilon^{\alpha\beta\gamma\tau}\varepsilon_{\mu\nu\rho\tau} = -6\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho]}$. This representation can help us visualize simple bivectors: they lie in the 2-d planes Span(p, q), called the *blades* [78]. A blade can be timelike, spacelike or lightlike according to whether it intersects, is outside of or tangential to the light-cone, as shown in fig. 6.2. The following statements follow:

- ${}^{\star}f_{\mu\nu}$ is also simple;

- blade of ${}^\star f_{\mu\nu} \perp$ blade of $f_{\mu\nu} \qquad \Rightarrow \qquad I_2 = 0$;
- $f_{\mu\nu}k^\nu = 0$, $\forall k_\mu \in$ blade of ${}^\star f_{\mu\nu}$.

Coming back to the force-free equation $f_{\mu\nu}J^\nu = 0$, one has the picture that J^μ lies in the blade of ${}^\star f_{\mu\nu}$. Moreover, in the case $f_{\varphi t} = 0$ one can in fact find p_μ and q_μ for (6.139); they are given by (in BL coordinates)

$$p_\mu \, \mathrm{d}x^\mu = f_{\theta r} \, \mathrm{d}r + f_{\theta\varphi} \, \mathrm{d}\varphi + f_{\theta t} \, \mathrm{d}t \tag{6.141}$$

$$q_\mu \, \mathrm{d}x^\mu = f_{\varphi r} \, \mathrm{d}r + f_{\varphi\theta} \, \mathrm{d}\theta. \tag{6.142}$$

It can be checked explicitly that $f_{\mu\nu} \propto p_{[\mu}q_{\nu]}$ using $I_2 = 0$. We also notices that the vector

$$k^\mu \partial_\mu = \partial_t + \omega \partial_\varphi \tag{6.143}$$

satisfies $k^\mu p_\mu = k^\mu q_\mu = 0$, so k^μ lies in the blade of ${}^\star f_{\mu\nu}$ and is a candidate for the current.[5] One can then try to specify some form of J^μ (e.g. being null) and find vectors p_μ, q_ν.

A general bivector can be written as the linear combination of a simple bivector and its dual

$$F_{\mu\nu} = \cos\alpha f_{\mu\nu} + \sin\alpha \, {}^\star f_{\mu\nu}, \tag{6.144}$$

for some function α. For example, the Kerr-Newman-AdS field has the form (formed with two of the orthonormal tetrad):

$$F = (\cos\alpha + \sin\alpha \, {}^\star)\omega^{(0)} \wedge \omega^{(1)}, \tag{6.145}$$

where

$$\cos\alpha = \frac{q_m r^2 - 2q_e ar\cos\theta - q_m a^2\cos^2\theta}{(r^2 + a^2\cos^2\theta)\sqrt{q_e^2 + q_m^2}} \tag{6.146}$$

$$\sin\alpha = \frac{-q_e r^2 - 2q_m ar\cos\theta + q_e a^2\cos^2\theta}{(r^2 + a^2\cos^2\theta)\sqrt{q_e^2 + q_m^2}}. \tag{6.147}$$

Here the simple bivector happens to be the wedge product of basis 1-forms. The operation of the form (6.144) on bivectors (not necessarily simple ones), sending $F \to \tilde{F} \equiv F\cos\alpha + {}^\star F\sin\alpha$, is called a *duality rotation* [79] and can be compactly

[5] Using $J^\mu \propto k^\mu$ in BZ's solution gives $\omega = \Omega_H$.

written using the anti-self-dual F^- as

$$\tilde{F}^- = e^{-i\alpha}F^- \qquad \Leftrightarrow \qquad \tilde{F} + i\,{}^\star\tilde{F} = e^{-i\alpha}(F + i\,{}^\star F). \tag{6.148}$$

Note that for $\alpha = \pi/2$ duality rotation becomes Hodge dual and they commute. On the other hand, due to the expansion (6.5) of F^- with NP variables as coefficients, one can regard the duality rotation as acting on the NP variables by the same rotation:

$$\phi_{0,1,2} \to e^{-i\alpha}\phi_{0,1,2}, \tag{6.149}$$

while keeping the bivector basis unchanged. It is then clear that the modified NP variables Φ_1 & Φ_2 also transform in the same way. In some sense, the expansion (6.5) implies that the NP variables themselves can be viewed as duality rotating the anti-self-dual bivector basis (plus a scaling by a factor of $|\phi_{0,1,2}|$).

