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COLLEGE OF SOCIAL SCIENCES AND PUBLIC POLICY

FINANCIAL ASSETS IN A HETEROGENEOUS AGENT GENERAL EQUILIBRIUM MODEL

WITH AGGREGATE AND IDIOSYNCRATIC RISK

By

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I dedicate this dissertation to my wife Lindsay,
and my children Evan and Brynlee,
whose love, support and patience made this research possible.

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ABSTRACT

The financial economics profession has determined that identical agents in a dynamic, stochastic, general equilibrium (DSGE) model does not provide price and trading dynamics realized in financial markets. There has been quite a bit of research over the last three decades extending heterogeneity to the Lucas asset pricing framework, to address this issue. Once the assumption of homogeneous agents is relaxed, the problem becomes increasingly complex due to a state space including the wealth distribution, continuation utilities, and wealth distribution dynamics. To establish a more computationally feasible model, specific modifications have been made such as heterogeneity in idiosyncratic shocks and not risk aversion, including aggregate or idiosyncratic risk (but not both), or assuming no growth in the economy (steady state).

In this research, I will define a DSGE model with heterogeneous agents. This heterogeneity will refer to differing CRRA utilities through risk aversion. The economy will have growth due to the assumed dividend process. Agents will face idiosyncratic and aggregate shocks in a complete markets setting. The framework of the provided algorithm will enable issues to be addressed beyond homogeneous agent models.

The numerical simulation results of this model provide considerable asset price volatility and high trading volume. These results occur even in the complete markets setting, where investors are expected to fully insure. Given these dynamics from the simulations of the algorithm, I demonstrate the ability to calibrate this model to address specific financial economic issues, such as the equity premium puzzle. More importantly this exercise will assume realistic agent parameters of risk aversion and discount factors, relative to economic theory.

CHAPTER 1

A SUMMARY OF THE RESEARCH ISSUES AND THE MAJOR CONCLUSIONS

In this dissertation I study the behavior of a dynamic, stochastic, general equilibrium (DSGE) model with assets and heterogeneous investors in a trading environment with both aggregate and idiosyncratic shocks. The initial question is simply how to solve such a model in the case where investors are fundamentally heterogeneous in the sense that they have differing utility functions rather than merely differing state vectors. This is a difficult problem because the state space includes the wealth distribution and investor continuation utilities, the wealth distribution dynamics, both of which are evolving through time. Consequently, the representative agent is also evolving through time so individual investors must continuously adjust their portfolios as the aggregate states evolve. The second set of questions concern how asset market dynamics respond to changes in fundamental parameters such as the distribution of risk aversion among the investors, the volatility of aggregate and idiosyncratic shocks, and the proportion of investor wealth held in financial assets. Of particular interest is the extent to which heterogeneity can be used to explain such financial modeling puzzles as the equity premium and risk-free rate puzzles.

My main contributions include the algorithm that I develop to solve the model and the development of a modeling platform that can be used to study issues beyond the scope of homogeneous agent models or models with either only aggregate or idiosyncratic shocks but not both. My model is able to generate substantial asset price volatility along with high trading volume in a dynamic *equilibrium* even when full insurance is available to investors. I also demonstrate that it is possible to calibrate the model to produce realistic equity premiums using investors with realistic levels of risk aversion.

In the remainder of this chapter I provide some motivation for my research questions and summarize the remaining chapters.

1.1   Motivation

The standard model for asset pricing in the financial economics literature is a dynamic, stochastic, general equilibrium (DSGE) model where agents maximize their discounted life-time utility, all markets clear at equilibrium prices, and individual expectations are consistent with future aggregate outcomes. The seminal Lucas (1978) asset pricing model pioneered this approach using a representative agent. One implication of representative agent equilibrium models is that there is no trading. However, in financial markets there is a considerable amount of volatility in both price and in trading volume. The recent financial crisis is just one example of these dynamics across many different asset classes. Although the profession was enthusiastic about the rational expectations equilibrium (REE) theory, questions quickly developed concerning the application of these models. The equity premium puzzle (Mehra and Prescott (1985), Kocherlakota (1996), Mehra and Prescott (2003), Mehra et al. (2003)), risk-free rate puzzle (Weil (1989)), and countercyclical equity premium (Fama and French (1989), Campbell and Cochrane (1999)) are all examples of empirical anomalies that could not be explained by a representative agent approach. Therefore, either markets are not operating at an equilibrium, or the specific DSGE model is not rich enough to capture the type of dynamics observed in financial data. Assuming that markets are at or very near equilibrium, this research focusses on extending the basic representative agent model to a heterogeneous agent framework.

1.2   Outline of the Dissertation

The thesis consists of five chapters. This chapter motivates the research question, outlines the thesis and summarizes the major results. Chapter 2 reviews the critical literature on DSGE models and the strengths and weaknesses of each of these approaches. Chapter 3 introduces my heterogeneous agent model and the algorithm used to solve it. Chapter 4 explores the properties of the model and compares them to the alternative approaches presented in Chapter 2. Finally, in Chapter 5, I use the model to re-examine the equity-premium and risk-free rate puzzles as an illustration of the possible applications of model.

1.2.1 Chapter 2: A Review of Approaches for Solving General Equilibrium Models with Financial Assets

In this chapter I review the contributions of four specific DSGE models to the financial economics literature: Lucas (1978), Krussell and Smith (1998), Judd et al. (2003), and Benninga and Mayshar (1993). The setup and assumptions of each model will be described, an equilibrium will be defined for each model as well as a description of the solution or algorithm, and for each model I will summarize the main contributions and criticisms of the modeling approach.

The Lucas Model. The first model discussed is the Lucas (1978) asset tree model. The standard version Lucas asset pricing model assumes an endowment economy with one consumable good. There is a population of infinitely lived utility maximizing agents who, in each time period, must decide whether to consume the endowment or trade it for claims to future endowments. The primary source of aggregate uncertainty in the model is derived from an exogenous stochastic dividend process. The very first assumption in the Lucas (1978) model is that the economy is populated by a single type of agent. This assumption infers that either one individual agent inhabits the economy, or that one agent can be a representative of a large number of identical consumers. In the case of a representative agent, the model provides a no-trade general equilibrium with the asset pricing function

$$P_t = E_t \left\{ m_{t+1} X_{t+1} \right\}, \tag{1.1}$$

where P_t is the time t price of the asset, m_{t+1} is the representative agent's stochastic discount factor, X_{t+1} is the the payoff (price and dividend) of the asset next period, and the conditional expectation is taken with respect to a known distribution of future shocks and conditional upon the information set available at time t.

This general equilibrium model laid the foundation for modern financial economics. The pricing equation (1.1) is the most common formula analyzed in financial asset pricing models, and is often referred to as the "fundamental asset pricing equation" in the financial economics literature (Cochrane (2005)). Mehra and Prescott (1985) calibrate this pricing function to observed financial data to unveil the well-known equity premium puzzle. Historically, equities have returned investors an average of 7% return, while risk-free assets have only returned an average of 0.8%. Using "reasonable" values for the parameters, the pricing function (1.1) is not found to be consistent with

these empirical facts. Consequently, the finance profession has been busy adjusting the model and pricing function to address these puzzles.

The Krussell-Smith Model. The well known macroeconomic model of Krussell and Smith (1998) is an ambitious attempt to generalize the Lucas model. The setup of the model included an aggregate production shock and an idiosyncratic employment shock. The markets are incomplete as there are more states than available assets (capital). The only asset available to preserve wealth is through individual ownership of portions of the aggregate capital stock. The model claims heterogeneity, which comes from consumers with different wealth and employment shocks, although agents have identical CRRA utility functions with a common risk aversion.

Due to the uninsurable idiosyncratic shocks, the wealth distribution and its dynamics are required state variables for this model, making it extremely difficult to solve (Peralta-Alva and Santos (2010)).

To simplify the computational problem Krussell and Smith assume that the distribution of wealth can be adequately approximated by a few critical moments such as the a mean of the distribution. In fact, in a model with no financial assets, Krussell and Smith argue that the mean alone of the wealth distribution is sufficient since the volatility of the aggregate capital stock is small relative to its mean. Unfortunately, as noted by Miao (2006), there is no proof of the existence of a solution to the Krussell-Smith model. Adding financial assets to the Krussell-Smith is difficult since at least four moments (mean, variance, skewness and kurtosis) are critical for asset pricing models.

The Judd, Kubler and Schmedders Model. A single type of consumer is a very restrictive assumption to the model. Judd et al. (2003) extend the Lucas asset pricing framework to include heterogeneity. A small number of types of infinitely lived agents maximize their discounted life-time constant relatively risk aversion (CRRA) utility function in an economy with complete financial markets. The heterogeneity of the model pertains to the risk aversion of each agent type. However, Judd et al. (2003) make two other key assumptions to mitigate the complexity. First, complete markets are assumed for the economy. Unlike Lucas (1978), investors are faced with as many unique assets as states available. The second assumption is that there is no growth in the economy, as the possible states at any point in time are the same at each date in time. More specifically, the Markov tree is simplified to a Markov "channel" so that the number of nodes in the tree does

not grow over time. Given these two assumptions, the wealth distribution reaches a steady-state and thus, at equilibrium, will not enter the state space. This, along with the complete markets assumption, allows a planner's problem approach to be applied to solve the model.

In the Judd et al. (2003) model agents only trade financial assets in the first period. An adjustment is made to their initial asset holdings and they hold that portfolio forever. The only way to induce trading in the model is through a specific assumption of the probabiltiy transition matrix. Judd et al. (2006) produce perpetual trading when there exists one state of the world at time $t + 1$ that cannot be reached from a state at time t (there exists a zero probability in the transition matrix) so that it takes at least two periods to reach the desired portfolio. This model provided a solution to a DSGE model with heterogeneous agents that populated an economy with aggregate and idiosyncratic shocks. However, the application of this model to policy and research is limited due to the simple dynamics. The reason perpetual trading does not exist in this model hinges on the assumption of a no growth, steady state economy. Without growth in the economy, results are not path dependent and each agent can fully insure after the initial period is revealed. In other terms, for this model there exists a representative agent who is identical in all periods.

The Benninga-Mayshar Model. The Benninga and Mayshar (1993) model relaxes the steady state economy assumption. Instead, the exogenous dividend process is path dependent, as the current dividend is the previous dividend multiplied by an observed growth rate,

$$D_t = (1 + g)D_{t-1}. \tag{1.2}$$

where the growth rate g is g_u in a positive state and g_d in a negative state. Idiosyncratic shocks are not considered in this model and the only source of wealth is the initial asset holdings. Given an assumption of complete markets (2 states, 2 assets), Benninga and Mayshar (1993) a planner's approach in the algorithm to solve Pareto efficiency. However, this representative agent's risk aversion is time inhomogeneous as γ^R changes over time and state. In fact, the representative agent does not exhibit CRRA utility even though both individual agents do. The difference between other DSGE models and their algorithm is Benninga and Mayshar (1993) solve the numerical solution as a finite horizon problem. Essentially, the planner can solve a finite horizon sequential problem by backward induction and in this way avoid the issue of including the wealth distribution and its dynamics as state variables in a recursive formulation. The implicit assumption is that a sufficiently long horizon finite period model provides a good approximation to the infinite horizon problem.

The numerical solution does provide perpetual trading over time. The wealth distribution is path dependent, as the less risk averse agent gains wealth when the positive stock dividend states are observed.

The Benninga and Mayshar (1993) model was calibrated to historical data. Surprisingly, the model was able to provide stock and bond returns equivalent to the findings in Mehra and Prescott (1985). The assumptions of the calibrated model, however, did not conform with classical economic theory. It was assumed that one agent had a discount factor greater than 1 (valued future consumption more than current), and the other agent was extremely risk averse with $\gamma > 30$. Therefore, the model does provide reasonable market dynamics but at the cost of unrealistic parameter assumptions.

1.2.2 Chapter 3: Solving Heterogeneous Agent General Equilibrium Models with Assets

Chapter 3 introduces a finite period, heterogeneous agent, DSGE model with aggregate and idiosyncratic shocks. A two period problem and *competitive equilibrium* will be defined. This agent decisions of consumption versus savings and investment allocation will be considered. This model will be extended to a general T period model. A competitive equilibrium will be defined and the algorithm used for this research will be described in detail.

The DSGE model I present in chapter 3 includes different assumptions combined from previous research. The heterogeneous agents will maximize a life-time CRRA utility. These agents will differ in their aversion to risk. The dividend growth will follow (1.2), with a probability of a positive or negative shock. There is a stochastic idiosyncratic employment shock for each agent. With a risky and risk-free asset available to the investor, the markets are dynamically complete. As in Benninga and Mayshar (1993), I use a finite horizon T period model. Given this assumption, each agent will consume their entire wealth in period T. Working backwards, the algorithm solves recursively from time $T, T - 1, ..., t = 0$. Numerical simulations are run with a large T so that the early nodes in the tree approximate an infinite horizon solution.

In this chapter I discuss the algorithm in detail, beginning with a simple 2 period model and working up to a general T-period model with financial assets. Although the basic algorithmic approach is not new, the details matter a great deal and have not been clearly dealt with in the literature. Specifically, most models use assets with no dividend growth and are in zero net supply

so that the aggregate endowment does not grow over time and is independent of the aggregate shock. Relaxing these assumptions is more complex than it might first appear.

1.2.3   Chapter 4: Properties of the Model

Chapter 4 considers the properties of the T period model described in Chapter 3. A homogeneous agent model will be considered and numerical simulations will be presented to verify results of the model to well known solutions of Lucas (1978) and Levhari and Srinivasan (1969). A heterogeneous agent model with aggregate shocks in a pure exchange economy will also be considered. A numerical simulation will highlight possible values to agent consumption, investment allocation, and wealth as the economy evolves over time. This example of aggregate shocks in a growth economy is a similar setup to the Benninga and Mayshar (1993) model. Again, consistent results with previous research help validate the model as additional assumptions are introduced. The final model setup will include a finite period, heterogeneous agent, complete markets setting, with aggregate and idiosyncratic risk in a growing economy.

Through multiple numerical simulations, this model was able to provide price and trading volatility in both assets. The wealth distribution adjusts over time and is path dependent. Agents are able to insure through complete markets, however, trade is required as aggregate and idiosyncratic shocks are revealed. One result of the model is that including idiosyncratic shocks in a growth economy increases trading volume, price volatility, and the equity premium. Table 1.1 shows some results for the model with only aggregate shocks compared to the model with aggregate and idiosyncratic shocks.

Table 1.1: As seen in the last two rows of the table, introducing idiosyncratic shocks to the model greatly increases trading volume.

Statistic	Aggregate Shock	Aggregate and Idiosyncratic Shock
$E_t[R_f]$	8.90%	8.84%
$E_t[R_s]$	9.31%	9.64%
σ_{R_s}	4.00%	8.01%
$E_t[R_s - R_f]$	0.40%	0.80%
$E_t[\frac{P_t}{D_t}]$	13.25	13.32
$\sigma_{\frac{P_t}{D_t}}$	0.29	0.67
$E_t[s_{t+1}^i - s_t^i]$	0.02	3.88
$\sigma_{s_{t+1}^i - s_t^i}$	0.01	8.51

Figure 1.1 presents two simulations. The aggregate shock only is presented on the left side. This example is essentially the pure exchange model of Benninga and Mayshar (1993). There is very minimal trading each period and returns are realized around two potential rates. If a positive (negative) shock occurs then the higher (lower) return is recognized. Unless there is a highly negative shock to the dividend, returns on the equity will always be positive. The Judd et al. (2003) model would provide similar results to the aggregate shock only, with as many returns possible as states of the world.

The histogram on the right displays returns observed in the model outlined in chapter 3. The inclusion of idiosyncratic shocks to both agents provided the distribution more resembling of a normal curve. The assumed idiosyncratic shock for this model was a multiple of the current dividend. That multiple was drawn for each agent from a uniform distribution of $[1.9, 2.1]$ at each state of the world. With the same aggregate shock provided in each plot, the idiosyncratic shocks were able to provide states of negative returns.
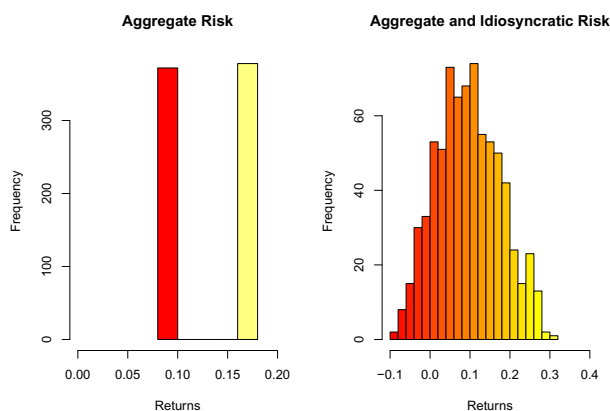
Figure 1.1: Histogram of the stock returns for aggregate shocks only (left) and aggregate and idiosyncratic shocks (right).

The numerical solutions to this model provide financial market dynamics not typically available in DSGE representative agent based models. Although this model has not been calibrated to specific historical data, the dynamics that result from suitable assumptions are relevant to financial markets. This model has the potential to inform policy and provide greater insight to some of the questions in financial economics.

1.2.4 Chapter 5: Heterogeneity and the Equity Premium Puzzle

The final chapter illustrate how the general model might be used to analyze real-world issues. I roughly calibrate the model to obtain results similar to those observed in financial markets and I specifically consider the equity premium puzzle. With a reasonable discount factor of $\beta = .98$, risk aversion parameters $\gamma^i < 10$, aggregate shocks consistent wtih annual U.S. consumption growth, and idiosyncratic shocks of roughly twice the size of the dividend, the model provided an expected equity premium of 2.5%. Although that is not the 6% reported in Mehra and Prescott (1985), the bond return was lowered to 2.7%. There were specific random paths that did generate a higher equity premium, with a lower bond return and higher equity return. Some of these results are summarized in Table 1.2.

Table 1.2: The market characteristics of the model are displayed for a specific path from the model with aggregate and idiosyncratic shocks.

Statistic	Average Paths	Specific Path
$E_t[R_f]$	2.74%	2.84%
$E_t[R_s]$	5.27%	7.68%
σ_{R_s}	18.10%	31.12%
$E_t[R_s - R_f]$	2.53%	4.85%
$E_t[\frac{P_t}{D_t}]$	76.6	57.27
$\sigma_{\frac{P_t}{D_t}}$	14.8	16.65
$E_t[s^i_{t+1} - s^i_t]$	2.84	7.36
$\sigma_{s^i_{t+1} - s^i_t}$	11.44	31.10

Numerous simulations of different parameters and initializations of the model will be compared. The discount factor, risk aversion, aggregate and idiosyncratic shocks were all modified during this process. The Lucas asset pricing model does not provide a low enough bond return or realize a large enough risk premium between the two asset returns. This exercise is not designed to solve the equity premium with exact numerical results of 7% equity returns and 0.8% bond returns. Instead, the goal is to apply assumptions in line with economic theory, yet obtain modeled results that contain dynamics consistent with empirical results while using intuitively plausible parameters.

Chapter five concludes with some possible extensions of the model such as introducing utility functions with habit formation to better model the risk-free rate and the risk premium separately, introducing addition investor types, and generalizing the idiosyncratic shocks.

CHAPTER 2

A REVIEW OF APPROACHES FOR SOLVING GENERAL EQUILIBRIUM MODELS WITH FINANCIAL ASSETS

2.1   Introduction

There are many questions and anomolies that still exist in financial economics. For instance, why are average stock returns so high relative to the average risk-free return? Mehra and Prescott (1985) and Campbell (1999) consider the historical difference observed in returns between equities and bonds. The conclusion is that the pricing function provided from agent based asset pricing models, calibrated with realistic parameters, does not reward equity investors to the magnitude historically observed in financial markets. Researchers have applied these Dynamic Stochastic General Equilibrium (DSGE) models to financial markets to address these issues. This chapter is designed as a review of specific DSGE models and their contribution in the financial asset pricing literature.

The seminal Lucas (1978) model is the most cited in the financial economics literature. Through the construction of a representative agent within incomplete markets, this consumption-based asset pricing model determined how assets are priced. Furthermore, this framework pioneered the use of DSGE models in financial economic research. Through aggregate shocks, the pricing function provides volatility in risky asset prices. However, this general equilibrium model concludes with a "one and done" trading environment. Investors are able to make one adjustment to their initial asset allocation and no longer need to trade. Therefore, although there is deviation in asset prices, portfolios remain constant. The limitations of the representative agent model are evident through inquiries about financial asset trading volatility as well as agent wealth distribution. Researchers conjectured that it would require at least heterogeneity to address these issues.

The models described in this chapter include heterogeneous agents that potentially trade financial assets after the initial time period. The first model considered includes idiosyncratic and aggregate uncertainty. The economy has growth and is not stationary, however, investors face fi-

nancial markets that are incomplete. Although there are multiple variations to this model, Krussell and Smith (1998) is the most well-known. The following two models are considered extensions of Lucas (1978). First, Judd et al. (2003) consider a stationary economy with aggregate and idiosyncratic risk. The heterogeneous agents can hedge these risks through a complete financial market. Benninga and Mayshar (1993) set up a similar model, however, allow growth in the aggregate economy. The heterogeneous agents can invest in a complete financial market, yet rely on their asset holdings for consumption as individual endowments do not exist. Each section will define a model and consider the assumptions made. An algorithm that can be used to solve the model will be provided in a step by step process. Finally, the contributions of the model to finanical economic theory will be considered.

2.2 The Lucas Asset Pricing Model

Consider an economy with N agents, a single type of consumption good C, one risky asset S, and one risk-free asset B. One could imagine this economy taking place on an island. The individuals on this island are identical and live forever. The island contains one type of fruit which the agents must eat in order to survive. The only way to obtain this fruit is from the only tree located on the island. This tree has a random process for producing the consumption good each period. The agents can purchase shares of the tree, which entitles them to future production. The agents also have the opportunity of trading current consumption for one unit guaranteed of next period's fruit. This fruit is not storable and only lasts one period. Each period the stock market clears and determines the price of the tree, which determines the agent's wealth. The agent then must decide what portion of his wealth to use for current consumption or investment.

The standard Lucas asset pricing model assumes an endowment economy with one consumable good. Agents are utility maximizers that live forever. Typically, these models generally make two key assumptions in order to obtain analytical solutions at equilibrium: (1) all agents are homogeneous, or the entire population can be represented by a single aggregate representative agent; (2) agents have sufficient information to determine future asset price distributions. The primary source of aggregate uncertainty in the model comes from an exogenous stochastic dividend process.

2.2.1 The Model

Each agent is assumed to have the following CRRA utility function which is continuous, twice differentiable, increasing in c_t, and strictly concave,

$$u(c_t) = \frac{(c_t)^{1-\gamma}}{1-\gamma}. \tag{2.1}$$

Where $\gamma > 0$ is the agent's degree of risk aversion. When $\gamma = 1$, the utility function is defined as $u(c_t) = ln(c_t)$.

In this economy a **financial market equilibrium** consists of individual stock holdings (s_t^i), bond holdings (b_t^i), consumption levels (c_t^i), stock spot prices (P_t), and bond prices (Q_t) such that:

- Each agent (i) maximizes his expected discounted lifetime utility

$$\max_{\{c_t^i, s_{t+1}^i, b_{t+1}^i\}} E_0 \sum_{t=0}^{\infty} \beta^t \, u(c_t^i) \tag{2.2}$$

subject to the constraints

$$c_t^i + P_t \, s_{t+1}^i + Q_t b_{t+1}^i \leq (D_t + P_t)s_t^i + b_t^i, \quad \text{and} \tag{2.3}$$

$$c_t^i \geq 0, \qquad \forall \; i, \, t. \tag{2.4}$$

- The stock, bond, and good markets clear

$$\sum_{i=1}^{N} s_t^i = 1,$$

$$\sum_{i=1}^{N} b_t^i = 0,$$

$$\sum_{i=1}^{N} c_t^i = D_t.$$

Let the dividend of the risky asset be D_t, and is assumed to follow the stochastic process

$$D_t = e^{x_t} D_{t-1}, \tag{2.5}$$

where

$$x_t = \mu + \xi_t, \quad \text{and} \quad \xi_t \sim iid \text{ N}(0,1). \tag{2.6}$$

Let w_t^i represent the agent's current period wealth. Notice that the agent's budget constraint (2.3) can be rewritten as

$$c_t^i + P_t s_{t+1}^i + Q_t b_{t+1}^i \leq w_t^i, \qquad \forall t. \tag{2.7}$$

Define α_t^i as the fraction of wealth invested in the risky asset, and $1 - \alpha_t^i$ as the proportion of wealth invested in the risk free asset,

$$\alpha_t^i = \frac{P_t s_{t+1}^i}{w_t^i - c_t^i}, \quad \text{and} \quad 1 - \alpha_t^i = \frac{Q_t b_{t+1}^i}{w_t^i - c_t^i}.$$

Given these defined variables, the budget constraint can be described as

$$c_t^i + \alpha_t^i(w_t^i - c_t^i) + (1 - \alpha_t^i)(w_t^i - c_t^i) \leq w_t^i, \qquad \forall t. \tag{2.8}$$

The returns each period on stock holdings and bond holdings ($R_{s,t+1}$ and $R_{b,t}$ respectively) will be defined as

$$R_{s,t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}, \quad \text{and} \quad R_{b,t} = \frac{1}{Q_t}.$$

Therefore, the individual portfolio returns are defined as

$$R_{p,t+1}^i = \alpha_t^i R_{s,t+1} + (1 - \alpha_t^i)R_{b,t}. \tag{2.9}$$

With the setup of this model (concave utility function and convex budget constraint), it is known that the budget constraint will bind. If the agent does not consume or invest his wealth it would be considered wasteful. Given the modification of the budget constraint, the following is provides consistent notation with Levhari and Srinivasan (1969):

$$w_{t+1}^i = R_{p,t+1}^i(w_t^i - c_t^i), \qquad \forall t. \tag{2.10}$$

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With a concave utility function, $\beta \in (0,1)$, a convex and compact budget constraint, the solution to the policy functions c_t^i and α_t^i are recursive (Mas-Colell et al. (1995)). This problem can be linked to the Bellman equation and set up as the following value function,

$$V(w,D) = \max_{\{c,\alpha\}} \left[u(c) + \beta E V(w',D') \right] \tag{2.11}$$

subject to the constraints

$$w' = R_p'(w-c) \quad \text{and} \tag{2.12}$$

$$w \geq c \geq 0. \tag{2.13}$$

Where,

$$R_p' = \alpha R_s' + (1-\alpha)R_b,$$

$$R_s' = \frac{D'+P'}{P}, \quad \text{and} \quad R_b = \frac{1}{Q}.$$

The primed variables denote period $t+1$ and the unprimed variables are t (Notice for notational purposes $R_p' = R_{p,t+1}$, $R_s' = R_{s,t+1}$, and $R_b = R_{b,t}$). When the agent wakes up on the island at time t, he is aware of his current holdings (s,b) and dividend (D). Given market clearing equilibrium spot prices (P,Q) the agent can determine current wealth (w). The state variables in the value function (2.11) are current wealth and dividend (w,d). Given the state of the world at time t and next periods expected returns, the agent must make the choice of both consumption (c) and allocation of investment (α). Due to the concavity of the utility function (2.1) the budget constraint (2.3) will bind with equality. Substituting the constraints, the value function becomes

$$V(w) = \max_{\{c,\alpha\}} \left[u(c) + \beta E V(R_p'(w-c)) \right]. \tag{2.14}$$

Notice the agent is given the state variable wealth and must choose between either consumption or investment. The current dividend is no longer needed as a state variable to the agent due to the specific *iid* dividend process given (2.5).

