Stochastic Ising Models at Zero Temperature on Various Graphs

by

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Abstract

In this thesis we study continuous time Markov processes whose state space consists of an assignment of $+1$ or -1 to each vertex x of a graph G. We will consider two processes, $\sigma(t)$ and $\sigma'(t)$, having similar update rules. The process $\sigma(t)$ starts from an initial spin configuration chosen from a Bernoulli product measure with density θ of $+1$ spins, and updates the spin at each vertex, $\sigma_x(t)$, by taking the value of a majority of x's nearest neighbors or else tossing a fair coin in case of a tie. The process $\sigma'(t)$ starts from an arbitrary initial configuration and evolves according to the same rules as $\sigma(t)$, except for some vertices which are frozen plus (resp., minus) with density ρ^+ (resp., ρ^-) and whose value is not allowed to change. Our results are for when $\sigma(t)$ evolves on graphs related to homogeneous trees of degree $K \geq 3$, such as finite or infinite stacks of such trees, while the process $\sigma'(t)$ evolves on $\mathbb{Z}^d, d \geq 2$. We study the long time behavior of these processes and, in the case of $\sigma'(t)$, the prevalence of vertices that are (eventually) fixed plus or fixed minus or flippers (changing forever). We prove that, if θ is close enough to 1, $\sigma(t)$ reaches fixation to $+1$ consensus. For $\sigma'(t)$ we prove that, if $\rho^+ > 0$ and $\rho^- = 0$, all vertices end up as fixed plus, while for $\rho^+ > 0$ and ρ^- very small (compared to ρ^+), the fixed minus and flippers together do not percolate.

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Introduction

In this work we study the long term behavior of continuous time Markov processes whose states assign either $+1$ or -1 (usually called a spin value) to each vertex x in a graph G. The graphs G we consider are either the lattice $\mathbb{Z}^2$ and finite thickness slabs of $\mathbb{Z}^2$, or are related to homogeneous trees of degree K and include finite and infinite stacks of homogeneous trees. These graphs will be specified in Chapter 1. In this thesis we consider a number of loosely related models. We will discuss three types of processes, a much studied one denoted $\sigma(t)$, a generalized version of $\sigma(t)$ denoted $\sigma^*(t)$, and then a modified one, denoted $\sigma'(t)$, in which some vertices are "frozen" – i.e. their spin values are not allowed to change. The modified process $\sigma'(t)$ basically corresponds to $\sigma(t)$ in a random environment where randomly selected vertices are frozen from time zero, some plus and some minus.

In this chapter we introduce the process $\sigma(t)$ and known results. The processes $\sigma^*(t)$ and $\sigma'(t)$ will be introduced in Section 1.2 and 1.3.1, respectively. Letting $\sigma(t) = \sigma(t, \omega)$, where ω is an element of the probability space Ω which will be defined later, we denote by $\sigma_x(t)$ the value of the spin at vertex $x \in G$ at time $t \geq 0$. Starting from a random initial configuration $\sigma(0) = \{\sigma_x(0)\}_{x \in G}$ drawn from the independent Bernoulli product measure

$$\mu_\theta(\sigma_x(0) = +1) = \theta = 1 - \mu_\theta(\sigma_x(0) = -1), \tag{1}$$

the system then evolves in continuous time according to an agreement inducing dynamics: at rate 1, each vertex changes its value if it disagrees with more than half of its neighbors, and tosses a fair coin in the event of a tie. This process corresponds to the zero-temperature limit of Glauber dynamics for a stochastic Ising model with ferromagnetic nearest neighbor interactions and no external magnetic field (see, e.g., [9] or [13]), having Hamiltonian

$$\mathcal{H} = -\sum_{\{x,y\}:d(x,y)=1} J_{x,y}\sigma_x\sigma_y, \tag{2}$$

where $d(x, y)$ denotes graph distance (i.e., the minimum number of edges) between x and y. The continuous time dynamics is defined by means of independent, rate 1 Poisson processes (clocks) assigned at each vertex x. If the clock at vertex x rings at time t and the change in energy

$$\Delta\mathcal{H}_x(\sigma) = 2\sum_{y:d(x,y)=1} J_{x,y}\sigma_x\sigma_y \tag{3}$$

is negative (resp., positive), a spin flip is done with probability 1 (resp., 0). To resolve the case of ties when $\Delta\mathcal{H}_x(\sigma) = 0$, each clock ring is associated to a fair coin toss and a spin flip is done with probability 1/2. For an introduction to the relation between our model and the Ising model with Glauber dynamics see Section 0.1.

Let $\mathbb{P}_{\text{dyn}}$ be the probability measure for the realization of the dynamics (clock rings and tie-breaking coin tosses), and denote by $\mathbb{P}_\theta = \mu_\theta \times \mathbb{P}_{\text{dyn}}$ the joint probability measure on the space Ω of initial configurations $\sigma(0)$ and realizations of the dynamics; an element of Ω will be denoted ω. Another possible source of randomness is the choice of coupling between spins, $J_{x,y}$. The most frequently studied models are the homogeneous ferromagnet, where $J_{x,y} = +1$ for all pairs of nearest neighbors

$\{x, y\}$, and disordered models, where a realization $\mathcal{J}$ of the $J_{x,y}$'s is chosen from an independent product measure $P_{\mathcal{J}}$ of some probability measure on the real line. Here we only consider uncoupled systems, thus we let $J_{x,y} = +1$ for all x, y throughout.

This process has been studied extensively in the physical and mathematical literature – primarily on graphs such as the hyper-lattice $\mathbb{Z}^d$ and the homogeneous tree of degree K, $\mathbb{T}_K$. A physical motivation, which corresponds to the symmetric initial spin configuration, is the behavior of a magnetic system following the a deep quench. A deep quench is when a system that has reached equilibrium at an initial high temperature T_1 is instantaneously subjected to a very low temperature T_2. Here we take $T_1 = \infty$ and $T_2 = 0$. For references on this and related problems see, e.g., [9] or [13]. The main focus in the study of this model is the formation and evolution of boundaries delimiting same spin cluster domains: these domains shrink or grow or split or coalesce as their boundaries evolve. This model is often referred to as a model of *domain coarsening*.

An interesting question is whether the system has a limiting configuration, or equivalently does every vertex eventually stop flipping? Whether

$$\lim_{t \to \infty} \sigma_x(t) \tag{4}$$

exists for almost every initial configuration, realization of the dynamics and for all $x \in G$ depends on the initial configuration and on the structure of the underlying graph G. We will refer to the existence of the limit (4) at a vertex x as **fixation** at x.

Nanda, Newman and Stein [9] investigated this question when $G = \mathbb{Z}^2$ and $\theta = \frac{1}{2}$

and found that in this case the limit does not exist, i.e., every vertex flips forever. Their work extended an old result of Arratia [1], who showed the same on $\mathbb{Z}$ for $\theta \neq 0$ or 1. It is an open problem to determine what happens for $d \geq 3$ when $\theta = \frac{1}{2}$. One important consequence of the methods of [9] is that $\sigma_x(\infty)$ does exist for almost every initial configuration, realization of the dynamics and every $x \in G$ if the graph is such that every vertex has an odd number of neighbors, such as for example $\mathbb{T}_K$ for K odd.

Another question of interest is whether sufficient bias in the initial configuration leads the system to reach consensus in the limit. I.e., does there exist $\theta_* \in (0,1)$, such that for $\theta \geq \theta_*$,

$$\forall x \in G, \mathbb{P}_\theta(\exists T = T(\sigma(0), \omega, x) < \infty \text{ so that } \sigma_x(t) = +1 \text{ for } t \geq T) = 1. \tag{5}$$

We will refer to (5) as **fixation to consensus** (of $+1$ or -1). Kanoria and Montanari [12] studied fixation to consensus on homogeneous trees of degree $K \geq 3$ for a process with synchronous time dynamics. Their process has the same update rules as ours, except that all vertices update simultaneously and at integer times $t \in \mathbb{N}$. For each K, Kanoria and Montanari defined the consensus threshold $\rho_*(K)$ to be the smallest bias in $\rho = 2\theta - 1$ such that the dynamics converges to the all $+1$ configuration, and proved upper and lower bounds for ρ_* as a function of K. Fixation to consensus was also investigated on $\mathbb{Z}^d$ for the asynchronous dynamics model. It was conjectured by Liggett [14] that fixation to consensus holds for all $\theta > \frac{1}{2}$. Fontes, Schonmann and Sidoravicius [7] showed this for all $d \geq 2$ and θ_* strictly less but very close to 1.

In [11] Howard investigated the Ising spin dynamics in detail on $\mathbb{T}_3$ and showed how fixation takes place. On this tree graph, vertices fixate in spin chains (defined as doubly infinite paths of vertices of the same spin sign). Though no spin chains are present at time 0 when $\theta = 1/2$, Howard showed that for any $\epsilon > 0$, there are (almost surely) infinitely many distinct $+1$ and -1 spin chains at time ϵ. He also showed the existence of a phase transition in θ: there exists a critical $\theta_c \in (0, \frac{1}{2})$ such that if $\theta < \theta_c$, $+1$ spin chains do not form almost surely, whereas if $\theta > \theta_c$ they almost surely form in finite time.

In [3] and [4], Damron, Kogan, Newman and Sidoravicius studied coarsening started from an unbiased initial configuration on finite width slabs of the form $\mathbb{Z}^2 \times \{0, \ldots, k-1\}$ with free and periodic boundary conditions and $k \geq 2$. Their work was motivated by the long standing open question to determine whether or not there are vertices that fixate for $d \geq 3$, and for which values of d. It has been implied by Spirin, Krapivsky and Redner in a computational physics paper [17] that some vertices do indeed fixate. The results of [3] and [4] on the slabs highlight the differences in long term behavior between $\mathbb{Z}^2$ and what is believed to occur on $\mathbb{Z}^3$ based on numerical results – see [10]. The authors showed that if $k = 2$ the system fixates with both free and periodic boundary conditions; if $k = 3$ with periodic boundary conditions the system also fixates; for all $k \geq 4$ with periodic boundary conditions some vertices fixate for large times and some do not, and the same holds for all $k \geq 3$ with free boundary conditions. I.e. for all $k \geq 3$ with free boundary conditions and for all $k \geq 4$ with periodic boundary conditions, with positive probability, there exist vertices which change spin sign forever; we call these vertices **flippers**. One interesting question, which remains open, is whether, for a slab on which the system does not fixate as $t \to \infty$, the set of flipping vertices percolates or alternatively consists of finite components surrounded by fixed vertices.

The main part of this thesis is motivated by the work of Howard, but for more general tree-related graphs. The other results and the definition of the σ' process are motivated by open questions which arose from [3] and [4]. One set of results, presented in Section 1.3.1, was done in collaboration with M. Damron, H. Kogan, C.M. Newman and V. Sidoravicius [6].

0.1 Zero-temperature Glauber dynamics

The Ising model is a model of magnetism in statistical mechanics. A magnetic system consists of a collection of particles, represented by the vertices of a graph $G = (V, E)$, which take one of two states, $+1$ and -1, also called spin values. The spin value of a particle at a vertex $x \in V$ corresponds to the magnetization of the particle and is denoted as σ_x. Pairs of nearby neighbors interact with each other, exerting influence on each others' spin values. The interactions between spins can be ferromagnetic, if spins favor matching values or antiferromagnetic, if spins favor opposite values, or of the spin glass type, if the interactions are both ferromagnetic and antiferromagnetic.

Common choices for G are the hyper-lattice $\mathbb{Z}^d$, the complete graph on N vertices, K_N, and the homogeneous tree of degree K, $\mathbb{T}_K$. The Ising model with no external magnetic field has the following probability measure P_β on the space of spin configurations $\sigma = \{\sigma_x\}_{x\in V}$:

$$P_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}, \tag{6}$$

where $\beta \geq 0$ is the inverse temperature, Z_β is a normalization constant and the Hamiltonian function $H(\sigma)$ is the energy of configuration σ,

$$\mathcal{H} = -\sum_{\{x,y\}:d(x,y)=1} J_{x,y}\sigma_x\sigma_y. \tag{7}$$

This measure, called a Gibbs distribution, assigns higher probability to lower energy spin configurations, especially for β large (low temperature). Spins interact with their nearest neighbors and this interaction, or coupling between x and y, is encoded in $J_{x,y}$. $J_{x,y}$ determines whether the model is a ferromagnet, an antiferromagnet or a spin glass. If $J_{x,y} \geq 0$ for all x, y, same spin configurations have lower energy and neighbors desire to be aligned; this is a ferromagnetic Ising model. If $J_{x,y} \leq 0$, neighboring spins desire to take opposite values; this is an Ising antiferromagnet. Typical choices for a ferromagnet are to take $J_{x,y} = 1$ for all x, y, as we have done in this thesis, or to draw the $J_{x,y}$'s independently from a probability distribution supported on $[0, \infty)$.