Under duality rotation, the invariant $I \equiv I_1 + iI_2$ transforms as

$$\tilde{I} = e^{-2i\alpha}I, \tag{6.150}$$

as can be seen by noting $I = 8(\phi_0\phi_2 - \phi_1^2)$ from (6.12). I can be used to classify the electromagnetic field. Any electromagnetic field, modulo a proper orthochronous Lorentz transformation on the NP tetrad at each spacetime point, falls into one of the following equivalence classes (the "Ruse-Synge" classes) [72]:

$$\text{Type A:} \qquad I \neq 0 \text{ (non-null)} \qquad \phi_0 = \phi_2 = 0, \quad \phi_1 \neq 0 \tag{6.151}$$

$$\text{Type B:} \qquad I = 0 \text{ (null)} \qquad \phi_0 = \phi_1 = 0, \quad \phi_2 \neq 0 \tag{6.152}$$

$$\text{Type C:} \qquad I = 0 \text{ (null)} \qquad \phi_1 = \phi_2 = 0, \quad \phi_0 \neq 0. \tag{6.153}$$

BZ's monopole solution ($I_1 > 0$, $I_2 = 0$) is of (a subtype of) Type A, while Brennan et al's null solution is of Type B or C. Note that a duality rotation, which acts on $\phi_{0,1,2}$ and I simply by the phase factor, does not send fields of one class to another. It does send a general field in Type A to the subtype mentioned above.

Duality rotations has the important property that it preserve the form of energy-momentum tensor. To see this, consider the following manifestly duality-rotation invariant expression

$${}^{\sim}(F_{\mu\nu} - i{}^\star F_{\mu\nu}){}^{\sim}(F_{\gamma\rho} + i{}^\star F_{\gamma\rho}) = \cancel{e^{i\alpha}e^{-i\alpha}}(F_{\mu\nu} - i{}^\star F_{\mu\nu})(F_{\gamma\rho} + i{}^\star F_{\gamma\rho}) \tag{6.154}$$

$$\Rightarrow \qquad {}^{\sim}(F_{\mu\nu}F_{\gamma\rho} + {}^{\star}F_{\mu\nu}{}^{\star}F_{\gamma\rho}) + i{}^{\sim}({}^{\star}F_{\mu\nu}F_{\gamma\rho} - F_{\mu\nu}{}^{\star}F_{\gamma\rho})$$
$$= (F_{\mu\nu}F_{\gamma\rho} + {}^{\star}F_{\mu\nu}{}^{\star}F_{\gamma\rho}) + i({}^{\star}F_{\mu\nu}F_{\gamma\rho} - F_{\mu\nu}{}^{\star}F_{\gamma\rho}) \quad (6.155)$$

Contracting both sides with $g^{\nu\rho}$ then yields $\tilde{T}_{\mu\gamma} = T_{\mu\gamma}$, with the imaginary part vanishing. Another way is by noticing that every component of $T_{(a)(b)}$ is in the form $-2\phi_0\bar{\phi}_1$ etc. and invoking (6.149). It thus follows that the conservation equation (6.138) is also invariant under duality rotation: $T^{\mu\nu}_{;\nu} = \tilde{T}^{\mu\nu}_{;\nu} = 0$. This fact can presumably be used to construct new solutions.

Duality rotation however does change the current J^μ. It usually mixes J^μ and the 'magnetic current' $L^\mu \equiv {}^{\star}F^{\mu\nu}_{;\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\tau}F_{[\rho\tau;\nu]}$, yielding

$$\tilde{J}^\mu - i\tilde{L}^\mu = [e^{i\alpha}(F^{\mu\nu} - i{}^{\star}F^{\mu\nu})]_{;\nu} = e^{i\alpha}(J^\mu - iL^\mu) + ie^{i\alpha}(F^{\mu\nu} - i{}^{\star}F^{\mu\nu})\alpha_{,\nu}. \quad (6.156)$$

An alternative expression is obtained by contracting (6.156) with $\alpha_{,\mu}$:

$$\tilde{J}^\mu\alpha_{,\mu} - i\tilde{L}^\mu\alpha_{,\mu} = e^{i\alpha}(J^\mu\alpha_{,\mu} - iL^\mu\alpha_{,\mu}). \quad (6.157)$$

With non-vanishing L^μ, the conservation equation becomes

$$0 = T^{\mu\nu}_{;\nu} = -F^{\mu\rho}J_\rho - {}^{\star}F^{\mu\rho}L_\rho. \quad (6.158)$$