Taking the first order condition with respect to c,

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$$\frac{\partial V(w)}{\partial c} : \quad u'(c) + \beta E[(-R'_p)V'(R'_p(w-c))] = 0,$$

$$u'(c) = \beta E[(R'_p)V'(R'_p(w-c))].$$

Applying the envelope condition,

$$\frac{\partial V(w)}{\partial w} : \quad V'(w) = \beta E[V'(w')(R'_p)],$$

$$V'(w) = u'(c).$$

When updating $V'(w)$,

$$V'(w') = u'(c').$$

Substitution provides the agent's Euler equation

$$u'(c) = \beta E[(R'_p)u'(c')]. \tag{2.15}$$

This is more commonly recognized as

$$1 = E[M'(R'_p)], \tag{2.16}$$

where $M' = \beta \frac{u'(c')}{u'(c)}$ is the stochastic discount factor.

Taking the first order condition with respect to α,

$$\frac{\partial V(w)}{\partial \alpha} : \quad \beta E[(R'_s - R'_b)(w-c)V'(R'_p(w-c))] = 0$$

$$\beta(w-c)E[(R'_s - R'_b)V'(w')] = 0.$$

Substitution provides

$$E[(R'_s - R'_b)u'(c')] = 0$$

which is equivalent to Danthine and Donaldson (2005)

$$E[M(R'_s - R'_b)] = 0, \tag{2.17}$$

in which M is the same stochastic discount factor defined above. Notice the Euler equation (2.16) will determine the agent's consumption versus investment decision. The other equation (2.17), will resolve the asset allocation choice. When both equations are satisfied, the agent will have maximized his discounted lifetime expected utility.

2.2.2 Representative Agent

The dividend process provides an infinite amount of possible states in the world. Given the asset choices are finite, including the stock and bond, an investor cannot perfectly hedge against risk. Therefore, this economy includes incomplete markets. To solve the model, Lucas (1978) considers the general equilibrium model described above with N identical infinitely lived agents. With the same initial holdings (s_0, b_0) and risk aversion parameter (γ), this problem can be set up with a representative agent. This specific assumption significantly simplifies the solution.

Assuming all agents are identical in wealth and risk, each agent maximizes (2.2), and the markets clear each period, it is easy to rationalize that the agent's stock and bond holdings are

$$s_t^i = s_{t+1}^i = \frac{1}{N}, \qquad \text{and} \tag{2.18}$$

$$b_t^i = b_{t+1}^i = 0, \qquad \forall \ i, \ t. \tag{2.19}$$

Notice that when $b_t = 0$ the agent has an $\alpha^i = 1$. Substituting (2.18) and (2.19) into the representative agent's budget constraint will provide $c_t = D_t$ for all t. Let the utility of agents be dfined by the CRRA utility function (2.1). Solving the first order condition would result in the same Euler equation found in (2.15). Setting $\alpha = 1$, the portfolio returns are simply the stock returns, $R_{p,t+1} = R_{s,t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}$. Through substitution,

$$P_t u'(c_t) = \beta E_t \left[(D_{t+1} + P_{t+1}) u'(c_{t+1}) \right], \tag{2.20}$$

obtaining the equilibrium pricing function

$$P_t = \beta E_t \left[(D_{t+1} + P_{t+1}) \frac{u'(c_{t+1})}{u'(c_t)} \right]. \tag{2.21}$$

Substituting $c_t = D_t$ and $u'(c_t) = c_t^{-\gamma}$, we obtain

$$P_t = \beta E_t \left[(D_{t+1} + P_{t+1}) \left(\frac{D_{t+1}}{D_t} \right)^{-\gamma} \right]. \tag{2.22}$$

Due to the given dividend process (2.5),

$$P_t = \beta E_t \left[(D_{t+1} + P_{t+1})(e^{x_{t+1}})^{-\gamma} \right]. \tag{2.23}$$

Many investors often like to use a price-dividend ratio to compare assets. In this example let $\nu_t = \frac{P_t}{d_t}$. Substituting into (2.22) provides

$$
\begin{aligned}
P_t &= \beta E_t \left[(D_{t+1} + P_{t+1}) \left(\frac{D_{t+1}}{D_t} \right)^{1-\gamma} \left(\frac{D_t}{D_{t+1}} \right) \right], \quad \text{or} \\
\nu_t &= \beta E_t \left[(1 + \nu_{t+1}) \left(\frac{D_{t+1}}{D_t} \right)^{1-\gamma} \right].
\end{aligned}
$$

The dividend process (2.5) provides the following:

$$\nu_t = \beta E_t \left[(1 + \nu_{t+1}) \left(e^{x_{t+1}(1-\gamma)} \right) \right]. \tag{2.24}$$

Let $m_{t+1} = \beta \left(e^{x_{t+1}(1-\gamma)} \right)$, and iterate on (2.24):

$$
\begin{aligned}
\nu_t &= E_t \left[m_{t+1}(1 + \nu_{t+1}) \right] \\
&= E_t m_{t+1} + E_t m_{t+1} \nu_{t+1} \\
&= E_t m_{t+1} + E_t m_{t+1} m_{t+2} + E_t m_{t+1} m_{t+2} \nu_{t+2} \\
&= \sum_{i=1}^{\infty} \left(E_t \prod_{j=1}^{i} m_{t+j} \right).
\end{aligned}
$$

Notice that if we assume x_t is normally distributed, $x_t \sim N\left(\mu_x, \sigma_x^2 \right)$,

$$
\begin{aligned}
E_t \prod_{j=1}^{i} m_{t+j} &= \beta^i e^{(1-\gamma)i\left[\mu_x + \frac{1}{2}(1-\gamma)\sigma_x^2 \right]} \\
\nu_t &= \sum_{i=1}^{\infty} \beta^i e^{(1-\gamma)i\left[\mu_x + \frac{1}{2}(1-\gamma)\sigma_x^2 \right]}.
\end{aligned}
$$

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This geometric series ν_t will converge (see Theorem 1 Burnside (1998)) as long as

$$0 \le \beta e^{(1-\gamma)\left[\mu_x + \frac{1}{2}(1-\gamma)\sigma_x^2\right]} < 1.$$

Given the parameters $(\beta, \gamma, \mu_x, \sigma_x)$, we will assume convergence to obtain

$$\nu_t = \frac{\beta e^{(1-\gamma)\left[\mu_x + \frac{1}{2}(1-\gamma)\sigma_x^2\right]}}{1 - \beta e^{(1-\gamma)\left[\mu_x + \frac{1}{2}(1-\gamma)\sigma_x^2\right]}} \qquad \forall t.$$

Therefore, with the dividend process assumed we have a constant price to dividend ratio ($\nu_t = \nu$). Notice, if the agent has log utility($\gamma = 1$) the pricing function simplifies to

$$P_t = \left(\frac{\beta}{1 - \beta}\right) D_t. \tag{2.25}$$

Finding a constant ν confirms that we do have $i.i.d.$ returns:

$$
\begin{aligned}
R_{t+1}^p &= \frac{D_{t+1} + P_{t+1}}{P_t} \\
&= \left(\frac{\frac{D_{t+1}}{D_{t+1}} + \frac{P_{t+1}}{D_{t+1}}}{\frac{P_t}{D_t}}\right)\left(\frac{D_{t+1}}{D_t}\right) \\
&= \left(\frac{\nu + 1}{\nu}\right) e^{x_{t+1}}.
\end{aligned}
$$

This section highlighted the genearal equilibrium of a special case Lucas model, in which returns are assumed to be $i.i.d.$. Employing the assumption of a representative agent provided a trivial solution to the pricing function. A major contribution of this research is the well known pricing function $P_t = E_t\left[M_{t+1}X_{t+1}\right]$ that is derived from (2.16). The price of an asset at time t is the expected stochastic dividend times the payoff. The financial asset spot prices are state specific. As a function of the aggregate harvest or dividend, price levels vary depending on the state of the world. However, at equilibrium trading financial assets does not exist among investors.

Given the recursive nature of the model, the problem is adjusted to use the Bellman equation (2.11). To obtain the equilibrium solution a value function iteration method is applied (see Stokey and Lucas (1989)). The most common criticism of the Lucas (1978) model is the lack of dynamics in trading and asset prices that is realized empirically. Furthermore, the representative or homogeneous agent model only provides a degenerate wealth distribution. The financial economics

literature has responded to these concerns by introducing heterogeneity to the model. Relaxing the homogeneous agent assumption, however, comes at the cost of increasing complexity to the equilibrium solution. The state space increases in size as the wealth distribution is required for (2.11). Krussell and Smith (1998) addresses this complex issue by assuming an approximation to the wealth distribution.

2.3   The Krussell-Smith Model

The goal of many agent based asset pricing models is to evaluate how policy or specific shocks may affect individual behavior. Adjustments made to agent consumption or investment decisions will have an impact on financial markets. Researchers can utilize these models to focus on how policy might affect asset prices, trading volume, and investors portfolio allocation choices.

The motivation of this model is to address heterogeneity among consumers in a complex environment. This complexity arises from five specific assumptions of the model. First, the infinitely-lived rational agents are utility maximizing. Second, the economy contains aggregate shocks through productivity. Third, investors face idiosyncratic employment shocks. Fourth, incomplete markets which do not provide full insurance. The final assumption introduced is essentially heterogeneity. These five assumptions of this model highlight an ambitious attempt to extend the Lucas asset pricing model.

Due to uninsurable idiosyncratic shocks, the entire wealth distribution and dynamics are required state variables, making it difficult to solve (Peralta-Alva and Santos (2010)). Accurate computation of these models becomes very difficult. In fact, a numerical approximation is applied to the wealth distribution, in order to determine a solution. To avoid these computational issues, quite often, models will drop one or more of these assumptions.

2.3.1   The Model

This section will set up and define the model following Krussell and Smith (1998). Unlike the rest of the models in this chapter, this economy incluldes households, a firm, and government. This is a production economy with aggregate productivity shocks and heterogeneous agents that face idiosyncratic employment shocks. The only asset in this economy is capital stock. Since only one asset is available to investors, markets are incomplete and individual agents cannot fully insure against risk.

The economy includes a continuum of ex ante infinitely lived agents uniformly distributed on the interval $I = [0,1]$. All agents have constant relative risk aversion (CRRA) preferences with risk aversion paramater $\gamma \geq 0$ and the discount factor β. Let $i \in I$ represent the individual agent. Define c_t^i as the individual level of consumption, and k_t^i as the level of capital owned by investor i. Capital depreciates at the rate δ, but also earns a rental rate of r_t. Therefore, the gross rate of return on capital is $(1 - \delta + r_t)$. Each agent realizes an idiosyncratic shock through employment ϵ_t^i. At each time period, the individual can be considered employed $\epsilon_t^i = 1$ or unemployed $\epsilon_t^i = 0$. The probability that one can be employed or unemployed each time period follows the transition probability matrix,

$$\pi(\epsilon' | \epsilon) = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix}.$$

Let L_t be the aggregate employment level. Each individual employed at time t receives a wage rate of w_t, for which they pay the government a tax rate τ_t. Those that are unemployed receive unemployment insurance μ, which is a percentage of the wage.

Therefore, the individual agent will maximize their lifetime utility,

$$\max_{\{c_t^i, k_{t+1}^i\}_{t=0}^{\infty}} U^i(c^i) = E\left\{ \sum_{t=0}^{\infty} \beta^t \frac{(c_t^i)^{(1-\gamma)}}{1-\gamma} \right\}, \tag{2.26}$$

subject to the period budget constraint,

$$c_t^i + k_{t+1}^i = (1 - \delta + r_t)k_t^i + [\epsilon_t^i(1 - \tau_t) + \mu(1 - \epsilon_t^i)]w_t, \tag{2.27}$$

and the non-negativity constraints,

$$c_t^i \geq 0, \quad \text{and} \tag{2.28}$$

$$k_t^i \geq 0, \quad \forall \ t. \tag{2.29}$$

Markets in this economy are perfectly competitive. Hence, the model assumes a single firm with a Cobb-Douglas production technology. There is an aggregate productivity shock a_t. The productivity shock is a_g in a good state and a_b in a bad state. The probability of a good state is ϕ, while $(1 - \phi)$ is the probability of a bad state. Let output Y_t be a function of aggregate labor L_t and capital K_t,

$$Y_t = a_t L_t^{(1-\alpha)} K_t^{\alpha}, \quad \alpha \in (0,1). \tag{2.30}$$

At profit maximization, the market clearing factor prices are

$$w_t = (1 - \alpha) a_t \left(\frac{K_t}{L_t} \right)^{\alpha}, \quad \text{and} \quad (2.31)$$

$$r_t = \alpha a_t \left(\frac{K_t}{L_t} \right)^{\alpha - 1}, \quad \forall \ t. \quad (2.32)$$

The government taxes the employed individuals and redistributes the funds to the unemployed. The model assumes a balanced budget, so that aggregate revenues equal aggregate unemployment insurance,

$$\tau_t w_t L_t = \mu w_t (1 - L_t). \quad (2.33)$$

Define $u_t = 1 - L_t$ as the unemployment rate at time t. Dividing (2.33) by the wage rate implies that the tax rate is

$$\tau_t = \frac{\mu u_t}{L_t}. \quad (2.34)$$

For this model, a stationary wealth distribution $F(k; a, \epsilon)$ exists. Each individual agent's investment decision is a function of the wealth distribution $k'(k, \epsilon; F)$. Given the recursivity of the household problem, the Bellman equation can be set up as

$$V(\epsilon, k, F) = \max_{c, k'} \left[\frac{c^{1-\gamma}}{1 - \gamma} + \beta E\{V'(\epsilon', k', F') | \epsilon, F\} \right], \quad (2.35)$$

subject to the budget constraint (2.27).

2.3.2 The Algorithm

This model includes both idiosyncratic and aggregate risk, which shifts the wealth distribution throughout time. As a state variable the entire wealth distribution F is required, however, it is complex and very difficult to calculate. Therefore, Krussell and Smith (1998) do not use the wealth distribution F to determine the choice variable k', but instead estimate the mean of the distribution. In theory, it is believed that the agents that hold a majority of the wealth will have similar savings rates. The individuals that do not hold as much wealth will not be able to contribute much to savings. Therefore, the entire distribution is not required in the state space in order to approximate (2.35). The assumption is made that the first moment of the wealth distribution does well at forecasting the next period's capital stock K. The following steps outline the algorithm used by Krussell and Smith (1998):

1. Initialize the distribution of capital, employment and the productivity shock to determine F_0. Given this distribution any initial moments from the K distribution can be determined.

2. The Krussell and Smith (1998) model assumes the aggregate law of motion for capital to follow

$$\ln K' = b_0(a) + b_1(a)\ln K. \tag{2.36}$$

In this model $b_1(a)$ is calculated as approximately 0.95, solved for the algorithm.

3. Given the assumed law of motion and approximating the wealth distribution, the household problem (2.35) is modified to

$$V(\epsilon, k, m) = \max_{c,k'} \left[\frac{c^{1-\gamma}}{1-\gamma} + \beta E\{V'(\epsilon', k', m')|\epsilon, m\} \right]. \tag{2.37}$$

Let m be an approximation of the capital distribution. In this case, m represents the first moment of the distribution of K. With the given law of motion for K (2.36) and initial values for $b_0(a)$ and $b_1(a)$, solve the household optimization problem and compute (2.37). Simulate the dynamics and obtain the sequence throughout time of the first moment of capital ($\bar{k}_t$).

4. Using OLS, calculate new values for $b_0(a)$ and $b_1(a)$. With the regression calculate any forecasting errors. If the errors are within desired tolerance, algorithm is complete. Otherwise, update values of $b_0(a)$ and $b_1(a)$ and iterate steps.

Krussell and Smith (1998) determine that the first moment is satisfactory in solving the model. If the model needs to be more precise, additional moments can be required to determine the functional form of the law of motion for capital. Although this computational assumption solves this specific model, there are some accuracy issues. The first moment is a fair approximation due to the specific model setup and consideration of productivity shock a. The assumed process of the shock creates little variation in the aggregate capital stock. The first moment of capital ($\bar{k}$, therefore, is a decent approximation. However, small errors in this assumption lead to substantial errors in equilibrium values of prices and capital stock. Badshah et al. (2013) address these errors that occur in these type of dynamic stochastic general equilibrium models. This research provides a quicker, more accurate algorithm to solving a similar model with idiosyncratic, but no aggregate shocks.

Krussell and Smith (1998) assume heterogeneity through a continuum of agents with different wealth and employment shocks. The risk aversion parameter, however, with CRRA utility is identical for all agents. Therefore, consumption versus investment behavior is similar among all

agents, just with different wealth levels and possible idiosyncratic shocks. Furthermore, there are no explicit assets in this economy. The aggregate capital has a low variance, enabling the mean approximation to be close.

It is important to understand that these general equilibrium models have a give-and-take functionality. Preferably, models should include assumptions that emulate the environment for which they are trying to explain. However, as certain assumptions are generalized, computational methods to solve the problem become more complex. For example, a homogeneous agent model often contains closed form solutions. The pricing function and agent decision rules are determined by a function. For a financial asset pricing model to include a practical solution and application to observed data, specific assumptions will need to be compromised.

2.4   The Judd-Kubler-Schmedders Model

Judd, Kubler and Schmedders, Judd et al. (2003), introduce heterogeneity through different risk aversion. The infinitely-lived heterogeneous agents inhabit an economy with a dynamically complete market structure. At any given point in time, these investors can fully insure across risk. Idiosyncratic and aggregate risk both exist, however, there are finite states of the world (steady state economy). The goal of this model is to introduce true heterogeneity in a Lucas framework, and address the trivial trading volume that resulted from a homogeneous agent model.

The homogeneous agent assumption simplifies the solution to a DSGE model. Investors are often considered to be identical, therefore, holding identical portfolios and consuming equal portions. Empirically, asset trading volume exists on a daily basis. Portfolios are often assessed for rebalancing. Many researchers believe that differences in tastes and preferences among investors accounts for trading among assets.

The motivation behind this model is to determine whether heterogeneous preferences alone induce trade in an asset pricing model. As an extension of Lucas (1978), the assumptions of the model remain relatively simple. In fact, the numerical methods used to solve the model are straightforward. The methodology was originally introduced by Negishi (1960).

2.4.1   The Model

In this economy there are N heterogeneous agents that are characterized by their risk aversion parameter γ. For this section it is assumed that agents posess a CRRA utility function. Time in

this economy is indexed by $t = 0, 1, ..., \infty$. There are potentially Z possible states that can occur in this steady state economy. Each state occurs with probability π_z. These probabilities are denoted through a Markov process Π.

There are J financial securities available to each agent. Each security has a state specific payoff d_z^j. The entire set of payoffs D are represented by a JxZ matrix. The assets are considered to be either infinitely-lived L or short-lived S, such that $J = L + S$. Agents purchase s_z^i shares of infinitely-lived assets and b_z^i shares of short-lived assets at a price of P_z and Q_z, respectively. Infinitely-lived assets are in unit supply and short-lived assets are in net zero supply. Equity and a one period discount bond are examples of an infinitely-lived and short-lived asset, respectively. There is idiosyncratic risk in this economy, as each agent receives a state specific endowment (e_z^i). For this set up, i denotes the individual agent and z represents the specific state.

Each agent will maximize their lifetime utility

$$\max_{\{c_t^i, s_{t+1}^i, b_{t+1}^i\}} U^i(c^i) = E \left\{ \sum_{t=0}^{\infty} \sum_{z=1}^{Z} \pi_z \beta^t \frac{(c_{t,z}^i)^{(1-\gamma^i)}}{1-\gamma^i} \right\} \tag{2.38}$$

subject to the time period budget constraints

$$c_{t,z}^i + P_{t,z} s_{t+1,z}^i + Q_{t,z} b_{t+1,z}^i \leq e_{t,z}^i + (P_{t,z} + D_{t,z}) s_t^i + b_t^i, \tag{2.39}$$

and positive consumption constraint

$$c_{t,z}^i > 0, \tag{2.40}$$

for all t, z, and i.

2.4.2   The Algorithm

Given the assumption of dynamically complete markets, Judd et al. (2003) determine the *competitive equilibrium* through solving the Pareto optimal allocations. A Planner's problem approach is applied with maximizing a social planner's utility. The planner maximizes a utility funciton that is defined by weighted individual utilities

$$\max_{\{c^i\}} U^*(C) = \sum_{t=0}^{\infty} \beta^t \left(\sum_{i=1}^{N} \alpha^i u^i(c^i) \right), \tag{2.41}$$

where α^i represents the planner (Negishi) weights. This is a constrained optimization problem, subject to the state specific resource constraint

$$\sum_{i=1}^{N} c_z^i \leq E_z + D_z, \tag{2.42}$$

where E_z and D_z denote the aggregate endowment and payoffs, respectively in that state. The following three steps describe the algorithm used by Judd et al. (2003):

1. The first step is to simultaneously compute the Negishi weights, Arrow security prices and consumption allocations for the representative agent. Taking first order conditions of the Planner's maximization problem and substituting the Lagrange multiplier gives the following equations:

$$u'(c_z^1) = \alpha^i u'(c_z^i), \qquad \forall \quad i \neq 1, \tag{2.43}$$

where z is the current state in the Markov process. Applying Walras' law allows the normalization of agent $i = 1$ to have a Negishi weight of $\alpha^1 = 1$. Since this is solved for all states and compared to all agents, there are $Z(N-1)$ equations. Let the Arrow security prices be the price of consumption,

$$p_z = u'(c_z^1).$$

Define the present value of lifetime agent consumption V_z^i and wealth W_z^i to be the recursive equations,

$$V_z^i = p_z c_z^i + \beta E\left\{V_z^i | z\right\},$$
$$W_z^i = p_z w_z^i + \beta E\left\{W_z^i | z\right\},$$

where $w_z^i = e_z^i + s^i D_z$. Given the recursivity of these equations, the solutions to these problems in matrix form are

$$V_z^i = [I_Z - \beta \Pi]^{-1}(p \circ c^i),$$
$$W_z^i = [I_Z - \beta \Pi]^{-1}(p \circ w^i),$$

where I_Z represents the Z dimension identity matrix. The life-time budget constraint requires discounted consumption to equate wealth (endowments and payoffs),

$$V_z^i = W_z^i, \qquad \forall \quad i, z.$$

Therefore, utilizing Walras' law across all agents there are $(N-1)$ equations,

$$[I_Z - \beta\Pi]^{-1}(p \circ (c^i - w^i)) = 0. \tag{2.44}$$

The market clearing condition insures aggregated wealth is consumed by the population,

$$\sum_{i=1}^{N} c_z^i = \sum_{i=1}^{N} w_z^i, \tag{2.45}$$

for all possible states z. Combining the nonlinear equations from (2.43), (2.44), and (2.45) provides $NZ + (N-1)$ equations. There are NZ agent consumption decisions and $N-1$ Negishi weights that will be solved.

2. Given agent consumption levels solved from the previous step, any financial asset can be priced using the Arrow security prices (p_z) defined above. The prices of the risky security P_z^j and risk free security Q_z^j are calculated

$$P_z^j = \frac{[I_z - \beta\Pi]^{-1}\beta\Pi(p \circ d^j)}{p_z}, \tag{2.46}$$

$$Q_z^j = \frac{\beta\Pi(p \circ d^j)}{p_z}. \tag{2.47}$$

3. Finally, given solved consumption decisions and asset prices, determine the individual asset holdings. Since the individual one period budget constraints must hold for all agents across all states, it is true that

$$s_z^{i,j}(P_x^j + d_x^j) + b_z^i(d_x^j) = s_y^{i,j}(P_x^j + d_x^j) + b_y^i(d_x^j),$$
$$s_y^{i,j}(P_y^j + d_y^j) + b_y^i(d_y^j) = c_y^i - e_y^i + P_y s_y^i + Q_y b^i,$$

for all $x, y, z \in Z$, where $y, z \neq x$. The first equation validates that no matter what state of the world occurred at time t, payoffs from a portfolio of risky and risk-free assets will be equivalent at a specific state in time $t+1$. The second equation is the one period individual budget constraint where current consumption plus investment is equivalent to portfolio payoff plus endowment. Let R_1 be the payoffs of each asset,

$$R_1 = (P^1 + d^1, P^2 + d^2, ..., d^{l+1}, ..., d^s).$$

Infinitely-lived assets J^l have a payoff of price and dividend, while short-term J^s assets expire and pay a single dividend. Combining R_1 with the previous two equations give

$$(s_y^i - s_z^i, b_y^i - b_z^i)R_1 = 0,$$
$$s_y^i d_y^j + b_y^i (d_y^j - Q_y) = c_y^i - e_y^i,$$

for all $y, z \in Z$. Define R_2 as R_1 minus the asset prices. Therefore,

$$R_2 = (d^1, d^2, ..., d^{l+1} - Q^{l+1}, ..., d^s - Q^s).$$

Combining R_2 and the previous two equations, individual portfolios of infinitely-lived and short-lived assets can be solved as

$$(s^i, b^i) = (c^i - e^i)R_2^{-1}. \tag{2.48}$$

Notice that the Judd et al. (2003) model is state dependent. If a different initial state of the world is chosen, different equilibrium prices and consumption bundles will be calculated. More precisely, the representative agent of this economy is dependent on the wealth distribution of the initial state. However, a representative agent exists due to complete markets. Without growth in the economy, the representative agent can be identified with risk aversion γ^R in CRRA utility.

The solution to this model provides trivial trading volume results. In this heterogeneous agent model with idiosyncratic and aggregate risks, agents are still able to self-insure across states and time. Agents trade the initial period and then hold on to their portfolios. No further rebalancing is required. Judd et al. (2006) determine that only under very specific assumptions of the Markov process does perpetual trading occur. In this scenario, zero probability of a specific state exists in at least one state of the world.

The model provides a numerical method for solving equilibrium asset prices, consumption bundles and portfolios in a dynamic, stochastic, general equilibrium (DSGE) heterogeneous agent model. The assumption which generates a no-trade general equilibrium is a zero growth steady state economy. There is a finite number of possible states, with a probability to realize each of those states at any given period. This assumption is different than the other models considered in this chapter. The Krussell and Smith (1998) model considered a continuum of states.

2.5   The Benninga-Mayshar Model

Similar to the previous model, Benninga and Mayshar (1993) address the trivial no trade equilibrium by extending heterogeneity to a Lucas tree type model. The goal of this research is to encourage dynamic asset trading, solely through differences in agents' preference to risk. This heterogeneity is ex-ante, unlike previous research that considers idiosyncratic risk ex-post with incomplete markets.

The motivation of the Benninga and Mayshar (1993) model is to better understand the equity premium puzzle. It has been acknowledged that homogeneous agent models do not provide enough dynamics with HARA class utility functions. Therefore, this model adds heterogeneity in risk preferences to an exchange economy to obtain price volatility, asset trading, and changes in the wealth distribution.

2.5.1   The Model

Let this complete market economy consist of N infinitely lived agents. There is a single consumption good c, which the agents must consume each time period t. There are z possible states that can occur, with Z_t representing the history of states up to time period t. The probability of each event occuring is denoted $\pi_{t,z}$. The probability of moving from state in time t to state in time $t+1$ will follow a Markov transition process Π.