In discrete-time Glauber dynamics, spins may be selected one at a time uniformly at random from V. When $|V| \to \infty$, after a time rescaling, this is equivalent to assigning independent, rate one, Poisson clocks to the vertices of G, and updating the spin at vertex x when its clock rings. The spin update will depend on the local field at x, which we define as

$$Z_x^t = \sum_{y:d(x,y)=1} J_{x,y}\sigma_y(t). \tag{8}$$

If the clock at vertex x rings at time t, then the new spin value, $\sigma_x(t^+)$, is given by

$$Prob(\sigma_x(t^+) = +1) = \frac{\exp[\beta Z_x^t]}{\exp[\beta Z_x^t] + \exp[-\beta Z_x^t]}, \tag{9}$$

$$Prob(\sigma_x(t^+) = -1) = \frac{\exp[-\beta Z_x^t]}{\exp[\beta Z_x^t] + \exp[-\beta Z_x^t]}. \tag{10}$$

Note that the above probability is the conditional distribution with respect to P_β of a $+1$ (resp. -1) spin at x, conditioned on the current values of all the other spins. This choice of transition probabilities guarantees that P_β is an invariant distribution of the process. We note that there are other versions of Glauber dynamics which also make P_β invariant.

For a general reference on Glauber dynamics see [13]. By letting $\beta \to \infty$ (which is equivalent to letting the temperature tend to zero) we obtain the dynamics described in the Introduction: $Prob(\sigma_x(t^+) = \text{sgn}(Z_x^t)) \to 1$ for $Z_x^t \neq 0$ and $Prob(\sigma_x(t^+) = +1) = Prob(\sigma_x(t^+) = -1) = \frac{1}{2}$ for $Z_x^t = 0$. This model, with initial spin configuration drawn from an independent, Bernoulli product measure, has been studied in [2], [3], [4], [7], [9], [10], [11], on various graphs and for both biased and unbiased initial conditions. The unbiased initial configuration given in Equation (1) corresponds to the infinite temperature limit of the Gibbs measure, thus describing a system that is instantaneously quenched from infinite temperature to zero temperature.

Chapter 1

Statements of theorems

In Sections 1.1 and 1.2 of this chapter we state the results we obtained for the process $\sigma(t)$, and its generalization, the process $\sigma^*(t)$. In Section 1.3 we motivate the definition of the stochastic process $\sigma'(t)$, and we define this process in Subsection 1.3.1.

1.1 Results for tree-related graphs

We begin with some notation an definitions of our graphs. Let S^∞ denote a doubly infinite stack of homogeneous trees of degree $K \geq 3$, i.e., the graph with vertex set $\mathbb{T}_K \times \mathbb{Z}$ and edge set specified below. The theorems presented in this section are fixation to consensus results on S^∞, semi-infinite and finite width stacks of trees for the process $\sigma(t)$ started from initial configuration with probability measure given in Equation (1). We express S^∞ as

$$S^{\infty} = \bigcup_{i=-\infty}^{\infty} S_i, \tag{1.1}$$

where $S_i = \mathbb{T}_K \times \{i\} = \{(u, i) : u \in \mathbb{T}_K, i \in \mathbb{Z}\}$, and think of this as a decomposition of the infinite stack S^{∞} into layers S_i. Let the edge set of S^{∞}, $\mathbb{E}^{\infty}$, be such that any two vertices $x, y \in S^{\infty}$ are connected by an edge $e_{xy} \in \mathbb{E}^{\infty}$ if and only if:

i $x = (u_x, i), y = (v_y, i) \in S_i$ for some i, and the corresponding u_x and v_y are adjacent vertices in $\mathbb{T}_K$; or

ii $x = (u_x, i)$ for some i and $y = (u_x, i + 1)$; or

iii $x = (u_x, i)$ for some i and $y = (u_x, i - 1)$.

For a more formal description of the Markov process than the one given in the Introduction, we associate to each vertex $x \in S^{\infty}$ a rate 1 Poisson process whose arrival times we think of as a sequence of clock rings at x. We will denote the arrival times of these Poisson processes by $\{\tau_{x,n}\}_{n=1,2,\ldots}$ and take the Poisson processes associated to different vertices to be mutually independent. We associate to the (x, n)'s independent Bernoulli(1/2) random variables with values $+1$ or -1, which will represent the fair coin tosses in the event of a tie. The process $\sigma(t)$ is associated to the probability space $(\Omega, \mathbb{P}_{\theta})$ defined in the Introduction.

The main result on tree-related graphs is the following theorem, which shows fixation to consensus for nontrivial θ; its proof is given in Chapter 3. Unlike Kanoria and Montanari [12], here we do not attempt to obtain good lower bounds on θ_*, but we restrict ourselves to proving fixation to $+1$ for θ close enough to 1 with the standard majority update rule: when its clock rings, each vertex updates to agree with the majority of its neighbors or tosses a fair coin in the event of a tie.

Theorem 1.1. *Given $K \geq 3$, there exists $\theta_* < 1$ such that for $\theta > \theta_*$ the process on S^∞ fixates to consensus.*

The same fixation to consensus result holds for the following graphs, as stated in Theorem 1.2 below, whose proof is also given in Chapter 3:

- Homogeneous trees $\mathbb{T}_K$ of degree $K \geq 3$.
- Finite width stacks of homogeneous trees of degree $K \geq 3$ with free or periodic boundary conditions. These are graphs, which we will denote by S_f^l and S_p^l, with vertex set $\mathbb{T}_K \times \{0, 1, \dots, l-1\}$ and edge set $\mathbb{E}_f$ and $\mathbb{E}_p$. $\mathbb{E}_f$ and $\mathbb{E}_p$ are defined similarly to the edge set $\mathbb{E}^\infty$ of S^∞: two vertices $x, y \in S_f^l$ are connected by an edge $e_{xy} \in \mathbb{E}_f$ if and only if either condition i above holds; or

    1. $1 \leq i \leq l-2$ and either condition ii or iii holds; or
    2. $x = (u_x, 0)$ and $y = (u_x, 1)$; or
    3. $x = (u_x, l-1)$ and $y = (u_x, l-2)$.

    Any two vertices $x, y \in S_p^l$ are connected by an edge $e_{xy} \in \mathbb{E}_p$ if and only if either condition i holds; or

    4. $1 \leq i \leq l-2$ and either condition ii or iii holds; or
    5. $x = (u_x, 0)$ and $y = (u_x, 1)$ or $y = (u_x, l-1)$; or
    6. $x = (u_x, l-1)$ and $y = (u_x, l-2)$ or $y = (u_x, 0)$.

- Semi-infinite stacks of homogeneous trees of degree $K \geq 3$ with free boundary conditions. These are graphs, which we will denoted by S^{semi}, with vertex set $\mathbb{T}_K \times \{0, 1, \ldots\}$ and edge set $\mathbb{E}^{\text{semi}}$. Two vertices $x, y \in S^{\text{semi}}$ are connected by an edge $e_{xy} \in \mathbb{E}^{\text{semi}}$ if and only either condition i holds; or

  8. $1 \leq i$ and either condition ii or iii holds; or
  9. $x = (u_x, 0)$ and $y = (u_x, 1)$.

Theorem 1.2. *Fix $K \geq 3$ and $l \geq 2$ and let G be one of the following graphs: $\mathbb{T}_K$, S^l_f, S^l_p or S^{semi}. There exists $\theta_* < 1$ such that for $\theta > \theta_*$ the process on G fixates to consensus.*

Our results have natural extensions to other dynamics. In the next section we state precisely a few theorems about such extensions for the graph S^∞, but meanwhile we discuss them more informally here. Let N_0 be the maximum number of neighbors of a vertex in the graph G; for some $M_0 > \frac{N_0}{2}$, we can change (arbitrarily) the update rules for those vertices whose number of $+1$ neighbors is strictly less than M_0, and the conclusions of Theorem 1.1 or 1.2 remain valid with the same proof. For large N_0, M_0 can be taken much larger than $\frac{N_0}{2}$. A special case of this type of extension of our results is to modify the update rule in the event of a tie: e.g., instead of flipping a fair coin, flip a biased coin with any bias $p \in [0, 1]$ or do nothing. We can also change from two-valued spins to any fixed number q of spin values, say $1, 2, \ldots, q$. The initial configuration is given by the measure $\nu(x \text{ is assigned color } i \text{ at time } 0) = \epsilon_i$ where $i \in \{1, \ldots, q\}$ and $\sum_i \epsilon_i = 1$ and the updating is done via the majority rule. We can think of color 1 as the $+1$ spin from before, and the other $q - 1$ colors together representing the -1 spin. If ϵ_1 is close enough to 1, we again obtain fixation to $+1$ consensus. All our results also apply to the discrete time, synchronous dynamics of [12].

1.2 Results for tree-related graphs with modified dynamics

In this section we consider a continuous time Markov process, started at $t = 0$, whose states assign one of q colors, $\{1, 2, \ldots, q\}$, to each of the vertices of the infinite stack S^∞ of K-trees for $K \geq 3$. The number of neighbors of each vertex in this graph is $K + 2$. We denote the stochastic process by $\sigma^*(t)$, and will treat color 1 as distinguished. The initial spin configuration is drawn from an independent product measure ν with

$$\nu(\sigma^*_x(0) = i) = \epsilon_i, \tag{1.2}$$

where $i \in \{1, \ldots, q\}$ and $\sum_i \epsilon_i = 1$. We associate to the vertices $x \in S^\infty$ independent rate 1 Poisson processes, and think about the arrival times of these processes as clock rings; when the clock at x rings, the color at x will be updated while the colors at all other vertices do not change. The update transition probabilities at each vertex x are determined only by the colors of the neighbors of x and the vertex x itself. Let $N_x = \{x\} \cup \{y : e_{xy} \in \mathbb{E}^\infty\}$ for each vertex x. We define V^x to be the (local), $|\{1, \ldots, q\}^{N_x}| \times |\{1, \ldots q\}|$ update transition matrix at x, which gives the transition probabilities for the color of x given the colors of its neighbors and itself; $V^x_{\alpha,j}$ is the probability that the color of x is j after the update given that the color of N_x is α (i.e., $\sigma^*_y = \alpha_y$ for each $y \in N_x$) before the update.

The following theorem is our general result about consensus, restricted to the graph S^∞. Analogous results hold for the other graphs considered in Theorem 1.2. The main hypothesis of the theorem simply means that, when sufficiently many neighbors of any vertex x have color 1, then x must update to color 1; all other transition probabilities can be defined arbitrarily.

Theorem 1.3. *Consider the above type of process on S^∞ with $K \geq 3$. Define an integer M_0, which for $K \geq 5$ we set as $M_0 = K - 1$, and for $K = 3$ or 4 as $M_0 = K$. Suppose that the update transition probabilities $V^x_{\alpha,j}$ have the property that $V^x_{\alpha,1} = 1$ whenever $|\{y : y \in N_x \backslash \{x\}, \sigma^*_y = +1\}| \geq M_0$. Then there exists $\epsilon^* < 1$ (depending on K) such that for $\epsilon_1 > \epsilon^*$ (see Equation 1.2) the process fixates to (color 1) consensus.*

Proof. The proof of the theorem when $q = 2$ becomes straightforward when doing the following mapping from the colors 1 and 2 of σ^* to the spin values ± 1 of σ: map color 1 to $+1$ and color 2 to -1. In this case the proof follows from the same arguments used to prove Theorem 1.1; these arguments are presented in Chapter 3.

The case $q > 2$ follows from the same arguments used to prove the case $q = 2$, by projecting spin 1 to spin $+1$ and spin i to spin -1, for $i = 2, \ldots, q$. Even though the projected process is in general no longer a Markov chain, the proof still follows from the same arguments. □

1.3 Results for $\mathbb{Z}^d$

The graphs we consider in this section are either the d-dimensional lattice $\mathbb{Z}^d$ or two-dimensional slabs of width k (with free boundary conditions), which we denote as Slab_k; these are graphs with vertex set $\mathbb{Z}^2 \times \{0, 1, \ldots, k-1\}$ $(k \geq 2)$ and edge set $\mathcal{E}_k = \{\{x, y\} : \|x - y\| = 1\}$, where $\| \cdot \|$ denotes the Euclidean distance.