Given a force-free solution ($f_{\mu\nu}$, $I_2 = 0$, $J^\mu \neq 0$) to the conservation equation, we are interested in finding a 'dual' solution $\tilde{F}_{\mu\nu}$ that is duality rotated from $f_{\mu\nu}$ and has different features, e.g. a non-degenerate vacuum solution with $\tilde{I}_2 \neq 0$, $\tilde{J} = 0$ and likely *not* stationary or axisymmetric. Conversely, it would also be interesting to duality rotate a known vacuum solution into a degenerate force-free solution. The duality rotation is guaranteed to exist with the angle given by [79]

$$\tan 2\alpha = -\tilde{I}_2/\tilde{I}_1, \quad (6.159)$$

as can be seen from (6.150). For physical reasons, we focus on the case $\tilde{L}^\mu = L^\mu = 0$. It is however helpful to list all possible configurations that solve the conservation equation, given in table 6.1. The interesting case to us according to the table is the duality rotation between Class 1(a) and Class 2(a).

As an example, we apply a duality rotation to the stationary and axisymmetric solution (6.60) for Brennan et al's null field configuration ($\phi_1 = \phi_2 = 0$). One requires

$F^{\mu\rho}J_\rho = 0 = {}^\star F^{\mu\tau}L_\tau$	$J^\mu = 0 = L^\mu$	$I_2 \neq 0$	$1(a)$
		$I_2 = 0$	$1(b)$
	$J^\mu \neq 0, \quad L^\mu = 0$	$I_2 = 0$	$2(a)$
	$J^\mu = 0, \quad L^\mu \neq 0$		$2(b)$
	$J^\mu \neq 0 \neq L^\mu$		$2(c)$
$F^{\mu\rho}J_\rho = -{}^\star F^{\mu\tau}L_\tau \neq 0$	$J^\mu \neq 0 \neq L^\mu$	$I_2 \neq 0$	$3(a)$
		$I_2 = 0$	$3(b)$

Table 6.1: Classification of field and current configurations obeying the conservation equation (6.158). Class 1 have vanishing (electric and magnetic) currents. Class 2 have at least one non-vanishing current and are degenerate solutions. Degenerate fields can also happen in other classes. Class 3 are more general solution not of interest.

that $\Im \tilde{J}_l = 0$:

$$0 = \Im \tilde{J}_l = \frac{\Re\Phi_1}{\sqrt{2}\Delta_r}\Big[\frac{\sin\alpha}{\sqrt{\Delta_\theta}\sin\theta}[\Xi\partial_\psi\alpha + \sin^2\theta(a\partial_v\alpha - J_l')] - \sqrt{\Delta_\theta}\cos\alpha\partial_\theta\alpha\Big] \tag{6.160}$$

where $J_l' \equiv J_l\sqrt{2}\Delta_r\sqrt{\Delta_\theta}/(\sin\theta\Re\Phi_1)$. Note that $\Re\tilde{J}_l = \Im\tilde{J}_l(\sin\alpha \to -\cos\alpha, \cos\alpha \to \sin\alpha)$. One also has $\tilde{J}_m = \bar{\tilde{J}}_{\bar{m}}(\tilde{\phi}_1 = \tilde{\phi}_2 = 0) = 0$ which gives $\partial_r\alpha = 0$. We then look at two special cases $\partial_\psi\alpha = \partial_v\alpha = 0$ and $\partial_\theta\alpha = 0$. Using (6.58) and (6.60) we find for the first case

$$\csc[\alpha(\theta)] = C_1\Delta_\theta\sin\theta S'(\theta), \qquad \tilde{J}_l = \frac{J_l}{\cos[\alpha(\theta)]}, \tag{6.161}$$

where C_1 is integration constant. For the second case, we find

$$\alpha(v) = C_2 v, \qquad \tilde{J}_l = 0, \tag{6.162}$$

where C_2 is integration constant. One sees that a time-dependent duality rotation maps the original solution to a vacuum solution. Some further examples are

- BZ's monopole solution
  We would like to see if the solution admits a dual vacuum description.

- Kerr-Newman-AdS field (given in (6.145))
  One directly sees that it is duality rotated from the degenerate field $\omega^{(0)} \wedge \omega^{(1)}$.