The payoff each period will be the state specific dividend $D_{t,z}$. Each agent will receive a portion of the dividend, determined by how many shares $s^i_{t,z}$ of ownership. These shares are considered ownership of all future dividends the tree (firm) produces. The shares are considered a risky financial asset, can be traded among the agents, and sum to one in aggregate. Agents are also able to borrow from each other $b^i_{t,z}$, however, the net supply of this risk free security is zero. Therefore,

$$\sum_{i=1}^{N} s^i_{t,z} = 1, \quad \text{and}$$
$$\sum_{i=1}^{N} b^i_{t,z} = 0.$$

The agents follow a CRRA utility function and are heterogeneous in the risk aversion parameter γ. The individual lifetime utility is definied as

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$$U^i(c^i) = \sum_{t=0}^{\infty} \sum_{z \in Z_t} \pi_{t,z} \beta^t \left(\frac{(c^i_{t,z})^{(1-\gamma^i)}}{1-\gamma^i} \right), \tag{2.49}$$

where i represents the individual agent, and β the discount factor.

Given complete markets, let $p_{t,z}$ be the Arrow-Debreu prices of one unit of consumption in period t and state z. An exogenously determined initial wealth distribution is required for this model. Let w^i represent the individual's weight of the initial economy's endowment. Therefore, agents will maximize (2.49) subject to the lifetime budget constraint

$$\sum_{t=0}^{\infty} \sum_{z \in Z_t} p_{t,z} c^i_{t,z} = w^i \left[\sum_{t=0}^{\infty} \sum_{z \in Z_t} p_{t,z} C_{t,z} \right]. \tag{2.50}$$

This model extends heterogeneity to a Lucas tree model in an exchange economy. Although individuals differ in risk preferences, a unique representative agent is identified to determine the equilibrium distribution of consumption. Many other asset pricing models make specific assumptions about the agent population, growth of the economy, or asset returns where a no-trade rational expectations equilibrium exists. The Lucas (1978) and Judd et al. (2003) are both examples.

2.5.2 The Algorithm

This section will briefly describe the algorithm used by Benninga and Mayshar (1993).

1. Choose an initial distribution of consumption c^i_0 (equivalently determining the initial wealth distribution w^i). Combining the consumption good market clearing equilibrium condition and definition of marginal rates of substitution provides

$$\sum_{i=1}^{N} c^i_0 \left[\frac{\beta^t}{q_{t,z}} \right]^{\frac{1}{\gamma^i}} = D_{t,z}, \tag{2.51}$$

where $q_{t,z}$ represents the stochastic discount factor. Given the initial consumptions, solve for all state specific prices. The Arrow-Debreu prices are now determined

$$p_{t,z} = \pi_{t,z} q_{t,z}. \tag{2.52}$$

2. At the rational expectation equilibrium, marginal rates of substitution must be equivalent to the state prices. Therefore, given the initial consumption distribution and state specific prices, agent consumption decisions can be solved using

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$$c^i_{t,z} = \beta^{\frac{t}{\gamma^i}} c^i_0 (q_{t,z})^{-\frac{1}{\gamma^i}}, \tag{2.53}$$

for all possible states.

3. Let the market value of the tree (firm) at each specific state be defined as the current dividend plus next periods market value. The state specific market value $M_{t,z}$ is calculated recursively as

$$M_{t,z} = D_{t,z} + \sum_{z \in Z} p_{t+1,z} M_{t+1,z}. \tag{2.54}$$

Let the state specific wealth $W^i_{t,z}$ of each agent be defined as the current consumption plus next periods wealth. Therefore, wealth is calculated recursively

$$W^i_{t,z} = c^i_{t,z} + \sum_{z \in Z} p_{t+1,z} W^i_{t+1,z}. \tag{2.55}$$

The state prices are contingent on the current state, and are calculated using the agent consumption decisions. Applying the first agent's ($i = 1$) parameters, the Arrow-Debreu prices are calculated

$$p_{t+1,z} = \beta \pi_{t+1,z} \left[\frac{c^1_{t+1,z}}{c^1_{t,z}} \right]^{-\gamma^1}. \tag{2.56}$$

These Arrow-Debreu prices also provide the risk free rate of return,

$$Rf_{t,z} = \left(\sum_{z \in Z} p_{t+1,z} \right)^{-1} - 1. \tag{2.57}$$

4. The final step in the algorithm determines agent portfolios. Since the financial assets are purchased at time t, asset holdings are constant at time $t + 1$ across all states z. With the return on risk free bonds known at time t, the only variance in agent wealth is due to the risky asset. Another way to think about it, is that if a representative agent owned all risky shares, his wealth would be equivalent to the market value of the tree ($W^i_{t,z} = M_{t,z}$). Therefore, agent stock holdings are calculated

$$s^i_{t,z} = \frac{W^i_{t+1,z} - W^i_{t+1,y}}{M_{t+1,z} - M_{t+1,y}}, \tag{2.58}$$

where y is a state in $t + 1$ different than z. With agent consumption, stock holdings, and state contingent prices determined, the bond holdings must satisfy the state specific budget constraints. Using wealth and market value, bond holdings are

$$b_{t,z}^i = \frac{M_{t+1,y} W_{t+1,z}^i - M_{t+1,z} W_{t+1,y}^i}{(M_{t+1,z} - M_{t+1,y})(1 + Rf_{t,z})}. \tag{2.59}$$

The algorithm described for this model is actually implemented using a finite horizon. In the limiting case, as the number of time periods T becomes large enough, the algorithm approximates an infinite horizon setup. At a specific final date $T + 1$, it is assumed that agent wealth is zero ($W_{T+1,z}^i = 0$). The period before each agent consumed their final wealth ($c_{T,z}^i = W_{T,z}^i$). Applying these assumptions, the algorithm solves the calculations in reverse time $t = T+1, T, ..., 0$ completing the "tree" or lattice. Therefore, the pricing function in the Benninga and Mayshar (1993) model is not a closed form solution.

This solution method uses complete markets to solve Arrow-Debreu prices in a finite horizon model. Given complete markets and initial consumption (wealth), a representative agent could be constructed. However, unlike Judd et al. (2003) the representative agent does not follow CRRA utility, even though the individual heterogeneous agents do. As states in the tree or lattice are revealed, expected growth in the dividend will typically provide the least risk averse agent with equity and wealth accumulation. As this process occurs, the representative agent has a decreasing γ^R. Proposition 5, in Benninga and Mayshar (1993) describes this result.

Krusell and Smith (2006) define five key characteristics for a theoretical model of the macroeconomy. The first two elements include agents that make rational and intertemporal investment and savings decisions. This model assumes agents maximize the lifetime utility (2.49). The third characteristic includes aggregate shocks in the model. Benninga and Mayshar (1993) includes a state specific dividend, which, from an aggregate perspective, represents the size of the economy. Following a Markov process enables upswings and downswings found in business cycles. The fourth characteristic is general equilibrium. This model contains asset prices that are endogenous to clear all markets. Finally, the model should contain heterogeneous agents. With the specific model setup defined, these five characteristics are met, and heterogeneity in risk preferences induces trade in financial assets. Agent portfolios of financial assets and wealth distribution adjust as different states of the economy are revealed throughout time.

This heterogeneous agent model design and algorithm provide price, volume, and wealth dynamics not recognized in the seminal Lucas (1978) model. However, this simulated approach comes at the cost of a closed form solution to the pricing function. With this simulation approach, Benninga and Mayshar (1993) address the equity premium puzzle Mehra and Prescott (1985), numerically through calibration. To obtain desired outcomes in asset returns, this model requires specific parameters to fall outside the expected theoretical ranges. At least one group of consumers in this heterogeneous agent model is assigned a discount factor greater than 1 (they would rather consume in the future than today), and another group is considered to be extremely risk averse ($\gamma > 35$). These calibration issues are identical to the findings from models with closed form solutions (Mehra and Prescott (2003)).

2.6   Conclusion

This chapter highlighted the contributions of four dynamic, stochastic, general equilibrium asset pricing models to the financial economics literature. The Lucas asset framework laid the foundation for financial asset pricing models. Although this model provided the fundamental pricing equation, criticism quickly arrived regarding the application. Krussell and Smith (1998) introduced a heterogeneous agent model in an incomplete market setting. The assumptions of this model were very ambitious, and quickly proved difficult to solve. Investors could not fully insure, and faced the aggregate and idiosyncratic risks. The wealth distribution adjusted throughout time as capital was exchanged among agents. Although this model provides interesting dynamics not found in the homogeneous agent case, it comes at the cost of a complex model with difficult numerical solutions. An "approximate aggregation" method is applied, however, accuracy of the results are a function of the specific setup and calibration of the model. Furthermore, heterogeneity is assumed through employment and wealth shocks, and not the characteristic assumption of risk aversion.

The next model reviewed, Judd et al. (2003), considered heterogeneous agents in a Lucas tree framework. The heterogeneity lies in risk averion in CRRA utility. The setup includes Z possible finite states of the world, creating a steady state economy. Agents experience both aggregate and idiosyncratic shocks, but can fully insure due to dynamically complete markets. Applying a Negishi weight methodology, given an initial state and wealth distribution, a planner's approach is used to solve the rational expectations equilibrium. In this model agents trade assets in the initial period

and hold this portfolio forever. Although agents are heterogeneous in risk preferences, the trivial trading result and price dynamics are quite similar to the homogeneous agent case.

Finally, Benninga and Mayshar (1993), also extends true heterogeneity to a Lucas tree agent based model. Markets are complete in this pure exchange model. However, the economy grows over time through an exogenous process. This assumed dividend growth process enables the number of possible states to grow exponentially over time. To solve the model a finite horizon approach is used. As a result, the pricing function is not a closed form solution.

These models contained many similar characteristics. Each contained discrete, infinite horizon economies, populated with rational heterogeneous agents. There exists a stochastic aggregate shock, potentially creating upswings and downswings in the economy. Each of these models have made significant contributions to finanical economics. Specific issues in the asset pricing field have been addressed, including but not limited to the equity premium puzzle, price and trading dynamics, and changes in the wealth distribution.

Although these models have provided insight into some of the key questions in financial economics, only further research will provide better understanding. Some researchers have approached these puzzles by stepping away from traditional general equilibrium theory. One example is the Santa Fe Institute's Artificial Stock Market (SFI-ASM). Arthur et al. (1997) consider a model with N heterogeneous agents. Investors choose a portfolio from many risky assets or a risk free bond that pays an exogenously determined constant rate. These agents are unaware of the dividend distribution, and apply predetermined decision rules based on asset prices. Throughout the simulation, agents use a learning mechanism to update their beliefs of the pricing function. Once all agents have determined the pricing function, the simulation has converged to an equilibrium. During this process agents are rebalancing their portfolios, prices are adjusting to clear the markets, and the wealth distribution is shifting. This model setup and myopic behavior, however, eliminates key elements necessary for a theoretical model to have practical policy implications, as defined in Krusell and Smith (2006).

After reviewing these different approaches, the question remains as how to efficiently solve a model with heterogeneity in differing utility functions. With growth in the economy, the problem becomes quite difficult given the wealth distribution, wealth dynamics, and continuation utilities. The next chapter will define in detail a dynamic, stochastic, general equilibrium asset pricing model.

This model will include heterogeneous agents in a growth economy. Agents will face idiosyncratic and aggregate shocks through a random endowment and dividend process. Specific assumptions will be addressed in order to determine a computationally feasible model. The remainder of this research will focus on the properties of this specific model and the possible applications it may have.

CHAPTER 3

SOLVING HETEROGENEOUS AGENT GENERAL EQUILIBRIUM MODELS WITH ASSETS

3.1   Introduction

Many agent based financial asset pricing models are solved under simplified assumptions. The most commonly applied assumption is the existance of a representative agent. This is implied by either allowing the agent population to be represented by an individual consumer or by imposing that all agents are identical. The seminal Lucas (1978) model is just one example. Given a representative agent economy, consumption is typically equivalent to any dividends from assets ($C = D$). Solving consumption determines the pricing kernel and ultimately asset prices. Finally, asset holdings are derived from the budgetary constraints in the economy.

Although assuming identical agents or a representative consumer simplifies solving the model, it is not indicative of heterogeneous consumers in an economy. If complete markets exist (as many assets available as states of the world), Arrow security prices can be determined and applied to price any asset. In this Arrow-Debreu ecoonomy agents insure across all possible states. If there is no growth in the model and the economy is stationary, a "one and done" scenario takes place in the initial period. Agents trade in the initial period and then hold the same portfolio of assets for the remaining time frame. A planner's approach is one methodology that can be applied to solve the heterogeneous agent model.

The goal of this chapter is to define a theoretically pleasing model, which has the potential to provide trading and price dynamics after the initial period. This model will include heterogeneous infinitely lived agents, that maximize a discounted lifetime constant relative risk aversion (CRRA) utility function. Markets are assumed to be dynamically complete, with as many possible states as available assets. The economy in this model will have the ability to grow over time. This model will require a different approach to solve, however, as agent specific wealth and portfolios will adjust over time.

This chapter will consider heterogeneous agent models with complete markets. The first model will be a general two period economy with Z possible states. With the assets available, each agent will choose their desired portfolio. For this model a set up of the economy will be described, a definition of the *competitive equilibrium* will be given, and the planner's problem will be addressed. An algorithm that can be used to solve the model for a one period and two period problem will be given. Additional time periods will then be included in the model for consideration. The time dimension will provide each agent a consumption versus savings decision. Therefore, a one, two, and finite T period models will be defined. The T period model will also include a algorithm to solve the model.

3.2 General Two Period Model

3.2.1 Introduction

This section will examine a general two period financial asset pricing model. The economy will be setup and a definition of *competitive equilibrium* will be given. An algorithm to solve the model will be provided using a social planner's approach. In this section a numerical example of a one period model will be considered. Agents initially solve an asset allocation problem and consume total wealth once a specific state is revealed. The second numerical example will include a two period model. Individual consumption will be required in the initial period of the model, forcing agents to make a consumption versus investment decision and an asset allocation choice in the initial period. Once the state is revealed in the second period, agents will consume their remaining wealth.

3.2.2 Model Setup

Let this two period economy be populated by N agents. There is a perishable good C available, which each agent derives utility through consumption. There are J different securities available with payoff D. These J financial securities will comprise of risky (J^l) and risk-free (J^s) assets, such that $J^l + J^s = J$. In the second period, Z possible states can occur with probability π_z. Given the assumption of complete markets, there are as many securities available as possible states ($J = Z$). The initial period is denoted $z = 0$ and is deterministic $\pi_0 = 1$. The subscript z and superscript i denote state and agent, respectively. Each agent is entitled to a state specific endowment e_z^i, which in aggregation E_z is the size of the economy.

Each agent will maximize their expected utility by choosing variables c_z^i and s_j^i,

$$\max_{\{c_z^i, s_j^i\}} U^i(c^i) = u^i(c_0^i) + \beta \sum_{z=1}^{Z} \pi_z u^i(c_z^i) \tag{3.1}$$

subject to the folowing budget and consumption constraints,

$$c_0^i + \sum_{j=1}^{J} P_j s_j^i \leq e_0^i, \tag{3.2}$$

$$c_z^i \leq e_z^i + \sum_{j=1}^{J} D_{j,z} s_j^i, \qquad \text{and} \tag{3.3}$$

$$c_z^i \geq 0, \qquad \forall \quad i \text{ and } z. \tag{3.4}$$

Many financial models consider current consumption to derive greater utility than future expected consumption. Therefore, β is defined as a discount factor in this model. The discount factor could be agent specific β^i, however, this chapter will assume a constant rate across agents. In the initial period each agent faces the constrained consumption versus savings decision (3.2): purchase shares s_j^i of securities at price P_j or consume good c_0^i, subject to their initial endowment e_0^i. In the second period a specific state is revealed and agents will consume a maximum of their endowment and payoff of portfolio (3.3). It is important to mention that all prices are denominated in terms of the consumption good available. If there were multiple consumption goods prices would be denominated in ratios of these goods. With just one consumption good available the normalization of price is equivalent to one unit of the consumption good.

3.2.3 Competitive Equilibrium

A *competitive equilibrium* in this two period economy includes consumption allocations $\{c_z^i\}_{i=1}^N$ and securities prices $\{P_j\}_{j=1}^J$ such that:

- Given P_j, each agent maximizes his expected utility (3.1) subject to budget constraint (3.2) and consumption constraints (3.3) and (3.4).

- The consumption good market clears at equilibrium with aggregate consumption equating endowments. The securities market clears with net zero supply. Thus,

$$\sum_{i=1}^{N} c_z^i = \sum_{i=1}^{N} e_z^i \qquad \forall \, z, \tag{3.5}$$

$$\sum_{i=1}^{N} s_j^i = 0, \qquad \forall \, z \text{ and } j. \tag{3.6}$$

The net zero supply of securities is not a required assumption. A positive supply of assets could be made available to investors. However, this would alter the size of the economy. For example, if the securities market had unit supply the following equations would need to be satisfied for a *competitive equilibrium*:

$$\sum_{i=1}^{N} c_z^i = \sum_{i=1}^{N} \left(e_z^i + s_j^i D_{j,z} \right),$$

$$\sum_{i=1}^{N} s_j^i = 1, \qquad \forall \quad t, \, z_t \text{ and } j.$$

To solve the *competitive equilibrium*, the constrained optimization problem can be set up using the following Lagrangian,

$$\mathcal{L}_{(c^i, s^i, \lambda^i)} = u^i(c^i) + \lambda_0^i(e_0^i - c_0^i - \sum_{j=1}^{J} P_j s_j^i) + \sum_{z=1}^{Z} \lambda_z^i(e_z^i + \sum_{j=1}^{J} D_{j,z} s_j^i - c_z^i). \tag{3.7}$$

For this problem the agent specific unknown choice variables are consumption and portfolio holdings. The following describes these variables and number of unknowns,

$$c_z^i : \qquad z = 0, ..., Z \qquad i = 1, ..., N \qquad (Z+1)N$$

$$s_j^i : \qquad j = 1, ..., J \qquad i = 1, ..., N \qquad JN$$

$$\lambda_z^i : \qquad z = 0, ..., Z \qquad i = 1, ..., N \qquad (Z+1)N.$$

This implies that each agent must solve $2(Z+1) + J$ unknowns given prices P_j. The market determines the remaining J unknown prices

$$P_j : \qquad j = 1, ..., J \qquad J.$$

Therefore, there are a total of $2N(Z+1)+J(N+1)$ unknowns in this economy. As an example, assume an economy has 2 agents ($N=2$), good or bad possible states ($Z=2$), and 2 available securities ($J=2$). For this simple scenario each agent has 8 unknown variables and the market solves for 2 prices for an aggregate of 18 unknowns in this economy. There are N more unkowns relative to the same model setup with one period (no consumption versus savings decision). These additional unknowns are the initial consumption for each agent c_0^i.

To solve this constrained optimization problem, take the first order condition of $\mathcal{L}$ with resepect to the choice variables and Lagrange multipliers. The first order conditions, constraints, and the number of equations are listed below

$$c_z^i: \quad \partial_{c_z^i} u^i(c^i) - \lambda_z^i \leq 0; \qquad c_z^i \geq 0; \qquad z = 0, ..., Z \qquad i = 1, ..., N \qquad (Z+1)N$$

$$s_j^i: \quad \lambda_0^i P_j = \sum_{z=1}^{Z} \lambda_z^i D_{j,z}; \qquad\qquad j = 1, ..., J \qquad i = 1, ..., N \qquad JN$$

$$\lambda_0^i: \quad e_0^i - c_0^i - \sum_{j=1}^{J} P_j s_j^i \geq 0; \quad \lambda_0^i \geq 0; \qquad i = 1, ..., N \qquad\qquad\qquad N$$

$$\lambda_z^i: \quad e_z^i + \sum_{j=1}^{J} D_{j,z} s_j^i - c_z^i \geq 0; \quad c_z^i \geq 0; \qquad z = 1, ..., Z \qquad i = 1, ..., N \qquad ZN.$$

The complementary slackness condition should also be considered. This includes the product of the first order conditions and constraints to be zero. For example, $c_z^i[\partial_{c_z^i} u^i(c^i) - \lambda_z^i] = 0$.

The Kuhn-Tucker conditions along with specific utility functions generate interior solutions and equalities for all equations above. The individual agent will solve $2(Z+1)+J$ equations. The market verifies that J different markets clear with net zero supply of all finite assets

$$\sum_{i=1}^{N} s_j^i = 0 \qquad j = 1, ..., J \qquad\qquad J.$$

This economy will have a total of $2N(Z+1)+J(N+1)$ equations, which is identical to the number of unknowns. There will be 18 equations in the simple economy described above with 2 agents, 2 states, and 2 securities. When solving this problem the Lagrange multipliers do not need to be solved, and can be substituted out of the first order conditions, eliminating $N(Z+1)$

unknowns and equations. Therefore, a total of $N(Z+1) + J(N+1)$ unknowns and equations are available to solve the *competitive equilibrium* in this economy.

Eliminating all Lagrangian multipliers from the first order conditions provides the following pricing function

$$P_j = \sum_{z=1}^{Z} \left(\frac{\partial_{c_z^i} u^i(c^i)}{\partial_{c_0^i} u^i(c^i)} \right) D_{j,z} \qquad \forall\, j. \tag{3.8}$$

The price of each security is equivalent to the sum of marginal utility in state z times the payoff $D_{j,z}$ normalized by the marginal utility in state zero. In the literature this pricing function is commonly referred to as the agent's Euler equation

$$1 = E[mR_j], \tag{3.9}$$

where $m = \beta \left(\frac{\partial_{c_z^i} u^i(c^i)}{\partial_{c_0^i} u^i(c^i)} \right)$ is the stochastic discount factor and $R_j = \frac{D_{j,z}}{P_j}$ is the return on security j.

The following outlines a three step process that could be implemented to solve a *competitive equilibrium*:

1. Solve each agent's utility maximization problem (3.1) to obtain demand functions for consumption $c_z^i(P, D, e)$ and portfolio holdings $s_z^i(P, D, e)$. It is important to note the agent's demand functions are dependent on the price, payoff, and endowment vectors.

2. Using the J asset market clearing conditions (3.20) solve for P_j.

3. Given the price vector P, solve for agent consumption allocations and asset holdings from demand functions.

With heterogeneous agents, the solution often involves a grid search over the price vector P. This approach can be difficult to solve under these conditions. One reason is that there can be a large number of unknowns, which are increasing in number of agents N and possible states Z. Second, there are usually not closed form solutions for consumption $c_z^i(P, D, e)$ and portfolio holdings $s_z^i(P, D, e)$. Finally, the demand curves for portfolio holdings are typically very steep with an unstable Jacobian.

3.2.4 Planner's Problem

Throughout this chapter markets are assumed to be complete. At any time period in these models, there are as many financial assets available J as possible states Z. Under these specific conditions, a planner's approach is an alternative to the grid search method when solving the *competitive equilibrium* described above. The following outlines the social planner's problem in the two period model.

The planner will maximize a weighted utility function

$$\max_{\{c^i\}} U^*(C) = \sum_{i=1}^{N} \alpha^i u^i(c^i), \tag{3.10}$$

subject to the resource constraints

$$\sum_{i=1}^{N} c_z^i \le \sum_{i=1}^{N} e_z^i, \qquad \forall \ z. \tag{3.11}$$

The weights on the individual agent utilities are denoted α^i. Given a specific set of α^i, the following Lagrangian can be set up to solve this constrained optimization problem:

$$\mathcal{L}_{(c_z^i, \mu_z)} = \sum_{i=1}^{N} \alpha^i u^i(c^i) + \sum_{z=0}^{Z} \mu_z \left(\sum_{i=1}^{N} e_z^i - \sum_{i=1}^{N} c_z^i \right). \tag{3.12}$$

The planner's approach provides $N(Z+1) + (Z+1)$ consumption and Lagrange multiplier unknowns that are from the following

$$c_z^i: \qquad z = 0, ..., Z \qquad i = 1, ..., N \qquad (Z+1)N$$

$$\mu_z: \qquad z = 0, ..., Z \qquad\qquad\qquad\qquad (Z+1).$$

Taking the first order conditions of (3.12) with respect to c_z^i and μ_z gives

$$c_z^i: \quad \alpha^i \partial_{c_z^i} u^i(c^i) - \mu_z \le 0; \quad \mu_z \ge 0; \qquad z = 0, ..., Z \qquad i = 1, ..., N \qquad (Z+1)N$$

$$\mu_z: \quad \sum_{i=1}^{N} e_z^i - \sum_{i=1}^{N} c_z^i \ge 0; \qquad c_z^i \ge 0; \qquad z = 0, ..., Z \qquad\qquad\qquad (Z+1).$$

The Kuhn-Tucker conditions along with specific utility functions gaurantee an interior solution. The Lagrangian multipliers μ_z are state specific and not agent specific. Therefore, the individual weights multiplied by state specific marginal utilities are equivalent for all agents

$$\alpha^i \partial_{c_z^i} u^i(c^i) = \alpha^k \partial_{c_z^k} u^k(c^k), \quad \forall \quad i \text{ and } k. \tag{3.13}$$

Through substitution of the resource constraints into (3.13) the Langrangian multipliers μ_z are eliminated. Therefore, there are $N(Z+1)$ equations and unknowns remaining. The solutions to this problem would provide agent consumption allocoations as a function of weights α^i.

In order to get on the contract curve given initial endowments (satisfying equation (3.13) and the resource constraint (3.11)), transfer payments are made across agents. These transfer functions are defined as

$$t^i(\alpha^i) = \sum_{z=0}^{Z} \alpha^i \partial_{c_z^i} u^i(c^i) \left(c_z^i - e_z^i\right), \qquad i = 1, ..., N. \tag{3.14}$$

These transfer functions are essentially agent lifetime budget constraints. Applying Walras' law generates a total of $N-1$ equations. If markets are complete ($J = Z$) the planner's problem will solve the *competitive equilibrium*. Given endowments, there are agent specific utility weights $\hat{\alpha}^i$ that will solve a *competitive equilibrium*. The following outlines the steps required to solve:

1. Set all transfer functions (3.14) equivalent to zero. This is the *Pareto Optimal* allocation for all agents in which no transfers are required. Due to Walras' law the transfer functions provide $N-1$ equations. Normalizing the first agent weight $\hat{\alpha}^1 = 1$ will leave $N-1$ unknowns.

2. Utilizing equation (3.13), the resource constraint (3.11), and the transfer functions (3.14), simultaneously solve agent utility weights $\hat{\alpha}^i$ and consumption allocations c_z^i. After combining these equations there are only $(N-1) + (Z+1)$ equations and unknowns.

3. With the equilibrium consumption allocations c_z^i determined, prices can be solved using the pricing equation (3.8) from the individual agent problem. Given consumption allocations, prices, endowments, and security payoffs, the agent asset holdings s_j^i can be solved using each agent's lifetime budget constraints and net zero supply of all assets.

It is important to differentiate between solving a *competitive equilibrium* with the individual agent's Euler equations in the above section and the planner's problem. Using the example of 2 agents ($N = 2$), good or bad possible states ($Z = 2$), and 2 available securities ($J = 2$) the individual agent problem solved 18 equations and unknowns $2N(Z+1) + J(N+1)$. The planner's problem, however, would only require solving 4 equations and unknowns $(Z+1) + (N-1)$.

The *competitive equilibrium* approach utilized agent demand equations as a function of price. These were derived from the individual agent Euler equations. Solving these equations in a heterogeneous agent model required a search over security prices to clear the asset and goods markets. With highly non-linear equations this process can become quite difficult. The social planner's methodology, however, prices are completely removed from the problem. Given initial endowments, weighted individual agent utilities are maximized. Weighted marginal utilities are equated to solve consumption allocations. Finally, prices and portfolio holdings are revealed.