Damron, Kogan, Newman and Sidoravicius (see [3] and [4]) studied the long term behavior of the coarsening dynamics on Slab_k, and answered the question presented in Equation (4) (for periodic as well as free boundary conditions). Here we only consider k's for which the limit in Equation (4) does not always exist, namely $k \geq 3$ (we do not consider $k \geq 4$ with periodic boundary conditions). In this case the system has both vertices that fixate (in fact some vertices are already fixed starting at $t = 0$) and vertices that change spin forever (flippers).

An interesting open question (see Section 1 of [3]) is whether, for a fixed k, the set of flipper vertices percolates rather than forming only finite components surrounded by fixed sites. It is not known whether the set of flipper vertices percolates or not. But, motivated by this question, we give an artificial construction of an infinite set of flipper vertices as a subset of Slab_3 with free boundary conditions. It is artificial in that the choice of such a final state is done by hand. The fact that all the vertices of this set flip infinitely often follows from Arratia [1]. We studied the admissible shapes of such a set in an effort to rule out certain configurations.

An infinite flipping region may appear if the slab stabilizes in a checkerboard-like pattern of $+1$'s and -1's. Figure 1.1 shows a possible state of the system at a very large time T in a box of fixed size around the origin, by which time the system restricted to this box has reached its final state. For simplicity we only show the configuration of the middle layer, $\mathbb{Z}^2 \times \{1\}$, of Slab_3.

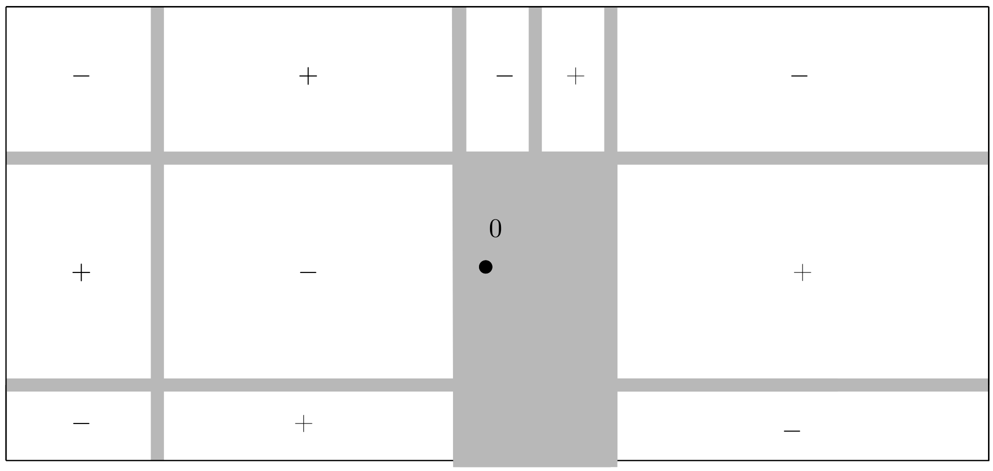

Figure 1.1: A configuration with an infinite subset of flipper vertices (gray) and fixed +1 and −1 vertices (white)

The gray area in Figure 1.1 represents the flipper vertices of this configuration and the white area represents the fixed vertices. The white rectangles labeled + and − represent two-dimensional pillars of same spin of the form

$$R = \{(x_1, x_2), \ldots, (x_1 + L, x_2)\} \times \{(x_1, x_2) \ldots (x_1, x_2 + M)\} \times \{0, 1, 2\}, \qquad (1.3)$$

for $(x_1, x_2) \in \mathbb{Z}^2$. The pillars with appropriate choices of (x_1, x_2) and of (L, M) depending on (x_1, x_2) form a distorted checkerboard pattern. These pillars, once formed, are stable (for $L, M \geq 1$) for all time with respect to the dynamics. One other condition we need to impose for the existence of the flipping region above is that, if $(x_1, x_2, 1)$ is a flipper, then $\sigma_{(x_1,x_2,2)} = +1$ and $\sigma_{(x_1,x_2,0)} = -1$. Notice that the flipping region can branch out and turn, and can have thickness greater than one.

We note that the only vertices that can be flippers on Slab_3 are $x \in \mathbb{Z} \times \{1\}$, since if $x \in \mathbb{Z} \times \{0\}$ or $x \in \mathbb{Z} \times \{2\}$, then x has only 5 neighbors and by Nanda, Newman, Stein [9] x fixates almost surely. This observation motivates the definition of a new model on $\mathbb{Z}^2$ with the usual coarsening dynamics but with some vertices whose spin is frozen for all time. If the initial configuration on Slab_3 is chosen according to a symmetric Bernoulli product measure, then by the Ergodic Theorem at time zero there are infinitely many pillar-like same-spin formations that are stable under the dynamics and are analogous to frozen vertices on $\mathbb{Z}^2$. A fairly general version of this new model will be presented in the following section on $\mathbb{Z}^d$ without referring to the motivating model on Slab_3.

1.3.1 $\mathbb{Z}^d$ with frozen vertices

In this subsection we define a new stochastic process on $\mathbb{Z}^d$, which we denote by $\sigma'(t)$. The initial configuration of this new process will be assigned as follows. Fix $\rho^+, \rho^- \geq 0$ with $\rho^+ + \rho^- \leq 1$ and pick three types of vertices (frozen plus, frozen minus and unfrozen) by i.i.d. choices with respective probabilities ρ^+, ρ^- and $1 - (\rho^+ + \rho^-)$. Once the frozen vertices have been chosen and assigned a spin value, the unfrozen vertices will be assigned spin values in one of two ways. In Theorem 1.4 below, they will be assigned according to an independent, identically distributed Bernoulli product measure, μ',

$$\mu'(\sigma'_x(0) = +1|x \text{ is not frozen }) = \theta^+, \tag{1.4}$$

$$\mu'(\sigma'_x(0) = -1|x \text{ is not frozen }) = \theta^- = 1 - \theta^+. \tag{1.5}$$

In Theorems 1.5 and 1.6, they will be assigned arbitrarily. (In other words, the theorems we prove will be valid for all choices of such spins values.) We will denote by $\mathbb{P} = \mathbb{P}_{\rho^+,\rho^-} \times \mathbb{P}'_{\text{dyn}}$ the overall probability measure where $\mathbb{P}_{\rho^+,\rho^-}$ is the distribution for the assignment of frozen plus, frozen minus and unfrozen vertices, and $\mathbb{P}'_{\text{dyn}}$ is the distribution of the following dynamics for $\sigma'(t)$. The continuous time dynamics is defined similarly to that of the $\sigma(t)$ process. Vertices are assigned independent, rate 1 Poisson clock processes and tie-breaking fair coins, and flip sign to agree with a majority of their neighbors; in the event of a tie the value is determined by tossing fair coins. Frozen vertices, however, never flip regardless of the configuration of their neighbors.

As usual, we are interested in the long term behavior of this model depending on the dimension d, the densities of frozen vertices, ρ^+, ρ^-, and the initial configuration of non-frozen vertices, which we denote by $\sigma'(0)$. Note that when $\rho^+ > 0$ and $\rho^- > 0$, almost surely there exist flipper vertices. To see this, consider the following configuration for the case $d = 2$, which has probability $(\rho^+)^2(\rho^-)^2$; the vertex labeled x in Figure 1.2 below has two frozen neighbors of spin $+1$ and two frozen neighbors of spin -1, and thus flips infinitely often. Similar flippers, as well as more complicated clusters of flippers, occur for any d.

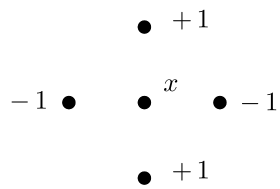

Figure 1.2: A vertex that flips infinitely often

This shows that $\sigma'_x(t)$ does not have a limit as $t \to \infty$ for some x, i.e., some vertices are flippers. When that is the case we are interested in such questions as whether the flipper vertices percolate. Of course, besides the flippers there are the collections of fixed plus and of fixed minus vertices and one may ask about the percolation of the other types of vertices or of the union of two of the three types. Theorem 1.6 below answers one such question.

The following theorems about fixation to consensus and the size of the flipping cluster were obtained in collaboration with M. Damron, H. Kogan, C.M. Newman and V. Sidoravicius [6]. They are stated in increasing order of difficulty of the proofs. The first two theorems are fixation to consensus results for the case of positive initial density of frozen +1's and zero initial density of frozen −1's. The third theorem is a more general result in which both $\rho^+, \rho^- > 0$, but we require ρ^- to be much smaller than ρ^+. For this more general case we obtain that the set of flippers together with (eventually) fixed vertices of spin −1 does not percolate. Complete proofs of the first two theorems will be given in Chapter 4, along with a sketch of the proof of the third theorem. A complete proof of the third theorem will be given in a paper in preparation.

Theorem 1.4. *Consider the stochastic process σ' on $\mathbb{Z}^2$, such that $\rho^+ > 0, \rho^- = 0$, and the initial configuration of non-frozen vertices is given by the Bernoulli product measure of Equations (1.4) with $\theta^+ = \theta^- = \frac{1}{2}$. Then the system fixates to $+1$ consensus.*

Theorem 1.5. *Consider the stochastic process σ' on $\mathbb{Z}^d$ for any d, with $\rho^+ > 0, \rho^- = 0$ and an **arbitrary** initial configuration of non-frozen vertices, $\sigma'(0)$. Then the system fixates to $+1$ consensus.*

Theorem 1.6. *Consider the stochastic process σ' on $\mathbb{Z}^d$ for any d with $\rho^+ > 0$ and $\rho^- > 0$ such that ρ^- is sufficiently small (depending on ρ^+ and d). For an **arbitrary** initial configuration of non-frozen vertices $\sigma'(0)$, the collection of (eventually) fixed -1's and flippers together do not percolate.*

Chapter 2

Preliminaries for tree-related graphs

In this chapter we state some definitions and show a proposition which will be an ingredient in the proof of Theorem 1.1. In order to prove Theorem 1.1 we will show that if we take θ close enough to 1, then already at time 0 there are stable structures of +1 vertices, which are fixed for all time. We will choose these structures to be subsets (denoted $\mathcal{T}_i$) of the layers S_i in the decomposition of S^∞ such that they are stable with respect to the dynamics. We will define a set $\mathcal{T}$ as the union of $\mathcal{T}_i$ for all i, and show that for θ close enough to 1, the complement of $\mathcal{T}$ is a union of almost surely finite components.

2.1 A set of fixed vertices in S^∞

Definition 2.1. *For i fixed, let $\mathcal{T}_i^{+,l}(t)$ be the union of all subgraphs H of S_i that are isomorphic to $\mathbb{T}_l$ with $\sigma_x(t) = +1, \forall x \in H$.*

We point out that $\mathcal{T}_i^{+,K-1}(t)$ is stable for $K \geq 5$, since every $x \in \mathcal{T}_i^{+,K-1}(t)$ has $K-1$ out of $K+2$ neighbors of spin $+1$ and $K - 1 > \frac{K+2}{2}$ for $K \geq 5$. Not only is this set stable with respect to the dynamics on S^∞ as in Theorem 1.1, but it's also stable with respect to the dynamics on S_i and the other graphs of Theorem 1.2. Let $\mathcal{T}$ represent the union of $\mathcal{T}_i^{+,K-1}(0)$ across all levels S_i, i.e.,

$$\mathcal{T} = \bigcup_{j=-\infty}^{\infty} \mathcal{T}_i, \tag{2.1}$$

where for shorthand notation, $\mathcal{T}_i = \mathcal{T}_i^{+,K-1}(0)$.

If $K \leq 4$, $\mathcal{T}$ as defined above is not stable with respect to the dynamics. In these cases the argument will be changed somewhat, as discussed in Chapter 3.

2.2 Asymmetric site percolation on $\mathbb{T}_K$

The goal of this subsection is to state and prove a geometric probability estimate, Proposition 2.1, which concerns asymmetric site percolation on $\mathbb{T}_K$ distributed according to the product measure μ_θ with:

$$\mu_\theta(\sigma_x = +1) = \theta = 1 - \mu_\theta(\sigma_x = -1), \forall x \in \mathbb{T}_K. \tag{2.2}$$

This equals the distribution of $\sigma(0, \omega)$ restricted to the layers S_i, and therefore applies to these graphs as well. The statement and proof of Proposition 2.1 require a series of definitions. The first of these defines graphical subsets of $\mathbb{T}_K$, whereas the second concerns probabilistic events for subgraphs of $\mathbb{T}_K$ that have a specific orientation. Later, in the proof of Theorem 1.1 which is given in Chapter 3, Proposition 2.1 will be applied to certain subsets of S_i.