- Wald's solution [80] in Kerr spacetime

$$F = B_0\Big[\frac{ar\sin^2\theta}{\Sigma} - \frac{ma(r^2-a^2\cos^2\theta)(1+\cos^2\theta)}{\Sigma^2}\Big]\omega^{(1)}\wedge\omega^{(0)}$$
$$+ B_0\frac{\sqrt{\Delta}r\sin\theta}{\Sigma}\omega^{(1)}\wedge\omega^{(3)} + B_0\frac{\sqrt{\Delta}a\sin\theta\cos\theta}{\Sigma}\omega^{(2)}\wedge\omega^{(0)}$$
$$\frac{B_0\cos\theta}{\Sigma}\Big[r^2+a^2-\frac{2mra^2(1+\cos^2\theta)}{\Sigma}\Big]\omega^{(2)}\wedge\omega^{(3)}. \quad (6.163)$$

The dual degenerate field would not simply be the wedge product of any two orthonormal basis.

Of course, the KNAdS and Wald solutions do not have any radial energy or angular momentum fluxes since $T^r_t = T^r_\varphi \equiv 0$. One could start with a known solution to the Einstein's equation with $T^r_t \neq 0$ and interpret the energy-momentum tensor as that of a degenerate electromagnetic field. This can be achieved in the framework of Rainich's "already unified theory" [79] which proposes the following relation between the spacetime geometry and a simple bivector $f_{\mu\nu}$

$$E_{\mu\alpha\nu\beta} = \mathcal{T}_{\mu\alpha\nu\beta} \equiv \frac{1}{2}(f_{\mu\alpha}f_{\nu\beta} + {}^\star f_{\mu\alpha}{}^\star f_{\nu\beta}), \quad (6.164)$$

where $E_{\mu\alpha\nu\beta}$ is the "semi-traceless" part of the Riemann tensor $R_{\mu\alpha\nu\beta}$ as in the "Ricci decomposition" of $R_{\mu\alpha\nu\beta}$ [51]

$$R_{\mu\alpha\nu\beta} = C_{\mu\alpha\nu\beta} + E_{\mu\alpha\nu\beta} + G_{\mu\alpha\nu\beta}. \quad (6.165)$$

Here

$$C_{\mu\alpha\nu\beta} = \text{Weyl tensor}, \qquad (\text{"fully traceless" part}) \quad (6.166)$$
$$E_{\mu\alpha\nu\beta} \equiv \frac{1}{2}(g_{\mu\nu}S_{\alpha\beta} + g_{\alpha\beta}S_{\mu\nu} - g_{\mu\beta}S_{\alpha\nu} - g_{\alpha\nu}S_{\mu\beta}) \quad (6.167)$$
$$= g_{\mu[\nu}S_{\beta]\alpha} - g_{\alpha[\nu}S_{\beta]\mu}, \qquad (\text{"semi-traceless" part}) \quad (6.168)$$
$$G_{\mu\alpha\nu\beta} \equiv \frac{1}{12}R(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}) = \frac{1}{6}R(g_{\mu[\nu}g_{\beta]\alpha}), \qquad (\text{"scalar" part}) \quad (6.169)$$

where

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda. \quad (6.170)$$

(6.164) contracts to give the Einstein's equations $S_{\mu\nu} = T_{\mu\nu}$. Now, with the requirement that $f_{\mu\nu}$ be simple, one can solve (6.164) for $f_{\mu\nu}$ to find [51]

$$f_{\mu\nu} f_{\alpha\beta} = E_{\mu\nu\alpha\beta} - (S_{\sigma\tau} S^{\sigma\tau})^{-1/2} E_{\mu\nu\rho\gamma} E_{\alpha\beta}{}^{\rho\gamma}, \qquad (6.171)$$

which basically says that one can construct a degenerate electromagnetic field from the geometry, using some other non-electromagnetic $T_{\mu\nu}$ to source the metric.

In conclusion, the search for techniques to efficiently tackle the force-free equations is by far still incomplete, despite that we have provided different prospectives. This is of course a reflection of the fact that the force-free equations represents a rather non-trivial problem and certainly deserves further exploration.