3.2.5 Algorithm

This section will consider two algorithms that could be applied to the two period asset pricing model. The first will be a one period model with two heterogeneous agents. The setup of this model is similar to the two period economy described above, except there is no initial consumption or endowments. In this model agents solve an asset allocation problem in the initial period. Once a state is revealed, each agent consumes their wealth. The second algorithm is a two heterogeneous agent scenario in the two period economy. Agents must determine a consumption versus savings decision and an asset allocation choice in the initial period. In the second period agents consume their wealth.

One Period Example. In this one period economy all agents will make an asset allocation decision before a stochastic state is revealed. With zero initial consumption and endowment, each agent will satisfy their individual constraint (3.2). Once the state is revealed all agents will consume their endowment and any dividends from purchased assets (3.3). For this example, assume the following constant relative risk aversion (CRRA) utility function

$$u(c_z^i) = \frac{(c_z^i)^{1-\gamma^i}}{1 - \gamma^i}, \tag{3.15}$$

which is continuous, twice differentiable, increasing in c_z^i, and strictly concave. Agents are heterogeneous in the risk aversion parameter γ^i. When $\gamma^i = 1$ the utility function is defined as $u(c_z^i) = ln(c_z^i)$.

In this economy there are two agents ($N = 2$), a possible positive or negative state ($Z = 2$), and two different securities ($J = 2$). The following outlines a process to solve the *Pareto Optimal* allocation:

1. Simultaneously solve α^2, c_1^1, and c_2^1 from the following three equations

$$(c_1^1)^{-\gamma^1} - \alpha^2(E_1 - c_1^1)^{-\gamma^2} = 0$$
$$(c_2^1)^{-\gamma^1} - \alpha^2(E_2 - c_2^1)^{-\gamma^2} = 0$$
$$(c_1^1)^{-\gamma^1}(e_1^1 - c_1^1) + (c_2^1)^{-\gamma^1}(e_2^1 - c_2^1) = 0.$$

   Note that the weight on the first agent's utility has been normalized $\alpha^1 = 1$. Also, through substitution it is implied that the resource constraint is met $c_z^2 = E_z - c_z^1$. The first two equations are derived from the first order conditions of the planner's problem. The third equation is the transfer function for agent 1.

2. Given the solved consumption allocations calculate the following security prices

$$P_j = \sum_{z=1}^{Z}(c_z^1)^{-\gamma^1}D_{j,z}.$$

3. Now determine agent 1 portfolio holdings by simultaneously solving

$$c_1^1 - e_1^1 = s_1^1 D_{1,1} + s_2^1 D_{2,1}$$
$$c_2^1 - e_2^1 = s_1^1 D_{1,2} + s_2^1 D_{2,2}.$$

   Note that the second agent portfolio holdings are the negative of agent 1 ($s_j^2 = -s_j^1$) due to the asset market clearing condition. If one agent is long a position the other agent must be short.

Two Period Example.   Consider an example in which each agent has the constant relative risk aversion (CRRA) utility function defined above (3.15). There are three states in this economy ($z = 0, 1, 2$) and two securities ($J = 1, 2$). The initial state ($z = 0$) is deterministic followed by a stochastic state. For this example let $z = 1$ be a positive state and $z = 2$ a negative state. These
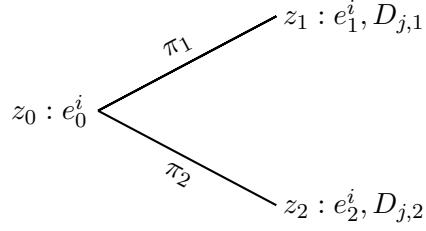
states will occur with probability π_1 and π_2, respectively. The figure below depicts the states in this economy.

Each agent receives e_0^i and determines how much to consume c_0^i and how much to invest $\sum_{j=1}^{2} P_j s_j^i$. Once the state is revealed each agent consumes their endowment and payoff of their securities $(e_z^i + D_{j,z} s_j^i$ for $z = 1, 2)$. The equations listed below outline an algorithm to solve the *competitive equilibrium*:

1. Simultaneously solve α^2, c_0^1, c_1^1, and c_2^1 from the following four equations

$$(c_0^1)^{-\gamma^1} - \alpha^2 (E_0 - c_0^1)^{-\gamma^2} = 0$$
$$(c_1^1)^{-\gamma^1} - \alpha^2 (E_1 - c_1^1)^{-\gamma^2} = 0$$
$$(c_2^1)^{-\gamma^1} - \alpha^2 (E_2 - c_2^1)^{-\gamma^2} = 0$$
$$(c_0^1)^{-\gamma^1}(e_0^1 - c_0^1) + \pi_1 (c_1^1)^{-\gamma^1}(e_1^1 - c_1^1) + \pi_2 (c_2^1)^{-\gamma^1}(e_2^1 - c_2^1) = 0.$$

Note that the weight on the first agent's utility has been normalized $\alpha^1 = 1$. Also, through substitution it is implied that the resource constraint is met $c_z^2 = E_z - c_z^1$. The first three equations are derived from the first order conditions of the planner's problem. The fourth equation is the lifetime transfer function for agent 1. If the transfer function of agent 1 equates to zero, the transfer function of agent 2 will as well (Walras' Law).

2. Given the solved consumption allocations above calculate the security prices

$$P_j = \sum_{z=1}^{2} \pi_z \left(\frac{c_z^1}{c_0^1} \right)^{-\gamma^1} D_{j,z}.$$

3. Now determine agent 1 portfolio holdings by simultaneously solving

$$c_1^1 - e_1^1 = s_1^1 D_{1,1} + s_2^1 D_{2,1}$$
$$c_2^1 - e_2^1 = s_1^1 D_{1,2} + s_2^1 D_{2,2}.$$

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The second agent portfolio holdings are the negative of the first agent ($s_j^2 = -s_j^1$) due to the asset market clearing condition. In this economy if one agent is long a security the other agent must be short.

3.2.6 Summary

This section setup a model, defined a *competitive equilibrium*, and provided a short algorithm to solve a one period heterogeneous agent problem. Without initial consumption, the problem is simply an asset allocation problem as each agent will consume whatever wealth they have after a state is revealed. Including initial consumption creates an additional decision rule as agents must decide how much to consume and how much to save. To arrive at the *competitive equilibrium*, the problem is computationally quicker with fewer equations and unknowns to solve for using the planner's approach. The next section will extend this model to include T periods.

3.3 General T Period Model

3.3.1 Introduction

This section will consider a finite period heterogeneous agent asset pricing model. Markets are dynamically complete in this economy, with as many available securities in each period as possible states. This model is an extension of the general two period model to any number of time periods. The notation and setup of this model will follow Ljungqvist and Sargent (2004).

3.3.2 Model Setup

Let this economy consist of N agents, J securities with payoff D, and Z possible states. Time periods are denoted t ($t = 0, 1, ..., T$). The J financial securities will consist of short lived j^s and long lived j^l (T period) assets. An example of a short lived asset would be a one period discount bond. The asset is purchased at time period t and a return is recognized next period $t+1$. A stock is an example of an long lived financial asset. The long lived security is available for all T periods with a payout or dividend disbursed each period. With dynamically complete markets, all agents are able to self insure for the next time period.

A stochastic state occurs at each time period $z_t \in Z$. The possible future states are path dependent, therefore, history of states up to time t is denoted $z^t = [z_0, z_1, ..., z_t]$. For this section I will assume that the same number of states are possible in each time period. Theoretically this does

not have to be the case, as the number of states could differ depending on specific state path history. Each agent is entitled to a state specific endowment $e_t^i(z_t)$ which occurs with probability $\pi(z_t)$. The aggregate endowment in each state is $E_t(z_t) = \sum_{i=1}^{N} e^i(z_t)$. Let β respresent the discount factor for all agents. The subscript t identifies the time period.

Each agent will maximize their lifetime expected utility,

$$\max_{\{c_t^i, s_{t+1,j}^i\}} U^i(c^i) = \sum_{t=0}^{T} \sum_{z_t} \beta^t \pi(z_t | z_{t-1}) u(c_t^i(z_t)), \tag{3.16}$$

subject to positive consumption and the time period budget constraints,

$$c_t^i(z_t) + \sum_{j=1}^{J} P_{t,j}(z_t) s_{t+1,j}^i \le e_t^i(z_t) + \sum_{j=1}^{J} \left(P_{t,j}(z_t) + D_{t,j}(z_t) \right) s_{t,j}^i, \tag{3.17}$$

for all i, t, and z_t. To solve the *competitive equilibrium* given this constrained optimazition problem, set up the Lagrangian and combine first order conditions to obtain the Euler equations.

3.3.3 Competitive Equilibrium

A *competitive equilibrium* in this T period economy includes consumption allocations $\{c_t^i\}_{i=1}^{N}$ and securities prices $\{P_{t,j}\}_j^J = 1$ such that:

- Given $P_{t,j}$, each agent maximizes his lifetime expected utility (3.16) subject to each state specific budget constraint (3.17).

- The consupmtion good market clears at equilibrium with aggregate consumption equating aggregate endowment plus the dividends of the long lived securities. The long lived securities market clears with unit supply. The short lived securities have zero net supply. Thus,

$$\sum_{i=1}^{N} c_t^i(z_t) = \sum_{i=1}^{N} \left(e_t^i(z_t) + \sum_{j=j^l} s_{t,j^l}^i D_{t,j^l}(z_t) \right), \tag{3.18}$$

$$\sum_{i=1}^{N} s_{t+1,j^l}^i = 1 \tag{3.19}$$

$$\sum_{i=1}^{N} s_{t+1,j^s}^i = 0, \qquad \forall \quad t, z_t \text{ and } j. \tag{3.20}$$

These steps would provide the pricing function

$$P_{t,j}(z_t) = \sum_{z_t} \left(\beta \pi_{t+1} \left(\frac{\partial_{c_{t+1}^i} u^i(c_{t+1}^i(z_{t+1}))}{\partial_{c_t^i} u^i(c_t^i(z_t))} \right) (P_{t+1,j}(z_{t+1}) + D_{t+1,j}) \right). \tag{3.21}$$

The price of each security is equivalent to the stochastic discount factor times the expected payoff of the security in the next period. This pricing function comes from the commonly known Euler equation $1 = E[mR]$. Given prices each agent would demand consumption $c_t^i(P, D, e)$ and securities $s_{t+1}^i(P, D, e)$. The prices and demand functions that clear the asset and goods market is the *competitive equilibrium*. This can be computationally difficult to solve given the number of equations and unknowns. Therefore, if markets are complete consider a planner's approach to solving this problem.

3.3.4 Planner's Problem

The planner will maximize the following utility

$$\max_{\{c^i\}} U^*(C) = \sum_{i=1}^{N} \alpha^i u^i(c^i), \tag{3.22}$$

where α^i is the weight on individual agent utilities.

With zero net supply of short lived assets and unit supply of long lived assets, the planner's optimization is constrained to the resource constraints

$$\sum_{i=1}^{N} c_t^i(z_t) \leq \sum_{i=1}^{N} e_t^i(z_t) + \sum_{j=j^l} s_{t,j^l}^i D_{t,j^l}(z_t), \qquad \forall \ t \text{ and } z_t.$$

The planner's problem Lagrangian is

$$\mathcal{L}_{(c_t^i(z_t), \mu_t(z_t))} = \sum_{t=0}^{T} \sum_{z_t} \left\{ \sum_{i=1}^{N} \beta^t \pi(z_t) \alpha^i u^i(c_t^i(z_t)) \right.$$

$$\left. + \mu_t(z_t) \sum_{i=1}^{N} \left(e_t^i(z_t) + \sum_{j=j^l} s_{t,j^l}^i D_{t,j^l}(z_t) - c_t^i(z_t) \right) \right\}.$$

The planner's approach provides $N(Z + 1) + (Z + 1)$ consumption and Lagrange multiplier unknowns that are from the following

$$
\begin{aligned}
c_z^i : & \quad z = 0, ..., Z \quad\quad i = 1, ..., N \quad\quad (Z + 1)N \\
\mu_z : & \quad z = 0, ..., Z \quad\quad\quad\quad\quad\quad\quad (Z + 1).
\end{aligned}
$$

Taking first order conditions of $\mathcal{L}$ with respect to $c_t^i(z_t)$ and $\mu_t(z_t)$ gives

$$
\begin{aligned}
c_t^i(z_t) : & \quad \beta^t \pi(z_t) \alpha^i \partial_{c_t^i} u^i(c_t^i(z_t)) - \mu_t(z_t) \leq 0, \quad \mu_t(z_t) \geq 0; \\
\mu_t(z_t) : & \quad \sum_{i=1}^{N} \left(e_z^i(z_t) + \sum_{j=j^l} s_{t,j^l}^i D_{t,j^l}(z_t) \right) - \sum_{i=1}^{N} c_t^i(z_t) \geq 0, \quad\quad \text{for all } t \text{ and } z_t.
\end{aligned}
$$

The Kuhn-Tucker conditions along with certain utility functions generate an interior solution. The Lagrangian multipliers $\mu_t(z_t)$ are not agent specific, allowing the first order condition of $\mathcal{L}$ with respect to $c_t^i(z_t)$ to hold for all agents. Therefore,

$$
\beta^t \pi(z_t) \alpha^i \partial_{c_t^i} u^i(c_t^i(z_t)) = \beta^t \pi(z_t) \alpha^k \partial_{c_t^k} u^k(c_t^k(z_t)), \quad\quad \forall \quad t, i \text{ and } k. \tag{3.23}
$$

3.3.5 T Period Algorithm

The following outlines an algorithm to solve the general T period model with two heterogeneous agents and two assets. The two assets consist of a stock and bond. For notational purposes, the risky asset is denoted s_t^i and the risk free bond is b_t^i (essentially replacing $s_{t,j}^i$). The prices of the stock and bond will be defined as $P_t(z_t)$ and $Q_t(z_t)$, respectively. Since markets are complete, there are two possible states. For this research, the states are different growth rates $g(z_t)$ observed in the dividend process

$$
D_t(z_t) = (1 + g(z_t))D_{t-1}. \tag{3.24}
$$

Consider an example in which each agent has the constant relative risk aversion (CRRA) utility function

$$
u(c_t^i(z_t)) = \frac{(c_t^i(z_t))^{1-\gamma^i}}{1 - \gamma^i},
$$

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which is continuous, twice differentiable, increasing in $c_t^i(z_t)$, and strictly concave. Agents are heterogeneous in the risk aversion parameter γ^i, and when $\gamma^i = 1$ the utility function is defined as $u(c_t^i(z_t)) = ln(c_t^i(z_t))$.

The initial endowment and asset holdings is known for each agent. An initial consumption c_0^i decision is made, and the rest of the endowment is allocated for the next period investment $P_0 s_1^i + Q_0 b_1^i$. In the next period $t + 1$ a new state is revealed. Agents at that point have a wealth of $e_1^i(z_1) + P_1 s_1^i + Q_1 b_1^i$. Another consumption $c_1^i(z_1)$ and investment $P_1 s_2^i + Q_1 b_2^i$ decision must be made. This process continues over the life-time of the agents. In the period $t = T$ a final state is revealed. Each agent consumes their final endowment and any investments remaining $e_T^i(z_T) + s_T^i D_T(z_T) + b_T^i$. The following outlines an algorithm to solve a T period *competitive equilibrium* with complete markets:

1. **Initialization:** The desired market paramaters must be set. Given the distribution of the dividend, determine the growth rates for each possible state z. Assign the probabilities $\pi(z)$ that correspond to these events. Assume an initial dividend D_0. Finally, determine how many time periods T this finite model will solve.

   Initialize the heterogeneous agent parameters. This includes the discount factors β^i, risk aversion parameters γ^i, and initial asset holdings s_0^i and b_0^i. Set the preferred agent specific endowemnt at each possible state e_t^i. Note that this could be a deterministic function or value for any state of the world, or an idiosyncratic shock that is drawn from a stochastic distribution.

   With these parameters set, a lifetime T lattice can be filled out with all possible states of the world, for multiple parameters. These lattice parameters includes the probality at time $t = 0$ of any state, the dividend process, endowments for each agents, and size of the economy (dividends plus endowments).

2. **Consumption:** Assign a value for the Pareto weight of agent 1. Since these are weights on the maximizing functions, any fixed number can be used as it will be relative to the other weights. For simplicity fix $\mu^1 = 1$. Guess the Pareto weight for agent 2, $\tilde{\mu}^2$.

   The marginal utilities of all agents are equivalent at equilibrium (3.23). Therefore, it must be that

   $$\frac{1}{\mu^1} \partial_{c_t^1} u^1(c_t^1(z_t)) = \frac{1}{\mu^2} \partial_{c_t^2} u^2(c_t^2(z_t)), \tag{3.25}$$

   for all time t and states z. With CRRA utility (3.15) and $\mu^1 = 1$,

$$\mu^2 (c_t^1(z_t))^{-\gamma^1} = (c_t^2(z_t))^{-\gamma^2}, \tag{3.26}$$

where the Pareto weight for each agent is defined as $\frac{1}{\mu^i}$. With the utility function always increasing in a non-storable consumption good, the agents consume the size of the economy (endowments and dividend). With two agents it must be that

$$c_t^2(z_t) = \sum_{i=1}^{2} e_t^i(z_t) + D_t(z_t) - c_t^1(z_t). \tag{3.27}$$

Therefore, with the guess of μ^2, substitution of (3.27) into (3.26) enables the calculation of agent 1 consumption through a non-linear solver. Solve consumption for agent 1 and agent 2 (3.27) for each state of the world.

3. **T Period Asset Holdings:** Since this is a finite solution, at time $t = T$ we know that each agent will consume their entire wealth. This includes their endowment, share of dividend, and all final debts are resolved. The final T budget constraint for each agent is

$$c_T^i(z_T) = s_T^i D_T(z_T) + b_T^i + e_T^i(z_T), \tag{3.28}$$

for all possible states. For each possible state at $T - 1$ there are two possible final states (positive or negative aggregate growth shock at time T). Therefore, for $T - 1$ nodes of the lattice there are two equations (3.28) and two unknown asset holdings s_T^i and b_T^i. Solve these two unknowns for all possible states and path dependent final asset holdings are determined.

4. **Remaining Asset Holdings:** The solved asset holdings at time T were purchased at time $T - 1$. Assume that at time T, $P_T = 0$ and $Q_T = 0$ since both assets are unavailable at time $T + 1$. Given the consumptions calculated and assumed $P_T = 0$, prices at time $T - 1$ can be calculated for any asset using the pricing function (3.21),

$$P_{T-1}(z_{T-1}) = \beta \sum_{z_T} \left(\pi_T \left(\frac{(c_T^i(z_T))}{c_{T-1}^i(z_{T-1})} \right)^{-\gamma^i} (D_T(z_T)) \right). \tag{3.29}$$

Therefore, with final asset holdings, consumption, and prices calculated, time $T - 1$ asset holdings can be calculated using the individual budget constraint (3.17). There are two equations and two unknowns ($s_{T-1}^i(z_{T-1})$ and $b_{T-1}^i(z_{T-1})$) for each possible state at time $T - 1$.

Given the solved prices and asset holdings at time $T - 1$, repeat price and portfolio allocation calculations for $T - 2$. Repeat this process until $t = 1$. The entire tree or lattice is completed for asset holdings, prices, and consumption for any state at any time period. These calculations are all contingent on the initial guess of $\tilde{\mu}^2$.

5. **Verify Agent's Initial Holdings:** Assume the initial bond holdings for agent 2 is correct $b_0^2 = 0$. Given the solved consumption levels and prices of assets, use the initial period budget constraint to solve for agent 2 initial stock holding $\tilde{s}_0^2$. To determine if the guess of μ^2 is correct, calculate the error of the calculated initial stock holding to the initialized value $(\epsilon = \tilde{s}_0^2 - s_0^2)$.

- If the ϵ is within a desired tolerance, $\tilde{\mu}^2 = \mu^2$. The algorithm has converged with the correct Pareto weights.

- If $\tilde{s}_0^2 < s_0^2$ and the ϵ is negative, lower the initial guess of $\tilde{\mu}^2$. Go back to the second step **Consumption**.

- If $\tilde{s}_0^2 > s_0^2$ and the ϵ is positive, increase the initial guess of $\tilde{\mu}^2$. Go back to the second step **Consumption**.

Typically the more risk averse agent will receive a heavier weight in the planner's problem. If agent 2 is the more risk averse agent, the grid search can be limited to $\mu^2 \in (0,1)$. For the simulations in later chapters, a bisection method was used to converge to the planner's weight. An example of the code, written in the R language can be found in Appendix A.

3.3.6 Summary

The two period heterogeneous agent model has been extended to include finite T periods. This model uses a planner's approach to solve Pareto optimal allocations. Once these allocations are assigned, it is possible to obtain prices and asset holdings in order to clear markets. The key to solving this model is the fact that each utility maximizing agent will consume their entire wealth in the final period. This gives a starting point to the recursive calculations required to find the unknowns in the pricing function (3.29) and budget constraint (3.28).

3.4   Conclusion

This section started with a one period model to consider asset allocation among heterogeneous agents through euler equation (3.9). Initial consumption was added to consider an additional choice of consumption versus savings. The algorithms described applied a planner's approach to obtain Pareto optimal allocations. Given the solved consumption, prices could be determined given the stochastic discount factor is the probability adjusted ratio of marginal utilities. This model was then extended to include T periods.

The notation of the general T period model was intended to remain consistent with Ljungqvist and Sargent (2004). Although the notation maybe consistent, there are differences in the assumptions of the model. First, there are no financial securities in the Ljungqvist and Sargent (2004) model. An Arrow-Debreu structure and sequential trading of $t + 1$ claims are considered. There is an idiosyncratic endowment shock, which is the only source of income. There is no storage of wealth, since the model does not include any long lived securites.

There are many similarities between this algorithm and the Judd et al. (2003). Each algorithm uses a planner's approach in a complete markets setting. However, in the infinite horizon model of Judd et al. (2003), if both agents have CRRA utility, there is a unique reprsentative agent specified with γ^R. Therefore, a rational expectations equilibrium is reached with a one and done trade. If the dividend process has growth, however, as defined in equation (3.24), the general T period algorithm will solve a representative agent that does not exhibit CRRA utility, as the γ^R will be state dependent. With growth in the economy, the assumption of a finite horizon model provides a terminal date T where prices and future investments can assumed to be zero $(P_T = Q_T = s^i_{T+1} = b^i_{T+1} = 0)$.

CHAPTER 4

PROPERTIES OF THE MODEL

4.1   Introduction

The previous chapter outlined a general T period model with heterogenous agents. Markets are dynamically complete with as many assets available at each time period as $t+1$ possible states. This model contains idiosyncratic as well as aggregate shocks. A step by step algorithm was provided for two heterogeneous agents in a two state economy. This chapter will analyze the properties of the model using numerical simulations, and put them into context with the finanical economic literature.

This chapter will start with a restrictive model and then relax certain assumptions. The first section introduces the T period model with identical agents. Numerical results are compared to the seminal research that introduced dynamic, stochastic, general equilibrium models to financial economics. Heterogeneity in risk aversion will be introduced in the next section. Findings will be compared to similar pure exchange models like Benninga and Mayshar (1993). Finally, idiosyncratic employment or wage shocks will be included in the final section. With path dependency in outcomes, ranges of possible outcomes will be presented along with the expected paths.

4.2   Homogeneous Agent Model

This section will analyze the T period model with two homogeneous agents. The purpose is to consider the properties of this model under the strongest assumptions. These finite lived agents will be identical in risk aversion as well as initial asset holdings. Agents consume what is observed by the dividend process ($C_t = D_t$), and there is no employment or endowment at each period.

Although there will not be price and trading dynamics in this environment, key characteristics of the model can be validated with the seminal findings of Lucas (1978) and Levhari and Srinivasan (1969). The wealth distribution, pricing function, consumption, and portfolio allocation decisions will be considered for the individual agent across time t.

4.2.1 Model Assumptions

Let the economy consist of two homogeneous agents in a pure exchange economy. Initially we will assume that each agent has log utility

$$u(c^i) = \ln(c^i). \tag{4.1}$$

There is a risky stock and risk-free bond available to investors. Both agents initially have the same asset holdings. Given unit supply for stocks and net zero supply for bonds, $s_0^1 = s_0^2 = 0.5$ and $b_0^1 = b_0^2 = 0$. The discount factor for each agent is $\beta^1 = \beta^2 = .90$. The dividend is the size of this pure exchange economy. The initial dividend is $D_0 = 1$. Each time period the dividend will have an equal probability of a positive or negative shock ($\pi_1 = \pi_2 = .5$, where 1 and 2 represent a positive and negative shock, respectively). The dividend will grow 5 percent in the positive state ($g_u = .05$) and decline 2 percent in the negative state ($g_d = -.02$). The remainder of this section will summarize the properties of this homogeneous agent example. The T period algorithm defined in chapter 2 will be applied with $T = 500$, unless stated otherwise.

4.2.2 Wealth Distribution

In the homogeneous agent setup, the agents remain identical throughout the time horizon. Consequently, the algorithm converges with the planner's weights being equal. Since the agents are identical, the wealth distribution, as shown in Figure 4.1 below, is 50% for each agent where wealth is defined as

$$w_{t,z}^i = (P_{t,z} + D_{t,z})s_{t,z}^i + b_{t,z}^i + e_{t,z}^i, \tag{4.2}$$

for all states and time periods.

It is important to note that the figures in this section are each a plot of 125,751 points. This is calculated as the $\sum_{t=1}^{501} t$. There is 1 determistic state at time $t = 0$, 2 possible states at time $t = 1$, and given the recombining lattice 3 possible states at time $t = 2$. Therefore, at each time period there are $t + 1$ possible states. These plots for the homogeneous agent case look like a line because all possible states at that time period are constant.
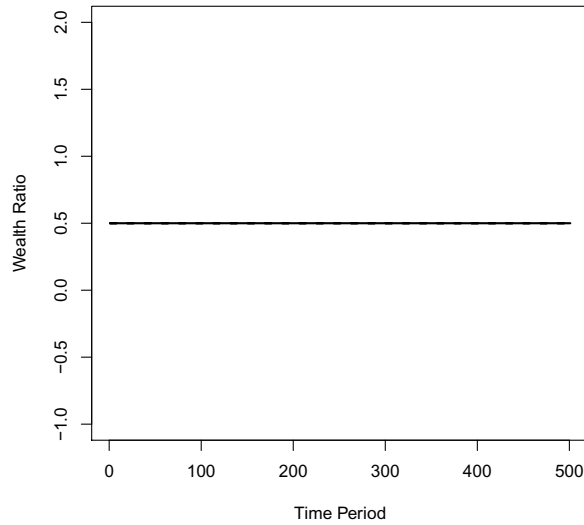
Figure 4.1: Agent 1 owns half of the shares of stock for all 500 periods no matter what shocks occur at each time period and the wealth distribution remains constant at 50% for each agent.

4.2.3 Pricing Function

Given the homogeneous agents with log utility setup, the infinite horizon Lucas model Lucas (1978) produces the pricing function

$$P_t = \left(\frac{\beta}{1-\beta}\right) D_t. \tag{4.3}$$

The price of a share of the risky asset in this special case is only dependent on the current dividend. Dividing both sides by the dividend gives the constant price to dividend ratio of $\frac{\beta}{1-\beta}$. With the parameterization of $\beta = .90$, the price to dividend ratio should be 9. Figure 4.2 displays the price to dividend ratio for the 500 time periods. The red line represents a simulation with $\beta = .90$, while the blue line represents a simulation with $\beta = .95$.