Definition 2.2. Certain rooted subtrees of $\mathbb{T}_K$ *Let x, y in $\mathbb{T}_K$ be two adjacent vertices, and denote by $A_y[x]$ the connected component of x in $\mathbb{T}_K \backslash \{y\}$ – see Figure 2.1.*

Let x, y, z be three adjacent vertices in $\mathbb{T}_K$, such that x and z are neighbors of y. Denote by $A_{x,z}[y]$ the connected component of y in $\mathbb{T}_K \setminus \{x \cup z\}$ – see Figure 2.2.

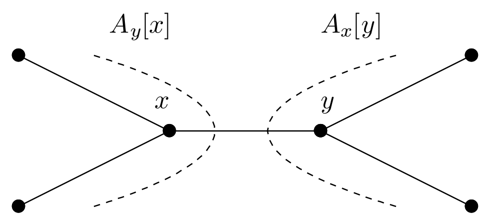

Figure 2.1: $A_y[x]$ and $A_x[y]$ are tree graphs whose roots have coordination number $K - 1$

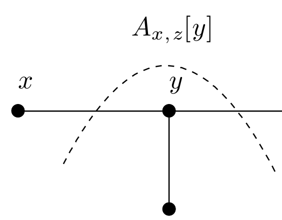

Figure 2.2: $A_{x,z}[y]$ is a tree graph whose root has coordination number $K - 2$

Definition 2.3. Random $(K-1)$-ary trees of spin $+1$ with a certain orientation

Let T be a deterministic subtree of $\mathbb{T}_K$ with at least two vertices, and v be a leaf of T; i.e., v has a neighbor v' in T and $(K-1)$ neighbors in $\mathbb{T}_K \setminus T$. $\mathrm{Tree}^+[v]$ *is the event that there is a subgraph H of $A_{v'}[v]$ isomorphic to $\mathbb{T}_{K-1}$ and containing v, such that $\sigma_u = +1, \forall u \in H$ – see Figure 2.3.*

*Let T be a deterministic subtree of $\mathbb{T}_K$ with at least five vertices, and v be a **2-point** of T (i.e., a vertex of T with exactly two neighbors in T) that is also **good** (i.e., both its neighbors in T are also 2-points of T). Let v', w be the two neighbors of v in T and let w' be w's other neighbor in T.* $\mathrm{Tree}^+[v,w]$ *is the event that there is a subgraph H of $A_{v',w}[v] \cup A_{v,w'}[w]$ isomorphic to $\mathbb{T}_{K-1}$ and containing v and w, such that $\sigma_u = +1, \forall u \in H$; here $A_{v',w}[v] \cup A_{v,w'}[w]$ is the graph with vertex set $\mathbb{V}_{A_{v',w}[v]} \cup \mathbb{V}_{A_{v,w'}[w]}$ and edge set $\mathbb{E}^{\infty}_{A_{v',w}[v]} \cup \mathbb{E}^{\infty}_{A_{v,w'}[w]} \cup e_{vw}$ – see Figure 2.4.*

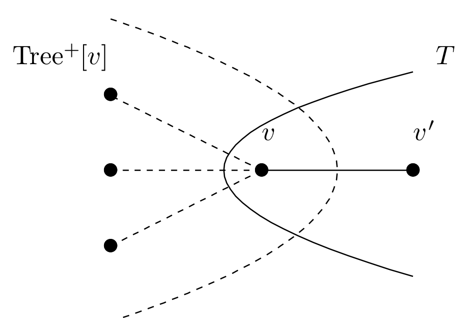

Figure 2.3: $\mathrm{Tree}^+[v]$ is a random $(K-1)$-ary tree of spin $+1$ that contains a leaf, v, of T

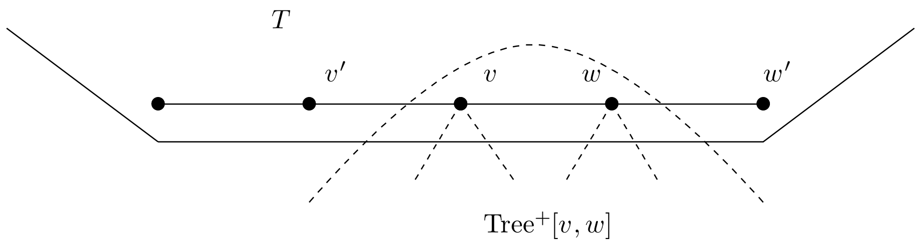

Figure 2.4: $\text{Tree}^+[v,w]$ is a random $(K-1)$-ary tree of spin $+1$ that contains a good two 2-point, v, of T, and one of its neighbors, w

For distinct leaves v of T, the events $\{\text{Tree}^+[v]\}_{v\in T}$ are defined on disjoint subsets of $\mathbb{T}_K$, and are therefore independent; they are also identically distributed. The same is true for $\{\text{Tree}^+[v,w]\}_{v,w\in T}$ for disjoint pairs $\{v,w\}$. The following is essentially the same as Definition 2.1, with the only difference being that the here we define the graph $\mathcal{T}^{+,l}$ on $\mathbb{T}_K$, whereas before we defined the same random graph on S_i.

Definition 2.4. *Let $\mathcal{T}^{+,l}$ be the union of all subgraphs H of $\mathbb{T}_K$ that are isomorphic to $\mathbb{T}_l$ with $\sigma_x = +1, \forall x \in H$.*

The next proposition estimates the probability that none of the vertices of a given set Λ belong to any random $(K-1)$-ary tree of spin $+1$ (see Definition 2.3). This proposition is a main ingredient in the proof of Theorem 1.1.

Proposition 2.1. *For any $\lambda \in (0,1), \exists \theta_\lambda \in (0,1)$ such that for $\theta \geq \theta_\lambda$ and any deterministic finite nonempty subset Λ of $\mathbb{T}_K$,*

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq \lambda^{|\Lambda|}. \tag{2.3}$$

Proof. Let T be the minimal spanning tree containing all the vertices of Λ. We will call the vertices of Λ the **special** vertices of T. Note that all the leaves of T are special vertices.

We first suppose $|\Lambda| \geq 2$; the case $|\Lambda| = 1$ will be handled at the end of the proof. By Lemma A.6 from the Appendix, there exist constants $\epsilon_1, \epsilon_2 \in (0, \infty)$ depending only on K, such that for each such tree T, one or both of the following is valid:

a) there are at least $\epsilon_1 |\Lambda|$ leaves v in T, with the events $\{\text{Tree}^+(v)\}$ mutually independent, and/or

b) there are at least $\frac{1}{2}\epsilon_2 |\Lambda|$ edges having endpoints v, w in T with v a good special 2-point, and the events $\{\text{Tree}^+(v, w)\}_{v,w}$ mutually independent.

Let us first suppose that a) holds. We claim that, for v any leaf of T,

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq [1 - \mu_\theta(\text{Tree}^+[v])]^{\epsilon_1 |\Lambda|}. \tag{2.4}$$

The claim follows from a string of inclusions. First,

$$\{\Lambda \cap \mathcal{T}^{+,K-1} = \oslash\} \subseteq \bigcap_{v \in T, v \text{ leaf of } T} \{v \notin \mathcal{T}^{+,K-1}\}. \tag{2.5}$$

But if v is a leaf of T, then

$$\{v \notin \mathcal{T}^{+,K-1}\} \subseteq \text{Tree}^+[v]^c, \tag{2.6}$$

so that

$$\bigcap_{v \in T, v \text{ leaf of } T} \{v \notin \mathcal{T}^{+,K-1}\} \subseteq \bigcap_{v \in T, v \text{ leaf of } T} \text{Tree}^+[v]^c. \tag{2.7}$$

Labeling $\epsilon_1|\Lambda|$ of the leaves in a) as v_j, we restrict the above intersection to the leaves v_j of T, so that

$$\bigcap_{v \in T, v \text{ leaf of } T} \text{Tree}^+[v]^c \subseteq \bigcap_{j=1}^{\epsilon_1|\Lambda|} \text{Tree}^+[v_j]^c. \tag{2.8}$$

Since the events $\text{Tree}^+[v_j]$ are mutually independent,

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq \prod_{j=1}^{\epsilon_1|\Lambda|} \mu_\theta(\text{Tree}^+[v]^c), \tag{2.9}$$

implying the claim.

Alternatively, suppose that b) holds. Now we claim that

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq [1 - \mu_\theta(\text{Tree}^+[v, w])]^{\frac{1}{2}\epsilon_2|\Lambda|}, \tag{2.10}$$

where v is a good special 2-point of T and w is one of v's neighbors. This claim also follows from a string of inclusions. First,

$$\{\Lambda \cap \mathcal{T}^{+,K-1} = \oslash\} \subseteq \bigcap_{\substack{\{v,w\} \in T, v,w \text{ adj.} \\ \text{and } v \text{ is a good special 2-point of } T}} \{\{v, w\} \notin \mathcal{T}^{+,K-1}\}. \tag{2.11}$$

If $\{v, w\}$ are adjacent and v is a good special 2-point of T, then

$$\{\{v, w\} \notin \mathcal{T}^{+,K-1}\} \subseteq \text{Tree}^+[v, w]^c. \tag{2.12}$$

As with the proof of the previous claim, we label $\frac{1}{2}\epsilon_2|\Lambda|$ of the pairs of vertices given in b) as $\{v_j, w_j\}$. Then

$$\{\Lambda \cap \mathcal{T}^{+,K-1} = \oslash\} \subseteq \bigcap_{j=1}^{\frac{1}{2}\epsilon_2|\Lambda|} \text{Tree}^+[v_j, w_j]^c. \tag{2.13}$$

The second claim follows from the mutual independence of the events $\text{Tree}^+[v_j, w_j]$. The two claims imply (2.3) for $|\Lambda| \geq 2$ by taking

$$\lambda > \lambda^*(\theta) = \min\left\{(1 - \mu_\theta(\text{Tree}^+[v])^{\epsilon_1}, (1 - \mu_\theta(\text{Tree}^+[v, w])^{\frac{1}{2}\epsilon_2}\right\}, \tag{2.14}$$

and using Lemma A.3 of Appendix A.1.

If $|\Lambda| = 1$, suppose the only vertex in Λ is 0, a distinguished vertex. Then

$$\mu_\theta(0 \notin \mathcal{T}^{+,K-1}) = 1 - K\text{Tree}^+[0] \tag{2.15}$$

with $\text{Tree}^+[0]$ is defined as in Appendix A.1. Then (2.3) follows in this case by taking $\lambda > \lambda^*(\theta) = 1 - K\text{Tree}^+[0]$ and using Equation (A.4) and Lemma A.1. This completes the proof.

□

Chapter 3

Proofs of results on tree-related graphs

In this chapter we present the proofs of Theorems 1.1 and 1.2. To prove these theorems we study the connected components of $S^\infty \setminus \mathcal{T}$ as a subgraph of S^∞, and show that if θ is close enough to 1 these connected components are finite almost surely. We will show that each of these finite connected components of -1 vertices shrinks and is eliminated in finite time, leading to fixation of all vertices to $+1$.

Definition 3.1. *For any $x \in S^\infty$, D_x is the connected component of x in $S^\infty \setminus \mathcal{T}$: D_x is the set of vertices $y \in S^\infty$ s.t. $x \overset{S^\infty\setminus\mathcal{T}}{\leftrightarrow} y$, i.e., there exists a path $(x_0 = x, x_1, \ldots, x_N = y)$ in S^∞ with every $x_j \notin \mathcal{T}$.*

Proposition 3.1. *Given K, there exists $\theta_* < 1$ such that for $\theta > \theta_*$, $S^\infty \setminus \mathcal{T}$ is a union of almost surely finite connected components.*

Proof. It suffices to show that D_0 is finite almost surely, where 0 is a distinguished vertex in S^∞. Since $\mathbb{E}_\theta\left[|D_0|\right] < \infty$ implies $D_0 < \infty$ a.s., it suffices to show $\mathbb{E}_\theta\left[|D_0|\right] < \infty$.