Chapter 7

Conclusion

This thesis has primarily studied the force-free magnetosphere in Kerr-AdS background, the family of metrics describing rotating black hole spacetimes that are asymptotically AdS. Such a configuration allows the rotational energy of the black hole to be extracted electromagnetically (in the form of a Poynting flux) via the Blandford-Znajek (BZ) process, originally formulated in asymptotically flat Kerr geometries. The motivation of the current study to consider asymptotically AdS geometries is from the AdS/CFT correspondence, under which processes in the AdS black hole backgrounds (the bulk) admit dual descriptions in terms of the boundary field theories. The BZ process is a manifestation of the existence of an ergosphere outside rotating black holes, which also underlies the Penrose process and superradiance. A clear dual field theory description of the bulk ergosphere has been lacking. By solving the electromagnetic field of the force-free magnetosphere, we wish to find a direct translation of the results to the boundary side using the standard AdS/CFT dictionary (in this case regarding the gauge field). To this end, we have generalized BZ's monopole solution to Kerr-AdS. The monopole field, though an idealized toy model, provides a sufficiently good approximation of the electromagnetic field produced by an accretion disk around the black hole, and captures the basic ingredients of the force-free magnetosphere capable of generating Poynting fluxes. This model however is restricted to the perturbative, slow rotation regime. Another task of this study thus involves finding other more general force-free solutions for arbitrary finite rotation. The recent advances in this direction are highlighted by the exact analytic solutions with null currents [76, 21]. Again we reproduce the analogous solutions in Kerr-AdS and also derive a few new solutions with non-null currents, though with additional assumptions on the fields.

To set the stage, we have reviewed in chapter 2 the energy extraction mechanism and particularly the BZ process in general terms, revealing some essential properties. Starting with the better known Penrose process and superradiance, we make it clear that energy extraction is possible due to the frame-dragging effect in rotating spacetimes which determines different local energy-defining Killing vectors at infinity and at the horizon and results in the notion of ergosphere. It is shown that an angular momentum extraction always occurs and in some sense is more fundamental. Especially from the viewpoint of a local observer on the horizon, the (positive) energy becomes ingoing while the angular momentum extraction persists. These points are further illustrated using thermodynamics where the ingoing energy on the horizon corresponds to the increasing of entropy. We also give a general analysis of the energy-momentum flux on the horizon and derived the first law incorporating the freedom in choosing the energy-defining Killing vector. We then turned to reviewing the essential properties of the force-free magnetosphere and BZ process, emphasizing that the energy extraction crucially depends on the toroidal component of the magnetic field which is due to the relevant rotation between the horizon and the boundary. Dynamically, the rotation of the spacetime induces an electric field driving the poloidal current which sources the toroidal magnetic field. This instructs us to look for a dual field theory description of the magnetosphere (or fundamentally the ergosphere of the rotating spacetime) by examining the falloffs of A_θ and A_r (related to toroidal magnetic field) or the poloidal current using the AdS/CFT dictionary. Chapter 3 is devoted to a detailed review of the AdS/CFT correspondence.

Before presenting the main results for the BZ process in Kerr-AdS, in chapter 4 we first derived the solution for a rotating monopole in the pure AdS spacetime, analogous to Michel's rotating monopole solution in flat spacetime which is used by BZ as the asymptotic configuration for their solution. A novelty we found in the AdS case is the ambiguity in specifying the monopole field; this includes an ambiguity from choosing different coordinate systems (the standard one and that associated with the horizon) and the freedom to add perturbations (subleading terms in the slow rotation limit). As a consequence, the angular velocity of the magnetic field lines is not uniquely fixed as in the Kerr case. These features seem all connected with the peculiarity of the timelike boundary of asymptotic AdS spacetimes.

In Chapter 5 we obtain analogues of the BZ split monopole solution for force-free magnetospheres around a slowly rotating Kerr-AdS black hole up to $\mathcal{O}(a^2)$. The field configuration is poloidal, and can be specified by A_φ, and thus the main differential equation to solve is that for $f(r)$ (or $f(z)$), the radial $\mathcal{O}(a^2)$ component of A_φ. In distinction to the Kerr case originally studied by Blandford and Znajek, the field angular velocity ω, as a parameter in the equation, is not uniquely determined solely from the (minimal) boundary conditions. However, it is directly related to the toroidal magnetic field B_T due to the horizon regularity constraint for B^φ in Kerr-Schild coordinates. Further constraints on ω do arise on imposing specific matching conditions at large radius with a rotating monopole in AdS. However, as anticipated in chapter 4, unlike the asymptotically flat space, these matching conditions are in turn non-unique due to the fact that the AdS boundary is only defined up to a conformal factor, and this allows a class of higher order multipole corrections. Matching to this full class of solutions re-introduces the freedom to vary the field angular velocity.