The histogram of stock returns are plotted in Figure 4.3. For $\beta = .90$, this degenerate distribution has two outcomes. If a negative state is revealed the return is 8.89%. If there is a positive aggregate shock, the return is roughly 16.67%. The expected return and variance at any point in time along the path is identical. Therefore, this exhibits returns that are independent and identically distributed.

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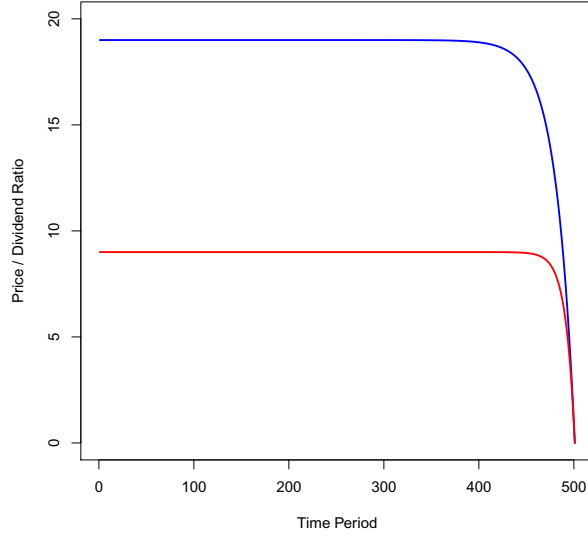
Figure 4.2: The price to dividend ratio is presented for both $\beta = .90$ and $\beta = .95$ in red and blue lines, respectively. As $t \to T$, $P_t \to 0$. The larger β is, the earlier in t it converges toward $P_T = 0$.
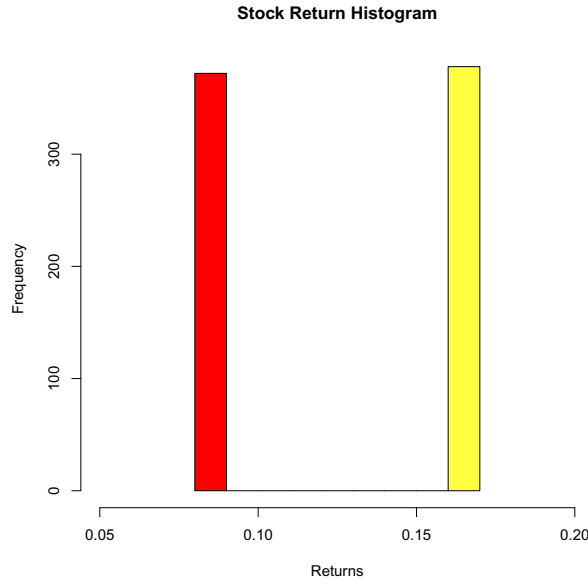
Figure 4.3: The returns of the stock are provided in the histogram. This model exhibits *iid* returns.

It is important to note that at the final period ($t = 500$), the asset value is worthless, $P_{500} = 0$. For the first few hundred time periods the price to dividend ratio is exactly 9 when $\beta = 0.9$ and 19 when $\beta = 0.95$. As t approaches the terminal time $T = 500$, the price to dividend ratio asymptotically approaches 0. The time period at which the price-dividend ratio begins to decay to zero depends upon the discount factor β—the higher is β, the earlier the decay to zero happens.

4.2.4 Consumption Plans

Since this is a pure exchange economy, the aggregate consumption must equal the dividend ($C_t = D_t$). Given identical agents, it is also true that each agent's consumption is equal. Therefore, with two identical agents, the individual agent consumption level is half the dividend ($c_t^i = \frac{D_t}{2}$). Also, the consumption level is path dependent—if the harvest is bountiful the agents consume more.

Let δ_i be the fraction of wealth that is consumed by individual i, ($\delta_t^i = \frac{c_t^i}{w_t^i}$). Levhari and Srinivasan (1969) examine an infinite horizon, pure exchange economy with a representative agent. The agent can invest his wealth to earn a random rate of return (r_t) that is independently and identically distributed, or the agent can consume. Given a strictly concave utility function, the representative agent in this model consumes a constant portion of wealth each period. Specifically, with CRRA utility (4.1), the proportion of wealth consumed is

$$\delta = 1 - \left(\beta E[r^{1-\gamma}] \right)^{\frac{1}{\gamma}}. \tag{4.4}$$

As an example, Levhari and Srinivasan (1969) show that log utility ($\gamma = 1$) reduces (4.4) to simply $\delta = 1 - \beta$. Figure 4.4 plots the proportion of wealth that is consumed each period for the homogeneous agent example. The red line represents a simulation with $\gamma = 1$, while the blue line displays δ for a homogeneous simulation with $\gamma = 3$. As a finite model ($T = 500$), each agent consumes their entire wealth in the last period. For the first few hundred time periods the individual consumes a constant 10% of their wealth ($1 - \beta$), when $\gamma = 1$. When $\gamma = 3$, the agent consumes approximately 12.3% of their wealth. As t approaches the terminal time T, the ratio of wealth consumed asymptotically approaches 1.

4.2.5 Agent Portfolio Allocation

The income that the agent does not consume is invested. Each investor has the choice of a risky stock or a risk-free bond. Due to identical agents, it is well known that portfolio holdings are
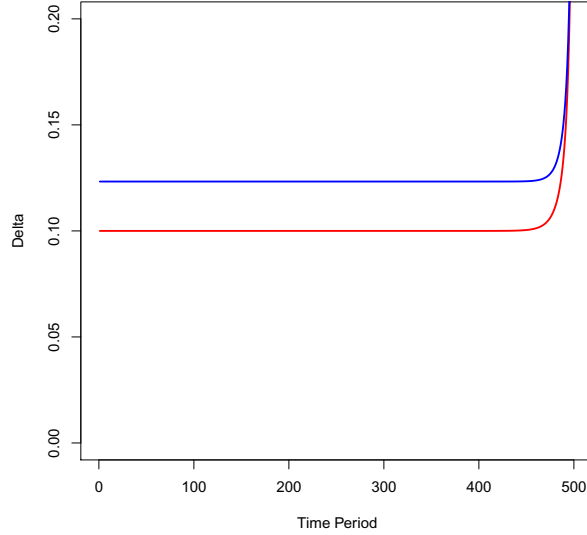
Figure 4.4: The ratio of wealth that is consumed by time period is presented here. The red line represents a $\gamma = 1$, while the blue line represents a more risk averse agent with $\gamma = 3$. Since this is a finite period model, at time T each agent consumes their entire wealth ($\delta_T = 1$). The δ remains constant before it asymptotically converges to 1.

identical. In order for markets to clear with two agents, each will own half a share of stock and zero bonds. This trivial result and the lack of dynamics motivates the need for heterogeneity.

Let α be defined as the portion of investment in the risky asset,

$$\alpha_t^i = \frac{P_t s_{t+1}^i}{w_t^i - c_t^i}. \tag{4.5}$$

and, the remaining portion of wealth is invested in the risk-free asset,

$$1 - \alpha_t^i = \frac{Q_t b_{t+1}^i}{w_t^i - c_t^i}. \tag{4.6}$$

Again, for this simplified model α is easy to compute. With the investor purchasing only stock, the degenerate result is $\alpha = 1$. Figure 4.5 depicts the homogeneous simulation. In the final period the agent consumes all wealth and no investment is made.

4.2.6 Conclusion

The purpose of this section was to highlight some features of the T period model. The homogeneous agent model provides a no-trade equilibrium. The properties described above are not path
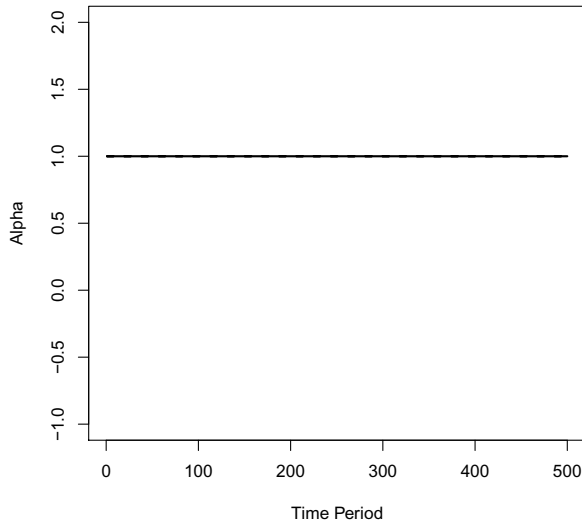
Figure 4.5: Proportion of risky investment by time period. In a homogeneous agent case there is nobody to borrow from $(b_t^i = 0)$. The only investment made is in the risky asset.

dependent, as all possible states reveal a constant value. Although the findings are trivial, they identify with the well known results of Lucas (1978), Levhari and Srinivasan (1969), and Duffie (1988). A representative agent in the Lucas framework provides the pricing function $P_t = E_t \{m_{t+1} X_{t+1}\}$. With log utility ($\gamma = 1$ in CRRA utility), this equation reduces to $P_t = \frac{\beta}{1-\beta}$. With a representative agent, if returns are independent and identically distributed Levhari and Srinivasan (1969) claims that a constant portion of wealth is consumed (δ). Specifically, when $\gamma = 1$, the constant portion of wealth reduces to $\delta = 1 - \beta$. Duffie (1988) finds that with homogeneous agents and no idiosyncratic shocks, δ is constant and returns are independent and identically distributed. All of these results were replicated with the finite T period model. This supports the validation of the T period methodology and algorithm.

The terminal assumption of the model forces $\delta = 1$ as agents consume their wealth in time T. The price to dividend ratio approaches zero since $P_T = 0$. This convergence towards zero occurs relatively earlier for larger discount factors. However, even with $\beta = .95$ and different γ, the asymptotic approach occured later in the model as $t \to T$.

4.3 Heterogeneous Agent Model with Aggregate Risk

In this section we will introduce heterogeneity among the agents into our numerical simulation of the T period model defined in the previous chapter. We begin with only aggregate shocks to the economy and in the following section we will add idiosyncratic shocks to the model.

The simulations will include two agents that are heterogeneous in risk preferences. With CRRA utility (2.1), let agent 1 be log utility ($\gamma^1 = 1$). Agent 2 is more risk averse with $\gamma^2 = 2$. Both consumers' have the same discount factor of $\beta = .90$.

The dividend growth of this economy is driven by a two state stochastic process and the dividend of the stock represents the entire size of the economy. A positive state occurs with probability π, while a declining growth state happens with probability $(1 - \pi)$. For this simulation, the positive and negative growth rates will be $g_u = .05$ and $g_d = -.02$, respectively, and each state occurs with equal probability, $\pi = .50$.

The simulation is initialized with a dividend of $D_0 = 1.0$. Each agent starts off with identical asset holdings with initial stock holdings of $s_0^1 = s_0^2 = 0.5$ and initial bond holdings of $b_0^1 = b_0^2 = 0$. This example will include $T = 500$ periods. Given the heterogeneity in risk preferences, each individual's consumption and portfolio allocation patterns will be examined. The wealth distribution and pricing function in this economy will also be considered.

4.3.1 Wealth Distribution

Every period, each agent's wealth is their share of the price and dividend of the stock plus the value of any bonds that were loaned or borrowed during the previous period, (4.2). The expected growth rate of the economy (dividend) is 1.5% in each period. Given the expected positive growth rate, the typical path of a growing economy will benefit the least risk averse agent. For this simulation agent 2 is relatively more risk averse than agent 1 as $\gamma^2 > \gamma^1$. Therefore, without any idiosyncratic shocks, it would be expected that agent 1 becomes relatively wealthier. Figure 4.6 displays the ratio of wealth over time for both agents and confirms this conjecture. The solid line represents the average wealth for all possible states that can occur each period. The dashed lines represent the maximum and minimum outcome in each time period.
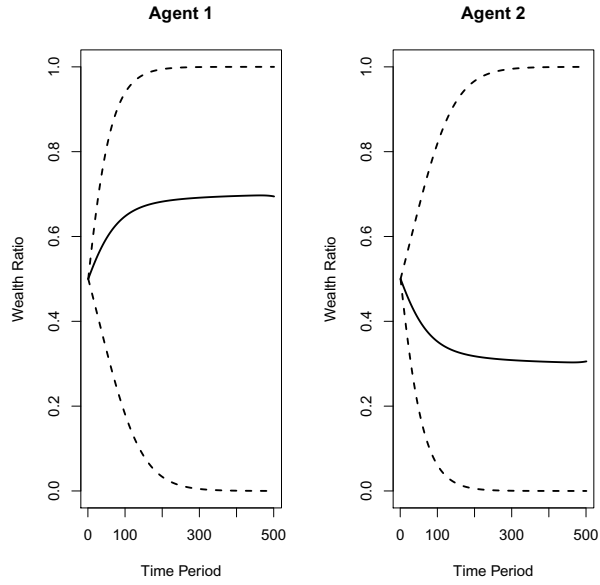
Figure 4.6: The ratio of agent wealth by time period. The solid line represents the average outcome, while the upper and lower dashed lines represent the maximum and minimum values, respectively.
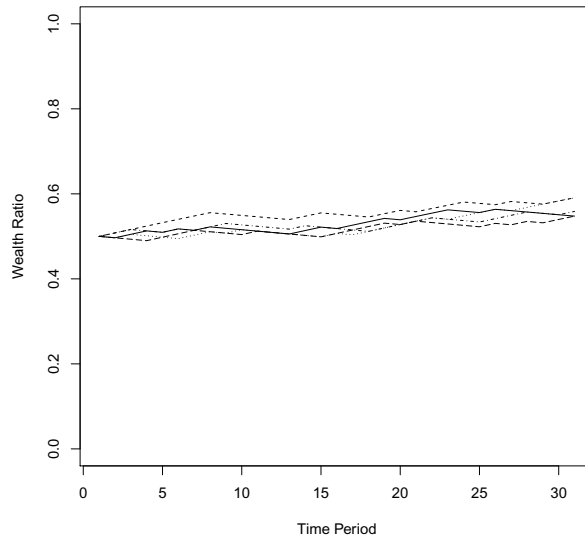
Figure 4.7: Agent 1 ratio of wealth through random paths. Since the expected growth rate of the dividend is positive, the less risk averse agent ($\gamma^1 = 1$) on average will accumulate wealth.

The wealth distribution is path dependent and it is possible for each agent to be relatively rich or poor. The wealth ratio for either agent is contained in the interval $(0, 1)$. On average, the less risk averse agent $(\gamma^1 = 1)$ becomes wealthier by owning approximately two thirds of the economy. This result is consistent with Benninga and Mayshar (1993) in that when the harvest is bountiful less risk averse agents obtain a higher proportion of the wealth distribution. Figure 4.7 represents wealth ratio for agent 1 given 5 random paths for 30 time periods. Initially owning half of wealth, the relatively less risk averse agent accumulates additional wealth over time. This result is a function of the the expected positive growth in the dividend process. If the dividend process declines, the more risk averse agent accumulates wealth. As an example, Figure 4.8 represents the wealth ratio for agent 1 when the 5 random paths observe the negative aggregate shock a majority of the time (over the thirty periods plotted, the positive shock was observed less than eight times).
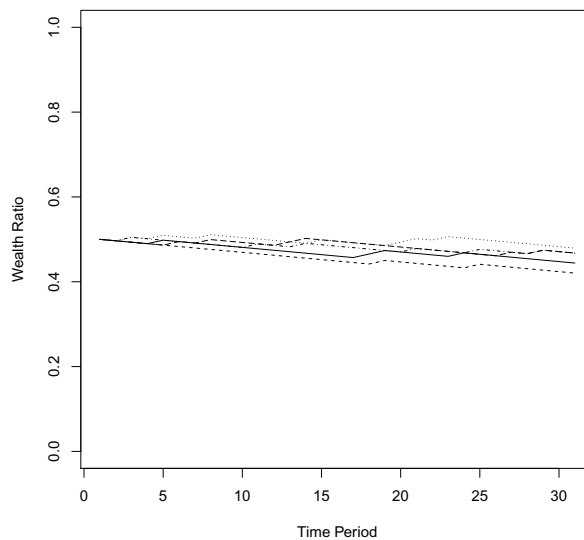
Figure 4.8: Agent 1 ratio of wealth through paths which the negative aggregate shock is observed a majority of the time. These outcomes are typically unlikely as t increases due to the expected growth in the dividend process.

4.3.2 Pricing Function

From equation (4.3), the pricing function for the homogeneous agent case with $\gamma = 1$ was $\left(\frac{\beta}{1-\beta}\right) D_t$. Given the assigned discount factor of $\beta = .9$, the price of the risky security is 9 times

the size of the dividend. When we introduce heterogeneity it is expected that the price to dividend ratio would be different as the wealth distribution changes through time. A more risk averse agent in the economy would have relatively lower demand for the risky asset, hence, leading to a relatively lower price.

Figure 4.9 plots the average, minimum, and maximum price to dividend ratio for all possible states over time. Since this is a finite horizon model, the price approaches zero as $t \to T$. Notice, unlike the homogeneous agent case, there is some variation in the price dividend ratios.

If all negative states occur in each time period (albeit the probability of this is approximately zero), the price to dividend ratio is approximately 8 at time $t = 300$. If this were to occur, the relatively higher risk averse agent $\gamma = 2$ would accumulate over 99% of total wealth. If all positive states occur, and agent 1 holds over 99% of the wealth, the price to dividend ratio approaches 9. When these extremes are observed, the model behaves essentially as a homogeneous agent of the wealthier agent.
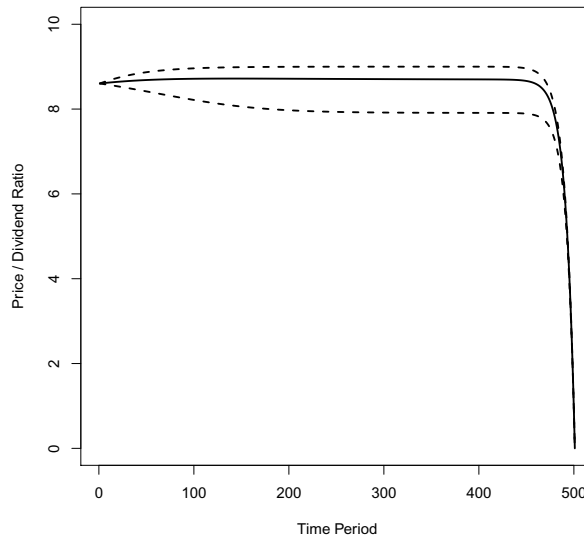
Figure 4.9: Price to dividend ratio by time period. The average outcome is represented by the solid line, while the maximum and minimum values are repesented by the respective dashed lines. Since this is a finite period model, $P_T = 0$.

Figure 4.10 depicts the price to dividend ratio for five random paths over thirty time periods. The red line is associated with all positive shocks. As this growth is observed agent 1 is becoming

wealthier. The blue line represents all negative shocks, which the more risk averse agent becomes relatively wealthier.
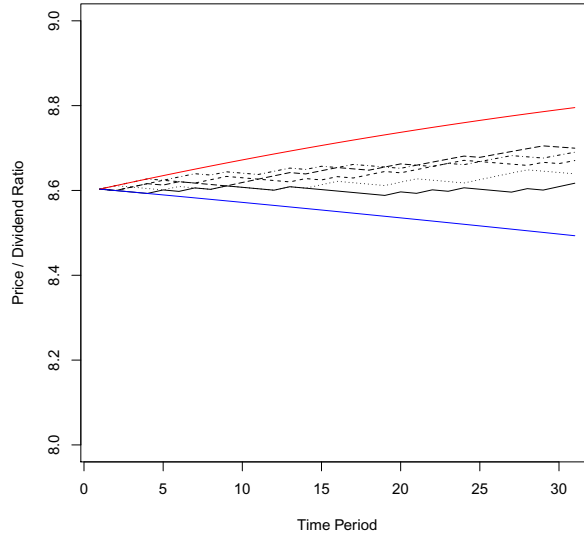
Figure 4.10: Price to dividend ratio through random paths. The red line represents the extreme case of positive shocks only, while the blue line indicates only negative shocks.

It is important to note that the calibrated parameters play a key role in specific characteristics of the model. For example, the price to dividend ratio in figures 4.2 and 4.9 remains relatively stable for the first 400 time periods before rapidly converging towards zero. This feature is crucially dependent on the discount factors of the agents. As investors are more patient through an increasing β, the price of the risky asset is higher (consider (4.3)). The increase in price creates a shorter time frame of stability as the convergence to zero at the terminal time T requires a larger drop in magnitude. Figure 4.11 shows an example of increasing the discount factor by keeping all other variables constant and letting $\beta^1 = \beta^2 = .95$.

At period $t = 300$, the minimum possible price to dividend ratio is approximately 15. The maximum the price to dividend ratio could be is 19. Again, this would be consistent with the $\gamma = 1$ homogeneous agent case for $\beta = .95$. It is important to note that as β increases the volatility of the price to dividend ratio increases. The difference between the extremes at $t = 300$ in Figure 4.9 was approximately 12.5%. This difference grew to approximately 26% in Figure 4.11.

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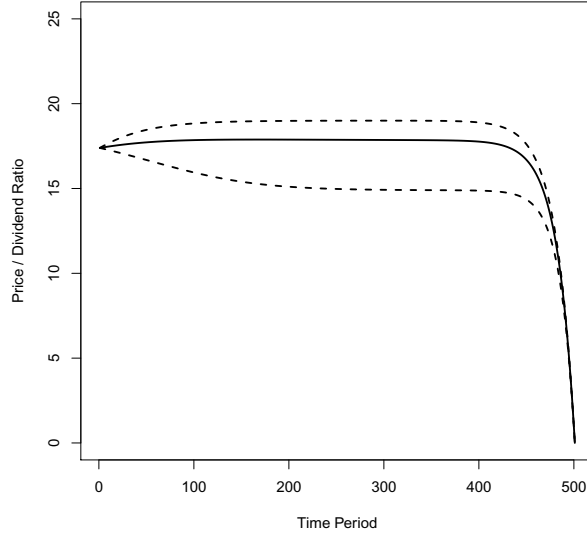
Figure 4.11: Price to dividend ratio by time period ($\beta = .95$). As β increases, the volatility of the price to dividend ratio increases.

4.3.3 Agent Consumption

For each time period investors must decide how much to consume or invest. The portion of wealth that an agent consumes in a single time period has been defined as δ_t. In the homogeneous agent case, with $\gamma = 1$, the agent consumed a constant portion of their wealth no matter what was the state of the world.

Figure 4.12 displays the portion of wealth consumed for both agents. Notice that the first investor, who has log utility ($\gamma^1 = 1$), consumes a constant $\delta = 1 - \beta$ of wealth across states. This well-known special case and is consistent with Levhari and Srinivasan (1969). As expected, as the simulation approaches the terminal period ($t \to T$) the consumption to wealth ratio converges to one. The second agent consumes a different portion of their wealth depending on the state of the economy. When the economy contracts the relatively higher risk averse agent consumes a greater portion of wealth. With an expanding economy the agent consumes a smaller portion. However, the agent that is more risk averse consumes a greater portion of their wealth relative to the less risk averse agent.

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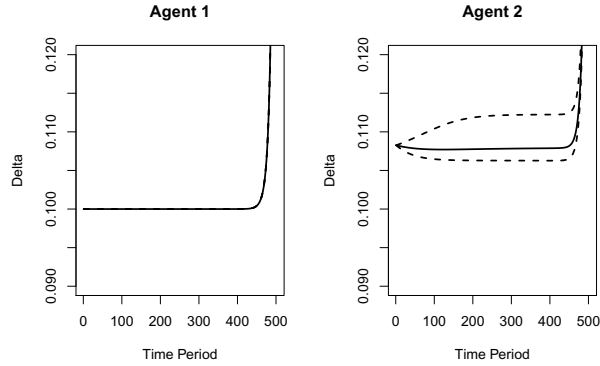
Figure 4.12: Agent ratio of wealth consumed by time period. The $\gamma = 1$ agent consumes a constant portion of wealth. Agent 2 on average, also consumes a relatively constant portion of wealth (solid line). The maximum and minimum outcomes adjust slightly, represented by the dashed lines.

4.3.4 Agent Portfolio Allocations

The remaining wealth that the agent does not consume $(1 - \delta)$ is invested in either the risky asset or the riskless bond. The variable α is the portion of investment allocated to the risky asset (4.5). Figure 4.13 depicts α for both agents over time. The solid line represents the average α_t^i over all states for each time period. The upper and lower dashed lines represent the maximum and minimum α_t, respectively for all states in each time period. The less risk averse agent $(\gamma^1 = 1)$ always purchases a larger portion of their investment in the risky asset than the less risk averse agent. The portion of investment in the risky asset for agent 1 is always in the range $\alpha_t^1 \in (1, 2)$. This implies that the less risk averse agent is willing to sell bonds to the other investor (borrowing) in order to leverage his portfolio in the risky asset. The more risk averse agent is willing to take the other side of the trade by purchasing bonds (lending) to ensure consumption next period. As positive aggregate shocks occur, agent 1 assumes more of the risky asset. This investor sells risk during periods of a contracting economy. These findings are identical to the findings in the pure exchange economy of Benninga and Mayshar (1993).

Figure 4.14 shows the shares of stocks and bonds for less risk averse Agent 1 (the log utility agent) for five random paths over thirty time periods. Since the expected growth path of dividends is positive, on average the less risk averse agent will accumulate a larger share of the risky asset through time. Although the states that occur in the economy have an equal probability to be
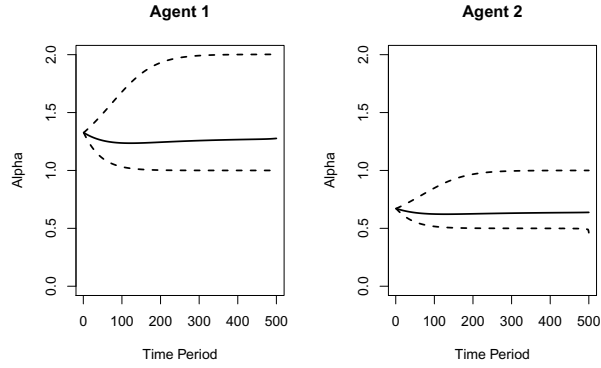
Figure 4.13: Agent proportion of risky investment by time period. The less risk averse agent will always have a higher portion of total investment in the risky asset.

expanding or contracting, the positive states have a larger magnitude. Therefore, Agent 1 purchases more shares of stock (shorts more bonds) in the positive state than he sells in the negative state. Since this economy has two types of investors, the more risk averse Agent 2 completes the stock and bond market with $s_t^2 = 1 - s_t^1$ and $b_t^2 = -b_t^1$.
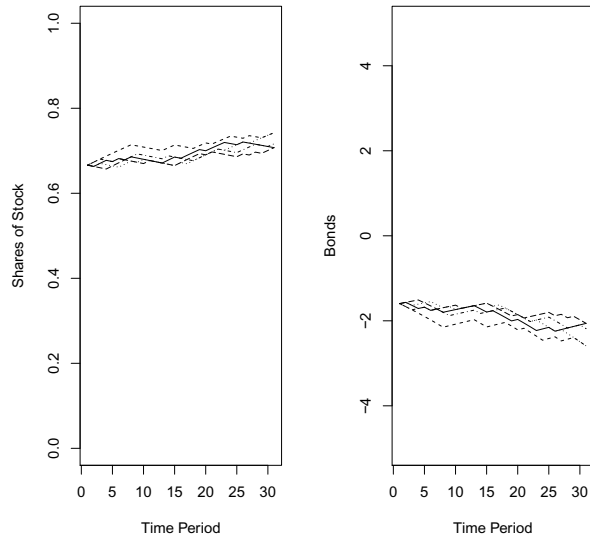
Figure 4.14: Agent 1 ($\gamma^1 = 1$) stock and bond holdings through random paths. The average trading volume between agents is minimal with approximately .004 shares.