Let γ_N represent any site self-avoiding path in S^∞ of length $|\gamma_N| = N \geq 0$ starting at 0, then by standard arguments

$$\mathbb{E}_\theta\left[|D_0|\right] \leq \sum_{N=0}^{\infty} \sum_{\gamma_N, |\gamma_N|=N} \mathbb{P}(\gamma_N \in D_0), \tag{3.1}$$

where by $\gamma_N \in D_0$ we mean that all the vertices of γ_N belong to D_0.

To show the sum is finite we need to bound $\mathbb{P}(\gamma_N \in D_0)$. Suppose the vertex set of γ_N is $\Lambda_1 \cup \ldots \cup \Lambda_J$, where for each $1 \leq i \leq J$, Λ_i is a nonempty subset of S_{l_i} for some $l_i \in \mathbb{Z}$ with the l_i distinct. We now apply Proposition 2.1 to Λ_i in each of the layers S_{l_i}, which are isomorphic to $\mathbb{T}_K$. This shows that for any $\lambda \in (0,1), \exists \theta_\lambda \in (0,1)$ such that for $\theta \geq \theta_\lambda$,

$$\mathbb{P}_\theta(\Lambda_i \cap \mathcal{T}_{l_i}^{+,K-1} = \oslash) \leq \lambda^{|\Lambda_i|}. \tag{3.2}$$

Since the Λ_i are subsets of distinct levels S_{l_i} of S^∞, the events $\{\Lambda_i \cap \mathcal{T}_{l_i}^{+,K-1} = \oslash\}$ are mutually independent. Therefore for $\theta \geq \theta_\lambda$,

$$\begin{aligned}
\mathbb{P}_\theta(\gamma_N \in D_0) &= \mathbb{P}_\theta\left(\{\Lambda_1 \cap \mathcal{T}_{l_1}^{+,K-1} = \oslash\} \cap \ldots \cap \{\Lambda_J \cap \mathcal{T}_{l_J}^{+,K-1} = \oslash\}\right) && (3.3)\\
&= \prod_{i=1}^{J} \mathbb{P}_\theta\left(\{\Lambda_i \cap \mathcal{T}_{l_i}^{+,K-1} = \oslash\}\right) && (3.4)\\
&\leq \lambda^{|\Lambda_1| + \ldots + |\Lambda_J|} && (3.5)\\
&= \lambda^N. && (3.6)
\end{aligned}$$

Equation (3.1) and the above bound on $\mathbb{P}_\theta(\gamma_N \in D_0)$ imply

$$\mathbb{E}_\theta\left[|D_0|\right] \le \sum_{N=0}^{\infty} \lambda^N \sum_{\gamma_N, |\gamma_N|=N} 1 \tag{3.7}$$

$$= \sum_{N=0}^{\infty} \rho(N)\lambda^N, \tag{3.8}$$

where $\rho(N)$ is the number of self-avoiding paths of length N starting at 0. It is easy to see that

$$\rho(N) \le (K+2)(K+1)^{N-1}. \tag{3.9}$$

Thus

$$\mathbb{E}_\theta\left[|D_0|\right] \le (K+2)\sum_{N=0}^{\infty}(K+1)^{N-1}\lambda^N. \tag{3.10}$$

The proof is finished by choosing $\theta_* = \theta_\lambda$ for $\lambda < \frac{1}{K+1}$.

□

Proof of Theorem 1.1 for $K \ge 5$. Taking θ_* as in Proposition 3.1, $S^\infty \setminus \mathcal{T}$ is a union of almost surely finite connected components:

$$S^\infty \setminus \mathcal{T} = \bigcup_i D_i, \tag{3.11}$$

where the D_i's are nonempty, disjoint and almost surely finite with $D_i = D_{x_i}$ for some x_i.

Fix any i; it suffices to show that D_i is eliminated by the dynamics in finite time. By this we mean that there exists $T_i < \infty$ such that for any $y \in D_i, \sigma_y(t) = +1, \forall t \geq T_i$, and so the droplet D_i fixates to +1. We proceed to show this.

For any set $B \subset S^\infty$, let

$$\partial B = \{x \in B \text{ such that there is an edge } e_{xy} \in \mathbb{E}^\infty \text{ with } y \in B^c\}. \tag{3.12}$$

$\partial(D_i^c) \subset \mathcal{T}$ so $\partial(D_i^c)$ is stable with respect to the dynamics and for any $x \in \partial(D_i^c)$, $\sigma_x(0) = +1$.

Since D_i is finite it contains a longest path, $p = (z, \dots, w)$. Since p cannot be extended to a longer path, z must have $K+1$ neighbors in D_i^c. When z's clock first rings, z flips to +1 and fixates for all later times. This argument can be extended to show D_i is eliminated (i.e., the −1 vertices are all flipped to +1) by the dynamics in finite time as follows. Consider the set of vertices in D_i which have not yet flipped to +1 by some time t, and take t to infinity. Suppose this limiting set is nonempty. Since this set is finite, it contains a longest path $\tilde{p} = (\tilde{z}, \dots, \tilde{w})$. But now $K+1$ of $\tilde{z}$'s neighbors have spin +1 as $t \to \infty$, implying that $\tilde{z}$ had no clock rings in $[T, \infty)$ for some finite T. This event has zero probability of occurring, which contradicts the supposition of a nonempty limit set. □

The proof of Theorem 1.1 for $K = 3$ and 4 is slightly different than for $K \geq 5$, since for $K = 3$ (respectively, $K = 4$) the $\mathcal{T}_i$'s of Definition 2.1 are not stable with respect to the dynamics: each vertex $v \in \mathcal{T}_i$ has 2 (resp., 3) neighbors of spin +1, which is always less than a strict majority. The proof for $K = 3$ or 4 requires a different decomposition of the space S^∞ and definition of stable subsets. With this purpose in mind, we express S^∞ as

$$S^\infty = \bigcup_{i=-\infty}^{\infty} \tilde{S}_i, \tag{3.13}$$

where $\tilde{S}_i = \mathbb{T}_k \times \{2i, 2i+1\} = \{(u, j) : u \in \mathbb{T}_K,\ \text{and } j = 2i \text{ or } 2i+1\}$ (see Equation (1.1) for a comparison). We call a vertex $x = (u, 2i)$ or its partner in $\tilde{S}_i$, $\hat{x} = (u, 2i+1)$, **doubly open** if both $\sigma_x(0) = +1$ and $\sigma_{\hat{x}}(0) = +1$; this occurs with probability θ^2. We proceed by defining a set of fixed vertices in S^∞ in the spirit of Section 2.1.

Definition 3.2. *For i fixed, let $\tilde{\mathcal{T}}_i^{+,l}$ be the union of all subgraphs H of $\tilde{S}_i$ that are isomorphic to $\mathbb{T}_l \times \{2i, 2i+1\}$ such that $\forall x \in H$, x is doubly open.*

It is easy to see that $\tilde{\mathcal{T}}_i^{+,K-1}$ is stable for $K = 3$ or 4 with respect to the dynamics on S^∞. Let $\tilde{\mathcal{T}}$ denote the union of $\tilde{\mathcal{T}}_i^{+,K-1}$ across all levels $\tilde{S}_i$, i.e.,

$$\tilde{\mathcal{T}} = \bigcup_{i=-\infty}^{\infty} \tilde{\mathcal{T}}_i, \tag{3.14}$$

where $\tilde{\mathcal{T}}_i = \tilde{\mathcal{T}}_i^{+,K-1}$.

Proof of Theorem 1.1 for $K = 3$ and 4. We map one independent percolation model, $\sigma_{(u,j)}(0)$ on S^∞ with parameter θ, to another one, $\tilde{\sigma}_{(u,i)}(0)$ on S^∞ with parameter θ^2, by defining $\tilde{\sigma}_{(u,i)}(0) = +1$ (resp., -1) if $(u, 2i)$ is doubly open (resp., is not doubly open). Propositions 2.1 and 3.1 applied to $\tilde{\sigma}$ imply that Proposition 3.1

with $\mathcal{T}$ replaced by $\tilde{\mathcal{T}}$ is valid for the $\tilde{\sigma}(0)$ percolation model. The rest of the proof proceeds as in the case for $K \geq 5$. □

Proof of Theorem 1.2. The proof proceeds analogously to that of Theorem 1.1, except that the conclusion of Proposition 3.1, that $S^\infty \setminus \mathcal{T}$ almost surely has no infinite components (for θ close to 1), is replaced by an analogous result for $G \setminus \mathcal{T}_G$ with an appropriately defined $\mathcal{T}_G$. We next specify a choice of $\mathcal{T}_G$ for each of the graphs G given in Section 1.1 and leave further details (which are straightforward given the proof of Proposition 3.1) to the reader.

For $G = \mathbb{T}_K$ with *any* $K \geq 3$, we simply label $\mathcal{T}_G = \mathcal{T}^{+,K-1}$ (see Definition 2.4). For $G = S^{\text{semi}}$, $\mathcal{T}_G$ depends on K like it did for $G = S^\infty$ - i.e., for $K \geq 5$, we take

$$\mathcal{T}_G = \bigcup_{i=0}^{\infty} \mathcal{T}_i^{+,K-1}, \tag{3.15}$$

and for $K = 3$ or 4 we take

$$\mathcal{T}_G = \bigcup_{i=0}^{\infty} \tilde{\mathcal{T}}_i^{+,K-1}. \tag{3.16}$$

For $G = S_f^l$ or S_p^l with $K \geq 5$, we take

$$\mathcal{T}_G = \bigcup_{i=0}^{l-1} \mathcal{T}_i^{+,K-1}. \tag{3.17}$$

For $G = S_f^l$ or S_p^l with $K = 3$ or 4, the choice of $\mathcal{T}_G$ depends on whether l is even or odd since in the odd case the layers cannot be evenly paired. If l is even, then we take

$$\mathcal{T}_G = \bigcup_{i=0}^{\frac{l-2}{2}} \tilde{\mathcal{T}}_i^{+,K-1}. \tag{3.18}$$

For l odd (and ≥ 3), we pair off the first $l-3$ layers and then use the final 3 layers to define $\tilde{\tilde{\mathcal{T}}}^{+,K-1}$ in which the use of doubly open sites for $\tilde{\mathcal{T}}^{+,K-1}$ is replaced by **triply open** sites; then we take

$$\mathcal{T}_G = \left(\bigcup_{i=0}^{\frac{l-3}{2}} \tilde{\mathcal{T}}_i^{+,K-1} \right) \cup \tilde{\tilde{\mathcal{T}}}^{+,K-1}. \tag{3.19}$$

□

Chapter 4

Proofs of results on $\mathbb{Z}^d$

In this chapter we prove the theorems stated in Section 1.3.1. We present the proofs in order of difficulty and only sketch the most general case, Theorem 1.6. Though Theorem 1.4 is a special case of Theorem 1.5, we present here two different proofs, namely a simple proof of Theorem 1.4 that only relies on arguments specific to the zero-temperature Ising model with Glauber dynamics and a proof of Theorem 1.5 that relies on Bootstrap Percolation arguments. Throughout this chapter we let $B_L = [-L, L]^d$ be the cube of side length $2L + 1$ centered at the origin, and $B_L(x) = x + [-L, L]^d$ be the translated cube centered at $x \in \mathbb{Z}^d$.

4.1 Proof of Theorem 1.4

The proof of Theorem 1.4 is based on the following lemma.

Lemma 4.1. *For any $\epsilon > 0, L < \infty$, there exists $T_L < \infty$ such that*

$$\mathbb{P}(\textit{at some time } t \in [0, T_L], \sigma'(t)|_{B_L} \equiv +1) > 1 - \epsilon. \tag{4.1}$$

Proof. Camia, de Santis, Newman [2] showed that, for any $\epsilon > 0, L < \infty$, there exists $T_L < \infty$ such that

$$\mathbb{P}_\theta(\text{at some time } t \in [0, T_L], \sigma(t)|_{B_L} \equiv +1) > 1 - \epsilon. \tag{4.2}$$

A comparison of the processes $\sigma'(t)$ and $\sigma(t)$ shows the same is true for $\sigma'(t)$. This uses a natural coupling between σ and σ' in which $\sigma'(t) \geq \sigma(t)$. We do not present this coupling in detail (we give the full proof of Theorem 1.5, which generalizes Theorem 1.4, in the next section), but note that $\sigma'(0) \geq \sigma(0)$, since when $\rho^+ > 0$, for any $x \in \mathbb{Z}^2$,

$$\mathbb{P}(\sigma'_x(t) = -1) < \mathbb{P}(\sigma_x(t) = -1). \tag{4.3}$$

If B_L is fully occupied in σ by time T_L, it is also fully occupied in σ'. $\square$

Proof of Theorem 1.4. For any $L \geq 1$, the probability that each box B_L has all four corners frozen plus is

$$\mathbb{P}(B_L \text{ has all corners frozen plus}) = (\rho^+)^4 > 0. \tag{4.4}$$

By the Law of Large Numbers, almost surely infinitely many boxes B_L have this property. Since both events $\{B_L \text{ has all corners frozen plus}\}$ and, by Lemma 1.4, {at some time $t \in [0, T_L], \sigma'(t)|_{B_L} \equiv +1$} occur with probability approaching 1 as $L \to \infty$, so does their intersection.