The Kerr-AdS geometry also has the property that the asymptotic timelike Killing vector is not unique, given by $K^\mu_{\Omega'}$ with a range of values for Ω', each of which is a candidate for the energy-defining Killing vector. Especially for large black holes with $r_H > \sqrt{al}$, $K^\mu_{\Omega_H}$ is globally timelike, i.e., there is no ergosphere associated with it. With the 'energy' flux into the black hole given by $\delta E \propto (\omega - \Omega')(\omega - \Omega_H)$ using $K^\mu_{\Omega'}$ as the energy-defining Killing vector, ω is constrained to $\Omega' < \omega < \Omega_H$ for energy extraction. The above noted ambiguity in ω has a less substantial effect on the energy flux, since one can always dial Ω' to change the sign of δE which is thus observer-dependent. What is more robust is the angular momentum flux $\delta L \propto \omega - \Omega_H$ which remains negative as long as $\omega < \Omega_H$.

In fact, there *are* two preferred choices for Ω', i.e. $\Omega' = \Omega_\infty, \Omega_H$. While only the first can make the first law of thermodynamics an exact differential, the second seems natural in the perspective of AdS/CFT correspondence, where large Kerr-AdS black holes with $r_H/l > 1$ holographically describe the thermodynamics of a strongly-interacting boundary field theory. The existence of a globally well-defined timelike Killing vector $K^\mu_{\Omega_H}$ external to the horizon suggests the absence of energy extraction through the Blandford-Znajek process ($\delta E \propto (\omega - \Omega_H)^2 \geq 0$) and is consistent with the stability argument. In this regime, we find that at least for slow rotation the force-free solution still exists with the arbitrariness in ω corresponding to the freedom in the dual field theory to rotate a magnetic field through a neutral plasma. In effect the boundary configuration is also 'force free', although the solution would need to be extended to higher order in 'a' to test this in a non-trivial manner. The distinctive features of the BZ process in the large black hole regime discussed here make it difficult to provide a more precise AdS/CFT description of the energy extraction process. Nonetheless, this question was one of the original motivations for this work, and it would be interesting to see if the dual field theory picture can be developed further, perhaps by extending the solutions described here away from the slow rotation limit.

Our small 'a' expansion was general enough to cover large, intermediate and small black hole regimes. For small black holes, we obtained an analytic solution to first order in the ratio r_H/l, which exhibits a radial Poynting flux with uniquely determined ω and for $r_H/l \to 0$ smoothly approaches the Blandford-Znajek configuration in an asymptotically flat Kerr background.

Research into force-free solutions in rotating black hole backgrounds has recently become more active. In Chapter 6 we have generalized the recent exact solutions [21] associated with null currents to the Kerr-AdS case, as well as generalized the current configuration to the non-null case, obtaining several solutions under certain assumptions on the field. While solving the force-free equations generally is still not possible, solutions in these simplified cases help us gain useful insights into the problem. For the solutions we obtained, we have given a thorough discussion of various properties, including the magnetic field, the energy densities and the causal nature of the currents. In particular, the currents all turn out to be spacelike. We used the NP formalism, which has proved an efficient tool during the course of the analysis. We also presented attempts to understand and formulate the force-free magnetosphere using different 'languages' (bivectors etc.) and hope this could lead to further useful results.

The BZ process features the interrelation of the curved spacetime and the electromagnetic field, though not in the full Einstein-Maxwell sense. More precisely, it is the rotational properties of the Kerr(-AdS) spacetime that are captured by the behavior of the force-free magnetosphere, especially the outgoing Poynting flux relying on the existence of an ergosphere. This is already made clear in the slow rotation limit by BZ's original split monopole ansatz. Searching for exact and analytic force-free solutions for finite rotation continues to constitute an interesting and challenging subject of research. The technique of the NP formalism seems promising in this respect. Existing studies on the relations of the geometry and the degenerate electromagnetic fields (bivectors) may also provide useful insights and routes into the problem. Finally, additional understanding may always be gained in the context of AdS/CFT correspondence that maps the nontrivial force-free solutions to descriptions in terms of the boundary dual field theory.