It is important to note that the overall trading volume observed is very minimal. On average, the trade volume ($E_t[s_{t+1}^i - s_t^i]$) for ten random paths is approximately .0043 shares. The standard deviation of these trades was approximately .0019. This trading volume is consistent with the pure exchange economy in Benninga and Mayshar (1993).

4.3.5   Conclusion

This simulation introduces heterogeneity and market risk to a pure exchange economy. Unlike the homogeneous agent example, there are now dynamics in the price of assets, trading volume, and wealth distribution. The properties of this model are path dependent.

A representative agent, if constructed, would have a γ^R adjusting over time. The typical path, as the economy grows, is that the less risk averse agent accumulates more wealth and equity, reducing the γ^R of a hypothetical representative agent. As time periods pass it would take many consecutive negative shocks for Agent 2 to hold a majority of wealth. For example, assume thirty time periods have occurred. Due to a recombining lattice, there are thirty one possible states of the world. Of these, Agent 2 will be wealthier in 9 possible states. However, the probability that any of these states of the world exist at $t = 30$ is just 1.2%. Agent 1 will hold a majority of the wealth the other 98.8% of the time. As time passes, this inequality of wealth grows exponentially. By the time 100 states have been revealed ($t = 99$), Agent 1 is wealthier in 99.99909% of the possible states. By the time 300 states have been revealed, Agent 1 owns more than 80% of wealth in 99.975% of the possible states. The five random paths depicted in Figures 4.7 and 4.14 highlight these trends.

This pure exchange economy inhabited by heterogeneous agents, introduced trading of financial assets, price volatility, and an adjusting wealth distribution over time periods. This model has not been calibrated, but trading volume from one period to the next ($t \rightarrow t + 1$) is not enough to match empirical findings. Again, similar to the homogeneous agent simulation, there is a sense of validation to the model and algorithm, as results of this simulation make theoretical sense and are consistent with Benninga and Mayshar (1993). Building on this specific model setup, the next section will consider an economy with heterogeneous agents that experience aggregate and idiosyncratic shocks.

4.4 Heterogeneous Agent Model with Aggregate and Idiosyncratic Risk

In this section we will consider the T period model inhabited by heterogeneous agents who face both aggregate and idiosyncratic wealth shocks. The agents face the same aggregate risk described in the previous section through a stochastic dividend. Additional idiosyncratic risk will be introduced through a random endowment. The stochastic endowment is drawn from a distribution at each state of the world. This is an extension of the Benninga and Mayshar (1993) framework which only considered aggregate shocks.

As discussed above, previous research has considered models with both aggregate and idiosyncratic risk, Judd et al. (2003), Krussell and Smith (1998), Ljungqvist and Sargent (2004), to name a few. However, these models either define heterogeneity to be differences in initial wealth or feature an economy without growth. These specific assumptions, in an infinite horizon setting, allow for a computationally feasible solution. This section will highlight the results of a finite horizon model, which provided enough time periods (T), approximates the results from infinite horizon models. We will analyze the wealth distribution, pricing function, consumption, and investor portfolio allocation decisions over time.

4.4.1 Model Setup

As before, our economy will be inhabited by two heterogeneous agents over a T period finite horizon model. The heterogeneous agents differ in risk aversion parameter γ and will have the CRRA utililty function consistent with (4.1). Agent 1 be log utility ($\gamma^1 = 1$) and agent 2 be more risk averse with $\gamma^2 = 2$. Each agent has the discount factor $\beta = .995$. This parameter is consistent with the Heer and Maussner (2005) calibration of the Krussell and Smith (1998) model which was calibrated to represent a time period of approximately 6 weeks.

The aggregate stochastic dividend process can take on either a positive or negative growth rate at each time period with equal probabilities. With the calibration of a time period representing roughly 6 weeks, let the positive growth rate be $g_u = 0.00625$ and the negative growth rate $g_d = -0.0025$. There is a positive expected growth rate of .1875% per period. This translates to approximately 1.64% on an annual basis.

In addition to the aggregate shocks the consumers face a stochastic endowment shock at each time period. For this example, let the endowment shock be a multiple of the current dividend $(e_t^i = \theta D_t)$. The shocks, defined as θ are independent of the aggregate dividend growth shock. This multiple is randomly selected from the uniform distribution $\theta \sim U(1.5, 2.5)$.

This calibration corresponds to a mean endowment equivalent to twice the aggregate dividend. Therefore, about 20% of an agent's wealth will be held in financial assets and the other 80% from an exogenous endowment (assuming an agent maintains $s_t^i = 0.5$). The total size of the economy at each point in time will be the sum of individual endowments and total dividend. Given the values of the defined parameters, the range of the size of the economy at time t willl be $[4D_t, 6D_t]$. This is definitely not calibrated to any reasonable value. The goal of this exercise is to understand the properties of the model.

The investors are initialized with identical stock and bond holdings $s_0^1 = s_0^2 = 0.5$ and $b_0^1 = b_0^2 = 0.0$, respectively. With idiosyncratic and aggregate shocks in a heterogeneous agent environment, dynamics of the model are path dependent and potentially highly volatile across states. For example, at any point in time net wealth can actually be negative as the investor is borrowing heavily to help smooth consumption. Recall that the planner's solution to this problem does not impose any borrowing constraints, because by construction, the life-time budget constraints of all consumers will be satisfied.

4.4.2 Wealth Distribution

As defined earlier in equation (4.2), the wealth of each agent is the value of stock holdings plus the agent's share of the current dividend, bond adjustments, and endowment (labor income). As a borrower, it is possible that wealth can be negative between time $t \in (1, T - 1)$. At time T agent's consume their entire wealth and consumption is restricted to always be positive. Although net wealth can be negative at a specific time, the life-time budget constraint must hold. In addition, all debts are to be paid back, eliminating any type of Ponzi schemes.
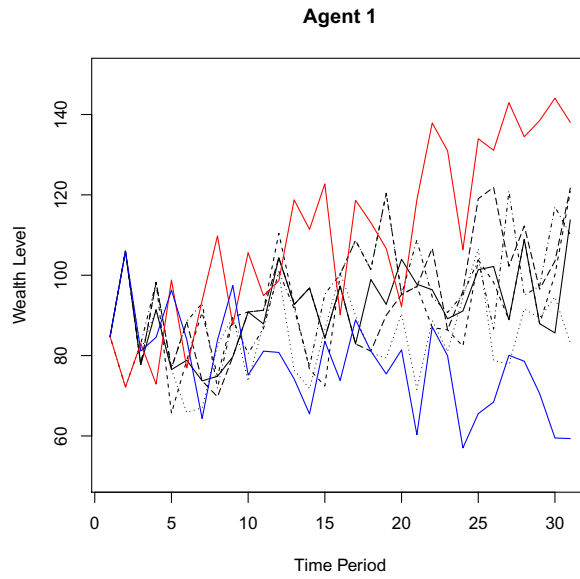
Agent 1

Figure 4.15: Agent 1 wealth levels through random paths. The red and blue lines represent the extreme case of positive and negative aggregate dividend shocks, respectively. Random paths can still provide higher or lower wealth levels due to the idiosyncratic shocks.
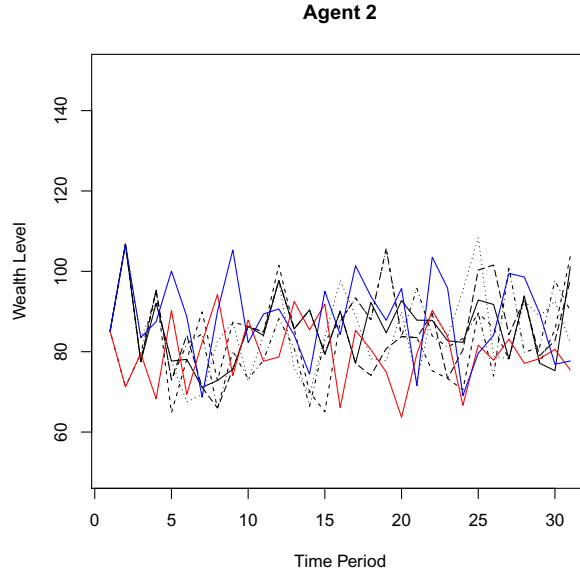
Agent 2

Figure 4.16: Agent 2 wealth levels through random paths. The red and blue lines represent the extreme case of positive and negative aggregate dividend shocks, respectively. Random paths can still provide higher or lower wealth levels due to the idiosyncratic shocks.

Figure 4.15 and 4.16 for agent 1 and 2, respectively, presents the wealth levels for each agent over five random paths for 30 time periods. The size of the economy is made up of the dividend and endowments which are the aggregate and idiosyncratic shocks, respectively. For this example, the extreme positive and negative dividend shocks (presented in the red and blue lines, respectively) do not necessarily equate to the extreme levels of wealth. In the pure exchange economy, without idiosyncratic shocks, the red line would have represented the highest path for agent 1 and the lowest path for agent 2.
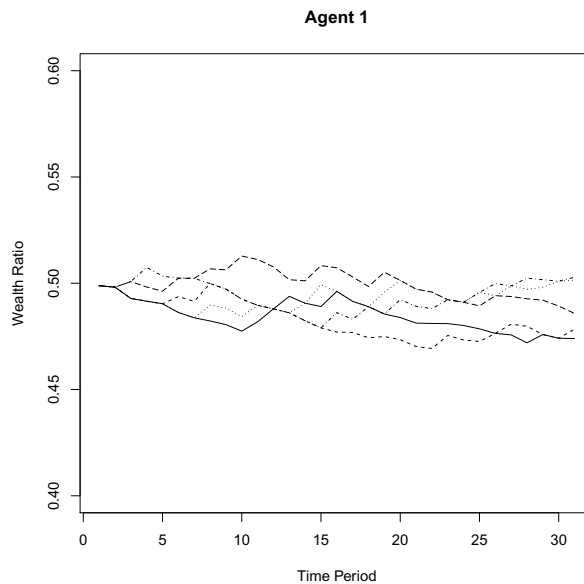
Agent 1

Figure 4.17: Agent 1 wealth ratio through random paths. Each of these paths have a cumulative positive growth rate for the dividend. However, Agent 1 is not relatively wealthier on all paths.

Figure 4.17 plots the wealth ratio of agent 1 ($\gamma^1 = 1$) for random paths. All of these paths contain cumulative growth in the dividend process. Even with a growing aggregate dividend it is possible for the less risk averse agent to be relatively poorer than the more risk averse agent. This outcome was not possible in the heterogeneous agent model with only aggregate shocks.

4.4.3 Pricing Function

Applying the homogeneous agent pricing function (4.3) to the higher discount factor calibrated in this simulation ($\beta = .995$) provides a price to dividend ratio of 199. Figure 4.18 depicts the

average, minimum, and maximum price to dividend ratio over all states at each time t. The average is represented as the solid line, while the maximum and minimum are the dashed lines accordingly. The average price to dividend ratio fluctuates in the early time periods and then converges over time towards zero as $t \to 500$. The possible states at each period provide plenty of variation with the minimum and maximum reaching over 50% from the average.
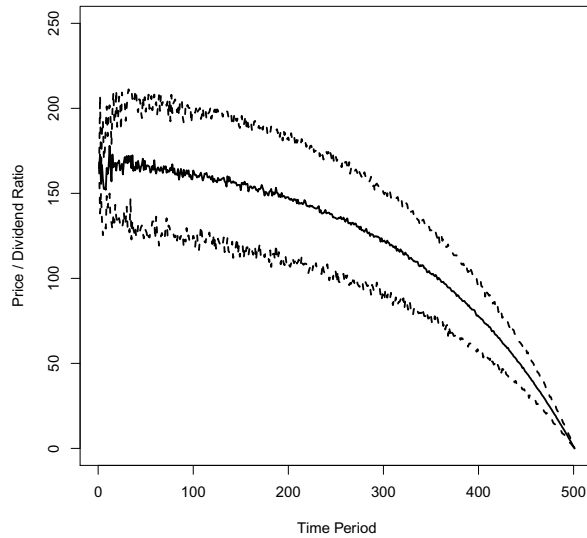
Figure 4.18: Price to dividend ratio by time period.

Figure 4.19 presents the price to dividend ratio of the five random paths. The large fluctuations in the price to dividend ratio is solely due to the differences from the idiosyncratic shocks. The minimum price to dividend ratio of these random paths corresponds to a total endowment of approximately $3.14D_t$. The maximum price to dividend ratio occurs with a total endowment of $4.93D_t$. Therefore, when agents receive a higher endowment, they are willing to pay more for the risky asset.

4.4.4 Agent Consumption

Consumption will always be positive for both agents in each period. Investors in an exchange economy maintained positive shares of stock in order to consume their share of dividends but when

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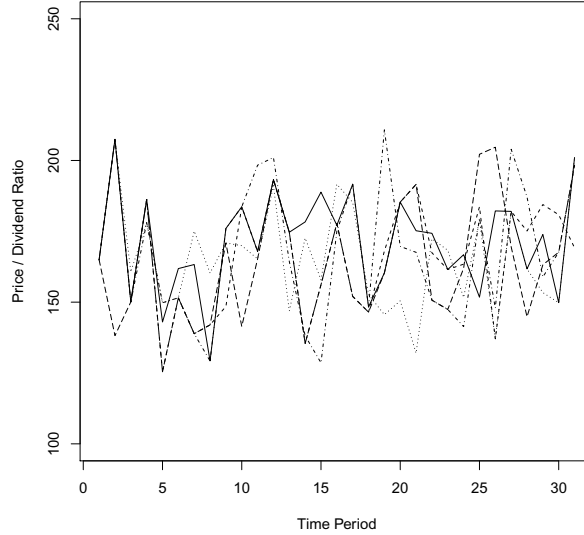
Figure 4.19: Price to dividend ratio through random paths. The extreme volatility of these random paths are due to the combination of a high discount factor β and range of idiosyncratic shocks $\theta \sim U[1.5, 2.5]$.

endowments are included in the model, agents have an alternative source of income for consumption and no longer have to rely on the dividends.

The portion of wealth consumed each period is defined as δ_t^i. Since wealth can be non-positive due to borrowing, δ potentially could be negative or even undefined. Figure 4.20 shows the portion of wealth consumed each period in five random paths for both agents. The first agent ($\gamma^1 = 1$) smooths portion of wealth consumed relatively more than the other agent. When agent 2 has a low endowment, consumption becomes a larger portion of his wealth.

4.4.5 Agent Portfolio Allocation

Figure 4.21 represents five random paths of risky asset holdings for agent 1. The idiosyncratic shocks provide much more volatility relative to the aggregate shocks only. On average, about .597 shares were traded amongst the agents per period. Even with the relatively less risk averse agent, there are paths where he purchased more than 1 share of the equity and other instances where he shorted the equity. Having a position greater than the supply is made possible with complete markets. The planner will verify that life-time budget constraints hold for each agent, and that all debts are paid by time T. Within this complete market framework, agents will be able to take
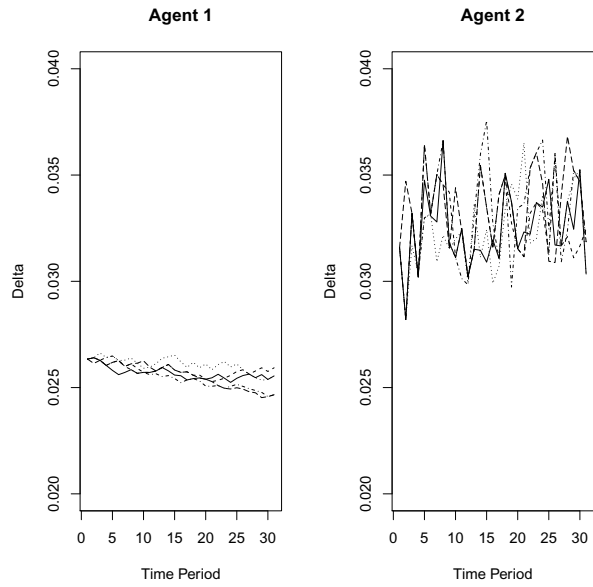
Figure 4.20: Agent ratio of wealth consumed by time period through random paths.
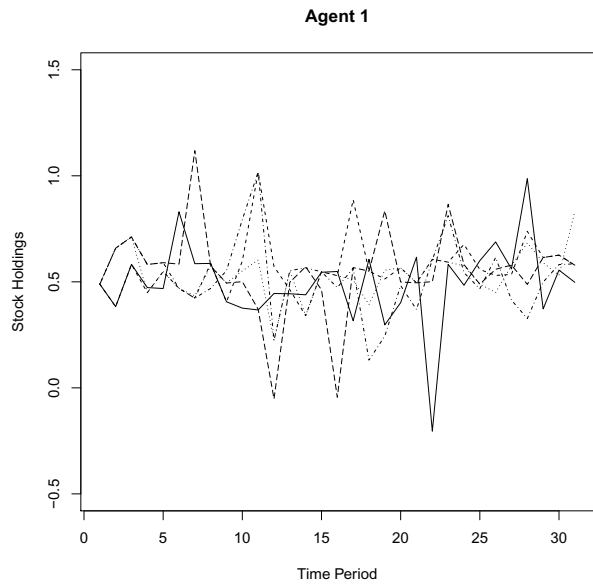
Figure 4.21: Equity holdings for agent 1 over random paths. On average, the trading volume was .597 shares per period.

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an aggressive long or short position in either asset if the other type of agent is willing to take the other side.

The volume of trades represent how many shares were exchanged during that period. The period could be calibrated to a day, six weeks, or a full year. This current exercise was not to calibrate to anything in particular, but present the potential dynamics of including idiosyncratic shocks.

4.4.6 Conclusion

This section considered numerical simulations of a dynamic, stochastic, general equilibrium model in complete markets. True heterogeneity in the utility functions was included with aggregate and idiosyncratic shocks. Furthermore, growth was included in the dividend process to provide a non-stationary economy. With a finite horizon model, some of the state space complexities were relaxed.

The properties of price, asset trading volume, wealth distribution, and consumption were considered in this section. Relative to models with additional restrictions, the properties of this model seem to hint at the potential for dynamics that could apply to financial economic questions such as the equity premium puzzle.

4.5 Summary

This chapter considered the properties of a dynamic, stochastic, general equilibrium model. The first section considered a homogeneous agent case. The simulated results were expected, as Lucas (1978), Levhari and Srinivasan (1969), and Duffie (1988) have provided theoretical results in detail. However, it helps validate the algorithms potential, including an approximation of the infinite horizon.

To add complexity, the second section included heterogeneous agents. With only aggregate risk, the numerical simulations did provide some dynamics. Trade existed in the model, resulting in a changing γ^R of a hypothetical representative agent. As states are revealed, the wealth distribution adjusts creating a different representative agent than the previous period. Therefore, even though the heterogeneous agents exhibited CRRA utility, the representative agent did not. This again was not a new contribution of the model, but again validated propositions and results of Benninga and Mayshar (1993).

Finally, idiosyncratic shocks were added to the model. Without specific calibration, the magnitude of the results should not be analyzed. However, the idiosyncratic shocks provided interesting dynamics not seen in the previous simulation. For example, it is possible for the risk averse agent to become relatively poorer in a growing economy. Additionally, the pricing function had much more volatility relative to more restrictive assumptions.

The next chapter will take the calibration of this model a step further. I will focus on addressing the possible applications of the model to financial economic questions.

CHAPTER 5

HETEROGENEITY AND THE EQUITY PREMIUM PUZZLE

5.1 Introduction

Mehra and Prescott (1985) examined historical U.S. equity returns, short term yields, and consumption growth. This research highlighted an observed equity premium in which average returns on equities was higher than returns on short term debt. This result is expected given the investor is taking on relatively more risk when investing in equities. Unfortunately, and what makes this a puzzle, is that when dynamic, stochastic, general equilibrium (DSGE) models have been utilized to identify with this, the magnitude of the modeled premium comes up short. To calibrate the equity and bond returns with historical data, the paramaterization required in a Lucas framework model are considered to be unrealistic or not in line with economic theory. For example, to obtain a lower rate of return on the risk free security, Mehra and Prescott (2003) show that investors must be extremely risk averse. With Constant Relative Risk Averson (CRRA), it is estimated that the risk aversion parameter γ would need to be greater than 30 depending on the discount factor β. The risk averse behavior that is exhibited from a $\gamma > 30$ is too extreme from a theoretical view.

Benninga and Mayshar (1993) also address the equity premium puzzle with two heterogeneous agents. The pure exchange economy can numerically obtain asset returns and volatility consistent with the empirical results defined in Mehra and Prescott (1985). The asset returns are calibrated identically to the historical ninety year empirical returns of approximately 7% for equity, and less than 1% for short term debt. The simulated standard deviation of the market was 18.75%, which was higher than the observed 16.54%. Given the simulated results of the Benninga and Mayshar (1993) model, it would appear that this DSGE framework can provide insight into the equity premium puzzle. However, consider the assumptions of the heterogeneous agents. One agent is relatively less risk averse with the risk aversion parameter $\gamma^1 = 3$. This agent is extremely patient given a discount factor $\beta^1 = 1.12$. This assumption is considered uncharacteristic of classical

economic theory. The agent would rather consume in the future compared to the current time period. The second agent would rather consume today as the discount factor is $\beta^2 = 0.93$. This agent, however, is extremely risk averse with $\gamma^2 = 36$. This assumption is consistent with the calibration of a representative agent. The combination of one agent's discount factor $\beta > 1$ and the other agent's extreme risk aversion of $\gamma > 30$ provides the model with a low yield on the risk free security.

This chapter will consider different numerical calibrations of the T period, heterogeneous agent model defined in chapter 3. The goal of this chapter is to consider the equity premium puzzle and whether a DSGE model can address these issues. The purpose will not be to match the exact observed returns of equities and short term debt. Instead, the role of idiosyncratic shocks in a truly heterogeneous agent environment will be addressed, to determine the potential of this finite period DSGE model. First, the selection of parameters and their significance will be introduced. Next, paramater values in the model will be adjusted and results compared. Finally, the opportunities of this model and possible research topics will be discussed.

5.2   Model Calibration

Mehra and Prescott (1985) examine 90 years of equity and short term debt returns from 1889-1978. Over that historical time period the Standard & Poor's 500 Index returned an average annual real return of 6.98%. The average yield on short term risk free securities over that time period was just 0.80%. The returns of the equity market provided a standard deviation of 16.54%. During that historical time period household consumption growth was approximately 1.8%. Benninga and Mayshar (1993) calibrated a heterogeneous agent, pure exchange, DSGE model to produce these empirical results. Where applicable, parameters for this simulation will remain consistent.

One of the typical assumptions of the consumption based asset pricing models, is that the consumption good is not storable. Therefore, in a pure exchange economy the aggregate consumption at each time period is equivalent to the size of the dividend ($C_t = D_t$). To control consumption growth, Benninga and Mayshar (1993) assume two possible states for the dividend. Either the dividend grows 5% from the previous amount, or declines by 2%. Therefore, the expected growth of the dividend is 1.5%, which is in line with the observed consumption growth. Unless noted

otherwise, let the positive state have growth $g_u = .05$ and the negative state $g_d = -.02$. One of these states will occur at each time period with equal probability $\pi_u = \pi_d = 0.5$.

The additional income through employment, wages, or some endowment to the agents each period is not storable as well. This increases the size of the economy, which agents desire to consume. Depending on the idiosyncratic shocks it can be difficult to target a specific consumption growth. It is assumed for these simulations that each agent's idiosyncratic shock will follow

$$e_t^i(z_t) = \theta_t^i(z_t)D_t^i(z_t), \tag{5.1}$$

where θ_t^i comes from the uniform distribution $[a^i, b^i]$. Since the expected value from the distribution will be $\frac{b^i - a^i}{2}$ for each agent, the size of the economy is expected to be a multiple of the dividend at each time period. This allows the income to remain at a consistent level relative to the dividend as the economy grows. With this assumed process of the idiosyncratic shock, the growth rate of the dividend will be consistent with the growth rate of aggregate consumption.

Unlike Benninga and Mayshar (1993), the discount factor for each agent will be $\beta < 1$ for these simulations. This will create current consumption to be more valuable (in terms of utility) relative to future consumption. For a representative agent with log utility ($\gamma = 1$), the price of the bond is a constant $Q = \beta$. Therefore, if trades existed in this homogeneous agent case, returns would be equivalent to $R_f = \frac{1}{Q}$. Increasing the discount factor β would lower the return of the bond, all else equal. This was the intention of a discount factor assumed to be greater than 1 for Benninga and Mayshar (1993). For the simulations in this T period model, assume the discount factor for both these agents will be $\beta^1 = \beta^2 = 0.95$, unless noted otherwise.

The dividend will be initialized with $D_0 = 1.0$. Unless defined otherwise, initial asset holdings will be identical for both agents ($s_0^1 = s_0^2 = 0.5$ and $b_0^1 = b_0^2 = 0$). To keep the simulations consistent, let the terminal period be $T = 500$. Each period is loosely calibrated to represent one year. The numerical solutions in Benninga and Mayshar (1993) consider the first thirty periods of possible outcomes. These simulations include the first seventy five periods for ten random paths (750 observations total). The standard deviations and means reported are the average standard deviations and means from the ten random paths. The approach of this model is to be used as an approximation to an infinite horizon model.

5.3   Idiosyncratic Shocks

The idiosyncratic shocks for each agent directly influence the size of the economy. Since the consumption good is non-storable, the agents will either consume their portion of the economy or trade it for financial assets. This section will consider adjusting the distribution range and levels of the idiosyncratic shock and the effects that it has on the prices, returns, and volatility of the equity market. To consider range, four simulations will be identified with different distributions of θ, adjusting a and b from the uniform distribution described in (5.1). To study the impact that income distributions have on the model, four simulations will be compared with heterogeneous agents receiving different distributions.

Let the simulations for this section include two heterogeneous agents with CRRA utility. The heterogeneity in this model occurs through different risk aversion parameters. Let agent 1 be less risk averse than agent 2, with $\gamma^1 = 2$, and $\gamma^2 = 5$. The baseline assumptions will be applied for the other agent parameters with $\beta^1 = \beta^2 = 0.95$, $s_0^1 = s_0^2 = 0.5$, and $b_0^1 = b_0^2 = 0$. The baseline economic growth assumptions and terminal date defined in the previous calibration section are also assumed.

The range of possible idiosyncratic shocks defined by $[a, b]$ in (5.1) effects market outcomes. Table 5.1 presents the results of four different simulations based on the range of the uniform distribution. The random income shocks have the largest variance with $[1.0, 3.0]$. The baseline for comparison is no idiosyncratic shock, and the endowment is simply two times the current dividend. For all four simulations, each agent expects to have an endowment at each time period of twice the current dividend $(e_t^i(z_t) = 2D_t(z_t))$. Therefore, on average the size of the economy, which includes stochastic agent endowments and dividend, for any time period is expected to be five times the realized dividend. The single period expected risk-free return $E_t[R_f]$, equity return $E_t[R_s]$, equity premium $E_t[R_s - R_f]$, price to dividend ratio $E_t\left[\frac{P_t}{D_t}\right]$, and volume of shares traded $E_t\left[|s_{t+1}^i - s_t^i|\right]$ is provided in table 5.1. Consumption growth for each agent $E_t[\Delta \log(c_t^i)]$ and growth in the economy $E_t[\Delta \log(D_t + \sum e_t^i)]$ will be provided. The associated standard deviations and planner's weight (Negishi) are also included. These statistics will be reported for the remaining simulations in this chapter. These calculations are based on ten random paths of the first seventy five periods. To maintain consistency of the same states, the same ten random paths were used in all four simulations.