If for some L, B_L has all four corners frozen plus and $\sigma'(t_0)|_{B_L} \equiv +1$ at some time t_0, then $\sigma'(t)|_{B_L} \equiv +1$ for all $t \geq t_0$. To see this note that, after time t_0, every vertex in B_L (except the frozen corners) has at least 3 neighbors whose spin value is $+1$, and thus cannot flip. Thus almost surely the origin fixates to $+1$. Of course, the same arguments apply to any translated cube $B_L(x) = B_L + x$, proving the result.

□

4.2 Proofs of Theorems 1.5 and 1.6

4.2.1 Bootstrap percolation

Following Section 2 of Fontes, Schonmann, Sidoravicius [7] we describe the bootstrap percolation process that assigns configurations $\{u, s\}^{\mathbb{Z}^d}$ to a subset of $\mathbb{Z}^d$; here u represents the *unstable* spin and s represents the *stable* spin at a vertex.

Definition 4.1. *The d-dimensional* $(u \to s)$ *bootstrap percolation process with threshold* γ*, defined in a finite or infinite volume* $\Lambda \subseteq \mathbb{Z}^d$*, starting from the initial configuration* $\eta_0 \in \{u, s\}^\Lambda$ *is a cellular automaton which evolves in discrete time* $t = 0, 1, 2, \ldots$ *and such that at each time unit* $t \geq 1$ *the current configuration is updated according to the following rules. For each* $x \in \Lambda$*,*

1 If $\eta_{t-1}(x) = s$*, then* $\eta_t(x) = s$*.*

2 If $\eta_{t-1}(x) = u$*, and at time* $t - 1$ *the vertex* x *has at least* γ *neighbors in* Λ *in* *state s, then $\eta_t(x) = s$; otherwise the spin at vertex x remains unchanged, i.e. $\eta_t(x) = u$.*

We will consider this process with threshold $\gamma = d$, as its evolution is close to our coarsening dynamics and assume the initial configuration to be chosen from an independent Bernoulli product measure $P(\eta_0(x) = s) = p$, for p small, on $\Lambda = \mathbb{Z}^d$.

Definition 4.2. *A configuration $\eta \in \{u, s\}^\Lambda$ **internally spans** a region $B_L(x) \subset \Lambda$, if the bootstrap percolation restricted to $B_L(x)$ started from $\eta_0 = \eta|_{B_L}$, ends up with all vertices of B_L in state s. We will denote $\eta|_{B_L}$ by η_L.*

The following proposition, an immediate consequence of results of Schonmann ([16]), provides a key ingredient to our proofs.

Proposition 4.2. [Schonmann] *If $p > 0$, then*

$$\lim_{L\to\infty} P(B_L \textit{ is internally spanned}) = 1. \tag{4.5}$$

4.2.2 Preliminary lemmas

We will make a comparison between the bootstrap percolation process on $\mathbb{Z}^d$ and our process σ' by mapping frozen plus spins to stable spins s, and all other spins to unstable spins u. The frozen minus spin is mapped to u as, even when $\rho^- > 0$, we will only bootstrap finite regions with no frozen minus vertices. Indeed, the following definitions will be applied in Lemmas 4.3 and 4.4 only to cubes with no frozen minus vertices to conclude that at some time t all spins in the cube will be $+1$.

We will say that a region $B_L(x)$ contains a *spanning subset* of frozen plus vertices if the configuration obtained by the above mapping internally spans $B_L(x)$ in the sense of bootstrap percolation.

Definition 4.3. *A cube, say $B_L(x)$, is* ***entrapped*** *if it contains a spanning subset of frozen plus vertices. It is* ***captured*** *if it is entrapped and all 2^d corners are frozen plus. It is M*-***captured*** *($M \in \{0, 1, \ldots, L\}$) if it is entrapped and for each of the 2^d corners $C^i, (i = 1, \ldots, 2^d)$, and each coordinate direction $j = 1, \ldots, d$ there is a frozen plus vertex within the cube of the form $C^{i,j} = C^i + me^j$ with $|m| \leq M$ (where $e^1, \ldots, e^d$ are the standard basis vectors of $\mathbb{R}^d$) – see Figure 4.1. Note that captured is the same as 0-captured.*

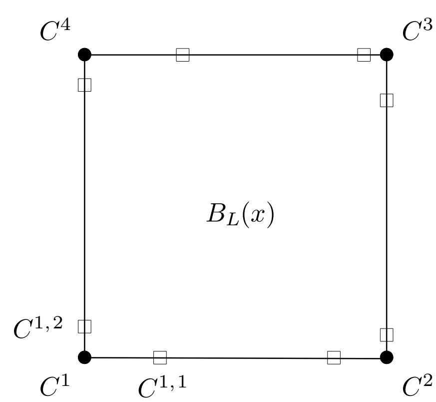

Figure 4.1: $B_L(x)$ is M-captured

The motivation for the definition of M-captured is that it will be used (in Lemma 4.8) to guarantee that with high probability most of the vertices in the cube B_L (at least for $1 \ll M \ll L$) will become fixed plus. Note that, although in the above definition $M = L$ is allowed, in Lemmas 4.7 and 4.8 we require $M < L$. But Lemma 4.6 shows that one can indeed choose $1 \ll M \ll L$. Lemmas 4.3, 4.4 and 4.5, presented next, will be used in the proof of Theorem 1.5.

Lemma 4.3. *Given L and a spanning subset η_L of frozen plus vertices that makes B_L entrapped, consider the σ' process in $\mathbb{Z}^d$ with initial spins in η_L taken as frozen plus and all others in $\mathbb{Z}^d \setminus \eta_L$ taken as minus but not frozen. Then*

$$\mathbb{P}(\text{at some time } t \in [0,1], \sigma'(t)|_{B_L} \equiv +1) > 0. \tag{4.6}$$

Proof. The proof follows easily from the following observation: since η_L is a spanning subset of B_L, the bootstrap percolation process (with frozen plus vertices playing the role of s-vertices) occupies all vertices of B_L in a finite number of steps. Since the threshold $\gamma = d$, we can, with a small but positive probability, arrange the clock rings for $t \in (0,1)$ and tie-breaking coin tosses of the coarsening dynamics to mimic the dynamics of bootstrap percolation (in a much longer discrete time interval). Thus $\sigma'(t)|_{B_L} \equiv +1$ for some $t \in [0,1]$ with positive probability. □

The next lemma strengthens the last one by showing that, if we allow the process to run until a large time, then with probability close to one all the vertices of B_L will flip to $+1$ before that time.

Lemma 4.4. *For any $\epsilon > 0, L < \infty$, there exists $T_L < \infty$ such that, if the frozen plus vertices in B_L are spanning and there are no frozen minus vertices in B_L, then* *for any choice of initial spins in* $\mathbb{Z}^d$ *(other than the frozen plus vertices in* B_L*)*

$$\mathbb{P}(\textit{at some time } t \in [0, T_L], \sigma'|_{B_L} \equiv +1) > 1 - \epsilon. \tag{4.7}$$

Proof. Pick a spanning subset η_L of frozen plus vertices that makes B_L entrapped. By Lemma 4.3 it follows that there is an $\epsilon' > 0$ such that, for any m and $\sigma'(m)$ (consistent with the frozen vertices in B_L),

$$\mathbb{P}(\text{at some time } t \in [m, m+1], \sigma'(t)|_{B_L} \equiv +1 | \sigma'(m)) \geq \epsilon'. \tag{4.8}$$

Let T_L be a large integer and, by repeatedly applying the last inequality to time intervals of the form $[m, m+1], 0 \leq m < T_L$, we have

$$\begin{aligned}\mathbb{P}(\text{at some time } t \in [0, T_L], \sigma'|_{B_L} \equiv +1) &\geq 1 - (1 - \epsilon')^{T_L(\eta_L)} && (4.9)\\ &> 1 - \epsilon && (4.10)\end{aligned}$$

providing $T_L(\eta_L)$ is sufficiently large (depending on ϵ' and hence on η_L). Finally, choose T_L to be the minimum of $T_L(\eta_L)$ over all finitely many possible spanning η_L's. □

Lemma 4.5. *If* $\rho^+ > 0$*, then*

$$\lim_{L \to \infty} \mathbb{P}(B_l \textit{ is captured for some } l \leq L) = 1. \tag{4.11}$$

Proof. Pick a sequence of increasing box sizes L_i such that $L_1 < L_2 < \ldots$, $L_i \to \infty$ so that, by Proposition 4.2 with $p = \rho^+$,

$$\mathbb{P}(B_{L_i} \text{ is not entrapped}) < \frac{1}{i^2}. \tag{4.12}$$

Thus by the Borel-Cantelli Lemma, almost surely, all but finitely many boxes B_{L_i} are entrapped. Now the probability that each box B_{L_i} has all corners frozen plus equals

$$\mathbb{P}(B_{L_i} \text{ has all corners frozen plus}) = (\rho^+)^{2^d}. \tag{4.13}$$

By the Law of Large Numbers, almost surely infinitely many boxes B_{L_i} have this property. But the intersection of two almost sure events is an almost sure event, therefore infinitely many boxes B_{L_i} are captured, which implies the conclusion of the lemma. □

The remaining lemmas and definition will be used in the proof of Theorem 1.6.

Lemma 4.6. *If $\rho^+ > 0$, then*

$$\lim_{M,L\to\infty} \mathbb{P}(B_L \textit{ is } M\textit{-captured}) = 1, \tag{4.14}$$

where M, L tend to infinity with no restriction other than $M \leq L$.

Proof. By Proposition 4.2, as in the proof of Lemma 4.5,

$$\lim_{L\to\infty} \mathbb{P}(B_L \text{ is entrapped}) = 1. \tag{4.15}$$

Now for any fixed L and $M \leq L$, let $A_{L,M}$ denote the event that there exist frozen plus spins within distance M from each of the 2^d corners of B_L in every one of the d coordinate directions, as in Definition 4.3. Thus the event that B_L is M-captured is the intersection of $A_{L,M}$ with the event that B_L is entrapped. The probability of the event that any specific collection of M vertices contains no frozen plus spins is $(1-\rho^+)^M$. It is then easy to see that

$$\mathbb{P}(A_{L,M}) \geq ([1-(1-\rho^+)^M]^d)^{2^d}, \tag{4.16}$$

and this tends to 1 as $M \to \infty$ (for fixed d). Since both the events $\{B_L \text{ is entrapped}\}$ and $A_{L,M}$ occur with probability approaching 1 as $L, M \to \infty$, so does their intersection, which completes the proof.

□

Definition 4.4. *Let B be a box of the form $B_L + x$. We say that B is M-**good** if B is M-captured and contains no frozen minus vertices. We define $B[M]$, the M-**trimming** of B as*

$$B \setminus \left(\bigcup_{i=1}^{2^d} \bar{C}^i(M) \right), \tag{4.17}$$

where each $\bar{C}^i(M)$ is a cube within B containing exactly M^d vertices including the i^{th} corner of B - see Figure 4.2 for the case$d = 2$.