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Appendix A

Force-free solutions in Kerr-Schild coordinates

The analysis in the main part of this thesis can also be carried out using Kerr-Schild (KS) coordinates (specified with a tilde) which use an alternate foliation $\Sigma_{\tilde{t}}$ that is horizon penetrating. The Kerr-AdS metric in this coordinate system is given by [81],

$$\begin{aligned}\mathrm{d}\,\tilde{s}^2 = {} & \frac{\Sigma}{\left(1+\frac{r^2}{l^2}\right)(r^2+a^2)}\mathrm{d}\,r^2 + \frac{\Sigma}{\Delta_\theta}\mathrm{d}\,\theta^2 + \frac{r^2+a^2}{\Xi}\sin^2\theta(\mathrm{d}\,\tilde{\varphi} + \frac{a}{l^2}\,\mathrm{d}\,\tilde{t})^2 - \frac{\left(1+\frac{r^2}{l^2}\right)\Delta_\theta}{\Xi}\mathrm{d}\,t^2 \\ & + \frac{2mr}{\Sigma}\left[\frac{\Sigma}{\left(1+\frac{r^2}{l^2}\right)(r^2+a^2)}\,\mathrm{d}\,r - \frac{a}{\Xi}\sin^2\theta\,\mathrm{d}\,\tilde{\varphi} + \frac{\Delta_\theta}{\Xi}\,\mathrm{d}\,\tilde{t}\right]^2, \qquad \text{(A.0.1)}\end{aligned}$$

where r & θ are the same as in BL coordinates and $\tilde{\varphi}$ & $\tilde{t}$ are related to the BL coordinates by the transformation [81],[1]

$$\mathrm{d}\,\tilde{\varphi} = \mathrm{d}\,\varphi + \frac{a\Xi}{\Delta_r\left(1+\frac{\Delta_r}{2mr}\right)}\,\mathrm{d}\,r = \mathrm{d}\,\varphi + \frac{2mar\Xi}{\Delta_r(r^2+a^2)\left(1+\frac{r^2}{l^2}\right)}\,\mathrm{d}\,r \qquad \text{(A.0.2)}$$

$$\mathrm{d}\,\tilde{t} = \mathrm{d}\,t + \frac{2mr}{\Delta_r\left(1+\frac{r^2}{l^2}\right)}\,\mathrm{d}\,r. \qquad \text{(A.0.3)}$$

Note that

$$\frac{\tilde{\varphi}_{,r}}{\tilde{t}_{,r}} = \frac{a\Xi}{(r^2+a^2)} \overset{r=r_H}{=\!=\!=} \Omega_H, \qquad \text{(A.0.4)}$$

[1] The KS coordinate ϕ in [81] is in fact $\phi = \tilde{\varphi} + \frac{a}{l^2}t$, associated with the non-rotating frame at infinity. Note that (A.0.2) does not reduce to the coordinate transformation used in [6] in the Kerr limit.

and thus the transformation $\{r, \theta, \tilde{\varphi}(\varphi, r), \tilde{t}(t, r)\}$ does not affect ZAMO 4-velocity components, i.e., $\tilde{K}^{\tilde{\mu}}_{\Omega_B} = K^{\mu}_{\Omega_B}$ (as long as $K^r_{\Omega_B} = 0$) and

$$\Omega_B = -\tilde{\beta}^{\varphi} - \frac{\tilde{h}_{r\varphi}}{\tilde{h}_{\varphi\varphi}} \tilde{\beta}^r, \tag{A.0.5}$$

$$\begin{aligned} K^{\mu}_{\Omega_B} K_{\Omega_B \mu} &= -\tilde{\alpha}^2 + \frac{\tilde{h}_{rr}\tilde{h}_{\varphi\varphi} - \tilde{h}^2_{r\varphi}}{\tilde{h}_{\varphi\varphi}} (\tilde{\beta}^r)^2, \\ &\overset{r=r_H}{=\!=\!=} 0 \neq \tilde{\alpha}^2 \tilde{n}_{\mu}\tilde{n}^{\mu} = -\tilde{\alpha}^2. \end{aligned} \tag{A.0.6}$$

Since $K^i_{\Omega_B} \neq -\tilde{\beta}^i$ and $K^{\mu}_{\Omega_B}$ is not parallel to $\tilde{\alpha}\tilde{n}^{\mu}$, the ZAMO is no longer a fiducial observer. On the horizon where $K^{\mu}_{\Omega_B}$ is the outgoing null generator, $\tilde{\alpha}\tilde{n}^{\mu}$ lies inside the light cone and is ingoing. The horizon condition (A.0.6) does not make any metric component singular. It is worth noting that for both BL and KS coordinates, $g^{rr} = 0$ on the horizon which is a null constant-r hypersurface.