Table 5.1: The market characteristics of the model are displayed from four different simulations. Each simulation is identified by $[a, b]$ for θ defined in (5.1).

Statistic	Idiosyncratic Shock [a, b]			
	[2.0, 2.0]	**[1.9, 2.1]**	**[1.5, 2.5]**	**[1.0, 3.0]**
$E_t[R_f]$	8.90%	8.84%	11.49%	20.24%
$E_t[R_s]$	9.31%	9.64%	14.80%	31.83%
σ_{R_s}	4.00%	8.01%	37.01%	85.77%
$E_t[R_s - R_f]$	0.40%	0.80%	3.30%	11.59%
$E_t[\frac{P_t}{D_t}]$	13.25	13.32	14.04	16.37
$\sigma_{\frac{P_t}{D_t}}$	.29	.66	3.09	6.72
$E_t[|s_{t+1}^i - s_t^i|]$	0.02	3.88	1.60	1.28
$\sigma_{|s_{t+1}^i - s_t^i|}$	.01	8.51	4.08	3.41
$E_t[\Delta \log(c_t^1)]$	1.95%	2.01%	1.94%	2.13%
$\sigma_{\Delta \log(c_t^1)}$	4.65%	5.73%	16.63%	31.38%
$E_t[\Delta \log(c_t^2)]$	0.78%	0.80%	0.78%	0.85%
$\sigma_{\Delta \log(c_t^2)}$	1.86%	2.29%	6.65%	12.55%
$E_t[\Delta \log(D_t + \sum e_t^i)]$	1.46%	1.50%	1.46%	1.57%
$\sigma_{\Delta \log(D_t + \sum e_t^i)}$	3.44%	4.26%	12.38%	23.21%
μ^2	0.04047594	0.041568244	0.044675646	0.040778576

As the interval between $[a, b]$ increases, the standard deviation of the return on equity and price to dividend ratio increases. The expected return on both financial assets also increases as greater uncertainty is included in the model. However, the expected equity return increases more than the bond return. This is represented in the equity premium, which is only .80% when the idiosyncratic shock can only vary 5% from the mean. That premium rises to 11.59% when the shock can be up to 50% higher or lower than $2D_t$.

Figure 5.1 is a histogram of equity returns for all four scenarios. This includes the first 75 time periods of all 10 random paths. The titles indicate which simulation each plot is associated with. When there are no idiosyncratic shocks, the size of the economy is always five times the size of the dividend. The only shock to the model is the aggregate positive or negative dividend shock. In this scenario there are essentially two returns of the equity, based on the aggregate shock that was drawn. With approximately a 5% return in the stock when the negative state occurs half the time, and a 13% return when the positive state is revealed the other half, the expected return on the stock is roughly 9%. This is consistent with Benninga and Mayshar (1993), with which two

returns are reported. As the difference in $[a, b]$ increases the distribution of equity returns expands. This is evident by σ_{Rs} in Table 5.1, which increases from 4% with no idiosyncratic shocks, up to 85.77% when the idiosyncratic shock can be 50% higher or lower than the mean. The skewness of the equity returns distribution with $\theta \sim U[1.9, 2.1]$ is 0.17. The kurtosis is approximately 2.46. The distribution of stock returns with $\theta \sim U[1.0, 3.0]$ are highly skewed with a calculated skewness of 1.79. This distribution is leptokurtic with excess kurtosis of 4.64.
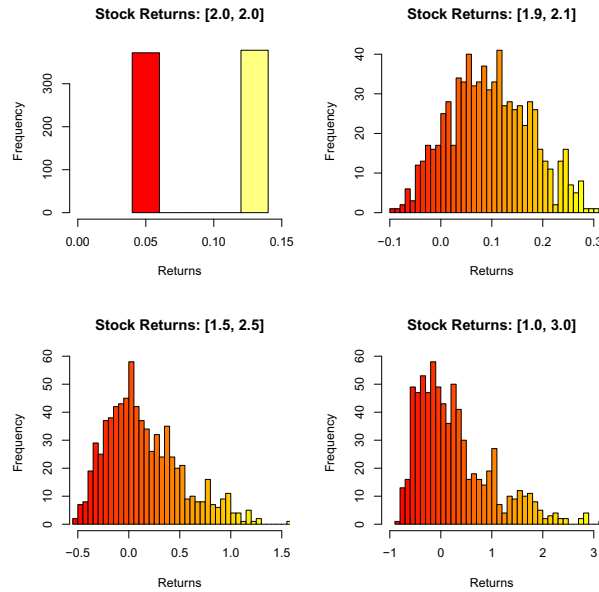
Figure 5.1: Histogram of the stock returns for the four simulations. The uniform distribution for $[a, b]$ is depicted above each graph.

The size of the economy is a multiple of the current dividend. The expected size of the economy for all of these simulations in Table 5.1 is $5D_t$. The dividend is expected to grow at 1.5% based on the assumed states. Therefore, the economy is also expected to grow at the same rate. Table 5.1 presents consistent expectations for the growth rates of the economy $\Delta \log(D_t + \sum e_t^i)$. Also in Table 5.1 is the average growth rate and standard deviation of each consumer for each simulation. The agent with greater risk aversion ($\gamma^2 = 5$) consistently has a smoother consumption path. This volatility and expected growth of one random path is presented in Figure 5.2. The consumption growth for each agent is represented by the blue line. Additionally, the red line indicates the agent endowment growth (idiosyncratic shock). The correlation between endowment and consumption

growth for the less risk averse agent ($\gamma^1 = 2$) is 74.31%. This correlation for the consumption smoothing agent 2 is slightly less at 69.81%.
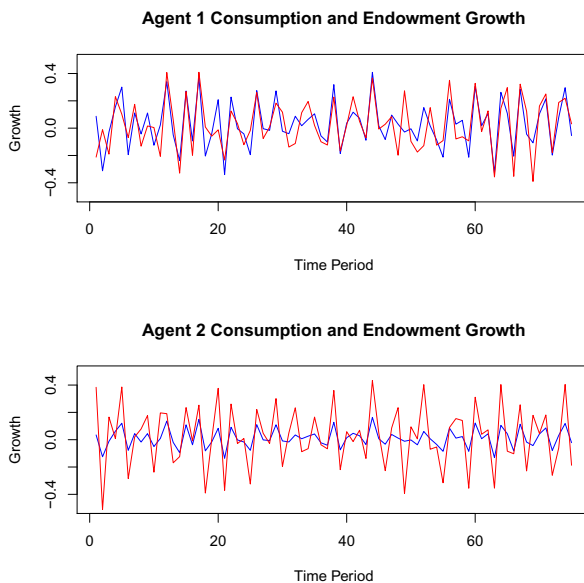
Figure 5.2: This figure plots the consumption growth (blue line) and endowment growth (red line) for each agent. This plot represents one of the ten random paths produced.

The expected volume of trades $E_t[s^i_{t+1} - s^i_t]$, standard deviation of trades, and the planner's weight of the second agent ($\frac{1}{\mu_2}$) do not seem to have a direct relationship with the minimum and maximum of the distribution for θ. It is possible that these differences are due to path dependency or lifetime endowment. For example, the largest trades occurred in the simulation with a uniform distribution of $[1.9, 2.1]$. Shares in this simulation were heavily traded when lower idiosyncratic shocks were observed by both agents, only to be followed by very high shocks the next time period $(t+1)$. It is important to note the importance of idiosyncratic risk relative to trade volume. With only aggregate shocks, trade is relatively minimal. This result is also consistent with the numerical solutions found in the pure exchange economy of Benninga and Mayshar (1993).

In a growth economy without idiosyncratic shocks, the relatively more risk averse agent typically decreases his share of total wealth over time as he trades away his risky asset holdings (4.6). However, with certain idiosyncratic shocks, this result is not always true. This is a feature of this

model not available in a pure exchange economy. Three additional simulations were constructed to compare heterogeneous agents with different distributions of endowment shocks.

The following depicts the setup of three separate numerical simulations with different distributions of idiosyncratic shocks.

Scenario	θ^1	θ^2
$\theta^1 > \theta^2$	[1.75, 2.25]	[.75, 1.25]
$\theta^1 = \theta^2$	[1.25, 1.75]	[1.25, 1.75]
$\theta^1 < \theta^2$	[.75, 1.25]	[1.75, 2.25]

Therefore, agent 1 will always have a larger idiosyncratic shock in scenario $\theta^1 > \theta^2$ and a lower shock in scenario $\theta^1 < \theta^2$. The expected size of the economy at any period is $4D_t$ for all three scenarios. The risk aversion parameters, discount factors, and initial holdings are consistent with the previous calibration in Table reftable1. The results of these three simulations are found in Table 5.2.

Table 5.2: The market characteristics of the model are displayed from three different simulations. Each simulation is identified by θ^i defined in equation (5.1).

Statistic	$\theta^1 > \theta^2$	$\theta^1 = \theta^2$	$\theta^1 < \theta^2$
$E_t[R_f]$	9.64%	10.59%	9.35%
$E_t[R_s]$	10.83%	11.34%	11.77%
σ_{R_s}	19.99%	21.41%	25.36%
$E_t[R_s - R_f]$	1.19%	0.75%	2.43%
$E_t[\frac{P_t}{D_t}]$	13.99	13.43	13.10
$\sigma_{\frac{P_t}{D_t}}$	1.75	1.84	1.95
$E_t[|s_{t+1}^i - s_t^i|]$	1.63	12.42	2.33
$\sigma_{|s_{t+1}^i - s_t^i|}$	3.24	64.03	5.12
$E_t[\Delta \log(c_t^1)]$	1.86%	1.98%	1.90%
$\sigma_{\Delta \log(c_t^1)}$	9.94%	10.64%	12.31%
$E_t[\Delta \log(c_t^2)]$	0.74%	0.79%	0.76%
$\sigma_{\Delta \log(c_t^2)}$	3.98%	4.26%	4.92%
$E_t[\Delta \log(D_t + \sum e_t^i)]$	1.51%	1.49%	1.31%
$\sigma_{\Delta \log(D_t + \sum e_t^i)}$	8.03%	7.95%	8.24%
μ^2	0.5414577	0.0834063	0.0153421

The returns of the risky asset increases as the relatively more risk averse agent receives an increasing portion of the overall endowment. However, that is not necessarily the case for the risk

free asset. In the $\theta^1 < \theta^2$ simulations, the returns on the bond decreased while the equity returns increased relative to the other scenarios. The price to dividend ratio decreases, but with higher standard deviation as the share of total endowment shifts to the more risk averse agent.

The expected consumption growth remains consistently higher for the first agent ($\gamma^1 = 2$). The standard deviation of consumption growth is relatively lower for the second agent, as consumption remains relatively smooth over time. The expected volume of equity trades and standard deviation again does not show a consistent relationship with endowment share.

The distribution of equity returns are displayed in Figure 5.3. The histograms for each scenario look similar, although the positive and negative tail ends extend further for scenario $\theta^1 < \theta^2$. Although these distributions are not identical to observed equity returns, it is important to consider distributions that provide skewness and kurtosis.
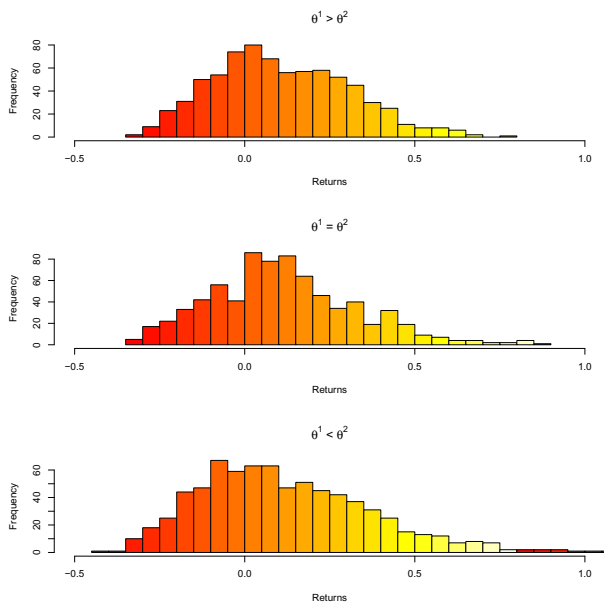
Figure 5.3: Histogram of the stock returns for three different simulations. Each simulation adjusts the endowment across agents. In the first graph, agent 1 receives a higher endowment. In the second histogram they get the same endowment process. Finally, agent 2 receives the higher endowment.

In both Table 5.1 and Table 5.2 it is easy to notice that through this simple calibration exercise the issue of the risk free asset over compensating investors relative to empirical data, still exists. In fact, the lowest rate of 9.31% is higher than the average equity returns over the ninety year

historical time period. However, it is important to recognize the reaction of the equity premium and other market characteristics. Given the equity premium increases with greater uncertainty in the idiosyncratic risk, an additional parameter is available to calibrate results.

5.4   Discount Factor

One additional parameter to consider is the discount factor β. Typically in financial economic theory it is assumed that $0 < \beta < 1$. Therefore, the agent will value the same level of consumption more if it is consumed now rather than later. For annual calibrations, it is common to see an assumed discount factor of $.95 \leq \beta < 1$. This section will consider the effects of adjusting β on a finite horizon, heterogeneous agent model.

To consider an annual calibration for this model, assume the dividend growth process of equal probabilities of $g_u = .05$ and $g_d = -.02$. Investors start the model with their initial asset holdings of $s_0^1 = s_0^2 = 0.5$, and $b_0^1 = b_0^2 = 0$. It is assumed that each agent has CRRA utility with risk aversion parameters $\gamma^1 = 2$ and $\gamma^2 = 5$. The idiosyncratic shock will follow the equation (5.1) with the uniform distribution $[1.5, 2.5]$. Three different simulations are considered in this section. The first simulation inlcudes the baseline $\beta = .95$. The second and third simulation assume investors are more patient with a higher discount factor of $\beta = .975$ and $\beta = .99$, respectively. Table 5.3 presents the results of ten random paths for each scenario.

The discount factor has an indirect relationship with the average equtiy and bond returns. As investors are assumed to be more patient, risky asset returns fall at a larger magnitude than the risk-free returns. With this numerical example, the equity premium decreased from 3.30% with $\beta = .95$ to 2.15% when $\beta = .99$. The price to dividend ratio and its standard deviation demonstrates a direct relationship with β. The more risk averse agent maintains a lower consumption growth and standard deviation across all three scenarios.

Figure 5.4 plots the time period results of one random path for the scenario in which $\beta = .95$. These graphs are designed to give an idea of how the asset market reacts to the aggregate and idiosyncratic shocks at each time period. The first graph is the price to dividend ratio. The very high standard deviation is quite apparent throughout the time periods.

Table 5.3: The market statistics of the model are displayed from three different simulations. Each simulation is identified by the discount factor β assumed for both agents.

Statistic	$\beta = 0.95$	$\beta = 0.975$	$\beta = 0.99$
$E_t[R_f]$	11.49%	8.74%	8.56%
$E_t[R_s]$	14.80%	11.98%	10.71%
σ_{R_s}	37.01%	37.55%	38.00%
$E_t[R_s - R_f]$	3.30%	3.25%	2.15%
$E_t[\frac{P_t}{D_t}]$	14.04	21.79	32.52
$\sigma_{\frac{P_t}{D_t}}$	3.09	4.80	7.63
$E_t[|s_{t+1}^i - s_t^i|]$	1.60	1.63	2.18
$\sigma_{|s_{t+1}^i - s_t^i|}$	4.08	3.37	6.12
$E_t[\Delta \log(c_t^1)]$	1.94%	1.84%	1.84%
$\sigma_{\Delta \log(c_t^1)}$	16.63%	16.92%	17.55%
$E_t[\Delta \log(c_t^2)]$	0.78%	0.73%	0.74%
$\sigma_{\Delta \log(c_t^2)}$	6.65%	6.77%	7.02%
$E_t[\Delta \log(D_t + \sum e_t^i)]$	1.46%	1.37%	1.35%
$\sigma_{\Delta \log(D_t + \sum e_t^i)}$	12.38%	12.47%	12.51%
μ^2	0.044675646	0.034652176	0.02502121

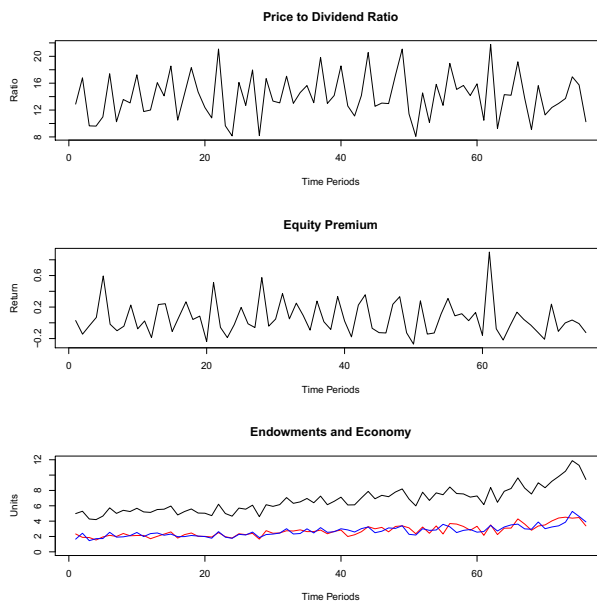Figure 5.4: The first graph plots the price to dividend ratio at each time period for one of the random paths from the scenario $\beta = .95$. The second graph plots the equity premium for the same random path. Finally, the third graph includes the agent endowments and total size of the economy. The red line represents the first agent $\gamma^1 = 2$, and the blue line represents the second agent with $\gamma^2 = 5$.

The second graph plots the equity premium $(R_s - R_f)$. This specific random path provides a relatively high average equity premium of 5.89%. The average returns for the stock and bond were 15.37% and 9.47%, respectively. The final graph in Figure 5.4 presents the idiosyncratic shocks (agent 1 represented by the red line and agent2 represented by the blue line) and total size of the economy $(D_t + \sum e_t^i)$ in consumption units at each time period.

5.5   Risk Aversion

The risk aversion parameter certainly has a direct effect on the returns of assets in the model. Increasing risk aversion in the model promotes demand for the risk-free asset, thus decreasing the return of the bond. Previous research has introduced extremely risk averse agents into the models to drive the bond return lower and increase the equity premium. This section will consider the effect that the risk aversion parameter has on the model. Results of multiple simulations with different risk aversion parameters will be presented.

The baseline calibrations defined earlier for β and the dividend process are assumed unless noted otherwise. The idiosyncratic shock will remain consistent for each of the simulations, applying the uniform distribution of $[1.5, 2.5]$ for equation (5.1). Table 5.4 presents the market statistics for three simulations. Each simulation is identified with the risk aversion parameters γ^1 and γ^2 assumed for the CRRA utility. There is not a specific set of risk aversion parameters that is widely accepted as calibrated to financial data. In a complete markets setting, Judd et al. (2003) compute a numerical example with $\gamma^1 = .5$ and $\gamma^2 = 4$. Benninga and Mayshar (1993) calibrates their model to empirical data with $\gamma^1 = 3$ and $\gamma^2 = 36$, which the latter has been argued to be improbable Mehra et al. (2003).

Considering the returns and equity premium, it seems that both increase with the level of risk aversion paramters. Although the bond return seems counterintuitive to economic theory, with two types of investors in the economy, and zero net supply, someone has to provide the other side of the trade. Therefore, its not only the level of the risk aversion, but also the spread between the two types of investors. The price to dividend ratio decreases as risk aversion is introduced in the model. The standard deviation of the price to dividend ratio, however, increases. The consumption growth is consistently higher for agent 1, the less risk averse agent. However, as risk aversion increases across the simulations, the consumption growth decreases for agent 1 and increases for agent 2.

Table 5.4: The market statistics of the model are displayed from three different simulations. Each simulation is identified by different risk aversion parameters noted as γ^1 and γ^2.

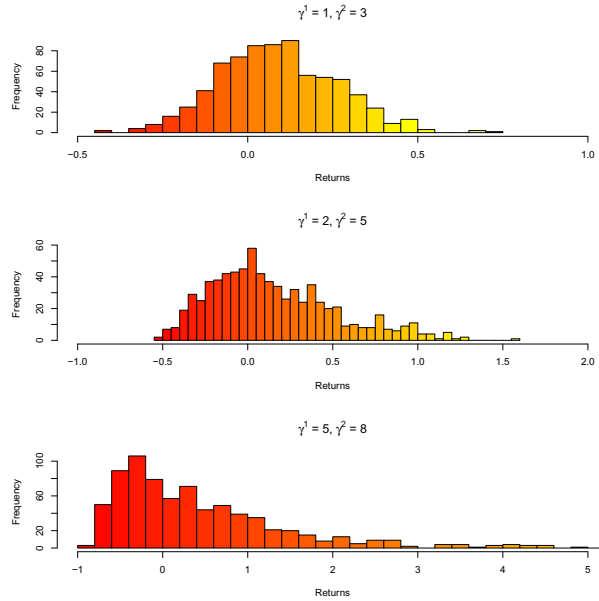| Statistic	$\gamma^1 = 1, \gamma^2 = 3$	$\gamma^1 = 2, \gamma^2 = 5$	$\gamma^1 = 5, \gamma^2 = 8$
$E_t[R_f]$	7.90%	11.49%	33.71%
$E_t[R_s]$	8.81%	14.80%	44.30%
σ_{R_s}	17.53%	37.01%	107.89%
$E_t[R_s - R_f]$	0.91%	3.30%	10.59%
$E_t[\frac{P_t}{D_t}]$	17.53	14.04	10.87
$\sigma_{\frac{P_t}{D_t}}$	2.00	3.09	5.65
$E_t[|s_{t+1}^i - s_t^i|]$	4.58	1.60	4.14
$\sigma_{|s_{t+1}^i - s_t^i|}$	12.45	4.08	23.46
$E_t[\Delta \log(c_t^1)]$	2.01%	1.94%	1.68%
$\sigma_{\Delta \log(c_t^1)}$	16.83%	16.63%	15.25%
$E_t[\Delta \log(c_t^2)]$	0.67%	0.78%	1.05%
$\sigma_{\Delta \log(c_t^2)}$	5.61%	6.65%	9.53%
$E_t[\Delta \log(D_t + \sum e_t^i)]$	1.43%	1.46%	1.41%
$\sigma_{\Delta \log(D_t + \sum e_t^i)}$	11.87%	12.38%	12.80%
μ^2	0.1058868	0.044675646	0.095153598

Figure 5.5: Histogram of the stock returns for three different simulations. Each simulation adjusts the risk aversion parameters across agents. The risk aversion parameters are labeled above each histogram.

The volume of equity trades and standard deviation does not show a consistent trend through risk aversion.

Figure 5.5 presents a histogram of the equity returns for each of these scenarios. The increasing standard deviation of these returns is apparent when considering the range of returns along the horizontal axis. The distribution of returns definitely adjusts as agent risk aversion increases. The scenario with the highest degree of risk aversion ($\gamma^1 = 5, \gamma^2 = 8$) exhibits a positively skewed distribution with returns extending much further in the positive tail. Although these simulations don't represent empirical returns, it is important to note the consequences of assumed parameters.

Three additional scenarios with different risk aversion parameters are considered in Table 5.5. For these specific scenarios the uniform distribution $[1.9, 2.1]$ for θ in equation (5.1) is assumed. This adjustment to $[a, b]$ is consistent with Table 5.1 that returns are relatively lower. This also minimizes the equity premium relative to the previous scenarios in Table 5.4. Increasing the investors risk aversion parameters increased the expected return of the risky asset and standard deviation. Consumption growth again is higher for the less risk averse agent, yet the difference of growth between the two agents is minimized as risk aversion increases. The volume of trades observed and planner's weight do not show consistent trends with adjusting γ.

Figure 5.6 presents the histogram of equity returns for the ten random paths. Each scenario identified by the risk aversion parameters in the title. The horizontal axis remains constant across all three scenarios to display the transformation of the distribution. The highest risk aversion parameters maintain a positively skewed distribution. However, the extreme positive returns are not as extraordinary as Figure 5.5, simply due to the idiosyncratic shock.

5.6   Variance of Growth in the Economy

The consumption growth in the U.S. was roughly 1.8% over the ninety year time frame referenced by Mehra and Prescott (1985). The simulations presented so far have assumed the same aggregate dividend process. At each time period there is a equal chance of a positive growth or negative growth state. Consistent with Benninga and Mayshar (1993), the positive shock has been 5% growth to the previously observed dividend and the negative shock decreases the dividend by 2%. This section will consider simulations of different variance in the aggregate dividend growth process.

Table 5.5: The market statistics of the model are displayed from three different simulations. Each simulation is identified by different risk aversion parameters noted as γ^1 and γ^2.

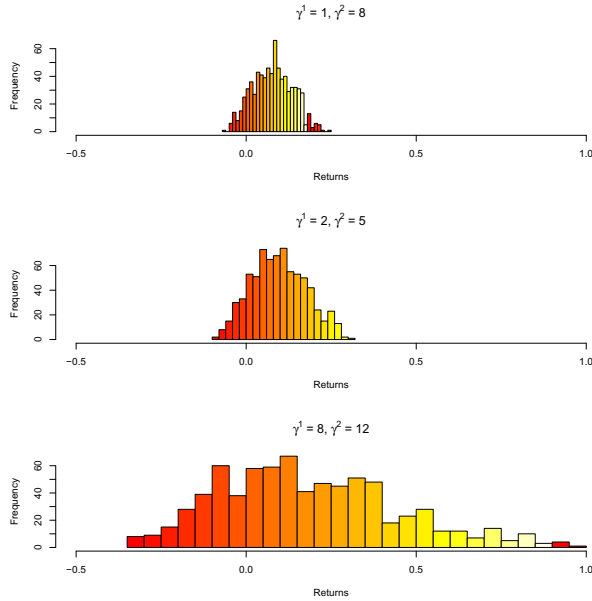| Statistic	$\gamma^1 = 1, \gamma^2 = 8$	$\gamma^1 = 2, \gamma^2 = 5$	$\gamma^1 = 8, \gamma^2 = 12$
$E_t[R_f]$	7.51%	8.84%	15.86%
$E_t[R_s]$	7.82%	9.64%	18.32%
σ_{R_s}	5.79%	8.01%	26.19%
$E_t[R_s - R_f]$	0.31%	0.80%	2.46%
$E_t[\frac{P_t}{D_t}]$	17.03	13.32	7.40
$\sigma_{\frac{P_t}{D_t}}$	.81	.67	1.16
$E_t[|s^i_{t+1} - s^i_t|]$	2.69	3.88	2.16
$\sigma_{|s^i_{t+1} - s^i_t|}$	9.50	8.51	7.15
$E_t[\Delta \log(c^1_t)]$	2.27%	2.01%	1.71%
$\sigma_{\Delta \log(c^1_t)}$	6.65%	5.73%	4.89%
$E_t[\Delta \log(c^2_t)]$	0.28%	0.80%	1.14%
$\sigma_{\Delta \log(c^2_t)}$	0.83%	2.29%	3.26%
$E_t[\Delta \log(D_t + \sum e^i_t)]$	1.47%	1.50%	1.46%
$\sigma_{\Delta \log(D_t + \sum e^i_t)}$	4.15%	4.26%	4.15%
μ^2	0.000412751	0.041568244	0.023722642

Figure 5.6: Histogram of the stock returns for three different simulations. Each simulation adjusts the risk aversion parameters across agents. The risk aversion parameters are labeled above each histogram.