Lemma 4.7. *Given $\rho^+ > 0$ and any $\epsilon > 0$, there exist $L < \infty$ and $M < L$ such that for all sufficiently small ρ^- (depending on d, L, M, ϵ and ρ^+),*

$$\mathbb{P}(B_L \text{ is } M\text{-good}) > 1 - \epsilon. \tag{4.18}$$

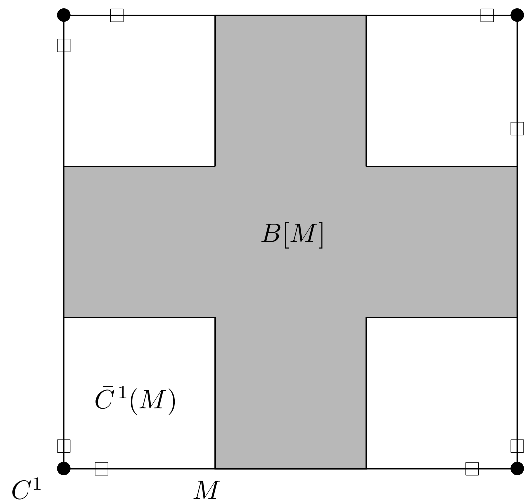

Figure 4.2: $B[M]$ (gray), the M-trimming of an M-good box B

Proof. By Lemma 4.6, we may choose M large enough and then L large so that

$$\mathbb{P}(B_L \text{ is M-captured}) \geq 1 - \frac{\epsilon}{2}. \tag{4.19}$$

We may also pick ρ^- small enough so that the probability that B_L contains any frozen minus vertices is less than $\frac{\epsilon}{2}$. Thus by the FKG inequality,

$$\begin{aligned} \mathbb{P}(B_L \text{ is M-good}) &\geq \left(1 - \frac{\epsilon}{2}\right)\left(1 - \frac{\epsilon}{2}\right) && (4.20) \\ &\geq 1 - \epsilon. && (4.21) \end{aligned}$$

□

Lemma 4.8. *If B_L is M-good with $M < L$, then with probability one all the vertices in $B[M]$ are (eventually) fixed plus.*

Proof. Since B_L is M-good, by Lemma 4.4 almost surely for some time t_0, $\sigma'(t_0)|_{B_L} \equiv +1$. In this case, if $M > 0$, $\sigma'(t)|_{B_L}$ need not stay identically $+1$ for $t > t_0$ because vertices near the corners can change from $+1$ to -1. But a moment's thought shows that the only vertices near a corner C^i that can change to -1 are those in a subset of the cube $\bar{C}^i(M)$ - see Definition 4.4. This is because the frozen plus vertices $C^{i,j}$ (from Definition 4.3) protect against the flipping of plus vertices beyond a rectangular parallelepiped contained in $\bar{C}^i(M)$. Note that because $M < L$, every vertex in $B_L[M]$ has at least $d + 1$ neighbors in $B_L[M]$. E.g., for $d = 2$ (see Figure 4.2), $B_L[M]$ is the union of two rectangles each of width at least three (hence at least two) so that every vertex in a rectangle has at least three neighbors in the rectangle. Thus $B_L[M]$ will have $\sigma'(t)|_{B_L[M]} \equiv +1$ for all $t \geq t_0$. Of course the same arguments apply to any translated cube $B_L(x) = B_L + x$ and to $B_L[M](x) = B_L[M] + x$. □

4.2.3 Proofs of main results

The first of the two theorems follows easily from Lemmas 4.4 and 4.5, as follows.

Proof of Theorem 1.5. If the box B_L has all 2^d corners frozen plus, and if at some time t_0, $\sigma'(t_0)|_{B_L} \equiv +1$, then $\sigma'(t)|_{B_L} \equiv +1$ for all $t \geq t_0$. This is because after time t_0 every vertex in B_L (other than the corners whose spin value is frozen) will have at least $d+1$ plus neighbors, so it won't flip sign. If B_L is also captured, then by Lemma 4.4, with probability one, $\sigma'(t)|_{B_L} \equiv +1$ will occur for some t. Now by Lemma 4.5, with probability one, B_L will be captured for some L and so $\sigma'(0)$ will fixate to $+1$. But the same argument can be translated to x and $B_L(x)$ for any $x \in \mathbb{Z}^d$. □

Proof of Theorem 1.6. Tile $\mathbb{Z}^d$ with cubes $C_L(y) = B_L((2L+1)y), y \in \mathbb{Z}^d$. Call y **good** if $C_L(y)$ is M-good. For disjoint cubes the events of being M-good are independent, so the collection of good y's or bad (i.e., not good) y's defines an independent percolation model. But we replace the usual nearest neighbor graph on $\mathbb{Z}^d$ with a graph G, still with vertex set $\mathbb{Z}^d$, but where y_1, y_2 are neighbors (with edge $\{y_1, y_2\}$) if $\|y_1 - y_2\|_\infty = 1$, so every y has $3^d - 1$ neighbors. This corresponds (e.g., for $d = 3$), to $C_L(y_1)$ and $C_L(y_2)$ sharing either a face or an edge or just a corner. The reason for changing the notion of neighbor is that a standard (i.e., using only nearest neighbor edges) cluster of fixed minus and flipper vertices can extend beyond the standard cluster of M-bad (i.e., not M-good) cubes into the G-cluster of M-bad cubes and beyond into the G-closure of that G-cluster.

Let $\mathcal{C}$ denote the cluster of bad vertices containing the origin in this independent site percolation model of bad sites in G, and let $\bar{\mathcal{C}}$ denote its closure (where we add good sites that are neighbors of bad sites in $\mathcal{C}$). The relevance for our σ' process is that the G-cluster $\mathcal{C}^*$ containing the origin, consisting of fixed minus together with flipper vertices, satisfies

$$\mathcal{C}^* \subseteq \bigcup_{y \in \bar{\mathcal{C}}} C_L(y). \tag{4.22}$$

This follows from Lemma 4.8. By standard percolation arguments, there is some $p^* > 0$ (e.g., $1/(3^d - 1)$ since $3^d - 1$ is the number of neighbors of any vertex in G), such that, if

$$\mathbb{P}(y \text{ is bad}) = \mathbb{P}(B_L \text{ is not M-good}) < p^*, \tag{4.23}$$

then there is no percolation of bad sites and $\mathbb{E}(|\bar{\mathcal{C}}|) < \infty$. To finish the proof we use Lemma 4.7 to choose ρ^- small enough so that inequality (4.23) is valid, and finally note that by inequality (4.22),

$$|\mathcal{C}^*| \leq |\bar{\mathcal{C}}|(2L+1)^d. \tag{4.24}$$

□

Appendix A

A.1 Galton-Watson lemmas

The goal of this section is to show that the quantity

$$\lambda^*(\theta) = \min\left\{(1-\mu_\theta(\mathrm{Tree}^+[v])^{\epsilon_1}, (1-\mu_\theta(\mathrm{Tree}^+[v,w])^{\frac{1}{2}\epsilon_2}\right\}, \tag{A.1}$$

which appears at the end of the proof of Proposition 2.1, converges to 0 as $\theta \to 1$. Here v is a leaf of a subtree T of $\mathbb{T}_K$, $\{v, w\}$ is a pair of adjacent vertices of T such that v is a good 2-point (as in Definition 2.3), and ϵ_1, ϵ_2 are fixed constants.

For this purpose we consider independent site percolation on $\mathbb{T}_K$ and let $a_1, a_2, \ldots, a_K$ denote the K neighbors of 0, a distinguished vertex in $\mathbb{T}_K$. We associate to each a_i a tree $A_0[a_i]$ (see Definition 2.2), for $i = 1, \ldots, K$ – see Figure A.1.

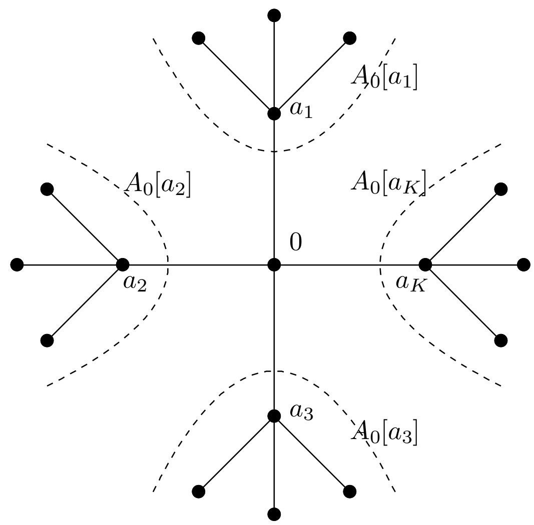

Figure A.1: K-ary tree with labeled vertices and branches

Let T be a subtree of $\mathbb{T}_K$ such that T contains a_1 and 0 is a leaf of T – see Figure A.2.

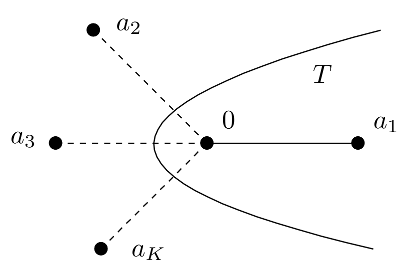

Figure A.2: T is a subtree of $\mathbb{T}_K$ and 0 is a leaf of T

Let b be one of the neighbors of a_2 (other than 0) and T' be a subtree of $\mathbb{T}_K$ containing $b, a_2, 0$ and a_1 (but not $a_3, \ldots, a_K$) such that a_2 is a good 2-point of T' – see Figure A.3.

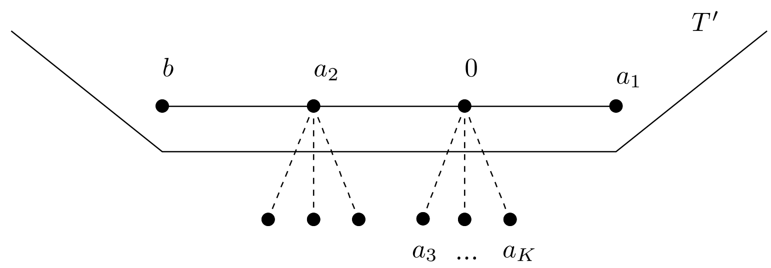

Figure A.3: T' is a subtree of $\mathbb{T}_K$ that contains $b, a_2, 0$ and a_1 such that a_2 is a good 2-point of T'

We consider the events $\text{Tree}^+[0]$ with respect to T and $\text{Tree}^+[a_2, 0]$ with respect to T' (see Definition 2.3) and estimate $\mu_\theta(\text{Tree}^+[0])$, $\mu_\theta(\text{Tree}^+[a_2, 0])$ by analyzing a related Galton-Watson process.

Definition A.1. *For any vertex $v \in \mathbb{T}_K$, let $C(v)$ denote the $+1$ spin cluster of v, that is, $C(v)$ is the set of vertices u in $\mathbb{T}_K$ such that the path from v to u (including v and u) includes only vertices w, with $\sigma_w = +1$.*

Let

$$Z_n = |v \in A_0[a_1] \cap C(a_1) : d(a_1, v) = n|, \tag{A.2}$$

where $d(a_1, v)$ represents the graph distance. $Z_0 = 1$ if and only if $\sigma_{a_1} = +1$, and in general Z_n is the number of vertices in $A_0[a_1]$ at distance n from a_1 that are in a_1's $+1$ spin cluster. Z_n is a Galton-Watson branching process with offspring distribution $\text{Bin}(K-1, \theta)$.

Let $\mathbb{T}^{\text{root}}[x]$ denote a tree with root x, such that x has coordination number $K-2$ and all the other vertices have coordination number $K-1$. The following definition is close to that of $\mathrm{Tree}^+[x]$ (see Definition 2.3 and Figure 2.3), except that here the $(K-1)$-ary tree in question is rooted.

Definition A.2. Random rooted $(K-1)$-ary trees of spin $+1$
Consider two vertices $v, v' \in \mathbb{T}_K$ such that v' is a neighbor of v. Let $\mathrm{Part}^+_{v'}[v]$ denote the event that there exists a subgraph H of $\mathbb{T}_K$ isomorphic to $\mathbb{T}^{\mathrm{root}}[v]$ and contained in $A'_v[v]$, such that for all $u \in H, \sigma_u = +1$.

Consider three vertices x, y, z such that x and z are neighbors of y. Let $\mathrm{Part}^+_{x,z}[y]$ be the event that there exists a subgraph H of $\mathbb{T}_K$ isomorphic to $\mathbb{T}^{\mathrm{root}}[x]$ and contained in $A_{x,z}[y]$ (see Figure 2.2), such that for all $u \in H, \sigma_u = +1$.