The transformation only affects the contravariant φ, t and covariant r components of tensorial objects $(g_{\mu\nu}, F_{\mu\nu}, dT_{\mu}, \dots)$; in particular, the following quantities are all invariant: functions of (r, θ) (e.g., $\det g_{\mu\nu}$), the derivatives ∂_{μ} (hence the conditions $(\dots)_{,t} = 0 = (\dots)_{,\varphi}$ and the bracket structure (5.26)) and the definition and value of ω. B^{φ} and B_T transform as

$$B^{\varphi} = \frac{F_{ru}}{\sqrt{-g}} = \frac{\tilde{F}_{ru} + \tilde{\varphi}_{,r}\tilde{F}_{\varphi u} + \tilde{t}_{,r} \overbrace{\tilde{F}_{tu}}^{=-\omega\tilde{F}_{\varphi u}}}{\sqrt{-g}} \tag{A.0.7}$$

$$= \tilde{B}^{\varphi} + (\omega\tilde{t}_{,r} - \tilde{\varphi}_{,r})\tilde{B}^r, \tag{A.0.8}$$

$$B_T = (g_{\varphi\varphi}g_{tt} - g^2_{\varphi t})B^{\varphi} \tag{A.0.9}$$

$$= \tilde{B}_T + (g_{\varphi\varphi}g_{tt} - g^2_{\varphi t})(\omega\tilde{t}_{,r} - \tilde{\varphi}_{,r})\tilde{B}^r. \tag{A.0.10}$$

B_T is r-independent and $\tilde{B}_T \sim \Delta_r \tilde{B}^{\varphi} = 0$ on the horizon (for regular $\tilde{B}^{\varphi}$), so

$$B_T = (g_{\varphi\varphi}g_{tt} - g^2_{\varphi t})(\omega\tilde{t}_{,r} - \tilde{\varphi}_{,r})\tilde{B}^r\big|_{r=r_H} \tag{A.0.11}$$

$$= -\frac{\Delta_{\theta}(1-u^2)}{\Xi}\frac{r_H^2 + a^2}{r_H^2 + a^2u^2}(\omega - \Omega_H)A_{\varphi,u} \tag{A.0.12}$$

$$= -(1-u^2)(\omega - \Omega_H)A_{\varphi,u} + \mathcal{O}(a^3). \tag{A.0.13}$$

Note that B_T's defined using u and θ differ by a sign, and $\sqrt{-g(\theta)} = \frac{\Sigma}{\Xi}\sin\theta$, $\sqrt{-g(u)} =$

$\frac{\Sigma}{\Xi}$.

Using the same ansatz (5.45)–(5.47) for $\{A_\varphi, \omega, \tilde{B}_T\}$ in the small 'a' expansion we find

$$\widetilde{dT}_\varphi = aC^2\left[\frac{\tilde{B}^{(1)}_{T,r}}{Cr^2} + 2ml^2(1-u^2)\frac{r^2(3r^2+l^2)\omega^{(1)} - 5r^2 - 3l^2}{r^6(r^2+l^2)^2}\right], \tag{A.0.14}$$

$$\widetilde{dT}_t = -\omega\widetilde{dT}_\varphi, \tag{A.0.15}$$

$$\widetilde{dT}_r = \frac{\tilde{B}_T r^2}{C(1-u^2)\Delta_0}\widetilde{dT}_\varphi = \frac{\tilde{B}^\varphi r^2}{C}\widetilde{dT}_\varphi, \tag{A.0.16}$$

$$\widetilde{dT}_u = (\text{expression involving 2nd derivatives of } A_\varphi), \tag{A.0.17}$$

where ω is constant. $\widetilde{dT}_\varphi$, $\widetilde{dT}_t$ and $\widetilde{dT}_u$ are the same as in BL coordinates; $\widetilde{dT}_r$ is now directly proportional to $\widetilde{dT}_\varphi$. The two independent equations are (A.0.14)=0 & (A.0.17)=0. Solving (A.0.14) = 0 for $\tilde{B}^{(1)}_T$ and imposing $\tilde{B}^{(1)}_T(r = r_1) = 0$ yields

$$\tilde{B}^{(1)}_T = -2Cml^2(1-u^2)\left[\frac{\omega^{(1)} - \frac{1}{r^2}}{r(r^2+l^2)} - \frac{\omega^{(1)} - \frac{1}{r_1^2}}{r_1(r_1^2+l^2)}\right]. \tag{A.0.18}$$

Substitution into (A.0.10) leads to

$$B_T = C(1-u^2)\Big(\omega - \frac{a}{r_1^2}\Big) + \mathcal{O}(a^3), \tag{A.0.19}$$

which agrees with (A.0.13). This fixes the sign ambiguity of B^c_T in (5.58).