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The calibration of this section will remain consistent with the baseline discount factors $\beta^1 = \beta^2 = 0.95$, and initial asset holdings $s_0^1 = s_0^2 = 0.5$, and $b_0^1 = b_0^2 = 0$. Agent 1 is assumed to be less risk averse than agent 2, with risk aversion parameters for CRRA utility $\gamma^1 = 2$ and $\gamma^2 = 5$. The idiosyncratic shock will follow equation (5.1), with the assumed uniform distribution $[1.5, 2.5]$ for each agent. The terminal period will be $T = 500$. The three different dividend growth assumptions will still maintain an expected growth path of 1.5%. The positive and negative growth rates will be adjusted to alter the volatility of the dividend path. The results from the different growth assumptions are highlighted in Table 5.6. Each scenario is identified by the assumed positive and negative growth rates g_u and g_d, respectively.

Table 5.6: The market statistics of the model are displayed from three different simulations. Each simulation is identified by the growth parameters of the dividend process noted as g_d and g_u.

Statistic	$g_d = -.01, g_u = .04$	$g_d = -.02, g_u = .05$	$g_d = -.03, g_u = .06$
$E_t[R_f]$	11.60%	11.49%	11.13%
$E_t[R_s]$	14.61%	14.80%	14.48%
σ_{R_s}	36.85%	37.01%	36.56%
$E_t[R_s - R_f]$	3.00%	3.30%	3.35%
$E_t[\frac{P_t}{D_t}]$	13.66	14.04	14.33
$\sigma_{\frac{P_t}{D_t}}$	3.12	3.09	3.12
$E_t[|s_{t+1}^i - s_t^i|]$	0.92	1.60	1.79
$\sigma_{|s_{t+1}^i - s_t^i|}$	1.83	4.08	3.33
$E_t[\Delta \log(c_t^1)]$	1.94%	1.94%	1.91%
$\sigma_{\Delta \log(c_t^1)}$	16.02%	16.63%	16.95%
$E_t[\Delta \log(c_t^2)]$	0.78%	0.78%	0.76%
$\sigma_{\Delta \log(c_t^2)}$	6.41%	6.65%	6.78%
$E_t[\Delta \log(D_t + \sum e_t^i)]$	1.45%	1.46%	1.44%
$\sigma_{\Delta \log(D_t + \sum e_t^i)}$	11.85%	12.38%	12.61%
μ^2	0.038092433	0.044675646	0.049891113

Increasing the variability in the possible dividends increases the risk premium. In the first scenario, a positive state revealed increases the current dividend by 4% over the previous, while the negative state decreases by 1%. With this dividend growth, the expected return of the equity is 14.61% and 11.6% for the return of the bond. An additional one percent is added and subtracted to the positive and negative growth rates, respectively. The average equity premium increases from 3.0% to 3.3% after this change in the dividend growth process. Increasing the magnitude
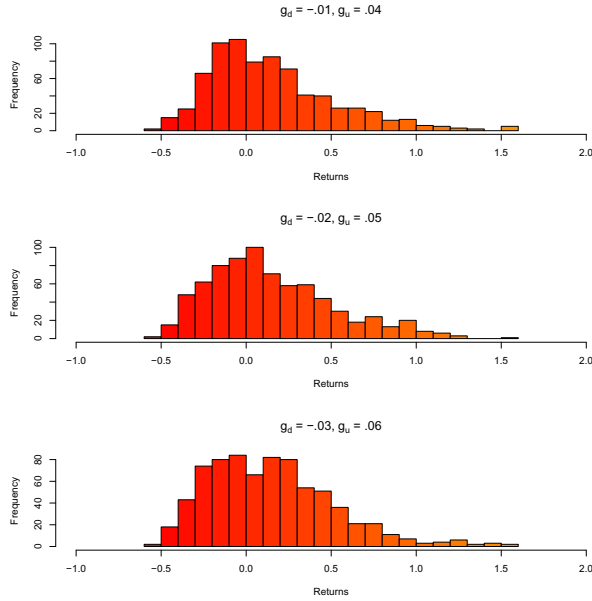
Figure 5.7: Histogram of the stock returns for three different simulations. Each simulation adjusts the growth rates for the dividend process. The growth assumptions are labeled above each histogram.

of the growth rate for each state by an additional one percent increases the equity premium to 3.35%. Throughout these three scenarios the average price to dividend ratio increases as variance in expected consumption growth increases. The average realized return on the risk-free bond also decreases as the difference between the positive growth rate and negative growth rate increases. The relatively more risk averse agent, agent 2, maintains consumption growth less than agent 1 for all scenarios. The standard deviation of consumption growth is also less for agent 2.

Figure 5.7 plots the distribution of the risky asset returns for each scenario. The scenarios are identified in each title by the negative and positive growth rates, g_d and g_u. The distributions look similar, with the tails approximately the same length.

5.7   Application to the Equity Premium Puzzle

This section considers an attempt to address real-world issues in financial economics. This exercise does not look to calibrate the model to exact returns observed, but to consider the potential of idiosyncratic shocks in a heterogeneous agent environment. This simulation includes two types of agents. The first agent is relatively less risk averse with $\gamma^1 = \frac{1}{4}$. This agent receives an idiosyncratic

shock each period $\theta^1 D_t$, where θ^1 is drawn from the uniform distribution $[1.3, 1.7]$. The second type of agent is much more risk averse, with $\gamma^2 = 8$. This type of agent receives a higher idiosyncratic shock of $[3.3, 3.7]$. Each agent has the same discount factor $\beta^1 = \beta^2 = 0.98$. Initial stock holdings $(s_0^1 = s_0^2 = 0.5)$ and bond holdings $(b_0^1 = b_0^2 = 0)$ are identical. The dividend grows 7% in a positive state and declines 4% in a negative state $(g_u = .07$ and $g_d = -.04)$. The finite period is $T = 500$. There are ten random paths over 75 periods. Table 5.7 presents the results for all ten paths and for one specific path. The distribution of returns is plotted in Figure 5.8.

The average equity premium of 2.53% is not equivalent to the 6% found historically. The average bond return is almost 2% higher. The standard deviation of returns is similar to Benninga and Mayshar (1993), slightly higher than empirical findings. One specific path approached a 4.8% equity premium. The distribution of equity returns is moderately skewed at 0.76. The returns distribution of an average random path is considered leptokurtic, with kurtosis of 5.05.

The paramaters and model setup that can obtain the exact historical results described in Mehra and Prescott (1985) are irrelevant. After the recent recession and "financial crisis", it is unlikely that the historical calculations would be identical to those calculated two decades earlier. However, certain financial questions are important to consider in order to progress the profession. The equity premium puzzle is just one of those anomolies. This chapter has considered four components of a DSGE model that can be calibrated and adjusted to research these specific questions.

Table 5.7: The market statistics of the model are displayed for a specific calibration. The All Paths represent the findings from all ten paths. The Specifc Path identifies just one random path from the simulation.

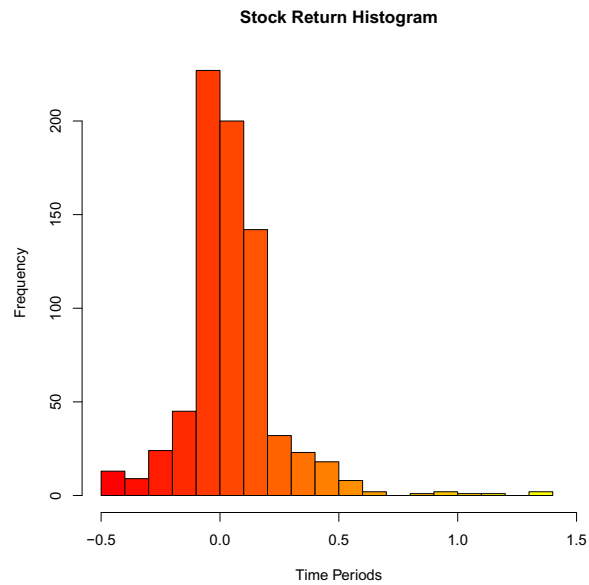| Statistic	All Paths	Specific Path
$E_t[R_f]$	2.74%	2.84%
$E_t[R_s]$	5.27%	7.68%
σ_{R_s}	18.10%	31.12%
$E_t[R_s - R_f]$	2.53%	4.85%
$E_t[\frac{P_t}{D_t}]$	76.60	57.27
$\sigma_{\frac{P_t}{D_t}}$	14.79	16.64
$E_t[|s^i_{t+1} - s^i_t|]$	2.84	7.36
$\sigma_{|s^i_{t+1}-s^i_t|}$	11.44	31.10
$E_t[\Delta \log(c^1_t)]$	7.17%	5.98%
$\sigma_{\Delta \log(c^1_t)}$	80.38%	155.06%
$E_t[\Delta \log(c^2_t)]$	0.22%	0.19%
$\sigma_{\Delta \log(c^2_t)}$	2.51%	4.85%
$E_t[\Delta \log(D_t + \sum e^i_t)]$	1.38%	0.72%
$\sigma_{\Delta \log(D_t+\sum e^i_t)}$	6.66%	7.14%
μ^2	0.000000255	0.000000255

Figure 5.8: Histogram of the stock returns for a specific example. The distribution of equity returns is moderately skewed at 0.76 and leptokurtic with a random path average kurtosis of 5.05.

5.8   Conclusion

This chapter was designed to reflect on the ability of a dynamic stochastic general equilibrium (DSGE) model to supply greater intuition into financial economic questions. Ideally this model would be capable of adapting to different situations, providing greater insight into policy and its effect on investors and financial markets. This specific DSGE model considered contains finte lived heterogeneous agents, in a complete markets setting, with non-stationary economic growth. The well known equity premium, where historically stocks have rewarded investors from a risk perspective much higher than risk free securities, will be evaluated with this finite DSGE model. Mehra and Prescott (1985) highlighted this anomoly and the poor performance that general equilibrium models have had providing explanations.

The goal of this chapter was not to calibrate the model to obtain the exact numerical equity premium reported by Mehra and Prescott (1985). This was an exercise that Benninga and Mayshar (1993) considered. Although they calibrated a DSGE model to duplicate identical asset returns, the result included parameters found to be implausable to the profession. Therefore, the objective of this chapter was to adjust the levers (parameters) of the model, and understand the response of investors decisions and financial markets. Could reasonable assumptions of this DSGE model provide similar dynamics observed in financial markets?

Basically there are four levers that could be adjusted through the initialization or setup of the model. Idiosyncratic shocks in this model added an additional assumption relative to the DSGE model of Benninga and Mayshar (1993). This additional process alone induced great differences in outcomes between the two models. First, a pure exchange economy without idiosyncratic shocks offers as many different equity returns as possible aggregate states. For example, with a positive or negative aggregate shock, the stock returns distribution is the degenerate distribution found in Figure 5.1. In that simulation a positive shock returned roughly 14% return, while the the negative shock observed roughly a 5% return. With the addition of idiosyncratic shocks, the equity returns distribution transforms to multiple observations. Many simulations exhibit a positively skewed distribution of equity returns.

The second contribution idiosyncratic shocks make in a DSGE model is trading volume. Table 5.1 presents trading volume with and without endowment or income shocks. Without any idiosyncratic risk there is very little trading. As the aggregate shocks are revealed through each time

period, the wealth distribution adjusts slighlty, creating a different representative agent (different γ^R) for a Pareto allocation. Trade of financial assets is required to obtain this equilibrium. This result is fundamentally the findings of Benninga and Mayshar (1993). Introducing idiosyncratic shocks completely alters the minimal trade outcome, as observed trading volume can typically exceed the total supply. As these models are calibrated to specific periods, such as quarterly or annual observations, these results of increased trading and higher volatilities might be consistent with empirical findings.

The main issue with agent based general equilibrium models and the equity premium puzzle is the risk free return. The results of most DSGE models consider the return on the bond too high relative to historical returns. The discount factor β is a parameter that has a direct relationship with bond prices. Table 5.3 confirms that the model with patient investors lowers both the bond and equity returns. Certain calibrated models have gone as far as assuming $\beta > 1$ in exchange for a lower return on the risk free asset. However, economic theory would suggest this is an improbable value to represent investor behavior.

Heterogeneity is the main reason for trade to take place in a DSGE model. Another parameter that can be adjusted at initialization of the model to alter market outcomes is γ, the CRRA parameter for risk aversion. Table 5.4 displays the trade offs of inducing higher risk aversion. The increased risk aversion across agents increases the equity premium. However, the cost to the model is increasing both equity and bond returns to extreme levels. This is another parameter that some research has questionably assumed in order to calibrate modeled results consistent with empirical findings. A $\gamma > 30$ with CRRA utility is considered so risk averse, that it cannot be identified with a typical investor.

Finally, the dividend growth process is a fourth control to the model that can be modified to change outcomes. Table 5.6 displays the results from three simulations with different dividend growth rates. The average price to dividend ratio increased as the variance of the dividend process increased. The equity premium also increased, and more importantly, did so with a lower return on the risk free asset. This exogenous dividend growth process drives consumption growth. This process leads to a non-stationary economy that separates this type of DSGE model from Judd et al. (2003).

The introduction of idiosyncratic shocks into a DSGE model is a necessity to get dynamics in price and trading. A discount factor within the range of $.95 \leq \beta < 1$ is consistent with economic theory depending on the time period of calibration. A more patient investor with a discount factor closer to one supports a lower risk free return. Heterogeneity is also an important component to obtaining desired dynamics from general equilibrium models. The exact distribution of γ for the CRRA utility is unkown, but typically an assumed $\gamma > 30$ is too risk averse. The specific dividend growth process in this DSGE model enables trading to occur with complete markets. Taking these results into account, an additional simulation was considered to address empirical results like the equity premium.

There are further extensions that could support further applications of this model. The T period algorithm described in chapter 3 can be extended to include multiple agents. Having additional heterogeneity in the model creates some computational complexity, however, I think the benefit of having multiple groups of agents will further the contribution.

A different utility function should be considered. As behavioral financial economics has emerged, habit formation or SP/A theory could help model some of the empirical results. Additionally, this model can be extended to include persistence of the aggregate or idiosyncratic shock.

APPENDIX A

EXAMPLE OF THE COMPUTER CODE

```
# Clean up memory
rm(list=ls())
graphics.off()

# Load required packages
require(combinat)

# Step 1: Initialize problem
T <- 500 # number of steps in lattice

# Agent parameters
N <- 2 # number of agents
gammas <- c(5,8) # risk aversions
b0 <- c(0,0) # initial bond holdings
s0 <- c(.5,.5) # initial stock holdings
betas <- c(.95,.95) # discount factors

# Market parameters
D0 <- 1 # initial dividend
gu <-  0.05 # dividend UP growth rate
gd <- -0.02 # dividend DOWN growth rate
probup <- 0.50 # probability of an UP state
S <- 1  # aggregate supply of stocks
B <- 0 # aggregate supply of bonds

# Initialize the probability and dividend lattices
# Note: R indices start at 1 rather than 0 so
# "today" or the zero node of the lattice will be
# element number 1 in the vectors and matrices and the T-th
# node will be elemet T+1.

Prob <- matrix(0,nrow=T+1,ncol=T+1)
Div <- Prob # Dividend lattice
Enda <- Prob # Endowment A
Endb <- Prob # Endowment B
Econ <- Prob # total size of economy
con1 <- Prob # agent 1 consumption
con2 <- Prob # agent 2 consumption
```

```
Ps1 <- Prob # Price of stock agent 1 consumption
Pb1 <- Prob # Price of bond agent 1 consupmtion
s1 <- Prob # agent 1 stock purchases (at time t)
s2 <- Prob # agent 2 stock purchases (at time t)
b1 <- Prob # agent 1 bond purchases (at time t)
b2 <- Prob # agent 2 bond purchases (at time t)

for (t in 0:T){
for (j in 0:t){
Prob[t+1,j+1] = nCm(t,j)*(probup^j)*((1-probup)^(t-j))
Div[t+1,j+1] = D0*((1+gu)^(j))*((1+gd)^(t-j))
x1 <- runif(1,1.5,2.5)
x2 <- runif(1,1.5,2.5)
Enda[t+1,j+1] = x1*Div[t+1,j+1]
Endb[t+1,j+1] = x2*Div[t+1,j+1]
Econ[t+1,j+1] = Enda[t+1,j+1]+Endb[t+1,j+1]+Div[t+1,j+1]
  }
   }


# Step 2: Let agent one pareto weight = 1 and guess pareto weight 2.
# Solve consumption
mus <- c(1.0,.09515355) # pareto weight agent 1, agent2
iter <- 10 # number of max iterations
tol <- 1e-12 # accepted tolerance of difference
step <- 1 # step size of mu guess
i = 1 # initialize iteration
diff <- array(0,iter) # initialize diff size
mu <- array(0,iter) # initialize mu guess size
mumax <- .09515364 # max guess for 2nd mu
mumin <- 0 # min guess for 2nd mu
diffs <- 100

while (i <= iter & abs(diffs) > tol) {

for (t in 0:T){
for (j in 0:t){
f <- function (c) ((c^(-gammas[1]))*mus[2]-
((Econ[t+1,j+1]-c)^(-gammas[2])))^2
con1[t+1,j+1] = as.numeric(optimize(f,c(0,Econ[t+1,j+1]),
tol=10^(-12))[1])
  }
  }
con2 <- Econ - con1 # Agent 2 consumption based on guess of mu

# Step 3: Check final period (stock and bond at T+1 = 0)
```

```
t = T
r = t-1
for (j in 0:r){
A <- matrix(1,nrow=2,ncol=2) #Agent 1
A[1,1] <- Div[t+1,j+2]
A[2,1] <- Div[t+1,j+1]
b <- matrix(0,nrow=2,ncol=1)
b[1,1] <- con1[t+1,j+2]-Enda[t+1,j+2]
b[2,1] <- con1[t+1,j+1]-Enda[t+1,j+1]
x <- matrix(0,nrow=2,ncol=1)
x <- solve(A)%*%b
s1[t,j+1] <- x[1,1]
b1[t,j+1] <- x[2,1]

C <- matrix(1,nrow=2,ncol=2) #Agent 2
C[1,1] <- Div[t+1,j+2]
C[2,1] <- Div[t+1,j+1]
d <- matrix(0,nrow=2,ncol=1)
d[1,1] <- con2[t+1,j+2]-Endb[t+1,j+2]
d[2,1] <- con2[t+1,j+1]-Endb[t+1,j+1]
y <- matrix(0,nrow=2,ncol=1)
y <- solve(C)%*%d
s2[t,j+1] <- y[1,1]
b2[t,j+1] <- y[2,1]
}

# Step 4: Work backwards to get stock and bond holdings
R = T-1

for (t in R:1){
for (j in t:1){
# Agent 1 stock and bond holdings
Qupup <- betas[1]*probup*((con1[t+2,j+2]^(-gammas[1]))/
(con1[t+1,j+1]^(-gammas[1])))
Qupdown <- betas[1]*(1-probup)*((con1[t+2,j+1]^(-gammas[1]))/
(con1[t+1,j+1]^(-gammas[1])))
Qdownup <- betas[1]*probup*((con1[t+2,j+1]^(-gammas[1]))/
(con1[t+1,j]^(-gammas[1])))
Qdowndown <- betas[1]*(1-probup)*((con1[t+2,j]^(-gammas[1]))/
(con1[t+1,j]^(-gammas[1])))
Psu <- Qupup*(Div[t+2,j+2]+Ps1[t+2,j+2])
+Qupdown*(Div[t+2,j+1]+Ps1[t+2,j+1])
Psd <- Qdownup*(Div[t+2,j+1]+Ps1[t+2,j+1])
+Qdowndown*(Div[t+2,j]+Ps1[t+2,j])
Pbu <- Qupup+Qupdown
```

```
Pbd <- Qdownup+Qdowndown
Ps1[t+1,j+1] <- Psu
Ps1[t+1,j] <- Psd
Pb1[t+1,j+1] <- Pbu
Pb1[t+1,j] <- Pbd
F <- matrix(0,nrow=2,ncol=1)
F[1,1] <- con1[t+1,j+1]+Ps1[t+1,j+1]*s1[t+1,j+1]
+Pb1[t+1,j+1]*b1[t+1,j+1]-Enda[t+1,j+1]
F[2,1] <- con1[t+1,j]+Ps1[t+1,j]*s1[t+1,j]
+Pb1[t+1,j]*b1[t+1,j]-Enda[t+1,j]
E <- matrix(1,nrow=2,ncol=2)
E[1,1] <- Ps1[t+1,j+1]+Div[t+1,j+1]
E[2,1] <- Ps1[t+1,j]+Div[t+1,j]
z <- matrix(0,nrow=2,ncol=1)
z <- solve(E)%*%F
s1[t,j] <- z[1,1]
b1[t,j] <- z[2,1]

# Agent 2 stock and bond holdings
FF <- matrix(0,nrow=2,ncol=1)
FF[1,1] <- con2[t+1,j+1]+Ps1[t+1,j+1]*s2[t+1,j+1]
+Pb1[t+1,j+1]*b2[t+1,j+1]-Endb[t+1,j+1]
FF[2,1] <- con2[t+1,j]+Ps1[t+1,j]*s2[t+1,j]
+Pb1[t+1,j]*b2[t+1,j]-Endb[t+1,j]
EE <- matrix(1,nrow=2,ncol=2)
EE[1,1] <- Ps1[t+1,j+1]+Div[t+1,j+1]
EE[2,1] <- Ps1[t+1,j]+Div[t+1,j]
zz <- matrix(0,nrow=2,ncol=1)
zz <- solve(EE)%*%FF
s2[t,j] <- zz[1,1]
b2[t,j] <- zz[2,1]
  }
    }

# Step 5: Check agent 2 initial holdings given the correct b0:
Qup2 <- betas[1]*probup*((con1[2,2]^(-gammas[1]))/
(con1[1,1]^(-gammas[1])))
Qdown2 <- betas[1]*(1-probup)*((con1[2,1]^(-gammas[1]))/
(con1[1,1]^(-gammas[1])))
Ps1[1,1] <- Qup2*(Ps1[2,2]+Div[2,2])+Qdown2*(Ps1[2,1]+Div[2,1])
Pb1[1,1] <- Qup2+Qdown2
s20 <- (con2[1,1]+Ps1[1,1]*s2[1,1]+Pb1[1,1]*b2[1,1]
-Endb[1]-b0[2])/(Ps1[1,1]+Div[1,1])
diffs <- s20 - s0[2]
```

```r
mu[i] <- mus[2]
diff[i] <- diffs

if (i == 1 & abs(diffs) > tol) {
mumin <- mu[i]
mus[2] <- mumax
i = i+1
}

if (i > 1 & abs(diffs) > tol) {
if (diff[i] < 0) {
mumax <- mu[i]
mus[2] <- (mumax+mumin)/2
i = i+1
}
if (diff[i] > 0) {
mumin <- mu[i]
mus[2] <- (mumax+mumin)/2
i = i+1
}
}
}
if (i > iter){
i <- iter
}

if (diff[i] < tol) {
print("Code successfully converged")
}

if (diff[iter] > tol){
print("Code did not converge - Increase mumax or iter")
}

# Once code has converged calculate alphas and deltas:

# Solve wealth for agent 1 and agent 2

w1 <- matrix(0,nrow=T+1,ncol=T+1)
w2 <- w1
d1 <- w1
d2 <- w1
a1 <- w1
a2 <- w1
```

```
# Initial Wealths and Alphas:
w1[1,1] <- (Ps1[1,1]+Div[1,1])*s0[1]+b0[1]+Enda[1,1]
w2[1,1] <- (Ps1[1,1]+Div[1,1])*s0[2]+b0[2]+Endb[1,1]
a1[1,1] <- (Ps1[1,1]*s1[1,1])/(w1[1,1]-con1[1,1])
a2[1,1] <- (Ps1[1,1]*s2[1,1])/(w2[1,1]-con2[1,1])
d1[1,1] <- con1[1,1]/w1[1,1]
d2[1,1] <- con2[1,1]/w2[1,1]

for (t in 1:T){
for (j in 1:t){
w1[t+1,j+1] = (Ps1[t+1,j+1]+Div[t+1,j+1])*s1[t,j]+b1[t,j]+Enda[t+1,j+1]
w2[t+1,j+1] = (Ps1[t+1,j+1]+Div[t+1,j+1])*s2[t,j]+b2[t,j]+Endb[t+1,j+1]
a1[t+1,j+1] <- (Ps1[t+1,j+1]*s1[t+1,j+1])/(w1[t+1,j+1]-con1[t+1,j+1])
a2[t+1,j+1] <- (Ps1[t+1,j+1]*s2[t+1,j+1])/(w2[t+1,j+1]-con2[t+1,j+1])
d1[t+1,j+1] <- con1[t+1,j+1]/w1[t+1,j+1]
d2[t+1,j+1] <- con2[t+1,j+1]/w2[t+1,j+1]
        }
   }

for (t in 1:T){
w1[t+1,1] = (Ps1[t+1,1]+Div[t+1,1])*s1[t,1]+b1[t,1]+Enda[t+1,1]
w2[t+1,1] = (Ps1[t+1,1]+Div[t+1,1])*s2[t,1]+b2[t,1]+Endb[t+1,1]
a1[t+1,1] <- (Ps1[t+1,1]*s1[t+1,1])/(w1[t+1,1]-con1[t+1,1])
a2[t+1,1] <- (Ps1[t+1,1]*s2[t+1,1])/(w2[t+1,1]-con2[t+1,1])
d1[t+1,1] <- con1[t+1,1]/w1[t+1,1]
d2[t+1,1] <- con2[t+1,1]/w2[t+1,1]
  }
```

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BIOGRAPHICAL SKETCH

Aaron Joseph Schmerbeck

Aaron Schmerbeck, son of Joseph and Beverly Schmerbeck, was born on the twentieth day of April, 1982, in Rochester, New York. After graduating from McQuaid Jesuit High School in Rochester, Aaron enrolled at the State University of New York at Fredonia. After two years of coursework, he transferred to St. John Fisher College in Rochester, NY. He graduated summa cum laude with a double major in Economics and Mathematics. In 2004, Aaron accepted the Quinn Assistantship at the Florida State University for study in Economics. In December 2007, Aaron was awarded a Master of Science degree in Economics. Aaron received his Ph.D. in the spring of 2014 for his research on heterogeneous agents in a dynamic, stochastic, general equilibrium model.

In the fall of 2008, Aaron accepted an Economist position with the multifamily research and model development team at Freddie Mac, in McLean, VA. There he helped produce and implement models that forecasted the performance, and evaluated risks of multifamily mortgages. In the spring of 2010, Aaron accepted a faculty position with the University of West Florida, in Pensacola, FL. Since then he has taught classes in Principles of Microeconomics and Managerial Finance. He currently works there as a Research Economist.

Aaron currently resides in Navarre, Florida. After completion of his studies, he would like to further his academic career or consider a change to the non academic finance sector. In his spare time, Aaron enjoys spedning time with his wife and two children, traveling, and enjoying the sunny beaches of Florida.