Define $\tau(\theta)$ as

$$\tau(\theta) = \mu_\theta(\mathrm{Part}^+_0[a_1]). \tag{A.3}$$

$\mathrm{Part}^+_0[a_i]$ and $\mathrm{Part}^+_0[a_j]$ are independent for $i \neq j$ by construction. The event $\mathrm{Tree}^+[0]$ is equivalent to the spin at 0 being spin $+1$ and the vertices $a_2, \ldots, a_K$ being the roots of $(K-1)$-ary trees of spin $+1$, so that

$$\mu_\theta(\mathrm{Tree}^+[0]) = \theta\, \tau(\theta)^{K-1}. \tag{A.4}$$

Lemma A.1. *$\tau(\theta) \to 1$ as $\theta \to 1$.*

Proof. The proof is a consequence of Proposition 5.30 from [15] (about occurrence of j-ary subtrees in Galton-Watson processes). □

Define $\tilde{\tau}(\theta)$ as

$$\tilde{\tau}(\theta) = \mu_\theta(\mathrm{Part}^+_{a_1,a_2}[0]). \tag{A.5}$$

The event $\mathrm{Tree}^+[a_2, 0]$ is equivalent to $\{\mathrm{Part}^+_{a_1,a_2}[0] \cap \mathrm{Part}^+_{b,0}[a_2]\}$, so that, by the independence of the events $\mathrm{Part}^+_{a_1,a_2}[0]$ and $\mathrm{Part}^+_{b,0}[a_2]$,

$$\mu_\theta(\mathrm{Tree}^+[a_2, 0]) = \tilde{\tau}(\theta)^2. \tag{A.6}$$

Lemma A.2. *$\tilde{\tau}(\theta) \to 1$ as $\theta \to 1$.*

Proof. This result follows as in the proof of Lemma A.1. □

Equations (A.4) and (A.6) imply that $\mu_\theta(\mathrm{Tree}^+[0])$ and $\mu_\theta(\mathrm{Tree}^+[a_2, 0])$ converge to 1 as $\theta \to 1$, which immediately implies:

Lemma A.3. *$\lambda^*(\theta) \to 0$ as $\theta \to 1$.*

A.2 Geometric lemmas

Let T be a finite tree with N vertices and maximal coordination number $\leq K$. $N_1 \leq N$ of T's vertices are labeled special, such that all of T's leaves are special vertices and the remaining special non-leaf vertices can have any coordination number $\leq K$. We remark that in Section 2.2, we start with $|\Lambda|$ special vertices in $\mathbb{T}_K$ and then T is the minimal subtree of $\mathbb{T}_K$ that contains all the special vertices.

Lemma A.4. *Let M_1 be the number of leaves in T, M_2 the number of 2-points (vertices with exactly two edges in T), …, M_K the number of K-points (vertices* *with exactly K edges in T); $M_1 + \ldots + M_K = N$. Then*

$$M_i \leq M_1 \tag{A.7}$$

for $i = 3, \ldots, K$.

Proof. The proof can be found, for example, as part of Theorem 8.1 in [8]. □

Definition A.3. *A* good *2-point in T is a 2-point both of whose neighbors are 2-points. A* bad *2-point is a 2-point that is not a good 2-point – see Figure A.4.*

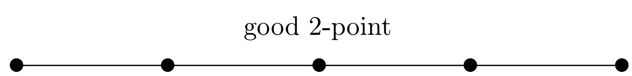

Figure A.4: A good 2-point

Lemma A.5. *There exist constants $\epsilon_1, \epsilon_2 \in (0, \infty)$, depending only on K, such that either:*

a) $M_1 \geq \epsilon_1 N_1$, and/or

b) there are at least $\epsilon_2 N_1$ special good 2-points.

Proof. By Lemma A.4

$$\sum_{i=3}^{K} M_i \leq (K-2)M_1, \tag{A.8}$$

and since $\sum_1^K M_i = N$,

$$
\begin{aligned}
(K-1)M_1 + M_2 &= M_1 + M_2 + (K-2)M_1 && \text{(A.9)} \\
&\geq \sum_{1}^{K} M_i = N. && \text{(A.10)}
\end{aligned}
$$

Thus either $(K-1)M_1 \geq \frac{N}{K}$ or $M_2 \geq \frac{N(K-1)}{K}$. In the first case, since $N \geq N_1$,

$$M_1 \geq \frac{N}{K(K-1)} \geq \frac{1}{K(K-1)} N_1, \tag{A.11}$$

and letting $\epsilon_1 = \frac{1}{K(K-1)}$ gives a).

In the second case $M_2 \geq \frac{N(K-1)}{K}$, and if a) is not valid with $\epsilon_1 = \frac{1}{K(K-1)}$, then $M_1 \leq \frac{N_1}{K(K-1)}$. To prove b) we need to count the various types of special vertices in T. The set of special vertices is comprised of:

- special good 2-points; let *Good* denote the set of such vertices,
- special bad 2-points; let *Bad* denote the set of such vertices,
- special leaves, special 3-points, ... , special K-points; let *Other* denote the set of such vertices.

Since $|\text{Good}| = N_1 - |\text{Bad}| - |\text{Other}|$, we need to upper bound $|\text{Other}|$ and $|\text{Bad}|$. By Lemma A.4,

$$
\begin{aligned}
|\text{Other}\,| &\leq M_1 + M_3 + \ldots + M_K && \text{(A.12)}\\
&\leq (K-2)M_1 && \text{(A.13)}\\
&\leq \frac{K-2}{K(K-1)}N_1. && \text{(A.14)}
\end{aligned}
$$

Now $|\text{Bad}| \leq |\{\text{all bad 2-points}\}|$ and it is easy to see that the latter is upper bounded by $M_1 + 3M_3 + \ldots + KM_K$. Thus by Lemma A.4,

$$
\begin{aligned}
|\text{Bad}\,| &\leq M_1 + 3M_3 + \ldots + KM_K && \text{(A.15)}\\
&\leq M_1(1 + 3 + \ldots + K) && \text{(A.16)}\\
&\leq \frac{1}{2}K(K-1)M_1 && \text{(A.17)}\\
&\leq \frac{1}{2}N_1, && \text{(A.18)}
\end{aligned}
$$

since $M_1 \leq \frac{N_1}{K(K-1)}$. Thus

$$
\begin{aligned}
|\text{Good}| &= N_1 - |\text{Bad}| - |\text{Other}| && \text{(A.19)}\\
&\geq N_1\left(1 - \frac{K-2}{K(K-1)} - \frac{1}{2}\right) && \text{(A.20)}\\
&= N_1\left(\frac{K^2 - 3K + 4}{2K(K-1)}\right). && \text{(A.21)}
\end{aligned}
$$

We let $\epsilon_2 = \frac{K^2-3K+4}{2K(K-1)} > 0$ and so $|\text{Good}| \geq \epsilon_2 N_1$. □

A.3 Probabilistic lemma

Consider site percolation on $\mathbb{T}_K$ distributed according to the product measure μ_θ with

$$\mu_\theta(\sigma_x = +1) = \theta = 1 - \mu_\theta(\sigma_x = -1), \forall x \in \mathbb{T}_K. \tag{A.22}$$

Let T be a finite subtree of $\mathbb{T}_K$ with $2 \leq N_1 \leq |T|$ of its vertices labeled special, such that all the leaves are special. As in Lemma A.5, in the following lemma ϵ_1 and ϵ_2 are strictly positive, finite and depend only on K. For the events $\text{Tree}^+[v]$ and $\text{Tree}^+[v,w]$, see Definition 2.3.

Lemma A.6. ***Disjoint events***

For each such tree T, one or both of the following is valid:

a) there are at least $\epsilon_1 N_1$ leaves v in T, with the events $\{\text{Tree}^+(v)\}$ mutually independent, and/or

b) there are at least $\frac{1}{2}\epsilon_2|\Lambda|$ edges having endpoints v, w in T with v a good special 2-point, and the events $\{\text{Tree}^+(v,w)\}_{v,w}$ mutually independent.

Proof. Lemma A.6.a follows from Lemma A.5.a, since for each of the $\epsilon_1 N_1$ leaves of T we can define an event $\text{Tree}^+[v]$, and these events depend on the spins of disjoint sets of vertices and are therefore mutually independent.

Otherwise, by Lemma A.5.b there are at least $N_3 = \epsilon_2 N_1$ good special 2-points in T. These are arranged into $p \geq 1$ nonempty maximal chains of adjacent vertices along T. We order the chains and let n_i denote the number of vertices in the i^{th}

chain, for $i = 1, \ldots, p$; $n_1, \ldots, n_p \geq 1$ and $n_1 + \ldots + n_p = N_3$. We also order the N_3 good special 2-points, $\{s_1, s_2, \ldots, s_{N_3}\}$, so that they are consecutively ordered in each chain.

Suppose $n_i = 1$ for some i, and the good special 2-point in this chain is s_i^*. Let w_i be one of s_i^*'s neighbors in T and consider the event $\text{Tree}^+[s_i^*, w_i]$. If $n_i = 2$, the i^{th} chain contains two adjacent special points $\{s_i^*(1), s_i^*(2)\}$ and we consider the event $\text{Tree}^+[s_i^*(1), s_i^*(2)]$. Generally for the i^{th} chain, we pair adjacent good special 2-points (other than the last if n_i is odd) so as to consider $\lfloor \frac{n_i+1}{2} \rfloor$ events $\text{Tree}^+[s_i^*(j), s_i^*(j+1)]$, where the last event is $\text{Tree}^+[s_i^*(n_i), w_i]$ if n_i is odd; these events involve disjoint sets of vertices and are therefore independent. Thus in total we can construct

$$\left\lfloor \frac{n_1 + 1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_p + 1}{2} \right\rfloor \geq \left\lfloor \frac{N_3}{2} \right\rfloor \tag{A.23}$$

mutually independent events. $\square$

Bibliography

[1] Arratia, Richard. "Site Recurrence for Annihilating Random Walks on $\mathbb{Z}^d$." *Annals of Probability* 11.3 (1983): 706-713. Print.

[2] Camia, Federico, Emilio De Santis, and Charles M. Newman. "Clusters and Recurrence in the Two-Dimensional Zero-Temperature Stochastic Ising Model." *Annals of Applied Probability* 12.2 (2002): 565-580. Print.

[3] Damron, Michael, Hana Kogan, Charles M. Newman, and Vladas Sidoravicius. "Fixation for Coarsening Dynamics in 2D Slabs." *Electronic Journal of Probability* 18.102 (2013): 1-20. Print.

[4] Damron, Michael, Hana Kogan, Charles M. Newman, and Vladas Sidoravicius. "Coarsening Dynamics in 2D Slabs." *Topics in Percolative and Disordered Systems* (2013). To appear.

[5] Cox, Theodore J., and David Griffeath. "Diffusive Clustering in the Two Dimensional Voter Model." *Annals of Probability* 14.2 (1986): 347-370. Print.

[6] Damron, Michael, Sinziana Eckner, Hana Kogan, Charles M. Newman, and Vladas Sidoravicius. "Coarsening Dynamics on $\mathbb{Z}^2$ with Frozen Vertices." In preparation.

[7] Fontes, Luis R., Roberto H. Schonmann, and Vladas Sidoravicius. "Stretched Exponential Fixation in Stochastic Ising Models at Zero Temperature." *Communications in Mathematical Physics* 228.3 (2002): 495-518. Print.

[8] Grimmett, Geoffrey. Percolation. New York, Berlin: Springer, 1999. Print.

[9] Nanda, Seema, Charles M. Newman, and Daniel L. Stein. "Dynamics of Ising Spins Systems at Zero Temperature." *On Dobrushin's way (From Probability Theory to Statistical Mechanics)*. Ed. R.A. Milnos, Senya Shlosman, and Yu. M. Suhov. American Mathematics Society Translations Series 2, 2000. 183-194. Print.

[10] Newman, Charles M., and Daniel L. Stein. "Zero-Temperature Dynamics of Ising Spin Systems Following a Deep Quench: Results and Open Problems." *Physica A* 279 (2000): 159-168. Print.

[11] Howard, Douglas. "Zero-Temperature Ising Spin Dynamics on the Homogeneous Tree of Degree Three." *Journal of Applied Probability* 37.3 (2000): 736-747. Print.

[12] Kanoria, Yashodhan, and Andrea Montanari. "Majority Dynamics on Trees and the Dynamics Cavity Method." *Annals of Applied Probability* 21.5 (2011):1694-1748. Print.

[13] Krapivsky, Pavel L., Sidney Redner, and Eli Ben-Naim. *A Kinetic View of Statistical Physics*. Cambridge University Press, 2010. Print.

[14] Liggett, Thomas. *Interacting Particle Systems*. New York, Berlin: Springer, 1985. Print.

[15] Lyons, Russell, and Yuval Peres. *Probability on Trees and Networks*. http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html.

[16] Schonmann, Roberto. "On the Behavior of Some Cellular Automata Related to Bootstrap Percolation." *Annals of Probability* 20.1 (1992): 174-193. Print.

[17] Spirin, Victor, Pavel L. Krapivsky, and Sydney. Redner, "Freezing in Ising Ferromagnets." *Physical Review E* 65.016119 (2001): 1-9. Print.